The Geodesic Gauss Map and Ruh-Vilms theorem for a Hypersurface in S^n

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Abstract

We are interested to work on normal homogeneous space and in this space we calculated Live-Civita connection and we derived a useful equation 2.17. The Ruh-Vilms theorem is a statement about the Gauss map for a submanifold of $\mathbb{R}^{n+1}$. Our aim is to prove, an isometrically immersed hypersurface $f : M \longrightarrow S^n$ has constant mean curvature if and only if the Gauss map of γ is harmonic. Here we provide a proof of the Ruh-Vilms result using Homogeneous geometry. First shown for curves in S^2, then proven for a hypersurface in the n-sphere by using symmetric space identification and results in 2.17.

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Author's Declaration

I declare that the work within this thesis is my own work (except where otherwise indicated).

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Introduction

The fundamental question is, "What is geometry?" still remained. There are two directions of development after Gauss. The first, is related to the work of B. Riemann, who conceived of a framework of generalizing the theory of surfaces of Gauss, from two to several dimensions. The new objects are called Riemannian manifold, where a notion of curvature is defined, and is allowed to vary from point to point, as in the case of a surface. Riemann brought the power of calculus into geometry in an emphatic way as he introduced metrics on the spaces of tangent vectors. The result is today called differential geometry.

In Chapter 1, We start with a brief review of Riemannian manifolds with a definition of connection, Levi-Civita connection and curvature tensor , and then we define the Lie group and some simple examples of the Lie group. We define the Lie algebra of a Lie group as the tangent space at the identity element of the group, and alternatively as the set of its one-parameter subgroups. The metrics which are important here are the bi-invariant metrics and with respect to such metrics we give formulas for the various type of curvatures.

In chapter 2 we define the notion of a homogeneous space and we discuss the reductive homogeneous spaces. Then we are discussing Riemannian Submersions and their Sectional Curvatures and also derive the curvature relation in Riemannian reductive homogeneous spaces. We define notion of a symmetric space and provide an example of sectional curvature of complex projective space as a homogeneous space. Then we develop Levi-Civita Connexions for normal homogeneous spaces and look at some of ONeill 's results.

In chapter 3, first we define the harmonic map and second fundamental form . Then we generalize Grassmann manifold and define the Gauss map. Then we state Ruh-Vilms theorem and prove the Ruh-Vilms theorem using the homogeneous geometry for the simple case of curves in S^2 . Here we use the main result of M. Obata that is: The Gauss map of immersions of Riemannian manifolds in space of constant curvature.

In the finial chapter I have proved the Ruh-Vilms theorem for a Hypersurface in the n- Sphere.

Chapter 1

Riemannian Geometry and Lie Groups: their application to homogeneous spaces

As a preface to understanding Riemannian homogeneous spaces, Riemannian submersions and their sectional curvature, we first review basic Riemannian Geometry principles, including vector fields, integral curves, connections, metrics and curvature. We will also consider the principles of Lie groups and Lie algebras, including Left- and Right-translation, Left-invariant vector fields and Killing forms.

1.1 Riemannian geometry

In what follows we will always consider smoothness on C^∞, that is:

Definition 1.1.1 *A map is said to be smooth if it has continuous partial derivatives of all orders.*

We denote the set of smooth maps $\mathbb{R}^m \longrightarrow \mathbb{R}^n$ by $C^\infty(\mathbb{R}^m, \mathbb{R}^n)$.

Definition 1.1.2 *Two functions f, g which are smooth about p are germ equivalent at p if $f = g$ on some open neighbourhood of p.*

The set of germ equivalence classes of locally smooth functions about p is denoted by $C_p^\infty(M)$.

Definition 1.1.3 *Let M be a manifold, $p \in M$, then a directional derivative (or tangent vector to M) at p is a real valued function $\xi : C_p^\infty(M) \longrightarrow \mathbb{R}$ that satisfies*

1. *$\xi(af + bg) = a\xi(f) + b\xi(g)$ and*
2. *$\xi(fg) = \xi(f)g(p) + f(p)\xi(g)$ for all $a, b \in \mathbb{R}$ and $f, g \in C_p^\infty(M)$.*

The set of all tangent vectors at p, denoted T_pM, is called the tangent space at p.

Definition 1.1.4 *For an n-dimensional manifold M, the tangent bundle of M is the disjoint union over all points of the tangent spaces at each point, $TM = \cup_{p\in M} T_pM$. The canonical projection is given by the map $\pi : TM \longrightarrow M$ given by $\pi(p, \xi) = p$. For each $p \in M$ the fibre over p is the pre-image $\pi^{-1}(p)$, the tangent space T_pM.*

A vector field X on a manifold M is the assignment of a tangent vector $X_p \in T_pM$ to each point $p \in M$: thus $X : M \longrightarrow TM$ with $X_p \in T_pM$. If X is a vector field on M and $f \in C^\infty(M)$, then Xf denotes the real-valued function on M given by $Xf(p) = X_p(f)$ for all $p \in M$.

Definition 1.1.5 *The vector field X is called smooth if the function Xf is smooth for all $f \in C^\infty(M)$.*

We denote the set of smooth vector fields by $\Gamma(TM)$.

We note that the function X defined above can be identified with a map $X : C^\infty(M) \longrightarrow C^\infty(M)$ satisfying

1. $X(af + bg) = aX(f) + bX(g)$ where $a, b \in \mathbb{R}$,
2. $X(fg) = X(f)g + fX(g)$ (Leibniz rule).

Definition 1.1.6 *A curve $\gamma : (-\epsilon, \epsilon) \longrightarrow M$ is an integral curve of $X \in \Gamma(TM)$ when $\gamma'(t) = X_{\gamma(t)}$ for all $t \in (-\epsilon, \epsilon)$.*

The velocity vector of γ is the vector $\gamma' = \frac{d\gamma}{dt}$. It can be shown that:

Proposition 1.1.7 *For every $\xi \in T_pM$ there exists a smooth curve $\gamma : (-\epsilon, \epsilon) \longrightarrow M$ such that $\gamma(0) = p$, $\xi f = \frac{d}{dt}(f \circ \gamma)|_{t=0}$ for all $f \in C_p^\infty(M)$.*

Proof [2] In this case ξ is often denoted byγ'.

A Riemannian metric on a smooth manifold M is an assignment to each $p \in M$ of an inner product $g_p =<, >_p$: $T_pM \times T_pM \longrightarrow \mathbb{R}$ (that is a symmetric bilinear, positive definite form) on the tangent space T_pM.

Definition 1.1.8 *A smooth manifold with a Riemannian metric g is called a Riemannian manifold, and is denoted by (M, g).*

Definition 1.1.9 *A connection ∇ on a smooth manifold M is a map*

$$\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$

satisfying

1. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,
2. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,
3. $\nabla_X(fY) = f\nabla_X Y + X(f)Y$ *(Leibniz rule)*

for all $X, Y, Z \in \Gamma(TM)$ and $f, g \in C^\infty(M)$.

Theorem 1.1.10 *Given a Riemannian manifold M, there exists a unique Levi-Civita connection such that*

1. $[X, Y] = \nabla_X Y - \nabla_Y X$

2. $X\text{<}Y, Z\text{>} = \text{<}\nabla_X Y, Z\text{>} + \text{<}Y, \nabla_X Z\text{>}$

for all $X, Y, Z \in \Gamma(TM)$*. This connection is characterized by the Koszul formula:*

$$\begin{aligned} 2\text{<}\nabla_X Y, Z\text{>} &= X\text{<}Y, Z\text{>} + Y\text{<}Z, X\text{>} - Z\text{<}X, Y\text{>} \\ &\quad -\text{<}X, [Y, Z]\text{>} + \text{<}Y, [Z, X]\text{>} + \text{<}Z, [X, Y]\text{>}. \end{aligned}$$

Proof [2]

Definition 1.1.11 *Let* $\gamma : (-\epsilon, \epsilon) \longrightarrow M$ *in* M *be a curve,* γ' *its velocity vector, and* X *a vector field along* γ*.* X *is said to be parallel along* γ *if* $\nabla_{\gamma'} X = 0$*. A smooth curve* γ*, is said to be geodesic if* γ' *is parallel along* γ*. i.e.* $\nabla_{\gamma'}\gamma' = 0$.

Definition 1.1.12 *Let (M,g) be a Riemannian manifold with Levi-Civita connection* ∇*. The Riemannian curvature tensor is the function*

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$

given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for $X, Y, Z \in \Gamma(TM)$*.*

Definition 1.1.13 *Let* V *be a 2-dimensional subspace of* T_pM *and let* $u, v \in V$ *be two linearly independent vectors which span* V*. Then we define the sectional curvature by:*

$$K(V) = \frac{g(R(u, v)v, u)}{g(u, u)g(v, v) - (g(u, v))^2}.$$

This is a number and is independent of basis u, v *chosen.*

Definition 1.1.14 *A smooth map* $\phi : M \longrightarrow N$ *between manifolds is*

1. *an immersion if* $d\phi_p$ *is one-to-one at each* $p \in M$*,*

2. *a submersion if* $d\phi_p$ *is onto at each* $p \in M$*,*

3. *an embedding if it is an immersion which is a diffeomorphism onto* $\phi(M)$*.*

4. *a homeomorphism is a map which is continuous, one-to-one, onto and the inverse map is continuous.*

5. *a map* $\phi : M \longrightarrow N$ *is a diffeomorphism if it is a homeomorphism and both* f *and* f^{-1} *are* C^∞*.*

1.2 Lie Groups

Definition 1.2.1 *Let G be a C^∞ manifold. Then G is called a Lie Group if*

1. *G is a Group*

2. *The Group operation $G \times G \longrightarrow G, (x, y) \mapsto xy^{-1}$ is C^∞ function.*

Let $M(n, \mathbb{R})$ be the set of all $n \times n$ real matrices: we will denote by the (i, j) entry of an $(n \times n)$ matrix $A = (a_{i,j})$ the points in the Euclidean space $\mathbb{R}^{n\times n}$ whose coordinates are $a_{1,1}, a_{1,2}, ..., a_{1,n}$. Topologically, therefore, $M(n, \mathbb{R})$ is simply the Euclidean n^2 space. By definition, the general linear group is given by $GL(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\}$. Since $\det A$ is a polynomial of degree n in the coordinates, it is a C^∞ function on $M(n, \mathbb{R})$, and the determinant function $\det : M(n, \mathbb{R}) \longrightarrow \mathbb{R}$ is continuous. Now $GL(n, \mathbb{R}) = M(n, \mathbb{R}) - \det^{-1}\{0\}$ is open since $\{0\}$ is closed, and hence $GL(n, \mathbb{R})$ is an open subset of $M(n, \mathbb{R})$. So, topologically $GL(n, \mathbb{R})$ is an open subset of a Euclidean space, and $GL(n, \mathbb{R})$ is therefore an n^2–dimensional manifold.

Matrix multiplication $(AB)_{i,j} = \sum_{k=1}^{n} a_{ik}b_{kj}$ is a polynomial in the coordinates of $GL(n, \mathbb{R})$ and is clearly C^∞. The inverse of A is $A^{-1} = \frac{1}{\det A}\text{adj}A$ whose coordinates are C^∞ function. Therefore, the inverse map $i : GL(n, \mathbb{R}) \longrightarrow GL(n, \mathbb{R})$ is also C^∞. This proves that $GL(n, \mathbb{R})$ is a Lie group.

Example 1.2.2 *The main examples of Lie groups are matrix groups:*

- *General linear group $GL(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\}$.*
- *Special linear group $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det A = 1\}$.*
- *Orthogonal group $O(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : AA^T = I\}$.*
- *Special Orthogonal group $SO(n, \mathbb{R}) = \{A \in O(n, \mathbb{R}) : \det A = 1\}$.*
- *Unitary group $U(n, \mathbb{C}) = \{A \in \mathbb{C}^{n\times n} : A\overline{A}^T = I\}$.*
- *Special unitary group $SU(n, \mathbb{C}) = \{A \in U(n, \mathbb{C}) : \det A = 1\}$.*

The product $G \times H$ of two Lie groups is itself a Lie group with the product manifold structure, and multiplication $(g_1, h_1).(g_2, h_2) = (g_1.g_2, h_1.h_2)$.

