

Rational Homotopy Theory and Derived Commutative Algebra

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Contents

Abstract and pre-introduction	5
Acknowledgements and background references	10
Chapter 1. Introduction	13
Rational Homotopy Theory in the sense of Sullivan	13
Derived Commutative Algebra and RHT	15
The Gorenstein condition	16
PL compactly supported forms in characteristic zero	19
A formality problem	23
Chapter 2. Rational Homotopy Theory in the sense of Sullivan	29
Chapter 3. Derived Commutative Algebra and RHT	37
Homotopy invariant formulations of conditions on rings	41
A generalisation of Regular Spaces	50
Chapter 4. The Gorenstein condition	63
Gorenstein Duality and the Local Cohomology Spectral Sequence	65
Examples of Infinite Dimensional Gorenstein Spaces	69
Chapter 5. PL Compactly supported forms in characteristic zero	77
PL bump functions	80
Two contravariant Mayer-Vietoris Sequences	82
The PL compactly supported de Rham Theorem	87
Chapter 6. Rational Homotopy Theory in the sense of Quillen	93
The Homotopy Lie Algebra	93
Chapter 7. A formality problem	101

Setting	101
Cofibrant replacement and Postnikov towers	104
Square-zero extensions, k-invariants and André-Quillen cohomology	108
Exterior algebras over wedges of spheres	115
Bibliography	127

Abstract and pre-introduction

Homotopy theory is now a rather old field by modern mathematical standards. Its earliest beginnings are perhaps in the late 19th century and the study of certain complex path integrals, and the realisation that certain quantities remained invariant under “continuous deformations” of the chosen path. The earliest rigorous treatment of what such “continuous deformations” should be comes from Jordan in the late 19th century, who gives the definition of what we now call a homotopy. Poincaré then advanced this in his “*Analysis situs*” in 1895, where he introduced the fundamental group and a notion of homology. The subject of Algebraic Topology then really began, and many advances were made in the abstract study of spaces with this new algebraic toolkit, such as Brouwer’s topological invariance of dimension. However it was not until 1932 that the first generalisation of the fundamental group to higher dimensions was given. Čech gave the original definition, which was rediscovered by Hurewicz in 1935, along with his eponymous theorem. The higher homotopy groups were quickly seen to be remarkably intuitive in their definition, however, it was quickly realised that they were, even for spheres, extremely difficult to compute. A way of simplifying the situation somewhat was arrived at by mathematicians such as Freudenthal and Whitehead, and this method was the beginnings of what is today called stable homotopy theory, and has broadly become the area in which much of today’s research in homotopy theory lives. However, even with this simplification, the stable homotopy groups of spheres are still immensely complicated, and to this day there is still no hope of understanding them in their totality (the stable homotopy groups of S^0 are currently known up to dimension 64). Around the same time of the 20th century, Serre developed what became known as “mod C theory”, a result of which showed that the homotopy groups of spheres are almost all finite. This was a remarkable and celebrated result, which eventually led to the development of the subject of this thesis, rational homotopy theory (RHT). This took many years to be envisioned, and finally with the work of Quillen, building on his 1967 work “Homotopical Algebra”, he published in 1969 his paper “Rational Homotopy

Theory”, which gave the foundation of an entirely new area of study in homotopy theory, and also developed much of the abstract language in which modern homotopy theory is phrased, such as model categories.

The main idea of rational homotopy theory is to study the homotopy theory of spaces “modulo torsion”, in the sense that rather than considering homotopy groups, you consider homotopy groups after tensoring with \mathbb{Q} , and rather than asking when spaces are homotopy equivalent, you ask when they are rationally homotopy equivalent (in a sense which will be made clear). As Serre had already shown that the homotopy groups of spheres become almost completely trivial after tensoring with \mathbb{Q} , it was suspected that rational homotopy theory in general should be much simpler than ordinary homotopy theory, which was indeed made precise in Quillen’s 1969 paper [36], where he gives a completely algebraic description of the rational homotopy category of (simply connected) spaces in terms of a homotopy category of certain differential graded Lie algebras. However, there was still the outstanding issue of computability. That is, if the theory is much simpler, one would hope that rational homotopy groups might actually be computable, at least under some conditions. A solution to this issue of computability was given by Sullivan in 1977 in [41], where he gives an alternative (in fact Koszul dual) description of the same rational homotopy category of (simply connected, \mathbb{Q} -finite) spaces studied by Quillen, this time showing it to be equivalent to the homotopy category of certain commutative differential graded algebras (CDGAs). These CDGAs were found to be much easier to compute with than differential graded Lie algebras, and in particular, Sullivan also showed how the rational homotopy groups of (formal) spaces could be read off from a minimal model for their rational cohomology ring, in effect making rational homotopy groups computable for an enormous class of examples. An important step taken in this paper was also the discovery of a commutative model for cochains on a space in characteristic zero; Sullivan’s model takes the form of a complex of piecewise linear polynomial differential forms on a simplicial set (analogous to differential forms on a manifold).

The work of Quillen was not just fruitful in its introduction of the new field of rational homotopy theory, but was also groundbreaking in its generalisation of what is considered a homotopy theory; Quillen's theory of model categories gave axioms which (almost completely in their original form) are still generally accepted today as good and reasonable axioms for a homotopy theory. Quillen and Sullivan used these to study the homotopy theories of simplicial sets, DG Lie algebras and DGAs respectively, but since, mathematicians have used the language of model categories to study the homotopy theories of spectra, schemes, coherent sheaves, stacks, and even just sets, or more ambitiously the homotopy theory of homotopy theories!

This close relationship between rational homotopy theory and commutative algebra led algebraists to be very interested in the subject, in particular Avramov and Foxby, who further studied the correspondence in collaboration with homotopy theorists such as Halperin, for example in [4]. The setting laid out in [4] was further expanded upon in [14] by Félix, Halperin and Thomas. This work looks at how certain classical notions relating to commutative rings (such as regularity, complete intersections and Gorenstein rings) can be translated into homotopy invariant notions for CDGAs, and consequently simply connected rational spaces. What is interesting is that for each notion, there are usually multiple ways of making such a translation, and thus the authors arrive at multiple notions of, for example, a “complete intersection space”, which are not always equivalent. Of the ones mentioned, the condition with the richest source of examples is the Gorenstein condition, of which all simply connected manifolds (and indeed any Poincaré duality space with finite dimensional rational cohomology) are examples. The Gorenstein condition is also very closely related to a duality condition (suitably termed Gorenstein duality), which is part of the focus of [13], where it is studied in the more general context of ring spectra. Our study of this area initially began with the hope of developing a similar theory for non simply connected spaces, which naturally led to considering what was in effect “equivariant versions” of the various conditions. In particular, given a discrete group G ,

we give a proposed notion of a simply connected regular G -space (in the language of [20]), and give a partial classification based on the actions on the homotopy groups. We also give some examples to illuminate the theory, and show where and how it differs from the simply connected theory in [20].

We also give some attention to the homotopy Gorenstein condition, most notably in the case where the rational cohomology is not finite dimensional, where we give a rich class of examples, and determine when their cohomology ring is Gorenstein (in the ring-theoretic sense). For completeness we give an exposition of rational Gorenstein duality for rational spaces, as appears in the unpublished appendix of [20].

In the next chapter we give of a compactly supported version of Sullivan’s PL polynomial forms on a simplicial set, and show that it models compactly supported singular cohomology. There have been similar attempts to develop characteristic zero compactly supported cohomology theories for spaces beyond manifolds, such as diffeological spaces, and one such treatment is by Haraguchi in [21]. His work differs from ours in that ours is in the context of Sullivan’s original work (where spaces are viewed as simplicial sets, and cochains are formal polynomials). A natural question following on from this (which we have not yet had the time to study further) is whether this compactly supported cohomology functor can be made into some kind of equivalence of “proper homotopy theories”, analagous to the equivalence in Sullivan’s original work. An axiomatic framework for proper homotopy theory is given for spaces by Baues and Quintero in [5], and if this could be reworked for simplicial sets and perhaps even commutative differential graded algebras, then this would be a first step towards viewing our work as an equivalence of proper homotopy theories. We suggest this as a possible avenue for extending our research.

The final chapter is concerned with a rational formality problem. It is a natural question when dealing with DGAs with a specified homology (up to isomorphism) to ask what possible quasi-isomorphism types of DGAs there are with the specified homology. This is what is meant by a formality

problem, with the obvious example of just the homology itself viewed as a DGA with zero differential being termed *formal*. Even when the specified homology is quite simple (for example polynomial on one variable), there can be interesting non-trivial quasi-isomorphism types emerging. Much of the inspiration and background for this chapter comes from Dugger and Shipley’s paper [10], where they work with connective, associative DGAs over \mathbb{Z} , and classify all quasi-isomorphisms of such DGAs with homology $\wedge_{\mathbb{F}_p}(g_n)$, for any prime p and generator g_n of degree $n \geq 1$ (when n is odd there is just the formal quasi-isomorphism type, and when n is even there is the formal one and another exotic one). Of course, working in the rational commutative case, we would expect to be able to ask what happens for more complicated specified homology types. We were initially interested in \mathbb{Q} -CDGAs with homology isomorphic to $k[x_{2n}]$, where k is a given \mathbb{Q} -algebra concentrated in degree 0, and x_{2n} is of some even degree $2n$. For reasons we will outline (based on the vanishing of the André-Quillen cohomology), when k is a complete intersection (ci) ring, all CDGAs with the given homology are formal, and so we looked at the case when k is perhaps the simplest example of a non-ci ring, that is, the “wedge of two 0-spheres” ($k = \mathbb{Q}[x, y]/(x^2, xy, y^2)$, with x and y in degree 0). Even in this case, we were not able to classify the quasi-isomorphism types with specified homology $k[x_{2n}]$, due to its non-finiteness as a module. However, under the assumption of a conjecture stated in the chapter, we were able to classify quasi-isomorphism types of CDGAs with homology isomorphic to $\wedge_k(x_n)$ (a 1-dimensional exterior algebra over k with generator in some degree $n \geq 1$). Moreover we were able to make a similar classification when k was a wedge of m copies of S^0 , for any $m \geq 2$ (that is, $k = \mathbb{Q}[t_1, \dots, t_m]/(t_i t_j | 1 \leq i, j \leq m)$ concentrated in degree 0). The computational part of the classification intricately involves the fact that the rational homotopy of a wedge of m spheres is a free graded Lie algebra on m generators, and a knowledge of the dimension of this graded Lie algebra in each degree. For a normal ungraded Lie algebra this would be given by the necklace polynomial, a fact which follows from the Poincaré-Birkhoff-Witt theorem, and so a key step was reformulating the necklace polynomial for the graded case using Poincaré-Birkhoff-Witt and generating functions.

Acknowledgements and background references

I would like to begin this section by thanking my supervisor John Greenlees for all his time, help, guidance, support, and company over the last 4 years. I estimate that we have had approximately 80 meetings in the 4 years I have been at Sheffield, and in each one John has provided me with great direction and suggestions for my research (and a variety of fascinating variants on the classical Rubik's cube to amuse myself with).

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Finally, there are many references and resources I have used as part of my mathematical education, both as an undergraduate and PhD student, that I feel have contributed to the projects I have worked on, and to the writing of this thesis. Citations are made in the body of my thesis when a specific result is called upon from a particular source, but the references mentioned here are the ones that I feel are not cited in my thesis to an extent which accurately represents their importance in my understanding of relevant concepts. These references are as follows.

For my understanding of general concepts in algebraic topology and homotopy theory, I referred to “Algebraic Topology” Hatcher [22].

For my understanding of categorical methods, I referred to “Categories for the working mathematician” Mac Lane [28], and was aided greatly in the first year of my PhD by talks in the category theory seminar at Sheffield by Eugenia Cheng, Nick Gurski and Simon Willerton.

For my learning about model categories and simplicial sets, I referred to “Model Categories” Hovey [26], “Homotopy theories and model categories” Dwyer and Spalinski [11], “Model Categories and Simplicial Methods” Goerss and Schemmerhorn [17], “Algebras and modules in monoidal model categories” Schwede and Shipley [37], “Simplicial Homotopy Theory” Goerss and Jardine [16].

For my learning about spectral sequences I referred to “Spectral Sequences in Algebraic Topology” Hatcher [23], “User’s Guide to Spectral Sequences” McCleary [33].

To gain some understanding of equivariant cohomology and equivariant homotopy theory, I referred to “Equivariant cohomology theories” Bredon [8], “Equivariant Homotopy and Cohomology Theory” May [32] and had many helpful conversations with my supervisor John Greenlees.

To gain a better understanding of the process of rationalisation, I referred to “Homotopy Limits, Completions and Localizations” Bousfield and Kan [7].

For an overview of homological algebra (beyond what I had learned during my undergraduate studies and Part III), I referred extensively to “An Introduction to Homological Algebra” Weibel [43], and also used it for background on regular, complete-intersection, Gorenstein and Cohen-Macaulay rings, their relationship to each other, and similar concepts.

As a way of introducing myself to rational homotopy theory for my Part III essay, I referred to “Rational homotopy theory: A brief introduction” Hess [24], “Rational Homotopy Theory” Félix, Halperin and Thomas [15], “PL de Rham theory and rational homotopy type” Bousfield and Gugenheim [6].

For an understanding of what happens to rational homotopy theory when simple connectivity/nilpotence assumptions are removed, I looked at “Rational homotopy theory for non-simply connected spaces” Gómez-Tato, Halperin and Tanré [18].

For an understanding of when the Eilenberg-Moore spectral sequence converges, I referred to “Strong convergence of the Eilenberg-Moore spectral sequence” Dwyer [12].

CHAPTER 1

Introduction

We will now briefly summarise the main results of this thesis.

Rational Homotopy Theory in the sense of Sullivan

In chapter 2 we review Rational Homotopy Theory in the sense of Sullivan, which will be fundamental to every subsequent chapter. The main results of his theory that underpin what we do are reformulated and expanded upon in [6], and so we shall state them as stated there, and cite appropriately.

Throughout this section, $cdga$ denotes the category of commutative differential graded algebras (CDGAs) over \mathbb{Q} (with standard grading conventions, meaning the differential reduces degrees). Bousfield and Gugenheim show in [6] that $cdga$ has a model structure lifted from the projective model structure on rational chain complexes, meaning that the fibrations are just the surjections, and the weak equivalences are the quasi-isomorphisms.

We say a CDGA A is *coconnected* if it is concentrated in non-positive degrees, and $H_0A \cong \mathbb{Q}$.

Similarly, $A \in cdga$ is of *finite-type* if H_mA is finite-dimensional for all m .

$sSet$ denotes the category of simplicial sets with the Serre model structure.

A connected simplicial set X is *rational* if its geometric realisation $|X|$ is a rational space, meaning that $\pi_m X$ is a rational vector space for all $m \geq 2$. X is *nilpotent* if $\pi_1|X|$ is a nilpotent group, and the action of $\pi_1|X|$ on the higher homotopy groups is nilpotent.

Moreover if X is rational, we say X is of *finite-type* if $\pi_m|X|$ is finite dimensional for each m .

Let C^* denote the singular cochains functor on simplicial sets. The main results of relevance to us are as follows.

THEOREM 1.1. ([6]) *There exists a functor $A^* : sSet^{op} \rightarrow \mathbb{Q} - dga$ whose image is contained in $cdga$, and a natural transformation $\rho : A^* \rightarrow C^*$ such that $H^*\rho_X : HA^*X \rightarrow HC^*X$ is a multiplicative isomorphism for any simplicial set X .*

This result is the fact that the singular cochain complex of a space (simplicial set) in characteristic zero has a strictly commutative model.

THEOREM 1.2. ([6]) *Let $sSet_{f,nil,rat}$ and $cdga_{f,coconn}$ denote the full subcategories of finite-type, nilpotent, rational simplicial sets and finite-type, rational, coconnected CDGAs. Then there exist functors*

$$F : cdga^{op} \rightarrow sSet$$

$$A : sSet \rightarrow cdga^{op}$$

which are adjoint, and induce equivalences of the homotopy categories

$$A : Ho(sSet_{f,nil,rat})^{op} \rightleftarrows Ho(cdga_{f,conn}) : F$$

The functor A in the second theorem is related but slightly different to the functor A in the first theorem (they differ by composition with a cofibrant replacement functor).

The second theorem is the statement that (under the given conditions), rational homotopy theory is “very algebraic”, in that the homotopy category of the given rational spaces is identified with the homotopy category of certain CDGAs.

Derived Commutative Algebra and RHT

In chapter 3 we describe the close relationship between the rational homotopy theory of simply connected spaces, and derived commutative algebra over \mathbb{Q} , and in particular, how it leads to homotopy invariant formulations of classical ring theoretic notions, such as Gorenstein, regular, and complete intersection. The main references for this section are [4] and [20]. In the former, the authors describe how the relationship works, and the latter gives the specific homotopy invariant formulations. In the case of the regular and complete intersection condition, there are various (equivalent) definitions of such rings, and interestingly, it is not trivially the case that the homotopy invariant formulations of these definitions are equivalent (and indeed, for the complete intersection condition, they are not). Establishing the extent to which they are equivalent is a large part of the focus of [20].

After recalling what we need from the various sources, we begin by considering a natural generalisation of the definition of an s -regular space (as given in [20]), to allow for possibly non-simply connected spaces.

In this section, G will denote an arbitrary discrete group, unless we introduce other assumptions.

For a space X , we denote its universal cover by \tilde{X} .

We begin by making the following definition.

DEFINITION 1.3. Let X be a rational, connected space with $\pi_1 X = G$. X is *s-regular* if there exist fibration sequences

$$S^{n_k} \rightarrow BG \rightarrow X_k, S^{n_{k-1}} \rightarrow X_k \rightarrow X_{k-1}, \dots, S^{n_0} \rightarrow X_1 \rightarrow X_0$$

where $X_0 = X$, and each fibration induces an isomorphism on π_1 of its source and target, for some positive integers n_0, \dots, n_k .

In [20], they define a regular space to be exactly as above, but are working with simply connected spaces only, and so $G = \{1\}$. An example of an s -regular space as defined above is the rationalisation of $BO(n)$, for any $n \geq 1$; in this case we would have $G = \mathbb{Z}/2$.

It is shown in [20] that any simply connected rational s -regular space must have the homotopy type of a product of even Eilenberg-Mac Lane spaces. In the non-simply connected case, the situation is more complicated, and we give the following sufficient conditions for s -regularity.

THEOREM 1.4. *Let X be a path connected rational space with nilpotent actions of the fundamental group on $\pi_n X$ for $n \geq 2$, and $\pi_* \tilde{X}$ finite dimensional over \mathbb{Q} and concentrated in even degrees below $2d$, for some minimal d . Suppose also that $H^{2k+1}(\pi_1 X; \mathbb{Q})$ being non-zero implies $k = 0$ or $k > d$. Then X is s -regular.*

We follow up this result with examples of when the theorem can fail due to the nilpotence conditions not being satisfied, and separately, examples of failure due to the condition on the cohomology of the fundamental group not being satisfied.

We then have the following result, which gives necessary conditions for a space to be s -regular. It is preceded by a review of the Jordan-Hölder theorem, as it is implicit in its formulation.

PROPOSITION 1.5. *Suppose X is a path connected rational s -regular space with $G := \pi_1 X$. Then for all n , $\pi_{2n} X$ has a composition series as a $\mathbb{Q}G$ -module, and the composition factors are all 1-dimensional over \mathbb{Q} (though not necessarily trivial).*

The Gorenstein condition

In chapter 4, we move on to discussing the Gorenstein condition, and in particular give a new class of examples of Gorenstein spaces with the property of being homotopy Gorenstein, but not having Gorenstein cohomology ring. In general, if a space has Gorenstein cohomology ring, then it is homotopy Gorenstein, however the converse is not always true, and specifically does

not always hold when the cohomology ring is not finite dimensional, and the examples given are of this type.

We begin the chapter by reviewing the h-Gorenstein condition, as presented in [20] (where h stands for “homotopy”), and some Morita theory, which gives us a model for the cellularisation of a DG-module. Another model for cellularisation is obtained from considering the unstable Koszul complex, and from this and the uniqueness of cellularisation up to homotopy, we can compare the two models for cellularisation and deduce the following duality property for h-Gorenstein CDGAs, which is the rational CDGA version of a result on Gorenstein Duality from [13].

THEOREM 1.6. (*Gorenstein Duality*)

*Suppose A is an h-Gorenstein CDGA of shift a with H^*A 1-connected and Noetherian. Then there is an equivalence*

$$\text{Cell}_{\mathbb{Q}}A \simeq \Sigma^a A^\vee$$

which implies we have a spectral sequence

$$H_I^*(H^*A) \implies \Sigma^a H^*(A)^\vee$$

The unstable Koszul complex is also the main ingredient in the construction of the local cohomology spectral sequence (the spectral sequence appearing in 1.6, which can be used to compute the local cohomology of the cohomology ring of an h-Gorenstein space/CDGA).

In the next part of the chapter, we look at a class of examples of rational h-Gorenstein spaces whose cohomology ring is not finite-dimensional. These examples are constructed as spaces X which fit into a fibration sequence of the form

$$X \rightarrow K(\mathbb{Q}^m, 2n) \xrightarrow{\gamma} K(\mathbb{Q}^2, 2kn)$$

for some map γ , any $m, n, k \geq 1$.

In particular, if a and b are the generators of the cohomology of the base space, then we get two homogenous polynomials $p = \gamma^*(a)$ and $q = \gamma^*(b)$,

whose terms live in degree $-2kn$. We note that there is the potential for confusion here over the word “degree” when referring to a polynomial, as it is unclear whether it means the degree that the actual terms of the polynomial live in as elements of a graded object, or whether it means the degree of the polynomial as an element of a polynomial ring. Thus if our language ever appears over-worded when talking about degree, it is probably an attempt for clarity on this issue.

Using the Eilenberg-Moore spectral sequence, we compute the following

PROPOSITION 1.7.

$$H^*X \cong \mathbb{Q}[u_1, \dots, u_m, \tau]/(p, q, \gcd(p, q)\tau, \tau^2)$$

where τ is of degree $-(4k - 2d)n + 1$, where d is the degree of the terms of the homogenous polynomial $\gcd(p, q)$.

We are then able to compute the local cohomology of H^*X using the local cohomology spectral sequence, and determine from the results precisely when X is h-Gorenstein. The local cohomology computation yields the following.

PROPOSITION 1.8. *Letting $g = \gcd(p, q)$, and $p' = p/g$ and $q' = q/g$ as before, we have that*

$$H_m^N(H^*X; H^*X) = \begin{cases} \Sigma^{-2n(2k-d-1)+m}(P/(p', q'))^\vee, & \text{if } N = m - 2, \\ \Sigma^{-2n(d-1)+m}(P/(g))^\vee \oplus \Sigma^{-2n(2k-1)+m+1}(P/(g))^\vee, & \text{if } N = m - 1, \\ 0, & \text{otherwise} \end{cases}$$

And using facts about the local cohomology of Gorenstein (in particular Cohen-Macaulay) rings, we can deduce the following.

PROPOSITION 1.9. *For any such p and q as above, H^*X is Gorenstein as a commutative ring if and only if p and q are coprime, or if they are equal up to multiplication by a unit.*

PL compactly supported forms in characteristic zero

The original motivation behind chapter 5 came from a question asked by Dan Petersen on Math Overflow, who was wishing to know if there was a way in which characteristic zero compactly supported cohomology could be constructed in a way similar to Sullivan’s PL de Rham theory, thus giving one way of defining a “suitable” compactly supported cohomology for spaces which aren’t necessarily manifolds (the word “suitable” meaning that it has the properties one would want of a cohomology theory, such as Mayer-Vietoris sequences). In terms of achieving the latter, a suitable compactly supported theory in characteristic zero has been constructed for diffeological spaces by Haraguchi in [21], and our work in chapter 5 constructs such a compactly supported theory instead for simplicial sets.

The actual definition of what the compactly supported theory should be is not hard to write down in terms of Sullivan’s original constructions. Let $\nabla(*, *)$ denote the simplicial CDGA A^*X of polynomial forms on standard simplices, as constructed in 2.1. Then the CDGA of polynomial q -forms on a simplicial set X is given by

$$A^qX := sSet(X, \nabla(*, q))$$

and so correspondingly, the CDGA A_c^*X of *compactly supported* polynomial forms on X is given by

DEFINITION 1.10.

$$A_c^qX := \{\Phi \in sSet(X, \nabla(*, q)) \mid \exists \text{ finite } K \subset X \text{ s.t. } \Phi|_{\langle X \setminus K \rangle} = 0\}$$

where for a collection of simplices $I \subset X$, $\langle I \rangle$ denotes the smallest sub-simplicial set containing I , and a finite simplicial set is one with only finitely many non-degenerate simplices.

Intuitively, one should think of a simplicial map $X \rightarrow \nabla(*, q)$ as a bit like a global section of a sheaf of functions on a space, except with open sets replaced by simplices: to each simplex, the simplicial map assigns a

polynomial q -form, and the fact that the map is a simplicial map (commutes with the face and degeneracy operators) can be thought of as ensuring that the assigned q -forms agree on any intersections. For the compactly supported version A_c^*X , the extra conditions on the global sections are just translated from what we would expect a compactly supported de Rham form to look like on a manifold (that is, vanishing outside a compact subset).

A remarkable discovery of Sullivan is that there is in fact a theory of integration for the polynomial forms on a simplex, which works completely analogously to the theory of integration of differential forms on smooth manifolds. Heuristically, to compute the integral $\int \omega$ of some $\omega \in \nabla(p, q)$ over Δ^p , one takes a fixed geometric realisation $|\Delta^p|$ of the simplex Δ^p , and then since ω is a polynomial form, and we are working over a field of characteristic 0, we can interpret ω as a form defined on $|\Delta^p|$, and integrate it (over the interior) in the usual way. The details of this method will be spelled out in 5.5. The formulation of Stokes' Theorem can be summarised as follows:

We can define a total differential

$$\partial : \nabla(p, q) \rightarrow \nabla(p-1, q)$$

for all p and q , by $\partial = \sum_{i=0}^p \partial_i$, satisfying $\partial d_p = d_{p-1} \partial$, where d_j denotes the exterior differential $d_j : \nabla(j, *) \rightarrow \nabla(j, *)$ of degree 1. ∂ is analagous to restricting a differential form on a manifold to a form on the boundary of the manifold. We then have

THEOREM 1.11. (*Stokes' Theorem (Proposition 1.4 of [6])*)

For any $\omega \in \nabla(p, p-1)$

$$\int d\omega = \int \partial\omega$$

One way we show the existence of a (contravariant) Mayer-Vietoris sequence for A_c^* is by showing that there is a notion of “bump functions” in the context we are working in.

Given a simplicial subset $L \subset X$, the *boundary* of L in X is denoted ∂L , and is defined to be $\langle X \setminus L \rangle \cap L$.

The *minimal neighbourhood* of L in X is denoted $\epsilon(L)$, and is defined to be the smallest subsimplicial set of X which contains L , and whose boundary in X does not intersect L . Note that since X contains L , and the boundary of X in X is empty, minimal neighbourhoods exist for every choice of X and $L \subset X$.

We prove the following.

THEOREM 1.12. (*Existence of PL bump functions*)

Let X be a simplicial set, and let $L \subset K \subset X$ be subsimplicial sets such that $\epsilon(L) \subset K$. Then there exists some $\phi \in A^0 X$ such that $\phi|_L = 1$ and $\phi|_{\langle X \setminus K \rangle} = 0$

The main ingredient for the proof of 1.12 is the extension property for $\nabla(*, *)$, which appears as Corollary 1.2 of [6], and as 5.4 in this document. It effectively says that if there exists a collection of q -forms defined on the boundary faces of a p -simplex (so all living in $\nabla(p-1, q)$), which all agree on the intersections of faces, then there always exists a q -form in $\nabla(p, q)$ which extends each of the forms on its faces. Moreover, it says that this extension respects addition of forms on each face of the p -simplex.

If $U, V \subset X$ are simplicial sets covering X , then we say that U and V have *good intersection* if $\epsilon(\langle V \setminus U \rangle) \subset V$. Using 1.12, we prove the following.

THEOREM 1.13. (*Contravariant Mayer-Vietoris sequence, version 1*)

Let X be a simplicial set with subsimplicial sets $U, V \subset X$ which cover X and have good intersection. Then there is a long exact sequence

$$\dots \leftarrow HA_c^n(U \cap V) \leftarrow HA_c^n U \oplus HA_c^n V \leftarrow HA_c^n X \leftarrow HA_c^{n-1}(U \cap V) \leftarrow \dots$$

We can in fact obtain the same contravariant Mayer-Vietoris sequence under different conditions involving local finiteness and properness of maps. A map $f : X \rightarrow Y$ of simplicial sets is *proper* if for any finite subsimplicial set $Z \subset Y$, the subsimplicial set $f^{-1}Z \subset X$ is finite. A simplicial set is *locally finite* if each of its simplices is a face of only finitely many non-degenerate simplices.

