Automorphism Groups of Homogeneous Structures

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To Palestine

To those who have lost their lives, loved ones, freedom, homeland, and olive trees

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At the time I have arrived to Leeds in 2013, I have not heard the term ‘model theory’. Today, four years later, I am submitting a PhD thesis in model theory. Something I find hard to imagine. If I want to single out one reason for how I was able to make it, it is Professor H. Dugald Macpherson. Attending his model theory course was the initial spark which attracted my attention to the field. To me the amount of mathematical knowledge I have learned from Professor Macpherson is precious and enormous. During the years I have received from him nothing but continuous support, guidance, and close supervision on my research and many other matters such as reading research papers, mathematical writing, attending conferences, giving talks, and organising seminars and conferences to name a few. I will express my deep gratitude to Professor Macpherson with a verse in Arabic poetry by the Egyptian poet Ahmed Shawqi. The verse says (قُم لِلمُعَلِّمِ وَفِّهِ التَبجيلا كادَ المُعَلِّمُ أَن يَكونَ رَسولا) which translates to “Rise up to the teacher and pay him his dues of respect and reverence; he (the teacher) could almost have been a prophet”[1].

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[1] K. Qattous. The image of the teacher in Arabic poetry. Intern. Journal of Arts and Commerce, 2014.

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Abstract

A homogeneous structure is a countable (finite or countably infinite) first order structure such that every isomorphism between finitely generated substructures extends to an automorphism of the whole structure. Examples of homogeneous structures include any countable set, the pentagon graph, the random graph, and the linear ordering of the rationals. Countably infinite homogeneous structures are precisely the Fraïssé limits of amalgamation classes of finitely generated structures. Homogeneous structures and their automorphism groups constitute the main theme of the thesis.

The automorphism group of a countably infinite structure becomes a Polish group when endowed with the pointwise convergence topology. Thus, using Baire Category one can formulate the following notions. A Polish group has generic automorphisms if it contains a comeagre conjugacy class. A Polish group has ample generics if it has a comeagre diagonal conjugacy class in every dimension. To study automorphism groups of homogeneous structures as topological groups, we examine combinatorial properties of the corresponding amalgamation classes such as the extension property for partial automorphisms (EPPA), the amalgamation property with automorphisms (APA), and the weak amalgamation property. We also explain how these combinatorial properties yield the aforementioned topological properties in the context of homogeneous structures.

The main results of this thesis are the following. In Chapter 3 we show that any free amalgamation class over a finite relational language has Gaifman clique faithful coherent EPPA. Consequently, the automorphism group of the corresponding free homogeneous structure contains a dense locally finite subgroup, and admits ample generics and the small index property. In Chapter 4 we show that the universal bowtie-free countably infinite graph admits generic automorphisms. In Chapter 5 we prove that Philip Hall's universal locally finite group admits ample generics. In Chapter 6 we show that the universal homogeneous ordered graph does not have locally generic automorphisms. Moreover we prove that the universal homogeneous tournament has ample generics if and only if the class of finite tournaments has EPPA.

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Notation and Terminology

'Countable' means finite or countably infinite.

The symbol $\mathcal{L}$ always denotes a countable first order language.

Most of the time, M, N are countably infinite $\mathcal{L}$-structures, while A, B, C are finite $\mathcal{L}$-structures. As a usual practice we use the same symbol to denote both the structure and its underlying set. We write $\bar{a} \in M$ when $\bar{a} = (a_1, \ldots, a_n)$ is an n-tuple of elements of M for some $n \in \omega$. The theory of a structure M is denoted by $\mathrm{Th}(M)$. The automorphism group of M is denoted by $\mathrm{Aut}(M)$. A common practice we do is to put $G = \mathrm{Aut}(M)$. The set of all partial automorphisms of M is denoted by $\mathrm{Part}(M)$.

Suppose that G is a group acting on a set M and let $\bar{a} \in M$. Then $G_{\bar{a}}$ denotes the pointwise stabiliser of $\bar{a}$ in G. Similarly, if $A \subseteq M$ then G_A or $\mathrm{Stab}_G(A)$ is the pointwise stabiliser of A in G. Furthermore, $\mathrm{Orb}_G(\bar{a})$ or $\bar{a}^G$ denote the orbit of $\bar{a}$ in M^n under the action of G. The set of all orbits of G in M^n is denoted by M^n/G. If $g_1, \ldots, g_n \in G$, then $\langle g_1, \ldots, g_n \rangle$ is the subgroup generated by $g_1, \ldots, g_n$.

A graph is a set with an irreflexive symmetric binary relation, denoted by E which stands for 'edge'. Trees are connected graphs with no cycles. Forests are graphs with no cycles. We sometimes denote an edge $\{u, v\}$ of a graph simply by uv.

EPPA is an abbreviation for the Extension Property for Partial Automorphisms (Definition 1.5.1). While APA stands for the Amalgamation Property with Automorphisms (Definition 2.1.1).

Chapter 1

Introduction

Homogeneous structures are mathematical structures characterised by the property that local isomorphisms extend to global automorphisms. This work belongs to the research area of homogeneous structures and their automorphism groups, which in its turn belongs to model theory, a branch of mathematical logic. Model theory aims to understand and classify mathematical structures and their definable subsets in terms of the logical formulas satisfied in those structures. Examples of such structures are groups, fields, vector spaces, graphs, and linear orders. The term 'model' stands for a structure which models or satisfies a collection of properties expressed using a formal language, mainly a first order language. The term 'model theory' was first proposed by Alfred Tarski in 1954, and adopted by Abraham Robinson in 1956. Core topics in model theory include homogeneity and categoricity, stability theory and its generalisations, o-minimality, and quantifier elimination. Furthermore, model theory has applications in diverse fields such as algebraic/Diophantine geometry, real/complex analysis, and number theory.

Classification is not the only theme in model theory. Another common activity in the field is the art of constructing structures to be models of a given set of logical sentences. Model theoretic constructions encompass Henkin constructions, ultraproducts, Ehrenfeucht-Mostowski constructions, Fraïssé constructions, and Hrushovski constructions. In this thesis we adopt the Fraïssé construction, as it is a powerful method of building homogeneous structures as we shall see in this chapter.

The area of homogeneous structures was initiated by the work of Roland Fraïssé in the early 1950s. It has evolved into a rich area connecting together several research branches in mathematics such as model theory, combinatorics, permutation group theory, descriptive set theory, topological dynamics, and theoretical computer science. Early results in the subject include the construction of $2^{\aleph_0}$ homogeneous directed graphs by Henson in 1972, the classification of finite homogeneous graphs by Gardiner, and independently by Golfand and Klin, in 1976, and the classification of countably infinite homogeneous graphs by Lachlan and Woodrow in 1980. For further knowledge about the pioneers, results, themes, and an overview of the subject, and its connection with other areas we advise the reader to consult Macpherson's survey [55].

Regarding the structure of the thesis, we discuss in this chapter Fraïssé's Theorem, and give a brief overview of some of the fields intertwined with the subject of homogeneous structures while paving the road to state the results of the thesis at the end of this chapter. In Chapter 2 we convey the details of how the extension property for partial automorphisms together with the amalgamation property with automorphisms form a sufficient condition for the existence of ample generics for homogeneous structures. We also present the Kechris-Rosendal characterisation of ample generics in the context of homogeneous structures, and finally motivate the importance of ample generics by discussing a handful of their group-theoretic consequences.

In Chapter 3 we focus on free homogeneous structures, those structures which arise as Fraïssé limits of free amalgamation classes. In Chapter 4 we study an instance of Cherlin-Shelah-Shi [15] universal graphs, namely the universal bowtie-free graph. In Chapter 5 we are concerned with the class of finite groups and their Fraïssé limit. In Chapter 6 we use the weak amalgamation property to investigate linear orders, ordered graphs, and tournaments. We conclude with Chapter 7 by presenting a list of open questions related to the thesis.

This chapter is entirely a review, and it has no new results.

1.1 Basic Model Theory

This section is based on Hodges [38], Marker [60], and Tent-Ziegler [74].

A *first order language* $\mathcal{L}$ is a set of symbols. Such symbols come in three kinds: constant symbols c_i, function symbols f_j, and relation symbols R_k. So in general, $\mathcal{L} = \{c_i : i \in I\} \cup \{f_j : j \in J\} \cup \{R_k : k \in K\}$ for some indexing sets I, J, K. Each function and relation symbol is associated with a nonzero natural number called the arity of the symbol. An *$\mathcal{L}$-structure* M is a set, which is also denoted by M, that has interpretations for the symbols of $\mathcal{L}$. More precisely, every constant symbol $c \in \mathcal{L}$ is interpreted as a distinguished element c^M of M, every function symbol $f \in \mathcal{L}$ with arity n is interpreted as a function $f^M : M^n \to M$, and finally every relation symbol $R \in \mathcal{L}$ of arity m is interpreted as an m-ary relation R^M on M, that is, as a subset of M^m. All first order languages contain a binary relation symbol '$=$' which is interpreted as equality. A language is called *relational* if it has no constant or function symbols. We write $M = \big(M, (c_i)_{i\in I}, (f_j)_{j\in J}, (R_k)_{k\in K}\big)$ when M is an $\mathcal{L}$-structure. A vast number of mathematical objects, like graphs, ordered structures, groups, rings, and fields, can be described as $\mathcal{L}$-structures for an appropriate language $\mathcal{L}$. In this thesis we work with countable languages.

Definition 1.1.1. Let M and N be $\mathcal{L}$-structures. An *$\mathcal{L}$-embedding* $\phi : M \to N$ is an injective map such that,

(i) for every $c \in \mathcal{L}$ we have that $\phi(c^M) = c^N$;

(ii) for every $f \in \mathcal{L}$ and any $\bar{a} \in M$ we have that $\phi(f^M(\bar{a})) = f^N(\phi(\bar{a}))$;

(iii) for every $R \in \mathcal{L}$ and any $\bar{a} \in M$ we have that $\bar{a} \in R^M$ if and only if $\phi(\bar{a}) \in R^N$.

An *$\mathcal{L}$-isomorphism* is a bijective $\mathcal{L}$-embedding. Let M and N be $\mathcal{L}$-structures. An *automorphism* of M is an $\mathcal{L}$-isomorphism from M onto itself. We say M is a *substructure* of N if $M \subseteq N$, and the inclusion map is an $\mathcal{L}$-embedding. We abuse notation and write $M \subseteq N$ when M is a substructure of N. Suppose that A is a subset of M. The *substructure of M generated by A* is the unique smallest substructure of M containing

A. If M itself is generated by finitely many of its elements we say that M is a *finitely generated structure*. A structure is *locally finite* if every finitely generated substructure is finite.

We use the language $\mathcal{L}$ to study and describe properties of $\mathcal{L}$-structures using the notions of $\mathcal{L}$-sentences and $\mathcal{L}$-formulas. An *$\mathcal{L}$-formula* [60, Definition 1.1.5] is a well-formed string of symbols from $\mathcal{L}$ together with the following symbols: comma, parentheses, infinitely many variable symbols $x, y, z, x_1, x_2, x_3, \ldots$, propositional connectives: negation $\neg$, conjunction $\wedge$, disjunction $\vee$, implication $\rightarrow$, the existential quantifier $\exists$, and the universal quantifier $\forall$. The term "first order" means that the variables range over the elements of a structure. Those formulas with no free variables, that is, every variable is quantified, are called *$\mathcal{L}$-sentences*. We write $\phi(x_1, \ldots, x_n)$ for a formula ϕ whose free variables are among $x_1, \ldots, x_n$. A formula without quantifiers is called a *quantifier free* formula.

Suppose that M is an $\mathcal{L}$-structure, $(a_1, \ldots, a_n) \in M^n$, and $\phi(x_1, \ldots, x_n)$ is an $\mathcal{L}$-formula. We write $M \models \phi(a_1, \ldots, a_n)$ to say that the formula ϕ is true in M when the free variable x_i is interpreted by the element $a_i \in M$. This interplay between structures and formulas can be made rigorous via *Tarski's definition of truth*—see [60, Definition 1.1.6]. Chang-Keisler [10] write "The truth definition is the bridge connecting the formal language with its interpretation by means of models." Every $\mathcal{L}$-sentence σ is either true or false in M. When $M \models \sigma$, we say M models or satisfies σ. Let T be a set of $\mathcal{L}$-sentences. We write $M \models T$ if for all $\sigma \in T$ we have that $M \models \sigma$, and say that M is a model of T. We say T is *consistent* if it has a model. More generally, a set $p(\bar{x})$ of $\mathcal{L}$-formulas in free variables $\bar{x}$ is said to be *consistent* if there are an $\mathcal{L}$-structure M and a tuple $\bar{a} \in M$ such that for every $\phi \in p$ we have $M \models \phi(\bar{a})$. We write $T \models \sigma$ if every model of T is a model of σ. An *$\mathcal{L}$-theory* is a consistent set of $\mathcal{L}$-sentences. A theory T is *complete* if for any $\mathcal{L}$-sentence σ either $\sigma \in T$ or $\neg\sigma \in T$. We now state one of the fundamental theorems of model theory of first order languages due to Gödel (1930) and Malcev (1936), see [74, Theorem 2.2.1].

Theorem 1.1.2 (Compactness Theorem)**.** *Suppose that T is a set of $\mathcal{L}$-sentences. If every finite subset of T has a model then T has a model.*

Definition 1.1.3. Let $\mathcal{L}$ be a first order language.

- A theory T has *quantifier elimination* if for every formula $\phi(\bar{x})$ there is a quantifier free formula $\phi^*(\bar{x})$ such that $T \models \forall \bar{x}\big(\phi(\bar{x}) \leftrightarrow \phi^*(\bar{x})\big)$.
- A *universal formula* is an $\mathcal{L}$-formula $\psi(\bar{y})$ of the form $\forall \bar{x}\phi(\bar{x}, \bar{y})$ where ϕ is a quantifier free formula. Similarly, an *existential formula* has the form $\exists \bar{x}\phi(\bar{x}, \bar{y})$ where ϕ is quantifier free.
- Let T be an $\mathcal{L}$-theory. We say that a set of $\mathcal{L}$-sentences Γ *axiomatises* T if for any $\mathcal{L}$-structure M, we have that $M \models T$ if and only if $M \models \Gamma$. We say that T is a *universal theory* if it is axiomatised by a set of universal sentences.
- A class $\mathcal{K}$ of $\mathcal{L}$-structures is *elementary* if there is a theory T such that $M \in \mathcal{K}$ if and only if $M \models T$ for every $\mathcal{L}$-structure M.

Let M be an $\mathcal{L}$-structure, and $A \subseteq M$. For every $a \in A$, let c_a denote a new constant symbol. We define $\mathcal{L}_A := \mathcal{L} \cup \{c_a : a \in A\}$; the new language obtained by adding the new constant symbols to the original language. The complete theory of an $\mathcal{L}$-structure M denoted by $\mathrm{Th}(M)$ is the set of all $\mathcal{L}$-sentences true in M. More generally, we define

$$\mathrm{Th}_A(M) = \{\sigma \in \mathcal{L}_A\text{-sentences} : M \models \sigma\}.$$

The set $\mathrm{Th}_M(M)$ is called the *elementary diagram* of M. We say that N is an *elementary extension* of M, and write $M \preceq N$, if M is a substructure of N and $N \models \mathrm{Th}_M(M)$.

Definition 1.1.4. Let M be an $\mathcal{L}$-structure, and let $A \subseteq M$. We say that a subset $X \subseteq M^n$ is A-*definable* if there is an $\mathcal{L}_A$-formula $\phi(\bar{x})$ such that $X = \{\bar{b} \in M^n : M \models \phi(\bar{b})\}$.

We now introduce the notion of a type. In the same way a theory describes a model using $\mathcal{L}$-sentences, a type uses $\mathcal{L}$-formulas to describe elements of a model of a theory.

Definition 1.1.5. Let M be an $\mathcal{L}$-structure, and $A \subseteq M$. We define an *n-type over A in M* to be a set $p(\bar{x})$ of $\mathcal{L}_A$-formulas in the free variables $\bar{x} = (x_1, ..., x_n)$ such that $p(\bar{x}) \cup \mathrm{Th}_A(M)$ is consistent.

An n-type $p(\bar{x})$ over A in M is called a *complete n-type* if for every $\mathcal{L}_A$-formula $\phi(\bar{x})$, we have either $\phi \in p$ or $\neg\phi \in p$. We denote by $S_n^M(A)$ the set of all complete n-types over A in M. For a tuple $\bar{a} \in M$, the *type of $\bar{a}$ over A in M* is the set $\text{tp}_M(\bar{a}/A) = \{\phi(\bar{x}) \in \mathcal{L}_A\text{-formulas} : M \models \phi(\bar{a})\}$. We write $\text{tp}_M(\bar{a})$ for $\text{tp}_M(\bar{a}/\emptyset)$, and clearly we have that $\text{tp}_M(\bar{a}/A) \in S_n^M(A)$. We say that $\bar{a} \in M^n$ *realises* $p(\bar{x})$ if $M \models \phi(\bar{a})$ for every $\phi(\bar{x}) \in p$. And we say M *omits* p if no $\bar{a} \in M^n$ realises p. A type may not be realised in a specific $\mathcal{L}$-structure M, however it will be realised in an elementary extension of M.

Proposition 1.1.6. [60, Proposition 4.1.3] *Let $p(\bar{x})$ be an n-type over A in M. Then, there is an elementary extension N of M which realises p.*

Alternatively, we may start with an $\mathcal{L}$-theory T, and define an *n-type of T* to be a set $p(\bar{x})$ of $\mathcal{L}$-formulas such that $p(\bar{x}) \cup T$ is consistent. A type $p(\bar{x})$ of T is *complete* if for every $\mathcal{L}$-formula $\phi(\bar{x})$, we have either $\phi \in p$ or $\neg\phi \in p$. We let $S_n(T)$ denotes the set of all complete n-types of T. Note that for a complete $\mathcal{L}$-theory T, a model $M \models T$, and $A \subseteq M$ we have that $S_n\big(\text{Th}_A(M)\big) = S_n^M(A)$.

Definition 1.1.7. Let p be an n-type of an $\mathcal{L}$-theory T. We say p is *isolated* if there is an $\mathcal{L}$-formula $\phi(\bar{x})$ such that $T \models \exists\bar{x}\phi(\bar{x})$ and for every $\psi(\bar{x}) \in p(\bar{x})$ we have that $T \models \forall\bar{x}(\phi(\bar{x}) \rightarrow \psi(\bar{x}))$.

The set $S_n^M(A)$ can be made into a topological space when endowed with the Stone topology, and thus called the *Stone space of types*. The Stone topology is generated by the basic open sets $[\phi]$ where $\phi(\bar{x})$ is an $\mathcal{L}_A$-formula and $[\phi] = \{p \in S_n^M(A) : \phi \in p\}$. Since each $p \in S_n^M(A)$ is a complete type, the complement of $[\phi]$ is $[\neg\phi]$, and thus $[\phi]$ is a clopen subset. It can be shown that an $\mathcal{L}$-formula ϕ isolates a complete type p if and only if $[\phi] = \{p\}$. So to say that p is isolated is equivalent to saying that the singleton $\{p\}$ is an open set in the Stone topology, hence the name isolated.

Lemma 1.1.8. [60, Lemma 4.1.8] *The Stone space $S_n^M(A)$ is a compact Hausdorff totally disconnected topological space.*

1.2 Homogeneity and Omega-categoricity

The main references of this section are Macpherson *et al.* [5, Chapter 14], Hodges [38, Chapter 6], Evans [25], and Macpherson [55].

We will present a method of constructing countably infinite structures from certain classes of finite structures. In 1954, Roland Fraïssé introduced a technique of constructing the rationals as a linear order from the class of all finite linear orders in infinitely many steps. His technique is known as Fraïssé's construction, and it constitutes the main method of construction in the thesis. In this section, $\mathcal{L}$ is a countable first order language. We now define what it means for a structure to be homogeneous.

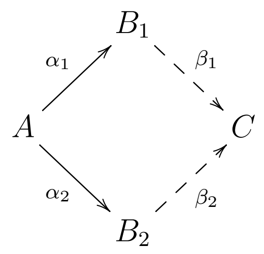
Definition 1.2.1. Let M be an $\mathcal{L}$-structure. We say that M is *homogeneous* if M is countable and every isomorphism between finitely generated substructures of M extends to an automorphism of M.

Definition 1.2.2. Let M an $\mathcal{L}$-structure. The *age* of M, denoted by $\text{Age}(M)$, is the class of all finitely generated structures which can be embedded in M.

Let $\mathcal{C}$ be a class of finitely generated $\mathcal{L}$-structures. We say that $\mathcal{C}$ has the *hereditary property (HP)* if whenever $B \in \mathcal{C}$ and A is a finitely generated substructure of B, then $A \in \mathcal{C}$. We say that $\mathcal{C}$ has the *joint embedding property (JEP)* if whenever $A_1, A_2 \in \mathcal{C}$, then there is $B \in \mathcal{C}$ such that A_1 and A_2 both embed into B. The age of any $\mathcal{L}$-structure satisfies both HP and JEP. Conversely, a countable class of finitely generated $\mathcal{L}$-structures with HP and JEP is the age of some countable structure [38, Theorem 6.1.1].

The next property, the amalgamation property, was formulated by Fraïssé, and it turned out that it has an important role to play in model theory. In [38, Chapter 5] Wilfrid Hodges writes "The idea of amalgamation is very powerful, and I have used it whenever I can."

We say that $\mathcal{C}$ has the *amalgamation property (AP)* if for all $A, B_1, B_2 \in \mathcal{C}$ and embeddings $\alpha_1 : A \to B_1$ and $\alpha_2 : A \to B_2$ there exists $C \in \mathcal{C}$ with embeddings $\beta_1 : B_1 \to C$ and $\beta_2 : B_2 \to C$ such that $\beta_1\alpha_1 = \beta_2\alpha_2$.



The amalgamation property of a class $\mathcal{C}$ guarantees that for all A, B_1, B_2 in $\mathcal{C}$ as above, there is a structure C, called the *amalgam* of B_1 and B_2 over A, which is also in the class $\mathcal{C}$ that contains a copy of B_1 and a copy of B_2 in a way that the part of B_1 isomorphic to A is glued with the corresponding part of B_2 as shown below in Figure 1.1. The embeddings α_1 and α_2 are like the arms of a maestro directing which vertex of B_1 is to be glued with which vertex of B_2.

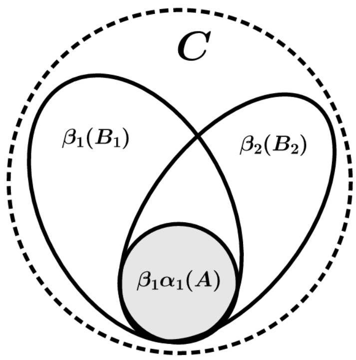


Figure 1.1: The amalgam C of B_1 and B_2 over A.

We note that when the language in hand is relational, every finitely generated structure is finite. We also may assume that the domain of the amalgam C is $\beta_1(B_1) \cup \beta_2(B_2)$, and so in this case we have that: $\max\{|B_1|, |B_2|\} \leq |C| \leq (|B_1| + |B_2| - |A|)$.

Definition 1.2.3. We call a class $\mathcal{C}$ of finitely generated $\mathcal{L}$-structures an *amalgamation class* if it contains countably many isomorphism types, closed under isomorphism, and has the hereditary property, the joint embedding property, and the amalgamation property.

We now introduce a stronger version of the amalgamation property which will play an important role in Chapter 3.

	Class of structures	**Language**	**HP**	**JEP**	**AP**
1	Finite graphs	$\{E\}$	Yes	Yes	Yes
2	Finite trees	$\{E\}$	No	Yes	Yes
3	Finite forests	$\{E\}$	Yes	Yes	No
4	Finite $\bowtie$-free graphs	$\{E\}$	Yes	Yes	No
5	Special $\bowtie$-free graphs	$\{E\}$	No	Yes	Yes
6	Finite linear orders	$\{<\}$	Yes	Yes	Yes
7	Finite groups	$\{1, \cdot, ^{-1}\}$	Yes	Yes	Yes
8	Finite fields	$\{0, 1, +, -, \cdot, ^{-1}\}$	Yes	No	Yes

Table 1.1: Examples of classes of finite structures and their properties. For graphs and linear orders see Examples 14(f) and 14(e) in [5], respectively. For $\bowtie$-free graphs see Chapter 4, and for groups see Chapter 5.

