# Classification of Countable Homogeneous 2-Graphs

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Submitted in accordance with the requirements for the degree of

*Doctor of Philosophy*

March 2011

## Acknowledgements

I would like to thank my very supportive supervisor, Professor John Truss, who has helped me enormously in getting this thesis ready. I would also like to thank more generally the staff and postgraduates in the School of Mathematics who have also been very supportive, as well as my parents who extremely kindly welcomed me back into the family home when I was in the closing stages of writing up this thesis.

This thesis would not have been possible without the funding provided by EPSRC through the Doctoral Training Grant scheme.

# Abstract

We classify certain families of homogeneous 2-graphs and prove some results that apply to families of 2-graphs that we have not completely classified.

We classify homogeneous 2-coloured 2-graphs where one component is a disjoint union of complete graphs and the other is the random graph or the generic $K_r$-free graph for some $r$. We show that any non-trivial examples are derived from a homogeneous 2-coloured 2-graph where one component is the complete graph and the other is the random graph or the generic $K_r$-free graph for some $r$; and these are in turn either generic or equivalent to one that minimally omits precisely one monochromatic colour-1 $(K_1, K_t)$ 2-graph for some $t < r$.

We also classify homogeneous 2-coloured 2-graphs $G$ where both components are isomorphic and each is either the random graph or the generic $K_3$-free graph; in both cases show that there is an antichain $\mathcal{A}$ of monochromatic colour-1 2-graphs all of the form $(K_s, K_t)$ (for some $s$ and $t$) such that $G$ is equivalent to the homogeneous 2-coloured 2-graph with the specified components that is generic subject to minimally omitting the elements of $\mathcal{A}$.

# Contents

# Chapter 1

# Introduction

We begin by giving a short summary of some of the literature that is relevant to the study of homogeneous structures and to homogeneous 2-graphs in particular. We do not define the terms at this stage; we do this in Chapter 2.

## 1.1 Literature

The study of homogeneous structures started with Roland Fraïssé. His principal work is summarised in the detailed and concise if somewhat idiosyncratic 1986 treatise *Théorie des relations* (Fraïssé (1986)) which summarises various properties that a relation, or a collection of relations, including the notion of homogeneity for a "relational structure". (The 1980s work is used as a convenient collated source though Fraïssé's principal papers were published in the mid 1950s.)

One of the earliest classifications of homogeneous structures to be completed was that of homogeneous graphs. This classification was completed over a period of years by many authors. Sheehan (1974/75) classified the finite **non-cubic** homogeneous graphs (that is, finite homogeneous graphs that do not contain vertices of degree 3). Later, Gardiner (1976) classified the remaining finite homogeneous graphs; his paper does not generally re-prove the results of Sheehan. The infinite triangle-free homogeneous graphs were classified in Woodrow (1979); later a rather more complicated classification of all homogeneous graphs was given in Lachlan & Woodrow (1980).

## 1. Introduction

We make free use of the classification of the homogeneous graphs. Generally we treat the classification as a "black box" but in Theorem 2.13, and perhaps elsewhere, we do make use of a variant of the method Lachlan and Woodrow used to complete their induction argument.

Some work was done on classifying some cases of homogeneous 2-graphs in Jenkinson (2006). His thesis concentrated mainly on classifying the homogeneous $n$-partite graphs up to $n = 5$ though in chapter 3 of his thesis he did prove some results about homogeneous 2-graphs where both components are $\overline{K_m}[K_n]$ and he also classified those homogeneous 2-graphs where either component is isomorphic to $C_5$ or $K_3 \times K_3$. He also stated a proof that there are continuum-many homogeneous $c$-coloured 2-graphs if $c \geq 3$, though this result was certainly known by Cherlin (who stated it without proof in Cherlin (1998)) and may be folklore.

The main part of Jenkinson's thesis has since been expanded by Seidel who has classified all the homogeneous $n$-partite graphs; this part of the classification has been consolidated in the paper by Jenkinson *et al.* (2011) which at the time of writing has been accepted for publication in the European Journal of Combinatorics. Work is progressing on classifying the *coloured* homogeneous $n$-partite graphs in Lockett & Truss (in preparation). A coloured $n$-partite graph is a very special case of an $n$-graph where all components are empty graphs; we therefore expect that a classification of the homogeneous $n$-graphs will be very much more difficult than a classification of the homogeneous coloured $n$-partite graphs.

Cherlin has done a considerable amount of work on homogeneous structures, and in Cherlin (1998) he gave an alternative proof of the classification of the infinite homogeneous graphs as well as classifying the homogeneous digraphs and the homogeneous $n$-tournaments. (The reader should note that Cherlin does not allow "bidirectional edges" in digraphs - if $x$ and $y$ are vertices in a digraph $D$ then $D$ cannot have edges $x \to y$ and $y \to x$. Some other authors allow such bidirectional edges, for example Lachlan (1982). The extra edge type does make a significant difference to the classification; for example there are many more homogeneous finite 4-type digraphs than there are homogeneous finite 3-type digraphs.)

Cherlin also stated a number of research problems, one of which is the following:

**Problem.** *Classify the homogeneous $n$-graphs and $n$-digraphs.*

We explain what an $n$-graph is in Chapter 2; an $n$-digraph is similar to an $n$-graph but with a digraph on each component instead of a graph. Cherlin remarks that this problem is "technically quite difficult". We agree. In this thesis we aim for a much more modest goal, namely to give an idea of how a classification of the homogeneous countable 2-graphs might go and to deal with at least some of the cases. The work was begun in chapter 3 of Jenkinson (2006); as we stated above, Jenkinson was only able to deal with a few highly special cases and mostly concentrated on classifying homogeneous complete multipartite graphs (which in turn he achieved only partial success with).

Cherlin's alternative proof of the classification of the homogeneous graphs did refer at several points to 2-graphs. We have not been able to isolate any results that we think would have been useful in our own classification, and generally this thesis does not rely on Cherlin (1998) in any significant way.

There have been several other classifications of classes of homogeneous structures. So far each have proceeded in a relatively ad-hoc way although certain techniques are seen in several of these classifications. Cherlin did hope that there would ultimately be a systematic classification of all homogeneous structures in an arbitrary finite relational language; while it may indeed be possible that such a classification could be found, it seems that it will be a long time before we see it.

An example of one of these other classifications is the classification of the *coloured* partial orders given in Torrezao de Sousa & Truss (2008). There has also been substantial work on classifications of structures satisfying properties that are related to, but different from, homogeneity; for example the weaker notions of $n$-homogeneity and $n$-transitivity. A survey of work in these areas can be found in Truss (2007).

In cases where the homogeneous structures to be classified are finite, Lachlan (1984) showed that there is a general structure to the classification. Specifically, there will be finitely many families of cases, and each family will consist either of a series of cases tending towards a common limit, or a single "sporadic" case. Of course the task still remains to determine specifically what these families consist of for any given class of structures.

We remind the reader that terminology in this area is somewhat confusing. The reader is warned that there is an unrelated notion of a structure called a "2-graph" meaning a set $T$ of (unordered) triples on domain $X$ such that every quadruple in $X$ contains an even number of triples in $T$. This notion has been developed in van Lint & Seidel (1966), Seidel (1973) and Taylor (1977). We mention this here mainly to prevent the reader being misled by papers describing this earlier notion. It is somewhat unfortunate that Cherlin chose the term "2-graph" (or in general "$n$-graph") to describe his (and our) notion but this is one of those historical accidents of mathematics that has to be lived with. The term "homogeneous" has been heavily overloaded over the years to describe various notions, some similar to the one we are using and some very different. We hope that these terms have at least been used in a mostly consistent manner within this thesis.

At various points in the thesis we will need to count the number of elements in a finite set, typically to show that one is larger than another to show that there is no isomorphism between them. In this thesis the combinatorial enumeration we need is normally very simple. Readers who wish to know more about combinatorial enumeration, especially those hoping to extend our results who need more precise estimates (we believe, for example, that more precise estimates would be needed to extend Theorem 4.11), should consult Goulden & Jackson (1983) for a detailed general background or Harary & Palmer (1973) for issues specifically related to graphs and structures based on graphs.

## 1.2 General intuition

Intuitively (we give a formal definition in Chapter 2), a 2-graph is a structure consisting of two graphs, called **components**, with a coloured edge (called a **cross-edge**) between each pair of points on opposite sides. We will be thinking diagrammatically, and so will speak of a **left component** and a **right component**. Swapping the two components, at least in general, gives a different, non-isomorphic 2-graph, so it makes sense to think of "left" and "right". In diagrams, the left component will actually be the left-hand oval in the diagram. If there are exactly two colours, we will normally call them "red" and "blue" and use these colours in diagrams. In cases where we need to number the colours, red will normally be colour 1 and blue will be colour 2.

The term "component" is also used in graph theory to describe a set of vertices so that there is a path between any two vertices in the set. For such a "component", we will use the term **connected component** to avoid confusion. Moreover, we often will avoid actually saying that a vertex is in the "left component" and instead say that it is **on the left** (and similarly for the right). The aim of this is again to avoid confusion.

Concretely, if somewhat artificially, we can visualise a 2-graph as follows: suppose there are two "island groups", called $A$ and $B$. Each "island group" contains islands (representing the connected components) and on each island there are boat landings (representing vertices) on the coast and edges between boat landings.[1] Between landings in **different** groups there are ferries, each operated by one of $c$ companies whose boats are identified by colours, the same colour for all boats run by the same company. (The ferry companies bid for the right to run each line; only one ferry company is allowed on one line, and the hypothetical government ensures that every line is given proper service.) There are no ferries **within** an island group. At each landing there is a sign stating which island group it is in, but all landings are otherwise anonymous. Homogeneity would then amount to the following – for any way one can label landings with two sets of finitely many numbered signs, one with the numbers in circles and the other with the numbers in squares, so that the network of the circles and the network of the squares are "the same", we can continue the numbering so that every landing is given both a circle and a square numbering, and the network of circles looks the same (together with numbering) as that of squares. (This is admittedly a rather contrived visualisation of the concept, and unfortunately we have not found a more natural one.)

Less concretely we might look at weaker notions than homogeneity. Informally, a relational structure $S$ is $n$-transitive if, for any pair $T_1, T_2$ of isomorphic substructures of $S$ of size $n$, there is an isomorphism $\alpha : T_1 \to T_2$ that extends to an automorphism of $T$ (i.e. there is an automorphism $\beta$ of $S$ such that $\beta(T_1) = T_2$). $S$ is $n$-homogeneous if every $\alpha : T_1 \to T_2$ (where $T_1, T_2$ are still a pair of isomorphic substructures of $S$ of size $n$) extends to an automorphism of $\alpha$. Note the difference – if $S$ is $n$-homogeneous it is certainly $n$-transitive, but $n$-transitivity is considerably

[1] It is not possible to go from one edge (i.e. path) to another except at a vertex (i.e. landing). If the connected component is complete (i.e. there are paths between all landings on a vertex) this stricture is irrelevant. Otherwise, if paths cross otherwise than at landings they are to be treated as if one has a bridge going over the other.

weaker in general. $S$ is **homogeneous** if it is $n$-homogeneous for every $n$; that is, for every pair $T_1, T_2$ of finite isomorphic substructures of $S$ and every isomorphism $\alpha : T_1 \to T_2$, there is an automorphism of $S$ extending $\alpha$. The generalisation of $n$-transitivity would in general be "high transitivity" – $S$ is highly transitive if, for every pair $T_1, T_2$ of finite isomorphic substructures of $S$, there is an automorphism $\beta$ of $S$ such that $\beta(T_1) = T_2$. In practice one does not speak of "high transitivity" as such but of more specific notions like "high vertex transitivity" or "high arc transitivity".

In this thesis we will classify:

- in chapter 3, all homogeneous 2-graphs with two cross-edge colours where one component is $\overline{K_m}[K_n]$ and the other is the generic $K_r$-free graph, called $\Gamma_r$ (where $m, n, r$ can be finite or infinite but $r \geq 3$; we abuse notation by writing $\Gamma_\infty$ for the "random graph"); and

- in chapter 4, all homogeneous 2-graphs with two cross-edge colours where either both components are isomorphic to $\Gamma_3$ or both components are isomorphic to $\Gamma_\infty$.

We explain what this notation means in Chapter 2 and give a precise statement of the classification in section 1.4.

Extending the classification to cover cases with two cross-edge colours where the two components are $\Gamma_r$ and $\Gamma_s$ respectively for values of $r$ and $s$ other than $r = s = 3$ and $r = s = \infty$ seems to be, at least in principle, practicable, though attempts to do so have to date been stymied by the number of cases that appear to have to be considered and by technical difficulties in so doing. We expect however that attempting to extend most of our results to cases with more than two cross-edge colours involves fundamental differences, and have so far not been able to say much about this situation. We say more about this in section 5.3.

We do not say much about 2-graphs of the form $(\overline{K_m}[K_p], \overline{K_n}[K_q])$, in part because this aspect of the classification is messy and not especially illuminating. I do plan to publish results relating to classification of these homogeneous 2-graphs at a later stage, and give a **statement** (without proof) of results I have been able to prove in section 5.2 (as Theorem 5.1).

## 1.3 Techniques used

Most of the methods we use are not especially novel in general. We generally apply a similar "bare-hands" style of argument to that used in Jenkinson (2006), though we have been able to apply these methods to much broader families of homogeneous 2-graphs than was possible in the earlier thesis and have been able to be rather more systematic in our treatment of 2-graphs than was possible in Jenkinson (2006). (Jenkinson (2006) concentrated mainly on homogeneous multipartite graphs and, for him, 2-graphs were something of an afterthought.)

As the treatment of 2-graphs in Jenkinson (2006) was not very systematic, I believe that this thesis can be considered to be the first semi-systematic attempt to classify at least some fairly-broad families of 2-graphs. As such many of the concepts I use are "new" in the sense of being applied to 2-graphs, though they are really only minor adaptations of concepts used generally in mathematics. The notion of collapsingness in Definition 2.7 is an example of this; it is really just an attempt to form some kind of "quotient" of a 2-graph, but I am not aware of any previous work on 2-graphs that has attempted to do this in anything like a systematic way. The notion of "equivalence" of 2-graphs is a similar example – it is a way of slightly weakening the notion of isomorphism to consider 2-graphs which are not the same but where any operation done on one can be easily and mechanically translated to an operation on the other. This procedure is common in mathematics, but again I am not aware of any previous formal attempt to apply it to 2-graphs.

The "copying argument" in Theorem 2.13 is an example of a different kind of translation. Lachlan & Woodrow (1980) uses a notion of "derivation" to express how each finite graph can be built up by amalgamation from so-called "basic" graphs. I use a slightly modified version of this notion to show how such a derivation can be translated to a derivation of a 2-graph from a sufficiently large family of "basic" 2-graphs. Again, while the idea in abstract is hardly new, I believe that the application to 2-graphs is at least somewhat novel.

## 1.4 Statement of main classification results

We now state the principal results of our classification. The notation and terminology will be defined later, principally in Chapter 2.

**Theorem 3.1** Let $G = (A, B, R)$ be a homogeneous non-collapsing 2-coloured 2-graph where $A \cong \overline{K_m}[K_n]$ for some $m, n \in \mathbb{N} \cup \{\infty\}$ and $B \cong \Gamma_r$ for some $r \in \mathbb{N} \cup \{\infty\}$ where $r \geq 3$. Then $mn = \infty$ and $G$ is equivalent to one of the following 2-graphs:

- $m = \infty$, $n = 1$ and $G$ is otherwise generic (i.e. embeds all finite 2-graphs satisfying these constraints);
- $m = \infty$, $n = 1$, the 2-graph $(K_1, K_k)^1$ is minimally omitted for some $k < r$, and $G$ is otherwise generic;
- $m = \infty$, $n = 2$, the 2-graphs $(K_2, K_1)^1$ and $(K_2, K_1)^2$ are minimally omitted, and $G$ is otherwise generic;
- $2 \leq m \leq \infty$, $n = \infty$ and $G$ is otherwise generic; or
- $2 \leq m \leq \infty$, $n = \infty$,the 2-graph $(K_1, K_k)^1$ is minimally omitted for some $k < r$, and $G$ is otherwise generic.

*Remark.* The proof that these five 2-graphs are indeed homogeneous is found in Propositions 3.7 and 3.8.

**Theorem 4.2** Let $G$ be a homogeneous 2-coloured $(\Gamma_r, \Gamma_r)$ 2-graph where $r = 3$ or $r = \infty$. Then there exists some antichain $\mathcal{A}$ of finite 2-graphs of the form $(K_m, K_n)^1$ (for various values of $m$ and $n$) such that $G$ is equivalent to some $G_{\mathcal{A}}$ (as defined in Proposition 4.1).

*Remark.* Proposition 4.1 states that $G_{\mathcal{A}}$ is the "most generic" 2-graph such that:

- $G_{\mathcal{A}}$ omits all elements of $\mathcal{A}$;
- $G_{\mathcal{A}}$ embeds every finite 2-graph of the form $(C, D)^1$ (where $C, D < \Gamma_r$) that does not embed any element of $\mathcal{A}$; and
- $G_{\mathcal{A}}$ embeds every finite 2-graph of the form $\left(\hat{C}, \hat{D}\right)^2$ (where $\hat{C}, \hat{D} < \Gamma_r$).

Proposition 4.1 also proves that this 2-graph is indeed homogeneous for every such antichain $\mathcal{A}$; moreover it proves **existence** for values of $r$ other than 3 and $\infty$, or where the two components of $G$ are $\Gamma_r$ and $\Gamma_s$ where $r \neq s$. We have not been able to prove **uniqueness** in these cases.

# Chapter 2

# Generalities

In this chapter, we discuss some fairly general results which we will be applying at several points in this thesis, and which it is therefore expedient to combine.

## 2.1 Basic definitions of graphs and graph theory

A **graph** is a structure $G = (V, E)$ where $E \subseteq V^{(2)}$ (that is, $E$ is a set of two-element subsets of $V$). We say that the elements of $V$ are **vertices** of $G$, and similarly that the elements of $E$ are **edges** of $G$. We write $V(G)$ for the set of vertices of $G$ and $E(G)$ for the set of edges of $G$. We will often be sloppy and identify a graph with its vertex set.

In this thesis, a **subgraph** of a graph $G$ is a graph $H = (W, F)$ such that $W \subseteq E$ and such that $(u, v) \in F$ if and only if $u, v \in W$ and $\{u, v\} \in E$. (This is usually called an *induced* subgraph of $G$, but for various reasons we almost never need to refer to the "standard" definition of subgraph, so where we say "subgraph" we mean "induced" subgraph. The standard definition of a subgraph of $G$ would be a graph $H' = (W, F)$ where $W \subseteq V$ and $H \subseteq F \cap W^{(2)}$; that is, the edges of $H'$ are a subset of the edges of $H$.)

Certain special graphs are given special names. The **complete** graph on $n$ vertices is the graph

$$K_n = (\{1, \dots, n\}, \{\{i, j\} : 1 \leq i < j \leq n\})$$

and where $n = \infty$ this amounts to

$$K_\infty = (\mathbb{N}, \{\{i, j\} : i, j \in \mathbb{N}, i < j\})$$

The **complement** of a graph $G$ is the graph

$$\overline{G} = (V(G), (V(G))^{(2)} \setminus E(G))$$

and the **empty** graph on $n$ vertices is

$$\overline{K_n} = (\{1, \ldots, n\}, \varnothing)$$

The **null** graph is the graph on zero vertices, namely

$$K_0 = (\varnothing, \varnothing)$$

At various points we will abuse notation by writing $\varnothing$ for the null graph.
A **path** in a graph $G$ between two vertices $x, y \in V(G)$ is a finite sequence of vertices

$$x_0 = x, x_1, \ldots, x_{n-1}, x_n = y$$

such that each $\{x_i, x_{i+1}\} \in E(G)$ when $0 \leq i < n$, and such that there are no $i \neq j$ such that $x_i = x_j$. A **cycle** in $G$ is a finite sequence of vertices

$$x_0 = x, x_1, \ldots, x_{n-1}, x_n$$

such that each $\{x_i, x_{i+1}\} \in E(G)$ when $0 \leq i < n$, such that $\{x_0, x_n\} \in E(G)$ and such that there are no $i \neq j$ such that $x_i = x_j$. Often we write such a sequence as

$$x_0, x_1, \ldots, x_n, x_0$$

to emphasise that we are talking about a cycle.
A graph $G$ is said to be **connected** if there is a path between any two vertices of $G$. A **connected component** of $G$ is a maximal connected subgraph of $G$. (This is usually just called a "component", but we use this term for another concept, so we are careful to speak only of the *connected* components of $G$.)
We write $P_n$ for the graph

$$P_n = (\{1, \ldots, n\}, \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\})$$

(that is, $P_n$ is a path consisting of $n$ vertices), and $C_n$ for the graph

$$C_n = (\{1, \ldots, n\}, \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{1, n\}\})$$

(that is, $C_n$ is a cycle consisting of $n$ vertices).
Two graphs $G$ and $H$ are **isomorphic** if and only if there is a bijection

$$\alpha : V(G) \to V(H)$$

such that, for all $x, y \in V(G)$, $\{x, y\} \in E(G)$ if and only if $\{\alpha(x), \alpha(y)\} \in E(H)$. Quite often we will not care about differences between isomorphic graphs and will regard them as "the same". Occasionally we will care about such distinctions. For example, this sort of distinction will be important in what we will be calling the "copying argument" in section 2.7. The **isomorphism class** of the graph $G$ is the proper class (not a set) of graphs isomorphic to $G$. We will use the term **isomorphism type** to loosely refer, given a specific labelled graph $G$, to an unlabelled graph $G^*$ that is isomorphic to $G$ and that is meant to typify all members of the isomorphism class containing $G$. (We will often abuse notation by not distinguishing between $G$ and $G^*$.)
We will be considering various unions of graphs (or, more often, isomorphism types of graphs) and need notations that describe the edges that exist between the two graphs in the union. We will need to use them when specifying structures we will later amalgamate, especially when (as will often be the case) previous steps have left us with structures that are not fully determined. This will be particularly important in chapter 4.
Given graphs $G$ and $H$ with disjoint vertex sets we will write $G + H$ for the graph

$$G + H = G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$$

and $G \boxplus H$ for the graph

$$G \boxplus H = (V(G) \cup V(H), E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\})$$

We write $G \sqcup H$ for any graph whose vertex set is the same as $V(G + H)$ and whose edge set is a superset of $E(G + H)$; we typically do this when the edge-types are either given arbitrarily or are the result of an earlier step in a chain of amalgamation arguments. Sometimes, when $G$ and $H$ are specified as graphs (**not** isomorphism types of graphs), we wish to refer to their union as identified entities, where $G$ and $H$ may overlap. In this case, we write $G \cup H$ for the normal union of $G$ and $H$ (i.e. $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$).

Note in particular that $G + H$ and $G \boxplus H$ are very different. $G \sqcup H$ is not fully specified and (in particular) $G + H$ and $G \boxplus H$ are cases of $G \sqcup H$. Examples can be seen in Figure 2.1.

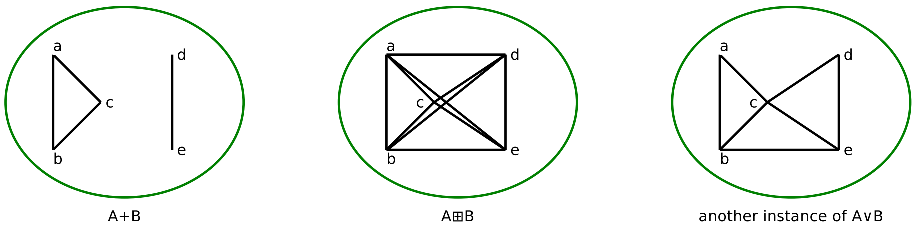


Figure 2.1: We show three different instances of disjoint unions between $A = (\{a, b, c\}, \{ab, ac, bc\})$ and $B = (\{d, e\}, \{de\})$.

We will often abuse notation by writing the edge $\{x, y\}$ as $xy$, or sometimes $(x, y)$. Moreover, we will often say $x \in G$ when we really mean $x \in V(G)$, or $yz \in G$ when we really mean $\{y, z\} \in E(G)$. The meaning will be clear by the context. In a further abuse of notation, we write $xy \cong x'y'$ to mean "$xy$ has the same type as $x'y'$" (i.e. both are edges or both are non-edges) and $xy \ncong x'y'$ to mean "$xy$ and $x'y'$ have different types" (i.e. one is an edge and the other is a non-edge). (We only use this notation when $x \neq y$ and $x' \neq y'$.)

Formally, $G+H$ only makes sense when $G$ and $H$ are disjoint. However, as a further abuse of notation, we will write $G + H$ for the graph $G' + H'$ where $G' \cong G$ and $H' \cong H$ are chosen to ensure that $V(G') \cap V(H') = \varnothing$. Similarly $G \boxplus H$ is $G' \boxplus H'$ (for the same $G'$ and $H'$). For $n \in \mathbb{N}$ and for a graph $G$ we will write $nG$ for the graph

$$G_1 + G_2 + \ldots G_n = \bigoplus_{i=1}^{n} G_i$$

where each $G_i$ (for $1 \leq i \leq n$) is isomorphic to $G$ and, for $1 \leq i \neq j \leq n$, $V(G_i) \cap V(G_j) = \varnothing$.

We will need to refer to the graph known as $K_3 \times K_3$, namely the **graph product** of two disjoint copies of $K_3$. We do not define the graph product in general. It is sufficient to state that $K_3 \times K_3$ is the graph with vertex set

$$\{(m, n) : m, n \in \{1, 2, 3\}\}$$

and edge set

$$\{((m,n),(m',n')) : m,m',n,n' \in \{1,2,3\}, m = m' \veebar n = n'\}$$

where $P \veebar Q$ means "exactly one of $P$ and $Q$ holds".
We will also need to refer to two families of graphs that can be described as **wreath products**, namely $K_m\left[\overline{K_n}\right]$ and $\overline{K_m}\left[K_n\right]$. $\overline{K_m}\left[K_n\right]$ is the graph consisting of $m$ disjoint copies of $K_n$ (if $m = \infty$ this means $\aleph_0$ disjoint copies of $K_n$). This graph can also be referred to as

$$mK_n = \underbrace{K_n + K_n + \ldots + K_n}_{m}$$

and both notations are used interchangeably. (Typically we use $\overline{K_m}\left[K_n\right]$ to describe the components, as defined later, of the 2-graphs we classify, and $mK_n$ to describe small instances of graphs of this form that arise in constructions.) $K_m\left[\overline{K_n}\right]$ is the complement of $\overline{K_m}\left[K_n\right]$; it is the complete $n$-partite graph where the parts each have size $m$. (In all cases $\infty$ is to be interpreted as $\aleph_0$ as all graphs and structures are countable unless specifically stated otherwise.)

## 2.2 Definitions of 2-graphs and general concepts of embedding and homogeneity

For any given positive integer value of $c$, a $c$-**coloured** 2-**graph** is a structure

$$\mathbf{G} = (G_1, G_2, R)$$

where each $G_i$ (called a **component** of $G$) is a graph $(V_i, E_i)$, where $V_i \cap V_j = \varnothing$ whenever $i \neq j$, and where $R$ is an ordered partition of $V_1 \times V_2$ into $c$ subsets (some of which may be empty). The elements of the partition $R$ are written $R_1, \ldots, R_c$. A **cross-edge** between $G_i$ and $G_j$ is an element of an element of $R_{i,j}$; we will often abuse notation by writing a cross-edge $(x,y)$ as $xy$.
We can consider a 2-graph to be a structure in the language

$$(L_1, L_2, E_1, E_2, C_1, \ldots, C_c)$$

where $L_1, L_2$ are unary relations denoting the two components; $E_1$ and $E_2$ are the edge relations on the components and $C_1, \ldots, C_c$ are the colours. The axioms will include sentences like

$$(\forall x)((L_1 x \wedge \neg L_2 x) \vee (\neg L_1 x \wedge L_2 x))$$

and, to take as an example the $c = 3$ case,

$$\begin{aligned}(\forall x, y)((L_1 x \wedge L_2 y) \quad \Rightarrow \quad & ((R_1(x,y) \wedge \neg R_2(x,y) \wedge \neg R_3(x,y)) \vee \\ & (\neg R_1(x,y) \wedge R_2(x,y) \wedge \neg R_3(x,y)) \vee \\ & (\neg R_1(x,y) \wedge \neg R_2(x,y) \wedge R_3(x,y)))\end{aligned}$$

Note that the language and the axioms depend on the number of colours.
Less formally, a 2-graph can be considered to be a coloured bipartite graph where we add a graph structure within each of the two "parts" (which we call components) and where we distinguish between the two components. A 2-graph, despite the name, is **not** itself a graph or even a coloured graph (but nor is it a digraph or a coloured digraph); this is because we impose an ordering on the components so that, in effect, all the cross-edges start in component 1 and end in component 2, so a 2-graph is a kind of "hybrid" between a graph and a digraph. In particular, the 2-graphs $(A, B, R)$ and $(B, A, R^{\leftrightarrow})$ (where $R^{\leftrightarrow}$ is formed from $R$ by swapping each cross-edge appearing in each element of $R$) are **not** in general isomorphic.
Two 2-graphs

$$\mathbf{G} = (G_1, G_2, R)$$

and

$$\mathbf{H} = (H_1, H_2, S)$$

are **isomorphic** if, for each $i$, there is an isomorphism

$$\alpha_i : G_i \to H_i$$

and, for each $k$ such that $1 \leq k \leq c$ and for each $x \in V(G_1)$ and $y \in V(G_2)$, $(x, y) \in R_k$ if and only if $(\alpha_1(x), \alpha_2(y)) \in S_k$. The isomorphism is given by

$$\alpha = \alpha_1 \cup \alpha_2$$

For some $i$ and for any graphs $A$ and $B$, the 2-graph $(A, B)^i$ is shorthand for the 2-graph

$$(A, B, (D_1, \ldots, D_c))$$

where $D_i = V(A) \times V(B)$ and $D_j = \varnothing$ whenever $j \neq i$.

We will often abuse notation by writing the 2-graph $(A, B, R)$ as simply $(A, B)$ when the colours of the cross-edges are implicit or unimportant, or when we are speaking of a member of a *family* of 2-graphs.

Let $S$ and $T$ be structures in some language (for example both may be graphs, or both may be 2-graphs with the same number of cross-edge colours). We say $S$ **embeds** in $T$ if there is a substructure $T'$ of $T$ such that $S$ is isomorphic to $T'$. Often we will be sloppy and write "$S$ is a substructure of $T$" or "$S \leq T$" when what we really mean is "$S$ embeds in $T$". We will say that $T$ **omits** $S$ if $S$ does not embed in $T$, and that $T$ **minimally omits** $S$ if $T$ omits $S$ but any proper substructure of $S$ embeds in $T$.

A countable structure $S$ is **homogeneous** if every isomorphism between finitely-generated substructures (which we will shorten to "finitely-generated partial isomorphism") of $S$ extends to an automorphism of $S$. If $S$ is a relational structure (that is, $S$ is a structure in a language $L$ with only relational symbols) then every finitely-generated substructure of $S$ is in fact finite and we will refer to the isomorphisms between finite substructures of $S$ as "finite partial isomorphisms" of $S$. (In fact this is also true if $L$ has finitely many constant symbols; we will not need to make use of this slight weakening of the hypotheses.)

## 2.3 Initial results

Let $G = (A, B, R)$ be a $c$-coloured 2-graph. We state and prove some simple results we will use in the rest of the thesis; in some cases the proofs are essentially trivial.

**Lemma 2.1.** *If $G$ is homogeneous then $A$ and $B$ are homogeneous graphs.*

*Proof.* We prove that $A$ is homogeneous; by symmetry, $B$ will also be homogeneous. We must show that every isomorphism $\alpha$ between substructures induced by finite subsets of $A$ extends to an automorphism of $A$. But $\alpha$ extends to an automorphism $\beta$ of $G$, and if $a \in A$ then $\beta(a) \in A$. Moreover, if $ab$ is an edge in $A$ then $\beta(a)\beta(b)$ is also an edge in $A$, and similarly if $ab$ is a non-edge in $A$ then $\beta(a)\beta(b)$ is a non-edge in $A$. Hence $\beta$ restricts to an automorphism $\gamma$ of $A$ which clearly extends $\alpha$. $\square$

*Remark.* We will usually omit the words "substructures induced by"; that is, we will normally identify a subset $C$ of the domain of $A$ with the substructure of $A$ induced by $C$.

Recall that Lachlan & Woodrow (1980) showed that the only infinite homogeneous graphs are:

- $\overline{K_m}[K_n]$ where either $m$ or $n$ (or both) are infinite;
- the complement of $\overline{K_m}[K_n]$, namely $K_m\left[\overline{K_n}\right]$ where either $m$ or $n$ (or both) are infinite;
- the generic $K_n$-free graph for finite $n \geq 3$ (which we write as $\Gamma_n$), proved to exist in Henson (1971) (where it was called $G_n$);
- the complement of $\Gamma_n$, namely the generic $\overline{K_n}$-free graph for finite $n \geq 3$ (which we write as $\overline{\Gamma_n}$); and
- the random graph (which we write as $\Gamma_\infty$) (easily seen to be equal to its complement).

Moreover, Gardiner (1976), following Sheehan (1974/75), proved that the only finite homogeneous graphs are:

- $\overline{K_m}[K_n]$ where $m, n \in \mathbb{N}$;
- its complement, namely $K_m\left[\overline{K_n}\right]$ where $m, n \in \mathbb{N}$;
- $C_5$ (equal to its complement); and
- $K_3 \times K_3$ (equal to its complement).

Hence Lemma 2.1, as simple as it is, already helps cut down the scope of our task and indeed makes it possible. The following result deals with the trivial monochromatic case and allows us to restrict to cases where there are at least two cross-edge colours that are both used.

**Proposition 2.2.** *If $c = 1$, then $G = (A, B, \{V(A) \times V(B)\})$ is homogeneous if and only if $A$ and $B$ are.*

*Proof.* If $G$ is homogeneous then $A$ and $B$ are by Lemma 2.1.
Suppose $A$ and $B$ are homogeneous and $c = 1$. Let $\alpha$ be an isomorphism between finite subsets of $G$. By homogeneity, the restrictions of $\alpha$ to $A$ and $B$ extend to automorphisms $\alpha_A, \alpha_B$ of these. Moreover, since any pair of points $(a, b)$ with $A \in A$ and $b \in B$ has the same colour of cross-edge between them, $\alpha_A \cup \alpha_B$ will be an automorphism of $G$ extending $\alpha$. □

We therefore will assume from now on that the 2-graphs we are trying to classify have at least two distinct colours of cross-edges.
We now states some conditions by which two non-isomorphic 2-graphs can nevertheless be said to be "equivalent" (that is, any property of one can be easily and mechanically translated to a property of the other), and prove that these operations preserve homogeneity. The analogue of this is that any graph is equivalent for our purposes to its complement (any property of a graph $G$ can be mechanically translated to a property of $\overline{G}$). We now list some other operations that allow a 2-graph to be translated to a different, but equivalent, 2-graph.

**Lemma 2.3.** *$G = (A, B, R)$ is homogeneous if and only if:*

1. *$(\overline{A}, B, R)$ is;*

2. *$(B, A, R^{\leftrightarrow})$ is, where*
$$R^{\leftrightarrow} = (R_1^{\leftrightarrow}, \ldots, R_c^{\leftrightarrow})$$
*and, for $1 \leq i \leq c$,*
$$R_i^{\leftrightarrow} = \{(b, a) : (a, b) \in R_i\}$$

3. *$(A, B, (R_{\sigma(1)}, \ldots, R_{\sigma(c)}))$ is, for each $\sigma \in S_c$ (where $S_c$ is the symmetric group on $c$ elements); and*

4. *$(A, B, (R_1, \ldots, R_c, \varnothing))$ is.*

*Sketch proof.* 1. The same maps work for $(A, B, R)$ and $(\overline{A}, B, R)$.

2. Reverse all relevant maps (e.g. if the original map $\alpha$ were such that $\alpha(a) = b$ for some $a, b \in A$, then the new map $\alpha'$ will still have $\alpha'(a) = b$ but this time $a$ and $b$ are in the *right* component).

3. The same maps work.

4. The same maps work.

□

**Definition 2.4.** *For each 2-graph $G$, let $[G]$ be the family of 2-graphs obtained from $G$ by closing under the operations in Lemma 2.3 (and their inverses). Two 2-graphs $G, H$ are* ***equivalent*** *if and only if $G \in [H]$. Explicitly, $G$ and $H$ are equivalent if and only if there is a chain $G_0 = G, G_1, \dots, G_k = H$ of 2-graphs such that, for each $i$, one of the following holds (where in each case $G_i = (A_i, B_i, R_i)$ and $R_i = (R_i^1, \dots, R_i^c)$):*

1. $G_{i+1} \cong G_i$;
2. $G_{i+1} = (\overline{A_i}, B_i, R_i)$;
3. $G_{i+1} = (B_i, A_i, R_i^{\leftrightarrow})$;
4. *for some* $\sigma \in S_c$, $G_{i+1} = (A_i, B_i, (R_i^{\sigma(1)}, \dots, R_i^{\sigma(c)}))$;
5. $G_{i+1} = (A_i, B_i, (R_i^1, \dots, R_i^c, \varnothing))$; *or*
6. $G_{i+1} = (A_i, B_i, (R_i^1, \dots, R_i^{c-1}))$ *(where* $R_i^c = \varnothing$*).*

*Remark.* We will classify 2-graphs *only* up to equivalence.

## 2.4 Perfect matchings

**Definition 2.5.** *A* ***perfect matching*** *in a 2-graph $G = (A, B, R)$ is a colour $i$ such that, for all $a \in A$, there is a unique $b \in B$ such that $(a, b) \in R_i$, and such that for all $b \in B$ there is a unique $a \in A$ such that $(a, b) \in R_i$.*

We will show that, for any homogeneous graph $A$, there is a 2-graph $G = (A, A, R)$ containing a perfect matching; indeed there is essentially only *one* such 2-graph (i.e. if $G' = (A, A, R')$ also contains a perfect matching then $G'$ is equivalent to $G$).
Given $A$, let $A'$ be a graph isomorphic to $A$, and define a 2-graph $P_A = (A, A', (R_1, S_2, S_3))$ so that $R_1, S_2, S_3$ have the following properties:

- $R_1$ induces an isomorphism from $A$ to $A'$; and
- for $a_1 \neq a_2 \in A$ and $b_1 \neq b_2 \in A'$ where $a_1b_1, a_2b_2 \in R_1$:

- $a_1b_2, a_2b_1 \in S_2$ if $a_1a_2$ is an edge; and
- $a_1b_2, a_2b_1 \in S_3$ otherwise (i.e. if $a_1a_2$ is a non-edge)

The following proposition shows that any homogeneous 2-graph containing a perfect matching must be equivalent to $P_A$ for some $A$.

**Proposition 2.6.** *Let $G = (A, B, R)$ be a homogeneous 2-graph in which $R_1 \in R$ is a perfect matching. Then:*

1. *$R_1$ defines an isomorphism from $A$ to either $B$ or $\overline{B}$, and*
2. *$G$ is equivalent to $P_A$.*

*Proof.* Since $R_1$ is a perfect matching, we can label the vertices of $A$ and $B$ as $\{a_i : i \in I\}$ and $\{b_i : i \in I\}$ respectively for some index set $I$. Suppose that the map given by

$$\alpha : (\forall i \in I) a_i \mapsto b_i$$

is neither an isomorphism nor an anti-isomorphism; that is, without loss of generality, there are $i, j, i', j'$ so that $a_ia_j \cong a_{i'}a_{j'} \cong b_ib_j$ and $a_{i'}a_{j'} \ncong b_{i'}b_{j'}$. (There is no assumption that $i \neq i'$ or that $j \neq j'$.) But then

$$\alpha : a_i \mapsto a_{i'}, a_j \mapsto a_{j'}$$

is a finite partial automorphism of $G$ which does not extend to an automorphism of $G$, since if it did then the extension $\beta$ would $b_i$ to $b_{i'}$ and $b_j$ to $b_{j'}$, which clearly cannot happen.
Hence either $A \cong B$ or $A \cong \overline{B}$. Without loss of generality we can assume that $A \cong B$. Suppose $G$ is not equivalent to $P_A$. Then one of the following occurs in $G$:

1. there exist $i, j$ so that $a_ib_j \ncong a_jb_i$, or
2. there exist $i, j, k$ so that $a_ia_j \cong a_ia_k$ but $a_ib_j \ncong a_ib_k$, or
3. there exist $i, j, k$ so that $a_ia_j \ncong a_ia_k$ but $a_ib_j \cong a_ib_k$.

For (1), the map

$$\beta : a_i \mapsto a_j, a_j \mapsto a_i$$

proves that $G$ could not be homogeneous, since it would extend to a map which interchanges $b_i$ with $b_j$, and $a_ib_j \ncong a_jb_i$.

For (2), the map

$$\gamma : a_i \mapsto a_i, a_j \mapsto a_k$$

is a finite partial automorphism, but it cannot extend since if it did the extension would map $b_j$ to $b_k$, but $a_i b_j \ncong a_i b_k$.
For (3),

$$\delta : a_i \mapsto a_i, b_j \mapsto b_k$$

is a finite partial automorphism, but again it does not extend since if it did then the extension would map $a_j$ to $a_k$ and $a_i a_j \ncong a_i a_k$.
The only remaining possibility is that $G$ is indeed equivalent to $P_A$, as required. □

It remains to verify that $P_A$ is indeed homogeneous. Let $\theta$ be a finite partial automorphism of $P$; we claim that it must extend to an automorphism of $P_A$. But there is a unique finite partial automorphism $\phi$ of $P_A$ obtainable from $\theta$ by extending the domain to

$$\{a_i, b_i : a_i \in \mathrm{dom}(\theta) \vee b_i \in \mathrm{dom}(\theta)\}$$

This similarly extends the domain of any partial automorphism of $P_A$, including infinite ones. The restriction of $\phi$ to $A$ extends to an automorphism $\psi$ of $A$, and this in turn extends to an automorphism $\omega$ of $P_A$ which extends $\theta$. Since this holds for any finite partial automorphism $\theta$, $P_A$ is homogeneous.

## 2.5 Collapsing

We introduce another important notion, that of a **collapsing** 2-graph. This notion will allow us to consider certain 2-graphs as being simply expanded versions of simpler ones.

**Definition 2.7.** *Let $G = (A, B, R)$ be a 2-graph in which $A = \bigoplus_{i \in I} A_i$ and each $A_i \cong K_n$ for some $n \in \mathbb{N} \cup \{\infty\}$. Then $G$ is said to **left-collapse** to a 2-graph $H = (C, B, S)$ if*

$$C = (\{c_i : i \in I\}, \varnothing)$$

*and, for every $i \in I$ and every $b \in B$ there is a $j$ so that, for every $a \in A_i$, $(a, b) \in R_j$, and $(c_i, b) \in S_j$.*
*There is an analogous notion of **right-collapsing**. $G$ is said to **collapse** to $H$ if $G$ either left-collapses to $H$ or $G$ right-collapses to $H$.*

This notion is too strong to fully encapsulate all the ways that a 2-graph $G$ can be an "expanded" version of another 2-graph $H$. However, it is a straightforward notion, and if $G$ collapses to $H$ then the following proposition reduces the question of whether or not $G$ is homogeneous to the question of whether or not $H$ is homogeneous. This will allow us to assume, in the rest of this thesis, that the homogeneous 2-graphs we work with are **not** collapsing.

**Proposition 2.8.** *Suppose $A = \bigoplus_{i \in I} A_i$ where each $A_i$ is a complete graph. If $G = (A, B, R)$ left-collapses to a 2-graph $H$, $G$ is homogeneous if and only if every $A_i$ has the same cardinality and $H$ is homogeneous.*

*Sketch proof.* If $|A_i| \neq |A_j|$ then $G$ is not homogeneous since it is not 1-homogeneous. Hence we assume $|A_i| = |A_1|$ for all $i$.
Suppose $G$ is homogeneous. Let $\alpha$ be a finite partial automorphism of $H$. There is a corresponding finite partial automorphism $\beta$ of $G$ (given by arbitrarily choosing $a_i \in A_i$ and putting $\beta(a_i) = a_j$ whenever $\alpha(c_i) = c_j$), which extends to an automorphism $\gamma$ of $G$. Since $\gamma$ is a map of components (on the left), it restricts to an automorphism $\delta$ of $H$ which clearly extends $\alpha$. Since we can do this for every $\alpha$, $H$ is homogeneous.
We now suppose that $H$ is homogeneous and seek to show that $G$ is also homogeneous. Let $\alpha$ be a finite partial automorphism of $G$. Now, for every $a, b \in A$, if $a, b \in \mathrm{dom}(\alpha)$ then $\alpha(a)$ and $\alpha(b)$ are in the same $A_j$ if and only if $a$ and $b$ are in the same $A_i$. Hence there is a finite partial automorphism $\beta$ of $H$ corresponding to $\alpha$, and $\beta$ extends to $\gamma \in \mathrm{Aut}(G)$. But there is an automorphism $\delta$ of $G$ corresponding to $\gamma$ and extending $\alpha$. Again, since we can do this for every $\alpha$, $G$ is homogeneous. □

## 2.6 Amalgamation

The main tool we will use throughout this thesis is that of **amalgamation**.
Let $L$ be a finite (or countable) language. (We will normally assume that $L$ is "relational"; that is, $L$ only contains relation symbols.) Let $M$ be a countable $L$-structure and let $I$ be a set. We say that the $I$-**age** of $M$, written $\mathcal{A}^I$, is the set of finitely-generated $L$-structures embedding in $M$ whose domain is a subset of $I$. (This is **not** the same, in general, as the set of finitely-generated *substructures* of $M$, even if $I$ is the domain of $M$.) Now $\mathcal{A}^I$ will always have the following two properties:

1. the **hereditary property** (HP): if $X \in \mathcal{A}^I$ and $Y$ is a substructure of $X$ then $Y \in \mathcal{A}^I$; and

2. the **joint embedding property** (JEP): if $X, Y \in \mathcal{A}^I$ then there is $Z \in \mathcal{A}^I$ such that $X$ and $Y$ are substructures of $Z$.

Note that the $I$-age is also closed under isomorphisms restricted to $I$. Hence the $I$-age of a structure $M$ with domain $I$ will **not** (in general) be the set of finitely-generated substructures of $M$ as this will not normally be closed under isomorphism. In cases where it is not important to distinguish between isomorphic structures, we refer simply to an **age**. The age of $M$ is the set of isomorphism types of $M$, and will also have HP and JEP. In section 2.7 we will use the notion of an $I$-age (not simply an age) as it allows for a slightly neater formulation (the embeddings will come out as simply being inclusions) which, in principle, might be easier to encode on (for example) a computer. Later in the thesis we will drop the ground set $I$ and simply refer to ages.

The age (or $I$-age) of a structure does not in general determine that structure up to isomorphism. For example, the following (isomorphism types of) graphs have the same age but are clearly not isomorphic:

1. $\overline{K_\infty}[K_\infty]$
2. $\overline{K_\infty}[K_\infty] + K_1$
3. $K_1 + K_2 + K_3 + \ldots$

It will however turn out that if $M$ is known to be *homogeneous* then the age of $M$ is sufficient to determine $M$ up to isomorphism (i.e. if $\mathcal{F}$ is the age of both $M$ and $M'$, and $M$ and $M'$ are homogeneous, then $M \cong M'$). (The same is of course true for $I$-ages.)

If $M$ is a homogeneous $L$-structure, and $\mathcal{A}$ is its age, then $\mathcal{A}$ has a property known as the **amalgamation property** (AP): if $X$ and $Y$ are finitely-generated substructures of $M$ (and so $X, Y \in \mathcal{A}$), and $W \in \mathcal{A}$ is such that there are embeddings $f_1 : W \to X$ and $f_2 : W \to Y$, then there are $Z \in \mathcal{A}$ and embeddings $g_1 : X \to Z, g_2 : Y \to Z$ so that $g_2 f_2 = g_1 f_1$. (In fact, in the $I$-age of $M$, there will be copies $X'$ of $X$ and $Y'$ of $Y$ so that $X' \cap Y' = W$, and $Z$ will then be $X' \cup Y'$. Moreover, by using the existence of $X'$ and $Y'$, either in $M$ or in $\mathcal{A}$, we can usually assume, at least when working with $I$-ages, that the embeddings are trivial.)

We say that an **amalgamation class** is a set $\mathcal{A}$ of isomorphism types of finitely-generated $L$-structures satisfying HP, JEP and AP; similarly, an $I$-**amalgamation class** is a set $\mathcal{A}^I$ of finitely-generated $L$-structures on ground set $I$ satisfying HP, JEP and AP and closed under isomorphism. Fraïssé (1986) tells us how the problem of classifying homogeneous $L$-structures reduces to that of classifying amalgamation classes. (A perhaps more accessible account of how this, and other results in this section, are proved can be found in Hodges (1997).)

**Theorem 2.9** (Fraïssé)**.** *Let $L$ be a countable language, and let $\mathcal{A}$ be a countable, non-empty amalgamation class. Then there is a countable homogeneous $L$-structure $M$, unique up to isomorphism, such that the age of $M$ is $\mathcal{A}$.*

In this thesis, we say that an $L$-structure $M$ is "generic subject to $P$", given a property $P$ about $M$ (or typically the age of $M$), if $M$ realises all finite $L$-structures that are consistent with $P$. There will be certain implicit restrictions that we will not always state explicitly (e.g. if $M$ is a 2-graph, the graph structure on the components of $M$ will often be treated as implicit). Where the implicit restrictions are the only ones that apply, we will say that $M$ is "fully generic" or "completely generic". Note in particular that in this thesis the "generic subject to $P$" is **not** simply the homogeneous $L$-structure with the largest amalgamation class whose members all satisfy $P$. For any given $P$, the "generic subject to $P$" may or may not exist, and may or may not be homogeneous. (In essence, this notion of "generic" is an assertion that the age of the generic is trivially determined by $P$ and there are no extra restrictions that have been neglected. We will say a little more about this point in Section 2.9, with specific reference to the 2-graph case.) We give two examples where the generic (in this sense) does exist; the most general structure is indeed what we might expect it to be in these cases.

**Corollary 2.10.** *1. The random graph is homogeneous.*

*2. For each $c$, the generic $c$-coloured 2-graph (i.e. the 2-graph that embeds all finite $c$-coloured 2-graphs) is homogeneous.*

*Proof.* The corresponding $\mathbb{N}$-ages would be, respectively:

1. the set $\mathcal{C}$ of all finite graphs whose vertices lie in $\mathbb{N}$, and
2. the set $\mathcal{D}_c$ of all finite $c$-coloured 2-graphs whose vertices lie in $\mathbb{N}$.