The unit circle S^1 is a Lie groups by considering S^1 in complex plane.

The n-torus $T^n = S^1 \times .... \times S^1$ is a Lie group of dimension n.

Definition 1.2.3 *A Lie algebra is a real (or complex) vector space V with the operation $[,] : V \times V \longrightarrow V$ that satisfies the following properties:*

- $[X, Y] = -[Y, X]$
- $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
  $[Z, aX + bY] = a[Z, X] + b[Z, Y]$
- *Jacobi identity,* $[X[Y, Z]] + [Y[Z, X]] + [Z[X, Y]] = 0$

Let V be a vector space over $\mathbb{R}$. If we define the bracket operation by $[x, y] = 0$ for all $x, y \in V$ we get a Lie algebra $(V, [,])$. This is an example of an *abelian* Lie algebra.

For each $g \in G$ we define the maps

$$\begin{array}{rcll} L_g : G & \longrightarrow & G, \quad L_g(a) = ga & \text{(left translation)}, \\ R_g : G & \longrightarrow & G, \quad R_g(a) = ag & \text{(right translation)}. \end{array}$$

These maps are smooth diffeomorphisms with inverses. The derivatives $(dL_g)_e, (dR_g)_e : \mathfrak{g} = T_eG \longrightarrow T_gG$ are vector space isomorphisms.

Definition 1.2.4 *A vector field X on a Lie group G is left-invariant when $dL_a(X) = X$ for all $a \in G$.*

If $dL_a(X(e)) = X(a)$ for all $g \in G$, we call X the left-invariant vector field on G generated by $X(e)$. The set $L(G)$ of left-invariant vector fields on G is obviously a vector space.

Lemma 1.2.5 *$L(G)$ is a vector space and the map $L(G) \longrightarrow T_eG$ given by $X \mapsto X_e$ is an isomorphism of vector spaces.*

Definition 1.2.6 *Let $\phi : M \longrightarrow N$ be any smooth map. Two vector fields $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$ are called ϕ- related when $d\phi(X_p) = Y_{\phi(p)}$ for all $p \in M$*

Corollary 1.2.7 *Vector fields $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$ are ϕ-related if and only if $X(g \circ \phi) = Y(g) \circ \phi$ for all $g \in C^\infty(N)$ $[g : N \longrightarrow \mathbb{R}]$*

Proof See[2] p.14

Proposition 1.2.8 *Suppose $\phi : M \longrightarrow N$ is a C^∞ map, $X, Y \in \Gamma(TM)$, and $\overline{X}, \overline{Y} \in \Gamma(TM)$ such that $\overline{X} = d\phi(X)$ and $\overline{Y} = d\phi(Y)$. Then*

$$[\overline{X}, \overline{Y}] = d\phi([X, Y]).$$

Proof Let $f : N \longrightarrow \mathbb{R}$ be a smooth function, then

$$\begin{aligned} [\overline{X}, \overline{Y}] &= d\phi(X)(d\phi(Y)(f)) - d\phi(Y)(d\phi(X)(f)) \\ &= X(d\phi(Y)(f) \circ \phi) - Y(d\phi(X)(f) \circ \phi) \\ &= X(Y(f \circ \phi)) - Y(X(f \circ \phi)) \\ &= [X, Y](f \circ \phi) \\ &= d\phi([X, Y])(f). \end{aligned}$$

Proposition 1.2.9 *Let $\phi : M \longrightarrow N$ be a smooth bijective map. If $X, Y \in \Gamma(TM)$ then*

1. *$d\phi(X) \in \Gamma(TN)$,*

2. *the map* $d\phi : \Gamma(TM) \longrightarrow \Gamma(TN)$ *is a Lie algebra homomorphism i.e* $d\phi[X, Y] = [d\phi(X), d\phi(Y)]$.

Proof The map ϕ is bijective implies that $d\phi(X)$ is a section of the tangent bundle. That $d\phi(X) \in \Gamma(TN))$ follows directly from the fact that

$$d\phi(X)(f) = X(f \circ \phi).$$

Part (2) follows directly from Proposition 1.2.8.

Proposition 1.2.10 *If* X *and* Y *are left-invariant vector fields on* G*, then so is* $[X, Y]$.

Proof For any diffeomorphism $\phi : M \longrightarrow N$ and $X, Y \in \Gamma(TM)$

$$d\phi[X, Y] = [d\phi(X), d\phi(Y)].$$

Then

$$dL_g[X, Y] = [dL_g(X), dL_g(Y)] = [X, Y]$$

since L_g is a diffeomorphism.

Definition 1.2.11 *A one-parameter subgroup of a Lie group* G *is smooth homomorphism* $\phi : (\mathbb{R}, +) \longrightarrow G$.

Thus $\phi : \mathbb{R} \longrightarrow G$ is a curve such that $\phi(s + t) = \phi(s)\phi(t)\ \forall s, t \in \mathbb{R}$. Moreover $\phi(0) = e, \phi(-t) = (\phi(t))^{-1}$ and $\phi(t)\phi(s) = \phi(s)\phi(t), \forall s, t$.

Proposition 1.2.12 *Each one- parameter subgroup of* G *is the maximal integral curve, starting at* e*, of a left invariant vector field* X*. In fact, every integral curve of* X *is a translate of this one parameter subgroup.*

Corollary 1.2.13 *For each* $X \in \mathfrak{g}$ *there exists a unique one-parameter subgroup* $\phi_X : \mathbb{R} \longrightarrow G$ *such that* $\phi'_X(0) = X$.

Definition 1.2.14 *Let* $\mathfrak{g}$ *be an Lie algebra and* $\mathfrak{h}$ *a vector subspace of* $\mathfrak{g}$

- $\mathfrak{h}$ *is called a Lie subalgebra of* $\mathfrak{g}$*, if* $[X, Y] \in \mathfrak{h}\ \forall X, Y \in \mathfrak{h}$.
- $\mathfrak{h}$ *is called an ideal in* $\mathfrak{g}$, $[X', X] \in \mathfrak{h}$ *whenever* $X \in \mathfrak{h}$ *and* $X' \in \mathfrak{g}$.

Definition 1.2.15 *The exponential map* $\exp : \mathfrak{g} \longrightarrow G$ *is defined by* $\exp(X) = \phi_X(1)$*, where* ϕ_X *is the unique one-parameter subgroup of* X.

Note: $\exp(tX) = \phi_{tX}(1) = \phi_X(t)$.

Let G be a Lie group. For $x, g \in G$, let the map $I_x : G \longrightarrow G$ be defined by $I_x = xgx^{-1}$. Then I_x is a Lie group homomorphism and $I_x = R_{x^{-1}} \circ L_x$ is diffeomorphism. It is called an inner automorphism of G.

Definition 1.2.16 *The adjoint representation of* G *is the homomorphism* $Ad : G \longrightarrow GL(\mathfrak{g})$ *given by* $Ad(x) = (dI_x)_e$.

Definition 1.2.17 *The adjoint representation of* $\mathfrak{g}$ *is the homomorphism* $ad : \mathfrak{g} \longrightarrow End(\mathfrak{g})$ *given by* $ad(X) = (dAd)_e(X)$.

Lemma 1.2.18 *The adjoint representation of* $\mathfrak{g}$ *satisfies* $ad_X(Y) = [X, Y]$ *for all* $X, Y \in \mathfrak{g}$.

Proof For any $g \in G$, $Y \in \mathfrak{g}$ the curve $\alpha(s) = g(\exp sY)g^{-1}$ is the one-parameter subgroup of G satisfying $\alpha' = Ad_g(Y)$. The Campbell - Hausdorff formula [12] implies that for any $t \in \mathbb{R}$, $X \in \mathfrak{g}$,

$$(\exp tX)(\exp sY)(\exp -tX) = \exp(sY + ts[X, Y] + O(t^2 s) + O(ts^2)).$$

This implies

$$Ad_{\exp tX}(Y) = Y + t[X, Y] + O(t^2),$$

$$\Rightarrow \qquad \lim_{t \longrightarrow 0} \frac{\{Ad_{\exp tX} Y - Y\}}{t} = [X, Y],$$

from which the result follows.

Let $\mathfrak{g}$ be a Lie algebra and $X \in \mathfrak{g}$. Then the map $ad_X : \mathfrak{g} \longrightarrow \mathfrak{g}$ defined by $ad_X(Y) = [X, Y]$. The ad_X is a linear operator, and by the Jacobi identity it is a Lie derivation and from defintion 1.2.3

$$ad_Z[X, Y] = [ad_Z X, Y] + [X, ad_Z Y]$$

and also

$$ad_{[X,Y]} = [ad_X, ad_Y].$$

Definition 1.2.19 *A Riemannian metric on a Lie group G is called left- invariant if* $<X, Y> = <dL_a(X), dL_a(Y)>$ *for all* $a \in G$, $X, Y \in \Gamma(TG)$. *The notion of right-invariant is defined similarly.*

Note: A metric on G that is both Left- invariant and Right- invariant is called a bi-invariant metric.

Lemma 1.2.20 *There is a one-to-one correspondence between bi-invariant metrics on* G *and Ad-invariant scalar products on* $\mathfrak{g}$*, that is* $<Ad_g X, Ad_g Y> = <X, Y>$ *for all* $g \in G$, $X, Y \in \mathfrak{g}$. *Furthermore, the last condition is equivalent to the relation*

$$<[X, Z], Y> = <X, [Z, Y]>.$$

Proof By definition,

$$Ad(g)Y = dI_g(Y) = dR_{g^{-1}} dL_g(Y) = dR_{g^{-1}} Y,$$

for all $g \in G$ and $X, Y \in \mathfrak{g}$ and hence, by using the right invariance,

$$<Ad_g X, Ad_g Y> = <dR_{g^{-1}} X, dR_{g^{-1}} Y> = <X, Y>.$$

Now, let $\exp(tX)$ be the flow of X. Then

$$\begin{aligned} <[X,Z],Y> &= <ad_X Z,Y> = <\tfrac{d}{dt}Ad(\exp tX)Z|_{t=0},Y> \\ &= \tfrac{d}{dt}<Ad(\exp tX)Z,Y>|_{t=0} \\ &= \tfrac{d}{dt}<Z,Ad(\exp(-tX))Y>|_{t=0} \qquad \text{(Ad-invariant)} \\ &= <Z,-ad_X Y> = -<Z,[X,Y]>, \end{aligned}$$

where we used the Ad- invariance of the inner product in the fourth equality. What we just proved is equivalent to $<[X,Z],Y> = <X,[Z,Y]>$.

Definition 1.2.21 *The Killing form of a Lie algebra $\mathfrak{g}$ is the function $B : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathbb{R}$ given by $B(X,Y) = tr(adX \circ adY)$.*

The Lie algebra $\mathfrak{g}$ is called semisimple, if its Killing form is non-degenerate.

Proposition 1.2.22 [2] *The Killing form has the following properties;*

1. *It is a symmetric bilinear form on $\mathfrak{g}$.*
2. *If $\mathfrak{g}$ is the algebra of G, then B is Ad-invariant, that is, $B(X,Y) = B(Ad_gX, Ad_gY)$ for all $g \in G$ and $X,Y \in \mathfrak{g}$.*
3. *G is semisimple and compact when B is negative definite.*
4. *Each ad(Z) is skew-symmetric with respect to B, that is, $B(ad_ZX,Y) = -B(X,ad_ZY)$ or $B([X,Z],Y) = B(X,[Z,Y])$.*

Theorem 1.2.23 [1] *A compact Lie group possesses a bi-invariant metric.*

Theorem 1.2.24 [1] *If $<,>$ is bi-invariant metric on a Lie group G then the Levi-Civita connection ∇ and Riemann curvature R, the following are satisfied by $X,Y,Z,W \in L(G)$,*

1. $\nabla_X Y = \frac{1}{2}[X,Y]$
2. $<R(X,Y)Z,W> = \frac{1}{4}(<[X,W],[Y,Z]> - <[X,Z],[Y,W]>)$
3. $<R(X,Y)Y,X> = \frac{1}{4}\|[X,Y]\|^2$
4. *One- parameter subgroups are geodesics.*

Proof

(1). Let $<,>$ be a left-invariant metric on G, and let X,Y are left-invariant vector fields. Then the function $<X,Y> : G \longrightarrow \mathbb{R}$ is constant on G.
Now $<X,Y>$ is constant, so $Z<X,Y> = 0$ for all $Z \in \mathfrak{g}$. From Koszul's formula the first three terms zero. So we get

$$2<\nabla_X Y,Z> = -<X,[Y,Z]> + <Y,[Z,X]> + <Z,[X,Y]>$$

By Lemma 1.2.20 the bi-invariance of $<,>$ implies that

$$<X,[Y,Z]> = -<Y,[X,Z]> = <[X,Y],Z>.$$

Hence $2<\nabla_X Y, Z> = <Z,[X,Y]>$, which gives $\nabla_X Y = \frac{1}{2}[X,Y]$.