Definition 1.2.4. Suppose that $\mathcal{L}$ is a relational language. Given finite $\mathcal{L}$-structures A, B_1, B_2 with $A \subseteq B_1$ and $A \subseteq B_2$, the *free amalgam* of B_1 and B_2 over A is the structure C whose domain is the disjoint union of B_1 and B_2 over A, and for every relation symbol $R \in \mathcal{L}$ we define $R^C := R^{B_1} \cup R^{B_2}$.

We say that a class $\mathcal{C}$ of structures over a fixed relational language has the *free amalgamation property* if $\mathcal{C}$ is closed under free amalgams. Moreover $\mathcal{C}$ is called a *free amalgamation class* if it is closed under substructures and isomorphism, and has both the joint embedding property and the free amalgamation property. Lastly, a homogeneous structure whose age is a free amalgamation class is called a *free homogeneous structure*. We now introduce Fraïssé's Theorem, which is one of the pillars on which the thesis rests. Accordingly, the reader may find its proof in Appendix A.

Theorem 1.2.5 (Fraïssé's Theorem [28]). *Suppose that $\mathcal{C}$ is an amalgamation class of finitely generated $\mathcal{L}$-structures. Then there is a unique, up to isomorphism, homogeneous $\mathcal{L}$-structure M such that* $\mathrm{Age}(M) = \mathcal{C}$. *Conversely, if N is a homogeneous $\mathcal{L}$-structure then* $\mathrm{Age}(N)$ *is an amalgamation class.*

So Fraïssé's Theorem provides a one to one correspondence between homogeneous structures and amalgamation classes. Given an amalgamation class $\mathcal{C}$, the unique homogeneous structure whose age is $\mathcal{C}$ is called the *Fraïssé limit* of $\mathcal{C}$.

Remark 1.2.6. In Appendix A, Lemma A.2 shows that a structure M being homogeneous is equivalent to the following property: whenever we have that $A \subseteq B$ in $\mathrm{Age}(M)$ and an embedding $f : A \to M$, then there is an embedding $g : B \to M$ extending f. In the thesis we refer to this property by the phrase 'by homogeneity'.

Class $\mathcal{C}$ of structures	**Fraïssé limit of $\mathcal{C}$**
Finite sets	The trivial structure $(\mathbb{N}, =)$
Finite linear orders	The rationals $(\mathbb{Q}, <)$
Finite graphs	The random graph
Finite Boolean algebras	The countable atomless Boolean algebra
Finite metric spaces with rational distances	The rational Urysohn metric space
Finite groups	Philip Hall's locally finite universal group

Table 1.2: Examples of Fraïssé limits

Note 1.2.7 (Classification results (I))**.** For the classification of countably infinite homogeneous graphs see Lachlan-Woodrow [51], for countable homogeneous partial orders see Schmerl [67], and for countably infinite homogeneous directed graphs see Cherlin [18].

We next introduce the notion of ω-categoricity and its connection with homogeneity.

Definition 1.2.8. A complete $\mathcal{L}$-theory T is called *ω-categorical* if T has a unique countably infinite model, up to isomorphism. An $\mathcal{L}$-structure M is ω-categorical if its theory is ω-categorical.

By a theorem of Cantor [5, Theorem 9.3], any model of cardinality $\aleph_0$ of the theory of dense linear orders without endpoints is isomorphic to $(\mathbb{Q}, <)$. Thus $\mathrm{Th}(\mathbb{Q}, <)$ is an ω-categorical theory.

Definition 1.2.9. [5, Definition 9.6] A group G acting on a set Ω is called *oligomorphic* on Ω if for every $n \in \omega$, the group G has finitely many orbits in its induced action on Ω^n.

In Section 1.3, we will treat automorphism groups as permutation groups. A structure M whose automorphism group is oligomorphic on its underlying set suggests that there are only a few different shapes of n-element subsets of M, as described by Peter Cameron in [7]. We now introduce a theorem which constitutes a bridge connecting together model theory and permutation group theory.

Theorem 1.2.10 (Ryll-Nardzewski, Engeler, Svenonius, 1959)**.** *Let M be a countably infinite $\mathcal{L}$-structure, and let $T = \mathrm{Th}(M)$. Then the following are equivalent:*

(i) The structure M is ω-categorical.

(ii) For all $n \in \mathbb{N}$, every n-type of T is isolated.

(iii) For all $n \in \mathbb{N}$, $S_n(T)$ is finite.

(iv) The automorphism group $\mathrm{Aut}(M)$ is oligomorphic on M.

(v) For all $n \in \mathbb{N}$, there are finitely many formulas $\phi(x_1, \ldots, x_n)$ up to equivalence modulo T.

Consult [38, Theorem 6.3.1] for a proof of Ryll-Nardzewski's Theorem. The proof leads to the corollary below which demonstrates a bidirectional translation between the language of model theory and the language of permutation group theory.

Corollary 1.2.11 ([25])**.** *Suppose that M is a countably infinite ω-categorical structure. Put $G = \mathrm{Aut}(M)$. Then the following hold.*

(i) For any $\bar{a}, \bar{b} \in M^n$, we have that $\mathrm{Orb}_G(\bar{a}) = \mathrm{Orb}_G(\bar{b})$ if and only if $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$.

(ii) Let $A \subseteq M$ be finite. Then a subset $\Delta \subseteq M^n$ is A-definable if and only if Δ is the union of G_A-orbits on M^n.

Corollary 1.2.12. [38, Corollary 6.3.2] *Suppose that T is an ω-categorical theory, and $M \models T$. Then M is locally finite. Additionally, there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for each $n \in \mathbb{N}$, if $A \subseteq M$ is a substructure generated by n elements, then $|A| \leq f(n)$.*

A structure which satisfies the 'additionally' part of the corollary above is said to be *uniformly locally finite*.

Theorem 1.2.13. [38, Corollary 6.4.2] *Suppose $\mathcal{L}$ is a finite first order language and M is a countably infinite $\mathcal{L}$-structure. Then the following are equivalent:*

(i) *M is homogeneous and uniformly locally finite.*

(ii) *$\mathrm{Th}(M)$ is ω-categorical and has quantifier elimination.*

In particular, any homogeneous structure over a finite relational language is ω-categorical and its theory has quantifier elimination (QE).

Proposition 1.2.14. [55, Proposition 3.1.6] *Suppose that M is an ω-categorical structure over a relational language. Then M is homogeneous if and only if $\mathrm{Th}(M)$ has QE.*

Example 1.2.15. Philip Hall's universal group is an example of a homogeneous locally finite group which is not ω-categorical, and so not uniformly locally finite. See Chapter 5 for more details.

Note 1.2.16 (Classification results (II))**.** For the classification of finite homogeneous groups see Cherlin-Felgner [12]. For partial classification results regarding countable homogeneous soluble groups see Cherlin-Felgner [11] and Saracino-Wood [66]. Groups of nilpotency class 2 and exponent 4 are uniformly locally finite. Saracino and Wood showed that there exist continuum many countable homogeneous (and so ω-categorical) groups of nilpotency class 2 and exponent 4.

Chapter 3 contains examples of homogeneous structures over finite relational languages. Droste and Macpherson [24, Theorem 1.4] used Fraïssé construction to prove that there are continuum many nonisomorphic countably infinite ω-categorical universal graphs which are not homogeneous. Also the universal bowtie-free graph of Chapter 4 is an ω-categorical structure which is not homogeneous. We conclude by noting that examples of homogeneous and ω-categorical structures are ubiquitous in Evans [25] and Macpherson [55].

1.3 Infinite Permutation Groups and Topology

The content of this section is based on Macpherson *et al.* [5, Chapter 2 and 6] and Cameron [7, Chapter 2].

The theory of infinite permutation groups is one of the newer branches of group theory, and it has established connections with model theory as we will see shortly. Let Ω be a finite or infinite set. A *permutation* on Ω is a bijection from Ω onto itself. The set of all permutations on Ω forms a group under function composition, called the *symmetric group* on Ω and denoted by Sym(Ω). A *permutation group on* Ω is a subgroup of Sym(Ω). An *infinite permutation group* is a permutation group on an infinite set. Permutation groups on Ω are precisely those groups which act faithfully on Ω. Any group can be seen as a permutation group on its underlying set by Cayley's Theorem.

Every permutation can be written as a product of disjoint finite or infinite cycles uniquely up to order of cycles and cyclic permutation of finite cycles. Given $h \in \mathrm{Sym}(\Omega)$, the *cycle type* of h is the sequence $\aleph_0^{k_0}\ 1^{k_1}\ 2^{k_2} \ldots n^{k_n} \ldots$, where each $k_n \in \omega + 1$, k_0 is the number of infinite cycles in h, and k_n is the number of finite cycles of length $n > 0$ in h. Two permutations $g, h \in \mathrm{Sym}(\Omega)$ are conjugate in Sym(Ω) if and only if they have the same cycle type. As our interest hovers around homogeneous structures and their automorphism groups, we focus on the case when Ω is countably infinite, and consequently replace it with the set of natural numbers $\mathbb{N}$. We note that $|\mathrm{Sym}(\mathbb{N})| = 2^{\aleph_0}$. Consult [5, Theorem 6.1] for a proof. We may assume that the underlying set of any countably infinite structure M is the set of natural numbers. Thus Aut$(M) \leq \mathrm{Sym}(\mathbb{N})$ is a permutation group on a countably infinite set.

Conversely, given a permutation group $G \leq \mathrm{Sym}(\mathbb{N})$ we may define a structure M on the set $\mathbb{N}$, called the *canonical relational structure* associated with G, such that $G \leq \mathrm{Aut}(M)$ and for every $n \in \omega$ we have that G and $\mathrm{Aut}(M)$ have the same orbits on M^n. The language of M is $\mathcal{L} = \{R_i^n : n \in \omega,\ i < k_n\}$ where R_i^n is an n-ary relation symbol, and $k_n \leq \aleph_0$ is the number of orbits of G on $\mathbb{N}^n$. The symbols are interpreted as the orbits, where $R_0^n, R_1^n, R_2^n, \ldots$ is a list of all the orbits of G on $\mathbb{N}^n$. The canonical structure is homogeneous over an infinite relational language in the sense defined in Section 1.2.

Our aim is to enrich permutation groups on a countably infinite set with a topology. Following [45, Section 9A], a *topological group* G is a group endowed with a topology such that both the group multiplication from $G \times G \to G$ and the inversion map from $G \to G$ are continuous maps. Consequently, the actions of left and right multiplication of any topological group G on itself are homeomorphisms of G. If a subgroup of a topological group is open, then all of its cosets are open as well. A distinguished subclass of topological groups is the class of Polish groups. Recall that a topological space is *separable* if it contains a countable dense subset.

Definition 1.3.1. [45, Definition 9.2]
A *Polish space* is a topological space whose topology is separable and completely metrisable. A *Polish group* is a topological group whose topology is Polish.

Example 1.3.2. [45, Section 9B] Examples of Polish groups.

(i) Any countable group with the discrete topology.
(ii) The additive group of real numbers $(\mathbb{R}, +)$ and the multiplicative group of nonzero real numbers $(\mathbb{R}^\times, \cdot)$ with the usual topology.
(iii) The *Cantor group* 2^ω, the unrestricted Cartesian product of ω many copies of the cyclic group of order 2 each endowed with the discrete topology.

We note that every uncountable Polish space has the cardinality of the continuum.

We will see more examples now. Let $G \leq \mathrm{Sym}(\mathbb{N})$. For a finite tuple $\bar{a} \in \mathbb{N}^n$, the *pointwise stabiliser* of $\bar{a}$ in G is the subgroup $G_{\bar{a}} = \{g \in G : g(\bar{a}) = \bar{a}\}$. We endow G with the topology generated by basic open sets of the form:

$$G(\bar{a}, \bar{b}) = \left\{g \in G : g(\bar{a}) = \bar{b}\right\} \text{ where } \bar{a}, \bar{b} \in \mathbb{N}^n, \text{ and } n \in \omega.$$

Notice that $G(\bar{a}, \bar{b}) = hG_{\bar{a}}$ for any $h \in G$ such that $h(\bar{a}) = \bar{b}$. So the basis of this topology consists of all cosets of pointwise stabilisers of finite tuples. This topology makes G into a topological group whose topology is separable. An instance of particular interest is $G = \mathrm{Aut}(M)$ for some countably infinite first order structure M. In this case and in the spirit of extending partial automorphisms (Section 1.5), the basic open set $G(\bar{a}, \bar{b})$ may be described as the set of automorphisms of M extending the finite partial automorphism $\phi : \bar{a} \to \bar{b}$.

The topology on Sym(ℕ) is also completely metrisable. Hence Sym(ℕ) is a Polish group. The topology on Sym(ℕ) is induced by the following complete metric $d : \mathrm{Sym}(\mathbb{N}) \times \mathrm{Sym}(\mathbb{N}) \to [0, 1]$. For any $f, g \in \mathrm{Sym}(\mathbb{N})$, the distance between f and g is defined to be:

$$d(f,g) = \begin{cases} 0, & \text{if } f = g; \\ 1/2^n, & \text{if } n = \min\{m \in \mathbb{N} : f(m) \neq g(m) \text{ or } f^{-1}(m) \neq g^{-1}(m)\}. \end{cases}$$

So the closer g and h are to each other, the longer is the initial segment of ℕ on which they agree. The reason to involve f^{-1} and g^{-1} in the definition above is to make d a complete metric on $\mathrm{Sym}(\mathbb{N})$. With respect to this metric, a sequence $(g_n)_{n\in\omega}$ of permutations converges to a permutation g if and only if for every $x \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $g_n(x) = g(x)$. That is why this topology is called the *pointwise convergence topology*.

We collect below some folkloric facts about subgroups of Sym(ℕ). See [7, Section 2.4].

Theorem 1.3.3. *The following hold.*

(i) Suppose that $G \leq \mathrm{Sym}(\mathbb{N})$. Then G is closed if and only if $G = \mathrm{Aut}(M)$ for some countably infinite first order structure M.

(ii) Suppose that $G \leq \mathrm{Sym}(\mathbb{N})$. Then a subgroup H of G is open in G if and only if for some finite tuple $\bar{a} \in \mathbb{N}$ we have $G_{\bar{a}} \leq H$.

(iii) (Cameron) Suppose that M is a countably infinite ω-categorical structure. Let $G = \mathrm{Aut}(M)$. Then any open subgroup of G is contained in finitely many subgroups of G.

(iv) *Suppose that G is a Polish group, and $H \leq G$. Then H is a Polish group if and only if H is closed in G.*

(v) $\mathrm{Aut}(M)$ *is a Polish group, for any countably infinite first order structure M.*

(vi) *Let M be countably infinite first order structure, and put $G = \mathrm{Aut}(M)$. Then $|G| \leq \aleph_0$ if and only if for some finite tuple $\bar{a} \in M^n$ we have that $G_{\bar{a}}$ is the identity.*

(vii) *Let $G \leq \mathrm{Sym}(\mathbb{N})$. A subgroup $H \leq G$ is dense in G if and only if H has the same orbits on $\mathbb{N}^n$ as G for all $n \in \omega$. That is, for all $g \in G$ and $\bar{a} \in \mathbb{N}^n$ there is $h \in H$ such that $h(\bar{a}) = g(\bar{a})$.*

(viii) *Any open subgroup of a topological group is also closed.*

(ix) *Any closed subgroup of a topological group with finite index is open.*

For the forward direction of statement (i), the canonical structure associated with G will do. For the proof of (iii) see [39, Lemma 2.4], and of (iv) see [2, Proposition 1.2.1].

1.4 Generic Automorphisms

Let X be a topological space, and $U \subseteq X$ be an open subset. We say a subset $D \subseteq X$ is *dense in* U if D intersects nontrivially every nonempty open subset of U. A subset $N \subseteq X$ is *nowhere dense* if N is not dense in any open subset $U \subseteq X$. Thus, a subset $N \subseteq X$ is nowhere dense if and only if every nonempty open subset $U \subseteq X$ has a nonempty open subset $V \subseteq U$ such that $N \cap V = \emptyset$ if and only if $X \setminus N$ contains a dense open subset if and only if the closure of N has an empty interior. In other words, a nowhere dense subset is characterised as a set which is "full of holes" as described by Oxtoby in [63, Chapter 1]. It follows that the class of nowhere dense subsets of some topological space is closed under subsets, taking closure, and finite unions. However, a countable union of nowhere dense sets is not necessarily nowhere dense. For example, in the real line $\mathbb{R}$ with the standard topology we can write the set of rationals $\mathbb{Q}$ as a countable union of singletons, but $\mathbb{Q}$ is dense in $\mathbb{R}$ nevertheless.

The last observation motivates the following definition introduced by Baire in 1899. A set $M \subseteq X$ is called *meagre* if it can be written as a countable union of nowhere dense sets. A meagre set is said to be of *first category*, and a set which is not meagre is said to be of *second category*. The complement of a meagre set is called *comeagre*. Note that a set is comeagre if and only if it contains a countable intersection of dense open sets. An important fact about comeagre sets in complete metric spaces is given by the *Baire Category Theorem*—see [45, Theorem 8.4].

Theorem 1.4.1 (The Baire Category Theorem)**.** *Suppose X is a complete metric space. Then every comeagre subset of X is dense in X. Equivalently, every nonempty open set of X is non-meagre.*

The class of comeagre subsets of a complete metric space does not contain the empty set, and is closed under supersets and countable intersection. Thus, the class of comeagre sets forms a *δ-filter*, and comeagreness gives a notion of largeness.

We will make use of the notion of comeagreness in our setting of automorphism groups of countably infinite structures. Suppose that $\mathcal{L}$ is a countable first order language. Let M be a countably infinite $\mathcal{L}$-structure. We have seen before that its automorphism group $\mathrm{Aut}(M)$ is a Polish group, and therefore the Baire Category Theorem holds for $\mathrm{Aut}(M)$. We now introduce one of the main definitions of the thesis.

Definition 1.4.2 (Truss [78])**.** Let M be a countably infinite $\mathcal{L}$-structure. We say that M has *generic automorphisms* if $\mathrm{Aut}(M)$ has a comeagre conjugacy class.

An automorphism of M whose conjugacy class is comeagre is called a generic automorphism. When two elements of $\mathrm{Aut}(M)$ are conjugate, they look alike in the sense that their action on M is the same up to relabelling the elements of M. Thus if M has generic automorphisms, it means that there exists a large set of automorphisms of M which look alike. By the Baire Category Theorem, $\mathrm{Aut}(M)$ may have at most one comeagre conjugacy class.

In [78] Truss provides a sufficient condition for the existence of generic automorphisms, and proves that the countably infinite set, the rationals $(\mathbb{Q}, <)$, the random graph, and other countable universal homogeneous edge-coloured graphs (with countably many colours) all have generic automorphisms. Furthermore, he gives an explicit description of generic elements in these cases. For example, in the first case an element $g \in \mathrm{Sym}(\mathbb{N})$ is generic if and only if g has cycle type $(\aleph_0)^0\ 1^{\aleph_0}\ 2^{\aleph_0} \ldots n^{\aleph_0} \ldots$. That is, generic elements of $\mathrm{Sym}(\mathbb{N})$ are those which have no infinite cycles and infinitely many finite cycles of length n for every nonzero $n \in \omega$.

Next, we mention some group-theoretic consequences of generic automorphisms.

Proposition 1.4.3. [55, Proposition 4.2.12] *Suppose that a Polish group G has a comeagre conjugacy class D. Then,*

(i) $G = D^2 = \{gh : g, h \in D\}$.

(ii) Every element of G is a commutator, so $G = G'$.

Proposition 1.4.4. [55, Prop. 4.2.12(i)] *Let G be an uncountable Polish group with a comeagre conjugacy class C. Then G has no proper normal subgroup of countable index.*

Proof. Suppose for a contradiction that N is a proper normal subgroup of G of countable index. If $C \cap N \neq \emptyset$, then by normality of N we have $C \subseteq N$. So N is comeagre. Let $g \in G \setminus N$, then gN is a comeagre coset disjoint from N, contradicting that the intersection of two comeagre sets is comeagre. If $C \cap N = \emptyset$, then N is meagre. So is every coset of N. Thus G being a countable union of meagre sets is meagre. This contradicts the Baire Category Theorem as any Polish space is non-meagre. ■

We next present some situations where generic automorphisms do not exist.

Example 1.4.5 (A structure without generic automorphisms)**.** Let $\mathcal{L} = \{E\}$ where E is a binary relation symbol. Let M be a countably infinite $\mathcal{L}$-structure where E is interpreted as an equivalence relation on M with two infinite equivalence classes. Then,

$$H := \{g \in \mathrm{Aut}(M) : g \text{ fixes both classes setwise}\} \lhd \mathrm{Aut}(M)$$

is a proper normal subgroup of $G := \mathrm{Aut}(M)$ of index 2. So by Proposition 1.4.4, the structure M does not have generic automorphisms. Nevertheless, after naming a point

$b \in M$ we get that $G_b = \mathrm{Aut}(M, E, b) \leq G$ is isomorphic to $\mathrm{Sym}(\mathbb{N}) \times \mathrm{Sym}(\mathbb{N})$, and the new structure has generic automorphisms.

Example 1.4.6 (Example 5.6 in [78]. Another structure without generic automorphisms)**.** Let $N := (\mathbb{Q}, \mathrm{Cyc})$ where Cyc is a ternary relation symbol interpreted as the cyclic order on the rationals. That is, for all $x, y, z \in \mathbb{Q}$ we have that $N \models \mathrm{Cyc}(x, y, z)$ if and only if $\mathbb{Q} \models (x \leq y \leq z) \vee (y \leq z \leq x) \vee (z \leq x \leq y)$. Intuitively, for distinct x, y, z, the statement $N \models \mathrm{Cyc}(x, y, z)$ means that you can walk on the line of rationals from left to right, starting at x, passing through y, and finishing at z, with the condition of arriving to each of x, y, z exactly once and with the ability to jump from $+\infty$ back to $-\infty$. See [5, Section 11.3.3] for more details. Now suppose that $G = \mathrm{Aut}(N)$ has a dense conjugacy class D. First, suppose there is $g \in D$ extending a transposition $(r\ s)$ for some distinct $r, s \in \mathbb{Q}$. Then every conjugate of g also extends some transposition. Consequently, no element of D may have a fixed point. Thus, $G_{(1)} \cap D = \emptyset$ contradicting the denseness of D. Otherwise, no element of D extends a transposition. But in this case we have $D \cap \{g \in G : g(1, 2) = (2, 1)\} = \emptyset$, again a contradiction. If we, however, consider an expansion of N by naming some element $a \in N$, then its automorphism group $G_a = \mathrm{Aut}(N, \mathrm{Cyc}, a) \leq G$ is isomorphic to $\mathrm{Aut}(\mathbb{Q}, <)$, and so the expansion has generic automorphisms.

In the previous two examples we had groups without generic automorphisms but contain open subgroups (G_b and G_a) with generic automorphisms. This observation motivates the following definition. For a group G and $g \in G$, the conjugacy class of g is denoted by g^G.

Definition 1.4.7 (Truss [78])**.** Let M be a countably infinite structure and put $G := \mathrm{Aut}(M)$. Then an automorphism $g \in G$ is called *locally generic* if there is some nonempty open subset $U \subseteq G$ such that $g^G \cap U$ is comeagre in U.

Towards a generalisation of the notion of generic automorphisms, recall that for a nonzero natural number n, the action of a group G by *diagonal conjugation* on the product G^n is given by:

$$g \cdot (h_1, \ldots, h_n) = (gh_1g^{-1}, \ldots, gh_ng^{-1}).$$

Moreover, the product of countably many complete metric spaces is a complete metric space [73, Proposition 2.1.31]. Thus, if G is a Polish group, then G^n equipped with the product topology is a Polish space. So comeagre subsets of G^n are dense.

Definition 1.4.8. Let M be a countably infinite $\mathcal{L}$-structure, and put $G = \mathrm{Aut}(M)$. We say that M has *n-generic automorphisms* if G has a comeagre orbit on G^n in its action by diagonal conjugation.