THEOREM 1.14. (*Contravariant Mayer-Vietoris sequence, version 2*)
 Suppose we have a pushout diagram

$$\begin{array}{ccc} W & \xrightarrow{f} & U \\ \downarrow \iota & & \downarrow h \\ V & \xrightarrow{g} & X \end{array}$$

of simplicial sets, where ι is an inclusion, f is proper and V is locally finite. Then g and h are proper maps, and there exists a long exact sequence

$$\dots \leftarrow HA_c^n(W) \leftarrow HA_c^n U \oplus HA_c^n V \leftarrow HA_c^n X \leftarrow HA_c^{n-1}(W) \leftarrow \dots$$

which is natural in all the variables in the pushout.

In particular we can deduce that if $U, V \subset X$ are subsimplicial sets which cover X , and one of U or V is locally finite, then a contravariant Mayer-Vietoris sequence, as above, exists.

In the final part of chapter 5 we prove a compactly supported version of the de Rham theorem, showing that the cohomology A_c^* is isomorphic to that given by compactly supported singular cohomology, and moreover, that the isomorphism is multiplicative. We begin by describing the usual de Rham theorem (2.2 and 3.4 of [6]). To do this, we first define a natural transformation $\rho : A^* \rightarrow C^*$ (where C^* denotes rational singular cochains) by

$$\langle \rho\omega, \sigma \rangle = \int \omega_\sigma$$

for any $\omega \in A^q X$ and $\sigma \in X_q$, for any simplicial set X . The de Rham theorem is then stated as

THEOREM 1.15. (*PL de Rham theorem. 2.2 and 3.4 of [6]*)
 ρ induces a multiplicative homology isomorphism

$$\rho_* : HA^* X \rightarrow HC^* X$$

for any simplicial set X .

We then are able to use this, and a simple argument using the five lemma, to prove a similar result for relative cohomology (in other words, the same result as the theorem above, but with X replaced by (X, A)). The importance of this for the compactly supported version is that we can prove the following characterisation of compactly supported cohomology.

LEMMA 1.16. *The canonical map*

$$\eta : HA_c^*X \rightarrow \operatorname{colim}_K(HA^*(X, \langle X \setminus K \rangle))$$

is an isomorphism of graded rings (where the colimit runs over all finite subsimplicial sets of X).

This then allows us to prove the PL compactly supported de Rham Theorem, which is as follows.

THEOREM 1.17. *(PL compactly supported de Rham Theorem)*

The restriction $\rho_c : A_c^*X \rightarrow C_c^*X$ induces a multiplicative isomorphism on cohomology, for any simplicial set X .

A formality problem

In the final chapter we look at a formality problem whose inspiration came from my supervisor's research in elliptic cohomology. The original problem was that of trying to classify \mathbb{Q} -CDGAs A augmented over a fixed \mathbb{Q} -algebra k , with homology isomorphic to $k[x]$, for some generator x in an even degree. It quickly became clear that if k is not a complete intersection, there could be many homotopy types of such objects, and finding a way of attacking the problem, even for the simplest examples of k being non-ci, proved difficult. So we looked instead at the simpler problem of classifying \mathbb{Q} -CDGAs A with homology isomorphic to $\Lambda_k(x_m)$, that is, an exterior algebra over k on a single generator x in any degree m . We now summarise the chapter and its results.

We begin by giving the relevant model structure on the category we are working in, which is the category of connective CDGAs over \mathbb{Q} . The model structure can be lifted directly from the projective model structure on the category $Ch_{\mathbb{Q}}$ of connective chain complexes over \mathbb{Q} (recall that the projective model structure has fibrations being surjective maps, and weak equivalences quasi-isomorphisms). Moreover, the cofibrant generation of the projective model structure can be lifted to CDGAs using the free functor

$$Ch_{\geq 0} \rightarrow cdga_{\geq 0}$$

These liftings are not typical of CDGAs in general, and indeed, there is no such way of lifting the projective model structure on chain complexes to CDGAs when working over the integers, or in positive characteristic. The issue comes from the fact that we are considering commutative DGAs, and that outside of characteristic 0, the free *commutative* DGA on the n disc chain complex $D(n) = \mathbb{Q} \langle x_{n-1}, y_n \mid dy = x \rangle$ is not acyclic, and thus applying the free functor to the generating acyclic cofibration $0 \rightarrow D(n)$ does not give a quasi-isomorphism. We stress this fact to make clear one of the many nice features of working in characteristic 0, as having a nice model structure on $cdga_{\geq 0}$ allows us to compute derived functors without using simplicial resolutions. The precise result we use to deduce the required model structure is

THEOREM 1.18. (*Transferred Model Structure, Theorem 3.3 of [9]*)

Let \mathfrak{C} be a cofibrantly generated model category, \mathfrak{D} a category, and suppose we have an adjunction

$$\mathfrak{D} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{U} \end{array} \mathfrak{C}$$

where F is left adjoint to U .

Moreover, call a map in \mathfrak{D} a fibration (resp. weak equivalence) if its image under U is a fibration (resp. weak equivalence) in \mathfrak{C} . Suppose now that F preserves small objects, and that any sequential colimit of pushouts of images under F of the generating acyclic cofibrations in \mathfrak{C} is a weak equivalence in \mathfrak{D} . Then the choice of fibrations and weak equivalences made above for \mathfrak{D}

determines a cofibrantly generated model structure on \mathfrak{D} , where the set of generating (acyclic) cofibrations in \mathfrak{D} is the set of images under F of the generating (acyclic) cofibrations in \mathfrak{C} .

Using this, we can prove the following.

THEOREM 1.19. *$cdga_{\geq 0}$ has a model structure whose fibrations are all surjective maps, and weak equivalences are all quasi-isomorphisms, and where the generating cofibrations are maps of the form $FS(n-1) \rightarrow FD(n)$, and the generating acyclic cofibrations are maps of the form $\mathbb{Q} \rightarrow FD(n)$, for all $n \geq 1$.*

As explained in the chapter, we are interested in the cases where k is not a complete intersection over \mathbb{Q} . We began by considering one of the simplest non-ci rings one could think of, that is

$$\mathbb{Q}[x, y]/(x^2, xy, y^2)$$

which led to considering the class of rings given by

$$k = \mathbb{Q}[t_1, \dots, t_n]/(t_i t_j | 1 \leq i, j \leq n) =: S^0 \vee \dots \vee S^0$$

for any number of variables $n \geq 2$.

As all the generators are concentrated in degree 0, we will often write this algebra as $S^0 \vee \dots \vee S^0$.

As stated, we are interested in the homotopy types of augmented \mathbb{Q} -CDGAs $A \rightarrow k$ with $H^*A \cong \Lambda_k(x_m)$, for any $m \geq 1$. In order to study such CDGAs A , we describe a theory of Postnikov extensions in $cdga_{\geq 0}$, which is developed in [10] for connective, associative DGAs over \mathbb{Z} , but much of the theory carries over to our context unchanged. Under this theory, we can rephrase the problem as asking for quasi-isomorphism classes of Postnikov m -extensions of k .

If $C \in cdga_{\geq 0}$ is acyclic in degrees m and above, a *Postnikov m -extension* of C is some $X \in cdga_{\geq 0}$, such that X is acyclic in degrees $m+1$ and above,

together with a map $f : X \rightarrow C$, which induces a multiplicative isomorphism on homology in degrees $\leq m - 1$. Intuitively, it can be thought of as the CDGA C extended by the H_0C -module H_mX .

We say that a *morphism* $(X, f) \rightarrow (Y, g)$ of Postnikov m -extensions of C is just a quasi isomorphism $X \rightarrow Y$ such that the obvious diagram commutes, and we write the category of Postnikov m -extensions X of C with $H_mX \cong M$ as $Pext_m(C; M)$.

Thus, we reduce our problem to computing the connected components of the category $Pext_m(C; M)$. We have the following result from [10] to begin down this road.

COROLLARY 1.20. (*3.10 of [10]*)

Let $C \in cdga_{\geq 0}$ be acyclic in degrees $m + 1$ and above, M be an H_0C -module, and $G = Aut_{H_0C}M$. Then we have a bijection

$$(Ho(cdga_{\geq 0}/C)(C, C \vee \Sigma^{m+1}M))/G \xrightarrow{\cong} \pi_0(Pext_m(C; M))$$

where $C \vee \Sigma^{m+1}M$ is the square-zero extension of C by $\Sigma^{m+1}M$ (see the full chapter for details).

The maps $C \rightarrow \Sigma^{m+1}M$ are the algebraic analogue of k -invariants in the Postnikov decomposition of a space, and the result effectively says that homotopy classes of these maps classify homotopy types of Postnikov extensions, after quotienting by automorphisms of M .

But $Ho(cdga_{\geq 0}/C)(C, C \vee \Sigma^{m+1}M)$ does not appear to be particularly computable, and so the final stage of this theory is to compare it to something purely algebraic that can be computed. This algebraic construction is in fact *André-Quillen cohomology* $AQ^{m+1}(C; M)$, which is intuitively the derived functor of derivations from C to M . The precise result is

THEOREM 1.21. (*Analogous to that which appears in [29]*)

Let $C \in cdga_{\geq 0}$ and let M be a C -module concentrated in a single degree.

Then there's an isomorphism

$$AQ^m(C; M) \cong Ho(cdga_{\geq 0}/C)(C, C \vee \Sigma^m M)$$

We have given a very straightforward proof of this result which works in our setting, and which is much simpler than proofs appearing in other contexts.

As such, for our original problem, we are interested in $AQ^{m+1}(k; k)$, and to compute this, we must compute a cofibrant replacement for $k = S^0 \vee \dots \vee S^0$ in $cdga_{\geq 0}/k$. Unfortunately, we only have a conjecture towards this, that we strongly believe to be true. For M a coconnected CDGA, let QM_{-j} denote its indecomposable quotient.

We now consider the coconnected CDGA

$$k' := \mathbb{Q}[w_1, \dots, w_n]/(w_i w_j)$$

where the s_j are concentrated in degree -2 . Then if $P' = (\Lambda V, d)$ denotes the minimal Sullivan algebra of k' with a given choice of basis including w_1, \dots, w_n , then we define $P(V)$ to be the free (connective) CDGA on $\Sigma^{-2}(V^\vee)$, with differential on $P(V)$ given by

$$(d_{P(V)})(s^{-2}x^\vee) = d_{P'}(x)$$

for any basis element x of V .

Now $P(V)_0 = \mathbb{Q}[s^{-2}w_1^\vee, \dots, s^{-2}w_n^\vee]$, and so there is a canonical map

$$P(V) \rightarrow k$$

for which we have the following conjecture.

CONJECTURE 1.22. *The canonical map $P(V) \rightarrow k$ is a cofibrant resolution in $cdga_{\geq 0}$. That is, $P(V)$ is cofibrant, and the map is a quasi-isomorphism.*

The importance of this conjecture is that we know a lot about k' , as it is given by $H^*(S^2 \vee \dots \vee S^2)$, which is known to be dual to the free graded Lie algebra on n letters. The conjecture is effectively saying that given a resolution of k' , we can take its indecomposables, shift them two degrees up, then dualise, and obtain a resolution of k . Simplifying further, it says that building a resolution of k is the same process as building a minimal resolution of k' . To prove the conjecture, it would suffice to prove that one can inductively build a minimal resolution of k that has purely quadratic differential, since then (under the reverse of the dualising process in the conjecture) this would correspond to a resolution of k' . We have been able to prove this in some cases, to the point of convincing us that the conjecture is true, but are still working on a complete proof. Let $\mathbb{Q}P^b$ denote rational projective space of dimension b . Assuming the conjecture, we are able to solve the original problem, and the main result is

THEOREM 1.23. *(Conditional on the conjecture) Let $k = \mathbb{Q}[t_1, \dots, t_n]$ with the generators concentrated in degree 0. Then the set of homotopy types of \mathbb{Q} -CDGAs A , augmented over k , for which $H_*A \cong \Lambda_k(x_m)$ is in bijection with $\mathbb{Q}P^{N_{mn}-1}$ (rational projective space of dimension $N_{mn} - 1$), where*

$$N_{mn} = (-1)^{m+2} \left(\frac{n}{m+2} \sum_{d|(m+2)} (-1)^{(m+2)/d} \mu(d) n^{(m+2)/d} + \frac{1}{m+1} \sum_{d|(m+1)} (-1)^{(m+1)/d} \mu(d) n^{(m+1)/d} \right)$$

CHAPTER 2

Rational Homotopy Theory in the sense of Sullivan

Sullivan’s original paper on Rational Homotopy Theory is [41], however a slightly more modern approach in terms of model categories is given in [6] by Bousfield and Gugenheim. The latter was the main reference for my Part III essay, and sets a lot of the foundation for my research area. The main result of both Sullivan’s work and theirs, is an equivalence of the homotopy category of rational (simply connected, or nilpotent) spaces with a certain homotopy category of CDGAs. In short, rational homotopy theory is “very algebraic”. Bousfield and Gugenheim use the language of model categories, which allows a lot of the required constructions to be formalised in a very general picture. We will give a quick overview of the adjunction, and a statement of the main Theorem.

For the entire chapter k denotes a field of characteristic 0, usually \mathbb{Q} , and $cdga$ is the category of commutative, differential graded algebras over k (with the forgetful model structure inherited from the projective model structure on cochain complexes, as given in [6]).

We will begin by giving an overview of how one can construct an algebra of polynomial differential forms on any simplicial set, over any field of characteristic zero. The construction is often called the Sullivan-de Rham complex. We begin by considering single simplices. Intuitively, one should think of the vertices of a p -simplex being the points $(1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1)$ in \mathbf{R}^{p+1} , and then the simplex is defined as the subspace

$$\{(t_0, \dots, t_p) \in \mathbf{R}^{p+1} : t_0 + \dots + t_p = 1\}$$

Then, since 0-forms are just continuous functions, the polynomial zero forms are precisely the span (as an algebra) of the coordinate functions on the

simplex, and just by basic calculus we obviously have $dt_0 + \dots + dt_p = 0$ also. If $\nabla(p, *)$ denotes the algebra of such polynomial differential forms on a p -simplex, then there are $p+1$ maps $\nabla(p, *) \rightarrow \nabla(p-1, *)$ given by the various restrictions to the boundary faces of the p -simplex, and similarly there are $p+1$ maps $\nabla(p, *) \rightarrow \nabla(p+1, *)$ which can each be thought of as adding a vertex to the p -simplex, and taking a degenerate extension of the given form. We make this precise now.

DEFINITION 2.1. Let $\nabla(*, *)$ denote the simplicial CDGA given by

$$(\nabla(*, *))_p := \nabla(p, *)$$

where $\nabla(p, *)$ is the CDGA given by polynomial differential forms on a p -simplex, where we denote by $\nabla(p, q)$ the module of q -forms on a p -simplex. To be precise, $\nabla(p, *)$ is the CDGA generated by t_0, \dots, t_p in degree 0, dt_0, \dots, dt_p in degree -1 , subject to the relations

$$t_0 + \dots + t_p = 1$$

$$dt_0 + \dots + dt_p = 0$$

and obviously we set $d(t_i) = dt_i$ for all i . The face maps are given by

$$\partial_i(t_m) = \begin{cases} t_{m-1}, & \text{if } i < m \\ 0, & \text{if } i = m \\ t_m, & \text{if } i > m \end{cases}$$

and the degeneracy maps are given by

$$s^i(t_m) = \begin{cases} t_{m+1}, & \text{if } i < m \\ t_m + t_{m+1}, & \text{if } i = m \\ t_m, & \text{if } i > m \end{cases}$$

Of course, each t_m on the left hand side is in a different dimension to the right hand side, but there is no inconsistency which arises from this, and the

dimensions can be extracted from context. The intuition for the face maps is given by restriction of functions: so in the second line for example, where $i = m$, this is saying that if you restrict the coordinate function corresponding to a vertex to the face opposite that vertex, the resulting function is zero, which is consistent with our choice of embedding of our simplices. The intuition for the degeneracy maps is given by linear interpolation of functions.

Bousfield and Gugenheim then show in [6] that $\nabla(p, *)$ is chain homotopic to k (concentrated in degree 0) for all p , and that $\nabla(*, q)$ is contractible as a simplicial k -module for all q , as one would expect, since a simplex is contractible, and so all closed forms on it should be exact.

They then make the following definition:

DEFINITION 2.2. Given a simplicial set K , let A^*K denote the CDGA given by

$$(A^*K)_{-n} = Hom_{sSet}(K, \nabla(*, n))$$

for each $n \geq 0$, with addition defined simplexwise by

$$(\Phi + \Psi)(\sigma) = \Phi(\sigma) + \Psi(\sigma)$$

and scalar multiplication and the algebra structure defined similarly. The differential is defined by

$$(d(\Phi))(\sigma) = d(\Phi(\sigma))$$

where the ∂_i are the face maps of $\nabla(*, n)$

Recalling that a morphism of simplicial sets is a natural transformation of the underlying functors, and hence commutes with the face and degeneracy operators, we can think of AK as being the DGA of polynomial forms on K , since $Hom_{sSet}(K, \nabla(*, n))$ can be thought of as all the possible ways of assigning an n -form to each simplex σ , in a way which agrees on the intersections of the simplices.

$A : sSet \rightarrow cdga$ is in fact a functor, and Bousfield and Gugenheim prove the following Theorem:

THEOREM 2.3. ([41]) *There exists a functor*

$$A : sSet^{op} \rightarrow cdga$$

and a natural transformation $\rho : A^ \rightarrow C^*$ such that*

$$H^* \rho_X : HA^*X \rightarrow HC^*X$$

is a multiplicative isomorphism for any simplicial set X .

We will frequently write A^* for A , for example when we want to emphasise the structure of A^*X as a CDGA. We will usually write AX to shorten notation when we are just concerned with the formal properties of A as a functor.

$A : sSet^{op} \rightarrow cdga$ is called the *De Rham functor*. It is in fact part of an adjunction, where its adjoint $F : cdga^{op} \rightarrow sSet$ is defined similarly by

$$(FB)_n = Hom_{cdga}(B, \nabla(n, *))$$

where the face maps are given by $(\partial_i(\sigma))(\omega) = \partial_i(\sigma(\omega))$, where the ∂_i on the right hand side is the i th face map of $\nabla(*, *)$, and the degeneracy maps are defined similarly. Hence, F is in fact a functor $cdga \rightarrow s(k - mod)$, where $s(k - mod)$ is the category of simplicial $k - modules$. In particular, the image of F is contained in $sAbGp$. Every simplicial Abelian group is in fact a Kan complex when viewed as a simplicial set, and so F takes all objects to fibrant (and cofibrant) objects. The correct terminology to describe the adjunction is that it is a contravariant adjunction on the right.

With no further alterations, we have a Quillen adjunction

$$A : sSet \rightleftharpoons cdga^{op} : F$$

meaning that (contravariantly) A and F both take cofibrations to fibrations. However, as it stands, it is not a Quillen equivalence, since the unit and counit $X \mapsto FAX$ and $B \mapsto AFB$ are not always weak equivalences, and so do not always become isomorphisms at the level of homotopy categories.

For the rest of this chapter, we suppose $k = \mathbf{Q}$. In order to make the induced adjunction of homotopy categories an equivalence, we need to work with certain (full) subcategories on each side, as described in the next proposition. Before stating it, we make the following definition.

DEFINITION 2.4. A coconnected CDGA M is *minimal* if M is free as a graded algebra, and the image of the differential of M is contained in the maximal ideal of M (that is, dx can be written as a sum of products of two or more elements of positive degree, for any $x \in M$).

It is proved in Bousfield and Gugenheim that minimal algebras are cofibrant, and that every CDGA is weakly equivalent to a minimal algebra, however, not functorially. Functorial cofibrant replacement in fact uses a larger class of algebras called *Sullivan algebras*, which we will not need to talk about. Regardless, cofibrant replacement by minimal algebras does become functorial when passing to the homotopy category.

PROPOSITION 2.5. ([6])

If X is a nilpotent rational space of finite \mathbf{Q} -type, and B is a connected CDGA of finite \mathbf{Q} -type, then the unit and counit maps $X \mapsto (\mathbb{R}F)(\mathbb{L}A)X$ and $B \mapsto (\mathbb{L}A)(\mathbb{R}F)B$ are weak equivalences.

REMARK 2.6. Since the adjunction is contravariant, both $\mathbb{R}F$ and $\mathbb{L}A$ are given by FQ_{cdga} and AQ_{sSet} , where Q_C denotes the functorial cofibrant replacement functor in any model category C . But since every object of $sSet$ is cofibrant, $\mathbb{L}AY = AY$ for any $Y \in sSet$, and so we are only concerned with taking cofibrant replacements of CDGAs. It turns out that the nicest choices of cofibrant replacement are often not functorial, but instead depend on a choice of generators for the given CDGA one is working with. One example of such non-functorial cofibrant replacements are Sullivan's *minimal models*,

which when computable, allow one to very easily read off the ranks of the rational homotopy groups of the simplicial set the minimal model represents.

The proof is in three stages, which are reviewed very nicely just after lemma 3.7 of [27]. The first step involves proving the proposition for $X = K(\mathbf{Q}, n)$ and $B = S(-n)$, the algebra with one generator in degree $-n$. This stage requires that $\pi_n(FB) \cong \text{Hom}(\pi^n B, \mathbf{Q})$ for $n \geq 2$ and B a cofibrant algebra, where $\pi^n B$ is defined to be the \mathbf{Q} -vector space spanned by the indecomposables of degree $-n$ in B . The fact that these “homotopy groups” of algebras satisfy a Whitehead theorem is also required, since the proof deduces that the counit map induces an isomorphism on these homotopy groups for all n , in order to prove that the counit is an isomorphism.

The second step involves using the Eilenberg-Moore spectral sequence to show that if the proposition holds for $X = V$, then it holds for the homotopy fibre of $V \rightarrow K(\mathbf{Q}, n)$, for any $n \geq 2$. Similarly, the Eilenberg-Moore spectral sequence is used to show that if the proposition holds for $B = V$, then it holds for the homotopy cofibre of $S(-n) \rightarrow V$, for any $n \geq 2$.

The third step uses the finite-type and nilpotent condition, in that every rational nilpotent space of finite-type has a Postnikov tower built in finitely steps from pullbacks of pathspace fibrations with base an Eilenberg-Mac Lane space, and every minimal CDGA of finite type is built in finitely many steps from pushouts of $S(-n) \rightarrow D(-(n+1))$ (where $D(-(n+1))$ has a single generator in degrees $-(n+1)$ and $-n$, and the differential takes the generator in degree $-n$ to the one in degree $-(n+1)$). By the previous two stages, the result is then proved. From this proposition, the following theorem is immediate:

THEOREM 2.7. ([6]) *Let $sSset_{f, nil, rat}$ and $cdga_{f, coconn}$ denote the full subcategories of finite-type, nilpotent, rational simplicial sets and finite-type, rational, coconnected CDGAs. Then the functors $\mathbb{R}F$ and $\mathbb{L}A = A$ induce contravariant equivalences of the homotopy categories of these subcategories.*

REMARK 2.8. We are avoiding discussing basepoints for spaces (and augmentations for CDGAs) throughout all of this chapter, and indeed, the previous proposition is in fact proved in the pointed/augmented case, and then the unpointed/unaugmented case follows. However, for this particular work the difference is unimportant, and there are no issues caused by choices of basepoint and augmentation, and so one can use whichever version of the results is most convenient.

To conclude this section, we briefly discuss its applications. Thus far, all the simplicial sets we have been working with have been rational, however, in practice spaces of interest are hardly ever rational from the outset. But notice that there is nothing in the definition of A which prevents it from being applied to non-rational simplicial sets. In fact, there is a *follow-up* theorem to 2.7, known as the *Sullivan de Rham localisation theorem*, which states that the unit map $X \rightarrow FAX$ is always a *rationalisation*, for any (not necessarily rational) connected simply connected space X of finite-type. This means that the unit induces an isomorphism on all homotopy groups after tensoring with \mathbf{Q} and is universal with respect to this property. Hence, we can use the fact that $\pi_n(FB) \cong \text{Hom}(\pi^n B, \mathbf{Q})$ for any cofibrant algebra B and $n \geq 2$, to deduce that

$$\pi_n(X) \otimes \mathbf{Q} \cong \text{Hom}(\pi^n M, \mathbf{Q})$$

where M is a minimal model for AX , and X is any connected nilpotent space of finite-type. In other words, we can calculate the rational homotopy groups of a space completely from a minimal model for its cochain complex. In practice, explicitly computing a minimal model for a CDGA B can be very hard when B is not *formal* (formal meaning that B is quasi-isomorphic to H^*B). This condition may at first appear rather tame, but there are very accessible examples of non-formal CDGAs, such as the rational cochain complex of the complement in \mathbb{R}^3 of the Borromean rings (or indeed any CDGA with a non-trivial Massey product). However, if B is formal, we can compute a minimal model by instead computing one for H^*B , which will be much easier, especially if B has finite \mathbf{Q} -type, as assumed above.

CHAPTER 3

Derived Commutative Algebra and RHT

In classical homological algebra, the objects of interest live in some Abelian category \mathcal{A} , and the derived category of \mathcal{A} is the category of chain complexes of objects of \mathcal{A} with chain maps as morphisms, localised at the class of quasi-isomorphisms. \mathcal{A} could for example be the category of modules over a commutative ring R . However in topological contexts, the rings that we work with often come with a natural grading and differential to begin with, such as the cochain complex on a space. This instead leads us to consider “graded homological algebra”, where instead of considering a category of modules over a ring R , we consider the category of differential graded modules (*DG*-modules) over a CDGA R . From this perspective, we have generalisations of *Ext* and *Tor* to differential graded objects, and various notions of projectivity for computing derived functors (it turns out that the categorical notion of projectivity is far too strong to be of any interest in differential graded contexts). These notions of projectivity nicely correspond to the notion of cofibrancy in a model category, and the chapter following this one talks of how we can define a model structure on the category of DG-modules over a DGA, using a general theory applying to modules over monoids in monoidal model categories. We summarise some of the theory below, the main reference being an unpublished paper by Avramov, Foxby and Halperin, but our intuition was also helped by parts of [30]. Since we always want our DGAs to be commutative, when we write C^*X for cochains on a space X , we really mean $A_{PL}SingX$, where A_{PL} is the polynomial de Rham functor from the previous chapter, and *Sing* is the singular chain functor from spaces to simplicial sets.

DEFINITION 3.1. Rather than modules over a ring, the main objects of study are *DG-modules*, which are in general, differential graded modules M

over a differential graded algebra R over a field k . Precisely, this means that M is an R -module when we view R just as a ring, and M also has a grading, and scalar multiplication of $m \in M$ by $r \in R$ satisfies $r.m \in M^{|r|+|m|}$. Finally, M has a differential d of degree -1 which satisfies

$$d(r.m) = d(r).m + (-1)^{|r|}rd(m)$$

where we use d to denote both the differential on R and on M .

Given a CDGA R , the category $DG - R - mod$ of differential graded R -modules is a symmetric monoidal category under the tensor product \otimes_R , which we now define (\otimes with no subscript just denotes the ordinary tensor product of complexes over the ground field k).

Let M and N be $DG - R$ -modules. Then both R_0 and R act on M and N , and so we have a morphism

$$\delta^{R,M,N} : R \otimes M \otimes N \rightarrow M \otimes N$$

given on generators by

$$r \otimes m \otimes n \mapsto (rm) \otimes n - (-1)^{|r||m|}m \otimes (rn)$$

we then set $M \otimes_R N = Coker(\delta^{R,M,N})$, where the cokernel is taken in the category of k -modules. $M \otimes_R N$ then inherits the structure of a $DG - R$ -module, where the differential is that induced from the differential on $M \otimes N$ (it is an easy check that the differential on $M \otimes N$ is zero on the image of $\delta^{R,M,N}$).

As one can see, the main difference between DG -modules and ordinary chain complexes over a ring, is that M_n is no longer an R -module, as it is not closed under the scalar multiplication given above. However, each M_n is an R_0 -module, and so when R is just a ring, by considering it as a DGA concentrated in degree 0, we recover the definition of a chain complex over a ring. Supposing R is a differential graded k -algebra for some (ungraded)

field k , and M is a DG-module over R , we can define the homology of M as a graded k -module in the usual way. In this generalised setting, H^*M does not inherit the structure of a graded R -module, however H^*M does inherit the structure of a graded H^*R -module.

We now give a well known example of where DG -modules appear.