In order to apply Theorem 2.9, it is sufficient to verify that $\mathcal{C}$ and each $\mathcal{D}_c$ are $\mathbb{N}$-amalgamation classes. By construction, the $\mathbb{N}$-ages satisfy the hereditary property, and are closed under isomorphisms restricted to the underlying domain. Given two finite graph (respectively finite $c$-coloured 2-graphs) called $G_1$ and $G_2$, the union $G_1 \cup G_2$ is clearly a finite graph (respectively finite $c$-coloured 2-graph) into which $G_1$ and $G_2$ embed, so the joint embedding property holds. Indeed, if $G_0$ embeds into $G_1$ and $G_2$, there are finite graphs (respectively finite $c$-coloured 2-graphs) $H_1$ and $H_2$ such that $H_0 = H_1 \cap H_2$ is isomorphic to $G_0$, and then $H_1 \cup H_2$ embeds $G_1$ and $G_2$ so that the embeddings of $G_0$ are consistent whether through $G_1$ or $G_2$. So in fact $\mathcal{C}$, and $\mathcal{D}_c$ for each $c$, satisfy the amalgamation property and so are $\mathbb{N}$-amalgamation classes as required. $\square$

In practice, at least for relational languages $L$, to check that an $I$-age $\mathcal{A}^I$ has AP it is sufficient to check that $\mathcal{A}^I$ has **two-point amalgamation**. That is, it is sufficient to verify that, if $X, Y \in \mathcal{A}$, $W = X \cap Y$, $X \setminus W = \{x\}$ and $Y \setminus W = \{y\}$, then there exists an $L$-structure $Z \in \mathcal{A}^I$ containing isomorphic copies $X'$ of $X$ and $Y'$ of $Y$ such that $X' \cap Y' \cong W$ and $X' \cup Y' = Z$. We express this diagrammatically with sets of overlapping ovals representing $X$ and $Y$, where the overlap is isomorphic to $W$ and where there is precisely one vertex, namely $y$, in the $Y$ ovals that is not in the $X$ ovals and precisely one vertex, namely $x$, in the $X$ ovals that is not in the $Y$ ovals, and aim to show that by assigning $xy$ to some relation and by taking the union we get a member of $\mathcal{A}^I$. In practice, if working with an age $\mathcal{A}$, to check that AP holds for types $X^*$ and $Y^*$ over $W^*$ (where $W^*$ embeds into $X^*$ and $Y^*$ and is one vertex smaller than each of $X^*$ and $Y^*$), we find representatives $W, X, Y$ in a sufficiently large $I$-age $\mathcal{A}^I$, obtain $Z$ as above and note that $Z^*$ is indeed in $\mathcal{A}$ and satisfies AP for $W^*, X^*, Y^*$. This will implicitly be the procedure we follow; in practice we will often not be this pedantic in the way we express it.

In a few cases it will be convenient to use "many-point" amalgamation, with more than two points outside the overlap. This will be equivalent to doing two-point amalgamation repeatedly and accepting any result that an amalgamation can give. We therefore will not say any more about the theoretical framework of "many-point" amalgamation.

## 2.7 The copying argument

We will describe one special, and somewhat complicated, use of amalgamation that will allow us to appeal to the work of earlier classification strategies in many of the cases we classify. As before we assume that our language $L$ is relational. Also, we will assume in this section that all structures have domain a subset of a fixed countably infinite set $I$, and use the notions of "$I$-age" and "$I$-amalgamation class" to allow us to (at least formally) work with actual identified structures and to make the embeddings explicit in the labellings of the structures. This allows us to ensure that all ways that structures can be amalgamated are considered; see for example Figure 2.2.

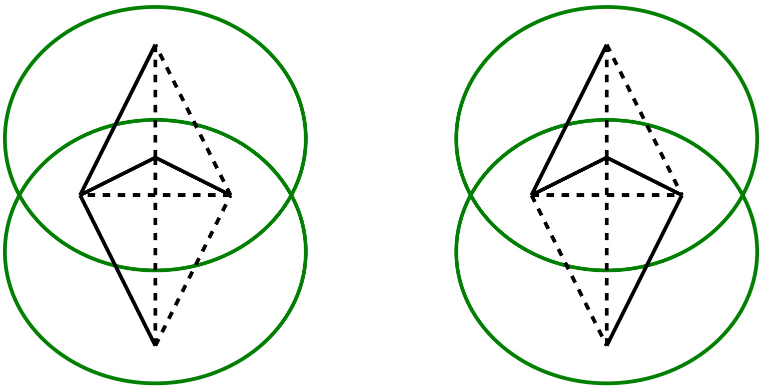

Figure 2.2: Care needs to be taken to ensure that all amalgamations are considered; in particular, these two possible amalgamations of two copies of $P_4$ over $P_3$ lead to very different results.

If, as is often the case, the set of all finite $L$-structures on some countable domain $I$ forms an $I$-amalgamation class (this is the case, for example, for graphs and for 2-graphs with any fixed number $c$ of cross-edge colours; see Corollary 2.10), then for any isomorphism-closed family $\mathcal{B}$ of so-called "basic" $L$-structures there will certainly be an $I$-amalgamation class $\mathcal{A}'$ containing all elements of $\mathcal{B}$.

Let $\tilde{\mathcal{A}}$ be the intersection of all $I$-amalgamation classes containing $\mathcal{B}$ (note that, in general, there is no reason to suppose that $\tilde{\mathcal{A}}$ is itself an $I$-amalgamation class). We need to see how each element of $\tilde{\mathcal{A}}$ is derived from the elements of $\mathcal{B}$ through a sequence of amalgamations, potentially taking several steps. In general when we amalgamate two structures we do not know what the result will be, though we know that it will be some element of a specified finite set (the goal will be to ensure that either this set has size 1 or that all of its elements embed the structure we are trying to obtain, but this may take many steps). The following definition gives a formal means by which we can specify how each element of $\tilde{\mathcal{A}}$ is derived ultimately from the elements of $\mathcal{B}$; it is similar to the definition given in Lachlan & Woodrow (1980) to encode how a finite graph can be built up from the "basic" graphs (the finite complete and empty graphs and the graphs $P_3$ and $\overline{P_3}$).

**Definition 2.11.** *Let $\mathcal{B}$ be an family of $L$-structures whose domains are finite subsets of $I$ (and which is closed under isomorphisms restricted to $I$), which we refer to as "basic" structures. We define an* ***amalgamation hierarchy*** *as follows:*

1. *let $\mathcal{C}_0 = \{\{B\} : B \in \mathcal{B}\}$ (i.e. there is a singleton for each "basic" structure in $\mathcal{B}$ at level 0, and nothing else).*

2. *given $\mathcal{C}_n$ for some $n$, obtain $\mathcal{C}'_n$ by closing $\mathcal{C}_n$ under taking finite unions and replacing elements with substructures: specifically, let $\mathcal{D}^0_n = \mathcal{C}_n$ and, for each $i \geq 0$, let $D^{i+1}_n$ be the smallest superset of $D^i_n$ such that:*

   *(a) if $X \in \mathcal{D}^i_n$, $G \in X$ and $H \leq G$, then $X \cup \{H\} \setminus \{G\} \in \mathcal{D}^{i+1}_n$; and*

   *(b) if $X, Y \in \mathcal{D}^i_n$ then $X \cup Y \in \mathcal{D}^{i+1}_n$;*

   *and then let $\mathcal{C}'_n = \bigcup_{i=0}^{\infty} \mathcal{D}^i_n$;*

3. *given $\mathcal{C}'_n$, let $\mathcal{C}_{n+1}$ be the smallest superset of $\mathcal{C}'_n$ such that, for each $U, V \in \mathcal{C}'_n$ where $U \neq V$, and for each $X \in U \setminus V$ and $Y \in V \setminus U$ where, if $Z = X \cap Y$, then $|X \setminus Z| = |Y \setminus Z| = 1$, if $P_1, \ldots, P_m$ are such that amalgamating $X$ and $Y$ over $Z$, must yield one of $P_1, \ldots, P_m$*[1] *then*

$$U \cup V \cup \{P_1, \ldots, P_m\} \setminus \{X, Y\} \in \mathcal{C}_{n+1}$$

We note that we allow closure under union and replacement by substructure within a level before we do the amalgamations. We do not close under union and substructures once we have performed the amalgamations in that level. Adding these

[1] $X$, $Y$ and $Z$ are actual structures, so we have defined a single amalgamation. If $X' \cong X$ and the domain of $X'$ lies in $I$ then $X'$ will appear in similar circumstances to $X$; hence all amalgamations that should be in the hierarchy will in fact be there.

closure operations would not make much difference to our arguments though it may change level numbers slightly. It **is** important that we don't allow closure under amalgamation in each level as this would collapse the hierarchy to a single level. Each set in each $\mathcal{C}_i$ in the hierarchy represents a set of structures, one of which must exist in every $I$-amalgamation class containing $\mathcal{B}$ (though we do not know which, nor that the choice need be independent of the choice of $I$-amalgamation class). However, if any $\mathcal{C}_i$ contains a singleton $\{X\}$ then $X$ must be in every amalgamation class containing $\mathcal{B}$. We can use this to define the notion of a *derivation*. Note that, in general, not every structure lying in an element of the hierarchy need have a derivation.

**Definition 2.12.** *Let $\mathcal{C} = \bigcup_{i=0}^{\infty} \mathcal{C}_n$, and let $\mathcal{A} = \{A : \{A\} \in \mathcal{C}\}$ (i.e. $\mathcal{A}$ is the set of elements of singletons of any $\mathcal{C}_n$). $\mathcal{A}$ is the set of structures **derived** from $\mathcal{B}$. A **derivation** of a graph $X \in \mathcal{A}$ is a finite generating substructure of*

$$(\mathcal{C}_0, \mathcal{C}_1, \ldots)$$

*(i.e. every set in the derivation is obtainable from ones in lower levels according to the rules (1) to (4) above, and $X$ is in some level of the derivation).*

An example of a derivation is the following derivation of $P_3 + K_1$ in $\Gamma_\infty$ (or indeed in $\Gamma_3$), shown in Figure 2.3:

- $C_0 = \{\{P_3\}, \{\overline{P_3}\}, \{\overline{K_3}\}\}$
- $C_1 = \{\{P_3 + K_1, P_4\}, \{P_3 + K_1, K_2 + \overline{K_2}\}\}$
- $C_2 = \{\{P_3 + K_1, T_1, T_2\}\}$
- $C_3 = \{\{P_3 + K_1\}\}$

In the above, $T_1$ is the graph:

$$(\{a, b, c, d, e\}, \{ab, bc, be, de\})$$

and $T_2$ is the graph:

$$(\{a, b, c, d, e\}, \{ab, ad, bc, cd, de\})$$

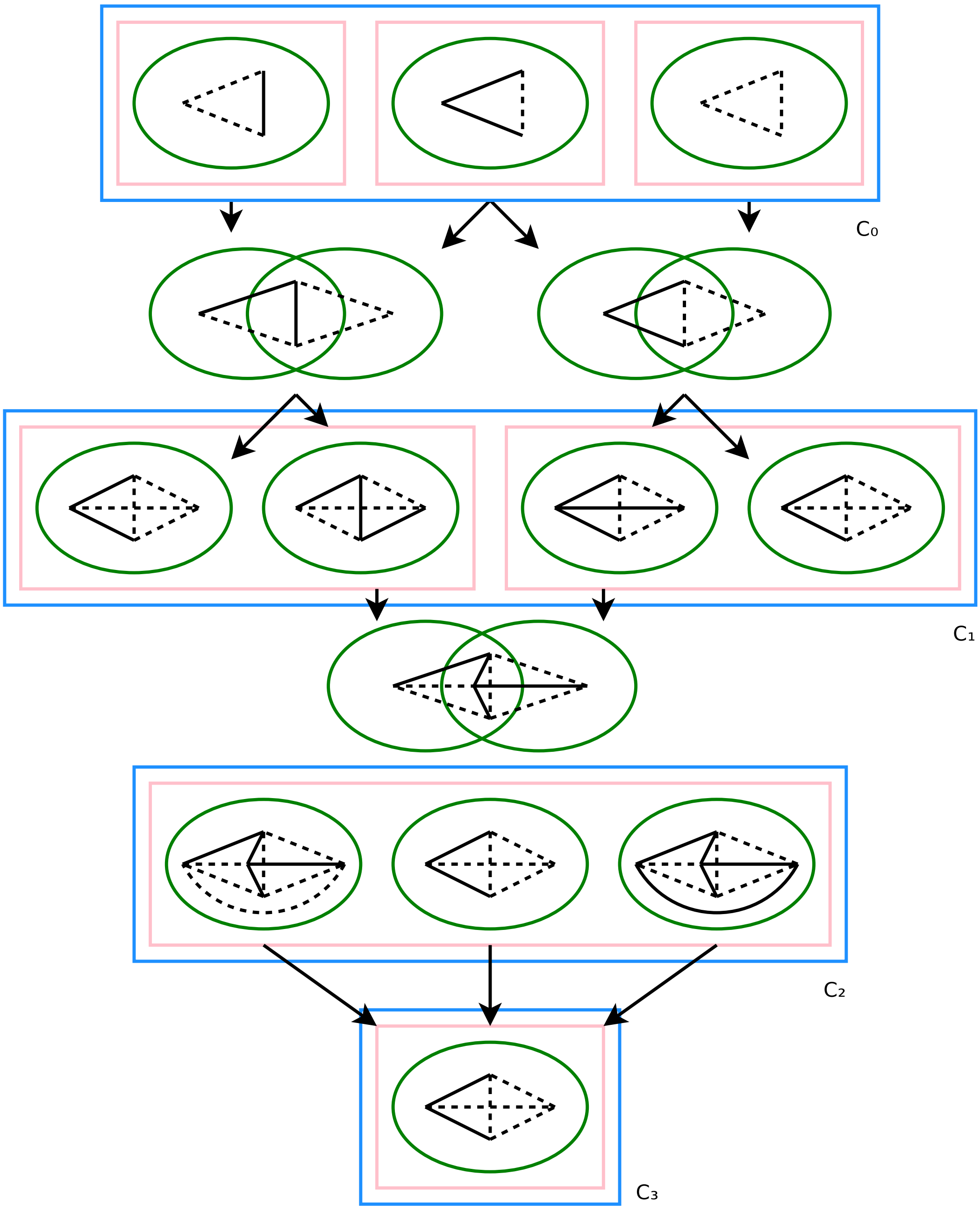


Figure 2.3: A derivation of $P_3 + K_1$. The light blue boxes indicate the levels, and at least one of the graphs within each pink box embeds at the level of the box.

(We have been slightly sloppy here and written these in terms of isomorphism types, not in actual graphs. The labelling of the other graphs in the derivation can be determined from the diagram.)

The idea is that we "know" that, for each element $U$ of $\mathcal{C}$, any $I$-amalgamation class containing $\mathcal{B}$ should have (at least) one element from $U$ (of course different classes might have different elements). We do not know that we have different elements of different elements of $\mathcal{C}$; if $U, V \in \mathcal{C}$ and $X \in U \cap V$ then there is no reason to suppose that we must have a second element from $U \cup V$. It is therefore important that $U$ and $V$ are distinct, and indeed that $Y \notin V$ and $X \notin U$. If we do not have this then we have nothing to amalgamate.

In general there is no reason to suppose that $\mathcal{A}$ is an $I$-amalgamation class or even that it contains anything other than $\mathcal{B}$. However Lachlan & Woodrow (1980) amounts to showing that if $\mathcal{B}$ is either

$$\mathcal{B}_\infty = \{P_3, \overline{P_3}\} \cup \{K_m, \overline{K_m} : m \in \mathbb{N}\}$$

or (for some finite $n$)

$$\mathcal{B}_n = \{P_3, \overline{P_3}, K_n\} \cup \{\overline{K_m} : m \in \mathbb{N}\}$$

then $\mathcal{A}$ is an $I$-amalgamation class (and since it contains $\mathcal{B}$ and no extraneous sets it will be the minimal $I$-amalgamation class containing all of $\mathcal{B}$). Our task is to translate the derivation of any specific graph into a derivation of a 2-graph with a given right component, and thus show that everything that ought to be derivable actually is.

**Theorem 2.13.** *Let $\mathcal{B}$ be a finite set of finite graphs such that the minimal $I$-amalgamation class containing every element of $\mathcal{B}$ is $\mathcal{A}$, and let $c$ be a positive integer. Let*

$$\mathcal{B}' = \{(B, D, R) : B \in \mathcal{B}, D \in \mathcal{D}, R \text{ arbitrary of size } c\}$$

*where $\mathcal{D}$ is a family of finite graphs such that if $D \in \mathcal{D}$ then $D + D \in \mathcal{D}$. Then there exists a minimal amalgamation class $\mathcal{A}'$ that contains all elements of $\mathcal{B}'$; moreover, $\mathcal{A}'$ contains $(A, D, R)$ for all $A \in \mathcal{A}$, all $D \in \mathcal{D}$ and all colourings $R$ (and $\mathcal{A}'$ contains nothing else).*

*Remark.* We can slightly weaken the requirement that if $D \in \mathcal{D}$ then $D + D \in \mathcal{D}$. For example, it is sufficient if $D \boxplus D \in \mathcal{D}$ whenever $D \in \mathcal{D}$. Having some disjoint union $D \sqcup D \in \mathcal{D}$ is **not** sufficient as the fundamental requirement is being able to ensure that the graphs on the right components are consistent.

*Proof.* Let $\mathcal{C}_n$ and $\mathcal{C}$ be as given in Definition 2.12.
Let $(A, D, R)$ be a 2-graph where $A \in \mathcal{A}$ and $D \in \mathcal{D}$. We want to find a derivation $\mathcal{H}'$ of $(A, D, R)$ from $\mathcal{B}'$. If we can do this for every such 2-graph, then the set of these 2-graphs will form an amalgamation class, which would be enough to prove the theorem.
Take a minimal derivation $\mathcal{H}$ of $A$ from $\mathcal{B}$; we seek to construct $\mathcal{H}'$ starting at the top level (the one that will contain $(A, D, R)$) and working down to level 0 that will contain only basic 2-graphs (elements of $\mathcal{B}'$; we will then show that $\mathcal{H}'$ is indeed a subset of $\mathcal{C}'$. At all stages we add sufficient elements to $\mathcal{H}'$ to close under finite unions and to close under the operation "replace $U \in \mathcal{H}'$ with $U \cup \{Y\} \setminus \{X\}$", where $X \in U$ and $Y$ is a substructure of $U$. We can therefore ignore these operations if they occur in $\mathcal{H}$, since we will have closed under them by the next stage anyway.
If the original derivation $\mathcal{H}$ were such that every amalgamation had only one possible outcome, we would be able simply to add $D$ on the right component and add the appropriate cross-edges. However, in general each amalgamation will have multiple outcomes, and the final result will only appear much later (indeed, even if it only had one outcome up to isomorphism, it may still have two or more outcomes with the desired result in different places). We will therefore need to add an appropriately coloured copy of $D$ for each amalgamation outcome ("appropriate" means appropriate to the desired final result), and propagate this down the chain.
Formally, suppose that:

- $X$ and $Y$ can be amalgamated over some $W$ to give one of $Z_1, \ldots, Z_k$;
- $\{U_1, \ldots, U_p, X\}, \{V_1, \ldots, V_q, Y\}, \{U_1, \ldots, U_p, V_1, \ldots, V_q, Z_1, \ldots, Z_k\} \in \mathcal{H}$;
- for some $i \geq 1$, $\{U_1, \ldots, U_p, X\}, \{V_1, \ldots, V_q, Y\} \in \mathcal{C}_i \setminus \mathcal{C}_{i-1}$; and
- by induction, some $\{U'_1, \ldots, U'_p, V'_1, \ldots, V'_q, Z'_1, \ldots, Z'_k\}$ is in $\mathcal{H}'$, where each $Z'_j = (Z_j, m_j D, R_j)$, each $U'_j = (U_j, \tilde{D}_j, \tilde{R}_j)$, and each $V'_j = (V_j, \hat{D}_j, \hat{R}_j)$.

Since if we amalgamate $X$ with $Y$ we get one of $Z_1, \ldots, Z_k$, define $P_1 = (X, mD, S_1)$ and $P_2 = (Y, mD, S_2)$, where $m = \sum_{j=1}^{k} m_j$ and each $S_1$ and $S_2$ is chosen so that the $j$th tranche of $m_j$ copies of $D$ would be coloured to $X$ and to $Y$ so that if the

amalgamation gives $Z_j$ then we have a copy of $Z'_j$. (This can always be defined.) Then add $\{P_1, P_2, U'_1, \ldots, U'_p\}$ to $\mathcal{H}'$.
This process continues until we reach level 0 in $\mathcal{H}$. But the structures here will have their left components be "basic" graphs (elements of $\mathcal{B}$) and their right components will consist of many copies of $D$. Hence, by definition of $\mathcal{D}$, they will be in $\mathcal{B}'$.
It remains to verify that $\mathcal{H}'$ is indeed a derivation. But it clearly has any and all necessary "union" and "substructure" steps, and we have constructed it in such a way as to have the "amalgamation" steps. Hence it is indeed a derivation producing $(A, D, R)$, as required. □

*Remark.* Figure 2.4 gives an example of how the derivation in Figure 2.3 can be translated to a derivation of the 2-graph

$$((abcd, \{ab, bc\}), x, (\{bx, cx\}, \{ax, dx\}))$$

where in this case we need only three copies of the $K_1$ on the right. By hypothesis every colouring of $(B, \overline{K_m})$ is in $\mathcal{B}'$ for every "basic" graph $B$ and every $m$.

## 2.8 Properties of graphs

We list some properties of certain homogeneous graphs we will use throughout this thesis.
We will often make use of the infinite Ramsey theorem. The simplest form of this, and the one we will use, can be written as follows:

**Theorem 2.14** (Ramsey)**.** *Let $X$ be an infinite graph. Either $K_\infty \leq X$ or $\overline{K_\infty} \leq X$.*

*Proof.* See Ramsey (1930). □

At various points we will need to "split" either $\Gamma_r$ or $\Gamma_\infty$ into two subgraphs, and want to show that one of the two subgraphs retains some genericity. For the random graph the proof is relatively simple and we present the proof here.

**Theorem 2.15.** *Let $X \cong \Gamma_\infty$ and let $(X_1, X_2)$ be a partition of the vertices of $X$. Then either $X_1 \cong \Gamma_\infty$ or $X_2 \cong \Gamma_\infty$.*

*Remark.* A rather abstract proof of this result is given in Fraïssé (1986), chapter 10, result 4.4. We give a more concrete proof here.

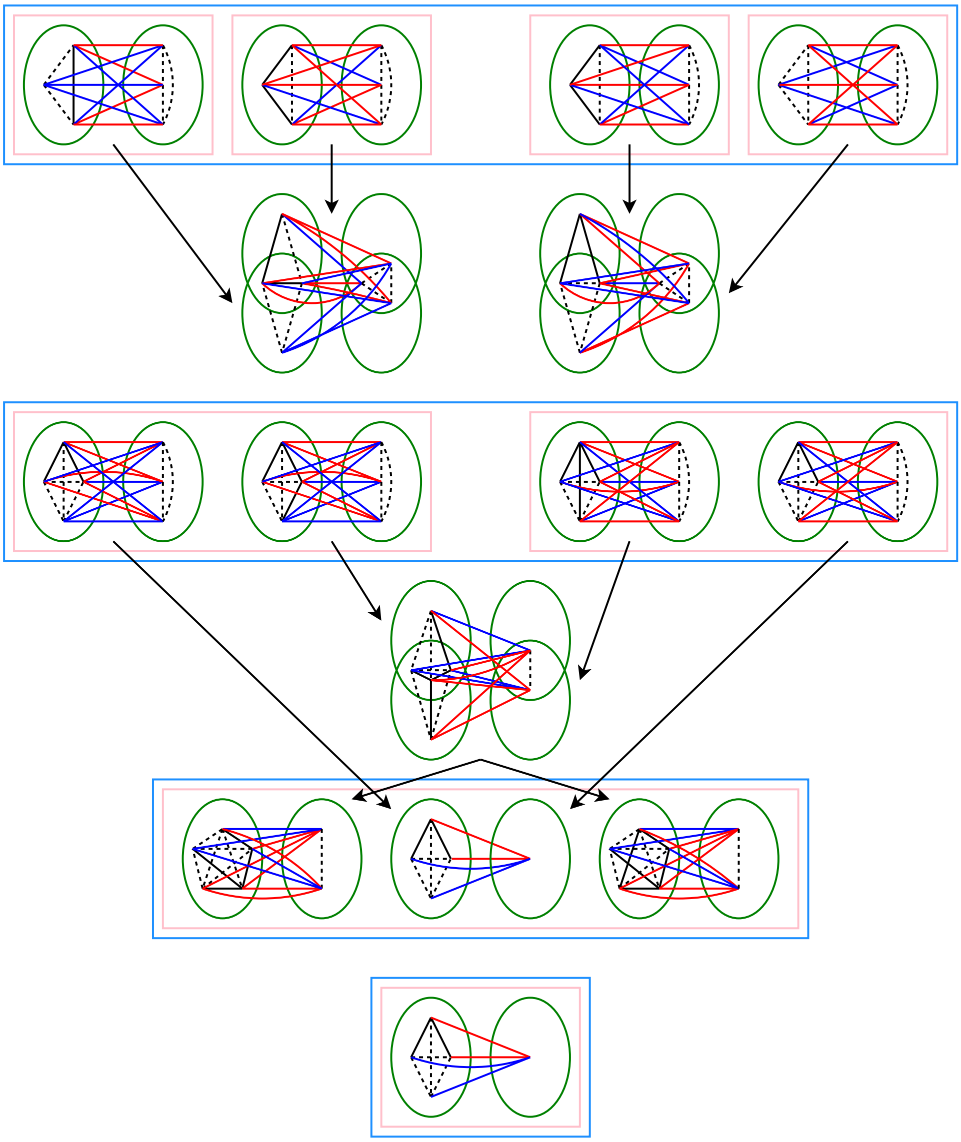

Figure 2.4: An example of a derivation of a $(P_3 + K_1, K_1)$ 2-graph; the cross-edge colours are obtained using the method in Theorem 2.13.

*Proof.* Suppose that $X_1 \not\cong \Gamma_\infty$, so that there is a finite graph $Y$ not realised in $X_1$. We want to show that an arbitrary finite graph $Z$ is realised in $X_2$.
We may assume that $Y$ is minimal, so that for every $y \in V(Y)$ the graph $Y \setminus \{y\}$ does embed in $X_1$; let $Y'$ be such a graph. Construct a graph $W = W_1 \cup W_2$ such that $W_1 \cap W_2 = \varnothing$, $W_1 \cong Y'$, $W_2 \cong Z$ and, for every $w \in W_2$, $W_1 \cup \{w\} \cong Y$.
Since $X$ is the random graph, $W$ embeds in $X$ in such a way that $W_1$ embeds in $X_1$. If any vertex of $W_2$ embeds in $X_1$ then $Y$ would too, which is a contradiction. So every vertex of the embedding $W_2$ must lie in $X_2$, and so $Z$ does also. □

El-Zahar & Sauer (1989) showed that we also cannot split a generic $K_r$-free graph into two simpler pieces. The proof is slightly complicated so we will merely give the statement here.

**Theorem 2.16** (El-Zahar and Sauer)**.** *Let $X \cong \Gamma_r$ for some $r$ (finite or infinite) and let $(X_1, X_2)$ be a partition of the vertices of $X$. Then there exists $i$ such that $X_i$ contains a copy of $\Gamma_r$.*

*Proof.* See El-Zahar & Sauer (1989). □

In many cases we will know independently that $X_1$ and $X_2$ are already homogeneous. In such cases we have the following corollary.

**Corollary 2.17.** *Let $X \cong \Gamma_r$ and let $(X_1, X_2)$ partition $X$ in such a way that each $X_i$ is a homogeneous graph. Then there exists $i$ such that $X_i$ is isomorphic to $\Gamma_r$.*

*Proof.* By Theorem 2.16, one of the $\Gamma_i$ contains (a copy of) $\Gamma_r$ as a subset. But $X_i$ is homogeneous, and so $X_i$ must be $\Gamma_s$ for some $s \geq r$, since these are the only homogeneous graphs containing $\Gamma_r$. If $r = \infty$ then the only possibility is that $X_i \cong \Gamma_\infty$. If $r$ is finite, then since $X$ omits $K_r$, $X_i$ also omits it. Hence $X_i \not\cong \Gamma_s$ for every $s > r$, and so $X_i \cong \Gamma_r$. □

## 2.9 Conventions

We conclude this chapter by listing certain notational and terminological conventions we will be using in this thesis. Some of these have been stated previously in the chapter and we restate them here, slightly less formally, for emphasis.
An **empty** graph is one with zero edges; it can have zero or more vertices. We will sometimes need to make use of graphs with zero vertices; such a graph is a

**null** graph, and by a slight abuse of notation we denote it by $\varnothing$. (For example, the generic $(\Gamma_3, \Gamma_\infty)$ 2-graph omits $(K_3, \varnothing)$.)

We use the terms **component** or **side** to refer to either of the distinguished sets of vertices in a 2-graph. For example, in the 2-graph $(A, B, R)$, the components are $A$ and $B$. We occasionally need to refer to "components" in the graph-theoretic sense: when we do, we call them **connected components**. We distinguish the two components by calling them "left" and "right", and in the 2-graph $(A, B, R)$, $A$ is the left component and $B$ is the right component. (In diagrams we will always have the left component on the left-hand side, and the right component on the right-hand side.)

For our purposes, within components there are two edge-types, called "edge" and "non-edge" (which can be thought of as "black edge" and "white edge"). Therefore, the term **subgraph** of a graph will always mean an *induced* subgraph. In normal graph theory $\overline{K_3}$ is a subgraph of $K_3$; for us this is not the case.

Where two edges (or cross-edges) $ab$ and $cd$ are known to have the same type, we will often write

$$ab \cong cd$$

to signify this. (This can be justified by considering the structures induced by $\{ab\}$ and $\{cd\}$; if $ab$ and $cd$ have the same type then these structures are isomorphic.) We occasionally abuse this notation further: if $ab$ and $xy$ are edges in different components we will still write $ab \cong xy$ as the graphs induced by $\{a, b\}$ and $\{x, y\}$ are isomorphic.

When we say that a 2-graph $G$, with given components and a given number of cross-edge types, is "generic subject to $P$" for some property $P$, we mean that $G$ realises every finite 2-graph with the same colour set as $G$ **except** those that are inconsistent with $P$ and those that $G$ cannot realise because their components do not embed into the components of $G$. We sometimes say "fully generic" or "completely generic" to mean that $P = \top$ (i.e. that $G$ realises all finite 2-graphs with the same colour set as $G$ whose components embed into the components of $G$). Given specified components and a specified number of cross-edge types, there may or may not be a homogeneous 2-graph that is "fully generic" or "generic subject to $P$" for any given $P$, and determining which values of $P$ (and which components and numbers of cross-edge types) do yield homogeneous 2-graphs is the main task of the classification.

Some authors use "generic" to mean that the amalgamation class is the maximal amalgamation class that actually exists whose members satisfy $P$ and which have the correct components and the correct number of cross-edge types. We do **not** use it in this sense. When we say that $G$ is generic subject to $P$, we are asserting that the maximal set of (isomorphism types of) 2-graphs consistent with $P$, the number of cross-edge types and the components **is** an amalgamation class. An example where this is relevant is the 2-coloured homogeneous $(\overline{K_\infty}[K_2], \Gamma_r)$ 2-graph (see Corollary 3.24): for each $r$ there is only one such 2-graph (up to equivalence), but it is not the "fully generic" in our sense since it must omit $(K_2, K_1)^1$ and $(K_2, K_1)^2$. We would normally say that it is the "generic omitting $(K_2, K_1)^1$ and $(K_2, K_1)^2$"; of course $(K_3, \varnothing)$, $(P_3, \varnothing)$ and $(\varnothing, K_r)$ are also (minimally) omitted, but we will usually not mention them as they are implicitly omitted by asserting that it is a $(\overline{K_\infty}[K_2], \Gamma_r)$ 2-graph.

Normally our 2-graphs have just two cross-edge types. When this happens, we normally call the cross-edge types "red" and "blue", where "red" is cross-type 1 and "blue" is cross-type 2: these are the colours we use in diagrams. In many cases we have incomplete information about the 2-graphs in the diagram. To encode this, there are two conventions we use:

- one colour (sometimes "white") for all cross-edges whose colours are unknown; or
- all cross-edges shown as the same colour are known to be the same colour (we do not know which).

If only one extra cross-edge colour is shown, we specify which of these conventions is being used where this is important.

We also use non-black edges (especially brown edges) within components to indicate that their natures are not fully determined; again, in each case where this is important we specify the significance of such edges.

# Chapter 3

# The $(\overline{K_m}[K_n], \Gamma_r)$ case

In this chapter we classify homogeneous 2-graphs of the form $(\overline{K_m}[K_n], \Gamma_r)$, where $m, n, r \in \mathbb{N} \cup \{\infty\}$ and $r \geq 3$.
We will prove the following:

**Theorem 3.1.** *Let $G = (A, B, R)$ be a homogeneous non-collapsing 2-coloured 2-graph where $A \cong \overline{K_m}[K_n]$ for some $m, n \in \mathbb{N} \cup \{\infty\}$ and $B \cong \Gamma_r$ for some $r \in \mathbb{N} \cup \{\infty\}$ where $r \geq 3$. Then $mn = \infty$ and $G$ is equivalent to one of the following 2-graphs:*

- *$m = \infty$, $n = 1$ and $G$ is otherwise generic (i.e. embeds all finite 2-graphs satisfying these constraints);*
- *$m = \infty$, $n = 1$, the 2-graph $(K_1, K_k)^1$ is minimally omitted for some $k < r$, and $G$ is otherwise generic;*
- *$m = \infty$, $n = 2$, the 2-graphs $(K_2, K_1)^1$ and $(K_2, K_1)^2$ are minimally omitted, and $G$ is otherwise generic;*
- *$2 \leq m \leq \infty$, $n = \infty$ and $G$ is otherwise generic; or*
- *$2 \leq m \leq \infty$, $n = \infty$,the 2-graph $(K_1, K_k)^1$ is minimally omitted for some $k < r$, and $G$ is otherwise generic.*

*Moreover, there does exist a homogeneous 2-graph for each of these cases.*

*Remark.* It is clear that the cases defined in Theorem 3.1 are genuinely different; that is, if $G_1$ and $G_2$ are homogeneous 2-coloured $(\overline{K_m}[K_n], \Gamma_r)$ 2-graphs satisfying different cases of Theorem 3.1 then $G_1 \ncong G_2$.

## 3.1 Initial results

The following proposition shows that there are no interesting cases if both $m$ and $n$ are finite.

**Proposition 3.2.** *There is no homogeneous 2-graph $G = (A, B, R)$ with cross-edges of more than one colour such that $A$ is a finite graph and $B \cong \Gamma_r$ ($r$ can be finite or infinite).*

*Proof.* Suppose that $G = (A, B, R)$ is a 2-graph where $A$ is a finite graph and $B \cong \Gamma_r$. We can partition $B$ into finitely many subsets $B_1, \dots, B_k$ such that, if $b, b' \in B_i$ for some $1 \leq i \leq k$, then, for all $a \in A$, $ab$ and $ab'$ are the same colour. Moreover, since at least two colours appear, by 1-transitivity we must have that $k \geq 2$.

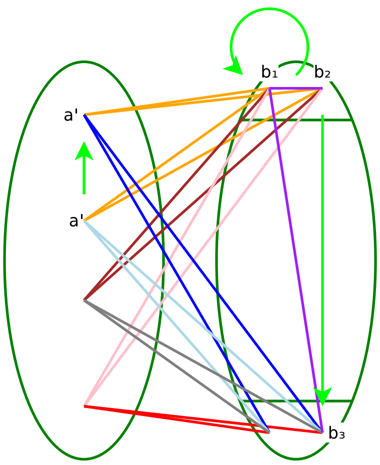


Figure 3.1: If $G = (A, \Gamma_r, R)$ were homogeneous where $R_1, R_2 \neq \varnothing$ and $A$ is finite, we can construct a contradiction.

Now $B$ is infinite, so by the pigeonhole principle some $B_i$ is infinite; without loss of generality assume that $i = 1$ and let $b_1, b_2 \in B_1$ be distinct vertices. Now $B$ and $\overline{B}$ are connected, so there is a $b_3 \in B \setminus B_1$ such that $b_1b_2$ and $b_1b_3$ have the same edge-type; the setup is as depicted in Figure 3.1. Consider the map

$$\alpha : b_1 \mapsto b_1, b_2 \mapsto b_3$$

By construction, $\alpha$ is an isomorphism $\{b_1, b_2\} \to \{b_1, b_3\}$. If $G$ were homogeneous, then $\alpha$ would extend to an automorphism $\beta$ of $G$. Hence, for all $a \in A$, $\beta(a)\beta(b_1) = \beta(a)b_1$ and $\beta(a)\beta(b_2) = \beta(a)b_3$ are the same colour. But there is a vertex $a' \in A$

such that $a'b_2$ (which is the same colour as $a'b_1$) and $a'b_3$ are **not** the same colour. Let $a = \beta^{-1}(a')$. Then

$$a'b_1 = \beta(a)\beta(b_1) \cong ab_1 \cong ab_2 \cong \beta(a)\beta(b_2) = a'b_3$$

since $\beta$ is an automorphism, but this contradicts the definition of $a'$. Hence $G$ is not homogeneous. □

Since there are no non-monochromatic cases if $mn$ is finite, we will assume in the rest of the chapter that either $m$ or $n$ is infinite. We do have to handle the three families of cases ($m$ finite, $n$ finite or both infinite) separately; for example, the $m = \infty, n = 2$ case is very different to the $m = 2, n = \infty$ one. It will also sometimes be convenient to distinguish the $m = \infty, n = 1$ case. The $m = 1, n = \infty$ case is of course equivalent to the $m = \infty, n = 1$ case, so we only need to classify one of these, and the one we will use is the $m = \infty, n = 1$ case.

We now let $G = (A, B, R)$ be a homogeneous 2-graph where $A \cong \overline{K_m}[K_n]$ and $B \cong \Gamma_r$. For each $a \in A$ and each $1 \leq i \leq |R|$, let

$$B_a^i = \{b \in B : (a, b) \in R_i\}$$

and, for each $b \in B$ let

$$A_b^i = \{a \in A : (a, b) \in R_i\}$$

We then have:

**Lemma 3.3.** *For all $a, b \in A$ and all $1 \leq i \leq |R|$, $B_a^i \cong B_b^i$.*

*Proof.* Let $\alpha$ be the map $a \mapsto b$. Since $G$ is homogeneous, $\alpha$ must extend to an automorphism $\beta$ of $G$. To show that $B_a^i \cong B_b^i$, it is enough to show that $\beta(B_a^i) = B_b^i$. For $x \in B$, we show that $x \in B_a^i$ if and only if $\beta(x) \in B_b^i$. But $x \in B_a^i$ if and only if $ax$ has colour $i$, which occurs if and only if $\beta(a)\beta(x) = b\beta(x)$ has colour $i$, which in turn happens if and only if $\beta(x) \in B_b^i$, as required. □

**Lemma 3.4.** *For every $a \in A$ and every $1 \leq i \leq |R|$, $B_a^i$ is homogeneous.*

*Proof.* Let $\gamma$ be an isomorphism between finite subsets of $B_a^i$. This extends to a map $\delta = \gamma \cup \{(a, a)\}$ which is a finite partial automorphism of $(a, B_a^i)$. Since $G$ is homogeneous, $\delta$ extends to an automorphism $\epsilon$ of $G$. But, for every $x \in B$, $ax$ and $\epsilon(a)\epsilon(x)$ always have the same colour. Hence $\epsilon$ must fix $B_a^i$. Since this holds for any isomorphism $\gamma$ between finite subsets of $B_a^i$, it follows that $B_a^i$ is a homogeneous graph. □

From this we obtain the following corollary.

**Corollary 3.5.** *For every $a \in A$, there is $1 \le i \le |R|$ such that $B_a^i$ is isomorphic to $\Gamma_r$.*

*Proof.* The $(B_a^i)$ partition $\Gamma_r$ into finitely many pieces, so by Theorem 2.15 if $r = \infty$, or Theorem 2.16 if $r < \infty$, one of them contains $\Gamma_r$. But all of the pieces are homogeneous, so by the proof of Corollary 2.17 the piece that contains $\Gamma_r$ must in fact be $\Gamma_r$, since it is clearly a subgraph of $\Gamma_r$ so cannot be $\Gamma_s$ for $s > r$. □

In general, it seems more difficult to prove directly that $B_a^i$ is always infinite. However, if $m = \infty$ and $n = 1$, proving that $B_a^i$ is infinite is easier, and it will turn out that in classifying the 2-coloured homogeneous $(\overline{K_m}[K_n], \Gamma_r)$ case we only need to prove that $B_a^i$ is infinite when $m = \infty$ and $n = 1$ (as it will turn out that all non-trivial cases are based on the $m = \infty, n = 1$ case; moreover, proving this will not need Lemma 3.6).

**Lemma 3.6.** *If $m = 1$ or $n = 1$, then for every $a \in A$ and every $1 \le i \le |R|$, $B_a^i$ is infinite.*

*Proof.* We may assume without loss of generality that $i = 1$. We recall that, by 1-transitivity on $A$, $B_a^i$ is non-empty.
First we show that $|B_a^1| \ge 2$. For if $|B_a^1| = 1$, then for every $a \in A$ there is a unique $x_a \in B$ so that $ax_a$ has colour 1 (and, by homogeneity, for every $y \in B$ there is $b \in A$ so that $y = x_b$). We want to find $a, b, c \in A$ so that $ab \cong ac$ (which is always true since $A$ has no structure[1]) but the map

$$\alpha : a \mapsto a, b \mapsto c$$

does not extend to $\beta \in \mathrm{Aut}(G)$; that is, we require that $x_a x_b \ncong x_a x_c$, since, for all $d$ and $e$ in $A$, if $\beta(d) = e$ then $\beta(x_d) = x_e$. But any choice of $x_a, x_b, x_c$ where $x_a x_b \ncong x_a x_c$ will do, and by genericity of $B$, and the fact that the map $a \mapsto x_a$ is onto, we can make such a choice where $a, b, c$ are distinct. Hence $|B_a^1| \ge 2$.
We now show that if $|B_a^1| \ge k$ for some finite $k \ge 2$, then $|B_a^1| \ge k + 1$, which will imply that $B_a^1$ is infinite. If not, choose $a, b, c \in A$ so that $|B_a^1 \setminus B_b^1| = 1$ but $|B_a^1 \setminus B_c^1| = 2$. (This is always possible since, by homogeneity, for every subset $D$ of

[1]If $m, n \ge 2$ this would of course not hold. We avoid this issue in a roundabout way by proving Lemma 3.6 only for the $m = 1$ (and $n = 1$) cases and also, independently of Lemma 3.6, proving that all non-trivial instances of other cases are simple variants of the $m = 1$ case.

$B$ isomorphic to $B_a^1$ there is $d \in A$ so that $D = B_d^1$, and by genericity we can find subsets of $B$ which have intersections of the right sizes.) Consider

$$\gamma : a \mapsto a, b \mapsto c$$

Since $A$ is complete or empty, $\gamma$ is always a finite partial isomorphism. But it does not extend to an automorphism of $G$, since if it did it would map $B_a^1 \cup B_b^1$ to $B_a^1 \cup B_c^1$, mapping a set of size $k+1$ to one of size $k+2$. This is a contradiction, so $|B_a^1| \neq k$ for any finite $k \geq 1$, so $k$ must be infinite. □

So far we have not had to restrict the number $c$ of colours of $G$. However, we will now have to restrict to cases where $G$ has only two cross-edge colours; that is, $R = (R_1, R_2)$. If $G$ has more than two cross-edge colours, then any classification would depend on knowledge of the various combinations of the $B_a^i$ (recall that, by Lemma 3.3, these will, up to isomorphism, be independent of the choice of $a \in A$). In principle this would be possible, but it would appear to need a classification of the 2-coloured homogeneous $(c-1)$-graphs, and even once this is done the number of such cases appears to be so large as to be unworkable.

By Corollary 2.17, we have that, for each $a \in A$, either $B_a^1 \cong \Gamma_r$ or $B_a^2 \cong \Gamma_r$, and by Lemma 3.3 the choice is the same for every $a \in A$. We will therefore assume, without loss of generality, that from now on $B_a^2 \cong \Gamma_r$, and consider the various possible isomorphism types of $B_a^1$ (which, by Lemma 3.3, is known to be a homogeneous graph).

We will show that the 2-graphs stated in Theorem 3.1 really are homogeneous, and then divide the rest of this chapter into proving uniqueness in three families of cases, which we consider separately. The families we consider are:

- $n$ finite, $m = \infty$;
- $m$ finite, $n = \infty$; and
- $m = n = \infty$.

A word of caution – we will typically handle the $\Gamma_r$ and $\Gamma_\infty$ cases together. We will often speak of omitting $(\varnothing, K_r)$; in the $\Gamma_\infty$ case this is to be interpreted as not omitting **any** $(\varnothing, K_k)$ for $k \in \mathbb{N}$. (In the $\Gamma_\infty$ case we will of course **realise** both $(\varnothing, K_\infty)$ and $(\varnothing, \overline{K_\infty})$.) We mention this severe abuse of notation here to avoid having to point out this difference every time. We will **not** have the luxury of being able to use this simplification in chapter 4 as there the $r = \infty$ and $r \neq \infty$ cases appear to need notably different techniques. (We will explain more about this at the points in chapter 4 where this point is relevant.)

## 3.2 Existence

We must verify that all of the 2-graphs listed in Theorem 3.1 really are homogeneous. There are two basic families – the family where $n$ is finite and the family where $n$ is infinite.

### 3.2.1 The $n \neq \infty$ family

The $n \neq \infty$ family of cases are those arising from extending the $(\overline{K_\infty}, \Gamma_r)$ case by increasing the size of components on the left. We will show later (in Lemma 3.9) that $n \leq 2$. We list the cases that can arise and show that in each case there is a homogeneous 2-coloured 2-graph (and they are all clearly not equivalent).

**Proposition 3.7.** *For each $r \geq 3$, there exist non-collapsing homogeneous 2-coloured $(\overline{K_\infty}[K_n], \Gamma_r)$ 2-graphs satisfying each one of the following lists of properties:*

1. *$n = 1$, and $G$ is generic (i.e. embeds all finite 2-graphs consistent with being of the specified form);*
2. *$n = 1$, and $G$ minimally omits $(K_1, K_k)^1$ and is otherwise generic (i.e. embeds all finite 2-graphs consistent with being of the specified form and omitting $(K_1, K_k)^1$); or*
3. *$n = 2$, and $G$ minimally omits $(K_2, K_1)^1$ and $(K_2, K_1)^2$ and is otherwise generic.*

*Proof.* We must prove that the ages of these structures are amalgamation classes. As usual we need only verify that each age has the two-point amalgamation property. The ages can be described respectively as:

1. all finite 2-coloured 2-graphs minimally omitting $(K_2, \varnothing)$ and $(\varnothing, K_r)$;
2. all finite 2-coloured 2-graphs minimally omitting $(K_2, \varnothing)$, $(\varnothing, K_r)$ and $(K_1, K_k)^1$; and

3. all finite 2-coloured 2-graphs minimally omitting $(K_3, \varnothing)$, $(P_3, \varnothing)$, $(\varnothing, K_r)$, $(K_2, K_1)^1$ and $(K_2, K_1)^2$;

In all these cases, we can always add non-edges on the right-hand side of any two-point amalgamation diagram. Moreover, if $n = 1$ we can (and indeed must) always add non-edges on the left-hand side of an amalgamation diagram, and always add blue cross-edges to a two-point amalgamation diagram. Hence there is no difficulty with the $n = 1$ case.

Suppose that $n = 2$. There is no difficulty if both added vertices are on the right-hand side of the two-point amalgamation diagram. If the added vertices are $a$ on the left and $z$ on the right, then either $a$ is joined to a single left-side vertex $b$ or $a$ is not joined to any left-side vertex $b$. In the latter case any cross-edge colours at all will suffice (since none of the restrictions are relevant).

Suppose then that $n = 2$, that the added points are $a$ and $z$, and that $a$ is joined to some $b$. The colour from $a$ to $z$ can simply be the reverse of the colour from $b$ to $z$ – this will certainly omit $(K_2, K_1)^1$ and $(K_2, K_1)^2$, and none of the other restrictions are relevant, so in fact the diagram can be completed and the two-point amalgamation property holds.

We are left with a diagram in the $n = 2$ case where both added vertices, $a$ and $b$, are on the left-hand side of the amalgamation diagram. We must show that this diagram can nevertheless be completed.

There are three choices for what to do between $a$ and $b$:

1. add a non-edge between $a$ and $b$;
2. add an edge between $a$ and $b$; or
3. identify $a$ and $b$.

We can add a non-edge between $a$ and $b$ unless there is some $c$ joined to both $a$ and $b$. In this case we cannot join $a$ and $b$, as this would give a $K_3$ on the left, and we cannot have a non-edge since this would give a $P_3$. So we have to identify $a$ and $b$. But, since $(K_2, K_1)^1$ and $(K_2, K_1)^2$ are omitted (by Lemma 3.10), the colours from $a$ to the right are exactly inverse to those from $c$ to the right, and these are in turn exactly inverse to the colours from $b$ to the right. Hence the colours from $a$ to the right are exactly the same, in the same order, to those from $b$ to the right. Furthermore, the only edges from $a$ or $b$ on the left are the ones to $c$; there is no $d \neq a, b, c$ so that $ad$ or $bd$ are edges. We can therefore identify $a$ and $b$. The class of finite graphs omitting $(\varnothing, K_r)$, $(K_3, \varnothing)$ and $(K_1, K_k)^1$ is therefore an amalgamation class (for each $2 \leq k \leq r, k \in \mathbb{N}$). $\square$

### 3.2.2 The $n = \infty$ family

Whenever $n = \infty$, the following proposition shows that all the cases we require to exist (i.e. those listed in Theorem 3.1) do indeed exist, and moreover the different cases are clearly not equivalent.

**Proposition 3.8.** *For each finite $s$ where $2 \leq s < r$, there exists a $\mathbf{\Gamma}_s^r \cong (\overline{K_m}[K_\infty], \Gamma_r)$ such that every $(K_\infty, \Gamma_r)$ in $G$ minimally omits precisely $(K_1, K_s)^1$. Moreover, there exists a $\mathbf{\Gamma}_r^r \cong (\overline{K_m}[K_\infty], \Gamma_r)$ such that every $(K_\infty, \Gamma_r)$ in $G$ is isomorphic to the fully generic.*

*Remark.* $\mathbf{\Gamma}_r^r$ is a degenerate case of $\mathbf{\Gamma}_s^r$. We define it separately in order to avoid difficulties when $r = s = \infty$ (where the normal definition breaks down).