A tuple $\bar{g} \in G^n$ whose diagonal conjugacy class is comeagre is called a *generic element* of G^n. We already know that the rationals $(\mathbb{Q}, <)$ has 1-generic automorphisms. However, by an unpublished note of Hodkinson or by [76, Theorem 2.4], the structure $(\mathbb{Q}, <)$ does *not* have 2-generic automorphisms. The work of Hodges, Hodkinson, Lascar, and Shelah in [39], and its continuation by Kechris and Rosendal in [47] has introduced the following definition. It is the chief definition of the thesis.

Definition 1.4.9. Let M be a countably infinite $\mathcal{L}$-structure, and put $G = \mathrm{Aut}(M)$. We say that M has *ample generics* if M has n-generic automorphisms for each $n > 0$.

The term ample generics is short for *ample homogeneous generic automorphisms*. The notion of ample generics makes sense for Polish groups in general. A Polish group G has ample generics if for each $n > 0$, the diagonal conjugation action of G on G^n has a comeagre orbit. Chapter 2 treats the notion of ample generics in more detail.

1.5 The Extension Property for Partial Automorphisms

In this section we will introduce part of the combinatorial flavour of the subject of homogeneous structures. Let $\mathcal{L}$ be a fixed countable first order language. Suppose that A is an $\mathcal{L}$-structure. A *partial automorphism* of A is an $\mathcal{L}$-isomorphism $p : U \to V$ where U, V are substructures of A. If U, V are finite, we say p is a finite partial automorphism. When $U = A = V$ then $p \in \mathrm{Aut}(A)$, and in this case we may call p a *total automorphism* for emphasis. We denote by $\mathrm{Part}(A)$ the set of all partial automorphisms of A.

In general, a partial automorphism of a *finite* structure may defy being extended to a total automorphism. Gardiner [30] and independently Golfand and Klin [31] proved the classification theorem of finite homogeneous graphs, which states that an exhaustive list of finite homogeneous graphs is the following:

1. The pentagon C_5.
2. The line graph $L(K_{3,3})$ of the complete bipartite graph $K_{3,3}$.
3. Any disjoint union of finitely many copies of the complete graph K_n.
4. Any finite complete multipartite graph with all parts have the same size. This is the complement of (3).

So any finite graph G outside the list above admits a partial automorphism which does not extend to an automorphism of G. But what will happen if we allow to extend partial automorphisms of G to automorphisms of a bigger finite graph which contain G as an induced subgraph? Hrushovski gave a positive answer as we will see shortly.

Definition 1.5.1 (The Extension Property for Partial Automorphisms)**.** A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is said to have the *extension property for partial automorphisms (EPPA)* if for every $A \in \mathcal{C}$, there exists $B \in \mathcal{C}$ containing A as a substructure such that every partial automorphism of A extends to an automorphism of B.

Said differently, $\mathcal{C}$ has EPPA if for every $A \in \mathcal{C}$, there exist $B \in \mathcal{C}$ and a map $\phi : \mathrm{Part}(A) \to \mathrm{Aut}(B)$ such that $A \subseteq B$ and for every $p \in \mathrm{Part}(A)$ we have that $p \subseteq \phi(p)$. Note that we require the extension B to belong to the class $\mathcal{C}$.

Definition 1.5.2. Suppose that A is an $\mathcal{L}$-structure. An extension B of A is called an *EPPA-extension* if every partial automorphism of A extends to an automorphism of B.

Definition 1.5.1 coincides with [40, Definition 8] and [35]. It is a stronger version of the notion of EPPA as defined in [36, p. 1986] which we call here weak EPPA.

Definition 1.5.3. [36, p. 1986] A class $\mathcal{C}$ of finite and infinite $\mathcal{L}$-structures has *weak EPPA* if for all finite $A \in \mathcal{C}$ whenever A has a (possibly infinite) EPPA-extension in $\mathcal{C}$, then A has a finite EPPA-extension also in $\mathcal{C}$.

Note that in the realm of homogeneous structures the two notions are equivalent. More precisely, suppose that $\mathcal{C}$ is an amalgamation class of finite structures, and M is its Fraïssé limit. Then $\mathcal{C}$ has EPPA if and only if $\mathcal{C} \cup \{M\}$ has weak EPPA.

Fact 1.5.4. ([35, Section 6]). Suppose that $\mathcal{C}$ is a class of finite structures with *EPPA* and the joint embedding property. Then $\mathcal{C}$ has the amalgamation property.

It is easy to see that the class of all finite sets (with no structure) has EPPA. In 1992, Hrushovski [41] showed that the class of all finite graphs has EPPA. The *Hrushovski Property* is a synonym for EPPA in literature. In [39], Hrushovski's result was used to establish the small index property (Definition 2.3.1) for the random graph. The proof of Hrushovski has a group-theoretic flavour. More recently, Herwig and Lascar [36, Section 4.1] provided a shorter simple combinatorial proof for Hrushovski's result. We will present their proof here for the amusement of the reader.

First we need some notation. Suppose that E is a finite set and $n \geq 1$. By $\Gamma(E, n)$ we mean the graph whose vertex set is the set of subsets of E containing exactly n elements. The edge relation is defined by setting two distinct vertices adjacent if their intersection is nonempty. The advantage of this construction is that every permutation of E induces an automorphism of $\Gamma(E, n)$, and all vertices have the same degree.

We also need a helpful observation. Every finite graph A can be embedded in a graph of uniform degree. To see this, let $n \geq 3$ be the least odd number greater than or equal to the maximum degree of vertices of A. To every vertex $v \in A$ with degree strictly less than n, add $n - \deg(v)$ new neighbours to v. We may assume that the number of new vertices is even, for if it were odd we may consider one of them to belong to A and add $n - 1$ new neighbours to it. At this point, all the vertices of A have degree n and the new vertices $u_1, u_2, \ldots, u_{2k}$ have degree 1. Take $2k$-many copies of the complete graph K_n. Let $\{e_1^i, e_2^i, \ldots, e_n^i\}$ be an enumeration of the i^{th} copy. Identify each e_1^i with u_i, and for each $1 \leq i \leq k$ and $2 \leq j \leq n$, add an edge between e_j^{2i-1} and e_j^{2i}. Now every vertex has degree n, and we are done.

Theorem 1.5.5 ([36], [41])**.** *The class of all finite graphs has EPPA.*

Proof. Suppose that A is a finite graph. By the above, we may assume that A is of uniform degree $n \geq 3$. Let E be the set of edges of A. Then A embeds in the graph $\Gamma(E, n)$ by sending a vertex v of A to the set of edges incident with v. This map is injective as $n > 1$.

The graph $\Gamma(E, n)$ is an EPPA-extension of A. For suppose that $p \in \mathrm{Part}(A)$. Then p induces a permutation ϕ on E defined in two steps. First, as A is of uniform degree, we may fix for each $u \in \mathrm{dom}(p)$ a bijection j_u from the set of neighbours of u to the set of neighbours of $p(u)$ such that j_u agrees with p. For each $e = \{u, v\}$ in E such that $u \in \mathrm{dom}(p)$ define $\phi(e) = \{p(u), j_u(v)\}$.

Second, complete ϕ to a permutation of E in any fashion. Because for each vertex $u \in \mathrm{dom}(p)$, the permutation ϕ sends the edges incident with u to those incident of $p(u)$, the automorphism of $\Gamma(E, n)$ induced by ϕ extends p. ■

Herwig and Lascar used the idea of the proof above to show the following generalisation.

Theorem 1.5.6. [36, Corollary 4.13] *Let $\mathcal{L}$ be a finite relational language. Then the class of all finite $\mathcal{L}$-structures has EPPA.*

More examples of classes which have EPPA include the class of all K_n-free graphs [35, Theorem 2], where K_n is the complete graph on n vertices; likewise, the class of all finite directed graphs omitting a fixed (possibly infinite) family of tournaments [35, Corollary 13], also see [34]. Such results are extended to any free amalgamation class in Chapter 3.

We now present another EPPA related result in [36]. Let $\mathcal{L}$ be a relational first order language, and let A, B be $\mathcal{L}$-structures. A *homomorphism* $h : A \to B$ is a map such that for every $R \in \mathcal{L}$ and $\bar{a} \in A$, if $A \models R(\bar{a})$, then $B \models R(h(\bar{a}))$. Suppose that $\mathcal{F}$ is a family of $\mathcal{L}$-structures called the forbidden structures. We say a structure A is *$\mathcal{F}$-free under homomorphisms* if there is no structure $F \in \mathcal{F}$ and homomorphism $h : F \to A$.

Theorem 1.5.7. [36, Theorem 3.2] *Let $\mathcal{L}$ be a finite relational language and $\mathcal{F}$ a finite set of finite $\mathcal{L}$-structures. Then the class of all finite and infinite structures which are $\mathcal{F}$-free (under homomorphisms) has weak EPPA.*

Remark 1.5.8. There are other notions of $\mathcal{F}$-freeness. In Chapter 3 we work with classes which are $\mathcal{F}$-free under embeddings, and in Chapter 4 we work with classes of graphs which are $\mathcal{F}$-free under injective homomorphisms (monomorphisms).

Solecki used the theorem above to show that the class of all finite metric spaces has EPPA [70, Theorem 2.1], that is, every finite metric space A embeds in a finite metric space B such that every partial isometry of A extends to an isometry of B. We now introduce a strengthening of EPPA due to Solecki [71].

Definition 1.5.9 (Coherent maps)**.**

- Let X be a set and P be a family of partial bijections between subsets of X. A triple (f, g, h) from P is called a *coherent triple* if $\mathrm{dom}(f) = \mathrm{dom}(h)$, $\mathrm{range}(f) = \mathrm{dom}(g)$, $\mathrm{range}(g) = \mathrm{range}(h)$ and $h = g \circ f$.

- Let X and Y be sets, and P and Q be families of partial bijections between subsets of X and between subsets of Y, respectively. A function $\phi : P \to Q$ is called a *coherent map* if for each coherent triple (f, g, h) from P, its image $\big(\phi(f), \phi(g), \phi(h)\big)$ in Q is coherent.

Definition 1.5.10 (Coherent EPPA)**.** A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is said to have *coherent EPPA* if for every $A \in \mathcal{C}$, there exist $B \in \mathcal{C}$ and a coherent map $\phi : \mathrm{Part(A)} \to \mathrm{Aut}(B)$ such that $A \subseteq B$ and every $p \in \mathrm{Part(A)}$ extends to $\phi(p) \in \mathrm{Aut}(B)$.

That is, the map ϕ in the definition above satisfies the following property: for all $f, g, h \in \mathrm{Part}(A)$ if $h = gh$, then $\phi(h) = \phi(g)\phi(h)$. Thus the slogan for coherent EPPA is *"the extension of the composition is the composition of the extensions"*. Solecki has strengthened a theorem by Herwig-Lascar to the following theorem, which we will utilise in Chapter 3.

Theorem 1.5.11. [71, Theorem 3.1] *Let $\mathcal{L}$ be a finite relational language. The class of all finite $\mathcal{L}$-structures has coherent EPPA.*

We motivate the notion of EPPA by describing the role it plays in several research areas. The property EPPA constitutes a main ingredient of a technique developed in [39] which is used in the thesis to establish the existence of ample generics for some homogeneous structures. This technique and many interesting group-theoretic consequences which follow from ample generics are covered in Chapter 2.

Herwig and Lascar [36] revealed interesting relations between EPPA and results related to the profinite topology on free groups. The *profinite topology* on a group is the topology whose basic open subsets are cosets of subgroups of finite index. Ribes and Zalesskii generalised a classical result due to M. Hall [32, Theorem 3.4] to the following.

Theorem 1.5.12 (Ribes-Zalesskii [65])**.** *Let F_n be the free group on n generators. Suppose that $H_1, H_2, \ldots, H_k$ are finitely generated subgroups of F_n. Then their product $H_1 H_2 \ldots H_k = \{h_1 h_2 \ldots h_k : h_i \in H_i\}$ is closed under the profinite topology.*

Herwig and Lascar showed in [36, Section 2.1] that the Ribes-Zalesskii theorem implies that the class of finite graphs has EPPA [41]. In the converse direction, they showed in [36, Section 3.1] that EPPA for the class of n-partitioned cycle-free graphs implies the Ribes-Zalesskii theorem. Furthermore, using Theorem 1.5.7 above they proved a property [36, Theorem 3.3] of the profinite topology generalising the Ribes-Zalesskii theorem. Such property was of interest to Almeida and Delgado—see [1].

On another route, Hodkinson and Otto [40] provided, via EPPA, a new simple proof of the 'finite model property' for the 'clique guarded fragment' of first order logic.

EPPA has more implications for the automorphism group of a homogeneous structure. If the age of a homogeneous structure has coherent EPPA, then its automorphism group contains a dense locally finite subgroup—see Chapter 3. Bhattacharjee and Macpherson [4] used coherent EPPA for the class of finite graphs to show that the automorphism group of the random graph contains a dense locally finite subgroup. Vershik [79] asks whether the automorphism group of the rational Urysohn metric space has a dense locally finite subgroup. Solecki in an unpublished note answered Vershik's question positively.

Conversely, if M is a locally finite homogeneous structure, and $\mathrm{Aut}(M)$ contains a dense locally finite subgroup, then $\mathrm{Age}(M)$ has EPPA—see Proposition 3.3.5 and [47, Proposition 6.4]. Before finishing, we give a situation where EPPA fails to hold.

Definition 1.5.13. [38, p. 285] A complete theory T has the *strict order property (SOP)* if there is a formula $\varphi(\bar{x}, \bar{y})$, where both $\bar{x}, \bar{y}$ are tuples of length n of free variables, such that for any model $M \models T$, the formula φ defines a partial order on M^n which contains arbitrarily long finite chains.

For example $\mathrm{Th}(\mathbb{Q}, <)$ has SOP witnessed by the formula $x < y$.

Proposition 1.5.14. *Suppose that M is a homogeneous ω-categorical structure such that $\mathrm{Th}(M)$ has SOP. Then $\mathrm{Age}(M)$ does not have EPPA.*

Proof. By SOP there is a definable (irreflexive) partial order $<$ on M^n with arbitrarily long finite chains. By ω-categoricity there are finitely many n-types of $\mathrm{Th}(M)$, say k many. We can find a chain $\bar{a}_1 < \bar{a}_2 < \ldots < \bar{a}_k < \bar{a}_{k+1}$ in M^n. By the pigeonhole principle, there are i, j such that $1 \leq i < j \leq k+1$ and $\mathrm{tp}(\bar{a}_i) = \mathrm{tp}(\bar{a}_j)$, and so the map $p : \bar{a}_i \to \bar{a}_j$ is a partial automorphism. For the sake of contradiction, suppose that $\mathrm{Age}(M)$ has EPPA, and consider $A = (\bar{a}_i \cup \bar{a}_j)$ which belongs to $\mathrm{Age}(M)$. By EPPA there is finite $B \in \mathrm{Age}(M)$ containing A such that p extends to some $f \in \mathrm{Aut}(B)$. By homogeneity, we may assume that $A \subseteq B \subseteq M$. But now we have infinitely many distinct elements $\bar{a}_i < \bar{a}_j < f(\bar{a}_j) < f^2(\bar{a}_j) < \ldots$ all in B, contradicting its finiteness. ∎

1.6 Main Results

In Chapter 3 we focus on free amalgamation classes. We use the work of Hodkinson and Otto on Gaifman clique faithful EPPA (Definition 3.1.3), and Solecki's strengthening (Theorem 1.5.11) of a theorem by Herwig and Lascar, to show the following result for free homogeneous structures.

Theorem. *Suppose that $\mathcal{C}$ is a free amalgamation class over a finite relational language. Let M be the Fraïssé limit of $\mathcal{C}$. Then $\mathcal{C}$ has Gaifman clique faithful coherent EPPA. Consequently,* $\mathrm{Aut}(M)$ *contains a dense locally finite subgroup, and has ample generics and the small index property.*

The proof of the theorem above in fact shows that if M is a homogeneous structure such that $\mathrm{Age}(M)$ has coherent EPPA, then $\mathrm{Aut}(M)$ contains a dense locally finite subgroup.

In Chapter 4 we work with the class of finite bowtie-free graphs. It is not an amalgamation class, however, it contains a cofinal subclass which has the free amalgamation property. Accordingly, we use a variant of the Fraïssé construction to construct a countably infinite universal bowtie-free graph isomorphic to the one studied by Cherlin-Shelah-Shi [15]. Using an argument of Ivanov, we show that working in this cofinal subclass we can extend a single partial automorphism to a total automorphism. We prove that this universal bowtie-free graph is not finitely homogenisable. Furthermore, we show the following.

Theorem. *The universal bowtie-free graph admits generic automorphisms.*

Results of Philip Hall show that the class of finite groups has EPPA. In Chapter 5 we strengthen this by showing that the class of finite groups has coherent EPPA. Hall also showed that there is a countably infinite homogeneous locally finite group embedding every finite group. We proceed in the chapter by showing that the class of finite groups has the amalgamation property with automorphisms (Definition 2.1.1), where the main tool used is the amalgamated free product of groups. This yields the following.

Theorem. *Philip Hall's universal locally finite group admits ample generics.*

In Chapter 6 we use the weak amalgamation property (Definition 2.2.4) as a tool to study classes of finite structures equipped with partial automorphisms. In particular, we consider linear orders, ordered graphs, and tournaments. We give another proof of Hodkinson's result that the countable dense linear ordering does not have 2-generic automorphisms. Furthermore we show the following.

Theorem. *The universal homogeneous ordered graph does not have locally generic automorphisms.*

Theorem. *The universal homogeneous tournament has ample generics if and only if the class of finite tournaments has EPPA.*

Chapter 2

Ample Generics

This chapter examines when ample generics exist in the context of homogeneous structures. In the first section, we present a technique to establish the existence of ample generics for homogeneous structures due to Hodges-Hodkinson-Lascar-Shelah [39]. In the second section, we give the Kechris-Rosendal characterisation [47] of the existence of ample generics. Finally, in the third section we discuss several group-theoretic consequences of ample generics. Recall that a Polish group G has ample generics if for each $n > 0$, the diagonal conjugation action of G has a comeagre orbit on G^n. A countably infinite structure M has ample generics if $\mathrm{Aut}(M)$ has ample generics—see Definition 1.4.9.

2.1 Existence of Ample Generics

Hodges, Hodkinson, Lascar, and Sheleh showed in [39] that the automorphism group of the random graph and the automorphism group of any ω-stable ω-categorical structure have the small index property (Definition 2.3.1). They provide a criterion for the existence of ample generics, and use ample generics to establish the small index property. We present here the details of their technique in the context of homogeneous structures. The approach is based on EPPA and the following strengthening of the amalgamation property.

Definition 2.1.1. We say that a class $\mathcal{C}$ of finite structures has the *amalgamation property with automorphisms (APA)* if whenever $A, B_1, B_2 \in \mathcal{C}$ with embeddings $\alpha_1 : A \to B_1$ and $\alpha_2 : A \to B_2$, then there is a structure $C \in \mathcal{C}$ with embeddings $\beta_1 : B_1 \to C$ and $\beta_2 : B_2 \to C$ such that $\beta_1\alpha_1 = \beta_2\alpha_2$ and whenever $f \in \mathrm{Aut}(B_1)$ and $g \in \mathrm{Aut}(B_2)$ such that $f\alpha_1(A) = \alpha_1(A)$, $g\alpha_2(A) = \alpha_2(A)$, and for every $a \in A$ we have that $\alpha_1^{-1} f\alpha_1(a) = \alpha_2^{-1} g\alpha_2(a)$, then there exists $h \in \mathrm{Aut}(C)$ which extends $\beta_1 f\beta_1^{-1} \cup \beta_2 g\beta_2^{-1}$.

The definition above stems from [39, Definition 2.8]. The amalgamation property with automorphisms says that the structure C above is not only amalgamating structures B_1 and B_2 over A, but also has the advantage that any automorphism of B_1 and any automorphism of B_2 which agree on A extend simultaneously to an automorphism of C. The aim is to show that if the age of a homogeneous structure M has EPPA and APA, then M has ample generics. Towards that end, we need a technical definition which determines the desired diagonal conjugacy class as in the definition of ample generics. We will show that the set Γ in the definition below is what we are looking for.

Definition 2.1.2. Let M be a homogeneous structure. Put $G = \mathrm{Aut}(M)$ and fix some nonzero $n \in \omega$.

- A tuple $(g_1, \ldots, g_n) \in G^n$ satisfies *condition* (I) if for all $a \in M$, the orbit of a under the group $\langle g_1, \ldots, g_n \rangle$ is finite.
- A tuple $(g_1, \ldots, g_n) \in G^n$ satisfies *condition* (II) if whenever $A \subseteq B \subseteq M$ are both finite, and $f_1, \ldots, f_n \in \mathrm{Aut}(B)$ are such that $f_i(A) = A$ and $f_i{\restriction}_A = g_i{\restriction}_A$ for all i, then there is $\widetilde{B} \subseteq M$ with $A \subseteq \widetilde{B}$, and $g_i(\widetilde{B}) = \widetilde{B}$, and an isomorphism $\alpha : \widetilde{B} \to B$ such that $\alpha(a) = a$ for each $a \in A$, and $\alpha\, g_i{\restriction}_{\widetilde{B}}\, \alpha^{-1} = f_i$ on B for all i.
- Define $\Gamma_{\mathrm{I}} = \big\{\bar{g} \in G^n : \bar{g} \text{ satisfies condition (I)}\big\}$.
- Define $\Gamma_{\mathrm{II}} = \big\{\bar{g} \in G^n : \bar{g} \text{ satisfies condition (II)}\big\}$.
- Define $\Gamma = \big\{\bar{g} \in G^n : \bar{g} \text{ satisfies conditions (I) and (II)}\big\}$.

See [39, Definition 2.2] for conditions (I) and (II).

The proof of the following lemma originates from the proof of [35, Proposition 7].

Lemma 2.1.3. *Suppose that M is a homogeneous structure such that* $\operatorname{Age}(M)$ *has EPPA. Let $G = \operatorname{Aut}(M)$. Then the subset $\Gamma_{\mathrm{I}} \subseteq G^n$ is comeagre for all nonzero $n \in \omega$.*

Proof. We assume M is a relational structure. Fix a nonzero $n \in \omega$. For $a \in M$, define

$$\Gamma_a = \{(g_1, \ldots, g_n) \in G^n : a^{\langle g_1, \ldots, g_n \rangle} \text{ is finite}\}.$$

We will show that Γ_a is both open and dense. First, we show that it is open, so let $(g_1, \ldots, g_n) \in \Gamma_a$, and put $H = \langle g_1, \ldots, g_n \rangle$. Consider the finite subset $A = a^H \subseteq M$, and observe that $(g_1, \ldots, g_n) \in g_1 G_A \times \ldots \times g_n G_A \subseteq \Gamma_a$. Therefore, Γ_a is open.

Now to show that Γ_a is dense, take any basic open set, say $\Delta = h_1 G_{A_1} \times \ldots \times h_n G_{A_n}$, where $h_i \in G$ and $A_i \subseteq M$ finite for all $1 \leq i \leq n$. Let $p_1 := h_1 \restriction_{A_1}, \ldots, p_n := h_n \restriction_{A_n}$ be finite partial automorphisms on M. Define the finite substructure $A = \{a\} \cup \left(\bigcup_{i=1}^n A_i\right) \cup \left(\bigcup_{i=1}^n p_i(A_i)\right)$. Using EPPA, we obtain a finite structure B such that $A \subseteq B$ and every p_i extends to an automorphism $\hat{p}_i$ of B. By homogeneity of M, we can find an isomorphic copy $\widetilde{B}$ of B in M such that $A \subseteq \widetilde{B}$, and every $\hat{p}_i$ extends to an automorphism $g_i \in G$. As $a \in \widetilde{B}$, we have that $a^{\langle g_1, \ldots, g_n \rangle}$ is finite and so $(g_1, \ldots, g_n) \in \Gamma_a \cap \Delta$. Therefore, $\Gamma_{\mathrm{I}} = \bigcap_{a \in M} \Gamma_a$ is a comeagre set. ■

Lemma 2.1.4. *Suppose that M is a homogeneous structure such that* $\operatorname{Age}(M)$ *has EPPA and APA. Let $G = \operatorname{Aut}(M)$. Then the subset $\Gamma_{\mathrm{II}} \subseteq G^n$ is comeagre for all nonzero $n \in \omega$.*

Proof. We assume M is a relational structure. Let $n \in \omega$ be nonzero. Fix finite $A \subseteq B \subseteq M$, and $\bar{f} = (f_1, \ldots, f_n)$ where $f_i \in \operatorname{Aut}(B)$ with $f_i(A) = A$. Define the subset $\Gamma_A^B(\bar{f}) \subseteq G^n$ as follows:

A tuple $(g_1, \ldots, g_n) \in \Gamma_A^B(\bar{f})$ if and only if whenever $g_i \restriction_A = f_i \restriction_A$ for all i, then there exists $\widetilde{B} \subseteq M$ with $A \subseteq \widetilde{B}$ and an isomorphism $\alpha : \widetilde{B} \to B$ fixing A pointwise such that $\alpha g_i \alpha^{-1} = f_i$ for all i.

