Spaces of Analytic Functions on the Complex Half-Plane

UNIVERSITY OF LEEDS

Andrzej Stanisław Kucik

Submitted in accordance with the requirements for the degree of Doctor of Philosophy

The University of Leeds

School of Mathematics

March 2017

The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

Acknowledgements

I would like to express my gratitude towards the School of Mathematics at the University of Leeds and the Engineering and Physical Sciences Research Council for their generous financial assistance, granting me an opportunity to undertake and complete the research presented in this thesis.

I wish to thank my parents and grandparents too, for all their support throughout my life, allowing me to achieve everything I have accomplished so far.

I would also like to praise all the teachers and professors guiding me through the process of my formal education for over twenty years now. Of them all, I would especially like to declare my gratefulness towards Professor Jonathan Partington, my doctoral supervisor, who has continuously inspired me during my research project, giving me new ideas to contemplate, meticulously correcting countless typographical errors I so heedlessly make, and helping me to perfect the proofs, by pointing out the flaws in my reasoning and argument presentation.

Abstract

In this thesis we present certain spaces of analytic functions on the complex half-plane, including the Hardy, the Bergman spaces, and their generalisation: Zen spaces. We use the latter to construct a new type of spaces, which include the Dirichlet and the Hardy–Sobolev spaces. We show that the Laplace transform defines an isometric map from the weighted $L^2(0, \infty)$ spaces into these newly-constructed spaces. These spaces are reproducing kernel Hilbert spaces, and we employ their reproducing kernels to investigate their features. We compare corresponding spaces on the disk and on the half-plane.

We present the notions of Carleson embeddings and Carleson measures and characterise them for the spaces introduced earlier, using the reproducing kernels, Carleson squares and Whitney decomposition of the half-plane into an abstract tree.

We also study multiplication operators for these spaces. We show how the Carleson measures can be used to test the boundedness of these operators. We show that if a Hilbert space of complex valued functions is also a Banach algebra with respect to the pointwise multiplication, then it must be a reproducing kernel Hilbert space and its kernels are uniformly bounded. We provide examples of such spaces. We examine spectra and character spaces corresponding to multiplication operators. We study weighted composition operators and, using the concept of causality, we link the boundedness of such operators on Zen spaces to Bergman kernels and weighted Bergman spaces. We use this to show that a composition operator on a Zen space is bounded only if it has a finite angular derivative at infinity. We also prove that no such operator can be compact.

We present an application of spaces of analytic functions on the half-plane in the study of linear evolution equations, linking the admissibility criterion for control and observation operators to the boundedness of Laplace–Carleson embeddings.

Contents

Chapter 1

Introduction

> Not thinking about anything is Zen. Once you know this, walking, standing, sitting, or lying down, everything you do is Zen.
>
> BODHIDHARMA, *The Zen Teachings of Bodhidharma*

Spaces of analytic functions are not only the cornerstone of functional analysis but they are also at the very core of modern mathematics (both pure and applied) as a whole. Thence their rôle and importance hardly require any introduction or explanation. Their multitude is a reflection of the multitude of questions that the twentieth century mathematics aspired to answer. And although many of them are sometimes tailored to a specific problem, there is an aspect that they persistently seem to share. That is, the domain of definition of their elements - the unit disk of the complex plane

$$\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| < 1\}$$

(and possibly its higher dimensional analogues). This is often imposed by the problem, which they are employed to solve, itself. But routinely it is also due to the fact that analytic functions expressed as power series centred at 0 are particularly nice to manipulate and effectively produce elegant results. This may give us a false impression that other domains are somehow less useful and hence do not deserve much attention. There is an exception from this unfortunate rule, and it is the case of perhaps the most famous canonical examples of spaces of analytic functions, that is, the Hardy and the Bergman spaces. Monographs dedicated to the study of these two types of spaces, apart from covering extensively the usual unit disk case, also give us some insight to the situation when the disk is replaced by a complex half-plane. But since this alternative approach is normally presented as a side-note, exercise or an optional chapter, it may even enhance the misconception that this version of these spaces is somehow less important, less useful or even artificial.

This thesis is solely dedicated to spaces of analytic functions on the open right complex half-plane

$$\mathbb{C}_+ := \{z \in \mathbb{C} \ : \ \mathrm{Re}(z) > 0\} .$$

Often in the literature we may see similarly defined spaces of functions, with the right complex half-plane replaced by the upper complex-half plane (i.e. $\mathrm{Im}(z) > 0$). This variant is dictated by the techniques used in the relevant proofs or sometimes it is just a matter of personal taste. The choice of the half-plane as the domain of definition, however, is far from being arbitrary, and the reasoning behind it, both theoretical and practical, shall be unveiled in due course.

And finally, the reader should bear in mind that, although this setting provides many interesting and useful results, the choice of the domain (and in particular the fact that it is an unbounded domain) causes numerous complications, and the problems, which can be easily solved in the unit disk case, are either much more difficult to tackle, have a different answer or still remain to be answered in the half-plane setting.

1.1 Zen spaces

1.1.1 Foundations

We shall start by considering so-called Zen spaces.

Definition 1.1.1 *Let $1 \leq p < \infty$, let $\tilde{\nu}$ be a positive regular Borel measure on $[0, \infty)$ (see § 52, p. 223 in [50]) satisfying the following Δ_2-condition:*

$$\sup_{x>0} \frac{\tilde{\nu}([0, 2x))}{\tilde{\nu}([0, x))} < \infty, \tag{Δ_2}$$

let λ be the Lebesgue measure on $i\mathbb{R}$, and let ν be a positive regular Borel measure on $\overline{\mathbb{C}_+} := [0, \infty) \times i\mathbb{R}$, given by $\nu := \tilde{\nu} \otimes \lambda$. The Zen space *corresponding to p and ν is defined to be the normed vector space*

$$A^p_\nu := \left\{ F : \mathbb{C}_+ \longrightarrow \mathbb{C},\ \textit{analytic} \ : \ \|F\|_{A^p_\nu} := \left(\sup_{\varepsilon>0} \int_{\overline{\mathbb{C}_+}} |F(z+\varepsilon)|^p \, d\nu(z) \right)^{\frac{1}{p}} < \infty \right\}.$$

These spaces were originally constructed in [51] and [52] by Zen Harper, and named after him in [61], where their definition appears in the form given above along with many fundamental results related to them. They also occur in [22], [64], [65], [66], [67], [61], [62] and [87].

The measures satisfying the (Δ_2)-condition (also known as the *doubling condition*) have been studied in the theory of harmonic analysis and partial differential equations (see [97] for an early reference). If $\tilde{\nu}(\{0\}) > 0$, then, by Hardy space theory, every function F in A^p_ν has a well-defined boundary function $\tilde{F}$ in $L^p(i\mathbb{R})$ (see Theorem in Chapter 8, p. 128, of [56]) and we can give meaning to the expression $\int_{\overline{\mathbb{C}_+}} |F(z)|^p \, d\nu(z)$. Hence we shall write

$$\|F\|_{A^p_\nu} = \left(\int_{\overline{\mathbb{C}_+}} |F(z)|^p \, d\nu(z) \right)^{\frac{1}{p}}.$$

Note that this expression also makes sense when $\tilde{\nu}(\{0\}) = 0$, since then F is still defined ν-a.e. on $\overline{\mathbb{C}_+}$. In this case we can of course write $\mathbb{C}_+$ instead of $\overline{\mathbb{C}_+}$.

The space A^p_ν is a Banach space, and if $p = 2$, it is clearly a Hilbert space (see Proposition 4.1, p. 61 in [87]). Examples of Zen spaces include Hardy spaces $H^p(\mathbb{C}_+)$ (when $\tilde{\nu} = \frac{1}{2\pi}\delta_0$, where δ_0 is the Dirac delta measure in 0) and weighted Bergman spaces $\mathcal{B}^p_\alpha(\mathbb{C}_+)$, $\alpha > -1$ (when $d\tilde{\nu}(r) = \frac{1}{\pi} r^\alpha\, dr$). Some other examples are discussed in [51]. We shall use the convention $\mathcal{B}^p_{-1}(\mathbb{C}_+) := H^p(\mathbb{C}_+)$.

1.1.2 Laplace transform isometry

One of the main tools in the analysis of Zen spaces is the fact that the Laplace transform defines an isometric map (and often we can even say: an isometry) from weighted L^2 spaces on the positive real half-line into (or respectively: onto) certain spaces of analytic functions on the complex plane, which we shall derive from the Zen spaces in the next section. But first, let us explain in detail what we understand by a weighted L^2 space and the Laplace transform.

Let $1 \leq p < \infty$ and let w be positive measurable function on $(0, \infty)$. By a *weighted* $L^p_w(0, \infty)$ *space* we mean the Lebesgue function space $L^p((0, \infty), \mu)$, where μ is a measure on $(0, \infty)$, given by $d\mu(t) = w(t)\, dt$. Or, in other words,

$$L^p_w(0, \infty) := \left\{ f : (0, \infty) \longrightarrow \mathbb{C} \ : \ \|f\|_{L^p_w(0,\infty)} := \left(\int_0^\infty |f(t)|^p w(t)\, dt \right)^{\frac{1}{p}} < \infty \right\}.$$

Corollary 1.1.2 *Let w be a positive measurable function on $(0, \infty)$. The subspace $L^1(0, \infty) \cap L^p_w(0, \infty)$ is dense in $L^p_w(0, \infty)$.*

Proof

Let $f \in L^p_w(0, \infty)$. We define

$$f_n(t) := \begin{cases} f(t) & \text{if } \frac{1}{n} \leq t \leq n, \\ 0 & \text{otherwise} \end{cases} \qquad (\forall n \in \mathbb{N}).$$

Clearly, $\{f_n\}_{n\in\mathbb{N}} \subset L^1(0,\infty) \cap L^p_w(0,\infty)$ and $f_n \overset{\text{pointwise}}{\longrightarrow} f$ as $n \longrightarrow \infty$. Thus, by Lebesgue's Dominated Convergence Theorem for $L^p_w(0,\infty)$, with $|f|$ as the dominating function (1.34 in [90], p. 26), we have

$$\lim_{n\to\infty} \|f - f_n\|_{L^p_w(0,\infty)} \overset{\text{def}^{\underline{\text{n}}}}{=} \lim_{n\to\infty} \left(\int_0^\infty |f - f_n|^p w(t)\, dt \right)^{\frac{1}{p}} = 0,$$

proving the claim. □

The *Laplace transform* ($\mathfrak{L}$) is the integral transform

$$\mathfrak{L}[f](z) := \int_0^\infty f(t) e^{-tz}\, dt, \tag{1.1}$$

taking a function f of a positive variable t to a function $\mathfrak{L}[f]$ of a complex variable z (see [102]). As mentioned before, our considerations will involve mainly functions belonging to weighted L^2 spaces on $(0,\infty)$, for which the integral (1.1) may not necessarily be convergent. The previous corollary allows us, however, to extend the definition of the Laplace transform in a natural way. Suppose that there exists a Banach space of analytic functions B such that for each g in $L^1(0,\infty) \cap L^p_w(0,\infty)$ we have

$$\mathfrak{L}[g] \in B \qquad \text{and} \qquad \|\mathfrak{L}[g]\|_B \leq \|g\|_{L^p_w(0,\infty)}. \tag{1.2}$$

Given $f \in L^p_w(0,\infty)$, we know that there exists a sequence $(f_n)_{n=0}^\infty$ of functions lying in $L^1(0,\infty) \cap L^p_w(0,\infty)$ and converging to f in the $L^p_w(0,\infty)$ norm, and hence we can define the (extended) Laplace transform to be the linear operator

$$\mathfrak{L} : L^p_w(0,\infty) \longrightarrow B \qquad f \longmapsto \mathfrak{L}[f] := \lim_{n\to\infty} \int_0^\infty f_n(t) e^{-tz}\, dt,$$

where the limit is taken with respect to the B norm. This limit exists since, by (1.2), convergence of $(f_n)_{n=0}^\infty$ in $L^1(0,\infty) \cap L^p_w(0,\infty)$ implies convergence of $(\mathfrak{L}[f_n])_{n=0}^\infty$ in B.

The next theorem is elementary but also crucial for studying certain properties of Zen spaces and their generalisations. For $n = 0$ and $A^2_\nu = H^2(\mathbb{C}_+)$ is known as Paley–Wiener theorem (see Theorem 1.4.1, § 1.4, p. 25 in [21]). It appeared in [61] for Zen spaces and

$n = 0$ (Proposition 2.3, p. 795), and earlier for some special cases in [29] (Theorem 2, p. 309) [33] (Theorem 1, p. 460), [51] (Theorem 2.5, p. 119) and [52]. The general case ($n \geq 0$) was stated in [65] and we present its proof below.

Theorem 1.1.3 (Paley–Wiener theorem for $\mathfrak{L}^{(n)}$ and Zen spaces - Theorem 1 in [65]) *The n^{th} derivative of the Laplace transform defines an isometric map*

$$\mathfrak{L}^{(n)} \, : \, L^2_{w_n}(0, \infty) \longrightarrow A^2_\nu,$$

where

$$w_n(t) := 2\pi t^{2n} \int_{[0,\infty)} e^{-2tx} \, d\tilde{\nu}(x) \qquad (t > 0). \tag{1.3}$$

Remark 1.1.4 *One of the properties of the Laplace transform, which are immediate from the definition, is that its n^{th} derivative can be expressed as*

$$\mathfrak{L}^{(n)}[f](z) := \frac{d^n}{dz^n} \int_0^\infty f(t) e^{-tz} \, dt = \int_0^\infty (-t)^n f(t) e^{-tz} \, dt \overset{\text{def.}}{=} \mathfrak{L}[(-\cdot)^n f](z),$$

for all $z \in \mathbb{C}$ and $f : (0, \infty) \longrightarrow \mathbb{C}$ such that the above integrals converge. Thus a corresponding linear map $\mathfrak{L}^{(n)} : L^2_{w_n}(0, \infty) \longrightarrow A^2_\nu$ can be defined by the density argument outlined above.

Remark 1.1.5 *The (Δ_2)-condition ensures that the integral in (1.3) converges for all $t > 0$. Indeed, for any $k \in \mathbb{N}_0$ and $x \geq 0$ we have*

$$\begin{aligned} \tilde{\nu}([2^k x, 2^{k+1} x)) &= \tilde{\nu}([0, 2^{k+1} x)) - \tilde{\nu}([0, 2^k x)) \\ &\overset{(\Delta_2)}{\leq} (R-1)\tilde{\nu}([0, 2^k x)) \\ &\overset{(\Delta_2)}{\leq} (R-1) R^k \tilde{\nu}([0, x)), \end{aligned} \tag{1.4}$$

where R is as defined in (Δ_2). And so

$$\begin{aligned} \int_{[0,\infty)} e^{-2tx} \, d\tilde{\nu}(x) &\leq \tilde{\nu}([0, 1)) + \sum_{k=0}^{\infty} e^{-2^{k+1}t} \tilde{\nu}([2^k, 2^{k+1})) \\ &\overset{(1.4)}{\leq} \tilde{\nu}([0, 1)) \left(1 + (R-1) \sum_{k=0}^{\infty} R^k e^{-2^{k+1}t} \right), \end{aligned} \tag{1.5}$$

and the series converges for any $t > 0$ by the D'Alembert Ratio Test.

The proof of Theorem 1.1.3 follows closely the proof of Proposition 2.3 in [61], using the elementary relation between the Laplace and the Fourier ($\mathfrak{F}$) transforms (see [13]), and that the latter defines an isometry (by the Plancherel theorem, see Theorem 1.4.2, § 1.4, p. 25 in [21]; also see [78]); and Fubini's Theorem for regular measures (Theorem 8.8, Chapter 8, p. 164 in [90]).

Proof

Let $f \in L^1 \cap L^2_{w_n}(0,\infty)$, $g_n(t) := t^n f(t)$ and $z = x + iy \in \mathbb{C}_+$. Then

$$
\begin{aligned}
\sup_{\varepsilon>0} \int_{\overline{\mathbb{C}_+}} \left|\mathfrak{L}^{(n)}[f](z+\varepsilon)\right|^2 d\nu(z) &= \sup_{\varepsilon>0} \int_{[0,\infty)} \int_{-\infty}^{\infty} \left|\mathfrak{L}^{(n)}[f](x+iy+\varepsilon)\right|^2 \, d\lambda(y) d\tilde{\nu}(x) \\
&= \sup_{\varepsilon>0} \int_{[0,\infty)} \|(-1)^n \mathfrak{L}[t^n f](x+\cdot+\varepsilon)\|^2_{L^2(i\mathbb{R})} \, d\tilde{\nu}(x) \\
&= \sup_{\varepsilon>0} \int_{[0,\infty)} \|\mathfrak{L}[g_n](x+\cdot+\varepsilon)\|^2_{L^2(i\mathbb{R})} \, d\tilde{\nu}(x) \\
&= \sup_{\varepsilon>0} \int_{[0,\infty)} \left\|\mathfrak{F}\left[e^{-(x+\varepsilon)\cdot} g_n\right]\right\|^2_{L^2(\mathbb{R})} \, d\tilde{\nu}(x) \\
&= \sup_{\varepsilon>0} \int_{[0,\infty)} 2\pi \left\|e^{-(x+\varepsilon)\cdot} g_n\right\|^2_{L^2(0,\infty)} \, d\tilde{\nu}(x) \\
&\overset{\text{Fubini's Th}^{\underline{\text{m}}}}{=} \sup_{\varepsilon>0} \int_0^{\infty} |g_n(t)|^2 \, 2\pi \int_{[0,\infty)} e^{-2(x+\varepsilon)t} \, d\tilde{\nu}(x) \, dt \\
&\overset{(1.3)}{=} \int_0^{\infty} |f(t)|^2 \, w_n(t) \, dt,
\end{aligned}
$$

and the result follows by the density of $L^1 \cap L^2_{w_n}(0,\infty)$ in $L^2_{w_n}(0,\infty)$. $\square$

1.2 $A^p(\mathbb{C}_+,\,(\nu_n)_{n=0}^m)$ and $A^2_{(m)}$ spaces

1.2.1 Definitions

Theorem 1.1.3 suggests a generalisation of Zen spaces. Namely, let $1 \le p < \infty$, $m \in \mathbb{N}_0 \cup \{\infty\}$, and let $(\tilde{\nu}_n)_{n=0}^m$ be a sequence of positive regular Borel measures on

$[0, \infty)$, each of which satisfies the (Δ_2)-condition. We then have a sequence of Zen spaces $(A^p_{\nu_n})_{n=0}^m$, where each $\nu_n = \tilde{\nu}_n \otimes \lambda$, and we can define a normed space

$$A^p(\mathbb{C}_+, (\nu_n)_{n=0}^m) := \left\{ F : \mathbb{C}_+ \longrightarrow \mathbb{C}, \text{analytic} \ : \ \|F\| := \left(\sum_{n=0}^m \left\| F^{(n)} \right\|^p_{A^p_{\nu_n}} \right)^{\frac{1}{p}} < \infty \right\}.$$

These spaces were introduced in [65]. They also appear [64] and [66].

Corollary 1.2.1 (Theorem 1 in [65] and [64]) *Let $w_{(m)}$ be a self-map on the set of positive real numbers given by*

$$w_{(m)}(t) := \sum_{n=0}^m w_n(t) < \infty, \quad \textit{where} \quad w_n(t) := 2\pi t^{2n} \int_{[0,\infty)} e^{-2tx}\, d\tilde{\nu}_n(x) \qquad (\forall t > 0). \tag{1.6}$$

Then the Laplace transform defines an isometric map

$$\mathfrak{L} : L^2_{w_{(m)}}(0, \infty) \longrightarrow A^2(\mathbb{C}_+, (\nu_n)_{n=0}^m).$$

Proof

Let $f \in L^2_{w_{(m)}}(0, \infty)$, and let $F := \mathfrak{L}[f]$. By Theorem 1.1.3 we have

$$\|F\|^2_{A^2(\mathbb{C}_+, (\nu_n)_{n=0}^m)} \overset{\text{def}^{\underline{n}}}{=} \sum_{n=0}^m \|F^{(n)}\|^2_{A^2_{\nu_n}} \overset{\text{Th}^{\underline{m}}1.1.3}{=} \sum_{n=0}^m \|f\|^2_{L^2_{w_n}(0,\infty)} \overset{(1.6)}{=} \|f\|^2_{L^2_{w_{(m)}}(0,\infty)}.$$

□

The following lemma was proved in [67] for $m = 0$, but we can easily adapt its proof to include other values of m.

Lemma 1.2.2 (Lemma 3 in [67] for $m = 0$) *Let $m \in \mathbb{N}_0$. There exists $C \geq 2$ such that*

$$w_{(m)}\left(\frac{t}{2}\right) \leq C w_{(m)}(t) \qquad (\forall t > 0). \tag{1.7}$$

Proof

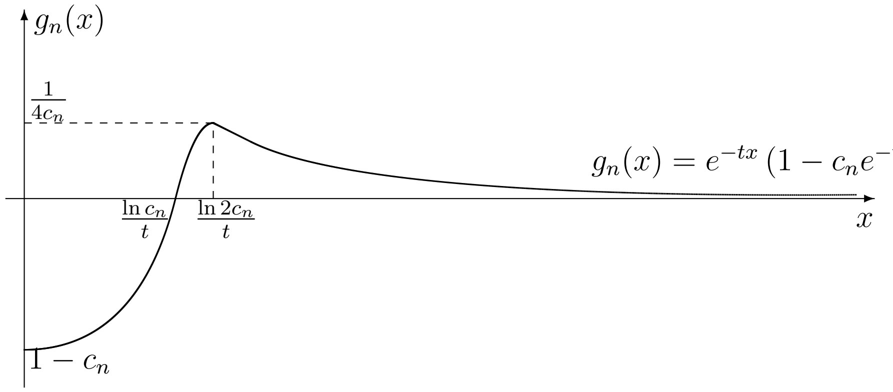
We want to show that for each $0 \leq n \leq m$ there exists $c_n \geq 2$ such that

$$\int_{[0,\infty)} e^{-tx}\, d\tilde{\nu}_n(x) \leq c_n \int_{[0,\infty)} e^{-2tx}\, d\tilde{\nu}_n(x), \tag{1.8}$$

or equivalently

$$\int_{[0,\infty)} e^{-tx}(1 - c_n e^{-tx})\, d\tilde{\nu}_n(x) \leq 0.$$

Consider the graph below.



It is clear that we need to have

$$-\int_{\left[0,\, \frac{\ln c_n}{t}\right)} e^{-tx}\left(1 - c_n e^{-tx}\right)\, d\tilde{\nu}_n(x) \geq \int_{\left[\frac{\ln c_n}{t},\, \infty\right)} e^{-tx}\left(1 - c_n e^{-tx}\right)\, d\tilde{\nu}_n(x).$$

Let R_n be the supremum we get from the (Δ_2)-condition for $\tilde{\nu}_n$, for all $0 \leq n \leq m$. Observe that for $c_n \geq 2$ we have

$$\frac{\ln 2c_n}{2t} \leq \frac{\ln c_n}{t} \qquad (\forall t > 0). \tag{1.9}$$

Thus

$$\begin{aligned} -\int_{\left[0,\, \frac{\ln c_n}{t}\right)} e^{-tx}\left(1 - c_n e^{-tx}\right)\, d\tilde{\nu}_n(r) &\overset{(1.9)}{\geq} \tilde{\nu}_n\left[0, \frac{\ln 2c_n}{2t}\right)\left(c_n e^{-\ln 2c_n} - e^{-\frac{\ln 2c_n}{2}}\right) \\ &\overset{(\Delta_2)}{\geq} \frac{\tilde{\nu}\left[0, \frac{\ln 2c_n}{t}\right)}{R_n}\left(\frac{1}{2} - \frac{1}{\sqrt{2c_n}}\right). \end{aligned}$$

Notice that we also have

$$-\tilde{\nu}_n\left[0,\frac{\ln c_n}{t}\right)\overset{(1.9)}{\leq}-\tilde{\nu}_n\left[0,\frac{\ln 2c_n}{2t}\right)\overset{(\Delta_2)}{\leq}-\frac{\tilde{\nu}_n\left[0,\frac{\ln 2c_n}{t}\right)}{R_n},\tag{1.10}$$

so

$$\begin{aligned}\int_{\left[\frac{\ln c_n}{t},\frac{\ln 2c_n}{t}\right)} e^{-tx}\left(1-c_ne^{-tx}\right)\,d\tilde{\nu}_n(r)&\leq\frac{1}{4c_n}\tilde{\nu}_n\left[\frac{\ln c_n}{t},\frac{\ln 2c_n}{t}\right)\\&=\frac{1}{4c_n}\left(\tilde{\nu}_n\left[0,\frac{\ln 2c_n}{t}\right)-\tilde{\nu}_n\left[0,\frac{\ln c_n}{t}\right)\right)\\&\overset{(1.10)}{\leq}\frac{R_n-1}{4R_nc_n}\tilde{\nu}_n\left[0,\frac{\ln 2c_n}{t}\right).\end{aligned}$$

If $c_n > R_n/2$, then we have

$$\begin{aligned}&\int_{\left[\frac{\ln 2c_n}{t},\infty\right)} e^{-xt}\left(1-c_ne^{-xt}\right)\,d\tilde{\nu}_n(r)\\&\quad\leq\sum_{k=0}^{\infty}\tilde{\nu}_n\left[2^k\frac{\ln 2c_n}{t},\,2^{k+1}\frac{\ln 2c_n}{t}\right)e^{-2^k\ln 2c_n}\left(1-c_ne^{-2^k\ln 2c_n}\right)\\&\quad\overset{(1.4)}{\leq}(R_n-1)\tilde{\nu}_n\left[0,\frac{\ln 2c_n}{t}\right)\sum_{k=0}^{\infty}\frac{R_n^k}{(2c_n)^{2^k}}\left(1-\frac{c_n}{(2c_n)^{2^k}}\right)\\&\quad\leq(R_n-1)\tilde{\nu}_n\left[0,\frac{\ln 2c_n}{t}\right)\frac{1}{2c_n}\sum_{k=0}^{\infty}\left(\frac{R_n}{2c_n}\right)^k\\&\quad=\frac{R_n-1}{2c_n-R_n}\tilde{\nu}_n\left[0,\frac{\ln 2c_n}{t}\right).\end{aligned}$$

Putting these inequalities together, we get

$$R_n\left(\frac{R_n-1}{2c_n-R_n}+\frac{R_n-1}{4R_nc_n}\right)\leq\frac{1}{2}-\frac{1}{\sqrt{2c_n}},$$

which holds for sufficiently large c_n, since the LHS approaches 0 and the RHS approaches $1/2$ as c_n goes *ad infinitum*. So if we choose such c_n, then the inequality in (1.8) is satisfied, and letting $C := \max_{0\leq n\leq m}\{c_n\}$ gives us the desired result. □

Note that Lemma 1.2.2 does not remain true if $m=\infty$. Take for example $\tilde{\nu}_0=\frac{\delta_0}{2\pi}$ and $\tilde{\nu}_n$ such that $d\tilde{\nu}_n(x)=\frac{2^{3n-1}(3n-1)!}{n!}x^{3n-1}\,dx$ for all $n\in\mathbb{N}$. Then $w_{(\infty)}=\sum_{n=0}^{\infty}t^{-n}/n!=e^{-t}$, and $w_{(\infty)}(t/2)\leq Cw_{(\infty)}(t)$ would mean that there exists $C\geq 2$ such that $t-\ln C\geq t/2$, which is clearly absurd.

And finally, by the space $A^2_{(m)}$ we shall denote the image of $L^2_{w_{(m)}}(0,\infty)$ under the Laplace transform, equipped with $A^2(\mathbb{C}_+, (\nu_n)_{n=0}^m)$ norm. An important example of a space of analytic functions on the complex half-plane, which is not a Zen space but is an $A^2_{(m)}$ space, is the Dirichlet space $\mathcal{D}(\mathbb{C}_+)$, which corresponds to the weight $1+t$ on the positive real half-line.

1.2.2 Reproducing kernels

Definition 1.2.3 *Let X be a set, let $\mathbb{K}$ be a field (real or complex) and let $\mathcal{H}$ be a Hilbert space whose elements are $\mathbb{K}$-valued functions on X. We say that $\mathcal{H}$ is a* reproducing kernel Hilbert space (RKHS) *if for every $x \in X$ the evaluation functional*

$$E_x : \mathcal{H} \longrightarrow \mathbb{K} \qquad f \longmapsto E_x(f) = f(x) \qquad (\forall f \in \mathcal{H})$$

is bounded. By the Riesz–Fréchet representation theorem (Theorem 6.8 in [105], § 6.1, p. 62) this is equivalent to the condition that for each $x \in X$ there exists a vector $k_x^{\mathcal{H}} \in \mathcal{H}$ (which we call the reproducing kernel at x*) such that*

$$f(x) = \left\langle f, k_x^{\mathcal{H}} \right\rangle_{\mathcal{H}} \qquad (\forall f \in \mathcal{H}).$$

The theory of reproducing kernels was originally laid out by Nachman Aronszajn in [9], and it is still a good reference on the subject. Another valuable source, which was recently published, is [84]. We can easily deduce from the definition that

$$\left\| k_x^{\mathcal{H}} \right\|_{\mathcal{H}}^2 \overset{\text{def}^{\text{n}}}{=} \left\langle k_x^{\mathcal{H}}, k_x^{\mathcal{H}} \right\rangle_{\mathcal{H}} \overset{\text{def}^{\text{n}}}{=} k_x^{\mathcal{H}}(x) \qquad (\forall x \in X), \tag{1.11}$$

$$k_x^{\mathcal{H}}(y) \overset{\text{def}^{\text{n}}}{=} \left\langle k_x^{\mathcal{H}}, k_y^{\mathcal{H}} \right\rangle_{\mathcal{H}} \overset{\text{def}^{\text{n}}}{=} \overline{\left\langle k_y^{\mathcal{H}}, k_x^{\mathcal{H}} \right\rangle_{\mathcal{H}}} \overset{\text{def}^{\text{n}}}{=} \overline{k_y^{\mathcal{H}}(x)} \qquad (\forall x, y \in X),$$

and if, for some x in X, $l_x^{\mathcal{H}}$ is also a reproducing kernel of $\mathcal{H}$, then

$$l_x^{\mathcal{H}}(y) \overset{\text{def}^{\text{n}}}{=} \left\langle l_x^{\mathcal{H}}, k_y^{\mathcal{H}} \right\rangle_{\mathcal{H}} \overset{\text{def}^{\text{n}}}{=} \overline{\left\langle k_y^{\mathcal{H}}, l_x^{\mathcal{H}} \right\rangle_{\mathcal{H}}} \overset{\text{def}^{\text{n}}}{=} \overline{k_y^{\mathcal{H}}(x)} = k_x^{\mathcal{H}}(y) \qquad (\forall y \in X).$$

The last property is sometimes called the *uniqueness property of reproducing kernels* and is often included in the definition, as it is also implied by the Riesz–Fréchet representation theorem.

The next theorem and its proof were sketched in [65] and we present its extended version below.

Theorem 1.2.4 *The $A^2_{(m)}$ space is a RKHS with reproducing kernels given by*

$$k_z^{A^2_{(m)}}(\zeta) := \int_0^\infty \frac{e^{-t(\overline{z}+\zeta)}}{w_{(m)}(t)}\,dt \qquad (\forall z,\, \zeta \in \mathbb{C}_+). \tag{1.12}$$

Proof

First, let us note that for each $z \in \mathbb{C}_+$, $k_z^{A^2_{(m)}}$ belongs to $A^2_{(m)}$. Indeed,

$$\begin{aligned} \left\| k_z^{A^2_{(m)}} \right\|^2_{A^2_{(m)}} &\overset{\text{Cor. 1.2.1}}{=} \int_0^\infty \frac{e^{-2t\operatorname{Re}(z)}}{w_{(m)}(t)}\,dt \\ &\overset{(1.6)}{\leq} \int_0^\infty \frac{e^{-2t\operatorname{Re}(z)}}{2\pi\tilde{\nu}_0\left(\left[0, \frac{1}{2}\operatorname{Re}(z)\right)\right)e^{-t\operatorname{Re}(z)}}\,dt \\ &= \frac{1}{2\pi\operatorname{Re}(z)\tilde{\nu}_0\left(\left[0, \frac{1}{2}\operatorname{Re}(z)\right)\right)}, \end{aligned} \tag{1.13}$$

which is finite, since, by the Δ_2-condition, $\tilde{\nu}_0\left(\left[0, \frac{1}{2}\operatorname{Re}(z)\right)\right) > 0$, for any $z \in \mathbb{C}_+$.

Next, we observe that, given F in $A^2_{(m)}$, there exists f in $L^1(0, \infty) \cap L^2_{w_{(m)}}(0, \infty)$ such that for any $z \in \mathbb{C}_+$ we have

$$\begin{aligned} F(z) = \mathfrak{L}[f](z) &\overset{\text{def}^{\text{n}}}{=} \int_0^\infty f(t)e^{-tz}\,dt = \int_0^\infty f(t)\overline{\frac{e^{-t\overline{z}}}{w_{(m)}(t)}}w_{(m)}(t)\,dt \\ &\overset{\text{def}^{\text{n}}}{=} \left\langle f,\, \frac{e^{-\cdot\overline{z}}}{w_{(m)}} \right\rangle_{L^2_{w_{(m)}}(0,\infty)} \\ &= \left\langle \mathfrak{L}[f],\, \mathfrak{L}\left[\frac{e^{-\cdot\overline{z}}}{w_{(m)}}\right] \right\rangle_{A^2_{(m)}} \\ &\overset{\text{def}^{\text{n}}}{=} \left\langle F,\, k_z^{A^2_{(m)}} \right\rangle_{A^2_{(m)}}. \end{aligned} \tag{1.14}$$

And by the Cauchy–Schwarz inequality

$$|F(z)| \overset{(1.14)}{=} \left\langle F, k_z^{A^2_{(m)}} \right\rangle_{A^2_{(m)}} \leq \|F\|_{A^2_{(m)}} \left\| k_z^{A^2_{(m)}} \right\|_{A^2_{(m)}} \overset{(1.13)}{<} \infty,$$

so the evaluation functional E_z is bounded on the dense subset $\mathfrak{L}(L^1(0, \infty) \cap L^2_{w_{(m)}}(0, \infty))$ of $A^2_{(m)}$, for all $z \in \mathbb{C}_+$. So it is also bounded on $A^2_{(m)}$. In particular, $A^2_{(m)}$ is a RKHS, and its kernels are of the form (1.12), which follows from (1.14) and the uniqueness property of reproducing kernels. □

Remark 1.2.5 *The equality*

$$\langle \mathfrak{L}[f], \mathfrak{L}[g] \rangle_{A^2_{(m)}} = \langle f, g \rangle_{L^2_{w_{(m)}}(0, \infty)} \qquad (\forall f, g \in L^2_{w_{(m)}}(0, \infty))$$

is a consequence of polarization identity*, i.e.*

$$4 \langle x, y \rangle_{\mathcal{I}} = \|x + y\|^2_{\mathcal{I}} - \|x - y\|^2_{\mathcal{I}} + i\|x + iy\|^2_{\mathcal{I}} - i\|x - iy\|^2_{\mathcal{I}},$$

which holds for all x, y in any inner product space $\mathcal{I}$ (Theorem 1.14 in [105], § 1.1, p. 9).

Remark 1.2.6 *By Proposition 4.1 from [87] (p. 61) we know that the evaluation functional is bounded on Zen spaces, and thus it must also be bounded on $A^p(\mathbb{C}_+, (\nu_n)_{n=0}^m)$ spaces. In particular, $A^2(\mathbb{C}_+, (\nu_n)_{n=0}^m)$ is a RKHS. If $P : A^2(\mathbb{C}_+, (\nu_n)_{n=0}^m) \longrightarrow A^2_{(m)}$ is the orthogonal projection, then*

$$k_z^{A^2_{(m)}} = P k_z^{A^2(\mathbb{C}_+, (\nu_n)_{n=0}^m)} \qquad (\forall z \in \mathbb{C}_+).$$

The following lemma appears in [67] for Zen spaces A^2_ν (assuming that the Laplace transform is a surjective map $\mathfrak{L} : L^2_{w_0}(0, \infty) \longrightarrow A^2_\nu$, where w_0 is as given in (1.3)), but it is easy to see that, by the virtue of Lemma 1.2.2, it also holds for $A^2_{(m)}$ spaces, provided that m is finite.