(2). By definition the Riemannian curvature is

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

For left-invariant vector fields X, Y, Z and W, $X<\nabla_Y Z, W> = 0$
Part (2) of Theorem 1.2.24 and part (1) of this Theorem imply

$$\begin{aligned} <\nabla_X\nabla_Y Z, W> &= -<\nabla_Y Z, \nabla_X W> &&= -\tfrac{1}{4}<[Y,Z],[X,W]> \\ -<\nabla_Y\nabla_X Z, W> &= <\nabla_X Z, \nabla_Y W> &&= \tfrac{1}{4}<[X,Z],[Y,W]> \\ -<\nabla_{[X,Y]}Z, W> &= -<\nabla_{[X,Y]}Z, W> &&= -\tfrac{1}{2}<[[X,Y],Z],W> \end{aligned}$$

Adding the above three equation we get,

$$<R(X,Y)Z,W> = -\frac{1}{4}<[Y,Z],[X,W]> + \frac{1}{4}<[X,Z],[Y,W]> - \frac{1}{2}<[[X,Y],Z],W>.$$

But by the Jacobi identity,

$$\begin{aligned} <[Z,[X,Y]],W> &= <[[Z,X],Y],W> + <[X,[Z,Y]],W> \\ &= <[Z,X],[Y,W]> - <[Z,Y],[X,W]> \end{aligned}$$

When we substitute this in the previous equation we get the answer.

(3). From part (2), the computation is straightforward by setting $Z = Y$, $W = X$. The first term cancels out, hence the result is obtained.

(4). Let γ be the one-parameter subgroup corresponding to the left-invariant vector field X, then $\nabla_{\gamma'}\gamma' = \nabla_X X = \frac{1}{2}[X,X] = 0$, from part (1). Hence γ is a geodesic.

Proposition 1.2.25 *Let G be a Lie group with bi-invariant metric then for any $X, Y, Z \in \mathfrak{g}$, the sectional curvature is given by*

$$K(X,Y) = \frac{1}{4}\frac{<[X,Y],[X,Y]>}{<X,X><Y,Y> - <X,Y>^2}$$

Proof By Theorem 1.2.24 part (3) and using the definition of sectional curvature, we can get the result.

Chapter 2

Homogeneous Spaces and Reductive Homogeneous Spaces

2.1 Group Action

An action of a group G on a set M is a homomorphism A from G to the group of diffeomorphisms of M, $A : G \longrightarrow \mathrm{Dif}f(M)$, written

$$A(g)(x) = g.x \quad g \in G, \quad x \in M.$$

We can also describe the action of G on M as a smooth mapping $A : G \times M \longrightarrow M$ such that

$$A(gh, x) = A(g, A(h, x)) \quad and \quad A(e, x) = x \quad \forall g, h \in G, \quad \forall x \in M.$$

Definition 2.1.1 *Let G be a Lie group and M be a manifold. Then a smooth action of G on M is an action A , that is a $G \times M \longrightarrow M$ smooth map between manifolds. The orbit of a point $x \in M$ is the set*

$$A(G)(x) = G.x = \{A(g)(x)|g \in G\} \subset M.$$

- *An action is said to be transitive if for any $x, y \in M$ there exists a $g \in G$ such that $y = g.x$.*
- *For each $x \in M$ the set $G_x = \{g \in G; g.x = x\}$ is called the isotropy group or stabiliser of x.*

Definition 2.1.2 *If G is a connected Lie group and H a closed subgroup, G/H is the space of cosets $\{gH\}$, and $\pi : G \longrightarrow G/H$ is defined by $g \mapsto gH$. G/H is called a homogeneous space.*

A homogeneous space is called reductive if there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $Ad(h)\mathfrak{m} \subset \mathfrak{m}$ for all $h \in H$, that is, $\mathfrak{m}$ is Ad(H)-invariant. $M = G/H$ is said to be naturally reductive, if there exists a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ satisfying

$$< [X, Y]_{\mathfrak{m}}, Z > + < [X, Z]_{\mathfrak{m}}, Y >= 0$$

for all $X, Y, Z \in \mathfrak{m}$, where $[X, Y]_{\mathfrak{m}}$ denotes the $\mathfrak{m}$- component of $[X, Y]$ and $<, >$ is the metric induced by g on $\mathfrak{m}$, by using the canonical identification $\mathfrak{m} \cong T_pM$ at $p \in M$.

2.2 Riemannian Submersions and their Sectional Curvatures

Let $(M^{n+k}, \overline{g})$ and N^n, g be Riemannian manifolds. A submersion is a smooth map $\pi : M^{n+k} \longrightarrow N^n$ whose derivative $d\pi$ is onto. Hence for each $p \in N^n$, $\pi^{-1}(p)$ is submanifold of M^{n+k}, of dimension n. Like with the metrics we shall use the notation $\overline{p}$ and p as well as $\overline{X}$ and X for points and vector fields that are π-related, i.e, $\pi(\overline{p}) = p$ and $d\pi(\overline{X}) = X$. The vertical distribution consists of the tangent spaces to the preimages $\pi^{-1}(p)$ and is therefore given by $V_{\overline{p}} = \ker d\pi_{\overline{p}} \subset T_{\overline{p}}M$. The horizontal distribution is the orthogonal complement $H_{\overline{p}} = (V_{\overline{p}})^{\perp} \subset T_{\overline{p}}M$. To say that π is a Riemannian submersion means that $d\pi : H_{\overline{p}} \longrightarrow T_pN$ is an isometry for all $\overline{p} \in M$. Given a vector field X on N we can always find a unique horizontal vector field. $\overline{X}$ on M, called the horizontal lift, is π related to X. Any vector in M can be decomposed into horizontal and vertical parts, i.e, $\xi = \xi^V + \xi^H$.

Remark 2.2.1 *Let $\beta : [0,1] \longrightarrow N$ be a smooth curve on N. We say that a smooth curve $\gamma(t)$ on M is a horizontal lift of β if*

- $\pi \circ \gamma(t) = \beta(t)$ *and*
- $\gamma'(t) \in H$ *for all t.*

Theorem 2.2.2 *Let X, Y, Z be vector fields on N with horizontal lifts $\overline{X}, \overline{Y}, \overline{Z}$ respectively. At each $p \in N$, for any $\overline{p} \in \pi^{-1}(p)$,*

$$K_p^N(X,Y) = K_{\overline{p}}^M(\overline{X},\overline{Y}) + \frac{3}{4}\|[\overline{X},\overline{Y}]_{\overline{p}}^V\|^2,$$

where $[\overline{X},\overline{Y}]_{\overline{p}}^V$ is a vertical vector field at a point $\overline{p}$.

Proof Let X, Y, Z be vector fields on N and let $\overline{X}, \overline{Y}, \overline{Z}$ be their horizontal lifts on M. Let T be a vertical vector field on M. Then $<\overline{X}, T> = 0$ and $[\overline{X}, T]$ is vertical, since $\overline{X}$ is π related to X and T is π related to the zero vector field on N. Thus

$$d\pi([\overline{X},T]) = [d\pi(\overline{X}), d\pi(T)] = [X,0] = 0.$$

By the Riemannian submersion property and lemma **??** we have

$$<[\overline{X},\overline{Y}],\overline{Z}> = <[X,Y],Z> \text{ and } <[\overline{X},T],\overline{Z}> = 0. \tag{2.1}$$

From the Koszul formula and Riemannian submersion property we get

$$\begin{aligned} <\overline{\nabla}_{\overline{X}}\overline{Y},\overline{Z}> &= \tfrac{1}{2}\{\overline{X}<\overline{Y},\overline{Z}> + \overline{Y}<\overline{X},\overline{Z}> - \overline{Z}<\overline{X},\overline{Y}> \\ &\quad + <[\overline{X},\overline{Y}],\overline{Z}> - <[\overline{X},\overline{Z}],\overline{Y}> - <[\overline{Y},\overline{Z}],\overline{X}>\} \qquad (2.2) \\ &= <\nabla_X Y, Z>. \end{aligned}$$

T is vertical, so

$$\begin{aligned} <\overline{\nabla}_{\overline{X}}\overline{Y},T> &= \tfrac{1}{2}\{\overline{X}<\overline{Y},T> + \overline{Y}<\overline{X},T> - T<\overline{X},\overline{Y}> \\ &\quad + <[\overline{X},\overline{Y}],T> - <[\overline{X},T],\overline{Y}> - <[\overline{Y},T],\overline{X}>\}, \end{aligned}$$

Since $<\overline{X},\overline{Y}> = <X,Y> \circ \pi$, $T<\overline{X},\overline{Y}> = 0$ and equation 2.1 we can get,

$$<\overline{\nabla}_{\overline{X}}\overline{Y},T> = \frac{1}{2}<[\overline{X},\overline{Y}],T>. \tag{2.3}$$

Thus by equations 2.2 and 2.3:

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{\nabla_X Y} + \frac{1}{2}[\overline{X},\overline{Y}]^V, \tag{2.4}$$

and by 2.1 and 2.3:

$$\begin{aligned} <\nabla_T\overline{X},\overline{Y}> &= <\overline{\nabla}_{\overline{X}}T,\overline{Y}> + <[T,\overline{X}],\overline{Y}> \\ &= -<\overline{\nabla}_{\overline{X}}\overline{Y},T> = -\tfrac{1}{2}<[\overline{X},\overline{Y}],T> = -\tfrac{1}{2}<[\overline{X},\overline{Y}]^V,T>. \end{aligned} \tag{2.5}$$

Now by 2.2 it is clear that

$$\overline{X}<\overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}> = X<\nabla_Y Z,W>, \tag{2.6}$$

and therefore

$$\begin{aligned} <\overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}> &= \overline{X}<\overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}> - <\overline{\nabla}_{\overline{Y}}\overline{Z},\overline{\nabla}_{\overline{X}}\overline{W}> \\ &= X<\nabla_Y Z,W> - <\nabla_Y Z,\nabla_X W> - \tfrac{1}{4}<[\overline{Y},\overline{Z}],[\overline{X},\overline{W}]> \\ &= <\nabla_X\nabla_Y Z,W> - \tfrac{1}{4}<[\overline{Y},\overline{Z}]^V,[\overline{X},\overline{W}]^V>. \end{aligned} \tag{2.7}$$

Also by 2.2 and 2.5,

$$\begin{aligned} <\overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z},\overline{W}> &= <\overline{\nabla}_{[\overline{X},\overline{Y}]^H}\overline{Z},\overline{W}> + <\overline{\nabla}_{[\overline{X},\overline{Y}]^V}\overline{Z},\overline{W}> \\ &= <\nabla_{[X,Y]}Z,W> - \tfrac{1}{2}<[\overline{Z},\overline{W}]^V,[\overline{X},\overline{Y}]^V>. \end{aligned} \tag{2.8}$$

Therefore, using 2.7 and 2.8,

$$\begin{aligned} <\overline{R}(\overline{X},\overline{Y})\overline{Z},\overline{W}> &= <\overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}> - <\overline{\nabla}_{\overline{Y}}\overline{\nabla}_{\overline{X}}\overline{Z},\overline{W}> - <\overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z},\overline{W}> \\ &= <R(X,Y)Z,W> + \tfrac{1}{4}<[\overline{X},\overline{Z}]^V,[\overline{Y},\overline{W}]^V> \\ &\quad -\tfrac{1}{4}<[\overline{Y},\overline{Z}]^V,[\overline{X},\overline{W}]^V> + \tfrac{1}{2}<[\overline{Z},\overline{W}]^V,[\overline{X},\overline{Y}]^V>. \end{aligned} \tag{2.9}$$

By setting $\overline{Z} = \overline{Y}$, $\overline{X} = \overline{W}$ we derive the result.

