Sums of Reciprocals and the Three Distance Theorem

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Abstract

This thesis is primarily concerned with sums of reciprocals in number theory. We first give a brief background and introduction to metric number theory, in particular the areas of Diophantine approximation and continued fractions. We then review sums of reciprocals and their significance and usefulness in certain fields of mathematics as a motivation for their study. The Three Distance theorem, also known as the Steinhaus conjecture, is then discussed and we use it to develop a new technique for obtaining bounds for sums of reciprocals, making constants explicit.

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Thanks goes out to my family and friends, all of whom I will always be grateful for. I would like to especially thank my father, Christopher, for his ever-present support and love.

Finally, I would like to dedicate this thesis to the memory of my late mother, Jaclyn, the greatest woman I know. Without her unconditional love I would not be the person I am today.

Author's Declaration

Some of the material presented within this thesis is part of an ongoing work where I am advised by Professor Victor Beresnevich at the University of York. We are doing this work with the intention of writing and publishing a paper.

Except where stated, I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for a degree or diploma or other qualification at this, or any other, University. All sources are acknowledged as References.

1 Introduction

1.1 Notation

The following standard notation will be used throughout this thesis:

Any real number α may be uniquely written as

$$\alpha = [\alpha] + \{\alpha\},$$

where we define the *integer part of* α as

$$[\alpha] = \max\{n \in \mathbb{Z} : n \leq \alpha\},$$

and define the *fractional part of* α as

$$\{\alpha\} = \alpha - [\alpha].$$

We define the *distance of* α *from the nearest integer* as

$$\|\alpha\| = \min\{|\alpha - n| : n \in \mathbb{Z}\} = \min\{\{\alpha\}, 1 - \{\alpha\}\} = \min\{\{\alpha\}, \{-\alpha\}\},$$

so note that $\{-\alpha\} = 1 - \{\alpha\}$ and always $0 \leq \|\alpha\| \leq \frac{1}{2}$.

$\#\{\cdot\}$ represents the cardinality of a set and is not to be confused with fractional part.

We will also use the Vinogradov notation. For functions $f(x), g(x)$ of a real variable x,

- $f(x) \ll g(x)$ is equivalent to $f(x) = O(g(x))$,
- $f(x) \gg g(x)$ is equivalent to $g(x) \ll f(x)$,
- $f(x) \asymp g(x)$ means that both $f(x) \ll g(x)$ and $f(x) \gg g(x)$ hold and we say that $f(x)$ and $g(x)$ are comparable.

1.2 A General Introduction to Diophantine Approximation

The set of real numbers, $\mathbb{R}$, is defined to be the completion of the set of rational numbers, $\mathbb{Q}$, in the natural topology. Therefore, by construction, $\mathbb{Q}$ is dense in the real line. Diophantine approximation is a branch of number theory that is concerned with the quantitative analysis of this property and its generalisations. The density means that every $\alpha \in \mathbb{R}$ can be approximated by a rational fraction to an arbitrarily close degree. For the purposes of this thesis, in this section we will give a general introduction to diophantine approximation and continued fractions, omitting proofs, which can be found in [13], [18], [32], [36], [40], [47], and [50]. Later in this section, we will also see that investigating the case where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ naturally leads to a discussion of continued fractions.

Density means that for any $\alpha \in \mathbb{R}$ and $\epsilon > 0$, there exists a rational number $\frac{p}{q}$, where $p, q \in \mathbb{Z}$, $q > 0$, such that $|\alpha - \frac{p}{q}| < \epsilon$. By choosing $p = [q\alpha]$ and $q = 1 + \left[\frac{1}{\epsilon}\right]$ we see that $|\alpha - \frac{p}{q}| < \frac{1}{q} \leq \epsilon$. The rate of approximation by $\frac{p}{q}$, which is $\frac{1}{q}$ in the above example can be made better. We see this from the well known theorem by Dirichlet.

Theorem 1 *(Dirichlet, 1842) For any $\alpha \in \mathbb{R}$ and $N \in \mathbb{N}$, there exist $p, q \in \mathbb{Z}$ with $1 \leq q \leq N$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qN}.$$

From the two inequalities in Dirichlet's Theorem, we have the following important consequence.

Theorem 2 *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Then there exists infinitely many integers p, q such that $gcd(p, q) = 1$, $q > 0$, and*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

A natural question to ask is if we can do any better. Hurwitz's Theorem gives us the answer.

Theorem 3 *(Hurwitz, 1891) For any $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, there are infinitely many integers p, q with $q > 0$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

The constant $\frac{1}{\sqrt{5}}$ is best possible.

There are multiple different ways of proving the above theorem, one of which uses the theory of continued fractions, which we will discuss next.

1.3 Continued Fractions

From now on, unless otherwise stated, we will always assume that α is an irrational number.

Definition 1 *A finite (simple/regular) continued fraction is an expression of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cdots + \cfrac{1}{a_k}}},$$

understood as a rational function of variables $a_0, \ldots, a_k$ and denoted by $[a_0; a_1, a_2, \ldots, a_k]$. An infinite (simple/regular) continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

which is understood as an infinite sequence of finite continued fractions $[a_0; a_1, a_2, \ldots, a_k]$, $n = 0, 1, 2, \ldots$ and denoted by $[a_0; a_1, a_2, \ldots]$.

In general, the variable a_k could be real, complex, or even a function of a single or multiple variables, but for the purposes of this thesis we will always assume a_k to be a positive integer, with the exception of a_0, which could be any integer. We now present some useful theorems, facts and properties of continued fractions.

Lemma 1 *Let* $\frac{p_k}{q_k} = [a_0; a_1, \ldots, a_k]$*, where* p_k *and* q_k *are coprime integers. Then*

$$p_k = a_k p_{k-1} + p_{k-2} \qquad \textit{and} \qquad q_k = a_k q_{k-1} + q_{k-2} \qquad \textit{for } k \geq 0, \tag{1}$$

where

$$p_{-1} = 1, \qquad q_{-1} = 0,$$

$$p_{-2} = 0, \qquad q_{-2} = 1.$$

In particular,

$$p_0 = a_0, \qquad\qquad q_0 = 1,$$

$$p_1 = a_1 a_0 + 1, \qquad q_1 = a_1.$$

Definition 2 *If* $[a_0; a_1, \ldots]$ *is a continued fraction expansion of* α*, then the rational number*

$$\frac{p_k}{q_k} = [a_0; a_1, \ldots, a_k] \qquad (k \geq 0)$$

is called the k*th convergent, and the integer* a_k *is called the* k*th partial quotient.*

Continued fractions can also be used as a means of representing real numbers.

Theorem 4 *To every real number* α*, there corresponds a continued fraction with value equal to* α*. This fraction is finite if* α *is rational, and is infinite and unique if* α *is irrational.*

Convergents allow us to state some useful properties of continued fractions for real numbers. First we define, for $k \geq 0$, the following quantities, which measure how the kth convergent approximates α:

$$D_k = q_k \alpha - p_k. \tag{2}$$

By (1) and the definition of D_k it clearly follows that

$$a_{k+1} D_k = D_{k+1} - D_{k-1} \qquad (k \geq 1). \tag{3}$$

Lemma 2 *For all* $k \geq 0$*, we have*

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k. \tag{4}$$

We also have the following lemma, which concerns the approximation properties of the convergents to α:

Lemma 3 *For* $k \geq 0$ *we have that*

$$\alpha = [a_0; a_1, \ldots, a_{k-1}, \alpha_k] \qquad \text{and} \qquad \alpha_k = [a_k; a_{k+1}, \ldots] = a_k + \frac{1}{\alpha_{k+1}}.$$

If p_k *and* q_k *are defined as in Lemma 1, then* $p_k, q_k \in \mathbb{Z}$ *and are coprime, and*

$$\alpha = \frac{\alpha_k p_{k-1} + p_{k-2}}{\alpha_k q_{k-1} + q_{k-2}} \qquad \text{for } k \geq 2.$$

Furthermore, the sequence $\frac{p_{2k}}{q_{2k}}$ *is strictly increasing and the sequence* $\frac{p_{2k+1}}{q_{2k+1}}$ *is strictly decreasing, so*

$$\alpha \leq \frac{p_k}{q_k} \quad \text{if } k \text{ is odd}, \qquad \alpha \geq \frac{p_k}{q_k} \quad \text{if } k \text{ is even}.$$

Both sequences converge to α *and we have that*

$$\frac{1}{2q_{k+1}} < \frac{1}{q_{k+1} + q_k} < |D_k| < \frac{1}{q_{k+1}}\,. \tag{5}$$

Since $q_k \geq 2$ for $k \geq 2$, we have from the above lemma that $D_k = (-1)^k \|q_k \alpha\|$ for all $k \geq 1$. Also we have that D_k alternates the sign, that is

$$D_k D_{k+1} < 0 \qquad (k \geq 0). \tag{6}$$

In particular, in view of (3) and (6) we have that

$$a_{k+1}|D_k| + |D_{k+1}| = |D_{k-1}| \qquad (k \geq 1). \tag{7}$$

One might raise the question of how well convergents approximate real numbers. It turns out that they are best approximations.

Definition 3 *Let $\alpha \in \mathbb{R}$. The integer $q > 0$ is called a best approximation to α if*

$$\forall\, q' \in \mathbb{N},\;\; q' < q \implies \|q'\alpha\| > \|q\alpha\|.$$

Lemma 4 *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. An integer $q > 0$ is a best approximation to α if and only if $q = q_k$ (where q_k denotes the denominator of the kth convergent to α) for some k.*

We also have the notion of badly approximable numbers.

Definition 4 *A real number α is called badly approximable if there is a constant $c > 0$ such that for $p, q \in \mathbb{Z}$ with $q > 0$, we have that*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^2}.$$

There are uncountably many badly approximable numbers, and the following lemmas show some properties they can be identified with, and also show how they are characterised by continued fractions.

Lemma 5 *A real number α is badly approximable if and only if*

$$\liminf_{q\to\infty} q\|q\alpha\| > 0.$$

Lemma 6 *A real irrational number $\alpha = [a_0; a_1, a_2, \ldots]$ is badly approximable if and only if the partial quotients in its continued fraction expansion are bounded, that is, there is a $C > 0$ such that*

$$a_k \leq C \qquad \textit{for all}\;\; k \in \mathbb{N}.$$

Lemma 7 *Any quadratic irrationality is badly approximable.*

The next natural question to ask would be whether other irrational algebraic numbers besides quadratic irrationals are badly approximable. We see an even stronger condition in the Thue-Siegel-Roth Theorem, also known as Roth's Theorem [48], a Fields Medal winning work.

Theorem 5 *(Roth, 1955) Suppose $\alpha \in \mathbb{R}$ is algebraic and irrational. Then for any $\epsilon > 0$ and coprime integers p, q, there are only finitely many solutions p, q, to*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}.$$

In other words, the above theorem means that every irrational algebraic $\alpha \in \mathbb{R}$ satisfies

$$\left|\alpha - \frac{p}{q}\right| > \frac{c(\alpha, \epsilon)}{q^{2+\epsilon}},$$

where the constant $c(\alpha, \epsilon) > 0$ depends on α and ϵ. Roth's Theorem was a result improving Liouville's work [39] in 1844 where instead of $2 + \epsilon$, the theorem was stated using degree n over the rationals. It is still a major open problem to determine whether real algebraic numbers of degree ≥ 3 are badly approximable, though it is believed that they are not.

1.4 Metric Diophantine Approximation

We now move on to some results in Metric Diophantine Approximation. A large number of the above results deal with variations of Dirichlet's Theorem. This discussion can be extended to general error functions.

Definition 5 *Given $\psi : \mathbb{N} \to (0, +\infty)$, the real number α will be called ψ-approximable or ψ-well approximable if*

$$\|q\alpha\| < \psi(q)$$

for infinitely many $q \in \mathbb{N}$. If $\psi(q) = q^{-\tau}$, then we say that α is τ-approximable. In what follows, $W(\psi)$ will denote the set of all ψ-approximable real numbers. In addition, given $\tau > 0$, $W(\tau)$ will denote the set of all τ-approximable real numbers.

We refer to the function ψ as an approximating function as it controls how well the rationals approximate the reals. It can be easily verified that $W(\psi)$ is invariant under translations by integers. As such, it is convenient for us to restrict $W(\psi)$ to the unit interval $\mathbb{I} := [0, 1)$, and we can do this without any loss of generality. We denote this new set as follows:

$$\mathcal{A}(\psi) = W(\psi) \cap [0, 1) = \{\alpha \in \mathbb{I} : \|q\alpha\| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}.$$

Investigating the size of the set $\mathcal{A}(\psi)$ leads Khintchine [33] to the following theorem, which gives a simple criterion for determining its measure. For more details on the subject, see [13], [28], [50], and [55].

Theorem 6 *(Khintchine, 1924) Let λ denote Lebesgue measure in $\mathbb{R}$. For any approximating function $\psi : \mathbb{N} \to (0, +\infty)$,*

$$\lambda(\mathcal{A}(\psi)) = \begin{cases} 0, & \text{if } \sum_{q=1}^{\infty} \psi(q) < \infty, \\ 1, & \text{if } \sum_{q=1}^{\infty} \psi(q) = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

1.5 Hausdorff Measures and Dimension

In order to state the next theorem, we require the concepts of Hausdorff measures and dimension. In what follows, a dimension function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a left-continuous function which is monotonic near the origin, and $f(r) \to 0$ as $r \to 0$. Let F be a subset of $\mathbb{R}^n$ and $\rho > 0$. Any finite or countable collection $\{B_i\}$ of balls of radius $r(B_i) \leq \rho$, for all i, such that

$$F \subset \bigcup_i B_i$$

is called a ρ-cover of F. For $\rho > 0$, define

$$\mathcal{H}^f_\rho(F) = \inf \left\{ \sum_i f(r(B_i)) : \{B_i\} \text{ is a } \rho\text{-cover for } F \right\},$$

where the infimum is taken over all ρ-covers of F. Then, the Hausdorff f-measure of F is defined as

$$\mathcal{H}^f(F) = \lim_{\rho \to 0} \mathcal{H}^f_\rho(F) = \sup_{\rho > 0} \mathcal{H}^f_\rho(F).$$

In the case where $f(r) = r^s$, for some $s \geq 0$, we write $\mathcal{H}^s(F) = \mathcal{H}^f(F)$ for the more common Hausdorff s-measure, also known as the s-dimensional Hausdorff measure, the measure $\mathcal{H}^0$ being the cardinality of F.

For $s \in \mathbb{N}$, we have that $\mathcal{H}^s(F)$ is a constant multiple of the Lebesgue measure of F in $\mathbb{R}^s$ and that this constant is 1 when $s = 1$. Hence if we know the s-dimensional Hausdorff measure of a set for each $s > 0$, we also know its n-dimensional Lebesgue measure for each $n \geq 1$. It can be easily verified that

$$\mathcal{H}^s(F) < \infty \quad \Longrightarrow \quad \mathcal{H}^{s'}(F) = 0 \qquad \text{if } s' > s.$$

Consequently, there exists a unique real point s_0 at which the Hausdorff s-measure drops from infinity to 0, that is,

$$\mathcal{H}^s(F) = \begin{cases} 0 & \text{if } s > s_0, \\ \infty & \text{if } s < s_0. \end{cases}$$

This point s_0 is called the Hausdorff dimension of F and is defined as

$$\dim_H F = \inf \{ s > 0 : \mathcal{H}^s(F) = 0 \} .$$

Note that at the critical value $s = s_0$, $\mathcal{H}^s(F)$ could be 0, ∞, or finite and positive.