*Proof.* We have to prove that the class of (isomorphism types of) finite 2-coloured 2-graphs of the form $H = (A_1 + \ldots + A_m, B)$ (where each $A_j \cong K_{i_j}$, where each $i_j < r$ and where $H$ also omits $(K_1, K_t)^1$ for finite $t \geq s$) is an amalgamation class. (If $r = s = \infty$, this amounts to realising every finite $(K_1, K_t)^1$.) Clearly the class has the hereditary property, and clearly it is closed under isomorphism. We therefore have to prove that the class has the amalgamation property, and it is sufficient to show that it has the two-point amalgamation property. If the two points are on the right we can clearly add a non-edge between them, and if one of the two points is on the left and the other on the right we can clearly add a cross-edge of colour 2. Therefore, the only potential difficulty is if the two new points are on the left (and if $m \geq 2$; if $m = 1$ we can simply add an edge).
Label the two points on the left $a$ and $b$. If there is a vertex $c$ on the left with edges from $c$ to both $a$ and $b$, we want to add an edge between $a$ and $b$; this is always possible in such circumstances. If there is no such $c$, but there is a $d$ with an edge $ad$ but no edge $bd$ (or vice versa), we want to add a non-edge between $a$ and $b$, which we can do since we do not need to increase the number of connected components on the left.
Suppose that for every vertex $e$ on the left (except $a$ and $b$), both $ae$ and $be$ are non-edges. In this case we want to join $a$ and $b$. The number of connected components on the left remains the same and all components are complete graphs. Hence the amalgamation is valid and the class is indeed closed under amalgamation. $\square$

*Remark.* Note that there is never a need to **identify** the added vertices. This is the key difference between this case and the case where $m = \infty$ and $n$ is finite.

## 3.3 $n$ finite; uniqueness

Having proved that all the 2-coloured 2-graphs in Theorem 3.1 do exist and are homogeneous, we turn our attention to showing that those are the only homogeneous 2-coloured $(\overline{K_m}[K_n], \Gamma_r)$ 2-graphs. In this section we look at those cases where $m = \infty$ and $n < \infty$.

We can fairly easily show that a number of potential cases, in particular those where $m = \infty$ and $3 \le n < \infty$ and which are not collapsing, cannot in fact arise. We show this using the following lemma.

**Lemma 3.9.** *There are **no** homogeneous non-collapsing 2-coloured $(\overline{K_\infty}[K_n], \Gamma_r)$ 2-graphs for any finite $n \ge 3$.*

*Proof.* Fix a connected component $a_0 \dots a_{n-1}$ in $A$, and divide $B$ into $B_0, \dots, B_{2^n-1}$ such that $B_i$ is red to all vertices in $a_j$ if $\lfloor i/2^j \rfloor \equiv 1 \pmod 2$, and blue to all vertices in $a_j$ otherwise (i.e. if $\lfloor i/2^j \rfloor \equiv 0 \pmod 2$).

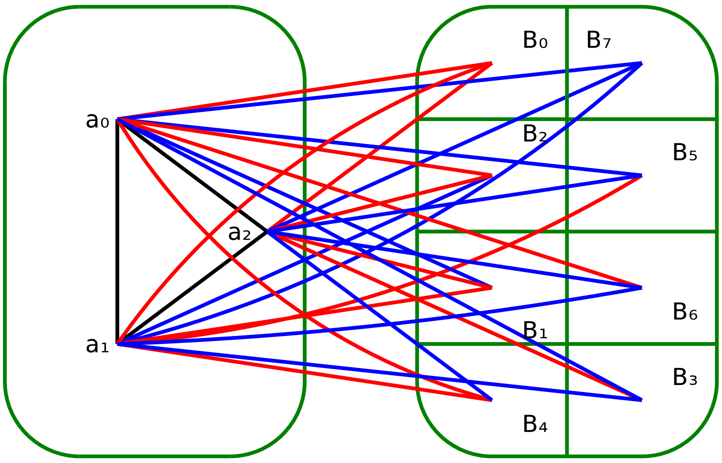


Figure 3.2: An illustration of how we divide $B$ in the $(\overline{K_\infty}[K_3], \Gamma_r)$ case (i.e. when $n = 3$).

Let

$$C = \{B_i : B_i \neq \varnothing\}$$

Either $C \subseteq \{B_0, B_{2^n-1}\}$ or, for all $B_{j_1}$ and $B_{j_2}$ in $C$,

$$(\{a_0, \ldots, a_{n-1}\}, B_{j_1}) \cong (\{a_0, \ldots, a_{n-1}\}, B_{j_2})$$

(for if $B_{j_1} \in C \setminus \{B_0, B_{2^n-1}\}$ then for every $B_{j_2} \in C$ there are $i_1, i_2$ so that, for some $x \in B_{j_1}$ and $y \in B_{j_2}$, $a_{i_1}x \cong a_{i_2}y$, which extends to an isomorphism as above) and we may assume the latter since the former (i.e. $C \subseteq \{B_0, B_{2^n-1}\}$) implies that $G$ is collapsing.

Now some $B_i \in C$ is isomorphic to $\Gamma_r$, and since $B_i \cong B_j$ for all $i$ and $j$ it follows that every $B_i \in C$ is isomorphic to $\Gamma_r$. Since $n \geq 3$, we can choose $B_{j_1}, B_{j_2} \in C$ so that both are identically coloured to $a_{n-1}$ and differently coloured to $a_0$ (this fails if $n = 2$). Pick $x \in B_{j_1}$ and $z \in B_{j_2}$. Since $B_{j_1} \cong \Gamma_r$ and since $\Gamma_r$ and $\overline{\Gamma_r}$ are connected, there certainly exists $y \in B_{j_1}$ such that $xy \cong xz$. Consider

$$\gamma : a_{n-1} \mapsto a_{n-1}, x \mapsto x, y \mapsto z$$

and its extension $\delta \in \operatorname{Aut} G$.

Now $\delta$ must fix the set

$$\{a_0, \ldots, a_{n-1}\}$$

as a set (not necessarily pointwise), so it must map each $B_j$ to some $B_{j'}$ (since the partition is defined by the set $\{a_0, \ldots, a_{n-1}\}$). But $\delta$ necessarily destroys this partition; since $\delta(x) = x$ and $\delta(y) = z$, $\delta(B_{j_1})$ cannot be equal to any $B_j$. Contradiction. □

Moreover, if $n = 2$ we will show that the only solution involves a type of "copying" (but not the "copying" of Theorem 2.13); the two vertices in each edge on the left have to be, in a sense, "mirror images" of each other. In particular, we will eventually prove that:

**Theorem.** *Up to equivalence, there is exactly one non-collapsing homogeneous 2-coloured $(\overline{K_\infty}[K_2], \Gamma_r)$ 2-graph $G$.*

Most of the proof of this result will be given after we complete the classification of the 2-coloured homogeneous $(\overline{K_\infty}, \Gamma_r)$ case (i.e. the $n = 1$ case). However, we will give the proof of the "easy" part here; that is, any such homogeneous 2-coloured $(\overline{K_\infty}[K_2], \Gamma_r)$ 2-graph must have a "mirror image" property.

**Lemma 3.10.** *Let $G = (\overline{K_\infty}[K_2], \Gamma_r)$ be a non-collapsing homogeneous 2-coloured 2-graph. Then $G$ is "quasi-collapsing" - that is, if $ab$ is an edge in the left component of $G$ and $x$ is in the right component of $G$ then $ax$ and $bx$ are of **different** colours.*

*Proof.* Let $G = (A, B, R)$ be a homogeneous 2-coloured $(\overline{K_\infty}[K_2], \Gamma_r)$ 2-graph. Let

$$A = (\{a_i, b_i : i \in \mathbb{N}^+\}, \{a_i b_i : i \in \mathbb{N}\})$$

(i.e. the only edges of $A$ are $a_i b_i$).
Suppose that $G$ is **not** quasi-collapsing; that is, suppose there is an edge $ab \in A$ and there are vertices $x, y, z \in B$ such that $(ab, x) \cong (ab, y)$ but $(ab, x) \ncong (ab, z)$. Then, by 1-transitivity, there is a vertex $y \in B$ and there are edges $a_1 b_1, a_2 b_2$ in $A$ such that $a_1 y \cong b_1 y$ but $a_2 y \ncong b_2 y$. We may assume without loss of generality that $a_1 y \cong a_2 y$. Consider

$$\delta : a_1 \mapsto a_2, y \mapsto y$$

This is a finite partial automorphism of $G$, so extends to some $\epsilon \in \mathrm{Aut}(H)$. But then $\epsilon(b_1) = b_2$ (since it must be joined to $a_2$) but by construction

$$b_1 y \cong a_1 y \cong a_2 y \ncong b_2 y$$

a contradiction. Hence if $G$ is not quasi-collapsing then it cannot be homogeneous. □

To classify the homogeneous non-collapsing $m = \infty, n \neq \infty$ case in full, since the $m = \infty, 3 \leq n < \infty$ case cannot arise, it is enough to classify the $m = \infty, n = 1$ case and then to show how that the $m = \infty, n = 2$ case arises naturally out of (one of) the $m = \infty, n = 1$ cases. We will now concentrate on the $m = \infty, n = 1$ case before briefly going back to conclude the classification of the $m = \infty, n = 2$ case.
There are two sub-cases of the $n = 1$ case that we will distinguish - the case where we omit some monochromatic $(1, D)$ 2-graph and the case where all monochromatic $(1, D)$ 2-graphs embed. We will now classify these separately.

### 3.3.1 $n = 1$; not all monochromatics embed

In this section we assume that $m = \infty$ and that $n = 1$. Our aim is to prove the following:

**Theorem 3.11.** *Let $G$ be a homogeneous $(\overline{K_\infty}, \Gamma_r)$ 2-coloured 2-graph such that $G$ omits a monochromatic 2-graph $(C, D)^1$. Then there exists $k$ such that:*

1. *if $r$ is finite, $G$ is the generic 2-coloured $(\overline{K_\infty}, \Gamma_r)$ 2-graph omitting $(K_1, K_k)^1$; or*

2. *if $r = \infty$, $G$ is either the generic 2-coloured $(\overline{K_\infty}, \Gamma_\infty)$ 2-graph omitting $(K_1, K_k)^1$ or the generic 2-coloured $(\overline{K_\infty}, \Gamma_\infty)$ 2-graph omitting $(K_1, \overline{K_k})^1$.*

Suppose that $G$ omits a 2-graph of the form $(1, D)^i$. Without loss of generality, we may assume that $i = 1$, and as such $D$ is omitted from (every) $B_a^1$. In principle, since $B$ is a homogeneous graph, there are therefore four possibilities for $D$:

1. $D = P_3$;
2. $D = \overline{P_3}$;
3. for some $k$, $D = K_k$; or
4. for some $k$, $D = \overline{K_k}$.

Since $B$ is infinite, we may assume that $B_a^1$ contains $\overline{K_\infty}$. (By construction $B_a^1$ contains $\overline{K_\infty}$ if $r$ is finite anyway, and if $r$ is infinite and $B_a^1$ omitted $\overline{K_k}$ then consider the complement.) If we can prove that $B_a^1$ contains both $P_3$ and $\overline{P_3}$, then $B_a^1$ will contain, and will therefore be equal to, $\Gamma_s$ for some $s \geq 3$. If $B_a^1$ omits both $P_3$ and $\overline{P_3}$ then it omits $K_2$ and will therefore be $\overline{K_\infty}$; by analogy we say that $\Gamma_2 = \overline{K_\infty}$. The following lemma shows that either both or neither of $P_3$ and $\overline{P_3}$ must embed in $B_1^1$, and so (in a slight abuse of notation) $B_a^1 \cong \Gamma_s$ for some $s \geq 2$.

**Lemma 3.12.** *If $n = 1$ and $B_a^1$ embeds $K_2$ then it embeds both $P_3$ and $\overline{P_3}$*

*Proof.* Suppose that $B_a^1$ does embed $\overline{P_3}$. We prove that it embeds $P_3$ as well. (The converse, showing that if $B_a^1$ contains $P_3$ then it contains $\overline{P_3}$, is similar; to do so we simply take complements of the right components of all amalgamation diagrams in this proof.)
Since $B$ is connected, there is an edge between some vertex $x$ in $B_a^1$ and some vertex $y$ in $B_a^2$. Hence the 2-graph

$$(\{a\}, \{x, y\}, (\{ax\}, \{ay\}))$$

does embed in $G$. Similarly, since $\overline{B}$ is connected,

$$(\{a\}, \varnothing, (\{ax\}, \{ay\}))$$

also embeds in $G$

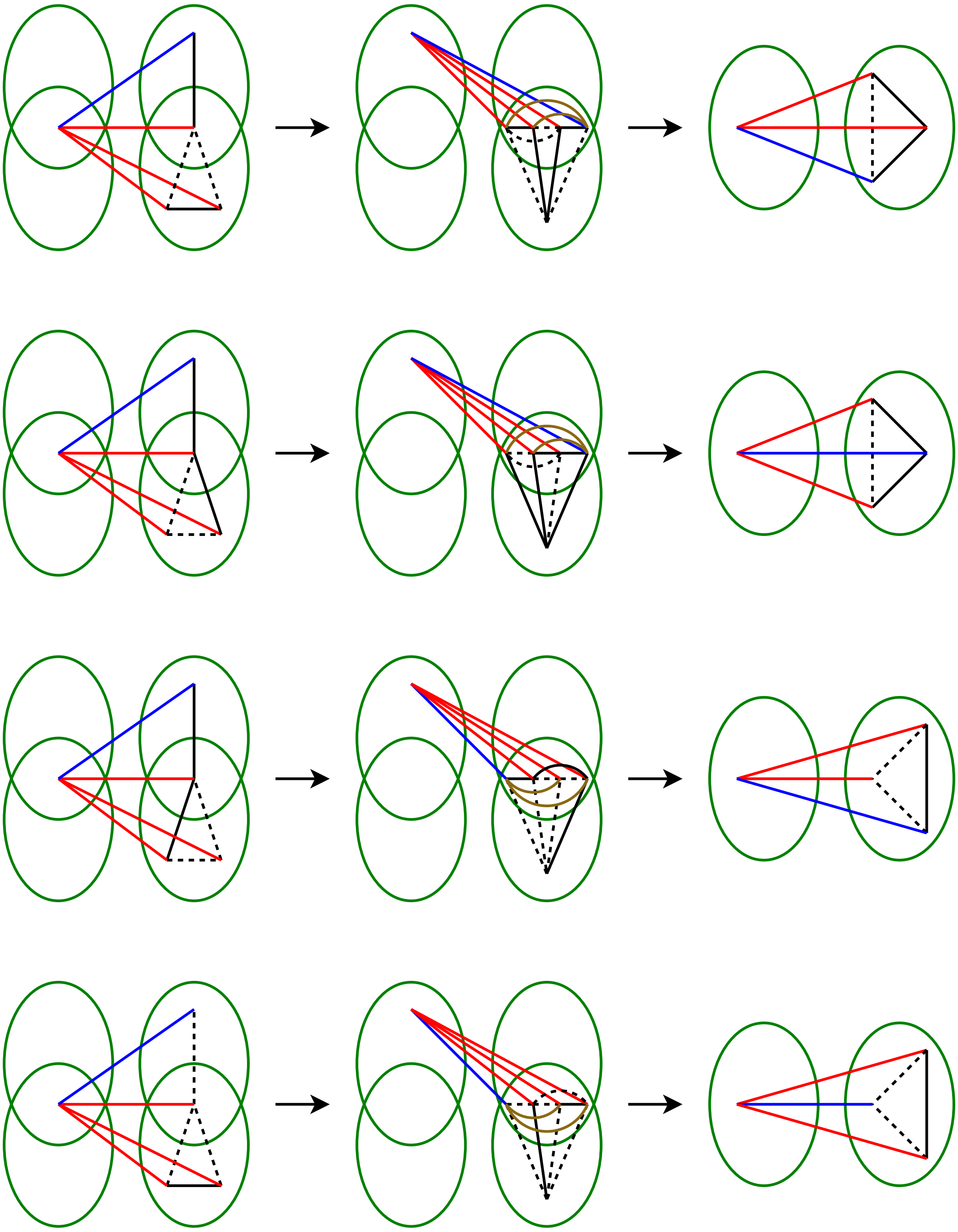

Figure 3.3: Obtaining the four "ingredients" needed to prove that $G$ embeds $(K_1, P_3)^1$ and $(K_1, \overline{P_3})^1$.

We will need four small 2-graphs, which we call "ingredients", to embed in $G$. We exhibit them, and show how they do indeed embed in $G$, in Figure 3.3.
We then use the amalgamation in Figure 3.4 to see that these "ingredients" are enough to show that $(K_1, P_3)^1$ embeds in $G$ (i.e. that $P_3 \leq B_a^1$). Each of the amalgamands in Figure 3.4 is in turn obtained by amalgamating two of the "ingredients" over $(\varnothing, P_3)$, and in so doing we must obtain non-edges on the left in both cases, allowing us to perform the amalgamation of Figure 3.4.

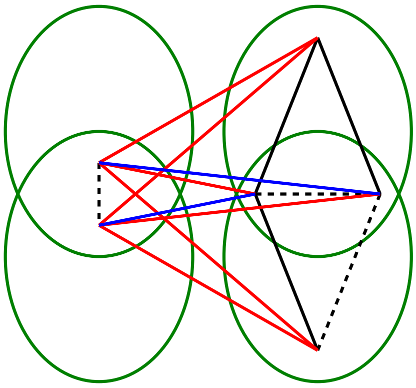

Figure 3.4: The ingredients in Figure 3.3 combine to form this amalgamation diagram, and whether we add an edge or a non-edge we must obtain $(K_1, P_3)^1$. (Note that we can obtain the top and bottom without identifying, and since $n = 1$ we have to have a non-edge, not an edge, on the left.)

□

**Corollary 3.13.** *There exists a finite $k \geq 2$ such that $B_a^1 \cong \Gamma_k$ (if $k = 2$ then $B_a^1 \cong \overline{K_\infty}$).*

We now prove that $G$ must embed all finite 2-graphs $(K_1, D)$ that do not themselves embed $(K_1, K_k)^1$.

**Lemma 3.14.** *Let $H = (K_1, D, R)$ be a finite 2-graph not embedding $(K_1, K_k)^1$. Then $H$ embeds in $G$.*

*Remark.* The idea here is similar to the proof we gave of Lemma 2.15. Note that this proof relies on the homogeneity of $B_a^1$ and $B_a^2$.

*Proof.* Let $D = D_1 \cup D_2$ so that, if we write $H = (b, D_1 \cup D_2)$, then $bd$ is red if $d \in D_1$ and blue if $d \in D_2$. Fix $a \in A$ and let $v = |D_2|$. Label the points of $D_2$ by $d_1, \ldots, d_v$.

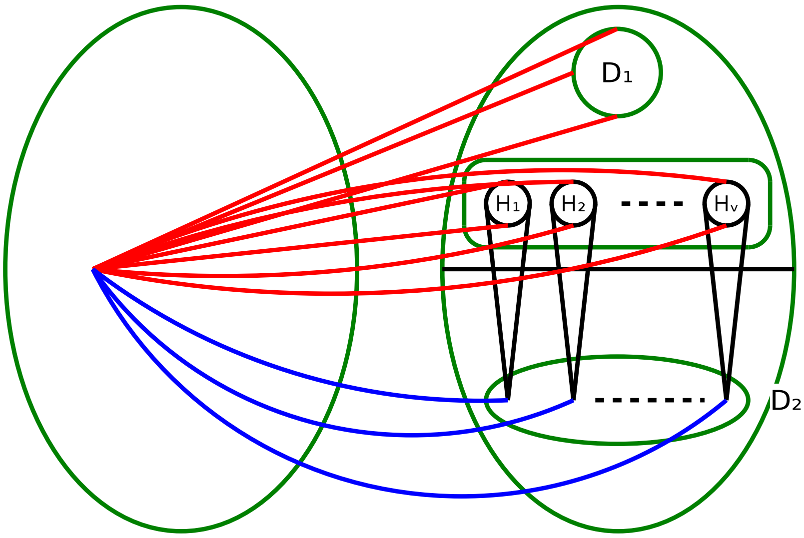


Figure 3.5: Why $G$ embeds $(K_1, D)$ of mixed colour whenever $D_1$ does not contain $K_k$.

Let $H' = vK_{k-1}$ and label the copies of $K_{k-1}$ in $H'$ by $H_1, \ldots, H_v$. Form a graph $J$ consisting of $H' \cup D$, with the only new edges being, for each $i$, those from (the copy of) $d_i$ to each vertex in (the copy of) $H_i$. (We still have the appropriate edges between $D_1$ and $D_2$.)
Note that $H' \cup D_1$ embeds in $B_a^1$ since $B_a' \cong \Gamma_k$. By genericity of $\Gamma_r$, $J$ must embed in $B$ in such a way that the copy of $H'$ in $J$ also embeds in $G$. But no copy of $d_i$ could be mapped into $B_a^1$, as if it were we would obtain a copy of $K_k$ in $B_a^1$, and this is impossible. Hence every $d_i$ is mapped into $B_a^2$, and so we obtain our copy of $H$ in $G$. □

We have proved that, for every finite graph $D < \Gamma_r$, $G$ realises every $H' = (K_1, D, R')$ that does not realise $(K_1, K_k)^1$. To complete the "uniqueness" part of our classification of the $n = 1$ case, we need to show that, for every finite $l \in \mathbb{N}$, $G$ realises every $H = (\overline{K_l}, D, R)$ not itself embedding $(K_1, K_k)^1$, and we do this using the following lemma.

**Lemma 3.15.** *Let $G$ be a 2-coloured homogeneous $(\overline{K_\infty}, \Gamma_r)$ 2-graph omitting $(K_1, K_k)^1$ for some $k < r$. Then $G$ embeds every finite 2-graph $H$ of the form $(\overline{K_p}, D)$ not embedding $(K_1, K_k)^1$.*

*Proof.* Increase $p$ inductively, the case $p = 1$ being a consequence of Lemma 3.14. Suppose $H = (A_1 + \ldots + A_p, D)$, where each $A_i$ is the 1-vertex graph $\{a_i\}$. Let

$$H_1 = H \setminus A_p \cup \{(\varnothing, z)\}$$

and let

$$H_2 = (H \cap (A_p, D)) \cup \{(\varnothing, z)\}$$

where $z$ is colour 1 to all of $A_p$ and colour 2 to all of $A_1, \ldots, A_{p-1}$.
By induction, both $H_1$ and $H_2$ embed in $G$. Amalgamate both over $(\varnothing, D \cup \{z\})$. We clearly cannot identify any vertex in $A_p$ with any vertex in any $A_i$ with $i \neq p$, and we cannot add edges. Therefore we have to obtain $H$, as required. □

*Proof of Theorem 3.11.* Immediate corollary of Lemmas 3.12, 3.14 and 3.15, Corollary 3.13 and other remarks in this section. □

### 3.3.2 $n = 1$; all monochromatics embed

Let $G$ be a homogeneous 2-coloured $(\overline{K_\infty}, \Gamma_r)$ 2-graph embedding every 2-graph of the form $(K_1, D)^i$ where $D$ is a finite graph (in $\Gamma_r$) and $i \in \{1, 2\}$. We will show that $G$ **must** be the generic 2-coloured $(\overline{K_m}[K_n], \Gamma_r)$ 2-graph (i.e. $G$ must realise any finite 2-coloured $(\overline{K_p}, D)$ 2-graph where $D < \Gamma_r$). We showed in Proposition 3.7 that there **is** such a 2-graph and that it is indeed homogeneous; here we prove that it is unique up to equivalence. That is, we prove:

**Theorem 3.16.** *Let $G$ be a homogeneous 2-coloured $(\overline{K_\infty}, \Gamma_r)$ 2-graph realising every 2-graph of the forms $(K_1, K_s)^i$ and $(K_1, \overline{K_s})^i$ for $s < r$ and $i \in \{1, 2\}$. Then $G$ is the generic 2-coloured $(\overline{K_\infty}, \Gamma_r)$ 2-graph.*

We will aim to use the "copying argument" (that is, Theorem 2.13). Therefore, it will be sufficient to show that $G$ embeds everything of the form

$$(\overline{K_p}, B)$$

where $p \in \mathbb{N}$ and

$$B \in \{P_3, \overline{P_3}\} \cup \{K_s : s < r\} \cup \{\overline{K_s} : s \in \mathbb{N}\}$$

In practice we will aim for slightly more. We will show that $G$ embeds everything of the form

$$(\overline{K_p}, B)$$

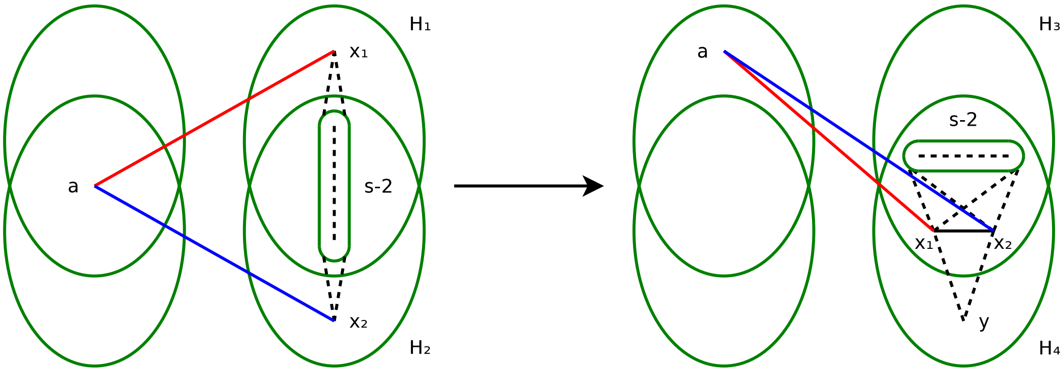


Figure 3.6: How we add a point to $(K_1, \overline{K_s})$; if the initial amalgamation gives an edge instead of a non-edge, we then amalgamate with a structure with null left component (as shown) to convert the edge back to a non-edge.

where $p \in \mathbb{N}$ and either

$$B = i_s K_s + \ldots + i_2 K_2 + i_1 K_1$$

or

$$B = j_3 P_3 + j_2 K_2 + j_1 K_1$$

for $i_1, \ldots, i_s, j_1, j_2, j_3 \in \mathbb{N}$ and $s < r$. We will show that this holds when $p = 1$, and then increase $p$ by using the method of Theorem 3.15. Note that this aspect of the argument is mostly transferable to the $(\Gamma_r, \Gamma_s)$ case in Chapter 4 (though, for technical reasons, we will not always rely on this).

**Lemma 3.17.** *$G$ embeds everything of the form $(K_1, \overline{K_s})$ for all $s \in \mathbb{N}$.*

*Proof.* We work by induction on $s$, and note that if $s \leq 1$ we know the result holds. Let $H = (a, x_1 \ldots x_s)$ where there are no edges. We may assume that $ax_1$ is red and $ax_2$ is blue (if $ax_i$ is the same colour for every $i$ then $H$ is already known to embed in $G$). Amalgamate

$$H_1 = (a, x_1 x_3 \ldots x_s)$$

with

$$H_2 = (a, x_2 x_3 \ldots x_s)$$

over

$$H_0 = (a, x_3 \ldots x_s)$$

as in the left-hand diagram in Figure 3.6 to give a product $H_3$ in which $x_1x_2$ is either an edge or a non-edge. ($H_1$ and $H_2$ embed in $G$ by induction on $s$.) If $x_1x_2$ is a non-edge we are finished. Otherwise, amalgamate $H_3$ with

$$H_4 = (\varnothing, x_1 \dots x_s y)$$

(which clearly embeds in $G$) over

$$(\varnothing, x_1 \dots x_s)$$

where the only edge is $x_1x_2$, as in the right-hand diagram in Figure 3.6. Then, whether $ay$ is red or blue we obtain $H$ (by discarding $x_1$ if $ay$ is blue, and by discarding $x_2$ if $ay$ is red). □

We can do something almost identical to show that $G$ also embeds $(K_1, K_s)$ for all colours when $s < r$.

**Lemma 3.18.** *$G$ embeds everything of the form $(K_1, K_s)$ whenever $s < r$.*

*Proof.* Work by induction on $s$, the cases where $s \leq 1$ being known. Let $H = (a, x_1 \dots x_s)$ where $x_ix_j$ is an edge whenever $i \neq j$. Assume, without loss of generality, that $ax_1$ is red and $ax_2$ is blue. Amalgamate

$$H_1 = (a, x_1x_3 \dots x_s)$$

with

$$H_2 = (a, x_2x_3 \dots x_s)$$

over

$$H_0 = (a, x_3 \dots x_s)$$

giving $H_3$ in which $x_1x_2$ is either an edge or a non-edge. ($H_1$ and $H_2$ embed in $G$ by induction.) If it is an edge we are done, so assume it is a non-edge. Amalgamate $H_3$ with

$$H_4 = (\varnothing, x_1 \dots x_s y)$$

over

$$(\varnothing, x_1 \dots x_s)$$

where $x_iy$ is an edge for all $i$; clearly $H_4$ embeds in $G$. Whether $ay$ is red or blue, we then see that $H$ embeds in $G$. □

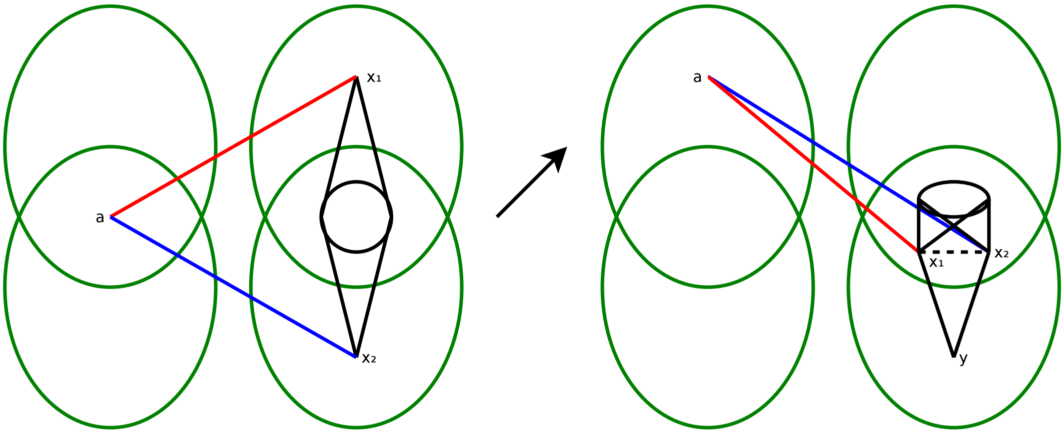


Figure 3.7: How we add a point to $(K_1, K_s)$; if the initial amalgamation gives a non-edge instead of an edge, we then amalgamate with a structure with null left component (as shown) to convert the non-edge back to an edge.

We now show that $G$ embeds $(K_1, \lambda K_s)$ for all $\lambda, s \in \mathbb{N}$ and $s < r$. In practice we prove slightly more, namely we show that $G$ embeds

$$(K_1, K_{s_1} + \ldots + K_{s_\lambda})$$

for $s_1, \ldots, s_\lambda < r$.

**Lemma 3.19.** *Let $H = (K_1, K_s + \overline{K_t})$ for some $s, t \in \mathbb{N}$ where $s < r$. Then $H$ embeds in $G$.*

*Proof.* We work by induction on $(s, t)$, cases where either $s = 1$ or $t = 0$ being trivial. Write $H = (a, x_1 \ldots x_s y_1 \ldots y_t)$, where $x_i x_j$ is an edge if and only if $i \neq j$, and these are the only edges on the right. There are three non-trivial cases to consider:

1. $ax_1$ is red and $ax_2$ is blue;
2. $ay_1$ is red and $ay_2$ is blue; or
3. $ax_i$ is red for all $i$, and $ay_j$ is blue for all $j$.

In case 1 we can apply Lemma 3.18, and in case 2 we can apply Lemma 3.17, in both cases with extraneous matter that does not affect the argument. So the only interesting case is case 3.

In case 3 we apply the amalgamations in Figure 3.8. $H_5$ would embed in $G$ by induction on $s$ with an increased value of $t$. To make this explicit, note that $H_1$ and

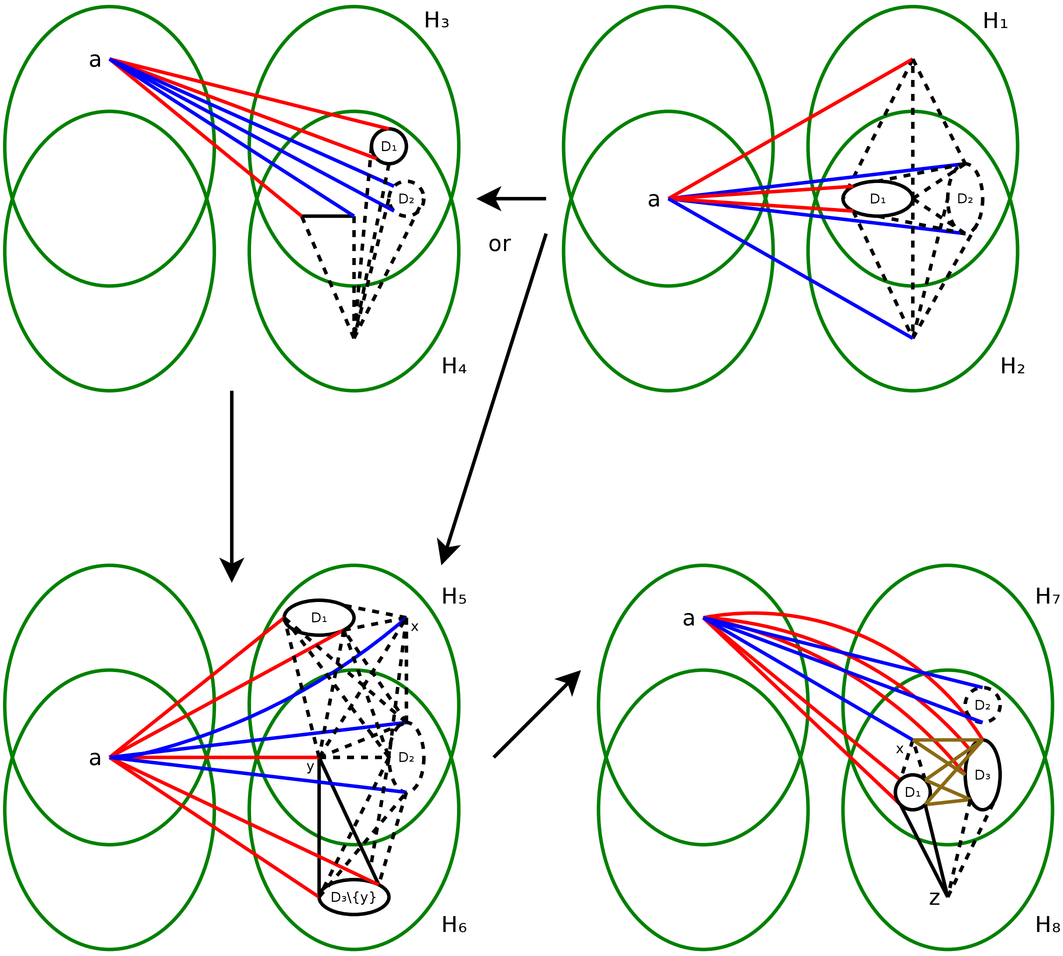


Figure 3.8: Obtaining $H \cong (K_1, K_s + K_t)$ in case 3 (the difficult case). Note that $D_1 \cong K_{s-1}$, $D_2 \cong \overline{K_{t-1}}$ and $D_3 \cong K_s$.

$H_2$ embed in $G$ by induction on $G$, and amalgamating them gives either $H_3$ or $H_5$, and if $H_3$ then since $H_4$ definitely embeds in $G$ we can amalgamate $H_3$ with $H_4$ to obtain $H_5$.

Now $H_6$ embeds in $G$ by induction on $t$. We wish to amalgamate $H_5$ with $H_6$. We have arranged things so that $y$ is joined to no vertices of $D_1$ and to all vertices of $D_3$. Hence amalgamation cannot cause any points to be identified, so we obtain $H_7$. $H_8$ clearly embeds in $G$, and, whether $az$ is red or blue, amalgamating $H_7$ with $H_8$ gives $H$ as required. □

**Lemma 3.20.** *Let* $H = (K_1, K_{s_1} + \ldots + K_{s_\lambda} + \overline{K_t})$ *for some* $\lambda, s_1, \ldots, s_\lambda, t \in \mathbb{N}$ *and*

*$2 \leq s_i < r$ for all $i$. Then $H$ embeds in $G$.*

*Proof.* We will work by induction on $s_1$ and on $t$.
If, within any complete connected component of the right-component of $H$, there is a red and a blue cross-edge from the left-hand vertex $a$, then we apply the method of Lemma 3.18 to apply induction on the size of that component.
Write $H = (a, D_1 + \ldots + D_\lambda + E)$ where $D_i \cong K_{s_i}$ and $E \cong \overline{K_t}$. We consider two cases:

1. $(a, D_i)$ is monochromatic red for every $i$ but there is $e \in E$ so that $(a, e)$ is blue; or

2. $(a, D_1)$ is monochromatic red and $(a, D_2)$ is monochromatic blue.

In the first case, if there exists $e, f \in E$ so that $(a, e)$ is blue and $(a, f)$ is red, then likewise we can apply the method of Lemma 3.17 and apply induction on $t$. So assume that $(a, e)$ is blue for all elements $e \in E$.
We then proceed inductively using the sequence of amalgamations in Figure 3.9. The inductions are on $s_1$ and on $t$; and the base cases will thus be when $t = 0$ and when $s_1 = 0$.
Now consider the second case, and in this case we will work by induction on $(s_1, s_2)$ and assume that, whenever either is smaller, we have all cases with, in particular, all values of $t$.
Proceed inductively using the sequence of amalgamations in Figure 3.10. □

We have thus far shown that $G$ embeds every 2-coloured 2-graph of the form

$$(K_1, i_s K_s + \ldots + i_1 K_1)$$

for all $s, i_1, \ldots, i_s \in \mathbb{N}$ where $s < r$. We also need every 2-coloured 2-graph of the form

$$(K_1, iP_3)$$

for all $i \in \mathbb{N}$. We will actually prove slightly more.

**Lemma 3.21.** *Let $H = (K_1, iP_3 + jK_2 + \overline{K_s})$, for some $i, j, s \in \mathbb{N}$. Then $H$ embeds in $G$.*

*Proof.* Let $H = (a, D_1 + D_2 + D_3)$ where $D_1 \cong iP_3$, $D_2 \cong jK_2$ and $D_3 \cong \overline{K_s}$. We wish to work by induction on $(i, j, s)$.

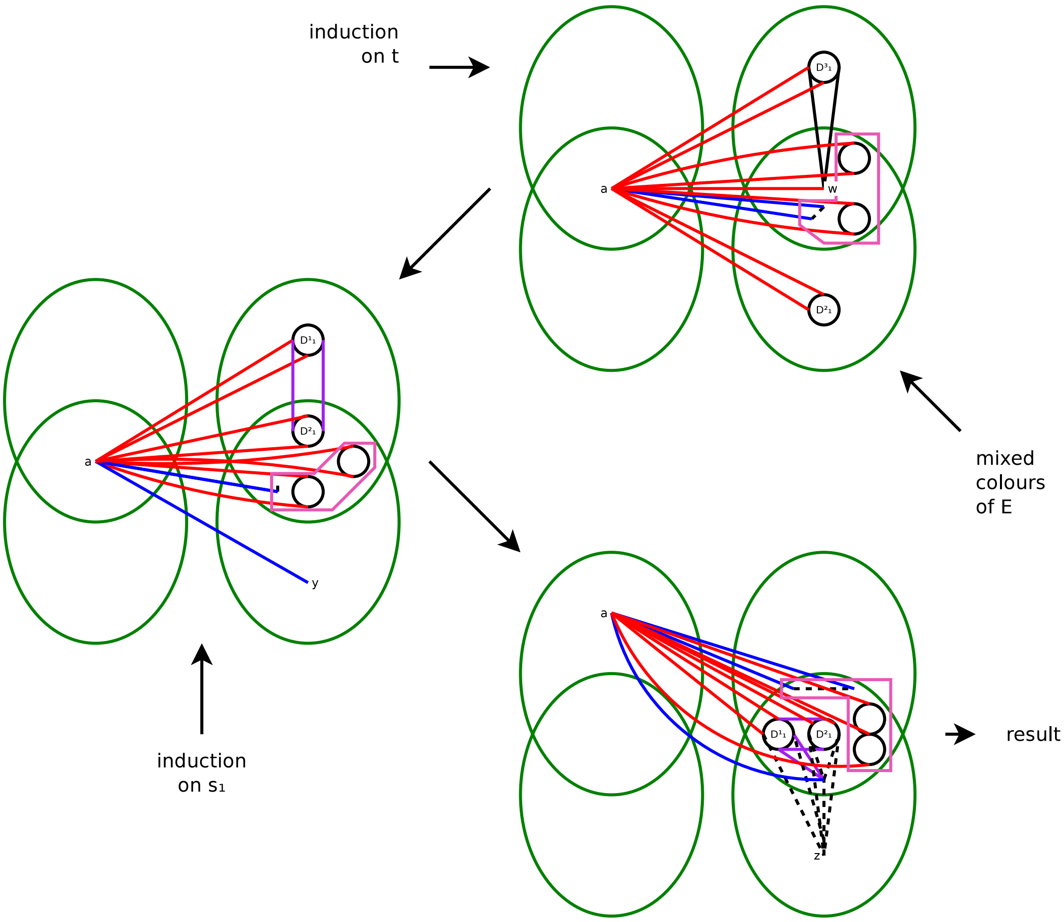


Figure 3.9: The inductive step for case 1 of Lemma 3.20. Note that $D_1^1 = D_1^3 \cup \{w\} \cong D_1^2 \cup \{z\} \cong D_1$. The contents of the pink boxes are carried forward unaltered throughout the sequence of amalgamations.

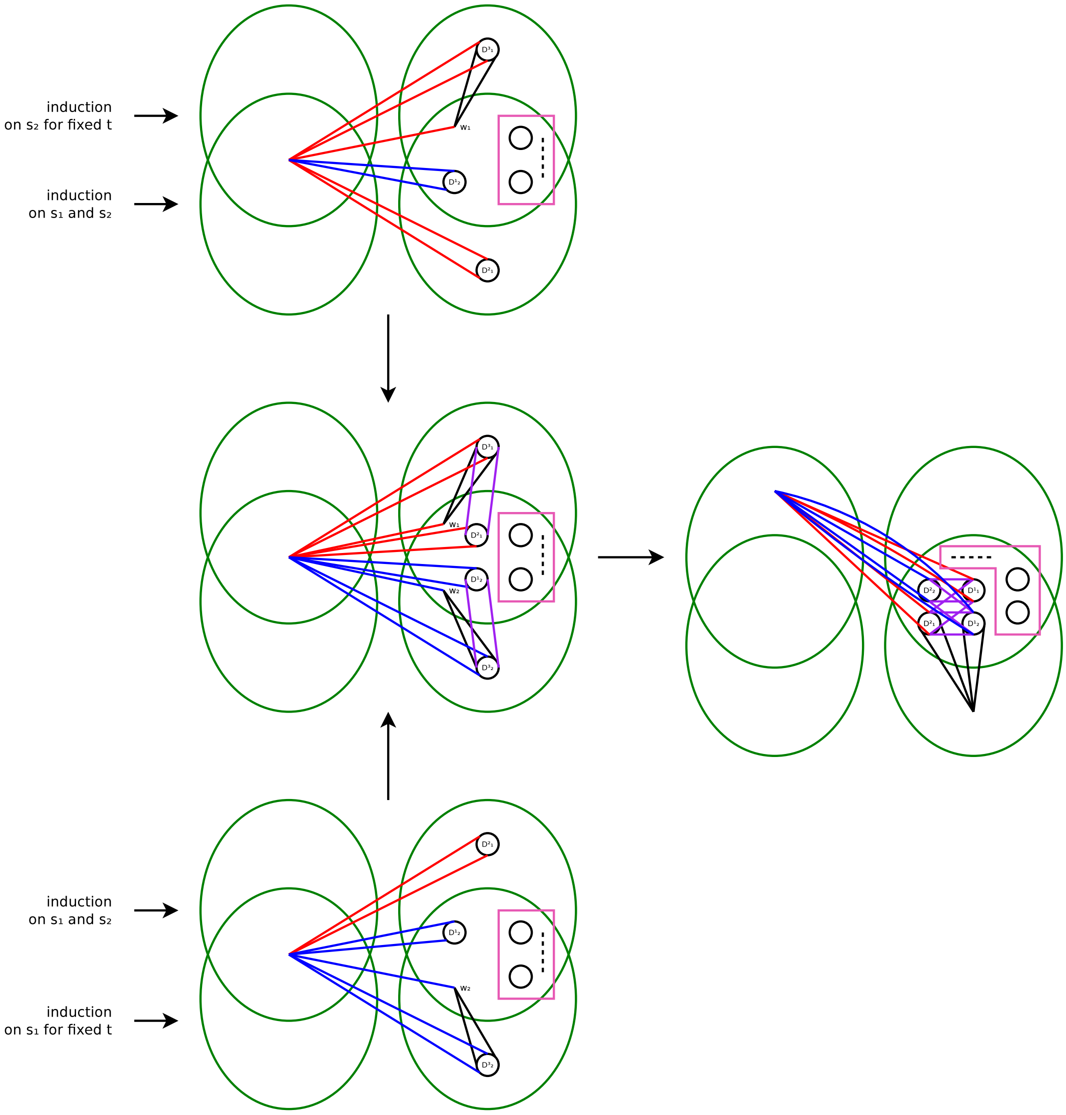


Figure 3.10: The inductive step for case 1 of Lemma 3.20. Note that $D_1^1 = D_1^3 \cup \{w_1\} \cong D_1^2 \cup \{z\} \cong D_1$ and that $D_2^2 \cong D_2^3 \cup \{w_2\} \cong D_2^1 \cup \{z\} \cong D_2$. The contents of the pink boxes are carried forward unaltered throughout the sequence of amalgamations.

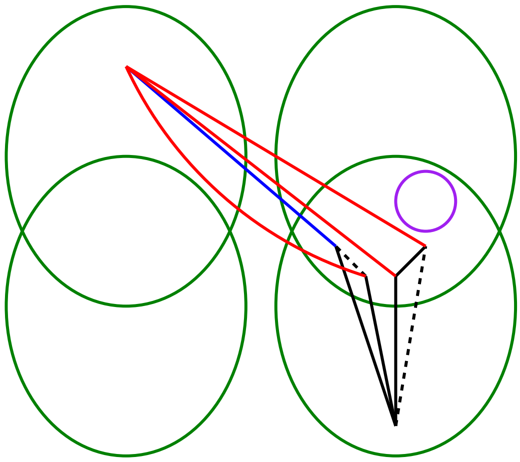

Figure 3.11: Reducing $i$ in case 1 of Lemma 3.21 (even if $D_2$ and $D_3$ are not monochromatic).

It might be that $D_2$ or $D_3$ is **not** monochromatic to $a$. If this is the case, we can reduce $j$ or $s$ respectively using arguments in earlier lemmas. The same applies if $D_2$ and $D_3$ are both monochromatic to $a$ but of different colours. We will therefore assume that $D_2$ and $D_3$ are both monochromatic blue to $a$.
There are therefore four cases to consider:

1. there is a $P_3$ $x_1x_2x_3$ in $D_1$ where $x_1x_2$ and $x_2x_3$ are edges and $ax_1, ax_2$ are red and $ax_3$ is blue;

2. there is a $P_3$ $x_1x_2x_3$ in $D_1$ where $x_1x_2$ and $x_2x_3$ are edges and $ax_1, ax_3$ are red and $ax_2$ is blue;

3. there are two instances of $P_3$ in $D_1$, namely $x_1x_2x_3$ and $y_1y_2y_3$, where $ax_1, ax_2, ax_3$ are red and $ay_1, ay_2, ay_3$ are blue; or

4. every instance $z_1z_2z_3$ of $P_3$ in $D_1$ is monochromatic red to $a$.

(The first and second cases do not depend on $D_2$ and $D_3$ being monochromatic **blue**, and so there is no loss of generality in restricting to these.)
In the first case, we reduce $i$ by using the amalgamations in Figure 3.11. Similarly, in the second case we reduce $i$ by using the slightly different amalgamations in Figure 3.12. Note that in both cases we increase $j$ by 1 and $s$ by 2.
Hence we can restrict to the third and fourth cases, where we may assume that every instance of $P_3$ in $D_1$ is individually monochromatic to $a$.

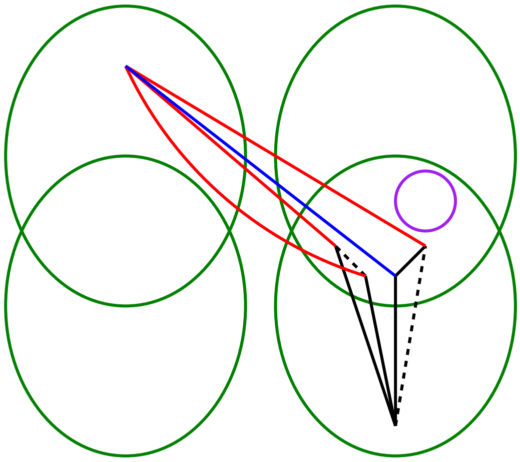

Figure 3.12: Reducing $i$ in case 2 of Lemma 3.21 (even if $D_2$ and $D_3$ are not monochromatic).

In the third case we will use the two-stage amalgamation procedure shown in Figure 3.13. Note that the inputs have extra isolated points, and of different colours, but this is not an issue since the arguments in Lemma 3.17 can deal with these.

The fourth case can be split into two sub-cases: $j \neq 0$ or $j = 0$. In both cases we aim to reduce $i$ (and possibly increase $j$ and/or $s$) **or** keep $i$ fixed *and* reduce one of $j$ and $s$. This will be sufficient to avoid circularity.

If $j \neq 0$ we will obtain the $(i, j, s)$ case from the $(i-1, j, s+3)$ and $(i, j-1, s+3)$ cases, as in Figure 3.14. Note that in this case the new $D_2'$ is of course **not** monochromatic, but this should not matter as we can use the arguments from our earlier results to obtain it.

If $j = 0$ we obtain the $(i, 0, s)$ case from a $(i-1, 0, s+3)$ and a $(i, 0, s-1)$ case, as in Figure 3.15, which again are obtainable inductively. This completes our inductive derivation of $H$. □

We can now formally conclude the proof of Theorem 3.16.