Using the above theorem we now derive the curvature relation in Riemannian reductive homogeneous spaces. Let the map $\pi : G \longrightarrow G/H$ be a Riemannian submersion. Let $\mathfrak{h}$ be the Lie algebra of H and $\mathfrak{m}$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$; i.e, $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \perp \mathfrak{h}$. Then Ad(H) leaves $\mathfrak{h}$, and by Lemma 1.2.20 also $\mathfrak{m}$, invariant. Restricting the inner product $<,>$ to $\mathfrak{m}$ induces a G- invariant Riemannian metric on G/H, which is called normal.

Corollary 2.2.3 *If the metric on G/H is normal, then for $X, Y \in \mathfrak{m} \subset TG$:*

$$K^{G/H}(d\pi(X), d\pi(Y)) = \|[X,Y]_{\mathfrak{h}}\|^2 + \frac{1}{4}\|[X,Y]_{\mathfrak{m}}\|^2.$$

Proof The map $\pi : G \longrightarrow G/H$ is a Riemannian submersion. By Theorem 2.2.2

$$K^{G/H}(d\pi(X), d\pi(Y)) = K^G(X,Y) + \frac{3}{4}\|[X,Y]_{\mathfrak{h}}\|^2, \tag{2.10}$$

where $d\pi : TG \longrightarrow T(G/H)$ and $TG = G \times T_eG = G \times \mathfrak{g} = \mathfrak{g}'$. The metric on G/H is normal, so $H \backsimeq \mathfrak{m}$, $V \backsimeq \mathfrak{h}$ then $\mathfrak{g}' = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \perp \mathfrak{h}$. Now we have to find $K^G(X,Y)$. Restricting the inner product $<,>$ to $\mathfrak{m}$, it is bi-invariant. Therefore, from Theorem 1.2.24 part (3),

$$\begin{aligned} K^G(X,Y) &= \tfrac{1}{4}\|[X,Y]\|^2 \\ &= \tfrac{1}{4}\|[X,Y]_{\mathfrak{h}}\|^2 + \tfrac{1}{4}\|[X,Y]_{\mathfrak{m}}\|^2. \end{aligned}$$

Substituting this into equation 2.10 we can get the answer.

2.3 Symmetric Spaces

Definition 2.3.1 *A Riemannian globally symmetric space is a connected Riemannian manifold M, if for each $p \in M$ there exists an isometry, $I_p : M \longrightarrow M$ such that*

- $I_p(p) = p$
- $(dI_p)_p = -(Id)_p$.

Let G be Lie group whose Lie algebra $\mathfrak{g}$ has an Ad-invariant non-degenerate scalar product $<,>$ and a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that $\mathfrak{h} \perp \mathfrak{m}$, $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. Let $\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$ be the linear map determined by
$\theta(X) = -X, \quad X \in \mathfrak{m}, \qquad \theta(U) = U, \quad U \in \mathfrak{h}$ Then

1. θ is an isometry of $<,>$.
2. $\theta[E,F] = [\theta E, \theta F] \qquad \forall E, F \in \mathfrak{g}$.
3. $\theta^2 = id$.

Conversely, suppose we start with a θ satisfying these conditions. Since $\theta^2 = id$ we can write $\mathfrak{g}$ as the linear direct sum of the $+1$ and -1 eigenspaces of θ, i.e. we define $\mathfrak{h} = \{U|\theta(U) = U\}$ and $\mathfrak{m} = \{X|\theta(X) = -X\}$. Since θ preserves the scalar product, eigenspaces corresponding to different eigenvalues must be orthogonal, and the bracket conditions on $\mathfrak{h}$ and $\mathfrak{m}$ follow automatically from their definition.
Finding such a θ is to find a diffeomorphism $\sigma : G \longrightarrow G$ such that

- G has a bi-invariant metric which is also preserved by σ,

- σ is an automorphism of G, i.e. $\sigma(ab) = \sigma(a)\sigma(b)$,
- $\sigma^2 = id$.

If we have such a σ, then $\theta = d\sigma_e$ satisfies our requirements. Furthermore, the set of fixed points of σ,

$$F = \{a \in G | \sigma(a) = a\}$$

is clearly a subgroup, which we could take as our subgroup, H. In fact, let F_0 denote the connected component of the identity in F , and let H be any subgroup satisfying$F_0 \subset H \subset F$. Then $M = G/H$ satisfies 1,2 and 3, such a space is called a normal symmetric space.

Theorem 2.3.2 [1] *Let G/H be a symmetric space. Then the sectional curvature is given by*

$$K(X,Y) = \|[X,Y]\|^2$$

for all $X, Y \in \mathfrak{m}$.

Example 2.3.3 [1] *The sectional curvature of complex projective space as a homogeneous space.*

Consider the unitary groups

$$U(n) = \{A \in Gl(n, \mathbb{C}); A\overline{A}^T = I\} \quad \text{and}$$

$$SU(n) = \{A \in U(n); det(A) = 1\}.$$

Their Lie algebras are,

$$\mathfrak{u}(n) = \{A \in \mathfrak{gl}(n, \mathbb{C}); \overline{A} + A^T = 0\} \quad \text{and}$$

$$\mathfrak{su}(n) = \{A \in \mathfrak{u}(n); tr(A) = 0\} \quad \text{respectively.}$$

The complex projective space is the homogeneous space $CP(n) = \frac{SU(n+1)}{U(n)}$, where $U(n)$ is a subgroup of SU(n+1) whose element have the form

$$\begin{bmatrix} U_{(n\times n)} & 0 \\ 0 & \omega \end{bmatrix}$$

where $\omega = \det(U)^{-1}$. The rule $<A, B> = -\frac{1}{2}trAB$, defines a bi-invariant metric on SU(n+1) that gives rise to the decomposition $\mathfrak{su}(n+1) = \mathfrak{m} + \mathfrak{u}(n)$, where $\mathfrak{m}$ consists of a matrix of the form:

$$\alpha = \begin{bmatrix} & & & & & \overline{\alpha_1} \\ & & & & & . \\ & & \mathbf{0}_(n \times n) & & & . \\ & & & & & . \\ & & & & & \overline{\alpha_n} \\ \alpha_1 & . & . & . & \alpha_n & 0 \end{bmatrix}$$

$\mathfrak{m}$ may be though of as a complex n-space, and multipling by i gives a real linear transformation $J : \mathfrak{m} \longrightarrow \mathfrak{m}$ such that $J^2 = -1$ and $<x, y> = <J(x), J(y)>$ $[\mathfrak{m}, \mathfrak{m}] \subset$

$\mathfrak{u}(n)$; so CP(n) is a symmetric spaces.
Consider $\|\alpha\| = \|\beta\| = 1$ and $<\alpha, \beta> = 0$; by corollary 2.2.3 the sectional curvature is,

$$K(\alpha, \beta) = \|[\alpha, \beta]\|^2.$$

We can write

$$\alpha\beta = \begin{bmatrix} (\beta_i \overline{\alpha_j})_{(n\times n)} & 0_{n\times 1} \\ 0_{1\times n} & \sum(\alpha_i \overline{\beta_i}) \end{bmatrix}$$

Similarly we find $\beta\alpha$, so that $[\alpha, \beta]$ is given by

$$[\alpha, \beta] = \begin{bmatrix} (\beta_i \overline{\alpha_j} - \alpha_i \overline{\beta_j})_{(n\times n)} & 0_{n\times 1} \\ 0_{1\times n} & \sum(\alpha_i \overline{\beta_i} - \beta_i \overline{\alpha_i}) \end{bmatrix}$$

Now $\|[\alpha, \beta]\|^2 = -\frac{1}{2} trac([\alpha, \beta]^2)$, giving:

$$\begin{aligned} \|[\alpha, \beta]\|^2 &= -\tfrac{1}{2} \sum(\beta_i \overline{\alpha_j} - \alpha_i \overline{\beta_j})(\beta_j \overline{\alpha_i} - \alpha_j \overline{\beta_i}) &&- \tfrac{1}{2} \sum(\alpha_i \overline{\beta_i} - \beta_i \overline{\alpha_i})(\alpha_j \overline{\beta_j} - \beta_j \overline{\alpha_j}) \\ &= \sum \alpha_i \overline{\alpha_i} \sum \beta_j \overline{\beta_j} &&- \tfrac{1}{2} \sum \beta_i \overline{\alpha_i} \sum \overline{\alpha_j} \beta_j \\ &\quad -\tfrac{1}{2} \sum \alpha_i \overline{\beta_i} \sum \overline{\beta_j} \alpha_j &&- \tfrac{1}{2} \sum(\alpha_i \overline{\beta_i} - \beta_i \overline{\alpha_i})(\alpha_j \overline{\beta_j} - \beta_j \overline{\alpha_j}) \\ &= \|\alpha\|^2 \|\beta\|^2 &&+ 3<J(\alpha), \beta>^2 \end{aligned}$$

where $\sum(\alpha_i \overline{\beta_i} - \beta_i \overline{\alpha_i}) = i<J(\alpha), \beta>$ and $\|\alpha\| = \|\beta\| = 1$. Hence

$$\|[\alpha, \beta]\|^2 = 1 + 3<J(\alpha), \beta>^2.$$

If $<J(\alpha), \beta> = 1$, then $K(\alpha, \beta) = 4$, while if $<J(\alpha), \beta> = 0$, $K(\alpha, \beta) = 1$, so the sectional curvature varies between 1 and 4 only.

2.4 Levi-Civita Connexions for Normal Homogeneous Spaces

Let G be a compact Lie group equipped with a bi-invariant Riemannian metric $<, >$. Let $\mathfrak{g}$ be its Lie algebra identified in the standard way with the tangent space at the identity T_eG. The tangent bundle TG can be trivialised by left invariant vector fields. We make this explicit by introducing the left Maurer-Cartan 1-form, defined as:

$$\omega : TG \longrightarrow \mathfrak{g} \quad ; \quad \omega(X) = \xi,$$

where $X \in T_gG$ is given by $X = \frac{d}{dt} g \exp(t\xi)|_{t=0} = dL_g(\xi)$. In other words, $\omega(X) \in T_eG$ is the left translation by g^{-1} of $X \in T_gG$, i.e. $\omega(X) = dL_{g^{-1}}(X)$ for all $X \in T_gG$.

The Maurer-Cartan form is a $\mathfrak{g}$-valued 1-form ω on G which gives a linear isomorphism $\omega_g : T_gG \longrightarrow \mathfrak{g}$ for all $g \in G$. The term left invariant refers to the fact that ω is invariant under left translation, which may be seen as follows: since $X \in T_gG$ implies that $dL_h(X) \in T_{hg}G$, we have

$$L_h^*\omega(X) = \omega(dL_h(X)) = dL_{(hg)^{-1}}(dL_h(X)) = dL_{g^{-1}}(X) = \omega(X).$$

This allows us to give an explicit expression for the left trivialisation, namely

$$TG \cong G \times \mathfrak{g}; \qquad X \mapsto (\pi(X), \omega(X)),$$

where $\pi : TG \longrightarrow G$ is the tangent bundle.
For any smooth vector field $X \in \Gamma(TG)$ we represent of $\omega(X) \in C^\infty(G, \mathfrak{g})$ by

$$\omega(X) : g \mapsto \omega_g(X_g).$$

This gives an isomorphism $\Gamma(TG) \cong C^\infty(G, \mathfrak{g})$. Moreover, for any $f \in C^\infty(G)$, $\omega(fX) = f\omega(X)$ for all $X \in \Gamma(TG)$.