Lemma 1.2.7 (Lemma 4 in [65]) *Let $m \in \mathbb{N}_0$. The normalised reproducing kernels $k_z^{A^2_{(m)}} / \left\| k_z^{A^2_{(m)}} \right\|$ tend to 0 weakly as z approaches infinity.*

Proof

Let $\zeta \in \mathbb{C}_+$. First, we consider pointwise limits. By Lemma 1.2.2 we have

$$\begin{aligned}
\lim_{\mathrm{Re}(z)\to\infty} \left| k_z^{A^2_{(m)}}(\zeta) \right| / \left\| k_z^{A^2_{(m)}} \right\|_{A^2_{(m)}} &\overset{(1.12)}{\leq} \lim_{\mathrm{Re}(z)\to\infty} \int_0^\infty \frac{e^{-t(\mathrm{Re}(z)+\mathrm{Re}(\zeta))}}{w_{(m)}(t)}\, dt / \left\| k_z^{A^2_{(m)}} \right\|_{A^2_{(m)}} \\
&\leq \lim_{\mathrm{Re}(z)\to\infty} \int_0^\infty \frac{e^{-t\,\mathrm{Re}(z)}}{w_{(m)}(t)}\, dt / \left\| k_z^{A^2_{(m)}} \right\|_{A^2_{(m)}} \\
&= 2 \lim_{\mathrm{Re}(z)\to\infty} \int_0^\infty \frac{e^{-2t\,\mathrm{Re}(z)}}{w_{(m)}(2t)}\, dt / \left\| k_z^{A^2_{(m)}} \right\|_{A^2_{(m)}} \\
&\overset{(1.7)}{\lessapprox} \lim_{\mathrm{Re}(z)\to\infty} \int_0^\infty \frac{e^{-2t\,\mathrm{Re}(z)}}{w_{(m)}(t)}\, dt / \left\| k_z^{A^2_\nu} \right\|_{A^2_{(m)}} \\
&\overset{(1.11),(1.12)}{=} \lim_{\mathrm{Re}(z)\to\infty} \left\| k_z^{A^2_{(m)}} \right\|^2_{A^2_{(m)}} / \left\| k_z^{A^2_{(m)}} \right\|_{A^2_{(m)}} \\
&= \lim_{\mathrm{Re}(z)\to\infty} \left\| k_z^{A^2_{(m)}} \right\|_{A^2_{(m)}} \\
&\overset{(1.11),(1.12)}{=} \lim_{\mathrm{Re}(z)\to\infty} \left(\int_0^\infty \frac{e^{-2t\,\mathrm{Re}(z)}}{w_{(m)}(t)}\, dt \right)^{\frac{1}{2}} = 0.
\end{aligned} \tag{1.15}$$

Let $0 < a < \infty$. Then

$$\left\| k_z^{A^2_{(m)}} \right\|^2_{A^2_{(m)}} \overset{(1.11),(1.12)}{=} \int_0^\infty \frac{e^{-2t\,\mathrm{Re}(z)}}{w_{(m)}(t)}\, dt \geq \int_0^\infty \frac{e^{-2at}}{w_{(m)}(t)} \overset{(1.11),(1.12)}{=} \left\| k_a^{A^2_{(m)}} \right\|^2_{A^2_{(m)}}, \tag{1.16}$$

whenever $\mathrm{Re}(z) \leq a$. Also,

$$\left| k_z^{A^2_{(m)}}(\zeta) \right| \overset{(1.12)}{\leq} \int_0^\infty \left| \frac{e^{-t\bar{\zeta}}}{w_{(m)}(t)} \right| dt = \int_0^\infty \frac{e^{-2t\,\mathrm{Re}(\zeta)/2}}{w_{(m)}(t)}\, dt \overset{(1.11),(1.12)}{=} \left\| k_{\frac{\zeta}{2}}^{A^2_{(m)}} \right\|^2_{A^2_{(m)}} < \infty,$$

so $e^{-\cdot\bar{\zeta}}/w_{(m)}(\cdot)$ is in $L^1(0, \infty)$, for all $\zeta \in \mathbb{C}_+$. Therefore, by the Riemann–Lebesgue Lemma for the Laplace transform (Theorem 1, p. 3 in [13]), we get

$$\lim_{z\to\infty} k_z^{A^2_{(m)}}(\zeta) \overset{(1.12)}{=} \lim_{z\to\infty} \mathfrak{L}\left[\frac{e^{-\cdot\bar{\zeta}}}{w_{(m)}(\cdot)} \right](z) = 0 \quad \Longrightarrow \quad \lim_{\substack{\mathrm{Im}(z)\longrightarrow\infty \\ 0<\mathrm{Re}(z)<a}} k_z^{A^2_{(m)}}(\zeta) = 0. \tag{1.17}$$

And thus

$$\lim_{\substack{\mathrm{Im}(z)\longrightarrow\infty \\ 0<\mathrm{Re}(z)<a}} \left|k_z^{A^2_{(m)}}(\zeta)\right| / \left\|k_z^{A^2_{(m)}}\right\|_{A^2_{(m)}} \overset{(1.16)}{\leq} \left\|k_a^{A^2_{(m)}}\right\|^{-1}_{A^2_{(m)}} \lim_{\substack{\mathrm{Im}(z)\longrightarrow\infty \\ 0<\mathrm{Re}(z)<a}} \left|k_z^{A^2_{(m)}}(\zeta)\right|_{A^2_{(m)}} \overset{(1.17)}{=} 0. \tag{1.18}$$

Now suppose, for contradiction, that the normalised reproducing kernels of $A^2_{(m)}$ do not converge to 0 pointwise. Then there exists $\delta > 0$ and a sequence of complex numbers $(z_n)_{n=0}^{\infty}$ such that $\mathrm{Re}(z) > 0$, $\lim_{n\to\infty} z_n = \infty$ and

$$\left|k_{z_n}^{A^2_{(m)}}(\zeta)\right| / \left\|k_{z_n}^{A^2_{(m)}}\right\|_{A^2_{(m)}} \geq \delta > 0 \qquad (\forall n \in \mathbb{N}_0).$$

But since $\lim_{n\to\infty} z_n = \infty$, there exists a subsequence $(z_{n_k})_{k=0}^{\infty}$ of $(z_n)_{n=0}^{\infty}$ such that $\lim_{k\to\infty} \mathrm{Re}(z_{n_k}) = \infty$ or $\mathrm{Re}(z_{n_k}) \leq a$, for some $a > 0$, and

$$\left|k_{z_{n_k}}^{A^2_{(m)}}(\zeta)\right| / \left\|k_{z_{n_k}}^{A^2_{(m)}}\right\|_{A^2_{(m)}} \geq \delta > 0 \qquad (\forall k \in \mathbb{N}_0),$$

which contradicts either (1.15) or (1.18). So we must have

$$\lim_{z\to\infty} \frac{k_z^{A^2_{(m)}}(\zeta)}{\left\|k_z^{A^2_{(m)}}\right\|_{A^2_{(m)}}} = 0 \qquad (\forall \zeta \in \mathbb{C}_+). \tag{1.19}$$

Now, let $F = \sum_{j=0}^{n} k_{\zeta_j}^{A^2_{(m)}} \in A^2_{(m)}$, for some $\{\zeta_j\}_{j=0}^n \subset \mathbb{C}_+$, $n \in \mathbb{N}_0$. Then

$$\lim_{z\to\infty} \left\langle k_z^{A^2_{(m)}} / \left\|k_z^{A^2_{(m)}}\right\|_{A^2_{(m)}}, F \right\rangle_{A^2_{(m)}} = \lim_{z\to\infty} \sum_{j=0}^{n} \frac{k_z^{A^2_{(m)}}(\zeta_j)}{\left\|k_z^{A^2_{(m)}}\right\|_{A^2_{(m)}}} = 0,$$

and the result follows, since the linear span of reproducing kernels is dense in $A^2_{(m)}$ (see Lemma 2.2., § 2.1, p. 17 in [84]). □

Example 1.2.8 *The reproducing kernels of the Hardy space $H^2(\mathbb{C}_+)$ are given by*

$$k_z^{H^2(\mathbb{C}_+)}(\zeta) \overset{(1.12)}{=} \frac{1}{\overline{z} + \zeta} \qquad (\forall z, \zeta \in \mathbb{C}_+),$$

while the reproducing kernels of the weighted Bergman spaces $\mathcal{B}^2_\alpha(\mathbb{C}_+)$, $\alpha > -1$, are of the form

$$k_z^{\mathcal{B}^2_\alpha(\mathbb{C}_+)}(\zeta) \overset{(1.12)}{=} \frac{2^\alpha(1+\alpha)}{(\overline{z}+\zeta)^{2+\alpha}} \qquad (\forall z,\, \zeta \in \mathbb{C}_+).$$

Definition 1.2.9 *For $\alpha \geq -1$, the normalised kernels*

$$K_\alpha(\zeta,\, z) := \frac{1}{(\overline{z}+\zeta)^{2+\alpha}} \qquad (\forall z, \zeta \in \mathbb{C}_+)$$

are sometimes called the Bergman kernels *for the open right complex half-plane.*

Lemma 1.2.10 *If $m \in \mathbb{N}_0$, then there exists $\alpha' \geq -1$ such that for all $z \in \mathbb{C}_+$ and $\alpha \geq \alpha'$, $K_\alpha(\cdot, z)$ is in $A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^\infty)$.*

Proof

For each $0 \leq n \leq m$, let R_n be the (Δ_2)-condition supremum for $\tilde{\nu_n}$. Choose $q > 0$ such that

$$2^q > \sup_{0 \leq n \leq m} R_n.$$

Define $g : [0, \infty) \longrightarrow (0, \infty)$ to be a step function given by

$$g(r) := \begin{cases} \mathrm{Re}(z)^{-q}, & \text{if } 0 \leq r < 1, \\ (2^k + \mathrm{Re}(z))^{-q}, & \text{if } r \in [2^k,\, 2^{k+1}),\ \forall k \in \mathbb{N}_0. \end{cases}$$

Then

$$\begin{aligned} \int_{[0,\,\infty)} g(r)\, d\tilde{\nu}_n(r) &\overset{\text{def}^{\underline{n}}}{=} \frac{\tilde{\nu}_n\left([0,\,1)\right)}{\mathrm{Re}(z)^q} + \sum_{k=0}^\infty \frac{\tilde{\nu}_n\left([2^k,\, 2^{k+1})\right)}{(2^k + \mathrm{Re}(z))^q} \\ &\overset{(1.4)}{\leq} \frac{\tilde{\nu}_n([0,\,1))}{\mathrm{Re}(z)^q} + (R_n - 1)\sum_{k=0}^\infty \frac{\tilde{\nu}_n([0,\, 2^k))}{(2^k + \mathrm{Re}(z))^q} \\ &\overset{(\Delta_2)}{\leq} \tilde{\nu}_n([0,\,1))\left(\frac{1}{\mathrm{Re}(z)^q} + (R_n - 1)\sum_{k=0}^\infty \left(\frac{R_n}{2^q}\right)^k\right) \\ &= \tilde{\nu}_n([0,\,1))\left(\frac{1}{\mathrm{Re}(z)^q} + \frac{2^q(R_n - 1)}{2^q - R_n}\right) \end{aligned} \tag{1.20}$$

for all $0 \leq n \leq m$. Let w_n be as given in (1.6), and let $\alpha \geq \alpha' := (q-3)/2$. It follows that

$$\begin{aligned}\int_0^\infty \left|t^{\alpha+1} e^{-t\overline{z}}\right|^2 w_n(t)\,dt &= 2\pi \int_{[0,\,\infty)} \int_0^\infty t^{2(\alpha+n+1)} e^{-2t(x+\mathrm{Re}(z))}\,dt\,d\tilde{\nu}_n(x) \\ &= \frac{\pi\Gamma(2\alpha+2n+3)}{2^{2\alpha+2n+2}} \int_{[0,\,\infty)} \frac{d\tilde{\nu}_n(x)}{(x+\mathrm{Re}(z))^{2\alpha+2n+3}} \\ &\leq \frac{\pi\Gamma(2\alpha+2n+3)}{2^{2\alpha+2n+2}} \int_{[0,\,\infty)} g(r)\,d\tilde{\nu}_n(r),\end{aligned}$$

which is finite for all $0 \leq n \leq m$, by (1.20). Here

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\,dt \qquad\qquad (\forall z \in \mathbb{C} \setminus (-\infty,\, 0])$$

is the gamma function (see Chapter 6 in [1]).

Consequently, by Theorem 1.1.3, we have

$$\mathfrak{L}\left[\frac{t^{\alpha+1}e^{-t\overline{z}}}{\Gamma(\alpha+2)}\right](\zeta) = \frac{1}{(\overline{z}+\zeta)^{\alpha+2}} = K_\alpha(\zeta,\, z) \in A^2_{(m)} \subseteq A^2(\mathbb{C}_+, (\nu_n)_{n=0}^m).$$

The general result for $1 \leq p < \infty$ follows, because

$$\frac{1}{(\overline{z}+\zeta)^\beta} \in A^2_\nu \qquad\qquad \Longleftrightarrow \qquad\qquad \frac{1}{(\overline{z}+\zeta)^{\beta p/2}} \in A^p_\nu.$$

$\square$

To see that Lemma 1.2.10 does not hold for $m = \infty$, let $\tilde{\nu}_n = \frac{\delta_0}{2\pi n!}$, for all $n \in \mathbb{N}_0$. Then $w_{(\infty)}(t) = e^{t^2}$ and evidently $t^{\alpha+1}e^{-t\overline{z}} \notin L^2_{w_{(\infty)}}(0,\, \infty)$, for any choice of $\alpha \geq -1$ and $z \in \mathbb{C}_+$.

1.3 Comparison of $\mathbb{D}$ and $\mathbb{C}_+$

1.3.1 The weighted Hardy and Bergman spaces

It is easy to verify that if $f(s) = \sum_{n=0}^{\infty} a_n s^n$ and $g(s) = \sum_{n=0}^{\infty} b_n s^n$ are analytic functions lying in H^2 (the (Hilbert) Hardy space on the disk), then

$$\langle f,\, g\rangle_{H^2} \stackrel{\text{def}^{\text{n}}}{=} \sup_{0<r<1} \int_0^{2\pi} f(re^{i\theta})\overline{g(re^{i\theta})}\, \frac{d\theta}{2\pi} = \sum_{n=0}^{\infty} a_n \overline{b_n} \stackrel{\text{def}^{\text{n}}}{=} \langle \boldsymbol{a},\, \boldsymbol{b}\rangle_{\ell^2}$$

and

$$\|f\|_{H^2} \stackrel{\text{def}^{\text{n}}}{=} \left(\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2\, \frac{d\theta}{2\pi} \right)^{\frac{1}{2}} = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{\frac{1}{2}} \stackrel{\text{def}^{\text{n}}}{=} \|\boldsymbol{a}\|_{\ell^2},$$

where $\boldsymbol{a} = (a_n)_{n=0}^{\infty}$ and $\boldsymbol{b} = (b_n)_{n=0}^{\infty}$. More generally, if these functions are both in $\mathcal{B}_\alpha^2$ (the weighted (Hilbert) Bergman space on the disk), for some $\alpha > -1$, then

$$\langle f,\, g\rangle_{\mathcal{B}_\alpha^2} \stackrel{\text{def}^{\text{n}}}{=} \int_{\mathbb{D}} f(s)\overline{g(s)}(1-|s|^2)^{\alpha}\, \frac{ds}{\pi} = \sum_{n=0}^{\infty} a_n \overline{b_n} B(n+1,\, 1+\alpha) \stackrel{\text{def}^{\text{n}}}{=} \langle \boldsymbol{a},\, \boldsymbol{b}\rangle_{\ell^2_{w_\alpha}}$$

and

$$\|f\|_{\mathcal{B}_\alpha^2} \stackrel{\text{def}^{\text{n}}}{=} \left(\int_{\mathbb{D}} |f(s)|^2 (1-|s|^2)^{\alpha}\, \frac{ds}{\pi} \right)^{\frac{1}{2}} = \left(\sum_{n=0}^{\infty} |a_n|^2 B(n+1,\, 1+\alpha) \right)^{\frac{1}{2}} \stackrel{\text{def}^{\text{n}}}{=} \|\boldsymbol{a}\|_{\ell^2_{w_\alpha}},$$

where

$$\ell^2_{w_\alpha} := \left\{ \boldsymbol{x} = (x_n)_{n=0}^{\infty} \;:\; \|\boldsymbol{x}\|_{\ell^2_{w_\alpha}} := \left(\sum_{n=0}^{\infty} |x_n|^2 B(n+1,\, 1+\alpha) \right)^{\frac{1}{2}} < \infty \right\}$$

and $B(\cdot,\, \cdot)$ is the beta function, that is

$$B(z,\, w) := \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \qquad (\forall z,\, w \in \mathbb{C} \setminus (-\infty,\, 0]).$$

(see § 6.2, p. 258 in [1]).

The Hardy spaces are discussed for example in [32], [56] and [75]; the weighted Bergman spaces in [34] and [53].

Therefore, in Hilbertian setting, we can associate the weighted Bergman spaces on the disk (including the Hardy space, which, again, we identify with $\mathcal{B}^2_{-1}$) with weighted sequence spaces ℓ^2. Similarly, using the Laplace transform and its isometric properties given in Theorem 1.1.3, we can establish a connection between $\mathcal{B}^2_\alpha(\mathbb{C}_+)$, $\alpha \geq -1$, and weighted L^2 spaces on $(0, \infty)$. Namely, let $f,\, g \in L^2_{t^{-1-\alpha}}(0, \infty)$, for some $\alpha \geq -1$, and let $F = \mathfrak{L}[f]$, $G = \mathfrak{L}[g]$. Then

$$\langle F,\, G\rangle_{\mathcal{B}^2_\alpha(\mathbb{C}_+)} \overset{\text{def}^{\text{n}}}{=} \int_{\mathbb{C}_+} F(z)\overline{G(z)}(\mathrm{Re}(z))^\alpha \,\frac{dz}{\pi} \overset{\text{Th}^{\text{m}}\, 1.1.3}{=} \int_0^\infty f(t)\overline{g(t)}\,\frac{\Gamma(1+\alpha)}{2^\alpha t^{1+\alpha}}\,dt,$$

if $\alpha > -1$, and

$$\langle F,\, G\rangle_{H^2(\mathbb{C}_+)} \overset{\text{def}^{\text{n}}}{=} \sup_{x>0} \int_{-\infty}^\infty F(x+iy)\overline{G(x+iy)}\,\frac{dy}{2\pi} \overset{\text{Th}^{\text{m}}\, 1.1.3}{=} \int_0^\infty f(t)\overline{g(t)}\,dt$$

otherwise. We can also observe an analogy between their respective reproducing kernels (cf. Example 1.2.8), as

$$k_s^{H^2}(\varsigma) = \frac{1}{1-\overline{s}\varsigma} = \sum_{n=0}^\infty (\overline{s}\varsigma)^n \qquad (\forall s,\, \varsigma \in \mathbb{D})$$

and

$$k_s^{\mathcal{B}^2_\alpha}(\varsigma) = \frac{1+\alpha}{(1-\overline{s}\varsigma)^{2+\alpha}} = \sum_{n=0}^\infty \frac{(\overline{s}\varsigma)^n}{B(n+1,\, 1+\alpha)} \qquad (\forall s,\, \varsigma \in \mathbb{D},\, \alpha > -1).$$

(see Proposition 1.4 in [53], § 1.1, p. 5). So we can view the weighted Bergman spaces defined on the unit disk/complex half-plane as discrete/continuous counterparts. We shall go one step further.

Definition 1.3.1 *Let $\mathcal{H}$ be a Hilbert space whose vectors are functions analytic on the unit disk of the complex plane. If the monomials $1,\, s,\, s^2, \ldots$ form a complete orthogonal set of non-zero vectors in $\mathcal{H}$, then $\mathcal{H}$ is called a* weighted Hardy space.

Remark 1.3.2 *These spaces are discussed in [27], p. 14. The condition that $\{1,\, s,\, s^2,\, \ldots\}$ is a complete orthogonal set of non-zero vectors in $\mathcal{H}$ is equivalent to* *the density of polynomials in $\mathcal{H}$. It is often assumed that $\|1\|_{\mathcal{H}} = 1$. In this case we write $\mathcal{H} = H^2(\boldsymbol{\beta})$, where $\boldsymbol{\beta} := (\beta_n)_{n=0}^{\infty}$ and $\beta_n := \|z^n\|_{\mathcal{H}}$. Orthogonality implies that*

$$\|f\|^2_{H^2(\boldsymbol{\beta})} = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 \qquad \left(\forall f = \sum_{n=0}^{\infty} a_n s^n \in H^2(\boldsymbol{\beta})\right)$$

and

$$\langle f, g \rangle^2_{H^2(\boldsymbol{\beta})} = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta_n^2,$$

for all $f(s) = \sum_{n=0}^{\infty} a_n s^n$, $g(s) = \sum_{n=0}^{\infty} b_n s^n \in \mathcal{H}$. Conversely, given a positive sequence $\boldsymbol{\beta} = (\beta_n)_{n=0}^{\infty}$ with $\beta_0 = 1$ and $\liminf_{n\to\infty} (\beta_n)^{1/n} \geq 1$, we can construct a corresponding weighted Hardy space.

So, given that certain conditions are satisfied, a weighted Hardy space has a parallel weighted sequence space, revealing its structure, and vice versa.

Moving from the discrete unit disk case to the continuous half-plane setting, we replace the variable s, $|s| < 1$, by e^{-z}, $|e^{-z}| < 1$; the power series with a discrete index variable $n \in \mathbb{N}_0$ by an integral over the positive real half-line with respect to a continuous variable $t > 0$; and a sequence $\boldsymbol{a}$ lying in some weighted sequence space ℓ^2 by a function f lying in some weighted L^2 space on $(0, \infty)$. In this way we get

$$F(z) = \int_0^{\infty} f(t) e^{-tz} \, dt \qquad (\mathrm{Re}(z) > 0),$$

that is the Laplace transform of f. It conspicuous that Zen spaces enjoy the same relation with weighted Lebesgue spaces on the positive real half-line (by Theorem 1.1.3) as the weighted Hardy spaces do with weighted sequence spaces ℓ^2. Yet, it would be false to claim that weighted Hardy spaces and Zen spaces are discrete/continuous counterparts, and even call the latter: the weighted Hardy spaces on the half-plane. Notice that the weight w_0 (defined in (1.3)) corresponding to a Zen space A^2_ν is always non-increasing, while in case of weighted Hardy spaces, we do allow the sequence weights to be increasing. An important example of a weighted Hardy space with an increasing weight is the Dirichlet space.

1.3.2 The Dirichlet spaces

The *(classical) Dirichlet space* $\mathcal{D}$ is defined to be the vector space of functions analytic on the unit disk of the complex plane with derivatives lying in the (unweighted) Bergman space $\mathcal{B}^2 := \mathcal{B}_0^2$. Or in other words

$$\mathcal{D} := \left\{ f : \mathbb{D} \longrightarrow \mathbb{C} \text{ analytic } : \mathcal{D}(f) := \|f'\|^2_{\mathcal{B}^2} \overset{\text{def}^{\text{n}}}{=} \int_{\mathbb{D}} |f'(s)|^2 \frac{ds}{\pi} < \infty \right\}.$$

The Dirichlet space is discussed for example in [7] and [35]. It is easy to show that if $f = \sum_{n=0}^{\infty} a_n s^n$, then $\mathcal{D}(f) = \sum_{n=1}^{\infty} n|a_n|^2$, and hence $\mathcal{D}$ is contained in H^2 (Theorem 1.1.2 and Corollary 1.1.3 in [35], § 1.1, pp. 1-2). We can define a semi-inner product on $\mathcal{D}$ by

$$\mathcal{D}(f,\, g) := \int_{\mathbb{D}} f'(s)\overline{g'(s)}\, \frac{ds}{\pi} \qquad (\forall f, g \in \mathcal{D}).$$

Clearly $\mathcal{D}(f,\, f) = \mathcal{D}(f)$, and $\mathcal{D}(\cdot)^{1/2}$ is a seminorm on $\mathcal{D}$. It is not a norm, since $\mathcal{D}(f) = 0$, whenever f is a constant function. The inner product and the norm on $\mathcal{D}$ are usually defined by

$$\langle f,\, g\rangle_{\mathcal{D}} := \langle f,\, g\rangle_{H^2} + \mathcal{D}(f,\, g) \qquad (\forall f, g \in \mathcal{D})$$

and

$$\|f\|_{\mathcal{D}} = \left(\|f\|^2_{H^2} + \mathcal{D}(f)\right)^{\frac{1}{2}} = \left(\sum_{n=0}^{\infty} (n+1)|a_n|^2\right)^{\frac{1}{2}} \qquad \left(\forall f(s) = \sum_{n=0}^{\infty} a_n s^n \in \mathcal{D}\right).$$

Therefore $\mathcal{D}$ is a RKHS with the kernel

$$k_s^{\mathcal{D}}(\varsigma) = \frac{1}{\varsigma\overline{s}} \log\left(\frac{1}{1-\varsigma\overline{s}}\right) = \sum_{n=0}^{\infty} \frac{(\varsigma\overline{s})^n}{n+1} \qquad (\forall (s,\, \varsigma) \in \mathbb{D}^2)$$

(2.3, p. 51 in [7]), and also a weighted Hardy space $H^2(\boldsymbol{\beta})$, where $\boldsymbol{\beta} = ((n+1)^{1/2})_{n=0}^{\infty}$.

Alternatively, we may define a norm on $\mathcal{D}$ by

$$|||f|||_{\mathcal{D}} := \left(|f(0)|^2 + \mathcal{D}(f)\right)^{\frac{1}{2}} = \left(|a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2\right)^{\frac{1}{2}} \qquad \left(\forall f(s) = \sum_{n=0}^{\infty} a_n s^n \in \mathcal{D}\right),$$

and it is equivalent to $\|\cdot\|_{\mathcal{D}}$. In a similar way (for $\alpha \geq -1$) we define the *weighted Dirichlet spaces*

$$\mathcal{D}_\alpha := \left\{ f : \mathbb{D} \longrightarrow \mathbb{C} \text{ analytic } \; : \; \mathcal{D}_\alpha(f) := \|f'\|^2_{\mathcal{B}^2_\alpha} < \infty \right\},$$

and

$$\|f\|_{\mathcal{D}_\alpha} := \left(|f(0)|^2 + \mathcal{D}_\alpha(f)\right)^{\frac{1}{2}} = \left(|a_0|^2 + \sum_{n=1}^{\infty} n^2 |a_n|^2 B(n,\, 1+\alpha)\right)^{\frac{1}{2}},$$

for all $f(s) = \sum_{n=0}^{\infty} a_n s^n \in \mathcal{D}_\alpha$. It is easy to see that $\mathcal{B}^2_\alpha = \mathcal{D}_{2+\alpha}$ (up to equivalent norms).

Our aim now is to define the Dirichlet spaces on the complex half-plane in such a way that they mirror the properties of the classical Dirichlet spaces listed above. We start by considering the *Dirichlet integral* on $\mathbb{C}_+$:

$$\int_{\mathbb{C}_+} |F'(z)|^2 \, \frac{dz}{\pi},$$

for some function F analytic on $\mathbb{C}_+$. It is a seminorm on the vector space of functions with derivatives lying in the (unweighted) Bergman space $\mathcal{B}^2(\mathbb{C}_+) := \mathcal{B}^2_0(\mathbb{C}_+)$, again, because it equals zero for all constant functions. The problem is that the constant functions do not belong to $H^2(\mathbb{C}_+)$, so in order to define the Dirichlet space with the usual choice of norm we need an extra condition, that is

$$\mathcal{D}(\mathbb{C}_+) := \left\{ F \in H^2(\mathbb{C}_+) \; : \; F' \in \mathcal{B}^2(\mathbb{C}_+) \right\}$$

with norm given by

$$\|F\|_{\mathcal{D}(\mathbb{C}_+)} := \left(\|F\|^2_{H^2(\mathbb{C}_+)} + \|F'\|^2_{\mathcal{B}^2(\mathbb{C}_+)} \right)^{\frac{1}{2}}$$

Clearly, $\mathcal{D}(\mathbb{C}_+) = A^2(\mathbb{C}_+, (\nu_0,\, \nu_1))$, where $\tilde{\nu}_0 = \frac{1}{2\pi}\delta_0$ and $d\tilde{\nu}_1(r) = \frac{1}{\pi} r \, dr$. This is a RKHS, with kernels given by

$$k_z^{\mathcal{D}(\mathbb{C}_+)}(\zeta) \overset{(1.12)}{=} \int_0^\infty \frac{e^{-t(\overline{z}+\zeta)}}{1+t} \, dt = e^{\overline{z}+\zeta} \Gamma\left(0, \overline{z}+\zeta\right) \qquad (\forall z,\, \zeta \in \mathbb{C}_+),$$

where $\Gamma(\cdot,\,\cdot)$ is the upper incomplete gamma function (§ 6.5, p. 260 in [1]).

Alternatively, we can also define

$$\mathcal{D}'(\mathbb{C}_+) := \left\{F : \mathbb{C}_+ \longrightarrow \mathbb{C} \text{ analytic } : \; F' \in \mathcal{B}^2(\mathbb{C}_+)\right\}$$

with

$$\|F\|_{\mathcal{D}'(\mathbb{C}_+)} := \left(|F(1)|^2 + \|F'\|^2_{\mathcal{B}^2(\mathbb{C}_+)}\right)^{\frac{1}{2}} \qquad (\forall F \in \mathcal{D}'(\mathbb{C}_+)).$$

And more generally, for $\alpha \geq -1$,

$$\mathcal{D}'_\alpha(\mathbb{C}_+) := \left\{F : \mathbb{C}_+ \longrightarrow \mathbb{C} \text{ analytic } : \; F' \in \mathcal{B}^2_\alpha(\mathbb{C}_+)\right\}$$

with

$$\|F\|_{\mathcal{D}'(\mathbb{C}_+)} := \left(|F(1)|^2 + \|F'\|^2_{\mathcal{B}^2(\mathbb{C}_+)}\right)^{\frac{1}{2}} \qquad (\forall F \in \mathcal{D}'(\mathbb{C}_+)).$$

The following theorem appeared in [64] for $\alpha = 0$ (*salvo errore et omissione* of including the factor $1/\pi$ in the kernel expression instead of the Bergman norm). Below we extend it to $\alpha \geq -1$ and furnish it with a complete (and corrected) proof.

Theorem 1.3.3 *The space $\mathcal{D}'_\alpha(\mathbb{C}_+)$, $\alpha \geq -1$, is a RKHS and its kernels are given by*

$$k_z^{\mathcal{D}'_\alpha(\mathbb{C}_+)}(\zeta) = \begin{cases} (\overline{z}+\zeta)\ln(\overline{z}+\zeta) - (\overline{z}+1)\ln(\overline{z}+1) - (1+\zeta)\ln(1+\zeta) + \ln 4 + 1, & \text{if } \alpha = -1, \\ \ln(1+\overline{z}) - \ln(\overline{z}+\zeta) + \ln(1+\zeta) - \ln 2 + 1, & \text{if } \alpha = 0, \\ \frac{2^\alpha}{\alpha}\left(\frac{1}{(\overline{z}+\zeta)^\alpha} - \frac{1}{(\overline{z}+1)^\alpha} - \frac{1}{(1+\zeta)^\alpha} + \frac{\alpha-1}{2^\alpha}\right) & \text{otherwise,} \end{cases} \tag{1.21}$$

for all $z,\,\zeta \in \mathbb{C}_+$. Here by $\ln(z)$ we mean $\int_1^z \frac{d\xi}{\xi}$, for any path of integration between 1 and z within $\mathbb{C}_+$ (see 4.1.1, p. 61 in [1]).

Proof

Firstly, we prove that that $\mathcal{D}'_\alpha(\mathbb{C}_+)$ is complete. If $(F_n)_{n=0}^\infty$ is a Cauchy sequence in $\mathcal{D}'_\alpha(\mathbb{C}_+)$, then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}_0$ such that

$$\|F_n - F_m\|_{\mathcal{D}'_\alpha(\mathbb{C}_+)} \overset{\text{defn}}{=} \left(|F_n(1) - F_m(1)| + \|F_n - F_m\|^2_{\mathcal{B}^2_\alpha(\mathbb{C}_+)}\right)^{\frac{1}{2}} < \varepsilon \qquad (\forall m,\, n \geq N).$$

Thus $(F_n(1))_{n=0}^{\infty}$ and $(F_n)_{n=0}^{\infty}$ are Cauchy sequences, in $\mathbb{C}$ and $\mathcal{B}^2_\alpha(\mathbb{C}_+)$ respectively. Since both these spaces are complete with respect to their norms, we can define

$$F(z) = \int_1^z \lim_{n\to\infty} F_n'(\zeta)\, d\zeta + \lim_{n\to\infty} F_n(1) \qquad\qquad (\forall z \in \mathbb{C}_+),$$

which clearly belongs to $\mathcal{D}'_\alpha(\mathbb{C}_+)$ and is the limit of $(F_n)_{n=0}^{\infty}$. So $\mathcal{D}'_\alpha(\mathbb{C}_+)$ is a Hilbert space.

Secondly, we show that functions given in (1.21) lie in $\mathcal{D}'_\alpha(\mathbb{C}_+)$, for all $\alpha \geq 1,\ z,\ \zeta \in \mathbb{C}_+$. Note that, by L'Hôpital's Rule, we have

$$\begin{aligned}
&\lim_{t\to 0^+} \frac{|e^{-t} - e^{-t\overline{z}}|^2}{t^2} \overset{\text{def}^{\text{n}}}{=} \lim_{t\to 0^+} \frac{e^{-2t} - 2e^{-t(1+\operatorname{Re} z)}\cos(t \operatorname{Im} z) + e^{-2t\operatorname{Re} z}}{t^2} \\
&= \lim_{t\to 0^+} \frac{e^{-2t} - e^{-t(1+\operatorname{Re}(z))}\left[(1+\operatorname{Re}(z))\cos(t\operatorname{Im}(z)) + \operatorname{Im}(z)\sin(t\operatorname{Im}(z))\right] + \operatorname{Re}(z)e^{-2t\operatorname{Re}(z)}}{-t} \\
&= \lim_{t\to 0^+}\left(2e^{-2t} - e^{-t(1+\operatorname{Re}(z))}\cos(t\operatorname{Im}(z))\left[(1+\operatorname{Re}(z))^2 - \operatorname{Im}(z)^2\right] + 2\operatorname{Re}(z)^2 e^{-2t\operatorname{Re}(z)}\right) \\
&= 2 - (1+\operatorname{Re}(z))^2 + \operatorname{Im}(z)^2 + 2\operatorname{Re}(z)^2 \\
&= (\operatorname{Re}(z)-1)^2 + \operatorname{Im}(z)^2,
\end{aligned}$$

for all $z \in \mathbb{C}_+$, so

$$t^{\alpha-1}|e^{-t} - e^{-t\overline{z}}|^2 = O(1) \qquad\qquad \text{as} \qquad\qquad t \longrightarrow 0^+,$$

and hence $t^\alpha(e^{-t} - e^{-t\operatorname{Re}(z)}) \in L^2_{t^{-\alpha-1}}(0,\infty)$, for all $\alpha \geq -1$ and $z \in \mathbb{C}_+$. It follows from Theorem 1.1.3 that

$$\begin{aligned}
\frac{d}{d\zeta}\left(k_z^{\mathcal{D}'_{-1}(\mathbb{C}_+)}(\zeta)\right) &= \ln(\overline{z}+\zeta) - \ln(1+\zeta) \\
&\overset{\text{def}^{\text{n}}}{=} \int_1^{\overline{z}} \frac{d\xi}{\xi+\zeta} \\
&= \int_1^{\overline{z}}\int_0^\infty e^{-t(\xi+\zeta)}\, dt\, d\xi \\
&= \int_0^\infty e^{-t\zeta}\int_1^{\overline{z}} e^{-t\xi}\, d\xi,\ dt \\
&= \int_0^\infty \frac{e^{-t} - e^{-t\overline{z}}}{t} e^{-t\zeta}\, dt \\
&\overset{\text{def}^{\text{n}}}{=} \mathfrak{L}\left[\frac{e^{-\cdot} - e^{-\cdot\overline{z}}}{\cdot}\right](\zeta) \in \mathcal{B}^2_{-1}(\mathbb{C}_+),
\end{aligned}$$

and if $\alpha > -1$, then

$$\begin{aligned}\frac{d}{d\zeta}\left(k_z^{\mathcal{D}'_\alpha(\mathbb{C}_+)}(\zeta)\right) &= 2^\alpha\left(\frac{1}{(1+\zeta)^{1+\alpha}} - \frac{1}{(\overline{z}+\zeta)^{1+\alpha}}\right)\\ &= \frac{2^\alpha}{\Gamma(1+\alpha)}\mathfrak{L}\left[(\cdot)^\alpha(e^{-\cdot} - e^{-\cdot\overline{z}})\right](\zeta) \in \mathcal{B}^2_\alpha(\mathbb{C}_+).\end{aligned}$$

Therefore $k_z^{\mathcal{D}'_\alpha(\mathbb{C}_+)}$ belongs to $\mathcal{D}'_\alpha(\mathbb{C}_+)$, for all $z \in \mathbb{C}_+$.

And finally, if $F \in \mathcal{D}'_\alpha(\mathbb{C}_+)$, then, by the Fundamental Theorem of Complex Calculus (Theorem 3.13, § 3.3, p. 95 in [18])

$$\begin{aligned}\left\langle F,\, k_z^{\mathcal{D}'_\alpha(\mathbb{C}_+)}\right\rangle_{\mathcal{D}'_\alpha(\mathbb{C}_+)} &\overset{\text{def}^{\text{n}}}{=} F(1) + \left\langle F',\, \left(k_z^{\mathcal{D}'_\alpha(\mathbb{C}_+)}\right)'\right\rangle_{\mathcal{B}^2_\alpha(\mathbb{C}_+)}\\ &= F(1) + \left\langle F',\, \int_1^{\overline{z}} k_\xi^{\mathcal{B}^2_\alpha(\mathbb{C}_+)}\, d\xi\right\rangle_{\mathcal{B}^2_\alpha(\mathbb{C}_+)}\\ &= F(1) + \int_1^z \left\langle F',\, k_\xi^{\mathcal{B}^2_\alpha(\mathbb{C}_+)}\right\rangle_{\mathcal{B}^2_\alpha(\mathbb{C}_+)} d\xi\\ &= F(1) + \int_1^z F'(\xi)\, d\xi\\ &= F(z),\end{aligned}$$

as required. □

Because $\mathcal{D}'(\mathbb{C}_+)$ contains constant functions, it cannot be represented as either $A^2(\mathbb{C}_+,\, (\nu_n)_{n=0}^m)$ or $A^2_{(m)}$, for any choice of measures $(\tilde{\nu}_n)_{n=0}^\infty$, and thus we shall adopt a convention that by the (unweighted) Dirichlet space we mean $\mathcal{D}(\mathbb{C}_+)$, as it suits better our discrete/continuous-disk/half-plane framework. We also define the *weighted Dirichlet* by

$$\mathcal{D}_\alpha(\mathbb{C}_+) := \left\{F \in H^2(\mathbb{C}_+) \;:\; F' \in \mathcal{B}^2_\alpha(\mathbb{C}_+)\right\},$$

with

$$\|F\|_{\mathcal{D}_\alpha(\mathbb{C}_+)} := \left(\|F\|^2_{H^2(\mathbb{C}_+)} + \|F'\|^2_{\mathcal{B}^2_\alpha(\mathbb{C}_+)}\right)^{\frac{1}{2}},$$

The motivation for this definition is as follows. Notice that, by Stirling's formula (6.1.37, p. 257 in [1]), $B(n,\, 1+\alpha) \approx \Gamma(1+\alpha)n^{-1-\alpha}$, so $\mathcal{D}_\alpha = H^2(\boldsymbol{\beta})$, where

$\beta = \sqrt{1+n^{1-\alpha}}$ (in fact, this is sometimes the definition of the weighted Dirichlet spaces, see for example [63] or [98]). So if we want $\mathcal{D}_\alpha(\mathbb{C}_+)$ to be a continuous version of $\mathcal{D}_\alpha$, we let $w_{(1)} = 1 + t^{1-\alpha}\Gamma(1+\alpha)/2^\alpha$ to get, via the Laplace transform, $A^2(\mathbb{C}_+, (\nu_0, \nu_1))$ precisely as above. In this case, however, we do not get any equality between $\mathcal{B}^2_\alpha(\mathbb{C}_+)$ and $\mathcal{D}_{2+\alpha}(\mathbb{C}_+)$. If $\alpha = -1$, then $\mathcal{D}_{-1}(\mathbb{C}_+) = H^{1,2}(\mathbb{C}_+)$, the Hardy–Sobolev space on the complex half-plane .

1.3.3 Hardy–Sobolev spaces

We can use the identity given in Remark 1.1.4 to define a *fractional Laplace transform*. Namely, for $r \geq 0$, we let

$$\mathfrak{L}^{(r)}[f](z) := \mathfrak{L}[e^{i\pi r}(\cdot)^r f](z) \stackrel{\text{defn}}{=} \int_0^\infty e^{i\pi r} t^r f(t) e^{-tz}\, dt,$$

for all $z \in \mathbb{C}_+$ and f such that the last integral converges. For some Banach spaces we can also use the density argument outlined in Subsection 1.1.2 to define a linear operator $\mathfrak{L}^{(r)}$. Note that the statement and the proof of Theorem 1.1.3 remain valid for non-integer values of $n \geq 0$.

Let $1 \leq p,\, q < \infty$, and let w and w' be measurable self-maps on $(0, \infty)$. Suppose that there exist Banach spaces B and B', whose vectors are functions analytic on $\mathbb{C}_+$, such that $\mathfrak{L}$ and $\mathfrak{L}^{(r)}$ are well-defined maps

$$\mathfrak{L} : L^p_w(0, \infty) \longrightarrow B,$$
$$\mathfrak{L}^{(r)} : L^q_{w'}(0, \infty) \longrightarrow B'.$$

If $F \in \mathfrak{L}\left(L^p_w(0, \infty)\right) \subseteq B$ and $\mathfrak{L}^{-1}F \in L^q_{w'}(0, \infty)$, then we will write

$$\frac{d^r}{dz^r}F(z) := F^{(r)}(z) := \mathfrak{L}^{(r)}\left[\mathfrak{L}^{-1}[F]\right](z) \qquad (\forall z \in \mathbb{C}_+),$$

and call it the *fractional derivative of F of degree r*.

Let $1 \leq p < \infty$, let μ be a positive regular Borel measure supported on some subset M of non-negative real numbers, which contains 0, let $\{A^p_{\nu_r}\}_{r\in M}$ be a family of Zen spaces such that, for some weights $\{w_r\}_{r\in M}$ on $(0,\,\infty)$, each pair $(A^p_{\nu_0},\, A^p_{\nu_r})$ is, with respect to the pair of weights $(w_0,\, w_r)$, like $(B,\, B')$ above. Then we define

$$A^p_\mu(\mathbb{C}_+,\, (\nu_r)_{r\in M}) := \left\{F \in \mathfrak{L}\left(L^p_{w_0}(0,\,\infty)\right) \subseteq A^p_{\nu_0} \;:\; \forall r \in M \quad \mathfrak{L}^{-1}[F] \in L^p_{w_r}(0,\,\infty)\right\},$$

and equip it with a norm

$$\|F\|_{A^p_\mu(\mathbb{C}_+,\,(\nu_r)_{r\in M})} := \left(\int_M \|F^{(r)}\|^p_{A^p_{\nu_r}}\, d\mu(r)\right)^{\frac{1}{p}}.$$

Observe that the space $A^2_{(m)}$ is a special case of $A^2_\mu(\mathbb{C}_+,\, (\nu_r)_{r\in M})$, corresponding to μ=$\sum_{n=0}^m \delta_n$, where δ_n is the Dirac measure in n .

We can also set

$$w_\mu(t) := 2\pi \int_M t^{2r} \int_{[0,\,\infty)} e^{-2xt}\, d\nu_r(x)\, d\mu(r) \qquad\qquad (\forall t > 0).$$

It follows from Theorem 1.1.3 (for $n \in [0,\,\infty)$) that the Laplace transform is an isometry between $L^2_{w_\mu}(0,\,\infty)$ and $A^2_\mu(\mathbb{C}_+,\, (\nu_r)_{r\in M})$, and that the latter is a RKHS with kernels given by

$$k_z^{A^2_\mu(\mathbb{C}_+,\,(\nu_r)_{r\in M})}(\zeta) := \int_0^\infty \frac{e^{-t(\overline{z}+\zeta)}}{w_\mu(t)}\, dt \qquad\qquad (\forall (z,\,\zeta) \in \mathbb{C}^2_+).$$

These two facts are proved in the same way as Corollary 1.2.1 and Theorem 1.2.4, replacing $\sum_{n=0}^m \cdot$ with $\int_M \cdot\, d\mu(r)$.

Definition 1.3.4 *Let* $1 \leq p < \infty$ *and let* $r > 0$*. The* Hardy–Sobolev *space on the open right complex half-plane is defined to be*

$$H^{r,p}(\mathbb{C}_+) := \left\{F \in H^p(\mathbb{C}_+) \;:\; F^{(r)} \in H^p(\mathbb{C}_+)\right\}$$

with

$$\|F\|_{H^{r,p}(\mathbb{C}_+)} := \left(\|F\|^p_{H^p(\mathbb{C}_+)} + \left\|F^{(r)}\right\|^p_{H^p(\mathbb{C}_+)}\right)^{\frac{1}{p}} \qquad\qquad (\forall F \in H^{r,p}(\mathbb{C}_+)).$$

Example 1.3.5 *Let* $1 \leq p < \infty,\ r > 0,\ \mu = \delta_0 + \delta_r$*, and let* $\tilde{\nu_0} = \tilde{\nu_1} = \frac{\delta_0}{2\pi}$*. Then*

$$H^{r,p}(\mathbb{C}_+) = A^p_\mu(\mathbb{C}_+,\ (\nu_1,\ \nu_2)),$$

and if $p = 2$*, then* $H^{r,p}(\mathbb{C}_+)$ *is a RKHS with reproducing kernels given by*

$$k_z^{H^{r,2}(\mathbb{C}_+)}(\zeta) = \int_0^\infty \frac{e^{-t(\overline{z}+\zeta)}}{1+t^{2r}}\, dt.$$

In the previous subsection it was said that $\mathcal{D}_{-1}(\mathbb{C}_+) = H^{2,2}(\mathbb{C}_+)$*, but in fact, if we allow equivalent norms, we have* $\mathcal{D}(\mathbb{C}_+)_{1-2\alpha} = H^{\alpha,2}(\mathbb{C}_+)$*, whenever* $0 < \alpha \leq 1$*.*

Hardy–Sobolev spaces on the open unit disk of the complex plane are defined in the similar way, using partial derivatives or sequence spaces. They are discussed for example in [2].