*Proof of Theorem 3.16.* Lemma 3.20 shows that $G$ realises every finite 2-coloured 2-graph of the form $(K_1, pK_s + \overline{K_q})$, and Lemma 3.21 shows that $G$ realises every finite 2-coloured 2-graph of the form $(K_1, pP_3 + \overline{K_q})$, for all finite $p$ and $q$ and all $s < r$. The proof of Theorem 3.15 allows us to move isolated vertices from the right component to the left component; it therefore follows that $G$ realises every finite 2-coloured 2-graph of the forms $(\overline{K_q}, pK_s)$ or $(\overline{K_q}, pP_3)$. By Theorem 2.13, $G$ realises

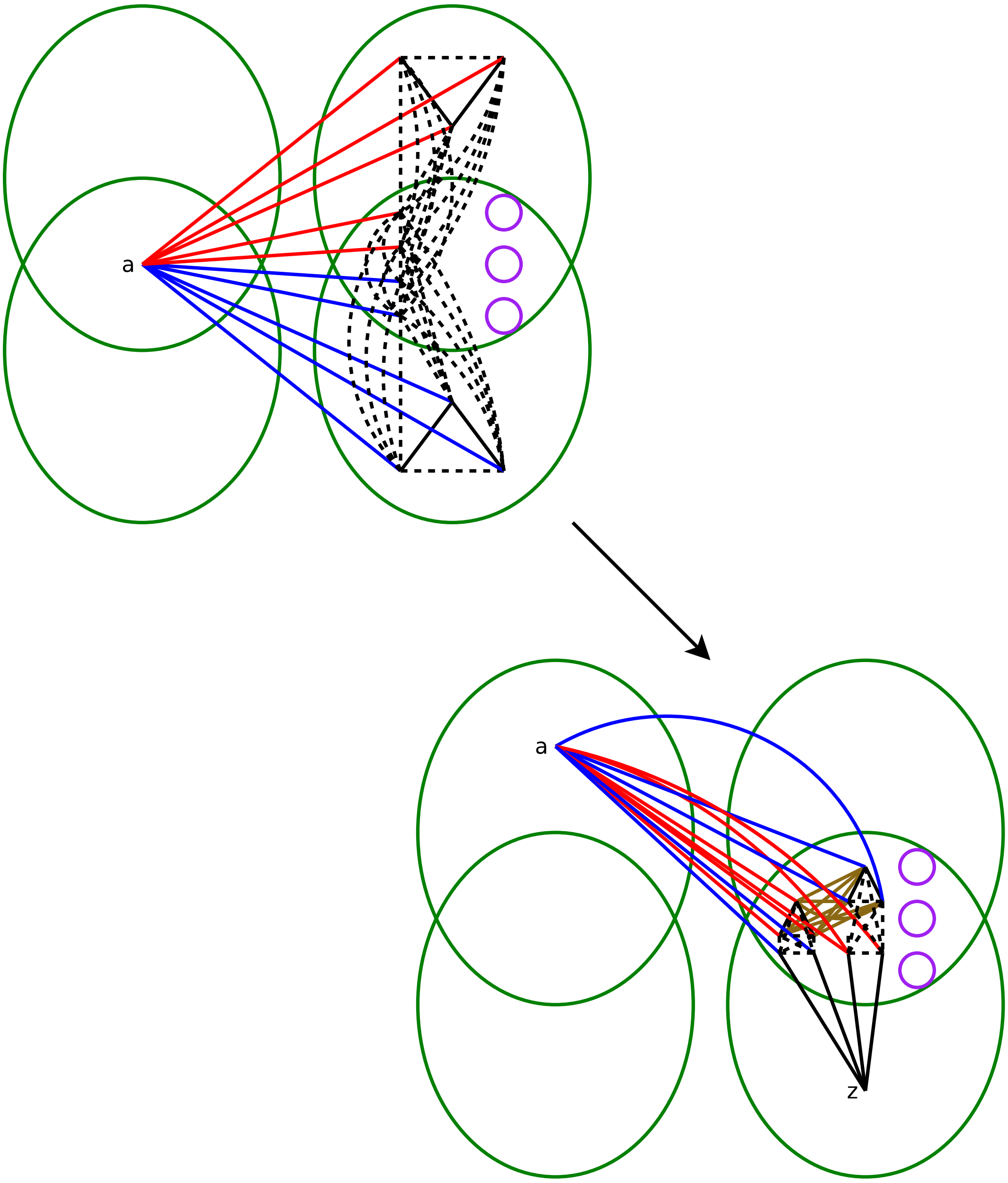


Figure 3.13: Reducing $i$ in case 3 of Lemma 3.21. This is a two-stage process. The inputs for the first case have one fewer copy of $P_3$ but have extra isolated points (which we can add using arguments in Lemma 3.17). Whatever 2-graph this amalgamation yields can be input into the second stage (and the other amalgamand here has empty left-hand side and so embeds in $G$ anyway) and we will get the required extra $P_3$ whether the "long diagonal" is red or blue.

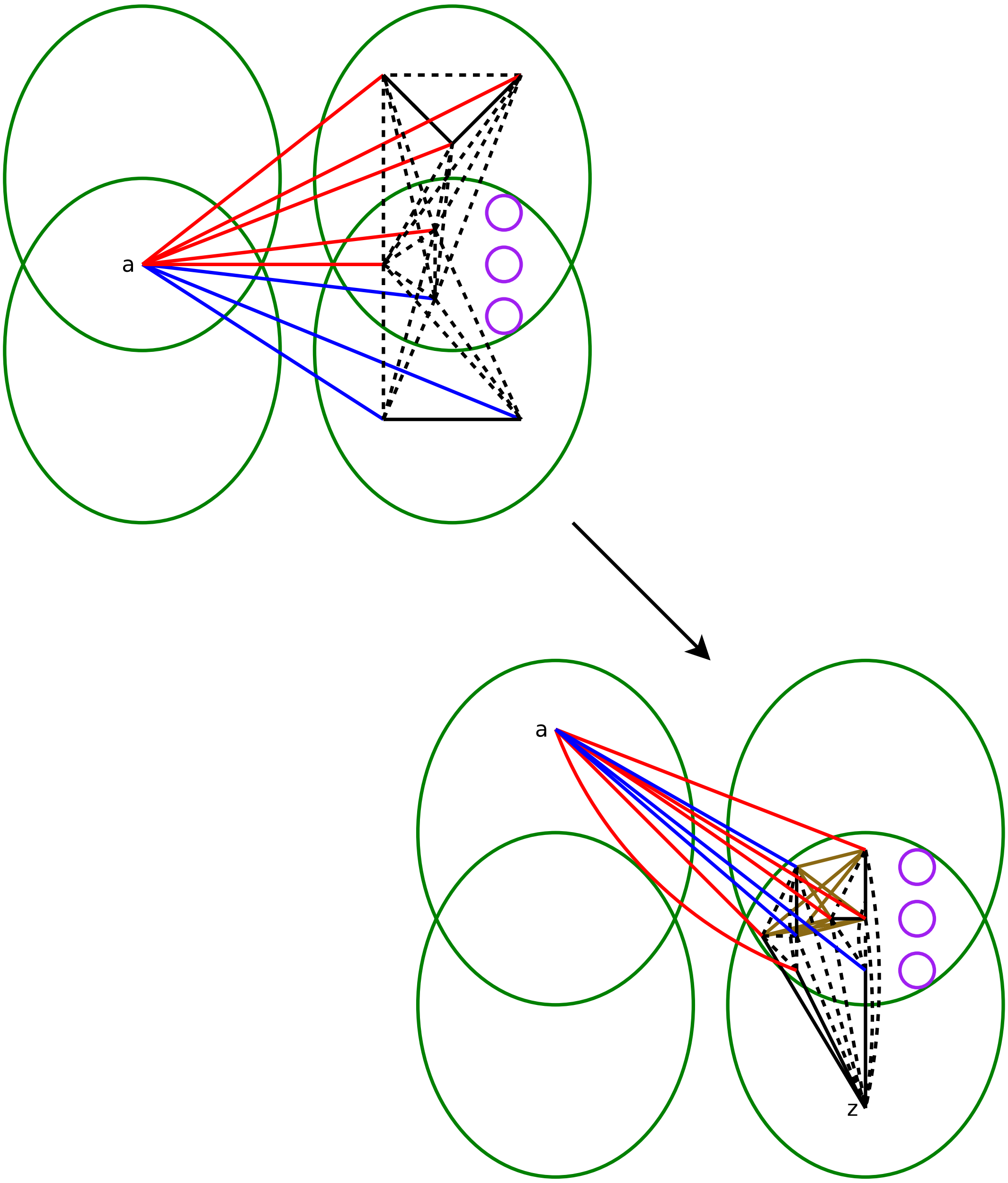


Figure 3.14: Reducing 4 in case 2 of Lemma 3.21 when $j \neq 0$.

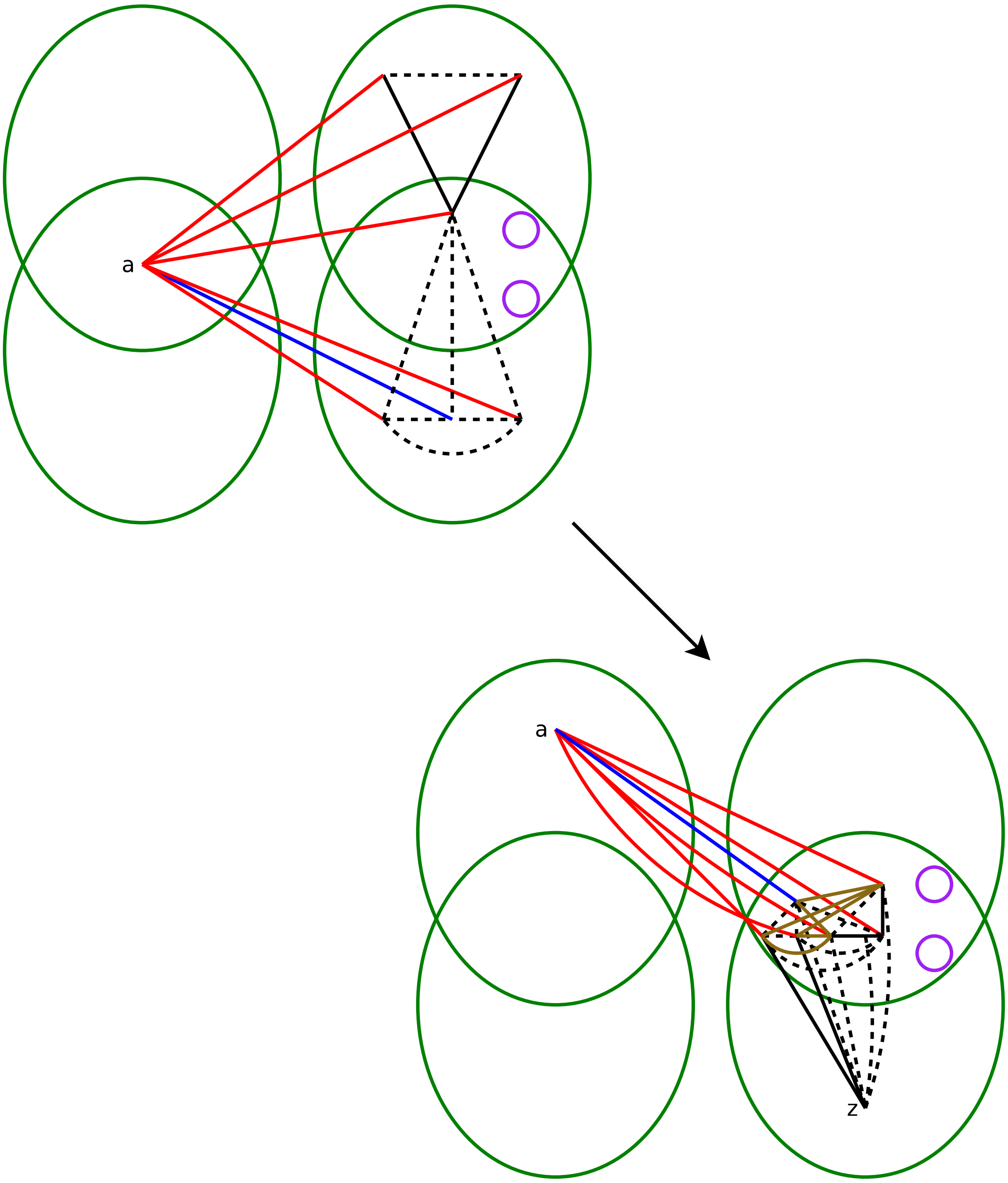


Figure 3.15: Reducing $i$ in case 4 of Lemma 3.21 when $j = 0$.

any finite 2-coloured 2-graph of the form $(\overline{K_q}, D)$, as required. ☐

### 3.3.3 $n = 2$; uniqueness

We showed in Lemma 3.10 that if $G$ is a homogeneous 2-coloured $(\overline{K_\infty}[K_2], \Gamma_r)$ 2-graph then $G$ must be "quasi-collapsing" (i.e. if $G = (A, B, R)$ then for every edge $ab$ in $A$ and all $x, y \in B$ we have that $(ab, x) \cong (ab, y)$).
Moreover:

**Lemma 3.22.** *Let $G = (A, B, R)$ be a non-collapsing homogeneous 2-coloured $(\overline{K_\infty}[K_2], \Gamma_r)$ 2-graph. Then, for every finite graph $D < R$, $G$ must realise $(K_1, D)^1$ and $(K_1, D)^2$.*

*Proof.* Suppose $G$ omits $(K_1, D)^1$ for some $D$. By Theorem 2.17, $G$ must realise $(K_1, D)^2$, say as $(a, X)$ for some $a \in A$ and some $X \cong D$ in $B$. But $a$ is joined to a (unique) vertex $b$ in $A$, and $G$ is quasi-collapsing so for every $x \in X$ $ax \ncong bx$. So $b$ is colour 1 to every vertex in $X$ and $G$ realises $(K_1, D)^1$ after all. Contradiction. ☐

We claim that $G$ realises $(K_1, D, R)$ for every finite $D < \Gamma_r$ and every valid choice $R$ of cross-edges. For this we aim to extend those results of section 3.3.2 that don't directly apply.
The results (Lemmas 3.17 to 3.21) that show that $G$ realises every

$$\left(K_1, \sum_{i=1}^{\lambda} K_{s_i} + \overline{K_t}\right)$$

and every

$$\left(K_1, \kappa P_3 + \sigma K_2 + \overline{K_\tau}\right)$$

work just as well here as they did in section 3.3.2. We do however need a little more care to be able to move isolated points from the right component to the left component (as we need to in order to apply the copying argument).

**Lemma 3.23.** *Let $G = (A, B, R)$ be a non-collapsing homogeneous 2-coloured $(\overline{K_\infty}[K_2], \Gamma_r)$ 2-graph realising every finite 2-graph of the form $(K_1, D + \overline{K_t})$ for some finite $D < \Gamma_r$ and for all $t \in \mathbb{N}$. Then $G$ realises every finite 2-graph of the form $(\overline{K_s}, D)$ for all $s \in \mathbb{N}$.*

*Proof.* We work by induction on $s$, the case $s = 1$ being trivial.
Let $H = (\overline{K_s}, D, R)$, and let $H' = (\overline{K_s}[K_2], D, R')$ be the (unique) extension of $H$ that extends each left-side vertex to an edge. Let $H_1 = (ab, D, R)$ be a sub-2-graph of $H$ where $ab$ is an edge, and let $H_2 = H \setminus \{a, b\}$. Moreover, let $D' = D + \{x, y\}$ and $H_1' = H_1 \cup \{x, y\}$ and $H_2 = H_2 \cup \{x, y\}$, where $ax, ay$ are red and $bx, by$ are blue, and where for all edges $cd$ in the left component of $H_2$ $cx, dy$ are red and $cy, dx$ are blue. By the induction hypothesis, both $H_1'$ and $H_2'$ embed in $G$.
Now amalgamate $H_1'$ and $H_2'$ over $(\varnothing, D')$. We clearly cannot add edges from $a$ or $b$ to any vertex in the left component of $H_2'$, and we cannot identify either since $a$ and $b$ are both differently coloured to $x$ and $y$ than any vertex in the left component of $H_2'$. So we have to add non-edges, and the amalgam must therefore contain a copy of $H$. □

Note that given a set of 2-graphs of the form $(\overline{K_s}, D)$ realised in $G$ we can compute precisely the set of 2-graphs of the form $(\overline{K_s}[K_2], D)$ that $G$ embeds. Hence:

**Corollary 3.24.** *Let $G$ be a homogeneous 2-coloured $(\overline{K_\infty}[K_2], \Gamma_r)$ 2-graph. Then $G$ is the generic 2-coloured $(\overline{K_\infty}[K_2], \Gamma_r)$ 2-graph omitting $(K_2, K_1)^1$ and $(K_2, K_1)^2$; specifically, $G$ minimally omits precisely the following 2-graphs: $(P_3, \varnothing)$, $(K_3, \varnothing)$, $(K_2, K_1)^1$ and $(K_2, K_1)^2$, and $(\varnothing, K_r)$ if $r < \infty$.*

## 3.4 $n = \infty$; uniqueness

In this section we aim to prove the following:

**Theorem 3.25.** *Let $G \cong (\overline{K_m}[K_\infty], \Gamma_r)$ be a homogeneous 2-graph, where $2 \leq m \leq \infty$. Then, up to equivalence, either:*

- *for some $2 \leq s < r$, $G$ is the 2-coloured homogeneous $(\overline{K_m}[K_\infty], \Gamma_r)$ 2-graph minimally omitting precisely $(1, K_s)^1$, which we write as $\mathbf{\Gamma_s^r}$, or where the $r$ is understood, $\mathbf{\Gamma_s}$; or*
- *$G$ is the 2-coloured homogeneous fully-generic $(\overline{K_m}[K_\infty], \Gamma_r)$ 2-graph $\mathbf{\Gamma_r^r}$ (or $\mathbf{\Gamma_r}$ if it is understood that we are working in $(\overline{K_m}[K_\infty], \Gamma_r)$).*

Write

$$G = (A_1 + A_2 + \ldots + A_m, B)$$

where $A_i \cong K_\infty$ and $B \cong \Gamma_r$. (This sum does not imply that $m$ is finite.)

**Lemma 3.26.** *One of the following holds: either*

1. *for all $i$ and $j$, $(A_i, B) \cong (A_j, B)$; or*
2. *for all $i$, $(A_i, B)$ is monochromatic.*

*Proof.* Suppose that some $(A_i, B)$ is **not** monochromatic; specifically, suppose that there are $a, b \in A_i$ and $x, y \in B$ such that $ax$ and $by$ are different colours. (There is no assumption that $a \neq b$ or that $x \neq y$.
Because there are only two cross-edge colours, for each $j \neq i$ there must be $c \in A_j$ and $z \in B$ such that $cz$ is the same colour as either $ax$ or $by$. Suppose without loss of generality that $ax \cong cz$. Then consider the finite partial automorphism

$$\alpha : a \mapsto c, x \mapsto z$$

which, by homogeneity, extends to $\beta \in \mathrm{Aut}(G)$. But $\beta$ must map $A_i$ to $A_j$, giving the required isomorphism $(A_i, B) \to (A_j, B)$. □

Of course, if every $(A_i, B)$ is monochromatic then $G$ is collapsing. We will show that if $G$ is **not** collapsing then every $(A_i, B)$ is homogeneous. We will need a lemma that allows us to split partial isomorphisms of $\Gamma_r$ into maps that each move only one point.

**Lemma 3.27.** *Let $H \cong \Gamma_r$ and let $C, D < H$ be finite subgraphs of $H$ such that there is an isomorphism*

$$\alpha : C \to D$$

*Then there exist $\kappa$, subgraphs $C_0, \ldots, C_\kappa$ of $H$ and isomorphisms $\beta_0, \ldots, \beta_{\kappa-1}$ such that:*

1. *$C_0 = C$;*
2. *$C_\kappa = D$;*
3. *for each $i$, $\beta_i : C_i \to C_{i+1}$; and*
4. *for each $i$, there exists $x_i \in C_i$ such that, for all $y \in C_i \setminus \{x_i\}$, $\beta_i(y) = y$.*

*Proof.* Label the points of $C$ by $c_1, \ldots, c_\lambda$, and the points of $D$ by $d_1, \ldots, d_\lambda$, so that $\alpha(c_i) = d_i$. Let $E_1, E_2, F < H$ be such that:

1. $E_1, E_2, F \cong C$;

2. $C \cap E_1 = C \cap E_2 = C \cap F = D \cap E_1 = D \cap E_2 = D \cap F = E_1 \cap E_2 = E_1 \cap F = E_2 \cap F = \varnothing$;

3. $E(C, F) = E(C, E_2) = E(E_1, E_2) = E(D, E_1) = E(D, F) = \varnothing$;

4. if the points of $E_i$ are labelled $e^i_j$, and the points of $F$ are labelled $f_j$ (so that $c_j \mapsto e^1_j, c_j \mapsto e^2_j, c_j \mapsto f_j$ are all isomorphisms), then $c_i e^1_j$, $f_i e^2_j$, $f_i e^1_j$ and $d_i e^2_j$ are edges if and only if $c_i c_j$ is an edge (which is true if and only if $d_i d_j$ is an edge).

We need to check that, if $r$ is finite, then there are no "rogue" copies of $K_r$. Now the only way we can get one is by using at least two out of $(C, D, E_1, E_2, F)$, and the only pairs between which there might be edges are

$$(C, D), (C, E_1), (D, E_2), (E_1, F), (E_2, F)$$

If there is a vertex in at least three of the sets, there will be non-edges, since any three out of $(C, D, E_1, E_2, F)$ will contain one of the pairs

$$(C, E_2), (C, F), (D, E_1), (D, F), (E_1, E_2)$$

Moreover, since $C$ and $D$ were given there will not be a $K_r$ in $(C, D)$. If there is a $K_r$ within one of the other pairs which could have one, without loss of generality it will be in $(C, E_1)$. But if there is a $K_r$ in

$$C \cup E_1 = \{c_1, \ldots, c_\lambda, e^1_1, \ldots, e^1_\lambda\}$$

then there will be one in

$$C = \{c_1, \ldots, c_\lambda\}$$

since we can replace each $e^1_i$ by $c_i$. But by assumption there is no $K_r$ in $C$.
The idea is to map $C$ into $E_1$, $E_1$ into $F$, $F$ into $E_2$ and $E_2$ into $D$, and to do these maps one point at a time. In this case $\kappa = 4\lambda$.
I claim that the maps

$$\alpha_i : \{e^1_1, \ldots, e^1_i, c_{i+1}, \ldots, c_\lambda\} \to \{e^1_1, \ldots, e^1_{i+1}, c_{i+2}, \ldots, c_\lambda\}$$

for $0 \leq i \leq \lambda - 1$ given by $\alpha_i(c_{i+1}) = e^1_{i+1}$, and which are the identity map elsewhere, are all isomorphisms. There are similar sequences of isomorphisms from $E_1$ into $F$, from $F$ into $E_2$ and from $E_2$ into $D$ which are essentially the same but in the respective sets.

To see that $\alpha_i$ is indeed an isomorphism, we need to know that it preserves edges and non-edges. We need only worry about edges or non-edges starting from $c_{i+1}$. That is, $c_{i+1}c_j$ should be an edge if and only if $e^1_{i+1}c_j$ is, and $c_{i+1}e^1_j$ should be an edge if and only if $e^1_{i+1}e^1_j$ is. By construction of the edges and non-edges between $C$ and $E_1$ these can be seen to be true, so $\alpha_i$ is indeed an isomorphism for each $i$. Thus there is a sequence of $\lambda$ isomorphisms from $C$ to $E_1$ that each move only one point such that their composition maps $C$ to $E_1$. By symmetry, the required sequence of length $4\lambda$ of isomorphisms from $C$ to $D$ that each move exactly one point exists. □

We now use Lemma 3.27 by assuming that any finite partial automorphisms of $(A_1, B)$ that do not extend to automorphisms of $(A_1, B)$ (and thus would falsify the homogeneity of $(A_1, B)$) are of the form

$$\alpha : x \mapsto y, z_1 \mapsto z_1, \ldots, z_\lambda \mapsto z_\lambda$$

**Theorem 3.28.** *If $(A_1, B)$ is not homogeneous then $G$ is collapsing.*

*Proof in the $\lambda = 0$ case.* Suppose that the inhomogeneity of $G$ is witnessed by the finite partial automorphism

$$\alpha : x \mapsto y$$

If $G$ is not collapsing then

$$(\exists i)(\exists z \in B)(\exists a, b \in A_i)(az \in R_1, bz \in R_2)$$

and since all restrictions are isomorphic,

$$(\forall i)(\exists a, b \in A_i)(\exists z \in B)(az \in R_1, bz \in R_2)$$

Let $x, y \in B$, and suppose that there are no $a, b \in A_1$ so that

$$\alpha : a \mapsto b, x \mapsto y$$

extends to an automorphism of $G$.
If it were the case that

$$(\exists i)(\exists a \in A_1)(ax, ay \in R_i)$$

then

$$\alpha_1 : a \mapsto a, x \mapsto y$$

would do, and if it were the case that

$$(\exists i)(\exists a \neq b \in A_1)(ax, by \in R_i)$$

then likewise

$$\alpha_2 : a \mapsto b, x \mapsto y$$

suffices. Therefore, we may assume that neither holds and, without loss of generality, for all $a \in A_1$, $ax$ is red and $ay$ is blue.
But $G$ is not collapsing, so $(A_1, B)$ is not collapsing, which means that we can choose $z \in B$ and $c, d \in A_1$ such that $cz \in R_1$ and $dz \in R_2$.
Consider

$$\beta : z \mapsto x, c \mapsto c$$

which maps $cz$ to $cx$, and both are red so this is a finite partial isomorphism, and extends to an automorphism $\gamma$ of $G$. But

$$dz \cong \gamma(d)\gamma(z) = \gamma(d)x = ex$$

for some $e \in A_1$, but $dz$ is blue and $ex$ is red for all $e \in A_1$. Contradiction.

□

*Proof in the $\lambda \geq 1$ case.* Let $x, y, z_1, \ldots, z_\lambda \in B$ be such that

$$\alpha : x \mapsto y, z_i \mapsto z_i$$

is a finite partial isomorphism $\{x, z\} \to \{y, z\}$ where $\lambda \geq 1$ (and thus a potential counterexample to the homogeneity of $(A_1, B)$; recall that by Lemma 3.27 we may assume that any counterexample is of this form). We will show that there is $a \in A_1$ such that

$$\beta : a \mapsto a, x \mapsto y, z_i \mapsto z_i$$

is an isomorphism $(a, xz) \to (a, yz)$ (and so $\alpha$ is not a counterexample after all).
If there is no such $a$, let

$$A_{1,1} = \{a \in A : ax \in R_1, ay \in R_2\}$$

and

$$A_{1,2} = \{a \in A : ax \in R_2, ay \in R_1\}$$

partition $A_1$. By the proof above in the $\lambda = 0$ case,

$$\alpha' : x \mapsto y$$

extends to an automorphism $\beta'$ of $G$ that fixes $A_1$ set-wise, so both $A_{1,1}$ and $A_{1,2}$ must be non-empty.

Let

$$B_1 = \{z \in B : (\forall a \in A_{1,1})(\forall b \in A_{1,2})(az \in R_1, bz \in R_2)\}$$

and

$$B_2 = \{z \in B : (\forall a \in A_{1,1})(\forall b \in A_{1,2})(az \in R_2, bz \in R_1)\}$$

Let $B_3 = B \setminus (B_1 \cup B_2)$. If $B_3 = \varnothing$, then since one of $A_{1,1}$ and $A_{1,2}$ is infinite, and we may assume that $A_{1,1}$ is infinite, if we let $a, b \in A_{1,1}$ and $c \in A_{1,2}$ the finite partial automorphism

$$\pi : a \mapsto a, b \mapsto c$$

should extend to an automorphism of $G$ that fixes $B_1$ and interchanges $B_1$ with $B_2$, a contradiction. Hence $B_3$ is non-empty.

Suppose $u, v, w \in B_1$ and $z \in B_3$ were such that $uv \cong wz$. (We assume that $w \neq u$ but not necessarily that $w \neq v$.) It must be that there are $b \in A_{1,1}$ and $c \in A_{1,2}$ such that $bz \cong cz$; without loss of generality we may assume this colour is red. Consider the finite partial automorphism

$$\gamma : u \mapsto w, v \mapsto z, b \mapsto b$$

of $G$ which extends to an automorphism $\delta$. Now $\delta$ must both fix $A_{1,1}$ set-wise (since $A_{1,1}$ is precisely the set of points to which $u$ and $w$ are red) and not fix $A_{1,1}$ set-wise (since it should map $A_{1,1}$, the set of points to which $v$ is red, to the set of points to which $z$ is red, which by definition is not $A_{1,1}$). Contradiction. So if $B_1$ has an edge then between every $u \in B_1$ and $v \in B_3$ is a non-edge, and between every $s, t \in B_1$ is an edge; and similarly if $B_1$ contains a non-edge. So $B_1$ is complete or empty. By a similar argument, so is $B_2$. Moreover, if $B_1$ and $B_2$ are both complete then no vertex of $B_3$ is in the same connected component of $B$ as $B_1$ or $B_2$, which is a contradiction since $B$ is connected. Similarly if $B_1$ and $B_2$ are both empty then $\overline{B}$ would fail to be connected, which is also false. So without loss of generality $B_1$ is complete and $B_2$ is empty.

Let $a \in A_{1,1}$ and $b \in A_{1,2}$ and consider

$$\eta : a \mapsto b, x \mapsto y$$

Now $\eta$ is a finite partial automorphism, so must extend to an automorphism $\zeta$. But $\zeta$ maps $A_{1,1}$ to $A_{1,2}$ and $B_1$ to $B_2$ (by the same argument as before), so $B_1 \cong B_2$ and hence both are singletons. But then this means that $y$ has degree at most 1 in $B$, and $x$ has co-degree at most 1 in $B$. Neither of these are possible in $B$, which is isomorphic to $\Gamma_r$ for some $r$. Contradiction.

The only remaining possibility is therefore that there is $a \in A_1$ such that $ax \cong ay$, as claimed, thus completing the proof of Theorem 3.25. $\square$

At this point we have proved that each $(A_i, B)$ has to be homogeneous; clearly they are all isomorphic. But, at least in principle, there are many ways by which these can be jumbled. We seek to prove that only the "most generic" version for each variant of $(A_i, B)$ is possible. (Recall that, without loss of generality, $(A_i, B)$ is either the fully generic $(K_\infty, \Gamma_r)$ or is the generic $(K_\infty, \Gamma_r)$ minimally omitting some $(K_1, K_s)^1$ for some $s < r$.)
To conclude the proof of uniqueness, we will involve a sequence of several nested inductions. Let $G$ be a homogeneous 2-coloured $(\overline{K_m}[K_\infty], \Gamma_r)$ 2-graph omitting $(K_1, K_s)^1$ and realising $(K_1, K_t)^1$ whenever $t < s$. (If $r = s = \infty$ this simply means that $G$ embeds every $(K_1, K_t)^1$ for $t \in \mathbb{N}$.) We prove that $G$ embeds every finite 2-graph of the form

$$H = (K_{i_1} + \ldots + K_{i_p}, D)$$

for all finite $D \in \Gamma_r$ and all finite $p \leq m$, subject to $H$ not realising $(K_1, K_s)^1$.

**Lemma 3.29.** *$G$ embeds every*

$$H = (\{b_1, \ldots, b_p\}, D, R) \cong (\overline{K_p}, D)$$

*such that no $(b_i, D)$ contains $(K_1, K_s)^1$.*

*Proof.* Clearly some

$$H_0 = (\{c_1, \ldots, c_p\}, D, R_0)$$

for **some** colours $R_0$ must embed in $G$. We aim to replace each $c_i$ with the corresponding $b_i$ in $H$. Specifically, we prove that if

$$H_i = (\{b_1, \ldots, b_i, c_{i+1}, \ldots, c_p\}, D, R_i)$$

embeds in $G$, then so does

$$H_{i+1} = (\{b_1, \ldots, b_{i+1}, c_{i+2}, \ldots, c_m\}, D, R_{i+1})$$

where each $b_j$ is coloured to $D$ as in $H$, and each $c_j$ as in $H_0$.
Amalgamate $H_i$ with

$$H_i' = (\{b_{i+1}, c_{i+1}\}, D, R_i')$$

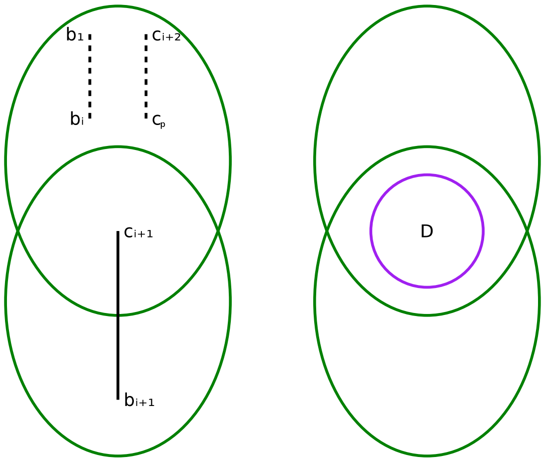


Figure 3.16: The method of interchanging a point with unknown colours to the right with one where these colours are correct.

over $(\{c_{i+1}\}, D)$, where $b_{i+1}c_{i+1}$ is an edge, where $b_{i+1}$ is coloured to $D$ as in $H$, and $c_{i+1}$ is coloured to $D$ as in $H_0$. Note that $H_i'$ does indeed embed in $G$, since the $(K_\infty, \Gamma_r)$ subgraphs of $G$ are all homogeneous, and so are determined by sections 4.1.1 and 4.1.2 and all of these must realise $H_i'$ if they realise the $(K_1, D)$ substructures of it. We therefore get a product $H_i''$, but cannot identify $b_{i+1}$ with any $b_j$ or $c_j$, nor can we add any edges since if we did we would get a $P_3$ on the left which we cannot have. Hence $H_i''$ has no new edges, and thus contains a copy of $H_{i+1}$. The induction proceeds. □

**Theorem 3.30.** *$G$ embeds every*

$$H = (A_1 + \ldots + A_p, D, R)$$

*not realising* $(K_1, K_s)^1$ *where each* $A_i \cong K_{j_i}$.

*Proof.* Assume without loss of generality that $|A_i| \geq |A_{i'}|$ whenever $i < i'$. Let $i$ be maximal such that $j_i > 1$; by Lemma 3.29 we are done if $j_i = 1$ for all $i$. We work by induction on $i$.
Label the vertices of $A_i$ by $a_1, \ldots, a_{j_i}$, and label the vertices of $A_j$ for $j > i$ by $b_j$. Let

$$H_1 = H \setminus (A_i \setminus \{a_1\})$$

and

$$H_2 = H \cap (A_i, D)$$

and amalgamate $H_1$ with $H_2$ over their intersection $(\{a_1\}, D)$. Note that $H_1$ embeds in $G$ and $H_2$ embeds by the classification of the homogeneous 2-coloured $(K_\infty, \Gamma_r)$ 2-graphs earlier in the chapter. Moreover, in the amalgam we cannot identify any vertices or join any $a_i$ with any other vertex. Hence the only possible product involves adding non-edges, and the induction proceeds. □

# Chapter 4

# The $(\Gamma_r, \Gamma_s)$ case

In this chapter we classify homogeneous 2-graphs $G = (A, B, R)$ of the form $(\Gamma_r, \Gamma_s)$. For similar technical reasons to those in Chapter 3, and which we reiterate later, we will assume that $G$ has only two cross-edge colours. We have only been able to classify those cases where $r = s = 3$ or $r = s = \infty$, but we have proved some lemmas that are of more general application.

Many of the results, particularly early in this chapter, are essentially the same as the corresponding results in Chapter 3, with the same proofs. In these cases we repeat the statement but refer the reader to the proof given in Chapter 3.

## 4.1 Existence

We first give a list of the cases we suspect are the only ones that exist (up to equivalence) and verify that they do indeed all exist.

Recall that an **antichain** of (isomorphism types of) 2-graphs is a set $\mathcal{A}$ of 2-graphs so that if $A, B \in \mathcal{A}$ then $A \not< B$ and $B \not< A$ (i.e. $A$ does not embed in $B$ and $B$ does not embed in $A$). Also recall that, if $A$ and $B$ are graphs and $i = 1, 2$, then $(A, B)^i$ is the (2-coloured) 2-graph whose components are $A$ and $B$ (in this order) and where all cross-edges have colour $i$.

**Proposition 4.1.** *Let $r, s \in \mathbb{N} \cup \{\infty\} \setminus \{0, 1, 2\}$. For every antichain $\mathcal{A}$ of monochromatic 2-graphs of the form $(K_m, K_n)$[1] where $m < r$ and $n < s$, there exists a homogeneous 2-coloured $(\Gamma_r, \Gamma_s)$ 2-graph $G_{\mathcal{A}}$ such that:*

- *$G_{\mathcal{A}}$ omits all elements of $\mathcal{A}$;*

- *$G_{\mathcal{A}}$ embeds every finite 2-graph of the form $(C, D)^1$ (where $C < \Gamma_r$ and $D < \Gamma_s$) that does not embed any element of $\mathcal{A}$; and*
- *$G_{\mathcal{A}}$ embeds every finite 2-graph of the form $\left(\hat{C}, \hat{D}\right)^2$ (where $\hat{C} < \Gamma_r$ and $\hat{D} < \Gamma_s$).*

*Proof.* Let $\mathcal{A}$ be such an antichain, and let $\mathcal{C}$ be the set of finite 2-graphs (whose domain is a subset of $\mathbb{N} \times \{1, 2\}$) such that, if $H = (C, D, R) \in \mathcal{C}$, then $C < \Gamma_r$, $D < \Gamma_s$ and $H$ does not embed any element of $\mathcal{A}$.
We need to check that $\mathcal{C}$ defines an amalgamation class. We claim that it will always be sufficient to do the following:

- add blue cross-edges when a cross-edge relation between vertices in different components is undetermined, and
- add non-edges on both sides whenever an edge relation within a component is undetermined.

Clearly $\mathcal{C}$ has the hereditary property and appropriate closure under isomorphism, so it is sufficient to check that it has the amalgamation property, and for this, as usual, it is sufficient to verify the two-point amalgamation property. Specifically, it is sufficient to verify that if the result of the amalgamation contains $(K_m, K_n)^1$ for some $m, n \in \mathbb{N}$ (and is therefore not in $\mathcal{C}$) then one of the amalgamands contained $(K_m, K_n)^1$ (and so was not in $\mathcal{C}$ originally).
Let the amalgamands be $H_1 = (A_1, B_1, R_1)$ and $H_2 = (A_2, B_2, R_2)$. There are three cases:

1. $B_1 = B_2$, $A_1 \setminus A_2 = \{a_1\}$ and $A_2 \setminus A_1 = \{a_2\}$;
2. $A_1 = A_2$, $B_1 \setminus B_2 = \{b_1\}$ and $B_2 \setminus B_1 = \{b_2\}$; and
3. $A_2 \setminus A_1 = B_1 \setminus B_2 = \varnothing$, $A_1 \setminus A_2 = \{a\}$, $B_2 \setminus B_1 = \{z\}$.

In the first two cases the amalgamation product does not embed any $(K_m, K_n)^1$ that did not embed in either of the amalgamands, as if it did then we would have to increase the size of a complete subgraph on one of the two sides, and this does not happen. Similarly, if we got a new $(K_m, K_n)^1$ in the third case then we would have had to add red across, which we don't (and adding blue clearly does give something in the class). Hence $\mathcal{C}$ is indeed an amalgamation class and thus defines a homogeneous 2-graph $G_{\mathcal{A}}$. □

## 4.2 General results

Our task throughout this chapter will be to prove that the instances in Proposition 4.1 are the only ones. We will accomplish this in the $r = s = 3$ and $r = s = \infty$ cases; that is we will prove the following:

**Theorem 4.2.** *Let $G$ be a homogeneous 2-coloured $(\Gamma_r, \Gamma_r)$ 2-graph where $r = 3$ or $r = \infty$. Then there exists some antichain $\mathcal{A}$ of finite 2-graphs of the form $(K_m, K_n)^1$ (for various values of $m$ and $n$) such that $G$ is equivalent to some $G_{\mathcal{A}}$ (as defined in Proposition 4.1).*

In this section we will prove some introductory results. Most of these results apply for all values of $r$ and $s$, however there is an important exception (namely Theorem 4.11) where for technical reasons we have had to restrict ourselves to the $r = s = 3$ and $r = s = \infty$ cases.
Let $G = (A, B, R)$ be a homogeneous 2-coloured 2-graph where $A \cong \Gamma_r$ and $B \cong \Gamma_s$ for some finite $r, s \geq 3$, and $R = (R_1, R_2)$. For each $a \in A$, let

$$B_a = \{b \in B : (a, b) \in R_1\}$$

and

$$B'_a = \{b \in B : (a, b) \in R_2\}$$

Similarly, for each $b \in B$ let

$$A_b = \{a \in A : (a, b) \in R_1\}$$

and

$$A'_b = \{a \in A : (a, b) \in R_2\}$$

As in Chapter 3, and by using the same proofs as we used there, we can show that $B_a$ is always homogeneous (given that $G$ is) and that, up to isomorphism, $B_a$ is independent of the choice of $a$.

**Lemma 4.3.** *For all $a, b \in A$, $B_a \cong B_b$.*

*Proof.* See Lemma 3.3. □

**Lemma 4.4.** *For all $a \in A$, $B_a$ is homogeneous.*

*Proof.* See Lemma 3.4. □

The proofs in Chapter 3 do not immediately extend to show that $B_a$ must always be infinite in the $(\Gamma_r, \Gamma_s)$ case. However we can adapt the proof used there and we do so now.

**Lemma 4.5.** *For all $a \in A$, $B_a$ is infinite.*

*Proof.* Suppose that $B_a$ is finite, say that $|B_a| = k$ for some finite $k$. We will show that every such value leads to a contradiction.
Suppose $k = 1$, so for each $a \in A$ there is a unique $b \in B$ such that $(a, b) \in R_1$. If, for each $b \in B$, there was a unique $a \in A$ such that $(a, b) \in R_1$, then we would have a perfect matching, which cannot happen as it contradicts Proposition 2.6. So take $b \in B$ and consider $A_b$. Consider two cases:

1. $A_b$ has no edges, but has at least two vertices; or

2. $A_b$ has an edge $a_1a_2$.

In the first case, since $\overline{\Gamma_r}$ is connected there is a non-edge $a_1a_3$ such that $a_1 \in A_b$ and $a_3 \in A'_b$. Let $a_2 \in A_b$ (so in particular $a_1a_2$ is also a non-edge) and consider

$$\alpha : a_1 \mapsto a_1, a_2 \mapsto a_3$$

Now $\alpha$ is an isomorphism $\{a_1, a_2\} \to \{a_1, a_3\}$, so it extends to an automorphism $\alpha'$ of $G$. But this automorphism would have to simultaneously fix $b$ and map it to the single vertex $b'$ in $B_{a_3}$, and this cannot be.
In the second case, take an edge $a_1a_2$ in $A_b$. Since $A$ is connected, there is a path from $a_1$ to some vertex $a_3 \in A'_b$ such that $a_3$ is the only vertex in the path not in $A_b$. There must therefore be a path $a_3a_4a_5$ where $a_4, a_5 \in A_b$. Consider

$$\beta : a_4 \mapsto a_4, a_5 \mapsto a_3$$

Now $\beta$ is an isomorphism $\{a_4, a_5\} \to \{a_4, a_3\}$, so it extends to an automorphism $\beta'$ of $G$. But this automorphism would have to simultaneously fix $b$ and map it to the single vertex $b'$ in $B_{a_3}$, and this cannot happen.
Hence $k \neq 1$, so $k \geq 2$. We aim to prove that no finite value of $k \geq 2$ is permissible. Let $a \in A$. It is clear that if $U < B$ and $U \cong B_a$ then there exists $a' \in A$ such that $U = B_{a'}$.

If we can find $a, b, c \in A$ such that $ab \cong ac$ but

$$[B_a, B_b] \ncong [B_a, B_c]$$

(which means that there is no isomorphism

$$\phi : B_a \cup B_b \to B_a \cup B_c$$

such that $\phi(B_a) = B_a$ and $\phi(B_b) = B_c$) then the finite partial automorphism

$$\theta : a \mapsto a, b \mapsto c$$

would extend to an automorphism $\phi$ of $G$ that fixes $B_a$ and maps $B_b$ to $B_c$. But by choice of $a, b, c$ this is impossible.
So there can be only as many types of $(B_a, B_x)$ as there are types of $(a, x)$, namely 2. But if $k \geq 3$ then there clearly must be at least three types of $(B_a, B_x)$ (since $|B_a \cap B_x|$ could have sizes 0, 1 or 2 at least, and there is at least one type for each size). So the only value of $k$ that could cause any difficulty is $k = 2$. If $k = 2$ and either $B_a \cong \overline{K_2}$ or $s \geq 4$, then there are still at least three types:

- $B_a \cap B_x = \varnothing$;
- $B_a = \{u, v\}$, $B_x = \{u, w\}$ and $vw$ is an edge; and
- $B_a = \{u, v\}$, $B_x = \{u, w\}$ and $vw$ is a non-edge.

Hence for any $k \geq 2$ there are more types of $(B_a, B_x)$ than of $(a, x)$, except possibly when $s = 3$ and $B_a \cong K_2$, a case which we will handle specially. But we have shown that this is impossible if $G$ is homogeneous. Hence $G$ cannot be homogeneous if $B_a$ is finite, unless $B_a \cong K_2$ and $s = 3$.
Suppose that $s = 3$ and $B_a \cong K_2$. In this case, of course, we cannot have an edge between $B_a$ and $B_x$ if they intersect. However, we could have up to two edges between $B_a$ and $B_x$ if their intersection is trivial. This therefore gives us four types of $(B_a, B_x)$, more than the two types of $(a, x)$. So even in this special case $G$ cannot be homogeneous. □

It is easy to see that the proofs of Lemmas 4.3, 4.4 and 4.5 work equally well if we reverse left and right; hence, for all $x, y \in B$, $A_x \cong A_y$ and $A_x$ is homogeneous and infinite. Similarly, $A'_x \cong A'_y$ and $A'_x$ is homogeneous and infinite, and, for all $a, b \in A$, $B'_a$ is homogeneous and infinite and $B'_a \cong B'_b$. We will sometimes prove results for, say, $B_a$ and then use the corresponding result for $A_x$ without further comment.

The following result allows us to use equivalence to reduce the number of cases we have to consider. As with Lemmas 4.3 and 4.4, the proof is the same as the proof of the corresponding result in Chapter 3.

**Lemma 4.6.** *For every $a \in A$, either $B_a$ or $B'_a$ is isomorphic to $\Gamma_s$.*

*Proof.* See Lemma 3.5. □

We therefore assume in the rest of the chapter that $B'_a \cong \Gamma_s$. We will now show that the only possible values of $B_a$ are $B_a \cong \Gamma_q$ for **some** $q \leq s$.

If $s \neq \infty$ then $B_a$ cannot omit $\overline{K_n}$ for any $n \in \mathbb{N}$ since then it would violate the Infinite Ramsey Theorem (Theorem 2.14); similarly, even if $s = \infty$, $B_a$ cannot simultaneously omit $\overline{K_n}$ and $K_m$ for any $m, n \in \mathbb{N}$. Hence we may assume that $B_a$ embeds $\overline{K_\infty}$.

It remains to verify that $B_a$ indeed cannot omit exactly one of $P_3$ and $\overline{P_3}$ (of course if it omits $K_2$ then it must omit $P_3$ and $\overline{P_3}$). This is similar to the situation in Lemma 3.12; the difference here is that we cannot simply make both $ab$s non-edges (as we could before in the $(\overline{K_\infty}, \Gamma_r)$ case, which was the prototype for all $(\overline{K_m}[K_n], \Gamma_r)$ cases).

**Lemma 4.7.** *Let $a \in A$. If $B_a$ contains $\overline{P_3}$ then it contains $P_3$.*

*Proof.* $B_a$ is homogeneous and embeds $\overline{P_3}$ and $\overline{K_\infty}$, so it contains $3K_2$, say with edges $x_1y_1, x_2y_2, x_3y_3$. Moreover, there exists $z_1$ in $B$ joined to $x_1$ but not to $y_1$, $x_2$, $y_2$, $x_3$ or $y_3$. Then if $az_1$ is red $P_3 \subseteq B_a$. Suppose instead that $az_1$ is blue and let $H = (a, x_1x_2x_3y_1y_2y_3z_1)$. The sequence of amalgamations in Figure 4.1 then shows that $B_a$ must contain $P_3$ after all. □

*Remark.* We can use a sequence of amalgamations where the graphs are the complements of the ones given here to show that if $B_a$ contains $P_3$ then it contains $\overline{P_3}$.

We have already shown that, for all $b \in B$, either $A_b$ or $A'_b$ is isomorphic to $\Gamma_r$. However, one potentially difficult case could arise if, for example, $A_b \cong \Gamma_r$ and $B'_a \cong \Gamma_s$ but $A'_b \not\cong \Gamma_r$ and $B_a \not\cong \Gamma_s$. The following result tells us that this is not the case.

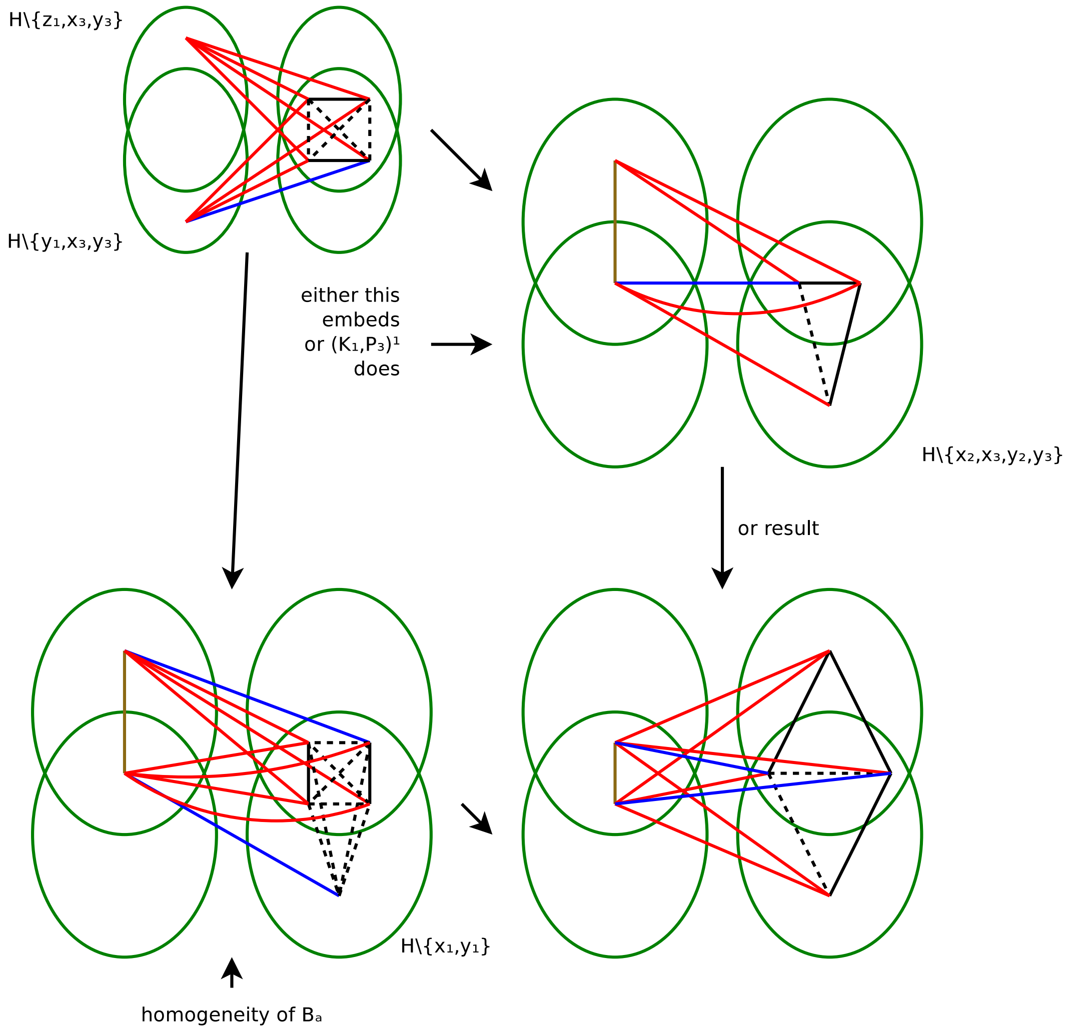


Figure 4.1: If $B_a$ contains $\overline{P_3}$ then it contains $P_3$ also. The amalgamations in this diagram depend on $H = (a, x_1y_1z_1x_2y_2x_3y_3)$ (where $H$ has edges $x_1y_1, x_1z_1, x_2y_2, x_3y_3$ and where $az_1$ is blue and all other cross-edges are red) embedding in $G$ as they use substructures of $H$ and results of earlier amalgamations in the sequence. $H \setminus \{z_1\}$ embeds in $G$ by homogeneity of $B_a$, and if $az_1$ were red we would have $P_3 \subset B_a$ immediately. We do not know whether the brown edge is an edge or a non-edge; we have arranged matters so that this is unimportant.