We will show that $\Gamma_A^B(\bar{f})$ is both dense and open. Take an element $(g_1, \ldots, g_n) \in \Gamma_A^B(\bar{f})$. If $g_i \restriction_A = f_i \restriction_A$ then there is $\widetilde{B}$ as above, otherwise take $\widetilde{B} = B$, and observe that $(g_1, \ldots, g_n) \in g_1 G_{\widetilde{B}} \times \ldots \times g_n G_{\widetilde{B}} \subseteq \Gamma_A^B(\bar{f})$. So $\Gamma_A^B(\bar{f})$ is an open set.

For density, let Δ be any basic nonempty open set. By applying EPPA in a similar fashion as in the previous lemma, we may assume that $\Delta = h_1 G_C \times \ldots \times h_n G_C$ where $h_i \in G$ and $C \subseteq M$ is some finite substructure containing A such that $p_i := h_i \restriction_C \in \operatorname{Aut}(C)$.

Suppose that there is some i such that $1 \leq i \leq n$ and f_i does *not* agree with p_i on A. Then any extensions $(g_1, \ldots, g_n) \in G^n$ of $(p_1, \ldots, p_n)$ will be in $\Delta \cap \Gamma_A^B(\bar{f})$. Otherwise, $f_i \restriction_A = p_i \restriction_A$. Now, by APA there is an amalgam D of B and C over A such that for all i there is $\delta_i \in \operatorname{Aut}(D)$ with $f_i \cup p_i \subseteq \delta_i$. By homogeneity we may assume that $C \subseteq D \subseteq M$, and here D contains a copy of B.

For each $1 \leq i \leq n$, let $g_i \in \operatorname{Aut}(M)$ be an extension of δ_i. Then $(g_1, \ldots, g_n) \in G^n$ belongs to $\Delta \cap \Gamma_A^B(\bar{f})$. Thus, $\Gamma_A^B(\bar{f})$ meets every nonempty open set. Thus

$$\Gamma_{\mathrm{II}} = \bigcap \left\{ \Gamma_A^B(\bar{f}) : A \subseteq B \subseteq M \text{ finite}, f_i \in \operatorname{Aut}(B), f_i(A) = A \right\}$$

is comeagre. ■

Theorem 2.1.5. *Suppose that M is a homogeneous structure such that $\operatorname{Age}(M)$ has both EPPA and APA. Then M has ample generics.*

Proof. Let $G = \operatorname{Aut}(M)$. We will show that for every nonzero $n \in \omega$, the subset $\Gamma \subseteq G^n$ is comeagre and contained in a single diagonal conjugacy class. By the previous two lemmas we have that $\Gamma = \Gamma_{\mathrm{I}} \cap \Gamma_{\mathrm{II}} \subseteq G^n$ is comeagre for all nonzero $n \in \omega$. It remains to show that $\Gamma \subseteq G^n$ is contained in a single orbit of the action of G by diagonal conjugation on G^n. Fix a nonzero $n \in \omega$, and take any two tuples $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ in $\Gamma \subseteq G^n$. That is, both tuples satisfy conditions (I) and (II). We will show they are conjugate by a back-and-forth argument. We will build a chain $\alpha_0 \subseteq \alpha_1 \subseteq \alpha_2 \subseteq \ldots$ of finite partial isomorphisms of M, where $\alpha_k : \widetilde{B}_k \to B_k$, such that $\alpha_k \circ g_i \restriction_{\widetilde{B}_k} \circ \alpha_k^{-1} = f_i \restriction_{B_k}$ for all $1 \leq i \leq n$ and $M = \bigcup_{k \in \omega} \widetilde{B}_k = \bigcup_{k \in \omega} B_k$.

Fix an enumeration $\{a_0, a_1, a_2, \ldots\}$ of the domain of M. Start with $B_0 = a_0^{\langle f_1, \ldots, f_n \rangle}$ which is finite by condition (I). Applying condition (II) for $(g_1, \ldots, g_n)$ with $A = \emptyset$ and B_0, we obtain $\widetilde{B}_0 \subseteq M$ with an isomorphism $\alpha_0 : \widetilde{B}_0 \to B_0$ such that $\alpha_0 \circ g_i \restriction_{\widetilde{B}_0} \circ \alpha_0^{-1} = f_i \restriction_{B_0}$ for all i.

Next, let $m \in \omega$ be the least such that $a_m \notin B_0$, and let $B_1 = B_0 \cup a_m^{\langle f_1,\ldots,f_n\rangle}$. By condition (I), B_1 is finite. By homogeneity, α_0 extends to some $\hat{\alpha}_0 \in \mathrm{Aut}(M)$, and so there is a copy $C_1 = \hat{\alpha}_0^{-1}(B_1)$ of B_1 such that $\widetilde{B}_0 \subseteq C_1$. Applying condition (II) for $(g_1,\ldots,g_n)$, $\widetilde{B}_0 \subseteq C_1$, and $f_i' := (\hat{\alpha}_0^{-1} \circ f_i \circ \hat{\alpha}_0)\restriction_{C_1} \in \mathrm{Aut}(C_1)$, we obtain a substructure $\widetilde{B}_1 \subseteq M$ with $\widetilde{B}_0 \subseteq \widetilde{B}_1$ and an isomorphism $\beta : \widetilde{B}_1 \to C_1$ such that $\beta(b) = b$ for any $b \in \widetilde{B}_0$ and $\beta \circ g_i\restriction_{\widetilde{B}_1} \circ \beta^{-1} = f_i'$ for all i. Now, define $\alpha_1 : \widetilde{B}_1 \to B_1$ to be $\alpha_1 = \hat{\alpha}_0 \circ \beta$. It remains to check that α_1 works as desired: we have that $\alpha_1 \circ g_i\restriction_{\widetilde{B}_1} \circ \alpha_1^{-1} = (\hat{\alpha}_0 \circ \beta) \circ g_i\restriction_{\widetilde{B}_1} \circ (\beta^{-1} \circ \hat{\alpha}_0^{-1}) = \hat{\alpha}_0 \circ f_i' \circ \hat{\alpha}_0^{-1} = f_i\restriction_{B_1}$ for all i, and $\alpha_0 \subseteq \alpha_1$ as well.

Next, let $m \in \omega$ be the least such that $a_m \notin \widetilde{B}_1$, and let $\widetilde{B}_2 = \widetilde{B}_1 \cup a_m^{\langle g_1,\ldots,g_n\rangle}$. By condition (I), we have that $\widetilde{B}_2$ is finite. Using condition (II) for $(f_1,\ldots,f_n)$ in a similar fashion as in the previous step we obtain a finite structure B_2 containing B_1 and an isomorphism $\alpha_2 : \widetilde{B}_2 \to B_2$, such that $\alpha_1 \subseteq \alpha_2$, and $\alpha_2 \circ g_i\restriction_{\widetilde{B}_2} \circ \alpha_2^{-1} = f_i\restriction_{B_2}$ for all i.

Continuing in this pattern, by adding new points to B_k when k is odd, and to $\widetilde{B}_k$ when k is even, we will build an automorphism $\alpha \in \mathrm{Aut}(M)$ where $\alpha = \bigcup_{i\in\omega} \alpha_i$ such that $(\alpha g_1 \alpha^{-1},\ldots,\alpha g_n \alpha^{-1}) = (f_1,\ldots,f_n)$. Therefore, the tuples $(f_1,\ldots,f_n)$ and $(g_1,\ldots,g_n)$ are conjugate. Therefore, G^n contains a comeagre diagonal conjugacy class, establishing that the structure M has ample generics. ■

2.2 Characterisation of Ample Generics

Kechris and Rosendal characterised the existence of ample generics for homogeneous structures in terms of JEP and the 'weak amalgamation property' for classes of finite structures equipped with partial automorphisms. Here we present these properties and some of the main results in [47]. We also explain their connection with EPPA and APA.

Definition 2.2.1. Suppose that $\mathcal{L}$ is a countable first order language, and $\mathcal{C}$ is an amalgamation class of finite $\mathcal{L}$-structures. Let $n \in \omega$. An *n-system over $\mathcal{C}$* is a tuple

$\langle A, p_1, \ldots, p_n \rangle$ where $A \in \mathcal{C}$ and each p_i is a partial automorphism of A. Denote by $\mathcal{C}^n$ the class of all n-systems over $\mathcal{C}$.

There is a natural notion of embedding between elements of $\mathcal{C}^n$.

Definition 2.2.2. Suppose that $S = \langle A, p_1, \ldots, p_n \rangle$ and $T = \langle B, f_1, \ldots, f_n \rangle$ are n-systems over $\mathcal{C}$. An *embedding* from S to T is an $\mathcal{L}$-embedding $\phi : A \to B$ such that for all $1 \leq i \leq n$ we have that $\phi(\mathrm{dom}(p_i)) \subseteq \mathrm{dom}(f_i)$, and $\phi(\mathrm{range}(p_i)) \subseteq \mathrm{range}(f_i)$, and $\phi \circ p_i \subseteq f_i \circ \phi$.

We say that $\mathcal{C}^n$ has the *joint embedding property* if whenever $S_1, S_2 \in \mathcal{C}^n$ there is $T \in \mathcal{C}^n$ which embeds both S_1 and S_2. Similarly, we may define the *amalgamation property* for $\mathcal{C}^n$. The joint embedding property characterises when the automorphism group of a homogeneous structure admits a dense diagonal conjugacy class.

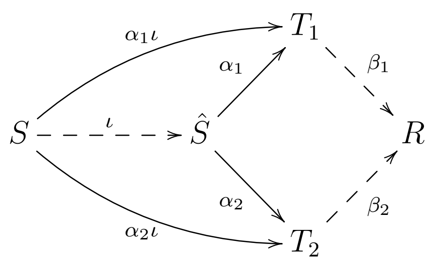
Theorem 2.2.3. [47, Theorem 2.11] *Let $\mathcal{C}$ be an amalgamation class and M be its Fraïssé limit. Put $G = \mathrm{Aut}(M)$. Then the following are equivalent:*

(i) G^n admits a dense diagonal conjugacy class.

(ii) $\mathcal{C}^n$ has the joint embedding property.

Suppose that $\mathcal{C}$ is an amalgamation class and M is its Fraïssé limit. Truss [78, Theorem 2.1] showed that if $\mathcal{C}^1$ has the joint embedding property and contains a cofinal subclass which has the amalgamation property, then M has generic automorphisms. In the converse direction, he showed [78, Theorem 2.2] that if M has generic automorphisms, then $\mathcal{C}^1$ has the joint embedding property. He then asked whether the existence of generic automorphisms is equivalent to some condition in terms of the joint embedding property and the amalgamation property of $\mathcal{C}^1$. Kechris and Rosendal answered the question by discovering a weaker version of the amalgamation property, using which they characterised when comeagre diagonal conjugacy classes exist. We also mention that Ivanov also proved a theorem [43, Theorem 1.2] in the direction of answering Truss' question in the context of ω-categorical structures where he used the term 'almost amalgamation property' for the weak amalgamation property.

Definition 2.2.4. We say that the class of n-systems $\mathcal{C}^n$ has the *weak amalgamation property* if for every system $S \in \mathcal{C}^n$ there is some $\hat{S} \in \mathcal{C}^n$ and embedding $\iota : S \to \hat{S}$ such that whenever $T_1, T_2 \in \mathcal{C}^n$ with embeddings $\alpha_1 : \hat{S} \to T_1$ and $\alpha_2 : \hat{S} \to T_2$, there is $R \in \mathcal{C}^n$ with embeddings $\beta_1 : T_1 \to R$ and $\beta_2 : T_2 \to R$ such that $\beta_1\alpha_1\iota = \beta_2\alpha_2\iota$.



The weak amalgamation property as demonstrated above states that R amalgamates T_1 and T_2 over S, rather than $\hat{S}$. Clearly, the amalgamation property implies the weak amalgamation property. Because once we have the amalgamation property, then in the definition above, for any $S \in \mathcal{C}^n$ we can take $\hat{S} = S$ and take R to be the amalgam of T_1 and T_2 over S.

Theorem 2.2.5. [47, Theorem 6.2] *Let $\mathcal{C}$ be an amalgamation class of finite structures and M be its Fraïssé limit. Then the following are equivalent:*

(i) M admits n-generic automorphisms.

(ii) $\mathcal{C}^n$ has the joint embedding property and the weak amalgamation property.

There is a distinguished subclass of the class of n-systems over $\mathcal{C}$. An n-system $\langle A, f_1, \ldots, f_n\rangle$ is called *complete* if each $f_i \in \mathrm{Aut}(A)$. Let $\bar{\mathcal{C}}^n$ denotes the class of all complete n-systems.

Lemma 2.2.6. *Suppose that $\mathcal{C}$ is an amalgamation class. Then,*

(i) The class $\mathcal{C}$ has APA if and only if the class $\bar{\mathcal{C}}^n$ of complete n-systems has the amalgamation property for all $n \in \omega$.

(ii) If $\mathcal{C}$ has EPPA and APA, then $\mathcal{C}^n$ has the weak amalgamation property for all $n \in \omega$.

Proof. **(i)** The forward direction is clear. For the reverse direction, assume that the class $\bar{\mathcal{C}}^n$ of complete n-systems has the amalgamation property for all $n \in \omega$. For simplicity we will assume that the embeddings are the inclusion maps, so suppose that $A, B_1, B_2 \in \mathcal{C}$ with $A \subseteq B_1$ and $A \subseteq B_2$. List all triples $\left(p_i, f_i, g_i\right)_{i=1}^{k}$ such that $p_i \in \mathrm{Aut}(A)$, $f_i \in \mathrm{Aut}(B_1)$, $g_i \in \mathrm{Aut}(B_2)$, and $f_i{\restriction}_A = p_i = g_i{\restriction}_A$. Now, $S = \langle A, p_1, \ldots, p_k \rangle$, $T_1 = \langle B_1, f_1, \ldots, f_k \rangle$, and $T_2 = \langle B_2, g_1, \ldots, g_k \rangle$ are complete k-systems such that S embeds in both T_1 and T_2. By assumption, the class $\bar{\mathcal{C}}^k$ has the amalgamation property. Thus, there is a complete k-system $\langle C, h_1, \ldots, h_k \rangle$ such that $B_1 \subseteq C$, $B_2 \subseteq C$, $A \subseteq B_1 \cap B_2 \subseteq C$, and $f_i \cup g_i \subseteq h_i$ for all $1 \leq i \leq k$. As each $h_i \in \mathrm{Aut}(C)$, this shows that the class $\mathcal{C}$ has APA.

(ii) Fix $n \in \omega$. By EPPA, the subclass $\bar{\mathcal{C}}^n$ of complete n-systems is cofinal (with respect to embeddings of n-systems) in the class $\mathcal{C}^n$. Let $S \in \mathcal{C}^n$. By cofinality of $\bar{\mathcal{C}}^n$, there is a complete n-system and embedding $\iota : S \to \hat{S}$. Suppose that $T_1, T_2 \in \mathcal{C}^n$ with embeddings $\alpha_1 : \hat{S} \to T_1$ and $\alpha_2 : \hat{S} \to T_2$. By cofinality again, there are complete n-systems $\hat{T}_1$ and $\hat{T}_2$ with embeddings $\iota_1 : T_1 \to \hat{T}_1$ and $\iota_2 : T_2 \to \hat{T}_2$. By the above and APA, we have that $\bar{\mathcal{C}}^n$ has the amalgamation property. Thus, there is $R \in \mathcal{C}^n$ with embeddings $\beta_1 : \hat{T}_1 \to R$ and $\beta_2 : \hat{T}_2 \to R$ such that $\beta_1(\iota_1\alpha_1) = \beta_2(\iota_2\alpha_2)$. Therefore, the class $\mathcal{C}^n$ has the weak amalgamation property. ∎

Kechris-Rosendal defined an automorphism $g \in \mathrm{Aut}(M)$ to be locally generic if its conjugacy class is non-meagre. We note below that this condition is equivalent to the one defined by Truss (Definition 1.4.7).

Fact 2.2.7. *Let H be a Polish group, and let $g \in H$. Then the following are equivalent:*

(i) The conjugacy class g^H is non-meagre in H.

(ii) There is a nonempty open subset $U \subseteq H$ such that $g^H \cap U$ is comeagre in U.

Proof. Suppose that g^H is non-meagre in H. Recall that a subset of a topological space is G_δ if it is a countable intersection of open sets. We invoke a result of Effros-Marker-Sami (see [2]) which states that any non-meagre orbit of a Polish group acting continuously on a Polish space is in fact G_δ. Thus g^H is a G_δ set, say $g^H = \bigcap_{n\in\omega} V_n$ where each V_n is open. As g^H is non-meagre, it is not nowhere dense, and so it is dense in some nonempty open $U \subseteq H$. Therefore, $g^H \cap U = \bigcap_{n\in\omega} (V_n \cap U)$ being dense G_δ in U is comeagre in U.

Conversely, if g^H is meagre in H. Then $g^H = \bigcup_{n\in\omega} A_n$ where each A_n is nowhere dense. Let $U \subseteq H$ be a nonempty open subset, then $A_n \cap U$ is nowhere dense in U for each $n \in \omega$, and so $g^H \cap U = \bigcup_{n\in\omega} (A_n \cap U)$ is meagre in U. ■

Towards a characterisation of the existence of locally generic automorphisms we have the following notion.

Definition 2.2.8 ([47])**.** Let $\mathcal{C}$ be an amalgamation class. We say that $\mathcal{C}^1$ satisfies the *local weak amalgamation property* if there is a 1-system $S \in \mathcal{C}^1$ such that the weak amalgamation property holds for the subclass of all $T \in \mathcal{C}^1$ into which S embeds.

Theorem 2.2.9. [47, Theorem 3.7] *Let $\mathcal{C}$ be an amalgamation class and M be its Fraïssé limit. Put $G = \mathrm{Aut}(M)$. Then the following are equivalent:*

(i) M has a locally generic automorphism.

(ii) $\mathcal{C}^1$ satisfies the local weak amalgamation property.

Hodges-Hodkinson-Lascar-Shelah [39] is a core motivating paper for the thesis. To state one of their results we need the following definition.

Definition 2.2.10. Let T be a complete theory in a countable language. We say that T is ω-*stable* if for all $M \models T$, and $A \subseteq M$ with $|A| = \aleph_0$, we have that $|S_n^M(A)| = \aleph_0$.

Example 2.2.11 (Corollary of [39])**.** Suppose that M is a countably infinite ω-stable ω-categorical structure. Then $\mathrm{Aut}(M)$ has an open subgroup with ample generics.

Example 2.2.12. The following structures have ample generics.

- The random graph [39], [41].
- The homogeneous K_n-free graph [35].
- The rational Urysohn metric space [70].

- Free homogeneous structures over finite relational languages [Chapter 3].
- Philip Hall's locally finite universal group [Chapter 5].

2.3 Consequences of the Existence of Ample Generics

The main references are Macpherson [55, Sections 5.2, 5.5] and Kechris-Rosendal [47].

We motivate the study of ample generics by presenting a number of group-theoretic consequences of their existence. Ample generics are considered a powerful tool to establish the small index property, uncountable cofinality, the Bergman property, Serre's property (FA), and automatic continuity. We begin our discussion with the first one. Suppose that G is a Polish group, and $H \leq G$ is an open subgroup. Then H has a countable index. For if the index of H were uncountable, then G is the union of uncountably many pairwise disjoint nonempty open sets. So any dense subset of G is uncountable, but this contradicts that G is separable. The converse statement deserves to be a definition in its own right.

Definition 2.3.1. A Polish group has the *small index property (SIP)* if every subgroup of small index ($< 2^{\aleph_0}$) is open. A countably infinite structure has SIP if its automorphism group has SIP.

Thus if M is a countably infinite structure with the small index property, then $H \leq G = \mathrm{Aut}(M)$ is open if and only if H has small index. As open sets of G are unions of cosets of basic open subgroups, we have in this situation that the group-theoretic structure of G determines the topology on G.

Example 2.3.2. Countably infinite structures with the small index property.

- The trivial structure $(\mathbb{N}, =)$ [68], [20].
- The countable dense linear ordering $(\mathbb{Q}, <)$ [77].
- Any countable 2-homogeneous tree [22].

- Any ω-categorical abelian group [27].
- Any countable ω-stable ω-categorical structure [39].
- The random graph [39].

The small index property was one of the first motives behind developing the technique of ample generics as is apparent from the title of [39]. By the work of Hodges, Hodkinson, Lascar, and Shelah in [39, Theorem 5.3], and Kechris and Rosendal in [47] we have the following result.

Theorem 2.3.3. [47, Theorem 6.9] *Suppose that G is a Polish group with ample generics. Then G has the small index property.*

The converse of the above theorem is not always true as shown by the following example.

Example 2.3.4. We have seen earlier that $(\mathbb{Q}, <)$ does not have 2-generic automorphisms, and so does not have ample generics. Nevertheless, $\mathrm{Aut}(\mathbb{Q}, <)$ has the small index property as shown by Truss [77].

Example 2.3.5. [55, p. 1613, 1616] We give an example by Cherlin and Hrushovski of an ω-categorical structure without the small index property, and hence without ample generics. Consider the infinite first order language $\mathcal{L} = \{E_n : n \in \omega\}$, where E_n is a $2n$-ary relation symbol. Let $\mathcal{C}$ be the class of all finite $\mathcal{L}$-structures in which each E_n is interpreted as an equivalence relation on n-tuples of distinct elements, and having at most two equivalence classes. One can check that $\mathcal{C}$ is an amalgamation class. Let M be its Fraïssé limit. The $\mathcal{L}$-structure M is ω-categorical, for M is homogeneous and for every $n \in \omega$ there are only finitely many isomorphism types of substructures of M of size n. Let $G = \mathrm{Aut}(M)$, and $F \leq G$ be the subgroup of all automorphisms of M which fix setwise each equivalence class of E_n for each $n \in \omega$. It can be shown that F is closed and normal in G and the quotient group G/F is isomorphic to the group 2^ω (the unrestricted Cartesian product of ω many copies of the cyclic group of order 2). Moreover, the quotient G/F, and hence G, has $2^{2^{\aleph_0}}$ subgroups of index 2. However, by Theorem 1.3.3(ii)-(iii), there are at most $\aleph_0$ many open subgroups of G. Thus, G has non-open subgroups of index 2.

The construction in the example above was used by Evans and Hewitt [26, Lemma 3.1] to show that any separable profinite group is a homomorphic image of the automorphism group of some ω-categorical structure. Another group-theoretic condition related to ample generics is the cofinality of a group.

Definition 2.3.6. Let G be a group which is not finitely generated. The *cofinality* of G is the smallest cardinal κ such that G can be written as a union of a chain of length κ of proper subgroups.

Kechris and Rosendal generalised results of [39] and showed that a Polish group with ample generics cannot be written as a union of a countable chain of non-open subgroups—see [47, Theorem 6.12]. Combined with Cameron's result [39, Lemma 2.4] that any open subgroup of the automorphism group of an ω-categorical structure is only contained in finitely many subgroups, we obtain the following.