Jarník [29] in 1929 and Besicovitch [12] in 1934 both independently determined the Hausdorff dimension of the set $\mathcal{A}(\tau) = W(\tau) \cap [0, 1)$, where $\tau > 0$.

Theorem 7 *(Jarník-Besicovitch)*

$$\dim_H \mathcal{A}(\tau) = \begin{cases} \frac{2}{\tau+1} & \textit{if } \tau > 1, \\ 1 & \textit{if } \tau \leq 1. \end{cases}$$

For $\tau \leq 1$, the result is trivial as $\mathcal{A}(\tau) = \mathbb{I}$ is simply a consequence of Dirichlet's Theorem. We therefore focus on the case when $\tau > 1$ to see that

$$\mathcal{H}^s(\mathcal{A}(\tau)) = \begin{cases} 0 & \text{if } s > \frac{2}{\tau+1}, \\ \infty & \text{if } s < \frac{2}{\tau+1}, \end{cases}$$

although we still have no information on $\mathcal{H}^s(\mathcal{A}(\tau))$ at $s = \dim_H \mathcal{A}(\tau)$. In a further study, Jarník [30] proves the following more general result which can be viewed as the Hausdorff measure analogue of Khintchine's Theorem. In [6], there is a less technical version of the theorem, which we present here.

Theorem 8 *(Jarník, 1931) Let $s \in (0,1)$ and $\psi : \mathbb{N} \to \mathbb{R}^+$. Then*

$$\mathcal{H}^s(\mathcal{A}(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{1-s}\psi(q)^s < \infty, \\ \infty & \text{if } \sum_{q=1}^{\infty} q^{1-s}\psi(q)^s = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

For $\tau > 1$, $\psi(q) = q^{-\tau}$, the above theorem not only implies that $\dim_H W(\tau) = \frac{2}{\tau+1}$ but also tells us that

$$\mathcal{H}^s(W(\tau)) = \infty \qquad \text{at } s = \frac{2}{\tau+1}.$$

Further, we see that the sum in the above theorem is now

$$\sum_{q=1}^{\infty} q^{1-s-s\tau}$$

and this sum converges if and only if $s > \frac{2}{\tau+1}$. Thus it is clear that the Jarník-Besicovitch Theorem follows from Jarník's Theorem. In fact, a consequence of a result due to Beresnevich and Velani [10] called the Mass Transference Principle, it turns out that Khintchine's Theorem implies the seemingly more general Jarník's Theorem. Use of the Mass Transference Principle even allows us to see that Dirichlet's Theorem implies the Jarník-Besicovitch Theorem.

2 The Three Distance Theorem

The main question we discuss in this section is the following: *given an $N \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, what can we say about the distribution of the points*

$$\{\alpha\}, \{2\alpha\}, \ldots, \{N\alpha\} \tag{8}$$

in the unit interval $[0,1)$*?* Equivalently this question can be posed using circle rotations on identifying $[0,1)$ with the unit circle. The unit interval $[0,1)$ can be embedded into the unit circle by the map $\alpha \mapsto e^{2\pi i\alpha}$. Under this embedding, 0 and 1 are identified with $0 = e^{2\pi i 0\alpha}$, and $\{n\alpha\}$ is identified with $e^{2\pi i n\alpha}$. Then the points $e^{2\pi i n\alpha}$ for $n = 0, \ldots, N$ partition the circle into $N+1$ arcs. The following statement conjectured by Hugo Steinhaus is widely known by various names, such as the Steinhaus conjecture, the three distance, three gap, three step, or three length theorem.

2.1 The Theorem

Theorem 9 *(The Three Distance Theorem) For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and any integer $N \geq 1$ the points $\{n\alpha\}$, for $n = 0, \ldots, N$ partition $[0,1]$ into $N+1$ intervals which lengths take at most 3 different values δ_A, δ_B and δ_C with $\delta_C = \delta_A + \delta_B$.*

There are various generalisations of the above fact and several independent proofs. It was originally conjectured by Hugo Steinhaus and then first proved by Sós ([53], [54]), followed by Świerckowski [57], Surányi [56], Halton [22], Slater [52], and also more recently by Van Ravenstein [44], Langevin [37], and Mayero [41]. One can find a survey of different approaches used by these authors in [37, 44, 52].

Remarkably, the length of the gaps as well as the number of gaps of every length can be exactly specified using the continued fraction expansion of α. Within this thesis we will use an even more precise statement that also specifies the order in which the intervals of various lengths appear.

Theorem 10 *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $[a_0; a_1, a_2, \ldots]$ be the continued fraction expansion of α, and $D_k = q_k\alpha - p_k$, where $\frac{p_k}{q_k}$ are the convergents to α. Then for any $N \in \mathbb{N}$ there exists a unique integer $k \geq 0$ such that*

$$q_k + q_{k-1} \leq N < q_{k+1} + q_k \tag{9}$$

and unique integers r and s satisfying

$$N = rq_k + q_{k-1} + s, \qquad 1 \le r \le a_{k+1}, \qquad 0 \le s \le q_k - 1, \tag{10}$$

such that the points $\{\alpha\}, \{2\alpha\}, \ldots, \{N\alpha\}$ partition $[0,1]$ into $N+1$ intervals, of which

$$N_A = N + 1 - q_k \qquad \text{are of length } \delta_A = |D_k|, \tag{11}$$

$$N_B = s + 1 \qquad \text{are of length } \delta_B = |D_{k+1}| + (a_{k+1} - r)|D_k|, \tag{12}$$

$$N_C = q_k - s - 1 \qquad \text{are of length } \delta_C = \delta_A + \delta_B. \tag{13}$$

Furthermore, the unique permutation $(n_1, \ldots, n_N)$ of $(1, \ldots, N)$ such that

$$0 < \{n_1\alpha\} < \{n_2\alpha\} < \ldots < \{n_{N-1}\alpha\} < \{n_N\alpha\} < 1 \tag{14}$$

is given by defining $n_0 = 0$ and

$$n_{i+1} = n_i + \Delta_i \qquad \text{for } i = 0, \ldots, N-1, \tag{15}$$

where

$$\Delta_i = \begin{cases} (-1)^k q_k & \text{if } \; n_i \in A\,, \\ (-1)^{k-1}\big(q_{k-1} + rq_k\big) & \text{if } \; n_i \in B\,, \\ (-1)^{k-1}\big(q_{k-1} + (r-1)q_k\big) & \text{if } \; n_i \notin (A \cup B) \end{cases} \tag{16}$$

and

$$\begin{aligned} A &:= \big\{ n \in \mathbb{Z} \cap [0,N] \;:\; 0 \le n + (-1)^k q_k \le N \big\}\,, \\ B &:= \big\{ n \in \mathbb{Z} \cap [0,N] \;:\; 0 \le n + (-1)^{k-1}\big(q_{k-1} + rq_k\big) \le N \big\}\,, \\ C &:= \big\{ n \in \mathbb{Z} \cap [0,N] \;:\; n \notin (A \cup B) \big\} \end{aligned} \tag{17}$$

are disjoint subsets of integers.

2.2 Proof of the Three Distance Theorem

Proof. Observe that $(q_k + q_{k-1})_{k \ge 0}$ is a strictly increasing sequence of integers starting from $q_0 + q_{-1} = 1$. Hence the existence of k satisfying (9) readily follows. Next, observe that $q_k \leq N - q_{k-1} < q_{k+1} + q_k - q_{k-1} = (a_{k+1} + 1)q_k$. Therefore, by division with remainder, r and s satisfying (10) exist and are unique. Now we verify that $A \cap B = \varnothing$. Indeed, if there existed $n \in A \cap B$, then in the case of even k we would have that

$$rq_k + q_{k-1} \overset{n\in B}{\leq} n \overset{n\in A}{\leq} N - q_k = rq_k + q_{k-1} + s - q_k \overset{s<q_k}{<} rq_k + q_{k-1}\,,$$

while in the case of odd k we would have that

$$q_k \overset{n\in A}{\leq} n \overset{n\in B}{\leq} N - (rq_k + q_{k-1}) = s < q_k.$$

In both instances we would get a contradiction, hence A and B are disjoint. In particular, it means that the sequence n_i defined by (15) is well defined. Now we show that

$$0 \leq n_i \leq N \qquad \text{for all} \quad 0 \leq i \leq N\,. \tag{18}$$

The proof is by induction. Clearly, $n_0 = 0$ satisfies the inequalities. Now suppose that $i < N$ and $0 \leq n_i \leq N$. We shall prove these inequalities for n_{i+1}. If $n_i \in A \cup B$ this claim follows immediately from the definition of A and B. Thus we will assume that $n_i \notin A \cup B$. Then, assuming k is even, we have that $N - q_k < n_i < q_{k+1} - (a_{k+1} - r)q_k$ and therefore

$$n_{i+1} = n_i - \big(q_{k-1} + (r-1)q_k\big) > N - q_k - \big(q_{k-1} + (r-1)q_k\big) =$$

$$= rq_k + q_{k-1} + s - q_k - \big(q_{k-1} + (r-1)q_k\big) = (r-1)q_k + s - (r-1)q_k = s \geq 0$$

while clearly $n_{i+1} < n_i \leq N$. Further, assuming k is odd, we get that

$$n_{i+1} = n_i + \big(q_{k-1} + (r-1)q_k\big) < q_k + \big(q_{k-1} + (r-1)q_k\big) = rq_k + q_{k-1} \leq N$$

while clearly $0 \leq n_i < n_{i+1}$. Thus (18) is indeed satisfied.

Further, observe that

$$
\begin{aligned}
N_A\delta_A + N_B\delta_B + N_C\delta_C &= (N-s)\delta_A + q_k\delta_B \\
&= (rq_k + q_{k-1})|D_k| + q_k(|D_{k+1}| + (a_{k+1}-r)|D_k|) \\
&= \big(a_{k+1}q_k + q_{k-1}\big)|D_k| + q_k|D_{k+1}| \overset{(1)}{=} q_{k+1}|D_k| + q_k|D_{k+1}| \\
&\overset{(6)}{=} |q_{k+1}D_k - q_kD_{k+1}| \overset{(2)}{=} |q_{k+1}(q_k\alpha - p_k) - q_k(q_{k+1}\alpha - p_{k+1})| \\
&= |-q_{k+1}p_k + q_kp_{k+1}| \overset{(4)}{=} 1\,.
\end{aligned} \tag{19}
$$

In particular, it meant that the quantities δ_A, δ_B and δ_C are all strictly less than 1.

Next we will show that for $0 \le i < N$

$$
\{(n_{i+1}-n_i)\alpha\} = \begin{cases} \delta_A & \text{if } n_i \in A, \\ \delta_B & \text{if } n_i \in B, \\ \delta_C & \text{if } n_i \notin A \cup B. \end{cases} \tag{20}
$$

Indeed, in the case $n_i \in A$ we have that $n_{i+1} - n_i = (-1)^k q_k$ and then

$$
\{(n_{i+1}-n_i)\alpha\} = \{(-1)^k q_k\alpha\} = \{(-1)^k(q_k\alpha - p_k)\} = \{|D_k|\} = |D_k|\,,
$$

since $|D_k| < 1$. In the case $n_i \in B$ we have that

$$
\{(n_{i+1}-n_i)\alpha\} = \{(-1)^{k-1}\big(q_{k-1}+rq_k\big)\alpha\} = \{(-1)^{k-1}\big(q_{k+1}-(a_{k+1}-r)q_k\big)\alpha\}
$$

$$
= \{(-1)^{k-1}\big(q_{k+1}\alpha - p_{k+1} - (a_{k+1}-r)(q_k\alpha - p_k)\big)\} = \{|D_{k+1}| + (a_{k+1}-r)|D_k|\} = \{\delta_B\} = \delta_B\,.
$$

Finally, if $n_i \notin A \cup B$,

$$
\{(n_{i+1}-n_i)\alpha\} = \{(-1)^{k-1}\big(q_{k-1}+(r-1)q_k\big)\alpha\} = \{(-1)^{k-1}\big(q_{k+1}-(a_{k+1}+1-r)q_k\big)\alpha\}
$$

$$
= \{(-1)^{k-1}\big(q_{k+1}\alpha - p_{k+1} - (a_{k+1}+1-r)(q_k\alpha - p_k)\big)\} = \{|D_{k+1}| + (a_{k+1}+1-r)|D_k|\} = \delta_C\,.
$$

Now, we prove (14). Let $C = \mathbb{Z} \cap [0,N] \setminus (A \cup B)$. For a start, note that $0 < \{n_1\alpha\}$ since α is irrational. The proof continues by induction. Suppose that $1 \le i < N$ and that

$$
0 < \{n_1\alpha\} < \cdots < \{n_i\alpha\}\,.
$$

This means that $n_0, n_1, \ldots, n_i$ are all different. Since they must be members of the disjoint sets A, B and C, by (19) and (20), we get that

$$\sum_{j=0}^{i}\{(n_{j+1}-n_j)\alpha\} < \#A\cdot\delta_A + \#B\cdot\delta_B + \#C\cdot\delta_C = N_A\delta_A + N_B\delta_B + N_C\delta_C = 1\,, \quad (21)$$

where $\#X$ is the cardinality of X. Note that for any real numbers x and y if $\{x\}+\{y\} < 1$ then $\{x\}+\{y\} = \{x+y\}$. Therefore,

$$\sum_{j=0}^{i}\{(n_{j+1}-n_j)\alpha\} = \left\{\sum_{j=0}^{i}(n_{j+1}-n_j)\alpha\right\} = \{(n_{i+1}-n_0)\alpha\} = \{n_{i+1}\alpha\}\,. \quad (22)$$

Similarly, $\sum_{j=0}^{i-1}\{(n_{j+1}-n_j)\alpha\} = \{n_i\alpha\}$ and thus $\{n_{i+1}\alpha\} = \{(n_{i+1}-n_i)\alpha\} + \{n_i\alpha\} > \{n_i\alpha\}$. This completes the proof of (14). Consequently, the integers $n_0, \ldots, n_N$ are all different and, since $n_0 = 0$, by (18), $(n_1, \ldots, n_N)$ is the required permutation of $(1, \ldots, N)$.

From (11), (12), (13) and (17), it is clear that

$$N_A = \#A, \qquad N_B = \#B, \qquad \text{and} \qquad N_C = \#C, \quad (23)$$

and also that

$$N_A + N_B + N_C = N + 1. \quad (24)$$

Now let N'_A, N'_B and N'_C be the number of intervals of length δ_A, δ_B and δ_C respectively that occur amongst the first N gaps, so

$$N'_A + N'_B + N'_C = N. \quad (25)$$

We can see by (15) and (16) that each of the first N gaps corresponds to unique integers lying in one of the sets A, B or C. Hence since A, B and C are disjoint subsets of integers, by (23), we have that

$$N'_A \le N_A, \qquad N'_B \le N_B, \qquad \text{and} \qquad N'_C \le N_C. \quad (26)$$

Further, by (24) and (25), of the three inequalities in (26) we must have equality for two of them, while the left hand side of the remaining inequality is exactly one less than the right hand side. Hence, by (19), the remaining gap between $\{n_N\alpha\}$ and 1, that is to say, the "$N+1$"st gap, must be exactly of length

$$\begin{aligned}&(N_A\delta_A + N_B\delta_B + N_C\delta_C) - (N'_A\delta_A + N'_B\delta_B + N'_C\delta_C)\\ &\quad = (\#A\cdot\delta_A + \#B\cdot\delta_B + \#C\cdot\delta_C) - (N'_A\delta_A + N'_B\delta_B + N'_C\delta_C).\end{aligned}$$

So if $N'_C < N_C$, then we have

$$N'_A = N_A, \qquad N'_B = N_B, \qquad \text{and} \qquad N'_C = N_C - 1 < N_C,$$

and the "$N+1$"st gap will be of length δ_C. Similarly, if instead $N'_A < N_A$, then the "$N+1$"st gap will be of length δ_A, and if $N'_B < N_B$, then it will be of length δ_B. Therefore there are exactly N_A, N_B, N_C gaps of length δ_A, δ_B, δ_C respectively, and we can see that (11), (12), and (13) follow. $\boxtimes$

Strictly speaking Theorem 10 is not new and can be assembled from published results. The closest versions can be found in [5], as well as in [1], in which it is also remarked that the above result can be reformulated in terms of n-Farey points.