In the remaining part of this thesis, we shall present our results mostly in terms of $A^2_{(m)}$ spaces, to keep the notation as simple as possible. But it is discernible that they normally remain valid for $A^2_\mu(\mathbb{C}_+,\ (\nu_r)_{r\in M})$ spaces too.

We summarise this chapter with tableaux containing all the spaces of functions analytic on the open right complex half-plane discussed so far, along with their basic properties.

1.4 Résumé of spaces of analytic functions on $\mathbb{C}_+$

Zen spaces A^p_ν
$A^p_\nu := \left\{F : \mathbb{C}_+ \to \mathbb{C},\ \text{analytic}\ :\ \sup_{\varepsilon>0} \int_{\overline{\mathbb{C}_+}} \lvert F(z+\varepsilon)\rvert^p \, d\nu(z) < \infty\right\}$
$\lVert F\rVert_{A^p_\nu} := \left(\int_{\overline{\mathbb{C}_+}} \lvert F(z)\rvert^p \, d\nu(z)\right)^{\frac{1}{p}}$
$\langle F,\, G\rangle_{A^2_\nu} := \int_{\overline{\mathbb{C}_+}} F(z)\overline{G(z)} \, d\nu(z)$
$\lVert \mathfrak{L}[f]\rVert_{A^2_\nu} = \left(\int_0^\infty \lvert f(t)\rvert^2 w(t)\, dt\right)^{\frac{1}{2}}$
$\langle \mathfrak{L}[f],\, \mathfrak{L}[g]\rangle_{A^2_\nu} = \int_0^\infty f(t)\overline{g(t)} w(t)\, dt$
$\left\lVert \mathfrak{L}^{(n)}[f]\right\rVert_{A^2_\nu} = \left(\int_0^\infty \lvert f(t)\rvert^2 t^{2n} w(t)\, dt\right)^{\frac{1}{2}}$
$\left\langle \mathfrak{L}^{(n)}[f],\, \mathfrak{L}^{(n)}[g]\right\rangle_{A^2_\nu} = \int_0^\infty f(t)\overline{g(t)} t^{2n} w(t)\, dt$
$w(t) := 2\pi \int_{[0,\,\infty)} e^{-2tx}\, d\tilde{\nu}(x)$

Hardy spaces $H^p(\mathbb{C}_+)$
$H^p(\mathbb{C}_+) := \left\{F : \mathbb{C}_+ \to \mathbb{C},\ \text{analytic}\ :\ \sup_{x>0} \int_{-\infty}^\infty \lvert F(x+iy)\rvert^p \, \frac{dy}{2\pi} < \infty\right\}$
$\lVert F\rVert_{H^p(\mathbb{C}_+)} := \left(\int_{-\infty}^\infty \lvert F(x+iy)\rvert^p \, \frac{dy}{2\pi}\right)^{\frac{1}{p}}$
$\langle F,\, G\rangle_{H^2(\mathbb{C}_+)} := \sup_{x>0} \int_{-\infty}^\infty F(x+iy)\overline{G(x+iy)} \, \frac{dy}{2\pi}$
$\lVert \mathfrak{L}[f]\rVert_{H^2(\mathbb{C}_+)} = \left(\int_0^\infty \lvert f(t)\rvert^2 \, dt\right)^{\frac{1}{2}}$
$\langle \mathfrak{L}[f],\, \mathfrak{L}[g]\rangle_{H^2(\mathbb{C}_+)} = \int_0^\infty f(t)\overline{g(t)}\, dt$
$\left\lVert \mathfrak{L}^{(n)}[f]\right\rVert_{H^2(\mathbb{C}_+)} = \left(\int_0^\infty \lvert f(t)\rvert^2 t^{2n}\, dt\right)^{\frac{1}{2}}$
$\left\langle \mathfrak{L}^{(n)}[f],\, \mathfrak{L}^{(n)}[g]\right\rangle_{H^2(\mathbb{C}_+)} = \int_0^\infty f(t)\overline{g(t)} t^{2n}\, dt$
$k_z^{H^2(\mathbb{C}_+)}(\zeta) = \frac{1}{\overline{z}+\zeta}$

Weighted Bergman spaces $\mathcal{B}^p_\alpha(\mathbb{C}_+), \alpha > -1$
$\mathcal{B}^p_\alpha(\mathbb{C}_+) := \left\{F : \mathbb{C}_+ \to \mathbb{C},\ \text{analytic}\ :\ \int_{-\infty}^\infty \int_0^\infty \lvert F(x+iy)\rvert^p x^\alpha \, \frac{dx\,dy}{\pi} < \infty\right\}$
$\lVert F\rVert_{\mathcal{B}^p_\alpha(\mathbb{C}_+)} := \left(\int_{-\infty}^\infty \int_0^\infty \lvert F(x+iy)\rvert^p x^\alpha \, \frac{dx\,dy}{\pi}\right)^{\frac{1}{p}}$
$\langle F,\, G\rangle_{\mathcal{B}^2_\alpha(\mathbb{C}_+)} := \int_{-\infty}^\infty \int_0^\infty F(x+iy)\overline{G(x+iy)} x^\alpha \, \frac{dx\,dy}{\pi}$
$\lVert \mathfrak{L}[f]\rVert_{\mathcal{B}^2_\alpha(\mathbb{C}_+)} = \left(\int_0^\infty \lvert f(t)\rvert^2 \, \frac{\Gamma(1+\alpha)}{2^\alpha t^{1+\alpha}}\, dt\right)^{\frac{1}{2}}$
$\langle \mathfrak{L}[f],\, \mathfrak{L}[g]\rangle_{\mathcal{B}^2_\alpha(\mathbb{C}_+)} = \int_0^\infty f(t)\overline{g(t)}\, \frac{\Gamma(1+\alpha)}{2^\alpha t^{1+\alpha}}\, dt$
$\left\lVert \mathfrak{L}^{(n)}[f]\right\rVert_{\mathcal{B}^2_\alpha(\mathbb{C}_+)} = \left(\int_0^\infty \lvert f(t)\rvert^2 t^{2n-1-\alpha} \frac{\Gamma(1+\alpha)}{2^\alpha}\, dt\right)^{\frac{1}{2}}$
$\left\langle \mathfrak{L}^{(n)}[f],\, \mathfrak{L}^{(n)}[g]\right\rangle_{\mathcal{B}^2_\alpha(\mathbb{C}_+)} = \int_0^\infty f(t)\overline{g(t)} t^{2n-1-\alpha} \frac{\Gamma(1+\alpha)}{2^\alpha}\, dt$
$k_z^{\mathcal{B}^2_\alpha(\mathbb{C}_+)}(\zeta) = \frac{2^\alpha(1+\alpha)}{(\overline{z}+\zeta)^{2+\alpha}}$
Conventions: $\mathcal{B}^p(\mathbb{C}_+) := \mathcal{B}^p_0(\mathbb{C}_+)$, $\mathcal{B}^p_{-1}(\mathbb{C}_+) := H^p(\mathbb{C}_+)$.

$A^p(\mathbb{C}_+, (\nu_n)_{n=0}^{\infty})$ spaces ($m \in \mathbb{N}_0 \cup \{\infty\}$)

$$A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^{m}) := \left\{F \in A^p_{\nu_0} \;:\; \forall 0 \le n \le m \; F^{(n)} \in A^p_{\nu_n}\right\}$$

$$\|F\|_{A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^{m})} := \left(\sum_{n=0}^{m} \left\|F^{(n)}\right\|^p_{A^p_{\nu_n}}\right)^{\frac{1}{p}}$$

$$\langle F,\, G\rangle_{A^2(\mathbb{C}_+,\, (\nu_n)_{n=0}^{m})} := \sum_{n=0}^{m} \left\langle F^{(n)},\, G^{(n)}\right\rangle_{A^2_{\nu_n}}$$

$A^2_{(m)}$ spaces

$$A^2_{(m)} := \mathfrak{L}\left(L^2_{w_{(m)}}(0,\, \infty)\right) \subseteq A^2(\mathbb{C}_+,\, (\nu_n)_{n=0}^{m})$$

$$w_{(m)}(t) := 2\pi \sum_{n=0}^{m} t^{2n} \int_{[0,\, \infty)} e^{-2tx}\, d\tilde{\nu}_n(x)$$

$$\|F\|_{A^2_{(m)}} := \|F\|_{A^2(\mathbb{C}_+,\, (\nu_n)_{n=0}^{m})}$$

$$\langle F,\, G\rangle_{A^2(m)} := \langle F,\, G\rangle_{A^2(\mathbb{C}_+,\, (\nu_n)_{n=0}^{m})}$$

$$\|\mathfrak{L}[f]\|_{A^2_{(m)}} = \left(\int_0^{\infty} |f(t)|^2 w_{(m)}(t)\, dt\right)^{\frac{1}{2}}$$

$$\langle \mathfrak{L}[f],\, \mathfrak{L}[g]\rangle_{A^2_{(m)}} = \int_0^{\infty} f(t)\overline{g(t)} w_{(m)}(t)\, dt$$

$$k_z^{A^2_{(m)}}(\zeta) = \int_0^{\infty} \frac{e^{-t(\overline{z}+\zeta)}}{w_{(m)}(t)}\, dt$$

Weighted Dirichlet spaces $\mathcal{D}_\alpha(\mathbb{C}_+)$, $\alpha \ge -1$

$$\mathcal{D}_\alpha(\mathbb{C}_+) := \{F \in H^2(\mathbb{C}_+) \;:\; F' \in \mathcal{B}^2_\alpha(\mathbb{C}_+)\}$$

$$\|F\|_{\mathcal{D}_\alpha(\mathbb{C}_+)} := \left(\|F\|^2_{H^2(\mathbb{C}_+)} + \|F'\|^2_{\mathcal{B}^2_\alpha(\mathbb{C}_+)}\right)^{\frac{1}{2}}$$

$$\langle F,\, G\rangle_{\mathcal{D}_\alpha(\mathbb{C}_+)} := \langle F,\, G\rangle_{H^2(\mathbb{C}_+)} + \langle F',\, G'\rangle_{\mathcal{B}^2_\alpha(\mathbb{C}_+)}$$

$$\|\mathfrak{L}[f]\|_{\mathcal{D}_\alpha(\mathbb{C}_+)} = \left(\int_0^{\infty} |f(t)|^2 \left(2^\alpha + t^{1-\alpha}\Gamma(1+\alpha)\right) \frac{dt}{2^\alpha}\right)^{\frac{1}{2}}$$

$$\langle F,\, G\rangle_{\mathcal{D}_\alpha(\mathbb{C}_+)} = \int_0^{\infty} f(t)\overline{g(t)} \left(2^\alpha + t^{1-\alpha}\Gamma(1+\alpha)\right) \frac{dt}{2^\alpha}$$

$$k_z^{\mathcal{D}_\alpha(\mathbb{C}_+)}(\zeta) = 2^\alpha \int_0^{\infty} \frac{e^{-t(\overline{z}+\zeta)}}{2^\alpha + t^{1-\alpha}\Gamma(1+\alpha)}\, dt$$

Conventions: $\mathcal{D}(\mathbb{C}_+) := \mathcal{D}_0(\mathbb{C}_+)$, $\mathcal{D}_{-1}(\mathbb{C}_+) := H^{1,2}(\mathbb{C}_+)$ (see below)

$$k_z^{\mathcal{D}(\mathbb{C}_+)}(\zeta) = e^{\overline{z}+\zeta}\Gamma\left(0, \overline{z}+\zeta\right)$$

Variant weighted Dirichlet spaces $\mathcal{D}'_\alpha(\mathbb{C}_+)$, $\alpha > -1$

$$\mathcal{D}'_\alpha(\mathbb{C}_+) := \{F : \mathbb{C}_+ \to \mathbb{C},\ \text{analytic} \;:\; F' \in \mathcal{B}^2_\alpha(\mathbb{C}_+)\}$$

$$\|F\|_{\mathcal{D}'_\alpha(\mathbb{C}_+)} := \left(|F(1)|^2 + \|F'\|^2_{\mathcal{B}^2_\alpha(\mathbb{C}_+)}\right)^{\frac{1}{2}}$$

$$\langle F,\, G\rangle_{\mathcal{D}'_\alpha(\mathbb{C}_+)} := F(1)\overline{G(1)} + \langle F',\, G'\rangle_{\mathcal{B}^2_\alpha(\mathbb{C}_+)}$$

$$k_z^{\mathcal{D}'_{-1}(\mathbb{C}_+)}(\zeta) = (\overline{z}+\zeta)\ln(\overline{z}+\zeta) - (\overline{z}+1)\ln(\overline{z}+1) - (1+\zeta)\ln(1+\zeta) + \ln 4 + 1$$

$$k_z^{\mathcal{D}'_\alpha(\mathbb{C}_+)}(\zeta) = \frac{2^\alpha}{\alpha}\left(\frac{1}{(\overline{z}+\zeta)^\alpha} - \frac{1}{(\overline{z}+1)^\alpha} - \frac{1}{(1+\zeta)^\alpha} + \frac{\alpha-1}{2^\alpha}\right),\ \alpha \in (-1,\, 0) \cup (0,\, \infty)$$

Convention: $\mathcal{D}'(\mathbb{C}_+) := \mathcal{D}'_0(\mathbb{C}_+)$

$$k_z^{\mathcal{D}'(\mathbb{C}_+)}(\zeta) = \ln(1+\overline{z}) - (\overline{z}+\zeta) + \ln(1+\zeta) - \ln 2 + 1$$

$A^p_\mu(\mathbb{C}_+,\, (\nu_r)_{r\in M})$ **spaces**
$\mathfrak{L}^{(r)}(L^p_{w_r}(0\,\infty)) \subseteq A^p_{(\nu_r)}, \qquad \|\mathfrak{L}^{(r)}[f]\|_{A^p_{\nu_r}} \leq \|f\|_{L^p_{w_r}(0\,\infty)}$
$A^p_\mu(\mathbb{C}_+,\, (\nu_r)_{r\in M}) := \left\{F \in \mathfrak{L}\left(L^p_{w_0}(0,\,\infty)\right) \subseteq A^p_{\nu_0} \,:\, \forall r \in M\; \mathfrak{L}^{-1}[F] \in L^p_{w_r}(0,\,\infty)\right\}$
$\|F\|_{A^p_\mu(\mathbb{C}_+,\,(\nu_r)_{r\in M})} := \left(\int_M \|F^{(r)}\|^p_{A^p_{\nu_r}}\, d\mu(r)\right)^{\frac{1}{p}}$
$\langle F,\, G\rangle_{A^2_\mu(\mathbb{C}_+,\,(\nu_r)_{r\in M})} := \int_M \left\langle F^{(r)},\, G^{(r)}\right\rangle_{A^2_{\nu_r}}\, d\mu(r)$
$\|\mathfrak{L}[f]\|_{A^2_\mu(\mathbb{C}_+,\,(\nu_r)_{r\in M})} = \left(\int_0^\infty \|f(t)\|^2 w_\mu(t)\, dt\right)^{\frac{1}{2}}$
$\langle \mathfrak{L}[f],\, \mathfrak{L}[g]\rangle_{A^2_\mu(\mathbb{C}_+,\,(\nu_r)_{r\in M})} = \int_0^\infty f(t)\overline{g(t)} w_\mu(t)\, dt$
$k_z^{A^2_\mu(\mathbb{C}_+,\,(\nu_r)_{r\in M})}(\zeta) = \int_0^\infty \frac{e^{-t(\overline{z}+\zeta)}}{w_\mu(t)}\, dt$
$w_\mu(t) := 2\pi \int_{r\in M} t^{2r} \int_{[0,\,\infty)} e^{-2tx}\, d\tilde{\nu}_r(x), d\mu(r)$

Hardy–Sobolev spaces $H^{r,p}(\mathbb{C}_+)$
$H^{r,p}(\mathbb{C}_+) := \left\{F \in H^p(\mathbb{C}_+) \,:\, F^{(r)} \in H^p(\mathbb{C}_+)\right\}$
$\|F\|_{H^{r,p}(\mathbb{C}_+)} := \left(\|F\|^p_{H^p(\mathbb{C}_+)} + \|F^{(r)}\|^p_{H^p(\mathbb{C}_+)}\right)^{\frac{1}{p}}$
$\langle F,\, G\rangle_{H^{r,2}(\mathbb{C}_+)} := \langle F,\, G\rangle_{H^2(\mathbb{C}_+)} + \left\langle F^{(r)},\, G^{(r)}\right\rangle_{H^2(\mathbb{C}_+)}$
$\|\mathfrak{L}[f]\|_{H^{r,2}(\mathbb{C}_+)} := \left(\int_0^\infty \|f(t)\|^2 (1+t^{2r})\, dt\right)^{\frac{1}{2}}$
$\langle \mathfrak{L}[f],\, \mathfrak{L}[g]\rangle_{H^{r,2}(\mathbb{C}_+)} = \int_0^\infty f(t)\overline{g(t)}(1+t^{2r})\, dt$
$k_z^{H^{r,2}(\mathbb{C}_+)}(\zeta) = \int_0^\infty \frac{e^{-t(\overline{z}+\zeta)}}{1+t^{2r}}\, dt$
$H^{r,2}(\mathbb{C}_+) = \mathcal{D}_{1-2r}(\mathbb{C}_+),\; 0 < r \leq 1$ (up to equivalent norms)

Chapter 2

Carleson Embeddings and Carleson Measures

> I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely, in your thoughts, advanced to the stage of science, whatever the matter may be.
>
> WILLIAM THOMSON, 1st Baron Kelvin, *Lecture on "Electrical Units of Measurement", Popular Lectures and Addresses Vol. I Constitution of Matter*

Definition 2.0.1 *Let* $1 \leq q < \infty$*, let* μ *be a positive Borel measure on* $\mathbb{C}_+$*, and let* X *be either* $A^p(\mathbb{C}_+, (\nu_n)_{n=0}^m)$ *or* $A^2_{(m)}$*. If there exists* $C(\mu) > 0$*, depending on* μ *only, such that*

$$\left(\int_{\mathbb{C}_+} |f|^q \, d\mu\right)^{\frac{1}{q}} \leq C(\mu)\|f\|_X \qquad (\forall f \in X), \tag{2.1}$$

then we will call the embedding

$$X \hookrightarrow L^q(\mathbb{C}_+,\, \mu) \tag{2.2}$$

a Carleson embedding*, and the expression in* (2.1) *will be called the* Carleson criterion*. If $p = q$ (or $q = 2$ for $X = A^2_{(m)}$) and the embedding* (2.2) *is bounded, then we will say that μ is a* Carleson measure *for X. The set of Carleson measures for X will be denoted by $CM(X)$ and we can define a positive function $\|\cdot\|_{CM(X)}$ on it by*

$$\|\mu\|_{CM(X)} := \inf C(\mu) \qquad (\forall \mu \in CM(X)),$$

where the infimum is taken over all constants $C(\mu)$ for which the Carleson criterion is satisfied. The values of p and q are normally explicitly given in the context.

The notion of a Carleson measure was introduced by Lennart Carleson in his proof of the Corona Theorem for H^∞ (the Hardy space of bounded holomorphic functions on the open unit disk, equipped with the supremum norm) in [19]. Carleson characterised there these measures for Hardy spaces H^p on the open unit disk of the complex plane. Lars Hörmander extended Carleson's result to the open unit ball of $\mathbb{C}^n$ in [58]; Joseph Cima and Warren Wogen in [24] and David Luecking in [73] characterised Carleson measures for the weighted Bergman spaces on the unit ball of $\mathbb{C}^n$; and, in [94], David Stegenga characterised them for the weighted Dirichlet spaces on $\mathbb{D}$. Carleson measures for Zen spaces have been described in [61] and we will partially extend this description to $A^2_{(m)}$ and $A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^m)$ spaces in this thesis. For now, we can immediately say that

$$CM(A^p_{\nu_0}) \subseteq CM(A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^m)) \subseteq CM(A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^{m'})),$$

whenever $m' \geq m$, because

$$A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^{m'}) \subseteq A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^{m}) \subseteq A^p_{\nu_0}$$

and

$$\|F\|_{A^p_{\nu_0}} \leq \|F\|_{A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^m)} \leq \|F\|_{A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^{m'})} \qquad (\forall F \in A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^{m'})).$$

For the same reason we also have

$$CM(A^2_{(m)}) \subseteq CM(A^2_{(m')}) \subseteq CM(A^2(\mathbb{C}_+,\, (\nu_n)_{n=0}^{m'})) \qquad (\forall m' \geq m).$$

Note that we assume that p in the definition of a Carleson embedding is the same as in the definition of the Zen space $A^p_{\nu_0}$ and related spaces, since $A^p_{\nu_0} \subset L^p(\mathbb{C}_+,\, \nu_0)$ (in the set sense).

The popularity of this research is a result of wide range of applications of this concept. In particular, in this thesis we will show how Carleson measures can be used to determine the boundedness of multiplication operators and weighted composition operators. We will also explain how the boundedness of Carleson embeddings can be employed in testing weighted infinite-time admissibility of control and observation operators. The results presented in this chapter have been published in [64] and [66].

2.1 Carleson embeddings

Before examining Carleson measures, we are going to consider Carleson embeddings in *sensu lato*, that is for general $1 \leq p,\, q < \infty$.

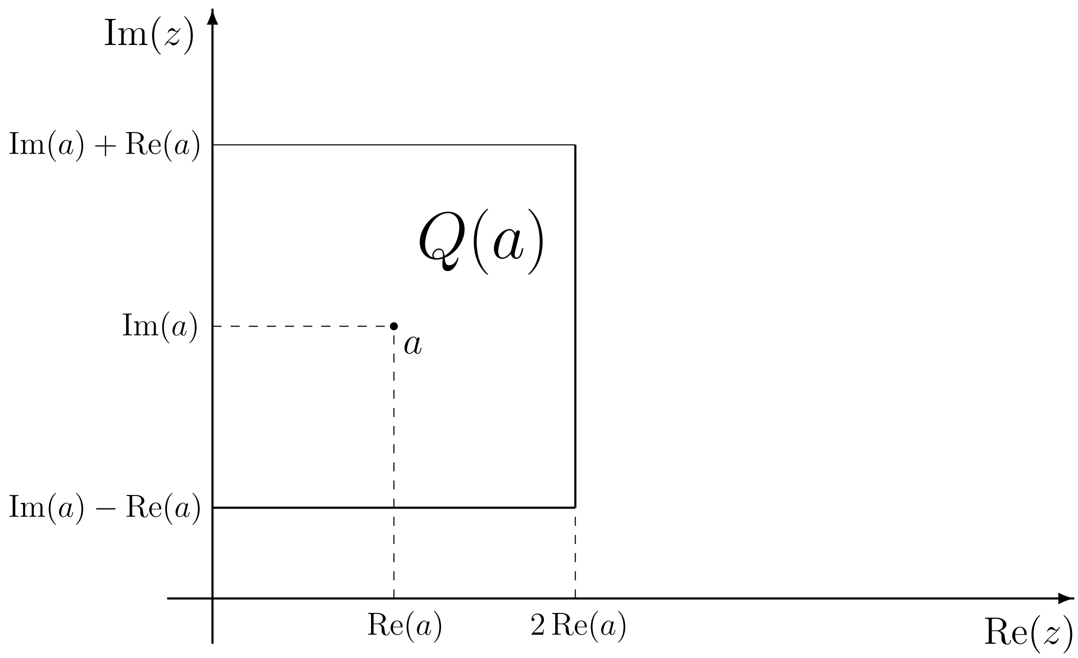
2.1.1 Carleson squares

We start by introducing the following notion.

Definition 2.1.1 *Let $a \in \mathbb{C}_+$. A* Carleson square centred at *a is defined to be the subset*

$$Q(a) := \{z \in \mathbb{C}_+ \;:\; 0 < \mathrm{Re}(z) \leq 2\,\mathrm{Re}(a),\, \mathrm{Re}(a) \leq \mathrm{Im}(z) - \mathrm{Im}(a) < \mathrm{Re}(a)\} \quad (2.3)$$

of the open right complex half-plane.



A related family of sets can also be defined for other domains, but they are seldom of rectangular shape, so very often they are referred to as *Carleson boxes*.

Theorem 2.1.2 (Theorem 3 in [66]) *Let* $1 \leq p,\, q < \infty$ *and* $m \in \mathbb{N}_0$. *If the embedding*

$$A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^m) \hookrightarrow L^q(\mathbb{C}_+,\, \mu) \tag{2.4}$$

is bounded, then there exists a constant $C(\mu) > 0$, *such that*

$$\mu(Q(a)) \leq C(\mu) \left[\sum_{n=0}^{m} \frac{\nu_n\left(\overline{Q(a)}\right)}{(\mathrm{Re}(a))^{np}} \right]^{\frac{q}{p}}, \tag{2.5}$$

for each Carleson square $Q(a)$.

Proof

Let $a \in \mathbb{C}_+$. Note that for all $z \in Q(a)$ we have

$$|z + \overline{a}| \overset{\text{def}^{\text{n}}}{=} \sqrt{(\mathrm{Re}(z) + \mathrm{Re}(a))^2 + (\mathrm{Im}(z) - \mathrm{Im}(a))^2} \overset{(2.3)}{\leq} \sqrt{10}\,\mathrm{Re}(a). \tag{2.6}$$

Choose $\gamma > \sup_{0\leq n\leq m}(\log_2(R_n) - np + 1)/p$, where R_n denotes the supremum from the (Δ_2)-condition for each $\tilde{\nu}_n$, $0 \leq n \leq m$. Then

$$\frac{\mu\left(Q(a)\right)}{(\sqrt{10}\,\mathrm{Re}(a))^{\gamma q}} \leq \int_{\mathbb{C}_+} \frac{d\mu(z)}{|z+\overline{a}|^{\gamma q}}. \tag{2.7}$$

On the other hand

$$|z+\overline{a}| \geq \sqrt{\mathrm{Re}(a)^2} = \mathrm{Re}(a) > \frac{\mathrm{Re}(a)}{2} \qquad (\forall z \in Q(a)).$$

Additionally, given $k \in \mathbb{N}_0$, for all $z \in Q(2^{k+1}\,\mathrm{Re}(a) + i\,\mathrm{Im}(a)) \setminus Q(2^k(\mathrm{Re}) + i\,\mathrm{Im}(a))$, with $0 < \mathrm{Re}(z) \leq 2^{k+1}\,\mathrm{Re}(a)$, we have

$$|z+\overline{a}| \geq \sqrt{\mathrm{Re}(a)^2 + (2^k\,\mathrm{Re}(a))^2} \geq 2^k\,\mathrm{Re}(a),$$

and if $2^{k+1}\,\mathrm{Re}(a) < \mathrm{Re}(z) \leq 2^{k+2}\,\mathrm{Re}(a)$, then we also have

$$|z+\overline{a}| \geq \sqrt{(2^{k+1}\,\mathrm{Re}(a) + \mathrm{Re}(a))^2} \geq 2^k\,\mathrm{Re}(a).$$

And

$$\begin{aligned}
&\nu_n\left(Q(2^{k+1}\,\mathrm{Re}(a) + i\,\mathrm{Im}(a)) \setminus Q(2^k(\mathrm{Re}) + i\,\mathrm{Im}(a))\right)\\
&\quad\leq \nu_n\left(Q(2^{k+1}\,\mathrm{Re}(a) + i\,\mathrm{Im}(a))\right)\\
&\quad\leq \tilde{\nu}_n\left(\left[0,\ 2^{k+2}\,\mathrm{Re}(a)\right)\right)\cdot 2^{k+1}\,\mathrm{Re}(a)\\
&\quad\overset{(\Delta_2)}{\leq} (2R_n)^{k+1}\,\tilde{\nu}_n\left([0,\ 2\,\mathrm{Re}(a))\right)\cdot 2\,\mathrm{Re}(a)\\
&\quad\leq (2R_n)^{k+1}\,\nu_n\left(\overline{Q(a)}\right).
\end{aligned} \tag{2.8}$$

Hence

$$\begin{aligned}
\sup_{\varepsilon>0}\int_{\mathbb{C}_+} \frac{d\nu_n(z)}{|z+\varepsilon+\overline{a}|^{(\gamma+n)p}} &\leq \left(\frac{2}{\mathrm{Re}(a)}\right)^{(\gamma+n)p} \nu_n\left(Q(a)\right)\\
&\quad+\sum_{k=0}^{\infty} \frac{\nu_n\left(Q(2^{k+1}\,\mathrm{Re}(a) + i\,\mathrm{Im}(a)) \setminus Q(2^k(\mathrm{Re}(a)) + i\,\mathrm{Im}(a))\right)}{(2^k\,\mathrm{Re}(a))^{(\gamma+n)p}}\\
&\overset{(2.8)}{\leq} \left(\frac{2}{\mathrm{Re}(a)}\right)^{(\gamma+n)p} \nu_n\left(\overline{Q(a)}\right)\left(1 + \sum_{k=0}^{\infty}\frac{(2R_n)^{k+1}}{2^{(k+1)(\gamma+n)p}}\right)\\
&\leq \left(\frac{2}{\mathrm{Re}(a)}\right)^{(\gamma+n)p} \nu_n\left(\overline{Q(a)}\right)\sum_{k=0}^{\infty}\left(\frac{R_n}{2^{(\gamma+n)p-1}}\right)^k
\end{aligned} \tag{2.9}$$

and the sum converges for all $0 \leq n \leq m$. Now, if the embedding is bounded, with a constant $C'(\mu) > 0$ say, then

$$\begin{aligned}
\mu(Q(a)) &\overset{(2.7)}{\leq} (\sqrt{10}\,\mathrm{Re}(a))^{\gamma q} \int_{\mathbb{C}_+} \frac{d\mu(z)}{|z+\overline{a}|^{\gamma q}} \\
&\overset{(2.4)}{\leq} C'(\mu)(\sqrt{10}\,\mathrm{Re}(a))^{\gamma q} \, \|K_{\gamma-2}(\cdot,\, a)\|^q_{A^p(\mathbb{C}_+,\,(\nu_n)_{n=0}^m)} \\
&\overset{\text{def}^{\text{n}}}{=} C'(\mu)(\sqrt{10}\,\mathrm{Re}(a))^{\gamma q} \left[\sum_{n=0}^m \sup_{\varepsilon_n>0} \int_{\overline{\mathbb{C}_+}} \frac{d\nu_n(z)}{\left|[(z+\varepsilon_n+\overline{a})^{\gamma}]^{(n)}\right|^p}\right]^{\frac{q}{p}} \\
&\leq C'(\mu)(\sqrt{10}\,\mathrm{Re}(a))^{\gamma q} \left[\sum_{n=0}^m \left(\prod_{l=1}^n (\gamma+l-1)\right) \sup_{\varepsilon_n>0} \int_{\overline{\mathbb{C}_+}} \frac{d\nu_n(z)}{|z+\varepsilon_n+\overline{a}|^{(\gamma+n)p}}\right]^{\frac{q}{p}} \\
&\overset{(2.9)}{\leq} C(\mu) \left[\sum_{n=0}^m \frac{\nu_n\left(\overline{Q(a)}\right)}{(\mathrm{Re}(a))^{np}}\right]^{\frac{q}{p}},
\end{aligned}$$

where

$$C(\mu) := 2^{q(n+3\gamma/2)} 5^{\gamma q/2} \left[\left(\prod_{l=0}^m (\gamma+l-1)\right) \max_{0\leq n\leq m} \sum_{k=0}^{\infty} \left(\frac{R_n}{2^{(\gamma+n)p-1}}\right)^k\right]^{\frac{q}{p}} C'(\mu),$$

(and we adopted the convention that the product $\prod(\gamma + l - 1)$ is defined to be 1, if the lower limit is a bigger number than the upper limit). □

Remark 2.1.3 *This theorem was initially proved for Carleson measures for Zen spaces (i.e. $p = q$ and $m = 0$) in [61] (Theorem 2.1, p. 787), in which case it is necessary as well as sufficient. That is, a positive Borel measure μ is a Carleson measure for a Zen space A^p_ν if and only if there exists a constant $C(\mu) > 0$ such that*

$$\mu(Q(a)) \leq C(\mu)\, \nu\left(\overline{Q(a)}\right) \qquad (\forall a \in \mathbb{C}_+).$$

The version for Carleson measures for $A^2(\mathbb{C}_+,\,(\nu_n)_{n=0}^m)$ (i.e. $p = q = 2$ and $m \in \mathbb{N}_0$) was proved in [64] (Theorem 2, p. 482), using a proof, very similar to the proof of Lemma 1.2.10 in this thesis. Note that the estimates in (2.9) *are essentially* *an alternative proof of Lemma 1.2.10. The theorem and its proof in the form presented above (i.e.* $1 \leq p, q < \infty$ *and* $m \in \mathbb{N}_0$*) were obtained in [66] (Theorem 3). Observe that the reason why the LHS of* (2.5) *involves a Carleson square, while the RHS comprises its closure, is due to the fact that* μ *is only defined on* $\mathbb{C}_+$*, whereas the proof relies on the* (Δ_2)*-condition, which requires the intervals to be left-closed. This was overlooked in [61] and [66].*

2.1.2 Carleson embeddings and trees

To find a sufficient condition for the Carleson criterion to be satisfied, we are going to use techniques involving abstract trees developed by Nicola Arcozzi, Richard Rochberg and Eric Sawyer to classify Carleson measures for analytic Besov spaces on $\mathbb{D}$ in [5], and for Drury–Arveson Hardy space and other Besov–Sobolev spaces on complex n-balls in [6].

Definition 2.1.4 *A* tree *is an undirected graph in which any two vertices are connected by exactly one path. We call the vertices of a tree leaves. If* T *is a tree with a partial order relation "*$\leq$*" defined on the set of its leaves, we will write* $v \in T$ *to denote that* v *is a leaf of* T*, and in general associate* T *with the set of its leaves, viewing its edges only as the structure underlying its ordering. Let* x, y *be two distinct leaves of* T. *If for all* $c \in T$ *such that* $y \leq c \leq x$ *we have* $x = c$ *or* $y = c$*, then we call* y *the* predecessor of x *and write* $y := x^-$. *For any* $\varphi : T \longrightarrow \mathbb{C}$ *we define the* primitive $\mathfrak{I}$ of φ at $x \in T$ *to be*

$$\mathfrak{I}\varphi(x) := \sum_{y \leq x} \varphi(y).$$

And finally, the shadow of x *is defined to be*

$$S(x) := \{y \in T \ : \ x \leq y\} \qquad \textit{and} \qquad S(-\infty) := T.$$

Trees and related concepts are discussed for example in [15] (§ I.2, pp. 8-14) and [31] (§ 1.5, pp. 13-16).

The next two lemmata and the ensuing theorem, concerning rootless trees and decomposition of $\mathbb{C}_+$, appeared in [64] and [66], and are adaptations of results initially proved in [5] for a tree with a root and the Whitney decomposition of the open unit disk of the complex plane (see Appendix J, p. 463 in [47]). In the proof of the next lemma, as well as in other places throughout this thesis, we reserve the symbol χ_E to denote the characteristic function on a set E.

Lemma 2.1.5 (Lemma 3, p. 488 in [64]) *Let T be a tree with a partial order defined on the set of its leaves, let $1 < p \leq q < \infty$, and let $p' = p/(p-1)$, $q' := q/(q-1)$ be the conjugate indices of p and q. Let also ω be a weight on T, and μ be a non-negative function on T. If there exists a constant $C(\mu, \omega) > 0$ such that for all $r \in T \cup \{-\infty\}$,*

$$\left(\sum_{x \in S(r)} \left(\sum_{y \in S(x)} \mu(y) \right)^{p'} \omega(x)^{1-p'} \right)^{\frac{q'}{p'}} \leq C(\mu, \omega) \sum_{x \in S(r)} \mu(x), \tag{2.10}$$

then there exists a constant $C'(\mu, \omega) > 0$ such that for all $\varphi : T \longrightarrow \mathbb{C}$,

$$\left(\sum_{x \in T} |\mathfrak{I}\varphi(x)|^q \mu(x) \right)^{\frac{1}{q}} \leq C'(\mu, \omega) \left(\sum_{x \in T} |\varphi(x)|^p \omega(x) \right)^{\frac{1}{p}}. \tag{2.11}$$

Proof

We can define $\tilde{\mu}$ and $\tilde{\omega}$ to be measures on the Borel algebra over T by

$$\tilde{\mu}(\{x\}) := \mu(x) \qquad \text{and} \qquad \tilde{\omega}(\{x\}) := \omega(x) \qquad (\forall x \in T).$$

Let $g \in L^p(T, \tilde{\omega})$. To prove this lemma we only need to show that if (2.10) holds, then

$$\|\mathfrak{I}g\|_{L^q(T, \tilde{\mu})} \leq C'(\mu, \omega) \|g\|_{L^p(T, \tilde{\omega})},$$

for all $g \geq 0$, in which case $\mathfrak{I}g$ is non-decreasing with respect to the order relation on T. Let

$$\Omega_k := \left\{ x \in T \ : \ \mathfrak{I}g(x) > 2^k \right\} = \bigcup_j S(r_j^k) \qquad (\forall k \in \mathbb{Z}),$$

where $\{r_j^k \in T \ : \ j = 1, \dots\}$ is the set of minimal points in Ω_k with respect to the partial order on T, if such points exist. Otherwise we define $r_1^k := -\infty$, and then

$$\Omega_k := \left\{x \in T \ : \ \mathfrak{I}g > 2^k\right\} \stackrel{\text{def}^{\text{n}}}{=} S(r_1^k) \stackrel{\text{def}^{\text{n}}}{=} S(-\infty) \stackrel{\text{def}^{\text{n}}}{=} T.$$

Let $E_j^k = S(r_j^k) \cap (\Omega_{k+1} \setminus \Omega_{k+2})$. Then for $x \in E_j^k$ we get

$$\mathfrak{I}(\chi_{S(r_j^k)} g)(x) = \sum_{r_j^k \le y \le x} g(y) = \mathfrak{I}g(x) - \mathfrak{I}g((r_j^k)^-) \ge 2^{k+1} - 2^k = 2^k, \tag{2.12}$$

where we adopt a convention that $\mathfrak{I}g((r_j^k)^-) := 0$, whenever $r_j^k = -\infty$. Thus we have,

$$\begin{aligned}
2^k \tilde{\mu}(E_j^k) &\stackrel{\text{def}^{\text{n}}}{=} 2^k \sum_{x \in E_j^k} \mu(x) \\
&\stackrel{(2.12)}{\le} \sum_{x \in E_j^k} \mathfrak{I}(\chi_{S(r_j^k)} g)(x) \mu(x) \\
&= \sum_{y \in S(r_j^k)} g(y) \sum_{x \in E_j^k,\, x \ge y} \mu(x) \\
&= \sum_{y \in S(r_j^k)} g(y) \sum_{x \ge y} \chi_{E_j^k}(x) \mu(x) \\
&= \sum_{y \in S(r_j^k) \cap (\Omega_{k+2}^c \cup \Omega_{k+2})} g(y) \sum_{x \ge y} \chi_{E_j^k}(x) \mu(x) \\
&= \sum_{y \in S(r_j^k) \cap \Omega_{k+2}^c} g(y) \sum_{x \ge y} \chi_{E_j^k}(x) \mu(x) + \sum_{y \in S(r_j^k) \cap \Omega_{k+2}} g(y) \underbrace{\sum_{x \ge y} \chi_{E_j^k}(x) \mu(x)}_{=0} \\
&= \sum_{y \in S(r_j^k) \cap \Omega_{k+2}^c} g(y) \sum_{x \ge y} \chi_{E_j^k}(x) \mu(x)
\end{aligned} \tag{2.13}$$

where Ω_{k+2}^c denotes the complement of Ω_{k+2} in T. Now,

$$\begin{aligned}
\sum_{x\in T} |\Im g(x)|^q \mu(x) &\le \sum_{k\in\mathbb{Z}} 2^{(k+2)q} \tilde{\mu}\left(\left\{x \in T \ : \ 2^{k+1} < \Im g(x) \le 2^{k+2}\right\}\right) \\
&\overset{\text{def}^{\text{n}}}{=} 2^{2q} \sum_{k\in\mathbb{Z}} 2^{kq} \tilde{\mu}\left(\Omega_{k+1} \setminus \Omega_{k+2}\right) \\
&\le 2^{2q} \sum_{k\in\mathbb{Z}} 2^{kq} \mu\left(\bigcup_j \left(S(r_j^k) \cap (\Omega_{k+1} \setminus \Omega_{k+2})\right)\right) \\
&\le 2^{2q} \sum_{k\in\mathbb{Z},\, j} 2^{kq} \tilde{\mu}(E_j^k) \\
&= 2^{2q} \left(\sum_{(k,j)\in E} 2^{kq} \tilde{\mu}(E_j^k) + \sum_{(k,j)\in F} 2^{kq} \tilde{\mu}(E_j^k)\right),
\end{aligned}$$

where

$$E := \left\{(k, j) \ : \ \tilde{\mu}(E_j^k) \le \beta \tilde{\mu}(S(r_k^j))\right\}, \tag{2.14}$$

$$F := \left\{(k, j) \ : \ \tilde{\mu}(E_j^k) > \beta \tilde{\mu}(S(r_k^j))\right\}, \tag{2.15}$$

for some $0 < \beta < 1 - 2^{-q}$. Let $\{x_k^n\}_{k,n} \subseteq T$ be a collection of distinct leaves of this tree such that $\bigcup_n \left\{x_n^k\right\} = \Omega_k \setminus \Omega_{k+1}$, for all $k \in \mathbb{Z} \setminus \{0\}$. Then

$$\begin{aligned}
\sum_{(k,j),k\ge 1} 2^{kq} \tilde{\mu}(S(r_j^k)) &\overset{\text{def}^{\text{n}}}{=} \sum_{k=1}^{\infty} 2^{kq} \tilde{\mu}(\Omega_k) \\
&= \sum_{k=1}^{\infty} \tilde{\mu}(\Omega_k \setminus \Omega_{k+1}) \sum_{l=1}^{k} 2^{lq} \\
&= \sum_{k=0}^{\infty} \tilde{\mu}\left(\bigcup_n \{x_n^k\}\right) \sum_{l=0}^{k-1} 2^{(k-l)q} \\
&\le \sum_{k=0}^{\infty} \sum_n \mu(x_n^k) \left|\Im g\left(x_n^k\right)\right|^q \sum_{l=0}^{k-1} 2^{-lq} \\
&\le \frac{1}{1 - 2^{-q}} \sum_{x\in T} |\Im g(x)|^q \mu(x) \\
&\overset{\text{def}^{\text{n}}}{=} \frac{1}{1 - 2^{-q}} \|\Im g\|_{L^q(T,\tilde{\mu})}^q .
\end{aligned} \tag{2.16}$$