EXAMPLE 3.2. ([4])

Suppose we have a homotopy pullback square in Top given by

$$\begin{array}{ccc} L & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

Then applying C^* to this diagram gives a homotopy pushout square of CDGAs:

$$\begin{array}{ccc} C^*L & \longleftarrow & C^*Y \\ \uparrow & & \uparrow \\ C^*X & \longleftarrow & C^*Z \end{array}$$

which means that $C^*L = C^*X \otimes_{C^*Z}^{\mathbb{L}} C^*Y$, where $\otimes^{\mathbb{L}}$ denotes the derived tensor product. So in this case, C^*Z , C^*X and C^*Y are DG-modules over C^*Z , despite also being DGAs themselves, and we have that

$$H^*Z = H^*(C^*X \otimes_{C^*Z}^{\mathbb{L}} C^*Y)$$

and the latter is what we define to be $Tor^{C^*Z}(C^*X, C^*Y)$. Under certain conditions on the original spaces, we can use a gadget called the Eilenberg-Moore spectral sequence to compute $Tor^{C^*Z}(C^*X, C^*Y)$. The E_2 page of this spectral sequence is given by $Tor^{H^*Z}(H^*X, H^*Y)$, and the said conditions ensure that it converges strongly to $Tor^{C^*Z}(C^*X, C^*Y)$. Sufficient conditions for this to converge are that all the spaces X, Y and Z are simply connected.

In order to compute derived functors in the category $DGM - R$ of DG - R -modules, we need to have a notion of cofibrant replacement, or at the very least, be able to take resolutions which we know are “sufficiently nice” to capture all the necessary homotopical information. In any category, an object P is said to be *projective* if $Hom(P, -)$ preserves epimorphisms. However, in categories of differential graded objects such as $DGM - R$, this tends to be too strong a condition to give a useful class of resolutions, and as is shown in Theorem 9.7.1 of [3], any projective object of $DGM - R$ is necessarily contractible, and so has trivial homology. Hence the usual categorical notion of projectivity could only give resolutions of objects which have trivial homology, which is far from all objects of the category! [3] gives various weakenings of the notion of projectivity (termed linear, homotopical and semi projectivity), with a nice diagram showing their relationship to each other on p.63 of [3]. However, when actually performing computations, we can usually avoid all the complications of these varied notions of projectivity by working with a smaller class of objects called *semi-free* DG -modules. Just as in the ungraded case, where every module in fact has a free resolution, and free objects are projective, in the graded case, every object in fact has a semi-free resolution, and semi-free objects satisfy all the weakened notions of projectivity given in [3]. The reason for the term semi-free, is that (similarly to the notion of categorical projectivity), the notion of freeness in the differential graded case gives a very restricted class of objects.

DEFINITION 3.3. A DG - R -module M is said to be *semi-free* if it has a *semi-basis*, that is, if it has a subset $E \subset M$ which spans M , and where $E = \bigcup_{u \geq 0} E^u$ is a union of disjoint graded subsets $E^u \subset M$ such that

$$d(E^u) \subset \langle \bigcup_{i < u} E^i \rangle$$

for all u , where the angled brackets denote taking the smallest DG - R -module spanned by the contents.

THEOREM 3.4. (8.3.2 of [3]) *Given a DG - R -module M , there is a surjective quasi-isomorphism $\epsilon : L \rightarrow M$, where L is a semi-free DG - R -module.*

With this at hand, we can now define Ext and Tor in the graded case from a computational viewpoint in the following way.

Given $DG - R$ -modules M and N , and a semi-free resolution $L \rightarrow M$, define

$$Tor^R(M, N) = H^*(L \otimes_R N)$$

and define

$$Ext_R(M, N) = H^*(hom(L, N))$$

In the latter case, $hom^{\mathbb{R}}(L, N)$ denotes the internal Hom object of $DGR - mod$ viewed as a symmetric monoidal category with monoidal product given by the tensor product of DG -modules as defined above. Explicitly

$$hom(L, N)^n = Hom_R(L, \Sigma^n N)$$

where Σ^n shifts everything down by n degrees. The differential is then given by

$$(d(f))(v) = d(f(v)) - (-1)^{|f|} f(dv)$$

In the case when R is a ring and M and N are just R -modules, by viewing everything as concentrated in degree 0 we recover the usual definitions of Ext and Tor (our semi-free resolution L becomes a free resolution of M in the classical sense, in that it is exact everywhere except at degree 0).

All of the above is flavoured towards computation, and derived functors in the graded setting.

Homotopy invariant formulations of conditions on rings

We now turn to derived commutative algebra, and its close relationship to rational homotopy theory. Such a relationship is to be expected from the equivalence in 2.7, but what makes it more interesting is that it allows the translation of algebraic concepts into the world of rational homotopy theory. In particular, classical concepts in commutative ring theory, such as the conditions of regular, complete intersection (ci), or Gorenstein rings can be given homotopy invariant formulations, leading to the concept of a regular (or

ci, or Gorenstein) space. We begin by looking at the regular condition, and so we recall various equivalent definitions of a regular commutative ring (indeed, all rings will be commutative and Noetherian unless otherwise stated). Also, when we write Hom objects or tensor products, these will all be assumed to be derived unless stated otherwise.

DEFINITION 3.5. Given a ring R and an R -module M , a *regular sequence* on M of length n is a sequence $x_1, \dots, x_n \in M$ such that for all $0 \leq i \leq n$, x_i is not a zero-divisor in $\frac{R}{(x_1, \dots, x_{i-1})}$. A *regular sequence on R* will always be understood to be a regular sequence on R as a module over itself.

DEFINITION 3.6. Given an object X of a category \mathfrak{C} and a cardinal κ , we say that X is κ -small in \mathfrak{C} if the functor $\mathfrak{C}(X, -) : \mathfrak{C} \rightarrow \text{Set}$ preserves sequential colimits of length λ , for any κ -filtered ordinal λ . (An ordinal λ is κ -filtered if it is a limit ordinal, and for any $A \subset \lambda$ with $|A| \leq \kappa$, we have $\sup(A) < \lambda$.) For the duration of this chapter only, we will say that X is *small* if it is κ -small for some *finite* cardinal κ (this is the definition of X being *finite* in \mathfrak{C} given in [26]).

DEFINITION 3.7. (Ideal theoretic regularity)

A local ring (R, \mathfrak{m}, k) is *regular* if \mathfrak{m} can be generated by a regular sequence on R .

THEOREM 3.8. (*Definitions of regularity*)

For a local Noetherian ring (R, \mathfrak{m}, k) , the following are equivalent:

1. R is regular.
2. $\text{Ext}_R(k, k)$ is finite dimensional over k .
3. Every finitely generated R -module is small in the derived category $\mathbf{D}(R)$.

The hardest part to prove is that 2 implies 1, which is part of the Auslander-Buchsbaum-Serre theorem (stating that a local ring is regular if its global dimension is finite), and is proved in [38]. It is claimed in [20] that the other parts of the proof are not hard, however we have decided to

include their proofs, as we were unable to source them anywhere in the literature. For an R -module M , we denote by $pd_R(M)$ the *projective dimension* of M , which is the infimum of the set of lengths of projective resolutions of M over R . We denote by $gldim(R)$ the *global dimension* of R , which is the supremum of the set of projective dimensions of R -modules.

Before proving 3.8 we need a few results. For all of the following results R will be a local ring with maximal ideal \mathfrak{m} and residue field k . We start with a result of Auslander from [2]:

THEOREM 3.9. (*Theorem 1 of [2]*)

For any ring R

$$gldim(R) = \sup_{B \in cyc(R)} (pd_R(B))$$

where $cyc(R)$ is the set of all R -modules which are generated by a single element.

The importance of 3.9 is that it means that the global dimension of any ring is fully determined by the projective dimensions of its finitely generated modules.

We will also need the following lemma.

LEMMA 3.10. (8.2 of [31])

Suppose R is local and Noetherian. Let M be a finitely generated R -module. Then M admits a free resolution

$$\dots \xrightarrow{d_n} F_n \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} M \rightarrow 0$$

such that for each n , the image of $F_n \rightarrow F_{n-1}$ is contained in $\mathfrak{m}F_{n-1}$. Such a resolution is termed minimal.

PROOF. We can choose a minimal set of finitely many generators for M , and take F_0 to be free with one basis element for each generator. Then we have the natural map $\epsilon : F_0 \rightarrow M$. Now by Nakayama's lemma, the map $F_0 \otimes_R k \rightarrow M \otimes_R k$ induced by ϵ must be an isomorphism, and hence $\ker(\epsilon) \subset \mathfrak{m}F_0$. Now to construct d_1 , take a minimal set of generators for

$\ker(\epsilon)$, and let F_1 be free with a basis element for each generator, and then take d_1 to be the natural map. Now since R is Noetherian and F_0 is finitely generated, the submodule $\ker(\epsilon)$ is finitely generated also, and so we can apply Nakayama's lemma as before, to deduce that $\ker(d_1) \subset \mathfrak{m}F_1$. In this way, we can then continue constructing the resolution inductively. \square

PROPOSITION 3.11. (8.3 of [31]) *Suppose R is local and Noetherian. Then*

$$\text{gldim}(R) = \text{pd}_R(k)$$

PROOF. Clearly $\text{pd}_R(k) \leq \text{gldim}(R)$, and so by 3.9, it suffices to prove that $\text{pd}_R(M) \leq \text{pd}_R(k)$, for any finitely generated R -module M . Indeed, we can take a minimal resolution $\{F_j\}_{j \geq 0}$ of M as in 3.10, and since the resolution is minimal, all the differentials on $F_* \otimes_R k$ are zero, and hence

$$0 = \text{Tor}_{\text{pd}_R(k)+1}^R(k, M) \cong \text{Tor}_{\text{pd}_R(k)}^R(M, k) = F_{\text{pd}_R(k)+1} / \mathfrak{m}F_{\text{pd}_R(k)+1}$$

and so by Nakayama's Lemma, $F_{n+1} = 0$. Hence $\text{gldim}(R) \leq \text{pd}_R(k)$. \square

We will also need to know about the *Koszul complex* corresponding to a regular sequence.

DEFINITION 3.12. For any $a \in R$, define $K(a)$ to be the complex

$$0 \rightarrow R \xrightarrow{a} R \rightarrow 0$$

where the leftmost copy of R is in degree 1 and the rightmost is in degree 0. For a set of elements $X = \{a_1, \dots, a_n\} \subset R$, define

$$K_X = K(a_1) \otimes \dots \otimes K(a_n)$$

so that K_X is the exterior DGA on n generators e_1, \dots, e_n , all of degree 1. K_X is free on the basis

$$\{e_{i_1} \wedge \dots \wedge e_{i_q} \mid 1 \leq i_1 < \dots < i_q \leq n\}$$

and has differential given by

$$d(e_{i_1} \wedge \dots \wedge e_{i_q}) = \sum_{p=1}^q (-1)^{p-1} a_{i_p} e_{i_1} \wedge \dots \wedge e_{i_{p-1}} \wedge e_{i_{p+1}} \wedge \dots \wedge e_{i_q}$$

The importance of the above construction for our purposes is the following result

THEOREM 3.13. (4.5 of [31])

If $X = a_1, \dots, a_m$ is a regular sequence on R , then K_X is a free resolution of $R/(a_1, \dots, a_m)$.

In the case where R is local, regular and $X = a_1, \dots, a_m$ generates the maximal ideal, K_X is a free resolution of k , and is called the *Koszul Complex* of R .

We now prove 3.8.

PROOF. First we prove that 1 implies 2.

Suppose R is regular. We begin by showing that $\text{gldim}(R)$ is finite. By 3.11 $\text{gldim}(R) = \text{pd}_R(k)$. Now by assumption, R is a regular local ring, and so has maximal ideal generated by a regular sequence, and by taking the Koszul Complex corresponding to this regular sequence, we get a finite projective resolution of k , and hence $\text{gldim}(R)$ is finite. This proves that 1 implies 2.

We now prove that 1 implies 3.

Suppose again that R is regular. Then once again, $\text{gldim}(R)$ is finite, and hence any R -module M has a finite free resolution. Now suppose that M is a finitely generated R -module, as in the hypothesis of definition 3 of 3.8. Then since R is regular and Noetherian, M has a finite free resolution $P_* \rightarrow M$, where each P_* has finite rank. We need to show that M is small in $\mathbf{D}(R)$ for which it suffices to show that for any set S and chain complexes N_s over R (one for each $s \in S$), that the map

$$\theta : \bigoplus_{s \in S} \text{Hom}(P, N_s) \rightarrow \text{Hom}(P, \bigoplus_{s \in S} N_s)$$

given by $f_{s_1} + \dots + f_{s_t} \mapsto f$, where $f(m) = f_{s_1}(m) + \dots + f_{s_t}(m)$, is an isomorphism of Abelian groups.

θ is clearly injective. To show that θ is surjective, we simply observe that since P is a finite complex of free modules of finite rank, any map $g : P \rightarrow \bigoplus_{s \in S} N_s$ must have image contained in $\bigoplus_{s \in S'} N_s$, for some finite subset $S' \subset S$, and hence there is a corresponding map in $\bigoplus_{s \in S} \text{Hom}(P, N_s)$, which is taken to g under θ .

We now prove that 3 implies 2.

Suppose that F_* is the minimal free resolution of k , and that it is of infinite length. By the minimality condition on the resolution, the complex $\text{Hom}_R(F_*, k) \cong \bigoplus_{n \geq 0} \text{Hom}_R(F_n, \Sigma^n k)$ has zero differential. By 3, the map

$$\theta : \bigoplus_{n \geq 0} \text{Hom}(F_*, \Sigma^n k) \rightarrow \text{Hom}(F_*, \bigoplus_{n \geq 0} \Sigma^n k)$$

is an isomorphism. Provided R is not a field, for each n , we can pick some $x_n \in F_n$ such that x_n is not in the kernel of d_n (and fields are regular, so the case of R being a field is trivial). Moreover, since the resolution is free, the subspace $\langle x_n \rangle$ of F_n is disjoint from the kernel of d_n . So we can define a map $f \in \text{Hom}(F_*, \bigoplus_{n \geq 0} \Sigma^n k)$ of degree 0 by setting $f(x_n) = 1$, for all n , and $f(x) = 0$ for x not in one of the subspaces $\langle x_n \rangle$. There can be no element in the domain of θ which maps to f , because the image of f is non-zero in infinitely many dimensions. This is a contradiction, and hence the resolution F_* must be finite, and hence the global dimension of R is finite, from which 2 follows.

That 2 implies 1 follows from the Auslander-Buchsbaum-Serre theorem. \square

These three formulations of the regular condition in fact carry over nicely into the derived setting.

DEFINITION 3.14. (Regular conditions on spaces)

Let X be a rational 1-connected space.

1. X is *s-regular* if there exist fibration sequences

$$S^{n_k} \rightarrow * \rightarrow X_k, S^{n_{k-1}} \rightarrow X_k \rightarrow X_{k-1}, \dots, S^{n_0} \rightarrow X_1 \rightarrow X_0$$

where $X_0 = X$.

2. X is *g-regular* if $H_*(\Omega X)$ is finite-dimensional over \mathbb{Q} .
3. X is *h-regular* if every object in

$$FG(X) := \{M \in C^*X\text{-mod} \mid H^*M \text{ is finitely generated as an } H^*X\text{-module}\}$$

is small in $\mathbf{D}(C^*X)$.

From the Morita theory given above, it is easy to see where the motivation for the *g-regularity* condition comes from: it corresponds to 2 of 3.8 in terms of the Ext algebra, and hence the *g* stands for “growth”. The correspondence works in the following way.

ΩX can be constructed as the homotopy pullback of the diagram

$$* \rightarrow X \leftarrow *$$

Hence $C^*(\Omega X)$ is the homotopy pushout of

$$\mathbb{Q} \leftarrow C^*X \rightarrow \mathbb{Q}$$

Rewriting this, we have

$$C^*(\Omega X) \simeq \mathbb{Q} \otimes_{C^*X} \mathbb{Q}$$

So applying $Hom_{\mathbb{Q}}(-, \mathbb{Q})$ to both sides (that is, dualising) gives that

$$C_*(\Omega X) \simeq Hom_{\mathbb{Q}}(\mathbb{Q} \otimes_{C^*X} \mathbb{Q}, \mathbb{Q})$$

and now applying the tensor-hom adjunction gives that

$$C_*(\Omega X) \simeq \text{Hom}_{C^*X}(\mathbb{Q}, \mathbb{Q})$$

Hence $H_*(\Omega X) \cong \text{Ext}_{C^*X}(\mathbb{Q}, \mathbb{Q})$, and hence we see the relationship to 2 of 3.8.

The h in h -regular stands for “homotopy”, and it corresponds to definition 3 of 3.8. We use $FG(X)$ as defined, because asking for all finitely generated C^*X -modules to be small is too weak to obtain a meaningful definition of regularity, and indeed there exist small objects which are not finitely generated over C^*X .

It is perhaps less clear how the s-regular condition emerges, and so we motivate it in the following way.

Observe that a local ring (R, \mathfrak{m}, k) is regular in the sense of definition 1 of 3.14 if and only if we have a sequence of monomorphisms of R -modules

$$R \xrightarrow{\cdot x_1} R, \frac{R}{(x_1)} \xrightarrow{\cdot x_2} \frac{R}{(x_1)}, \dots, \frac{R}{(x_1, \dots, x_{n-1})} \xrightarrow{\cdot x_n} \frac{R}{(x_1, \dots, x_{n-1})}$$

If we now suppose that R was a CDGA, then we can take homotopy cofibres of the above maps in the category of DG- R -modules, and obtain cofibre sequences

$$\frac{R}{(x_1, \dots, x_j)} \xrightarrow{\cdot x_{j+1}} \frac{R}{(x_1, \dots, x_j)} \rightarrow \frac{R}{(x_1, \dots, x_{j+1})}$$

for each j . Hence

$$\frac{R}{(x_1, \dots, x_{j+1})} \simeq \frac{R}{(x_1, \dots, x_j)} \oplus^{\Sigma^{|x_j|-1}} \left(\frac{R}{(x_1, \dots, x_j)} \right)$$

with differential twisted by the map $\cdot x_{j+1}$. So we now consider the homotopy cofibre sequence of CDGAs given by

$$\frac{R}{(x_1, \dots, x_j)} \rightarrow \frac{R}{(x_1, \dots, x_{j+1})} \rightarrow \frac{R}{(x_1, \dots, x_{j+1})} \otimes_{\frac{R}{(x_1, \dots, x_j)}} k$$

and we observe that

$$\frac{R}{(x_1, \dots, x_{j+1})} \otimes_{\frac{R}{(x_1, \dots, x_j)}} k \simeq \left(\frac{R}{(x_1, \dots, x_j)} \oplus \Sigma^{|x_j|-1} \left(\frac{R}{(x_1, \dots, x_j)} \right) \right) \otimes_{\frac{R}{(x_1, \dots, x_j)}} k$$

and this is the same as $k \oplus \Sigma^{|x_j|-1}k$. Hence when $R = C^*X$ and X is connected, the above cofibration sequence translates to a fibration sequence of spaces $S^{|x_j|-1} \rightarrow X_{i+1} \rightarrow X_i$, hence motivating the definition of s-regularity.

In order to show that the three definitions in 3.14 are equivalent for 1-connected rational spaces, [20] first classify all rational g-regular spaces. As they observe, having to use such a classification feels somewhat unsatisfactory, and indeed, when considering the three definitions in characteristic p , no such simple classification exists, and it is strongly suspected (though has not actually been shown) the s-regular condition is strictly stronger than the others. However, due to the simplicity of the rational case, it is not hard to see that s-regular implies g-regular, since if X is s-regular, $\Omega X_k \simeq S^{n_k}$, and an easy inductive argument gives that $H_*(\Omega X_i)$ is finite dimensional for all i . Because $C_*(\Omega X) \simeq \text{Hom}_{C^*X}(\mathbb{Q}, \mathbb{Q})$ and $\mathbb{Q} \in FG(X)$, it is also immediate that h-regular implies g-regular. For the other implications the following classification is used.

THEOREM 3.15. (4.3 of [20])

*A 1-connected rational space X is g-regular if and only if π_*X is finite dimensional and concentrated in even degrees.*

Note that in the rational context, the conclusion of 3.15 implies that such an X must be a product of even Eilenberg-Mac Lane spaces, and hence the theorem immediately gives us that g-regular implies s-regular.

PROOF. Rationally, ΩX is always a product of Eilenberg-Mac Lane spaces, and hence by the Kunneth Theorem $H_*(\Omega X)$ is finite dimensional if and only if these Eilenberg-Mac Lane spaces are odd (since odd Eilenberg-Mac Lane spaces are spheres with finite dimensional cohomology, and even Eilenberg-Mac Lane spaces have polynomial cohomology).

□

From 3.15 the authors obtain the following result.

PROPOSITION 3.16. *(4.4 of [20]) If X is g -regular and H^*M is finitely generated over H^*X , then M is small.*

PROOF. Since X is g -regular, we can apply 3.15 and the argument is given in the proof of 4.4 of [20]. □

COROLLARY 3.17. *For a 1-connected rational space X , the s -regular, g -regular and h -regular conditions are all equivalent.*

REMARK 3.18. The complete intersection and Gorenstein conditions on rings can also be formulated in a homotopy invariant way for spaces. In the case of the complete intersection condition, there are many different formulations, and they are not all equivalent, even in the rational case. The complete intersection condition for rational spaces is the main focus of [20], and we refer the interested reader there. The Gorenstein condition will make an appearance in Chapter 5, where we will give the definition, and a new class of examples with infinite dimensional cohomology.

A generalisation of Regular Spaces

We now discuss an extended notion of s -regularity which makes sense for non-simply connected spaces. Unlike rational s -regularity in the simply connected case, as described in the previous section, there are interesting examples coming from the action of the fundamental group on the higher homotopy groups, or also from possibly non-trivial k -invariants. A natural question to ask would be whether the other notions of regularity can be interpreted in a similar way for non-simply connected spaces, and we have not found what we consider to be a sensible way of doing this. It should be noted that the s -regular condition for simply connected spaces has the illusion of being a very “structural” condition, in that it requires a decomposition of the space in question using iterated fibrations. However, we know from the previous section that it is equivalent to a much more concrete growth condition on the cohomology. The notion of s -regularity given here will in

fact be much more structural in nature, and we give examples illuminating this.

Localisation of spaces. This very short section is just to outline what we mean by the terms “rational space” or “rationalisation”, when the space in question is not necessarily simply connected. We use localisation in the sense of [18]. We will say a space X is *rational* if its universal cover is rational in the classical sense. When we say the *rationalisation* of a space X , we mean the space defined by

$$X \otimes^G \mathbb{Q} := (\tilde{X} \otimes \mathbb{Q})/G$$

where $- \otimes \mathbb{Q}$ denotes the classical rationalisation of a simply connected space in the sense of [41] (or many other sources), and G is the fundamental group of X , which acts on \tilde{X} in the usual way, and so acts on $\tilde{X} \otimes \mathbb{Q}$. Since classical rationalisation is functorial, we can form the quotient $X \otimes^G \mathbb{Q}$, which also has fundamental group G , and the action of G on the higher homotopy is simply obtained by tensoring the action on the integral homotopy by \mathbb{Q} . Our only exception to the above is that in 3.20 and where obvious, S^1 will denote the rational circle, for which we use any model of $K(\mathbb{Q}, 1)$. This is consistent with the meaning of S^1 in [20], when they discuss the s-regular condition.

Regular Spaces.

DEFINITION 3.19. For simply connected G -spaces X and Y , and any simply connected space F , we say $F \rightarrow X \rightarrow Y$ is a *1-fibration sequence* if $X \rightarrow Y$ is a G -map and $F \rightarrow X \rightarrow Y$ is a fibration sequence when X and Y are viewed simply as non-equivariant spaces.

This definition describes a state of affairs which occurs when we wish to lift a fibration sequence $F \rightarrow X \rightarrow Y$ to a “ G -fibration sequence”

$$F' \rightarrow \tilde{X} \rightarrow \tilde{Y}$$

involving the universal covers \tilde{X} and \tilde{Y} of X and Y respectively, where $G = \pi_1 X = \pi_1 Y$. Under these conditions, F' will not in general have a G -action coming naturally from the original sequence, hence we do not strictly

get a G -fibration sequence, but instead what we have termed a 1-fibration sequence.

DEFINITION 3.20. Let X be a path connected space with $\pi_1 X = G$. X is *s-regular* if there exists fibration sequences

$$S^{n_k} \rightarrow BG \rightarrow X_k, S^{n_{k-1}} \rightarrow X_k \rightarrow X_{k-1}, \dots, S^{n_0} \rightarrow X_1 \rightarrow X_0$$

where $X_0 = X$, and each fibration induces an isomorphism on π_1 of its source and target, for some positive integers n_0, \dots, n_k .

REMARK 3.21. The “s” in s-regular stands for *structurally*, and is to distinguish the above notion of regularity from ones we may study in the future (and which are well understood in [20] in the simply connected case). To motivate the definition from the simply connected case, if we lift the fibration sequences of 3.20 to universal covers, we get 1-fibration sequences

$$S^{n_k} \rightarrow EG \rightarrow \tilde{X}_k, S^{n_{k-1}} \rightarrow \tilde{X}_k \rightarrow \tilde{X}_{k-1}, \dots, S^{n_0} \rightarrow \tilde{X}_1 \rightarrow \tilde{X}_0$$

and if we view these non-equivariantly, we recover the definition of \tilde{X} being s-regular in the sense of [20] (since EG is contractible by definition). Thus our definition can be considered a generalisation of that in [20], with extra rigidity coming from a possibly non-trivial fundamental group.

We can immediately give a large class of examples of non-simply connected regular spaces.

THEOREM 3.22. *Let X be a path connected space with nilpotent actions of the fundamental group on $\pi_n X$ for $n \geq 2$, and $\pi_* \tilde{X}$ finite dimensional over \mathbb{Q} and concentrated in even degrees below $2d$, for some minimal d . Suppose also that $H^{2k+1}(\pi_1 X; \mathbb{Q})$ being non-zero implies $k = 0$ or $k > d$. Then X is s-regular.*

REMARK 3.23. From 3.22 we can see that any path connected space X with $\pi_* \tilde{X}$ finite dimensional over \mathbb{Q} and concentrated in even degrees, $\pi_1 X$ finite, and $\pi_1 X$ acting trivially on the higher homotopy groups must be s-regular (finite groups have trivial rational cohomology, and so the condition

on the rational cohomology of the fundamental group is met in this case). So for G a finite group, any principal G -bundle $\tilde{X} \rightarrow X \rightarrow BG$ will be s-regular provided \tilde{X} has even homotopy.

For the proof of 3.22 we will need the following lemma, proved as Proposition 8^{bis}.2. of [33].

LEMMA 3.24. *If $F \xrightarrow{i} X \xrightarrow{f} Y$ is a fibration sequence with F 1-connected and X and Y connected (so that $\pi_1 X = \pi_1 Y = G$), then the long exact sequence of Abelian groups*

$$\dots \rightarrow \pi_n F \rightarrow \pi_n X \rightarrow \pi_n Y \rightarrow \pi_{n-1} F \rightarrow \dots \rightarrow \pi_2 F \rightarrow \pi_2 X \rightarrow \pi_2 Y \rightarrow 0$$

is also a long exact sequence of $\mathbb{Q}G$ -modules, where G acts on $\pi_n X$ and $\pi_n Y$ by the usual action of the fundamental group on higher homotopy, and G acts on $\pi_n F$ by monodromy.

We now prove the theorem.

PROOF. Let $G = \pi_1 X$. We argue by induction on the total dimension of $\pi_*(\tilde{X})$. Clearly BG is regular. Let $2m$ be the lowest dimension in which \tilde{X} has non-zero homotopy. Then since X is nilpotent, it has a Postnikov decomposition where each fibration decomposes as a composition of principal fibrations. The decomposition up to the $2m$ th section of this Postnikov tower looks like:

$$\begin{array}{ccccccc} & & & & & K(V, 2m) & \\ & & & & & \downarrow & \\ \dots & \longrightarrow & P_{2m,1}X & \longrightarrow & P_{2m,q}X & \longrightarrow & P_{2m,q-1}X & \longrightarrow & \dots & \longrightarrow & P_{2m,1}X & \xrightarrow{f} & BG \\ & & & & & & & & & & & & \downarrow \psi & \\ & & & & & & & & & & & & & K(V, 2m+1) \end{array}$$

where the staircase at the right of the diagram is a fibration sequence (of course such sequences exist for each map in the tower, but we have suppressed the rest). Now since f is principal, the action of G on V is trivial,

and since V is a rational vector space, the hypothesis on the rational cohomology of BG implies that ψ is nullhomotopic, and hence f is a trivial fibration, and so

$$P_{2m,1}X \simeq K(V, 2m) \times BG$$

Hence we can replace f with a composition of principal (and trivial) fibrations

$$P_{2m,1}X \xrightarrow{p_2} K_b \dots \xrightarrow{p_3} K_1 \xrightarrow{p_1} BG$$

each with fibre $K(\mathbb{Q}, 2m)$, such that the limit of the resulting Postnikov tower is unchanged. So

$$K_1 \simeq K(\mathbb{Q}, 2m) \times BG$$

and so composing the projection map $K \rightarrow K(\mathbb{Q}, 2m)$ with the canonical map $X \rightarrow K$ gives a map $X \rightarrow K(\mathbb{Q}, 2m)$ which induces a quotient map $\pi_{2m}X \rightarrow \mathbb{Q}$, where G acts trivially on the generator represented by \mathbb{Q} . So taking successive fibres gives a fibration sequence

$$S^{2m-1} \simeq K(\mathbb{Q}, 2m-1) \rightarrow X' \rightarrow X \rightarrow K(\mathbb{Q}, 2m)$$

where the homotopy equivalence on the left exists because we are working rationally. The long exact sequence in homotopy for this fibration gives the short exact sequence

$$0 \rightarrow \pi_{2m}X' \rightarrow \pi_{2m}X \rightarrow \mathbb{Q} \rightarrow 0$$

and so $\pi_{2m}X'$ is of dimension one lower than $\pi_{2m}X$, and so to complete the inductive argument, it suffices to check that X' is nilpotent. But by 3.24, $\pi_{2m}X'$ is a subrepresentation of $\pi_{2m}X$, and hence $\pi_{2m}X'$ is nilpotent. \square

For the next result about s -regular spaces, we briefly review composition series and the Jordan-Hölder theorem. All definitions and results concerning these purely algebraic facts are classical, and full details can be found in one form or another in a vast array of places in the literature. We include them purely for the purposes of clarifying our conventions.