Theorem 2.3.7 ([47])**.** *Suppose that M is a countably infinite ω-categorical structure with ample generics. Then $\mathrm{Aut}(M)$ has uncountable cofinality.*

Bergman [3] showed that $\mathrm{Sym}(\mathbb{N})$ has the Bergman property (see below). Kechris and Rosendal found that the existence of ample generics has ties to the Bergman property, and they extended Bergman's result to a stronger property and to a larger class of automorphism groups of first order structures.

Definition 2.3.8. Let G be a group which is not finitely generated.

(i) We say that G has the *Bergman property* if whenever $G = \langle U \rangle$ for $U \subseteq G$ and $1 \in U = U^{-1}$, then there is $k \in \omega$ such that $G = U^k := \{u_1 \dots u_k : u_i \in U\}$.

(ii) Let $k \in \omega$. We say that G has the *k-Bergman property* if for any countable chain $(U_n)_{n\in\omega}$ of subsets of G such that $G = \bigcup_{n\in\omega} U_n$, there is $n \in \omega$ such that $G = U_n^k$.

Theorem 2.3.9. [47, Theorem 6.19] *Suppose that M is a countably infinite ω-categorical structure with ample generics. Then $\mathrm{Aut}(M)$ has the 21-Bergman property.*

For more on the cofinality, 'strong cofinality', and the Bergman property for certain permutation groups see Droste-Göbel [21, Proposition 2.2], Droste-Holland [23], and Macpherson-Neumann [56].

We conclude this section with Serre's property (FA). A *tree* is a connected graph with no cycles. A group G *acts without inversions* on a tree T if G acts by automorphisms on T and for all $g \in G$ there is no edge uv of T such that $g(u, v) = (v, u)$.

Definition 2.3.10 ([69]). A group G has *property (FA)* if whenever G acts on a tree T without inversions, then there is a vertex $v \in T$ such that for all $g \in G$ we have that $g(v) = v$, that is to say, G has a global fixed point.

A free product with amalgamation $G = G_1 \star_A G_2$ (Definition 5.2.3) is *trivial* if $G = G_1$ or $G = G_2$. Serre [69] proved that an uncountable group G has property (FA) if and only if all of the following conditions hold:

(i) G is not a non-trivial free product with amalgamation.

(ii) $\mathbb{Z}$ is not a homomorphic image of G.

(iii) G has uncountable cofinality.

Theorem 2.3.11 (Macpherson-Thomas [54]). *Suppose that G is a Polish group with a comeagre conjugacy class. Then G is not a non-trivial free product with amalgamation.*

Moreover, if G is a Polish group with a comeagre conjugacy class, then by Theorem 1.4.4 we have that $\mathbb{Z}$ is not a homomorphic image of G. Thus, using Theorem 2.3.7 above we fulfil the conditions characterising property (FA) and obtain the following.

Theorem 2.3.12. [47, Corollary 1.9] *Suppose that M is a countably infinite ω-categorical structure with ample generics. Then $\mathrm{Aut}(M)$ has property (FA)* .

Chapter 3

Free Homogeneous Structures

The central object of study in this chapter is a *free homogeneous first order structure* over a finite relational language. Recall that a relational structure is *homogeneous* if it is countable and every partial isomorphism between any two of its finite substructures extends to a total automorphism. Moreover, it is called free homogeneous if its age has the *free amalgamation property*, which will be examined in Section 3.2. We collect the main results of this chapter in the following statement.

Theorem. *Any free amalgamation class over a finite relational language has Gaifman clique faithful coherent EPPA. Consequently, the automorphism group of the corresponding free homogeneous structure contains a dense locally finite subgroup, and has ample generics and the small index property.*

One example of a free homogeneous structure is the *random graph*; it is the unique homogeneous countably infinite graph which embeds all finite graphs. Bhattacharjee and Macpherson showed that the automorphism group of the random graph has a dense locally finite subgroup, and they asked whether it is possible to generalise their [4, Lemma 1.2] about extending partial automorphisms of finite graphs. Here, we generalise their result to the case of free amalgamation classes—see Theorem 3.2.8.

The construction of the dense locally finite subgroup of $\mathrm{Aut}(M)$ in the main theorem above relies on the extension property for partial automorphisms (EPPA)—Definition 1.5.1; more accurately, on coherent EPPA (Definition 1.5.10), where in addition to extending finite partial isomorphisms, coherent EPPA has the advantage that the composition of the extensions of any two partial automorphisms is the extension of the composition of the original partial automorphisms. The desired subgroup of $\mathrm{Aut}(M)$ is dense with respect to the pointwise convergence topology. Recall that a subset $\Gamma \subseteq \mathrm{Aut}(M)$ is dense if and only if for every $g \in \mathrm{Aut}(M)$ and every finite $A \subseteq M$ there is an $h \in \Gamma$ such that $g{\restriction}_A = h{\restriction}_A$. For more information on $\mathrm{Aut}(M)$ as a topological group see Section 1.3.

Let $\mathcal{L}$ be a finite relational language. In Section 3.1 we strengthen [40, Theorem 9] by proving that the class of all finite $\mathcal{L}$-structures has Gaifman clique faithful coherent EPPA. In Section 3.2 we extend this result to free amalgamation classes over $\mathcal{L}$. Finally, in Section 3.3 we use coherent EPPA to construct a dense locally finite subgroup of the automorphism group of a free homogeneous structure. We obtain ample generics and the small index property for free homogeneous structures as well.

3.1 Extending Partial Automorphisms

An adaptation by Solecki to the proof of Herwig-Lascar [36, Corollary 4.13] leads to the following result.

Theorem 3.1.1. [71, Theorem 3.1] *Let $\mathcal{L}$ be a finite relational language. The class of all finite $\mathcal{L}$-structures has coherent EPPA.*

Hodkinson and Otto [40] proved a Gaifman clique constrained strengthening of EPPA building on the work of Herwig and Lascar. In this section we show that the strengthened EPPA they proved can be made coherent when more conditions are demanded in their construction.

Definition 3.1.2 ([40])**.** Let $\mathcal{L}$ be a relational language, and A be an $\mathcal{L}$-structure.

- The *Gaifman graph* of A, denoted by $\mathrm{Gaif}(A)$, is the graph whose vertex set is the domain of A, and whose edge relation is defined as: two vertices $u, v \in A$ are adjacent if and only if there is an n-ary relation $R \in \mathcal{L}$ and $(a_1, a_2, \ldots, a_n) \in A$ such that $u, v \in \{a_1, a_2, \ldots, a_n\}$ and $A \models R(a_1, a_2, \ldots, a_n)$.
- A substructure $Q \subseteq A$ is a *Gaifman clique* if it is a clique in $\mathrm{Gaif}(A)$.

Suppose that C is an EPPA-extension of some $\mathcal{L}$-structure A. Consider the substructure $B \subseteq C$ whose underlying set is $B = \bigcup\{g(A) : g \in \mathrm{Aut}(C)\}$. Then, B is also an EPPA-extension of A, and additionally has the property that every point $b \in B$ can be sent to A by some $g \in \mathrm{Aut}(B)$. We call such extension a *point faithful EPPA-extension*. Can we do more in terms of faithfulness?

Definition 3.1.3. [40, Definition 8] A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is said to have *Gaifman clique faithful EPPA* if for every $A \in \mathcal{C}$, there exists an EPPA-extension $B \in \mathcal{C}$ of A such that for every Gaifman clique $Q \subseteq B$ there is $g \in \mathrm{Aut}(B)$ such that $g(Q) \subseteq A$.

Theorem 3.1.4 (Hodkinson-Otto [40]). *Let $\mathcal{L}$ be a finite relational language. The class of all finite $\mathcal{L}$-structures has Gaifman clique faithful EPPA.*

Our aim in this section is to show that the extension procedure for partial automorphisms given in the proof of the theorem above can be made coherent. We follow the terminology and ideas presented in [40]. The proof of Theorem 3.1.4 goes as follows: start with any finite $\mathcal{L}$-structure A, obtain an EPPA-extension B of A, say by Theorem 1.5.6. The obstacle at this point for Gaifman clique faithfulness would be if some cliques in B cannot be sent to A by an automorphism of B. In Hodkinson's terminology, call such cliques "*false cliques*". Then using B construct a structure C extending A which preserves EPPA and in which all false cliques are destroyed.

We now present the details and adapt the construction to fulfil our aim, namely to show that the class of all finite $\mathcal{L}$-structures has Gaifman clique faithful *coherent* EPPA.

Fix a finite $\mathcal{L}$-structure A.

Let $B \supseteq A$ be a coherent EPPA-extension guaranteed by Theorem 3.1.1. If $A = B$ we are done, so suppose that $A \neq B$. A subset $u \subseteq B$ is called *large* if there is no $g \in \mathrm{Aut}(B)$ such that $g(u) \subseteq A$. Otherwise, the subset u is called *small*. Define,

$$\mathcal{U} := \{u \subseteq B \mid u \text{ is large}\}.$$

Notice that false cliques and the domain of B are large sets, and the image of a large set under an automorphism of B is also large. Given a finite set X, by $[X]$ we denote the set $\{0, 1, 2, \ldots, |X| - 1\} \subseteq \mathbb{N}$.

Definition 3.1.5 ([40])**.** Let $b \in B$. A map $\chi_b : \mathcal{U} \to [B]$ is called a *b-valuation* if for all $u \in \mathcal{U}$ it satisfies: (i) $\chi_b(u) = 0$ if and only if $b \notin u$, and (ii) $1 \leq \chi_b(u) < |u|$ if otherwise.

The domain of the extension C of A given by Theorem 3.1.4 is,

$$C := \big\{(b, \chi_b) \mid b \in B,\ \chi_b \text{ is a } b\text{-valuation}\big\}.$$

Note. When we write $(b, \chi_b) \in C$, we mean that $b \in B$ and χ_b is *some* b-valuation. For the same $b \in B$, there will in general be many different b-valuations denoted by χ_b.

Definition 3.1.6 ([40])**.** A subset $S \subseteq C$ is called *generic* if for any two distinct points $(a, \chi_a), (b, \chi_b) \in S$:

- $a \neq b$, and
- for all $u \in \mathcal{U}$, if both $a, b \in u$, then $\chi_a(u) \neq \chi_b(u)$.

Note that if $S \subseteq C$ is generic, then any subset of S is generic. Define the *projection map*:

$$\pi : C \to B \text{ where } \pi(b, \chi_b) = b.$$

Fact 3.1.7 ([40])**.** *If $S \subseteq C$ is generic, then $\pi(S)$ is a small subset of B.*

Proof. Let $S \subseteq C$ be a generic subset, and suppose that $u := \pi(S) \subseteq B$ is large. As S is generic, $\pi{\restriction}_S : S \to u$ is a bijection. We now define a map $\theta : u \to [u] \setminus \{0\}$ by setting $\theta(b) = \chi_b(u)$ where $b \in u$ and $\pi^{-1}(b) = (b, \chi_b) \in S$. Again, as S is generic, θ is injective, but this contradicts that $|u| = |[u]|$. ■

We now make C into an $\mathcal{L}$-structure in a way that all the π-fibres in C of large subsets of B are forbidden from being cliques in C. This is where all false cliques are killed.

For every n-ary relation symbol $R \in \mathcal{L}$ and n-tuple $\big((b_1, \chi_1), (b_2, \chi_2), \ldots, (b_n, \chi_n)\big) \in C$, define $C \models R\big((b_1, \chi_1), (b_2, \chi_2), \ldots, (b_n, \chi_n)\big)$ if and only if

(i) the set $\{(b_1, \chi_1), (b_2, \chi_2), \ldots, (b_n, \chi_n)\}$ is a generic subset of C, and

(ii) $B \models R(b_1, b_2, \ldots, b_n)$.

Note. From this point onward in this section, the structures A, B, and C above are fixed.

We include the proof of the following proposition for the convenience of the reader.

Proposition 3.1.8 ([40]). *The original structure A embeds in C.*

Proof. We will define an embedding $\nu : A \to C$ as follows. Any large subset $u \in \mathcal{U}$ is not a subset of A. Otherwise the identity automorphism of B violates that u is a large subset. Thus, $|u \cap A| < |u|$. For each $u \in \mathcal{U}$ fix an enumeration of

$$u \cap A = \{a_1^u, a_2^u, \ldots, a_n^u\}$$

where $n < |u|$. Now for each $a \in A$ we define an a-valuation $\chi_a : \mathcal{U} \to \mathbb{N}$.

$$\chi_a(u) = \begin{cases} 0, & \text{if } a \notin u, \\ i \text{ such that } a = a_i^u, & \text{if } a \in u. \end{cases}$$

Now for each $a \in A$ we define $\nu(a) = (a, \chi_a)$. The set $\nu(A)$ is a generic subset of C, and it follows that $\nu : A \to C$ is an $\mathcal{L}$-embedding. ■

Below we will just use A for both structures $A \subseteq B$ and $\nu(A) \subseteq C$, as it is clear from the context which one we mean. Also keep in mind that A is a generic subset of C.

Definition 3.1.9. Let $p \in \mathrm{Part}(C)$ be a partial automorphism of C, and let $g \in \mathrm{Aut}(B)$. We say that p is *g-compatible* if $\pi \circ p = g \circ \pi$, that is, for all $(b, \chi_b) \in \mathrm{dom}(p)$ we have that $p(b, \chi_b) = \big(g(b), \chi_{g(b)}\big)$, where $\chi_{g(b)}$ is some $g(b)$-valuation.

We use the freedom of choice given in [40] in constructing the extension $\hat{p}$ of the lemma below to make additional constraints, namely the ordering, in their construction which will be needed later on to make the extension procedure of partial automorphisms coherent.

Lemma 3.1.10. *Suppose that $g \in \mathrm{Aut}(B)$, and let $p \in \mathrm{Part}(C)$ be a g-compatible partial automorphism with generic domain and range. Then p extends to some g-compatible $\hat{p} \in \mathrm{Aut}(C)$.*

Proof. As $\mathrm{dom}(p)$ is a generic set, for any $b \in \pi(\mathrm{dom}(p))$ there is only one b-valuation χ_b such that $(b, \chi_b) \in \mathrm{dom}(p)$. So we can write (b, χ_b) for elements of $\mathrm{dom}(p)$ without ambiguity. Similarly, as $\mathrm{range}(p)$ is generic and p is g-compatible, we write $p(b, \chi_b) = (g(b), \chi_{g(b)})$, where $\chi_{g(b)}$ is some $g(b)$-valuation determined by the map p.

Fix a large set $u \in \mathcal{U}$. We will define a permutation θ^p_u of the set $[u] = \{0, 1, 2, \ldots, |u|-1\}$ which fixes 0. First, for every element $(b, \chi_b) \in \mathrm{dom}(p)$, where its image under p is $p(b, \chi_b) = (g(b), \chi_{g(b)})$, define:

$$\theta^p_u\big(\chi_b(u)\big) := \chi_{g(b)}\big(g(u)\big).$$

After that, by using the well-ordering of the natural numbers extend θ^p_u to a total permutation of the set $[u]$, fixing 0, by sending elements from the subset $[u] \setminus \big\{\chi_b(u) : (b, \chi_b) \in \mathrm{dom}(p)\big\}$ to the subset $[u] \setminus \big\{\chi_{g(b)}(g(u)) : (b, \chi_b) \in \mathrm{dom}(p)\big\}$ in an order-preserving manner.

For each $u \in \mathcal{U}$, define the corresponding permutation θ^p_u of the set $[u]$. Now we are ready to define the extension $\hat{p}$ on C. Let $(c, \chi_c) \in C$ be any point. Define,

$$\hat{p}(c, \chi_c) := \big(g(c), \chi_{g(c)}\big)$$

where $\chi_{g(c)}$ is a $g(c)$-valuation given by:

$$\chi_{g(c)}\big(g(u)\big) := \theta_u^p\big(\chi_c(u)\big) \text{ for each } u \in \mathcal{U}.$$

By definition, $\hat{p}$ is g-compatible. Now we check that $\hat{p}$ extends p. So let $(b, \chi_b) \in \operatorname{dom}(p)$ and let its image be $p(b, \chi_b) = (g(b), \chi_{g(b)})$. Suppose that $\hat{p}(b, \chi_b) = \big(g(b), \psi_{g(b)}\big)$ where $\psi_{g(b)}$ is a $g(b)$-valuation given by the definition above. Then $\psi_{g(b)}\big(g(u)\big) = \theta_u^p\big(\chi_b(u)\big) = \chi_{g(b)}\big(g(u)\big)$ for each $u \in \mathcal{U}$. Thus $\psi_{g(b)} = \chi_{g(b)}$, and so $\hat{p}\restriction_{\operatorname{dom}(p)} = p$.

We check that $\hat{p}$ is bijective. Suppose that $\hat{p}(b, \chi_b) = \hat{p}(c, \chi_c)$ for some $(b, \chi_b), (c, \chi_c) \in C$. Then $(g(b), \chi_{g(b)}) = (g(c), \chi_{g(c)})$ as given above. So $g(b) = g(c)$, and by injectivity of g we get that $b = c$. We also have that $\chi_{g(b)} = \chi_{g(c)}$. So $\chi_b(u) = (\theta_u^p)^{-1}(\chi_{g(b)}(g(u)) = (\theta_u^p)^{-1}(\chi_{g(c)}(g(u)) = \chi_c(u)$ for each $u \in \mathcal{U}$. Thus $\hat{p}$ is injective. Now for surjectivity, suppose that we are given $(b, \chi_b) \in C$. Let $c := g^{-1}(b)$ and define a c-valuation χ_c as follows $\chi_c(u) := (\theta_u^p)^{-1}(\chi_b(g(u)))$ for each $u \in \mathcal{U}$. Then $\hat{p}(c, \chi_c) = (b, \chi_b)$.

Finally, $\hat{p}$ preserves generic subsets of C, that is, $S \subseteq C$ is generic if and only if $\hat{p}(S) \subseteq C$ is generic. To see this, let $S \subseteq C$ be a generic set. We will show that $\hat{p}(S)$ is generic. Choose two distinct points $\hat{p}(a, \chi_a) = (g(a), \chi_{g(a)})$ and $\hat{p}(b, \chi_b) = (g(b), \chi_{g(b)})$ in $\hat{p}(S)$, where $(a, \chi_a), (b, \chi_b) \in S$. As $\hat{p}$ is bijective, $(a, \chi_a), (b, \chi_b)$ are distinct, and as S is generic, $a \neq b$. As g is bijective, $g(a) \neq g(b)$. For the second point in the definition of genericity, suppose that $u \in \mathcal{U}$ and $g(a), g(b) \in g(u)$. As S is generic, $\chi_a(u) \neq \chi_b(u)$. So $\chi_{g(a)}(g(u)) = \theta_u^p(\chi_a(u)) \neq \theta_u^p(\chi_b(u)) = \chi_{g(b)}(g(u))$.

The observation above together with that $\hat{p}$ is g-compatible and the definition of the structure on C above yields that $\hat{p} \in \operatorname{Aut}(C)$. ■

Notation. We would like to fix some notation. Let $g \in \operatorname{Aut}(B)$, and $p \in \operatorname{Part}(C)$ be g-compatible partial automorphism with generic domain and range. For $u \in \mathcal{U}$, we denote by θ_u^p the permutation of the set $[u]$, fixing 0, as constructed in the proof of Lemma 3.1.10 above. That is, for every point $(b, \chi_b) \in \operatorname{dom}(p)$ and its image

$p(b, \chi_b) = (g(b), \chi_{g(b)})$, define $\theta_u^p\big(\chi_b(u)\big) = \chi_{g(b)}\big(g(u)\big)$. And then extend it to the rest of $[u]$ in an order-preserving way.

Lemma 3.1.11 ([40]). *Let $p \in \mathrm{Part}(A)$, then p extends to an automorphism $\hat{p} \in \mathrm{Aut}(C)$ where $\hat{p}$ is the automorphism defined in the proof of Lemma 3.1.10.*

Proof. Let $p \in \mathrm{Part}(A) \subseteq \mathrm{Part}(C)$. By Theorem 3.1.1, the partial automorphism p has an extension $g \in \mathrm{Aut}(B)$, and clearly p is g-compatible. As A is a generic subset of C, we have that both $\mathrm{dom}(p)$, $\mathrm{range}(p) \subseteq A$ are also generic subsets. Now, apply Lemma 3.1.10 on p to get a g-compatible extension $\hat{p} \in \mathrm{Aut}(C)$. ■

It is in the proof of the next lemma where we really use that B is a coherent EPPA-extension of A as given by Theorem 3.1.1.

Lemma 3.1.12. *The map from $\mathrm{Part}(A) \to \mathrm{Aut}(C)$ as defined in Lemma 3.1.11 which sends $p \mapsto \hat{p}$ is coherent.*

Proof. We will show that the image of a coherent triple in $\mathrm{Part}(A)$ is a coherent triple in $\mathrm{Aut}(C)$ under the map $p \mapsto \hat{p}$ defined in Lemma 3.1.11. Suppose that $p_2, p_1, q \in \mathrm{Part}(A)$, and (p_2, p_1, q) is a coherent triple. That is, $\mathrm{dom}(p_2) = \mathrm{dom}(q)$, $\mathrm{range}(p_2) = \mathrm{dom}(p_1)$, $\mathrm{range}(p_1) = \mathrm{range}(q)$, and $q = p_1 \circ p_2$. Recall that A is a substructure of both B and C. By Theorem 3.1.1 there are $g_2, g_1, h \in \mathrm{Aut}(B)$ extending p_2, p_1, q, respectively. Moreover, (g_2, g_1, h) constitutes a coherent triple, that is, $h = g_1 \circ g_2$. Notice that p_2 is g_2-compatible, p_1 is g_1-compatible, and q is h-compatible. Now let $\hat{p}_2, \hat{p}_1, \hat{q} \in \mathrm{Aut}(C)$ be the g_2-compatible, g_1-compatible, and h-compatible extensions of $p_2, p_1, q \in \mathrm{Part}(A)$, respectively, as constructed in Lemma 3.1.11 above. We will show that $\hat{q} = \hat{p}_1 \circ \hat{p}_2$.

Now let $(b, \chi_b), (c, \chi_c) \in C$ be any two points. Here, χ_b is some b-valuation, and χ_c is some c-valuation. By the construction of $\hat{p}_2$ and $\hat{p}_1$ we get that,

$$\hat{p}_2(b, \chi_b) = \big(g_2(b), \chi_{g_2(b)}\big) \text{ where } \chi_{g_2(b)}\big(g_2(u)\big) = \theta_u^{p_2}\big(\chi_b(u)\big) \text{ for } u \in \mathcal{U},$$

and

$$\hat{p}_1(c, \chi_c) = \big(g_1(c), \chi_{g_1(c)}\big) \text{ where } \chi_{g_1(c)}\big(g_1(v)\big) = \theta_v^{p_1}\big(\chi_c(v)\big) \text{ for } v \in \mathcal{U}.$$

On the one hand, we want to find the value of $\hat{p}_1\big(\hat{p}_2(b,\chi_b)\big)$. So using the above by taking $c = g_2(b)$, $\chi_c = \chi_{g_2(b)}$ and $v = g_2(u)$ we get the following:

$$\hat{p}_1\big(\hat{p}_2(b,\chi_b)\big) = \hat{p_1}\big(g_2(b), \chi_{g_2(b)}\big) = \big(g_1g_2(b), \chi_{g_1(g_2(b))}\big) = \big(h(b), \chi_{h(b)}\big)$$

where for each $u \in \mathcal{U}$ we have that,

$$\chi_{h(b)}\big(h(u)\big) = \chi_{g_1(g_2(b))}\big(g_1g_2(u)\big) = \theta^{p_1}_{g_2(u)}\Big(\chi_{g_2(b)}\big(g_2(u)\big)\Big) = \theta^{p_1}_{g_2(u)} \circ \theta^{p_2}_u\big(\chi_b(u)\big).$$

On the other hand, we have that

$$\hat{q}(b,\chi_b) = \big(h(b), \psi_{h(b)}\big) \text{ where}$$

$$\psi_{h(b)}\big(h(u)\big) = \theta^q_u\big(\chi_b(u)\big) \text{ for } u \in \mathcal{U}.$$

Therefore, we reach our desired result if we show that $\chi_{h(b)} = \psi_{h(b)}$, which follows from showing that

$$\theta^q_u\big(\chi_b(u)\big) = \theta^{p_1}_{g_2(u)} \circ \theta^{p_2}_u\big(\chi_b(u)\big)$$

for any $(b,\chi_b) \in C$ and $u \in \mathcal{U}$. Recall that $\theta^q_u, \theta^{p_1}_{g_2(u)}, \theta^{p_2}_u$ are all permutations of the set $[u] = \{0, 1, 2, \ldots, |u| - 1\}$, all fixing 0.