Remark. Since $\frac{1}{q_k}$ decreases as q_k increases, it is clear from (5) that $|D_{k+1}| < |D_k|$. Hence, $\delta_B < \delta_A$ if and only if $r = a_{k+1}$, otherwise, δ_A will be the smallest length.

Remark. As α is irrational, the three lengths are distinct. In fact, while the two lengths δ_A and δ_B always appear, gaps of the largest length, δ_C, exist if and only if $s < q_k - 1$. There are infinitely many integers N for which there are only two lengths. The structure and the transformation rules for the partitioning in the two length intervals are studied in detail in [44]. Other in-depth studies involving the two length case can also be found in [14, 15, 16, 45, 51].

2.3 An Illustration

To help visualise how the Three Distance Theorem works, Dave Richeson made an interesting GeoGebra applet [46] which one can use to see how gaps form on the unit circle by varying α and N. This inspired Nick Hamblet to write a code for an interactive Sage notebook [23], resulting in a version of the applet which gives more details.

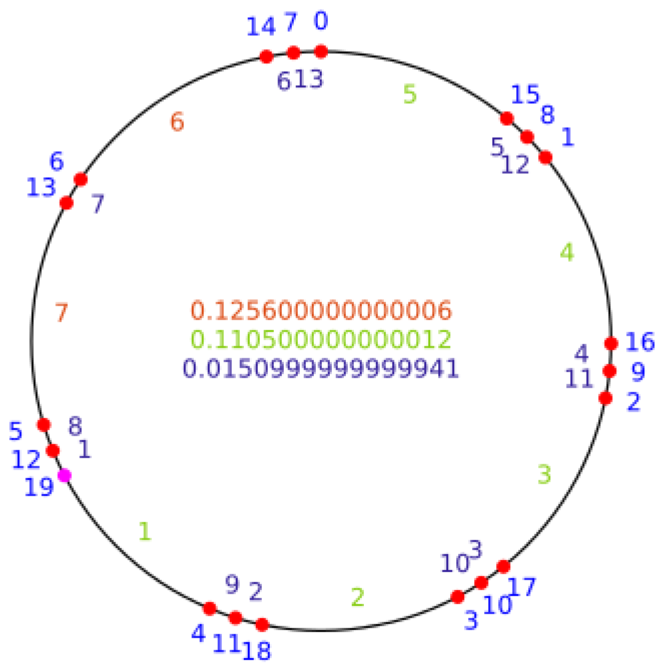

Figure 1: Illustration of the Three Distance Theorem based on $N = 20$, $\alpha \approx \{\pi\}$.

In Figure 1 we present an illustration of how the Three Distance Theorem behaves using Hamblet's code. The figure is based on $N = 20$, $\alpha \approx \{\pi\}$, where the list of numbers in the middle of the circle are the three possible gap lengths in decreasing order. The numbers along the outer side of the circle represent the order of the points $\{n_i\alpha\}$, which travel clockwise from $0 = \{n_N\alpha\}$, located at the top of the circle. The numbers along the inside of the circle are colour-coded to the list of gap lengths, and they tell us the order in which the gaps appear.

It is interesting to note that the gaps seem to form in clusters, which raises the question of whether this phenomenon is an intrinsic property of the gaps or a mere coincidence. This naturally leads us to investigate the structure of the gaps in greater detail.

2.4 More on Gaps Structure

Corollary 1 *For $0 \leq i \leq N-1$ the gap between $\{n_{i+1}\alpha\}$ and $\{n_i\alpha\}$ is*

$$\{n_{i+1}\alpha\} - \{n_i\alpha\} = \{(n_{i+1} - n_i)\alpha\} = \begin{cases} \delta_A & \text{if } n_i \in A\,, \\ \delta_B & \text{if } n_i \in B\,, \\ \delta_C & \text{otherwise} \end{cases}$$

while

$$1 - \{n_N\alpha\} = \begin{cases} \delta_A & \text{if } k \text{ is odd}\,, \\ \delta_B & \text{if } k \text{ is even.} \end{cases}$$

Furthermore, it is always true that $\{n_1\alpha\} \neq 1 - \{n_N\alpha\}$.

Proof. The first assertion of Corollary 1 readily follows from the proof of Theorem 10, in particular, from (20) and (22).

Now we prove that $\{n_1\alpha\} \neq 1 - \{n_N\alpha\}$. This follows from the fact that $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Let α be an irrational number and assume for a contradiction that $\{n_1\alpha\} = 1 - \{n_N\alpha\}$. This gives us

$$n_1\alpha - p_1 = \{n_1\alpha\} = 1 - \{n_N\alpha\} = 1 - n_N\alpha + p_N$$

for some $p_1, p_N \in \mathbb{Z}$, which tells us that

$$\alpha = \frac{1 + p_1 + p_N}{n_1 + n_N} \in \mathbb{Q},$$

a contradiction.

Finally, we prove the second assertion of Corollary 1. By Definition 3, we know that the convergents $\frac{p_k}{q_k}$ are best approximations to α. So since $n_0 = 0$, by the definitions of A, B, C in (17), we have that

$$\{n_1\alpha\} = \begin{cases} \delta_A & \text{if } k \text{ is even,} \\ \delta_B & \text{if } k \text{ is odd.} \end{cases} \tag{27}$$

First assume k is odd and consider $n = q_k$. Then

$$\{n\alpha\} = \{q_k\alpha\} = q_k\alpha - p_k + 1$$

for some $p_k \in \mathbb{N}$, since $\{n\alpha\} \in [0, 1]$ and by Lemma 3, $q_k\alpha - p_k < 0$ if k is odd. We therefore have

$$1 - \{n\alpha\} = |D_k| = \delta_A.$$

Hence $1 - \{n_N\alpha\} \leq 1 - \{n\alpha\} = \delta_A < \delta_C$. Furthermore, (27) and the last assertion of Corollary 1 tells us that $1 - \{n_N\alpha\} \neq \{n_1\alpha\} = \delta_B$. Thus $1 - \{n_N\alpha\} \neq \delta_B$ or δ_C and it must be δ_A.

If instead k is even, consider $n = rq_k + q_{k-1}$. Then

$$\{n\alpha\} = \{(rq_k + q_{k-1})\alpha\} = rq_k\alpha - rp_k + q_{k-1}\alpha - p_{k-1} + 1.$$

Note that $\{n\alpha\} \in [0, 1]$ and, by Lemma 3, $r(q_k\alpha - p_k) > 0$ as k is even and $q_{k-1}\alpha - p_{k-1} < 0$ as $k - 1$ is odd. Therefore by (7), we have

$$\begin{aligned} 1 - \{n\alpha\} &= |D_{k-1}| - r|D_k| \\ &= |D_{k+1}| + (a_{k+1} - r)|D_k| = \delta_B. \end{aligned}$$

Hence $1 - \{n_N\alpha\} \leq 1 - \{n\alpha\} = \delta_B < \delta_C$. Since again by (27) and the last assertion of Corollary 1 we have that $1 - \{n_N\alpha\} \neq \{n_1\alpha\} = \delta_A$, therefore we must have that $1 - \{n_N\alpha\} = \delta_B$. ⊠

Corollary 2 *Let $\delta_A, \delta_B, \delta_C$, as well as a_k and r be defined as in Theorem 10. Then*

- *there are no more than $r+1$ consecutive gaps of length δ_A,*
- *there is no more than one consecutive gap of length δ_B,*
- *if $r=1$, there are no more than a_k+1 consecutive gaps of length δ_C, and if $r>1$, there is no more than one consecutive gap of length δ_C.*

Proof. Assume k even. Then Δ_i is positive if $n_i \in A$, and negative if $n_i \in B$ or C, where Δ_i and n_i are defined as in Theorem 10. By the definition of the sets A, B, C and the fact that $N = rq_k + q_{k-1} + s$, we will have that

$$\begin{aligned} A &= \mathbb{Z} \cap [0, N - q_k], \\ C &= \mathbb{Z} \cap [N - q_k + 1, N - s - 1], \\ B &= \mathbb{Z} \cap [N - s, N]. \end{aligned} \tag{28}$$

Suppose that $n_i, n_{i+1}, \ldots, n_{i+l_A} \in A$. This corresponds to having $l_A + 1$ consecutive gaps of length δ_A. Using (16) and (28) gives us

$$0 \leq n_i + l_A q_k \leq N - q_k.$$

Then since $n_i \geq 0$, using the expression for N and the bound for s from Theorem 10, we obtain

$$l_A \leq \left[\frac{N - q_k}{q_k}\right] = \left[\frac{rq_k + q_{k-1} + s - q_k}{q_k}\right] \leq \left[\frac{rq_k + q_{k-1} - 1}{q_k}\right] = r. \tag{29}$$

Therefore, the number of consecutive gaps of length δ_A is no more than $r+1$.

Similarly, now suppose $n_i, \ldots, n_{i+l_B} \in B$, that is, we have $l_B + 1$ consecutive gaps of length δ_B. Using (16) and (28) gives us

$$n_i - l_B(q_{k-1} + rq_k) \geq N - s.$$

Hence, using the fact that $n_i \leq N = rq_k + q_{k-1} + s$, we get that

$$N - l_B(N - s) \geq N - s,$$

whence

$$l_B + 1 \leq \left[\frac{N}{N-s}\right] = 1 + \left[\frac{s}{rq_k + q_{k-1}}\right] = 1$$

since $s < q_k$ and $r \geq 1$. Therefore $l_B = 0$ and we never have more than one consecutive gap of length δ_B.

Finally, suppose $n_i, \ldots, n_{i+l_C} \in C$ and we have $l_C + 1$ consecutive gaps of length δ_C. Using (16) and (28) gives

$$n_i - l_C(q_{k-1} + (r-1)q_k) \geq N - q_k + 1.$$

Then since $n_i \leq N - s - 1$, we obtain

$$(N - s - 1) - (N - q_k + 1) \geq l_C(q_{k-1} + (r-1)q_k).$$

Now recall from an earlier remark that gaps of length δ_C exist if and only if $0 \leq s \leq q_k - 2$, so we have

$$l_C \leq \left[\frac{q_k - 2 - s}{q_{k-1} + (r-1)q_k}\right] \leq \left[\frac{q_k - 2}{q_{k-1} + (r-1)q_k}\right].$$

Note that $q_{k-1} + (r-1)q_k = N - q_k - s > 0$ and hence the above division is legitimate. So if $r > 1$, then $l_C = 0$ and we never have more than one consecutive gap of length δ_C. If $r = 1$, then

$$l_C \leq \left[\frac{q_k - 2}{q_{k-1}}\right] = \left[\frac{a_k q_{k-1} + q_{k-2} - 2}{q_{k-1}}\right] \leq a_k$$

since $q_{k-2} < q_{k-1}$, and we therefore have no more than $a_k + 1$ consecutive gaps of length δ_C.

Now assume k odd. In this case the proof is analogous to when k is even but is included for completeness. Let Δ_i and n_i be defined as in Theorem 10. Then Δ_i is negative if $n_i \in A$, and positive if $n_i \in B$ or C. By the definition of the sets A, B, C and the fact that $N = rq_k + q_{k-1} + s$, we have that

$$\begin{aligned} B &= \mathbb{Z} \cap [0, s], \\ C &= \mathbb{Z} \cap [s+1, q_k - 1], \\ A &= \mathbb{Z} \cap [q_k, N]. \end{aligned} \tag{30}$$

Suppose that $n_i, n_{i+1}, \ldots, n_{i+l_A} \in A$. This corresponds to having $l_A + 1$ consecutive gaps of length δ_A. Using (16) and (30), we have

$$q_k \leq n_i - l_A q_k \leq N.$$

Then since $n_i \leq N$, we use the expression for N and the bound for s from Theorem 10 to obtain

$$l_A + 1 \leq \left[\frac{N}{q_k}\right] = \left[\frac{rq_k + q_{k-1} + s}{q_k}\right] = r + 1 + \left[\frac{q_{k-1} - 1}{q_k}\right] = r + 1.$$

So $l_A \leq r$ and the number of consecutive gaps of length δ_A is therefore not more than $r + 1$.

Similarly, suppose $n_i, \ldots, n_{i+l_B} \in B$, that is we have $l_B + 1$ consecutive gaps of length δ_B. Using (16) and (30) gives

$$0 \leq n_i + l_B(q_{k-1} + rq_k) \leq s.$$

Then as $n_i \geq 0$, using the bound for s from Theorem 10, we get

$$l_B \leq \left[\frac{s}{q_{k-1} + rq_k}\right] \leq \left[\frac{q_k - 1}{q_{k-1} + rq_k}\right] = 0.$$

Therefore, there is no more than one consecutive gap of length δ_B.

Finally, suppose that $n_i, \ldots, n_{i+l_C} \in C$ so we have $l_C + 1$ consecutive gaps of length δ_C. Using (16) and (30) gives

$$n_i + l_C(q_{k-1} + (r-1)q_k) \leq q_k - 1.$$

Then since $n_i \geq s + 1$, we have

$$(s + 1) + l_C(q_{k-1} + (r-1)q_k) \leq q_k - 1.$$

Therefore,

$$l_C \leq \left[\frac{q_k - 2 - s}{q_{k-1} + (r-1)q_k}\right] \leq \left[\frac{q_k - 2}{q_{k-1} + (r-1)q_k}\right].$$

Then by exactly the same argument as the case for when k is even, we see that if $r > 1$, then $l_C = 0$ and we never have more than one consecutive gap of length δ_C. Further, if $r = 1$, then $l_C \leq a_k$ and we have no more than $a_k + 1$ consecutive gaps of length δ_C. $\boxtimes$

3 Sums of Reciprocals

In this section we discuss sums of reciprocals and their significance and uses in various areas of mathematics. Let $N \in \mathbb{N}$. We mainly focus on sums of the form

$$T_N(\alpha, \gamma) := \sum_{n=1}^{N} \frac{1}{\{n\alpha - \gamma\}}$$

and

$$R_N(\alpha, \gamma) := \sum_{n=1}^{N} \frac{1}{\|n\alpha - \gamma\|}$$

where α is irrational and $\gamma \in \mathbb{R}$ such that $\{n\alpha - \gamma\}, \|n\alpha - \gamma\| \neq 0 \ \forall n \in \mathbb{N}$.