DEFINITION 3.25. Let R be a ring and M an R -module. A finite sequence of submodules

$$0 = M_0 \subset M_1 \subset \dots \subset M_k = M$$

of M is a *normal series* if $M_j \neq M_{j+1}$ for all j . It is a *composition series* if moreover M_{j+1}/M_j is irreducible for all j (an irreducible R -module being one with no non-zero proper R -submodules). The R -modules M_{j+1}/M_j are called the *composition factors* of the series.

The following two results are standard theory.

LEMMA 3.26. *If M is a Noetherian and artinian R -module, then any normal series for M can be refined to become a composition series. In particular, a composition series for M exists.*

THEOREM 3.27. (*Jordan-Hölder*)

For any ring R and R -module M with a composition series, all composition series of M have the same length, and the same composition factors, up to isomorphism and permutation.

Thus we can simply talk of the *composition factors* of M , whenever M has a composition series.

COROLLARY 3.28. *For any group G , field k , and kG -module V which is finitely generated over k , V has a composition series, and all composition series of V have the same composition factors up to kG -isomorphism and permutation.*

PROOF. Any chain of kG -submodules of V is also a chain of k -vector subspaces of V , and hence is finite. Hence V is artinian and Noetherian as a kG -module, so we can apply 3.27 to obtain the result.

□

We now apply the above theory of composition series to yield the following result.

PROPOSITION 3.29. *Suppose X is an s -regular space with $G := \pi_1 X$. Then for all n , $\pi_{2n} X$ has a composition series, and the composition factors are all 1-dimensional over \mathbb{Q} (though not necessarily trivial).*

PROOF. Since s-regular spaces have finite dimensional homotopy over \mathbb{Q} , by 3.28, $\pi_{2n}X$ has a composition series for all n . Suppose that for some n , $\pi_{2n}X$ has a composition factor V of \mathbb{Q} -dimension strictly greater than 1. Let $S^m \rightarrow X' \rightarrow X$ be the first fibration sequence in a regular decomposition for X . We will show that V must be a composition factor of $\pi_{2n}X'$.

If $m \neq 2n$ or $2n - 1$, then $\pi_{2n}X' \cong \pi_{2n}X$ as $\mathbb{Q}G$ -modules, and so they have the same composition factors.

If $m = 2n$, then the long exact sequence for the fibration has a segment

$$0 \rightarrow U \rightarrow \pi_{2n}X' \rightarrow \pi_{2n}X \rightarrow 0$$

where U is a 1-dimensional representation, and the sequence is a short exact sequence of $\mathbb{Q}G$ -modules. Now by 3.26, the inclusion $U \subset \pi_{2n}X'$ can be refined to a composition series

$$0 \subset U = U_0 \subset U_1 \subset \dots \subset U_k = \pi_{2n}X'$$

Now quotienting this series by U gives the composition series

$$0 \subset U_1/U \subset \dots \subset \pi_{2n}X'/U \cong \pi_{2n}X$$

of $\pi_{2n}X$. Now by 3.27 there exists some j such that $(U_j/U)/(U_{j-1}/U) \cong V$, and hence $U_j/U_{j-1} \cong V$; in other words, V is a composition factor of $\pi_{2n}X'$.

Finally, if $m = 2n - 1$, then the long exact sequence for the fibration has a segment

$$0 \rightarrow \pi_{2n}X' \rightarrow \pi_{2n}X \rightarrow U' \rightarrow 0$$

where U' is a 1-dimensional representation. By a similar argument as the $m = 2n$ case above, the inclusion $\pi_{2n}X' \subset \pi_{2n}X$ can be refined to a composition series

$$0 \subset W_1 \subset \dots \subset W_{b-1} \subset W_b = \pi_{2n}X' \subset \pi_{2n}X$$

(since U' is 1-dimensional, there can be no proper intermediate subrepresentation between $\pi_{2n}X'$ and $\pi_{2n}X$). So since V is a composition factor of $\pi_{2n}X$, and has dimension greater than 1, by 3.8 there exists $1 \leq t \leq b$ such that $W_t/W_{t-1} \cong V$ as $\mathbb{Q}G$ -modules, and hence V occurs as a quotient in the composition series of $\pi_{2n}X'$ given by $\{W_i\}_i$.

So since X is s-regular, we deduce (by working along the fibrations in its decomposition) that $\pi_{2n}(BG) = 0$ has V as a quotient in its composition series, which is a contradiction, and so completes the proof. \square

REMARK 3.30. In particular 3.29 implies that for an s-regular space X , any irreducible subrepresentation of $\pi_{2n}X$ must be 1-dimensional over \mathbb{Q} .

The proposition above gives some criteria by which we can determine when spaces are not s-regular in the sense of this chapter. It also shows that some conditions on actions of the fundamental group are necessary for 3.22 to hold (nilpotence of actions is a potential condition, although the example of $BO(2n)$ given later shows that it is not always necessary). However, one could still wonder whether the conditions on the cohomology of the fundamental group in the theorem are necessary. For example, a natural question would be whether there exists a space with higher homotopy groups all 1-dimensional and concentrated in finitely many even degrees, but which is not s-regular (see the example 3.31 below). In general, for spaces which do not have exotic actions of the fundamental group on their homotopy groups which disallow them being s-regular by 3.29, no good method is known for showing that such spaces are not s-regular. This is because it requires showing that certain fibration sequences do not exist, which without a good source of invariants at hand, is a very hard problem. The issue comes from the possible existence of even dimensional spherical fibrations in the decomposition of an s-regular space, and such fibrations kill homotopy in one degree and create homotopy in another, and it is not clear that such a process is always unnecessary in the decomposition of an s-regular space. However, we have been able to make some progress with simple cases. The following is an example of a space which has trivial actions of its fundamental group on its

higher homotopy groups, and higher homotopy concentrated in even degrees, yet is not s -regular.

EXAMPLE 3.31. Consider the space X with $G := \pi_1 X = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, $\pi_2 X = \mathbb{Q}$ and zero in all other degrees, with k -invariant given by the map $BG \rightarrow K(\mathbb{Q}, 3)$ corresponding to the cochain given by the orientation class of $BG = S^1 \times S^1 \times S^1$.

PROPOSITION 3.32. *The X defined in the example above is not s -regular.*

PROOF. The first step in showing this is to show that if $S^m \rightarrow X' \rightarrow X$ is the first step in an s -regular decomposition of X , then m cannot be even: indeed, if m were even, then since X has no homotopy in degrees above 2, and S^m has homotopy in degree $2m - 1$, the long exact sequence for the fibration gives an exact sequence

$$0 \rightarrow \mathbb{Q} \rightarrow \pi_{2m-1} X' \rightarrow 0$$

But since X is s -regular, so is X' , and hence \tilde{X}' has homotopy concentrated in even degrees, and hence $\pi_{2m-1} X' = 0$, which is a contradiction. Hence m cannot be even.

A similar argument using the long exact sequence of the fibration shows that if m is odd, then it must be 1, or in other words, any decomposition of X must be of length 1, meaning we have a single fibration $S^1 \rightarrow BG \rightarrow X$. We will now show using the Serre spectral sequence that the total space of such a fibration cannot have cohomology equal to that of BG , which completes the argument. Suppose that such a fibration exists. Observe that

$$C^* X \simeq \mathbb{Q}[x] \otimes \wedge(a, b, c)$$

where x is codegree 2, and a, b and c are codegree 1, and $dx = abc$. Hence the E_2 -page of the Serre spectral sequence for the fibration

$$S^1 \rightarrow BG \rightarrow X$$

is concentrated in the 0th and 1st row, and all the entries in these rows are \mathbb{Q} -vector spaces of dimension 3, except in the zeroth column where both entries are 1 dimensional. Observe that the differential

$$d : \mathbb{Q} \langle \omega \rangle \rightarrow \mathbb{Q} \langle ab, bc, ca \rangle$$

leaving the $(0, 1)$ -entry has image of dimension at most 1. Hence, because the spectral sequence must converge to $H^*(BG)$ (which is of dimension 3 in degree 2), the differential

$$d : \mathbb{Q} \langle \omega a, \omega b, \omega c \rangle \rightarrow \mathbb{Q} \langle xa, xb, xc \rangle$$

leaving the $(1, 1)$ -entry must be non-zero. But $d(\omega a) = (d\omega)a = 0$, because $d\omega$ is a linear combination of ab, bc and ca and $a^2 = abc = 0$ in cohomology. Similarly, $d(\omega b) = d(\omega c) = 0$, and so this differential is zero, which is a contradiction.

□

For completeness, we could now ask if there are spaces with even higher homotopy, which do not have obstructions of the kind above (more precisely, they satisfy the cohomological conditions on the fundamental group of 3.22), and satisfy the conclusion of 3.29, but yet still fail to be s-regular.

REMARK 3.33. As shown on page 89 of [39], the action of $\pi_0(O(2)) = \mathbb{Z}/2$ on $\pi_1(O(2)) = \mathbb{Q}$ is non-trivial (precisely, the non-identity element acts as -1), and thus the action of $\pi_1(BO(2))$ on $\pi_2(BO(2))$ is non-trivial. This generator in $\pi_1(O(2)) = \mathbb{Q}$ reappears in $\pi_{2n-1}(O(2n)) = \mathbb{Q}$ for each n , and so $\pi_{2n}(BO(2n))$ is acted on non-trivially by $\pi_1(BO(2n)) = \mathbb{Z}/2$ for all n .

EXAMPLE 3.34. For all n , $BO(n)$ is an example of an s-regular space with fundamental group $\mathbb{Z}/2$ (since we have the fibration sequences

$$S^{n-1} \rightarrow BO(n-1) \rightarrow BO(n)$$

for all n). As we observe later, $BO(2n-1)$ is rationally a product of EM-spaces, as all the actions of its fundamental group are trivial, and it has no

non-trivial k -invariants. For $BO(2n)$, we have the fibration sequence

$$S^{2n-1} \rightarrow BO(2n-1) \rightarrow BO(2n)$$

where n is odd and not 1. The non-identity element of $\pi_1(BO(2n)) = \mathbb{Z}/2$ acts as -1 on the unstable generator of $\pi_{2n}(BO(2n))$ (the latter is two dimensional if n is even, and one dimensional if n is odd, as it is in this case), which is a non-trivial, and hence non-nilpotent action, since $\mathbb{Z}/2$ is finite. From the long exact sequence of the fibration, we see that there is an isomorphism of $\mathbb{Q}(\mathbb{Z}/2)$ -modules $\pi_{2n}(BO(2n)) \rightarrow \pi_{2n-1}(S^{2n-1})$, and so the monodromy action is also non-trivial. Hence $BO(2n)$ (for any n) is an example of a space which is s-regular but not nilpotent.

In the simply connected case, it is in fact true that all s-regular spaces are a finite product of even Eilenberg-Mac Lane spaces, and so the class of regular spaces is rather restricted from a homotopical viewpoint in this case. In our non-simply connected case, the corresponding statement would be that for a space X with fundamental group G , X is s-regular if and only if the \tilde{X} is a G -equivariant product of even Eilenberg-Mac Lane spaces. However, even when the action of the fundamental group is trivial on all higher homotopy, this result fails due to the existence of possibly non-trivial equivariant k -invariants. However, when the fundamental group is finite and its actions are nilpotent (equivalently trivial), we can recover the classification result, since the rational cohomology of finite groups is purely torsion, and so vanishes rationally. The details are spelled out in the next proposition below:

DEFINITION 3.35. A connected space X is *simple* if $\pi_1 X$ is Abelian, and its action on $\pi_n X$ is trivial for all $n \geq 2$.

PROPOSITION 3.36. *If X is a simple space with finite fundamental group G , and \tilde{X} is rational with $\pi_* \tilde{X}$ concentrated in even degrees, then X is a product of Eilenberg-Mac Lane spaces.*

PROOF. Let

$$K(M, 2n) \rightarrow X_{2n} \rightarrow X_{2n-2} \xrightarrow{k} K(M, 2n+1)$$

be the fibration sequence of the $2n$ th Postnikov section of X , where $M = \pi_{2m}X$ is a \mathbb{Q} -vector space. Now

$$k \in [X_{2n-2}, K(M, 2n+1)] \cong H^{2n+1}(X_{2n-2}; M)$$

and we have the fibration sequence $\tilde{X}_{2n-2} \rightarrow X_{2n-2} \rightarrow BG$, and since the action of G on the higher homotopy is trivial, the Serre spectral sequence (SSS) for this fibration has E_2 -page given by

$$E_2^{p,q} = H^p(G; H^q(\tilde{X}_{2n-2}; M)).$$

Now since \tilde{X} has homotopy concentrated in even degrees, it is a finite product of even Eilenberg-Mac Lane spaces, and hence so is \tilde{X}_{2n-2} , and so since even EM-spaces have even rational cohomology, the Kunneth theorem shows that \tilde{X}_{2n-2} has even rational cohomology. Now since the rational cohomology of finite groups is trivial, the only non-zero column in the SSS, is the 0th column, and in this column, the only non-zero entries are of even height, and all the differentials on this page and all subsequent pages vanish, and hence $H^*(X_{2n-2}; M)$ is concentrated in even degrees, and so $H^{2n+1}(X_{2n-2}; M) = 0$, and hence k is nullhomotopic. This shows that the k -invariants of all the higher homotopy are rationally trivial. The k -invariant $BG \rightarrow K(\pi_2 X, 3)$ is trivial since $H^3(BG; \mathbb{Q}) = 0$, and hence X is a product of Eilenberg-Mac Lane spaces.

□

The following example shows that it is possible for a space to be s-regular, with trivial actions of the fundamental group G on its higher homotopy groups, whilst its universal cover is not a G -equivariant product of EM-spaces.

EXAMPLE 3.37. Consider $G = \mathbb{Z}$. We construct a space X with $\pi_1 X = \mathbb{Z}$, $\pi_2 X = \mathbb{Q}$, $\pi_4 X = \mathbb{Q}$, and zero in all other degrees, with trivial action of $\pi_1 X$ on all the homotopy groups. By the Kunneth theorem

$$H^*(\mathbb{C}P^\infty \times B\mathbb{Z}; \mathbb{Q}) = H^*(\mathbb{C}P^\infty; \mathbb{Q}) \otimes H^*(S^1; \mathbb{Q})$$

and so in particular $H^5(\mathbb{C}P^\infty \times B\mathbb{Z}; \mathbb{Q})$ is non-zero ($\mathbb{C}P^\infty$ is a $K(\mathbb{Q}, 2)$ rationally). So consider the Postnikov decomposition defined by any non-trivial k -invariant

$$X_2 = K(\mathbb{Q}, 2) \times B\mathbb{Z} \rightarrow K(\mathbb{Q}, 5)$$

Then the resulting space X is not a product of Eilenberg-Mac Lane spaces, although X is regular by 3.22, and has trivial actions of the fundamental group on all homotopy groups.

CHAPTER 4

The Gorenstein condition

In this chapter we look at the homotopy Gorenstein (written henceforth h-Gorenstein) condition on spaces, which follows the same philosophy as the conditions of regularity defined in the previous chapter, in that it is a homotopy invariant notion on spaces/CDGAs, which is induced from an algebraic notion. A nice aspect of the h-Gorenstein condition is that even rationally, there is a wealth of examples (such as any finite Poincaré duality space). We will give the basic definitions and machinery, then prove a duality property for simply connected rational h-Gorenstein spaces. Finally, we will look at a class of examples of h-Gorenstein spaces which do not have finite dimensional rational cohomology. This fact means that they do not necessarily have Gorenstein cohomology ring (in the purely ring-theoretic sense), and we say precisely when this happens for the given class of examples.

We make the following conventions for this chapter.

k will always denote a field of characteristic 0.

Given a CDGA A over k , and DG- A -modules M and N , $\text{Hom}_A(M, N)$ will denote the DG- A -module of A -homomorphisms from M to N of varying degree (such homomorphisms need not respect the various differentials).

Felix, Halperin and Thomas in their paper [14] adapt the usual definition of a local Gorenstein ring to the context of simply connected spaces. They make the following definition.

DEFINITION 4.1. An augmented CDGA A over k is *h-Gorenstein of shift a* if $\text{Hom}_A(k, A) \simeq \Sigma^a k$.

DEFINITION 4.2. A space X is *h-Gorenstein of shift a* over k if $C^*(X; k)$ is h-Gorenstein over k .

REMARK 4.3. The usual definition of a Gorenstein local ring R is a ring of finite injective dimension over itself. If R is Noetherian and k denotes the residue field of R , then R is Gorenstein if and only if $\text{Ext}_R^*(k, R)$ is concentrated in a single degree, and is isomorphic to k in that degree. Moreover this degree must be the Krull dimension of the ring. This is the motivation for the definition above.

The point of defining it in this way is that the definition is *homotopy invariant*, meaning that if we have a quasi-isomorphism $A \rightarrow A'$ of CDGAs over k , then A is h-Gorenstein of shift a if and only if A' is.

Felix, Halperin and Thomas prove

THEOREM 4.4. (*“Gorenstein Ascent” as in [14]*)

If $F \rightarrow E \rightarrow B$ is a fibration sequence of 1-connected spaces with $H^(F; \mathbb{Q})$ finite dimensional, and both F and B are Gorenstein over \mathbb{Q} of shifts f and b respectively, then E is Gorenstein over \mathbb{Q} of shift $f + b$.*

This result allows us construct many classes of examples of h-Gorenstein spaces from some obvious starting ones, such as polynomial rings with coefficients in \mathbb{Q} (these are regular and local, hence Gorenstein). We will present a class of examples of this kind later.

To illustrate the importance of the examples given later, we now give some known facts about finite dimensional Gorenstein spaces. From now on $k = \mathbb{Q}$, and all (co)homology is with coefficients in \mathbb{Q} .

THEOREM 4.5. *Let A be a CDGA such that $H^0 A \cong \mathbb{Q}$, $H^1 A = 0$, and $H^* A$ is finite dimensional over \mathbb{Q} . Then the following are equivalent*

1. *A is h-Gorenstein.*
2. *$H^*(A)$ is a Gorenstein ring.*
3. *$H^*(A)$ satisfies Poincaré duality.*

PROOF. The result follows from 3.6 of [14].

□

Gorenstein Duality and the Local Cohomology Spectral Sequence

The Gorenstein condition is closely related to a duality property, as described at length in various contexts in [13]. We present a version here in the rational context. We first recall the notion of cellular objects in model categories. We will give it in terms of DG-modules, however the basic theory is common to all model categories. We will then describe two constructions of the cellularisation of DG-modules. This is all described in detail in section 4 of [19] (the cited paper is an earlier unpublished version of [20]).

DEFINITION 4.6. For a CDGA A and a class of objects C in the homotopy category of DG- A -modules, a DG- A -module M is said to be C -cellular if it is in the smallest class of objects containing C which is closed under homotopy colimits and weak equivalences.

A map $M \rightarrow N$ of DG- A -modules is a C -equivalence if for all $L \in C$

$$\text{Hom}_A(L, M) \rightarrow \text{Hom}_A(L, N)$$

is a weak equivalence (a quasi-isomorphism).

A map $M \rightarrow N$ is a C -cellular approximation if it is a C -equivalence and M is C -cellular.

Such an M is unique up to weak equivalence, and is called the C -cellularisation of N .

We will describe different constructions of cellularisation, but when we are only concerned with cellularisation up to homotopy type, we will write $\text{Cell}_C(N)$ for the cellularisation of N .

For our purposes, we will really only need to concern ourselves with when $C = \{\mathbb{Q}\}$, and so from now on this will be the assumed context.

The first model of \mathbb{Q} -cellularisation comes from Morita Theory. As described in 4.1 and 4.2 of [19], there is an adjunction

$$\text{Hom}_{C^*X}(\mathbb{Q}, -) : C^*X\text{-mod} \rightarrow \text{mod} - C_*(\Omega X) : (-) \otimes_{C_*(\Omega X)} \mathbb{Q}$$

and when X is 1-connected, the Eilenberg-Moore and Rothenberg-Steenrod theorems give equivalences

$$C_*(\Omega X) \simeq \text{Hom}_{C^*X}(\mathbb{Q}, \mathbb{Q})$$

and

$$C^*X \simeq \text{Hom}_{C_*(\Omega X)}(\mathbb{Q}, \mathbb{Q})$$

respectively, and thus the adjunction gives an equivalence of the simply connected subcategories of the respective homotopy categories. They then prove the following

PROPOSITION 4.7. (4.2 of [19])

*If H^*X is Noetherian, then the counit of the adjunction*

$$\text{Hom}_{C^*X}(\mathbb{Q}, M) \otimes_{C_*(\Omega X)} \mathbb{Q} \rightarrow M$$

is a \mathbb{Q} -cellularisation of M .

We now give an alternative model for cellularisation, using local cohomology as defined by Grothendieck, and presented in [19] and [13]. As is typical in derived commutative algebra, we will have a classical ring theoretic formulation, with an adaptation to DGAs.

Suppose first that R is a commutative ring with an ideal $I = (x_1, \dots, x_n)$, and let N be any R -module.

DEFINITION 4.8. The *local cohomology* of N over R is

$$H_I^*(R; N) := H^*\left(\left(R \rightarrow R\left[\frac{1}{x_1}\right]\right) \otimes_R \dots \otimes_R \left(R \rightarrow R\left[\frac{1}{x_n}\right]\right) \otimes_R N\right)$$

where each $R \rightarrow R[\frac{1}{x_i}]$ is a complex concentrated in degrees 0 and 1.

When $N = R$ we write $H_I(R) := H_I(R; R)$.

Now suppose A is a CDGA over \mathbb{Q} . For any $0 \neq [x] \in H^*A$, write $\Gamma_x A$ for the homotopy fibre of the map $A \rightarrow A[\frac{1}{x}]$. One model for $\Gamma_x A$ is the square zero extension $A \vee \Sigma^{-1}A[\frac{1}{x}]$ which in degree $-m$ is defined to be

$A_{-m} \oplus A[\frac{1}{x}]_{-(m-1)}$, with multiplication given by

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, a_1b_2 + (-1)^{|b_1|}b_1a_2)$$

(using fact that $A[\frac{1}{x}]$ is canonically an A -bimodule). Note that this particular model has an obvious filtration as an A -module given by

$$0 \subset \Sigma^{-1}A[\frac{1}{x}] \subset \Gamma_x A$$

Now for any ideal $I = ([x_1], \dots, [x_n])$ in H^*A , and any DG- A -module M write

$$\Gamma_I M := \Gamma_{x_1} A \otimes_A \otimes_A \dots \otimes_A \Gamma_{x_n} A \otimes_A M$$

When I is the maximal ideal, we simply write ΓM for the above.

It turns out that, up to equivalence, $\Gamma_I M$ depends only on the ideal I (in fact it depends only on the radical of I). In particular, the homotopy type of each $\Gamma_{x_i} A$ does not depend on the choice of representative of $[x_i]$.

We observe that ΓM has a natural filtration of length $n + 1$, coming from the tensor product of the filtrations of each individual term, and we have a spectral sequence for calculating the cohomology of ΓM corresponding to this filtration as follows.

LEMMA 4.9. (4.3 of [19])

There's a spectral sequence

$$H_I^*(H^*A; H^*M) \implies H^*(\Gamma M)$$

and to make this of relevance, we have

PROPOSITION 4.10. (9.3 of [13])

The canonical map

$$\Gamma M \rightarrow M$$

is a \mathbb{Q} -cellularisation.

In particular, by uniqueness of cellularisation up to homotopy, if $A = C^*X$ and X is simply connected we have that

$$\Gamma M \simeq \text{Hom}_A(\mathbb{Q}, M) \otimes_{C_*(\Omega X)} \mathbb{Q}$$

and this allows us to formulate Gorenstein duality. Before doing so, we remind the reader that we are working completely rationally, and so the formulation we give of Gorenstein duality is in fact encoded by the arguments in [13], although we give a purely rational statement and proof for clarity.

THEOREM 4.11. (*Gorenstein Duality*)

*Suppose A is an h -Gorenstein CDGA of shift a with H^*A 1-connected and Noetherian. Then there is an equivalence*

$$\text{Cell}_{\mathbb{Q}}A \simeq \Sigma^a A^\vee$$

and so by 4.9 we have a spectral sequence (the local cohomology spectral sequence)

$$H_I^*(H^*A) \implies \Sigma^a H^*(A)^\vee$$

PROOF. Since A is h -Gorenstein, we have an equivalence $\mathbb{Q} \rightarrow \Sigma^a \text{Hom}_A(\mathbb{Q}, A)$. Now consider the ring $\epsilon = \text{Hom}_A(\mathbb{Q}, \mathbb{Q})$. For any DG- A -module M , $\text{Hom}_A(\mathbb{Q}, M)$ has a natural right ϵ -module structure, and so since A is h -Gorenstein, this gives a right ϵ -module structure on \mathbb{Q} . Now observe that we have the chain of equivalences of DG- A -modules

$$\text{Hom}_A(\mathbb{Q}, A) \simeq \Sigma^a \mathbb{Q} \simeq \text{Hom}_A(\mathbb{Q}, \Sigma^a A^\vee)$$

(where the second is just the tensor-hom adjunction), and because A is 1-connected, ϵ is connected, and hence there is a unique right ϵ -module structure on \mathbb{Q} . Thus the two ϵ -module structures that $\Sigma^a \mathbb{Q}$ inherits from being equivalent to $\text{Hom}_A(\mathbb{Q}, A)$ and $\text{Hom}_A(\mathbb{Q}, \Sigma^a A^\vee)$ are the same. Hence $\text{Hom}_A(\mathbb{Q}, A)$ and $\text{Hom}_A(\mathbb{Q}, \Sigma^a A^\vee)$ are equivalent as ϵ -modules. So we can

apply $(-)\otimes_{\epsilon}\mathbb{Q}$ to both sides of this equivalence, and then since A is Noetherian, we can apply 4.7 to deduce that

$$\text{Cell}_{\mathbb{Q}}(A) \simeq \text{Cell}_{\mathbb{Q}}(\Sigma^a A^{\vee})$$

but because A is coconnected, A^{\vee} is bounded below, and hence A^{\vee} is in fact \mathbb{Q} -cellular as an A -module (see 3.17 of [13]). This proves the first part of the result, and the second part is now evident from the first part, 4.9 and 4.10.

□

Examples of Infinite Dimensional Gorenstein Spaces

In this section we construct a class of examples of non \mathbb{Q} -finite h-Gorenstein spaces, and give conditions under which their cohomology ring is (and is not) Gorenstein. Note that throughout this section we use standard grading conventions, in that the cohomology groups of spaces are concentrated in non-positive degrees, and the suspension Σ^b increases degrees by b (and so reduces codegrees by b).

Consider the fibration sequence

$$X \rightarrow K(\mathbb{Q}^m, 2n) \xrightarrow{\gamma} K(\mathbb{Q}^2, 2kn)$$

Now

$$H^*(K(\mathbb{Q}^2, 2kn)) \cong \mathbb{Q}[a, b]$$

$$H^*(K(\mathbb{Q}^m, 2n)) \cong \mathbb{Q}[u_1, \dots, u_m]$$

where a and b are both of degree $-2kn$, and the u_i are of degree $-2n$. Now $p := \gamma^*(a)$ and $q := \gamma^*(b)$ are both homogenous polynomials in u_1, \dots, u_m of the same polynomial degree $k \geq 1$ (and so their terms have degree $-2kn$).