Fix any $(b,\chi_b) \in C$ and $u \in \mathcal{U}$. Let $m = \chi_b(u) \in [u]$. Recall that $\mathrm{dom}(p_2) = \mathrm{dom}(q)$, $\mathrm{range}(p_2) = \mathrm{dom}(p_1)$, $\mathrm{range}(p_1) = \mathrm{range}(q)$ are all generic sets as they are subsets of the generic set $A \subseteq C$, and so we can write their elements in the form (c,χ_c) without ambiguity, where χ_c is some c-valuation.

Case 1. Suppose that $m = \chi_c(u)$ for some $(c,\chi_c) \in \mathrm{dom}(p_2) = \mathrm{dom}(q)$.

The point $p_2(c,\chi_c) = \big(g_2(c), \chi_{g_2(c)}\big)$ belongs to $\mathrm{range}(p_2) = \mathrm{dom}(p_1)$ and so $p_1 \circ p_2(c,\chi_c) = p_1\big(g_2(c), \chi_{g_2(c)}\big) = \big(g_1g_2(c), \chi_{g_1g_2(c)}\big) = \big(h(c), \chi_{h(c)}\big)$, where $\chi_{h(c)}$ is an $h(c)$-valuation (see the diagrams below). Using this information and the way $\theta^{p_1}_{g_2(u)}$ and $\theta^{p_2}_u$ were constructed, we get that,

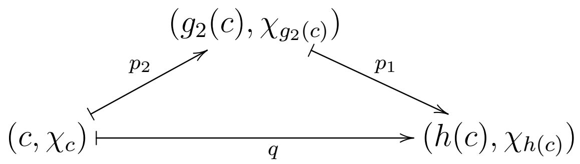
$$\begin{aligned}\theta^{p_1}_{g_2(u)} \circ \theta^{p_2}_u(m) = \theta^{p_1}_{g_2(u)} \circ \theta^{p_2}_u(\chi_c(u)) &= \theta^{p_1}_{g_2(u)}\big(\chi_{g_2(c)}(g_2(u))\big) \\ &= \chi_{g_1g_2(c)}(g_1g_2(u)) = \chi_{h(c)}(h(u)).\end{aligned}$$

As $q = p_1 \circ p_2$ we have $q(c, \chi_c) = p_1 \circ p_2(c, \chi_c) = \big(h(c), \chi_{h(c)}\big)$, and so by construction of θ_u^q we get,

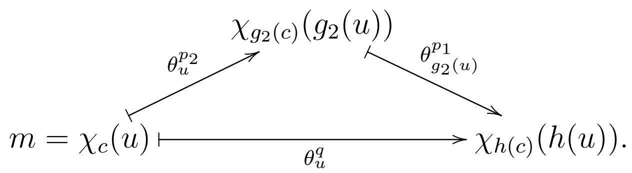
$$\theta_u^q(m) = \theta_u^q(\chi_c(u)) = \chi_{h(c)}(h(u)).$$

Therefore, $\theta_{g_2(u)}^{p_1} \circ \theta_u^{p_2}(m) = \theta_u^q(m)$ when $m = \chi_c(u)$ for some $(c, \chi_c) \in \mathrm{dom}(p_2)$.

The following commutative diagrams illustrates the above computations:



and



Case 2. Suppose that $m \neq \chi_c(u)$ for all $(c, \chi_c) \in \mathrm{dom}(p_2) = \mathrm{dom}(q)$.

In this case, the permutation $\theta_u^{p_2}$ was defined in an order-preserving way. So suppose that $m \in [u] = \{0, 1, \ldots, |u| - 1\}$ is the i^{th} element such that $m \neq \chi_c(u)$ for all $(c, \chi_c) \in \mathrm{dom}(p_2)$. Then $n := \theta_u^{p_2}(m)$ is the i^{th} element of $[u]$ such that $n \neq \chi_{g_2(c)}(g_2(u))$ for all $(c, \chi_c) \in \mathrm{dom}(p_2)$. Note that $[g_2(u)] = [u]$ as g_2 is a bijection, meaning that $\theta_{g_2(u)}^{p_1}$ is also a permutation of the set $[u]$. Finally, as $\mathrm{range}(p_2) = \mathrm{dom}(p_1)$ we get that $k := \theta_{g_2(u)}^{p_1} \circ \theta_u^{p_2}(m) = \theta_{g_2(u)}^{p_1}(n)$ is the i^{th} element of $[u]$ such that $k \neq \chi_{h(c)}(h(u))$ for all $(c, \chi_c) \in \mathrm{dom}(p_2)$.

Now suppose that $\theta_u^q(m) = k'$, then by construction of θ_u^q and as $\mathrm{dom}(p_2) = \mathrm{dom}(q)$ we have that k' is the i^{th} element of $[u]$ such that $k' \neq \chi_{h(c)}(h(u))$ for all $(c, \chi_c) \in \mathrm{dom}(q)$. Thus, $k = k'$, and so $\theta_{g_2(u)}^{p_1}\theta_u^{p_2}(m) = \theta_u^q(m)$ when $m \neq \chi_c(u)$ for all $(c, \chi_c) \in \mathrm{dom}(p_2)$.

Therefore, we have shown that $\theta^q_u\big(\chi_b(u)\big) = \theta^{p_1}_{g_2(u)} \circ \theta^{p_2}_u\big(\chi_b(u)\big)$ for any $(b, \chi_b) \in C$ and $u \in \mathcal{U}$, implying that $\chi_{h(b)} = \psi_{h(b)}$ and so we get that $\hat{p}_1 \circ \hat{p}_2 = \hat{q}$. So the map $p \mapsto \hat{p}$ from $\mathrm{Part}(A)$ to $\mathrm{Aut}(C)$ is coherent. ■

Theorem 3.1.13. *Let $\mathcal{L}$ be a finite relational language. The class of all finite $\mathcal{L}$-structures has Gaifman clique faithful coherent EPPA.*

Proof. Let A be a finite $\mathcal{L}$-structure. By Theorem 3.1.1, there is an extension B of A in which every element of $\mathrm{Part}(A)$ extends to an element of $\mathrm{Aut}(B)$ such that the corresponding map is coherent. From B construct the $\mathcal{L}$-structure $C = \big\{(b, \chi_b) \mid b \in B,\ \chi_b \text{ is a } b\text{-valuation}\big\}$ as described above in this section. By 3.1.11 every element of $\mathrm{Part}(A)$ extends to an element of $\mathrm{Aut}(C)$. By [40] every clique in C is the image of a clique in A under an automorphism of C. Furthermore, by Lemma 3.1.12, the extension map from $\mathrm{Part}(A)$ to $\mathrm{Aut}(C)$ is coherent. ■

3.2 Free Amalgamation Classes and Coherent EPPA

Our aim in this section is to apply the Gaifman clique faithful coherent EPPA result of the previous section to free amalgamation classes. The relationship between these two notions is that every free amalgamation class is a class which forbids a fixed family of Gaifman cliques—see Lemma 3.2.7 below. Let $\mathcal{L}$ be a finite relational language, and let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures. Recall that $\mathcal{C}$ is called an amalgamation class if it is closed under substructures and isomorphism, and has both the joint embedding property and the amalgamation property—see Section 1.2.

Definition 3.2.1. Given finite $\mathcal{L}$-structures A, B_1, B_2 with $A \subseteq B_1$ and $A \subseteq B_2$, the *free amalgam* of B_1 and B_2 over A is the structure C whose domain is the disjoint union of B_1 and B_2 over A, and for every relation symbol $R \in \mathcal{L}$ we define $R^C := R^{B_1} \cup R^{B_2}$.

As a result we have the following two observations on the free amalgam C. First, when

B_1, B_2 are viewed as subsets of C we have that $B_1 \cap B_2 = A$. Second, there is no relation symbol $R \in \mathcal{L}$ and a tuple $\bar{c} \in C$ such that $\bar{c}$ meets both $B_1 \setminus A$ and $B_2 \setminus A$, and $C \models R(\bar{c})$.

Definition 3.2.2. A class $\mathcal{C}$ of structures over a fixed relational language has the *free amalgamation property* if $\mathcal{C}$ is closed under taking free amalgams.

Note that the free amalgamation property implies the amalgamation property. The class $\mathcal{C}$ is called a *free amalgamation class* if it is closed under substructures and isomorphism, and has both the joint embedding property and the free amalgamation property. Lastly, the Fraïssé limit of a free amalgamation class is called a *free homogeneous structure*.

Example 3.2.3. The following are examples of free homogeneous structures.

1. The random graph [8].
2. The universal homogeneous K_n-free graph [55, Example 2.2.2].
3. The universal homogeneous directed graph.
4. The continuum many Henson digraphs [34].
5. The universal homogeneous k-hypergraph [75].
6. The universal homogeneous tetrahedron-free 3-hypergraph, where a tetrahedron is a complete 3-hypergraph on four vertices [49, Definition 2.3].
7. The Fraïssé limit of the class of all finite 3-hypergraphs such that every subset of size 4 contains at most two 3-hyperedges.

In situations where we have a binary relation which is either transitive or total, one expects free amalgamation to fail. For example the classes of all finite partial orders, linear orders, tournaments, and structures with an equivalence relation do not have the free amalgamation property. Here is another example.

Example 3.2.4 (An amalgamation class which is not free)**.** A 3-hypergraph H is called a *two-graph* if every subset of H of size 4 has an even number of 3-hyperedges—see [55, Example 2.3.1.4]. Let $\mathcal{C}$ be the class of all finite two-graphs. An instance of free amalgamation failure is the following: let $B_1 = \{a, b\}$, $B_2 = \{a, u, v\}$ with hyperedge

auv. Then the free amalgam of B_1 and B_2 over $\{a\}$ is a 3-hypergraph of size 4 with exactly one hyperedge, and so is not in $\mathcal{C}$. However $\mathcal{C}$ has the amalgamation property. One can show that by first taking the free amalgam, and then adding an extra hyperedge in the right place to sets of size 4 with an odd number of hyperedges.

In the literature one can find a number of interesting results about a free homogeneous structure M over a finite relational language. It was shown in Macpherson-Tent [57] that if $G = \mathrm{Aut}(M)$ acts transitively on M and $G \neq \mathrm{Sym}(M)$ then G is a simple group. Furthermore, if M is ω-categorical, then M has *weak elimination of imaginaries*. Ivanov [44] proved that M has generic automorphisms. Consequently, by Theorem 2.3.11 we have that G is not a non-trivial free product with amalgamation. For more results on free homogeneous structures see [55].

Definition 3.2.5. Let $\mathcal{L}$ be a first order language, and $\mathcal{F}$ be a family of $\mathcal{L}$-structures.

- We say a structure A is *$\mathcal{F}$-free under embeddings* if there is no structure $F \in \mathcal{F}$ and embedding $g : F \to A$.
- Denote by $\mathrm{Forb}_e(\mathcal{F})$ the class of all finite $\mathcal{L}$-structures which are $\mathcal{F}$-free under embeddings.

Definition 3.2.6. Let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures. A finite $\mathcal{L}$-structure F is called *forbidden* in $\mathcal{C}$ if $F \notin \mathcal{C}$. Moreover F is called *minimal forbidden* in $\mathcal{C}$ if $F \notin \mathcal{C}$ and for any $v \in F$ we have that $(F \setminus \{v\})$ is in $\mathcal{C}$.

One can observe that every finite structure F which is forbidden in $\mathcal{C}$ contains a minimal forbidden substructure. For if F were not a minimal forbidden structure, there is a vertex $v \in F$, such that $F \setminus \{v\}$ is still forbidden in $\mathcal{C}$. We keep repeating this process until we find a substructure $F' \subseteq F$ which is minimal forbidden.

Note that the class $\mathrm{Forb}_e(\mathcal{F})$ has the hereditary property. Conversely, suppose that $\mathcal{C}$ is a class of finite $\mathcal{L}$-structures closed under isomorphism and having the hereditary property. Let $\mathcal{F}$ be the family of all finite structures which are minimal forbidden in $\mathcal{C}$. Then

$\mathcal{C} = \mathrm{Forb_e}(\mathcal{F})$. To see this, first suppose that $A \in \mathrm{Forb}_e(\mathcal{F})$ but $A \notin \mathcal{C}$. So A is forbidden in $\mathcal{C}$, and hence contains some minimal forbidden structure. This contradicts that A is $\mathcal{F}$-free. So $\mathrm{Forb}_e(\mathcal{F}) \subseteq \mathcal{C}$. For the other direction, supposing that $A \in \mathcal{C}$ but $A \notin \mathrm{Forb_e}(\mathcal{F})$, there is some $F \in \mathcal{F}$ and an embedding $g : F \to A$. As $\mathcal{C}$ has the hereditary property, $F \in \mathcal{C}$, contradicting F a forbidden structure. So $\mathcal{C} \subseteq \mathrm{Forb_e}(\mathcal{F})$.

Lemma 3.2.7. *Suppose that $\mathcal{C}$ is a class of finite structures over a relational language $\mathcal{L}$. The class $\mathcal{C}$ is a free amalgamation class if and only if $\mathcal{C} = \mathrm{Forb_e}(\mathcal{F})$ for some family $\mathcal{F}$ of Gaifman cliques.*

Proof. Suppose that $\mathcal{C}$ is a free amalgamation class. By the above $\mathcal{C} = \mathrm{Forb_e}(\mathcal{F})$ where $\mathcal{F}$ is the family of all finite structures which are minimal forbidden in $\mathcal{C}$. We claim that every element $Q \in \mathcal{F}$ is a Gaifman clique. If not, then there are two elements $u, v \in Q$ which do not satisfy any relation of $\mathcal{L}$. Let $Q_u = Q \setminus \{u\}$ and $Q_v = Q \setminus \{v\}$. By minimality of Q, both Q_u and Q_v belong to $\mathcal{C}$. Moreover, $Q_{uv} := Q \setminus \{u, v\}$ belongs to $\mathcal{C}$ too, as $\mathcal{C}$ has the hereditary property. By the free amalgamation property of $\mathcal{C}$, we get that Q which is the free amalgam of Q_u and Q_v over Q_{uv} is in $\mathcal{C}$, contradicting $Q \in \mathcal{F}$. Therefore, every $Q \in \mathcal{F}$ is a Gaifman clique.

For the reverse direction, suppose that $\mathcal{C} = \mathrm{Forb_e}(\mathcal{F})$ for some collection $\mathcal{F}$ of Gaifman cliques. Let $A, B_1, B_2 \in \mathcal{C}$ such that $A \subseteq B_1$ and $A \subseteq B_2$. Let C be the free amalgam of B_1 and B_2 over A. We claim that $C \in \mathcal{C}$. If C were not in $\mathcal{C}$, then there is a Gaifman clique $Q \in \mathcal{F}$ and embedding $g : Q \to C$. Moreover, there are two vertices $u, v \in Q$ with $u \in B_1 \setminus A$ and $v \in B_2 \setminus A$. But u and v are related by some $R \in \mathcal{L}$, contradicting C a free amalgam. ■

Theorem 3.2.8. *Let $\mathcal{C}$ be a free amalgamation class of finite structures over a finite relational language. Then $\mathcal{C}$ has Gaifman clique faithful coherent EPPA.*

Proof. By Lemma 3.2.7, we have that $\mathcal{C} = \mathrm{Forb_e}(\mathcal{F})$ for some family $\mathcal{F}$ of Gaifman cliques. Let $A \in \mathcal{C}$, and consider the Gaifman clique faithful coherent EPPA-extension B of A guaranteed by Theorem 3.1.13. We already know that every $p \in \mathrm{Part}(A)$ extends to some $\hat{p} \in \mathrm{Aut}(B)$, and the map $p \mapsto \hat{p}$ is coherent. It remains to show that $B \in \mathcal{C}$. Suppose for the sake of a contradiction that $B \notin \mathcal{C}$, then there is some Gaifman clique $Q \in \mathcal{F}$ such that $Q \subseteq B$. By Gaifman clique faithfulness, there is $g \in \mathrm{Aut}(B)$ such that $g(Q) \subseteq A$. This means A contains a forbidden structure, contradicting $A \in \mathcal{C}$. Thus, $B \in \mathcal{C}$ and we are done. ■

We formulate, by means of the next definition and proposition, the technique we have used above in a more general setting.

Definition 3.2.9. Let $\mathcal{F}$ be a family of finite $\mathcal{L}$-structures. A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is said to have *$\mathcal{F}$-faithful EPPA* if for every $A \in \mathcal{C}$, there exists an EPPA-extension $B \in \mathcal{C}$ of A such that for every $F \in \mathcal{F}$ with $F \subseteq B$ there is $g \in \mathrm{Aut}(B)$ such that $g(F) \subseteq A$.

Proposition 3.2.10. *Suppose that the class of all finite $\mathcal{L}$-structures has $\mathcal{F}$-faithful (coherent) EPPA. Then the class* $\mathrm{Forb}_e(\mathcal{F})$ *has (coherent) EPPA.*

We now proceed towards the existence of ample generics for free homogeneous structures. Note that any free amalgamation class $\mathcal{C}$ has APA—see Definition 2.1.1. For suppose that $A, B_1, B_2 \in \mathcal{C}$ with $A \subseteq B_1$ and $A \subseteq B_2$. Let $C \in \mathcal{C}$ be the free amalgam of B_1 and B_2 over A. Suppose that $f \in \mathrm{Aut}(B_1)$ and $g \in \mathrm{Aut}(B_2)$ such that $f\restriction_A = g\restriction_A$. Then $h := f \cup g \in \mathrm{Aut}(C)$. Thus, using Theorem 3.2.8 and Theorem 2.1.5 we obtain the following.

Theorem 3.2.11. *Suppose that M is a free homogeneous structure over a finite relational language. Then M has ample generics.*

Any homogeneous structure over a finite relational language is ω-categorical. Therefore, based on the discussion contained in Section 2.3 we infer the following.

Corollary 3.2.12. *Suppose that M is a free homogeneous structure over a finite relational language. Then* $\mathrm{Aut}(M)$ *has the small index property, uncountable cofinality, 21-Bergman property, and Serre's property (FA).*

Coherent EPPA gives rise to the following group-theoretic observation which we will use in the next section. Suppose that B is a coherent EPPA-extension of A. So there is a coherent map $\phi : \mathrm{Part}(A) \to \mathrm{Aut}(B)$ such that $p \subseteq \phi(p)$ for all $p \in \mathrm{Part}(A)$. In particular, the restriction $\phi : \mathrm{Aut}(A) \to \mathrm{Aut}(B)$ is a group embedding (monomorphism) such that $g \subseteq \phi(g)$ for all $g \in \mathrm{Aut}(A)$. This observation is closely related to [4, Lemma 1.2] where A and B are finite graphs.

3.3 A Dense Locally Finite Subgroup

By now we know that free amalgamation classes have coherent EPPA. So what implications does this fact have for the automorphism group G of a free homogeneous structure? We will find out that G contains a dense locally finite subgroup.

The following lemma is a generalisation of [4, Lemma 1.2(i)].

Lemma 3.3.1. *Let $\mathcal{C}$ be a free amalgamation class of finite $\mathcal{L}$-structures. Let $B \in \mathcal{C}$, and $b \in B$. Put $A = B \setminus \{b\}$. Then there is a structure $C \in \mathcal{C}$ with $B \subseteq C$, and a group embedding $\phi : \mathrm{Aut}(A) \to \mathrm{Aut}(C)$ such that $g \subseteq \phi(g)$ for each $g \in \mathrm{Aut}(A)$.*

Proof. Put $H := \mathrm{Aut}(A)$. For each $h \in H$, let b_h be a new element and $B_h := A \cup \{b_h\}$ be an $\mathcal{L}$-structure such that $\chi_h := h \cup (b, b_h) : B \to B_h$ is an isomorphism. Take the free amalgam C of all $(B_h : h \in H)$ over A. This yields $C = A \cup \{b_h : h \in H\}$ and $C \in \mathcal{C}$. We identify B with B_{1_A}, and b with b_{1_A}, where 1_A is the identity map on A, so $A \subseteq B \subseteq C$.

We now define a group embedding $\phi : \mathrm{Aut}(A) \to \mathrm{Aut}(C)$. For $g \in H = \mathrm{Aut}(A)$, we define $\hat{g} \in \mathrm{Aut}(C)$ extending g as follows. For each $a \in A$, put $\hat{g}(a) := g(a)$, and for each $b_h \in \{b_h : h \in H\}$, put $\hat{g}(b_h) := b_{gh}$. Finally, define $\phi(g) := \hat{g}$. It remains to check that $\hat{g} \in \mathrm{Aut}(C)$. If $h, h' \in H$ are distinct, then by free amalgamation, $b_h, b_{h'}$ are not related by any relation of the language. Suppose that $R \in \mathcal{L}$ and $\bar{a} \in A$, then:

$$\begin{aligned} C \models R(b_h, \bar{a}) \Leftrightarrow B_h \models R(b_h, \bar{a}) \Leftrightarrow B \models R(b, h^{-1}(\bar{a})) \\ \Leftrightarrow B_{gh} \models R(b_{gh}, g(\bar{a})) \Leftrightarrow C \models R(b_{gh}, g(\bar{a})) \\ \Leftrightarrow C \models R(\hat{g}(b_h), \hat{g}(\bar{a})). \end{aligned}$$

The second equivalence holds as $\chi_h : B \to B_h$ is an isomorphism. The same argument works for any permutation of the arguments of $R \in \mathcal{L}$. ■

Remark 3.3.2. We remark that Lemma 3.3.1 follows from coherent EPPA of free amalgamation classes, however, the proof given above is direct. More generally, suppose that $\mathcal{C}$ is a class of finite structures which has coherent EPPA. Let $B \in \mathcal{C}$, and $A = B \setminus \{b\}$ for some $b \in B$. Take a coherent EPPA-extension $C \in \mathcal{C}$ of B. Then the coherent extension procedure gives a group embedding $\phi : \mathrm{Aut}(A) \to \mathrm{Aut}(C)$ such that any $g \in \mathrm{Aut}(A)$ extends to $\phi(g) \in \mathrm{Aut}(C)$.

We are ready to prove a theorem about the automorphism group of a free homogeneous structure, which generalises [4, Theorem 1.1].

Theorem 3.3.3. *Suppose that $\mathcal{C}$ is a free amalgamation class over a finite relational language, and M its Fraïssé limit. Then $\mathrm{Aut}(M)$ contains a dense locally finite subgroup.*

Proof. We will build a chain $A_0 \subseteq A_1 \subseteq \ldots \subseteq A_i \subseteq A_{i+1} \subseteq \ldots$ of finite substructures of M such that $M = \bigcup_{i\in\omega} A_i$, and simultaneously we build a directed system $G_0 \to \ldots \to G_i \xrightarrow{\phi_i} G_{i+1} \to \ldots$ of finite groups such that for each $i \in \omega$ we have that $G_i \leq \mathrm{Aut}(A_i)$ and the map $\phi_i : G_i \to G_{i+1}$ is a group embedding such that $\phi_i(g)$ extends g for every $g \in G_i$. Then, the dense locally finite subgroup of $\mathrm{Aut}(M)$ will be $G = \varinjlim G_i$, the direct limit of the directed sequence $(G_i)_{i\in\omega}$.

Enumerate the domain of $M = \{a_i \mid i \in \omega\}$, and let $\{(\bar{a}_i, \bar{b}_i) \mid i \in \omega\}$ be a list of all pairs $(\bar{a}, \bar{b})$ of finite sequences of M where $\bar{a} = (a_1, a_2, \ldots, a_n)$ and $\bar{b} = (b_1, b_2, \ldots, b_n)$ such that the map $a_i \mapsto b_i$ is an $\mathcal{L}$-isomorphism. Here, the role of $(\bar{a}_i, \bar{b}_i)$ is to ensure that the resultant subgroup G is dense in $\mathrm{Aut}(M)$. Start by putting $A_0 = \{a_0\}$ and $G_0 = \mathrm{Aut}(A_0)$. Suppose stage i has been completed and we have constructed a finite substructure $A_i \subseteq M$ and a group $G_i \leq \mathrm{Aut}(A_i)$. We will proceed to construct stage $i+1$ in three steps.