Remark. It is easily seen that $R_N(\alpha, \gamma)$ can be related to the sum

$$S_N(\alpha, \gamma) := \sum_{n=1}^{N} \frac{1}{n\|n\alpha - \gamma\|}$$

where α is irrational, $N \in \mathbb{N}$ and $\gamma \in \mathbb{R}$ such that $\|n\alpha - \gamma\| \neq 0 \ \forall n \in \mathbb{N}$. This is done using the well known Abel's summation formula or partial summation formula: given sequences (a_n) and (b_n) with $n \in \mathbb{N}$,

$$\sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N} (a_n - a_{n+1})(b_1 + \ldots + b_n) + a_{N+1}(b_1 + \ldots + b_N), \tag{31}$$

which yields

$$S_N(\alpha, \gamma) = \sum_{n=1}^{N} \frac{R_N(\alpha, \gamma)}{n(n+1)} + \frac{R_N(\alpha, \gamma)}{N+1}.$$

Although the sum $S_N(\alpha, \gamma)$ is not a main focus in this thesis, it has a multitude of applications, especially to Khintchine type theorems in metric Diophantine Approximation (see [8, 35, 49] for more details).

3.1 Background

These sums are extremely useful and their bounds have therefore been the subject of extensive study for a long time, with bounds obtained through a wide variety of methods. One of the earliest recorded applications of these sums was in 1922 by Hardy and Littlewood [24, 25, 26] to determine an accurate approximation for the number of lattice-points in polygons. Their method used assumptions on the continued fraction expansion of α and was later refined by Haber and Osgood in [21] where they prove the following theorems.

Theorem 11 *If $t \geq 1$ and $A > 1$, and M and r are fixed positive numbers, then*

$$\sum_{n=k+1}^{[AK]} \|n\alpha\|^{-t} > \begin{cases} CK \log K, & t = 1\,, \\ CK^{1+\frac{(t-1)}{r}}, & t > 1, \end{cases}$$

if the convergents $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ of α satisfy

$$q_{i+1} < Mq_i^r$$

and α is either irrational or is a rational number whose denominator (when α is expressed in lowest terms) is greater than AK. $C = C(t, A, M, r)$ is independent of α and of K.

Theorem 12 *If t and r are any real numbers greater than 1, then there is a constant C and an irrational number α whose convergents $\frac{p_i}{q_i}$ satisfy*

$$q_{i+1} < Mq_i^r$$

(for some fixed M) such that

$$\sum_{n=K+1}^{[AK]} \|n\alpha\|^{-t} < CK^{1+\frac{(t-1)}{r}}$$

for arbitrarily large values of K.

Bounds for the homogeneous case, that is, $\gamma = 0$, were also obtained by Lang in [36] where he proved the following.

Definition 6 *Let $g \geq 1$ be an increasing positive function, and $B_0 \geq 10$ a positive integer. We say that α is of principal cotype $\leq g$ for all numbers $\geq B_0$ if given a number $B \geq B_0$, there exists a convergent $\frac{p_i}{q_i}$ to α such that $B < q_i \leq Bg(B)$.*

Remark. In particular, it is clear that if $\frac{p}{q}$ and $\frac{p'}{q'}$ are two successive convergents to α, and $q \geq B_0$, then $q' \leq qg(q)$.

Theorem 13 *Let α be of principal cotype $\leq g$ for all numbers $\geq B_0$. Then for all integers $N \geq B_0$ we have*

$$\sum_{n=1}^{N} \frac{1}{\{n\alpha\}} \leq 2N \log N + 20Ng(N) + K_0,$$

where

$$K_0 \leq \sum_{n=1}^{B_0 g(B_0)} \frac{1}{\{n\alpha\}}.$$

Remark. This same estimate holds if $\{n\alpha\}$ is replaced by $\|n\alpha\|$.

From Theorem 13 we have that

$$R_N(\alpha, 0) \ll N \log N + Ng(N). \tag{32}$$

Lang further remarks that this estimate is essentially best possible and that both the terms $N \log N$ and $Ng(N)$ are necessary. Hence there exist functions g, irrational numbers α, and arbitrarily large N such that the inequality (32) can be reversed.

Although (32) might not be ideal for all choices of α and N, we see that the estimate is accurate when α is badly approximable. We have, when α is badly approximable,

$$N \log N \ll R_N(\alpha, 0) \ll N \log N$$

for all $N > 1$. This can be seen as a result from Hardy and Littlewood's [24, 25] counting of lattice-points in certain polygons, later refined by Haber and Osgood (see

Theorem 11, 12 above) and also from Lê and Vaaler's paper in [38] where they used a different method involving Fourier analysis to obtain a more precise inequality:

$$\sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{1}{2\|n\alpha\|} \geq (N+1)\log(N+1) - N$$

for all $\alpha \in \mathbb{R}/\mathbb{Z}$ and sufficiently large N. In particular, this tells us that

$$R_N(\alpha, 0) \gg N \log N \tag{33}$$

holds for all N regardless of the properties of the irrational number α. This is further evidenced by Beresnevich, Haynes and Velani in [8] where they prove the following corollary.

Corollary 3 *For any irrational number α and any integer $N \geq 2$,*

$$\sum_{n=1}^{N} \frac{1}{\|n\alpha\|} \geq N \log N + N \log\left(\frac{e}{2}\right) + 2$$

and

$$\sum_{n=1}^{N} \frac{1}{n\|n\alpha\|} \geq \frac{1}{2}\left(\log N\right)^2.$$

In [38], Lê and Vaaler then go on to show that the bound (33) cannot be significantly improved and is in fact best possible for a large class of α, and also that for every sufficiently large N and $0 < \epsilon < 1$ there exists a subset $X_{N,\epsilon}$ of $\alpha \in [0,1]$ of Lebesgue measure greater than or equal to $1-\epsilon$ such that

$$R_N(\alpha, 0) \ll_\epsilon N \log N$$

for all $\alpha \in X_{N,\epsilon}$.

Schmidt [49] has shown that for any $\gamma \in \mathbb{R}$ and any $\epsilon > 0$,

$$(\log N)^2 \ll S_N(\alpha, \gamma) \ll (\log N)^{2+\epsilon}$$

for almost all $\alpha \in \mathbb{R}$. More precisely, for the homogeneous case $\gamma = 0$, we have the following result, proved in [8].

Theorem 14 *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\alpha = [a_0; a_1, a_2, \ldots]$, $N \in \mathbb{N}$ and let $m = m(N, \alpha)$ denote the largest non-negative integer satisfying $q_m \leq N$, where $\frac{p_k}{q_k}$ are the continued fraction convergents to α. Then, for all sufficiently large N,*

$$\max\left\{\frac{1}{2}(\log N)^2, A_{m+1}\right\} \leq S_N(\alpha, 0) \leq 33(\log N)^2 + 10A_{m+1},$$

where

$$A_k = \sum_{i=1}^{k} a_i$$

denotes the sum of the first $k \in \mathbb{N}$ partial quotients of α.

The inhomogeneous case, that is $\gamma \neq 0$, for the sums have not been well documented till recently [8] and we will establish bounds for them later in this thesis using a new method.

For the rest of this section, we will concern ourselves with the usefulness of these sums for problems in the metric theory of Diophantine Approximation, as well as in Dynamical Systems, in particular Uniform Distribution, and we will present a brief summary of examples on their applications in these areas so as to provide a sort of motivation for studying these sums.

3.2 Uniform Distribution

The Theory of Uniform Distribution modulo 1 is classically concerned with the distribution of fractional parts of real numbers on the unit interval $[0, 1]$. It is deeply rooted in Diophantine approximations and has connections to number theory, measure theory, ergodic theory, and probability theory, to name a few. For more complete proofs of results in this section, refer to Kuipers and Niederreiter's book [35], which provides a comprehensive and useful introduction to the subject.

3.2.1 Preliminary Results

Definition 7 *The sequence* (x_n) *of real numbers is said to be uniformly distributed modulo* 1 *if for every pair* a, b *of real numbers with* $0 \leq a < b < 1$*, we have*

$$\lim_{N\to\infty} \frac{\#\{n : 1 \leq n \leq N, \{x_n\} \in [a,b]\}}{N} = b - a.$$

Remark. Equidistributed and uniformly distributed mod 1 are some other common but equivalent ways of saying uniformly distributed modulo 1.

Definition 8 *Let* $x_1, \ldots, x_N$ *be a finite sequence of real numbers. Then the discrepancy of the given sequence is defined as*

$$D_N = D_N(x_1, \ldots, x_N) = \sup_{0\leq a<b<1} \left| \frac{\#\{x_n \in [a,b] : 1 \leq n \leq N\}}{N} - (b-a) \right|.$$

Now consider the sequence $(n\alpha)$, $n = 1, 2, \ldots$ for α an irrational number. This is a special class of uniformly distributed modulo 1 sequences where the discrepancy of $(n\alpha)$ is dependent on the arithmetical properties of α. We will show how sums of reciprocals simplify obtaining the bounds for the discrepancy for $(n\alpha)$ in the most general case. Note that sharper bounds for the discrepancy can be obtained depending on the properties of α. For more details, see [3, 4, 24, 25, 31, 35, 42].

3.2.2 Weyl's Criterion

Theorem 15 *(Weyl's Criterion) The sequence* (x_n)*,* $n = 1, 2, \ldots$ *is uniformly distributed modulo* 1 *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0$$

for all integers $h \neq 0$.

Whether or not the sequence $(n\alpha), n = 1, 2, \ldots$ is uniformly distributed modulo 1 is conditional on the rationality or irrationality of α.

If α is rational, then it is clear that $\{n\alpha\}$ can only take on finitely many values on the unit interval $[0, 1]$. In particular, for a reduced rational number $\alpha = \frac{p}{q}$, the only possible values for $\{n\alpha\}$ are

$$\left\{\frac{p}{q}\right\}, \left\{\frac{2p}{q}\right\}, \dots, \left\{\frac{(q-1)p}{q}\right\}.$$

Hence by definition, $\{n\alpha\}$, and therefore the sequence $(n\alpha)$ for $\alpha \in \mathbb{Q}$ cannot be uniformly distributed modulo 1.

If α is irrational, then we apply Weyl's Criterion. First, by the geometric series

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r}$$

with $r = e^{2\pi i h\alpha}$, we have

$$\left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h n\alpha} \right| = \left| \frac{1}{N} \left(\frac{1 - e^{2\pi i h(N+1)\alpha}}{1 - e^{2\pi i h\alpha}} - 1 \right) \right| = \left| \frac{1}{N} \left(\frac{e^{2\pi i h\alpha} - e^{2\pi i h(N+1)\alpha}}{1 - e^{2\pi i h\alpha}} \right) \right|. \tag{34}$$

By applying the triangle inequality, we obtain

$$\left| e^{2\pi i h\alpha} - e^{2\pi i h(N+1)\alpha} \right| \leq \left| e^{2\pi i h\alpha} \right| + \left| -e^{2\pi i h(N+1)\alpha} \right| \leq 2. \tag{35}$$

Then for an integer $h \neq 0$, $1 - e^{2\pi i h\alpha} \neq 0$, and we have

$$\left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h n\alpha} \right| \leq \left| \frac{1}{N} \left(\frac{2}{1 - e^{2\pi i h\alpha}} \right) \right| \to 0 \qquad \text{as } N \to \infty,$$

therefore

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h n\alpha} = 0$$

for all integers $h \neq 0$, which by Weyl's Criterion, tells us that the sequence $(n\alpha)$ is uniformly distributed modulo 1. A consequence of this fact is that $\{n\alpha\}, n = 1, 2, \dots$ is dense in the unit interval $[0, 1]$ for irrational α.

3.2.3 Erdös-Turán Inequality

An effective technique used to obtain upper bounds for discrepancy is to estimate it in terms of the exponential sums that occur in Weyl's Criterion. One such way is through the Erdös-Turán Inequality proved in [17] in 1948, which is a very useful tool for doing this, and can be viewed as a quantitative form of Weyl's Criterion for uniform distribution. There are many different forms of the Erdös-Turán Inequality, but here we present the following statement found in [35] which contains explicit constants.

Theorem 16 *(Erdös-Turán Inequality) For any finite sequence $x_1, \ldots, x_N$ of real numbers and any positive integer M, we have*

$$D_N(x_1, \ldots, x_N) \leq \frac{6}{M+1} + \frac{4}{\pi} \sum_{h=1}^{M} \left(\frac{1}{h} - \frac{1}{M+1} \right) \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} \right| .$$

Remark. It is easier to work with and apply this theorem in the following form: There exists an absolute constant C such that

$$D_N \leq C \left(\frac{1}{M} + \frac{1}{N} \sum_{h=1}^{M} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right) \tag{36}$$

for any real numbers $x_1, \ldots, x_N$ and any positive integer M. This inequality holds for arbitrary natural numbers M, N. We now prove the following Lemma.

Lemma 8 *The discrepancy D_N of the sequence $(n\alpha)$, $n = 1, \ldots, N$, with α irrational, satisfies*

$$D_N \leq C \left(\frac{1}{M} + \frac{1}{N} \left(33(\log M)^2 + 10 A_{m+1} \right) \right)$$

for any positive integer M, where C is an absolute constant and A_{m+1} is defined as in Theorem 14.

Proof. From (36) we have

$$D_N \leq C \left(\frac{1}{M} + \frac{1}{N} \sum_{h=1}^{M} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi i h n \alpha} \right| \right) \tag{37}$$

for any positive integer M. Now using the geometric series

$$\sum_{n=0}^{N} r^n = \frac{1-r^{N+1}}{1-r}$$

with $r = e^{2\pi i h\alpha}$, we have

$$\left|\sum_{n=1}^{N} e^{2\pi i h n\alpha}\right| = \left|\frac{1-e^{2\pi i h(N+1)\alpha}}{1-e^{2\pi i h\alpha}} - 1\right| \leq \left|\frac{2}{e^{2\pi i h\alpha}-1}\right|, \tag{38}$$

by a similar argument to (34) and (35). This then gives us

$$\left|\frac{i}{i}\right|\left|\frac{2}{e^{\pi i h\alpha}(e^{\pi i h\alpha}-e^{-\pi i h\alpha})}\right| = \left|\frac{1}{ie^{\pi i h\alpha}}\right|\left|\frac{1}{\sin \pi h\alpha}\right| = \frac{1}{|\sin \pi h\alpha|}. \tag{39}$$

Now let p be some integer. By definition, we know that either $\|h\alpha\| = h\alpha - p$ or $\|h\alpha\| = p - h\alpha$, and so $|\sin \pi h\alpha| = \sin \pi \|h\alpha\|$.

Now recall that always $\|h\alpha\| \leq \frac{1}{2}$ and note that $\sin \pi x \geq 2x$ for $0 \leq x \leq \frac{1}{2}$. Hence

$$\frac{1}{|\sin \pi h\alpha|} = \frac{1}{\sin \pi \|h\alpha\|} \leq \frac{1}{2\|h\alpha\|}$$

for all $h \geq 1$. Combining this result with (37), (38), (39) and applying the upper bound of Theorem 14, we obtain

$$D_N \leq C\left(\frac{1}{M} + \frac{1}{2N}\sum_{h=1}^{M}\frac{1}{h\|h\alpha\|}\right) \leq C\left(\frac{1}{M} + \frac{1}{N}\Big(33(\log M)^2 + 10A_{m+1}\Big)\right)$$

for any positive integer M, where A_{m+1} is defined as in Theorem 14 and C is an absolute constant. $\boxtimes$

Generally, the exponential sums are usually estimated using well-known methods from analytic number theory, but in the case of the sequence $(n\alpha)$, using the bounds for sums of reciprocal substantially simplifies matters.