PROPOSITION 4.12.

$$H^*X \cong \mathbb{Q}[u_1, \dots, u_m, \tau]/(p, q, \gcd(p, q)\tau, \tau^2)$$

where τ is of degree $-(4k - 2d)n + 1$, where d is the polynomial degree of $\gcd(p, q)$

PROOF. Let $A = \mathbb{Q}[a, b]$ and $\mathbf{U} = \{u_1, \dots, u_m\}$. Then the Eilenberg-Moore spectral sequence for the above fibration, is given by

$$E_2^{*,*} = \text{Tor}_*^A(\mathbb{Q}, \mathbb{Q}[\mathbf{U}]) \Rightarrow H^*X$$

where a and b act on $\mathbb{Q}[\mathbf{U}]$ by the polynomials p and q respectively. Now in order to compute the E_2 -page, we use a Koszul resolution of \mathbb{Q} over A , which is

$$\mathbb{Q} \leftarrow A \xleftarrow{\theta_1} \Sigma^{-2kn}(A \oplus A) \xleftarrow{\theta_2} \Sigma^{-4kn}A \leftarrow 0$$

where $\theta_1(\alpha) = a$ and $\theta_1(\beta) = b$, and $\theta_2(\alpha \wedge \beta) = a\beta - b\alpha$ (where α, β and $\alpha \wedge \beta$ are the canonical generators of $\Sigma^{-2kn}(A \oplus A)$ and $\Sigma^{-4kn}A$). Tensoring this resolution by $\mathbb{Q}[\mathbf{U}]$ gives the complex

$$0 \leftarrow \mathbb{Q}[\mathbf{U}] \xleftarrow{\phi_1} \Sigma^{-2kn}(\mathbb{Q}[\mathbf{U}] \oplus \mathbb{Q}[\mathbf{U}]) \xleftarrow{\phi_2} \Sigma^{-4kn}(\mathbb{Q}[\mathbf{U}]) \leftarrow 0$$

where $\phi_1(g, h) = pg + qh$ and $\phi_2(f) = (-fq, fp)$. Denote this (double) complex by C . Let $p' = p/\gcd(p, q)$ and $q' = q/\gcd(p, q)$, and let $-2dn$ be the degree of $\gcd(p, q)$. Then since $\mathbb{Q}[\mathbf{U}]$ is a domain,

$$\ker(\phi_1) = \Sigma^{-2kn} \langle (-fq', fp') | f \in \mathbb{Q}[\mathbf{U}] \rangle$$

and

$$\text{im}(\phi_2) = \Sigma^{-2kn} \langle (-fq, fp) | f \in \mathbb{Q}[\mathbf{U}] \rangle$$

Hence

$$H_1(C) \cong \Sigma^{-(4k-2d)n}(\mathbb{Q}[\mathbf{U}]/(\gcd(p, q)))$$

Now clearly $H_0(C) \cong \mathbb{Q}[\mathbf{U}]/(p, q)$, and since $\mathbb{Q}[\mathbf{U}]$ is a domain, $\ker(\phi_2) = 0$, and hence $H_2(C) = 0$. Hence there is no room for differentials on the E_2 -page of the EM-spectral sequence, and so we see that

$$H^*X \cong \mathbb{Q}[u_1, \dots, u_m, \tau]/(p, q, \gcd(p, q)\tau, \tau^2)$$

where τ is of degree $-(4k - 2d)n + 1$.

□

We could have in fact determined the rational cohomology of X by computing its Sullivan model. To do this, observe that X fits in to the fibration sequence

$$K(\mathbb{Q}^2, 2kn - 1) \rightarrow X \rightarrow K(\mathbb{Q}^m, 2n)$$

and so to find the Sullivan model (M_X, d) for X we find by taking the following “twisted product” (where $|u_j| = -2n$ and $|\alpha| = |\beta| = -(2kn - 1)$)

$$M_X := \mathbb{Q}[u_1, \dots, u_m] \otimes \mathbb{Q}[\alpha, \beta]/(\alpha^2, \beta^2)$$

as a graded algebra, but where the differential satisfies

$$d(\alpha) = \gamma^*(a) = p, \text{ and } d(\beta) = \gamma^*(b) = q$$

and is 0 on the other generators.

Now let $\tau = \frac{q}{\gcd(p,q)} \otimes \alpha - \frac{p}{\gcd(p,q)} \otimes \beta \in M_X$. Then $d(\tau) = 0$, τ lives in degree $-(4k - 2d)n + 1$ (where d is the polynomial degree of $\gcd(p, q)$) and all the cycles in M_X are generated by u_1, \dots, u_m and τ . So upon taking homology, we recover the result of 4.12.

We now look at the local cohomology spectral sequence for H^*X . We will denote the local cohomology of a module M over a ring R with respect to an ideal I by $H_I^*(R; M)$. The local cohomology of a local ring R is defined to be $H_{\mathfrak{m}}^*(R; R)$, where \mathfrak{m} is the maximal ideal. We now compute the local cohomology of H^*X .

PROPOSITION 4.13. *Letting $g = \gcd(p, q)$, and $p' = p/g$ and $q' = q/g$ as before, and $P := \mathbb{Q}[u_1, \dots, u_m]$, we have that*

$$H_{\mathfrak{m}}^N(H^*X; H^*X) = \begin{cases} \Sigma^{-2n(2k-d-1)+m}(P/(p', q'))^\vee, & \text{if } N = m - 2, \\ \Sigma^{-2n(d-1)+m}(P/(g))^\vee \oplus \Sigma^{-2n(2k-1)+m+1}(P/(g))^\vee, & \text{if } N = m - 1, \\ 0, & \text{otherwise} \end{cases}$$

PROOF. Let $R = H^*X$ and $\bar{P} = P/(p, q)$. Then

$$R \cong \bar{P} \oplus \Sigma^{-(4k-2d)n+1}(\bar{P}/(g))$$

as a P -module, and we have a chain of morphisms

$$P \rightarrow \bar{P} \rightarrow R$$

where each target is finitely generated as a module over its source (that is, the morphisms are *finite*) and hence

$$H_{\mathfrak{m}}^*(R; R) \cong H_{\mathfrak{m}}^*(\bar{P}; R) \cong H_{\mathfrak{m}}^*(P; R) \cong H_{\mathfrak{m}}^*(P; \bar{P}) \oplus \Sigma^{-(4k-2d)n+1} H_{\mathfrak{m}}^*(P; \bar{P}/(g))$$

where the first isomorphism is induced from the finite map of rings $\bar{P} \rightarrow R$, the second isomorphism is induced from the finite map of rings $P \rightarrow \bar{P}$, and the third isomorphism comes from the description of R as a P -module given above, and the fact that local cohomology commutes with direct sums in its second argument.

So from now on, we will assume all local cohomology is over the ring P , so will suppress the first argument. In order to calculate $H_{\mathfrak{m}}^*(\bar{P})$ and $H_{\mathfrak{m}}^*(\bar{P}/(g))$, we use the three short exact sequences of P modules

$$0 \rightarrow (g) \rightarrow P \rightarrow P/(g) \rightarrow 0 \quad (\text{S1})$$

(note $P/(g) = \bar{P}/(g)$),

$$0 \rightarrow \Sigma^{-(4k-2d)n} P \rightarrow \Sigma^{-2kn}(P \oplus P) \xrightarrow{\phi} (p, q) \rightarrow 0 \quad (\text{S2})$$

(where $\phi(f, h) = fp + hq$), and

$$0 \rightarrow (p, q) \rightarrow P \rightarrow \bar{P} \rightarrow 0 \quad (\text{S3})$$

Label these short exact sequences by $S1$, $S2$ and $S3$ respectively. The long exact sequence in local cohomology coming from $S1$ is

$$0 \rightarrow H_{\mathfrak{m}}^{m-1}(P/(g)) \rightarrow \Sigma^{-2dn} H_{\mathfrak{m}}^m(P) \rightarrow H_{\mathfrak{m}}^m(P) \rightarrow H_{\mathfrak{m}}^m(P/(g)) \rightarrow 0$$

where $H_{\mathfrak{m}}^{m-1}(P) = 0$ because P is a polynomial ring on m generators, and so has no local cohomology outside of degree m . The right most entry is zero because the Krull dimension of each entry in the short exact sequence is $\leq m$, and so there can be no non-zero local cohomology in degrees above m .

Now $H_{\mathfrak{m}}^m(P/(g)) = 0$, because the Krull dimension of $P/(g)$ is at most $m-1$. So using the fact that P satisfies Gorenstein duality as in 4.11, the exact sequence becomes (after dualising)

$$0 \leftarrow H_{\mathfrak{m}}^{m-1}(P/(g))^{\vee} \leftarrow \Sigma^{2dn-2n-m} P \leftarrow \Sigma^{-2n-m} P \leftarrow 0$$

where the right most non-trivial map is multiplication by g , and hence

$$H_{\mathfrak{m}}^{m-1}(P/(g)) \cong \Sigma^{-2n(d-1)+m}(P/(g))^{\vee}$$

and is zero in all other degrees. The long exact sequence for $S2$ is (after dualising)

$$0 \leftarrow H_{\mathfrak{m}}^{m-1}(p, q)^{\vee} \leftarrow \Sigma^{2n(2k-d-1)-m} P \leftarrow \Sigma^{2n(k-1)-m}(P \oplus P) \leftarrow H_{\mathfrak{m}}^m(p, q)^{\vee} \leftarrow 0$$

and the middle map sends (f, h) to $fp' + hq'$, and so has kernel generated by the single element $(q', -p')$, and image the ideal (p', q') (the zeros appear at either end of the sequence for the same reasons as for the long exact sequence of $S1$). Hence

$$\begin{aligned} H_{\mathfrak{m}}^m(p, q) &\cong \Sigma^{-2n(d-1)+m} P^{\vee} \\ H_{\mathfrak{m}}^{m-1}(p, q) &\cong \Sigma^{-2n(2k-d-1)+m}(P/(p', q'))^{\vee} \end{aligned}$$

The long exact sequence of S_3 is computed similarly, and gives that

$$H_{\mathfrak{m}}^{m-2}(\bar{P}) \cong H_{\mathfrak{m}}^{m-1}(p, q) \cong \Sigma^{-2n(2k-d-1)+m}(P/(p', q'))^{\vee}$$

$$H_{\mathfrak{m}}^m(\bar{P}) = 0$$

$$H_{\mathfrak{m}}^{m-1}(\bar{P}) \cong \Sigma^{-2n(d-1)+m}(P/(g))^{\vee}$$

Now collecting all these calculations together yields the result. □

We can now recover the shift of R from 4.13. It is given by $-2n(2k-1) + m + 1 - (m-1) = -2n(2k-1) + 2$, which is consistent with shift predicted by 2.4. There are other nice facts to be seen from the local cohomology, for example, if p and q were coprime (ie, $g = \gcd(p, q)$ is a unit), then $\bar{P} = P/(p, q)$ would have dimension $m-2$, and so we would expect $H_{\mathfrak{m}}^{m-1}(\bar{P})$ to be zero, and indeed the above is consistent with this.

Whenever one is considering h -Gorenstein spaces, it is a natural question to ask whether their cohomology rings are classically Gorenstein as commutative rings (this is always the case if the cohomology is finite dimensional). The computation in 4.13 now allows us to do this in full generality for this class of examples

PROPOSITION 4.14. *For any such p and q as above, H^*X is Gorenstein as a commutative ring if and only if p and q are coprime, or if they are equal up to multiplication by a unit.*

PROOF. The fact that X is h -Gorenstein follows from 4.4, as does the shift (the shift can also be read off from our computation as above).

We now turn to when H^*X is actually Gorenstein in the classical sense as a commutative ring. Clearly, if $\gcd(p, q)$ is a unit, then H^*X is a complete intersection ring, hence Gorenstein. In the case where $\gcd(p, q)$ is a non-unit, consider first the case where p and q are not unit multiples of each other. In this case, observe that since p and q are homogenous and of the same non-zero degree, $p' := p/\gcd(p, q)$ and $q' := q/\gcd(p, q)$ do not generate P as an

ideal. We know from the calculation of H^*X in 4.12 that it has dimension at most $m - 1$, and from the local cohomology calculation in 4.13 that it has dimension at least $m - 1$, and so it has dimension $m - 1$. But by the calculation above

$$H_{\mathfrak{m}}^{m-2}(H^*X) \cong \Sigma^{(4k-2d-2)n-m}(P/(p', q'))^\vee$$

which is non-zero in this case, and hence the local cohomology spectral sequence does not collapse, which means H^*X is not Cohen-Macaulay, and hence not Gorenstein.

Now consider the case where p and q are equal up to multiplication by a unit. In this case, we have

$$H^*X \cong \mathbb{Q}[u_1, \dots, u_m, \tau]/(p, \tau^2) \cong \mathbb{Q}[u_1, \dots, u_m]/(p) \otimes_{\mathbb{Q}} \wedge(\tau)$$

Now $\wedge(\tau)$ is finitely generated as a \mathbb{Q} -module, and both $\mathbb{Q}[u_1, \dots, u_m]$ and $\wedge(\tau)$ are Gorenstein. Now Theorem 2 of [42] states that the tensor product of two Gorenstein rings $C \otimes_R D$ over a Gorenstein ring R , with C flat over R , and D finitely generated over R , is Gorenstein, and hence in this case H^*X is Gorenstein as required. □

REMARK 4.15. 4.14 effectively says that the cohomology ring of examples of this type is almost always Gorenstein.

When dealing with more than two polynomials in the original example, it is not clear how to compute the kernel of ϕ_1 in the computation of the cohomology of X , as it involves considerations of the greatest common divisors of more than two polynomials.

CHAPTER 5

PL Compactly supported forms in characteristic zero

For this chapter, given a simplicial set X , and a collection of simplices S of X , we denote by $\langle S \rangle$ the simplicial subset of X generated by S . If X is a simplicial set, and we write $K \subset X$, then unless otherwise stated, we will assume that K is a subsimplicial set, rather than just a collection of simplices in X (although on at least one occasion it will simply be the latter).

The initial question that led to this chapter was a Math Overflow question by Dan Petersen. He wanted to know if Sullivan's method of PL polynomial forms had a compactly supported analogue: in particular giving an explicit commutative model for compactly supported cohomology, that applies to all CW complexes, rather than just manifolds. An unfortunate feature of the category of manifolds is that many colimits do not exist in it, and this led authors to consider generalisations, the earliest of which was Chen's *Chen spaces* in 1973. This was followed by Souriau's *diffeological spaces* in 1980. The categories of both are complete and cocomplete, and have many other nice categorical properties (locally cartesian closed, weak subobject classifier). This interest beyond the category of manifolds means it is both interesting and helpful to see how much of the theory applying to manifolds can be carried over in some way. In particular, in [21], Haraguchi develops a theory of compactly supported cohomology for diffeological spaces. Our work in this chapter can be viewed as a simplicial analogue of Haraguchi's work.

We also feel that this work has potentially deep proper homotopical implications. Sullivan's original polynomial de Rham complex gives (under an array of conditions) an equivalence of two homotopy theories, and so one might expect that a compactly supported de Rham complex could give an

equivalence between certain “proper homotopy theories”. The sense in which “proper homotopy theory” would be meant is not completely clear, although the work of Baues and Quintero in [5] seems appropriate, and would need to be reworked for simplicial sets. The fundamental change in moving from homotopy theory to proper homotopy theory is that the notion of a model structure is no longer applicable, as the category of spaces with proper maps is far from having a natural model structure (it doesn’t even have a terminal object). Instead, the category of spaces and proper maps has the structure of a *cofibration category*, and this is the axiomatic framework in which [5] largely works. We have not had the time to pursue these homotopical questions, but think this would be a very interesting future project.

We will now begin the chapter by setting up the basic notions, and reminding the reader of important earlier constructions.

Recall the simplicial CDGA over k denoted by $\nabla(*, *)$, defined in 2.1.

DEFINITION 5.1. A simplicial set X is *finite* if it has only finitely many non-degenerate simplices, and is *locally finite* if every simplex is a face of only finitely many non-degenerate simplices.

DEFINITION 5.2. Let X be a simplicial set. $\nabla(*, *)$ allows the authors of [6] to define the CDGA A^*X of polynomial forms on X , by

$$A^q X := sSet(X, \nabla(*, q))$$

Correspondingly, we define the CDGA $A_c^* X$ of *compactly supported* polynomial forms on X by

$$A_c^q X := \{\Phi \in sSet(X, \nabla(*, q)) \mid \exists \text{ finite } K \subset X \text{ s.t. } \Phi|_{\langle X \setminus K \rangle} = 0\}$$

Given $\phi \in A^q X$ and a q -simplex σ of X , we will often write $\phi|_\sigma$ in place of $\phi(\sigma)$.

REMARK 5.3. $A^q X$ can be thought of as all possible ways of assigning a q -form to each simplex of X , in a manner which is compatible with the

face and degeneracy operators. Thus $A^q X$ is analagous to the global sections of a sheaf of functions on a space. It is easily checked that $A^* X$ and $A_c^* X$ are both well defined CDGAs. However, if X is not finite, the latter does not have any unit element as non-zero constant 0-forms are not compactly supported. This can be re-interpreted as the fact that the category of spaces with proper continuous maps has no terminal object.

We now give what will be an essential property of $\nabla(*, q)$ for our purposes. We will refer to this property as the *extension property*. The geometric interpretation of this property is that if one has a q -form defined on the boundary of simplex, then that form can be extended to a q -form on the entire simplex. Expressing this for $\nabla(*, q)$, the property states that if we are given forms $\omega_0, \dots, \omega_p \in \nabla(p-1, q)$ that model a form on the boundary of a p -simplex (the necessary condition for this is that $\partial_i \omega_j = \partial_{j-1} \omega_i$, for all i and j), then there exists a form $\omega \in \nabla(p, q)$ with $\partial_i \omega = \omega_i$, for all i . In addition, this can be done in a way which is linear with respect to addition of forms.

We now state the extension property described above in its precise form.

PROPOSITION 5.4. (*The Extension Property. Corollary 1.2 of [6]*)

There exists a naturally defined function

$$E : \{(w_0, \dots, w_p) \mid w_k \in \nabla(p-1, q) \text{ for all } k, \partial_i w_j = \partial_{j-1} w_i \text{ for all } i \leq j\} \rightarrow \nabla(p, q)$$

such that

$$\partial_i(E(w_0, \dots, w_p)) = w_i$$

for all i , and

$$E(w_0, \dots, w_p) + E(w'_0, \dots, w'_p) = E(w_0 + w'_0, \dots, w_p + w'_p)$$

5.4 is the *extension property* for $\nabla(*, *)$, and geometrically means that if we have a form defined over the boundary of a simplex, we can extend it to the entire simplex.

Bousfield and Gugenheim in [6] go on to prove that $\nabla(p, *)$ is acyclic (the Poincaré lemma), and they also define a process of formal integration of forms over a simplex, which we review now.

DEFINITION 5.5. Suppose $w \in \nabla(p, p)$ is given by $w = f(t_1, \dots, t_p)dt_1 \dots dt_p$, where f is a polynomial. Let $|\Delta^p|$ denote the standard p -simplex in \mathbb{R}^p given by $0 \leq t_i \leq 1$ and $0 \leq t_1 + \dots + t_p \leq 1$. Then since k has characteristic 0 we can compute $\int_{|\Delta^p|} f dt_1 \dots dt_p$ term by term as an integral over \mathbb{R} , the answer being a polynomial with coefficients in k , and so we define

$$\int w := \int_{|\Delta^p|} f dt_1 \dots dt_p$$

There is a total differential

$$\partial : \nabla(p, q) \rightarrow \nabla(p-1, q)$$

defined by $\partial = \sum_{i=0}^p (-1)^i \partial_i$, satisfying $\partial d = d\partial$. ∂ should be thought of as simply restricting a form to its boundary. We now have the following

PROPOSITION 5.6. (*Stokes' Theorem (Proposition 1.4 of [6])*)

For any $w \in \nabla(p, p-1)$

$$\int dw = \int \partial w$$

PL bump functions

One way of obtaining a Mayer-Vietoris sequence for A_c^* is to first show that we have a PL analogue of bump functions. The situation is made somewhat simpler than for smooth manifolds, since our functions can be piecewise smooth, however we are of course restricted within this to using piecewise polynomial functions.

DEFINITION 5.7. Let X be a simplicial set, and $K \subset X$ a simplicial subset. Define the *minimal neighbourhood* of K in X to be

$$\epsilon(K) := \langle \{ \sigma \in X \mid \partial_{i_1} \dots \partial_{i_m} \sigma \in K \text{ for some } \{i_j\} \} \rangle$$

EXAMPLE 5.8. Consider when X is the standard tessellation of the plane using equilateral 2-simplices. This is in fact a simplicial complex, but we can view it as a simplicial set where the faces of each non-degenerate simplex are also non-degenerate. In this case, if we take $L \subset X$ to be a single vertex v , then $\epsilon(L)$ is a hexagon with 6 non-degenerate 2-simplices, all meeting v .

THEOREM 5.9. (*Existence of PL bump functions*) *Let X be a simplicial set, and $L \subset K \subset X$ be subsimplicial sets such that $\epsilon(L) \subset K$. Then there exists some $\phi \in A^0(X)$ such that $\phi|_L = 1$ and $\phi|_{\langle X \setminus K \rangle} = 0$.*

PROOF. It suffices to prove the theorem in the case that $K = \epsilon(L)$, because the ϕ constructed in the proof for the case $K = \epsilon(L)$ will satisfy $\phi|_L = 1$ and $\phi|_{\langle X \setminus K \rangle} = 0$, for any K as in the statement of the theorem. So we need to construct ϕ so that $\phi|_{\langle X \setminus \epsilon(L) \rangle} = 0$. We begin by defining ϕ on $\langle \epsilon(L) \setminus L \rangle$. For each $m \geq 0$, denote the set of non-degenerate m -simplices of $\langle \epsilon(L) \setminus L \rangle$ by $\Sigma^m := \{\sigma_\alpha | \alpha \in I_m\}$, for some indexing set I_m . For each $\gamma \in I_0$, define

$$\phi|_{\sigma_\gamma^0} = \begin{cases} 1, & \text{if } \sigma_\gamma^0 \in L \\ 0, & \text{otherwise} \end{cases}$$

Now suppose ϕ has been defined on all simplices of Σ^k , for each $k < n$, in such a way that $\partial_i(\phi|_{\sigma_\gamma^k}) = \phi|_{\partial_i \sigma_\gamma^k}$, for all i , for all $\gamma \in I_k$ and $k \leq n$, and also that

$$\phi|_{\sigma_\gamma^k} = \begin{cases} 1, & \text{if } \sigma_\gamma^k \in L \\ 0, & \text{if for all } s \geq 0, \text{ and for all } i_1, \dots, i_s, \text{ we have } \partial_{i_1} \dots \partial_{i_s} \sigma_\gamma^k \notin L \end{cases}$$

Then ϕ can be naturally extended to be defined on all degenerate simplices of $\langle \epsilon(L) \setminus L \rangle$ of dimension $\leq n$. Then for all $\gamma \in I_n$ we can define

$$\phi|_{\sigma_\gamma^n} = \begin{cases} 1, & \text{if } \sigma_\gamma^n \in L \\ 0, & \text{if for all } s \geq 0, \text{ and for all } i_1, \dots, i_s, \text{ we have } \partial_{i_1} \dots \partial_{i_s} \sigma_\gamma^n \notin L \\ \text{any extension,} & \text{otherwise} \end{cases}$$

Where in the last case, such an extension exists by the extension property 5.4. Now for all $\sigma \in \langle X \setminus \epsilon(L) \rangle$, define $\phi|_\sigma = 0$. To show this gives a well-defined extension of ϕ , we need to check that it agrees with the face and degeneracy operators, and that it agrees with the above definition of ϕ on $\epsilon(L)$. Suppose $\sigma \in \langle X \setminus \epsilon(L) \rangle \cap \epsilon(L)$. Assume first that σ is non-degenerate: as $\sigma \in \langle X \setminus \epsilon(L) \rangle$, there exists some $\tau \in X \setminus \epsilon(L)$ such that $\gamma_1 \gamma_2 \dots \gamma_N \tau = \sigma$, where each γ_j is some face or degeneracy map (and potentially $N = 0$). Now we can use the simplicial identities reorder the γ_j so that $s_{n_1} \dots s_{n_t} \partial_{i_1} \dots \partial_{i_s} \tau = \sigma$ (where $t+s = N$), and now since σ was assumed to be non-degenerate, we must have $\partial_{i_1} \dots \partial_{i_s} \tau = \sigma$. Now since $\tau \notin \epsilon(L)$, the equation relating σ and τ means that for all $q \geq 0$ and all p_1, \dots, p_q , we have $\partial_{p_1} \dots \partial_{p_q} \sigma \notin L$ and hence $\phi|_\sigma = 0$ is well-defined. If σ were instead degenerate, then since $\langle X \setminus \epsilon(L) \rangle \cap \epsilon(L)$ is a subsimplicial set, there would exist some non-degenerate $\sigma' \in \langle X \setminus \epsilon(L) \rangle \cap \epsilon(L) \subset \epsilon(L)$ such that $s_{j_1} \dots s_{j_l} \sigma' = \sigma$, and so since ϕ agrees with the face and degeneracy operators on $\epsilon(L)$, setting $\phi|_\sigma = 0$ is well-defined. Now suppose only that $\sigma \in \langle X \setminus \epsilon(L) \rangle$. Then if $\partial_i \sigma \notin \epsilon(L)$ (respectively $s_j \sigma \notin \epsilon(L)$) then $\partial_i(\phi|_\sigma) = \phi|_{\partial_i \sigma} = 0$ (resp. $s_j(\phi|_\sigma) = \phi|_{s_j \sigma} = 0$). So supposing $\partial_i \sigma \in \epsilon(L)$ (resp. $s_j \sigma \in \epsilon(L)$), we have that $\partial_i \sigma \in \langle X \setminus \epsilon(L) \rangle \cap \epsilon(L)$ (resp. $s_j \sigma \in \langle X \setminus \epsilon(L) \rangle \cap \epsilon(L)$), and so by the above argument $\partial_i(\phi|_\sigma) = \phi|_{\partial_i \sigma} = 0$ (resp. $s_j(\phi|_\sigma) = \phi|_{s_j \sigma} = 0$). Hence $\phi \in A^*X$ satisfies the required conditions.

□

Two contravariant Mayer-Vietoris Sequences

Any reasonable compactly supported cohomology theory should have Mayer-Vietoris sequences associated to it. We give two ways of obtaining such sequences, one using 5.9 and imposing a condition on the intersection (version 1), and the other using a decomposition of our given simplicial set as a pushout, with various conditions on the maps (version 2). In both cases we obtain a contravariant Mayer-Vietoris sequence.

DEFINITION 5.10. Let X be a simplicial set with two subsimplicial sets $U, V \subset X$ which cover X . Then U and V are said to have *good intersection* if $\epsilon(\langle V \setminus U \rangle) \subset V$.

LEMMA 5.11. *Let X be a simplicial set with subsimplicial sets $U, V \subset X$ which cover X . Then*

$$\epsilon(\langle V \setminus U \rangle) \subset V \Leftrightarrow \epsilon(\langle U \setminus V \rangle) \subset U$$

and hence the notion of a good intersection is symmetric.