First step. We ensure that $a_{i+1} \in A_{i+1}$. Suppose that $a_{i+1} \notin A_i$, and put $B = A_i \cup \{a_{i+1}\} \subseteq M$. By Lemma 3.3.1, there is $C \in \mathcal{C} = \mathrm{Age}(M)$ such that $B \subseteq C$, and there is a group embedding $\phi : G_i \to \mathrm{Aut}(C)$ with $\phi(g)$ extending g for every $g \in G_i$. By homogeneity of M we can think of C as a substructure of M containing B. Put $G^C := \phi(G_i) \leq \mathrm{Aut}(C)$. Otherwise, if $a_{i+1} \in A_i$, then put $C = A_i$ and $G^C := G_i$.

Second step. We ensure that $\bar{a}_i \cup \bar{b}_i \subseteq A_{i+1}$. Starting with C and G^C, and by iteratively applying the first step, we construct a finite structure $D \in \mathcal{C}$ such that $C \subseteq D$ and D contains all the coordinates of the tuples $\bar{a}_i$ and $\bar{b}_i$, and there is a group embedding $\psi : G^C \to \mathrm{Aut}(D)$ such that $\psi(g)$ extends g for every $g \in G_C$. Put $G^D := \psi(G^C) \leq \mathrm{Aut}(D)$. At this point we have that $A_i \subseteq C \subseteq D \subseteq M$, and group isomorphisms $G_i \xrightarrow{\phi} G^C \xrightarrow{\psi} G^D$.

Third step. We ensure that G_{i+1} contains some element h with $h(\bar{a}_i) = \bar{b}_i$. By Theorem 3.2.8 the class $\mathcal{C}$ has coherent EPPA. So starting with $D \in \mathcal{C}$, we obtain a finite structure $A_{i+1} \in \mathcal{C}$ such that $D \subseteq A_{i+1}$, and every partial automorphism of D extends to an automorphism of A_{i+1}. Thus, the partial automorphism $\bar{a}_i \to \bar{b}_i$ of D extends to an automorphism $h \in \mathrm{Aut}(A_{i+1})$. Moreover, as the process of extending partial automorphisms given by Theorem 3.2.8 is coherent we get a group embedding $\chi : G^D \to \mathrm{Aut}(A_{i+1})$ such that $\chi(g)$ extends g for every $g \in G^D$. Finish by putting $G_{i+1} := \langle \chi(G^D), h \rangle \leq \mathrm{Aut}(A_{i+1})$.

The first step above ensures that $M = \bigcup_{i\in\omega} A_i$. The second and third steps provide that $G = \varinjlim G_i$ is a dense subgroup of $\mathrm{Aut}(M)$. Finally, the finiteness of each G_i implies that G is locally finite. ■

By Remark 3.3.2, the proof of Theorem 3.3.3 gives the following result for homogeneous structures.

Theorem 3.3.4. *Suppose that M is a homogeneous locally finite structure such that $\mathrm{Age}(M)$ has coherent EPPA. Then $\mathrm{Aut}(M)$ contains a dense locally finite subgroup.*

We have seen that coherent EPPA leads to the existence of a dense locally finite subgroup. The following lemma treats the opposite direction, see [47, Proposition 6.4] for a more general statement.

Proposition 3.3.5. *Let M be a homogeneous relational structure. Suppose that $\mathrm{Aut}(M)$ has a dense locally finite subgroup. Then $\mathrm{Age}(M)$ has EPPA.*

Proof. Let $\Gamma \leq \mathrm{Aut}(M)$ be a dense locally finite subgroup. Fix $A \in \mathrm{Age}(M)$. We may assume that $A \subseteq M$. Let $\mathrm{Part}(A) = \{p_1, \ldots, p_n\}$ be the set of all partial automorphisms of A. By the homogeneity of M there are $f_1, \ldots, f_n \in \mathrm{Aut}(M)$ such that $p_i \subseteq f_i$. As Γ is dense, we may assume that each $f_i \in \Gamma$. Consider the finite subgroup $H = \langle f_1, \ldots, f_n \rangle \leq \Gamma$, and define the finite substructure $B = \bigcup_{h \in H} h(A)$ of M. Clearly, $B \in \mathrm{Age}(M)$. As H is a group we have $h(B) = B$ for all $h \in H$, that is, B is H-invariant. Therefore, each $f_i \restriction_B$ belongs to $\mathrm{Aut}(B)$ and extends p_i. ■

Corollary 3.3.6. *Suppose that M is an ω-categorical structure such that $G = \mathrm{Aut}(M)$ has a dense locally finite subgroup. Then $\mathrm{Th}(M)$ does not have the strict order property.*

Proof. We may assume that M is homogeneous for we can pass to its Morleyisation (see the proof of Theorem 4.2.8) without changing the automorphism group. By Proposition 3.3.5, we get that $\mathrm{Age}(M)$ has EPPA. By Proposition 1.5.14, $\mathrm{Th}(M)$ does not have the strict order property. ■

Example 3.3.7 (A class with EPPA but not APA)**.** Consider a structure (M, E) where M is a countably infinite set, and E is a binary relation symbol interpreted as an equivalence relation with two equivalence classes, both infinite. Let (A_n, E) be a structure of size $2n$ where E is interpreted as an equivalence relation with two equivalence classes, each of size n. We may assume that $A_n \subseteq A_{n+1} \subseteq M$ for each $n \in \omega$. Then $M = \bigcup\limits_{n \in \omega} A_n$. We also have that $H := \bigcup\limits_{n \in \omega} \mathrm{Aut}(A_n)$ is a dense locally finite subgroup of $\mathrm{Aut}(M)$. Thus, Proposition 3.3.5 implies that $\mathrm{Age}(M)$ has EPPA. However, by Example 1.4.5, M does not have generic automorphisms, and so $\mathrm{Age}(M)$ does not have APA.

Remark 3.3.8. We think it might be possible to show that if M is a free homogeneous structure, then $\mathrm{Aut}(M)$ contains a dense locally finite *simple* subgroup H, so strengthening Theorem 3.3.3. The proposal is to construct H as in the proof of Theorem 3.3.3, so $H = \bigcup_{i \in \omega} H_i$ where $H_i \leq H_{i+1}$ and $H_i \leq \mathrm{Aut}(A_i)$ for some finite $A_i \subseteq M$, and additionally ensure that each H_i is a simple group. So H, being a union of an increasing sequence of simple groups, is itself a simple group. The candidate for H_i is $\mathrm{Alt}(n)$, the alternating group of degree n, for some $n \geq 5$. To achieve this, we need to check that we may use alternating groups instead of symmetric groups in the proof of Herwig-Lascar [36, Lemma 4.9] as such groups induce the desired automorphisms on the extension in the definition of EPPA—see the note below Definition 4.6 in [36]. We also note that in Section 3.1 the group acting on the structure C which ensures EPPA by Hodkinson-Otto [40] is isomorphic to a subgroup of $\mathrm{Aut}(B)$ where the existence of B is guaranteed by the aforementioned work of Herwig-Lascar.

Example 3.3.9. We give an example of a free amalgamation class which cannot be written as a class which forbids a family of structures under homomorphisms, that is, in the Herwig-Lascar sense—see Theorem 1.5.7. Let $\mathcal{L}$ be the language of 3-hypergraphs, that is, $\mathcal{L}$ contains one ternary relation symbol R. A 3-hypergraph is an $\mathcal{L}$-structure such that R is interpreted as an irreflexive symmetric ternary relation. A 3-tuple which satisfies R is called a hyperedge. Let Q be a 3-hypergraph on four vertices with exactly 3 hyperedges. Let $\mathcal{C}$ be the class of all finite 3-hypergraphs which forbid Q under embeddings. The class $\mathcal{C}$ is a free amalgamation class, and so has EPPA by Theorem 3.2.8 above. Recall that a tetrahedron T is a complete 3-hypergraph on four vertices, and note that $T \in \mathcal{C}$. Now suppose that there is a finite set of finite $\mathcal{L}$-structures such that $\mathcal{C}$ is the class of all finite structures which are $\mathcal{F}$-free under homomorphisms. Then as $Q \notin \mathcal{C}$, there is $F \in \mathcal{F}$ and a homomorphism $h : F \to Q$. Let $\alpha : Q \to T$ be a bijective map. Then α is a homomorphism, and so $\alpha h : F \to T$ is a homomorphism too. So T is not $\mathcal{F}$-free, contradicting that $T \in \mathcal{C}$.

Chapter 4

The Universal Bowtie-free Graph

The main research problem from which this chapter stems is the problem of existence of a countably infinite universal graph which forbids finitely many finite graphs as subgraphs, rather than just as induced subgraphs. The first examples of such universal graphs are the random graph and the universal homogeneous K_n-free graph. We focus on the case of a bowtie-free universal graph, where a bowtie ($\bowtie$) is the graph consisting of two triangles glued at one common vertex. A bowtie-free universal graph was first proved to exist by Komjáth [48], a result which was not attainable via the Fraïssé amalgamation technique at the time. Such an obstacle provided the motivation behind the combinatorial theory developed by Cherlin, Shelah, and Shi [15] which established the existence of an ω-categorical universal bowtie-free graph $\mathcal{U}_{\bowtie}$ and other universal graphs via the algebraic closure operator. Their theory and the uniqueness of $\mathcal{U}_{\bowtie}$ is discussed in Section 4.1. Hubička and Nešetřil [42] are also interested in bowtie-free graphs; they wrote that the class of finite bowtie-free graphs "plays a key role in the context of both Ramsey theory and model theory in the area related to universality and homogeneity. It is the interplay of these two fields which makes this example interesting and important". In Section 4.2 we extend an amalgamation lemma in [42] regarding a cofinal subclass of the class of all finite bowtie-free graphs. Consequently, via a variation of Fraïssé's amalgamation technique, we obtain a universal bowtie-free graph isomorphic to $\mathcal{U}_{\bowtie}$. Moreover, we have:

Theorem. *The universal bowtie-free graph $\mathcal{U}_{\bowtie}$ admits generic automorphisms.*

4.1 Universal Graphs with Forbidden Subgraphs

In this section we present the model theoretic approach developed in Cherlin-Shelah-Shi [15] to the problem of existence of a universal graph with forbidden subgraphs. Let $\mathcal{F}$ be a family of finite graphs, viewed as 'forbidden' graphs. A graph G is called *$\mathcal{F}$-free* if no graph in $\mathcal{F}$ is isomorphic to a (not necessarily induced) subgraph of G. It is $\mathcal{F}$-free with respect to injective homomorphisms in terms of Section 1.5. Denote by $\mathcal{C}_{\mathcal{F}}$ the class of all *countable* (finite and countably infinite) $\mathcal{F}$-free graphs. A graph $G \in \mathcal{C}_{\mathcal{F}}$ is *universal* for $\mathcal{C}_{\mathcal{F}}$ if every graph in $\mathcal{C}_{\mathcal{F}}$ is isomorphic to an *induced* subgraph of G. For graphs G, H, by $G \subseteq H$ we mean that G is an induced subgraph of H.

We collect below some positive and negative results regarding the existence of a countable universal graph. We first describe a graph generalising the bowtie. Given a collection $K_{n_1}, K_{n_2}, \dots, K_{n_k}$ of complete graphs, their *bouquet* $K_{n_1} + K_{n_2} + \dots + K_{n_k}$ is the graph formed by taking the free amalgam of the given complete graphs over one common vertex. The bouquet $K_3 + K_3$ is called the bowtie. Moreover, a graph is 2-*connected* if it is connected, and remains connected after deleting any vertex together with the edges incident with it.

Example 4.1.1.

(i) (Rado [64]). The class $\mathcal{C}_{\emptyset}$ of all countable graphs has a universal element.

(ii) (Cherlin-Shi [16]). Suppose that $\mathcal{F}$ is a finite set of cycles. Then there is a countable universal $\mathcal{F}$-free graph if and only if $\mathcal{F} = \{C_3, C_5, C_7, \dots, C_{2k+1}\}$ for some $k \geq 1$.

(iii) (Komjáth [48]). There is a countable universal bowtie-free graph.

(iv) (Cherlin-Tallgren [17]). Let $F = K_m + K_n$ be a bouquet where $m \leq n$. Then there is a countable universal F-free graph if and only if $1 \leq m \leq 5$ and $(m, n) \neq (5, 5)$.

(v) (Komjáth [48]). Let $m, n \geq 3$. If $F = m \cdot K_n$, the bouquet of m-many copies of K_n, then there is no F-free countable universal graph.

(vi) (Cherlin-Komjáth [13]). There is no countable universal C_n-free graph for $n \geq 4$. Here C_n is a cycle of length n.

(vii) (Füredi-Komjáth [29]). If F is a finite, 2-connected, but not complete graph, then there is no countable universal F-free graph.

We work with the language of graphs $\mathcal{L} = \{E\}$. Denote by $T_{\mathcal{F}}$ the theory of the class $\mathcal{C}_{\mathcal{F}}$. That is, the theory $T_{\mathcal{F}}$ is the set of all $\mathcal{L}$-sentences true in all members of $\mathcal{C}_{\mathcal{F}}$. Note that $T_{\mathcal{F}}$ is a universal theory. Recall Definition 1.1.3.

Definition 4.1.2. [15, Definition 2]

(i) Let H be a graph, and $G \subseteq H$ an induced subgraph. We say that G is *existentially closed* in H if for every existential sentence $\exists \bar{x} \phi(\bar{x})$ with parameters from G we have that if $H \models \exists \bar{x} \phi(\bar{x})$ then $G \models \exists \bar{x} \phi(\bar{x})$.

(ii) A graph $G \in \mathcal{C}_{\mathcal{F}}$ is *existentially closed* in $\mathcal{C}_{\mathcal{F}}$ if G is existentially closed in every graph $H \in \mathcal{C}_{\mathcal{F}}$ containing G.

(iii) Denote by $\mathcal{E}_{\mathcal{F}}$ the class of all existentially closed graphs in $\mathcal{C}_{\mathcal{F}}$. And let $T^{ec}_{\mathcal{F}}$ be the theory of the class $\mathcal{E}_{\mathcal{F}}$.

Remark 4.1.3. A graph $G \subseteq H$ being existentially closed in H is equivalent to the following condition: if $A \subseteq B$ are finite graphs such that $A \subseteq G$ and $B \subseteq H$ then there is an embedding $f : B \to G$ such that $f\restriction_A$ is the identity.

The notions above are not special for graphs. For example, the existentially closed elements in the class of fields are the algebraically closed fields. The existentially closed elements in the class of ordered fields are the real closed fields. Dense linear orders are existentially closed in the class of linear orders. Existentially closed first order structures appear in model theory in Abraham Robinson's work on model complete theories—see [59, Chapter 3], [10, Section 3.5], and [37]. A theory T is said to be *model complete* if whenever $M, N \models T$ and $M \subseteq N$, then $M \preceq N$. Robinson's Test [59, Theorem 3.2.1] states that the following are equivalent for an $\mathcal{L}$-theory T:

(i) T is model complete.

(ii) Whenever $M, N \models T$ with $M \subseteq N$, then M is existentially closed in N.

(iii) Every $\mathcal{L}$-formula is equivalent to an existential formula modulo T.

(iv) Every $\mathcal{L}$-formula is equivalent to a universal formula modulo T.

Suppose that $\mathcal{K}$ is an elementary class (Definition 1.1.3) of $\mathcal{L}$-structures which is closed under unions of chains. Then every element $M \in \mathcal{K}$ can be extended to an element $\bar{M} \in \mathcal{K}$ which is existentially closed in $\mathcal{K}$ [10, Lemma 3.5.7]. Let $\mathcal{E}(\mathcal{K})$ be the subclass of all existentially closed structures in $\mathcal{K}$. Then $\mathcal{E}(\mathcal{K})$ may not be an elementary class. Eklof and Sabbagh proved that the class of existentially closed groups is not elementary [59, Theorem 3.5.7].

Proposition 4.1.4. [10, Proposition 3.5.15] *Let $\mathcal{K}$ be an elementary class of $\mathcal{L}$-structures closed under unions of chains. Let $T := \mathrm{Th}(\mathcal{K})$ and $T^{ec} := \mathrm{Th}(\mathcal{E}(\mathcal{K}))$. Then T^{ec} is model complete if and only if $\mathcal{E}(\mathcal{K})$ is elementary.*

We get back to our setting of graphs. Cherlin, Shelah, and Shi proved the following which in view of the proposition above shows that $T^{ec}_{\mathcal{F}}$ is model complete when $\mathcal{F}$ is finite.

Theorem 4.1.5. [15, Theorem 1] *Let $\mathcal{F}$ be a finite family of finite graphs. Then a countable graph $G \in \mathcal{E}_{\mathcal{F}}$ if and only if $G \models T^{ec}_{\mathcal{F}}$. Moreover, if every $F \in \mathcal{F}$ is connected, then $T^{ec}_{\mathcal{F}}$ is a complete theory.*

Example 4.1.6. [15, Example 4] Let $\mathcal{F} = \{S_3\}$ where S_3 is a star of degree 3, that is, a graph of 4 vertices where one vertex is adjacent to the other three, and there are no more edges. Then $T_{\mathcal{F}}$ is the theory of graphs in which every vertex has degree at most 2. And $T^{ec}_{\mathcal{F}}$ is the theory of graphs in which every vertex has degree 2, and which contain infinitely many cycles C_n for each $n \geq 3$. Let $\mathbb{Z}$ be the 2-way infinite path, that is, vertices are the integers, and every n is adjacent to $n+1$. Then a countable model of $T^{ec}_{\mathcal{F}}$ is characterised up to isomorphism by the number of its connected components isomorphic to $\mathbb{Z}$. Let $G_k \models T^{ec}_{\mathcal{F}}$ be the countable model with k-many components isomorphic to $\mathbb{Z}$. Then $\mathcal{E}_{\mathcal{F}} = \{G_k : k \in \omega + 1\}$. Moreover $G_\omega \in \mathcal{C}_{\mathcal{F}}$ is a universal $\mathcal{F}$-free graph. Remember that the members of $\mathcal{C}_{\mathcal{F}}$ and $\mathcal{E}_{\mathcal{F}}$ are countable.

Definition 4.1.7. Suppose that M is an $\mathcal{L}$-structure, and let $A \subseteq M$. The *algebraic closure* $\mathrm{acl}_M(A)$ of A in M is the union of all finite A-definable subsets of M.

Theorem 4.1.8. [15, Theorem 3] *Let $\mathcal{F}$ be a finite family of connected finite graphs. Then the following are equivalent.*

(i) The theory $T^{ec}_{\mathcal{F}}$ is ω-categorical.

(ii) For any finite $A \subseteq M \models T^{ec}_{\mathcal{F}}$, we have that $\mathrm{acl}_M(A)$ is finite.

Proposition 4.1.9. [15, Proposition 1] *Let $G \in \mathcal{E}_{\bowtie}$ be a countable existentially closed bowtie-free graph, and let $A \subseteq G$ be finite. Then $|\mathrm{acl}_G(A)| \leq 4|A|$.*

As every graph $G \in \mathcal{C}_{\mathcal{F}}$ embeds in some graph $\bar{G} \in \mathcal{E}_{\mathcal{F}}$, we have that $\mathcal{C}_{\mathcal{F}}$ contains a universal element if and only if $\mathcal{E}_{\mathcal{F}}$ contains a universal element. Therefore, by the last two theorems and proposition above we have that $\mathcal{E}_{\bowtie} = \{G \text{ graph} : G \models T^{ec}_{\bowtie} \text{ and } |G| = \aleph_0\}$ contains exactly one element; an ω-categorical existentially closed universal bowtie-free graph. We denote this universal bowtie-free graph by $\mathcal{U}_{\bowtie}$.

4.2 Bowtie-free Graphs

Let $\mathcal{L} = \{E\}$ be the language of graphs. Recall that a bowtie ($\bowtie$) is the graph formed by freely amalgamating two triangles over one common vertex. A graph is called bowtie-free if it has no (not necessarily induced) subgraph isomorphic to the bowtie. Also $\mathcal{C}_{\bowtie}$ is the class of all countable bowtie-free graphs. Let $\mathcal{C}^0_{\bowtie}$ denotes the class of all finite bowtie-free graphs. Notice that a graph is bowtie-free if and only if it has no induced subgraph isomorphic to a graph B where $\bowtie \subseteq B \subseteq K_5$. So in view of Definition 3.2.5, we have that $\mathcal{C}^0_{\bowtie} = \mathrm{Forb}_e(\mathcal{F})$ for $\mathcal{F} = \{B \text{ graph} : \bowtie \subseteq B \subseteq K_5\} \cup \{A_1, A_2\}$ where A_1, A_2 are $\mathcal{L}$-structures such that $A_1 = \{a\}$ with $E(a, a)$, and $A_2 = \{a, b\}$ with $a \neq b$ and with $E(x, y)$ holding if and only if $x = a$ and $y = b$.

Following Hubička and Nešetřil in [42], a *chimney* is the free amalgam of *two* or more triangles over one common edge. Moreover, we expand this terminology as follows. We call the vertices of the common edge *base vertices*, and the rest we call them *tip vertices*. We also call the number of tip vertices the *height* of the chimney. Any chimney contains exactly two base vertices, and at least two tip vertices.

Fact 4.2.1 ([15], [42])**.** Suppose that G is a finite connected bowtie-free graph such that every edge is contained in some triangle. If $K_4 \subseteq G$, then $G = K_4$. Otherwise, G is a chimney or a triangle.

Definition 4.2.2. A bowtie-free graph is called *special* if every vertex is contained either in a K_4 or in a chimney.

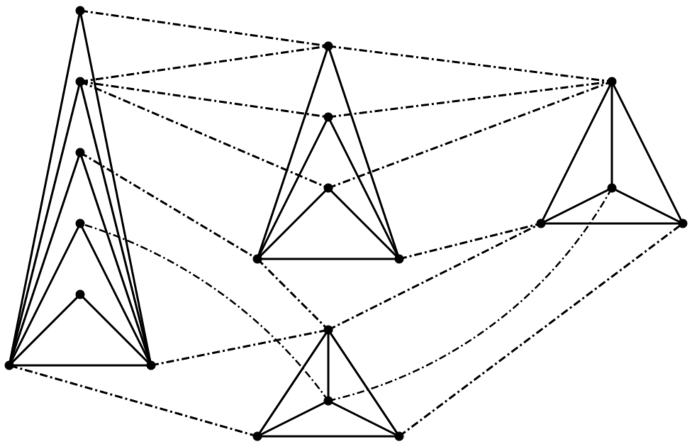


Figure 4.1: A special bowtie-free graph. Any solid edge lies in some triangle, while a dashed edge does not.

The definition of special bowtie-free graphs is due to [42], though they call them 'good' instead. Let $\mathcal{C}_{\bowtie}^{sp}$ denotes the class of all finite special bowtie-free graphs. It should be noted that every vertex of a special bowtie-free graph lies in a triangle, and a triangle is bowtie-free, but not special.

Fact 4.2.3 ([42])**.** Let G be a special bowtie-free graph. By deleting all the edges of G which do not lie in any triangle, we obtain a disjoint union of copies of K_4 and chimneys.

Therefore, any finite special bowtie-free graph can be constructed in two stages. First, take a disjoint union of finitely many graphs $H_1, H_2, \ldots, H_n$ where each one is either a chimney or copy of K_4. Second, to add an extra edge $e = \{u, v\}$, we must have that $u \in H_i$, $v \in H_j$ for distinct i, j, and ensure that the edge e will not create a new triangle; otherwise a bowtie will appear.

Clearly the class $\mathcal{C}^0_{\bowtie}$ of finite bowtie-free graphs has the joint embedding property. However $\mathcal{C}^0_{\bowtie}$ does not have the amalgamation property as shown in the figure below. Hence, owing to Fact 1.5.4 we deduce that $\mathcal{C}^0_{\bowtie}$ does not have EPPA. Nevertheless $\mathcal{C}^0_{\bowtie}$ contains a cofinal subclass with the free amalgamation property.