Similarly,

$$
\begin{aligned}
\sum_{(k,j),k<1} 2^{kq}\tilde{\mu}(S(r_k^j)) &= \sum_{k=-\infty}^{0} 2^{kq}\tilde{\mu}(\Omega_k) \\
&= \left(\sum_{l=0}^{\infty} 2^{-lq}\right)\left(\tilde{\mu}(\Omega_0) + \sum_{k=1}^{\infty} 2^{-kq}\tilde{\mu}(\Omega_{-k}\setminus\Omega_{-k+1})\right) \\
&\leq \frac{1}{1-2^{-q}}\left(\sum_j \mu(r_j^0)\left|\Im g\left(r_j^0\right)\right|^q + \sum_{k=1}^{\infty} 2^{-kq}\mu\left(\bigcup_n \{x_n^{-k}\}\right)\right) \\
&= \frac{1}{1-2^{-q}}\left(\sum_j \mu(r_j^0)\left|\Im g\left(r_j^0\right)\right|^q + \sum_{k=1}^{\infty}\sum_n \mu\left(\{x_n^{-k}\}\right)|\Im g(x_n^{-k})|^q\right) \\
&\leq \frac{2}{1-2^{-q}}\sum_{x\in T}|\Im g(x)|^q\mu(x) \\
&\overset{\text{def}^{\text{n}}}{=} \frac{1}{1-2^{-q}}\|\Im g\|^q_{L^q(T,\tilde{\mu})}.
\end{aligned}
\tag{2.17}
$$

So

$$
2^{2q}\sum_{(k,j)\in E}\tilde{\mu}(E_j^k)2^{kq} \overset{(2.14),(2.16),(2.17)}{\leq} \frac{2^{2q+2}}{1-2^{-q}}\beta\|\Im g\|^q_{L^q(T,\mu)}. \tag{2.18}
$$

For the sum indexed by F we have

$$
\begin{aligned}
\sum_{(k,j)\in F}\tilde{\mu}(E_j^k)2^{kq} &\overset{(2.13)}{\leq} \sum_{(k,j)\in F}\tilde{\mu}(E_j^k)^{1-q}\left|\sum_{y\in S(r_j^k)\cap\Omega^c_{k+2}} g(y)\sum_{x\geq y}\chi_{E_j^k}(x)\mu(x)\right|^q \\
&\overset{(2.15)}{\leq} \beta^{1-q}\sum_{(k,j)\in F}\tilde{\mu}(S(r_j^k))^{1-q}\left|\sum_{y\in S(r_j^k)\cap\Omega^c_{k+2}} g(y)\sum_{x\geq y}\chi_{E_j^k}(x)\mu(x)\right|^q \\
&\overset{\text{Hölder}}{\leq} \beta^{1-q}\sum_{(k,j)\in F}\tilde{\mu}(S(r_j^k))^{1-q} \\
&\quad\cdot\left(\sum_{y\in S(r_j^k)\cap\Omega^c_{k+2}}\left|\sum_{x\geq y}\chi_{E_j^k}(y)\mu(y)\right|^{p'}\omega(y)^{1-p'}\right)^{\frac{q}{p'}}\left(\sum_{y\in S(r_j^k)\cap\Omega^c_{k+2}}|g(y)|^p\,\omega(y)\right)^{\frac{q}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \beta^{1-q} \sum_{(k,j)\in F} \left(\sum_{x\in S(r_j^k)} \mu(x) \right)^{1-q} \\
&\quad \cdot \left(\sum_{x\in S(r_j^k)} \left(\sum_{y\in S(x)} \mu(y) \right)^{p'} \omega(y)^{1-p'} \right)^{\frac{q'(q-1)}{p'}} \left(\sum_{y\in S(r_j^k)\cap\Omega_{k+2}^c} |g(y)|^p \, \omega(y) \right)^{\frac{q}{p}} \\
&\overset{(2.10)}{\leq} \beta^{1-q} C(\mu,\, \omega)^{q-1} \sum_{(k,j)} \left(\sum_{y\in S(r_j^k)\cap\Omega_{k+2}^c} |g(y)|^p \, \omega(y) \right)^{\frac{q}{p}} \\
&\overset{q\geq p}{\leq} \beta^{1-q} C(\mu,\, \omega)^{q-1} \left(\sum_{(k,j)} \sum_{y\in S(r_j^k)\cap\Omega_{k+2}^c} |g(y)|^p \, \omega(y) \right)^{\frac{q}{p}} \\
&= \beta^{1-q} C(\mu,\, \omega)^{q-1} \left(\sum_{k\in\mathbb{Z}} \sum_{y\in \Omega_k\cap\Omega_{k+2}^c} |g(y)|^p \, \omega(y) \right)^{\frac{q}{p}} \\
&= \beta^{1-q} C(\mu,\, \omega)^{q-1} \left(\sum_{k\in\mathbb{Z}} \sum_{y\in \Omega_k\setminus\Omega_{k+2}} |g(y)|^p \, \omega(y) \right)^{\frac{q}{p}} \\
&\leq 2^{q/p} \beta^{1-q} C(\mu,\, \omega)^{q-1} \left(\sum_{x\in T} |g(x)|^p \, \omega(y) \right)^{\frac{q}{p}} \\
&\overset{\text{def}^{\text{n}}}{=} 2^{q/p} \beta^{1-q} C(\mu,\, \omega)^{q-1} \|g\|^q_{L^p(T,\tilde{\omega})}.
\end{aligned}
\tag{2.19}
$$

Therefore we can conclude that

$$
\|\mathfrak{I}g\|^q_{L^q(T,\mu)} \overset{(2.18),\,(2.19)}{\leq} \frac{2^{2q+2}}{1-2^{-q}} \beta \|\mathfrak{I}g\|^q_{L^q(\mu)} + 2^{q/p} \beta^{1-q} C^{q-1} \|g\|^q_{L^p(T(\zeta))},
$$

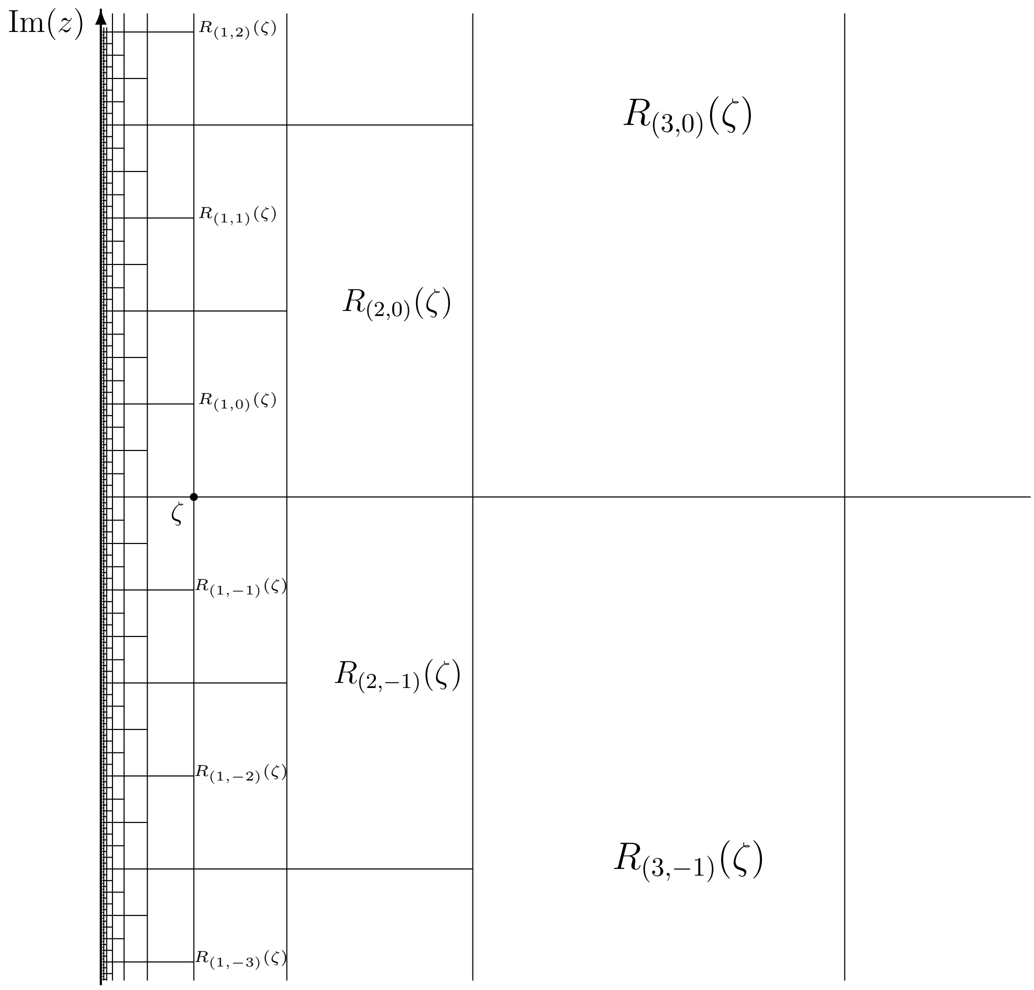
and since

$$
\beta < \frac{1-2^{-q}}{2^{2q+2}},
$$

we get the desired result. $\square$

Consider the following (Whitney) decomposition of the open right complex half-plane. Given $\zeta \in \mathbb{C}_+$, for any $(k, l) \in \mathbb{Z}^2$ let

$$R_{(k,l)}(\zeta) := \left\{ z \in \mathbb{C}_+ \ : \ 2^{k-1} < \frac{\mathrm{Re}(z)}{\mathrm{Re}(\zeta)} \leq 2^k,\ 2^k l \leq \frac{\mathrm{Im}(z) - \mathrm{Im}(\zeta)}{\mathrm{Re}\,\zeta} < 2^k(l+1) \right\}.$$



We can view each element of the set of rectangles $\{R_{(k,l)}(\zeta) : (k,\, l) \in \mathbb{Z}^2\}$ as a vertex of an abstract graph. If we have that $x,\, y \in \{R_{(k,l)}(\zeta) : (k,\, l) \in \mathbb{Z}^2\}$ and $\overline{x} \cap \overline{y}$ is a vertical segment in $\mathbb{C}_+$, then we can say there is an edge between x and y. With this convention, these vertices and edges form an abstract tree, which we shall denote by $T(\zeta)$. Let $A(\cdot)$ be a positive function on the set leaves of $T(\zeta)$ assigning to each of them the area of the corresponding rectangle from $\{R_{(k,l)}(\zeta) : (k,\, l) \in \mathbb{Z}^2\}$. We can define a partial order on $T(\zeta)$ by considering the unique path between each pair $x,\, y \in T(\zeta)$; if for each leaf c lying on this path, $A(x) \geq A(c) \geq A(y)$, then $x \leq y$.

Definition 2.1.6 *A positive weight* $\omega : \mathbb{C}_+ \longrightarrow (0,\, \infty)$ *is called* regular *if for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $\omega(z_1) \leq \delta\omega(z_2)$*, whenever* z_1 *and* z_2 *are within (Poincaré) hyperbolic right half-plane distance* ε*, i.e. when*

$$d_H(z_1,\, z_2) \stackrel{def^n}{=} \cosh^{-1}\left(1 + \frac{(\mathrm{Re}(z_1) - \mathrm{Re}(z_2))^2 + (\mathrm{Im}(z_1) - \mathrm{Im}(z_2))^2}{2\,\mathrm{Re}(z_1)\,\mathrm{Re}(z_2)}\right) \leq \varepsilon.$$

Lemma 2.1.7 (Lemma 4, p. 492 in [64]) *Let* $\omega : \mathbb{C}_+ \longrightarrow (0,\, \infty)$ *be regular, let* μ *be a positive Borel measure on* $\mathbb{C}_+$*. If there exists a constant* $C(\mu, \omega) > 0$*, such that for all* $a \in \mathbb{C}_+$ *we have*

$$\left(\int_{Q(a)} \frac{\mu(Q(a) \cap Q(z))^{p'}}{(\mathrm{Re}(z))^2}\omega(z)^{1-p'}\, dz\right)^{\frac{q'}{p'}} \leq C(\mu,\, \omega)\mu(Q(a)), \tag{2.20}$$

then there exists a constant $C'(\mu,\, \omega) > 0$ *such that*

$$\left(\sum_{\beta \geq \alpha}\left(\sum_{\gamma \geq \beta}\mu(\gamma)\right)^{p'} \tilde{\omega}(\beta)^{1-p'}\right)^{\frac{q'}{p'}} \leq C'(\mu,\, \omega)\sum_{\beta \geq \alpha}\mu(\beta),$$

for all $\alpha \in T(\zeta)$*. Here* $\tilde{\omega}(\beta)$ *is defined to be* $\omega(z_\beta)$*, for some fixed* $z_\beta \in \beta \subset \mathbb{C}_+$*, for each* $\beta \in T(\zeta)$*.*

Proof

Let $a \in \mathbb{C}_+$. We can choose $\zeta \in \mathbb{C}_+$ such that there exists $\alpha \in T(\zeta)$ for which we have

$$Q(a) = \bigcup_{\beta \geq \alpha} \beta \qquad \text{and} \qquad \mu(Q(a)) = \sum_{\beta \geq \alpha}\mu(\beta). \tag{2.21}$$

Given $\beta \geq \alpha$, let $(k,\, l) \in \mathbb{Z}^2$ be such that $\beta = R_{(k,\, l)}(\zeta)$ and let

$$U(\beta) := \left\{z \in \mathbb{C}_+ \,:\, 2^{k-1} < \frac{\mathrm{Re}(z)}{\mathrm{Re}(\zeta)} \leq 2^k,\, \left|\mathrm{Im}(z) - \mathrm{Im}(\zeta) - 2^k\left(l + \frac{1}{2}\right)\mathrm{Re}(\zeta)\right| < \mathrm{Re}(z) - 2^{k-1}\right\}.$$

Now

$$\bigcup_{\gamma \geq \beta} \gamma \subseteq Q(z),$$

whenever $z \in U(\beta) \subset \beta \geq \alpha$, and also

$$\bigcup_{\gamma \geq \beta} \gamma \subseteq Q(a) \cap Q(z). \tag{2.22}$$

We also have that for any z_1 and z_2 in β

$$d_H(z_1,\, z_2) \leq \cosh^{-1}\left(1 + \frac{2^{2k-2} + 2^{2k}}{2^{2k-2}}\right) = \cosh^{-1}\left(\frac{7}{2}\right),$$

which does not depend on the choice of $\beta \in T(\zeta)$, so there exists $\delta > 0$ such that

$$\begin{aligned}
C(\mu,\, \omega) \sum_{\beta \geq \alpha} \mu(\beta) &\overset{(2.21)}{=} C(\mu,\, \omega)\mu(Q(a)) \\
&\overset{(2.20)}{\geq} \left(\int_{Q(a)} \frac{(\mu(Q(a)) \cap Q(z))^{p'}}{(\mathrm{Re}(z))^2} \omega(z)^{1-p'}\, dz\right)^{\frac{q'}{p'}} \\
&\overset{(2.21)}{\geq} \delta^{q'/p} \left(\sum_{\beta \geq \alpha} \omega(\beta)^{1-p'} \int_{\beta} \frac{(\mu(Q(a) \cap Q(z)))^{p'}}{(\mathrm{Re}(z))^2}\, dz\right)^{\frac{q'}{p'}} \\
&\geq \delta^{q'/p} \left(\sum_{\beta \geq \alpha} \omega(\beta)^{1-p'} \int_{U(\beta)} \frac{(\mu(Q(a) \cap Q(z)))^{p'}}{(\mathrm{Re}(z))^2}\, dz\right)^{\frac{q'}{p'}} \\
&\overset{(2.22)}{\geq} \delta^{q'/p} \left(\sum_{\beta \geq \alpha} \omega(\beta)^{1-p'} (\mu(\bigcup_{\gamma \geq \beta} \gamma))^{p'}\right)^{\frac{q'}{p'}} \\
&= \delta^{q'/p} \left(\sum_{\beta \geq \alpha} \left(\sum_{\gamma \geq \beta} \mu(\gamma)\right)^{p'} \omega(\beta)^{1-p'}\right)^{\frac{q'}{p'}},
\end{aligned}$$

as required. □

Theorem 2.1.8 *Let $1 < p \leq q < \infty$ and let μ be a positive Borel measure on $\mathbb{C}_+$. If ω is a regular weight such that*

$$\int_{\mathbb{C}_+} |F'(z)|^p (\mathrm{Re}(z))^{p-2} \omega(z)\, dz \leq \|F\|^p_{A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^m)}, \tag{2.23}$$

for all $F \in A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^m)$ *and there exists a constant* $C(\mu,\, \omega) > 0$ *such that*

$$\left(\int_{Q(a)} \frac{\mu(Q(a) \cap Q(z))^{p'}}{(\mathrm{Re}(z))^2} \omega(z)^{1-p'}\, dz \right)^{\frac{q'}{p'}} \leq C(\mu,\, \omega)\mu(Q(a)) \qquad (\forall a \in \mathbb{C}_+), \quad (2.24)$$

then the embedding

$$A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^m) \hookrightarrow L^q(\mathbb{C}_+,\, \mu)$$

is bounded.

Proof

Given $F \in A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^m)$, for each $\alpha \in T(\zeta)$ let $\mathrm{diam}(\alpha)$ denote the diameter of α (as a rectangle in $\mathbb{C}_+$), and let $w_\alpha,\, z_\alpha \in \overline{\alpha} \subset \mathbb{C}_+$ be such that

$$z_\alpha := \sup_{z \in \alpha}\{|F(z)|\} \qquad \text{and} \qquad w_\alpha := \sup_{w \in \alpha}\{|F'(w)|\}.$$

Define weights $\tilde{\mu}$ and $\tilde{\omega}$ on $T(\zeta)$ by

$$\tilde{\mu}(\alpha) := \mu(\alpha) \qquad \text{and} \qquad \tilde{\omega}(\alpha) := \omega(z_\alpha),$$

for all $\alpha \in T(\zeta)$. Let also

$$\begin{aligned} r_\alpha &:= \mathrm{Re}(w_\alpha)/4, \\ \Phi(\alpha) &:= F(z_\alpha), \\ \varphi(\alpha) &:= \Phi(\alpha) - \Phi(\alpha^-), \end{aligned}$$

for all $\alpha \in T(\zeta)$. Denote the open unit ball with radius r_α and centred at w_α by $B_{r_\alpha}(w_\alpha)$. Note that $\mathfrak{I}\varphi = \Phi$. This is because if F is in $A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^m)$, then it is in the Zen space $A^p_{\nu_0}$, and hence in the Hardy space $H^p(\mathbb{C}_+)$ (or its shifted version, see [87], p. 61), and hence

$$\lim_{\alpha \to -\infty} |F(z_\alpha)| = \lim_{\mathrm{Re}(z) \to \infty} |F(z)| = 0.$$

Since (2.24) holds, we can apply Lemmata 2.1.5 and 2.1.7 to φ, μ and ω, and by the Fundamental Theorem of Calculus, Mean-Value Property (Theorem 1.6 in [10], p. 6) and Hölder's inequality we get

$$
\begin{aligned}
\int_{\mathbb{C}_+} |F|^q \, d\mu &= \sum_{\alpha \in T(\zeta)} \int_\alpha |F|^q \, d\mu \\
&\leq \sum_{\alpha \in T(\zeta)} |\Phi(\alpha)|^q \tilde{\mu}(\alpha) \\
&\lessapprox \left(\sum_{\alpha \in T(\zeta)} |\varphi(\alpha)|^p \tilde{\omega}(\alpha) \right)^{\frac{q}{p}} \\
&\overset{\text{def}^{\text{n}}}{=} \left(\sum_{\alpha \in T(\zeta)} |\Phi(\alpha) - \Phi(\alpha^-)|^p \tilde{\omega}(\alpha) \right)^{\frac{q}{p}} \\
&\overset{\substack{\text{Fundamental Thm} \\ \text{of Calculus}}}{\leq} \left(\sum_{\alpha \in T(\zeta)} \left| \int_{z_{\alpha^-}}^{z_\alpha} F'(w) \, dw \right|^p \tilde{\omega}(\alpha) \right)^{\frac{q}{p}} \\
&\lessapprox \left(\sum_{\alpha \in T(\zeta)} \text{diam}(\alpha)^p (|F'(w_\alpha)| + |F'(w_{\alpha^-})|)^p \tilde{\omega}(\alpha) \right)^{\frac{q}{p}} \\
&\lessapprox \left(\sum_{\alpha \in T(\zeta)} \text{diam}(\alpha)^p |F'(w_\alpha)|^p \tilde{\omega}(\alpha) \right)^{\frac{q}{p}} \\
&\overset{\substack{\text{Mean-Value} \\ \text{Property}}}{\leq} \left(\sum_{\alpha \in T(\zeta)} \text{diam}(\alpha)^p \left| \frac{1}{\pi r_\alpha^2} \int_{B_{r_\alpha}(w_\alpha)} F'(z) \, dz \right|^p \tilde{\omega}(\alpha) \right)^{\frac{q}{p}} \\
&\overset{\text{Hölder}}{\leq} \left(\sum_{\alpha \in T(\zeta)} \frac{\text{diam}(\alpha)^p}{(\pi r_\alpha^2)^{p(1-1/p')}} \int_{B_{r_\alpha}(w_\alpha)} |F'(z)|^p \, dz \tilde{\omega}(\alpha) \right)^{\frac{q}{p}} \\
&\lessapprox \left(\sum_{\alpha \in T(\zeta)} \text{diam}(\alpha)^{p-2} \int_{\bigcup_{\beta \in T(\zeta) \,:\, \beta \cap B_{r_\alpha}(w_\alpha) \neq \varnothing}} |F'(z)|^p \, dz \, \tilde{\omega}(\alpha) \right)^{\frac{q}{p}} \\
&\lessapprox \left(\sum_{\alpha \in T(\zeta)} \int_{\bigcup_{\beta \in T(\zeta) \,:\, \beta \cap B_{r_\alpha}(w_\alpha) \neq \varnothing}} |F'(z)|^p \frac{\omega(z)}{(\text{Re}(z))^{2-p}} \, dz \right)^{\frac{q}{p}} \\
&\lessapprox \left(\sum_{\alpha \in T(\zeta)} \int_\alpha |F'(z)|^p \frac{\omega(z)}{(\text{Re}(z))^{2-p}} \, dz \right)^{\frac{q}{p}} \\
&\leq \|F\|^q_{A^p(\mathbb{C}_+, (\nu_n)_{n=0}^m)},
\end{aligned}
$$

as required. □

2.2 Carleson measures for Hilbert spaces

Let us now look at the Carleson measures for the Hilbert spaces that we have discussed in Chapter 1. That is, we assume that $p = q = 2$ throughout this section.

2.2.1 Kernel criteria

The Hilbert spaces introduced in the previous chapter are all reproducing kernel Hilbert spaces. This gives us the advantage of being able to test the Carleson criterion on a set of functions which has particularly nice properties. In fact, it is possible to give a simple and complete characterisation of Carleson measures for any RKHS of L^2 functions, using just the reproducing kernels. The following result has been obtained by Nicola Arcozzi, Richard Rochberg and Eric Sawyer in [6].

Lemma 2.2.1 (Lemma 24, p. 1145 in [6]) *Let $\mathcal{H}$ be a reproducing kernel Hilbert space of functions on X, with reproducing kernels $\{k_x^{\mathcal{H}}\}_{x\in X}$. A positive Borel measure μ is a Carleson measure for $\mathcal{H}$ if and only if the linear map*

$$f(\cdot) \longmapsto \int_X \operatorname{Re}\left(k_x^{\mathcal{H}}(\cdot)\right) f(x)\, d\mu(x)$$

is bounded on $L^2(X, \mu)$.

In the definition of the $A^2_{(m)}$ spaces we require that both a function and its derivative(s) lie in some specific space(s). This is clearly a generalisation of the idea behind the definition of the Dirichlet space (on the disk or otherwise), therefore we can use the techniques which were successful for the Dirichlet space case in order to examine the properties of $A^2_{(m)}$ spaces and related structures. For example, the following two results, which appeared in [64], are adaptations of Theorem 5.2.2 and Theorem 5.2.3 from [35] (pp. 76-77).

Lemma 2.2.2 (Lemma 1, p. 479 in [64]) *Let μ be a positive Borel measure on $\mathbb{C}_+$, then*

$$\sup_{\|F\|_{A^2_{(m)}}\leq 1} \int_{\mathbb{C}_+} |F(z)|^2 \, d\mu(z) = \sup_{\|G\|_{L^2(\mathbb{C}_+,\,\mu)}\leq 1} \left| \int_{\mathbb{C}_+} \int_{\mathbb{C}_+} k_z^{A^2_{(m)}}(\zeta)\, G(\zeta)\overline{G(z)}\, d\mu(\zeta)\, d\mu(z) \right|. \tag{2.25}$$

Proof

First, let us assume that the LHS of (2.25) is finite (i.e. μ is a Carleson measure for $A^2_{(m)}$). Let

$$\iota : A^2_{(m)} \longrightarrow L^2(\mathbb{C}_+,\,\mu) \qquad \text{and} \qquad \iota^* : L^2(\mathbb{C}_+,\,\mu) \longrightarrow A^2_{(m)}$$

denote respectively the inclusion map and its adjoint. The LHS of (2.25) is evidently equal to the square of the norm of ι. Furthermore, for each $g \in L^2(\mathbb{C}_+,\,\mu)$, we have

$$\iota^* G(z) = \left\langle \iota^* G,\, k_z^{A^2_{(m)}} \right\rangle_{A^2_{(m)}} = \left\langle G,\, k_\zeta^{A^2_{(m)}} \right\rangle_{L^2(\mathbb{C}_+,\,\mu)} \overset{\text{def}^{\underline{n}}}{=} \int_{\mathbb{C}_+} G(\zeta) k_z^{A^2_{(m)}}(\zeta)\, d\mu(\zeta), \tag{2.26}$$

and thus

$$\begin{aligned} \|\iota^* G\|^2_{A^2_{(m)}} &\overset{\text{def}^{\underline{n}}}{=} \langle \iota^* G,\, \iota^* G \rangle_{A^2_{(m)}} \\ &\overset{\text{def}^{\underline{n}}}{=} \langle \iota\iota^* G,\, G \rangle_{A^2_{(m)}} \\ &\overset{(2.26)}{=} \int_{\mathbb{C}_+} \int_{\mathbb{C}_+} k_z^{A^2_{(m)}}(\zeta)\, G(\zeta)\overline{G(z)}\, d\mu(\zeta)\, d\mu(z), \end{aligned}$$

so the expression (2.25) is equivalent to $\|\iota\|^2 = \|\iota^*\|^2$, hence it must be true. Now, if the LHS of (2.25) it is not finite, we let

$$\Omega_r := \left\{ x + iy \in \mathbb{C}_+ \;:\; \frac{1}{r} \leq x \leq r,\; |y| \leq r \right\} \subset \mathbb{C}_+ \qquad (r \geq 1).$$

Then, by the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\int_{\mathbb{C}_+} |F|^2 \, d\mu|_{\Omega_r} &\leq \mu(\Omega_r) \sup_{z \in \Omega_r} |F(z)|^2 \\ &= \mu(\Omega_r) \sup_{z \in \Omega_r} \left| \left\langle F, \, k_z^{A^2_{(m)}} \right\rangle_{A^2_{(m)}} \right|^2 \\ &\leq \mu(\Omega_r) \sup_{z \in \Omega_r} \left\| k_z^{A^2_{(m)}} \right\|^2 \|F\|^2_{A^2_{(m)}} \\ &\overset{(1.12)}{=} \mu(\Omega_r) \left\| k_{1/r}^{A^2_{(m)}} \right\|^2 \|F\|^2_{A^2_{(m)}} \, , \end{aligned} \tag{2.27}$$

for all F in $A^2_{(m)}$. This means that $\mu|_{\Omega_r}$ (i.e. the restriction of μ to Ω_r) is a Carleson measure for $A^2_{(m)}$, so we can use the first part of the proof (that is, we know that (2.25) holds for $\mu|_{\Omega}$), to get

$$\sup_{\|F\|_{A^2_{(m)}} \leq 1} \int_{\mathbb{C}_+} |F(z)|^2 \, d\mu|_{\Omega_r}(z) = \sup_{\|G\|_{L^2(\mathbb{C}_+, \, \mu)} \leq 1} \left| \int_{\mathbb{C}_+} \int_{\mathbb{C}_+} k_z^{A^2_{(m)}}(\zeta) \, G(\zeta) \overline{G(z)} \, d\mu|_{\Omega_r}(z) \, d\mu|_{\Omega_r}(\zeta) \right| ,$$

where the RHS is at most equal to the RHS of (2.25) and the LHS tends to infinity as r approaches infinity, so the RHS of (2.25) must not be finite. □

Proposition 2.2.3 (Proposition 1, p. 480 in [64]) *If*

$$\sup_{z \in \mathbb{C}_+} \int_{\mathbb{C}_+} \left| k_z^{A^2_{(m)}}(\zeta) \right| \, d\mu(\zeta) < \infty, \tag{2.28}$$

then μ is a Carleson measure for $A^2_{(m)}$.

Proof

Let

$$M := \sup_{z \in \mathbb{C}_+} \int_{\mathbb{C}_+} \left| k_z^{A^2_{(m)}}(\zeta) \right| \, d\mu(\zeta). \tag{2.29}$$

Then for all $G \in L^2(\mathbb{C}_+,\, \mu)$

$$\begin{aligned}\left| \int_{\mathbb{C}_+} \int_{\mathbb{C}_+} k_z^{A^2_{(m)}}(\zeta)\, G(\zeta)\overline{G(z)}\, d\mu(z)\, d\mu(\zeta) \right| &\overset{\text{Schwarz}}{\leq} \left(\int_{\mathbb{C}_+} \int_{\mathbb{C}_+} \left| k_\zeta^{A^2_{(m)}}(z) \right| |G(\zeta)|^2\, d\mu(z)\, d\mu(\zeta) \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{\mathbb{C}_+} \int_{\mathbb{C}_+} \left| k_z^{A^2_{(m)}}(\zeta) \right| |G(z)|^2\, d\mu(z)\, d\mu(\zeta) \right)^{\frac{1}{2}} \\ &\overset{(2.29)}{\leq} M \|G\|^2_{L^2(\mathbb{C}_+,\, \mu)}. \end{aligned} \tag{2.30}$$

Therefore

$$\begin{aligned}\int_{\mathbb{C}_+} \left(\frac{|H(z)|}{\|H\|_{A^2_{(m)}}} \right)^2 d\mu(z) &\leq \sup_{\|F\|_{A^2_{(m)}} \leq 1} \int_{\mathbb{C}_+} |F(z)|^2\, d\mu(z) \\ &\overset{(2.25)}{=} \sup_{\|G\|_{L^2(\mathbb{C}_+,\, \mu)} \leq 1} \left| \int_{\mathbb{C}_+} \int_{\mathbb{C}_+} k_z^{A^2_{(m)}}(\zeta)\, G(\zeta)\overline{G(z)}\, d\mu(z)\, d\mu(\zeta) \right| \\ &\overset{(2.30)}{\leq} M, \end{aligned}$$

for all $H \in A^2_{(m)}$, as required. □

Remark 2.2.4 *It is an elementary observation that these two results can be applied to more general RKHS of functions on some set* Ω*, provided that* Ω *can be written as a union of sets*

$$\Omega = \bigcup_{r \in E} \Omega_r, \qquad \Omega_r \subseteq \Omega r' \qquad (\forall r \leq r';\ r,\, r' \in E \subseteq \mathbb{R}),$$

on which the reproducing kernels are uniformly bounded, so that estimates similar to those in (2.27) *can be obtained.*

2.2.2 Carleson measures for Dirichlet spaces

We are now going to turn our attention to the Dirichlet spaces. The study of the Dirichlet space on the unit disk of the complex plane can be dated back to at least Arne Beurling's doctoral thesis from 1933 (see [12]), and many of the methods developed for this type of space can be successful employed to spaces like $A^2_{(m)}$. Nevertheless there are limitations. For example, David Stegenga's elegant characterisation of Carleson measures for $\mathcal{D}$ in terms of so-called logarithmic capacity (see [94]) relies on the fact that $\mathbb{D}$ is a bounded domain and that $\mathcal{D}$ can be equipped with two equivalent norms, which on the half-plane, as we have seen in Chapter 1, give rise to two distinct spaces of functions. Instead, we can use the results presented above to describe the Carleson measures for the Dirichlet spaces on the complex half-plane, although we should bear in mind that they can only provide a partial characterisation of Carleson measures in this instance.

We can immediately state the following.

Proposition 2.2.5 *Let μ be a positive Borel measure on $\mathbb{C}_+$.*

1. *If for each $a \in \mathbb{C}_+$*
$$\mu(Q(a)) = O(\mathrm{Re}(a)), \tag{2.31}$$
then μ is a Carleson measure for $\mathcal{D}_\alpha(\mathbb{C}_+)$.

2. *If μ is a Carleson measure for $\mathcal{D}_\alpha(\mathbb{C}_+)$, then*
$$\mu(Q(a)) = O(\mathrm{Re}(a) + \mathrm{Re}(a)^{\alpha+2}),$$
for all $a \in \mathbb{C}_+$.

Proof

If (2.31) holds for every $a \in \mathbb{C}_+$, then, by Remark 2.1.3, μ is a Carleson measure for $H^2(\mathbb{C}_+)$, so it must also be a Carleson measure for $\mathcal{D}_\alpha(\mathbb{C}_+)$. Part 2. follows from Theorem 2.1.2. □

Observe that, if $f \in L^1(0,\infty) \cap L^2_{t^{1-\alpha}}(0,\infty)$, then

$$\begin{aligned}\|\mathfrak{L}[f]\|^2_{\mathcal{D}'_\alpha(\mathbb{C}_+)} &= \left|\int_0^\infty f(t)e^{-t}\,dt\right|^2 + \int_0^\infty |f(t)|^2 t^{1-\alpha}\,dt \\ &\overset{\text{Schwarz}}{\leq} \int_0^\infty |f(t)|^2(1+t^{1-\alpha}) \\ &\approx \|\mathfrak{L}[f]\|^2_{\mathcal{D}_\alpha(\mathbb{C}_+)},\end{aligned}$$

so $CM(\mathcal{D}'_\alpha(\mathbb{C}_+)) \subseteq CM(\mathcal{D}_\alpha(\mathbb{C}_+))$, for all $\alpha \geq -1$. This inclusion is proper, since whenever μ is a Carleson measure for $\mathcal{D}'_\alpha(\mathbb{C}_+)$, we must have

$$\mu(\Omega) \leq \int_{\mathbb{C}_+} |1|^2\,d\mu \leq C(\mu)\|1\|^2_{\mathcal{D}'_\alpha(\mathbb{C}_+)} = C(\mu),$$

for every $\Omega \subset \mathbb{C}_+$ and some $C(\mu) > 0$, not depending on Ω. In other words, μ is a bounded measure on $\mathbb{C}_+$, whereas $\delta_0 \otimes \lambda$ is clearly an unbounded measure on $\mathbb{C}_+$, which belongs to $CM(\mathcal{D}_\alpha(\mathbb{C}_+))$.

Theorem 2.2.6 *Let μ be a positive Borel measure on $\mathbb{C}_+$.*

1. *The measure μ is a Carleson measure for $\mathcal{D}'_{-1}(\mathbb{C}_+)$ if and only if there exists a constant $A(\mu) > 0$ such that*

$$\begin{aligned}\sup_{x>0}\int_{-\infty}^\infty &\left|\int_{\mathbb{C}_+} G(\zeta)\left[\ln(x+iy+\overline{\zeta}) - \ln(x+iy+1)\right]\,d\mu(\zeta)\right|^2 dy \\ &\leq A(\mu)\int_{\mathbb{C}_+}|G(\zeta)|^2\,d\mu(\zeta) - \left|\int_{\mathbb{C}_+} G(\zeta)\,d\mu(\zeta)\right|^2,\end{aligned}$$

for all $G \in L^2(\mathbb{C}_+,\mu)$.

2. *The measure μ is a Carleson measure for $\mathcal{D}'_\alpha(\mathbb{C}_+)$, $\alpha > -1$ if and only if there exists a constant $B(\mu) > 0$ such that*

$$\begin{aligned}\int_{\mathbb{C}_+}&\left|\int_{\mathbb{C}_+} G(\zeta)\left[(z+\overline{\zeta})^{-\alpha-1} - (z+1)^{-\alpha-1}\right]\,d\mu(\zeta)\right|^2 \operatorname{Re}(z)^\alpha\,dz \\ &\leq B(\mu)\int_{\mathbb{C}_+}|G(\zeta)|^2\,d\mu(\zeta) - \left|\int_{\mathbb{C}_+} G(\zeta)\,d\mu(\zeta)\right|^2,\end{aligned}$$

for all $G \in L^2(\mathbb{C}_+,\mu)$.

3. The measure μ is a Carleson measure for $\mathcal{D}(\mathbb{C}_+)$ if and only if there exists a constant $C(\mu) > 0$ such that

$$\int_{\mathbb{C}_+} \left| \int_{\mathbb{C}_+} \frac{G(\zeta)}{z+\overline{\zeta}}\, d\mu(\zeta) \right|^2 \frac{dz}{e^{2\operatorname{Re}(z)}} \leq C(\mu) \int_{\mathbb{C}_+} |G|^2\, d\mu,$$

for all $G \in L^2(\mathbb{C}_+,\, \mu)$.