PROOF. Suppose $\epsilon(\langle U \setminus V \rangle) \not\subset U$. Then by the minimality of $\langle - \rangle$,

$$\{\sigma \in X \mid \partial_{i_1} \dots \partial_{i_m} \sigma \in \langle U \setminus V \rangle \text{ for some } \{i_j\}\} \not\subset U$$

and hence there exists $\sigma \in X \setminus U$ such that $\partial_{i_1} \dots \partial_{i_s} \sigma \in \langle U \setminus V \rangle$. So since U and V cover X , $\sigma \in V$ and

$$\partial_{i_1} \dots \partial_{i_s} \sigma = (Xg)\gamma$$

for some $g \in \text{Mor}(\Delta)$, and $\gamma \in U \setminus V$. Now since $\sigma \in \langle V \setminus U \rangle$, $(Xg)\gamma \in \langle V \setminus U \rangle$. Now Xg is a composition of face and degeneracy operators, and using the simplicial identities, we can always write $Xg = s_{i_1} \dots s_{i_k} \partial_{t_1} \dots \partial_{t_m}$. Hence $\partial_{i_k} \dots \partial_{i_1} (Xg)\gamma = \partial_{t_1} \dots \partial_{t_m} \gamma \in \langle V \setminus U \rangle$, and hence $\gamma \in \epsilon(\langle V \setminus U \rangle)$. But $\gamma \notin V$, and hence $\epsilon(\langle V \setminus U \rangle) \not\subset V$. The converse follows by symmetry. \square

THEOREM 5.12. (*Contravariant Mayer-Vietoris sequence, version 1*) *Let X be a simplicial set with subsimplicial sets $U, V \subset X$ which cover X and have good intersection. Then there is a long exact sequence*

$$\dots \leftarrow HA_c^n(U \cap V) \leftarrow HA_c^n U \oplus HA_c^n V \leftarrow HA_c^n X \leftarrow HA_c^{n-1}(U \cap V) \leftarrow \dots$$

PROOF. Denote the obvious inclusions by $\iota^U : U \cap V \rightarrow U$, $\iota^V : U \cap V \rightarrow V$, $j^U : U \rightarrow X$, $j^V : V \rightarrow X$. For any inclusion $\iota : Y \rightarrow Z$ of simplicial sets, there is an induced map $\iota_* : A_c^* Z \rightarrow A_c^* Y$ given by restriction: indeed, all

we must show is that such restrictions vanish on all but finitely many non-degenerate simplices of Y , which follows from that fact that a simplex of Y is degenerate in Y if and only if it is degenerate in Z . We claim we have a short exact sequence

$$0 \rightarrow A_c^k X \xrightarrow{\theta_1} A_c^k U \oplus A_c^k V \xrightarrow{\theta_2} A_c^k(U \cap V) \rightarrow 0$$

for all k . Define θ_1 by $\theta_1(\omega) = (j_*^U \omega, -j_*^V \omega)$. Define $\theta_2(\omega_1, \omega_2) = \iota_*^U \omega_1 + \iota_*^V \omega_2$. Now θ_1 is injective, because U and V cover X . To show θ_2 is surjective, let $\omega \in A_c^k(U \cap V)$. By the good intersection hypothesis, and 5.9, there exists some $\phi \in A^k X$ such that $\phi|_{\langle U \setminus V \rangle} = 1$ and $\phi|_{\langle X \setminus U \rangle} = 0$. Let $\phi_U = \phi$ and $\phi_V = 1 - \phi$. Then these two functions form a partition of unity subordinate to the cover $\{U, V\}$, and so $\theta_2(\phi_V \omega|_U, \phi_U \omega|_V) = \omega$. Exactness at the middle term follows easily from the fact that we can glue forms which agree on their intersection. Hence the sequence is exact for all k , and the long exact sequence now follows as standard. □

There are in fact alternative conditions under which we can deduce a similar result. For this, we recall the definition of a proper map of simplicial sets.

DEFINITION 5.13. A map $f : X \rightarrow Y$ of simplicial sets is *proper* if for any finite subsimplicial set $Z \subset Y$, the subsimplicial set $f^{-1}(Z) \subset X$ is finite.

REMARK 5.14. It is easily seen that all inclusions of simplicial sets are proper, and that any map $f : X \rightarrow Y$ is proper if and only if $f^{-1}(\langle \sigma \rangle)$ contains only finitely many non-degenerate simplices, for each non-degenerate simplex $\sigma \in Y$.

THEOREM 5.15. (*Contravariant Mayer-Vietoris sequence, version 2*)
Suppose we have a pushout diagram

$$\begin{array}{ccc}
W & \xrightarrow{f} & U \\
\downarrow \iota & & \downarrow h \\
V & \xrightarrow{g} & X
\end{array}$$

of simplicial sets, where ι is an inclusion, f is proper and V is locally finite. Then g and h are proper maps, and there exists a long exact sequence

$$\dots \leftarrow HA_c^n(W) \leftarrow HA_c^n U \oplus HA_c^n V \leftarrow HA_c^n X \leftarrow HA_c^{n-1}(W) \leftarrow \dots$$

which is natural in all the variables in the pushout.

PROOF. Since an inclusion of simplicial sets is a cofibration in the Kan-Quillen model structure and cofibrations are preserved under pushouts, h is an inclusion (and hence is proper). To show g is proper, we use the structure of the pushout of simplicial sets, that is, $X \cong U \coprod_W V$ naturally. For the rest of this proof we will identify X and $U \coprod_W V$. Now let $K \subset X$ be a finite subsimplicial set, and $\sigma \in K$ any simplex. Then suppose $\sigma = f(\tau)$, some $\tau \in W$. Then claim that $g^{-1}(\sigma) = \iota(f^{-1}(\sigma))$. Indeed, since h is an inclusion, $f(\iota^{-1}(g^{-1}(\sigma))) = \{\sigma\}$, hence $\iota(W) \cap g^{-1}(\sigma) \subset \iota(f^{-1}(\sigma))$, but since $\sigma = f(\tau)$, $g^{-1}(\sigma) \subset \iota(W)$, hence $g^{-1}(\sigma) \subset \iota(f^{-1}(\sigma))$. For the reverse direction, observe that by commutativity of the pushout, $g\iota(f^{-1}(\sigma)) = \sigma$, and hence $\iota(f^{-1}(\sigma)) \subset g^{-1}(\sigma)$, and so we have the claimed equality. This equality extends to show that $\iota(f^{-1}(K \cap U)) = g^{-1}(K \cap U)$, and hence, $\iota(f^{-1}(K \cap U)) \cup g^{-1}(K \setminus U) = g^{-1}(K)$. But g is injective on $g^{-1}(X \setminus U)$, and hence $g^{-1}(K \setminus U)$ has only finitely many non-degenerate simplices (non-degenerate in V), as does $\iota(f^{-1}(K \cap U))$, because f is proper, and hence g is proper.

We now prove the existence of the stated long exact sequence. Since all the maps in the pushout diagram are proper, they all induce maps on $A_c(-)$ in the opposite direction. By the proof of 14.1 of [6], there exists a short

exact sequence of complexes

$$0 \rightarrow AX \xrightarrow{(Ah, Ag)} AU \oplus AV \xrightarrow{A\iota - Af} AW \rightarrow 0$$

We also have the sequence

$$0 \rightarrow A_c X \xrightarrow{(Ah, Ag)} A_c U \oplus A_c V \xrightarrow{A\iota - Af} A_c W \rightarrow 0$$

which we claim is exact also. Notice that the maps in the second sequence are well-defined by properness. (Ah, Ag) is injective in the second sequence, since it is just a restriction of the map in the first sequence. Now

$$\ker(A^n \iota - A^n f) = \{(\Phi, \Theta) \in A_c^n \oplus A_c^n V \mid A^n f(\Phi) = A^n \iota(\Theta)\}$$

so given any $(\Phi, \Theta) \in \ker(A^n \iota - A^n f)$, define $\Psi \in A^n X$ by

$$\Psi|_\sigma = \begin{cases} \Phi|_\sigma, & \text{if } \sigma \in U \\ \Theta|_\sigma, & \text{if } \sigma \in V \end{cases}$$

To show this is well defined, if $\sigma = f(\tau) \in U$, then

$$\Theta|_\tau = A^n f(\Phi)|_\tau = \Phi|_{f(\tau)} = \Phi|_\sigma$$

as required. So $(Ah, Ag)(\Psi) = (\Phi, \Theta)$, and hence the sequence is exact at the middle term. To show that $A^n \iota - A^n f$ is surjective in the second sequence, it suffices to show that $A^n \iota : A_c^n U \rightarrow A_c^n W$ is surjective (since $A_c^n U$ is naturally contained in $A_c^n U \oplus A_c^n V$). Indeed, suppose $\omega \in A_c^n W$. Then by the extension property 5.4 for A , there exists some extension $\omega_0 \in A^n V$. Now since V is locally finite and $\text{supp}(\omega)$ is finite, $\epsilon(\text{supp}(\omega))$ is also finite, and so by 5.9 there exists a bump function $\psi \in A^0(V)$ with $\psi|_{\text{supp}(\omega)} = 1$ and $\psi|_{\langle V \setminus \epsilon(\text{supp}(\omega)) \rangle} = 0$, and hence $\psi\omega_0 \in A_c^n V$ is an extension of ω as required. \square

The PL compactly supported de Rham Theorem

DEFINITION 5.16. For a simplicial set X with subsimplicial set A , we define the *relative polynomial de Rham complex* $A^*(X, A)$ by

$$A^q(X, A) = \{\Phi \in sSet(X, \nabla(*, q)) \mid \Phi|_A = 0\}$$

In order to prove a de Rham theorem, we will need to use a model of singular cohomology which behaves well with respect to integration, which means we really don't want to have to think about degenerate simplices. Thus we will use the *normalized* (or *Moore*) complex of a simplicial Abelian group. We will quickly say precisely what this is.

DEFINITION 5.17. For a simplicial set X , the *chain complex* C_*X of X is defined by

$$C_n X = k[X_n]$$

with differential given by the alternating sum of the face maps

$$\sum_{i=0}^n (-1)^i \partial_i : C_n X \rightarrow C_{n-1} X$$

NC_*X will (temporarily) denote the *normalised chain complex* of X . This is defined by

$$NC_n X = k[X_n] / D(k[X_n])$$

where $D(Y_n)$ for a simplicial group Y denotes the subgroup of Y_n generated by the degenerate simplices. The differential is induced by the differential on C_*X (it is standard that this is well defined on the quotients by the degenerate simplices).

We define the corresponding (normalised) cochain complex as the dual over k of the (normalised) chain complex, and for $A \subset X$, we define $C^*(X, A)$ to be the subobject of C^*X of cochains which vanish on all simplices of A , and similarly for $NC^*(X, A)$.

PROPOSITION 5.18. (*Eilenberg, Mac Lane. Appears as Thm. 2.4 in chap. III of [16]*)

For a simplicial set X there is a natural inclusion $NC_*X \rightarrow C_*X$, which is a chain homotopy equivalence. Hence by dualising, there is a natural chain homotopy equivalence $C^*X \rightarrow NC^*X$.

This model allows Bousfield and Gugenheim to construct the de Rham natural transformation $A^* \rightarrow C^*$ using the integration we defined earlier. We will repeat this now.

Write $\rho : A^* \rightarrow C^*$ for the natural transformation given by

$$\langle \rho\omega, \sigma \rangle = \int \omega|_\sigma$$

where $\omega \in A^qX$ and $\sigma \in X_q$, for any simplicial set X .

Observe that if σ is degenerate, then $\sigma = s_j\sigma'$, for some σ' , and $\omega|_\sigma = s_j\omega|_{\sigma'} = 0$, because $\omega|_{\sigma'} \in \nabla(q-1, q) = 0$. Hence $\rho_X(\omega)$ vanishes on degenerate simplices, and so ρ in fact maps into NC^*X . It is easy to check that ρ is a well defined natural transformation.

We now have the PL de Rham theorem

THEOREM 5.19. (2.2 and 3.4 of [6])
 ρ induces a multiplicative homology isomorphism

$$\rho_*HA^*X \rightarrow HNC^*X \simeq HC^*X$$

for any simplicial set X .

An easy corollary is

COROLLARY 5.20. *For any pair of simplicial sets (X, A) with $A \subset X$, the chain map $\rho_{(X,A)} : A^*(X, A) \rightarrow NC^*(X, A)$, defined by restricting ρ_X , induces a multiplicative homology isomorphism*

$$\rho_* : HA^*(X, A) \rightarrow HNC^*(X, A) \cong HC^*(X, A)$$

PROOF. We have the following diagram of chain complexes, with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^*(X, A) & \longrightarrow & A^*X & \longrightarrow & A^*A & \longrightarrow & 0 \\
& & \downarrow \rho_{(X,A)} & & \downarrow \rho_X & & \downarrow \rho_A & & \\
0 & \longrightarrow & NC^*(X, A) & \longrightarrow & NC^*X & \longrightarrow & NC^*A & \longrightarrow & 0
\end{array}$$

and so the vertical maps given by ρ induce a map of long exact homology sequences, and ρ_X and ρ_A induce homology isomorphisms by the de Rham theorem (2.2 of [6]), and so by the five lemma, $\rho_{(X,A)}$ also induces a homology isomorphism.

To see that the isomorphism is multiplicative, observe that it factors as the composition

$$HA^*(X, A) \rightarrow HA^*(X/A, *) \rightarrow HC^*(X/A, *) \rightarrow HC^*(X, A)$$

where the first and last maps are the isomorphisms by the induced canonical multiplicative maps on the level of chains. The fact that the middle map is multiplicative follows from the non-compactly supported de-Rham theorem, and the fact that the reduced homologies of X/A are canonically isomorphic to the respective homologies, except in degree 0 where they are both 0.

□

From now on we will identify NC^* and C^* , and their compactly supported versions also.

We now show that A_c^* has an alternative construction as a direct limit of relative cohomology groups (analogous to the well known construction of C_cX in this way).

Let X be a simplicial set. Observe that the collection of finite simplicial subsets of X forms a directed system, since it is closed under finite unions. Moreover, if $L \subset K \subset X$ are finite simplicial subsets, we get an evident induced map $HA^*(X, \langle X \setminus L \rangle) \rightarrow HA^*(X, \langle X \setminus K \rangle)$, and so the collection $\{HA^*(X, \langle X \setminus K \rangle) | K \subset X \text{ is finite}\}$ is naturally a directed

system also, and so we can form the direct limit

$$\operatorname{colim}_K(HA^*(X, \langle X \setminus K \rangle))$$

(or indeed we can form the direct limit of the $A^*(X, \langle X \setminus K \rangle)$ as K ranges over all finite subsimplicial sets of X . We note as well that we consider this as a colimit in the category of graded Abelian groups, or graded rings, and the underlying complex will be the same. We now have the following results

LEMMA 5.21. *The natural map*

$$\eta : HA_c^*X \rightarrow \operatorname{colim}_K(HA^*(X, \langle X \setminus K \rangle))$$

is an isomorphism of graded rings.

PROOF. η is defined as follows. Given $\omega \in A_c^q X$, there exists some finite $K \subset X$ such that $\omega|_K = 0$, and so ω naturally belongs to some

$$A^*(X, \langle X \setminus K \rangle)$$

and is also a cocycle in this complex. Hence ω naturally represents some element of the direct limit in the lemma, which is what we define $\eta(\omega)$ to be. To show η is well defined on the level of homology, suppose $\omega = \omega' + d\gamma$, for some $\gamma \in A_c^{q-1}X$. Then ω and ω' are both cocycles in some common $A^*(X, \langle X \setminus K \rangle)$, and K can also be chosen so that γ belongs to the complex, and it is clear that $\omega = \omega' + d\gamma$ in $A^*(X, \langle X \setminus K \rangle)$ also, and hence ω and ω' are cohomologous in $\operatorname{colim}_L(A^*(X, \langle X \setminus L \rangle))$, and so since homology commutes with direct limits, they represent the same element of the direct limit in the lemma, hence η is well defined.

Showing that η is injective and surjective just amounts to noticing that any cochain in $\omega \in A_c^*X$ is closed if and only if its image in

$$\operatorname{colim}_K(A^*(X, \langle X \setminus K \rangle))$$

is closed (and again using that homology commutes with direct limits). The fact that it is multiplicative follows easily from how the multiplication on the colimit is defined.

□

It is standard that the same two lemmas hold with A^* replaced by C^* . Hence we now have

THEOREM 5.22. (*PL compactly supported de Rham Theorem*)
*The restriction $\rho_c : A_c^*X \rightarrow C_c^*X$ induces a multiplicative isomorphism on cohomology.*

PROOF. By Lemma 5.21 and the variant for C^* , we have isomorphisms

$$HA_c^*X \rightarrow \operatorname{colim}_K(HA^*(X, \langle X \setminus K \rangle))$$

$$HC_c^*X \rightarrow \operatorname{colim}_K(HC^*(X, \langle X \setminus K \rangle))$$

and so by the relative de Rham theorem (and naturality of the above isomorphisms), ρ_c is a homology isomorphism.

□

CHAPTER 6

Rational Homotopy Theory in the sense of Quillen

Quillen introduced the area of rational homotopy theory, and provided a model in terms of differential graded Lie algebras. Unlike Sullivan's CDGA model, it is not particularly easy to compute with, as the Lie bracket is given by the Whitehead product on homotopy classes, which can be rather cumbersome to work with. However, there is a correspondence between the homotopy Lie algebra and a minimal model for the cochain algebra, coming from Koszul duality, which can in fact be exploited (as we shall do in the final chapter). This section is purely expository, with main reference being [36].

The Homotopy Lie Algebra

DEFINITION 6.1. A *differential graded Lie algebra* (DGLA) is a chain complex (L, d) with a map $L \otimes L \rightarrow L$ denoted $x \otimes y \mapsto [x, y]$ satisfying a derivation condition, and the graded anti symmetry and graded Jacobi identities as follows

$$d[x, y] = [dx, y] + (-1)^{|x|}[x, dy]$$

$$[x, y] = -(-1)^{|x||y|}[y, x]$$

$$(-1)^{|x||z|}[[x, y], z] + (-1)^{|y||x|}[[y, z], x] + (-1)^{|z||y|}[[z, x], y] = 0$$

for all $x, y, z \in L$. A DGLA L will be called *r-reduced* if $L_q = 0$ for $q < r$.

A *quasi-isomorphism* of DGLAs is a map which is an isomorphism on homology.

We denote the category of q -reduced DGLAs over \mathbb{Q} by $dgLie_{\mathbb{Q}}^q$.

THEOREM 6.2. (*Appendix B of [36]*)

$dgLie_0$ is a model category with fibrations the surjective maps and weak equivalences quasi-isomorphisms.

DEFINITION 6.3. For a connected space X , the *Homotopy Lie Algebra* $L(X)$ is a graded Lie algebra (with zero differential) defined by

$$L_q = \pi_{q+1}X \otimes_{\mathbb{Z}} \mathbb{Q}$$

and so has a natural map $\tau : \pi_{q+1}X \rightarrow L_q(X)$ given by $\tau(x) = x \otimes 1$. The Lie bracket on $L(X)$ is defined by

$$[\tau\alpha, \tau\beta] = (-1)^{|\alpha|}\tau[\alpha, \beta]$$

where the brackets on the right hand side denote the usual Whitehead product of homotopy classes $\alpha, \beta \in \pi_*X$. Thus the graded symmetry and graded Jacobi identities follow from the properties of the Whitehead product, and so $L(X)$ is a graded Lie algebra.

$L(-)$ thus defines a functor $L(-) : Top_{\mathbb{Q}}^1 \rightarrow gLie^0$, where we denote by $Top_{\mathbb{Q}}^q$ the category of $(q-1)$ -connected rational spaces, and $gLie^0$ is the full subcategory of $dgLie^0$ of graded Lie algebras with zero differential.

Denote the category of q -reduced rational cocommutative differential graded coalgebras by $dgCoalg_{\mathbb{Q}}^q$. There's a model structure (see Theorem 3.1 of [25]) on $dgCoalg_{\mathbb{Q}}^{-\infty}$ with injections as cofibrations and quasi-isomorphisms as weak equivalences. There's a functor

$$H_*(-; \mathbb{Q}) : Top_{\mathbb{Q}} \rightarrow gCoalg$$

given by rational singular homology. In this context, Quillen's main result is

THEOREM 6.4. (*Theorem 1 of [36]*) *There are the following equivalences of categories*

$$Ho(Top_{\mathbb{Q}}^2) \xrightarrow{\lambda} Ho(dgLie_{\mathbb{Q}}^1) \xrightarrow{\mu} Ho(dgCoalg_{\mathbb{Q}}^2)$$

with natural isomorphisms

$$L(-) \rightarrow \pi_*(\lambda(-)) \text{ and } H_*(-; \mathbb{Q}) \rightarrow H(\mu\lambda(-))$$

The functors are given by a long chain of equivalences, outlined in detail at the end of Chapter 1 of [36]. We will review and discuss them briefly. The functors are as follows

$$Top_{\mathbb{Q}}^2 \begin{array}{c} \xleftrightarrow{|\cdot|} \\ \xleftarrow{E_2\text{Sing}} \end{array} sSet_{\mathbb{Q}}^1 \begin{array}{c} \xleftrightarrow{\Omega} \\ \xleftarrow{\bar{W}} \end{array} sGp_{\mathbb{Q}}^1 \begin{array}{c} \xleftrightarrow{\bar{Q}} \\ \xleftarrow{\mathbb{G}} \end{array} sCHA_{\mathbb{Q}}^1 \begin{array}{c} \xleftrightarrow{\bar{U}} \\ \xleftarrow{\text{Prim}} \end{array} sLiealg_{\mathbb{Q}}^1 \begin{array}{c} \xleftrightarrow{N^*} \\ \xleftarrow{N} \end{array} dgLiealg_{\mathbb{Q}}^1 \begin{array}{c} \xleftrightarrow{\text{Lieprim}} \\ \xleftarrow{C} \end{array} dgCoalg_{\mathbb{Q}}^2$$

and when derived, these become equivalences of categories of the homotopy categories of the subcategories above.

We will now spend some time describing the chain of functors and categories above.

$sSet_{\mathbb{Q}}^2$ is the category of simplicial sets with a single vertex and edge, and whose geometric realisation is a rational space. The model structure comes from the usual Quillen model structure on the category $sSet$ of simplicial sets, with weak equivalences as maps whose geometric realisation is a weak equivalence, and cofibrations as injections.

$sGp_{\mathbb{Q}}^1$ is the category of simplicial groups with a single vertex, and whose geometric realisation is a rational space, with rational fundamental group also. Note that we do not have to worry about the fact that the given space may not be simply connected, because it will also be a topological group, and hence its fundamental group will be Abelian, and act trivially on the higher homotopy groups. The model structure has weak equivalences as maps whose geometric realisation is a weak equivalence of spaces, and fibrations maps f for which $N_q f$ is surjective for $q > 0$ (where N_q is the q th normalized chains functor defined in the previous section).

$sCHA_{\mathbb{Q}}^1$ is the category of simplicial complete Hopf algebras R over \mathbb{Q} such that $R_0 \cong \mathbb{Q}$. The model structure has weak equivalences being those

maps which are weak equivalences of simplicial sets, and cofibrations retracts of free maps.

$sLiealg_{\mathbb{Q}}^1$ is the category of rational simplicial Lie algebras with a single vertex. Its model structure has weak equivalences as maps which induce an isomorphism on homotopy groups, and cofibrations retracts of free maps.

REMARK 6.5. The various model structures are deduced from theorem 4.1 of [36]. The only fact of ultimate importance is what their weak equivalences are, as this alone determines the homotopy category.

The functors $| - |$ and $Sing$ are the geometric realisation and singular chain functors respectively, and give a Quillen equivalence between Top and $sSet$. The functor $E_2 : sSet^1 \rightarrow sSet^2$ is given by defining E_2X to be the subcomplex of X given by the simplices of X whose 1-dimensional faces are all degenerate.

The functors Ω and \bar{W} are loop group and classifying object functors respectively. ΩX is the *Kan Loop Group* and can be defined as a simplicial group with $(\Omega X)_n = F(X_{n+1} \setminus s_0X_n)$, where $F(Y)$ denotes the free group on a set Y . For a detailed presentation of the construction see Chapter V, section 5 of [16]. On the construction of $\bar{W}G$, given a simplicial group G , one defines a simplicial set WG by

$$(WG)_n = G_n \times G_{n-1} \times \dots \times G_0$$

with face maps

$$\partial_i(g_n, g_{n-1}, \dots, g_0) = \begin{cases} (\partial_i g_n, \partial_{i-1} g_{n-1}, \dots, \partial_0 g_{n-i} g_{n-i-1}, g_{n-i-2}, \dots, g_0), & \text{for } i < n \\ (\partial_n g_n, \partial_{n-1} g_{n-1}, \dots, \partial_1 g_1), & \text{for } i = n \end{cases}$$

and degeneracy maps

$$s_j(g_n, g_{n-1}, \dots, g_0) = (s_j g_n, s_{j-1} g_{n-1}, \dots, s_0 g_{n-1}, 1, g_{n-i-1}, \dots, g_0)$$

where 1 denotes the unit in any of the groups G_k .

There is a natural left G -action $G \times WG \rightarrow WG$ on WG given by

$$(h, (g_n, \dots, g_0)) \mapsto (hg_n, g_{n-1}, \dots, g_0)$$

and $\bar{W}G$ is defined as the quotient of WG by this action.

The functors \tilde{Q} and \mathbb{G} are given (in each degree) by group completion and taking group-like elements respectively. More precisely, given a group G , one can define a complete Hopf algebra $\tilde{Q}G$ by taking the Hopf algebra $\mathbb{Q}G$, with augmentation ideal I (that is, the kernel of the counit $\mathbb{Q}G \rightarrow \mathbb{Q}$), and taking the completion of $\mathbb{Q}G$ with respect to the filtration

$$\mathbb{Q}G \supset I \supset I^2 \supset \dots$$

to obtain a *complete* Hopf Algebra. In other words

$$\tilde{Q}G := \lim(\dots \rightarrow \mathbb{Q}G/I^2 \rightarrow \mathbb{Q}G/I \rightarrow \mathbb{Q}G)$$

As stated, \tilde{Q} is then extended to a functor on simplicial groups in the obvious degree-wise fashion.

Turning to the functor \mathbb{G} , for any Hopf algebra A with antipode $S : A \rightarrow A$, an element $x \in A$ is called *group-like* if its comultiplication is $x \otimes x$, and the set of all group-like elements form a group, where $x^{-1} = S(x)$. This defines a functor from (complete) Hopf algebras to groups, and this is extended in the obvious degree-wise fashion to a functor on simplicial complete Hopf algebras.

The functors \tilde{U} and $Prim$ are the universal enveloping algebra and primitives functors respectively, extended degree-wise to simplicial objects. Precisely, given a Lie algebra \mathfrak{g} , its universal enveloping algebra $U\mathfrak{g}$ is a Hopf algebra, and so can be completed with respect to powers of its augmentation ideal (as above for $\mathbb{Q}G$), to obtain a complete Hopf algebra $\tilde{U}\mathfrak{g}$.

Turning to $Prim$, if A is a Hopf algebra, an element $x \in A$ is called *primitive* if its comultiplication is $x \otimes 1 + 1 \otimes x$, and the set of all primitives forms a

Lie algebra $\text{Prim}(A)$ with bracket given by

$$[x, y] = xy - yx$$

for any $x, y \in \text{Prim}(A)$ (it is easy to check that $xy - yx$ is necessarily primitive).

The functor N is the normalized chain complex functor described in the previous Chapter, and N^* is its left adjoint. For a simplicial Lie algebra L , the Lie bracket on NL is defined by using the Eilenberg-Zilber shuffle map $\nabla_{L,L} : NL \otimes NL \rightarrow N(L \otimes L)$; given $x, y \in NL$, $\nabla_{L,L}(x \otimes y)$ is a linear combination of tensor products of elements of L , and hence we can apply the Lie bracket of L to it to obtain something in NL . For more details see Chapter 4 of [36].

The functor LiePrim takes a cocommutative dg coalgebra to the Lie algebra of primitives in its cobar construction (the cobar construction of a dg coalgebra yields a dg Hopf algebra, and hence one can take its primitives).

The functor C takes a dg-Lie algebra to (a dg version of) a complex for calculating its homology coalgebra. To be precise, we give a brief construction (the *Chevalley-Eilenberg construction*) of C from [35] a dg-Lie algebra $(L, [-, -], d_L)$

$$C(L) := S'(\Sigma^1 L)$$

where $S'(V)$ denotes the free symmetric coalgebra on a vector space V . In general, for a graded vector space V , $S'(V)$ is the coalgebra generated by the elements of the tensor algebra $T(V)$ of the form

$$v, (v')(v'') := v' \otimes v'' + (-1)^{(|v'|+1)(|v''|+1)} v'' \otimes v', (v''')(v'''')(v'''''), \dots$$

for any $v, v', v'', \dots \in V$.

The differential on $C(L)$ is given on generators by

$$d_{C(L)}(\Sigma^1 v) = -\Sigma^1 d_L(v)$$

and

$$d_{C(L)}((\Sigma^1 v)(\Sigma^1 v')) = \Sigma^1[v, v']$$

for any $v, v' \in L$. $d_{C(L)}$ is defined to be zero on all other generators, and the resulting differential is a coderivation.

In terms of the differential *Tor* functor defined earlier we have that

$$H_*(C(L)) \cong \text{Tor}_{U(L)}(\mathbb{Q}, \mathbb{Q})$$

which comes from the fact that there is a natural inclusion $C(L) \rightarrow BU(L)$ which is a quasi-isomorphism ($BU(L)$ denotes the bar construction on the universal enveloping algebra of $U(L)$, which gives a resolution of \mathbb{Q} as a $U(L)$ -module). This is due to Moore [34].

For the following chapter, we will only really be concerned with the passage from a space to its dg-Lie model. Moreover, the particular space X we will be interested will be *coformal*, meaning that its Lie model is in fact homotopy equivalent to its homotopy Lie algebra $L(X)$ (or equivalently that the Sullivan minimal model of X has purely quadratic differential). However, Quillen's picture gives extra insight into the general framework of rational homotopy theory, and we felt it helpful to summarise it in full.

CHAPTER 7

A formality problem

Setting

We work in the category of connective CDGAs over \mathbb{Q} , henceforth denoted $cdga_{\geq 0}$. Note that our DGAs will almost never be connected. $cdga_{\geq 0}$ is a cofibrantly generated model category with fibrations as surjections and weak equivalences as quasi-isomorphisms. The model structure can be obtained by considering the projective model structure on $Ch_{\geq 0}$ (rational connective chain complexes), and applying the following result.