**Lemma 4.8.** *Let $a \in A$ and $b \in B$. If $B'_a \cong \Gamma_s$ and $B_a \ncong \Gamma_s$, then $A'_b \cong \Gamma_r$.*

*Proof.* Suppose not, so that $G$ minimally omits $(1, Q)^1$ and $(P, 1)^2$ for some finite graphs $P < \Gamma_r$ and $Q < \Gamma_s$. Certainly $G$ does embed $(P, 1)^1$ since $A_b \cong \Gamma_r$. It will be enough if we can show that $G$ embeds $(P, Q')^1$ for some subgraph $Q' < Q$ where $|Q \setminus Q'| = 1$, since then we can show that each point of $P$ must be blue to the remaining point $q$ in $Q$. This can be seen in Figure 4.2.

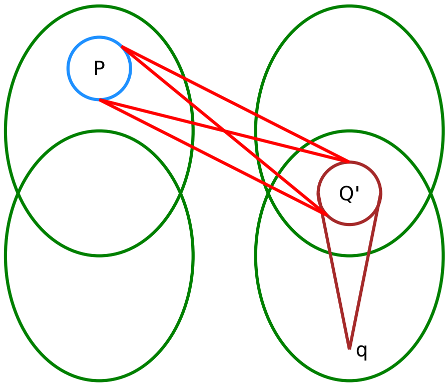


Figure 4.2: Obtaining $(1, Q)^1$ or $(P, 1)^2$ in $G$; if for any $p \in P$ $pq$ is red then we get $(1, Q)^1$, otherwise we get $(P, 1)^1$.

To obtain $(P, Q')^1$, note that we can label the points of $Q'$ by $q_1, \ldots, q_k$. Split $A$ into $2^k$ pieces according to how they are coloured to $Q'$ (i.e. within a piece $A_i$, if $a, b \in A_i$ and $q \in Q'$ then $aq \cong bq$). By Theorem 2.16 (or Theorem 2.15 if $r = \infty$), one of these pieces must contain $\Gamma_r$. If any piece other than the "all-red" piece contains $\Gamma_r$, then $G$ would realise $(\Gamma_r, 1)^2$, so $A'_b \cong \Gamma_r$. Otherwise the "all-red" piece contains $\Gamma_r$, so $G$ realises $(\Gamma_r, Q')^1$ and hence $(P, Q')^1$ embeds in $G$. □

We want to extend Lemma 4.5. For $C < A$ let

$$B_C = \{b \in B : (\forall c \in C)(c, b) \in R_1\}$$

and similarly for $D < B$ let

$$A_D = \{a \in A : (\forall d \in D)(a, d) \in R_1\}$$

The following is an easy extension of Lemma 4.4.

**Lemma 4.9.** *If $C < A$ is finite then $B_C$ is homogeneous.*

*Proof.* Let $\alpha$ be a finite partial automorphism of $B_C$; we require an automorphism $\beta$ of $B_C$ extending $\alpha$. But $\gamma = \alpha \cup 1_C$ is a finite partial automorphism of $G$, so must extend to an automorphism $\delta$ of $G$ which fixes $B_C$ set-wise. Then $\beta = \delta|_{B_C}$ is the required automorphism of $B_C$ that extends $\alpha$. □

The following lemma rules out certain potential forms of $B_C$ when $C < A$ is finite. Specifically, we show that $B_C$ is either empty, complete or isomorphic to either $\Gamma_t$ or $\overline{\Gamma_t}$ for some $t$; we do this by showing that it cannot be any of the other homogeneous graphs.

**Lemma 4.10.** *Let $C < A$ be finite. If $B_C$ is either finite or isomorphic to $\overline{K_m}[K_n]$ or $K_m[\overline{K_n}]$ for some $m, n \in \mathbb{N} \cup \{\infty\}$, then $B_C$ is complete or empty.*

*Proof.* Let $C = \{c_1, \ldots, c_k\}$. If $B_C$ is not complete, empty or isomorphic to $\Gamma_t$ or $\overline{\Gamma_t}$ for some $t$, then by Lemma 4.9 it is one of the following:

1. $C_5$,
2. $K_3 \times K_3$,
3. $\overline{K_m}[K_n]$ where $m, n \geq 2$ or
4. $K_m[\overline{K_n}]$ where $m, n \geq 2$.

Suppose that $B_C$ is finite. Let $xyz$ be such that $xy \cong yz$ but $xy \ncong xz$. Let $w \in B \backslash B_C$ be joined to $x$ and no other vertex of $B_C$ (this exists by genericity of $B$). Consider

$$\alpha_1 : w \mapsto w, y \mapsto z, c_1 \mapsto c_1, \ldots, c_k \mapsto c_k$$

Now $\alpha_1$ is a finite partial automorphism, so must extend to an automorphism $\beta_1$. But $\alpha_1$ fixes $B_C$ set-wise, so fixes $x$ pointwise (since $w$ is joined only to $x$ in $B_C$). But $xy$ is an edge and $\beta_1(x)\beta_1(y) = xz$ is a non-edge. Contradiction.
Now suppose that $B_C \cong \overline{K_m}[K_n]$ for some $m, n \geq 2$. The above paragraph shows that $m$ and $n$ cannot both be finite. If $m = \infty$ and $n < \infty$, suppose that there are connected components $D_1$ and $D_2$ in $B_C$, that $x, y \in D_1$, that $z \in D_2$ and that $w \in B \setminus B_C$ is joined to $x \in D_1$ and no other vertex in $D_1 \cup D_2$. Now consider

$$\alpha_2 : w \mapsto w, y \mapsto z, c_1 \mapsto c_1, \ldots, c_k \mapsto c_k$$

Again $\alpha_2$ is a finite partial automorphism so must extend to an automorphism $\beta_2$. But $\alpha_2(B_C) = B_C$ and moreover:

- $\beta_2(D_1) = D_2$ (since $y \in D_1$ is mapped to $z \in D_2$), but
- $\beta_2(D_1) \neq D_2$ (since $wx$ is an edge but $\beta(w)\beta(x) = w\beta(x)$ cannot be an edge if $\beta(x) \in D_2$)

giving a contradiction.
Now suppose $B_C \cong \overline{K_m}[K_\infty]$ (i.e. $n = \infty$) where $m \geq 2$; this implies that $s = \infty$. Let $x_1x_2$ and $y_1y_2$ be two edges in different connected components of $B_C$ and let $C' < A$ be such that $C' \cong C$ and

$$B_{C'} \cap B_C = \{x_1, x_2, y_1, y_2\}$$

Let $d_1, \ldots, d_k$ be the vertices of $C'$, and let $w \in B \setminus (B_C \cup B_{C'})$ be joined to $x_1$ and not to $x_2$, $y_1$ or $y_2$. Consider

$$\alpha_3 : w \mapsto w, x_2 \mapsto y_2, c_1 \mapsto c_1, d_1 \mapsto d_1, \ldots, c_k \mapsto c_k, d_k \mapsto d_k$$

Now $\alpha_3$ is a finite partial automorphism of $G$ and so extends to an automorphism $\beta_3$. By fixing $C$ and $C'$ pointwise, $\beta_3$ fixes

$$B_C \cap B_{C'} = \{x_1, x_2, y_1, y_2\}$$

set-wise. But then fixing $w$ fixes $x_1$ and hence also $x_2$, but $\beta_3(x_2) = y_2$. Contradiction.
We have shown that $B_C$ cannot be isomorphic to $\overline{K_m}[K_n]$ for any $m, n \geq 2$. By taking complements it will also follow that $B_C$ is not isomorphic to $K_m[\overline{K_n}]$. ☐

Proving that $B_C$ is infinite (or null) when $C$ is finite is more difficult. We will however prove this if $r = s = 3$ or $r = s = \infty$.

**Theorem 4.11.** *If $C < A$ is a finite graph and either $r = s = 3$ or $r = s = \infty$, then $B_C$ is infinite or null (of size zero).*

*Proof in the $r = s = \infty$ case.* If $|C| = 1$ then this is true (and indeed $B_a$ is not the null graph for any $a \in A$ if $G$ is not monochromatic). So assume that $|C| \geq 2$ and that $B_C$ is finite but not the null graph. Write $n = |C|$.
By homogeneity, it can easily be seen that for any $C' \cong C$ in $A$, then $B_{C'} \cong B_C$. Similarly, for any graph $D \cong B_C$ in $B$, there is $C'' \cong C$ in $A$ so that $D = B_{C''}$. If there are $C_1, C_2, C_3, C_4 \leq A$, all isomorphic to $C$ (and not necessarily distinct), then if

$$[C_1, C_2] \cong [C_3, C_4]$$

(that is, if there is an isomorphism $\alpha$ mapping $C_1 \cup C_2$ to $C_3 \cup C_4$ such that $\alpha(C_1) = C_3$ and $\alpha(C_2) = C_4$), then

$$[B_{C_1}, B_{C_2}] \cong [B_{C_3}, B_{C_4}]$$

since by homogeneity $\alpha$ must extend to an automorphism $\beta$ of $G$ such that $\beta(B_{C_1}) = B_{C_3}$ and $\beta(B_{C_2}) = B_{C_4}$.
Therefore, since if $B_C$ is finite then every type of $[B_{C_1}, B_{C_2}]$ that could exist in $B$ must be realised, there are at least as many types of $[C, C']$ as there are of $[B_C, B_{C'}]$. Similarly, there are at least as many types of $[B_C, B_{C'}]$ as there are of $[A_{B_C}, A_{B_{C'}}]$. Since $B_C$ is **known** to be complete or empty, **if** $A_{B_C}$ is finite (and we will return later to consider the possibility that it is infinite) then it must be that $|B_C| \geq |A_{B_C}|$ (given that $r = s = \infty$; for other values the situation is slightly more complex). Hence:

- $A_{B_C}$ is finite, and so complete or empty, so
- $C$ is complete or empty, and no larger than $A_{B_C}$, therefore
- we also have that $n = |C| \geq |B_C|$, and hence
- $n = |C| = |B_C|$ and $C = A_{B_C}$.

The remaining cases are:

1. there is a "perfect matching" between types of $[C, C']$ and types of $[B_C, B_{C'}]$; or
2. $A_{B_C}$ is infinite.

If $A_{B_C}$ is infinite, let $D_1, D_2 \cong A_{B_C}$ be disjoint subsets of $A$ such that $C < A_{B_C}$. Let $C_1 < D_1 \backslash C$ and $C_2 < D_2$ be isomorphic to $C$. Let $\phi$ be a finite partial automorphism of $G$ that fixes $C$ and maps $C_1$ to $C_2$. Then $\phi$ should extend to $\psi \in \mathrm{Aut}(G)$. But $\psi(B_C) = B_C$ and $\psi(B_{C_1}) = B_{C_2} \neq B_{C_1}$, which is a contradiction since $B_{C_1} = B_C$.
Suppose there is a perfect matching between types of $[C, C']$ and types of $[B_C, B_{C'}]$. We show that there must also be a perfect matching between these and types of $(C, B_{C'})$. Suppose instead that there are $C_1, C_2, C_3, C_4 \leq A$, all isomorphic to $C$ (and not necessarily distinct) so that

$$[C_1, C_2] \cong [C_3, C_4]$$

but

$$(C_1, B_{C_2}) \ncong (C_3, B_{C_4})$$

Consider an isomorphism $\gamma : C_1 \mapsto C_3, C_2 \mapsto C_4$; this ought to extend to an automorphism $\delta$ of $G$ such that $\delta(C_1) = C_3$ and $\delta(B_{C_2}) = B_{C_4}$, which clearly cannot hold. Hence there are at least as many types of $[C, C']$ as of $(C, B_{C'})$, and similarly there are at least as many types of $(C, B_{C'})$ as there are of $[B_C, B_{C'}]$.

Recall that, since $A$ and $B$ are the random graph, every possible type of $[C, C']$ or $[B_C, B_{C'}]$ must in fact be realised. (Indeed, even if $r \neq \infty$ or $s \neq \infty$, it is still true that every possible type of $[C, C']$ that omits $K_r$, and every type of $[B_C, B_{C'}]$ that omits $K_s$, must be realised.) We will show, for a contradiction, that there are strictly fewer possible types of $(C, B_{C'})$ than there are possible types of $[C, C']$; then there will certainly be fewer realised types of $(C, B_{C'})$ than of $[C, C']$, destroying the perfect matching.

But there can only be as many types of $(C, B_{C'})$ as there are bipartite graphs where the parts have sizes $|C|$ and $|B_C|$ respectively. Since $|C| = |B_C|$ (as we are in the $r = s = \infty$ case), and since $r = s = \infty$ so the types of $[C, C']$ can realise any finite complete graph, we have at most as many types of $(C, B_{C'})$ as we do types of $[C, C']$ where $C \cap C' = \varnothing$. But there is at least one more type of $[C, C']$, namely $[C, C]$, and this destroys any prospect of a perfect matching. □

*Sketch proof in the $r = s = 3$ case.* If $r = s = 3$, proceed as above when $C$ and $B_C$ are empty (note that the equality of $|C|$ and $|B_C|$ does hold here as no $K_3$ could possibly embed in any $[C, C']$). In the case where one of $C$ and $B_C$ is complete (and assume without loss of generality that $C$ is complete), then since $|C| = 2$ there are precisely five types of $[C, C']$, all realised. But $|B_C| \geq 2$ so there are at least 10 types of $[B_C, B_{C'}]$, all realised, and the required perfect matching of types cannot exist.

If $C$ is complete (rather than empty) and non-null, it must be $K_2$. There are then precisely five types of $[C, C']$; the three shown in Figure 4.3. However, at least six of the seven possible types of $(C, B_{C'})$ must be realised. Hence the required perfect matching of types cannot exist. □

*Remark.* More information about the number of isomorphism types of bipartite graphs is given in chapter 4.3 of Harary & Palmer (1973). We only needed a rather crude estimate in the $r = s = \infty$ case; we would probably need to be much more precise in cases where either or both are finite. (When $r = s = 3$ we were able to count these types precisely; the same could be done if we were given fixed finite values of $r$ and $s$.)

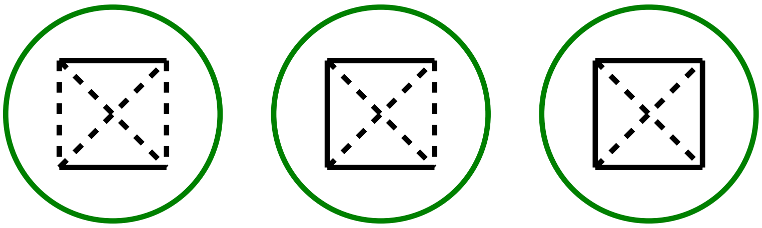

Figure 4.3: The three types of $[K_2, K_2]$ in $\Gamma_3$ where the copies of $K_2$ do not intersect.

We have now determined which all-red finite substructures can be minimally omitted in $G$ (specifically, up to equivalence, they must be of the form $(K_p, K_q)^1$ for some $p < r$ and $q < s$) and as such $G$ must minimally omit an antichain of structures of this form. Proving that $G$ minimally omits no other finite 2-coloured 2-graphs is the subject of the remainder of the chapter.

Note that this is one of the main reasons we must restrict ourselves to classifying 2-coloured $(\Gamma_r, \Gamma_s)$ 2-graphs. The problem is not with the results proved so far in this section; we can define

$$B_C^i = \{x \in B : (\forall a \in C) ax \in R_i\}$$

and apply the same arguments in almost the same way. The problem is that we cannot rely on the fact that the possible combinations of values (for a given finite $C < A$) of $B_C^i$ are easy to determine. This is a technical rather than a fundamental difficulty (since in principle a classification of the 2-coloured homogeneous $(n-1)$-graphs would be sufficient to obtain the combinations of $(B_C^i : 1 \leq i \leq n)$), but it is one we have not been able to resolve.

In cases where $r$ is finite, it is possible to exploit this fact in certain situations to obtain colourings $(\overline{K_m}, D)$ whenever the constituent $(K_1, D)$ substructures embed in $G$. More specifically:

**Lemma 4.12.** *Suppose that $r$ is finite. Suppose that every colouring of $(K_1, nD)$ embeds in $G$ for every $n \in \mathbb{N}$ and some finite graph $D$. Then every colouring of $(\overline{K_m}, D)$ embeds in $G$ for every $m \in \mathbb{N}$. Moreover, if $G$ omits $(K_1, K_p)^1$ and embeds every $(K_1, nD)$ not embedding $(K_1, K_p)^1$ then $G$ embeds every $(mK_1, D)$ not embedding $(K_1, K_p)^1$.*

*Remark.* This result is stated here in this form because it is potentially useful in classifying cases for values of $r$ and $s$ other than those we have been able to consider in detail, and so for extending the results in this thesis.

*Proof.* We work by induction on $m$. Specifically, we show that if, for some $D$ and every $p \in \mathbb{N}$, every colouring of $(\overline{K_m}, pD)$ embeds in $G$, then any required colouring $H$ of $(\overline{K_{m+1}}, nD)$ embeds in $G$. By assumption this is **true** whenever $m = 1$.
The idea is to keep adding points on the left-hand side until we are forced to add non-edges, and then single out an appropriately large $\overline{K_m}$. If there are sufficiently many copies of $D$ on the right-hand side, we will be able to single out one with the right colours to the left as long as it is conceivable a correct colouring does exist (which we ensure with the condition in the statement of the lemma).
Let $H_1, H_2 \cong (\overline{K_m}, k_0 D)$ (where the colours are still to be decided). Amalgamate over $(\overline{K_{m-1}}, kD)$ to obtain either the desired $H$ or

$$H_3, H_4 \cong (K_2 + \overline{K_{m-1}}, k_1 D)$$

and in general, given

$$H_{2i-1}, H_{2i} \cong (K_i + \overline{K_{m-1}}, k_i D)$$

amalgamate $H_{2i-1}$ and $H_{2i}$ over an intersection

$$(K_{i-1} + \overline{K_{m-1}}, \hat{k}D)$$

to obtain either $H$ or

$$H_{2i+1}, H_{2i+2} \cong (K_{i+1} + \overline{K_{m-1}}, k_{i+1} D)$$

until we reach the point where $i = n - 1$ at which point we could not possibly add an edge on the left.
Now $k_{m-1} = 2$ (one copy to avoid identifying, one copy for the result) and $k_i = 2k_{i+1} + 2$ (one set for $H_{2i+1}$, one set for $H_{2i+2}$, one copy for if we happen to add a non-edge on the left and one extra copy to be sure we do not identify).
We need to choose colours to do two things:

- avoid identifying the two points not in the intersection of the amalgamands, and

- choose the colours appropriately.

The colours are chosen recursively. At the $i = m-1$ stage, one copy of $D$ is coloured so as to give $H$, and the other is coloured in such a way as to avoid identifying (blue to the points in the middle and one point outside, red to the other point outside). At the $i$th stage we duplicate both sets of colours at the $(i+1)$th stage, and add the two sets we need at the $i = m-1$ stage. These propagate to the $i = 1$ stage.
$H_1$ and $H_2$ are well-defined, and by assumption they both embed in $G$. By proceeding in this way so does $H$. □

In the rest of this chapter, we will show that, in the following four sub-cases, any homogeneous 2-coloured 2-graph must be one of the cases shown by Proposition 4.1 to exist:

1. Not all monochromatics embed in $G \cong (\Gamma_r, \Gamma_s)$ where $r = s = 3$;
2. All monochromatics embed in $G \cong (\Gamma_r, \Gamma_s)$ where $r = s = 3$; and
3. Not all monochromatics embed in $G \cong (\Gamma_r, \Gamma_s)$ where $r = s = \infty$;
4. All monochromatics embed in $G \cong (\Gamma_r, \Gamma_s)$ where $r = s = \infty$.

Throughout the rest of the chapter, a finite 2-coloured 2-graph $H = (C, D, R)$ will be called **legal** if $C < \Gamma_r$, $D < \Gamma_s$ and $H$ does not embed any element of the antichain (if any) of which all elements are asserted to be minimally omitted from $G$.

## 4.3 Not all monochromatics embed, $r = s = 3$

$G$ minimally omits an antichain $\mathcal{A}$ of monochromatic 2-graphs. The results in section 4.2 imply that, up to equivalence, all elements of $\mathcal{A}$ are monochromatic red and both components are complete. Hence the only possible values of $\mathcal{A}$ are:

1. $\mathcal{A}_1 = \{(K_1, K_2)^1, (K_2, K_1)^1\}$;
2. $\mathcal{A}_2 = \{(K_1, K_2)^1\}$; and
3. $\mathcal{A}_3 = \{(K_2, K_2)^1\}$

We show in each case that $G$ is generic subject to minimally omitting the elements of the relevant antichain; that is, we show that $G$ realises every finite 2-coloured 2-graph $H$ such that:

1. $H$ omits all elements of the antichain; and

2. both components of $H$ omit $K_3$.

### 4.3.1 $\{(K_1, K_2)^1, (K_2, K_1)^1\}$ minimally omitted

To show that $G$ is generic (subject to minimally omitting $(K_1, K_2)^1$ and $(K_2, K_1)^1$), Theorem 2.13 states that it is sufficient to show that any finite 2-coloured 2-graph $H \cong (B, D)$ that satisfies the following two properties embeds in $G$:

- $H$ does not embed $(K_1, K_2)^1$ or $(K_2, K_1)^1$, and

- $B$ is either $P_3$, $\overline{P_3}$ or $\overline{K_m}$ for some finite $m$.

To show that any $(K_1, D)$ 2-graph **not** realising $(K_1, K_2)^1$ embeds in $G$, we use the same proofs as we used for for the corresponding results in Chapter 3 and we do not need to give them again here. Given that $G$ realises any $(K_1, D)$ that omits $(K_1, K_2)^1$, Lemma 4.12 tells us that $G$ realises every $H \cong (\overline{K_m}, D)$ that omits $(K_1, K_2)^1$.

The following result shows that $G$ realises any legal finite monochromatic blue 2-graph.

**Lemma 4.13.** *For all finite* $C, D < \Gamma_3$*,* $G$ *realises* $(C, D)^2$*.*

*Proof.* Let $G = (A, B, R)$ and assume that $C < A$. Split $B$ into subsets $B_0, B_1, \ldots, B_k$ according to the colours of each vertex to $C$ (i.e. for all $i$, all $a \in C$ and all $x, y \in B_i$, $ax \cong ay$, but for all $i \neq j$ there exist $a \in C$, $x \in B_i$ and $y \in B_j$ such that $ax \ncong ay$), where $B_0$ is monochromatic blue to $C$. By Theorem 2.16, there exists $i$ such that $B_i$ contains $\Gamma_3$, but if any vertex of $C$ is red to the vertices of $B_i$, then $B_i$ cannot contain an edge. Hence only $B_0$ can contain $\Gamma_3$, and since some $B_i$ contains $\Gamma_3$ it must therefore be $B_0$. It follows that $G$ realises $(C, D)^2$ for any finite $C, D < \Gamma_3$. □

*Remark.* The idea we use here of splitting either $B$ or $D$ into subsets where all vertices of the same subset have the same pattern of colours to some finite subgraph $C$ of $A$ is one we use throughout this chapter.

Since results in section 4.2 tell us which finite $(C, D)^1$ 2-graphs embed in $G$, and Lemma 4.13 says that every finite $(C, D)^2$ embeds if $C, D < \Gamma_3$, and since we have determined which finite $(\overline{K_m}, D)$ 2-graphs embed in $G$, we have reduced the task of classifying $G$ to that of showing that $G$ realises all legal mixed-colour finite 2-graphs of the forms $(P_3, D)$ and $(\overline{P_3}, D)$. Before we do this, we will first show that $G$ realises any legal mixed-colour finite 2-graph of the form $(K_2, D)$; this is needed as an intermediate step towards obtaining all legal 2-graphs of the forms $(P_3, D)$ and $(\overline{P_3}, D)$.

**Lemma 4.14.** *G realises any "legal" finite 2-graph of the form $H = (K_2, D)$.*

*Proof.* Let $H = (ab, D, R)$ (where $ab$ is an edge) and let $D = D_0 \sqcup D_1 \sqcup D_2 \sqcup D_3$ such that all vertices of $D_0$ and $D_1$ are red to $a$, all vertices of $D_0$ and $D_2$ are red to $b$, and all other cross-edges are blue. For $H$ to be legal, $D_0 = \varnothing$ and $D_1$ and $D_2$ cannot contain any edges. We must show that any such $H$ does embed in $G$.
If only $D_3$ is non-empty, then $H$ embeds by Lemma 4.13, so we may assume without loss of generality that $D_1$ is non-empty. We prove the lemma in two parts:

1. we show that $H$ is realised if $D_2 = \varnothing$; then

2. we show that $H$ is realised when both $D_1$ and $D_2$ are non-empty.

In the first case (where $D_2 = \varnothing$), amalgamate $(a, D)$ with $(ab, D_3)$ over $(a, D_3)$. By the above results the top and bottom embed in $G$, and $b$ cannot be red to any point in $D_1$ (since $a$ is, and if both are then $(K_2, K_1)^1$ would be realised). Hence $H$ must embed in $G$. This amalgamation can be seen in the left-hand diagram in Figure 4.4.
In the second case (where $D_2 \neq \varnothing$), we may assume (by applying the previous case) that $(ab, D_1 \cup D_3)$ is realised in $G$. Amalgamate it with $(b, D)$ (which also embeds in $G$) over $(b, D_1 \cup D_3)$. Again $a$ cannot be red to any point in $D_2$ since $b$ is and both cannot be, so $H$ must embed in $G$. This amalgamation can be seen in the right-hand diagram in Figure 4.4. ☐

**Lemma 4.15.** *G realises any "legal" finite 2-graph of the form $H = (P_3, D)$.*

*Proof.* Write $H = (a_0a_1a_2, D)$ where the edges are $a_0a_1$ and $a_1a_2$, and partition $D$ into 8 subsets $D_0, \dots, D_7$ so that $a_i$ is red to all of $D_j$ if $\lfloor j/2^i \rfloor \equiv 0 \pmod 2$ and blue to all of $D_j$ if $\lfloor j/2^i \rfloor \equiv 1 \pmod 2$. Note that $D_0$, $D_1$ and $D_4$ are empty and every other $D_j$, other than $D_7$, contains no edges.
We split into two cases:

1. the case where only $D_5$ and $D_7$ are non-empty; and

2. cases where at least one of $D_2$, $D_3$ and $D_6$ are non-empty.

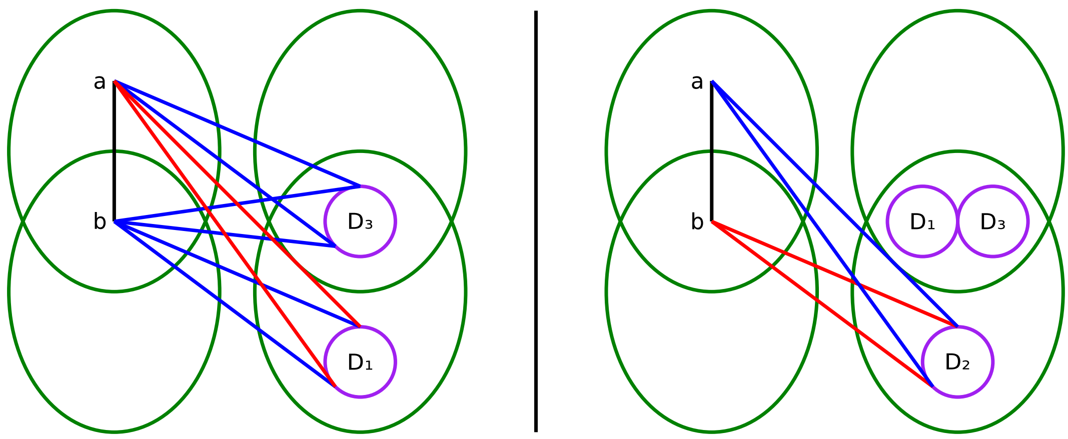


Figure 4.4: The amalgamation diagrams needed for the first (shown on the left) and second (shown on the right) cases of Lemma 4.14.

In the first case, amalgamate $(a_0a_1a_2, D_7)$ with $(a_0a_1, D_5 \cup D_7)$ over $(a_0a_1, D_7)$. Both amalgamands embed in $G$, and $a_2$ cannot be red to any vertex of $D_5$, so it must be blue, so we obtain $H$. This amalgamation can be seen in the left-hand diagram in Figure 4.5.

For the second case, assume that we have first done all instances of the first case. Amalgamate $(a_0a_1a_2, D_5 \cup D_7)$ with $(a_0a_2, D)$ over $(a_0a_2, D_5 \cup D_7)$. Again, both amalgamands embed in $G$, and if $a_1$ were red to any vertex in $D_2 \cup D_3 \cup D_6$ we would obtain $(K_2, K_1)^1$ which we don't. Hence $a_1$ is blue to all of $D_2 \cup D_3 \cup D_6$ and so we must obtain $H$. This amalgamation can be seen in the right-hand diagram in Figure 4.5. □

**Lemma 4.16.** *G realises any "legal" finite 2-graph of the form* $H = (\overline{P_3}, D)$.

*Proof.* Write $H = (a_0a_1a_2, D)$ where the edge is $a_0a_1$, and split $D$ into 8 pieces $D_0, \ldots, D_7$ so that $a_i$ is red to all of $D_j$ if $\lfloor j/2^i \rfloor \equiv 0 \pmod 2$ and blue to all of $D_j$ if $\lfloor j/2^i \rfloor \equiv 1 \pmod 2$. Note that $D_0$ and $D_4$ are empty and every other $D_j$, other than $D_7$, contains no edges.

This time we need to divide into three cases:

1. cases where only $D_2$, $D_3$ and $D_7$ can be non-empty, and
2. cases where at least one of $D_1$ and $D_5$ is non-empty, but $D_6$ is still empty; and
3. cases where $D_6$ is non-empty.

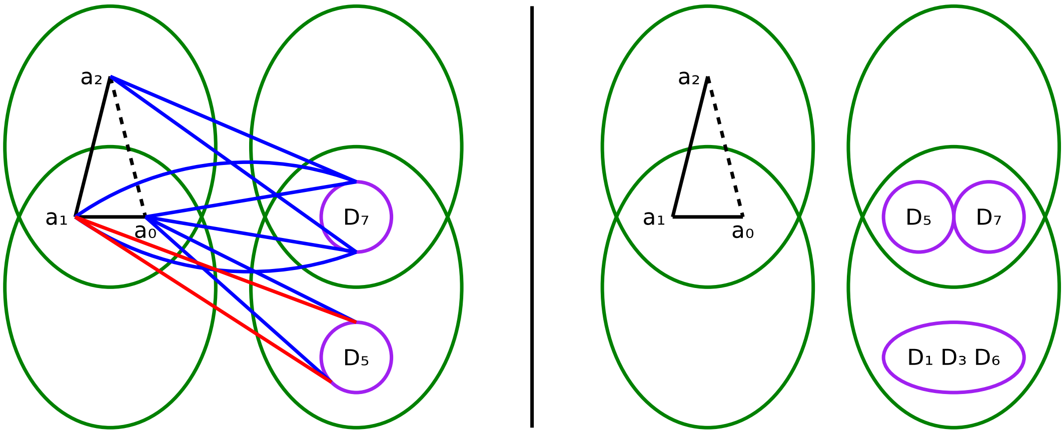


Figure 4.5: The amalgamation diagrams needed for the first (left) and second (right) cases in Lemma 4.15.

In the first case, amalgamate $(a_0a_1a_2, D_7)$ with $(a_0a_2, D)$ over $(a_0a_2, D_7)$. Both amalgamands embed in $G$. By construction, no vertex of $D \setminus D_7 = D_2 \cup D_3$ can be red to $a_1$, or else we would obtain $(K_2, K_1)^1$ which we don't. Hence $H$ embeds in $G$. This amalgamation can be seen in the left-hand diagram in Figure 4.6.

In the second case, we may assume that $(a_0a_1a_2, D_2 \cup D_3 \cup D_7)$ embeds in $G$ by applying the first case. Amalgamate it with $(a_1a_2, D)$, which also embeds in $G$, over $(a_1a_2, D_2 \cup D_3 \cup D_7)$. No vertex of $D_1 \cup D_5$ can be red to $a_1$, or else we obtain $(K_2, K_1)^1$ in $G$. Hence they must all be blue and $H$ embeds in $G$. This amalgamation can be seen in the right-hand diagram in Figure 4.6.

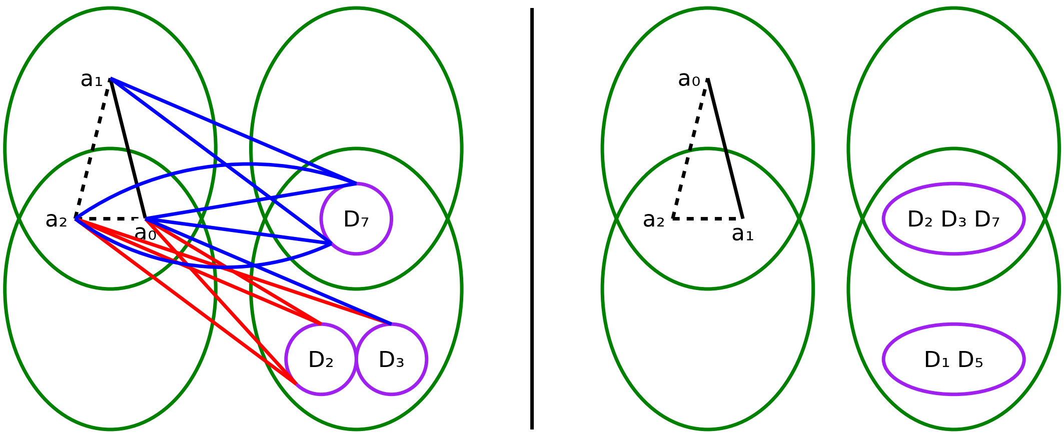


Figure 4.6: The amalgamation diagrams needed for the first (left) and second (right) cases in Lemma 4.16.

In the final case, where $D_6$ is non-empty, we define two auxiliary 2-graphs $H_1 = (a_0a_1b, D)$ and $H_2 = (a_1a_2b, D)$ where $b$ is joined to $a_0$ and $a_2$, but not to $a_1$, and $b$ is red to all vertices of $D_5$ (if any) and blue to all other vertices of $D$. Then $H_1$ does embed in $G$ by use of the first two cases, and $H_2$ clearly does by Lemma 4.15. So amalgamate over $(a_1b, D)$, and note that $a_0a_2$ must be a non-edge in the amalgam (they can't be identified since $D_6$ is non-empty and vertices of $D_6$ are red to $a_0$ and blue to $a_2$, and they can't be joined since then $a_0a_2b$ would be a $K_3$). Hence we must again obtain $H$, as required. This amalgamation is shown in Figure 4.7. □

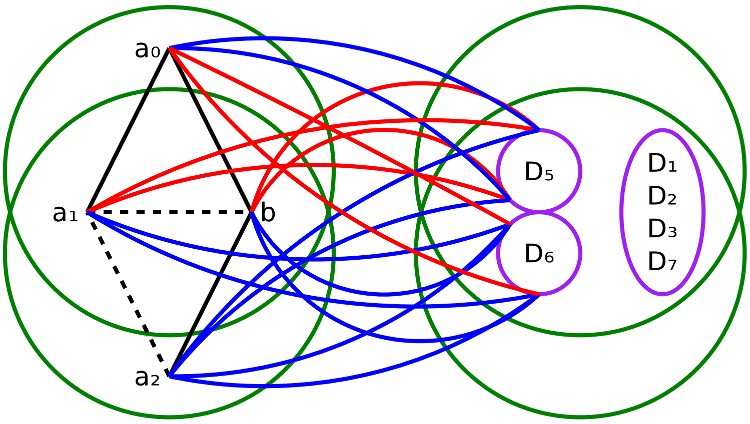


Figure 4.7: The amalgamation diagram needed for the third case of Lemma 4.16.

Hence $G$ must embed every "legal" $(B, D)$ where

$$B \in \{P_3, \overline{P_3}\} \cup \{\overline{K_m} : m \in \mathbb{N}\}$$

and by Theorem 2.13 $G$ is therefore uniquely determined.

### 4.3.2 $\{(K_1, K_2)^1\}$ minimally omitted

As in the previous section, $G$ realises every finite 2-coloured 2-graph of the form $(\overline{K_m}, D)$ with $D < \Gamma_3$ that omits $(K_1, K_2)^1$. It is therefore sufficient to show that $G$ realises every finite 2-coloured 2-graph of the form $(B, D)$ which omits $(K_1, K_2)^1$ and where $D < \Gamma_3$ and $B$ is either $P_3$ or $\overline{P_3}$. Moreover, we can easily extend Lemma 4.13 to show that $G$ must realise any $(C, D)^2$ where $C, D < \Gamma_3$, since the proof of that result did not use the fact that $(K_2, K_1)^1$ was minimally omitted.

We therefore have to transfer Lemmas 4.14, 4.15 and 4.16 to this new situation where $(K_2, K_1)^1$ is realised.

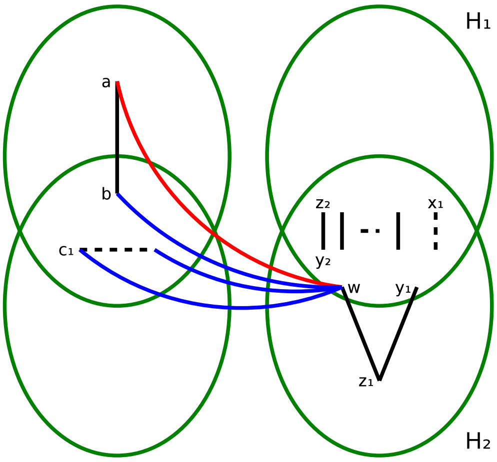


Figure 4.8: The amalgamation needed for Lemma 4.17. $H_2$ embeds because the left component has no edges, and $H_1$ embeds by induction on $l$ for increased $n$. Only cross-edges to $w$ are shown.

**Lemma 4.17.** *$G$ realises any "legal" finite 2-graph of the form $H = (K_2+\overline{K_m}, lK_2+\overline{K_n})$.*

*Proof.* We prove that any

$$H = (abc_1 \dots c_m, x_1 \dots x_n y_1 \dots y_n z_1 \dots z_l, R)$$

embeds where $H$ is such that the edges are $ab$ and, for each $i$, $y_i z_i$. We may also assume that $az_1$ is blue.

Let

$$H_1 = (abc_1 \dots c_m, wx_1 \dots x_n y_1 \dots y_l z_2 \dots z_l)$$

and

$$H_2 = (bc_1 \dots c_m, wx_1 \dots x_n y_1 \dots y_l z_1 \dots z_l)$$

where $wz_1$ is an edge and where $aw$ is red and $bw$ and $c_i w$ are blue. $H_1$ and $H_2$ are shown in Figure 4.8.

Now $H_1$ embeds in $G$ by induction on $n$ and $H_2$ embeds in $G$ by Lemma 4.12 (if $l = 1$ then both $H_1$ and $H_2$ embed in $G$ by this lemma). So amalgamate $H_1$ with $H_2$ over their intersection

$$(bc_1 \dots c_m, wx_1 \dots x_n y_1 \dots y_l z_2 \dots z_l)$$

and note that $az$ cannot be red (since $aw$ is) so it has to be blue, thus yielding $H$. □

Now all we have to do is replace each $K_2$ with a $P_3$. We do so using the following two lemmas.

**Lemma 4.18.** *Suppose, for some finite graphs $D$ and $E$, that $G$ realises every*

$$H' = (D + K_2, E + K_1)$$

*that omits $(K_1, K_2)^1$. Then $G$ realises every*

$$H = (D + P_3, E)$$

*omitting $(K_1, K_2)^1$.*

*Proof.* Suppose we wish to show that $G$ realises

$$H = (D + P_3, E, R)$$

where the $P_3$ is labelled $abc$ and $ab, bc$ are edges. Let

$$H_1 = (D + \{a, b\}, E + \{z\})$$

and

$$H_2 = (D + \{b, c\}, E + \{z\})$$

where $az$ is red and $bz$ and $cz$ are blue. Now $H_1$ and $H_2$ embed in $G$ by hypothesis, and so we can amalgamate them over their intersection

$$(D + \{b\}, E + \{z\})$$

But we cannot identify $a$ with $c$ (because $az$ is red and $cz$ blue), and we cannot join $a$ and $c$ (as then we would have a $K_3$ in $G$, which we don't). So we must add a non-edge and obtain $H$.
This process is illustrated in the left-hand amalgamation diagram in Figure 4.9. □

**Lemma 4.19.** *Suppose, for some finite graphs $D$ and $E$, that $G$ realises every*

$$H' = (D + K_1, E + K_2)$$

*that omits $(K_1, K_2)^1$. Then $G$ realises every*

$$H = (D, E + P_3)$$

*omitting $(K_1, K_2)^1$.*

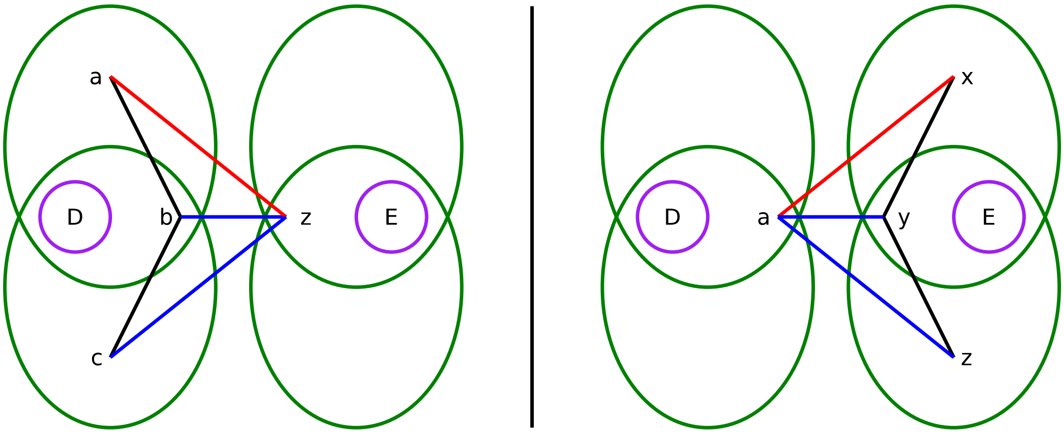


Figure 4.9: The amalgamations required for the inductive step in changing a $K_2$ on the left component of $H$ to a $P_3$: Lemma 4.18 needs the first amalgamation and Lemma 4.19 needs the second amalgamation.

*Proof.* Suppose we wish to show that $G$ realises

$$H = (D, E + P_3, R)$$

where the $P_3$ is labelled $xyz$ and $xy, yz$ are edges. Let

$$H_1 = (D + \{a\}, E + \{x, y\})$$

and

$$H_2 = (D + \{a\}, E + \{y, z\})$$

where $ax$ is red and $ay$ and $az$ are blue. Now $H_1$ and $H_2$ embed in $G$ by hypothesis, and so we can amalgamate them over their intersection

$$(D + \{a\}, E + \{y\})$$

But we cannot identify $x$ with $z$ (because $ax$ is red and $az$ blue), and we cannot join $x$ and $z$ (as then we would have a $K_3$ in $G$, which we don't). So we must add a non-edge and obtain $H$.

We see this in the right-hand amalgamation diagram in Figure 4.9. □

The above results combine to give us the following result that concludes our classification of the homogeneous 2-coloured $(\Gamma_3, \Gamma_3)$ 2-graphs minimally omitting $(K_1, K_2)^1$:

**Theorem 4.20.** *Let $G$ be a homogeneous 2-coloured $(\Gamma_3, \Gamma_3)$ 2-graph minimally omitting $(K_1, K_2)^1$. Then $G$ is the generic homogeneous 2-coloured $(\Gamma_3, \Gamma_3)$ 2-graph minimally omitting $(K_1, K_2)^1$.*

*Proof.* $G$ realises every $(P_3+mK_1, nP_3)$ omitting $(K_1, K_2)^1$. We can apply Theorem 2.13 to show that $G$ realises every $(D, nP_3)$ (where $D$ is a finite $K_3$-free graph) that omits $(K_1, K_2)^1$, and apply Theorem 2.13 again to show that $G$ must in fact realise every $H = (D, E)$, where $D$ and $E$ are finite $K_3$-free graphs, that omits $(K_1, K_2)^1$. □

### 4.3.3 $\{(K_2, K_2)^1\}$ minimally omitted

If $(K_2, K_2)^1$ is minimally omitted, this must mean that $(K_1, K_2)^1$ and $(K_2, K_1)^1$ are embedded in $G$. It follows that every monochromatic $(K_1, D)$ and $(D, K_1)$ embeds in $G$. The proofs given in section 3.3.2 show that $G$ realises every $(K_1, D, R)$ where $D \cong nP_3$ for some $n \in \mathbb{N}$.

**Lemma 4.21.** *$G$ realises every colouring of $(K_2+\overline{K_m}, K_2+\overline{K_n})$ where the $(K_2, K_2)$ is not monochromatic red.*

*Proof.* Let

$$H = (abc_1 \dots c_m, x_1 \dots x_n yz, R)$$

where the only edges are $ab$ and $yz$. Without loss of generality we can assume that $az$ is blue.
Let

$$H_1 = (abc_1 \dots c_m d, wx_1 \dots x_n y)$$

and

$$H_2 = (bc_1 \dots c_m d, wx_1 \dots x_n yz)$$

where in addition $ad$ and $wz$ are edges and $aw$, $dw$ and $dz$ are red. Now both $K_1$ and $K_2$ have an edge-free graph on one or other component, so both embed in $G$. Amalgamate them over their intersection

$$(bc_1 \dots c_m d, wx_1 \dots x_n)$$

as in Figure 4.10, and note that $az$ cannot be red, so must be blue, and if it is blue we must obtain $H$. Hence $H$ embeds in $G$. □

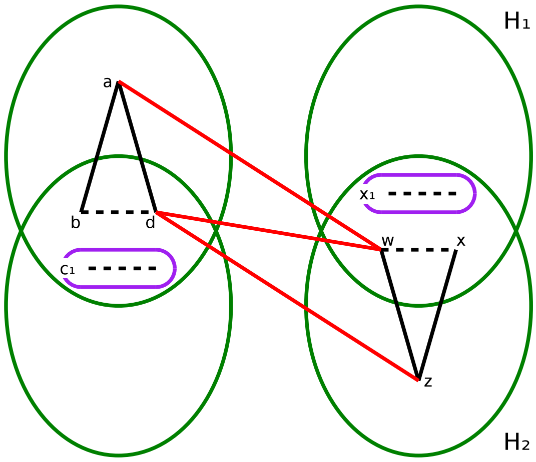


Figure 4.10: The amalgamation diagram for Lemma 4.21; we have omitted all cross-edges except the ones we added as red $(aw, dw, dz)$.

We would like to extend Lemma 4.21 to show, by induction on $m$, that $G$ realises every instance of

$$(K_2 + lK_1, mK_2 + nK_1)$$

with no all-red $(K_2, K_2)^1$. This needs a little care; specifically, we need to do our induction through $P_3$. We now state the lemma we will use to convert a $K_2$ to a $P_3$. (We gave the proof of a corresponding result in section 4.3.2 and as this proof also applies in this case we do not need to repeat it.)

**Lemma 4.22.** *Suppose, for some finite graphs $D$ and $E$, that $G$ realises every*

$$H' = (D + K_2, E + K_1)$$

*that omits $(K_2, K_2)^1$. Then $G$ realises every*

$$H = (D + P_3, E)$$

*omitting $(K_2, K_2)^1$.*

*Proof.* See Lemma 4.18; the proof is the same. □

We now extend Lemma 4.21 using the following lemma.

**Lemma 4.23.** *Suppose, for some finite graphs $D$ and $E$, that $G$ realises every*

$$H' = (P_3 + D, E + K_1)$$

*that omits* $(K_2, K_2)^1$. *Then* $G$ *realises every*

$$H = (K_2 + D, E + K_2)$$

*omitting* $(K_2, K_2)^1$.

*Proof.* Suppose we want to show that $G$ realises

$$H = (D + \{a, b\}, E + \{y, z\})$$

where $az$ is blue and $ab, yz$ are edges. Let

$$H_1 = (D + \{a, b, c\}, E + \{x, y\})$$

and

$$H_2 = (D + \{b, c\}, E + \{x, y, z\})$$

where $bc$ and $xy$ are non-edges and $ac$ and $xz$ are edges, and $ax$, $cx$ and $cz$ are red. By Lemma 4.22, $H_1$ embeds in $G$, and, by Lemma 4.12, $H_2$ embeds in $G$. So we can amalgamate $H_1$ with $H_2$ over their intersection

$$(D + \{b, c\}, E + \{y, z\})$$

as in Figure 4.11, and $az$ cannot be red so must be blue, giving us $H$. □

Finally we can successively convert each edge on the right to a $P_3$ using the following lemma. As the proof is the same as for the corresponding result in section 4.3.2, we only give the statement here.

**Lemma 4.24.** *Suppose, for some finite graphs* $D$ *and* $E$, *that* $G$ *realises every*

$$H' = (D + K_1, E + K_2)$$

*that omits* $(K_2, K_2)^1$. *Then* $G$ *realises every*

$$H = (D, E + P_3)$$

*omitting* $(K_2, K_2)^1$.