3.3 Multiplicative Diophantine Approximation

It is possible to extend the theory of Diophantine approximations to higher dimensions. This allows us to do simultaneous approximation, that is, to approximate a set of numbers $x_1, \ldots, x_n$ by fractions $\frac{p_1}{q}, \ldots, \frac{p_n}{q}$ with a common denominator q, or equivalently, to make $\|qx_1\|, \ldots, \|qx_n\|$ simultaneously small. This section concerns an expansion of that idea. For more complete proofs of results in this section, refer to [11, 13, 27, 50].

3.3.1 Preliminary Results

We begin with generalisations of Theorems found in Section 1 to higher dimensions and simultaneous approximation in $\mathbb{R}^n$.

Theorem 17 *(Theorem 1 in $\mathbb{R}^n$) Let $(i_1, \ldots, i_n)$ be any n-tuple of numbers satisfying*

$$0 < i_1, \ldots, i_n < 1 \quad \textit{and} \quad \sum_{t=1}^{n} i_t = 1.$$

Then, for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and $N \in \mathbb{N}$, there exists $q \in \mathbb{Z}$ such that

$$\max\left\{ \|qx_1\|^{\frac{1}{i_1}}, \ldots, \|qx_n\|^{\frac{1}{i_n}} \right\} < N^{-1} \quad \textit{and} \quad 1 \leq q \leq N.$$

This can be viewed as the higher dimensional version of Dirichlet's Theorem (Theorem 1). We have the following important analogue of Theorem 2 as a consequence.

Theorem 18 *(Theorem 2 in $\mathbb{R}^n$) Let $(i_1, \ldots, i_n)$ be any n-tuple of real numbers satisfying*

$$0 < i_1, \ldots, i_n < 1 \quad \textit{and} \quad \sum_{t=1}^{n} i_t = 1.$$

Let $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then there exist infinitely many integers $q > 0$ such that

$$\max\left\{ \|qx_1\|^{\frac{1}{i_1}}, \ldots, \|qx_n\|^{\frac{1}{i_n}} \right\} < q^{-1}.$$

We have the notion of badly approximable points for $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

Definition 9 *(Badly approximable points in $\mathbb{R}^n$) Define the set of $(i_1, \ldots, i_n)$-badly approximable points by the set $(x_1, \ldots, x_n) \in \mathbb{R}^n$ such that there exists a positive constant $c(x_1, \ldots, x_n) > 0$ so that*

$$\max\left\{ \|qx_1\|^{\frac{1}{i_1}}, \ldots, \|qx_n\|^{\frac{1}{i_n}} \right\} > c(x_1, \ldots, x_n)\, q^{-1} \qquad \forall q \in \mathbb{N}.$$

We also have the notion of simultaneously ψ-well approximable points for $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

Definition 10 *(ψ-well approximable points in $\mathbb{R}^n$) Given $\psi : \mathbb{N} \to (0, +\infty)$, let $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and let $\mathbb{I}^n = [0,1)^n$ denote the unit cube in $\mathbb{R}^n$. A point $(x_1, \ldots, x_n) \in \mathbb{I}^n$ is ψ-well approximable points if there exist infinitely many rational points $\left(\frac{p_1}{q}, \ldots, \frac{p_n}{q}\right)$ with $q \in \mathbb{N}$, such that the inequalities*

$$\left| x_i - \frac{p_i}{q} \right| < \frac{\psi(q)}{q}$$

are simultaneously satisfied for $1 \leq i \leq n$.

Many problems in metrical Diophantine approximation can be phrased in terms of the set

$$\mathcal{A}_n(\psi) = \left\{ (x_1, \ldots, x_n) \in \mathbb{I}^n : \max_{1 \leq i \leq n} \|qx_i\| < \psi(q) \text{ for infinitely many } q \in \mathbb{N} \right\}, \tag{40}$$

where $n \geq 1$ is an integer, and $\psi : \mathbb{N} \to (0, +\infty)$ is often referred to as an approximating function. Note that while more generally we work with $\mathbb{R}^n$ rather than $\mathbb{I}^n$, we work over the unit cube here for convenience. In fact this is not restrictive as the set is invariant under translation by integer vectors. $\mathcal{A}_n(\psi)$ denotes the set of simultaneously ψ-well approximable points $(x_1, \ldots, x_n) \in \mathbb{I}^n$ and allows us to state a theorem on simultaneous approximation of Khintchine's, which he extended from his one-dimensional case (see [13, 34]).

Theorem 19 *(Khintchine in $\mathbb{R}^n$) Let λ_n denote n-dimensional Lebesgue measure. For any approximating function $\psi : \mathbb{N} \to (0, +\infty)$,*

$$\lambda_n(\mathcal{A}_n(\psi)) = \begin{cases} 0, & \text{if } \displaystyle\sum_{q=1}^{\infty} \psi(q)^n < \infty, \\ 1, & \text{if } \displaystyle\sum_{q=1}^{\infty} \psi(q)^n = \infty \text{ and } \psi \text{ is monotonic.} \end{cases}$$

Remark. The convergence case is a consequence of the Borel-Cantelli Lemma and hence the monotonicity condition for the convergence case is not required.

Remark. Similar to the one-dimensional version of Khintchine's Theorem, there was originally a stronger monotonic condition. For $n \geq 2$, it turns out that the monotonicity condition can be dropped entirely. This is due to a theorem of Gallagher [20].

3.3.2 Littlewood's Conjecture

Multiplicative Diophantine approximation is a very active field of research currently. In particular, the long standing and famous conjecture of Littlewood from the nineteen thirties. It has been the focus of much interest and attention (see [2, 43, 58] and references within) and tremendous progress has been made, but the problem remains very much open to this day.

Similar to (40), many problems in multiplicative Diophantine approximation concern the following set, which we define as such: given an approximating function $\psi : \mathbb{N} \to (0, +\infty)$, let

$$\mathcal{A}_n^{\times}(\psi) = \left\{ (x_1, \ldots, x_n) \in \mathbb{I}^n : \prod_{i=1}^{n} \|qx_i\| < \psi(q) \text{ for infinitely many } q \in \mathbb{N} \right\}$$

denote the set of multiplicative ψ-well approximable points $(x_1, \ldots, x_n) \in \mathbb{I}^n$. We present this set in terms of $\mathbb{I}^n$ rather than $\mathbb{R}^n$ for convenience. However, note that similar to (40), this is not restrictive. In fact, the results and proofs given in the rest of this section are exactly the same regardless of whether we work in $\mathbb{R}^n$ or $\mathbb{I}^n$.

As a consequence of Theorem 18, it can be easily seen that for any pair of real numbers $(\alpha, \beta) \in \mathbb{I}^2$, there exist infinitely many $q \in \mathbb{N}$ such that

$$\|q\alpha\|\|q\beta\| \leq q^{-1}.$$

Now for an arbitrary $\epsilon > 0$, we want to determine if this statement holds if q^{-1} on the right hand side of the inequality is replaced by ϵq^{-1}. This leads us to Littlewood's conjecture.

Conjecture 1 *(Littlewood's Conjecture) For all* $(\alpha, \beta) \in \mathbb{I}^2$,

$$\liminf_{q\to\infty} q\|q\alpha\|\|q\beta\| = 0.$$

Equivalently, Littlewood's conjecture states that for an arbitrary $\epsilon > 0$ and any pair of real numbers $(\alpha, \beta) \in \mathbb{I}^2$, there exist infinitely many rational points $\left(\frac{p_1}{q}, \frac{p_2}{q}\right)$ such that

$$\left|\alpha - \frac{p_1}{q}\right|\left|\beta - \frac{p_2}{q}\right| < \frac{\epsilon}{q^3}.$$

3.3.3 Gallagher's Theorem

Investigating the metrical side of Littlewood's conjecture leads us to the following result of Gallagher [19], which can be seen as the multiplicative analogue of Khintchine's simultaneous approximation theorem (Theorem 19).

Theorem 20 *(Gallagher, 1962) Let* $\psi : \mathbb{N} \to (0, +\infty)$ *be a monotonically decreasing function. Then*

$$\lambda_n(\mathcal{A}_n^\times(\psi)) = \begin{cases} 0, & \text{if } \sum_{q=1}^{\infty} \psi(q) \log^{n-1} q < \infty, \\ 1, & \text{if } \sum_{q=1}^{\infty} \psi(q) \log^{n-1} q = \infty. \end{cases}$$

Remark. The monotonicity condition for the convergence case can be dropped if the convergence condition is replaced by $\sum_{q=1}^{\infty} \psi(q) |\log \psi(q)|^{n-1} < \infty$. (See [7] for more details.)

3.3.4 Gallagher on Fibers

In [8], Beresnevich, Haynes, and Velani obtained and proved the following convergence and divergence fiber versions of Gallagher's Theorem. In summary, for $\alpha \in \mathbb{I}$, the points of interest $(\alpha_1, \alpha_2) \in \mathbb{I}^2$ are forced to all lie on a line given by $\{\alpha\} \times \mathbb{I}$. The two theorems naturally complement each other and their proofs are similar. In this section we will give a proof of the convergence version of the theorem, and demonstrate how sums of reciprocals are essential to the proof.

Theorem 21 *(Divergence result) Let $\alpha \in \mathbb{I}$ and $\psi : \mathbb{N} \to (0, +\infty)$ be a monotonically decreasing*

$$\sum_{q=1}^{\infty} \psi(q) \log q = \infty\,, \tag{41}$$

and such that

$$\exists\, \delta > 0 \qquad \liminf_{n\to\infty} q_n^{3-\delta}\psi(q_n) \geq 1, \tag{42}$$

where q_n denotes the denominators of the convergents of α. Then for almost every $\beta \in \mathbb{I}$, there exists infinitely many $q \in \mathbb{N}$ such that

$$\|q\alpha\|\|q\beta\| < \psi(q). \tag{43}$$

Remark. Define the Diophantine exponent of approximation of $\alpha \in \mathbb{R}$ as

$$\tau(\alpha) = \sup\{\tau > 0 : \|q\alpha\| < q^{-\tau} \text{ for infinitely many } q \in \mathbb{N}\}.$$

We have that condition (42) is not particularly restrictive and holds for all α with $\tau(\alpha) < 3$. Further, by the Jarník-Besicovitch Theorem (see [6, 12, 29]), it follows that the complement has relatively small Hausdorff dimension $\dim_H\{\alpha \in \mathbb{R} : \tau(\alpha) \geq 3\} = \frac{1}{2}$.

Theorem 22 *(Convergence result) Let $\alpha \in \mathbb{I}$ be an irrational real number and let $\psi : \mathbb{N} \to (0, +\infty)$ be such that*

$$\sum_{n=1}^{\infty} \psi(n) \log n < \infty. \tag{44}$$

Furthermore, assume either of the following two conditions:

(i) *$n \mapsto n\psi(n)$ is decreasing and*

$$\sum_{n=1}^{N} \frac{1}{n\|n\alpha\|} \ll (\log N)^2 \qquad \text{for all } N \geq 2; \tag{45}$$

(ii) *$n \mapsto \psi(n)$ is decreasing and*

$$\sum_{n=1}^{N} \frac{1}{\|n\alpha\|} \ll N \log N \qquad \textit{for all } N \geq 2. \tag{46}$$

Then for almost all $\beta \in \mathbb{I}$, there exist only finitely many $n \in \mathbb{N}$ such that

$$\|n\alpha\|\|n\beta\| < \psi(n). \tag{47}$$

Remark. It has been shown in [8] that for almost all $\alpha \in \mathbb{R}$, (45) holds but (46) fails.

Remark. While we have given the homogeneous statement of the theorem here, note that the theorem and its proof remain the same for its inhomogeneous statement. That is, for $\gamma, \delta \in \mathbb{R}$, $S_N(\alpha, \gamma)$ replaces $S_N(\alpha, 0)$ in (45), $R_N(\alpha, \gamma)$ replaces $R_N(\alpha, 0)$ in (46), and instead of (47) we have $\|n\alpha - \gamma\|\|n\beta - \delta\| < \psi(n)$. Unfortunately, the same cannot be said of the divergence case and we know nothing regarding the inhomogeneous version of Theorem 21.

Proof. (Proof of Theorem 22) We want to simplify the multiplicative problem of (47) into a one-dimensional metrical one which is much easier to deal with. We start by rewriting (47) as

$$\|n\beta\| < \Psi_\alpha(n), \qquad \text{where} \qquad \Psi_\alpha(n) = \frac{\psi(n)}{\|n\alpha\|}. \tag{48}$$

If $\Psi_\alpha(n) \geq \frac{1}{2}$, then (48) is trivially satisfied. Hence we restrict our attention to $\Psi_\alpha(n) < \frac{1}{2}$.

Now let E_n be the set of $\beta \in \mathbb{I}$ satisfying (48). One can easily verify that

$$\lambda_1(E_n) = 2\Psi_\alpha(n).$$

Now if

$$\sum_{n=1}^{\infty} \Psi_\alpha(n) < \infty,$$

then the well known Borel-Cantelli Lemma from probability theory implies that the set of β's such that (48) and hence (47) is satisfied infinitely often, is of Lebesgue measure zero. This is an equivalent way of restating the assertion in (47).

Remark. The converse is not necessarily true. Divergence with Borel-Cantelli Lemma does not guarantee full measure. Fortunately, in the case of proving Theorem 21, this is not a problem as we already know through some other means that $\mathcal{A}_n^{\times}(\psi)$ satisfies a zero-one law (also known as zero-full law) with respect to the measure λ_n. That is, $\lambda_n(\mathcal{A}_n^{\times}(\psi)) = 0$ or 1 (for more details on this, see [7, 9]). So if we had wanted to obtain full measure and hence satisfy (43), we would require divergence. Note that it is not possible to use Khintchine's Theorem in $\mathbb{R}^n$ to achieve this as $\Psi_\alpha(n)$ is not a monotonic function of n. A different approach, similar to what we will use in proving Theorem 22 is needed.

Going back to the proof of Theorem 22, we see that an equivalent way of stating the theorem is as follows. In the setting of the (x, y)-plane, let L_x denote the line parallel to the y-axis which passes through the point $(x, 0)$. Then, for $\alpha \in \mathbb{I}$, Theorem 22 states that

$$\lambda_1(\mathcal{A}_2^{\times}(\psi) \cap L_\alpha) = 0$$

if ψ satisfies (44) and either one of (45) or (46).

To prove the theorem, we want to show that

$$\sum_{n=1}^{\infty} \psi(n) \log n < \infty \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} \Psi_\alpha(n) < \infty. \tag{49}$$

when either one of conditions (45) or (46) hold.

Case (i) Assume that for $\alpha \in \mathbb{I}$ an irrational number and $\psi : \mathbb{N} \to (0, +\infty)$ where $n \mapsto n\psi(n)$ is decreasing, the conditions (44) and (45) hold.