THEOREM 7.1. (*Transferred Model Structure, Theorem 3.3 of [9]*)
Let \mathfrak{C} be a cofibrantly generated model category, \mathfrak{D} a category, and suppose we have an adjunction

$$\mathfrak{D} \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{U} \end{array} \mathfrak{C}$$

where F is left adjoint to U . Moreover, call a map in \mathfrak{D} a fibration (resp. weak equivalence) if its image under U is a fibration (resp. weak equivalence) in \mathfrak{C} . Suppose now that F preserves small objects, and that any sequential colimit of pushouts of images under F of the generating acyclic cofibrations in \mathfrak{C} is a weak equivalence in \mathfrak{D} . Then the choice of fibrations and weak equivalences made above for \mathfrak{D} determines a cofibrantly generated model structure on \mathfrak{D} , where the set of generating (acyclic) cofibrations in \mathfrak{D} is the set of images under F of the generating (acyclic) cofibrations in \mathfrak{C} .

For the rest of this chapter, we will be using the following adjunction

$$cdga_{\geq 0} \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{U} \end{array} Ch_{\geq 0}$$

where U is the forgetful functor, and F takes a chain complex (M, d_M) to the free commutative CDGA on M . Precisely, (FM, d_{FM}) can be defined as

the tensor algebra

$$FM := (\oplus_{k \geq 0} M^{\otimes k}) / (a \otimes b + (-1)^{|a||b|} b \otimes a)$$

where $M^{\otimes k}$ denotes the k th tensor power of M , with $M^0 := \mathbb{Q}$, and the differential on FM is defined inductively by the Leibniz rule

$$d_{FM}(a \otimes b) = d_M(a)b + (-1)^{|a|} a d_M(b)$$

We now use the previous theorem to prove the following.

THEOREM 7.2. *$cdga_{\geq 0}$ has a model structure whose fibrations are all surjective maps, and weak equivalences are all quasi-isomorphisms, and where the generating cofibrations are maps of the form $FS(n-1) \rightarrow FD(n)$, and the generating acyclic cofibrations are maps of the form $\mathbb{Q} \rightarrow FD(n)$, for all $n \geq 1$.*

PROOF. We have the adjunction

$$cdga_{\geq 0} \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{U} \end{array} Ch_{\geq 0}$$

given above. We will first prove that F preserves small objects.

Suppose $A \in Ch_{\geq 0}$ is small. Recall that by definition this means there exists some regular cardinal κ such that for all κ -filtered ordinals λ and all λ -sequences $J : \lambda \rightarrow Ch_{\geq 0}$, the canonical map

$$\operatorname{colim}_{\beta < \lambda} (\operatorname{Hom}(A, J(\beta))) \rightarrow \operatorname{Hom}(A, \operatorname{colim}_{\beta} (J(\beta)))$$

is a bijection (where the Hom sets are just the sets of degree 0 chain maps of complexes). Now let λ be any κ -filtered ordinal, and $J : \lambda \rightarrow cdga_{\geq 0}$ a λ sequence in $cdga_{\geq 0}$. We are required to show that the canonical map

$$\operatorname{colim}_{\beta < \lambda} (\operatorname{Hom}(FA, J(\beta))) \rightarrow \operatorname{Hom}(FA, \operatorname{colim}_{\beta} (J(\beta)))$$

is a bijection.

The only fact we need to use is that the forgetful functor U commutes with filtered colimits, since this then (by smallness of A) means that we have a bijection

$$\operatorname{colim}_{\beta < \lambda} (\operatorname{Hom}(A, UJ(\beta))) \rightarrow \operatorname{Hom}(A, U\operatorname{colim}_{\beta} (J(\beta)))$$

and then using the adjunction, we obtain the desired bijection.

We now prove the condition that any sequential colimit of pushouts of images under F of the generating acyclic cofibrations in \mathfrak{C} is a weak equivalence.

The only generating acyclic cofibrations in the projective model structure on $Ch_{\geq 0}$ are the maps $a_n : 0 \rightarrow D(n)$ for all $n \geq 1$, where

$$D(n) = \langle x_{n-1}, y_n \mid dy = x \rangle$$

and after applying F , these become the maps

$$Fa_n : \mathbb{Q} \rightarrow FD(n)$$

Now since \mathbb{Q} has characteristic zero, $FD(n)$ has homology only in degree 0 (given by the unit), and hence Fa_n is a weak equivalence. Translating the statement we are required to prove, we see that it is equivalent to the statement that for all $A \in cdga_{\geq 0}$, the canonical map

$$A \rightarrow \operatorname{colim} (A \otimes FD_{n_1} \rightarrow A \otimes FD_{n_1} \otimes FD_{n_2} \rightarrow \dots) = A \otimes_{\mathbb{Q}} (\otimes_{j < \lambda} FD(n_j))$$

is a weak equivalence, where λ is some ordinal and the $n_j \geq 1$ are arbitrary. The tensor product of two acyclic CDGAs is acyclic, and so we need only give special attention to the case when λ is a limit ordinal. In this case, suppose $\omega \in \otimes_{j < \lambda} FD(n_j)$ represents some homology class of positive degree. Then since the tensor product is generated by elements with all but finitely many entries being 1, ω is the sum of finitely terms each of which has only finitely many non-unit entries, and hence ω in fact lives in $\otimes_{1 \leq \alpha \leq m} FD_{n_\alpha}$, for

some finite m , and ω must be a cycle in this complex. Since the complex is acyclic and ω is in positive degree, ω must be a boundary, and hence also a boundary in $\otimes_{j<\lambda} FD(n_j)$. This proves that $\otimes_{j<\lambda} FD(n_j)$ is acyclic, and hence that the map

$$A \rightarrow A \otimes_{\mathbb{Q}} (\otimes_{j<\lambda} FD(n_j))$$

is a weak equivalence as desired. □

REMARK 7.3. It is well known that when working with CDGAs over the integers (or even fields of positive characteristic), there is no model structure of the kind in 7.2. One reason for this is that over such rings, the corresponding objects $FD(n)$ are not acyclic. In order for them to be acyclic, one must drop the commutativity condition, and work with all DGAs, where the functor F is replaced by the functor which assigns to a chain complex the free (non-commutative) DGA on its generators. Allowing these alterations, the model structure of the type in 7.2 does exist over the integers, fields of positive characteristic, and even any commutative ring. To be precise, if R is any commutative ring, the category $R - dga$ of associative DGAs over R has a model whose fibrations are the surjective maps, and whose weak equivalences are the quasi-isomorphisms. This is proved in 2.3.11 of [26].

Cofibrant replacement and Postnikov towers

DEFINITION 7.4. We denote by $FS(n)$ the free commutative object of $cdga_{\geq 0}$ on a single generator x in degree n , with zero differential. We denote by $FD(n+1)$ the free commutative object on generators y and z in degrees n and $n+1$ respectively, with $dz = y$. Since the model structure is transferred from chain complexes, we have generating cofibrations given by the inclusions $\iota^{(n)} : FS(n) \rightarrow FD(n+1)$ for all n .

Suppose we have $C \in cdga_{\geq 0}$. There is a functorial cofibrant replacement of C given by the model structure transferred from chain complexes, but it is quite unwieldy. Thus we use the following construction, which gives a smaller resolution that will be more suitable for computations later on. This is the

same construction used in [10], where they work with associative DGAs over \mathbb{Z} (in particular, the following construction does not require us to be working over \mathbb{Q} , nor does it require us to be commutative).

Pick generators for H_*C as a \mathbb{Q} -algebra, then pick representatives in C for these generators; let ZC denote the \mathbb{Q} -module generated by these cycles, and let Q_0C denote the free \mathbb{Q} algebra on ZC with zero differential. There is a canonical map $f_0 : Q_0C \rightarrow C$ which is surjective on homology. Let n denote the smallest degree in which $H_*(f_0)$ has non-zero kernel. We must first treat the case where $n = 0$. In this case pick generators for $\ker(H_0(f_0))$ as a \mathbb{Q} -algebra, and denote them by $\{t_i\}_{i \in I}$. Then for each $i \in I$, we have a map $g_i : FS_i(n) \rightarrow Q_0C$ which sends the generator x_i of $FS_i(n)$ to t_i . We can arrange all these maps into a diagram

$$\begin{array}{ccc} \coprod_i FS_i(n) & \xrightarrow{\coprod_i g_i} & Q_0C \\ \downarrow \coprod_i \iota_i^{(n)} & & \downarrow j_0 \\ \coprod_i FD_i(n+1) & \longrightarrow & Q_1C \end{array}$$

where we have denoted by Q_1C the pushout of the diagram. Observe that since cofibrations are preserved by coproducts and pushouts, $j_0: Q_0C \rightarrow Q_1C$ is a cofibration. We can define a map

$$h : \coprod_i FD_i(n) \rightarrow C$$

by sending y_i to $f_0(t_i)$ and z_i to any element α with $d\alpha = f$ (such an element must exist, since t_i is in the kernel of $H_*(f_0)$). The maps h and f_0 are compatible with the pushout diagram, and hence there exists a universal map $f_1 : Q_1C \rightarrow C$. Now the kernel of f_1 is concentrated in degrees 1 and above, and so if we replaced Q_1C by Q_0C and f_1 by f_0 , we would be in the case $n > 0$ from the above. So now assume $n > 0$. Then the only difference in how we proceed is that now $\ker(H_n(f_0))$ is no longer a \mathbb{Q} -algebra,

but instead is a $(Q_0C)_0$ -module. So we pick generators for $\ker(H_n(f_0))$ as a $(Q_0C)_0$ -module, and denote them $\{t_i\}_{i \in I}$, and perform the same construction as above, obtaining Q_1C with its canonical map $Q_1C \rightarrow C$. We now let $Q_\infty C$ denote the direct limit of

$$Q_0C \rightarrow Q_1C \rightarrow \dots$$

Since cofibrations are closed under transfinite compositions, each Q_jC , and hence $Q_\infty C$, is cofibrant, and the map $Q_\infty C \rightarrow C$ is a quasi-isomorphism, since it is homology surjective in all degrees, and homology injective by construction.

REMARK 7.5. In the strict model categorical sense, $Q_\infty C$ is not a cofibrant replacement of C in general, because $Q_\infty C \rightarrow C$ is not always a fibration (ie not always surjective), though this does not matter for our purposes.

REMARK 7.6. Note that since we are working with commutative DGAs, the coproduct is just the tensor product, however we use the symbol \coprod to emphasise the generality of the above construction beyond commutative DGAs, and will switch to \otimes only dealing with statements specific to commutative DGAs.

We refer the reader to [10] for the details of the theory of Postnikov towers for DGAs. They work in the non-commutative setting, but this part of the theory carries over without obstacle. We review the main points below.

DEFINITION 7.7. Given an object $(C, d) \in \text{cdga}_{\geq 0}$, an *nth Postnikov section* of C is a CDGA X with a map $C \rightarrow X$ such that $H_i X = 0$, for all $i \geq n + 1$, and $H_i C \rightarrow H_i X$ is an isomorphism for all $i \leq n$.

There are two common functorial constructions of *nth Postnikov sections* that are used, each with advantages. The first has the advantage of simplicity,

and is denoted $\mathbb{P}_n C$, and defined by

$$(\mathbb{P}_n C)_i = \begin{cases} C_i & \text{if } i \leq n-1 \\ C_i/(d(C_{i+1})) & \text{if } i = n \\ 0 & \text{if } i \geq n+1 \end{cases}$$

This gives a well-defined DGA.

The downside of this model is that $C \rightarrow \mathbb{P}_n C$ is not a cofibration. We can construct an alternative functorial model, denoted $P_n C$, by adjoining cycles (in the same way as the construction of cofibrant replacements above), however in order for it to be functorial, we cannot simply pick generators for the cycles in each degree, and so must adjoin new generators for *all* the cycles in each degree. This means that the model is extremely large, however $C \rightarrow P_n C$ will be a cofibration. The construction is as follows.

For any cycle z in degree $n+1$, there is a unique map $FS(n+1) \rightarrow C$ which sends the generator of $FS(n+1)$ to z . We can then construct the pushout of the diagram

$$\coprod FD(n+2) \leftarrow \coprod FS(n+1) \rightarrow C$$

where the coproducts run over all cycles in degree $n+1$. We denote this pushout $L_{n+1} C$, and then we define $P_n C$ to be the colimit of

$$C \rightarrow L_{n+1} C \rightarrow L_{n+2} L_{n+1} C \rightarrow \dots$$

Since cofibrations are closed under pushouts and transfinite compositions, $C \rightarrow P_n C$ is a cofibration.

PROPOSITION 7.8. (3.3 of [10])

For any $C \in cdga_{\geq 0}$, and any n th Postnikov section X of C , there is a quasi-isomorphism $P_n C \rightarrow X$.

DEFINITION 7.9. There are canonical maps $P_{n+1}C \rightarrow P_nC$ which are compatible with the maps $C \rightarrow P_nC$, and the sequence

$$\dots \rightarrow P_{n+1}C \rightarrow P_nC \rightarrow \dots \rightarrow P_0C$$

is termed the *Postnikov tower* of C .

Square-zero extensions, k -invariants and André-Quillen cohomology

If we have constructed a Postnikov tower up to some degree n , we may wish to know all the possible ways (up to homotopy) of constructing the next stage. As with the theory of Postnikov towers for spaces, these are classified by homotopy classes of certain maps out of the n th section called *k -invariants*. For DGAs, it turns out that all k -invariants can be viewed in a very nice form, as a *square-zero extension*. We now review how this works, again with main reference being [10].

DEFINITION 7.10. Let $C \in \text{cdga}_{\geq 0}$, and let M be a C -module. Then M has a right C -module structure also, given by $m.c := (-1)^{|c||m|}c.m$. We denote by $C \vee M$ the *square-zero extension* of C , by M , which is the object of $\text{cdga}_{\geq 0}$ whose underlying chain complex is $C \oplus M$, where the algebra structure comes from the left and right module structures, and the requirement that $m.m' = 0$, for all $m, m' \in M$. Translating this, it means that

$$(c, m).(c', m') = (cc', cm' + (-1)^{|m|}mc')$$

for all $c, c' \in C$ and $m, m' \in M$.

Now for any $C \in \text{cdga}_{\geq 0}$ we have $\mathbb{P}_0C = H_0C$, and so since $H_{n+1}C$ is a H_0C -module, it inherits a \mathbb{P}_nC -module structure from the canonical maps $\mathbb{P}_nC \rightarrow \mathbb{P}_0C \rightarrow H_0C$. This means we can consider square-zero extensions of the form $\mathbb{P}_nC \vee \Sigma^{n+2}H_{n+1}C$, which appear frequently from now on.

k-invariants. Continuing from the previous section, there is a canonical map

$$\gamma_n : P_n C \rightarrow \mathbb{P}_n C \vee \Sigma^{n+2} H_{n+1} C$$

where $H_{n+1} C$ is considered as being a module concentrated in degree 0 for the above square-zero extension. The map γ is defined as the identity in degrees less than n , the quotient map in degree n , zero in degrees $n + 1$ and all degrees above $n + 2$. In degree $n + 2$, it is defined as follows: γ_n is zero on C_{n+2} , and if $x \in (P_n C)_{n+2}$ was adjoined to kill $z \in C_{n+1}$, then x is mapped to the class $[z]$ in $\Sigma^{n+2} H_{n+1} C$.

DEFINITION 7.11. Let $cdga_{\geq 0}/(\mathbb{P}_n C)$ denote the model category of connective, rational CDGAs augmented over $\mathbb{P}_n C$. If $Ho(cdga_{\geq 0}/\mathbb{P}_n C)$ denotes its homotopy category, then the homotopy class of γ_n in

$$Ho(cdga_{\geq 0}/\mathbb{P}_n C)(P_n C, \mathbb{P}_n C \vee \Sigma^{n+2} H_{n+1} C)$$

is called the *n*th *k*-invariant of C .

PROPOSITION 7.12. (3.5 of [10])

The sequence

$$P_{n+1} \rightarrow P_n C \xrightarrow{\gamma} \mathbb{P}_n C \vee \Sigma^{n+2} H_{n+1} C$$

is a homotopy fibre sequence in $Ho(cdga_{\geq 0}/\mathbb{P}_n C)$.

This proposition shows that the homotopy class of γ depends only on the quasi-isomorphism type of $P_{n+1} C$.

It has been somewhat inconvenient up until now that we have had to use two different models for *n*th Postnikov sections. However, observe that since every object of $cdga_{\geq 0}$ is fibrant, it is right proper, and right properness of a model category is equivalent to the slice categories $cdga_{\geq 0}/X$ over any object X being preserved (up to Quillen equivalence) by weak equivalences $X \rightarrow Y$ of the base objects (see the blog post “The mysterious nature of right properness” on the n-category cafe). Hence we have a Quillen equivalence

$$cdga_{\geq 0}/P_n C \rightarrow cdga_{\geq 0}/\mathbb{P}_n C$$

and hence we can identify

$$Ho(cdga_{\geq 0}/P_n C)(P_n C, P_n C \vee \Sigma^{n+2} H_{n+1} C)$$

with

$$Ho(cdga_{\geq 0}/\mathbb{P}_n C)(P_n C, \mathbb{P}_n C \vee \Sigma^{n+2} H_{n+1} C)$$

the former of which being what we will use from now on.

Extensions and André Quillen cohomology.

DEFINITION 7.13. ([10])

Let C be an object of $cdga_{\geq 0}$, with $H_i C = 0$ for $i \geq n + 1$. A *Postnikov $(n+1)$ -extension* of C is an object X of $cdga_{\geq 0}$ together with a map

$$f : X \rightarrow C$$

in $cdga_{\geq 0}$, such that $H_i(f)$ is an isomorphism for $i \leq n$, and the canonical map $X \rightarrow P_{n+1} X$ is a quasi-isomorphism. A *map $(X, f) \rightarrow (Y, g)$* of Postnikov $(n + 1)$ -extensions of C is a quasi-isomorphism $X \rightarrow Y$ which is compatible with f and g .

The following result tells us that quasi-isomorphism classes of Postnikov extensions with a specified isomorphism are classified by homotopy classes of certain square-zero extensions.

PROPOSITION 7.14. (3.9 of [10])

Let $C \in cdga_{\geq 0}$ with $C \rightarrow P_n C$ a quasi-isomorphism, and let M be some $H_0 C$ -module. Let $Pext_{n+1}^{can}(C; M)$ be the category whose objects are triples (X, f, θ) , where (X, f) is a Postnikov $(n + 1)$ -extension of C , and $\theta : H_{n+1} X \rightarrow M$ is a quasi-isomorphism. The morphisms of $(X, f, \theta) \rightarrow (Y, g, \phi)$ in $Pext_{n+1}^{can}(C; M)$ are quasi-isomorphisms $X \rightarrow Y$ which are compatible f, g, θ and ϕ . Then we have a bijection

$$\pi_0(Pext_{n+1}^{can}(C; M)) \xrightarrow{\cong} Ho(cdga_{\geq 0}/C)(C, C \vee \Sigma^{n+2} M)$$

The superscript “*can*” on $Pext$ is to indicate that we have fixed a canonical isomorphism of each object with M as an H_0C -module. An immediate corollary is

COROLLARY 7.15. (*3.10 of [10]*)

Let C and M be as above, and let $G = Aut_{H_0C}M$. Then let $Pext_{n+1}(C; M)$ denote the category of Postnikov $(n + 1)$ -extensions X of C which satisfy $H_{n+1}X \cong M$ as H_0C -modules (but where we do not fix a specified isomorphism), and with morphisms just maps of Postnikov $(n + 1)$ -extensions. Then we have a bijection

$$(Ho(cdga_{\geq 0}/C)(C, C \vee \Sigma^{n+2}M))/G \xrightarrow{\cong} \pi_0(Pext_{n+1}(C; M))$$

We are now ready for the entire point of reviewing this material, which was to be able to specify the form of André-Quillen cohomology we are going to be using.

DEFINITION 7.16. Let $C \in cdga_{\geq 0}$ and let M be a C -module. Then a *derivation* f from C to M of *degree* $|f|$ is a morphism $f : C \rightarrow \Sigma^{-|f|}M$ of the underlying chain complexes over \mathbb{Q} , such that the following graded Leibniz rule holds

$$f(ab) = (fa).b + (-1)^{|a||f|}a.(fb)$$

We denote by $Der_*(C; M)$ the chain complex of such derivations (which will usually be concentrated in non-positive degrees for our purposes), where the differential δ is defined in the same way as for the internal hom of chain complexes

$$\delta(f) := d_M f - (-1)^{|f|} f d_C$$

It is straightforward to check that $\delta(f)$ is indeed a derivation of degree $|f| - 1$.

DEFINITION 7.17. Let $X \in cdga_{\geq 0}$, and M be an X -module, and let Ch denote the model category of unbounded chain complexes over \mathbb{Q} with the projective model structure (for a good explanation of the existence of this

model structure, see [40]). Then derivations define a functor

$$Der_*(-; M) : (cdga_{\geq 0}/X)^{op} \rightarrow Ch$$

which is a left Quillen functor (that is, it preserves cofibrations and acyclic cofibrations), and hence it has a left derived functor

$$AQ^{-*}(-; M) : (Ho(cdga_{\geq 0}/X))^{op} \rightarrow Ch$$

and for any C augmented over X , $AQ^*(C; M)$ is called the *André-Quillen cohomology* of C with coefficients in M .

REMARK 7.18. Note that depending on the degrees in which M is concentrated, $Der_*(C; M)$ may be non-zero in both positive and negative degrees, which is why we are forced to use the category of unbounded chain complexes in the definition. We will be most interested in the case when M is concentrated in degree 0, in which case $Der_*(C; M)$ will be concentrated in non-positive degrees.

REMARK 7.19. Note that the augmentation over X is what allows us to define $Der_*(C; M)$ (or $Der_*(P, M)$, where $P \rightarrow C$ is a cofibrant replacement), since M must be a C -module for it to make sense.

REMARK 7.20. Since $AQ^*(-; M)$ is a left derived functor on $(cdga_{\geq 0}/X)^{op}$, we can compute $AQ^*(C; M)$ as $Der_{-*}(P; M)$, where $P \rightarrow C$ is a cofibrant replacement in $cdga_{\geq 0}/X$.

We now arrive at the main result, which links André-Quillen cohomology and k -invariants of CDGAs. The analogous statement is made in 3.14 of [10] with a proof referenced in [29] and private communication of the authors of [10] with M.Mandell. However for our purposes, and because of the specific definition of the derivation complex we have been able to use, the result is more straightforward to prove directly, as we do not need to consider the cotangent complex, as is used in [10] and [29].

THEOREM 7.21. (*Analogous to that which appears in [29]*)
 Let $C \in cdga_{\geq 0}$ and let M be a C -module concentrated in a single degree.

Then there's an isomorphism

$$AQ^n(C; M) \cong Ho(cdga_{\geq 0}/C)(C, C \vee \Sigma^n M)$$

PROOF. Fix a cofibrant replacement $q : P \rightarrow C$ of C . Since C is the terminal object q is also the augmentation of P . Let

$$\delta : Der_*(P; M) \rightarrow Der_{*-1}(P; M)$$

denote the differential. We will first define the functions in question, then check they are inverse, then check they are well defined on homotopy/homology classes.

Given a map of CDGAs $\gamma : P \rightarrow C \vee \Sigma^n M$, we can define $\alpha(\gamma) : P \rightarrow \Sigma^n M$ as the composition

$$P \xrightarrow{\gamma} C \vee \Sigma^n M \xrightarrow{p} \Sigma^n M$$

where the last map is the obvious projection. It is easy to check using the definition of the multiplication in a square zero extension that p is a derivation of degree 0 (where $\Sigma^n M$ has the obvious $(C \vee \Sigma^n M)$ -module structure), and hence that $\alpha(\gamma)$ is a derivation of degree n . $\delta(\alpha(\gamma)) = 0$ follows from the hypothesis that M has zero differential (this is an easy check). Hence, assuming it is well defined on homotopy classes of maps (we show this later), α defines a map from the right to the left of the objects in the theorem.

Suppose now we have a derivation $f : P \rightarrow \Sigma^n M$ such that $\delta f = 0$ (ie, $fd_C = 0$). Then we can use the augmentation $q : P \rightarrow C$ of P to define a map

$$\beta(f) : P \rightarrow C \vee \Sigma^n M$$

by $\beta(x) = (q(x), f(x))$. That $\beta(f)$ is a map of CDGAs again follows easily from the Leibniz rule and definition of the product in a square zero extension. Hence β (provided it is well defined on homology classes) defines a map from left to the right of the objects in the theorem, and α and β are clearly inverse.

So it only remains to show that α and β are well defined on homotopy

and homology classes respectively. We treat β first. In order to show β is well defined we can use a choice of path object for $C \vee \Sigma^n M$ in $cdga_{\geq 0}/C$. Thankfully there is a nice choice available, defined as follows:

Recall that a path object for some X in a model category \mathcal{C} is an object PX which fits into a sequence

$$X \xrightarrow{j} PX \rightarrow X \times X$$

whose composition is the diagonal map, and for which j is a weak equivalence. Let $I = \langle b_1, b_2, a \rangle$ be the chain complex over \mathbb{Q} generated by two elements b_1 and b_2 in degree 0, and one element a in degree 1 with $da = b_1 - b_2$. For any C -module N , we can consider the chain complex $hom_{\mathbb{Q}}(I, N)$, given by the internal hom object in the category of chain complexes. This also inherits a C -module structure. We have the sequence

$$C \vee N \xrightarrow{j} C \vee hom_{\mathbb{Q}}(I, N) \xrightarrow{(p_1, p_2)} C \vee N \oplus_C C \vee N$$

where $p_1(c, f) := (c, f(b_1))$, $p_2(c, f) := (c, f(b_2))$ and $j(c, x) := (c, j_x)$, where $j_x(b_1) = j_x(b_2) = x$ and $j_x(a) = 0$.

$C \vee N \oplus_C C \vee N$ denotes the pushout of the diagram

$$C \vee N \leftarrow C \rightarrow C \vee N$$

which is the coproduct square of $C \vee N$ in $cdga_{\geq 0}/C$. We claim that this sequence makes $C \vee hom_{\mathbb{Q}}(I, N)$ a path object for $C \vee N$. This follows from the fact that $hom_{\mathbb{Q}}(I, N)$ is a path object for N in the category of chain complexes. Note that all the objects in the sequence are canonically augmented over C , and so this does indeed define a path object for $cdga_{\geq 0}/C$. In fact, in $cdga_{\geq 0}/C$, $C \vee M$ is also a *good* path object (in the sense of [10]), because the map

$$hom_{\mathbb{Q}}(I, M) \rightarrow (C \vee M) \oplus_C (C \vee M)$$

is a fibration (ie surjective).

With the above construction at hand, suppose $f, g : P \rightarrow \Sigma^n M$ are derivations which represent the same element in homology. Then $f - g = \delta h$, for some derivation h of degree $-(n - 1)$, and using this we can define a homotopy

$$H : P \rightarrow C \vee \text{Hom}_{\mathbb{Q}}(I, \Sigma^n M)$$

from $\beta(f)$ to $\beta(g)$ by setting $H(x) = (q(x), \theta_x)$, where $\theta_x(a) := hx$, $\theta_x(b_1) := fx$ and $\theta_x(b_2) := gx$.

We now turn to α . Suppose that $\gamma, \lambda : P \rightarrow C \vee \Sigma^n M$ are homotopic. By the fact that $C \vee \text{hom}_{\mathbb{Q}}(I, \Sigma^n M)$ is a good path object, the fact that all objects are fibrant, the fact that P is cofibrant, and abstract nonsense, we can deduce that γ and λ are homotopic through $C \vee \text{hom}_{\mathbb{Q}}(I, \Sigma^n M)$ (see 4.23 of [10]). That is, there exists a homotopy $H : P \rightarrow C \vee \text{hom}_{\mathbb{Q}}(I, \Sigma^n M)$ from γ to λ (ie, $p_1 H = \gamma$ and $p_2 H = \lambda$). Then we define a derivation

$$g : p \rightarrow \Sigma^{n-1} M$$

of degree $-(n - 1)$ by

$$g(x) = (pH(x))(a)$$

where p is the projection $p : C \vee \Sigma^n M \rightarrow \Sigma^n M$, and a is the element of I in degree 1. It is easily checked that g is a derivation, and (by using again the fact that M is concentrated in a single degree) $\delta g = \gamma - \lambda$.

□

Exterior algebras over wedges of spheres

Perhaps the most immediate application of the above material is to classify exterior algebras (of various flavours). More precisely, for k any commutative \mathbb{Q} -algebra concentrated in degree 0, we ask the following:

Classify all objects $C \in \text{cdga}_{\geq 0}$ (up to quasi isomorphism) such that $H_* C \cong k[x_m]/x^2$, where $m \geq 1$.