Proof

To prove part 1., note that μ is a Carleson measure for $\mathcal{D}'_{-1}(\mathbb{C}_+)$ if and only if the adjoint of the inclusion map $\iota^* : L^2(\mathbb{C}_+,\, \mu) \hookrightarrow \mathcal{D}'_{-1}(\mathbb{C}_+)$ is bounded, that is, there exists $A(\mu) > 0$ such that

$$\|\iota^* G\|^2_{\mathcal{D}'_\alpha(\mathbb{C}_+)} \leq A(\mu) \|G\|^2_{L^2(\mathbb{C}_+,\, \mu)}, \tag{2.32}$$

for all $G \in L^2(\mathbb{C}_+,\, \mu)$. Also

$$\iota^* G(z) \overset{\text{def}^{\text{n}}}{=} \left\langle \iota^* G,\, k_z^{\mathcal{D}'_{-1}(\mathbb{C}_+)} \right\rangle_{\mathcal{D}'_{-1}(\mathbb{C}_+)} \overset{\text{def}^{\text{n}}}{=} \left\langle G,\, k_z^{\mathcal{D}'_{-1}(\mathbb{C}_+)} \right\rangle_{L^2(\mathbb{C}_+,\, \mu)} \tag{2.33}$$

for all $z \in \mathbb{C}_+$ and $G \in L^2(\mathbb{C}_+,\, \mu)$. And so

$$\begin{aligned} A(\mu)\|G\|^2_{L^2(\mathbb{C}_+,\, \mu)} &\overset{(2.32)}{\geq} |\iota^* G(1)|^2 + \|(\iota^* G)'\|^2_{H^2(\mathbb{C}_+)} \\ &\overset{(2.33)}{=} \left| \int_{\mathbb{C}_+} G(\zeta)\, d\mu(z) \right|^2 \\ &\quad + \sup_{x>0} \int_{-\infty}^{\infty} \left| \int_{\mathbb{C}_+} G(\zeta) \left[\ln(x+iy+\overline{\zeta}) - \ln(x+iy+1) \right] d\mu(\zeta) \right|^2 \frac{dy}{\pi} \end{aligned}$$

as required. The proof of part 2. is analogous. And similarly, for part 3, we have

$$\begin{aligned} B(\mu)\|G\|^2_{L^2(\mathbb{C}_+,\, \mu)} &\geq \left\| \left\langle G,\, k_\cdot^{\mathcal{D}(\mathbb{C}_+)} \right\rangle_{L^2(\mathbb{C}_+,\, \mu)} \right\|^2_{\mathcal{D}(\mathbb{C}_+)} \\ &= \int_0^\infty \left| \mathfrak{L}^{-1} \left[\int_{\mathbb{C}_+} G(\zeta) \int_0^\infty \frac{e^{-\tau(z+\overline{\zeta})}}{1+\tau}\, d\tau\, d\mu(\zeta) \right](t) \right|^2 (1+t)\, dt \\ &= \int_0^\infty \left| \int_{\mathbb{C}_+} G(\zeta) e^{-t\overline{\zeta}}\, d\mu(\zeta) \right|^2 \frac{dt}{1+t}. \end{aligned}$$

Now,

$$\frac{1}{1+t} = 2\int_0^\infty e^{-2x(t+1)}\, dx = 2\pi \int_0^\infty e^{-2xt} \frac{dx}{\pi e^{2x}} \qquad (\forall t > 0),$$

and the measure $\tilde{\nu}$ on $\mathbb{C}_+$, given by $d\tilde{\nu}(x) = e^{-2x}dx/\pi$ satisfies the (Δ_2)-condition:

$$\sup_{x>0} \frac{\tilde{\nu}[0,\, 2x)}{\tilde{\nu}[0,\, x)} = \sup_{x>0} \frac{\int_0^{2x} e^{-2r}\, dr}{\int_0^{x} e^{-2r}\, dr} = \sup_{x>0} \frac{1-e^{-4x}}{1-e^{-2x}} = \sup_{x>0}(1+e^{-2x}) = 2,$$

hence by Theorem 1.1.3

$$\begin{aligned} D\|G\|^2_{L^2(\mathbb{C}_+,\,\mu)} &\geq \int_{\mathbb{C}_+} \left| \int_0^\infty \int_{\mathbb{C}_+} G(\zeta) e^{-t\overline{\zeta}}\, d\mu(\zeta) e^{-tz}\, dt \right|^2 \frac{dz}{\pi e^{2\,\mathrm{Re}(z)}} \\ &= \int_{\mathbb{C}_+} \left| \int_{\mathbb{C}_+} \frac{G(\zeta)}{z+\overline{\zeta}}\, d\mu(\zeta) \right|^2 \frac{dz}{\pi e^{2\,\mathrm{Re}(z)}}. \end{aligned}$$

□

Theorem 2.2.7 *Let μ be a positive Borel measure on $\mathbb{C}_+$. If there exists a constant $C(\mu) > 0$ such that for all $a \in \mathbb{C}_+$ we have*

$$\int_{\mathbb{C}_+} \left(\frac{\mu(Q(a) \cap Q(z))}{\mathrm{Re}(z)} \right)^2 dz \leq C(\mu)\mu(Q(a)),$$

then μ is a Carleson measure for $\mathcal{D}(\mathbb{C}_+)$. Conversely, if μ is a Carleson measure for $\mathcal{D}(\mathbb{C}_+)$, then there exists a constant $C'(\mu) > 0$ such that for all $a \in \mathbb{C}_+$ we have

$$\int_{\mathbb{C}_+} \left(\frac{\mu(Q(a) \cap Q(z))}{e^{\mathrm{Re}(z)}\,\mathrm{Re}(z)} \right)^2 dz \leq C'(\mu)\mu(Q(a)).$$

Proof

The first part is essentially Theorem 2.1.8 applied with $p = q = 2$ and $\omega \equiv 1$. The second part follows from the previous theorem applied to $G = \chi_{Q(a)}$. In this case we get

$$\mu(Q(a)) \gtrapprox \int_{\mathbb{C}_+} \left| \int_{Q(a)} \frac{d\mu(\zeta)}{z+\overline{\zeta}} \right|^2 \frac{dz}{e^{2\,\mathrm{Re}(z)}}.$$

Now

$$\mathrm{Re}\left(\frac{1}{z+\overline{\zeta}} \right) = \frac{\mathrm{Re}(z)+\mathrm{Re}(\zeta)}{|z+\overline{\zeta}|^2} \geq 0, \tag{2.34}$$

so for any $z \in \mathbb{C}_+$,

$$\begin{aligned}\left|\int_{Q(a)} \frac{d\mu(\zeta)}{z+\overline{\overline{\zeta}}}\right| &\geq \operatorname{Re}\left(\int_{Q(a)} \frac{d\mu(\zeta)}{z+\overline{\overline{\zeta}}}\right) \\ &\overset{(2.34)}{\geq} \int_{Q(a)\cap Q(z)} \frac{\operatorname{Re}(z)+\operatorname{Re}(\zeta)}{(\operatorname{Re}(z)+\operatorname{Re}(\zeta))^2+|\operatorname{Im}(z)-\operatorname{Im}(\zeta)|^2}\, d\mu(\zeta) \\ &\overset{(2.3)}{\geq} \int_{Q(a)\cap Q(z)} \frac{\operatorname{Re}(z)}{10(\operatorname{Re}(z))^2}\, d\mu(\zeta) \\ &= \frac{\mu(Q(a)\cap Q(z))}{10\operatorname{Re}(z)},\end{aligned}$$

and the result follows. □

Chapter 3

Weighted composition operators

Never compose anything unless the not composing of it becomes a positive nuisance to you.

GUSTAVUS THEODORE VON HOLST, cited in Imogen Clare Holst's
The Music of Gustav Holst

Let V be a vector space over a field $\mathbb{K}$ (real or complex), consisting of all $\mathbb{K}$-valued functions on a set Ω, let $U \subseteq V$ be another $\mathbb{K}$-linear space, let φ be a self map of Ω and let $h : \Omega \longrightarrow \mathbb{K}$. The *weighted composition operator*, corresponding to h and φ, is defined to be the linear map $W_{h,\varphi} \in \mathscr{L}(U, V)$ (i.e. the space of all linear maps from U to V), given by

$$u \longmapsto W_{h,\varphi}u := h \cdot (u \circ \varphi) \qquad (\forall u \in U).$$

If $\varphi = \mathrm{Id}_\Omega$ (the identity map on Ω), then we will write $M_h := W_{h,\mathrm{Id}_\Omega}$ and call it the *multiplication operator* on U corresponding to h. If $M_h(U) \subseteq U$, then we will call h a *multiplier* of U and define the *algebra of multipliers* of U to be

$$\mathscr{M}(U) := \{h : \Omega \longrightarrow \mathbb{K} \ : \ \forall u \in U \quad M_h(u) \in U\}$$

If U is a Banach space and $M_h \in \mathscr{B}(U)$ (i.e. the space of bounded linear operators on U), then we may equip the algebra of multipliers of U with the norm

$$\|h\|_{\mathscr{M}(U)} := \|M_h\|_{\mathscr{B}(U)} \overset{\text{def}^{\text{n}}}{=} \sup_{\|u\|_U \leq 1} \|hu\|_U \qquad (\forall h \in \mathscr{M}(U)).$$

If $h \equiv 1$, then we will write $C_\varphi := W_{1,\varphi}$ and call it the *composition operator* on U corresponding to φ. If U is a RKHS, then we immediately get the following well-known lemma.

Lemma 3.0.8 *Let $W_{h,\varphi}$ be a bounded weighted composition operator on a RKHS $\mathcal{H}$ of functions defined on a set Ω, with a reproducing kernel $k_x^{\mathcal{H}}$, for all $x \in \Omega$. Then*

$$W_{h,\varphi}^* k_x^{\mathcal{H}}(y) = \overline{h(x)} k_{\varphi(x)}^{\mathcal{H}}(y) \qquad (\forall x,\, y \in \Omega).$$

Proof

For all $u \in \mathcal{H}$ we have

$$\left\langle u,\, W_{h,\varphi}^* k_x^{\mathcal{H}} \right\rangle_{\mathcal{H}} = \left\langle W_{h,\varphi} u,\, k_x^{\mathcal{H}} \right\rangle = h(x) u(\varphi(x)) = \left\langle u,\, \overline{h(x)} k_{\varphi(x)}^{\mathcal{H}} \right\rangle_{\mathcal{H}}.$$

□

In this chapter we are going to discuss multiplication operators, composition operators and weighted composition operators for Banach spaces of analytic functions on the open complex half-plane. It is self-evident that if $W_{h,\varphi}$ is a bounded operator on some of these spaces then h must be analytic and φ must also be analytic, given that h is not a zero function. The results presented below were published in [65] and [67].

3.1 Multiplication operators

3.1.1 Multipliers

The following lemma is standard and well-known for general reproducing kernel Hilbert spaces. The $A^2_{(m)}$ version appeared in [65] (Lemma 2).

Lemma 3.1.1 *If h is a multiplier of a RKHS $\mathcal{H}$ of functions defined on a set Ω, then h is bounded and*

$$\sup_{x\in\Omega} |h(x)| \leq \|h\|_{\mathscr{M}(\mathcal{H})} .$$

Proof

Let $h \in \mathscr{M}(\mathcal{H})$. Then M_h^* is a bounded operator on $\mathcal{H}$, so its eigenvalues are bounded, and of modulus no bigger than $\|M_h\|_{\mathscr{B}(\mathcal{H})}$. So, by Lemma 3.0.8, the values of h are bounded and of modulus no more than $\|h\|_{\mathscr{M}(\mathcal{H})}$. $\square$

In the remaining part of this section we assume that $\varphi = \mathrm{Id}_{\mathbb{C}_+}$ and that $h : \mathbb{C}_+ \longrightarrow \mathbb{C}$ is a holomorphic map.

Multipliers of the Hardy or weighted Bergman spaces on the disk are the bounded holomorphic functions on the open unit disk (see for example Proposition 1.13, p. 19 in [3]). Multipliers of the Dirichlet space on the disk can be characterised in terms of Carleson measures for $\mathcal{D}$ (see Theorem 5.1.7, p. 74 in [35]). It appears that the multipliers of the corresponding spaces on the half-plane can be classified in an analogous way. The relevant statement, concerning $A^2_{(m)}$ spaces, and its proof have been published in [65] (Theorem 2). Below we extend it to $A^p(\mathbb{C}_+, \, (\nu_n)_{n=0}^m)$ spaces.

Theorem 3.1.2

1. *$\mathscr{M}(A^2_{(0)}) = \mathscr{M}(A^p_\nu) = H^\infty(\mathbb{C}_+)$ (the Hardy space of analytic functions bounded on $\mathbb{C}_+$), and*

$$\|h\|_{\mathscr{M}(A^2_{(0)})} = \|h\|_{\mathscr{M}(A^2(\mathbb{C}_+,\,(\nu)^m_{n=0}))} = \|h\|_{H^\infty(\mathbb{C}_+)} \qquad (\forall h \in H^\infty(\mathbb{C}_+)).$$

2. *Suppose that $\tilde{\nu}_n(\{0\}) = 0$, for all $n \in \mathbb{N}_0$, $n \leq m$. If for all $0 \leq k \leq n \leq m$, $\mu_{n,\,k}$, given by $d\mu_{n,\,k}(z) := \left|h^{(k)}\right|^p d\nu_n$, is a Carleson measure for $A^p_{\nu_{n-k}}$, and*

$$\sup_{0\leq n\leq m} \sum_{k=n}^{m} \begin{pmatrix} k \\ k-n \end{pmatrix}^p (k+1)^p \|\mu_{k,\,k-n}\|_{A^p_{\nu_n}} < \infty, \tag{3.1}$$

then $h \in \mathscr{M}(A^p(\mathbb{C}_+,\,(\nu_n)^m_{n=0}))$.

3. *$h \in \mathscr{M}(A^2(\mathbb{C}_+,\,(\nu_n)^m_{n=0}))$ if and only if h is bounded on $\mathbb{C}_+$ and there exists a sequence $(c_n) \in \ell^1$ such that for all $F \in A^2(\mathbb{C}_+,\,(\nu_n)^m_{n=0})$ and all $1 \leq n \leq m$, we have*

$$\sup_{\varepsilon>0} \int_{\mathbb{C}_+} \left|\sum_{k=1}^{n} \binom{n}{k} h^{(k)}(z+\varepsilon) F^{(n-k)}(z+\varepsilon)\right|^2 d\nu_n \leq |c_n|\, \|F\|^2_{A^2(\mathbb{C}_+,\,(\nu_n)^m_{n=0})}\,. \tag{3.2}$$

In particular, if $m = 1$ and $\tilde{\nu}_1(\{0\}) = 0$, then $h \in \mathscr{M}(A^2(\mathbb{C}_+,\,(\nu_0,\,\nu_1)))$ if and only if $|h'(z)|^2\, d\nu_1$ is a Carleson measure for $A^2(\mathbb{C}_+,\,(\nu_0,\,\nu_1))$.

Proof

1. Suppose first that h is bounded on $\mathbb{C}_+$. Then, for all $F \in A^2_\nu$, we have

$$\sup_{\varepsilon>0} \int_{\mathbb{C}_+} |h(z+\varepsilon)F(z+\varepsilon)|^2\, d\nu \leq \|h\|^p_{H^\infty(\mathbb{C}_+)} \|F\|^p_{A^p_\nu},$$

so h is a multiplier of A^2_ν, and hence it is also a multiplier of $A^2_{(0)}$. The converse is proved in Lemma 3.1.1. The norm expression of h in $\mathscr{M}(A^2_\nu)$ or (equivalently) in $\mathscr{M}(A^2_{(m)})$ can be easily deduced.

2. Let $F \in A^p(\mathbb{C}_+,\, (\nu_n)_{n=0}^m)$. Then

$$
\begin{aligned}
\|M_h F\|^p_{A^p(\mathbb{C}_+,\,(\nu_n)_{n=0}^m)} &\overset{\text{def}^{\text{n}}}{=} \sum_{n=0}^m \int_{\mathbb{C}_+} \left|(hF)^{(n)}\right|^p \, d\nu_n \\
&\overset{\text{Leibniz rule}}{=} \sum_{n=0}^m \int_{\mathbb{C}_+} \left| \sum_{k=0}^n \binom{n}{k} h^{(k)} F^{(n-k)} \right|^p \, d\nu_n \\
&\leq \sum_{n=0}^m (n+1)^p \sum_{k=0}^n \binom{n}{k}^p \int_{\mathbb{C}_+} \left|F^{(n-k)}\right|^p \left|h^{(k)}\right|^p \, d\nu_n \\
&\leq \sum_{n=0}^m (n+1)^p \sum_{k=0}^n \binom{n}{k}^p \|\mu_{n,\,k}\|_{CM(A^p_{\nu_{n-k}})} \left\|F^{(n-k)}\right\|^p_{A^p_{\nu_{n-k}}} \\
&= \sum_{n=0}^m \left(\sum_{k=n}^m \binom{k}{k-n}^p (k+1)^p \|\mu_{k,\,k-n}\|_{A^p_{\nu_n}} \right) \left\|F^{(n)}\right\|^p_{A^p_{\nu_n}} \\
&\overset{(3.1)}{\lessapprox} \|F\|^p_{A^p(\mathbb{C}_+,\,(\nu_n)_{n=0}^m)}.
\end{aligned}
$$

3. Suppose that (3.2) holds for some h. Then

$$
\begin{aligned}
\|hF\|^2_{A^2(\mathbb{C}_+,\,(\nu_n)_{n=0}^m)} &\overset{\text{def}^{\text{n}}}{=} \sum_{n=0}^m \sup_{\varepsilon_n>0} \int_{\mathbb{C}_+} \left| \frac{d^n}{dz^n} \left(h(z+\varepsilon_n)F(z+\varepsilon_n)\right) \right|^2 \, d\nu_n(z) \\
&\leq \sum_{n=0}^m \sup_{\varepsilon_n>0} \int_{\mathbb{C}_+} \left| \sum_{k=0}^n \binom{n}{k} h^{(k)}(z+\varepsilon_n) F^{(n-k)}(z+\varepsilon_n) \right|^2 \, d\nu_n(z) \\
&\overset{(3.2)}{\leq} 2 \sum_{n=0}^m \left(\|h\|^2_{H^\infty(\mathbb{C}_+)} \int_{\overline{\mathbb{C}_+}} \left|F^{(n)}\right|^2 \, d\nu_n + |c_n| \, \|F\|^2_{A^2(\mathbb{C}_+,\,(\nu_n)_{n=0}^m)} \right) \\
&= 2 \left(\|h\|^2_{H^\infty(\mathbb{C}_+)} + \|(c_n)\|_{\ell^1} \right) \|F\|^2_{A^2(\mathbb{C}_+,\,(\nu_n)_{n=0}^m)} \,.
\end{aligned}
$$

Thus h is a multiplier of $A^2(\mathbb{C}_+,\, (\nu_n)_{n=0}^m)$.

Conversely, suppose that $h \in \mathscr{M}(A^2(\mathbb{C}_+,\, (\nu_n)_{n=0}^m))$. Then

$$\begin{aligned}
&\sup_{\varepsilon>0}\int_{\mathbb{C}_+}\left|\sum_{k=1}^{n}\binom{n}{k}h^{(k)}(z+\varepsilon)F^{(n-k)}(z+\varepsilon)\right|^2\,d\nu_n\\
&\quad=\sup_{\varepsilon>0}\int_{\mathbb{C}_+}\left|(hF)^{(n)}(z+\varepsilon)-hF^{(n)}(z+\varepsilon)\right|^2\,d\nu_n\\
&\quad\leq 2\left(\|hF\|^2_{A^2(\mathbb{C}_+,\,(\nu_n)_{n=0}^m)}+\|h\|^2_{H^\infty(\mathbb{C}_+)}\|F\|^2_{A^2_{\nu_0}}\right)\\
&\quad\leq 2\left(\|M_h\|^2+\|h\|^2_{H^\infty(\mathbb{C}_+)}\right)\|F\|^2_{A^2(\mathbb{C}_+,\,(\nu_n)_{n=0}^m)}\,.
\end{aligned}$$

□

Remark 3.1.3 *If* $\tilde{\nu}_n(\{0\})\neq 0$*, for some* n*, then the condition in part 2. of the theorem, saying that* $\mu_{n,k}$ *has to be a Carleson measure for* $A^p_{\nu_{n-k}}$*, can be replaced by*

$$\sup_{\varepsilon>0}\int_{\mathbb{C}_+}\left|h^{(k)}(z+\varepsilon)F^{(n-k)}(z+\varepsilon)\right|^p\,d\nu(z)\lessapprox\|F^{(n-k)}\|^p_{A^p_{n-k}},$$

for all $F\in A^p(\mathbb{C}_+,\,(\nu_n)_{n=0}^m)$*, provided that the inequality constants satisfy an estimate similar to this in* (3.1)*. It is a trivial observation than the statement of part 3. remains valid if we replace* $A^2(\mathbb{C}_+,\,(\nu_n)_{n=0}^m)$ *by* $A^2_{(m)}$*.*

3.1.2 Banach algebras

Although the descriptions of the multipliers of the three classical spaces of analytic functions - Hardy, Bergman and Dirichlet - are virtually the same in the disk and the half-plane settings, there is one major difference between them. If $\mathcal{H}$ is a RKHS, then $\mathscr{M}(\mathcal{H})$ is a unital Banach subalgebra of $\mathscr{B}(\mathcal{H})$, which is closed in the weak operator topology (Corollary 5.24, p. 79 in [84]). Since 1 lies in each weighted Hardy space $H^2(\boldsymbol{\beta})$, $h\cdot 1$ must also lie in $H^2(\boldsymbol{\beta})$, for all $h\in\mathscr{M}(H^2(\boldsymbol{\beta}))$. That is, the unital Banach algebra $\mathscr{M}(H^2(\boldsymbol{\beta}))$ is a subset of $H^2(\boldsymbol{\beta})$. A comparable inclusion can never be stated for $A^p(\mathbb{C}_+,\,(\nu_n)_{n=0}^m)$ or $A^2_{(m)}$ spaces, because they do not contain constant functions, which are clearly their multipliers. But maybe the reverse inclusion could be true, e.g. $A^2_{(m)} \subseteq \mathscr{M}(A^2_{(m)})$? This would imply that the space $A^2_{(m)}$ is a complete normed algebra (although it may not necessarily be a Banach algebra, and it certainly cannot be unital). We shall verify this possibility in this subsection.

Theorem 3.1.4 (Theorem 3 in [64]) *Let $\mathcal{H}$ be a Hilbert space of complex-valued functions defined on a set Ω. Suppose that $\mathcal{H}$ is also a Banach algebra with respect to pointwise multiplication. Then $\mathcal{H}$ is a RKHS, and if k_x is the reproducing kernel of $\mathcal{H}$ at $x \in \Omega$, then*

$$\sup_{x\in\Omega} \|k_x\|_{\mathcal{H}} \leq 1, \tag{3.3}$$

and consequently all elements of $\mathcal{H}$ are bounded.

Proof

First, note that the evaluation functional $E_x : \mathcal{H} \longrightarrow \mathbb{C},\ f \xmapsto{E_x} f(x)$ is bounded for every $x \in \Omega$, since it is a multiplicative functional on a Banach algebra $\mathcal{H}$ and hence $\|E_x\| \leq 1$ (see [16], § 16, Proposition 3, p. 77), so $\mathcal{H}$ is a RKHS. Let k_x denote the reproducing kernel of $\mathcal{H}$ at $x \in \Omega$. Then we have

$$\|k_x\|^2_{\mathcal{H}} = |k_x(x)| \leq \sup_{y\in\Omega} |k_x(y)| \,. \tag{3.4}$$

Also, by the Cauchy–Schwarz inequality and the fact that $\mathcal{H}$ is a Banach algebra, we get

$$\begin{aligned} |k_x(y)| \, \|k_y\|^2_{\mathcal{H}} &\overset{(1.11)}{=} |k_x(y)k_y(y)| \\ &= \left|\langle k_x k_y ,\, k_y\rangle_{\mathcal{H}}\right| \\ &\leq \|k_x k_y\|_{\mathcal{H}} \, \|k_y\|_{\mathcal{H}} \\ &\leq \|k_x\|_{\mathcal{H}} \, \|k_y\|^2_{\mathcal{H}} \,, \end{aligned}$$

and since it holds for all $x, y \in \Omega$, after cancelling $\|k_y\|^2_{\mathcal{H}}$ and taking the supremum, we get

$$\sup_{y\in\Omega} |k_x(y)| \leq \|k_x\|_{\mathcal{H}} \,. \tag{3.5}$$

From (3.4) and (3.5) we get

$$\|k_x\|^2_{\mathcal{H}} \overset{(3.4)}{\leq} \sup_{y\in\Omega} |k_x(y)| \overset{(3.5)}{\leq} \|k_x\|_{\mathcal{H}} \tag{3.6}$$

and consequently

$$\|k_x\|_{\mathcal{H}} \overset{(3.6)}{\leq} 1, \tag{3.7}$$

for all $x \in \Omega$. And for any $f \in \mathcal{H}$ we also have

$$\sup_{x\in\Omega} |f(x)| = \sup_{x\in\Omega} |\langle f,\, k_x\rangle| \overset{(3.7)}{\leq} \|f\|_{\mathcal{H}}\,.$$

□

Theorem 3.1.5 (Theorem 4 in [65]) *If $A^2_{(m)}$ is a Banach algebra, then*

$$\int_0^\infty \frac{dt}{w_{(m)}(t)} \leq 1, \tag{3.8}$$

and therefore

$$L^2_{w_{(m)}}(0,\,\infty) \subseteq L^1(0,\,\infty) \qquad \text{and} \qquad A^2_{(m)} \subseteq \mathscr{M}(A^2_{(m)}) \cap H^\infty(\mathbb{C}_+) \cap \mathscr{C}_0(i\mathbb{R}),$$

where $\mathscr{C}_0(i\mathbb{R})$ is the space of functions continuous on $i\mathbb{R}$ and vanishing at infinity. Conversely, if for all $t > 0$

$$\left(\frac{1}{w_{(m)}} * \frac{1}{w_{(m)}}\right)(t) \leq \frac{1}{w_{(m)}(t)}, \tag{3.9}$$

then $A^2_{(m)}$ is a Banach algebra. Here $*$ denotes the convolution operation.

Proof

Suppose that $A^2_{(m)}$ is a Banach algebra, then, by the previous theorem,

$$\int_0^\infty \frac{dt}{w_{(m)}(t)} = \sup_{z\in\mathbb{C}_+} \int_0^\infty \frac{e^{-2\,\mathrm{Re}(z)t}}{w_{(m)}(t)}\,dt \overset{(1.11),\,(1.12)}{=} \sup_{z\in\mathbb{C}_+} \left\| k_z^{A^2_{(m)}} \right\|^2_{A^2_{(m)}} \overset{(3.3)}{\leq} 1. \tag{3.10}$$

By Schwarz' inequality we also get

$$|F(z)| = \left| \int_0^\infty f(t)e^{-tz}\, dt \right| \leq \int_0^\infty |f(t)|\, dt \overset{(3.10)}{\leq} \left(\int_0^\infty |f(t)|^2 \, w_{(m)}(t)\, dt \right)^{\frac{1}{2}},$$

for all $F = \mathfrak{L}[f] \in \mathfrak{L}(L^1(0,\, \infty) \cap L^2_{w_{(m)}}(0,\, \infty))$ and $z \in \mathbb{C}_+$. On the boundary we have

$$F(iy) = \int_0^\infty f(t)e^{-ity}\, dt \in \mathscr{C}_0(i\mathbb{R}),$$

by the Riemann–Lebesgue Lemma. The converse follows from the fact that pointwise multiplication in $A^2_{(m)}$ is equivalent to convolution $(*)$ in $L^2_{w_{(m)}}(0,\, \infty)$, (via the Laplace transform) for which the sufficient condition to be a Banach algebra was given in [80] and in [17] (Lemma 8.11, p. 42), and the proof is quoted here. Suppose that (3.9) holds for all $t > 0$. Using Schwarz' inequality and that $(L^1(0, \infty), *)$ is a Banach algebra (see [28], § 4.7, p. 518), we get

$$\begin{aligned}
\|f * g\|^2_{L^2_{w_{(m)}}(0,\,\infty)} &\overset{\text{def}^{\text{n}}}{=} \int_0^\infty \left| \int_0^t f(\tau) g(t-\tau)\, d\tau \right|^2 w_{(m)}(t)\, dt \\
&\leq \int_0^\infty \int_0^t |f(\tau)|^2 \, w_{(m)}(\tau)\, |g(t-\tau)|^2 \, w_{(m)}(t-\tau)\, d\tau \\
&\quad \times \int_0^t \frac{d\tau}{w_{(m)}(\tau) w_{(m)}(t-\tau)} \, w_{(m)}(t)\, dt \\
&= \int_0^\infty (|f|^2 \, w_{(m)} * |g|^2 \, w_{(m)})(t) \left(\frac{1}{w_{(m)}} * \frac{1}{w_{(m)}} \right)(t)\, w_{(m)}(t)\, dt \\
&\overset{(3.9)}{\leq} \left\| |f|^2 \, w_{(m)} \right\|_{L^1(0,\,\infty)} \left\| |g|^2 \, w_{(m)} \right\|_{L^1(0,\,\infty)} \\
&\overset{\text{def}^{\text{n}}}{=} \|f\|^2_{L^2_{w_{(m)}}(0,\,\infty)} \, \|g\|^2_{L^2_{w_{(m)}}(0,\,\infty)}
\end{aligned}$$

for all $f,\, g$ in $L^2_{w_{(m)}}(0,\, \infty)$, and hence $A^2_{(m)}$ is a Banach algebra. □

This theorem shows that no Zen space A^2_ν can be a Banach algebra, since the weight

$$w(t) \overset{\text{def}^{\text{n}}}{=} 2\pi \int_{[0,\,\infty)} e^{-2xt}\, \tilde{\nu}(x) \qquad (\forall t > 0)$$

is never increasing. It is not clear which weights $w_{(m)}$ (if any) satisfy (3.9), so we are now going to investigate Banach algebras contained in $A^2_{(m)}$ spaces in order to produce an alternative sufficient condition for an $A^2_{(m)}$ space to be a Banach algebra. The next theorem was proved in [65] for $p = 2$ (Theorem 5) and we present its generalised version below.

Theorem 3.1.6 *Let* $1 \leq p < \infty$.

1. $A^p_\nu \cap H^\infty(\mathbb{C}_+)$ *is a Banach algebra with respect to the norm given by*

$$\|F\|_{A^p_\nu \cap H^\infty(\mathbb{C}_+)} := \|F\|_{H^\infty(\mathbb{C}_+)} + \|F\|_{A^p_\nu} \qquad (\forall F \in A^p_\nu \cap H^\infty(\mathbb{C}_+)).$$

2. *Suppose that for all* $1 \leq k < n \leq m-1 < \infty$ *the embedding*

$$L^1([0,\,\infty),\,\tilde{\nu}_{n-k}) \;\hookrightarrow\; L^1([0,\,\infty),\,\tilde{\nu}_n) \tag{3.11}$$

is bounded with norm 1. Then

$$\mathscr{A}^p_m := \bigcap_{n=0}^{m-1} \left\{F \in A^2_{(m)} \;:\; F^{(n)} \in H^\infty(\mathbb{C}_+)\right\}$$

is a Banach algebra with respect to the norm given by

$$\|F\|_{\mathscr{A}_m} := \sum_{n=0}^{m-1} \frac{\left\|F^{(n)}\right\|_{H^\infty(\mathbb{C}_+)}}{n!} + \sum_{n=0}^{m} \frac{\left\|F^{(n)}\right\|_{A^2_{\nu_n}}}{n!} \qquad (\forall F \in \mathscr{A}_m).$$

Proof

Those are clearly Banach spaces. For all F and G in $A^p_\nu \cap H^\infty(\mathbb{C}_+)$ we have

$$\begin{aligned}
\|FG\|_{A^p_\nu \cap H^\infty(\mathbb{C}_+)} &\overset{\text{def}^{\text{n}}}{=} \|FG\|_{H^\infty(\mathbb{C}_+)} + \|FG\|_{A^p_\nu} \\
&\leq \|F\|_{H^\infty(\mathbb{C}_+)} \|G\|_{H^\infty(\mathbb{C}_+)} + \|F\|_{H^\infty(\mathbb{C}_+)} \|G\|_{A^p_\nu} \\
&\leq \left(\|F\|_{H^\infty(\mathbb{C}_+)} + \|F\|_{A^p_\nu}\right)\left(\|G\|_{H^\infty(\mathbb{C}_+)} + \|G\|_{A^p_\nu}\right) \\
&\overset{\text{def}^{\text{n}}}{=} \|F\|_{A^p_\nu \cap H^\infty(\mathbb{C}_+)} \|G\|_{A^p_\nu \cap H^\infty(\mathbb{C}_+)}\,,
\end{aligned}$$

proving 1. To prove 2., let F and G be in $\mathscr{A}^p_m$, and let

$$\begin{aligned}
f_n &= \frac{\left\|F^{(n)}\right\|_{H^\infty(\mathbb{C}_+)}}{n!} \quad \text{for } 0 \le n < m \qquad \text{and} \qquad f_m = 0,\\
f'_n &= \frac{\left\|F^{(n)}\right\|_{A^p_{\nu_n}}}{n!},\\
g_n &= \frac{\left\|G^{(n)}\right\|_{H^\infty(\mathbb{C}_+)}}{n!} \quad \text{for } 0 \le n < m \qquad \text{and} \qquad g_m = 0,\\
g'_n &= \frac{\left\|G^{(n)}\right\|_{A^p_{\nu_n}}}{n!}.
\end{aligned}$$

By (3.11) we have

$$\int_{\overline{\mathbb{C}_+}} |F^{(n-k)}|^p \, d\nu_n \lessapprox \int_{\overline{\mathbb{C}_+}} |F^{(n-k)}|^p \, d\nu_{n-k},$$

and thus

$$\begin{aligned}
\|FG\|_{\mathscr{A}^p_m} &\overset{\text{def}^{\underline{\text{n}}}}{=} \sum_{n=0}^{m-1} \frac{\left\|(FG)^{(n)}\right\|_{H^\infty(\mathbb{C}_+)}}{n!} + \sum_{n=0}^{m} \frac{\left\|(FG)^{(n)}\right\|_{A^p_{\nu_n}}}{n!}\\
&\overset{\text{Minkowski's}}{\le} \sum_{n=0}^{m-1} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \left\|F^{(n-k)}\right\|_{H^\infty(\mathbb{C}_+)} \left\|G^{(k)}\right\|_{H^\infty(\mathbb{C}_+)}\\
&\qquad + \sum_{n=0}^{m} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\int_{\overline{\mathbb{C}_+}} \left|F^{(n-k)} G^{(k)}\right|^p \, d\nu_n\right)^{\frac{1}{p}}\\
&\overset{(3.11)}{\le} \sum_{n=0}^{m-1} \sum_{k=0}^{n} \frac{\left\|F^{(n-k)}\right\|_{H^\infty(\mathbb{C}_+)}}{(n-k)!} \frac{\left\|G^{(k)}\right\|_{H^\infty(\mathbb{C}_+)}}{k!} + \|F\|_{A^p_{\nu_0}} \|G\|_{H^\infty(\mathbb{C}_+)}\\
&\qquad + \sum_{n=1}^{m} \frac{1}{n!} \sum_{k=0}^{n-1} \binom{n}{k} \left\|F^{(n-k)}\right\|_{A^p_{\nu_{n-k}}} \left\|G^{(k)}\right\|_{H^\infty(\mathbb{C}_+)}\\
&\qquad + \sum_{n=1}^{m} \frac{1}{n!} \|F\|_{H^\infty(\mathbb{C}_+)} \left\|G^{(n)}\right\|_{A^p_{\nu_n}}\\
&\overset{\text{def}^{\underline{\text{n}}}}{=} \sum_{n=0}^{m-1} \sum_{k=0}^{n} f_{n-k} g_k + f'_0 g_0 + \sum_{n=1}^{m} \sum_{k=0}^{n-1} f'_{n-k} g_k + f_0 \sum_{n=1}^{m} g'_n\\
&\le \sum_{n=0}^{m} \sum_{k=0}^{n} (f_{n-k} g_k + f'_{n-k} g_k + f_{n-k} g'_k + f'_{n-k} g'_k)
\end{aligned}$$

$$\begin{aligned}
&\overset{\text{def}^{\text{n}}}{=} \left[\sum_{n=0}^{m} (f_n + f'_n)\right] \left[\sum_{n=0}^{m} (g_n + g'_n)\right] \\
&\overset{\text{def}^{\text{n}}}{=} \|F\|_{\mathscr{A}^p_m} \|G\|_{\mathscr{A}^p_m} ,
\end{aligned}$$

as required. □

Remark 3.1.7 *The algebras in Theorem 3.1.6 are modelled after A.2.4., p. 300, from [88].*

Theorem 3.1.8 (Theorem 6 in [65]) *Let $m \in \mathbb{N}$, and, for all $0 \leq n \leq m$, let w_n be as in* (1.6). *Assume that the embedding*

$$L^1([0, \infty), \tilde{\nu}_{n-k}) \hookrightarrow L^1([0, \infty), \tilde{\nu}_n)$$

is bounded for all $1 \leq k < n \leq m-1$. If

$$\int_0^\infty \frac{dt}{w_{m-1}(t) + w_m(t)} \leq 1, \tag{3.12}$$

then there exists a constant $C > 0$ such that $\left(A^2_{(m)}, C\,\|\cdot\|_{A^2_{(m)}}\right)$ is a Banach algebra.

Proof

Given $0 \leq n \leq m$, let

$$w'_{(m-n)}(t) := 2\pi \sum_{k=n}^{m} w_k(t) \qquad (\forall t > 0),$$

and let

$$B^2_{(m-n)} := \mathfrak{L}\left(L^2_{w'_{(m-n)}}(0, \infty)\right),$$

that is, $B^2_{(m-n)}$ is a truncated $A^2_{(m)}$ space, with first n weights/measures removed. So if $F \in A^2_{(m)}$, then $F^{(n)}$ lies in $B^2_{(m-n)}$, for all $0 \leq n \leq m$; and for all $z \in \mathbb{C}_+$ we have

$$\left|F^{(n)}(z)\right|^2 = \left|\left\langle F^{(n)}, k_z^{B^2_{(m-n)}}\right\rangle\right|^2 \overset{(1.11),(1.12)}{\leq} \left\|F^{(n)}\right\|^2_{B^2_{(m-n)}} \int_0^\infty \frac{e^{-2t\,\mathrm{Re}(z)}}{w_n(t) + \ldots + w_m(t)}\, dt,$$

so clearly

$$\left\|F^{(n)}\right\|^2_{H^\infty(\mathbb{C}_+)} \leq \left\|F^{(n)}\right\|^2_{B^2_{(m-n)}} \sup_{z\in\mathbb{C}_+} \int_0^\infty \frac{e^{-2t\operatorname{Re}(z)}}{w_{m-1}(t)+w_m(t)}\,dt \overset{(3.12)}{\leq} \|F\|^2_{A^2_{(m)}}, \quad (3.13)$$

for all $0 \leq n \leq m-1$. Now let F, $G \in A^2_{(m)}$ and let $d\nu'_n := (n!)^2 d\nu_n$. Then for any $0 \leq k \leq m$ we have

$$\begin{aligned}
\left\|(FG)^{(k)}\right\|^2_{A^2_{\nu_k}} &\leq \sum_{n=0}^{m-1} \frac{\|(FG)^{(n)}\|_{H^\infty(\mathbb{C}_+)}}{n!} + \sum_{n=0}^{m} \frac{\|(FG)^{(n)}\|_{A^2_{\nu'_n}}}{n!} \\
&\overset{\text{Th}^{\underline{\text{m}}}\ 3.1.6}{\leq} \left(\sum_{n=0}^{m-1} \frac{\|F^{(n)}\|_{H^\infty(\mathbb{C}_+)}}{n!} + \sum_{n=0}^{m} \frac{\|F^{(n)}\|_{A^2_{\nu'_n}}}{n!}\right) \\
&\quad \cdot \left(\sum_{n=0}^{m-1} \frac{\|G^{(n)}\|_{H^\infty(\mathbb{C}_+)}}{n!} + \sum_{n=0}^{m} \frac{\|G^{(n)}\|_{A^2_{\nu'_n}}}{n!}\right) \\
&\leq \left(1 + \sum_{n=0}^{m-1} \frac{1}{n!}\right)^2 \left(\sum_{n=0}^{m} \|F^{(n)}\|_{A^2_{\nu_n}}\right) \left(\sum_{n=0}^{m} \|G^{(n)}\|_{A^2_{\nu_n}}\right) \\
&\lessapprox \|F\|^2_{A^2_{(m)}} \|G\|^2_{A^2_{(m)}},
\end{aligned}$$

summing the above expression over all k between 0 and m and taking the square root proves the claim. Or, to be precise, by multiplying the weights by appropriate constants, we can assure that $A^2_{(m)}$ is a Banach algebra. □

Corollary 3.1.9 (Corollary 1 in [65]) *$A^2_{(1)}$ is a Banach algebra (after possibly adjusting its norm/weights) if and only if*

$$\int_0^\infty \frac{dt}{w_{(1)}(t)} < \infty.$$

Proof

It follows from Theorems 3.1.5 and 3.1.8. □

Example 3.1.10 *The integral*

$$\int_0^\infty \frac{dt}{1+t^{1-\alpha}} < \infty$$

if and only if $\alpha < 0$, so $\mathcal{D}_\alpha(\mathbb{C}_+)$ is a Banach algebra (after re-normalisation) if and only if $-1 \leq \alpha < 0$.

Thus the question that we have asked at the beginning of this subsection has an affirmative answer, i.e. there exist spaces $A^2_{(m)}$ which are (Banach algebras) contained within the set of their multipliers. A statement for weighted Dirichlet spaces on the disk, similar to this in Example 3.1.10, appeared in [98], and because weighted Hardy spaces always contain their set of multipliers, we then have the equality $\mathscr{M}(\mathcal{D}_\alpha) = \mathcal{D}_\alpha,\ -1 \leq \alpha < 0$.

Hilbert spaces which are also Banach algebras are *rarae aves* of function spaces, and hence they are seldom studied in much detail. For an early deliberation on this concept see [4].

3.1.3 Spectra of multipliers

Definition 3.1.11 *Let A be an algebra over the field of complex numbers. The* spectrum *of an element $a \in A$ is the set*

$$\sigma(A,\, a) := \{c \in \mathbb{C} \ :\ (a-c) \textit{ is not invertible in } A\}$$

if A is unital, and

$$\sigma(A,\, a) := \{0\} \cup \{c \in \mathbb{C} \ :\ a + cb - ab \neq 0,\ \forall b \in A\}\,.$$

otherwise. The resolvent set $\rho(A,\, a)$ of a is defined to be the complement of $\sigma(A,\, a)$ in $\mathbb{C}$. If the choice of algebra A is unambiguous, we will write simply $\sigma(a)$ and $\rho(a)$, for all $a \in A$. The spectral radius, r(a), of a *is defined by*

$$r(a) := \sup\{c \in \sigma(A,\, a)\}\,.$$

If A is a Banach algebra, then the spectral radius formula (or Gel'fand's formula) states that

$$r(a) = \lim_{n\to\infty} \|a^n\|_A^{1/n} \qquad (\forall a \in A)$$

(see Proposition 8, § 2, p. 11 and Theorem 8, § 5, p. 23 in [16]). If A is a commutative Banach algebra, then the maximal ideal space *(or the* character space*, or the* carrier space*), denoted by $\mathfrak{M}(A)$ is the set of all multiplicative linear functionals (non-zero algebra homomorphisms/characters) on A. We then have that*

$$r(a) = \sup_{\varphi\in\mathfrak{M}(A)} |\phi(a)| \qquad (\forall a \in A) \tag{3.14}$$

(see Theorem 5, § 17, p. 83 in [16]).

Theorem 3.1.12 (Theorem 7 in [65]) *If $h \in \mathscr{M}(A^2_{(m)})$, then*

$$\overline{h(\mathbb{C}_+)} \subseteq \sigma(\mathscr{M}(A^2_{(m)}), h),$$

with equality at least for $m \le 1$.