Then to show (49), we use partial summation (31) to see that

$$\begin{aligned}\sum_{n=1}^{N}\Psi_\alpha(n) &= \sum_{n=1}^{N} n\psi(n)\left(\frac{1}{n\|n\alpha\|}\right)\\ &= \sum_{n=1}^{N}\Big(n\psi(n)-(n+1)\psi(n+1)\Big)\left(\sum_{k=1}^{n}\frac{1}{k\|k\alpha\|}\right)\\ &\quad + (N+1)\psi(N+1)\left(\sum_{k=1}^{N}\frac{1}{k\|k\alpha\|}\right).\end{aligned}$$

Now since $n \mapsto n\psi(n)$ is monotonic, we apply (45), and then use the fact that $\sum_{k=1}^{n}\frac{\log k}{k} \asymp (\log n)^2$ to obtain

$$\begin{aligned}\sum_{n=1}^{N}\Psi_\alpha(n) &\ll \sum_{n=1}^{N}\Big(n\psi(n)-(n+1)\psi(n+1)\Big)(\log n)^2 + (N+1)\psi(N+1)(\log N)^2\\ &\asymp \sum_{n=1}^{N}\Big(n\psi(n)-(n+1)\psi(n+1)\Big)\sum_{k=1}^{n}\frac{\log k}{k} + (N+1)\psi(N+1)\sum_{k=1}^{N}\frac{\log k}{k}\\ &= \sum_{n=1}^{N}\psi(n)\log n < \infty \qquad \text{as } N\to\infty\end{aligned}$$

by (44). Then since we have the bound

$$\sum_{n=1}^{\infty}\Psi_\alpha(n) \ll \sum_{n=1}^{\infty}\psi(n)\log n, \tag{50}$$

the left hand side of of (50) must converge as well and we have shown that (49) is true.

Case (ii) Assume that for $\alpha \in \mathbb{I}$ an irrational number and $\psi : \mathbb{N} \to (0, +\infty)$ where $n \mapsto \psi(n)$ is decreasing, the conditions (44) and (46) hold.

Then to show (49), we use partial summation (31) to see that

$$\begin{aligned}\sum_{n=1}^{N}\Psi_\alpha(n) &= \sum_{n=1}^{N}\psi(n)\left(\frac{1}{\|n\alpha\|}\right)\\ &= \sum_{n=1}^{N}\Big(\psi(n)-\psi(n+1)\Big)\left(\sum_{k=1}^{n}\frac{1}{\|k\alpha\|}\right) + \psi(N+1)\left(\sum_{k=1}^{N}\frac{1}{\|k\alpha\|}\right).\end{aligned}$$

Now since $n \mapsto \psi(n)$ is monotonic, we apply (46), and then use the fact from Stirling's approximation that $\sum_{m=1}^{n} \log m = \log(n!) \asymp n \log n$ to obtain

$$\begin{aligned}
\sum_{n=1}^{N} \Psi_\alpha(n) &\ll \sum_{n=1}^{N} \Big(\psi(n) - \psi(n+1)\Big) n \log n + \psi(N+1) N \log N \\
&\asymp \sum_{n=1}^{N} \Big(\psi(n) - \psi(n+1)\Big) \sum_{k=1}^{n} \log k + \psi(N+1) \sum_{k=1}^{N} \log k \\
&= \sum_{n=1}^{N} \psi(n) \log n < \infty \qquad \text{as } \; N \to \infty
\end{aligned}$$

by (44). Then since we have the bound

$$\sum_{n=1}^{\infty} \Psi_\alpha(n) \ll \sum_{n=1}^{\infty} \psi(n) \log n, \tag{51}$$

the left hand side of of (51) must converge as well and we have shown that (49) is true.

$\boxtimes$

4 Homogeneous Bounds

We have thus far shown the usefulness of sums of reciprocals and provided motivation for their study. This subject is nothing new and bounds have been obtained for these sums using a wide variety of methods. In this section and the next, we will present in detail results obtained using a new method we developed. This technique uses the Three Distance Theorem to estimate the sums, resulting in explicit constants for the bounds.

To obtain bounds for sums of reciprocals we first need to understand the distribution of the points $\{n_i\alpha\}$. By the Three Distance Theorem, these points partition $[0,1]$ into intervals of at most three different lengths(gaps). In Corollary 1 and 2, we determine how these gaps are distributed on the unit line, and equivalently, the unit circle, by working out the distribution of the n_i's. This enables us to prove the following theorems.

4.1 Bounds for $\sum\{n\alpha\}^{-1}$

We first estimate the sum $\sum_{n=1}^{N}\{n\alpha\}^{-1}$.

Theorem 23 *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\alpha = [a_0; a_1, a_2, \ldots]$, $N \in \mathbb{N}$. Let N, r, q_k, δ_A and δ_B be defined as in Theorem 10, where $\frac{p_k}{q_k}$ are the continued fraction convergents to α. Then*

(i) *if k is even, and $\delta_A < \delta_B$, then*

$$\sum_{\substack{1\leq n\leq N\\ n\not\equiv 0(mod\ q_k)}} \frac{1}{\{n\alpha\}} < 3N\left(\log q_k + 1\right)$$

and

$$\sum_{\substack{1\leq n\leq N\\ n\equiv 0(mod\ q_k)}} \frac{1}{\{n\alpha\}} < 2q_{k+1}\left(\log\left(\frac{N}{q_k}+1\right)+1\right);$$

(ii) *if k is odd, and $\delta_A < \delta_B$, then*

$$\sum_{\substack{1\leq n\leq N\\ n\not\equiv q_{k-1}(mod\ q_k)}} \frac{1}{\{n\alpha\}} < 3N\left(\log q_k + 1\right),$$

and for $1 \leq r \leq \frac{a_{k+1}}{2}$, we have that

$$\sum_{\substack{1\leq n\leq N\\ n\equiv q_{k-1}(mod\ q_k)}} \frac{1}{\{n\alpha\}} < 12N,$$

while for $\frac{a_{k+1}}{2} < r < a_{k+1}$, we have that

$$\sum_{\substack{1\leq n\leq N\\ n\equiv q_{k-1}(mod\ q_k)}} \frac{1}{\{n\alpha\}} < 4N\left(\log\left(\frac{2N}{q_k}\right)+1\right);$$

(iii) *if k is odd, and $\delta_B < \delta_A$, then*

$$\sum_{\substack{1\leq n\leq N\\ n\neq q_{k+1}}} \frac{1}{\{n\alpha\}} < 4N(\log N + 1)$$

and

$$\sum_{\substack{1\leq n\leq N\\ n=q_{k+1}}} \frac{1}{\{n\alpha\}} < 2q_{k+2};$$

(iv) *if k is even, and $\delta_B < \delta_A$, then*

$$\sum_{\substack{1 \le n \le N \\ n \not\equiv 0, q_k (mod\ q_{k+1})}} \frac{1}{\{n\alpha\}} < 4N(\log N + 1)$$

and

$$\sum_{\substack{1 \le n \le N \\ n \equiv 0, q_k (mod\ q_{k+1})}} \frac{1}{\{n\alpha\}} < 4N.$$

Proof. Let α and N as above be given and let n_i be the same as in Theorem 10. In particular, $\{n_i\alpha\} < \{n_{i+1}\alpha\}$ for any $0 \le i \le N-1$. Since $n_1, \ldots, n_N$ is a permutation of $1, \ldots, N$, we have that

$$\sum_{n=1}^{N} \frac{1}{\{n\alpha\}} = \sum_{i=1}^{N} \frac{1}{\{n_i\alpha\}}, \tag{52}$$

where the sum on the right hand side is of decreasing terms. The desired estimates will be obtained by carefully estimating $\{n_i\alpha\}$ from below using Theorem 10 and Corollary 2.

Case (i) First assume k even and $\delta_A < \delta_B$, that is $r \ne a_{k+1}$. By Corollary 2, amongst any $r+2$ consecutive gaps, there will be at least one of length δ_B or δ_C. Therefore amongst the first $i \le N$ gaps there will be at least t gaps of lengths at least $\min\{\delta_B, \delta_C\} = \delta_B$, while the other gaps are of length at least δ_A, where

$$t = \left[\frac{i}{r+2}\right] \le (s+1) + (q_k - s - 1) = q_k$$

since t is at most equal to the number of δ_B and δ_C gaps combined. This gives us the following lower bound for $\{n_i\alpha\}$:

$$\{n_i\alpha\} = \sum_{\ell=1}^{i} \underbrace{\Big(\{n_\ell\alpha\} - \{n_{\ell-1}\alpha\}\Big)}_{\text{gaps}} \ge \delta_B t + \delta_A(i-t).$$

Now by the division algorithm, we have that

$$i = t(r+2) + j, \text{ where } 0 \le j \le r+1,$$

where the pair (t, j) is in one to one correspondence with i. Thus we have

$$\{n_i\alpha\} \geq t\delta_B + t(r+1)\delta_A + j\delta_A, \tag{53}$$

which allows us to estimate our sum (52) as follows

$$\sum_{i=1}^{N} \frac{1}{\{n_i\alpha\}} \leq \sum_{t=1}^{q_k} \sum_{j=0}^{r+1} \frac{1}{t\delta_B + t(r+1)\delta_A + j\delta_A} + \sum_{j=1}^{r+1} \frac{1}{j\delta_A}. \tag{54}$$

Now consider the first term on the right hand side of (54). Using the explicit values for δ_A and δ_B given in Theorem 10, we have

$$\begin{aligned} \sum_{t=1}^{q_k} \sum_{j=0}^{r+1} \frac{1}{t\delta_B + t(r+1)\delta_A + j\delta_A} &\leq \sum_{t=1}^{q_k} \sum_{j=0}^{r+1} \frac{1}{(\delta_B + (r+1)\delta_A)t} \\ &< \sum_{t=1}^{q_k} \frac{r+2}{(a_{k+1}+1)|D_k|t} \end{aligned} \tag{55}$$

since omitting terms in the denominator only increases the sum. Applying (5) and the well known fact that $\sum_{n=1}^{k} \frac{1}{n} < \log k + 1$, (55) becomes

$$\begin{aligned} \left(\frac{r+2}{(a_{k+1}+1)|D_k|}\right) \sum_{t=1}^{q_k} \frac{1}{t} &< \frac{r+2}{(a_{k+1}+1)|D_k|}(\log q_k + 1) \\ &< \frac{(r+2)(q_{k+1}+q_k)}{a_{k+1}+1}(\log q_k + 1). \end{aligned} \tag{56}$$

Now from Theorem 10, $rq_k + q_{k-1} \leq N$ and $1 \leq r \leq a_{k+1}$, so

$$\begin{aligned} \frac{(r+2)(q_{k+1}+q_k)}{a_{k+1}+1} &= r\left(1 + \frac{2}{r}\right) \frac{(a_{k+1}+1)q_k + q_{k-1}}{a_{k+1}+1} \\ &\leq r\left(1 + \frac{2}{r}\right)\left(q_k + \frac{q_{k-1}}{r+1}\right) < 3N, \end{aligned}$$

hence from (56) we have

$$\frac{(r+2)(q_{k+1}+q_k)}{a_{k+1}+1}(\log q_k + 1) < 3N(\log q_k + 1). \tag{57}$$

Similarly, we use (5) to estimate the second term on the right hand side of (54). We have that $r + 1 \leq \frac{N}{q_k} + 1$, so

$$\sum_{j=1}^{r+1} \frac{1}{j\delta_A} < \frac{1}{|D_k|}(\log(r+1) + 1) < 2q_{k+1}\left(\log\left(\frac{N}{q_k} + 1\right) + 1\right). \tag{58}$$

Therefore, combining (52), (54), (57), and (58), for k even, $\delta_A < \delta_B$, we have

$$\sum_{n=1}^{N} \frac{1}{\{n\alpha\}} < 3N(\log q_k + 1) + 2q_{k+1}\left(\log\left(\frac{N}{q_k} + 1\right) + 1\right).$$

Finally, notice that the second term on the right hand side of (54) corresponds to the first $r+1$ consecutive δ_A gaps. By (16) and Theorem 10, these are precisely all the $n_i \leq N$ which are multiples of q_k. Therefore, (55), (56), and (57) imply the first assertion of Theorem 23(i), and (58) implies the second assertion of Theorem 23(i).

Case (ii) Now assume k is odd and $\delta_A < \delta_B$. Since k is odd, by (27) and Corollary 1, $\{n_1\alpha\} = \delta_B \neq 1 - \{n_N\alpha\}$. Now δ_B is not the smallest gap and hence would not affect the sum too much. If we start counting after the first gap, by Corollary 2, amongst every $r+2$ consecutive gaps, there will be at least one of length δ_B or δ_C. In light of the additional δ_B gap at the beginning, for $1 \leq i \leq N$, similar to the case when k even, we have at least t gaps of length at least $\min\{\delta_B, \delta_C\} = \delta_B$, while the other gaps are of length at least δ_A, where

$$t = \left[\frac{i-1}{r+2}\right] \leq (s+1) + (q_k - s - 1) - 1 = q_k - 1.$$

We thus have the following lower bound for $\{n_i\alpha\}$:

$$\{n_i\alpha\} \geq \delta_B + \delta_B\left[\frac{i-1}{r+2}\right] + \delta_A\left((i-1) - \left[\frac{i-1}{r+2}\right]\right).$$

By the division algorithm, we get

$$i - 1 = t(r+2) + j, \text{ where } 0 \leq j \leq r+1,$$

where the pair (t, j) is in one to one correspondence with i. Hence we have

$$\{n_i\alpha\} \geq \delta_B + t\delta_B + t(r+1)\delta_A + j\delta_A,$$

which allows us to rewrite our sum as

$$\begin{aligned}\sum_{i=1}^{N} \frac{1}{\{n_i\alpha\}} &\leq \sum_{t=0}^{q_k-1}\sum_{j=0}^{r+1} \frac{1}{\delta_B + t\delta_B + t(r+1)\delta_A + j\delta_A} \\ &= \sum_{j=0}^{r+1} \frac{1}{\delta_B + j\delta_A} + \sum_{t=1}^{q_k-1}\sum_{j=0}^{r+1} \frac{1}{\delta_B + t\delta_B + t(r+1)\delta_A + j\delta_A}.\end{aligned}$$

Omitting terms in the denominator, we simplify the above to

$$\sum_{i=1}^{N} \frac{1}{\{n_i\alpha\}} \leq \frac{1}{\delta_B} + \sum_{j=1}^{r+1} \frac{1}{\delta_B + j\delta_A} + \sum_{t=1}^{q_k} \frac{r+2}{(\delta_B + (r+1)\delta_A)t}. \tag{59}$$

Now consider the third term on the right hand side of (59). Using exactly the same argument as for (55), (56), and (57), we have

$$\begin{aligned} \sum_{t=1}^{q_k} \frac{r+2}{(\delta_B + (r+1)\delta_A)t} &< \frac{r+2}{(a_{k+1}+1)|D_k|} \sum_{t=1}^{q_k} \frac{1}{t} \\ &< 3N\left(\log q_k + 1\right). \end{aligned} \tag{60}$$

It is easily verified that for positive and fixed A and B,

$$\sum_{n=1}^{k} \frac{1}{A + Bn} = \frac{1}{B} \sum_{n=\frac{A}{B}+1}^{\frac{A}{B}+k} \frac{1}{n} < \frac{1}{B} \log\left(\frac{kB}{A} + 1\right). \tag{61}$$

Thus we use (61) to estimate the second term on the right hand side of (59) and see that

$$\sum_{j=1}^{r+1} \frac{1}{\delta_B + j\delta_A} < \frac{1}{\delta_A} \log\left((r+1)\frac{\delta_A}{\delta_B} + 1\right). \tag{62}$$

First assume $1 \leq r \leq \frac{a_{k+1}}{2}$. Then using the basic estimate that $\log(1+x) \leq x$ for $x \geq 0$, (62) gives

$$\frac{1}{\delta_A} \log\left((r+1)\frac{\delta_A}{\delta_B} + 1\right) \leq \frac{r+1}{\delta_B}.$$