Now let $H \subset G$ be a closed subgroup with Lie algebra $\mathfrak{h}$. Set $N = G/H$ and give it the metric for which $\pi^N : G \longrightarrow G/H$ is a Riemannian submersion.
TG has an orthogonal decomposition into subbundles

$$TG \cong H + V$$

where $V = \ker d\pi^N$ and $H = V^\perp$.

We know that, for $\mathfrak{m} = \mathfrak{h}^\perp \subset \mathfrak{g}$,

$$\begin{array}{ccccc} \omega & : & H & \longrightarrow & G \times \mathfrak{m} \\ \omega & : & V & \longrightarrow & G \times \mathfrak{h} \end{array}$$

are isometric isomorphisms of vector bundles. A vector field $\overline{X} \in \Gamma(TG)$ is horizontal whenever $\overline{X} \in \Gamma(H)$. We say it is a basic vector field if it is the horizontal lift of a vector field $X \in \Gamma(N)$.

Lemma 2.4.1 *A vector field $\overline{X}$ is basic if and only if at each $g \in G$, $\omega_g(\overline{X}_g) \in \mathfrak{m}$ and $\omega_{gh}(\overline{X}_{gh}) = Adh^{-1}\omega(X_g)$ for all $h \in H$.*

Proof Let $\overline{X}$ be basic, so that it is a horizontal vector field $\overline{X} \in \Gamma(H)$. This implies that for every $g \in G$, $\omega_g(\overline{X}_g) \in \mathfrak{m}$. If $\overline{X}$ is a horizontal vector field, so is $dR_h\overline{X}$, $h \in H$.

Now

$$\begin{aligned}\omega_{gh}(\overline{X}_{gh}) &= \omega_{gh}(dR_h\overline{X}_g)\\ &= dL_{(gh)^{-1}}(dR_h\overline{X}_g)\\ &= dL_{h^{-1}}dL_{g^{-1}}dR_h\overline{X}_g\\ &= dL_{h^{-1}}dR_h dL_{g^{-1}}(\overline{X}_g)\\ &= dL_{h^{-1}}dR_h\omega(\overline{X}_g)\\ &= Adh^{-1}\omega(\overline{X}_g) \qquad \forall h,g\in G.\end{aligned}$$

The converse is straightforward.

Remark 2.4.2 *A vector field $\overline{X}$ is left invariant precisely when $\omega(\overline{X})$ is constant, i.e when $\omega(\overline{X}_{gh}) = \omega(\overline{X}_g)$ for all g and h.*

Thus $\overline{X}$ is basic left invariant if and only if

$$Adh^{-1}\omega(\overline{X}_g) = \omega(\overline{X}_g) \quad \forall g, \forall h.$$

This can only happen if there exists an $\xi \in \mathfrak{m}$ that is fixed by Ad_H. So we cannot generally expect basic vector fields to be left invariant.

The Levi-Civita connexion on G can be thought of as a map

$$\begin{array}{rcccc}\nabla^G & : & \Gamma(TG) & \longrightarrow & \Omega^1(TG)\\ & & Y & \mapsto & \nabla Y\end{array}$$

i.e, $\nabla_X Y : TG \longrightarrow TG \quad ; \quad X \mapsto \nabla_X Y$, is a linear map.
Using $TG \cong G\times\mathfrak{g}$ this converts into

$$\begin{array}{rcccc}\nabla^G & : & C^\infty(G,\mathfrak{g}) & \longrightarrow & \Omega^1(\mathfrak{g})\\ & & \xi & \mapsto & d\xi + \frac{1}{2}[\omega,\xi]\end{array}.$$

i.e, $\nabla^G_X\xi = X\xi + \frac{1}{2}[\omega(X),\xi]$.
Here the tangent vector X acts on $C^\infty(G,\mathfrak{g})$ as a derivation via

$$X\xi = (X\xi^j)e_j$$

where $\{e_j\}$ is any basis for $\mathfrak{g}$, so each $\xi^j \in C^\infty(G)$.

More generally we can say, to prove this formula

$$\begin{aligned}\nabla^G &= d + \tfrac{1}{2}[\omega,\cdot]\\ &= d + \tfrac{1}{2}ad\omega\end{aligned}$$

we only need to show two things,

1. $\nabla^G_X(f\xi) = Xf.\xi + f\nabla^G_X\xi \quad \forall f \in C^\infty(G)$ i.e, this is a connexion on $\mathfrak{m}$, and,

2. $\nabla^G_X\xi = \frac{1}{2}[\omega(X), \xi]$ when ξ is constant. This proves ∇^G agrees with the formula for the Levi-Civita connexion on left invariant vector fields.

The proof of (1) is ; define

$$\begin{aligned} d_X : C^\infty(G, \mathfrak{g}) &\longrightarrow C^\infty(G, \mathfrak{g}) \\ f\xi &\mapsto X(f\xi) \quad \text{where } f \in C^\infty(G). \end{aligned}$$

Thus,

$$\begin{aligned} d_X(f\xi) &= X(f\xi) \\ &= Xf.\xi + f.X\xi \\ d_X(f\xi) + \tfrac{1}{2}[\omega(X), f\xi] &= Xf.\xi + f.X\xi + \tfrac{1}{2}[\omega(X), f\xi] \\ \nabla^G_x(f\xi) &= Xf.\xi + f(X\xi + \tfrac{1}{2}[\omega(X), \xi]) \\ &= Xf.\xi + f\nabla^G_X\xi. \end{aligned}$$

The proof of (2) is ; define

$$\begin{aligned} \nabla_X : \Gamma(TG) &\longrightarrow \Gamma(TG) \\ \xi &\mapsto \nabla_X\omega(Y). \end{aligned}$$

Thus,

$$\begin{aligned} \nabla_X\xi &= \nabla_X\omega(Y) \\ &= \omega(\nabla_X Y) \\ &= \omega(\tfrac{1}{2}[X, Y]) \quad \text{since, } X, Y \text{are left invariant vector fields} \\ &= \tfrac{1}{2}[\omega(X), \omega(Y)] \\ &= \tfrac{1}{2}[\omega(X), \xi]. \end{aligned}$$

To calculate the Levi-Civita connexion for N we need following O'Neill lemma.

Lemma 2.4.3 [*3*] *Let $\pi^N : G \longrightarrow N$ be a Riemannian submersion with horizontal subbundle $H \subset TG$ and horizontal projection*

$$\mathcal{H} : TG \longrightarrow H.$$

For $X, Y \in \Gamma(TN)$ with horizontal lift $\overline{X}, \overline{Y}, \mathcal{H}(\nabla^G_{\overline{X}}\overline{Y})$ is the horizontal lift of $\nabla^N_X Y$. Hence $\nabla^N_X Y = d\pi(\nabla^G_{\overline{X}}\overline{Y})$.

Proof Let X, Y, Z are vector fields on N and $\overline{X}, \overline{Y}, \overline{Z}$ are corresponding basic vector fileds on $H \subset G$. By definition,

$$< \overline{X}, \overline{Y} >=< X, Y > \circ\pi.$$

$\mathcal{H}[\overline{X}, \overline{Y}]$ is the basic vector field corresponding to $[X, Y]$. Now

$$\begin{aligned} \overline{X} < \overline{Y}, \overline{Z} > &= \overline{X}\{< Y, Z > \circ\pi\} \\ &= (d\pi\overline{X}) < Y, Z > \\ &= X < Y, Z > \circ\pi, \end{aligned}$$

and

$$\begin{aligned} < \overline{X}, [\overline{Y}, \overline{Z}] > &= < \overline{X}, \mathcal{H}[\overline{X}, \overline{Y}] > \\ &= < X, [Y, Z] > \circ\pi. \end{aligned}$$

From the Koszul formula,

$$< \nabla^G_{\overline{X}}\overline{Y}, \overline{Z} >=< \nabla^N_X Y, Z > \circ\pi.$$

Therefore $d\pi(\nabla^G_{\overline{X}}\overline{Y}) = \nabla^N_X Y$.

Corollary 2.4.4 *For any normal homgeneous space N of G, gives $X, Y \in \Gamma(TN)$ with horizontal lift $\overline{X}, \overline{Y}$ and $\omega(\overline{X}) = \xi, \omega(\overline{Y}) = \eta$ then $\nabla^N_X Y$ can be calculated from*

$$\omega(\mathcal{H}(\nabla^G_{\overline{X}}\overline{Y})) = \overline{X}\eta + \frac{1}{2}[\xi, \eta]_{\mathfrak{m}}.$$

In particular, if $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ (for example, when N is a symmetric space)

$$\omega(\mathcal{H}(\nabla^G_{\overline{X}}\overline{Y})) = \overline{X}\eta.$$

Proof The objects ξ and η are defined by $\omega(\overline{X}) = \xi$, $\omega(\overline{Y}) = \eta$ where $\overline{X}, \overline{Y} \in \Gamma(TG)$. Horizontal projection

$$\mathcal{H} : TG \longrightarrow H \qquad \tilde{\mathcal{H}} : G \times \mathfrak{g} \longrightarrow G \times \mathfrak{m}$$

we can write $\omega(\mathcal{H}(X)) = \tilde{\mathcal{H}}(\omega(X))$.
Now

$$\begin{aligned} \omega(\mathcal{H}(\nabla^{TG}_{\overline{X}}\overline{Y})) &= \tilde{\mathcal{H}}(\omega(\nabla^{TG}_{\overline{X}}\overline{Y})) \\ &= \tilde{\mathcal{H}}(\overline{X}\eta + \tfrac{1}{2}[\xi, \eta]) \\ &= (\overline{X}\eta)_{\mathfrak{m}} + \tfrac{1}{2}[\xi, \eta]_{\mathfrak{m}} \\ &= \overline{X}\eta + \tfrac{1}{2}[\xi, \eta]_{\mathfrak{m}}. \end{aligned}$$

If $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ then $[\xi, \eta]_{\mathfrak{m}} = 0$ this implies $\omega(\mathcal{H}(\nabla^{TG}_{\overline{X}}\overline{Y})) = \overline{X}\eta$.

2.5 The tangent bundle

[14] A direct isomorphism with a subbundle of the trivial bundle $G/H \times \mathfrak{g}$: explain as follows.
The subgroup H has a right action on $G \times \mathfrak{m}$ by

$$(g, \xi) \mapsto (gh, \mathrm{Ad}h^{-1}\cdot)\xi, h \in H.$$

The quotient, denoted by $G \times_H \mathfrak{m}$, has a fibre linear isomorphism

$$\hat{\beta} : T(G/H) \longrightarrow G \times_H \mathfrak{m}, \tag{2.11}$$

whose inverse is the map

$$G \times_H \mathfrak{m} \longrightarrow T(G/H); \quad [g, \xi]_H \mapsto \frac{d}{dt} g e^{t\xi} H|_{t=0}.$$

Here $[g, \xi]_H$ denotes the equivalence class corresponding to the H-orbit of the point $(g, \xi) \in G \times \mathfrak{m}$. Hence $[g, \xi]_H = [g', \xi']_H$ precisely when there exists $h \in H$ for which $g' = gh$ and $\xi' = \mathrm{Ad}h^{-1} \cdot \xi$. This identifies vector fields over G/H with H-equivariant maps

$$\xi : G \longrightarrow \mathfrak{m}; \quad \xi(gh) = \mathrm{Ad}h^{-1} \cdot \xi(g), \forall h \in H. \tag{2.12}$$

The bundle $G \times_H \mathfrak{m}$ can be embedded as a subbundle into $G/H \times \mathfrak{g}$ by

$$G \times_H \mathfrak{m} \longrightarrow G/H \times \mathfrak{g}; \quad [g, \xi] \mapsto (gH, \mathrm{Ad}g \cdot \xi), \tag{2.13}$$

we denote the image subbundle by $[\mathfrak{m}]$. We will let $\beta : T(G/H) \longrightarrow \mathfrak{g}$ denote the corresponding Lie algebra valued 1-form, i.e,

$$\beta(X_{\pi(g)}) = \mathrm{Ad}g \cdot \xi \Longleftrightarrow \hat{\beta}(X_{\pi(g)}) = [g, \xi]_H,$$

where $\pi : G \longrightarrow G/H$ is the quotient map. Notice that when H is the trivial subgroup, β is the left Maurer-Cartan 1-form $\omega : TG \longrightarrow \mathfrak{g}$.