Proof

Let $h \in \mathscr{M}(A^2(\mathbb{C}_+, (\nu_n)_{n=0}^m))$. We have that $(h-c)^{-1} \in H^\infty(\mathbb{C}_+)$, for some $c \in \mathbb{C}$, if and only if $\inf_{z\in\mathbb{C}_+} |h(z) - c| > 0$, and consequently $\sigma(H^\infty(\mathbb{C}_+), h) = \overline{h(\mathbb{C}_+)}$. If $c \in \sigma(H^\infty(\mathbb{C}_+), h)$, then $(h-c)^{-1} \notin H^\infty(\mathbb{C}_+) \supseteq \mathscr{M}(A^2(\mathbb{C}_+, (\nu_n)_{n=0}^m))$, so clearly $\overline{h(\mathbb{C}_+)} = \sigma(H^\infty(\mathbb{C}_+), h) \subseteq \sigma(\mathscr{M}(A^2(\mathbb{C}_+, (\nu_n)_{n=0}^m)), h)$.

For the reverse inclusion, when $m \le 1$, recall that $\mathscr{M}(A^2(\mathbb{C}_+, (\nu_n)_{n=0}^m)) = H^\infty(\mathbb{C}_+)$, and also observe that if $h^{-1} \in H^\infty(\mathbb{C}_+) = \mathscr{M}(A^2_{\nu_0})$, then

$$\begin{aligned}\int_{\overline{\mathbb{C}_+}} \left|\left(\frac{F}{h}\right)'\right|^2 d\nu_1 &= \int_{\overline{\mathbb{C}_+}} \left|\frac{F'}{h} - \frac{h'F}{h^2}\right|^2 d\nu_1 \\ &\leq 2\|h^{-1}\|^2_{H^\infty(\mathbb{C}_+)} \left(\int_{\overline{\mathbb{C}_+}} |F'|^2\, d\nu_1 + \|h^{-1}\|^2_{H^\infty(\mathbb{C}_+)} \int_{\overline{\mathbb{C}_+}} |h'F|^2\, d\nu_1\right) \\ &\overset{(3.2)}{\lessapprox} \|h^{-1}\|^2_{H^\infty(\mathbb{C}_+)} \left(1 + \|h^{-1}\|^2_{H^\infty(\mathbb{C}_+)}\right) \|F\|^2_{A^2(\mathbb{C}_+,\,(\nu_0,\,\nu_1))},\end{aligned}$$

that is, $h^{-1} \in \mathscr{M}(A^2(\mathbb{C}_+,\, (\nu_0,\, \nu_1)))$. □

Theorem 3.1.13 (Theorem 8 in [65]) *Suppose that $A^2_{(m)}$ is a Banach algebra and that for each $a > 0$ there exists $K(a) > 0$ such that $w_{(m)}(t) \leq K(a)e^{at}$, for all $t > 0$. Let $\pi : \mathfrak{M}(\mathscr{M}(A^2_{(m)})) \longrightarrow \overline{\mathbb{D}}$ (the closed unit disk of the complex plane) be given by*

$$\pi(\varphi) = \varphi\left(\frac{1-z}{1+z}\right) \qquad\qquad (\varphi \in \mathfrak{M}(\mathscr{M}(A^2_{(m)})))$$

(that is, π is the Gel'fand transform of the function $(1-z)/(1+z)$, see Chapter V, § 1, p. 184 in [45]). Then

1. *π is surjective.*
2. *If $m \leq 1$ or for all $1 \leq k < n \leq m-1 < \infty$ the embedding*

$$L^1([0,\, \infty),\, \tilde{\nu}_{n-k}) \hookrightarrow L^1([0,\, \infty),\, \tilde{\nu}_n)$$

is bounded, then π is injective over the open unit disk $\mathbb{D}$ and $(\pi|^{\mathbb{D}})^{-1}$ (that is, the inverse of the restriction of π to $\mathbb{D}$ in its image) maps $\mathbb{D}$ homeomorphically onto an open subset $\Delta \subset \mathfrak{M}(\mathscr{M}(A^2_{(m)}))$.

Proof

First, note that, for all $a \in \mathbb{C}_+$, $(z+a)^{-1}$ is in $A^2_{(m)}$, since

$$\int_0^\infty \left|e^{-at}\right|^2 w_{(m)}(t)\, dt \leq K(a) \int_0^\infty e^{-t\,\mathrm{Re}(a)}\, dt = \frac{K(a)}{\mathrm{Re}(a)},$$

so $e^{-at} \in L^2_{w_{(m)}}(0,\,\infty)$, and hence

$$\frac{1-z}{1+z} = \frac{2}{1+z} - 1 \in A^2_{(m)} + \mathbb{C} \subseteq \mathscr{M}(A^2_{(m)}).$$

We know that $\sigma\left(\mathscr{M}(A^2_{(m)}),\, \frac{1-z}{1+z}\right) \supseteq \sigma\left(H^\infty(\mathbb{C}_+),\, \frac{1-z}{1+z}\right)$. If $\left(\frac{1-z}{1+z} - c\right)^{-1} \in H^\infty(\mathbb{C}_+)$, for some $c \in \mathbb{C}$, then

$$\left(\frac{1-z}{1+z} - c\right)^{-1} = \frac{1+z}{1-c-z(1+c)} = \frac{1}{1+c}\left[-\frac{2}{1+c}\underbrace{\left(z - \frac{1-c}{1+c}\right)^{-1}}_{\in A^2_{(m)}} - 1\right]$$

is a multiplier of $A^2_{(m)}$. So we actually have

$$\sigma\left(\mathscr{M}(A^2_{(m)}), \frac{1-z}{1+z}\right) = \sigma\left(H^\infty(\mathbb{C}_+), \frac{1-z}{1+z}\right),$$

and thus

$$|\pi(\varphi)| \leq \sup_{\varphi \in \mathfrak{M}(\mathscr{M}(A^2_{(m)}))} \left|\varphi\left(\frac{1-z}{1+z}\right)\right| \overset{(3.14)}{=} r\left(\frac{1-z}{1+z}\right) = 1.$$

Because the evaluation homomorphisms are in $\mathfrak{M}(\mathscr{M}(A^2_{(m)}))$, every point of the open unit disk is in the image of π. Also, $\mathfrak{M}(\mathscr{M}(A^2_{(m)}))$ is a compact Hausdorff space (see Theorem 2.5, Chapter I, § 2, p. 4 in [43]), so its image under π must also be compact (Theorem 2.10, p. 38 in [90]), and thus π is surjective.

For the second part, let $|c| < 1$ and suppose that $\pi(\varphi) = c$. Then for any $F \in A^2_{(m)}$ vanishing at $\kappa = \frac{1-c}{1+c} \in \mathbb{C}_+$, we have $F = \frac{z-\kappa}{z+\overline{\kappa}}G$, with $G \in H^\infty(\mathbb{C}_+)$ (see [75], p. 293). Let $\overline{B_r(\kappa)}$ be the closed ball, centred at κ, with radius $r > 0$. Choose r small enough to get $\overline{B_r(\kappa)} \subset \mathbb{C}_+$, then

$$\int_{\overline{\mathbb{C}_+}} |G(z)|^2\, d\nu_0(z) = \int_{\overline{B_r(\kappa)}} |G(z)|^2\, d\nu_0 + \int_{\overline{\mathbb{C}_+} \setminus \overline{B_r(\kappa)}} \left|\frac{z+\overline{\kappa}}{z-\kappa}F(z)\right|^2\, d\nu_0(z).$$

The first integral is finite, since G is bounded on $\mathbb{C}_+$ and $\overline{B_r(\kappa)}$ is compact. The second integral is also finite, since $\frac{z+\overline{\kappa}}{z-\kappa}$ is bounded on $\overline{\mathbb{C}_+} \setminus \overline{B_r(\kappa)}$. Let

$$C := \sup_{z \in \overline{\mathbb{C}_+} \setminus \overline{B_r(\kappa)}} \left|\frac{z+\overline{\kappa}}{z-\kappa}\right|.$$

Then we have

$$\begin{aligned}\int_{\overline{\mathbb{C}_+}} |G'|^2 \, d\nu_1 &= \int_{\overline{B_r(\kappa)}} |G'|^2 \, d\nu_1 + \int_{\overline{\mathbb{C}_+} \setminus \overline{B_r(\kappa)}} \left| F'(z)\frac{z+\overline{\kappa}}{z-\kappa} - F(z)\frac{2\operatorname{Re}\kappa}{(z-\kappa)^2} \right|^2 \, d\nu_1(z) \\ &\leq \int_{\overline{B_r(\kappa)}} |G'|^2 \, d\nu_1 + 2C^2 \|F'\|^2_{A^2_{\nu_1}} \\ &\quad + (2C^2 \operatorname{Re}(\kappa))^2 \|F\|^2_{H^\infty(\mathbb{C}_+)} \int_{\mathbb{C}_+} \left| \frac{d\nu_1(z)}{(z+\overline{\kappa})^2} \right|^2 \, d\nu_1(z)\end{aligned}$$

which is also finite, since $|G'|^2$ is continuous, $\overline{B_r(\kappa)}$ is compact and $(z+\overline{\kappa})^{-1} \in A^2_{(m)}$ implies $(z+\overline{\kappa})^{-2} \in A^2_{\nu_1}$. If $n > 1$, then

$$\begin{aligned}\int_{\overline{\mathbb{C}_+}} \left|G^{(n)}\right|^2 \, d\nu_n &= \int_{\overline{B_r(\kappa)}} \left|G^{(n)}\right|^2 \, d\nu_n \\ &\quad + \int_{\overline{\mathbb{C}_+} \setminus \overline{B_r(\kappa)}} \left| \sum_{k=0}^{n} \binom{n}{k} F^{(n-k)}(z) \left(\frac{z+\overline{\kappa}}{z-\kappa}\right)^{(k)} \right|^2 \, d\nu_n(z) \\ &\lessapprox \int_{\overline{B_r(\kappa)}} \left|G^{(n)}\right|^2 \, d\nu_n + \sum_{k=1}^{n-1} \left\|F^{(n-k)}\right\|^2_{A^2_{\nu_{n-k}}} \\ &\quad + \|F\|^2_{H^\infty(\mathbb{C}_+)} \left\|(z+\overline{\kappa})^{-n}\right\|^2_{A^2_{\nu_n}} < \infty.\end{aligned}$$

Therefore, in either case, $G \in A^2_{(m)}$. Let

$$H := \underbrace{-\frac{(1+z)(1+\kappa)}{2(z+\overline{\kappa})}}_{\in \mathscr{M}(A^2_{(m)})} G \in A^2_{(m)}.$$

Then

$$\varphi(F) = \varphi\left(\frac{1-z}{1+z} - c\right)\varphi(H) = 0. \tag{3.15}$$

Let $h \in \mathscr{M}(A^2_{(m)})$ be such that $h(\kappa) = 0$. Then $h(z)/(z+1)$ lies in $A^2_{(m)}$ and vanishes at κ, so

$$0 \overset{(3.15)}{=} \varphi\left(\frac{h}{z+1}\right) = \varphi(h)\varphi\left(\frac{1}{1+z}\right) = \frac{\varphi(h)}{2}\varphi\left(\frac{1-z}{1+z}+1\right) = \varphi(h)\frac{c+1}{2},$$

so φ must in fact be the evaluation homomorphism, proving injectivity. For the remaining part, let $\Delta := (\pi|^{\mathbb{D}})^{-1}(\mathbb{D})$. Then π maps Δ homeomorphically onto $\mathbb{D}$, since the topology of Δ is the weak topology defined by Gel'fand transforms of functions from $\mathscr{M}(A^2_{(m)})$, and the topology of $\mathbb{D}$ is the weak topology defined by bounded functions in $\mathscr{M}(A^2_{(m)})$. $\square$

This theorem shows the existence of an analytic disk in $\mathfrak{M}(\mathscr{M}(A^2_{(m)}))$. It is a natural question to ask whether this disk is dense therein, or in other words, does the Corona Theorem hold in this setting. *Dictatum erat*, Lennart Carleson had proved the Corona Theorem for $\mathfrak{M}(H^\infty) = \mathfrak{M}(\mathscr{M}(\mathcal{B}^2_\alpha))$, $\alpha \geq -1$. The Corona Theorem is valid for $\mathscr{M}(\mathcal{D})$ too (see [100]), but whether it is also the case for $\mathscr{M}(A^2_{(m)})$ still remains to be established.

3.2 Weighted composition operators

Weighted composition operators for Hardy spaces H^p have been discussed for example in [26] and [42], for weighted Bergman spaces $\mathcal{B}^2_\alpha$, $\alpha \geq -1$ in [41] and [68], for the Dirichlet space $\mathcal{D}$ in [23] and for weighted Hardy spaces $H(\boldsymbol{\beta})$ in [22]. This last reference also includes a discussion on weighted composition operators for Zen spaces, which was continued in [67], and we present some of the results from that paper here and in the next section. We also outline some minor, partial extensions to $A^p(\mathbb{C}_+, (\nu_n)_{n=0}^\infty)$ space, which have not been published yet.

Throughout this section we assume that $h : \mathbb{C}_+ \longrightarrow \mathbb{C}$ and $\varphi : \mathbb{C}_+ \longrightarrow \mathbb{C}_+$ are analytic functions.

3.2.1 Bergman kernels, Carleson measures and boundedness of weighted composition operators

In [26] M. D. Contreras and A. G. Hernández-Díaz gave a necessary and sufficient condition for a weighted composition operator $W_{h,\varphi}$ to be bounded on H^p. We modify their proof to show that a similar condition also characterises the boundedness of weighted composition operators for Zen spaces.

Lemma 3.2.1 *Let* ν *be a positive Borel measure on* $\mathbb{C}_+$ *and let* g *be a non-negative measurable function on* $\mathbb{C}_+$*. Then*

$$\int_{\mathbb{C}_+} g \, d\mu_{\nu,\,h,\,\varphi,\,p} = \int_{\mathbb{C}_+} |h|^p (g \circ \varphi) \, d\nu, \tag{3.16}$$

where $\mu_{\nu,\,h,\,\varphi,\,p}$ *is given by*

$$\mu_{\nu,\,h,\,\varphi,\,p}(E) = \int_{\varphi^{-1}(E)} |h|^p \, d\nu, \tag{3.17}$$

for each Borel set $E \subseteq \mathbb{C}_+$*.*

Proof

Let $(E_i)_i$ be a countable collection of disjoint Borel subsets of $\mathbb{C}_+$ such that $\bigcup_i E_i = \mathbb{C}_+$ and suppose that $g(z) = \sum_i c_i \chi_{E_i}(z)$ is a simple non-negative measurable function. Then

$$\begin{aligned}
\int_{\mathbb{C}_+} g \, d\mu_{\nu,\,h,\,\varphi,\,p} &= \sum_i c_i \mu_{\nu,\,h,\,\varphi,\,p}(E_i) \\
&\overset{(3.17)}{=} \sum_i c_i \int_{\varphi^{-1}(E_i)} |h|^p \, d\nu \\
&= \sum_i c_i \int_{\mathbb{C}_+} |h|^p \chi_{\varphi^{-1}(E_i)} \, d\nu \\
&= \int_{\mathbb{C}_+} |h|^p \sum_i c_i \chi_{\varphi^{-1}(E_i)} \, d\nu \\
&= \int_{\mathbb{C}_+} |h|^p (g \circ \varphi) \, d\nu.
\end{aligned}$$

If g is not simple, then we can find a sequence of simple functions such that

$$0 \leq g_0(z) \leq g_1(z) \leq \ldots \leq g_n(z) \leq \ldots \qquad (\forall z \in \mathbb{C}_+),$$

which converges to g pointwise for each $z \in \mathbb{C}_+$, and thus, by Lebesgue's Monotone Convergence Theorem (1.26, p. 21 in [90])

$$\lim_{n\to\infty} \int_{\mathbb{C}_+} g_n \, d\mu_{\nu,\,h,\,\varphi,\,p} = \int_{\mathbb{C}_+} g \, d\mu_{\nu,\,h,\,\varphi,\,p}.$$

And similarly, $(|h(z)|^p(g_n \circ \varphi)(z))_{n=0}^{\infty}$ is a non-decreasing sequence of non-negative measurable functions converging pointwise to $|h(z)|^p(g \circ \varphi)(z)$, for each $z \in \mathbb{C}_+$, such that

$$\lim_{n\to\infty} \int_{\mathbb{C}_+} |h|^p(g_n \circ \varphi) \, d\nu = \int_{\mathbb{C}_+} |h|^p(g \circ \varphi) \, d\nu,$$

from which the desired result follows easily. $\square$

For the rest of this section we assume that $\tilde{\nu}(\{0\}) = \tilde{\nu}_0(\{0\}) = \tilde{\nu}_1(\{0\}) = 0$.

Theorem 3.2.2 *The weighted composition operator $W_{h,\,\varphi}$ is bounded on a Zen space A^p_ν if and only if $\mu_{\nu,\,h,\,\varphi,\,p}$ is a Carleson measure for A^p_ν.*

Proof

Given $F \in A^p_\nu$, by the previous lemma (applied with $g = |F|^p$), we get

$$\int_{\mathbb{C}_+} |h \cdot (F \circ \varphi)|^p \, d\nu \overset{(3.16)}{=} \int_{\mathbb{C}_+} |F|^p \, d\mu_{\nu,\,h,\,\varphi,\,p} \leq C \int_{\mathbb{C}_+} |F|^p \, d\nu$$

(for some $C > 0$, not depending on F), if and only if $\mu_{\nu,\,h,\,\varphi,\,p}$ is a Carleson measure for A^p_ν, or equivalently if and only if $W_{h,\,\varphi}$ is bounded on A^p_ν. $\square$

An analogous condition in case when $\tilde{\nu}(\{0\}) > 0$ is given in Theorem 2.2, p. 227 in [26]. From Lemma 3.2.1 we can also deduce a partial result for $A^p(\mathbb{C}_+, \, (\nu_0, \, \nu_1))$.

Corollary 3.2.3 *If $\mu_{\nu_0,\,h,\,\varphi,\,p}$ and $\mu_{\nu_1,\,h',\,\varphi,\,p}$ are both Carleson measures for $A^p_{\nu_0}$, and $\mu_{\nu_1,\,h\varphi',\,\varphi,\,p}$ is a Carleson measure for $A^p_{\nu_1}$, then $W_{h,\,\varphi}$ is bounded on $A^p(\mathbb{C}_+,\,(\nu_0,\,\nu_1))$.*

Proof

It follows from the fact that

$$\begin{aligned}\|W_{h,\,\varphi}F\|^p_{A^p(\mathbb{C}_+,\,(\nu_0,\,\nu_1))} &= \int_{\mathbb{C}_+} |h \cdot (F \circ \varphi)|^p \, d\nu_0 + \int_{\mathbb{C}_+} |h' \cdot (F \circ \varphi) + h\varphi' \cdot (F' \circ \varphi)|^p \, d\nu_1 \\ &\leq \int_{\mathbb{C}_+} |F|^p \, d\mu_{\nu_0,\,h,\,\varphi,\,p} + 2^{p-1} \int_{\mathbb{C}_+} |F|^p \, d\mu_{\nu_1,\,h',\,\varphi,\,p} \\ &\quad + 2^{p-1} \int_{\mathbb{C}_+} |F'|^p \, d\mu_{\nu_1,\,h\varphi',\,\varphi,\,p}.\end{aligned}$$

□

A similar statement can also be given for $A^p(\mathbb{C}_+,\,(\nu_n)_{n=0}^m)$, for $1 < m < \infty$, using the general Leibniz rule and Faà di Bruno's formula (see [40]).

Theorem 3.2.4 *The weighted composition operator $W_{h,\varphi}$ is bounded on a Zen space A^p_ν if and only if there exists $\alpha \geq -1$ such that*

$$\Lambda(\alpha) := \sup_{z \in \mathbb{C}_+} \frac{\left\| h \cdot \left(k_z^{\mathcal{B}^2_\alpha(\mathbb{C}_+)} \circ \varphi \right) \right\|_{A^p_\nu}}{\left\| k_z^{\mathcal{B}^2_\alpha(\mathbb{C}_+)} \right\|_{A^p_\nu}} < \infty. \tag{3.18}$$

Proof

By Lemma 1.2.10 we know that there exists $\alpha \geq -1$ such that $k_z^{\mathcal{B}^2_\alpha(\mathbb{C}_+)}$ is in A^p_ν, for all $z \in \mathbb{C}_+$, and if $W_{h,\,\varphi}$ is bounded on A^p_ν, then $\Lambda(\alpha)$ must be finite. Conversely, if $\Lambda(\alpha)$ is finite, for some $\alpha \geq -1$, then, by Theorem 2.1, p. 787 from [61], $\mu_{\nu,\,h,\,\varphi,\,p}$ must be a Carleson measure for A^p_ν, and hence, by Lemma 3.2.1, $W_{h,\,\varphi}$ must be bounded on A^p_ν.

□

Remark 3.2.5 *If $p > 1$ and $A^p_\nu = \mathcal{B}^p_\alpha(\mathbb{C}_+)$ for some $\alpha > -1$, then, by estimate* (2.9)*, we have*

$$\|K_\alpha(\cdot\, z)\|^p_{\mathcal{B}^2_\alpha(\mathbb{C}_+)} \leq \left(\frac{2}{\mathrm{Re}(z)}\right)^{(2+\alpha)p} \nu\left(\overline{Q(z)}\right) \sum_{k=0}^{\infty} \left(\frac{2^{1+\alpha}}{2^{(2+\alpha)p-1}}\right)^k < \infty,$$

and hence we can substitute this index α into the equation (3.18)*. In this case we have that $W_{h,\varphi}$ is bounded on $B^p_\alpha(\mathbb{C}_+)$ if and only if*

$$\sup_{z\in\mathbb{C}_+} \frac{\left\|h \cdot \left(k_z^{\mathcal{B}^2_\alpha(\mathbb{C}_+)} \circ \varphi\right)\right\|_{\mathcal{B}^p_\alpha(\mathbb{C}_+)}}{\left\|k_z^{\mathcal{B}^2_\alpha(\mathbb{C}_+)}\right\|_{\mathcal{B}^p_\alpha(\mathbb{C}_+)}} < \infty. \tag{3.19}$$

In particular, if $p = 2$, then (3.19) *is equivalent to*

$$\sup_{z\in\mathbb{C}_+} |h(z)| \left(\frac{\mathrm{Re}(z)}{\mathrm{Re}(\varphi(z))}\right)^{\alpha+2} < \infty,$$

by Lemma 3.0.8.

3.2.2 Causality

Definition 3.2.6 *Let w be a positive measurable function on $(0, \infty)$. We say that $A : L^2_w(0,\infty) \longrightarrow L^2_w(0, \infty)$ is a* causal operator *(or a* lower-triangular operator*), if for each $T > 0$ the closed subspace $L^2_w(T, \infty)$ is invariant for A. If there exists $\alpha > 0$ such $AL^2_w(T, \infty) \subseteq L^2_w(T + \alpha, \infty)$, for all $T > 0$, then we say that A is* strictly causal.

The following lemma was proved in [22] (Theorem 3.2, p. 1091) for unweighted and in [66] for weighted L^2 spaces on $(0, \infty)$

Lemma 3.2.7 (Lemma 2 in [67]) *Let w be a positive, non-increasing, measurable function on $(0, \infty)$. Suppose that $A : L^2_w(0,\infty) \longrightarrow L^2_w(0, \infty)$ is a causal operator and D is the operator of multiplication by a strictly positive, monotonically increasing function d. Then*

$$\left\|D^{-1}AD\right\|_{L^2_w(0,\infty)} \leq \|A\|_{L^2_w(0,\infty)}\,. \tag{3.20}$$

Proof

First, suppose that A is strictly causal for some $\alpha > 0$. For $z \in \overline{\mathbb{C}_+}$ define $\Omega(z) = D^{-z}AD^z$, where D^z is the operator of multiplication by the complex function d^z. For each $N \in \mathbb{N}$ such that $N \geq \log_2 \alpha$ let

$$X_N := \text{span}\left\{e_k := \chi_{(k/2^N,\,(k+1)/2^N)} \;:\; k \in \mathbb{Z} \text{ and } 1 \leq k \leq 2^{2N}\right\}$$

be a subspace of $L^2_w(0,\,\infty)$. Clearly, $\bigcup_{N \geq \log_2 \alpha} X_N$ is a dense subspace of $L^2_w(0,\,\infty)$. Let $P_N : L^2_w(0,\,\infty) \longrightarrow X_N$ denote the orthogonal projection and define $\Omega_N(z) = \Omega(z)P_n$. For all $1 \leq k \leq 2^{2N}$, $\Omega_N(z)$ maps each e_k to $d^{-z}Ad^z e_k$ and

$$\begin{aligned}
\left\|d^{-z}Ad^z e_k\right\| &\leq \left(d\left(\frac{k}{2^N} + \alpha\right)\right)^{-\operatorname{Re}(z)} \|Ad^z e_k\| \\
&\leq \left(d\left(\frac{k}{2^N} + 2^{-N}\right)\right)^{-\operatorname{Re}(z)} \|A\| \left(d\left(\frac{k+1}{2^N}\right)\right)^{\operatorname{Re}(z)} \|e_k\| \\
&\leq \|A\|\|e_k\|,
\end{aligned}$$

since d is increasing and A is strictly causal (i.e. $Ad^z e_k$ is supported on $[k/2^N + \alpha,\ \infty)$). X_N is finite dimensional, so $\Omega_N(z)$ is bounded independently of z, because $\|\Omega_N(z)\| \leq \|\Omega(z)|_{X_N}\|$. By the maximum principle (also known as Phragmén–Lindelöf Principle, see 6.2, p. 117 in [89]) we also have that

$$\|\Omega_N(1)\| \leq \sup_{\operatorname{Re}(z) \geq 0} \|\Omega_N(z)\| \leq \sup_{\operatorname{Re}(z) = 0} \|\Omega(z)\| = \|A\|,$$

and the result holds on $L^2_w(0,\,\infty)$, since $\bigcup_{N \geq \log_2 \alpha} X_N$ is a dense set therein.

If A is not strictly causal, then let S_α denote the right shift by α. In this case the operator $S_\alpha A$ is strictly causal and, by the above, we have

$$\|D^{-1}S_\alpha AD\| \leq \|S_\alpha A\| = \|A\|.$$

Let $d_\alpha(t) = d(\alpha + t)$. Then for each $f \in L^2_w(0,\,\infty)$ we have $\|D^{-1}S_\alpha\| = \|d_\alpha^{-1} f\|$, and $|d_\alpha^{-1} f\sqrt{w}|$ increases to $|d^{-1} f\sqrt{w}|$ almost everywhere as $\alpha \longrightarrow 0$, because the monotonically decreasing function d^{-1} is continuous almost everywhere, and hence the result follows from Lebesgue's Monotone Convergence Theorem. $\square$

We will say that an operator $B : A^2_{(m)} \longrightarrow A^2_{(m)}$ is causal if the the corresponding isometric operator $\mathfrak{L}^{-1}B\mathfrak{L} : L^2_{w_{(m)}}(0,\infty) \longrightarrow L^2_{w_{(m)}}(0,\infty)$ is causal on $L^2_{w_{(m)}}(0,\infty)$

$$\begin{tikzcd} L^2_w(0,\infty) \arrow[r, "\mathfrak{L}^{-1}B\mathfrak{L}"] \arrow[d, "\mathfrak{L}"] & L^2_w(0,\infty) \\ A^2_\nu \arrow[r, "B"'] & A^2_\nu \arrow[u, "\mathfrak{L}^{-1}"'] \end{tikzcd}$$

Theorem 3.2.8 *Suppose that the weighted composition operator $W_{h,\varphi}$ is bounded and causal on $A^2_{(0)}$. Then there exists $\alpha' \geq 0$ such that for each $\alpha \geq \alpha'$ $W_{h,\varphi}$ is bounded on the weighted Bergman space $\mathcal{B}^2_\alpha(\mathbb{C}_+)$, and*

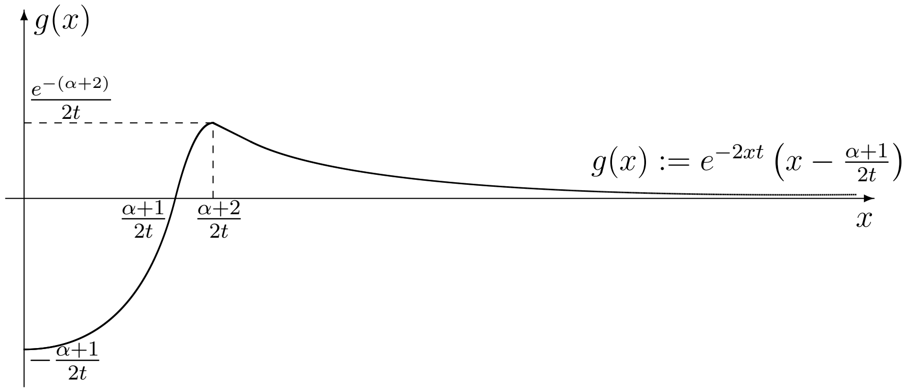
$$\|W_{h,\varphi}\|_{\mathcal{B}^2_\alpha(\mathbb{C}_+)} \leq \|W_{h,\varphi}\|_{A^2_{(0)}} . \tag{3.21}$$

Proof

The first part of this proof is conducted in essentially the same manner as the proof of Lemma 1.2.2. Let $L^2_{w_0}(0,\infty)$, $L^2_{v_\alpha}(0,\infty)$ be the spaces corresponding to $A^2_{(0)}$ and $\mathcal{B}^2_\alpha(\mathbb{C}_+)$ respectively (i.e. $v_\alpha(t) = 2^{-\alpha}\Gamma(\alpha+1)t^{-\alpha-1}$). We want to show that $\sqrt{w_0(t)/v_\alpha(t)}$ is an increasing function, that is, we must have

$$\begin{gathered} w_0'(t)v_\alpha(t) \geq w_0(t)v_\alpha'(t) \\ -2\pi\int_{[0,\infty)} 2xe^{-2tx}\,d\tilde{\nu}_0(x)\cdot\frac{\Gamma(\alpha+1)}{2^\alpha t^{\alpha+1}} \geq -2\pi\int_{[0,\infty)} e^{-2tx}\,d\tilde{\nu}_0(x)\cdot\frac{\Gamma(\alpha+2)}{2^\alpha t^{\alpha+2}} \\ \int_{[0,\infty)} e^{-2tx}\left(x-\frac{\alpha+1}{2t}\right)d\tilde{\nu}_0(x) \leq 0. \end{gathered}$$

Consider the graph:



We clearly need to have

$$-\int_{[0,\ \frac{\alpha+1}{2t})} e^{-2xt}\left(x - \frac{\alpha+1}{2t}\right)\, d\tilde{\nu}_0(x) \geq \int_{[\frac{\alpha+1}{2t},\infty)} e^{-2tx}\left(x - \frac{\alpha+1}{2t}\right)\, d\tilde{\nu}_0(x).$$

Observe that for $\alpha \geq 0$ we have

$$\frac{\alpha+2}{4t} \leq \frac{\alpha+1}{2t}. \tag{3.22}$$

Let R be defined for $\tilde{\nu}_0$ as in (Δ_2). Then we have

$$\begin{aligned}
-\int_{[0,\ \frac{\alpha+1}{2t})} e^{-2xt}\left(x - \frac{\alpha+1}{2t}\right)\, d\tilde{\nu}_0(x) &\geq \tilde{\nu}_0\left(\left[0,\ \frac{\alpha+2}{4t}\right)\right)\alpha\frac{e^{-\frac{\alpha+2}{2}}}{2t} \\
&\overset{(\Delta_2)}{\geq} \frac{\tilde{\nu}_0\left(\left[0,\ \frac{\alpha+2}{2t}\right)\right)}{2Rt}\alpha e^{-\frac{\alpha+2}{2}},
\end{aligned}$$

and

$$\begin{aligned}
\int_{[\frac{\alpha+1}{2t},\ \frac{\alpha+2}{2t})} e^{-2tx}\left(x - \frac{\alpha+1}{2t}\right)\, d\tilde{\nu}_0(x) &\leq \tilde{\nu}_0\left(\left[\frac{\alpha+1}{2t},\ \frac{\alpha+2}{2t}\right)\right)\frac{e^{-(\alpha+2)}}{2t} \\
&= \tilde{\nu}_0\left(\left[0,\ \frac{\alpha+2}{2t}\right)\right)\frac{e^{-(\alpha+2)}}{2t} \\
&\quad - \tilde{\nu}_0\left(\left[0,\ \frac{\alpha+1}{2t}\right)\right)\frac{e^{-(\alpha+2)}}{2t} \\
&\leq \frac{\tilde{\nu}_0\left(\left[0,\ \frac{\alpha+2}{2t}\right)\right)}{2Rt}(R-1)e^{-(\alpha+2)},
\end{aligned}$$

because

$$-\tilde{\nu}_0\left(\left[0,\, \frac{\alpha+1}{2t}\right)\right) \overset{(3.22)}{\leq} -\tilde{\nu}_0\left(\left[0,\, \frac{\alpha+2}{4t}\right)\right) \overset{(\Delta_2)}{\leq} -\frac{\tilde{\nu}_0\left(\left[0,\, \frac{\alpha+2}{2t}\right)\right)}{R}.$$

And

$$\begin{aligned}
\int_{\left[\frac{\alpha+2}{2t},\,\infty\right)} e^{-2tx}\left(x-\frac{\alpha+1}{2t}\right) d\tilde{\nu}_0(x) &\leq \sum_{n=0}^{\infty} \tilde{\nu}_0\left(\left[2^n\frac{\alpha+2}{2t},\, 2^{n+1}\frac{\alpha+2}{2t}\right)\right) \\
&\quad \cdot \left(e^{-2^n(\alpha+2)}\frac{2^n(\alpha+2)-\alpha-1}{2t}\right) \\
&\overset{(\Delta_2)}{\leq} \frac{\tilde{\nu}_0\left(\left[0,\, \frac{\alpha+2}{2t}\right)\right)}{2t}(R-1)(\alpha+2)e^{-(\alpha+2)} \\
&\quad \cdot \sum_{n=0}^{\infty}\left(2Re^{-(\alpha+2)}\right)^n \\
&= \frac{\tilde{\nu}_0\left(\left[0,\, \frac{\alpha+2}{2t}\right)\right)}{2t}(R-1)e^{-(\alpha+2)}\frac{\alpha+2}{1-2Re^{-(\alpha+2)}}.
\end{aligned}$$

Collecting these inequalities we get

$$e^{-\frac{\alpha+2}{2}}\frac{R-1}{\alpha}\left(1+\frac{R(\alpha+2)}{1-2Re^{-(\alpha+2)}}\right) \leq 1,$$

which is true for all $\alpha \geq \alpha'$, for some α' sufficiently large. Now, let A be an operator on $L^2_{v_\alpha}(0,\,\infty)$ induced by $W_{h,\varphi}$ acting on $\mathcal{B}^2_\alpha(\mathbb{C}_+)$, and let D be the isometric operator from $L^2_{w_0}(0,\,\infty)$ to $L^2_{v_\alpha}(0,\,\infty)$ of multiplication by $\sqrt{w_0(t)/v_\alpha(t)}$. Consider the following commutative diagram:

$$\begin{array}{ccc}
L^2_{w_0}(0,\,\infty) & \xrightarrow{D} & L^2_{v_\alpha}(0,\,\infty) \\
\Big\downarrow{\scriptstyle D^{-1}AD} & & \Big\downarrow{\scriptstyle A} \\
L^2_{w_0}(0,\,\infty) & \xleftarrow{D^{-1}} & L^2_{v_\alpha}(0,\,\infty)
\end{array}$$

Therefore, by Lemma 3.2.7, we have

$$\|W_{h,\varphi}\|_{\mathcal{B}^2_\alpha(\mathbb{C}_+)} = \|A\|_{L^2_{v_\alpha}(0,\,\infty)} = \left\|D^{-1}AD\right\|_{L^2_{w_0}(0,\,\infty)} \overset{(3.20)}{\leq} \|A\|_{L^2_{w_0}(0,\,\infty)} = \|W_{h,\varphi}\|_{A^2_{(0)}},$$

as required. □

Using essentially the same strategy we can prove a similar statement for $A^2_{(m)}$ spaces, for $m > 0$.

Theorem 3.2.9 *Suppose that $W_{h,\varphi}$ is causal. If $W_{h,\varphi}$ is bounded on $A^2_{(m)}$ and for some $\alpha \geq -1$ we have*

$$(1+\alpha)w_{(m)}(t) + tw'_{(m)}(t) \geq 0 \qquad (\forall t > 0), \tag{3.23}$$

then it is also bounded on $\mathcal{B}^2_\alpha(\mathbb{C}_+)$, and

$$\|W_{h,\varphi}\|_{\mathcal{B}^2_\alpha(\mathbb{C}_+)} \leq \|W_{h,\varphi}\|_{A^2_{(m)}}. \tag{3.24}$$

Conversely, if there exists $\alpha \geq -1$ such that $W_{h,\varphi}$ is bounded on $\mathcal{B}^2_\alpha(\mathbb{C}_+)$ and

$$\alpha w_{(m)}(t) + tw'_{(m)}(t) \leq 0 \qquad (\forall t > 0), \tag{3.25}$$

then $W_{h,\varphi}$ is bounded on $A^2_{(m)}$ and

$$\|W_{h,\varphi}\|_{A^2_{(m)}} \leq \|W_{h,\varphi}\|_{\mathcal{B}^2_\alpha(\mathbb{C}_+)}. \tag{3.26}$$

Proof

If there exists $\alpha \geq -1$ such that (3.23) holds, then $w_{(m)}(t)t^{1+\alpha}$ is increasing, and we get (3.24). Conversely, if (3.25) holds, then $(w_{(m)}(t)t^{1+\alpha})^{-1}$ is increasing, and we get (3.26). Then the result follows from Lemma 3.2.7. □

Let $a > 0$. We define a holomorphic map $\psi_a : \mathbb{C}_+ \longrightarrow \mathbb{C}_+$ by $\psi_a(z) = az$, for all $z \in \mathbb{C}_+$.

Proposition 3.2.10 (Corollary 3.4, p. 1094 in [22]) *Let $a > 0$. If $W_{h,\varphi}$ is bounded on $H^2(\mathbb{C}_+)$, then it is also bounded on each $A^2_{(0)}$, and*

$$\|W_{h,\varphi}\|_{A^2_{(0)}} \leq \|C_{\psi_a}\|_{A^2_{(0)}} \|C_{\psi_{1/a}}\|_{H^2(\mathbb{C}_+)} \|W_{h,\varphi}\|_{H^2(\mathbb{C}_+)}. \tag{3.27}$$

We can use Theorem 3.2.8 to state an inverted version of the above proposition.

Corollary 3.2.11 (Corollary 2 in [67]) *Let $a > 0$. If $W_{h,\varphi}$ is bounded on some $A^2_{(0)}$ space, then there exists $\alpha' \geq 0$ such that for all $\alpha \geq \alpha'$, we have*

$$\|W_{h,\varphi}\|_{\mathcal{B}^2_\alpha(\mathbb{C}_+)} \leq \|C_{\psi_a}\|_{\mathcal{B}^2_\alpha(\mathbb{C}_+)} \|C_{\psi_{1/a}}\|_{A^2_{(0)}} \|W_{h,\varphi}\|_{A^2_{(0)}}.$$

3.3 Composition operators

In this section we assume that φ is an analytic self-map of $\mathbb{C}_+$. The study of composition operators goes back to a paper [81] of Eric Nordgren. Composition operators for spaces of analytic functions on the disk are discussed extensively in [27]. In particular, as a consequence of the Littlewood subordination principle, every composition operator is bounded on Hardy spaces H^2 on the disk (see Theorem 8.3.2 in [21], p. 220). This is not the case for Hardy spaces on the complex half-plane. A description of bounded composition operators for $H^2(\mathbb{C}_+)$ corresponding to a rational symbol φ was given in [36]. In [77] Valentin Matache has shown that a composition operator C_φ is bounded on $H^2(\mathbb{C}_+)$ if and only if φ has a finite angular derivative at infinity. In [38] Samuel Elliott and Michael Jury have simplified Matache's proof and extended the result to $H^p(\mathbb{C}_+)$, also showing that if C_φ is bounded on $H^p(\mathbb{C}_+)$ then the norm of C_φ equals the p^{th} root of the angular derivative of φ at infinity. In [37] Sam Elliott and Andrew Wynn have shown that the condition for C_φ to be bounded on $\mathcal{B}^2_\alpha(\mathbb{C}_+)$, $\alpha \geq -1$ is the same as in the Hardy space case, and derived the expression for its norm. Recently, Riikka Schroderus has characterised spectra of fractional composition operators on the Hardy and weighted Bergman spaces of the half-plane (see [92]).

3.3.1 Boundedness

In this section we will cite boundedness conditions from [37] and [38], and will use the results from the previous section and [22] to show that this conditions remain the same if we replace weighted Bergman spaces with $A^2_{(0)}$ spaces.

Definition 3.3.1 *A sequence of points $z_n = x_n + iy_n \in \mathbb{C}_+$ is said to approach ∞ non-tangentially if $\lim_{n\longrightarrow\infty} x_n = \infty$ and $\sup_{n\in\mathbb{N}} |y_n|/x_n < \infty$. We also say that φ fixes infinity non-tangentially if $\varphi(z_n) \longrightarrow \infty$ whenever $z_n \longrightarrow \infty$ non-tangentially, and write*

$\varphi(\infty) = \infty$. *If it is the case and also the non-tangential limit*

$$\lim_{z\to\infty} \frac{z}{\varphi(z)} \tag{3.28}$$

exists and is finite, then we say that φ has a finite angular derivative at infinity and denote the above limit by $\varphi'(\infty)$.