Then since $r \leq \frac{a_{k+1}}{2}$, using (5) and the definitions of δ_B and N, we have

$$\frac{1}{\delta_B} + \frac{r+1}{\delta_B} < \frac{r+2}{(a_{k+1} - r)|D_k|} < \frac{2(r+2)2q_{k+1}}{a_{k+1}} = 4r\left(1 + \frac{2}{r}\right)\left(q_k + \frac{q_{k-1}}{a_{k+1}}\right) < 12N,$$

hence if $1 \leq r \leq \frac{a_{k+1}}{2}$, the first two terms on the right hand side of (59) becomes

$$\frac{1}{\delta_B} + \sum_{j=1}^{r+1} \frac{1}{\delta_B + j\delta_A} < 12N. \tag{63}$$

Now assume $\frac{a_{k+1}}{2} < r \leq a_{k+1} - 1$. Notice that $\frac{\delta_A}{\delta_B} < 1$, so (62) gives

$$\frac{1}{\delta_A} \log\left((r+1)\frac{\delta_A}{\delta_B} + 1\right) < \frac{1}{|D_k|} \log(a_{k+1} + 1).$$

Since $r \leq \frac{N}{q_k}$, hence $a_{k+1} + 1 \leq \frac{2N}{q_k}$. Therefore, using (5),

$$\frac{1}{|D_k|} \log(a_{k+1} + 1) < 2q_{k+1} \log\left(\frac{2N}{q_k}\right).$$

As r is large, $r > \frac{a_{k+1}}{2}$, using the definition of N we have $N > \frac{a_{k+1}}{2} q_k + q_{k-1}$, so

$$q_{k+1} < a_{k+1} q_k + q_{k-1} + q_{k-1} < 2N.$$

Trivially, $\frac{1}{\delta_B} < \frac{1}{\delta_A} = \frac{1}{|D_k|}$, so if $\frac{a_{k+1}}{2} < r < a_{k+1}$, the first two terms on the right hand side of (59) becomes

$$\frac{1}{\delta_B} + \sum_{j=1}^{r+1} \frac{1}{\delta_B + j\delta_A} < 2q_{k+1}\left(\log\left(\frac{2N}{q_k}\right) + 1\right) < 4N\left(\log\left(\frac{2N}{q_k}\right) + 1\right). \tag{64}$$

Therefore, combining (59), (60), (63), and (64), for k odd, $\delta_A < \delta_B$, we have when $1 \leq r \leq \frac{a_{k+1}}{2}$,

$$\sum_{n=1}^{N} \frac{1}{\{n\alpha\}} < 3N(\log q_k + 1) + 12N,$$

and when $\frac{a_{k+1}}{2} < r < a_{k+1}$,

$$\sum_{n=1}^{N} \frac{1}{\{n\alpha\}} < 3N(\log q_k + 1) + 4N\left(\log\left(\frac{2N}{q_k}\right) + 1\right).$$

Finally, since k is odd, the first gap will be δ_B. By (16), this is when $n_1 = q_{k-1} + rq_k$, and similarly, a δ_A gap is when $n_1 = q_k$. Notice that the first two terms on the right hand side of (59) correspond to the first $r+2$ gaps. These are precisely all the $n_i \leq N$ such that $n_i = q_{k-1} + rq_k + jq_k \equiv q_{k-1} \pmod{q_k}$. Therefore, it follows that (60) implies the first assertion of Theorem 23(ii) whereas (63) and (64) imply the second and third assertions of Theorem 23(ii) respectively.

Case (iii) Assume $\delta_B < \delta_A$, that is, $r = a_{k+1}$. We first assume k odd. By Corollary 2, amongst any 2 consecutive gaps, there will be at most one gap of length δ_B.

Since $\delta_B < \delta_A < \delta_C$, this distribution of gaps will give a sharper upper bound for the sum (52). Therefore amongst the first $i \leq N$ gaps there will be at most t gaps of length at least $\min\{\delta_A, \delta_C\} = \delta_A$, while the other gaps are of length δ_B, where

$$t \leq \left\lceil \frac{i}{2} \right\rceil \leq (N + 1 - q_k) + (q_k - s - 1) = N - s$$

since t is at most equal to the number of δ_A and δ_C gaps combined. This gives the following lower bound for $\{n_i\alpha\}$:

$$\{n_i\alpha\} = \sum_{\ell=1}^{i} (\{n_\ell\alpha\} - \{n_{\ell-1}\alpha\}) \geq \delta_A t + \delta_B(i - t).$$

By the division algorithm, we have

$$i = 2t + j, \text{ where } 0 \leq j \leq 1,$$

where the pair (t, j) is in one to one correspondence with i. Thus we have

$$\{n_i\alpha\} \geq t\delta_A + t\delta_B + j\delta_B, \tag{65}$$

which allows us to estimate our sum (52) as follows

$$\begin{aligned} \sum_{i=1}^{N} \frac{1}{\{n_i\alpha\}} &\leq \frac{1}{\delta_B} + \sum_{t=1}^{N-s} \sum_{j=0}^{1} \frac{1}{t\delta_A + t\delta_B + j\delta_B} \\ &\leq \frac{1}{\delta_B} + \sum_{t=1}^{N-s} \sum_{j=0}^{1} \frac{1}{t\delta_A + t\delta_B}. \end{aligned} \tag{66}$$

Now consider the second term on the right hand side of (66). Using the explicit values for δ_A and δ_B in Theorem 10 gives

$$\sum_{t=1}^{N-s} \sum_{j=0}^{1} \frac{1}{t\delta_A + t\delta_B} = \sum_{t=1}^{N-s} \frac{2}{t(|D_k| + |D_{k+1}|)} < \frac{2}{|D_k|} \sum_{t=1}^{N-s} \frac{1}{t}. \tag{67}$$

Since $r = a_{k+1}$, then from Theorem 10, $q_{k+1} \leq q_{k+1} + s = N$, so using (5),

$$\frac{2}{|D_k|} < 2(q_{k+1} + q_k) < 4N, \tag{68}$$

hence

$$\frac{2}{|D_k|}\sum_{t=1}^{N-s}\frac{1}{t} < 4N(\log(N-s)+1) = 4N(\log q_{k+1}+1) < 4N(\log N+1). \tag{69}$$

Now using (5), we see that

$$\frac{1}{\delta_B} = \frac{1}{|D_{k+1}|} < 2q_{k+2}. \tag{70}$$

Therefore, combining (66) - (70), for k odd, $\delta_B < \delta_A$, we have

$$\sum_{n=1}^{N}\frac{1}{\{n\alpha\}} < 4N(\log N+1) + 2q_{k+2}.$$

Finally, since k is odd, by Corollary 1, the very first gap will be δ_B. Notice then that this δ_B gap corresponds to the first term on the right hand side of (66). By Theorem 10, this happens precisely when $n_i = q_{k+1}$ since $|D_{k+1}| = \{q_{k+1}\alpha\}$. Therefore, (67) and (69) imply the first assertion of Theorem 23(iii), and (70) implies the second assertion of Theorem 23(iii).

Case (iv) Assume k is even and $\delta_B < \delta_A$. Since k is even, by (27) and Corollary 1, $\{n_1\alpha\} = \delta_A \neq 1 - \{n_N\alpha\}$. Now δ_A is not the smallest gap and hence would not affect the sum too much. We use a similar argument as in Case (iii), except we start counting after the first gap. This means that amongst the first $i \leq N$ gaps there will be at most t gaps of length at least $\min\{\delta_A, \delta_C\} = \delta_A$, while the other gaps are of length δ_B, where

$$t \leq \left[\frac{i-1}{2}\right] \leq (N+1-q_k) + (q_k - s - 1) - 1 = N - s - 1.$$

This gives the following lower bound for $\{n_i\alpha\}$:

$$\{n_i\alpha\} \geq \delta_A + \delta_A\left[\frac{i-1}{2}\right] + \delta_B\left((i-1) - \left[\frac{i-1}{2}\right]\right).$$

By the division algorithm, we have

$$i - 1 = 2t + j, \text{ where } 0 \leq j \leq 1,$$

where the pair (t, j) is in one to one correspondence with i. Thus we have

$$\{n_i\alpha\} \geq \delta_A + t\delta_A + t\delta_B + j\delta_B,$$

which allows us to estimate our sum (52) as follows

$$\begin{aligned}\sum_{i=1}^{N}\frac{1}{\{n_i\alpha\}} &\leq \sum_{t=0}^{N-s-1}\sum_{j=0}^{1}\frac{1}{\delta_A+t\delta_A+t\delta_B+j\delta_B}\\ &\leq \frac{1}{\delta_A}+\frac{1}{\delta_A+\delta_B}+\sum_{t=1}^{N-s-1}\sum_{j=0}^{1}\frac{1}{t\delta_A+t\delta_B}. \end{aligned}\tag{71}$$

Now consider the third term on the right hand side of (71). Similar to (67), (68), and (69), we have

$$\sum_{t=1}^{N-s-1}\sum_{j=0}^{1}\frac{1}{t\delta_A+t\delta_B} < \frac{2}{|D_k|}\sum_{t=1}^{N-s}\frac{1}{t} < 4N(\log N+1). \tag{72}$$

Now by (68), the first two terms on the right hand side of (71) gives

$$\frac{1}{\delta_A}+\frac{1}{\delta_A+\delta_B} < \frac{2}{\delta_A} = \frac{2}{|D_k|} < 4N. \tag{73}$$

Therefore, combining (71) - (73), for k even, $\delta_B < \delta_A$, we have

$$\sum_{n=1}^{N}\frac{1}{\{n\alpha\}} < 4N(\log N+1)+4N. \tag{74}$$

Finally, as $\delta_B < \delta_A$, notice that since $\delta_A = |D_k| = \{q_k\alpha\}$ and $\delta_B = |D_{k+1}| = \{q_{k+1}\alpha\}$, the first two terms on the right hand side of (71) correspond precisely to when $n_i = q_k$ and $n_i = q_{k+1}$. Therefore, the first and second assertions of Theorem 23(iv) follow from (72) and (73) respectively. ⊠

4.2 Bounds for $\sum \|n\alpha\|^{-1}$

We now estimate the sum $\sum_{n=1}^{N}\|n\alpha\|^{-1}$. To do so, we first require the following lemma:

Lemma 9 *For $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and $N \in \mathbb{N}$, let $\gamma \in \mathbb{R}$ such that $\{n\alpha-\gamma\}, \|n\alpha-\gamma\| > 0$. Then*

$$\sum_{n=1}^{N}\frac{1}{\|n\alpha-\gamma\|} < \sum_{n=1}^{N}\frac{1}{\{n\alpha-\gamma\}}+\sum_{n=1}^{N}\frac{1}{1-\{n\alpha-\gamma\}}-N$$

and

$$\sum_{n=1}^{N} \frac{1}{\|n\alpha - \gamma\|} \geq \sum_{n=1}^{N} \frac{1}{\{n\alpha - \gamma\}} + \sum_{n=1}^{N} \frac{1}{1 - \{n\alpha - \gamma\}} - 2N.$$

Proof. Notice that $\|n\alpha - \gamma\| \leq \frac{1}{2}$ always, and it will be equal to either $\{n\alpha - \gamma\}$ or $1 - \{n\alpha - \gamma\}$. First we assume that $\|n\alpha - \gamma\| = \{n\alpha - \gamma\} \leq \frac{1}{2}$. This implies that $1 - \{n\alpha - \gamma\} \geq \frac{1}{2}$ and hence

$$1 < \frac{1}{1 - \{n\alpha - \gamma\}} \leq 2.$$

If instead we assume $\|n\alpha - \gamma\| = 1 - \{n\alpha - \gamma\} \leq \frac{1}{2}$, then similarly $\{n\alpha - \gamma\} \geq \frac{1}{2}$ and

$$1 < \frac{1}{\{n\alpha - \gamma\}} \leq 2.$$

This means that regardless of whether $\|n\alpha - \gamma\|^{-1}$ equals to $\{n\alpha - \gamma\}^{-1}$ or $(1 - \{n\alpha - \gamma\})^{-1}$, the other quantity will always be bounded between 1 and 2. We therefore have

$$1 < \frac{1}{\{n\alpha - \gamma\}} + \frac{1}{1 - \{n\alpha - \gamma\}} - \frac{1}{\|n\alpha - \gamma\|} \leq 2. \tag{75}$$

Summing over N and simplifying then gives us

$$\sum_{n=1}^{N} \frac{1}{\|n\alpha - \gamma\|} < \sum_{n=1}^{N} \frac{1}{\{n\alpha - \gamma\}} + \sum_{n=1}^{N} \frac{1}{1 - \{n\alpha - \gamma\}} - N$$

and

$$\sum_{n=1}^{N} \frac{1}{\|n\alpha - \gamma\|} \geq \sum_{n=1}^{N} \frac{1}{\{n\alpha - \gamma\}} + \sum_{n=1}^{N} \frac{1}{1 - \{n\alpha - \gamma\}} - 2N.$$

⊠

To prove the next theorem, notice that the estimates for the sum $\sum_{n=1}^{N}(1-\{n\alpha\})^{-1}$ can be obtained using the same argument as for $\sum_{n=1}^{N}\{n\alpha\}^{-1}$. By our argument using Corollary 1, since we start counting the gaps from the opposite end of the unit interval, the first gap will change. In fact, it can be easily verified that only the parity of k changes. What this means is that the bounds for $\sum(1 - \{n\alpha\})^{-1}$ are identical to the bounds for $\sum\{n\alpha\}^{-1}$ with the only difference being a switch in the parity of k for all the various cases in Theorem 23.