*Proof.* The proof of Lemma 4.19 also works here. □

We are now in a position where we can combine the above results and conclude our classification of the homogeneous 2-coloured $(\Gamma_3, \Gamma_3)$ 2-graphs minimally omitting $(K_2, K_2)^1$

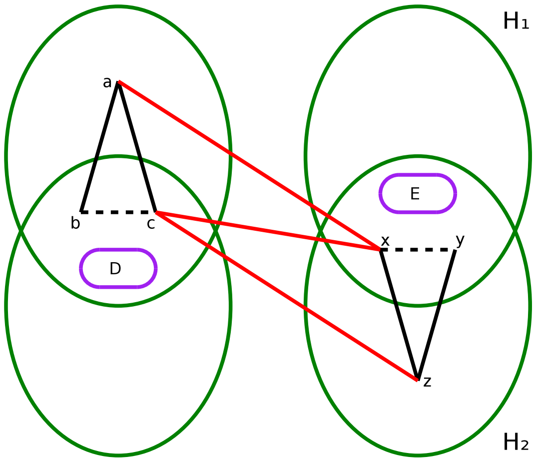


Figure 4.11: The amalgamation needed in Lemma 4.23 to convert a vertex on the right component to an edge at a price of reducing the $P_3$ on the left component to an edge. The only real difference between this diagram and Figure 4.10 is that $D$ and $E$ are now not just sets of isolated vertices.

**Corollary 4.25.** *Let $G$ be a homogeneous 2-coloured $(\Gamma_3, \Gamma_3)$ 2-graph minimally omitting $(K_2, K_2)^1$. Then $G$ is the generic homogeneous 2-coloured $(\Gamma_3, \Gamma_3)$ 2-graph minimally omitting $(K_2, K_2)^1$.*

*Proof.* $G$ realises every $(P_3+mK_1, nP_3)$ omitting $(K_2, K_2)^1$. We can apply Theorem 2.13 to show that $G$ realises every $(D, nP_3)$ omitting $(K_2, K_2)^1$ for any finite graph $D$, and apply Theorem 2.13 to show that $G$ must in fact embed every $(D, E)$ that omits $(K_2, K_2)^1$ where $D$ and $E$ are finite graphs. □

## 4.4 All monochromatics embed - $r = s = 3$

In this section, suppose that $G$ embeds all finite monochromatic 2-graphs $(C, D)^i$ where $i \in \{1, 2\}$, $C < \Gamma_r$ and $D < \Gamma_s$. We will show that $G$ is in fact generic; that is, we show that $G$ embeds **every** finite 2-graph $(C, D, S)$ where $C < \Gamma_r$, $D < \Gamma_s$ and $S = (S_1, S_2)$ is a partition of $V(C) \times V(D)$.

To do this we will use the copying argument (Theorem 2.13): that is, we show that every 2-colouring of $(\overline{K_m}, nP_3)$ embeds in $G$ for every $m, n \in \mathbb{N}$, and that every 2-colouring of $(P_3, nP_3)$ and $(\overline{P_3}, nP_3)$ embeds in $G$ for every $n \in \mathbb{N}$. We will then apply the copying argument twice; once on the left to obtain every colouring of

$(C, nP_3)$ for every finite $C < \Gamma_3$ and every $n \in \mathbb{N}$, and then once on the right to obtain every colouring of $(C, D)$ for every finite $C, D < \Gamma_3$.
We will therefore prove that, for every $m, n, p \in \mathbb{N}$, every colouring of $(P_3 + \overline{K_m}, pP_3 + \overline{K_n})$. We will work by means of alternation, starting with 2-graphs of the form $(K_2 + \overline{K_m}, K_2 + \overline{K_n})$ – the idea is to:

- change the $K_2$ on the left to a $P_3$, while keeping the right-hand side fixed, then
- add an edge on the right while reducing the $P_3$ on the left to a $K_2$.

We first need to prove explicitly that $G$ does indeed embed every 2-coloured 2-graph of the form $(K_2, K_2)$. This proof also shows that $G$ embeds any 2-colouring of $(K_2 + \overline{K_m}, K_2 + \overline{K_n})$ if the embedded $(K_2, K_2)$ is not monochromatic.

**Lemma 4.26.** *$G$ embeds every 2-colouring of* $(K_2, K_2)$.

*Remark.* This argument, by adding extra points on the left and right as necessary, also gives us every colouring of $(K_2 + \overline{K_m}, K_2 + \overline{K_n})$ *as long as* the $(K_2, K_2)$ is **not** monochromatic.

*Proof.* Let $H = (ab, xy, R)$ be a 2-graph of the form $(K_2, K_2)$. If $H$ is monochromatic it embeds in $G$ by assumption, so assume, for some $p, q \in \{a, b\}$ and $r, s \in \{x, y\}$, that $pr$ is red and $qs$ is blue. (There is no assumption that $p \neq q$ or that $r \neq s$. However, in practice both will hold, and the diagrams are drawn on the basis that $p \neq q$ and $r \neq s$.)
Construct the amalgamation diagram in Figure 4.12. By Lemma 4.12 both amalgamands do embed in $G$. (Note that for the corresponding result in the $r = s = \infty$ case this **cannot** be taken for granted. We will return to this point later.) The diagram is constructed so that if $cw$ is blue then $(cd, uw)$ is a copy of $(pq, rs)$, and if $cw$ is red then $(ce, vw)$ is a copy of $(pq, rs)$. (This type of argument will be implicit throughout the chapter and will not usually be written out with this level of detail.) Hence $(pq, rs)$ embeds in $G$, as required. □

We next need to show that $G$ embeds every 2-colouring of $H = (K_2, K_2 + \overline{K_n})$ for every $n \in \mathbb{N}$; we did the case $n = 0$ in Lemma 4.26. In order to do this, we need to go through $(P_3, K_2 + \overline{K_n})$ (but we only need certain cases of this, and it will turn out that we can show those cases embed in $G$ without circularity). We can assume

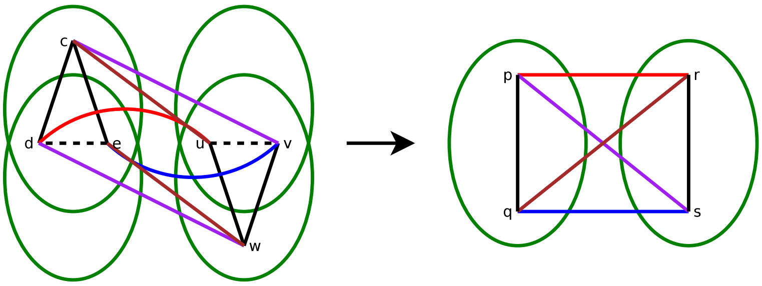


Figure 4.12: We desire $(pq, rs)$ (the false colours of $ps$ and $qr$ are just to identify these cross-edges in the amalgamation diagram, and cross-edges without specified colours can be taken as arbitrary). Note that whether $cw$ is red or blue we necessarily get $(pq, rs)$.

that the $(K_2, K_2)$ in $H$ is monochromatic; if it isn't, the Remark to Lemma 4.26 suffices to show that $H$ embeds in $G$.

We first do the base case ($n = 1$); that is, we show that every 2-colouring of $(K_2, K_2 + K_1)$ embeds in $G$.

**Lemma 4.27.** *$G$ embeds every 2-coloured $H = (K_2, K_2 + K_1, R)$.*

*Proof.* By Lemma 4.26 and the Remark, we may assume that the $(K_2, K_2)$ is monochromatic red but that $H$ is not monochromatic. Let $H = (ab, xyz)$ where the only edges are $ab$ and $xy$, and distinguish two cases:

1. $z$ is red to $a$ and blue to $b$ (or vice versa); or

2. $xz$ and $yz$ are blue.

For the first case, use the left-hand amalgamation in Figure 4.13 (if $az$ is red use $a, c, x, y, z$ and if $az$ is blue use $a, b, x, y, z$); for the second, use the other two amalgamations (the lower amalgamand in the second amalgamation is given by Lemma 4.12). □

We now inductively increase $n$; for this we need to go through $P_3$. Let $H = (ab, xyz_1 \dots z_n)$, where the only edges are $ab$ and $xy$. Suppose that $ax, ay, bx, by$ are all red. We consider the three cases separately:

1. $az_i$ and $bz_i$ are blue for all $i$;

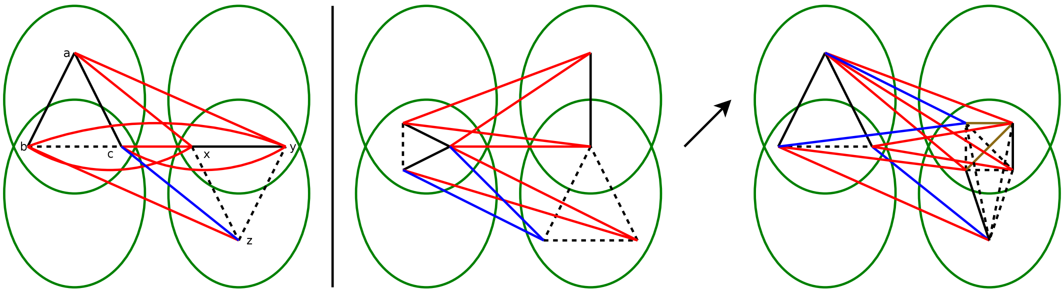


Figure 4.13: Obtaining $(K_2, K_2 + K_1)$ in $(\Gamma_3, \Gamma_3)$ where the $(K_2, K_2)$ is monochromatic red. Cross-edge types not specified can be made red.

2. $az_i \cong bz_i$ for all $i$, but (without loss of generality) $az_1$ is red and $az_2$ is blue; and

3. without loss of generality, $az_1$ is red and $bz_1$ is blue.

We consider these cases in this order. Note that later cases need a simpler amalgamation diagram but more data.

**Lemma 4.28.** *For all $n$, let $H_n$ be the 2-graph*

$$((\{a, b\}, \{ab\}), (\{x, y, z_1, \ldots, z_n\}, \{xy\}), (\{ax, ay, bx, by\}, \{az_i, bz_i : 1 \leq i \leq n\})$$

*(that is, $H_n$ is of the form $(K_2, K_2 + \overline{K_n})$ where the $(K_2, K_2)$ part is monochromatic red and all other cross-edges are blue). Suppose that, for some $n \geq 1$, $H_n$ embeds in $G$. Then so does $H_{n+1}$.*

*Proof.* We first show that

$$H'_n = (abc, xyz_1 \ldots z_n)$$

embeds in $G$, where $bc$ is an edge, $ac$ is a non-edge, $cx$ is red and $dy$ and every $cz_i$ and $dz_i$ is blue. For this, amalgamate as in Figure 4.14; note that the top amalgamand embeds by the induction hypothesis, and the bottom amalgamand embeds by the Remark to Lemma 4.26.
Then amalgamate $H'_n$ with the $(P_3, \overline{K_{n+3}})$ 2-graph $H''_n$ (over their intersection), as in Figure 4.15. This creates a product $\widetilde{H_n}$.
Finally, amalgamate $\widetilde{H_n}$ with the $(\overline{K_2}, D)$ 2-graph in Figure 4.16 over their intersection. This will necessarily produce $H_{n+1}$ in $G$. □

We now inductively obtain the other two cases of $(K_2, K_2 + \overline{K_n})$. We do these using the following lemma.

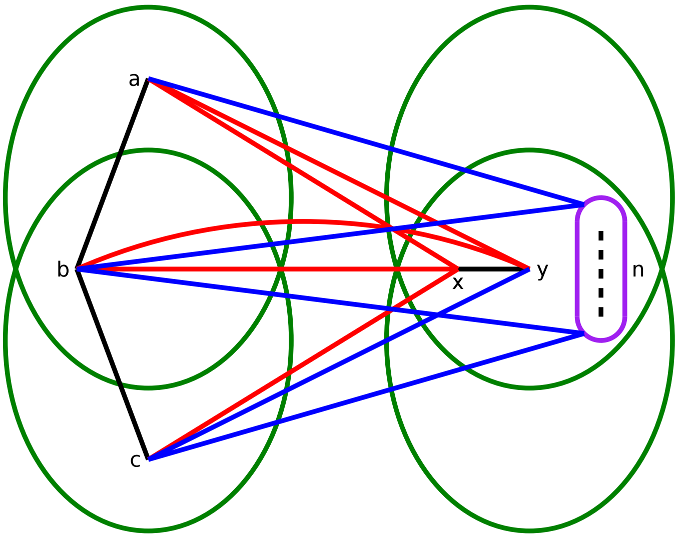


Figure 4.14: Obtaining $H'_n$ in Lemma 4.28. The top amalgamand embeds by the induction hypothesis; the bottom embeds by the Remark to Lemma 4.26.

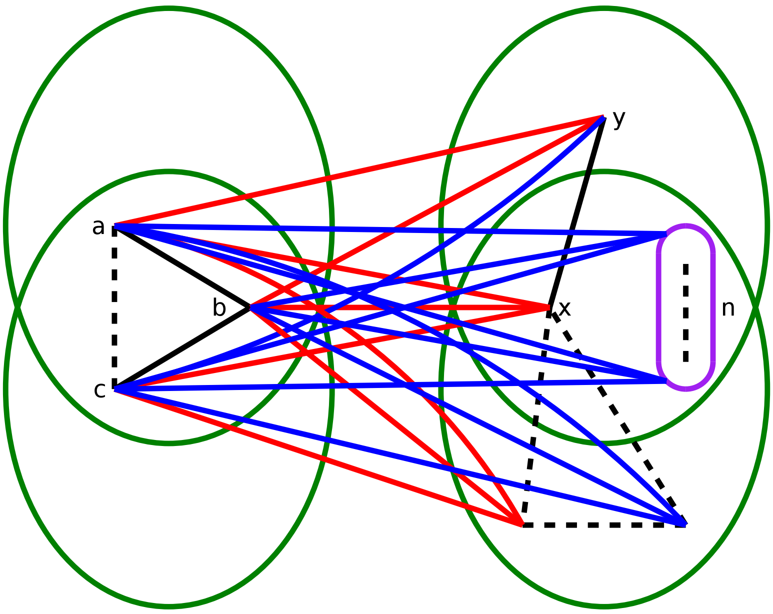


Figure 4.15: Obtaining $\widetilde{H_n}$ in Lemma 4.28. The top amalgamand is $H'_n$; the bottom embeds by Lemma 4.12.

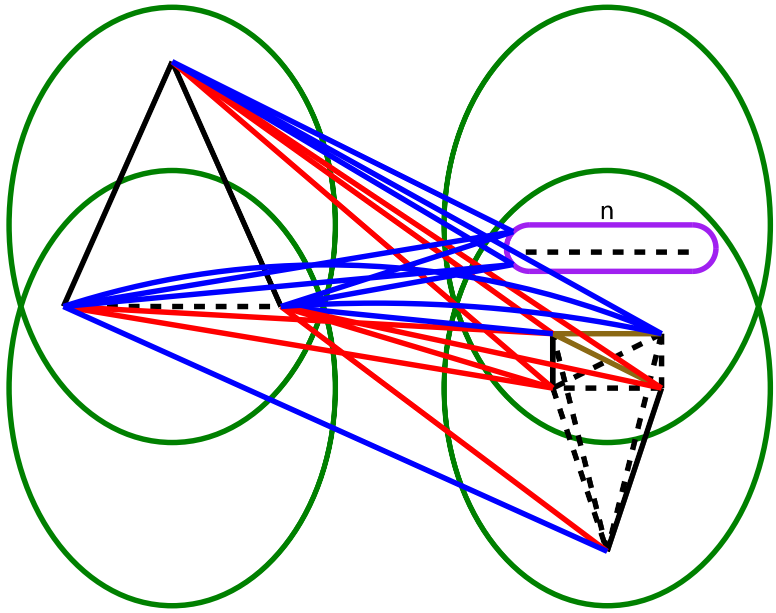


Figure 4.16: Obtaining $H_{n+1}$ in Lemma 4.28. The top amalgamand is $\widetilde{H_n}$; the bottom embeds by Lemma 4.12.

**Lemma 4.29.** *Fix $n \geq 2$ and let $H$ be a 2-graph either of the form*

$$((\{a, b\}, \{ab\}), (\{x, y, z_1, \ldots, z_n\}, \{xy\}), (R_1, R_2))$$

*where $\{ax, ay, bx, by, az_1\} \subseteq R_1$ and*

- ***either*** *$bz_1 \in R_2$,*
- ***or*** *$bz_1 \in R_1$ and $az_2, bz_2 \in R_2$.*

*Suppose that $G$ embeds every 2-graph of the form*

$$((\{c, d\}, \{cd\}), (\{u, v, w_1, \ldots, w_{n'}\}, \{uv\}), (S_1, S_2))$$

*where $n' < n$ and where $cu, cv, du, dv \in S_1$. Then $H$ also embeds in $G$.*

*Remark.* It is somewhat more convenient to do the induction for these cases together than to do them separately. They do both require rather different amalgamations.

*Proof.* If $H$ has $bz_1 \in R_2$ then induction proceeds as in Figure 4.17.
If $H$ has $bz_1 \in R_1$ and $az_2, bz_2 \in R_2$ then induction proceeds as in Figure 4.18. □

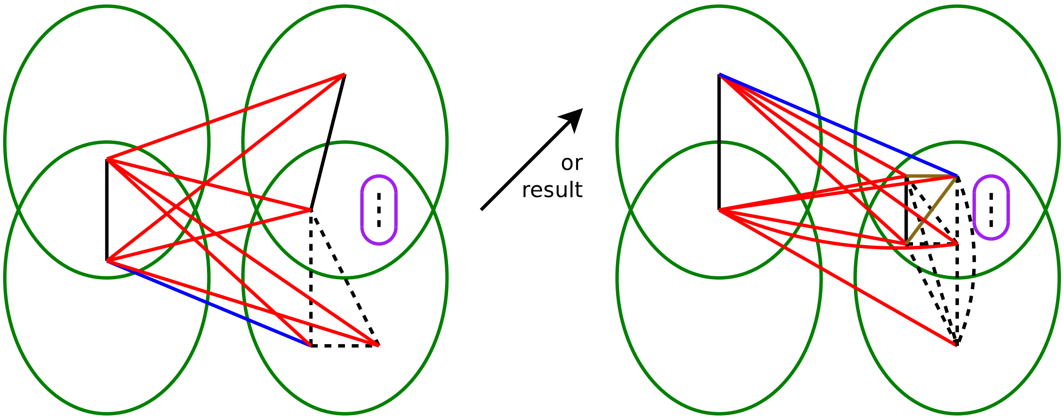


Figure 4.17: The inductive step for showing that $H \cong (K_2, K_2 + \overline{K_n})$ in Lemma 4.29 embeds in $G$ if $bz_1 \in R_2$. The purple oval contains one fewer isolated point than $H$.

At this point we are in a position to present the lemma that allows us to change a $K_2$ on the left to a $P_3$ on the left. Note that its extreme simplicity is due to our assumption that $s = 3$. The price we pay is that one of the isolated vertices on the right is "used up" - since we can have arbitrarily many of these, though, this is acceptable.

**Lemma 4.30.** *If, for some $n$ and some graph $D$, every colouring of*

$$(K_2, D + \overline{K_{n+1}})$$

*embeds in $G$, then so does every colouring of*

$$(P_3, D + \overline{K_n})$$

*Proof.* Suppose we want

$$H = (abc, D + E)$$

where $ab, bc$ are edges and $ac$ is a non-edge, and where $E \cong \overline{K_n}$. Let

$$H_1 = H + (\varnothing, z) \setminus (c, \varnothing)$$

and

$$H_2 = H + (\varnothing, z) \setminus (a, \varnothing)$$

where $az, bz$ are red and $bz$ is blue. By assumption $H_1$ and $H_2$ embed in $G$. Amalgamate over their intersection and note that $a$ and $c$ can neither be joined nor identified, so we necessarily obtain $H$. $\square$

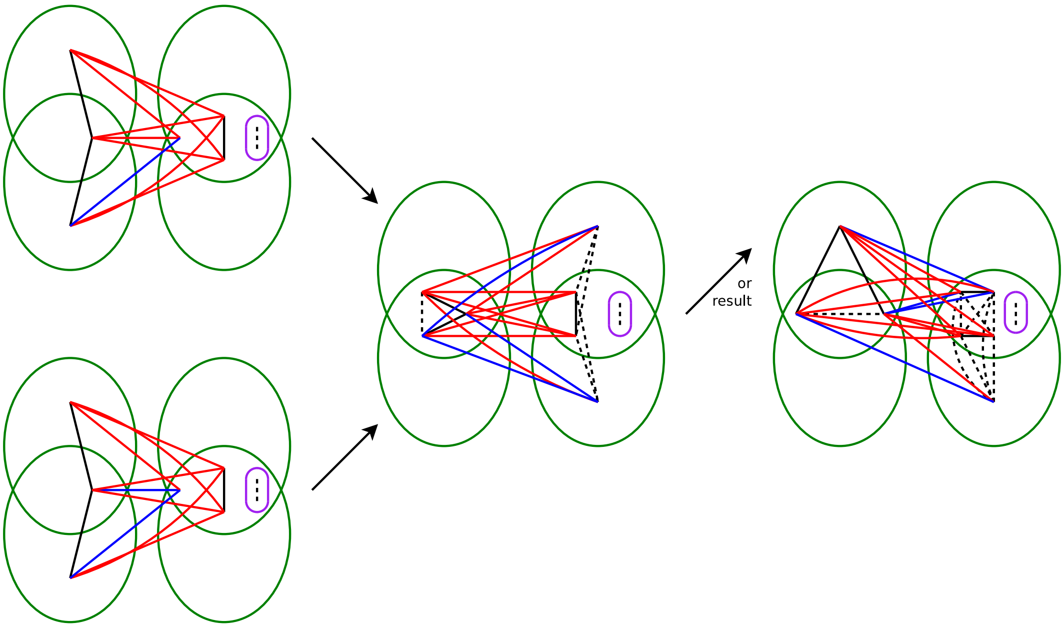


Figure 4.18: The inductive step for showing that $H \cong (K_2, K_2 + \overline{K_n})$ in Lemma 4.29 embeds in $G$ if $bz_1 \in R_1$ and $az_2, bz_2 \in R_2$. The purple oval contains **two** fewer isolated points than $H$.

There is a counterpart to Lemma 4.30 that allows us to change each $K_2$ on the right to a $\overline{P_3}$. For this we will need to be able to add singletons on the left, as each time we change a $K_2$ to a $P_3$ we "use up" one of these singletons.

Before we make further changes to the left-hand side, we will obtain every colouring of $(K_2, mK_2 + \overline{K_n})$. Distinguish three cases:

1. every $(K_2, K_2)$ is monochromatic red;
2. every $(K_2, K_2)$ is monochromatic, but some are red and some are blue; and
3. there is a non-monochromatic $(K_2, K_2)$.

In case 1 it is sufficient to apply Lemmas 4.28 and 4.29 with the extra edges. In case 3 it is sufficient to apply the Remark to Lemma 4.26 on the non-monochromatic $(K_2, K_2)$. So this means that we only have to consider case 2. But this can be handled easily using the induction shown in Figure 4.19. In this induction, the 2-graphs whose left components are $P_3$ are obtained using Lemma 4.30.

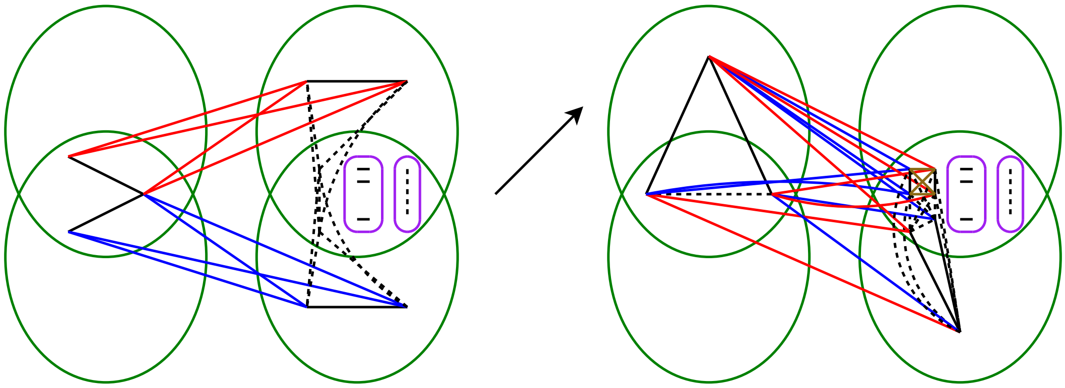

Figure 4.19: The inductive step for obtaining $H \cong (K_2, mK_2 + \overline{K_n})$ in $G \cong (\Gamma_3, \Gamma_3)$ when every embedded $(K_2, K_2)$ in $H$ is monochromatic but some are monochromatic red and some are monochromatic blue.

We now need to add singletons to the left. We will begin with the base case - obtaining all colourings of $(K_2+K_1, K_2+K_1)$. We only need to consider cases of this where the $(K_2, K_2)$ is monochromatic red as if the $(K_2, K_2)$ is not monochromatic then we can simply use the Remark to Lemma 4.26, and the "monochromatic red" and "monochromatic blue" cases are symmetric. These cases will be obtained using the following lemma.

**Lemma 4.31.** *$G$ realises all colourings of $H = (K_2 + K_1, K_2 + K_1)$ where the $(K_2, K_2)$ is monochromatic red.*

*Proof.* Let $H = (abc, xyz)$ where the edges are $ab$ and $xy$. If $c$ is red to all of $x, y, z$, or if $z$ is red to all of $a, b, c$, we can simply apply the arguments in Lemma 4.27 with an extra all-red point (reversing the sides if necessary). So assume that both $c$ and $z$ have a blue neighbour.

The sequence of amalgamations in Figure 4.20 allows us to handle a case where either $c$ or $z$ has a red and a blue to the edge on the other side. (We do this in the case where $cx$ and $az$ are blue and $bz, cy, cz$ are red; the other cases are similar.)

Similarly, we can handle a case where $az, bz, cx, cy$ are blue by the method shown in Figure 4.21.

Note that in these diagrams we have shown $cz$ as red; if it is blue, then in fact the same amalgamations will apply. However, this does leave the case where $cz$ is the **only** blue cross-edge. In this case we will need to apply the amalgamations in Figure 4.22; these are straightforward because we can obtain using a sequence

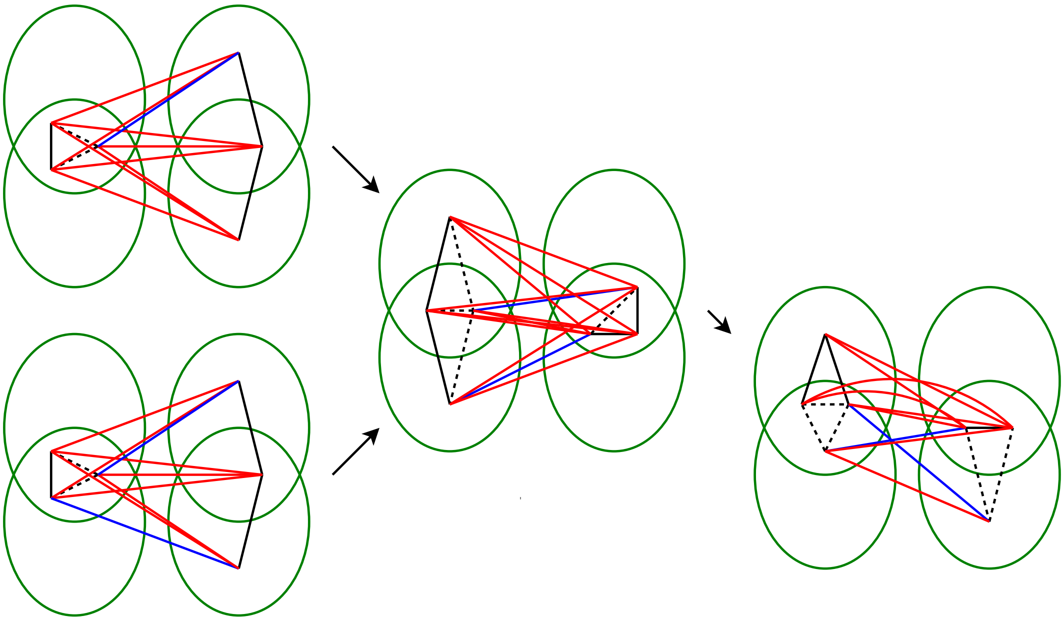

Figure 4.20: Obtaining $(K_2 + K_1, K_2 + K_1)$ where $cx$ is blue and $cy$ and $cz$ are red. We show this for the case where $az$ is also blue and $bz$ red; the other instances are handled similarly. (Recall that the $(K_2 + K_1, K_2)$ 2-graphs shown are known to embed in $G$ by Lemma 4.27.)

of amalgamations whose initial amalgamands are monochromatic 2-graphs and 2-graphs with empty (i.e. edge-less) graphs on one side. $\square$

In general, we wish to obtain $(K_2 + \overline{K_l}, mK_2 + \overline{K_n})$ for all $l, m, n \in \mathbb{N}$. This means we need to extend Lemmas 4.28 and 4.29 and the above remarks about $(K_2, mK_2 + \overline{K_n})$. But it is easy to see that, for fixed $l$, we can carry out all of these inductions in just the same way as we did in the $l = 0$ case; the only difficult case would be when $n = 1$ for each value of $l$. But we can also carry out the inductions for Lemmas 4.28 and 4.29 (when $m = 1$) for fixed $n$ and increasing $l$, so in fact even the $l = 1$ case poses no difficulty (given the $l = n = 1$ case, which we have from Lemma 4.31).
Hence $G$ does realise every $(K_2 + lK_1, mK_2 + nK_1)$, and indeed every $(P_3 + lK_1, mK_2 + nK_1)$. It remains to adapt Lemma 4.30 to a result that converts a $K_2$ on the right to a $P_3$, but in fact the amalgamation used there works here if we reverse the sides (we simply use up one left-side point to avoid identifying, and this is acceptable since we can have arbitrarily many isolated vertices in the left component).

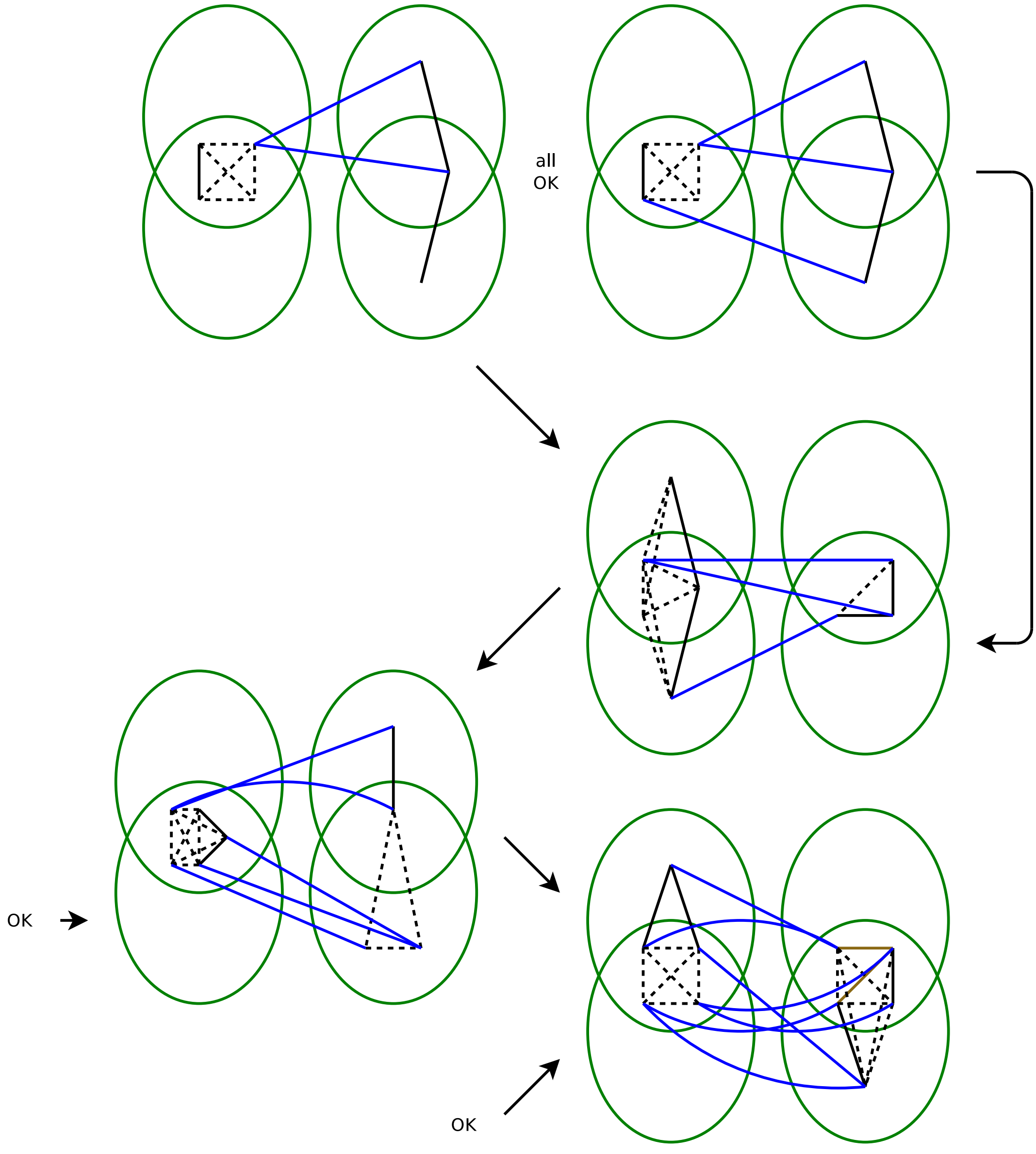


Figure 4.21: Obtaining $(K_2 + K_1, K_2 + K_1)$ where $az$, $bz$, $cx$ and $cy$ are blue and $cz$ is red. All cross-edges not shown are red. The $(K_2 + \overline{K_2}, K_2)$ 2-graphs shown embed in $G$ as a result of earlier lemmas.

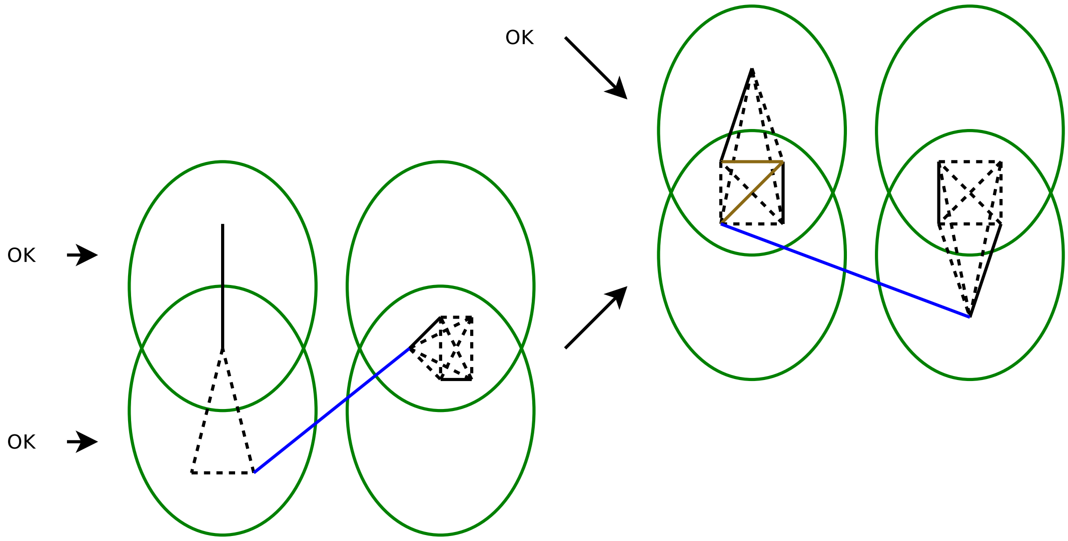


Figure 4.22: Obtaining $(K_2+K_1, K_2+K_1)$ where $cz$ is blue and all other cross-edges are red. All cross-edges not shown are red.

So $G$ realises every $(P_3 + lK_2, mP_3 + nK_1)$ for all $l, m, n \in \mathbb{N}$. Hence

**Theorem 4.32.** *Let $G$ be a 2-coloured homogeneous $(\Gamma_3, \Gamma_3)$ 2-graph embedding every finite monochromatic 2-graph $(C, D)^i$ where $C, D < \Gamma_3$ and either $i = 1$ or $i = 2$. Then $G$ is the generic 2-coloured homogeneous $(\Gamma_3, \Gamma_3)$ 2-graph.*

*Proof.* $G$ realises every $(P_3 + lK_2, mP_3 + nK_1)$ for all $l, m, n \in \mathbb{N}$. Apply Theorem 2.13 to show that $G$ realises every $(D, mP_3+nK_1)$ for every finite $D < \Gamma_3$, then apply Theorem 2.13 again to show that $G$ realises every finite $(C, D, R)$ where $|R| = 2$ and $C, D < \Gamma_3$. □

*Remark.* This completes the proof of Theorem 4.2 in the case where $r = s = 3$ and $\mathcal{A} = \varnothing$.

## 4.5 Not all monochromatics embed, $r = s = \infty$

Throughout this section, suppose that there is a non-empty antichain $\mathcal{A}$ of finite monochromatic 2-graphs $(C_j, D_j)^i$ (i.e. all of the same colour) such that for finite graphs $C, D < \Gamma_\infty$ and this value of $i$, the 2-graph $(C, D)^i$ is omitted from $G$ if and only if it embeds a member of $\mathcal{A}$. We will assume that $i = 1$ (i.e. all the 2-graphs in $\mathcal{A}$ are monochromatic red). We will prove that Theorem 4.2 holds for this case; that is, we will show that $G$ is the generic homogeneous 2-coloured $(\Gamma_\infty, \Gamma_\infty)$ 2-graph minimally omitting precisely the elements of $\mathcal{A}$. In doing so we will use a variation of the technique we used in Lemma 3.14.

We will therefore prove the following, which will be sufficient for this case of Theorem 4.2:

**Theorem 4.33.** *Let $H = (E, F, S)$ be a finite 2-graph that omits all elements of $\mathcal{A}$ and is not monochromatic red. Then $H$ embeds in $G$.*

*Proof.* Split $H$ into an "all-red" part and a residue. Let $(E_1, F_1)$ be such that

$$E_1 = \{e \in E : (\forall x \in F)(ex \in R_1)\}$$

and

$$F_1 = \{x \in F : (\forall e \in E)(ex \in R_1)\}$$

and let $E_2 = E \setminus E_1$ and $F_2 = F \setminus F_1$.

It is sufficient to prove that if every $H' = (E', F', S')$ such that $|E_2'| < |E_2|$ or $|E_2'| = |E_2|$ and $|F_2'| < |F_2|$ (where $E_1', E_2', F_1', F_2'$ are defined by analogy to $E_1, E_2, F_1, F_2$) embeds in $G$ then so does $H$.

The aim is to enlarge $H$ to some $H'$ that "almost" contains a copy of some element of $\mathcal{A}$; more precisely, $H'$ will embed $H$ and have a cross-edge such that if this cross-edge were red instead of blue then $H'$ would embed an element of $\mathcal{A}$. For this, we work by lexicographic induction on $(|E_2|, |F_2|)$ (i.e. we work by induction on $|E_2|$, and when this size is fixed work by induction on $|F_2|$). If $|E_2| = 0$ or $|F_2| = 0$ then $H$ is monochromatic red, so embeds in $G$ by assumption. Therefore it is enough to handle the inductive step.

If $|E_1| = 1$ or $|F_1| = 1$ the inductive step is handled in Lemma 4.35, and otherwise in Lemma 4.34. Once these lemmas have been proved, the proof of Theorem 4.33 will be complete. □

**Lemma 4.34.** *Let $\mathcal{A}$ be an antichain of finite monochromatic-red 2-graphs where both components are complete, such that every 2-graph $L = (C, D, S) \in \mathcal{A}$ has $|C| \geq 2$ and $|D| \geq 2$. Let $H = (E, F, R)$ be a finite 2-graph not embedding any element of $\mathcal{A}$, and let $E_1, E_2, F_1, F_2$ be defined as above. Suppose that, for any $H' = (E', F', R')$ such that if $E_1', E_2', F_1', F_2'$ are defined analogously then either $|E_2'| < |E_2|$ or $|E_2'| = |E_2|$ and $|F_2'| < |F_2|$, $H'$ embeds in $G$. Then $H$ also embeds in $G$.*

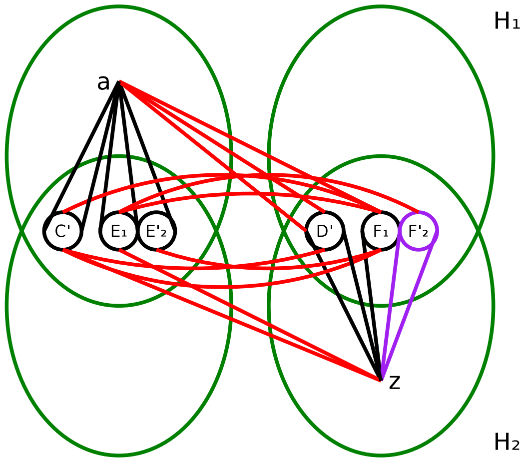


Figure 4.23: The inductive step for obtaining $H = (E, F, R)$ if $G \cong (\Gamma_\infty, \Gamma_\infty)$ omits $\mathcal{A}$ and $|C|, |D| \geq 2$; note that any blue cross-edges are in $(\{a\} \cup E_2', F_2' \cup \{z\})$ and the size of one component of this reduces in each amalgamand, which is sufficient for the induction to work, even though the sizes of the amalgamands do increase.

Let $(C, D)^1$ be in $\mathcal{A}$; that is, let it be minimally omitted from $G$. By assumption $|C| \geq 2$ and $|D| \geq 2$.

We want to amalgamate a 2-graph of the form

$$H_1 = (\{a\} \sqcup E_1 \sqcup C' \sqcup E_2', D' + (F_1 \sqcup F_2'))$$

with a 2-graph of the form

$$H_2 = (C' + (E_1 \sqcup E_2'), D' \sqcup F_1 \sqcup F_2' \sqcup \{z\})$$

over their intersection

$$H_0 = (C' + (E_1 \sqcup E_2'), D' + (F_1 \sqcup F_2'))$$

as in Figure 4.23, where:

- $C'$ is chosen so that $\{a\} \cup C' \cong C$;
- $E_2'$ is chosen so that $E_2' \cup \{a\} \cong E_2$;
- $D'$ is chosen so that $\{z\} \cup D' \cong D$;
- $F_2'$ is chosen so that $F_2' \cup \{z\} \cong F_2$;

and where the cross-edges are chosen so that:

- $a$ is red to all of $D' \cup F_1$;
- $z$ is red to all of $C' \cup E_1$;
- all of $C' \cup E_1$ is red to all of $D' \cup F_1$;
- all of $E_1$ is red to all of $F_2'$;
- all of $F_1$ is red to all of $E_2'$; and
- if $az$ were blue then $(\{a\} \cup E_1 \cup E_2', \{z\} \cup F_1 \cup F_2') \cong H$.

If suitable $H_1$ and $H_2$ are realised in $G$ then $H$ is automatically realised in $G$, since $az$ cannot possibly amalgamate to a red cross-edge (as then we would obtain $(C, D)^1$, which is omitted), so $az$ must be blue.

The idea is to use induction to obtain $H_1$ and $H_2$. We have not yet fully specified these two 2-graphs so we can make all unspecified cross-edges red (i.e. all cross-edges from $C'$ to $F_2'$ and all cross-edges from $D'$ to $E_2'$), and let any edge types that are not already specified be non-edges (this avoids accidentally creating any new elements of $\mathcal{A}$). Note that we add $C'$ and $D'$ to be disjoint from, and have no edges to, $E \setminus \{a\}$ and $F \setminus \{b\}$; as these have red cross-edges to all vertices of the other component, this avoids accidentally creating any elements of $\mathcal{A}$. (This doesn't work if $|C| = 1$; we will describe later what we do if this is the case.)

Once we have defined $H_1$ and $H_2$, we find that $H_2$ is just an instance of the result we are trying to obtain where $E_2'$ is the "new $E_2$"; if $|E_2'| = 0$ then $H_2$ is monochromatic so embeds since it does not embed any member of $\mathcal{A}$, otherwise it embeds in $G$ by induction on $|E_2|$. Similarly, we find that $H_1$ is an instance of the result where $E_2$ has remained the same and $F_2'$ is the "new $F_2$", so, since it omits all members of $\mathcal{A}$, it embeds in $G$ either by induction (if $|F_2'| > 0$) or by monochromaticity (if $|F_2'| = 0$).

The argument in Lemma 4.34 has the technical deficiency of needing two vertices in both $C$ and $D$. In these cases we need an alternative argument; the technique we sketch is similar to that we used in Lemma 3.14 and so we need only sketch it here.

**Lemma 4.35.** *Let $\mathcal{A}$ be an antichain of finite monochromatic-red 2-graphs where both components are complete, such that there is a 2-graph $L = (C, D, S) \in \mathcal{A}$ where*

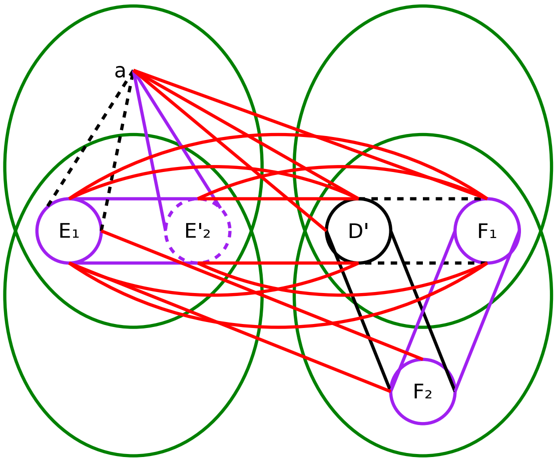


Figure 4.24: Obtaining $H = (E, F)$ if $G \cong (\Gamma_\infty, \Gamma_\infty)$ minimally omits $(K_1, D)^1$. The top amalgamand embeds because it is monochromatic red, and the bottom amalgamand embeds by induction on $|E_2|$. $a$ cannot be red to anything in $F_2$ so must be blue to all of $F_2$, giving $H$.

$|C| = 1$. *Let $H = (E, F, R)$ be a finite 2-graph not embedding any element of $\mathcal{A}$, and let $E_1, E_2, F_1, F_2$ be defined as above. Suppose that, for any $H' = (E', F', R')$ such that if $E'_1, E'_2, F'_1, F'_2$ are defined analogously then either $|E'_2| < |E_2|$ or $|F'_2| < |F_2|$, $H'$ embeds in $G$. Then $H$ also embeds in $G$.*

*Remark.* The $|D| = 1$ case is essentially the same as the $|C| = 1$ case we present.

*Sketch proof.* Let $H'$ be a 2-graph whose left component is $(\{a\} + E_1) \sqcup E'_2)$, where $\{a\} \cup E'_2 \cong E_2$. whose right component is $D' \sqcup (F_1 \sqcup F_2)$, where $D'$ is such that $D' \boxplus K_1 \cong D$ and where every vertex of $D'$ is joined to all of $F_1$ and none of $F_2$, and where $H' \setminus (a, D)$ is correctly coloured and $a$ is red to all of $D' \cup F_1$. By induction, $H'$ embeds in $G$. If $a$ were red to any vertex in $F_2$ then $(K_1, D)^1$ would embed in $G$, which is not the case, so $a$ is blue to all of $D$ giving the required copy of $H$. We can see this expressed as an amalgamation diagram in Figure 4.24. □

We have therefore shown that any homogeneous 2-coloured $(\Gamma_\infty, \Gamma_\infty)$ 2-graph $G$ that minimally omits the elements of an antichain $\mathcal{A}$ of finite monochromatic-red 2-graphs, each with both components complete, must be the generic homogeneous 2-coloured $(\Gamma_\infty, \Gamma_\infty)$ minimally omitting precisely the elements of $\mathcal{A}$, thus proving Theorem 4.2 for this case.

## 4.6 All monochromatics embed - $r = s = \infty$

Let $G$ be a homogeneous 2-coloured 2-graph of the form $(A, B, R)$ where $A, B \cong \Gamma_\infty$ such that $G$ realises every $(C, D)^1$ and $(C, D)^2$ for any finite graphs $C$ and $D$. Clearly, if we can show that $G$ realises every 2-graph $H = (C, D, S)$ where $C$ and $D$ are finite graphs and $|S| = 2$, then $G$ **must** be the fully-generic 2-coloured $(\Gamma_\infty, \Gamma_\infty)$ 2-graph.

In fact we don't have to do this for all finite graphs $C$ and $D$ in this section, as we have done some of the work already. By using the copying argument, Theorem 2.13, if we can show that $G$ realises any 2-graph $(C, D, S)$ where $D$ is a finite graph, $|S| = 2$ and $C$ is a **basic** finite graph (specifically $P_3$, $\overline{P_3}$ or $K_n$ or $\overline{K_n}$ for some finite $n$), then $G$ will realise any 2-graph $(E, D, S')$ where $|S'| = 2$ and $E$ is any finite graph. This reduces the amount of work we have to do. Moreover, symmetry will allow us to deduce that $(\overline{C}, D, S)$ is realised whenever $(C, D, S)$ is. Hence we only need to show that $G$ realises $H = (C, D, S)$ whenever $D$ is a finite graph, $|S| = 2$ and either $C = K_n$ or $C = P_3$.

The strategy will be somewhat different than the strategies we have used in earlier cases. Specifically, although we will only use basic graphs on the *left* components of our finite 2-graphs, we will **not** simply consider basic graphs on the *right* components. The structure of $\Gamma_\infty$ gives us certain freedoms that allow us to assume that certain things exist that we could not do so easily if we were working in $\Gamma_r$ for some *finite* value of $r$. (For similar reasons, some of the proofs in Chapter 3 would be simpler in, for example, the $(K_\infty, \Gamma_\infty)$ case than in the $(K_\infty, \Gamma_r)$ case for finite $r$. In that case, of course, the $r < \infty$ case turned out to be simple enough for us to consider the $r < \infty$ and $r = \infty$ case together. This does not appear to be true in the $(\Gamma_r, \Gamma_s)$ case.)

Specifically, we will prove the following:

**Theorem 4.36.** *Let $G = (A, B, R)$ be a homogeneous 2-coloured 2-graph where $A, B \cong \Gamma_\infty$ such that $G$ realises every 2-graph of the form $(E, F)^i$ for all finite graphs $E$ and $F$ and for each $i \in \{1, 2\}$. Then $G$ realises all finite 2-coloured 2-graphs of the form $(C, D, S)$ where $D$ is a finite graph and $C$ is either $P_3$ or a finite complete graph.*

We prove Theorem 4.36 in stages: the case where $C = K_1$ is covered in section 4.6.1 (specifically, Lemma 4.37), the case where $C = K_2$ in section 4.6.3 and the case where $C = P_3$ in section 4.6.6. The remaining cases where $C$ is a complete graph are handled by induction on $|C|$; the proof is in section 4.6.5.

### 4.6.1 $|C| = 1$

The following lemma proves Theorem 4.36 when $C = K_1$.