Proposition 3.3.2 (Julia–Carathéodory Theorem in $\mathbb{C}_+$ - Proposition 2.2, p. 491 in [38]) *Let φ be an analytic self-map on $\mathbb{C}_+$. The following are equivalent:*

1. $\varphi(\infty)$ *and* $\varphi'(\infty)$ *exist;*
2. $\sup_{z\in\mathbb{C}_+} = \frac{\mathrm{Re}(z)}{\mathrm{Re}(\varphi(z))} < \infty$*;*
3. $\limsup_{z\to\infty} = \frac{\mathrm{Re}(z)}{\mathrm{Re}(\varphi(z))} < \infty$.

Moreover, the quantities in 2. and 3. are both equal to $\varphi'(\infty)$.

Theorem 3.3.3 (Theorem 3.1, p. 492 in [38] and Theorem 3.4, p. 377 in [37]) *The composition operator C_φ is bounded on $\mathcal{B}^2_\alpha(\mathbb{C}_+)$, $\alpha \geq -1$, if and only if φ has finite angular derivative at infinity, in which case*

$$\|C_\varphi\| = (\varphi'(\infty))^{\frac{2+\alpha}{2}}.$$

Note that the above boundedness condition can also be deduced from Remark 3.2.5 if $\alpha > -1$.

Proposition 3.3.4 (Proposition 3.5, p. 1094 in [22]) *Let $a > 0$ and let $\psi_a(z) = az$. Then*

$$\|C_{\psi_a}\|_{A^2_{(0)}} = \sqrt{\sup_{t>0} \frac{w_0(at)}{aw_0(t)}}. \tag{3.29}$$

In the remaining part of this section we will use the *Nevanlinna representation* of a holomorphic function $\varphi : \mathbb{C}_+ \longrightarrow \mathbb{C}_+$:

$$\varphi(z) = az + ib + \int_{\mathbb{R}} \left(\frac{1}{it+z} + \frac{it}{1+t^2} \right) d\mu(t) = az + ib + \int_{\mathbb{R}} \frac{1+itz}{it+z} \frac{d\mu(t)}{1+t^2}, \quad (3.30)$$

where $a \geq 0$, $b \in \mathbb{R}$ and μ is a non-negative Borel measure measure on $\mathbb{R}$ satisfying the following growth condition:

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty$$

(see 5.3 in [89], p. 84). Clearly

$$a = \lim_{\mathrm{Re}(z) \to \infty} \frac{\varphi(\mathrm{Re}(z))}{\mathrm{Re}(z)}.$$

Theorem 3.3.5 (Theorem 4 in [67]) *The composition operator C_φ is bounded on $A^2_{(0)}$ if and only if φ has a finite angular derivative at infinity. If C_φ is bounded, then*

$$\varphi'(\infty) \inf_{t>0} \frac{w_0(t)}{w_0(\varphi'(\infty)t)} \leq \|C_\varphi\|^2_{A^2_{(0)}} \leq \varphi'(\infty) \sup_{t>0} \frac{w_0(t/\varphi'(\infty))}{w_0(t)}.$$

Proof

Suppose, for contradiction, that C_φ is bounded on $A^2_{(0)}$, but φ does not have a finite angular derivative at infinity. By Proposition 3.3.2 we know that for each $n \geq 1$ there must exist $z_n \in \mathbb{C}_+$ such that

$$\frac{\mathrm{Re}(z_n)}{\mathrm{Re}(\phi(z_n))} > n. \quad (3.31)$$

Now,

$$\|C^*_\varphi\|^2 \geq \frac{\left\| k^{A^2_{(0)}}_{\varphi(z_n)} \right\|^2_{A^2_{(0)}}}{\left\| k^{A^2_{(0)}}_{z_n} \right\|^2_{A^2_{(0)}}} \overset{(1.11),(1.12)}{=} \frac{\int_0^\infty \frac{e^{-2t\,\mathrm{Re}(\varphi(z_n))}}{w_0(t)}\, dt}{\int_0^\infty \frac{e^{-2t\,\mathrm{Re}(z_n)}}{w_0(t)}\, dt} \overset{(3.31)}{\geq} \frac{\int_0^\infty \frac{e^{-2t\,\mathrm{Re}(z_n)/n}}{w_0(t)}\, dt}{\int_0^\infty \frac{e^{-2t\,\mathrm{Re}(z_n)}}{w_0(t)}\, dt}. \quad (3.32)$$

Since w_0, *qua definitione*, is non-increasing, we have that $w_0(nt) \leq w_0(t)$, for all $n \geq 1$, and consequently

$$\|C_\varphi^*\|^2 \overset{(3.32)}{\geq} \frac{\int_0^\infty \frac{e^{-2t\operatorname{Re}(z_n)}}{w_0(nt)} n\, dt}{\int_0^\infty \frac{e^{-2t\operatorname{Re}(z_n)}}{w_0(t)}\, dt} \geq n \frac{\int_0^\infty \frac{e^{-2t\operatorname{Re}(z_n)}}{w_0(t)}\, dt}{\int_0^\infty \frac{e^{-2t\operatorname{Re}(z_n)}}{w_0(t)}\, dt} = n,$$

for all $n \geq 1$, which is absurd, as it contradicts the boundedness of C_φ. So, if C_φ is bounded, then φ has a finite angular derivative at infinity and

$$\varphi'(\infty) \overset{\text{def}^{\text{n}}}{=} \lim_{\substack{z \longrightarrow \infty \\ \text{nontangentially}}} \frac{z}{\varphi(z)} = \lim_{\operatorname{Re}(z) \to \infty} \frac{\operatorname{Re}(z)}{\varphi(\operatorname{Re}(z))} = a^{-1},$$

where $0 < a < \infty$ is defined as in (3.30). Conversely, if φ has a finite angular derivative at infinity, then, by Theorem 3.3.3, C_φ is bounded on the Hardy space $H^2(\mathbb{C}_+)$, and, by Proposition 3.2.10, we get that it is also bounded on $A^2_{(0)}$ with

$$\|C_\varphi\|_{A^2_{(0)}} \leq \|C_{\psi_a}\|_{A^2_{(0)}} \|C_{\psi_{1/a}}\|_{H^2(\mathbb{C}_+)} \|C_\varphi\|_{H^2(\mathbb{C}_+)},$$

where $\psi_a(z) = az$. We can evaluate the RHS of this inequality using Theorem 3.3.3 and Proposition 3.3.4 to get

$$\|C_\varphi\|^2_{A^2_{(0)}} \leq \sup_{t>0} \frac{w_0(at)}{aw_0(t)} \cdot a \cdot \varphi'(\infty) = \varphi'(\infty) \sup_{t>0} \frac{w_0(t/\varphi'(\infty))}{w_0(t)}.$$

By Corollary 3.2.11 we also know that if C_φ is bounded on $A^2_{(0)}$, then there exists $\alpha > 0$ such that

$$\|C_\varphi\|_{A^2_{(0)}} \geq \|C_{\psi_a}\|_{\mathcal{B}^2_\alpha(\mathbb{C}_+)} \|C_{\psi_{1/a}}\|_{A^2_{(0)}} \|C_\varphi\|_{A^2_{(0)}}..$$

Again, we can evaluate the RHS of this inequality using Theorem 3.3.3 and Proposition 3.3.4 to get

$$\|C_\varphi\|^2_{A^2_{(0)}} \geq \varphi'(\infty)^{\alpha+2} a^{\alpha+1} \inf_{t>0} \frac{w_0(t)}{w_0(\varphi'(\infty)t)} = \varphi'(\infty) \inf_{t>0} \frac{w_0(t)}{w_0(\varphi'(\infty)t)}.$$

□

3.3.2 Compactness

In [76] Valentin Matache has shown that there exist no compact composition operator on $H^2(\mathbb{C}_+)$. This result was also obtained in [38] and extended to weighted Bergman spaces on the half-plane in [37]. In this subsection we will show that this is also the case for general $A^2_{(0)}$ spaces.

Definition 3.3.6 *The* essential norm of an operator, *denoted* $\|\cdot\|_e$ *is the distance in the operator norm from the set of compact operators (see [11]).*

Theorem 3.3.7 (Theorem 5 in [67]) *There is no compact composition operator on* $A^2_{(0)}$.

Proof

Let C_φ be a bounded operator on $A^2_{(0)}$. For any $\delta > 0$ we can choose a compact operator Q such that $\|C_\varphi\|_e + \delta \geq \|C_\varphi - Q\|$. By Lemma 1.2.7, the sequence $k_z^{A^2_{(0)}}/\|k_z^{A^2_{(0)}}\|$ tends to 0 weakly, as z approaches infinity, so $Q^*\left(k_z^{A^2_{(0)}}/\|k_z^{A^2_{(0)}}\|\right) \longrightarrow 0$, and consequently

$$\begin{aligned}
\|C_\varphi\|_e + \delta &\geq \|C_\varphi - Q\| \\
&\geq \limsup_{z\longrightarrow\infty} \frac{\left\|(C_\varphi - Q)^* k_z^{A^2_{(0)}}\right\|_{A^2_{(0)}}}{\left\|k_z^{A^2_{(0)}}\right\|_{A^2_{(0)}}} \\
&= \limsup_{z\longrightarrow\infty} \frac{\left\|C_\varphi^* k_z^{A^2_{(0)}}\right\|_{A^2_{(0)}}}{\left\|k_z^{A^2_{(0)}}\right\|_{A^2_{(0)}}} \\
&= \limsup_{z\longrightarrow\infty} \frac{\left\|k_{\varphi(z)}^{A^2_{(0)}}\right\|_{A^2_{(0)}}}{\left\|k_z^{A^2_{(0)}}\right\|_{A^2_{(0)}}}.
\end{aligned}$$

Suppose, for contradiction, that C_φ is compact, then the last quantity above must be equal to 0, and hence the limit of $\|k_{\varphi(z)}^{A^2_{(0)}}\|/\|k_z^{A^2_{(0)}}\|$ exists and is also equal to 0. That is, for each $\varepsilon > 0$ there exists $z_0 \in \mathbb{C}_+$ such that

$$\frac{\left\|k_{\varphi(z)}^{A^2_{(0)}}\right\|_{A^2_{(0)}}}{\left\|k_z^{A^2_{(0)}}\right\|_{A^2_{(0)}}} \overset{(1.11),(1.12)}{=} \sqrt{\frac{\int_0^\infty \frac{e^{-2t\operatorname{Re}(\varphi(z))}}{w_0(t)}dt}{\int_0^\infty \frac{e^{-2t\operatorname{Re}(z)}}{w_0(t)}dt}} < \varepsilon, \tag{3.33}$$

for all $z \in \mathbb{C}_+$ with $|z| \geq |z_0|$. Since C_φ is bounded and

$$\varphi'(\infty) = \limsup_{z\longrightarrow\infty} \frac{\operatorname{Re}(z)}{\operatorname{Re}(\varphi(z))},$$

for any $0 < \kappa < \varphi'(\infty)$, there exists a sequence $(z_j)_{j=1}^\infty$ with $|z_j| \geq |z_0|$, for all $j \geq 0$, such that

$$\frac{\operatorname{Re}(z)}{\operatorname{Re}(\varphi(z))} > \kappa, \qquad \left(\forall z \in \{z_j\}_{j=1}^\infty\right) \tag{3.34}$$

Let $\psi(z) = \kappa z$. If $z \in \{z_j\}_{j=1}^\infty$, then

$$\|C_\psi\|^2 \geq \frac{\left\|k_{\psi(\varphi(z))}^{A^2_{(0)}}\right\|^2_{A^2_{(0)}}}{\left\|k_{\varphi(z)}^{A^2_{(0)}}\right\|^2_{A^2_{(0)}}} \overset{(1.11),(1.12)}{=} \frac{\int_0^\infty \frac{e^{-2t\kappa\operatorname{Re}(\varphi(z))}}{w_0(t)}dt}{\int_0^\infty \frac{e^{-2t\operatorname{Re}(\varphi(z))}}{w_0(t)}dt} \overset{(3.34)}{\geq} \frac{\int_0^\infty \frac{e^{-2\operatorname{Re}(z)}}{w_0(t)}dt}{\int_0^\infty \frac{e^{-2t\operatorname{Re}(\varphi(z))}}{w_0(t)}dt} \overset{(3.33)}{>} \frac{1}{\varepsilon^2},$$

which is absurd. So $\|C_\varphi\|_e > 0$, and consequently C_φ is not compact. $\square$

Chapter 4

Laplace–Carleson embeddings and weighted infinite-time admissibility

Nous devons donc envisager l'état présent de l'univers comme l'effet de son état antérieur et comme la cause de celui qui va suivre. Une intelligence qui, pour un instant donné, connaîtrait toutes les forces dont la nature est animée, et la situation respective des êtres qui la composent, si d'ailleurs elle était assez vaste pour soumettre ces données à l'Analyse, embrasserait dans la même formule les mouvements des plus grands corps de l'univers et ceux du plus léger atome: rien ne serait incertain pour elle et l'avenir, comme le passé serait présent à ses yeux.[1]

PIERRE-SIMON DE LAPLACE, 1er marquis de Laplace, *Essai philosophique sur les probabilités*

[1]We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes

As it was hinted at the very beginning of this thesis, the spaces of analytic functions, which we portrayed in previous chapters, apart from undeniably interesting theoretical aspects, also possess an important practical attribute. Videlicet, they can be used to test admissibility of observation and control operators for linear evolution equation systems.

The results presented in this chapter have been published in [66].

4.1 Control and observation operators for semigroups of linear operators

4.1.1 Semigroups

Definition 4.1.1 *Let X be a Banach space, let $\{\mathbb{T}_t\}_{t\geq 0} \subseteq \mathscr{B}(X)$, and let I denote the identity operator on X. If*

- $\mathbb{T}_0 = I$,
- $\mathbb{T}_t\mathbb{T}_\tau = \mathbb{T}_{t+\tau} \qquad (\forall t,\, \tau \geq 0)$,

then forms $\{\mathbb{T}_t\}_{t\geq 0}$ a (one parameter) semigroup with respect to the operation of composition of operators. If we additionally have that

- $\lim_{t\to 0^+} \mathbb{T}_t x = x \qquad (\forall x \in X)$,

then we call $\{\mathbb{T}_t\}_{t\geq 0}$ a strongly continuous semigroup *(or C_0-semigroup) on X.*

Semigroups of linear operators were informally considered for the first time by Joseph-Louis Lagrange in [69] by considering the expression

$$\sum_{n=0}^{\infty} \frac{t^n f^{(n)}(x)}{n!} = \exp\left(t\frac{d}{dt}\right) f(x),$$

before the theory of linear operators was even invented. Another source of this notion could be found in Augustin-Louis Cauchy's *Cours d'Analyse* [20], Chapitre V, pp. 103-122, where he considered a functional equation

$$\varphi(x+y) = \varphi(x) \cdot \varphi(y).$$

And last but not least, in the context which is closest to the linear evolution equations that we consider later, we can also find it in Giuseppe Peano's solution to the system of ordinary differential equations with constant coefficients

$$\begin{aligned} \frac{dx_1(t)}{dt} &= \alpha_{1,1}x_1(t) + \cdots + \alpha_{1,n}x_n(t) \\ \vdots \quad & \quad \vdots \qquad\qquad\qquad \vdots \\ \frac{dx_n(t)}{dt} &= \alpha_{n,1}x_1(t) + \cdots + \alpha_{n,n}x_n(t) \end{aligned}$$

in [86], where the solution was found to be

$$\boldsymbol{x}(t) = e^{t\boldsymbol{\alpha}}\boldsymbol{x}(0),$$

where

$$\boldsymbol{x}(t) := \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad \boldsymbol{\alpha} := \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n} \end{pmatrix} \qquad \text{and} \qquad e^{t\boldsymbol{\alpha}} := \sum_{n=0}^{\infty} \frac{\boldsymbol{\alpha}^n t^n}{n!},$$

and, as Peano proved, the series is convergent. In the modern setting, the theory of semigroups arose from works of Jacques Hadamard [49] and Marshall Stone [96]. The contemporary codification of semigroups can be found for example in [39], [30], [46] and [85].

Definition 4.1.2 *Let $\{\mathbb{T}_t\}_{t\geq 0}$ be a semigroup on a Banach space X over the field of complex numbers. The linear operator A, defined by*

$$\begin{aligned} D(A) &:= \left\{ x \in X \; : \; \lim_{t\to 0^+} \frac{\mathbb{T}_t x - x}{t} \textit{ exists} \right\} \\ Ax &:= \lim_{t\to 0^+} \frac{\mathbb{T}_t x - x}{t} \qquad\qquad (\forall x \in D(A)), \end{aligned}$$

is called the infinitesimal generator *of the semigroup* $\mathbb{T}_t$ *and* $D(A)$ *is the* domain *of A.*

Infinitesimal generators for one-parameter semigroups were introduced by Carl Einar Hille in [54] and Kōsaku Yosida in [104].

Example 4.1.3 *If* $A \in \mathscr{B}(X)$*, then we can define*

$$\mathbb{T}_t := e^{tA} := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \qquad (\forall t \geq 0). \tag{4.1}$$

Certainly, this infinite series converges and

$$\left\| e^{tA} \right\| \leq e^{t\|A\|} \qquad (\forall t \geq 0),$$

so $\{\mathbb{T}_t\}_{t\geq 0}$ *is a* C_0*-semigroup. We also have that*

$$\|\mathbb{T}_t - I\| = t\|A\|e^{t\|A\|} \tag{4.2}$$

and

$$\left\| \frac{\mathbb{T}_t - I}{t} - A \right\| \leq \|A\| \|\mathbb{T}_t - I\|,$$

so A *is the infinitesimal generator for* $\{\mathbb{T}_t\}_{t\geq 0}$*. Semigroups which satisfy* (4.2) *are called* uniformly continuous *semigroups. It can be proved that the only uniformly continuous semigroups are those defined by* (4.1) *(see [91], p. 359).*

4.1.2 Linear evolution equations

Let A be the infinitesimal generator of a strongly continuous semigroup $\{\mathbb{T}_t\}_{t\geq 0}$ and let X, Y and U be Banach spaces. Consider the following linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) = x_0 \\ y(t) = Cx(t), \end{cases} \tag{4.3}$$

where $x : [0, \infty) \longrightarrow X$, $y : [0, \infty) \longrightarrow Y$, $u : [0, \infty) \longrightarrow U$ are maps; and $B : U \longrightarrow X$, $C : X \longrightarrow Y$ are (unbounded) operators. The spaces X, Y and U are called the *state*, the *output* and the *input* spaces respectively, and $x(t)$, $y(t)$ and $u(t)$ are the state, the output and the input at time t, correspondingly.

Time and change are unquestionably amongst the oldest philosophical conceptions, which, in Western tradition, can be traced at least to the Presocratics (see [59]). The first to distinguish between time and change was probably Aristotle in his *Physics*, Book IV, Chapters 10-14 (see [8] and [25]). We obviously owe the mathematical foundations of time-change related concepts to Sir Isaac Newton (see [79]). And even the shortest description of later developments in this area would certainly be beyond the scope of this thesis. Observe that if $B \equiv 0$, then the semigroup property of $\{\mathbb{T}_t\}_{t\geq 0}$ forces systems described by (4.3) to be entirely deterministic (justifying the choice of the quotation at the beginning of this chapter); something that Newton would surely reject. Without going into any argument on how realistic this setting is, we would like to point out that even a professedly probabilistic model such as the unitary time evolution of the individual state following Schrödinger's equation

$$\dot{\Psi}(t) = \frac{-i}{\hbar} H\Psi(t),$$

where $\hbar$ is the Planck constant and H is a Hamiltonian (see [70], § 1.5, p. 15), evolves deterministically in the absence of measurement. The modern theory of linear evolution equations is discussed for example in [93] and [101].

A common minimal assumption is that $B \in \mathscr{B}(U, X_{-1}(A))$ (i.e. the Banach space of bounded linear functionals from U into $X_{-1}(A)$) and $C \in \mathscr{B}(X_1(A), Y)$, where $X_1(A)$ denotes $D(A)$ equipped with the graph norm (see [83], p. 19) and $X_{-1}(X)$ is the completion of X with respect to the norm given by

$$\|x\|_{X_{-1}(A)} := \left\|(\beta I - A)^{-1}x\right\|_X, \qquad (\forall x \in X),$$

for some fixed $\beta \in \rho(A)$ (see § 2.10, pp. 59-65 in [101]).

The solution to the system (4.3) has a formal representation

$$(x(t),\, y(t)) = \left(\mathbb{T}_t x_0 + \int_0^t \mathbb{T}_{t-\tau} B u(\tau)\, d\tau,\; C\mathbb{T}_t x_0 + C \int_0^t \mathbb{T}_{t-\tau} B u(\tau)\, d\tau \right)$$

(see Proposition 2.6, p. 73 in [74]), which leads us to the notion of *admissibility*.

4.1.3 Admissibility

In this subsection we assume that A, B, C, $D(A)$, $\{\mathbb{T}_t\}_{t\geq 0}$ u, U, x, X, X_{-1}, X_1, y, Y are defined as in the previous subsection.

Definition 4.1.4 *Let* $1 \leq p < \infty$*, and let* $B \in \mathscr{B}(U, X_{-1}(A))$*. The control operator* B *is said to be* finite-time L^p admissible *for* $\{\mathbb{T}_t\}_{t\geq 0}$ *if and only if for some* $t > 0$ *we have*

$$\int_0^t \mathbb{T}_{t-\tau} B u(\tau)\, d\tau \in X \qquad (\forall u(t) \in L^p([0,\, \infty), U)),$$

and consequently there exists a constant $m_t > 0$ *such that*

$$\left\| \int_0^t \mathbb{T}_{t-\tau} B u(\tau)\, d\tau \right\|_X \leq m_t \|u\|_{L^p([0,\,\infty),\, U)} := m_t \left(\int_0^\infty \|u(\tau)\|_U^p \, d\tau \right)^{\frac{1}{p}}$$

for all $u(t) \in U$*. If the constant* m_t *can be chosen independently of* $t > 0$*, then we say that* B *is* (infinite-time) L^p-admissible *for* $\{\mathbb{T}_t\}_{t\geq 0}$*, in this case there exists a constant* $m > 0$ *such that*

$$\left\| \int_0^\infty \mathbb{T}_t B u(t)\, dt \right\|_X \leq m \|u\|_{L^p([0,\,\infty),\, U)} \qquad (\forall u(t) \in L^p([0,\, \infty), U)).$$

The notion of admissibility for control operators was introduced in [55]. Since then it has appeared in omnifarious contexts in various publications. Particularly valuable and concise treaties on this matter are [60] and Chapter 4 in [101].

Definition 4.1.5 *Let* $1 \leq p < \infty$*, let* $C \in (X_1(A), Y)$*, and assume that* $B = 0$*. The observation operator* C *is said to be* finite-time L^p-admissible *for* $\{\mathbb{T}_t\}_{t\geq 0}$ *if and* *only if for some* $t > 0$ *there exists a constant* $k_t > 0$ *such that*

$$\left(\int_0^t \|C\mathbb{T}_\tau x_0\|_Y^p \, d\tau\right)^{\frac{1}{p}} \leq k_t \|x_0\|_X \qquad (\forall x_0 \in D(A)).$$

If k_t *can be chosen independently of* $t > 0$*, then we say that* C *is* (infinite-time) L^p-admissible. *In this case, clearly, there exists* $m > 0$ *such that*

$$\|C\mathbb{T}.x_0\|_{L^p[(0,\infty),Y]} := \left(\int_0^\infty \|C\mathbb{T}_t x_0\|_Y^p \, dt\right)^{\frac{1}{p}} \leq k\|x_0\|_X \qquad (\forall x_0 \in D(A)).$$

It is clear from these definitions that there is a duality between the admissibility of control and observation operators. Namely, if X and Y are reflexive Banach spaces (see [72], p. 219), then B is an L^p-admissible control operator for a semigroup $\{\mathbb{T}_t\}_{t\geq 0}$ if and only if B^* is an $L^{p'}$-admissible observation for the adjoint semigroup $\{\mathbb{T}_t^*\}_{t\geq 0}$. This is presented in [101] (Theorem 4.4.4 in § 4.4, p. 127) for X being a Hilbert space and $p = 2$, and in [93] (§ 10.2, pp. 572-576) for general case.

The above definitions can be amended, replacing L^p norms with weighted L^p norms. That is, the control operator $B \in \mathscr{B}(U, X_{-1}(A))$ is said to be *(infinite-time)* L^p_w*-admissible* for $\{\mathbb{T}_t\}_{t\geq 0}$ if and only if there exists a constant $M > 0$ such that

$$\left\|\int_0^\infty \mathbb{T}_t Bu(t)\, dt\right\|_X \leq M\|u\|_{L^p_w[(0,\infty),U]} := \left(\int_0^\infty \|u(t)\|_U^p w(t)\, dt\right)^{\frac{1}{p}},$$

for all u in U. And similarly, the observation operator $C \in (X_1(A), Y)$ is said to be *(infinite-time)* L^p_w*-admissible* for $\{\mathbb{T}_t\}_{t\geq 0}$ if and only if if there exists a constant $K > 0$ such that

$$\|C\mathbb{T}.x_0\|_{L^p_w[(0,\infty),Y]} := \left(\int_0^\infty \|C\mathbb{T}_t x_0\|_Y^p w(t)\, dt\right)^{\frac{1}{p}} \leq K\|x_0\|_X \qquad (\forall x_0 \in D(A)).$$

Weighted infinite-time admissibility is debated in [48], [103], [62] and [66].

4.1.4 Laplace–Carleson embeddings and weighted infinite-time admissibility

Definition 4.1.6 *Let $1 \leq q < \infty$, and let X be a Banach space over a field $\mathbb{K}$ (real or complex). A sequence $(\phi_n)_{n=0}^{\infty}$ of vectors in X is called a* Schauder basis *for X if for all $x \in X$ there exists a unique sequence of scalars $(\alpha_n)_{n=0}^{\infty}$ in $\mathbb{K}$ such that*

$$x = \sum_{n=0}^{\infty} \alpha_n b_n,$$

with the series converging in the norm of X (see Definition 2.5.5, p. 72 in [82]). If there exist constants $0 < c \leq C < \infty$ such that for any sequence of scalars $(\beta_n)_{n=0}^{\infty} \in \ell^q$ we have

$$c \left(\sum_{n=0}^{\infty} |\beta_n|^q \right)^{\frac{1}{q}} \leq \left\| \sum_{n=0}^{\infty} \beta_n \phi_n \right\|_X \leq C \left(\sum_{n=0}^{\infty} |\beta_n|^q \right)^{\frac{1}{q}},$$

*then we say that $(\phi_n)_{n=0}^{\infty}$ is a q-*Riesz basis *for X.*

For the remaining part of this thesis we assume that A is an infinitesimal generator of a strongly continuous semigroup $\{\mathbb{T}_t\}_{t\geq 0}$ on a Banach space X, with a q-Riesz basis $(\phi_n)_{n=0}^{\infty}$, consisting of eigenvectors of A, with corresponding eigenvalues $(\lambda_n)_{n=0}^{\infty}$, each of which lies in the open left complex half-plane

$$\mathbb{C}_- := \{z \in \mathbb{C} \ : \ \mathrm{Re}(z) < 0\}.$$

This means that

$$\mathbb{T}_t \phi_n = e^{\lambda_n t} \phi_n,$$

(Lemma 1.9, Chapter II, p. 55 in [39]), and that we can identify X with the sequence space ℓ^q. We shall also assume that $U = Y = \mathbb{C}$.

The following two theorems, proved in [62], link admissibility of control and observation operators with Laplace–Carleson embeddings (that is, Carleson embeddings induced by the Laplace transform). These results were presented there for weighted L^2 spaces and unweighted L^p spaces on $(0, \infty)$, but the proofs remain valid for weighted L^p spaces too.

Theorem 4.1.7 (Theorem 2.1, p. 1301 in [62]) *Let $1 \leq p, q < \infty$. Let w be a measurable self-map on $(0, \infty)$, and let B be a bounded linear map from $\mathbb{C}$ to $X_{-1}(A)$ corresponding to the sequence $(b_k)_{k=0}^{\infty}$. The control operator B is L^p_w-admissible for $\{\mathbb{T}_t\}_{t\geq 0}$, that is, there exists a constant $m > 0$ such that*

$$\left\| \int_0^\infty \mathbb{T}_t Bu(t)\, dt \right\|_X \leq m \|u\|_{L^p_w(0,\,\infty)} \overset{defn}{=} m \left(\int_0^\infty |u(t)|^p w(t)\, dt \right)^{\frac{1}{p}},$$

for all $u \in L^p_w(0, \infty)$, if and only if the Laplace transform induces a continuous mapping from $L^p_w(0, \infty)$ into $L^q(\mathbb{C}_+, \mu)$, where μ is the measure given by $\sum_{k=1}^\infty |b_k|^q \delta_{-\lambda_k}$.

Note that for $1 < p < \infty$ and X a reflexive Banach space, we can associate the dual space of $L^p_w(0, \infty)$ with $L^{p'}_{w^{-p'/p}}(0, \infty)$ via the pairing

$$\langle f, g\rangle = \int_0^\infty f(t)g(t)\, dt \qquad\qquad (f \in L^p_w(0, \infty),\ g \in L^{p'}_{w^{-p'/p}}(0, \infty)).$$

Weighted admissibility and duality is presented in [48] (Remark 1.4, p. 2097). The duality argument there is given in terms of $w(t) = t^\alpha$, but it is easy to see that it remains true for any weight w.

Theorem 4.1.8 (Theorem 2.2, p. 1301 in [62]) *Let C be a bounded linear map from $X_1(A)$ to $\mathbb{C}$. The observation operator C is L^p_w-admissible for $\{\mathbb{T}_t\}_{t\geq 0}$, that is, there exists a constant $K > 0$ such that*

$$\|C\mathbb{T}_\cdot x\|_{L^p_w(0,\,\infty)} \overset{defn}{=} \left(\int_0^\infty |C\mathbb{T}_t x(t)|^p w(t)\, dt \right)^{1/p} \leq K\|x\|_X \qquad (\forall x \in \mathcal{D}(A)),$$

if and only if the Laplace transform induces a continuous mapping from $L^{p'}_{w^{-p'/p}}(0, \infty)$ into $L^{q'}(\mathbb{C}_+, \mu)$, where μ is the measure given by $\sum_{k=1}^\infty |c_k|^{q'} \delta_{-\lambda_k}$, $c_k := C\phi_k$, for all $k \in \mathbb{N}$, and $q' := q/(q-1)$ is the conjugate index of q.

4.2 Laplace–Carleson embeddings

4.2.1 Carleson measures for Hilbert spaces and weighted admissibility

The notion of admissibility was originally coined for the state space being a Hilbert space and $p = q = 2$ ([55]) . We know (by Theorems 1.1.3 and 4.1.7) that testing the admissibility criterion in this case is equivalent to testing the Carleson criterion for the Hardy space $H^2(\mathbb{C}_+)$. In this subsection we will generalise it to weighted L^2-admissibility, using the results from Chapter 2.

Proposition 4.2.1 (Proposition 1 and Corollary 1 in [66]) *Let $B \in \mathscr{B}(\mathbb{C}, X_{-1}(A))$ be a control operator corresponding to a sequence $(b_n)_{n=0}^{\infty}$.*

1. *The control operator B is $L^2_{w_{(m)}}$-admissible if and only if the linear map*

$$(a_l)_{l=0}^{\infty} \longmapsto \left(\sum_{k=0}^{\infty} a_k b_k \int_0^{\infty} \frac{\mathrm{Re}(e^{t(\overline{\lambda_k}+\lambda_l)})}{w_{(m)}(t)}\, dt \right)_{l=0}^{\infty}$$

is bounded on ℓ^2.

2. *If*

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left| b_k b_l \int_0^{\infty} \frac{\mathrm{Re}(e^{t(\overline{\lambda_k}+\lambda_l)})}{w_{(m)}(t)}\, dt \right|^2 < \infty, \tag{4.4}$$

then B is $L^2_{w_{(m)}}$-admissible.

3. *If B is $L^2_{w_{(m)}}$-admissible, then there exists a constant $C > 0$ such that*

$$\sum_{k \in E} \sum_{l \in E} \left| b_k b_l \int_0^{\infty} \frac{\mathrm{Re}(e^{t(\overline{\lambda_k}+\lambda_l)})}{w_{(m)}(t)}\, dt \right|^2 \leq C \sum_{k \in E} |b_k|^2,$$

for all $E \subseteq \mathbb{N}_0$.

Proof

1. This is just Theorem 4.1.7 and Lemma 2.2.1 applied to $\mathcal{H} = A^2_{(m)}$, $X = \mathbb{C}_+$ and $\mu = \sum_{k=0}^{\infty} |b_k|^2 \delta_{-\lambda_k}$. Note that $f \in L^2(\mathbb{C}_+, \mu)$ if and only if $(f(-\lambda_k)|b_k|)_{k=0}^{\infty} = (a_k)_{k=0}^{\infty}$, for some $(a_k)_{k=0}^{\infty} \in \ell^2$.

2. Let $(a_k)_{k=0}^{\infty} \in \ell^2$. By Cauchy's inequality we have

$$\sum_{l=0}^{\infty} |b_l|^2 \left| \sum_{k=0}^{\infty} a_k b_k \int_0^{\infty} \frac{\mathrm{Re}(e^{t(\overline{\lambda_k}+\lambda_l)})}{w_{(m)}(t)}\, dt \right|^2 \overset{\text{Cauchy}}{\leq} \|(a_k)_{k=0}^{\infty}\|^2 \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \left| b_k b_l \int_0^{\infty} \frac{\mathrm{Re}(e^{t(\overline{\lambda_k}+\lambda_l)})}{w_{(m)}(t)}\, dt \right|^2,$$

   so by part 1. we get that B is $L^2_{w_{(m)}}$-admissible.

3. This is part 1. applied to χ_E.

□

In Chapter 3 we have shown that there exist weights $w_{(m)}$ such that the corresponding spaces $A^2_{(m)}$ are Banach algebras with respect to pointwise multiplication (or, equivalently, $L^2_{w_{(m)}}(0, \infty)$ are Banach algebras with respect to the convolution operation). Thus we can state the following.

Proposition 4.2.2 (Proposition 2 in [66]) *Suppose that $L^2_{w_{(m)}}(0, \infty)$ is a Banach algebra with respect to convolution. If $(b_k)_{k=0}^{\infty} \in \ell^2$, then the control operator B is $L^2_{w_{(m)}}$-admissible.*

Proof

Using Theorem 3.1.5 we get

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left|b_k b_l \int_0^{\infty} \frac{\operatorname{Re}(e^{t(\overline{\lambda_k}+\lambda_l)})}{w_{(m)}(t)}\,dt\right|^2 &\leq \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left|b_k b_l \int_0^{\infty} \frac{e^{t(\overline{\lambda_k}+\lambda_l)}}{w_{(m)}(t)}\,dt\right|^2 \\ &\overset{\text{Schwarz}}{\leq} \left(\sum_{k=0}^{\infty}|b_k|^2 \int_0^{\infty} \frac{e^{2t\operatorname{Re}(\lambda_k)}}{w_{(m)}(t)}\,dt\right)^2 \\ &\overset{(3.8)}{\leq} \|(b_k)_{k=0}^{\infty}\|_{\ell^2}^4 < \infty,\end{aligned}$$

and the result follows by part 2. of the previous proposition. □

4.2.2 Laplace–Carleson embeddings for sectorial measures

Testing the boundedness of a Laplace–Carleson embedding for arbitrary $1 \leq p,\, q < \infty$ is generally very difficult. Nonetheless, we can obtain some partial results, if we impose some conditions on the support of the measure we are testing.

Proposition 4.2.3 (Proposition 3 in [66]) *Let* $1 < p < \infty$, $1 \leq q < \infty$, *let* w *be a measurable self-map on* $(0,\, \infty)$, *and suppose that* μ *be a positive Borel measure supported on* $(0,\, \infty)$. *If the Laplace–Carleson embedding*

$$\mathfrak{L} : L^p_w(0,\, \infty) \hookrightarrow L^q(\mathbb{C}_+,\, \mu)$$

is well-defined and bounded, then there exists $C(\mu) > 0$ *such that*

$$\mu(I) \leq C(\mu)\left(\int_0^{\infty} \frac{e^{-|I|p't}}{w^{\frac{1}{p-1}}(t)}\,dt\right)^{-\frac{q}{p'}},$$

for all intervals $I = (0,\, |I|]$, *provided that the integral on the right exists.*

Proof

Let $0 < x \leq |I|$ and $a > 0$. Then

$$\left|\mathfrak{L}\left[\frac{e^{-\cdot a}}{w^{\frac{1}{p-1}}(\cdot)}\right](x)\right| \overset{\text{def}^{\text{n}}}{=} \int_0^\infty \frac{e^{-t(a+x)}}{w^{\frac{1}{p-1}}(t)}\,dt \geq \int_0^\infty \frac{e^{-t(a+|I|)}}{w^{\frac{1}{p-1}}(t)}\,dt. \tag{4.5}$$

And hence

$$\begin{aligned}
\mu(I) &\overset{(4.5)}{\leq} \left(\int_0^\infty \frac{e^{-t(a+|I|)}}{w^{\frac{1}{p-1}}(t)}\,dt\right)^{-q} \int_I \left|\mathfrak{L}\left[\frac{e^{-\cdot a}}{w^{\frac{1}{p-1}}(\cdot)}\right](x)\right|^q d\mu(x)\\
&\leq \left(\int_0^\infty \frac{e^{-t(a+|I|)}}{w^{\frac{1}{p-1}}(t)}\,dt\right)^{-q} \int_{\mathbb{C}_+} \left|\mathfrak{L}\left[\frac{e^{-\cdot a}}{w^{\frac{1}{p-1}}(\cdot)}\right](x)\right|^q d\mu(x)\\
&\leq C\left(\int_0^\infty \frac{e^{-t(a+|I|)}}{w^{\frac{1}{p-1}}(t)}\,dt\right)^{-q} \left\|\frac{e^{-\cdot a}}{w^{\frac{1}{p-1}}(\cdot)}\right\|^q_{L^p_w(0,\,\infty)}\\
&= C\left(\int_0^\infty \frac{e^{-t(a+|I|)}}{w^{\frac{1}{p-1}}(t)}\,dt\right)^{-q} \left(\int_0^\infty \frac{e^{-apt}}{w^{\frac{p}{p-1}}(t)} w(t)\,dt\right)^{\frac{q}{p}}\\
&= C\left(\int_0^\infty \frac{e^{-t(a+|I|)}}{w^{\frac{1}{p-1}}(t)}\,dt\right)^{-q} \left(\int_0^\infty \frac{e^{-apt}}{w^{\frac{1}{p-1}}(t)}\,dt\right)^{\frac{q}{p}},
\end{aligned}$$

where $C > 0$ is the constant from the Laplace–Carleson embedding. Choosing $a = |I|/(p-1)$ gives us the desired result. □

Theorem 4.2.4 (Theorem 5 in [66]) *Given $0 < a \leq b < \infty$, let*

$$S_{(a,\,b]} := \{z \in \mathbb{C}_+ \ : \ a < \operatorname{Re}(z) \leq b\}.$$

If there exists a partition

$$P : 0 < \ldots \leq x_{-n} \leq \ldots \leq x_{-1} \leq x_0 \leq x_1 \leq \ldots \leq x_n \leq \ldots \qquad (n \in \mathbb{N})$$

of $(0,\,\infty)$ and a sequence $(c_n) \in \ell^1_{\mathbb{Z}}$ (the ℓ^1 sequence space indexed with $\mathbb{Z}$) such that

$$\mu(S_{(x_n,\,x_{n+1}]}) \leq |c_n| \left(\int_0^\infty \frac{e^{-p'tx_n}}{w^{\frac{1}{p-1}}(t)}\,dt\right)^{-\frac{q}{p'}} \qquad (\forall n \in \mathbb{Z}),$$

then the Laplace–Carleson embedding

$$\mathfrak{L} : L^p_w(0,\, \infty) \hookrightarrow L^q(\mathbb{C}_+,\, \mu)$$

is well-defined and bounded.