Theorem 24 *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\alpha = [a_0; a_1, a_2, \ldots]$, $N \in \mathbb{N}$, and let $m = m(N, \alpha)$ be the largest integer satisfying $q_m \leq N$, where $\frac{p_k}{q_k}$ are the continued fraction convergents to α. Then*

$$\sum_{n=1}^{N} \frac{1}{\|n\alpha\|} < 8N \log N + (12a_{m+1} + 17)N.$$

Proof. From Lemma 9 with $\gamma = 0$ we have

$$\sum_{n=1}^{N} \frac{1}{\|n\alpha\|} < \sum_{n=1}^{N} \frac{1}{\{n\alpha\}} + \sum_{n=1}^{N} \frac{1}{1 - \{n\alpha\}} - N. \tag{76}$$

Recall that the estimates for the sum $\sum_{n=1}^{N}(1 - \{n\alpha\})^{-1}$ are the same as those for $\sum_{n=1}^{N}\{n\alpha\}^{-1}$, with the parity of k switched. Hence notice that the sum

$$\sum_{n=1}^{N} \frac{1}{\{n\alpha\}} + \sum_{n=1}^{N} \frac{1}{1 - \{n\alpha\}}$$

remains the same regardless of the parity of k. Let $m = m(N, \alpha)$ be the largest integer satisfying $q_m \leq N$. First assume $\delta_A < \delta_B$. By (9), $q_k < q_k + q_{k-1} \leq N$ so in this case $m = k$. Recall from (64) that the estimate for all $r < a_{k+1}$ in Theorem 23(ii) is

$$\sum_{n=1}^{N} \frac{1}{\{n\alpha\}} < 3N(\log q_k + 1) + 2q_{k+1}\left(\log\left(\frac{2N}{q_k}\right) + 1\right),$$

which when combined with Theorem 23(i) gives

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{\|n\alpha\|} &< \sum_{n=1}^{N} \frac{1}{\{n\alpha\}} + \sum_{n=1}^{N} \frac{1}{1 - \{n\alpha\}} - N \\
&< 6N \log q_k + 5N + 2q_{k+1}\left(\log\left(\frac{N}{q_k} + 1\right) + 1\right) + 2q_{k+1}\left(\log\left(\frac{2N}{q_k}\right) + 1\right) \\
&< 6N \log N + 5N + 4q_{k+1} \log\left(\frac{2N}{q_k} + 1\right) + 4q_{k+1}. \qquad (77)
\end{aligned}$$

Then since $\log(1 + x) \leq x$ for $x \geq 0$,

$$4q_{k+1} \log\left(\frac{2N}{q_k} + 1\right) < 4q_k(a_{k+1} + 1)\frac{2N}{q_k} = (8a_{k+1} + 8)N. \tag{78}$$

In addition, since $q_{k+1} < (a_{k+1}+1)q_k < (a_{k+1}+1)N$, (77) becomes

$$\sum_{n=1}^{N} \frac{1}{\|n\alpha\|} < 6N\log N + (12a_{m+1}+17)N. \tag{79}$$

Now instead assume $\delta_B < \delta_A$ so $r = a_{k+1}$. Therefore $q_{k+1} \leq q_{k+1}+s = N$ and $m = k+1$ in this case. Similarly, using (86), adding cases (iii) and (iv) from Theorem 23 together gives

$$\sum_{n=1}^{N} \frac{1}{\|n\alpha\|} < \sum_{n=1}^{N} \frac{1}{\{n\alpha\}} + \sum_{n=1}^{N} \frac{1}{1-\{n\alpha\}} - N < 8N\log N + 7N + 4q_{k+2}. \tag{80}$$

We have $q_{k+2} < (a_{k+2}+1)q_{k+1} \leq (a_{k+2}+1)N$, so (80) becomes

$$\sum_{n=1}^{N} \frac{1}{\|n\alpha\|} < 8N\log N + (4a_{m+1}+11)N. \tag{81}$$

Taking the worst of each estimate from (79) and (81), it is therefore clear that regardless of whether δ_A or δ_B is the smaller quantity, we will have

$$\sum_{n=1}^{N} \frac{1}{\|n\alpha\|} < 8N\log N + (12a_{m+1}+17)N.$$

$\boxtimes$

5 Inhomogeneous Bounds

5.1 Bounds for $\sum\{n\alpha-\gamma\}^{-1}$

Let $\gamma \in \mathbb{R}$ and suppose that $\{n\alpha-\gamma\} \neq 0 \ \forall n \in \mathbb{N}$. For $N \in \mathbb{N}$ we want to obtain upper bounds for the sum

$$\sum_{n=0}^{N} \frac{1}{\{n\alpha-\gamma\}}.$$

We can again do this through the use of Corollary 4. The argument is similar to the one used in estimating the homogeneous bounds, except that the introduction of γ changes the point where we start counting our gaps from.

Corollary 4 *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\alpha = [a_0; a_1, a_2, \ldots]$, $N \in \mathbb{N}$. Fix $n_0 \in \mathbb{Z}$, $0 \le n_0 \le N$ and let $m = m(N, \alpha)$ be the largest integer satisfying $q_m \le N$, where $\frac{p_k}{q_k}$ are the continued fraction convergents to α. Then*

$$\sum_{\substack{0 \le n \le N \\ (n-n_0) \not\equiv 0, q_{m-1} (mod\ q_m)}} \frac{1}{\{(n-n_0)\alpha\}} < 4N\left(\log q_m + 1\right)$$

and

$$\sum_{\substack{0 \le n \le N \\ (n-n_0) \equiv 0, q_{m-1} (mod\ q_m) \\ n \ne n_0}} \frac{1}{\{(n-n_0)\alpha\}} < 4q_{m+1}\left(\log\left(\frac{2N}{q_m} + 1\right) + 1\right).$$

Proof. Since $\{(n-n_0)\alpha\} = \{n\alpha - n_0\alpha\}$, it is clear the same argument for Theorem 23 applies, except we start counting the gaps from $\{n_0\alpha\}$ instead of 0. As $\{n_0\alpha\}$ could be anywhere on the unit interval, we lose any information about the first gap that the parity of k provides. Hence combining the worst estimates of Theorem 23(i) and (ii), we have for $\delta_A < \delta_B$,

$$\sum_{\substack{0 \le n \le N \\ (n-n_0) \not\equiv 0, q_{k-1} (\mathrm{mod}\ q_k)}} \frac{1}{\{(n-n_0)\alpha\}} < 3N\left(\log q_k + 1\right) \tag{82}$$

since there are now fewer terms in the sum, and also from (64), for all $r < a_{k+1}$,

$$\begin{aligned}\sum_{\substack{0 \le n \le N \\ (n-n_0) \equiv 0, q_{k-1} (\mathrm{mod}\ q_k) \\ n \ne n_0}} \frac{1}{\{(n-n_0)\alpha\}} &< 2q_{k+1}\left(\log\left(\frac{N}{q_k} + 1\right) + 1\right) + 2q_{k+1}\left(\log\left(\frac{2N}{q_k}\right) + 1\right) \\ &< 4q_{k+1}\left(\log\left(\frac{2N}{q_k} + 1\right) + 1\right), \end{aligned} \tag{83}$$

since there are now more terms in the sum. Similarly, from (69), combining the worst estimates of Theorem 23(iii) and (iv), we have for $\delta_B < \delta_A$,

$$\sum_{\substack{0 \le n \le N \\ (n-n_0) \not\equiv 0, q_k (\mathrm{mod}\ q_{k+1})}} \frac{1}{\{(n-n_0)\alpha\}} < 4N\left(\log q_{k+1} + 1\right) \tag{84}$$

and

$$\sum_{\substack{0\le n\le N\\(n-n_0)\equiv 0,q_k(\text{mod }q_{k+1})\\n\ne n_0}}\frac{1}{\{(n-n_0)\alpha\}}<2q_{k+2},\tag{85}$$

since from (73),

$$\frac{2}{|D_k|}<2(q_{k+1}+q_k)\le 2(a_{k+2}q_{k+1}+q_k)=2q_{k+2}.\tag{86}$$

Note that when $\delta_A<\delta_B$, by (9), $q_k<N$ so $m=k$. If instead $\delta_B<\delta_A$, then $q_{k+1}\le q_{k+1}+s=N$ so $m=k+1$. Hence, combining the worst estimates of (82) - (85) gives

$$\sum_{\substack{0\le n\le N\\(n-n_0)\not\equiv 0,q_{m-1}(\text{mod }q_m)}}\frac{1}{\{(n-n_0)\alpha\}}<4N\left(\log q_m+1\right)$$

and

$$\sum_{\substack{0\le n\le N\\(n-n_0)\equiv 0,q_{m-1}(\text{mod }q_m)\\n\ne n_0}}\frac{1}{\{(n-n_0)\alpha\}}<4q_{m+1}\left(\log\left(\frac{2N}{q_m}+1\right)+1\right).$$

$\boxtimes$

Corollary 4 allows us to easily estimate $\sum\{n\alpha-\gamma\}^{-1}$. We have the following theorem as a consequence:

Theorem 25 *Let $\alpha\in\mathbb{R}\backslash\mathbb{Q}$, $\alpha=[a_0;a_1,a_2,\ldots]$, $N\in\mathbb{N}$, and let $\gamma\in\mathbb{R}$. Fix n_γ such that*

$$\{n_\gamma\alpha-\gamma\}=\min_{0\le n\le N}\left(\{n\alpha-\gamma\}\right),$$

and let $m=m(N,\alpha)$ be the largest integer satisfying $q_m\le N$, where $\frac{p_k}{q_k}$ are the continued fraction convergents to α. Then

$$\sum_{\substack{0\le n\le N\\(n-n_\gamma)\not\equiv 0,q_{m-1}(mod\ q_m)}}\frac{1}{\{n\alpha-\gamma\}}<4N\left(\log q_m+1\right)$$

and

$$\sum_{\substack{0\le n\le N\\(n-n_\gamma)\equiv 0,q_{m-1}(mod\ q_m)\\ n\ne n_\gamma}} \frac{1}{\{n\alpha-\gamma\}} < 4q_{m+1}\left(\log\left(\frac{2N}{q_m}+1\right)+1\right).$$

Proof. We use the same argument to the one used in the proof of Theorem 23. Notice the only difference is that the introduction of γ means that the starting point where we count our gaps from could be anywhere on the unit interval. There will therefore exist an additional first gap of uncontrollable size, possibly less than $\min\{\delta_A,\delta_B,\delta_C\}$, thus any information given by the parity of k concerning the first gap is now lost. To define this first gap, we first assume $\{n\alpha-\gamma\}>0$, so $\{\gamma\}$ does not coincide with any of the existing $n_i\alpha$. Next fix n_γ such that

$$\{n_\gamma\alpha-\gamma\} = \min_{0\le n\le N}\left(\{n\alpha-\gamma\}\right),$$

then disregard the first gap, and estimate every subsequent gap of $\{n\alpha-\gamma\}$ from above by $\{n\alpha-n_\gamma\alpha\}$ since $\{n\alpha-\gamma\}\ge\{n\alpha-n_\gamma\alpha\}$. Thus we have

$$\sum_{\substack{0\le n\le N\\ n\ne n_\gamma}}\frac{1}{\{n\alpha-\gamma\}} \le \sum_{\substack{0\le n\le N\\ n\ne n_\gamma}}\frac{1}{\{n\alpha-n_\gamma\alpha\}} = \sum_{\substack{0\le n\le N\\ n\ne n_\gamma}}\frac{1}{\{(n-n_\gamma)\alpha\}},$$

to which we apply Corollary 4 with $n_0=n_\gamma$ to see that

$$\sum_{\substack{0\le n\le N\\(n-n_\gamma)\not\equiv 0,q_{m-1}(\mathrm{mod}\ q_m)}} \frac{1}{\{n\alpha-\gamma\}} < 4N\left(\log q_m+1\right)$$

and

$$\sum_{\substack{0\le n\le N\\(n-n_\gamma)\equiv 0,q_{m-1}(\mathrm{mod}\ q_m)\\ n\ne n_\gamma}} \frac{1}{\{n\alpha-\gamma\}} < 4q_{m+1}\left(\log\left(\frac{2N}{q_m}+1\right)+1\right).$$

⊠

5.2 Bounds for $\sum\|n\alpha-\gamma\|^{-1}$

To estimate $\sum\|n\alpha-\gamma\|^{-1}$, we require a similar approach to Theorem 24.

Theorem 26 *Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\alpha = [a_0; a_1, a_2, \ldots]$, $N \in \mathbb{N}$, and let $\gamma \in \mathbb{R}$. Fix n_γ such that*

$$\|n_\gamma \alpha - \gamma\| = \min_{0\le n\le N} \left(\|n\alpha - \gamma\|\right),$$

and let $m = m(N, \alpha)$ be the largest integer satisfying $q_m \le N$, where $\frac{p_k}{q_k}$ are the continued fraction convergents to α. Then

$$\sum_{\substack{n=0\\ n\ne n_\gamma}}^{N} \frac{1}{\|n\alpha - \gamma\|} < 8N\log N + (28a_{m+1} + 35)N.$$

Proof. Fix n'_γ such that

$$\{n'_\gamma \alpha - \gamma\} = \min_{0\le n\le N} \left(\{n\alpha - \gamma\}\right).$$

Then Theorem 25 tells us that

$$\sum_{\substack{n=0\\ n\ne n'_\gamma}}^{N} \frac{1}{\{n\alpha - \gamma\}} < 4N\left(\log q_m + 1\right) + 4q_{m+1}\left(\log\left(\frac{2N}{q_m} + 1\right) + 1\right). \tag{87}$$

Similar to $\sum\{n\alpha - \gamma\}^{-1}$, we use Theorem 25 to estimate $\sum(1 - \{n\alpha - \gamma\})^{-1}$, since the only difference is that we count the gaps in the opposite direction, which does not affect the sum in any way. In this case we fix n''_γ such that

$$1 - \{n''_\gamma \alpha - \gamma\} = \min_{0\le n\le N} \left(1 - \{n\alpha - \gamma\}\right) = \min_{0\le n\le N} \left(\{\gamma - n\alpha\}\right).$$

Note that $\{-n\alpha\} = 1 - \{n\alpha\}$ and also that we do not necessarily have $n''_\gamma = n'_\gamma$. Then we obtain

$$\sum_{\substack{n=0\\ n\ne n''_\gamma}}^{N} \frac{1}{(1 - \{n\alpha - \gamma\})} < 4N\left(\log q_m + 1\right) + 4q_{m+1}\left(\log\left(\frac{2N}{q_m} + 1\right) + 1\right). \tag{88}$$

From (75) and Lemma 9, summing instead from 0 to N, we have

$$\sum_{n=0}^{N} \frac{1}{\|n\alpha - \gamma\|} < \sum_{n=0}^{N} \frac{1}{\{n\alpha - \gamma\}} + \sum_{n=0}^{N} \frac{1}{1 - \{n\alpha - \gamma\}} - N - 1. \tag{89}$$

Fix n_γ such that

$$\|n_\gamma\alpha - \gamma\| = \min_{0\le n\le N}\|n\alpha-\gamma\| = \min\left\{\{n'_\gamma\alpha-\gamma\}, \{\gamma - n''_\gamma\alpha\}\right\},$$

therefore $n_\gamma \in \{n'_\gamma, n''_\gamma\}$. Assume without loss of generality that $n_\gamma = n'_\gamma$. Then as $\{n'_\gamma\alpha\}$ and $\{n''_\gamma\alpha\}$ are on opposite sides of $\{\gamma\}$, $\|n''_\gamma\alpha-\gamma\|$ is at least half the length of the smallest gap. If $\delta_A < \delta_B$, then $m = k$ and using (5),

$$\frac{1}{\|n''_\gamma\alpha-\gamma\|} \le \frac{2}{\delta_A} = \frac{2}{|D_k|} < 4q_{k+1} = 4q_{m+1},$$

and if $\delta_B < \delta_A$, then $m = k+1$, so

$$\frac{1}{\|n''_\gamma\alpha-\gamma\|} \le \frac{2}{\delta_B} = \frac{2}{|D_{k+1}|} < 4q_{k+2} = 4q_{m+1}.$$

Therefore, combining the above with (87), (88), and (89), we have

$$\sum_{\substack{n=0\\ n\ne n_\gamma}}^{N} \frac{1}{\|n\alpha-\gamma\|} < 8N\left(\log q_m + 1\right) + 8q_{m+1}\left(\log\left(\frac{2N}{q_m}+1\right)+1\right) - N + 4q_{m+1}.$$

We have $q_m \le N$, and recall from (78) and (79) that $q_{m+1} < (a_{m+1}+1)N$ and

$$4q_{m+1}\log\left(\frac{2N}{q_m}+1\right) < (8a_{m+1}+8)N,$$

so we have

$$\sum_{\substack{n=0\\ n\ne n_\gamma}}^{N} \frac{1}{\|n\alpha-\gamma\|} < 8N\log N + (28a_{m+1}+35)N.$$

⊠

6 Conclusion

This technique gives explicit constants for the bounds for sums of reciprocals. While both asymptotic bounds and bounds with explicit constants have been obtained via a wide variety of methods, (see Section 3 and references within) the main advantage of our method is in its simplicity. It primarily relies on a powerful version of the Three Distance Theorem which describes the distribution of gaps on the unit circle, and does not require much else.

Although we have only given results for upper bounds in this thesis, in principle, the same ideas apply in using this technique to obtain lower bounds and asymptotic formulas, and it can be done without too much difficulty. Further work can also be done in generalising the technique to higher dimensions, for instance, for sums of the form $\sum_{n\leq N} n^{-s}\|n\alpha-\gamma\|^{-t}$ where s,t are non-negative integers.

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