**Lemma 4.37.** *Let $D = D_1 \cup D_2$ be a finite graph where $D_1 \cap D_2 = \varnothing$. Let $H = (a, D, R)$ be a 2-coloured 2-graph where, for each $d \in D$, $(a, d) \in R_i$ if and only if $d \in D_i$. Then $H$ embeds in $G$.*

*Proof.* We work by induction on $\min(|D_1|, |D_2|)$; if this minimum is zero then, by assumption, $H$ is already known to embed in $G$. We may assume, without loss of generality, that $|D_1| \leq |D_2|$.
Let $S_1, S_2, T_1, T_2, z$ be such that $S_1 \cong T_1 \cup \{z\} \cong D_1$, $S_2 \cup \{z\} \cong T_2 \cong D_2$ and

$$S_1 \cup S_2 \cup \{z\} \cong T_1 \cup T_2 \cup \{z\} \cong D$$

.

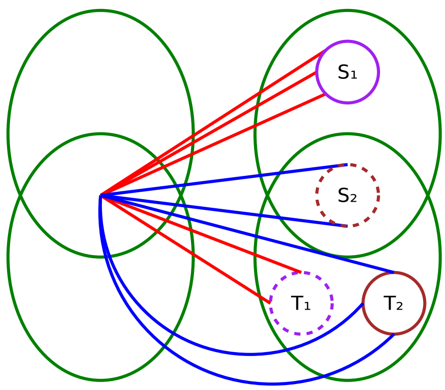


Figure 4.25: The key step in obtaining $(K_1, D)$ for any $D$ in $(\Gamma_\infty, \Gamma_\infty)$.

Amalgamate $(a, S_1 \cup S_2)$ with $(a, S_2 \cup T_1 \cup T_2)$ over their intersection $(a, S_2)$, as in Figure 4.25. This gives a 2-graph $H' = (a, S)$. Now $(a, S_1 \cup S_2)$ has one fewer blue than $H$, while $(a, S_2 \cup T_1 \cup T_2)$ has one fewer red than $H$, so both embed in $G$. Therefore we can certainly amalgamate the two. We need to avoid identifying any vertex of $S_1$ with any vertex of $T_1$ (we cannot identify vertices in $S_1$ with vertices

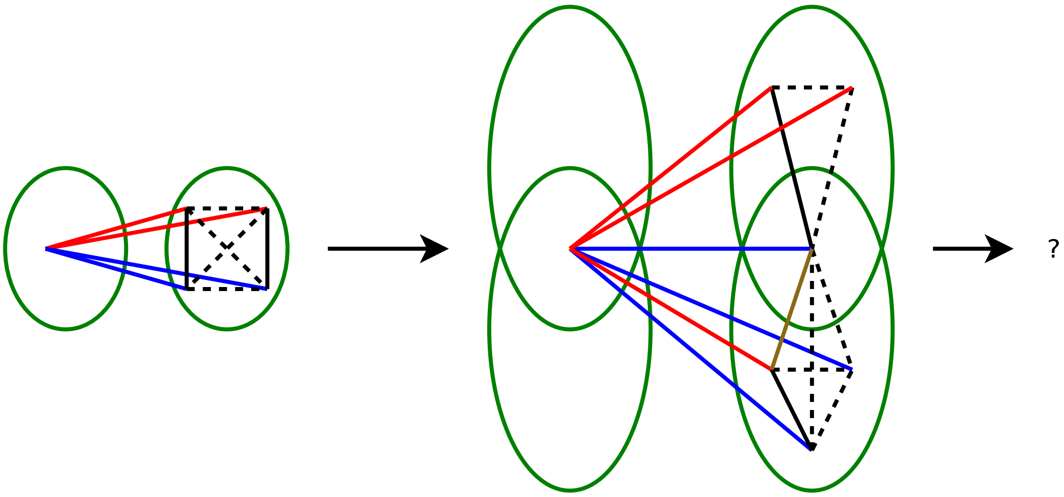


Figure 4.26: The problematic $s = t = 2$ case - if we use this method, whether we make the brown edge an edge or a non-edge we still end up identifying the bottom red point with one of the top points. (The other cases where $s = t = 2$ are straightforward.)

in $T_2$ since they are differently coloured to $a$). We do however have a free choice of how $S_2$ should be joined to $T_1 \cup T_2$. If there are more isomorphism types of $(x \cup S_2)$ for $x \notin S_2$ than there are vertices in $S_1$, one of them is not represented, so join every vertex in $T_1$ to $S_2$ using one of these unrepresented classes.
Let $s = |S_1|$ and $t = |T_2|$, so $|S_2| = t - 1$ and $|T_1| = s - 1$. Now there are $2^{t-1} \geq t$ labelled combinations of edges from a vertex in $S_1$ to $S_2$, and if $t > s$ or if $t = s \geq 3$ then there are strictly more than $s$ such combinations and can assign one such type from $S_2$ to the vertices of $T_1$. Similarly if $s > t$ we can swap $S_1$ with $T_2$ and $S_2$

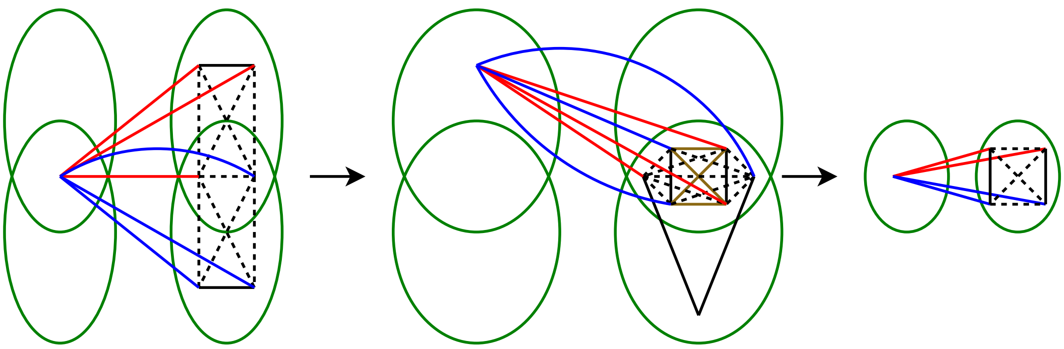

Figure 4.27: An ad-hoc way of dealing with the problematic case

and $T_1$ and get a similar result. So we are finished unless $s = t \geq 2$. Moreover, if $s = t = 1$ then either the top or the bottom is monochromatic so this problem does not arise. Hence we need only worry about the $s = t = 2$ case. That this is a problem can be seen in Figure 4.26, but we can handle this case in an ad-hoc way illustrated in Figure 4.27.
We now have $H' = (a, S)$, where

$$S = (S_1 \cup S_2) \cup (T_1 \cup T_2)$$

Amalgamate $H'$ with $(\varnothing, S \cup \{z\})$ over the intersection $(\varnothing, S)$. If $az$ is red, $(a, T_1 \cup T_2 \cup \{z\}) \cong H$, and if $az$ is blue then $(a, S_1 \cup S_2 \cup \{z\}) \cong H$. In either case we therefore obtain our required copy of $H$. □

### 4.6.2 General strategy for $|C| > 1$

Our strategy is to show, by induction on $n$, that $G$ realises every $H = (K_n, D, S)$ where $D$ is a finite graph and $|S| = 2$ (i.e. we show that if $G$ realises every $(K_m, D', S')$ where $m < n$, $D'$ is a finite graph and $|S'| = 2$ then it realises $H$) and then showing separately that $G$ realises every $(P_3, D, S)$ (where $D$ is a finite graph and $|S| = 2$). In this section we set out our plan for how we will do this; the details will occupy the remainder of section 4.6.
Now, for this $H$, let $D_1, \ldots, D_k$ partition $D$ such that:

- each $D_i$ is non-empty; and
- for all $x, y \in D$, $cx \cong cy$ for every $c \in C$ if and only if there exists $i$ such that $x, y \in D_i$.

(A word of warning. In later sections we will often, but *not* always, use a **different** ordering of the $D_j$ to the one we use here based on the colours from each vertex to $C$. We don't do that here as it would obscure the description of the strategy.)
We then work by induction on $k$. If $k = 1$ then either $H$ is monochromatic, in which case we are done, or at any rate each vertex in $D$ has the same pattern to $C$, a case which can be dealt with reasonably simply; we give this proof in Lemma 4.42.
Otherwise $k \geq 2$. There is an "easy" case and a "hard" case. (By "easy" we mean that it is relatively easy to reduce it to a smaller case, of course; the smaller cases may themselves be in the "hard" category.) The "easy" case for $C = K_n$ is

where, for some $i \neq j$ and for some labelling $c_1, \ldots, c_k$ of the vertices of $C$, for some (hence all) $x \in D_i$ and $y \in D_j$ we have that $c_1x \not\cong c_1y$ but there is a permutation $\sigma \in S(\{2, \ldots, k\})$ such that, for all $\iota$ such that $2 \leq \iota \leq k$, $c_\iota x \cong c_{\sigma(\iota)}y$. (There will be an "easy" case when $C = P_3$ but the permutation in that case has to take account of the graph structure; i.e. not all permutations will suffice.) If we are in the "easy" case, it will turn out to be easier to deduce the embeddability of $H$ in $G$ from graphs $H' = (K_n, D', S')$ with smaller values of $k$. The "hard" case is where there is no pair $(D_i, D_j)$ such that this property holds.
In the "hard" case we will order the $D_i$ by size (i.e. $|D_1| \leq |D_2| \leq \ldots \leq |D_k|$) and typically work with $D_1$ and $D_2$ (either reducing $|D_1|$ and possibly increasing $|D_2|$, or reducing $|D_2|$ while keeping $|D_1|$ constant). If there is a non-empty $D_3$ (and hence $|D_3| \geq |D_2| \geq |D_1|$), then we will add a vertex to $D_3$ while reducing either $|D_1|$ or $|D_2|$. This will eventually reduce the number of non-empty sets $D_i$ until there are only two remaining. Each step in this reduction involves three amalgamations:

1. add another copy of $C \setminus \{a\}$ to the left component, controlling only the colours to the vertex added to $D_3$;

2. form two copies of the $(D_i)$ by amalgamating over the enlarged left component and the single vertex added to $D_3$; and

3. amalgamate over a 2-graph which we obtain using Lemma 4.41 so that adding a red to the long diagonal uses one copy of $C \setminus \{a\}$ and the $(D_i)$ and adding a blue uses the other copies.

This leaves the case where there are precisely two non-empty sets $D_i$. In this case the induction works by **either** reducing $|D_1|$ and almost doubling $|D_2|$, or by reducing $|D_2|$ and leaving $|D_1|$ fixed. Each step in this reduction involves three amalgamations:

1. add another copy of $C \setminus \{a\}$ to the left component, controlling the colours to $D_2^1$ (a copy of $D_2$ with one vertex deleted) – the right components are either $D_1^1 \sqcup D_2^1$ (where $D_1^1$ is a copy of $D_1$) or $D_2^1 \sqcup D_1^2 \sqcup D_2^2$ (where $D_1^2$ is a copy of $D_1$ with one vertex deleted, and $D_2^2$ is a copy of $D_2$);

2. form two copies of $(D_1, D_2)$ by amalgamating over the enlarged left component and $D_2^1$; and

3. amalgamate over a 2-graph which we obtain using Lemma 4.41 so that adding a red to the long diagonal uses one copy of $C \setminus \{a\}$ and the $(D_i)$ and adding a blue uses the other copies.

We make this more precise and fill in the details in the following sections.
Before we do this we make a final notational remark: in the statement of several of the lemmas that follow, we will say that $H = (C, D, R)$ embeds in $G$ if every 2-graph $H' = (C', D', R')$ that satisfies certain properties embeds in $G$. In the proof of these lemmas, the graph $C$ is fixed in advance and so the proof is only for this value of $C$. We will refer to the family $(D'_j)$ where each $D'_j$ is the analogue of $D_j$ in $D'$ (i.e. vertices of $D'_j$ are coloured to $C$ the same way as vertices of $D_j$ are).

### 4.6.3 Details for $C = K_2$

We now show that $G$ realises every 2-coloured $H = (C, D, R)$ where

$$C = (\{a, b\}, \{ab\})$$

and $D$ is a finite graph. We will do this in rather more detail than we will do later in cases where $C \cong K_n$ for $n \geq 3$.
We first need to obtain the "base case" of the induction, namely the cases where $D = K_2$ and $D = \overline{K_2}$.

**Lemma 4.38.** *$G$ realises every 2-coloured 2-graph $H$ of the form $(K_2, K_2)$.*

*Proof.* We may assume without loss of generality that a vertex on the left of $H$ has both a red and a blue cross-edge from it. Use the amalgamations shown in Figure 4.28. Note that the four initial 2-graphs are of the form $(D, K_1)$ or $(K_1, D)$ so embed in $G$ by Lemma 4.37. □

Let

$$D = \bigcup_{i,j} D_{2i+j}$$

where $j+1$ is the colour from every vertex of $D_{2i+j}$ to $a$ and $i+1$ is the colour from every vertex of $D_{2i+j}$ to $b$.

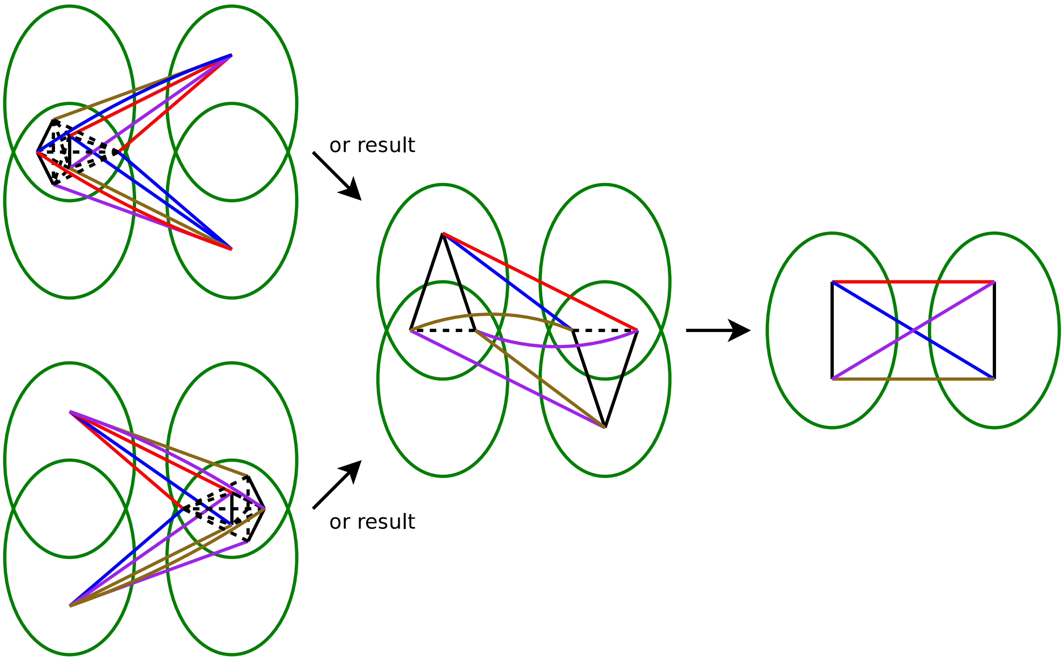


Figure 4.28: The amalgamations needed to obtain $H \cong (K_2, K_2)$ in $G$. The colours of cross-edges not shown are unimportant as long as they are consistent, and they can easily be chosen to make them consistent.

We first state the "easy" case. We give the statement in a form that works for larger values of $n$ (and, with a little extra care, can be made to work also for $C \cong P_3$ as we will see later in section 4.6.6).

**Lemma 4.39.** *Let $H$ be a 2-graph $(C, D, R)$ where*

$$C = (\{a_0, \dots, a_{n-1}\}, \{a_i a_j : 0 \le i < j \le k-1\})$$

*where $D_0, \dots, D_{2^n-1}$ are as above and where, for some $p$ and $q$ differing in precisely one binary digit, $1 \le |D_p| \le |D_q|$. Then $G$ realises $H$ if $G$ realises every $(K_n, D', R')$ such that $|D'_r| = 0$ whenever $|D_r| = 0$ and either $|D'_p| < |D_p|$ or $|D'_p| = |D_p|$ and $|D'_q| < |D_q|$.*

*Proof in the $n = 2$ case.* Without loss of generality, we may assume that $p = 0$ and $q = 1$.
It is sufficient to find 2-graphs

$$H_1 \cong (\{a, b\}, D_0^1 \sqcup D_1^1 \sqcup D_0^2 \sqcup D_1^2 \sqcup D_2^1 \sqcup D_3^1 \sqcup D_2^2 \sqcup D_3^2)$$

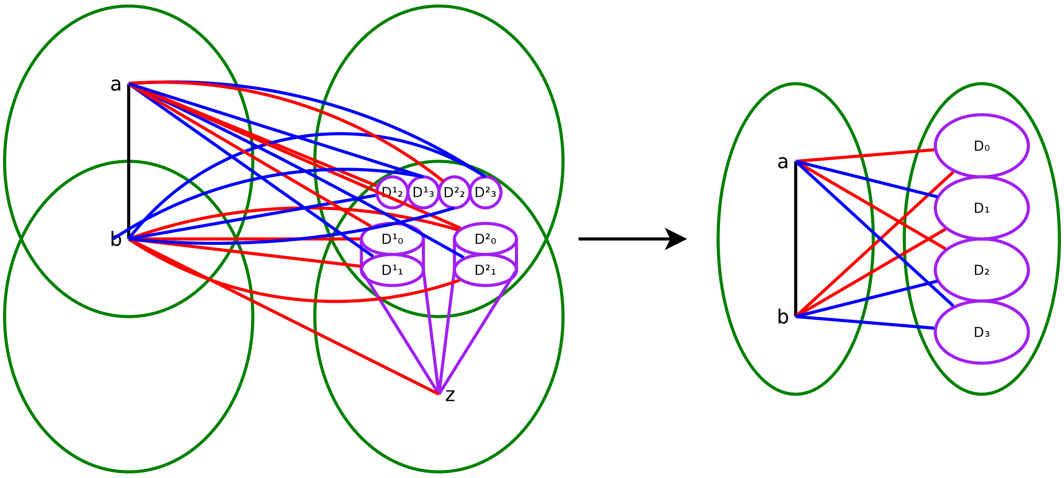


Figure 4.29: The final amalgamation needed for Lemma 4.39.

and

$$H_2 \cong (\{b\}, D_0^1 \sqcup D_1^1 \sqcup D_0^2 \sqcup D_1^2 \sqcup D_2^1 \sqcup D_3^1 \sqcup D_2^2 \sqcup D_3^2 \sqcup \{z\})$$

embedding in $G$, where

$$D_0^1 \cong D_0^2 \cup \{z\} \cong D_0$$

and

$$D_1^1 \cup \{z\} \cong D_1^2 \cong D_1$$

and

$$D_0^1 \cup D_1^1 \cup D_2^1 \cup D_3^1 \cup \{z\} \cong D_0^2 \cup D_1^2 \cup D_2^2 \cup D_3^2 \cup \{z\} \cong D$$

and where the colours are such as to ensure that when we amalgamate $H_1$ with $H_2$ over their intersection we obtain $H$ whether $az$ is red or blue. We see this amalgamation in Figure 4.29.

Showing that $H_2$ embeds in $G$ is straightforward; we have it as a consequence of Lemma 4.37. So we only need to show that $H_1$ embeds in $G$.

In the typical case, we let

$$H_3 = H_1 \setminus (D_0^1)$$

and

$$H_4 = H_1 \setminus (D_0^2 \cup D_1^2)$$

which **do** embed in $G$ (by assumption), and amalgamate these over their intersection, as in Figure 4.30. In **most** cases there are enough choices of edge-types between $D_1^1$

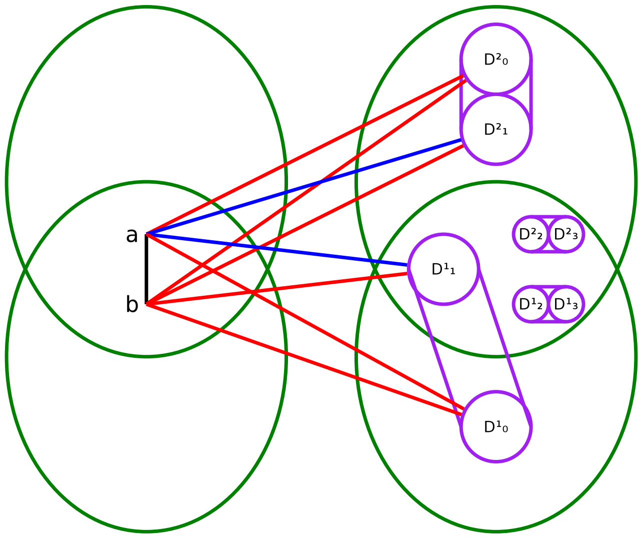


Figure 4.30: The amalgamation needed to obtain $H_1$ for Lemma 4.39.

and $D_0^2$ to ensure that we can choose one that avoids identifying any vertices, and any choice of edge-types across the amalgamation gives a satisfactory $H_1$.
When are there **not** enough edge-types? Recall that we assumed that $|D_0| \leq |D_1|$, and that at most $|D_0|$ edge-types are realised from $D_1^1 \to D_0^1$ while there are $2^{|D_1^1|}$ acceptable types from $D_1^1$ to $D_0^2$. If $|D_0| < |D_1|$ then $2^{|D_1^1|} > |D_0^1|$, and the same applies if $|D_0| = |D_1| \geq 3$. If $|D_0| = |D_1| = 1$ then $D_0^2$ will be empty so there will be no question of identifying. Thus the only problematic case is when $|D_0| = |D_1| = 2$. If in practice we have two edges or two non-edges from $D_1^1$ to $D_0^1$ then we use the other type from $D_1^1$ to $D_0^2$ and avoid identifying that way. Otherwise we have an edge and a non-edge. We can handle these using a separate amalgamation argument that we deal with in the next section in Lemma 4.43. □

Lemma 4.39 disposes of the "easy" instances of the $C = K_2$ (and $C = \overline{K_2}$) case. The remaining cases are thus (up to equivalence):

1. only $D_1$ is non-empty;
2. only $D_1$ and $D_2$ are non-empty; and
3. only $D_0$ and $D_3$ are non-empty.

When only $D_1$ is non-empty, it is easy to reduce $|D_1|$ inductively. We see how this is done diagrammatically in Figure 4.31. Note that every 2-graph in the diagram is a valid part of a derivation of $H$; either it is the result of amalgamating structures earlier in the diagram, or it follows from the induction hypothesis, or it embeds in $G$ because of Lemma 4.37. We come back to this idea for larger graphs $C$ in Lemma 4.42.

This leaves the cases where $D_1$ and $D_2$ are the only non-empty $D_i$ (and both are non-empty), and the cases where $D_0$ and $D_3$ are the only non-empty $D_i$ (again, both are non-empty; if only one is non-empty then $H$ is monochromatic and so already known to embed in $G$).

Suppose that $D_0, D_3 \neq \varnothing$ and $D_1 = D_2 = \varnothing$. Further suppose without loss of generality that $|D_0| \leq |D_3|$. We will aim to **either** reduce $|D_0|$ **or** reduce $|D_3|$ while keeping $|D_0|$ fixed (i.e. to move closer to the monochromatic situation). More precisely, we show the following:

**Lemma 4.40.** *Let $H = (C, D, R)$ be such that $C = K_2$ and $D = D_0 \cup D_3$ (where $C$ is red to all of $D_0$ and blue to all of $D_3$), where $|D_0| \leq |D_3|$. Suppose that $G$ realises every $H' = (C, D', R')$ where $D = D'_0 \cup D'_3$, where $C$ is red to all of $D'_0$ and blue to all of $D'_3$, and where*

- *either $|D'_0| < |D_0|$;*
- *or $|D'_0| = |D_0|$ and $|D'_3| < |D_3|$.*

*Then $G$ realises $H$.*

*Proof.* We work by induction, the inductive step being shown in Figure 4.32 (note that the steps that require us to assume that certain structures of the form $(P_3, E)$ embed **are** valid, as these structures are monochromatic), and the base case ($|D_0| = 0$) being trivial. Note that since

$$2^{|D_3^2 - 1|} \geq |D_3^2| \geq |D_0^1|$$

then unless $|D_0^1| = |D_3^2| = 2$ there are sufficient possible types from $D_3^1$ to each vertex of $D_0^2$ that we can assign one that doesn't clash with the types from $D_3^1$ to the vertices of $D_0^1$. Also note that the induction is valid - if we write $k = |D_0|$ and $l = |D_3|$, then we effectively obtain $(k, l)$ from $(k, l-1)$ (which is fine even if $k = l$) and from $(k-1, 2l-1)$ (which again is fine by the statement of the lemma).

The special case is where $k = l = 2$; in fact, it can be seen that this case is only a problem if each vertex in $D_0$ has an edge and a non-edge to $D_3$ and each vertex in

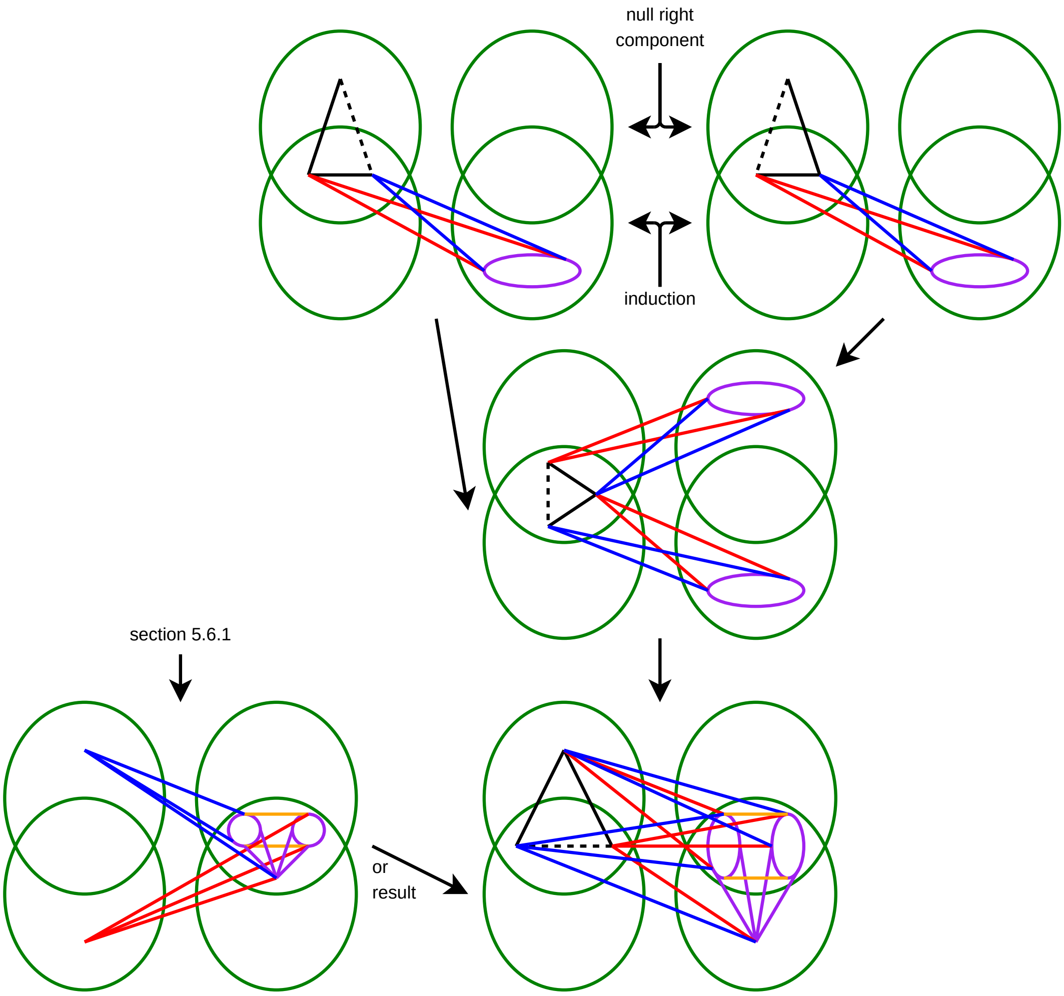


Figure 4.31: The inductive step in the procedure for obtaining $H = (C, D, R)$ if $D = D_1$.

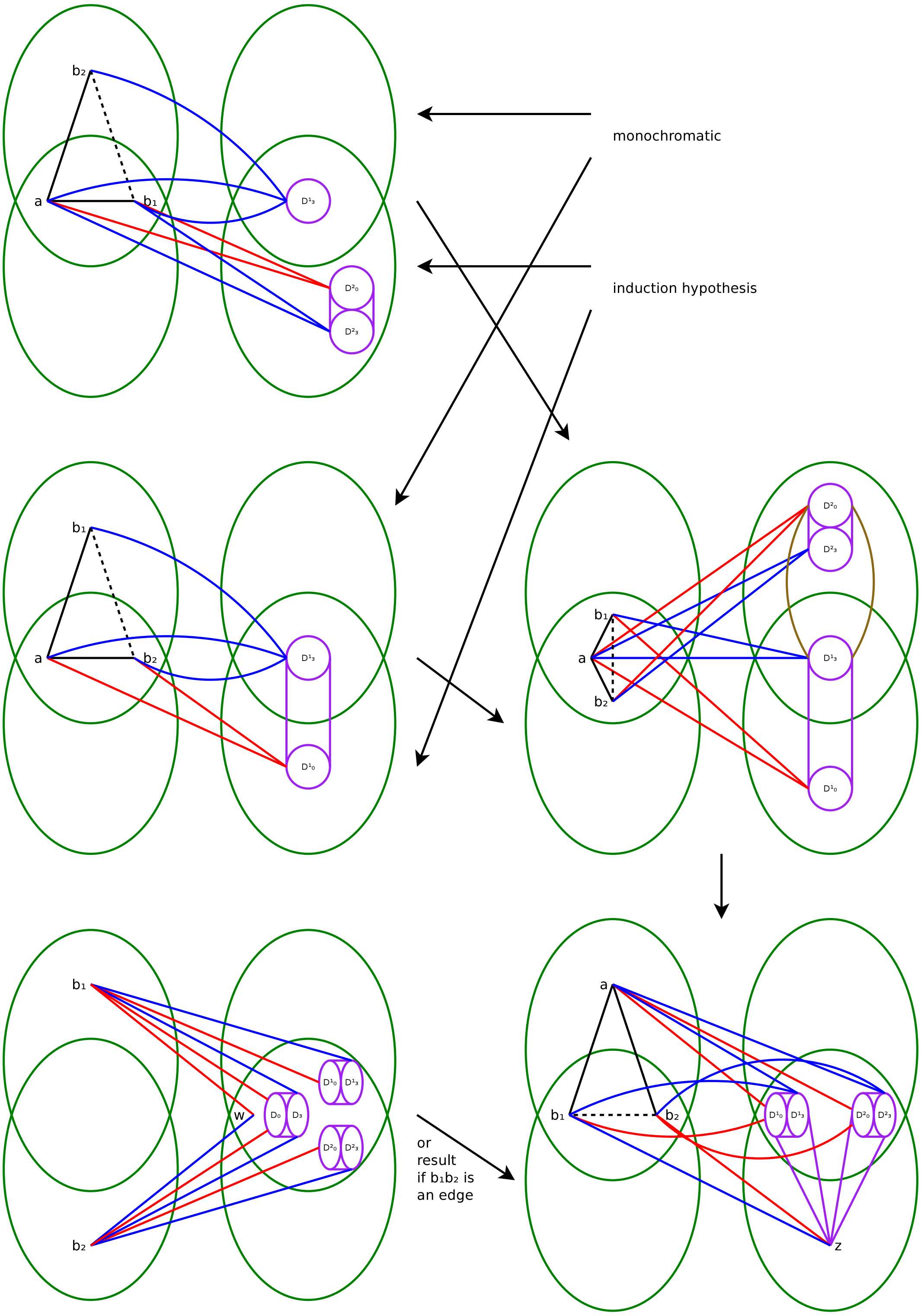


Figure 4.32: The inductive step for Lemma 4.40.

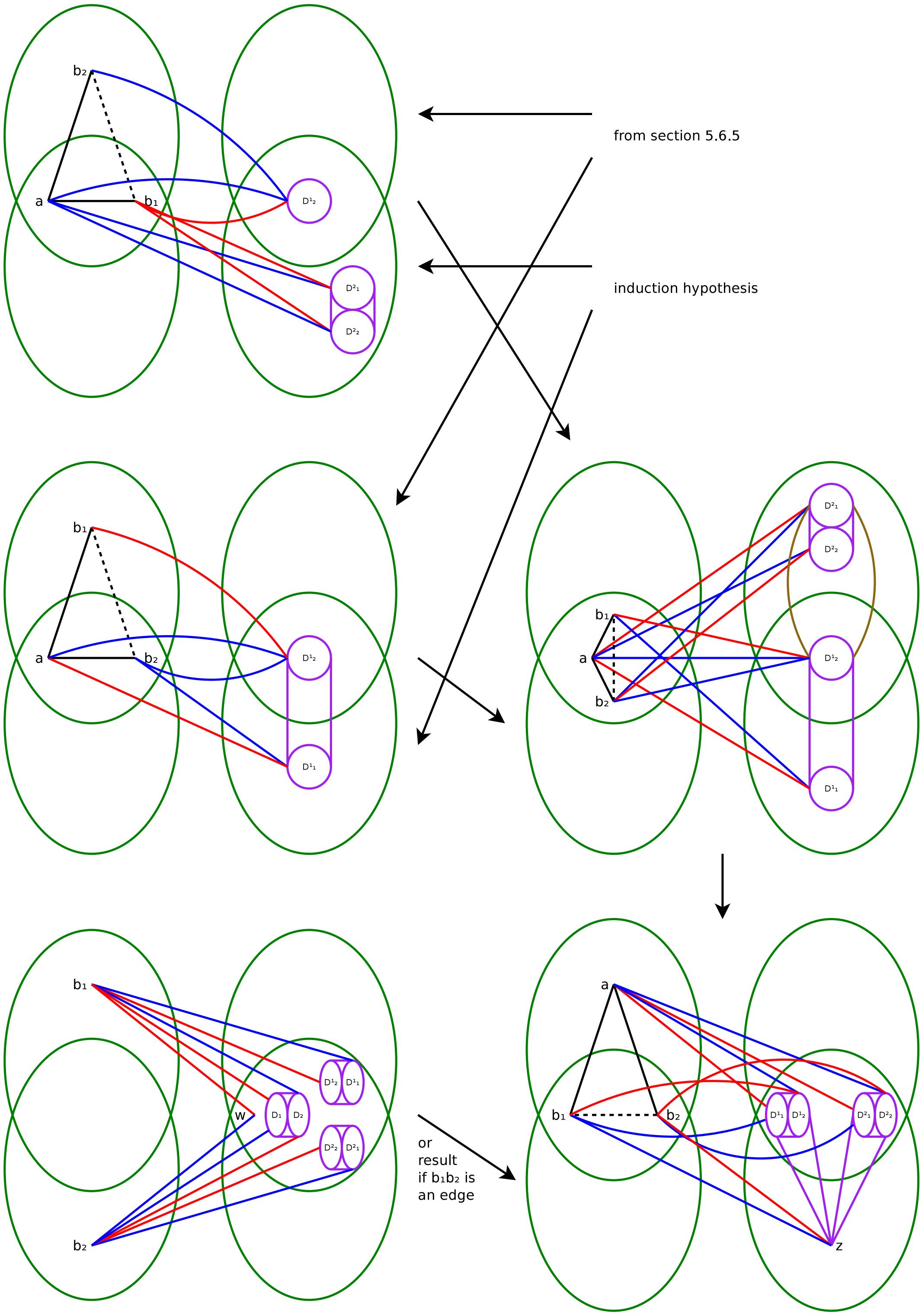


Figure 4.33: The induction step if $D_1, D_2 \neq \varnothing$, similar to the one in Figure 4.32. To make this work we need to use Lemma 4.42.

$D_3$ has an edge and a non-edge to $D_0$. But this special case is dealt with in the next section in Lemma 4.43. □

For the $D_1, D_2 \neq \varnothing$ case, assume that $|D_1| \leq |D_2|$. We wish to use the sequence of amalgamations shown in Figure 4.33. The base case is where $|D_1| = 0$, and we handled this using Figure 4.31. There is a structure $\hat{H}$ of the form $(P_3, D_2^1)$ (where all cross-edges are determined) in this sequence that requires care to prove that it does embed in $G$. It is, however, possible to produce $\hat{H}$ without any circularity, and we describe how to do this in Lemma 4.42. Again there is a special case that we will deal with in Lemma 4.43.

### 4.6.4 Two special cases

There are two special cases which arose in section 4.6.3 and need to be dealt with. As they will arise again in sections 4.6.5 and 4.6.6 it is prudent to deal with them here. We also need a lemma about the "bottom amalgamand" in the final step in each amalgamation (and specifically about the left component of this amalgamand) which we now present.

**Lemma 4.41.** *Let $H = (C, D, R)$ where $C$ is isomorphic to one of $K_n$, $P_3$, $K_n[\overline{K_2}]$ or either of*

- *$(\{a, b, c, d, e\}, \{ab, ac, ad, ae\})$, or*
- *$(\{a, b, c, d, e\}, \{ab, ad, bc, bd, be, cd, ce, de\})$.*

*Let $D_1$ and $D_2$ be proper substructures of $D$, and let $D'$ be a disjoint union of $D_1$ and $D_2$. There is a 2-graph*

$$H' = \big(C_1 \sqcup C_2, D' \sqcup \{z\}, R'\big)$$

*such that:*

- *$(C_1, D_1 \cup \{z\})$ and $(C_2, D_2 \cup \{z\})$ are proper substructures of $H$;*
- *a vertex $a$ can be added to $C_1$ in such a way that if $az$ were red then $(C_1 \cup \{a\}, D_1 \cup \{z\}) \cong H$;*
- *a vertex $a'$ can be added to $C_2$ in such a way that if $az$ were red then $(C_2 \cup \{a'\}, D_2 \cup \{z\}) \cong H$; and*

- *$G$ realises either $H$ or $H'$.*

*Remark.* The idea here is similar to that of Lemma 4.12. Note that the key point is that the graph we get on the left, if not $C$, is meant to be determined only by $C$ and the choice of substructures $C_1$ and $C_2$, or if there is any arbitrariness we can at least make the same choice every time we have these particular graphs.

*Proof in the case $C = K_n$.* The idea of the proof is to obtain $H'$ while maintaining some control over the graph structure by adding one point of $C_2$ at a time to $C_1$, in such a way that we either obtain $H$ or get successively closer to $H'$. At stage $i$ the points not amalgamated over will be $c_i$ and $d_i$ in such a way as to yield $H$ immediately if $c_i d_i$ amalgamates to an edge, and so that after the $(n-1)$th stage we will either have $H$ or a suitable $H'$ (with **known** graph on the left component). For the graphs on the left, we begin by amalgamating $K_{n-1}$ with $K_{n-1}$ over $K_{n-2}$; this will either produce $K_n$ or a graph

$$H_1 \cong \overline{K_2 + \overline{K_{n-2}}}$$

(we will describe later how we avoid identifying any pairs of vertices.) Then amalgamate $H_1$ with itself over a subgraph of size $n-1$ containing $a$ and $b$. Again we obtain either $K_n$ or

$$H_2 \cong \overline{2K_2 + \overline{K_{n-3}}}$$

At stage $i$ we amalgamate $H_{i-1}$ over $H_{i-1}$ which gives us $K_n$ or

$$H_i \cong \overline{iK_2 + \overline{K_{n-i}}}$$

and eventually we get a graph

$$H_{n-1} \cong \overline{(n-1)K_2} \cong K_{n-1}\left[\overline{K_2}\right]$$

(or $K_n$), so we have the required left component for the lemma.

To avoid identifying and to ensure we have the correct right components and cross-edge colours, we work in the other direction (i.e. we work out what right component we need at the last stage and propagate this inductively to earlier stages). If we require $X$ on the right with colours $R$ if we have $K_n$ on the left, and $Y$ on the right with colours $S$ otherwise, we let

$$H^1_{n-2}, H^2_{n-2} \cong (H, \{z\} + X + Y)$$

with cross-edge colours chosen to avoid identifying (by making $z$ red to one of the points not amalgamated over and blue to the other) and to ensure the required

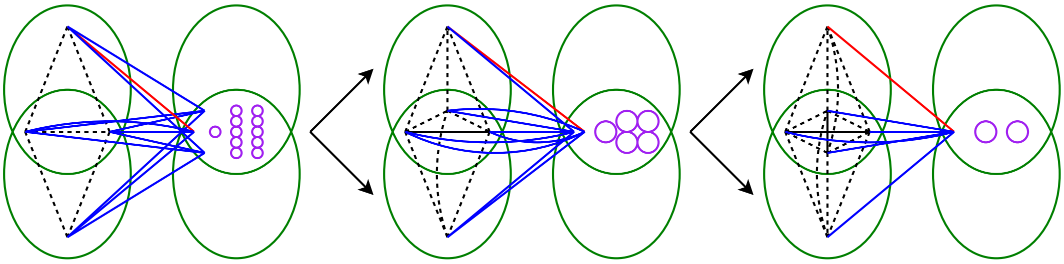

Figure 4.34: Obtaining $C = \overline{K_4}$ or $C' = \overline{K_3}[K_2]$ in Lemma 4.41. We show this for an **empty** graph rather than a complete one for clarity.

results. Similarly at the $i$th stage there will be a graph $Y_i$ and colours $R_i$ that feed into the next stage if we add a non-edge, and so let

$$H_i^1, H_i^2 \cong (H, \{z\} \boxplus X + Y_i)$$

with cross-edge colours again chosen to feed in correctly and to avoid identifying. This process proceeds downwards to $H_0^1$ and $H_0^2$ which are of the form $(K_{n-1}, D)$ for some graph $D$ so embed in $G$ by induction on $n$. □

*Sketch proof in the case $C \neq K_n$.* Suppose now that $|V(C)| = n$ but that $C$ is not complete (and we may assume that $C$ is not empty either). Let $a_1, a_2 \in C$ and let $C_1, C_2$ be such that $C_1$ and $C_2$ are not isomorphic to $C$ but $C_1 \sqcup \{a_1\}$ and $C_2 \sqcup \{a_2\}$ are. We will find a graph $C' = C_1 \sqcup C_2$ such that the choice of edges between (the copies of) $C_1$ and $C_2$ does not depend on the right-hand side of any amalgamation arguments, and such that $G$ realises either $H$ or the desired $H' = (C', D', R')$. The method is similar to the method in the $C = K_n$ case; here we need only describe what we need to do to obtain $C'$ in such a way that its structure is independent of $D$, $D'$, $R$ and $R'$. The reader is warned that we have had to assign rather idiosyncratic labels for the graphs we use and they typically do not fit into coherent series.
We need only consider cases where $C \cong K_n[\overline{K_2}]$ or where $C$ is isomorphic to one of three specific graphs.
We handle the most symmetric case first. If $C \cong K_n[\overline{K_2}]$ then

$$C_1, C_2 \cong K_{n-1}[\overline{K_2}] \boxplus K_1$$

This case has certain symmetries so we describe how to build it in a similar manner to how we dealt with the $C = K_n$ case. Again we only mention how the amalgamations work on the left component. Let

$$H_0 \cong K_{n-1}[\overline{K_2}]$$

and

$$H_1 \cong K_{n-1}[\overline{K_2}] \boxplus K_1$$

and begin by amalgamating $H_1$ with itself over $H_0$; if we get an edge this is $C$, otherwise the product is

$$H_2 \cong \overline{K_{n-1}[\overline{K_2}] + \overline{K_2}}$$

Similarly we amalgamate this with itself over $H_1$ and obtain $C$ or

$$H_3 \cong \overline{K_{n-2}[\overline{K_2}] + P_3 + K_1}$$

In general

$$H_{2m} \cong \overline{K_{n-m}[\overline{K_2}] + (m-1)P_3 + \overline{K_2}}$$

and

$$H_{2m+1} \cong \overline{K_{n-m-1}[\overline{K_2}] + mP_3 + K_1}$$

so we eventually obtain either $C$ or a graph containing two copies of $C_1$ disjointly, as required.

We now handle the three special cases. We will make assumptions based on at what point in the later sections we need Lemma 4.41 for that particular value of $C$.

If $C \cong P_3$ we need to consider three cases, which we describe in terms of the amalgamations on the left component. (If we obtain $P_3$ at any stage we can stop as this is one of the two cases we are trying to reduce to.)

1. $C_1 \cong C_2 \cong K_2$: amalgamate $K_2$ with $K_2$ over $K_1$, then $K_3$ with $K_3$ over $K_2$;

2. $C_1 \cong C_2 \cong \overline{K_2}$: amalgamate $K_2$ with $\overline{K_2}$ over $K_1$, then $wxy$ with $xyz$ over $xy$ where the edges are $wx, yz$; and

3. $C_1 \cong K_2; C_2 \cong \overline{K_2}$: amalgamate $K_2$ with $\overline{K_2}$ over $K_1$, then $wxy \cong K_2 + K_1$ with $xyz \cong K_2 + K_1$ over $xy$ where the edges are $wx, xz$.

If

$$C = (\{a, b, c, d, e\}, \{ab, ac, ad, ae\}) \cong F_5$$

then $C_1$ and $C_2$ are each isomorphic to either $\overline{K_4}$ or to

$$F_4 = (\{a, b, c, d\}, \{ab, ac, ad\})$$

and we describe the three cases in terms of the amalgamations on the left component. Note that since we only need this $C$ if we are trying to show that $(P_3, D, R)$ is realised, and it will turn out that at that part of the argument we can assume that $(K_n, D', R')$ and $(\overline{K_n}, D', R')$ embed for every $D'$ and $R'$, we can simplify some of our arguments.

1. $C_1 \cong C_2 \cong \overline{K_4}$: $(\overline{K_8}, D, R)$ is known to embed so there is nothing to prove in this case;
2. $C_1 \cong F_4, C_2 \cong \overline{K_4}$: amalgamate $F_4$ with $\overline{K_4}$ over $\overline{K_3}$, then $F_4 + K_1$ with $\overline{K_5}$ over $\overline{K_4}$, then $F_4 + \overline{K_2}$ with $\overline{K_6}$ over $\overline{K_5}$, then $F_4 + \overline{K_3}$ with $\overline{K_7}$ over $\overline{K_6}$. (The $(\overline{K_n}, D)$ are known to embed so this sequence is indeed valid.)
3. $C_1 \cong C_2 \cong F_4$: by the previous argument we have $(F_4 + \overline{K_3}, D', R')$ or $(F_5, D', R')$ for any desired $D'$ and $R'$, and if we obtain $F_5$ there is nothing further to prove, so assume that we obtain $F_4 + \overline{K_3}$ and amalgamate it with itself over $\overline{K_6}$ (where every vertex in the overlap has an edge to precisely one vertex outside); this gives $F_5$ or $F_4 + F_4$.

If

$$C = (\{a, b, c, d, e\}, \{ab, ae, bc, bd, be, cd, ce, de\}) \cong E_5$$

then $C_1$ and $C_2$ are each isomorphic to either $K_4$, to

$$D_4 \cong (\{a, b, c, d\}, \{ab, bc, bd, cd\})$$

or to

$$E_4 \cong (\{a, b, d, e\}, \{ab, ae, bd, be, de\})$$

and we describe the six cases in terms of the amalgamations on the left component. (If we obtain $E_5$ at any stage we stop.) Again we can assume that every $(K_n, D', R')$ embeds since we only need this $C$ to obtain $(P_3, D)$ at a point where we have already shown that every $(K_n, D', R')$ embeds.

1. $C_1 \cong C_2 \cong K_4$: $(K_8, D, R)$ is already known to embed;
2. $C_1 \cong K_4, C_2 \cong D_4$: amalgamate $D_4$ with $K_4$ over $K_3$, then the product with $K_5$ over $K_4$, then the product with $K_6$ over $K_5$, then the product with $K_7$ over $K_6$;
3. $C_1 \cong K_4, C_2 \cong E_4$: amalgamate $E_4$ with $K_4$ over $K_3$, then $E_4 \boxplus K_1$ with $K_5$ over $K_4$, then $E_4 \boxplus K_2$ with $K_6$ over $K_5$, then $E_4 \boxplus K_3$ with $K_7$ over $K_6$;

4. $C_1 \cong C_2 \cong D_4$: either we have $E_5$ or we have a graph $D_7$ containing $K_3$ and $D_4$ disjointly; amalgamate two copies of this over $K_6$ so that the two copies have their embedded copies of $D_4$ use disjoint subsets of the overlap;

5. $C_1 \cong C_2 \cong E_4$: either we have $E_5$ or we have a graph $E_7$ containing $K_3$ and $E_4$ disjointly; amalgamate two copies of this over $K_6$ so that the two copies have their embedded copies of $E_4$ use disjoint subsets of the overlap; and

6. $C_1 \cong D_4, C_2 \cong E_4$: we either have $E_5$ or we have $D_7$ and $E_7$; amalgamate these over $K_6$ so that the $E_4$ in $E_7$ uses a disjoint $K_3$ from the $D_4$ in $D_7$.

□

**Lemma 4.42.** *Let $H = (C, D, R)$ where, for all $x, y \in D$ and all $a \in C$, $ax \cong ay$. Then $H$ embeds in $G$.*

*Remark.* We only prove this result for those graphs $C$ for which we proved Lemma 4.41. If that were proved for all finite graphs $C < \Gamma_\infty$ then this proof of Lemma 4.42 would work for every finite $C$.

*Proof.* We may assume that any $\tilde{H} \cong (C', D')$ (for any $C' < C$ and any finite graph $D'$) embeds in $G$. Moreover, we may assume that any $\hat{H} \cong (C, D'')$ (for any graph $D''$ such that $|D''| < |D|$) embeds in $G$; the case where $|D| = 1$ was done by Lemma 4.37. We require

$$H_1 = (\{a\} \sqcup C_1 \sqcup C_2, D_1 \sqcup D_2)$$

and

$$H_2 = (C_1 \sqcup C_2, D_1 \sqcup D_2 \sqcup \{z\})$$

where

- $\{a\} \sqcup C_1 \cong \{a\} \sqcup C_2 \cong C$;
- $\{z\} \sqcup D_1 \cong \{z\} \sqcup D_2 \cong D$;
- $a$ is red to all of $D_1$ and blue to all of $D_2$; and
- if $az$ is red then $(\{a\} \sqcup C_1, \{z\} \sqcup D_1) \cong H$ while if $az$ is blue then $(\{a\} \sqcup C_2, \{z\} \sqcup D_2) \cong H$.

If these embed them we amalgamate $H_1$ with $H_2$ over $H_1 \cap H_2$ and will then inevitably obtain $H$ embedding in $G$.