Theorem 2.5.1 *Let $X, Y \in \Gamma(TN)$ and set $\xi = \beta(X)$, $\eta = \beta(Y)$. Then*

$$\beta(\nabla^N_X Y) = P_{\mathfrak{m}}(X\eta - \frac{1}{2}[\xi, \eta]), \tag{2.14}$$

where $P_{\mathfrak{m}} : N \times \mathfrak{g} \longrightarrow [\mathfrak{m}]$ is the orthogonal projection and we think of X as derivation on Lie algebra valued function.

Let $X, Y \in \Gamma(TM)$ and let ∇^f denote the covariant derivative along f. Recall that $\nabla^f_X f_* Y$ maps M to $\Gamma(TN)$. The map f pulls back the $\mathfrak{g}$- valued 1-form β on N to one on M. Let $\xi = f^*\beta(X) = \beta(f_*X)$ and $\eta = f^*\beta(Y)$,then $\xi, \eta : M \longrightarrow \mathfrak{g}$. Applying this to above equations

$$\beta(\nabla^f_X f_* Y) = P_{\mathfrak{m}}(X\eta - \frac{1}{2}[\xi, \eta]). \tag{2.15}$$

It is also useful to know how to calculate this using a local frame $F : U \longrightarrow G$ of $f : U \longrightarrow N$, where $U \subset M$ is some open set on which a frame exists, i.e, $f = \pi \circ F$. Let $\alpha : TM \longrightarrow \mathfrak{g}$ denote the pull-back $F^*\omega$. It is easy to see that

$$f^*\beta(X) = \mathrm{Ad}F \cdot \alpha(X)_{\mathfrak{m}}. \tag{2.16}$$

Substituting this into 2.14 gives

$$\beta(\nabla^f_X f_* Y) = \mathrm{Ad}F(X\alpha(Y)_{\mathfrak{m}} + [\alpha(X)_{\mathfrak{h}}, \alpha(Y)_{\mathfrak{m}}] + \frac{1}{2}[\alpha(X)_{\mathfrak{m}}, \alpha(Y)_{\mathfrak{m}}]_{\mathfrak{m}}). \tag{2.17}$$

This simplyfies when N is a symmetric space for then the final term vanishes.

Remark 2.5.2 *When G is a matrix Lie group we note that $\alpha = F^{-1}dF$.*

Chapter 3

Harmonic Maps and the Ruh - Vilms Theorem

3.1 Harmonic Maps

Let $(M,g),(N,h)$ be smooth Riemannian manifold with $\dim M = m$ and $\dim N = n$. Let $\phi : M \longrightarrow N$ be smooth map. The energy integral is

$$E(\phi) = \frac{1}{2}\int_M \operatorname{Tr} h(\phi_\star, \phi_\star) d\sigma,$$

where $\phi_\star$ is the differential of ϕ, Tr is the trace with respect to the metric g, and $d\sigma$ is the volume element of the metric g.

This Riemannian map is said to be harmonic if it is a critical point of the energy integral. i.e,

$$\frac{d}{dt}|_{t=0} E(\phi_t) = 0.$$

For more details see [4]

Definition 3.1.1 *The second fundamental form is a quadratic form on the tangent space with values in the normal bundle and it can be defined by* $\mathrm{II}(X,Y) = (\nabla_X Y)^\perp$, *where* $(\nabla_X Y)^\perp$ *denotes the orthogonal projection of covariant derivative* $\nabla_X Y$ *onto the normal bundle.*

For an isometric immersion of Riemannian manifolds $\phi : M \longrightarrow N$ the second fundamental form is the symmetric bilinear form

$$\begin{aligned} \mathrm{II} : \Gamma(TM) \times \Gamma(TM) &\longrightarrow \Gamma(TM^\perp) \\ \mathrm{II}(X,Y) &= \nabla^N_{d\phi(X)}(d\phi(Y)) - d\phi(\nabla^M_X Y). \end{aligned}$$

Here X and Y are vector fields on M and $d\phi(X)$ denotes the vector field along ϕ which assigns to each $p \in M$ the vector $d\phi_p(X_p) \in T_{\phi(p)}N$.

Definition 3.1.2 *A map* $\phi : M \longrightarrow N$ *is harmonic if and only if* $\tau(\phi) = 0$. *Where* $\tau(\phi) = tr_g \mathrm{II}$ *is called tension field of* ϕ.

Remark 3.1.3 *The tension field $\tau(\phi)$ of the map ϕ (relative to a given g and h) is the trace of the second fundamental form so*

$$\tau(\phi) = trace_g \mathrm{II}, \qquad i.e, \tau(\phi) = \sum_{j=1}^{m} \mathrm{II}(e_j, e_j)$$

where $e_1, ..., e_m$ is an orthonormal frame for T_pM.

Definition 3.1.4 *The mean curvature of the immersion is average of the tension field $\tau(\phi)$. i.e, $H = \frac{1}{m}\tau(\phi)$.*

The immersion ϕ has constant mean curvature when H is parallel with respect to the induced connexion on the normal bundle $TM^{\perp}$. The immersion is minimal if and only if its mean curvature zero, i.e, $H = 0$.

Let M be a submanifold of N with induced metric and d_M and d_N denote the distance functions in M and N. Then $d_N(p,q) \leq d_M(p,q)$ for any $p, q \in M$. M is said to be totally geodesic in N if the exponential map of N, restricted to $TM \subseteq TN$, maps TM into M, or equivalently if the second fundamental form of M in N vanishes identically on M. If M is totally geodesic in N and both M and N are complete, then $d_M(p,q) = d_N(p,q)$ for all $p, q \in M$.

Theorem 3.1.5 [4] *If $(M,g), (N,h)$ and (P,k) are three smooth manifolds and maps $\phi : M \longrightarrow N$, $\psi : N \longrightarrow P$ then*

- $\nabla d(\psi \circ \phi) = d\psi \circ \nabla d\phi + \nabla d\psi(d\phi, d\phi)$,
- $\tau(\psi \circ \phi) = d\psi \circ \tau(\phi) + trace \nabla d\psi(d\phi, d\phi)$.

In particular, if ϕ and ψ are totally geodesic, so is $\psi \circ \phi$, and if ϕ is harmonic and ψ totally geodesic, then $\psi \circ \phi$ is harmonic. Note that the composition of two harmonic maps is not harmonic in general.

Proof Let $X, Y \in \Gamma(TM)$,

$$\begin{aligned}
\nabla d(\psi \circ \phi)(X,Y) &= \nabla_X(d\psi \circ d\phi(Y)) - d(\psi \circ \phi)(\nabla_X Y) \\
&= (\nabla_{d\phi \cdot X} d\psi)(d\phi(Y)) + d\psi(\nabla_X(d\phi \cdot Y)) - d\psi \circ d\phi(\nabla_X Y) \\
&= \nabla d\psi(d\phi(X), d\phi(Y)) + d\psi(\nabla d\phi(X,Y)).
\end{aligned}$$

Taking traces both sides yields the second formula.

Lemma 3.1.6 [4] *Let Figure 3.1.1 be a commutative diagram, where ρ and π are Riemannian submersions, with $\psi_* T^H M \subset T^H N$, where $T^H_x M$ is the orthogonal complement of* $\ker d\rho(x)$. *Assume that one of the following conditions is satisfied:*

1. $\psi_*(TM) \subset T^H N$,
2. *π has totally geodesic fibres,*
3. *for all $z \in P$, $\rho^{-1}(z) \longrightarrow \pi^{-1}(\phi(z))$ is a Riemannian fibration with minimal fibres.*

Then $\tau(\pi \circ \psi) = \pi_* \tau(\psi)$, so that $\pi \circ \psi$ is harmonic if and only if $\tau(\psi)$ is vertical.

$$\begin{array}{ccc} M & \xrightarrow{\psi} & N \\ \downarrow \rho & & \downarrow \pi \\ P & \xrightarrow{\phi} & Q \end{array}$$

Figure 3.1.1: Commutative diagram for Lemma 3.1.6.

3.2 The Gauss Map

First we define the Grassmann manifold. The Grassmann manifold $Gr(p, n)$ is the set of p-dimensional planes that pass through the origin in $\mathbb{E}^n$. If we take the oriented p-dimensional planes, then we get the oriented Grassmann manifold $Gr^+(p, n)$.

Another way of generalizing Grassmann manifold: the sets of totally geodesic p-dimensional submanifolds in the spaces of constant curvature S^n and H^n are the spherical Grassmann manifold $SGr(p, n)$ and the hyperbolic Grassmann manifold $HGr(p, n)$.

We consider the manifold $Gr(p, n)$. The group $O(n)$ acts transitively on it. $O(p) \times O(n-p)$ is the isotropy group of the point $W \in Gr(p, n)$, where $O(p)$ acts in the p-dimensional plane W and $O(n-p)$ acts in its orthogonal complement. Thus $Gr(p, n)$ is the homogeneous space $O(n)/O(p) \times O(n-p)$.
We introduce a coordinate system in $\mathbb{E}^n$ and choose the plane $W_o : x^{p+1} = ... = x^n = 0$.

Then the isotropy group is a matrix group of the form

$$\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$$

$U \in O(p), V \in O(n-p)$. We now define an involutory automorphism $\sigma : O(n) \longrightarrow O(n)$ thus,

$$\sigma(A) = SAS^{-1}, \text{where } S = \begin{pmatrix} -I_p & 0 \\ 0 & I_{n-p} \end{pmatrix}.$$

This makes $Gr(p, n)$ into a symmetric space. The canonical decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is $\mathfrak{g} = \mathfrak{o}(n)$ the space of skew-symmetric $n \times n$-matrices,

$$\mathfrak{h} = \mathfrak{o}(p) + \mathfrak{o}(n-p) = \{\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}, U \in \mathfrak{o}(p), V \in \mathfrak{o}(n-p)\}$$

$$\mathfrak{m} = \{\begin{pmatrix} 0 & -X^T \\ X & 0 \end{pmatrix}, X \text{ is an } (p \times (n-p)) - \text{matrix}\}$$

$\mathfrak{m}$ is the $p(n-p)$-dimensional tangent space to $Gr(p, n)$.

In generally we define the Gauss Map as follows. Let M be an immersed oriented hypersurface in Euclidean space . At each $x \in M$ there is a unique (positive) unit normal vector ν, and hence a well defined Gauss map

$$\nu : M \longrightarrow S^n$$

assigning to each point $x \in M$ its unit normal vector, $\nu(x)$. Here S^n denotes the unit sphere, the set of all unit vectors in $\mathbb{R}^{n+1}$. The normal vector, $\nu(x)$ is orthogonal to the tangent space to M at x.

Let M be a regular p-dimensional surface in Euclidean space. At every point of the surface M we construt the p-dimensional tangent plane and carry all these planes by parallel transport to the origin $0 \in \mathbb{E}^n$. We get a certain subset in $Gr(p,n)$ called the Grassmann image (Gauss map) of the surface M; the map $\gamma : M \longrightarrow Gr(p,n)$ is the Grassmann map.

The Gauss map on Grassmannian manifold generalised as follows. Let $\phi : M \longrightarrow \mathbb{R}^n$ be an immersion. The Gauss map $\gamma : M \longrightarrow Gr(p,n)$ associated to this immersion. A point $x \in M$ the image $\gamma(x)$ is p-dimensional plane tangent to $\phi(M)$ at $\phi(x)$.

Theorem 3.2.1 [6] **Ruh-Vilms** *Let (M, g) be a Riemannian manifold, let $\phi : M \longrightarrow \mathbb{R}^n$ be an isometric immersion in Euclidean space, and let Gauss map (Grassmann map) $\gamma : M \longrightarrow Gr(p,n)$. Then $\tau(\gamma) = \nabla^{\perp} H$ is the derivative of the mean curvature vector under the normal connexion.*

3.3 Curves in S^2

In this section we are going to prove the Ruh-Vilms theorem using the homogeneous geometry for the simple case.