Since we are working rationally, if k is a complete intersection over \mathbb{Q} , the question is actually easy to solve, using the following classical result

THEOREM 7.22. (1 of [1])

Let A be a commutative, Noetherian ring and B a commutative A -algebra of finite-type. Then B is a complete intersection over A if and only if $AQ^n(B; M) = 0$ for all $n \geq 2$, and for all B -modules M .

Using this result, we see in particular that if k is a complete intersection then $AQ^n(k; k) = 0$ for $n \geq 2$, and hence that any CDGA C with the required homology must be formal.

So the cases of interest are those where k is not a complete intersection. Perhaps the simplest examples of these are

$$S^0 \vee \dots \vee S^0 := \mathbb{Q}[t_1, \dots, t_n]/(t_i t_j | 1 \leq i, j \leq n)$$

and so we will answer the question for k being of this form.

For this we will need a way of describing a minimal resolution of k in arbitrarily large degrees, which unfortunately we only have as a conjecture at the moment. This will involve a particular correspondence between minimal CDGAs and graded Lie algebras, so we will recall this now.

Minimal CDGAs and graded Lie algebras. Throughout this subsection, if V is a graded vector space, $A = (\bigwedge V, d)$ denotes the minimal CDGA on V , concentrated in degrees ≤ -2 . We will denote by d_2 the quadratic part of d , that is $d_2 : A \rightarrow V \wedge V$ takes any a to the quadratic part of da (da lives in the decomposables of A).

DEFINITION 7.23. Define the *homotopy Lie algebra* of A to be the graded vector space L given by $L = \text{Hom}(\Sigma^1 V, \mathbb{Q})$ (so L is concentrated in degrees ≥ 1). We will construct the Lie bracket below.

We have the canonical pairing

$$\langle -, - \rangle : V \otimes \Sigma^1 L \rightarrow \mathbb{Q}$$

where $\langle v, sx \rangle := (-1)^{|v|} sx(v)$, and sx denotes the suspension of an element x of L . This extends to a linear map

$$\bigwedge^k V \otimes \Sigma^1 L \otimes \dots \otimes \Sigma^1 L \rightarrow \mathbb{Q}$$

where

$$\langle v_1 \wedge \dots \wedge v_k; sx_k, \dots, sx_1 \rangle := \sum_{\sigma \in S_k} \epsilon(\sigma) \langle v_{\sigma(1)}; sx_1 \rangle \dots \langle v_{\sigma(k)}; sx_k \rangle$$

So we define a Lie bracket $[-, -] : L \otimes L \rightarrow L$, which is determined by the formula

$$\langle v; s[x, y] \rangle = -(-1)^{|y|} \langle d_2 v; sx, sy \rangle$$

for all $x, y \in L$ and $v \in V$.

For the proof of the symmetry and Jacobi identities see 21 (e) of [15].

Technical Lemmas and the main result. We will now compute a resolution of $k = \mathbb{Q}[t_1, \dots, t_n]/(t_i t_j | i, j \geq 0)$. To do this we will conjecture a result that allows us to work with coconnected CDGAs rather than connective ones. We begin with the following definition.

DEFINITION 7.24. Suppose C is any CDGA (connective or otherwise), and is augmented over some local \mathbb{Q} -algebra (S, \mathfrak{m}) of finite type, and that $\epsilon : C \rightarrow S$ is the augmentation. We define the ideal $I = \epsilon^{-1}(\mathfrak{m})$, and moreover define the ideal of *decomposables* of C relative to S to be ideal

$$Dec^\epsilon(C) := I^2$$

and call

$$QC_*^\epsilon = I/I^2$$

the *indecomposable quotient* of C over S . When the augmentation $\epsilon : C \rightarrow D$ is evident, we will simply write QC_* for the above.

REMARK 7.25. When concerned with decomposables of coconnected CDGAs, we will have $S = \mathbb{Q}$ and thus $QC_* = C_{>0}/(C_{>0})^2$. This is how indecomposables appear in rational homotopy theory, and if we have a minimal Sullivan algebra (M, d) (that is, coconnected, free and $d(M) \subset (C_{>0})^2$), the differential induces a map

$$d : QM_* \rightarrow QM_* \wedge QM_* + QM_* \wedge QM_* \wedge QM_* + \dots$$

For connective CDGAs, we will have $S = k = S^0 \vee \dots \vee S^0$, and C will usually be free, meaning that C_0 is a polynomial ring, and the augmentation just quotients out by the relations in k .

We now consider the coconnected CDGA

$$k' := \mathbb{Q}[w_1, \dots, w_n]/(w_i w_j)$$

where the s_j are concentrated in degree -2 . Then if $P' = (\Lambda V, d)$ denotes the minimal Sullivan algebra of k' with a given choice of basis including w_1, \dots, w_n , then we define $P(V)$ to be the free (connective) CDGA on $\Sigma^{-2}(V^\vee)$, with differential on $P(V)$ given by

$$(d_{P(V)})(s^{-2}x^\vee) = d_{P'}(x)$$

for any basis element x of V .

Now $P(V)_0 = \mathbb{Q}[s^{-2}w_1^\vee, \dots, s^{-2}w_n^\vee]$, and so there is a canonical map $P(V) \rightarrow k$, for which we have the following conjecture.

CONJECTURE 7.26. *The canonical map $P(V) \rightarrow k$ is a cofibrant resolution in $\text{cdga}_{\geq 0}$. That is, $P(V)$ is cofibrant, and the map is a quasi-isomorphism.*

LEMMA 7.27. *Let $P' \rightarrow k'$ be the minimal resolution. Then for any indecomposable $u \in QP'_{-r}$, and choice of generator w in P'_{-2} , there exists some indecomposable $v \in QP'_{-(r+1)}$ such that $d_2(v) = \alpha w \wedge u + g$, for some g containing no terms with w , and some $\alpha \neq 0$.*

PROOF. Fix u and w as in the statement of the lemma, and pick a basis v_1, \dots, v_n for QP'_{-2} which contains w , and a basis u_1, u_2, \dots, u_t for QP'_{-r} where $u = u_1$. Let $L(P')$ denote the homotopy Lie algebra of P' . Consider the vector subspace W of $L(P')_r$ spanned by words of the form $\{[lw, ly] | y \in QP'_{-r}\}$, where for x an indecomposable, lx denotes the corresponding element of $L(P')$. Observe that since $L(P')$ is a free graded Lie algebra, $[lw, ly]$ is non-zero for any $0 \neq y \in QP'_{-r}$. Now W is spanned by $\{[lw, lu_i] | 1 \leq i \leq t\}$, and so there's a basis B_0 of the form

$$\{[lw, lu_{i_k}] | 1 \leq k \leq N\}$$

of some size N , and we can construct it so that it contains $[lw, lu]$. Now we can extend this to a basis B for L_r . Denoting the dual DGA element by $\delta_{[lw, lu]}$, by definition $\langle \delta_{[lw, lu]}; s[lw, lu] \rangle = 1$, and so $d_2(\delta_{[lw, lu]})$ has a term with $w \wedge u$. Hence we can assume that $d_2(\delta_{[lw, lu]})$ has total term in w given by $\alpha w \wedge u + \beta w \wedge u'$, for some $u' \in QP'_{-r}$ and $\alpha \neq 0$. Given this, $\langle \delta_{[lw, lu]}; s[lw, lu'] \rangle = \beta$. Now by construction of the basis B , $[lw, lu']$ is a linear combination of words of the form $[lw, lu_i]$, and so if β were non-zero, then $\langle \delta_{[lw, lu]}; s[lw, lu_i] \rangle$ would have to be non zero for some $u_i \neq u$ in the basis for QP'_{-r} , which is a contradiction, and hence $\beta = 0$. Hence $d_2(\delta_{[lw, lu]}) = \alpha w \wedge u + g$, where g contains no terms involving w , and so $\delta_{[lw, lu]}$ is the required indecomposable v from the statement of the lemma.

□

COROLLARY 7.28. (*Dependent upon Conjecture 7.26*) *Suppose now we have a minimal resolution $P' \rightarrow k'$ which we have converted (as in 7.26) to a minimal resolution $P \rightarrow k$. Then for any $f \in \text{Der}_{-t}(P; k) \cong \text{Hom}_{\mathbb{Q}}(QP_t, k)$ of degree $-t$, f is a cycle if and only if its image is contained in the ideal (t_1, \dots, t_n) . Hence $Z_{-t}(\text{Der}_*(P; k)) \cong \text{Hom}_{\mathbb{Q}}(QP_t, \mathbb{Q}^n)$.*

PROOF. Suppose $f \in \text{Der}_{-t}(P; k) \cong \text{Hom}_{\mathbb{Q}}(QP_t, k)$. We will show that f is a cycle if and only if its image is contained in the maximal ideal of k . f is a cycle if and only if $(\delta f)(v) = 0$, for all $v \in QP_{t+1}$, which happens if and only if $f(d_P v) = 0$. Now by Lemma 7.27, for any $u \in QP_t$ and variable

$t_j \in P_0$, there exists some $v \in QP_{t+1}$ with $d_P(v) = \alpha t_j u + g$, where $\alpha \neq 0$ and g has no terms involving t_j . Hence, if $f(d_P v) = 0$ for all $v \in QP_{t+1}$, then

$$t_j f(u) = 0$$

for all $u \in QP_t$, which means that $f(u) \in (t_1, \dots, t_n) \subset k$, for all $u \in QP_t$ (that is that $f(u)$ is contained in the maximal ideal of k .)

Conversely if $f(u) \in (t_1, \dots, t_n)$, for all $u \in QP_t$, then since $d_P v$ is quadratic for all $v \in QP_{t+1}$ (by Conjecture 7.26), all the terms of $f(d_P v)$ must be products of two terms of the maximal ideal of k and hence $f(d_P v) = 0$, for all $v \in QP_{t+1}$, and hence f is a cycle.

So we have shown that f is a cycle if and only if $f(u) \in (t_1, \dots, t_n) \subset k$, for all $u \in QP_t$, and hence $Z_{-t}(Der_*(P; k)) \cong Hom_{\mathbb{Q}}(QP_t, \mathbb{Q}^n)$.

□

PROPOSITION 7.29. *(Dependent upon Conjecture 7.26) Suppose we have a minimal resolution $P' \rightarrow k'$, which we have converted (as in 7.26) to a minimal resolution $P \rightarrow k$. Then the space of boundaries $B_{-t}(Der_*(P; k))$ is (non-canonically) isomorphic to QP_{t-1} as a \mathbb{Q} -vector space.*

PROOF. Let $f \in Der_{-(t-1)}(P; k)$ be any derivation of degree $-(t-1)$, and pick a basis f_1, \dots, f_s for QP_t . For each i , $(\delta(f))(f_i)$ is some \mathbb{Q} -linear combination of terms of the form $t_j f(u)$, where t_j is any variable and $u \in QP_{-(t-1)}$ (note that any non-quadratic terms must map to zero under any derivation). By Lemma 7.27, for any basis v_1, \dots, v_b of $QP_{-(t-1)}$, each $t_a v_w$ appears as a term in some $d(f_r)$, and hence each $t_a f(v_w)$ appears as a term in $(\delta(f))(f_i)$. So since k is isomorphic to \mathbb{Q}^{n+1} as a \mathbb{Q} -module, the space

$$B_{-t}(Der_*(P; k))$$

can simply be expressed as a subspace of the space of $(n+1) \times s$ matrices with coefficients in \mathbb{Q} (in fact, by 7.28, we can express it as a subspace of $n \times s$ matrices) which, by the above observation, has dimension b as required.

□

The above conjectured results would put us nearly in position to complete the calculation. The final step is in calculating the dimension of the space of words of length n in a free graded Lie of algebra on m generators in degree 1. As with the formula for ungraded Lie algebras, the formula follows from (a graded version of) the Poincaré-Birkhoff Witt Theorem. However we could not find the precise construction required in the literature, and so we outline it below. This computation is completely rigorous and is not dependent on a conjecture:

Let V be a graded vector space with b_k generators in degree k . Then the graded Poincaré-Birkhoff-Witt Theorem states that

$$T(V) \cong U(L(V)) \cong S(L(V))$$

where $T(V)$ is the tensor algebra on V , $L(V)$ is the free graded Lie algebra on V , and $U(L(V))$ is the universal enveloping algebra of $L(V)$, and the second isomorphism is only an isomorphism of graded vector spaces. Let $d_k = \dim(L(V)_k)$. Then taking Poincaré series of $T(V)$ and $S(L(V))$ gives

$$\frac{1}{1 - \sum_{k \geq 0} b_k t^k} = \prod_{k \geq 0} \frac{(1 + t^{2k+1})^{d_{2k+1}}}{(1 - t^{2k})^{d_{2k}}}$$

LEMMA 7.30. *With the above notation, if we have $\frac{1}{1 - \sum_{k \geq 0} b_k t^k} = \exp(\sum_{k \geq 1} \frac{p_k}{k} t^k)$, for some coefficients p_k , then for all $k \geq 1$*

$$d_k = (-1)^k \frac{1}{k} \sum_{s|k} (-1)^{k/s} \mu(s) p_{k/s}$$

where $\mu(-)$ is the Möbius function.

PROOF. This is a simple argument using the Möbius inversion formula. Taking \log of both sides gives

$$\sum_{k \geq 1} \frac{p_k}{k} t^k = \sum_{k \geq 0} (d_{2k+1} \log(1 + t^{2k+1}) - d_{2k} \log(1 - t^{2k}))$$

then using the logarithm expansion, equating coefficients, and reformulating slightly gives

$$p_k = (-1)^k \sum_{s|k} (-1)^s s d_s$$

and now applying the Möbius inversion formula gives

$$(-1)^k k d_k = \sum_{s|k} \mu(s) (-1)^{k/s} p_{k/s}$$

which is the desired result. □

COROLLARY 7.31. *If $b_k = 0$ for $k > 1$, then writing $n = b_1$, we have that*

$$d_k = (-1)^k \frac{1}{k} \sum_{s|k} (-1)^{k/s} \mu(s) n^{k/s}$$

PROOF. We must express $\frac{1}{1-nt}$ in the form $\exp(\sum_{k \geq 1} \frac{p_k}{k} t^k)$ for some coefficients p_k . Setting these to be equal and taking logarithms gives

$$\sum_{k \geq 1} \frac{p_k}{k} t^k = \log(1 + nt) - \log(1 - n^2 t^2)$$

and using the series expansion for logarithms gives immediately that we must have $p_k = n^k$ for $k \geq 1$. □

Now with the conjectured results, and the formula in 7.31, we conclude the following

THEOREM 7.32. *(Dependent upon Conjecture 7.26)*

Let k be as above on n generators t_1, \dots, t_n . Then the set of quasi-isomorphism types of \mathbb{Q} -CDGAs A augmented over k , for which $H_ A \cong k[x]/x^2$, where x is in degree m , is in bijection with $\mathbb{Q}P^{N_{mn}-1}$ (rational projective space of dimension $N_{mn} - 1$) where*

$$N_{mn} = (-1)^{m+2} \left(\frac{n}{m+2} \sum_{d|(m+2)} (-1)^{(m+2)/d} \mu(d) n^{(m+2)/d} + \frac{1}{m+1} \sum_{d|(m+1)} (-1)^{(m+1)/d} \mu(d) n^{(m+1)/d} \right)$$

PROOF. The proof given here will be conditional on Conjecture 7.26.

Let $P \rightarrow k$ be a resolution of k as in 7.28. By 7.28 for all t we have

$Z_{-t}(\text{Der}_*(P; k)) \cong \text{Hom}_{\mathbb{Q}}(QP_t, \mathbb{Q}^n)$, and so by 7.31 and Conjecture 7.26

$$\dim_{\mathbb{Q}}(Z_{-t}(\text{Der}_*(P; k))) = n \dim_{\mathbb{Q}}(QP_t) = nd_{t+1}$$

Similarly, by 7.29 and 7.31, for all t we have

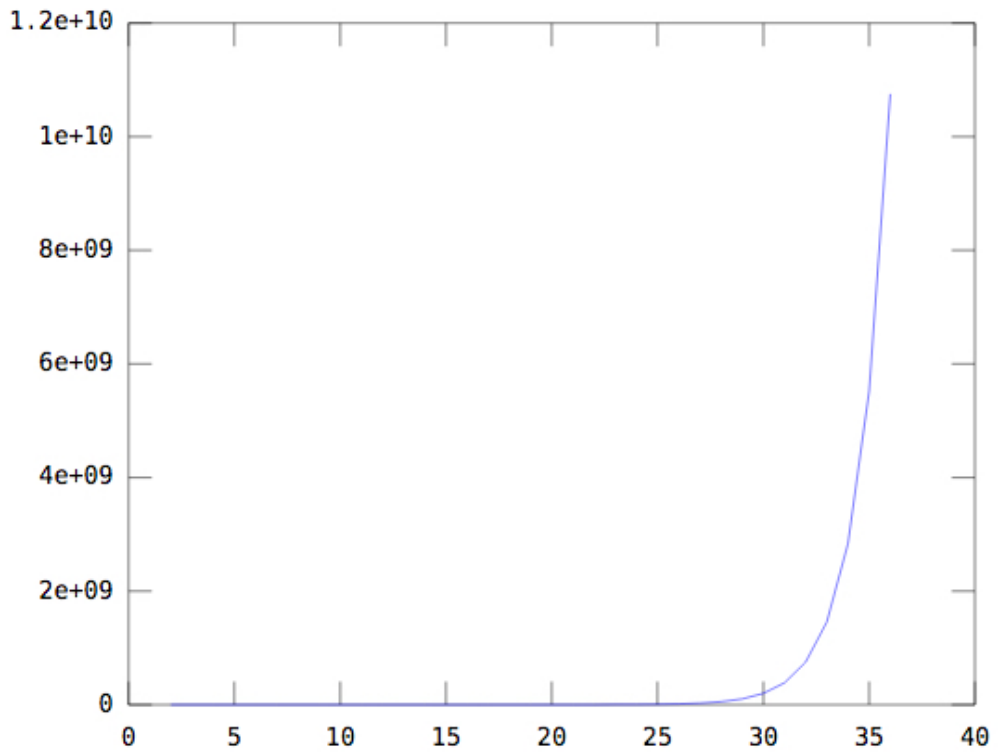
$$\dim_{\mathbb{Q}}(B_{-t}(\text{Der}_*(P; k))) = \dim_{\mathbb{Q}}(QP_{t-1}) = d_t$$

Hence the dimension of $AQ^t(k; k)$ is $nd_{t+1} - d_t$, for all t (because the homology of the derivation complex is just a quotient of vector spaces).

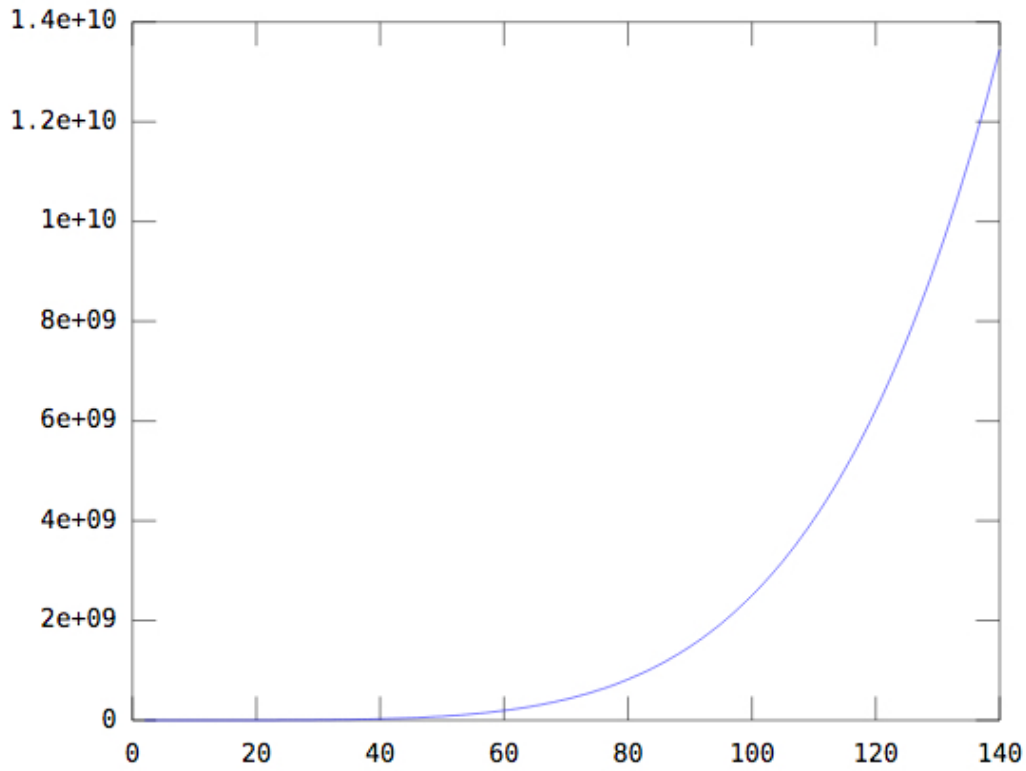
The generator x is in degree m , and so the set of quasi-isomorphism types is classified by $AQ^{m+1}(k; k)$, after quotienting by automorphisms of k . By the calculation above, the dimension of $AQ^{m+1}(k; k)$ is N_{mn} (as written in the statement of the theorem). The automorphisms of k as a module over itself are just given by multiplication by units, and hence the quasi-isomorphism types of the desired algebras are in bijection with $\mathbb{Q}P^{N_{mn}-1}$ as stated.

□

In order to try to indicate the speed with which N_{mn} grows with m and n , we have the following plots (overleaf).

FIGURE 1. N_{m2} against m

In Figure 1, we have fixed n (the number of variables t_1, \dots, t_n) to be 2, and plotted the values of N_{m2} against varying values of m (the dimension of the exterior homology generator x).

FIGURE 2. N_{2n} against n

In Figure 2 we have fixed the dimension m of the exterior generator x to be 2, and plotted N_{2n} against varying values of the number of variables t_1, \dots, t_n .

As is to be expected from the terms in the formula for N_{mn} , it grows much faster with m than with n .

Bibliography

- [1] M. André. Homologie des Algèbres Commutatives. *Springer-Verlag*, 1974.
- [2] M. Auslander. On the Dimension of Modules and Algebras. III. Global Dimension. *Nagoya Math. Journal*, 9:67–77, 1955.
- [3] L.L. Avramov, H-B. Føxby, and S. Halperin. Differential Graded Homological Algebra. *In preparation, privately communicated*, 2003.
- [4] L.L. Avramov and S. Halperin. Through the Looking Glass: A Dictionary between Rational Homotopy Theory and Local Algebra. *Lecture Notes in Mathematics*, 1183(4):1–27, 1986.
- [5] H-J. Baues and A. Quintero. *Infinite Homotopy Theory*. Springer, 2001.
- [6] A.K. Bousfield and V.K.A.M. Gugenheim. On PL de Rham Theory and Rational Homotopy Type. *Memoirs of the American Mathematical Society*, 8(179), 1976.
- [7] A.K. Bousfield and D.M. Kan. *Homotopy Limits, Completions and Localizations*, volume 304 of *Lecture Notes in Mathematics*. Springer, 1972.
- [8] G.E. Bredon. Equivariant Cohomology Theories. *Bulletin of the AMS*, 73(2):266–268, 1967.
- [9] S.E. Crans. Quillen Closed Model Structures for Sheaves. *Journal of Pure and Applied Algebra*, 101(1):35–57, 1995.
- [10] D. Dugger and B. Shipley. Topological Equivalences for Differential Graded Algebras. *Advances in Mathematics*, 212(1):37–61, 2007.
- [11] W. Dwyer and J. Spalinski. Homotopy Theories and Model Categories. *Handbook of Algebraic Topology*, pages 73–126, 1995.
- [12] W.G. Dwyer. Strong Convergence of the Eilenberg-Moore Spectral Sequence. *Pergamon Press*, 13:255–265, 1974.
- [13] W.G. Dwyer, J.P.C. Greenlees, and S. Iyengar. Duality in Algebra and Topology. *Advances in Mathematics*, 200(2):357–402, 2006.
- [14] Y. Félix, S. Halperin, and J-C. Thomas. Gorenstein Spaces. *Advances in Mathematics*, 71(1):92–112, 1988.

- [15] Y. Félix, S. Halperin, and J-C. Thomas. *Rational Homotopy Theory*, volume 205 of *Graduate Texts in Mathematics*. Springer, 2001.
- [16] P.G. Goerss and J.F. Jardine. *Simplicial Homotopy Theory*. Modern Birkhäuser Classics, 2009.
- [17] P.G. Goerss and K. Schemmerhorn. Model Categories and Simplicial Methods. *Lecture Notes from the Summer School on “Interactions between Homotopy Theory and Algebra”*, *Contemporary Mathematics*, 2007.
- [18] A. Gómez-Tato, S. Halperin, and D. Tanré. Rational Homotopy Theory for Non-Simply Connected Spaces. *Transactions of the American Mathematical Society*, 352(4):1493–1525, 2000.
- [19] J.P.C. Greenlees, K. Hess, and S. Shamir. Complete Intersections in Rational Homotopy Theory. arXiv:0906.3247 [math.AT] (unpublished), 2009.
- [20] J.P.C. Greenlees, K. Hess, and S. Shamir. Complete Intersections in Rational Homotopy Theory. *Journal of Pure and Applied Algebra*, 217(4):636–663, 2013.
- [21] T. Haraguchi. Long Exact Sequences for de Rham cohomology of Diffeological Spaces. *arXiv:1404.1127*, 2014.
- [22] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.
- [23] A. Hatcher. *Spectral Sequences in Algebraic Topology*. unpublished. available on A. Hatcher’s Cornell webpage, 2004.
- [24] K. Hess. Rational Homotopy Theory: A brief introduction. *Lecture Notes from the Summer School on “Interactions between Homotopy Theory and Algebra”*, *Contemporary Mathematics*, 436, 2007.
- [25] V. Hinich. DG Coalgebras as Formal Stacks. *Journal of Pure and Applied Algebra*, 162(2-3):209–250, 2001.
- [26] M. Hovey. *Model Categories*, volume 63 of *Mathematical Surveys and Monographs (AMS)*. American Mathematical Society, 1999.
- [27] A.P.M. Kupers. Sullivan’s Approach to Rational Homotopy Theory. *Notes from Stanford’s Thursday Seminar*, <http://web.stanford.edu/~kupers/sullivansapproach.pdf>.
- [28] S. Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer, 1971.
- [29] A. Lazarev. Homotopy Theory of A_∞ ring spectra and applications to MU-modules. *K-Theory*, 24(3):243–281, 2001.
- [30] J. Lurie. *Higher Algebra*. eBook. Harvard University, 2015.

- [31] J.P. May. Cohen-Macaulay and Regular Local Rings. *uchicago web paper*
<http://www.math.uchicago.edu/~may/MISC/RegularLocal.pdf>.
- [32] J.P. May. *Equivariant Homotopy and Cohomology Theory*. Number 91 in CBMS Regional Conference Series in Mathematics. AMS, 2015.
- [33] J. McCleary. *User's Guide to Spectral Sequences*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2nd edition, 2001.
- [34] J.C. Moore. Differential Homological Algebra. *Actes du Congrès International Math.*, Tome 1:335–339, 1970.
- [35] J. Neisendorfer. Lie algebras, Coalgebras and Rational Homotopy Theory for Nilpotent Spaces. *Pacific Journal of Mathematics*, 74:429–460, 1978.
- [36] D.G. Quillen. Rational Homotopy Theory. *The Annals of Mathematics*, 90(2):205–295, 1969.
- [37] S. Schwede and B.E. Shipley. Algebras and Modules in Monoidal Model Categories. *Proceedings of the LMS*, 80(2):491–511, 2000.
- [38] J.P. Serre. Sur la Dimension Homologique des Anneaux et des Modules Noethériens. *Proceedings of the International Symposium on Algebraic Number Theory, Tokyo*, pages 175–189, 1955.
- [39] N.E. Steenrod. *The Topology of Fibre Bundles*, volume 14 of *Princeton Mathematical Series*. Princeton University Press, 1999.
- [40] N.P. Strickland. The Model Structure for Chain Complexes (unpublished url).
<https://neil-strickland.staff.shef.ac.uk/research/chmc.pdf>.
- [41] D. Sullivan. Infinitesimal Computations in Topology. *Publications mathématiques de l'I.H.É.S.*, 47(1):269–331, 1977.
- [42] K. Watanabe, T. Ishikawa, S. Tachibana, and K. Otsuka. On Tensor Products of Gorenstein Rings. *J.Math. Kyoto Univ.*, 9(3):413–423, 1969.
- [43] C.A. Weibel. *An Introduction to Homological Algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1994.