Now, whatever the other colours and whatever the edge types on the right may be, $H_2$ embeds in $G$ by Lemma 4.41. So we need only concern ourselves with constructing $H_1$.
Let

$$H_3 = (\{a\} \sqcup C_1 \sqcup C_2, D_1)$$

and

$$H_4 = (\{a\} \sqcup C_1 \sqcup C_2, D_2)$$

Now $H_3 \setminus C_2$ and $H_4 \setminus C_1$ embed in $G$ by induction, and since $G$ is generic, it follows that $H_3$ and $H_4$ will embed for some (unknown) choice of cross-edge colours wherever not specified. So amalgamate these over their intersection

$$(\{a\} \sqcup C_1 \sqcup C_2, \varnothing)$$

and note that, since $a$ is differently coloured to $D_1$ and $D_2$, we cannot identify any pairs of vertices, so $D_1$ and $D_2$ embed in $G$ disjointly (with unknown edges between them), and this is sufficient for $H_1$ to also embed. □

**Lemma 4.43.** *Let $H = (C, D, R)$ where:*

- *$C \cong K_n$ for some $n$ or $C \cong P_3$;*
- *$D = D_1 \sqcup D_2$;*
- *$|D_1| = |D_2| = 2$ (we write $V(D_1) = \{x_1, y_1\}$ and $V(D_2) = \{x_2, y_2\}$);*
- *for all $i$ and all $a \in A$, $ax_i \cong ay_i$;*
- *there is $a \in A$ such that $ax_1 \ncong ax_2$; and*
- *$x_1x_2$ and $y_1y_2$ are edges and $x_1y_2$ and $y_1x_2$ are non-edges.*

*Then $H$ embeds in $G$.*

*Proof.* We require

$$H_1 = (\{a\} \sqcup C_1 \sqcup C_2, \{x_1^1, y_1^1, y_2^1, x_1^2, x_2^2, y_1^2\})$$

and

$$H_2 = (C_1 \sqcup C_2, \{x_1^1, y_1^1, y_2^1, x_1^2, x_2^2, y_1^2, z\})$$

where:

- $\{a\} \sqcup C_1 \cong \{a\} \sqcup C_2 \cong C$;
- $x_1^1 z \cong x_1^2 x_2^2 \cong x_1 x_2$;
- $y_1^1 y_2^1 \cong y_1^2 z \cong y_1 y_2$; and
- if $az$ is red then $(\{a\} \sqcup C_1, \{x_1^1, z, y_1^1, y_2^1\}) \cong H$ while if $az$ is blue then $(\{a\} \sqcup C_2, \{x_1^2, x_2^2, y_1^2, z\}) \cong H$.

That $H_2$ (or $H$) embeds in $G$ follows from Lemma 4.41.
For $H_1$, let

$$H_3 = (a \sqcup C_1 \sqcup C_2, \{x_1^1, y_1^1, y_2^1, y_1^2\})$$

and

$$H_4 = (a \sqcup C_1 \sqcup C_2, \{x_1^2, x_2^2, y_1^2, y_2^1\})$$

and note that if these embed in $G$ then we can amalgamate them over their intersection

$$(a \sqcup C_1 \sqcup C_2, \{x_1^1, y_1^2\})$$

to obtain $H_1$ as required.
It remains to obtain $H_3$ and $H_4$. We do $H_3$; the proof for $H_4$ is essentially the same. Let

$$H_5 = (a \sqcup C_1 \sqcup C_2, \{x_1^1, y_1^2\})$$

and

$$H_6 = (a \sqcup C_1, \{x_1^1, y_1^1, y_2^1, y_1^2\})$$

If we make $x_1^1 y_1^2$ an edge then $H_5$ embeds by the argument for $(K_2, D)$ with the sides reversed as long as $C > K_2$; if $C = K_2$ then we note that we have enough of the argument to give us every $(P_3, K_2)$ so $H_5$ embeds. For $H_6$ we note that we have an instance of $(C, D')$ where $D'$ is a four vertex graph that is **not** the awkward case, and so it embeds by the arguments we use for whichever value of $C$ we work in. Hence $H_3$ embeds in $G$, and so does $H_4$ by similarity, and *a fortiori* $H_1$ and $H$. □

### 4.6.5 Details for $C = K_n$ for $n \geq 3$

Suppose now that $C \cong K_n$ for some $n \geq 3$ and suppose that $V(C) = \{a_0, a_1, \ldots, a_{n-1}\}$. We will proceed by induction on $n$: that is, we will show:

**Theorem 4.44.** *Let $D$ be a finite graph, and let $H = (C, D, R)$ be a 2-coloured 2-graph. Suppose that, if $H' = (C', D', R')$ for any finite graph $D'$ and for $C' \cong K_{n'}$ where $n' < n$, then $H'$ embeds in $G$. Then $H$ also embeds in $G$.*

Let $D_0, \ldots, D_{2^n-1}$ partition $D$ in such a way that, for each $i$ and $j$, $a_i$ is blue to all vertices of $D_j$ if the $i$th binary digit of $j$ is 1; that is, if

$$\lfloor j/2^i \rfloor \equiv 1 \pmod 2$$

and $a_i$ is red to all other vertices of $D$.

In section 4.6.2 we stated that there is an "easy" case and a "hard" case. The easy case is the $n \geq 3$ version of Lemma 4.39, which we state again for convenience. As the proof is similar to how it was when $n = 2$, we will merely sketch the points where there are differences.

**Lemma 4.39** Let $H$ be a 2-graph $(C, D, R)$ where

$$C = (\{a_0, \ldots, a_{n-1}\}, \{a_i a_j : 0 \leq i < j \leq k-1\})$$

where $D_0, \ldots, D_{2^n-1}$ are as above and where, for some $p, q$ differing in precisely one binary digit, $1 \leq |D_p| \leq |D_q|$. Then $G$ realises $H$ if $G$ realises every $(K_n, D', R')$ such that $|D'_r| \leq |D_r|$ whenever $r \neq p, q$ and either $|D'_p| < |D_p|$ or $|D'_p| = |D_p|$ and $|D'_q| < |D_q|$.

*Sketch proof in the $n \geq 3$ case.* We work similarly to how we did in the proof of Lemma 4.39 when $n = 2$. We will write this sketch proof as though $p = 0$ and $q = 1$; the ideas are the same for other values. We seek to find embedded in $G$

$$H_1 \cong \left( \{a_0, \ldots, a_{n-1}\}, D_0^1 \cup D_1^1 \cup D_0^2 \cup D_1^2 \cup \left( \bigcup_{i=2}^{k} D_i^1 \cup D_i^2 \right) \right)$$

and

$$H_2 \cong \left( \{a_1, \ldots, a_{n-1}\}, D_0^1 \cup D_1^1 \cup D_0^2 \cup D_1^2 \cup \left( \bigcup_{i=2}^{k} D_i^1 \cup D_i^2 \right) \cup \{z\} \right)$$

where

$$D_0^1, D_0^2 \cup \{z\} \cong D_0$$

and

$$D_1^1 \cup \{z\}, D_1^2 \cong D_1$$

and

$$\bigcup_{i=0}^{k-1} D_i^1 \cup \{z\} \cong \bigcup_{i=0}^{k-1} D_i^2 \cup \{z\} \cong D$$

and where the colours are such as to ensure that when we amalgamate $H_1$ with $H_2$ over their intersection we obtain $H$ whether $a_0 z$ is red or blue.
Now $H_2$ is realised in $G$ by induction on $n$, so we only have to show that $H_1$ is realised in $G$. But we can use the same technique we used in Lemma 4.38 to obtain $H_1$ in those cases where either $|D_p| < |D_q|$ or $|D_p| = |D_q| \neq 2$.
Moreover, if there exists $r$ such that $|D_r| \geq |D_q|$ then we can instead amalgamate

$$H_3 = \left(\{a_1, \ldots, a_{n-1}\}, \{w\} \cup \bigcup_{i=0}^{k} D_i^1\right)$$

with

$$H_4 = \left(\{a_1, \ldots, a_{n-1}\}, \{w\} \cup \bigcup_{i=0}^{k} D_i^2\right)$$

over their intersection

$$(\{a_1, \ldots, a_{n-1}\}, \{w\})$$

where $w$ is joined to every vertex of every $D_j^1$ and not to any vertex of any $D_j^2$, and is coloured to match the colours from $D_r$ - $H_3$ and $H_4$ embed in $G$ by the induction hypothesis, and the presence of $w$ prevents any pairs of vertices being identified, so we obtain $H_1$.
This leaves us to deal with just one "awkward" case. But the awkward case is just the one we handled in Lemma 4.43. □

We now consider the "hard" cases. The easiest of these arises when only one $D_i$ is non-empty; that is, for all $a \in C$ and for all $x, y \in D$, $ax \cong ay$.

**Lemma 4.45.** *Let $H = (C, D, R)$ be such that, for all $a \in C$ and for all $x, y \in D$, $ax \cong ay$. Then $H$ embeds in $G$.*

*Proof.* Induction on $|C|$; the case $|C| = 1$ follows from Lemma 4.37.
It is sufficient to find $H_1$ and $H_2$ embedding in $G$ such that

$$H_1 = (C_1 \cup C_2 \cup \{a\}, D^1 \cup D^2)$$

and

$$H_2 = (C_1 \cup C_2, D^1 \cup D^2 \cup \{z\})$$

where

$$C_1 \cup \{a\} \cong C_2 \cup \{a\} \cong C$$

and

$$D^1 \cup \{z\} \cong D^2 \cup \{z\} \cong D$$

as then we can simply amalgamate $H_1$ with $H_2$ over their intersection to obtain $H$ (whether $az$ is red or blue).
For $H_1$, amalgamate

$$H_3 = (C_1 \cup C_2 \cup \{a\}, D^1)$$

with

$$H_4 = (C_1 \cup C_2 \cup \{a\}, D^2)$$

over their intersection

$$(C_1 \cup C_2 \cup \{a\}, \varnothing)$$

Note that $H_3$ and $H_4$ do embed in $G$ with the required colours: $H_3 \setminus C_2$ and $H_4 \setminus C_1$ do by induction, and we can add $C_2$ and $C_1$ respectively since we don't care about the colours. (The edge type between $C_1$ and $C_2$ will be determined later; for this step any edge type is equally good.) Hence $H_1$ embeds in $G$.
For $H_2$, amalgamate

$$H_5 = (C_1, D^1 \cup D^2 \cup \{a\})$$

with

$$H_6 = (C_2, D^1 \cup D^2 \cup \{a\})$$

over their intersection

$$(\varnothing, D^1 \cup D^2 \cup \{a\})$$

and with the edge type between $D^1$ and $D^2$ as above. $H_5$ and $H_6$ do embed in $G$. Because of $a$, we cannot identify any vertex of $C_1$ with any vertex of $C_2$, and any combination of edge-types between $C_1$ and $C_2$ is acceptable and yields $H_2$, and hence $H$, embedding in $G$. □

Assume now that at least two sets $D_p, D_q$ are non-empty. As long as there is a third non-empty set $D_r$, we will choose $p$ and $q$ so as to minimise $|D_p|$ (while ensuring it is non-empty), and then to minimise $|D_q|$ (while ensuring it is non-empty). (That is, we choose the two smallest non-empty sets $D_p$ and $D_q$, and make $D_p$ the smaller.) Suppose now that at least three of the $D_i$ are non-empty, say $|D_p| \leq |D_q| \leq |D_r|$. In the following lemma we will either reduce $|D_p|$ by 1 and fix $|D_q|$ or reduce $|D_q|$ by 1 and fix $|D_p|$, at a price of increasing $|D_r|$ by 1.

**Lemma 4.46.** *Let $H = (C, D, R)$ such that*

$$D = D_0 \cup \ldots \cup D_{2^n-1}$$

*(as above) and such that $|D_p| \leq |D_q| \leq |D_r| \leq |D_s|$ for some $p, q, r$ and for all $s \neq p, q, r$ such that $|D_s| > 0$. Suppose that $G$ realises every $(C, D')$ where*

$$D' = D'_0 \cup \ldots \cup D'_{2^n-1}$$

*(with the same numbering scheme), where $D'_i = \varnothing$ whenever $D_i = \varnothing$, and where either $|D'_p| < |D_p|$ and $|D'_q| = |D_q|$ or $|D'_p| = |D_p|$ and $|D'_q| < |D_q|$. Then $G$ also realises $H$.*

*Proof.* If $G$ embeds compatible

$$H_1 = \left(\{a\} \boxplus (C_1 \sqcup C_2), D^1 \sqcup D^2\right)$$

and

$$H_2 = \left(C_1 \sqcup C_2, \left(D^1 \sqcup D^2\right) \cup \{z\}\right)$$

where

$$(\{a\} \boxplus C_1, D^1 \cup \{z\}) \cong H$$

if $az$ is red, and

$$(\{a\} \boxplus C_2, D^2 \cup \{z\}) \cong H$$

if $az$ is blue, then we can amalgamate $H_1$ with $H_2$ over their intersection and will inevitably obtain $H$ in $G$. By Lemma 4.41 we have $H_2$ with a definitely known graph on the left (or $H$ outright), so we only need to obtain $H_1$. Note that $D^i_j \cong D_j$ except that $D^1_p \cup \{z\} \cong D_p$ and $D^2_q \cup \{z\} \cong D_q$.
If we can find

$$H_3 = \left(\{a\} \boxplus (C_1 \sqcup C_2), \{w\} \cup \left(\bigcup_j D^1_j\right)\right)$$

and

$$H_4 = \left(\{a\} \boxplus (C_1 \sqcup C_2), \{w\} \cup \left(\bigcup_j D^2_j\right)\right)$$

where $w$ is joined to all of every $D^1_j$, not joined to all of every $D^2_j$, and is coloured as in $D_r$ to both parts, then we can amalgamate $H_3$ with $H_4$ over their intersection to obtain $H_1$. But we can obtain $H_3$ by amalgamating

$$H_5 = (\{a\} \boxplus (C_1 \sqcup C_2), \{w\})$$

(which certainly exists in $G$) with

$$H_6 = \left( \{a\} \boxplus C_1, \{w\} \bigcup_j D_j^1 \right)$$

(which exists in $G$ by induction) over their intersection $(\{a\} \boxplus C_1, \{w\})$, and similarly we can obtain $H_4$. Hence $H$ is realised in $G$. □

We are now in a position to formally conclude the proof of Theorem 4.44.

*Proof of Theorem 4.44.* Let $D_0, \ldots, D_{2^n-1}$ partition $D$ as above. If two non-empty $D_i, D_j$ have $i$ and $j$ differing in precisely one binary digit, $H$ embeds using Lemma 4.39. If $H$ has precisely one non-empty $D_j$, $H$ embeds using Lemma 4.45. If $H$ has at least three non-empty $D_j$s, $H$ embeds using Lemma 4.46. We have therefore reduced to the case where **precisely** two of the $D_i$ are non-empty (and where Lemma 4.39 does not apply).
If the two sets are $D_0$ and $D_{2^n-1}$ then, other than if they are graphs of size 2, there are no complications if we simply follow the method we used in Lemma 4.40 in the $n = 2$ case. Even if this is not the case we can still use this method provided that we can obtain the required structures of the form $(K_{n-1}[\overline{K_2}], D')$ where every vertex of $D'$ has the same pattern of colours to the left component. But these can be obtained using Lemma 4.42. The rest of the proof follows the same arguments as in the $n = 2$ case and, as with that case, uses Lemma 4.41 to obtain the "bottom" amalgamand and Lemma 4.43 to deal with the case where $|D_i| = |D_j| = 2$ and each vertex of either has an edge and a non-edge to the other. □

### 4.6.6 Details for $C = P_3$

We now consider the case where $C = P_3$. Write

$$C = (\{a, b, c\}, \{ab, bc\})$$

so that $b$ is the "middle" of the $P_3$ and $a$ and $c$ the two ends. Recall that we need to show that:

**Theorem 4.47.** *For each finite graph $D$, every 2-coloured 2-graph $H = (C, D, R)$ embeds in $G$.*

The method of handling these cases is similar to that we used in sections 4.6.3 and 4.6.5, as outlined in section 4.6.2. We do have to take a little extra care because of the graph structure of a $P_3$, and these complexities will typically result in having extra sub-cases.

As with the other sections, there is an "easy" case and a "hard" case. We now handle the "easy" case (again, in the usual sense of reducing an "easy" instance to cases that are smaller in some way which allows non-circular induction). Write $D = \coprod_i D_i$ where, for all $i$, all $a \in C$ and $x, y \in D_i$, $ax \cong ay$, and where for all $i \neq j$ there are $a \in A, x \in D_i, y \in D_j$ such that $ax \ncong ay$. We do **not** yet impose any ordering on the $D_i$; we will do this later in the section.

**Lemma 4.48.** *Let $H = (C, D, R) \cong (P_3, D)$ be a 2-graph such that $D = \bigcup_{i=0}^{7} D_i$ where, for all $i$, for all $x, y \in D_i$ and for all $p \in \{a, b, c\}$, $px \cong py$ and, for all $i \neq j$, there exists $q \in \{a, b, c\}$ such that, for all $x \in D_i$ and $y \in D_j$, $qx \ncong qy$. Suppose moreover that $D_0, D_1$ are such that there exists **exactly** one $p \in \{a, b, c\}$ such that, for all $x \in D_i$ and $y \in D_j$, $px \ncong py$, and that $|D_0| \leq |D_1|$. Then, if $H' = (C, D', R')$ embeds in $G$ whenever $H$ is such that, for $D'_i$ corresponding to $D_i$, $|D'_i| = 0$ whenever $|D_i| = 0$ and **either** $|D'_0| \leq |D_0|$ (with no condition on $|D'_1|$) **or** $|D'_0| = |D_0|$ and $|D'_1| \leq |D_1|$, $H$ also embeds in $G$.*

*Remark.* Note that in this result $D_0$ need not be monochromatic red to $C$, although we will show it this way in diagrams.

*Proof.* There are essentially two cases of this lemma, corresponding to $p = b$ and $p = c$. (We may ignore the $p = a$ case as it is essentially the same as $p = c$.) We may assume that $G$ embeds all 2-coloured 2-graphs of the form $(K_2, D)$.

Let $Z$ be any graph containing the following disjointly:

- two copies $\tilde{D}^0$ and $\tilde{D}^1$ of $D \setminus \{D_0, D_1\}$,
- a point $z$,
- a copy $D_0^0$ of $D_0$,
- a copy $D_1^1$ of $D_1$,
- a copy $D_0^1$ of $D_0 \setminus \{x\}$ (that is, $D_0^1 \cup \{z\} \cong D_0$), and
- a copy $D_1^0$ of $D_1 \setminus \{x\}$ (that is, $D_1^0 \cup \{z\} \cong D_1$),

in such a way that

$$D_0^0 \cup D_1^0 \cup \{z\} \cup \tilde{D}^0 \cong D_0^1 \cup D_1^1 \cup \{z\} \cup \tilde{D}^1 \cong D$$

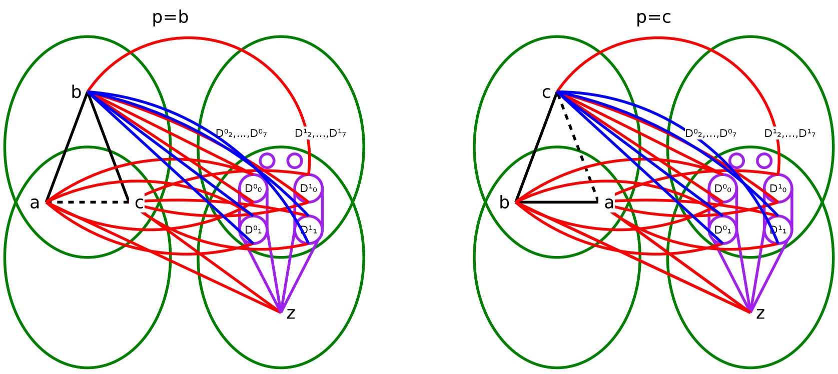


Figure 4.35: The $H_1$ and $H_2$ needed for obtaining the "easy" case of $(P_3, D)$. In this illustration $p$ is red to $D_0$ and blue to $D_1$, while both vertices of $C \setminus \{p\}$ are red to all vertices of both $D_0$ and $D_1$.

We want to amalgamate

$$H_1 = (C, Z \setminus \{z\})$$

with

$$H_2 = (C \setminus \{p\}, Z)$$

in such a way that, whether $pz$ is red or blue, if $H_1$ and $H_2$ embed in $G$ then so does $H$, as shown in Figure 4.35 (where one particular instance of the $p = b$ and $p = c$ cases is shown). But this is clear since $z$ is correctly coloured to all vertices of $C \setminus \{p\}$, and all other cross-edge colours are automatic. Moreover, by Section 4.6.3, $H_2$ is realised in $G$ for **every** finite graph $Z$. So we need only find some $H_1$ embedding in $C$ with **some** $Z$ that satisfies the conditions above.
Let

$$H_3 = (C, D_1^0 \cup D_0^1 \cup D_1^1 \cup \tilde{D}^0 \cup \tilde{D}^1)$$

and

$$H_4 = (C, D_0^0 \cup D_1^0 \cup \tilde{D}^0 \cup \tilde{D}^1)$$

(with the edge relationship between $D_1^0$ and $D_0^1 \cup D_1^1$ to be specified). If we can amalgamate $H_3$ with $H_4$ over their intersection without identifying any pairs of vertices, the graph we get on the right-hand side will certainly be suitable for $Z$. So the only issue is making this impossible. As before, there is no difficulty unless

$|D_0| = |D_1| = 2$ and each vertex in $D_0$ has an edge and a non-edge to $D_1$ (and vice versa), as in all other cases, as before, we can find some combination of edges that avoids any risk of identifying any vertex in $D_0^1$ with any vertex in $D_0^0$. But this case can be handled using Lemma 4.43. □

If precisely one of the $D_i$ sets is non-empty, then $H$ embeds by Lemma 4.42. We can now therefore assume that every $(K_2, D)$ embeds in $G$ (since the dependency of those cases on $(P_3, D)$ cases was only on those where only one $D_i$ is non-empty). We can even assume that every $(K_n, D)$ and every $(\overline{K_n}, D)$ embeds in $G$, as these were handled in section 4.6.5. We will further assume that at least two of the $D_i$ are non-empty. We distinguish the case where at least three of the $D_i$ are non-empty from the case where precisely two of the $D_i$ are non-empty; the following two lemmas each handle one of these two cases.

**Lemma 4.49.** *Suppose that there exist $i, j, k$ such that $1 \leq |D_i| \leq |D_j| \leq |D_k|$ and suppose that $G$ realises every $H' = (C, D', R')$ where $D'_l = \varnothing$ whenever $D_l = \varnothing$ and where:*

- *either $|D'_i| < |D_i|$ and $|D'_j| \leq |D_j|$,*
- *or $|D'_i| \leq |D_i|$ and $|D'_j| < |D_j|$.*

*Then $H$ embeds in $G$.*

*Sketch proof.* The proof is similar to that of Lemma 4.46, so we need only point out areas of caution.
We do need to be careful about the graph on the left component. We may assume that either $a_0$ or $a_1$ is differently coloured to $D_i$ and $D_j$.
If $a_0$ is differently coloured to $D_i$ and $D_j$, the main amalgamation diagram will be as in the left diagram in Figure 4.36, and we we can assume that the left component of the bottom amalgamand $H_2$ is $K_4$. Similarly, if $a_a$ is differently coloured to $D_i$ and $D_j$ we can assume that the left component of $H_2$ is $\overline{K_4}$, as in the right diagram in Figure 4.36. Hence, without loss of generality, the bottom amalgamand embeds in $G$ without needing Lemma 4.41.
To obtain $H_1$ we work as in Lemma 4.46. □

**Lemma 4.50.** *Suppose that there exist $i, j$ such that $1 \leq |D_i| \leq |D_j|$ and, for all $k \neq i, j$, $|D_k| = 0$. Suppose that $G$ realises every $H = (C, D', R')$ where $D'_k = \varnothing$ unless $k = i$ or $k = j$, and:*

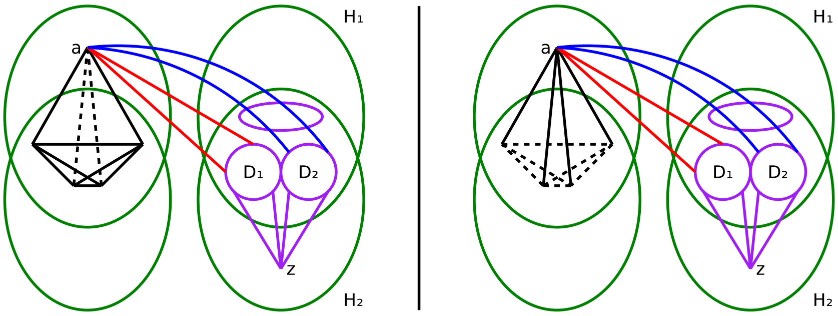


Figure 4.36: The main amalgamation diagrams for the first (left) and second (right) cases of Lemma 4.49. In this diagram we assume that $i = 1, j = 2, k = 3$.

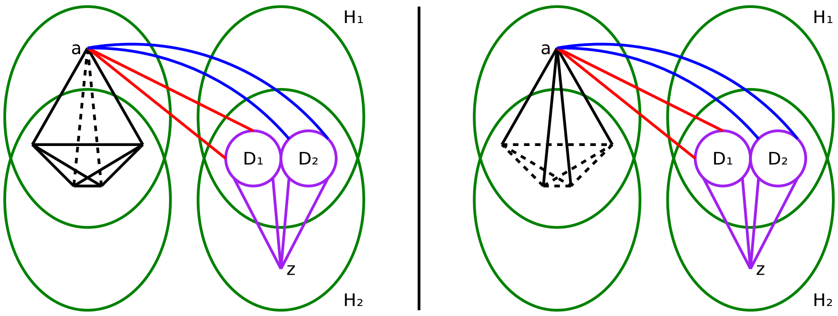


Figure 4.37: The main amalgamation diagrams for the first (left) and second (right) cases of Lemma 4.50. In this diagram we assume that $i = 1, j = 2, k = 3$.

- *either* $|D'_i| < |D_i|$,
- *or* $|D'_i| = |D_i|$ *and* $|D'_j| < |D_j|$.

*Then* $H$ *embeds in* $G$.

*Sketch proof.* The proof is similar to that in the $C = K_2$ case, so we need only point out areas where we have to be slightly more careful here than in the $C = K_2$ case. As in Lemma 4.49 we do need to be careful about the graph on the left component. We **can** assume that either $a_0$ or $a_1$ is differently coloured to $D_i$ and $D_j$.
If it is $a_0$ that is differently coloured to $D_i$ and $D_j$, the main amalgamation diagram will be as in the left diagram in Figure 4.37. Note that the left component of the bottom amalgamand $H_2$ can be assumed to be $K_4$. Similarly, if it is $a_1$ that is differently coloured to $D_i$ and $D_j$, we can assume that the left component of $H_2$ is $\overline{K_4}$, as in the right diagram in Figure 4.37.

For $H_1$ we work as before, using Lemma 4.42 with the left component being either

$$(\{a, b, c, d, e\}, \{ab, ac, ad, ae\})$$

or

$$(\{a, b, c, d, e\}, \{ab, ad, bc, bd, be, cd, ce, de\})$$

and using Lemma 4.43 as usual when $|D_i| = |D_j| = 2$ and each has an edge and a non-edge to the other. □

We have therefore proved Theorem 4.47.

# Chapter 5

# Summary and future work

## 5.1 Summary of classification results

In this section, we summarise the principal results of this thesis and also some results by other authors that are key to understanding this classification.

Let $G$ be a homogeneous 2-graph and write $G = (A, B, R)$. If $|R| = 1$ then $G$ is homogeneous if and only if $A$ and $B$ are homogeneous graphs. We will therefore assume that $|R| \geq 2$.

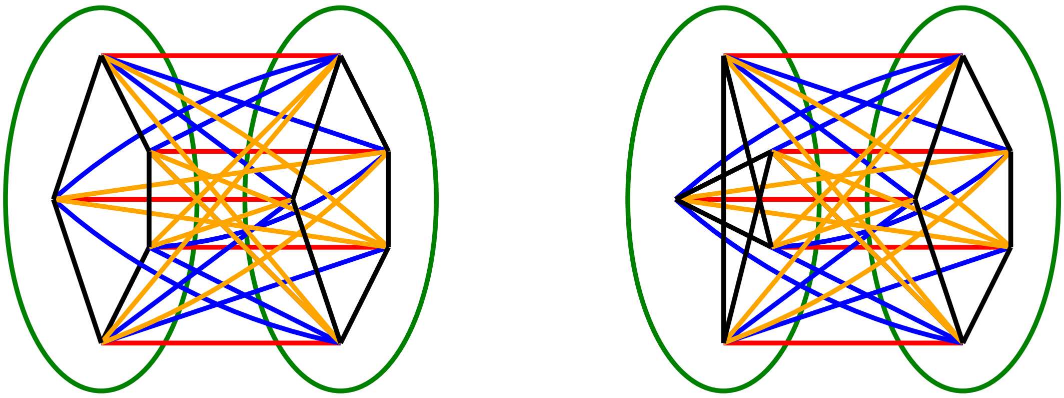

Figure 5.1: The two equivalent non-isomorphic instances of $(C_5, C_5)$.

Jenkinson (2006) showed that $A \cong C_5$ if and only if $B \cong C_5$, and if $A \cong B \cong C_5$ then $G$ is either monochromatic or equivalent to the 2-graph given by Proposition 2.6. There are two non-isomorphic equivalent instances of this shown in Figure 5.1. Jenkinson also showed that $A \cong K_3 \times K_3$ if and only if $B \cong K_3 \times K_3$, and if this holds then again $G$ is either monochromatic or equivalent to the 2-graph given by Proposition 2.6, of which there are also two equivalent non-isomorphic instances.

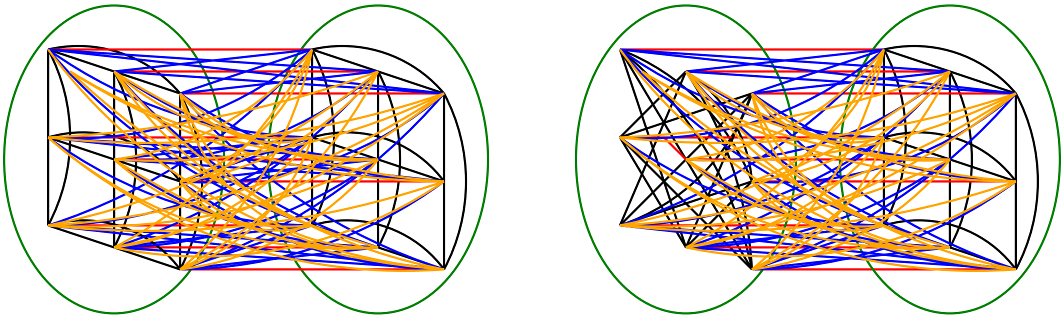

Figure 5.2: The two equivalent non-isomorphic instances of $(K_3 \times K_3, K_3 \times K_3)$. Since there are 81 cross-edges these diagrams are necessarily somewhat unclear.

Moreover, Jenkinson also observed, and we restated in Lemma 2.1, that $A$ and $B$ must be homogeneous graphs, and if either is $C_5$ or $K_3 \times K_3$ then Jenkinson fully described $G$. By this and using equivalence, we only needed to consider those cases where:

1. $A \cong \overline{K_m}[K_p]$ and $B \cong \overline{K_n}[K_q]$, for some $m, n, p, q \in \mathbb{N} \cup \{\infty\}$, where $m, n \geq 1$ and $p, q \geq 2$;
2. $A \cong \overline{K_m}[K_n]$ and $B \cong \Gamma_r$, for some $m, n, r \in \mathbb{N} \cup \{\infty\}$, where $m \geq 1$,$n \geq 2$ and $r \geq 3$; or
3. $A \cong \Gamma_r$ and $B \cong \Gamma_s$, for some $r, s \in \mathbb{N} \cup \{\infty\}$, where $r, s \geq 3$.

We showed in Proposition 2.8 that if $G$ is left-collapsing (i.e. $A = A_1 + A_2 + \ldots + A_m$ (there is no implication that $m$ is finite) where each $A_i \cong K_p$ for some $p \in \mathbb{N}$, and where, for every $i$, every $x \in B$ and every $a, b \in A_i$, $ax$ and $bx$ are the same colour) then $G$ is homogeneous if and only if the $(\overline{K_m}, B)$ 2-graph to which it collapses is homogeneous. A similar result holds if $G$ is right-collapsing. Hence we restricted ourselves to non-collapsing homogeneous 2-graphs.

The first of the families of cases listed above (where $A \cong \overline{K_m}[K_p]$ and $B \cong \overline{K_n}[K_q]$) has mostly been ignored in this thesis, though I have made some remarks about these cases in section 5.2.

If $A \cong \overline{K_m}[K_n]$ and $B \cong \Gamma_r$, and $G$ has exactly two cross-edge colours and is not collapsing, Theorem 3.1 shows that $G$ is homogeneous if and only if it is equivalent to one of the following:

- $m = \infty$, $n = 1$ and $G$ is otherwise generic;
- $m = \infty$, $n = 1$, the 2-graph $(K_1, K_k)^1$ is minimally omitted for some $k < r$, and $G$ is otherwise generic;
- $m = \infty$, $n = 2$, the 2-graphs $(K_2, K_1)^1$ and $(K_2, K_1)^2$ are minimally omitted, and $G$ is otherwise generic;
- $2 \leq m \leq \infty$, $n = \infty$ and $G$ is otherwise generic; or
- $2 \leq m \leq \infty$, $n = \infty$,the 2-graph $(K_1, K_k)^1$ is minimally omitted for some $k < r$, and $G$ is otherwise generic.

If $A \cong B \cong \Gamma_3$ and $G$ has exactly two cross-edge colours and is not collapsing, $G$ is homogeneous if and only if it is equivalent to one of the following:

- generic omitting $\{(K_1, K_2)^1, (K_2, K_1)^1\}$;
- generic omitting $\{(K_1, K_2)^1\}$;
- generic omitting $\{(K_2, K_2)^1\}$; or
- generic.

If $A \cong B \cong \Gamma_\infty$ and $G$ has exactly two cross-edge colours and is not collapsing, $G$ is homogeneous if and only if there is an antichain $\mathcal{A}$ of 2-graphs of the form $(K_m, K_n)^1$ such that $G$ is equivalent to the two-coloured $(\Gamma_\infty, \Gamma_\infty)$ minimally omitting precisely the members of $\mathcal{A}$.
(The "if" part (existence) of the preceding two cases was proved in Proposition 4.1. The "only if" part (uniqueness) is Theorem 4.2, and its proof occupies the bulk of chapter 4.)
These results amount to a complete classification of 2-coloured homogeneous $(\overline{K_m}[K_n], \Gamma_r)$ 2-graphs, for all values of $m, n, r$ where $r \geq 3$, and also of 2-coloured homogeneous $(\Gamma_3, \Gamma_3)$ and $(\Gamma_\infty, \Gamma_\infty)$ 2-graphs. The fact that the resulting classification has such limited scope is somewhat disappointing, but reflects the technical difficulties in getting even to the stage we have currently reached.

## 5.2 Notes on the $(\overline{K_m}[K_p], \overline{K_n}[K_q])$ case

I will now give a brief summary of the status of the $(\overline{K_m}[K_p], \overline{K_n}[K_q])$ classification (where $m$, $n$, $p$ and $q$ are each either a positive integer or equal to infinity (i.e. $\aleph_0$, since all structures considered are countable)). As the classification is not complete and the proofs of the cases so far dealt with are tedious and not very enlightening, I have not said much about this case so far in this thesis. I will therefore now merely give a **statement** of the stage which I have reached (namely, Theorem 5.1) and a statement of the main result of chapter 3 of Jenkinson (2006) on 2-graphs on this form (namely, Theorem 5.2). I intend that a complete classification of this case will be published separately once it is ready.

Theorem 5.1 amounts to a classification of the 2-coloured homogeneous $(\overline{K_m}[K_p], \overline{K_n}[K_q])$, and of all finite homogeneous $(\overline{K_m}[K_p], \overline{K_n}[K_q])$ 2-graphs; moreover, some other families of cases are also completely classified by Theorem 5.1. Note that we rely on Lemma 2.3, Proposition 2.8 and other results from chapter 2; as a result, not all equivalent combinations are explicitly mentioned and certain cases are omitted from the statement (e.g. the $m = p = 1$ case and the $p = q = 1$ case).

**Theorem 5.1.** *Let $G$ be a homogeneous, non-collapsing $(\overline{K_m}[K_p], \overline{K_n}[K_q])$ 2-graph. Then:*

1. *it is not the case that $mp$ is finite and $nq$ is infinite, or vice versa;*

2. *it is not the case that $m = n = \infty$ and $2 < p < q < \infty$;*

3. *it is not the case that $m = q = \infty$, $n \geq 2$ and $p > 2$;*

4. *if all of $m$, $n$, $p$ and $q$ are finite, and at least three of $m$, $n$, $p$ and $q$ are greater than or equal to 2, then all four are equal to 2, and moreover $G$ is equivalent to the 2-graph* **K** *in figure 5.3:*

5. *if $m = 1$, $n, q \geq 2$ and $p = nq = \infty$ then, for some partition of the colour set $C$ into subsets $C_1, \ldots, C_r$, $G$ is equivalent to the 2-graph $G_{C_1,\ldots,C_r}$ that realises a 2-graph $H$ of the form $(K_1, K_2)$ if and only if both cross-edges of $H$ lie in the same $C_i$ and that is generic subject to this stipulation;*

6. *if $m = n = \infty$ and $p = q = 2$, then either:*

   - *$G$ is the 2-graph $P_{\overline{K_\infty}[K_2]}$ given by Proposition 2.6, or*

- *there is a finite set $C$ of $(K_2, K_2)$ 2-graphs, none of which has three different cross-edge colours or three cross-edges of the same colour, and no two of which have a cross-edge in common, such that $G$ is generic subject to all $(K_2, K_2)$ sub-2-graphs of $G$ being isomorphic to elements of $C$;*

7. *if $m = n = \infty$ and $3 \leq p = q < \infty$, then every $(K_p, K_q)$ restriction in $G$ must be homogeneous (which implies that $G$ is $P_{\overline{K_\infty}[K_p]}$ from Proposition 2.6);*

8. *if $m = q = \infty$, $n \geq 2$ and $p = 2$, and $G$ has precisely two cross-edge types, then $G$ minimally omits only $(P_3, \varnothing)$, $(\varnothing, P_3)$, $(K_{n+1}, \varnothing)$ (if finite), $(\varnothing, K_3)$, $(K_2, K_1)^1$ and $(K_2, K_1)^2$;*

9. *if $p = q = \infty$ and $2 \leq m \leq n < \infty$, every $(K_\infty, K_\infty)$ restriction in $G$ must be homogeneous (and then the form of $G$ is given by Theorem 5.2);*

10. *if $n = p = q = \infty$ and $2 \leq m < \infty$ and $G$ has precisely two cross-edge colours, then all $(K_\infty, K_\infty)$ restrictions in $G$ are isomorphic;*

11. *if $m = n = p = q = \infty$ and $G$ has precisely two cross-edge types, $G$ is either generic (i.e. realising all finite 2-coloured 2-graphs where either component is a finite induced subgraph of $(\overline{K_\infty}[K_\infty])$) or is generic subject to omitting the two 2-graphs of the forms $(K_2, K_2)$ where one edge-type appears once and the other three times.*

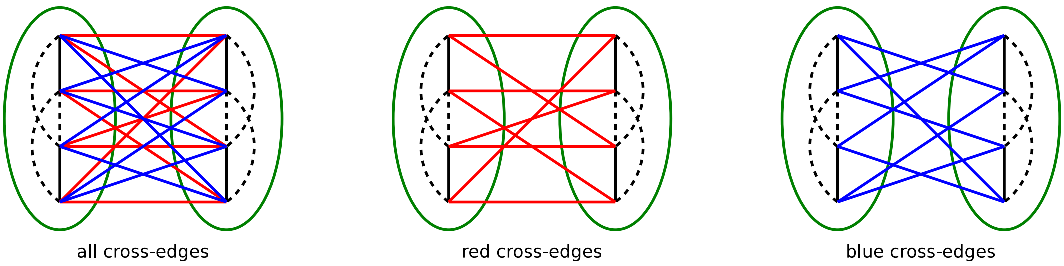


Figure 5.3: The only non-trivial example of a finite homogeneous $(\overline{K_m}[K_p], \overline{K_n}[K_q])$ 2-graph. For clarity we show the red and blue cross-edges separately.

Recall that Jenkinson (2006) proved the following about homogeneous $(\overline{K_m}[K_n], \overline{K_p}[K_q])$ 2-graphs, which we restate here for convenience without proof:

**Theorem 5.2.** *Let $G$ be a c-coloured $(\overline{K_m}[K_p], \overline{K_n}[K_q])$ homogeneous 2-graph where $p, q \geq 3$. Write $G = (A_1 + \ldots + A_m, B_1 + \ldots + B_n, R)$ where each $A_i$ and $B_j$ is a maximal clique. Suppose that, for all $i$ and $j$,*

$$(A_i, B_j, (R_1 \cap V(A_i) \times V(B_j), \ldots, R_c \cap V(A_i) \times V(B_j))$$

*is homogeneous. Then $G$ is equivalent to one of the following:*

1. *the 2-graph $P_{\overline{K_m}[K_p]}$ of Proposition 2.6;*
2. *a 2-graph where, for all $i, j$, $(A_i, B_j) \cong (A_1, B_1)$ and this is either monochromatic or the generic in two or more colours;*
3. *a 2-graph where $m = n$ and, for all $i$, $(A_i, B_i) \cong (A_1, B_1)$ and for all $i \neq j$, $(A_i, B_j) \cong (A_1, B_2) \ncong (A_1, B_1)$, and moreover $(A_1, B_1)$ and $(A_1, B_2)$ are each either monochromatic or generic, and there is no overlap in the colour sets;*
4. *a 2-graph satisfying the following properties:*
   - *$m = n = \infty$;*
   - *each $(A_i, B_j)$ is monochromatic or generic, and the isomorphism classes of restrictions can be labelled by $Q_1, Q_2, \ldots, Q_N$;*
   - *if $(A_i, B_j) \ncong (A_{i'}, A_{j'})$ then the two have no cross-edges in common;*
   - *if $X_1, X_2$ are finite, disjoint subsets of $\{A_1, A_2, \ldots\}$, and $Y_1, Y_2$ are finite, disjoint subsets of $\{B_1, B_2, \ldots\}$, then, for every $1 \leq k \leq N$:*
     - *there is an $i$ so that $A_i \notin X_1 \cup X_2$ and, for every $j$ so that $B_j \in Y_1 \cup Y_2$, $(A_i, B_j) \cong Q_k$ if $B_j \in Y_1$, and*
     - *there is a $j$ so that $B_j \notin Y_1 \cup Y_2$ and, for every $i$ so that $A_i \in X_1 \cup X_2$, $(A_i, B_j) \cong Q_k$ if $A_i \in X_1$.*

*Remark.* It is fairly easy to verify that these 2-graphs are indeed homogeneous; in each case there will be an age and it can be verified that the amalgamation property holds.

The catalogue of **finite** non-collapsing $(\overline{K_m}[K_n], \overline{K_p}[K_q])$ 2-graphs, up to equivalence, is rather brief:

1. $P_{\overline{K_m}[K_n]}$ for each finite $m$ and $n$
2. **K** (as in 5.3)

The set of known **infinite** non-collapsing $(\overline{K_m}[K_n], \overline{K_p}[K_q])$ 2-graphs (up to equivalence) is however less easy to succinctly state, and moreover I believe that this catalogue is **not** complete, and do not have a conjecture as to its full extent.

## 5.3 Future work

Cherlin (1998) set out an ambitious programme for the systematic classification of homogeneous structures **in general**. It is probably unrealistic to immediately embark on a general classification of this nature at this stage. Nevertheless there are some significantly less ambitious steps one might take in attempting to extend the results of this thesis.

The definition of a 2-graph given in chapter 2 can be extended to cases where there are more than two components; the resulting structure is an $n$-graph. More precisely, for any given finite positive integer values of $c$ and $n$, a $c$-**coloured** $n$-**graph** is a structure

$$\mathbf{G} = (G_1, \ldots, G_n, R_{1,2}, \ldots, R_{n-1,n})$$

where each $G_i$ is a graph $(V_i, E_i)$, where $V_i \cap V_j = \varnothing$ whenever $i \neq j$, and where each $R_{i,j}$ (for $i < j$) is an ordered partition of $V_i \times V_j$ into $c$ subsets (some of which may be empty). We write $R^1_{i,j}, \ldots, R^c_{i,j}$ for the elements of the partition $R_{i,j}$. A **cross-edge** between $G_i$ and $G_j$ is an element of an element of $R_{i,j}$; we will often abuse notation by writing a cross-edge $(x, y)$ as $xy$. A $c$-coloured $n$-digraph is similar to a $c$-coloured $n$-graph, but each $G_i$ is a digraph (definitions vary on whether it is permitted to have undirected edges).

As a more immediate though still technically difficult goal, Cherlin set the task of finding a classification of the homogeneous $n$-graphs and $n$-digraphs. This thesis is essentially a first attempt to systematically classify the homogeneous 2-graphs. Even that goal proved too ambitious and we have largely restricted ourselves to the homogeneous **2-coloured** 2-graphs (and even then not all of those).

It seems that it should be possible to generalise the techniques in Chapter 4 to classify the 2-coloured homogeneous $(\Gamma_r, \Gamma_s)$ 2-graphs for all $3 \leq r, s \leq \infty$. It appears that, with some hard work and careful bookkeeping, the proof we set out for the $r = s = 3$ case should be adaptable for larger, finite values of $r$ and $s$ (though we have so far been unable to make this generalisation). Similarly, the $r < \infty, s = \infty$

case should be conquerable using a mixture of the techniques in the $r = s = 3$ case, the techniques in the $r = s = \infty$ case and some hard work and care in bookkeeping. In either case we would need to extend the work of Theorem 4.11 for other values of $r$ and $s$. For specific values (if for example this was the only step missing from a classification of the homogeneous 2-coloured $(\Gamma_4, \Gamma_4)$ 2-graphs) this can be done if necessary by brute force. A general proof of Theorem 4.11 in these other cases appears to need some fairly careful combinatorial estimates.

There are a number of other natural possibilities for extending the work in this thesis and we describe some of them below.

### 5.3.1 The ultimate extension of the bipartite case

At several points we prove, in various guises, variations of the following two results:

**Lemma 5.3.** *Let $G$ be an $n$-coloured bipartite graph with both parts infinite (i.e. a 2-graph where both components are $\overline{K_\infty}$). Then if $G$ is homogeneous, no vertex (on either part) can have finite degree of any colour unless that degree is 0 or 1.*

**Theorem 5.4.** *If $G$ is a homogeneous $n$-coloured bipartite graph where both parts are infinite, then $G$ is equivalent to one of the following:*

1. *monochromatic (i.e. $n = 1$);*
2. *perfect matching in one colour and the complement of a perfect matching in one other colour (which implies that $n = 2$); or*
3. *generic in $n \geq 2$ colours.*

*Moreover all of these are homogeneous $n$-coloured bipartite graphs.*

We have needed to reprove variants of these results in part because in many cases it is difficult to systematically translate a finite partial automorphism of a structure $S$ into a second structure $T$ derived in some way from $S$. It has occurred to us that these various reiterations should be susceptible to generalisation, and efforts should be made to find general formulations, particularly of Lemma 5.3.

**Problem 5.5.** *Find the most general formulations of Lemma 5.3 and Theorem 5.4.*

### 5.3.2 Extensions of the notion of "collapsing"

In Definition 2.7 we defined a notion of "collapsingness" for 2-graphs. The intention of this definition was to allow us to eliminate quotients from our classification (since a collapsing 2-graph $G$ is homogeneous if and only if the 2-graph $G'$ to which $G$ collapses is homogeneous). Unfortunately, the definition did not encompass various types of "quasi-quotient" that arose. We need to generalise the notion of collapsingness to encompass these weakenings.

**Problem 5.6.** *Find a general and uniform weakening of the notion of collapsingness.*

### 5.3.3 Sizes of intermediate amalgamands

Generally we have not been especially careful about how large our intermediate amalgamands are. In some cases, for example in the proof of Lemma 4.12 where we sought structures of the form $(\overline{K_m}, D)$ embedding in $(\Gamma_r, \Gamma_s)$ for finite graphs $D$ and finite $r$, the sizes of the right components of the intermediate amalgamands increase exponentially with increasing $m$, and the sizes of the left components of the intermediate amalgamands also increase with $r$ as well as with $m$.

It is almost certainly possible to find tighter amalgamation arguments than the ones we use (that is, arguments where the intermediate amalgamands are smaller and/or using fewer intermediate amalgamands). It should be possible to investigate whether or not there are asymptotic bounds on the number of amalgamations, and the maximum sizes of the intermediate amalgamands, needed to obtain some $H = (C, D, R)$ embedding in, for example, a generic 2-coloured $(\Gamma_r, \Gamma_s)$ 2-graph, and to determine what such bounds may be or at least "bound the bounds" (i.e. find functions $f_1$ and $f_2$ of $k$, $r$ and $s$ such that, if the maximum number of amalgamations needed to obtain a legal $(C, D, R)$ where $\max(|C|, |D|) = k$ is $f(k, r, s)$ then $f(k, r, s) \in O(f_1(k, r, s))$ and $f(k, r, s) \in \Omega(f_2(k, r, s))$, and similarly for the maximum size of the amalgamands).

### 5.3.4 Other potential scope for extension

It may be possible to find versions of our results that work in the 2-digraph case (compare with the classification of the countable 2-tournaments given in Cherlin

(1998)). In such a case there would be three or four possible edge types within each component (depending on whether or not we allow an edge in both directions; Cherlin's classification does not but Lachlan (1982) does), and there would be some other technical complications. It is possible that our approach might not be very fruitful (just as Cherlin in his classification of the homogeneous digraphs did not simply extend the approach in Lachlan & Woodrow (1980) to classifying homogeneous graphs).

It might also be possible to extend some of our results to the $n$-graph (or even the $n$-digraph) cases for some $n \geq 3$. We can consider an $n$-graph as consisting of a family of overlapping 2-graphs (i.e. for each pair of components the restriction of the $n$-graph to these two components is a 2-graph). Jenkinson *et al.* (2011) classified the 2-coloured $n$-partite graphs; we can think of an $n$-partite graph as a special case of an $n$-graph where all components are edge-free graphs.

Extending to cases where there are infinitely many cross-edge relations is unlikely to lead to worthwhile results. I can think of three ways one might try to do such an extension, and all three are unsatisfactory.

1. Include a function that gives the cross-edge colour between any two vertices in different components – this requires care to determine precisely what "finitely generated" would mean.

2. Include a relation for each cross-edge colour – if this were done then the statement that "there is a colour between each pair of vertices in different components" cannot be expressed finitistically, so we do not have an $L_{\omega\omega}$ theory.

3. Include a relation for all but one cross-edge colour – this leaves a colour that is not definable, and makes it impossible to define certain finite 2-graphs that we would want to show are embedded or omitted.

The only approach to this question that appears to have any chance of success seems to be making the language non-relational by adding a function giving the colour between any two vertices, and even then it is questionable whether the resulting theory would resemble in any way the theories that arise when there are a (known) finite number of cross-edge colours. For this reason, attempting to extend to cases where there are more than two cross-edge colours appears unlikely to be fruitful.

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