The unit sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$, $S^2 = \{U \in \mathbb{R}^3 | \ \|U\| = 1\}$. The orbit of the point e_3 in $\mathbb{R}^3$ is the whole unit sphere $S^2 \subset \mathbb{R}^3$; so

$$S^2 = \{ge_3 \ \ | \ \ g \in SO(3)\}.$$

The unit sphere bundle $US^2 \subset TS^2 \subset T\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$,

$$US^2 = \{(U,V) \in \mathbb{R}^3 \times \mathbb{R}^3 \ \ | \ \ U.U = V.V = 1, U.V = 0\},$$

where V is a subspace of $\mathbb{R}^3$. Given a unit vector $U \in S^2$, there exist an oriented orthonormal basis $u_1, u_2, u_3 \in \mathbb{R}^3$ these form the columns of an orthogonal matrix $g \in SO(3)$ for which $ge_3 = U$ thus

$$US^2 = \{(ge_3, ge_2) \ \ | \ \ g \in SO(3)\}$$

by taking columns u_1, u_2, u_3 with $u_3 = U$, $u_2 = V$.
Let $g \in SO(3)$, $ge_3 = e_3$ if and only if e_3 is the last column of g in the mean time other columns of g must be an orthonormal set of vectors. So isotropy group of $SO(3)$ at e_3 is $SO(2)$. Let

$$K = \{g \in SO(3) | ge_3 = e_3\}$$

then

$$K \cong SO(2) \ \ \text{since}, K = \{g \in SO(3); g = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}\}$$

Let
$$S = \{g \in SO(3) | ge_2 \wedge ge_3 = e_2 \wedge e_3 \subset K\}$$
and $SO(1)$ is the trivial group so $S = \{I\}$ where I is the 3 by 3 identity matrix.
$$\text{Since,} \quad S = \{g \in SO(3); g = \begin{pmatrix} * & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}$$
also we can identify
$$S = \{g \in SO(3) | ge_1 = e_1\}.$$
The space of geodesics is $Gr(2,3)$, this is the set of subspaces spanned by two orthonormal vectors in $\mathbb{R}^3$
$$Gr(2,3) = \{ge_2 \wedge ge_3 | g \in SO(3)\}.$$
We can simply illustrate all the above result in the following way – see Figure 3.3.2.

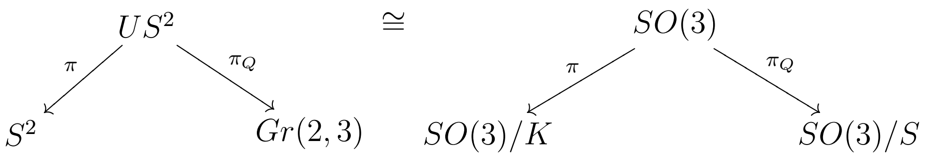

Figure 3.3.2: Isomorphism map.

Let $f(t)$ be a parametrise immersed curve on S^2 and its unit vector $\dot{f}(t) = \frac{df}{dt}$. Then the oriented unit normal to $f(t)$ in TS^2 is $\nu = f(t) \times \dot{f}(t)$. We can interpret this in the matrix form
$$F : \mathbb{R} \longrightarrow SO(3) \quad F = (\dot{f}(t), \nu(t), f(t)).$$
This is equivalent to the left
$$\mu : \mathbb{R} \longrightarrow US^2 \text{defined by} \mu(t) = (f(t), \nu(t)).$$

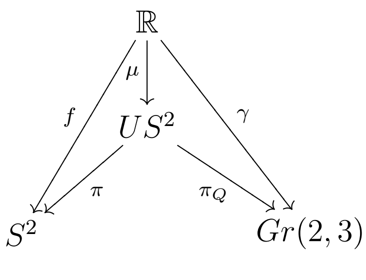

Figure 3.3.3: Commutative triangles

From Figure 3.3.3 it follows that $\pi \circ \mu = f$ and $\gamma = \pi_Q \circ \mu$ which is geodesic Gauss map.
Now we are going to prove the following two results.

1. Tension field of μ: $\tau(\mu)$ is π_Q- horizontal for the bi-invariant metric on $SO(3)$ and $\pi_{Q^*} \circ \tau(\mu) = \tau(\gamma)$. Hence γ is harmonic if and only if μ is harmonic.

2. $\nabla^{\perp}\tau(f) = 0$ if and only if $\tau(\mu) = 0$

By definition

$$\begin{aligned}\tau(\mu) &= \mathrm{II}_{\mu}(\tfrac{d}{dt}, \tfrac{d}{dt}) \quad \text{since} \quad \dim \mathbb{R} = 1 \\ &= \nabla^{\mu}_{\frac{d}{dt}} \mu_* \tfrac{d}{dt} - \mu_*(\nabla^{\mathbb{R}}_{\frac{d}{dt}} \tfrac{d}{dt}) \\ &= \nabla^{\mu}_{\frac{d}{dt}} \mu_* \tfrac{d}{dt}.\end{aligned}$$

Now we need to find ∇^{μ}; by using left Maurer-Cartan 1-form ω,

$$\mu^{-1}TG \cong \mathbb{R} \times \mathfrak{g} \ ; \ \text{where } G = SO(3) \ , \ \mathfrak{g} = \mathfrak{so}(3).$$

We can obtained the matric

$$<A, B> = -\frac{1}{2} trac(AB) \ \text{ for } \ A, B \in \mathfrak{so}(3).$$

Now

$$\begin{aligned}\omega(\nabla^{\mu}_{\frac{d}{dt}} \mu_* \tfrac{d}{dt}) &= \tfrac{d}{dt}(F^{-1}\dot{F}) + \tfrac{1}{2}[F^{-1}\dot{F}, F^{-1}\dot{F}] \\ &= \tfrac{d}{dt}(F^{-1}\dot{F}).\end{aligned}$$

We know that $F.F^t = 1$ implies $F^t = F^{-1}$, so

$$F^{-1}\dot{F} = \begin{pmatrix} \dot{f}.\ddot{f} & \dot{f}.\dot{\nu} & \dot{f}.\dot{f} \\ \nu.\ddot{f} & \nu.\dot{\nu} & \nu.\dot{f} \\ f.\ddot{f} & f.\dot{\nu} & f.\dot{f} \end{pmatrix}$$

But $|f| = |\dot{f}| = |\nu| = 1$ and also we can get, $\dot{f}.f = \dot{\nu}.\nu = \ddot{f}.\dot{f} = 0$, $\dot{f}.\dot{f} = 1$ and $\nu.\dot{f} = 0$. Since $F^{-1}\dot{F}$ must be skew-symmetric, this means

$$\begin{aligned}\omega(\mathrm{II}_{\mu}(\tfrac{d}{dt}, \tfrac{d}{dt})) &= \tfrac{d}{dt} \begin{pmatrix} 0 & -\nu.\ddot{f} & 1 \\ \nu.\ddot{f} & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\tfrac{d}{dt}(\nu.\ddot{f}) & 0 \\ \tfrac{d}{dt}(\nu.\ddot{f}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \omega(\tau(\mu)).\end{aligned}$$

The Lie algebra of S is $\mathfrak{s}$. Since $\mathfrak{s}$ is the zero Lie algebra, $\mathfrak{s}^{\perp} = \mathfrak{so}(3)$. Then

$$\mathfrak{s}^{\perp} = \{ \begin{pmatrix} 0 & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3)\}.$$

As we seen the above calculation show that $\omega(\tau(\mu))$ takes valuses in $\mathfrak{s}^{\perp}$ so $\tau(\mu)$ is horizontal for $\pi_Q : SO(3) \longrightarrow SO(3)/S$. Since π_Q is a Riemannian submersion and has geodesic fibres by lemma 3.1.6 implies that $\pi_{Q^*} \circ \tau(\mu) = \tau(\gamma)$. So if γ is harmonic if and only if μ is harmonic and $\frac{d}{dt}(\nu.\ddot{f}) = 0$.

Now we calculate $\nabla^{\perp}_{\frac{d}{dt}}\tau(f)$. First we calculate $\tau(f)$

$$\begin{aligned}
\tau(f) &= \mathrm{II}_f(\tfrac{d}{dt}, \tfrac{d}{dt}) \\
&= (\nabla^f_{\frac{d}{dt}} f_* \tfrac{d}{dt})^{\perp} \\
&= (\tfrac{d}{dt}(\dot{f}).\nu)\nu \\
&= (\ddot{f}.\nu)\nu.
\end{aligned}$$

Take the derivative both side under the normal connexion,

$$\begin{aligned}
\nabla^{\perp}_{\frac{d}{dt}}\tau(f) &= (\nabla_{\frac{d}{dt}}((\ddot{f}.\nu)\nu))^{\perp} \\
&= ((\tfrac{d}{dt}(\ddot{f}.\nu))\nu + (\ddot{f}.\nu)\dot{\nu})^{\perp}.
\end{aligned}$$

But $\dot{\nu}.\nu = 0$ which implies

$$\nabla^{\perp}_{\frac{d}{dt}}\tau(f) = ((\frac{d}{dt}(\ddot{f}.\nu))\nu)^{\perp}.$$

So if γ is harmonic if and only if $\nabla^{\perp}_{\frac{d}{dt}}\tau(f) = 0$. Hence f is constant mean curvature.

Chapter 4

Proof of the Ruh-Vilms Theorems

4.1 Hypersurfaces in S^n

In this chapter we prove the Ruh-Vilms theorem 3.2.1 for a Hypersurface in the n-Sphere.

Let the map $f : M \hookrightarrow S^n$ be an isometrically embedded hypersurface, oriented by the unit normal field $\nu \in \Gamma(TM^\perp)$. Let the map $\gamma : M \longrightarrow Gr(2, n+1)$ be its geodesic Gauss map

$$\gamma(p) = \mathrm{Span}\{\nu(p), f(p)\} \subset \mathbb{R}^{n+1}.$$

We can identify $S^n \cong G/K$ and $Gr(2, n+1) \cong G/H$, where

$$\begin{aligned} G &= SO(n+1) \\ K &= \{g \in G : ge_{n+1} = e_{n+1}\} \cong SO(n) \\ H &= \{g \in G : \mathrm{Span}\{ge_n, ge_{n+1}\} = \mathrm{Span}\{e_n, e_{n+1}\}\} \cong SO(n-1) \times SO(2) \end{aligned} .$$

Let $\mathfrak{g}, \mathfrak{k}$ and $\mathfrak{h}$ be the Lie algebra of G, K and H respectively, and define $\mathfrak{m} = \mathfrak{k}^\perp$ and $\mathfrak{p} = \mathfrak{h}^\perp$ where $\mathfrak{m}$ and $\mathfrak{p}$ are identified with the tangent space $T(G/K)$ and $T(G/H)$ respectively. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{m} = \mathfrak{h} + \mathfrak{p}$ and these are symmetric space decompositions. i.e $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$. With the usual symmetric space identifications

$$\begin{aligned} \beta_{\mathfrak{m}} &: TG/K \longrightarrow [\mathfrak{m}] \subset G/K \times \mathfrak{g} \\ \beta_{\mathfrak{p}} &: TG/H \longrightarrow [\mathfrak{p}] \subset G/H \times \mathfrak{g}. \end{aligned}$$

we have

$$\beta_{\mathfrak{m}}(df) : TM \longrightarrow [\mathfrak{m}]$$

and

$$\xi \mapsto AdF.\alpha_{\mathfrak{m}}(\xi),$$

where $F : M \longrightarrow G$ is a local frame, and $\alpha = F^{-1}dF$ is the pull back of the left Maurer-Cartan 1-form on G. Explicitly, we choose a frame suitable to f and γ as follows: first we locally fix an orthonormal frame $E_1, ..., E_{n-1}$ for TM. We then define the map $f : M \longrightarrow \mathbb{R}^{n+1}$ by

$$F = (f_1, ..., f_{n-1}, \nu, f)$$

for $f_j = E_j f = f_* E_j$, and for convenience set $f_n = \nu$.