Proof

For any$z \in S_{(x_k,\, x_{k+1}]}$ and $f \in L^p_w(0,\, \infty)$ we have

$$|\mathfrak{L}f(z)| \leq \int_0^\infty e^{-tx_n}|f(t)|\, dt \overset{\text{Hölder}}{\leq} \left(\int_0^\infty \frac{e^{-p'tx_n}}{w^{\frac{1}{p-1}}(t)}\, dt\right)^{\frac{1}{p'}} \|f\|_{L^p_w(0,\, \infty)}, \tag{4.6}$$

so

$$\begin{aligned}\int_{\mathbb{C}_+} |\mathfrak{L}f|^q\, d\mu &\overset{(4.6)}{\leq} \|f\|^q_{L^p_w(0,\, \infty)} \sum_{n=-\infty}^{\infty} \left(\int_0^\infty \frac{e^{-p'tx_n}}{w^{\frac{1}{p-1}}(t)}\, dt\right)^{\frac{q}{p'}} \mu(S_{(x_n,\, x_{n+1}]}) \\ &\leq \|(c_n)\|_{\ell^1_{\mathbb{Z}}} \|f\|^q_{L^p_w(0,\, \infty)}.\end{aligned}$$

□

Definition 4.2.5 *Let* $1 \leq p \leq \infty$*, and let* $f \in L^p(\mathbb{R})$*. We define the* maximal function *of* f *to be*

$$Mf(x) := \sup_{r>0} \frac{1}{2r} \int_{|y|\leq r} |f(x-y)|\, dy.$$

The maximal function of f is finite almost everywhere (Theorem 1, § 3.1 in [95]). This theorem also states that

$$\|Mf\|_{L^p(0,\, \infty)} \lesssim \|f\|_{L^p(0,\, \infty)} \qquad (\forall f \in L^p(\mathbb{R})). \tag{4.7}$$

Lemma 4.2.6 (Lemma 1 in [66]) *Let* $1 \leq p < \infty$*, and let* $f \in L^p_w(0,\, \infty)$*. Then for all* $x > 0$ *and any partition*

$$P\, :\, 0 \leq \ldots \leq t_{-k} \leq \ldots \leq t_0 = 1 \leq t_1 \leq \ldots \leq t_k \leq \ldots \qquad (k \in \mathbb{N}_0)$$

of $(0,\,\infty)$*, with* $\inf_{k\in\mathbb{N}} t_{-k}=0$*, we have*

$$\int_0^\infty e^{-\frac{t}{x}}|f(t)|\,dt \le \Theta(P,w,x)xMg(x), \tag{4.8}$$

where

$$g(t)=\begin{cases} w^{1/p}(t)f(t), & \textit{if } t>0\\ 0 & \textit{if } t\le 0,\end{cases}$$

$g\in L^p(\mathbb{R})$*, and*

$$\Theta(P,w,x)=2\left[\sum_{k=-\infty}^{-1}\frac{e^{-t_k^*}}{w^{\frac{1}{p}}(t_k^*x)}(1-t_k)+\sum_{k=0}^{\infty}\frac{e^{-t_k^*}}{w^{\frac{1}{p}}(t_k^*x)}(t_{k+1}-1)\right],$$

where each t_k^* *is such that*

$$\frac{e^{-t_k^*}}{w^{\frac{1}{p}}(t_k^*x)}\ge\frac{e^{-t}}{w^{\frac{1}{p}}(tx)} \qquad (\forall t\in(t_k,\,t_{k+1})).$$

Proof

Let $r_k:=\max\{|1-t_k|,\,|1-t_{k+1}|\}$, for each k. Given $x>0$, we have

$$\begin{aligned}\int_0^\infty e^{-\frac{t}{x}}|f(t)|\,dt &= x\int_0^\infty e^{-t}|f(tx)|\,dt \le x\sum_{k=-\infty}^{\infty}\frac{e^{-t_k^*}}{w^{\frac{1}{p}}(t_k^*x)}\int_{t_k}^{t_{k+1}}|g(tx)|\,dt\\ &=\sum_{k=-\infty}^{\infty}\frac{e^{-t_k^*}}{w^{\frac{1}{p}}(t_k^*x)}\int_{(1-t_{k+1})x}^{(1-t_k)x}|g(x-y)|\,dy\\ &\le\sum_{k=-\infty}^{\infty}\frac{e^{-t_k^*}}{w^{\frac{1}{p}}(t_k^*x)}\frac{r_kx}{r_kx}\int_{|y|\le r_kx}|g(x-y)|\,dy\\ &\le 2\left[\sum_{k=-\infty}^{\infty}\frac{e^{-t_k^*}}{w^{\frac{1}{p}}(t_k^*x)}r_k\right]xMg(x).\end{aligned}$$

To get the required result, note that if $k\le -1$, then $t_{k+1}\le t_0=1$, and hence

$$1-t_k\ge 1-t_{k+1}\ge 0 \qquad\Longrightarrow\qquad r_k=|1-t_k|=1-t_k,$$

otherwise $t_{k+1} > 1$, so $t_k \geq 1$, and thus

$$0 \geq 1 - t_k \geq 1 - t_{k+1} \qquad \Longrightarrow \qquad r_k = |1 - t_{k+1}| = t_{k+1} - 1.$$

□

The following theorem has been proved in [61] (Theorem 3.3, p. 801) for the unweighted $L^p(0, \infty)$ case and in [66] for the weighted case.

Theorem 4.2.7 (Theorem 6 in [66]) *Let $1 < p \leq q < \infty$, let μ be a positive Borel measure on $\mathbb{C}_+$ supported only inside the sector*

$$\mathcal{S}(\theta) := \{z \in \mathbb{C}_+ \ : \ |\arg(z)| < \theta\},$$

for some $0 \leq \theta < \pi/2$, and let $\alpha < p - 1$. For an interval $I = (0, |I|) \subset \mathbb{R}$ we define

$$\Delta_I := \{z \in \mathcal{S}(\theta) \ : \ \mathrm{Re}(z) \leq |I|\}.$$

The Laplace–Carleson embedding

$$\mathfrak{L} : L^p_{t^\alpha}(0, \infty) \hookrightarrow L^q(\mathbb{C}_+, \mu)$$

is well-defined and bounded if and only if there exists a constant $C(\mu) > 0$ such that

$$\mu(\Delta_I) \leq C(\mu)|I|^{\frac{q}{p'}\left(1 - \frac{\alpha}{p-1}\right)}, \tag{4.9}$$

for all intervals $I = (0, |I|) \subset \mathbb{R}$.

Proof

Suppose first that (4.9) holds. Let

$$T_n := \left\{z \in \mathcal{S}(\theta) \ : \ 2^{n-1} < \mathrm{Re}(z) \leq 2^n\right\} \subset \Delta_{(0,\, 2^n)} \qquad (n \in \mathbb{Z}),$$

and let also $x_n = 2^{-n+1}$. Clearly

$$\mathcal{S}(\theta) = \bigcup_{n\in\mathbb{Z}} T_n \qquad \text{and} \qquad \mu(T_n) \leq \mu(\Delta_{(0,2^n)}) \overset{(4.9)}{\leq} C(\mu) x_n^{-\frac{q}{p'}\left(1-\frac{\alpha}{p-1}\right)}.$$

By the previous lemma we have that

$$|\mathfrak{L}f(z)| \leq \int_0^\infty e^{-\frac{t}{x_n}} |f(t)|\, dt \overset{(4.8)}{\leq} \Theta(P, t^\alpha, x_n) x_n Mg(x_n), \tag{4.10}$$

for all $z \in T_n$ (Θ and g are defined as in Lemma 4.2.6). Note that the choice of t_k^* does not depend on x_n, since

$$\frac{e^{-t_k^*}}{(t_k^* x_n)^{\frac{\alpha}{p}}} \geq \frac{e^{-t}}{(t x_n)^{\frac{\alpha}{p}}} \;\; \forall t \in (t_k,\, t_{k+1}) \quad \Longleftrightarrow \quad \frac{e^{-t_k^*}}{(t_k^*)^{\frac{\alpha}{p}}} \geq \frac{e^{-t}}{t^{\frac{\alpha}{p}}} \;\; \forall t \in (t_k,\, t_{k+1}),$$

and there exists a partition P of $(0,\, \infty)$, for which $\Theta(P, t^\alpha, x_n)$ converges (since $\alpha < p$), so, fixing P, we can set $D_\Theta := x_n^{\frac{\alpha}{p}} \Theta(P, t^\alpha, x_n)$, which, by the definition of Θ, is a constant depending on P and α only. Thus we have

$$\begin{aligned}
\int_{\mathcal{S}(\theta)} |\mathfrak{L}f|^q \, d\mu &\overset{(4.10)}{\leq} D_\Theta \sum_{n=-\infty}^{\infty} \left(x_n^{1-\frac{\alpha}{p}} Mg(x_n) \right)^q \mu(T_n) \\
&\leq C(\mu) D_\Theta \sum_{n=-\infty}^{\infty} x_n^{q\left(1-\frac{\alpha}{p}\right) - \frac{q}{p'}\left(1-\frac{\alpha}{p-1}\right)} Mg(x_n)^q \\
&= C(\mu) D_\Theta \sum_{n=-\infty}^{\infty} x_n^{q\left(1-\frac{\alpha}{p}-\frac{1}{p'}+\frac{\alpha}{p}\right)} Mg(x_n)^q \\
&= C(\mu) D_\Theta \sum_{n=-\infty}^{\infty} (x_n Mg(x_n)^p)^{\frac{q}{p}} \\
&\leq C(\mu) D_\Theta \left(\sum_{n=-\infty}^{\infty} x_n Mg(x_n)^p \right)^{\frac{q}{p}} \\
&\overset{(4.7)}{\lesssim} \|g\|_{L^p(0,\infty)}^q \\
&= \|f\|_{L^p_{t^\alpha}(0,\infty)}^q .
\end{aligned}$$

Now suppose that the converse is true. For each $z \in \Delta_I$ we have $|z| \leq |I| \sec(\theta)$, so

$$\left| \mathfrak{L}\left[\frac{e^{-|I|\sec(\theta)t}}{t^{\frac{\alpha}{p-1}}} \right](z) \right| = \frac{\Gamma\left(1-\frac{\alpha}{p-1}\right)}{|z+|I|\sec(\theta)|^{1-\frac{\alpha}{p-1}}} \geq \frac{\Gamma\left(1-\frac{\alpha}{p-1}\right)}{(2|I|\sec(\theta))^{1-\frac{\alpha}{p-1}}}.$$

And therefore we have

$$\begin{aligned}
\mu(\Delta_I) &\lessapprox |I|^{q\left(1-\frac{\alpha}{p-1}\right)} \int_{\mathcal{S}(\theta)} \left| \mathfrak{L}\left[\frac{e^{-|I|\sec(\theta)t}}{t^{\frac{\alpha}{p-1}}}\right](z)\right|^q \, d\mu(z) \\
&\lessapprox |I|^{q\left(1-\frac{\alpha}{p-1}\right)} \left\| \frac{e^{-|I|\sec(\theta)t}}{t^{\frac{\alpha}{p-1}}} \right\|^q_{L^p_{t^\alpha}(0,\infty)} \\
&= |I|^{q\left(1-\frac{\alpha}{p-1}\right)} \left(\int_0^\infty \frac{e^{-|I|p\sec(\theta)t}}{t^{\frac{\alpha}{p-1}}} \, dt \right)^{\frac{q}{p}} \\
&\lessapprox |I|^{q\left(1-\frac{\alpha}{p-1}\right)} |I|^{-\frac{q}{p}\left(1-\frac{\alpha}{p-1}\right)} \\
&= |I|^{\frac{q}{p'}\left(1-\frac{\alpha}{p-1}\right)},
\end{aligned}$$

as required. □

Corollary 4.2.8 (Corollary 2 in [66]) *Let $1 < p \leq q < \infty$, let μ be a positive Borel measure on $\mathbb{C}_+$ supported only inside the sector $\mathcal{S}(\theta)$, $0 \leq \theta < \pi/2$. Suppose that*

$$\sup_{t>0} \frac{t^\alpha}{w(t)} < \infty,$$

for some $\alpha < p-1$. If, for some family of intervals $(I_n)_{n\in\mathbb{Z}} = ((0,\, 2^n|I_0|))_{n\in\mathbb{Z}}$, there exists a constant $C(\mu) > 0$ such that

$$\mu(\Delta_{I_n}) \leq C(\mu)(|I_n|)^{\frac{q}{p'}\left(1-\frac{\alpha}{p-1}\right)} \qquad (\forall n \in \mathbb{Z}),$$

then the Laplace–Carleson embedding

$$\mathfrak{L} : L^p_w(0,\,\infty) \longrightarrow L^q(\mathbb{C}_+,\,\mu)$$

is well-defined and bounded.

Proof

By the previous theorem we get that

$$\int_{\mathcal{S}(\theta)} |\mathfrak{L}f|^q \, d\mu \lessapprox \|f\|^q_{L^p_{t^\alpha}(0,\infty)} \leq \left(\sup_{t>0} \frac{t^\alpha}{w(t)}\right)^{\frac{q}{p}} \|f\|^q_{L^p_w(0,\infty)}.$$

□

Corollary 4.2.9 (Corollary 3 in [66]) *Let B and μ be defined as in Theorem 4.1.7, let $1 < p \leq q < \infty$ and $\alpha < p-1$, and suppose that there exists $0 < \theta < \pi/2$ such that*

$$\operatorname{Im}(-\lambda_k) < \operatorname{Re}(-\lambda_k)\tan\theta \qquad (\forall k \in \mathbb{N}).$$

Then the control operator B is $L^p_{t^\alpha}$-admissible if and only if there exists a constant $C(\mu) > 0$ such that

$$\sum_{k\in E} |b_k|^q \leq C(\mu) \max_{k\in E} \left[\operatorname{Re}(-\lambda_k)\right]^{\frac{q}{p'}\left(1-\frac{\alpha}{p-1}\right)} \qquad (\forall E \subseteq \mathbb{N}).$$

Example 4.2.10 *Consider the following one-dimensional heat PDE on the interval $[0,\, 1]$:*

$$\begin{cases} \frac{\partial z}{\partial t}(\zeta,\, t) = \frac{\partial^2 z}{\partial \zeta^2}(\zeta,\, t) \\ \frac{\partial z}{\partial \zeta}(0,\, t) = 0 \\ \frac{\partial z}{\partial \zeta}(1,\, t) = u(t) \\ z(\zeta, 0) = z_0(\zeta) \end{cases} \qquad \zeta \in (0,1),\ t \geq 0.$$

According to Example 3.6 in [62], this system can be expressed in the form (4.3) *with $X = \ell^2$, $Ae_n = -n^2\pi^2 e_n$ (where (e_n) is the canonical basis for ℓ^2), and $b_n = 1$, for each $n \in \mathbb{N}$. For $1 < p \leq 2$ and $\alpha < p-1$, by the previous corollary, we know that B is $L^p_{t^\alpha}$-admissible if and only if $p \geq \frac{4}{3}(\alpha+1)$.*

4.2.3 Sectorial Carleson measures for $A^2_{(m)}$ spaces

Using methods similar to those used in the previous subsection, we could find a sufficient condition for a sectorial measure to be Carleson for an $A^2_{(m)}$ space.

Theorem 4.2.11 (Theorem 7 in [66]) *Let μ be a positive Borel measure supported only in the sector $\mathcal{S}(\theta)$, $0 < \theta < \pi/2$. If there exists an interval $I \subset i\mathbb{R}$, centred at 0, and* *a constant $C(\mu) > 0$ such that*

$$\mu\left(Q(2^k|I|)\right) \leq C(\mu)\left[\left(\nu_0\left(Q(2^k|I|)\right)\right)^{-\frac{1}{2}} + \left(\sum_{n=0}^{m}\frac{\nu_n\left(Q(2^k|I|)\right)}{(2^k|I|)^{2n}}\right)^{-\frac{1}{2}}\right]^{-2}, \qquad (4.11)$$

for all $k \in \mathbb{Z}$, then μ is a Carleson measure for $A^2_{(m)}$.

Proof

For all $t,\ x > 0$ we have

$$\begin{aligned} w_{(m)}(tx) &\stackrel{\text{def}^{\text{n}}}{=} 2\pi\sum_{n=0}^{m}(tx)^{2n}\int_0^\infty e^{-2rtx}\,d\tilde{\nu}_n(r) \\ &\geq 2\pi\sum_{n=0}^{m}t^{2n}2^{2n}\left(\frac{x}{2}\right)^{2n}e^{-t}\tilde{\nu}_n\left[0,\ \frac{1}{2x}\right) \\ &\geq 2\pi\sum_{n=0}^{m}t^{2n}\left(\frac{x}{2}\right)^{2n}e^{-t}\frac{\tilde{\nu}_n\left[0,\ \frac{2}{x}\right)}{R_n^2}, \end{aligned}$$

where each R_n is the supremum obtained from the (Δ_2)-condition, corresponding to $\tilde{\nu}_n$. Clearly, we have that

$$w_{(m)}(tx) \geq 2\pi e^{-t}\frac{\tilde{\nu}_0\left[0,\ \frac{2}{x}\right)}{R_0^2}, \qquad (\forall t, x > 0),$$

and

$$w_{(m)}(tx) \geq 2\pi\sum_{n=0}^{m}\left(\frac{x}{2}\right)^{2n}e^{-t}\frac{\tilde{\nu}_n\left[0,\ \frac{2}{x}\right)}{R_n^2}, \qquad (\forall x > 0,\ t \geq 1).$$

Let

$$P\ :\ 0 = \ldots = t_{-k} = \ldots = t_{-1} < t_0 = 1 \leq t_1 \leq \ldots \leq t_k \leq \ldots, \qquad (k \in \mathbb{N}),$$

be a partition of $[0,\ \infty)$, and let $x_k = 2^{-k+1}|I|^{-1}$, $k \in \mathbb{Z}$. Then

$$\begin{aligned} \Theta(P,\ w_{(m)}, x_k) &\stackrel{\text{def}^{\text{n}}}{=} 2\left[\frac{e^{-t^*_{-1}}}{\sqrt{w_{(m)}(t^*_{-1}x_k)}} + \sum_{l=0}^{\infty}\frac{e^{-t^*_l}}{\sqrt{w_{(m)}(t^*_l x_k)}}(t_{l+1}-1)\right] \\ &\leq \sqrt{\frac{2}{\pi}}\left[\frac{R_0}{\sqrt{\tilde{\nu}_0\left[0,\ \frac{2}{x_k}\right)}} + \frac{\sum_{l=0}^{\infty}e^{-\frac{t_l}{2}}t_{l+1}}{\sqrt{\sum_{n=0}^{m}\left(\frac{x_k}{2}\right)^{2n}\frac{\tilde{\nu}_n\left[0,\ \frac{2}{x_k}\right)}{R_n^2}}}\right]. \end{aligned}$$

And by Lemma 4.2.6 we get that for any $z \in T_k$

$$
\begin{aligned}
|\mathfrak{L}f(z)| &\leq \sqrt{\frac{2}{\pi}} \left[\frac{R_0}{\sqrt{\tilde{\nu}_0 \left[0,\ \frac{2}{x_k}\right)}} + \frac{\sum_{l=0}^{\infty} e^{-\frac{t_l}{2}} t_{l+1}}{\sqrt{\sum_{n=0}^{m} \left(\frac{x_k}{2}\right)^{2n} \frac{\tilde{\nu}_n \left[0,\ \frac{2}{x_k}\right)}{R_n^2}}} \right] x_k M g(x_k) \\
&= \sqrt{\frac{2}{\pi}} \left[\frac{R_0}{\sqrt{\frac{1}{x_k} \tilde{\nu}_0 \left[0,\ \frac{2}{x_k}\right)}} + \frac{\sum_{l=0}^{\infty} e^{-\frac{t_l}{2}} t_{l+1}}{\sqrt{\sum_{n=0}^{m} \left(\frac{x_k}{2}\right)^{2n-1} \frac{\tilde{\nu}_n \left[0,\ \frac{2}{x_k}\right)}{R_n^2}}} \right] \sqrt{x_k} M g(x_k) \\
&\lessapprox \left[\left(\nu_0 \left(Q(2^k|I|)\right)\right)^{-\frac{1}{2}} + \left(\sum_{n=0}^{m} \frac{\nu_n \left(Q(2^k|I|)\right)}{(2^k|I|)^{2n}} \right)^{-\frac{1}{2}} \right] \sqrt{x_k} M g(x),
\end{aligned}
$$

so for any $\mathfrak{L}f = F \in A^2_{(m)}$ we have

$$
\int_{\mathbb{C}_+} |F|^2 \, d\mu = \int_{\mathcal{S}(\theta)} |\mathfrak{L}f|^2 \, d\mu \lessapprox \sum_{k=-\infty}^{\infty} x_k (M g(x_k))^2 \lessapprox \|f\|^2_{L^2_{w_{(m)}}(0,\infty)} = \|F\|^2_{A^2_{(m)}},
$$

as required. □

Appendix

A Index of notation

$=:$ - the LHS is defined to be the RHS

$\stackrel{\text{def}^{\text{n}}}{=}$ - the equality between the LHS and the RHS follows from the definition

$\stackrel{P}{=}$ / $\stackrel{P}{\leq}$ - the equality/inequality follows from the property P

$\lessapprox$ - the LHS is less than or equal to the RHS up to a constant factor, not depending on variables on either side of the inequality

$\mathbb{N}$ - set of positive integers; counting numbers, i.e. $\{1, 2, 3, \ldots\}$

$\mathbb{N}_0$ - set of non-negative integers; natural numbers, i.e. $\{0, 1, 2, \ldots\}$

$\mathbb{Z}$ - set of integers; whole numbers, i.e. $\{\ldots, -1, 0, 1, \ldots\}$

$\mathbb{R}$ - set of real numbers; real line

$\mathbb{C}$ - set of complex numbers; complex plane

$A^2_{(m)}$ - pp. 11, 31

A^p_ν - Zen space, pp. 3, 30

$A^p(\mathbb{C}_+, (\nu_n)_{n=0}^m)$ - pp. 8, 31

$A^p_\mu(\mathbb{C}_+, (\nu_r)_{r\in M})$ - pp. 27, 32

$\mathscr{A}^p_m$ - p. 68

$B(\cdot, \cdot)$ - beta function, p. 18

$B_r(z)$ - open ball in $\mathbb{C}$, centred at z, with radius $r > 0$, p. 48

$\overline{B_r(z)}$ - closed ball in $\mathbb{C}$, centred at z, with radius $r > 0$, p. 75

$\mathcal{B}^2$ - (unweighted Hilbert) Bergman space on the open unit disk of the complex plane, p. 21

$\mathcal{B}^2_\alpha$ - weighted (Hilbert) Bergman space on the open unit disk of the complex plane, p. 18

$\mathcal{B}^2(\mathbb{C}_+)$ - (unweighted Hilbert) Bergman space on the open right complex half-plane, pp. 22, 30

$\mathcal{B}^p_\alpha(\mathbb{C}_+)$ - weighted Bergman space on the open right complex half-plane, pp. 3, 30

$\mathscr{B}(U)$ - Banach algebra of bounded linear operators on a Banach space U, p. 60

$\mathscr{B}(X, Y)$ - Banach space of bounded linear functionals from a Banach space X to a Banach space Y, p. 97

C_0-semigroup - strongly continuous semigroup, p. 94

C_φ - composition operator corresponding to symbol φ, p. **??**

$CM(B)$ - set of Carleson measures for a Banach space B, p. 34

$\| \cdot \|_{CM(B)}$ - p. 34

$\mathscr{C}_0(i\mathbb{R})$ - vector space of functions continuous on $i\mathbb{R}$ and vanishing at infinity, p. 66

$\mathbb{C}_+$ - open right complex half-plane, p. 2

$\overline{\mathbb{C}_+}$ - p. 3

$\chi(E)$ - characteristic function of a set E, p. 40

$d_H(z_1, z_2)$ - (Poincaré) hyperbolic right half-plane distance, p. 46

$D(A)$ - domain of an infinitesimal generator A, p. 95

$\mathcal{D}(\cdot)$ - Dirichlet integral on the open unit disk of the complex plane, p. 21

$\mathcal{D}(\cdot, \cdot)$ - Dirichlet semi-inner product, p. 21

$\mathcal{D}$ - (classical) Dirichlet space on the open unit disk of the complex plane, p. 21

$\mathcal{D}_\alpha$ - weighted Dirichlet space on the open unit disk of the complex plane, p. 22

$\mathcal{D}(\mathbb{C}_+)$ - (unweighted) Dirichlet space on the open right complex half-plane, pp. 11, 22, 31

$\mathcal{D}_\alpha(\mathbb{C}_+)$ - weighted Dirichlet space on the open right complex half-plane, pp. 25, 31

$\mathcal{D}'(\mathbb{C}_+)$ - (unweighted) variant Dirichlet space on the open right complex half-plane,

Bibliography

[1] M. ABRAMOWITZ, I. A. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards (1964).

[2] P. R. AHERN, J. BRUNA, *Maximal and Area Integral Characterizations of Hardy–Sobolev Spaces in the Unit Ball of* $\mathbb{C}^n$, Revista Matemática Iberoamericana **4** (1), European Mathematical Society Publishing House (1988), pp. 123-153.

[3] H. A. ALSAKER, *Multipliers of the Dirichlet space*, Master's Thesis in Mathematical Analysis, Department of Mathematics, University of Bergen, Norway (2009).

[4] W. A. AMBROSE, *Structure theorems for a special class of Banach algebras*, Transactions of the American Mathematical Society **57** (3), (1945), pp. 364-386.

[5] N. ARCOZZI, R. ROCHBERG, E. T. SAWYER, *Carleson measures for analytic Besov spaces*, Revista Matemática Iberoamericana **18** (2), European Mathematical Society Publishing House (2002), pp. 443-510.

[6] N. ARCOZZI, R. ROCHBERG, E. T. SAWYER, *Carleson measures for the Drury–Arveson Hardy space and other Besov–Sobolev spaces on complex balls*, Advances in Mathematics **218** (4), (2008), pp. 1107-1180.

[7] N. ARCOZZI, R. ROCHBERG, E. T. SAWYER, B. D. WICK, *The Dirichlet space: a survey*, New York Journal of Mathematics **17a**, (2011), pp. 45-86.

[8] ARISTOTLE, *Physics*, English translation by P. H. Wicksteed and F. M. Cornford, Harvard University Press (2014).

[9] N. ARONSZAJN, *Theory of reproducing kernels*, Transactions of the American Mathematical Society **68** (3), (1950), pp. 337-404.

[10] S. J. AXLER, P. S. BOURDON, W. RAMEY, *Harmonic Function Theory*, Graduate Text in Mathematics **137**, Springer (2007).

[11] S. J. AXLER, N. JEWELL, A. L. SHIELDS, *The essential norm of an operator and its adjoint*, Transactions of the American Mathematical Society **261** (1), (1980), pp. 159-167.

[12] A. C-A. BEURLING, *Études sur un problème de majoration, Thèse pur le doctorat, Upsal 1933*, The Collected Works of Arne Beurling **1**, Complex Analysis, Contemporary Mathematicians, Birkhäuser (1989).

[13] S. BOCHNER, K. CHANDRASEKKHARAN, *Fourier Transform*, Annals of Mathematics Studies **19**, Princeton University Press (1949).

[14] BODHIDHARMA, *The Zen teaching of Bodhidharma, Translated and with an Introduction by Red Pine*, North Point Press (1987).

[15] B. BOLLOBÁS, *Modern Graph Theory*, Graduate Texts in Mathematics **184**, Springer (1998).

[16] F. F. BONSALL, J. DUNCAN, *Complete Normed Algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete **80**, Springer-Verlag (1973).

[17] A. A. BORICHEV, H. HEDENMALM, *Completeness of translates in weighted spaces on the half-line*, Acta Mathematica **174**, Institut Mittag-Leffler (1995), pp. 1-84.

[18] J. BRUNA, J. CUFÍ, *Complex Analysis*, EMS Textbooks in Mathematics, European Mathematical Society (2013).

[19] L. A. E. CARLESON, *Interpolations by Bounded Analytic Functions and the Corona Problem*, Annals of Mathematics **76** (3), (1962), pp. 547-559.

[20] A-L. CAUCHY, *Cours d'analyse de l'École royale polytechnique. Première Partie. Analyse Algébraique* (1821).

[21] I. CHALENDAR, J. R. PARTINGTON, *Modern Approaches to the Invariant-Subspace Problem*, Cambridge Texts in Mathematics **188**, Cambridge University Press (2011).

[22] I. CHALENDAR, J. R. PARTINGTON, *Norm Estimates for Weighted Composition Operators on Spaces of Holomorphic Functions*, Complex Analysis and Operator Theory **8**, (2014), pp. 1087-1095.

[23] I. CHALENDAR, E. A. GALLARDO-GUTIÉRREZ, J. R. PARTINGTON, *Weighted composition operators on the Dirichlet space: boundedness and spectral properties*, Mathematische Annalen **363** (3), (2015), pp. 1265-1279.

[24] J. A. CIMA, W. R. WOGEN, *Carleson Measure Theorem for the Bergman Space on the Ball*, Journal of Operator Theory **7** (1), (1982), pp. 157-165.

[25] U. COOPE, *Time for Aristotle: Physics IV.10-14*, Clarendon Press (2005).

[26] M. D. CONTRERAS, A. G. HERNÁNDEZ-DÍAZ, *Weighted Composition Operators on Hardy Spaces*, Journal of Mathematical Analysis and Applications **263** (1), (2001), pp. 224-233.

[27] C. COWEN, B. MACCLUER, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press (1995).

[28] H. G. DALES, *Banach Algebras and Automatic Continuity*, London Mathematical Society Monographs New Series **24**, Oxford Science Publications, Claredon Press (2000).

[29] N. DAS, J. R. PARTINGTON, *Little Hankel operators on the half plane*, Integral Equations and Operator Theory **20** (3), Birkhäuser Basel (1994), pp. 306-324.

[30] E. B. DAVIES, *One-parameter semigroups*, London Mathematical Society Monographs **15**, Academic Press (1980).

[31] R. DIESTEL, *Graph Theory*, Graduate Texts in Mathematics **173**, Springer (2005).

[32] P. L. DUREN, *Theory of H^p spaces*, Pure and applied mathematics **38**, New York: Academic Press (1970).

[33] P. L. DUREN, E. A. GALLARDO-GUTIÉRREZ, A. MONTES-RODRÍGUEZ, *A Paley–Wiener theorem for Bergman spaces with application to invariant subspaces*, Bulletin of the London Mathematical Society **39** (3), (2007), pp. 459-466.

[34] P. L. DUREN, A. SCHUSTER, *Bergman spaces*, Mathematical Surveys and Monographs **100**, American Mathematical Society (2004).

[35] O. EL-FALLAH, K. KELLAY, J. MASHREGHI, T. RANSFORD, *A Primer on the Dirichlet Space*, Cambridge Tracts in Mathematics **203**, Cambridge University Press (2014).

[36] S. J. ELLIOTT, *Adjoints of composition operators on Hardy spaces of the half-plane*, Journal of Functional Analysis **256** (12), Elsevier (2009), pp. 4162-4186.

[37] S. J. ELLIOTT, A. WYNN, *Composition operators on weighted Bergman spaces of a half-plane*, Proceedings of the Edinburgh Mathematical Society **54**, (2011), pp. 373-379.

[38] S. J. ELLIOTT, M. T. JURY, *Composition operators on Hardy spaces of the half-plane*, Bulletin of the London Mathematical Society **44** (3), (2012), pp. 489-495.

[39] K-J. ENGEL, R. NAGEL, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics **194**, Springer (2000).

[40] F. FAÀ DI BRUNO, *Note sur une nouvelle formule de calcul differentiel*, The Quarterly Journal of Pure and Applied Mathematics **1**, (1857), pp. 359-360.

[41] E. A. GALLARDO-GUTIÉRREZ, R. KUMAR, J. R. PARTINGTON, *Boundedness, Compactness and Schatten-class Membership of Weighted Composition Operators*, Integral Equations and Operator Theory **67** (4), Springer Basel (2010), pp. 467-479.

[42] E. A. GALLARDO-GUTIÉRREZ, J. R. PARTINGTON, *A generalization of the Aleksandrov operator and adjoints of weighted composition operators*, Annales de l'Institut Fourier **63** (2), (2013), pp. 373-389.

[43] T. W. GAMELIN, *Uniform Algebras*, Prentice–Hall Series in Modern Mathematics, Prentice-Hall (1969).

[44] D. J. H. GARLING, *Inequalities: A journey into linear analysis*, Cambridge University Press (2007).

[45] J. B. GARNETT, *Bounded Analytic Functions*, Pure and Applied Mathematics, A Series of Monographs and Textbooks **96**, Academic Press (1981).

[46] J. A. GOLDSTEIN, *Semigroups of linear operators and applications*, Oxford Mathematical Monographs, Oxford Science Publications, Oxford University Press (1985).

[47] L. GRAFAKOS, *Classical Fourier Analysis*, Graduate Texts in Mathematics **249**, Springer (2008).

[48] B. H. HAAK, P. C. KUNSTMANN, *Weighted Admissibility and Wellposedness of Linear Systems in Banach Spaces*, SIAM Journal on Control and Optimization (6) 45, Society for Industrial and Applied Mathematics (2007), pp. 2094-2118.

[49] J. S. HADAMARD, *Le principe de Huygens et prolongement analytique*, Bulletin de la Société Mathématique de France **52**, (1924) pp. 241-278.

[50] P. R. HALMOS, *Measure Theory*, The University Series in Higher Mathematics, D. Van Nostrand Company, Inc. (1950).

[51] Z. HARPER, *Boundedness of Convolution Operators and Input-Output Maps Between Weighted Spaces*, Complex Analysis and Operator Theory **3** (1), Birkhäuser Basel (2009) pp. 113-146.

[52] Z. HARPER, *Laplace Transform Representations and Paley–Wiener Theorems for Functions on Vertical Strips*, Documenta Mathematica 15, (2010), pp. 235-254.

[53] H. HEDENMALM, B. KORENBLUM, K. ZHU, *Theory of Bergman Spaces*, Graduate Text in Mathematics **199**, Springer (2000).

[54] C. A. HILLE, *Representation of One-Parameter Semi-Groups of Linear Transformations*, Proceedings of the National Academy of Sciences of the United States of America **28** (5), pp. 175-178.

[55] L. F. HO, D. L. RUSSELL, *Admissible Input Elements for Systems in Hilbert Space and a Carleson Measure Criterion*, SIAM Journal on Control and Optimization **21** (4), Society for Industrial and Applied Mathematics (1983), pp. 614-640.

[56] K. HOFFMAN, *Banach Spaces of Analytic Functions*, Prentice Hall (1962).

[57] I. C. HOLST, *The music of Gustav Holst*, Oxford University Press (1951).

[58] L. V. HÖRMANDER, *L^p Estimates for (Pluri-) Subharmonic Functions*, Mathematica Scandinavica **20**, (1967), pp. 65-78.

[59] E. HUSSEY, *The Presocratics*, Bristol Classical Press (2010).

[60] B. JACOB, J. R. PARTINGTON, *Admissibility of Control and Observation Operators for Semigroups: A Survey*, Current Trends in Operator Theory and its Applications, Operator Theory: Advances and Applications **149**, pp. 199-221.

[61] B. JACOB, J. R. PARTINGTON, S. POTT, *On Laplace–Carleson embedding theorems*, Journal of Functional Analysis **264** (3), (2013), pp. 783-814.

[62] B. JACOB, J. R. PARTINGTON, S. POTT, *Applications of Laplace–Carleson Embeddings to Admissibility and Controllability*, SIAM Journal on Control and Optimization **52** (2), Society for Industrial and Applied Mathematics (2014), pp. 1299-1313.

[63] R. KERMAN, E. SAWYER, *Carleson measures and multipliers of the Dirichlet-type spaces*, Transactions of the American Mathematical Society **309** (1), (1988), pp. 87-98.

[64] A. S. KUCIK, *Carleson measures for Hilbert spaces of analytic functions on the complex half-plane*, Journal of Mathematical Analysis and Applications **445** (1), Elsevier (2017), pp. 476-497.

[65] A. S. KUCIK, *Multipliers of Hilbert Spaces of Analytic Functions on the Complex Half-Plane*, Operators and Matrices **11** (2), Ele-Math (2017), pp. 435-453.

[66] A. S. KUCIK, *Laplace–Carleson embeddings and weighted infinite-time admissibility*, Mathematics of Control, Signals, and Systems (in press), Springer (2017), available at `https://arxiv.org/abs/1606.05479`.

[67] A. S. KUCIK, *Weighted composition operators on spaces of analytic functions on the complex half-plane*, Complex Analysis and Operator Theory (in press), Springer (2017), available at `http://rdcu.be/s48T`.

[68] R. KUMAR, J. R. PARTINGTON, *Weighted Composition Operators on Hardy and Bergman Spaces*, Operator Theory - Advances and Applications **153**, Birkhäuser Verlag Basel (2004), pp. 157-167.

[69] J-L. LAGRANGE, *Sur une nouvelle espèce de calcul relatif à la différentiation et à l'intégration des quantités variables*, Nouveaux mémoires de l'Académie royale des sciences et belles-lettres de Berlin (1772), pp. 441-476.

[70] O. L. DE LANGE, R. E. RAAB, *Operator Methods in Quantum Mechanics*, Oxford Science Publications, Clarendon Press (1991).

[71] P-S. DE LAPLACE, *Essai philosophique sur les probabilités* (1840).

[72] E. R. LORCH, *On a calculus of operators in reflexive vector spaces*, Transactions of the American Mathematical Society **45** (2), (1939), pp. 217-234.

[73] D. H. LUECKING, *A Technique for Characterizing Carleson Measures on Bergman Spaces*, Proceedings of the American Mathematical Society **87** (4), (1983), pp. 656-660.

[74] J. MALINEN, O. J. STAFFANS, G. WEISS *When is a linear system conservative?*, Quarterly of Applied Mathematics **64** (1), (2006), pp. 61-91.

[75] J. MASHREGHI, *Representation Theorems in Hardy Spaces*, London Mathematical Society Student Texts **74**, Cambridge University Press (2009).

[76] V. MATACHE, *Composition operators on Hardy spaces of a half-plane*, Proceedings of the American Mathematical Society **127** (5), (1999), pp. 1483-1491.

[77] V. MATACHE, *Weighted Composition Operators on H^2 and Applications*, Complex Analysis and Operator Theory **2** (1), Birkhäuser Basel (2008), pp. 169-197.

[78] M. G. MITTAG-LEFFLER, M. PLANCHEREL, *Contribution à l'étude de la représentation d'une fonction arbitraire par les intégrales définies*, Rendiconti del Circolo Matematico di Palermo **30** (1), (1910), pp. 289-335.

[79] I. NEWTON, *Philosophiæ Naturalis Principia Mathematica*, Jussu Societatis Regiæ ac Typis Josephi Streater. Prostat apud plures Bibliopolas (1687).

[80] N. K. NIKOLSKII, *Spectral synthesis for a shift operator and zeros in certain classes of analytic functions smooth up to the boundary*, Soviet Mathematics (translated from Doklady Akademii Nauk SSSR) (1) **11**, (1970), pp. 206-209.

[81] E. A. NORDGREN, *Composition Operators*, Canadian Journal of Mathematics, **20**, Canadian Mathematical Society (1968), pp. 442-449.

[82] J. R. PARTINGTON, *Interpolation, Identification, and Sampling*, London Mathematical Society Monographs New Series **117**, Oxford Science Publications, Clarendon Press (1997).

[83] J. R. PARTINGTON, *Linear Operators and Linear Systems*, London Mathematical Society Student Texts **60**, Cambridge University Press (2004).

[84] V. I. PAULSEN, M. RAGHUPATHI, *An Introduction to the Theory of Reproducing Kernel Hilbert Spaces*, Cambridge studies in advanced mathematics **152**, Cambridge University Press (2016).

[85] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences **44**, Springer-Verlag (1983).

[86] G. PEANO, *Integrazione per serie delle equazioni differenziali lineari*, Atti della Reale Accademia delle scienze di Torino **22**, (1887), pp. 293-302.

[87] M. M. PELOSO, M. E. SALVATORI *On some spaces of holomorphic functions of exponential growth on a half-plane*, Concrete Operators **3** (1), De Gruyter (2016), pp. 52-67.

[88] C. E. RICKART, *General Theory of Banach algebras*, Robert E. Krieger Publishing Company (1960 original, 1974 reprint).

[89] M. ROSENBLUM, J. L. ROVNYAK, *Topics in Hardy Classes and Univalent Functions*, Birkhäuser Advanced Texts, Birkhäuser Verlag (1994).

[90] W. RUDIN, *Real and Complex Analysis*, McGraw-Hill Series in Higher Mathematics, WCB/McGraw-Hill (1966), third edition published in 1987.

[91] W. RUDIN, *Functional Analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill (1973), second edition published in 1991.

[92] R. M. SCHRODERUS, *Spectra of linear fractional composition operators on the Hardy and weighted Bergman spaces of the half-plane*, Journal of Mathematical Analysis and Applications **447** (2), Elsevier (2017), pp. 817-833.

[93] O. J. STAFFANS, *Well-Posed Linear Systems*, Encyclopedia of Mathematics and Its Applications **103**, Cambridge University Press (2005).

[94] D. A. STEGENGA, *Multipliers of the Dirichlet space*, Illinois Journal of Mathematics **24** (1), (1980), pp. 113-139.

[95] E. M. STEIN, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Monographs in Harmonic Analysis **3**, Princeton University Press (1993).

[96] M. H. STONE, *Linear transformations in Hilbert Spaces: III. Operational Methods and Group Theory*, Proceedings of the National Academy of Sciences **16** (2), (1930), pp. 172-175.

[97] J-O. STRÖMBERG, A. TORCHINSKY, *Weights, sharp maximal functions and Hardy spaces*, Bulletin of the American Mathematical Society (New Series) **3** (3), (1980), pp. 1053-1056.

[98] G. D. TAYLOR, *Multipliers on D_α*, Transactions of the American Mathematical Society **123** (1), (1966), pp. 229-240.

[99] W. THOMSON, *Popular Lectures and Addresses Vol. I Constitution of Matter*, Nature Series, MacMillan and Co. (1889).

[100] T. T. TRENT, *A Corona Theorem for Multipliers on Dirichlet Space*, Integral Equations and Operator Theory (1) **49**, Birkhäuser Verlag Basel (2004), pp. 123-139.

[101] M. TUCSNAK, G. WEISS, *Observation and Control for Operator Semigroups*, Birkhäuser Advanced Texts, Birkhäuser Verlag (2009).

[102] D. V. WIDDER, *The Laplace Transform*, Princeton University Press (1941).

[103] A. WYNN, *α-Admissibility of Observation Operators in Discrete and Continuous Time*, Complex Analysis and Operator Theory **4** (1), (2010), pp. 109-131.

[104] K. YOSIDA, *On the differentiability and the representation of one-parameter semi-group of linear operators*, Journal of Mathematical Society of Japan **1** (1), (1948), pp. 15-21.

[105] N. J. YOUNG, *An introduction to Hilbert space*, Cambridge Mathematical Textbooks, Cambridge University Press (1988).