We can write $\alpha = F^{-1}dF$ in two block decomposed forms. In the first we take $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$

$$\alpha_{\mathfrak{k}} = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}$$

where $A = (f_i \cdot df_j)$ $1 \leqslant i, j \leqslant n$. i.e. $n \times n$ matrix of 1-forms. Here

$$\alpha_{\mathfrak{m}} = \begin{pmatrix} \mathbf{0}_n & & v \\ & & 0 \\ -v^t & 0 & 0 \end{pmatrix}$$

where $v = (f_i.df)$ $1 \leqslant i \leqslant n-1$ and we have used $f_n \cdot df = \nu \cdot df = 0$. In the second we take $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$

$$\alpha_{\mathfrak{h}} = \begin{pmatrix} B & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{0} & 0 & 0 \end{pmatrix}$$

where $B = (f_i \cdot df_j)$ $1 \leqslant i, j \leqslant n-1$. i.e. $(n-1) \times (n-1)$ matrix of 1-forms. Here

$$\alpha_{\mathfrak{p}} = \begin{pmatrix} \mathbf{0}_{n-1} & u & v \\ -u^t & 0 & 0 \\ -v^t & 0 & 0 \end{pmatrix}$$

where $u = (f_i \cdot d\nu)$ $1 \leqslant i \leqslant n-1$. Hence

$$A = \begin{pmatrix} B & u \\ -u^t & 0 \end{pmatrix}$$

and we have the natural isomorphisms

$$\mathbb{R}^n \longrightarrow \mathfrak{m} \;\; ; \qquad w \mapsto \begin{pmatrix} \mathbf{0}_n & w \\ -w^t & 0 \end{pmatrix}$$

$$\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1} \longrightarrow \mathfrak{p} \;\; ; \;\; (w_1 w_2) \mapsto \begin{pmatrix} \mathbf{0}_{n-1} & w_1 & w_2 \\ -w_1^t & 0 & 0 \\ -w_2^t & 0 & 0 \end{pmatrix}$$

which we will often make use of it.

Our aim is to prove the Ruh-Vilms theorem for these hypersurfaces, i.e. to show that $\tau(\gamma) = 0$ iff $\nabla^\perp \tau(f) = 0$, where $\tau(f)$ is the tension field:

$$\tau(f) = \mathrm{tr}\mathrm{II}_f \in \Gamma(TM^\perp), \;\; \mathrm{II}_f(X, Y) = \nabla^f_X f_* Y - f_*(\nabla^M_X Y),$$

and similarly for $\tau(\gamma)$.

Lemma 4.1.1 $\tau(f) = [\sum_{j=1}^{n-1}(\nu \cdot E_j E_j f)]\nu$

Proof Equation (2.17) gives

$$\beta_{\mathfrak{m}}(\nabla^f_X f_* Y) = \mathrm{Ad}F(X\alpha(Y)_{\mathfrak{m}} + [\alpha(X)_{\mathfrak{k}}, \alpha(Y)_{\mathfrak{m}}].$$

Now $X\alpha_{\mathfrak{m}}(Y)$ identifies with

$$\begin{aligned} X\begin{pmatrix} v(Y) \\ 0 \end{pmatrix} &= X\begin{pmatrix} f_i \cdot Yf \\ 0 \end{pmatrix} \; 1 \le i \le n-1 \\ &= \begin{pmatrix} Xf_i \cdot Yf + f_i \cdot XYf \\ 0 \end{pmatrix}. \end{aligned}$$

Similarly $[\alpha(X)_{\mathfrak{k}}, \alpha(Y)_{\mathfrak{m}}]$ identifies with

$$\begin{aligned} A(X)v(Y) &= (f_i \cdot Xf_j)(f_j \cdot Yf) \; 1 \le i,j \le n, f_n \cdot df = 0 \\ &= (\textstyle\sum_j (f_i \cdot Xf_j)(f_j \cdot Yf)). \end{aligned}$$

Now $f_i \cdot f_j = \delta_{ij} \Longrightarrow f_i \cdot Xf_j = -Xf_i \cdot f_j$, so

$$\begin{aligned} \textstyle\sum_j (f_i \cdot Xf_j)(f_j \cdot Yf) &= \textstyle\sum_j (-Xf_i \cdot f_j)(f_j \cdot Yf) \\ &= -Xf_i \cdot Yf \;\; 1 \le i \le n \end{aligned}$$

Note $-Xf_n \cdot Yf = f_n \cdot XYf$ since $f_n \cdot Xf = 0$. So $\beta_{\mathfrak{m}}(\nabla^f_X f_* Y)$ identifies with

$$\begin{pmatrix} f_1 \cdot XYf \\ \cdot \\ \cdot \\ f_{n-1} \cdot XYf \\ \nu \cdot XYf \end{pmatrix} = (f_i \cdot XYf) \; 1 \le i \le n.$$

Since

$$\nabla^M_X Y = \textstyle\sum_{j=1}^{n-1} (\nabla^M_X Y \cdot E_j) E_j$$

it follows that

$$df(\nabla^M_X Y) = \textstyle\sum_{j=1}^{n-1} (\nabla^M_X Y \cdot E_j) f_j,$$

but since ∇^M is the induced connection on M we have

$$\nabla^M_X Y \cdot E_j = XYf \cdot f_j \quad 1 \le j \le n-1.$$

Thus $\beta_{\mathfrak{m}}(f_* \nabla^M_X Y)$ identifies with the $\mathbb{R}^n$–valued function

$$\begin{pmatrix} f_1 \cdot XYf \\ \cdot \\ \cdot \\ f_{n-1} \cdot XYf \\ 0 \end{pmatrix}$$

Hence $\beta_{\mathfrak{m}}(\nabla^f_X f_* Y - f_* \nabla^M_X Y)$ identifies with

$$\begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \nu \cdot XYf \end{pmatrix}.$$

In other words, $\mathrm{II}_f(X,Y) = (\nu \cdot XYf)\nu$ and hence $\mathrm{tr}\mathrm{II}_f = \sum_{j=1}^{n-1} (\nu \cdot E_j E_j f)]\nu$.

Lemma 4.1.2 $\beta_{\mathfrak{p}}(\mathrm{II}_\alpha(X,Y)) = \begin{pmatrix} f_i \cdot XY\nu - XYf.E_i & 0 \end{pmatrix}$
Here we are using the identification $\mathfrak{p} \longrightarrow \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}$ *above.*

Proof

$$\begin{aligned} \mathrm{II}_\gamma(X,Y) &= \nabla^\gamma_X \gamma_* Y - \gamma_*(\nabla^M_X Y) \\ \beta_{\mathfrak{p}}(\nabla^\gamma_X \gamma_* Y) &= \mathrm{Ad}F \cdot (X\alpha_{\mathfrak{p}}(Y) + [\alpha_{\mathfrak{h}}(X), \alpha_{\mathfrak{p}}(Y)]) \end{aligned}$$

A calculation as before gives

$$\begin{aligned} \beta_{\mathfrak{p}}(\nabla^\gamma_X \gamma_* Y) &= \begin{pmatrix} f_i \cdot XY\nu & f_i \cdot XYf \end{pmatrix} \\ \beta_{\mathfrak{p}}(\gamma_* \nabla^M_X Y) &= \begin{pmatrix} XYf \cdot E_i\nu) & XYf \cdot f_i \end{pmatrix}. \end{aligned}$$

It follows that

$$\beta_{\mathfrak{p}}(\mathrm{II}_\gamma(X,Y)) = \begin{pmatrix} f_i \cdot XY\nu - XYf \cdot E_i\nu & 0 \end{pmatrix}.$$

We therefore have

$$\mathrm{II}_\gamma(X,Y) : TM \longrightarrow TM^\perp$$
$$\mathrm{II}_\gamma(X,Y)Z = (Zf \cdot XY\nu - XYf \cdot Z\nu)\nu.$$

and define $S(X,Y,Z) = \mathrm{II}_\gamma(X,Y)Z$

Lemma 4.1.3 $S(X,Y,Z)$ *is totally symmetric.*

Proof It suffices to show that

- $S(X,Y,Z) = S(Y,X,Z)$
- $S(X,Y,Z) = S(X,Z,Y)$

The first statement follows immediately from $\mathrm{II}_\gamma(X,Y) = \mathrm{II}_\gamma(Y,X)$, i.e the second fundamental form is symmetric. To show the second we note that $\mathrm{II}_f(Y,Z) = Yf \cdot Z\nu$ and $\mathrm{II}_f(Y,Z) = \mathrm{II}_f(Z,Y)$, giving

$$\begin{aligned} Yf \cdot Z\nu &= Zf \cdot Y\nu \\ \Rightarrow \quad XYf \cdot Z\nu + Yf \cdot XZ\nu &= XZf \cdot Y\nu + Zf \cdot XY\nu \\ \Rightarrow \quad Zf \cdot XY\nu - XYf \cdot Z\nu &= Yf \cdot XZ\nu - XZf \cdot Y\nu \\ \text{i.e,} \qquad S(X,Y,Z) &= S(X,Z,Y). \end{aligned}$$

Lemma 4.1.4 $(\nabla_X \mathrm{II}_f)(Y,Z) = S(X,Y,Z)$.

Proof Here ∇_X is the covariant derivative on $\mathrm{II}_f : TM \times TM \longrightarrow TM^\perp$. By definition

$$(\nabla_X \mathrm{II}_f)(Y,Z) = \nabla^\perp_X[\mathrm{II}_f(Y,Z)] - \mathrm{II}_f(\nabla^M_X Y, Z) - \mathrm{II}_f(Y, \nabla^M_X Z).$$

Here $\nabla^\perp$ is the connection on $TM^\perp$. Now

$$\nabla^\perp_X \nu = (X\nu \cdot \nu)\nu + (X\nu \cdot f)f = 0,$$

so

$$\begin{aligned} \nabla^{\perp}_X[\mathrm{II}_f(Y,Z)] &= -\nabla^{\perp}_X(Yf\cdot Z\nu)\nu \\ &= -(XYf\cdot Z\nu + Yf\cdot XZ\nu)\nu. \end{aligned}$$

Thus

$$\begin{aligned} (\nabla_X \mathrm{II}_f)(Y,Z) &= [-XYf\cdot Z\nu - Yf\cdot XZ\nu + \nabla^M_X Yf\cdot Z\nu + Yf\cdot\nabla^M_X Z\nu]\nu \\ &= [-XYf\cdot Z\nu + Zf\cdot\nabla^M_X Y\nu + Yf\cdot(\nabla^M_X Z\nu - XZ\nu)], \end{aligned}$$

but since Zf is tangent to M we have

$$Zf\cdot\nabla^M_X Yf = Zf\cdot XYf$$

and hence

$$(\nabla_X \mathrm{II}_f)(Y,Z) = S(X,Y,Z).$$

It follows that

$$(\nabla_X \mathrm{II}_f)(Y,Z) = \mathrm{II}_\gamma(X,Y)Z = \mathrm{II}_\gamma(Y,Z)X$$

and therefore

$$\tau(\gamma) = \mathrm{tr}\mathrm{II}_\gamma \in \Gamma(TM^*\otimes TM^{\perp}),$$

i.e. $\mathrm{tr}\mathrm{II}_\gamma : TM \longrightarrow TM^{\perp}; X \mapsto \sum_{i=1}^{n-1}\mathrm{II}_\gamma(E_i,E_i)X$.

The Ruh- Vilms theorem will follow from the next, quite general, result.

Proposition 4.1.5 *Let $h: TM\times TM \longrightarrow TM^{\perp}$ be a bilinear form with values in the normal bundle. Then*

$$\begin{aligned} \nabla^{\perp}_X trh &= tr\nabla_X h \\ i.e, \nabla^{\perp}_X \textstyle\sum_{i=1}^{n-1} h(E_i,E_i) &= \textstyle\sum_{i=1}^{n-1}(\nabla_X h)(E_i,E_i). \end{aligned}$$

Proof See [13]

By proposition 4.1.5 and lemma 4.1.4 we have

$$\tau(\gamma)X = \mathrm{tr}\nabla_X\mathrm{II}_f = \nabla^{\perp}_X\mathrm{tr}\mathrm{II}_f = \nabla^{\perp}_X\tau(f)$$

i.e. $\tau(\gamma) = \nabla^{\perp}\tau(f)$.

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