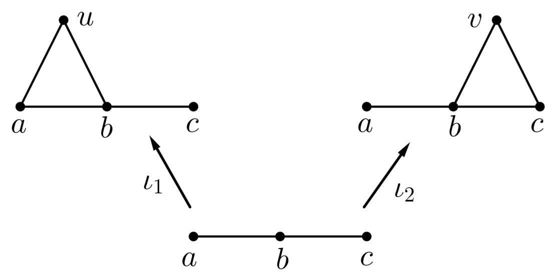


Figure 4.2: Any amalgam of the diagram above contains a bowtie.

Lemma 4.2.4 ([42])**.** *The subclass $\mathcal{C}^{sp}_{\bowtie}$ of special bowtie-free graphs is cofinal in the class $\mathcal{C}^0_{\bowtie}$ of finite bowtie-free graphs. That is, any finite bowtie-free graph is an induced subgraph of a special bowtie-free graph.*

Proof. Let G be a bowtie-free graph. Suppose $v \in G$ is a vertex that is neither contained in a K_4 nor in a chimney. If v is not contained in a triangle, then add a new copy of K_4 and identify v with one of its vertices. Otherwise, v is part of a triangle, say vxz, of G. In this case, add a new vertex u together with edges uv, ux, and uz, making $vxzu$ isomorphic to a K_4. One can show that neither of these two actions will introduce a bowtie. Repeat this process until a special bowtie-free graph has been constructed. ■

The following proposition is of a more general form than [42, Lemma 3.1] where special bowtie-free graphs are amalgamated over their induced subgraph on bases of chimneys and copies of K_4.

Proposition 4.2.5. *The class $\mathcal{C}_{\bowtie}^{sp}$ of all finite special bowtie-free graphs has the free amalgamation property.*

Proof. Suppose A, B_1, B_2 are finite special bowtie-free graphs such that $A \subseteq B_1$ and $A \subseteq B_2$. Let C be the free amalgam of B_1 and B_2 over A. We will show that $C \in \mathcal{C}_{\bowtie}^{sp}$. By free amalgamation, any triangle in C either lives entirely in B_1 or entirely in B_2. For the sake of contradiction, suppose C has a bowtie $T = \{a, b, c, u, v\}$ as a subgraph where c is the common vertex of degree four, and abc and cuv are triangles. As B_1 and B_2 are bowtie-free, we have that T is neither contained in B_1 nor in B_2. First, the vertex c must be in A, otherwise one of the triangles abc or cuv will meet both $B_1 \setminus A$ and $B_2 \setminus A$. Second, as the two triangles cannot both be in B_1 nor both in B_2, suppose without loss of generality that abc lives in B_1 with $a \in B_1 \setminus A$, and cuv lives in B_2 with $u \in B_2 \setminus A$.

By the hypothesis, A is a special bowtie-free graph, so the vertex c is either contained in a K_4 of A, or in a chimney of A. Supposing the former, then the triangle abc together with any triangle in A which contains c but not b in the K_4 will form a bowtie inside B_1, contradicting that B_1 is bowtie-free. So c must be contained in a chimney M of A. There are six possibilities in this situation, based on whether c is a tip or a base vertex of M. All lead to a contradiction.

Case 1: Suppose that c is a tip vertex of M, and $b \in M$. Then b must be a base vertex of M as it is connected to c, and so the triangle abc with any triangle of M not containing c will form a bowtie in B_1, a contradiction.

Case 2: Suppose that c is a tip vertex of M, and $b \notin M$. Then the triangle abc with the triangle in M containing c form a bowtie in B_1, a contradiction.

Case 3: Suppose that c is a base vertex of M and $b \notin M$. Then the triangle abc together with any triangle in M will form a bowtie in B_1, a contradiction.

Case 4: Suppose that c is a base vertex of M and b is a tip vertex of M. Then the triangle abc with another triangle of M not containing b will form a bowtie in B_1, a contradiction.

Case 5: Suppose that b, c are the base vertices of M, and $v \notin M$. Then the triangle cuv together with any triangle of M will form a bowtie in B_2, a contradiction.

Case 6: Suppose that b, c are the base vertices of M, and $v \in M$. So v must be a tip vertex of M. In this case, the triangle cuv together with a triangle of M not containing v will form a bowtie in B_2, a contradiction.

Hence, the free amalgam C is bowtie-free. Now we show C is special. Any vertex $v \in C$ is either in B_1 or B_2. Say $v \in B_1$. As B_1 is special, the vertex v lies either in a K_4 or in a chimney of B_1. If v were in a K_4 of B_1, then it will be in the same K_4 in C. Otherwise, if v were in a chimney of B_1, then v will be in chimney of C, possibly of greater height, containing the original chimney. Therefore C is a special bowtie-free graph. ■

Now we apply an argument by Ivanov [44, Theorem 3.1] to obtain the following result on extending partial automorphisms. We call such argument the 'necklace argument'.

Proposition 4.2.6. *Suppose that $G \in \mathcal{C}_{\bowtie}^{sp}$ is a finite special bowtie-free graph, and $(p : U \to V) \in \mathrm{Part}(G)$ with $U, V \in \mathcal{C}_{\bowtie}^{sp}$. Then there is $K \in \mathcal{C}_{\bowtie}^{sp}$ such that $G \subseteq K$ and p extends to some $f \in \mathrm{Aut}(K)$.*

Proof. By the previous proposition, $\mathcal{C}_{\bowtie}^{sp}$ has the free amalgamation property. The idea of constructing the desired graph K is to form a 'necklace' whose beads are isomorphic copies of G, and in which the range of p in one bead is amalgamated with the domain of p in the consecutive bead. Start with the triple $G_0 := G, U_0 := U, p_0 := p$. Let (G_1, U_1, p_1) be a new copy of (G_0, U_0, p_0). Take the free amalgam $G_0 \cup G_1 \in \mathcal{C}_{\bowtie}^{sp}$ of G_0 and G_1 identifying $p_0(U_0)$ with U_1. One can check that in $G_0 \cup G_1$, the maps p_0, p_1 agree on $U_0 \cap U_1$. So using the isomorphism between G_0 and G_1 we can extend $p_0 \cup p_1$ to a map $g_1 : G_0 \to G_1$ in $\mathrm{Part}(G_0 \cup G_1)$.

Let (G_2, U_2, p_2) be a new copy of (G_1, U_1, p_1). Form the free amalgam $G_0 \cup G_1 \cup G_2$ in $\mathcal{C}_{\bowtie}^{sp}$ of $G_0 \cup G_1$ and G_2 identifying $p_1(U_1)$ with U_2. Using the isomorphism between G_1 and G_2, extend the map $p_0 \cup p_1 \cup p_2$ to a map $g_2 : G_0 \cup G_1 \to G_1 \cup G_2$ in $\mathrm{Part}(G_0 \cup G_1 \cup G_2)$.

We continue this construction until we reach $n \in \omega$ such that the length of any complete cycle of p divides n, and n is strictly greater than the length of any partial cycle of p. At this point, we have that $\bar{G} = G_0 \cup \ldots \cup G_n$ in $\mathcal{C}^{sp}_{\bowtie}$ and a map $g := g_n : G_0 \cup \ldots \cup G_{n-1} \to G_1 \cup \ldots \cup G_n$ in $\mathrm{Part}(\bar{G})$ extending $p_0 \cup \ldots \cup p_n$. By the choice of n, we have that (i) for all $a \in G_0 \cap G_n$ we have that $g^n(a) = a$, and (ii) $G_0 \cap G_n = G_1 \cap G_n = \{a \in U_0 : g^k(a) = a \text{ for some } k > 0\}$. Point (i) implies that p_0 and p_n agree on $U_0 \cap U_n$. Point (ii) says that $G_0 \cap G_n = G_1 \cap G_n$ contains exactly the points which are in complete cycles of p. At this point, half of the necklace has been constructed.

Claim. The induced subgraph on $G_0 \cup G_n \subseteq \bar{G}$ belongs to $\mathcal{C}^{sp}_{\bowtie}$.

Proof of the claim. As $G_0 \cup G_n$ is the free amalgam of G_0 and G_n over $G_0 \cap G_n$, and both $G_0, G_n \in \mathcal{C}^{sp}_{\bowtie}$, it is enough to show that $G_0 \cap G_n \in \mathcal{C}^{sp}_{\bowtie}$. By point (ii) we have that $v \in G_0 \cap G_n$ if and only if v belongs to a complete cycle of p. Fix some $v \in G_0 \cap G_n$, then there is a complete k-cycle, say $(v = v_0, v_1, v_2, \ldots, v_{k-1})$ where $v_i = p^i(v)$ and $v = p^k(v)$ for some $k < \omega$ and $0 \leq i < k$. As $v_0 \in U$ and $U \in \mathcal{C}^{sp}_{\bowtie}$, there are two cases. First case: $v_0 \in Q_0 \subseteq U = \mathrm{dom}(p)$ where $Q_0 \cong K_4$. Because $\mathrm{range}(p) = V \in \mathcal{C}^{sp}_{\bowtie}$ as well, there are (not necessarily distinct) copies $Q_0, Q_1, \ldots, Q_{k-1}$ of K_4 such that $v_i \in Q_i$, and each $Q_i \subseteq U$, and $p(Q_i) = Q_{i+1}$ where addition is performed modulo k. This means all vertices in $Q_0 \cup Q_1 \cup \ldots \cup Q_{k-1}$ are in complete cycles of p. So $v \in Q_0 \subseteq G_0 \cap G_n$. Second case: $v \in M \subseteq U$ where M is a chimney. We may assume that M is a maximal such chimney. Then similarly as in the first case, we get that all the vertices in M belong to complete cycles of p. So $v \in M \subseteq G_0 \cap G_n$. So every vertex in $G_0 \cap G_n$ either belongs to a K_4 or a chimney which is contained in $G_0 \cap G_n$. Thus $G_0 \cap G_n$ is a special bowtie-free graph, and so is $G_0 \cup G_n$, establishing the claim.

Take a new copy $\bar{H} = H_0 \cup H_1 \cup \ldots \cup H_n$ of $\bar{G}$, and let $h : H_0 \cup \ldots \cup H_{n-1} \to H_1 \cup \ldots \cup H_n$ be the corresponding copy of g. Here $\bar{H}$ is the other half of the necklace. Let $\beta := g^n{\restriction}_{G_0} : G_0 \to G_n$ be the isomorphism induced by g^n. Using β and the isomorphism between $\bar{G}$ and $\bar{H}$, construct the free amalgam $K \in \mathcal{C}^{sp}_{\bowtie}$ of $\bar{G}$ and $\bar{H}$ over $G_0 \cup G_n$ where

G_0 is identified with H_n, and G_n is identified with H_0. Let $f := g \cup h$. Points (i) and (ii) guarantee that, under this identification, the restriction of g to $G_0 \cup G_n$ agrees with the restriction of h to $H_0 \cup H_n$. So f is a well-defined map, and moreover, f is a permutation of K. Finally, as $g \in \mathrm{Part}(\bar{G})$ and $h \in \mathrm{Part}(\bar{H})$ agree on $\mathrm{dom}(g) \cap \mathrm{dom}(h)$ in K, and K is a free amalgam of $\bar{G}$ and $\bar{H}$, we have that $f = g \cup h \in \mathrm{Aut}(K)$, and clearly f extends p. ■

So, the class $\mathcal{C}^{sp}_{\bowtie}$ of all finite special bowtie-free graphs has the free amalgamation property. Moreover, the class $\mathcal{C}^{sp}_{\bowtie}$ is closed under disjoint unions, and so it has the joint embedding property. However $\mathcal{C}^{sp}_{\bowtie}$ is not closed under induced subgraphs. In this situation, we can apply a slight variation of Fraïssé's Theorem which does not require the class of finite structures in hand to have the hereditary property. More precisely, we apply Kueker-Laskowski [50, Theorem 1.5] to the 'smooth class' $(\mathcal{C}^{sp}_{\bowtie}, \subseteq)$ and obtain the following.

Theorem 4.2.7. *There is a unique, up to isomorphism, graph $\mathcal{U}_{\bowtie}$ such that:*

(i) The graph $\mathcal{U}_{\bowtie} = \bigcup_{i \in \omega} G_i$ where $G_i \in \mathcal{C}^{sp}_{\bowtie}$ and $G_i \subseteq G_{i+1}$ for all $i \in \omega$.

(ii) Every $H \in \mathcal{C}^{sp}_{\bowtie}$ embeds into $\mathcal{U}_{\bowtie}$.

(iii) Every finite isomorphism $f : G \to H$ where $G, H \in \mathcal{C}^{sp}_{\bowtie}$ and $G, H \subseteq \mathcal{U}_{\bowtie}$ extends to an automorphism of $\mathcal{U}_{\bowtie}$.

We know that $\mathcal{C}^{sp}_{\bowtie}$ is cofinal in $\mathcal{C}^{o}_{\bowtie}$. Consequently, by Kueker-Laskowski [50, Lemma 2.4], $\mathcal{U}_{\bowtie}$ of Theorem 4.2.7 above is an existentially closed model of the universal theory $T_{\bowtie}$, that is, $\mathcal{U}_{\bowtie} \in \mathcal{E}_{\bowtie}$. By Cherlin-Shelah-Shi [15] the theory of existentially closed bowtie-free graphs is ω-categorical. Therefore, the graph $\mathcal{U}_{\bowtie}$ is isomorphic to the ω-categorical universal countable bowtie-free graph introduced at the end of the previous section.

We aim now to describe the algebraic closure of a finite induced subgraph of the universal bowtie-free graph $\mathcal{U}_{\bowtie}$. In [15], an edge in $\mathcal{U}_{\bowtie}$ is called a *special edge* if it lies in two triangles of $\mathcal{U}_{\bowtie}$. It was shown in [15, Proposition 1] that: (i) Every triangle in $\mathcal{U}_{\bowtie}$

contains a special edge. (ii) If a vertex $v \in \mathcal{U}_{\bowtie}$ lies in a triangle T, but not in a special edge of T, then v lies in unique triangle. (iii) If a vertex $v \in \mathcal{U}_{\bowtie}$ lies in two special edges then v lies in some $Q \cong K_4$, and thus any triangle containing v is contained in Q. It was shown further that for a finite induced subgraph $A \subseteq \mathcal{U}_{\bowtie}$,

$$\mathrm{acl}_{\mathcal{U}_{\bowtie}}(A) = A \cup \bigcup\big\{e \in \mathcal{U}_{\bowtie} \text{ special edge} : e \text{ lies in a triangle } T \text{ with } T \cap A \neq \emptyset\big\} \quad (\dagger)$$

In (†) and below, we identify an edge e with the corresponding set of the two vertices incident with e.

As $\mathcal{U}_{\bowtie}$ is existentially closed, one can see that every vertex $v \in \mathcal{U}_{\bowtie}$ lies in some triangle. By (i) and (iii) every triangle T in $\mathcal{U}_{\bowtie}$ either contains exactly one special edge or contains three special edges. In the former case, (ii) implies that T lies in a chimney. In the latter case, T lies in some K_4. So to sum up, *every vertex and every triangle in $\mathcal{U}_{\bowtie}$ lies in a chimney or a K_4*. Also note that in a chimney, there is only one special edge, namely the edge between the two base vertices. And in a K_4 all edges are special edges.

Suppose that $v \in \mathcal{U}_{\bowtie}$. By the above v could be one of three types: it belongs to a K_4, a tip vertex of a chimney, or a base vertex of a chimney. Owing to (†) we have the following. If $v \in Q \cong K_4$ then $\mathrm{acl}_{\mathcal{U}_{\bowtie}}(v) = Q$. Otherwise v lies in a chimney. If v is a tip vertex, then $\mathrm{acl}_{\mathcal{U}_{\bowtie}}(v)$ is the unique triangle containing v. If v is a base vertex, then $\mathrm{acl}_{\mathcal{U}_{\bowtie}}(v)$ is the unique special edge containing v. Moreover, it follows from (†) that the algebraic closure is *disintegrated*, that is, the algebraic closure of a set is the union of the algebraic closure of its singletons. Therefore, for a finite $A \subseteq \mathcal{U}_{\bowtie}$ we have that $\mathrm{acl}_{\mathcal{U}_{\bowtie}}(A)$ is either a base of a chimney, a triangle in a chimney, a special bowtie-free graph, or a union of sets of these types.

Theorem 4.2.8. *The universal bowtie-free graph $\mathcal{U}_{\bowtie}$ admits generic automorphisms.*

Proof. We want to show that $\mathrm{Aut}(\mathcal{U}_{\bowtie})$ contains a comeagre conjugacy class via the Kechris-Rosendal characterisation—Theorem 2.2.5. To do so we pass to the Morleyisation $\widetilde{\mathcal{U}}_{\bowtie}$ of $\mathcal{U}_{\bowtie}$. Here $\widetilde{\mathcal{U}}_{\bowtie}$ is an expansion of $\mathcal{U}_{\bowtie}$ in the language $\widetilde{\mathcal{L}} = \{R_\phi : \phi \ \mathcal{L}\text{-formula}\}$ where $\mathcal{L}$ is the language of graphs, and R_ϕ is a relation symbol of arity equal to the number of free variables in ϕ. Moreover, the new relation symbols are interpreted as: $\widetilde{\mathcal{U}}_{\bowtie} \models R_\phi(\bar{a})$ if and only if $\mathcal{U}_{\bowtie} \models \phi(\bar{a})$ for all $\bar{a} \in \mathcal{U}_{\bowtie}$. It turns out that $\mathrm{Aut}(\widetilde{\mathcal{U}}_{\bowtie}) = \mathrm{Aut}(\mathcal{U}_{\bowtie})$, and $\mathrm{Th}(\widetilde{\mathcal{U}}_{\bowtie})$ has quantifier elimination [38, Theorem 2.6.5]. Thus, by Proposition 1.2.14 we have that $\widetilde{\mathcal{U}}_{\bowtie}$ is a homogeneous $\widetilde{\mathcal{L}}$-structure.

We now show that the class of 1-systems over the amalgamation class $\mathrm{Age}(\widetilde{\mathcal{U}}_{\bowtie})$ has the weak amalgamation property. So let $A \in \mathrm{Age}(\widetilde{\mathcal{U}}_{\bowtie})$ and $(p : U \to V) \in \mathrm{Part}(A)$. We may assume that $A \subseteq \widetilde{\mathcal{U}}_{\bowtie}$. By homogeneity of $\widetilde{\mathcal{U}}_{\bowtie}$, the partial automorphism p extends to some $f \in \mathrm{Aut}(\widetilde{\mathcal{U}}_{\bowtie})$. Let $\bar{A} = \mathrm{acl}_{\widetilde{\mathcal{U}}_{\bowtie}}(A)$, and $\bar{U} = \mathrm{acl}_{\widetilde{\mathcal{U}}_{\bowtie}}(U)$ and $\bar{V} = \mathrm{acl}_{\widetilde{\mathcal{U}}_{\bowtie}}(V)$. Note that $\bar{U}, \bar{V} \subseteq \bar{A}$. By the discussion prior to this theorem, we may assume (after first increasing the universe of $\bar{A}$ slightly if necessary) that the reducts of $\bar{A}, \bar{U}, \bar{V}$ to $\mathcal{L}$ are special bowtie-free graphs. Moreover, the restriction of f on $\bar{U}$ gives a partial automorphism $(\bar{p} : \bar{U} \to \bar{V}) \in \mathrm{Part}(\bar{A})$. By applying the necklace argument (Proposition 4.2.6) to the graph reduct of $\bar{A}$ and $\bar{p} \in \mathrm{Part}(\bar{A})$, we obtain a special bowtie-free graph K with $g \in \mathrm{Aut}(K)$ such that $\bar{A}\restriction_{\mathcal{L}} \subseteq K$ and $\bar{p} \subseteq g$. By Theorem 4.2.7(ii), we have that $\bar{A}\restriction_{\mathcal{L}} \subseteq K \subseteq \mathcal{U}_{\bowtie}$. Let $\bar{K} \in \mathrm{Age}(\widetilde{\mathcal{U}}_{\bowtie})$ be the expansion of K to $\widetilde{\mathcal{L}}$, that is, equip K with the induced structure when it is viewed as a subset of $\widetilde{\mathcal{U}}_{\bowtie}$.

Now suppose that $\langle \bar{B}_1, h_1 \rangle$ and $\langle \bar{B}_2, h_2 \rangle$ are two 1-systems over $\mathrm{Age}(\widetilde{\mathcal{U}}_{\bowtie})$ extending $\langle \bar{K}, g \rangle$. By the previous paragraph we may assume that the reducts B_1, B_2 of $\bar{B}_1, \bar{B}_2$, respectively, to $\mathcal{L}$ are special bowtie-free graphs, and also we may assume that $h_1 \in \mathrm{Aut}(B_1)$ and $h_2 \in \mathrm{Aut}(B_2)$. Let C be the free amalgam of B_1 and B_2 over K, which is also a bowtie-free graph. So $C \in \mathrm{Age}(\mathcal{U}_{\bowtie})$ by Theorem 4.2.7(ii). Let $\bar{C} \in \mathrm{Age}(\widetilde{\mathcal{U}}_{\bowtie})$ be the expansion of C to $\widetilde{\mathcal{L}}$, that is, equip C with the induced structure from $\widetilde{\mathcal{U}}_{\bowtie}$. Then the 1-system $\langle \bar{C}, h_1 \cup h_2 \rangle$ amalgamates $\langle \bar{B}_1, h_1 \rangle$ and $\langle \bar{B}_2, h_2 \rangle$ over $\langle \bar{K}, g \rangle$, and so over $\langle A, p \rangle$. Therefore, the class of all 1-systems over $\mathrm{Age}(\widetilde{\mathcal{U}}_{\bowtie})$ has the weak amalgamation property.

As the class of special bowtie-free graphs is closed under disjoint unions, we can use an argument similar to the one in the previous paragraph by taking $\bar{K}$ to be empty and replace the free amalgam by a disjoint union to show that the class of all 1-systems over $\mathrm{Age}(\widetilde{\mathcal{U}}_{\bowtie})$ has the joint embedding property. Therefore, by Theorem 2.2.5, the automorphism group

$\mathrm{Aut}(\widetilde{\mathcal{U}_{\bowtie}}) = \mathrm{Aut}(\mathcal{U}_{\bowtie})$ contains a comeagre conjugacy class. That is, the universal bowtie-free graph $\mathcal{U}_{\bowtie}$ has generic automorphisms. ∎

In the first paragraph of the proof above, we passed to a homogeneous expansion of $\mathcal{U}_{\bowtie}$ using the Morleyisation technique. To do so we expanded the language of graphs to an infinite relational language. We conclude this chapter by showing that the universal bowtie-free graph is not homogeneous over a finite relational language using an idea in an example in Cherlin-Lachlan [14, p. 819].

Definition 4.2.9. [19, Definition 1.6] Let $\mathcal{L}$ be a finite relational language, and M be a countably infinite $\mathcal{L}$-structure. We say that M is *finitely homogenisable* if there is a finite relational language $\tilde{\mathcal{L}} \supseteq \mathcal{L}$ and an $\tilde{\mathcal{L}}$-structure $\tilde{M}$ such that $\tilde{M}$ is an expansion of M, and $\tilde{M}$ is homogeneous, and $\mathrm{Aut}(M) = \mathrm{Aut}(\tilde{M})$.

Remark 4.2.10. Let $\mathcal{L}$ be a finite relational language with maximum arity k, and $\bar{a}, \bar{b}$ be finite $\mathcal{L}$-structures of same size. Then if every k-subtuple of $\bar{a}$ is isomorphic to its corresponding k-subtuple of $\bar{b}$ then $\bar{a}$ is isomorphic to $\bar{b}$.

Lemma 4.2.11. *The universal bowtie-free graph $\mathcal{U}_{\bowtie}$ is not finitely homogenisable.*

Proof. Suppose $\mathcal{U}_{\bowtie}$ is finitely homogenisable. Let $\mathcal{L} = \{E\}$ be the language of graphs, and $\tilde{\mathcal{L}}$ be the finite relational language of the homogeneous expansion $\tilde{\mathcal{U}}_{\bowtie}$ of $\mathcal{U}_{\bowtie}$. Let k be the maximum arity of the symbols in $\tilde{\mathcal{L}}$. For $1 \leq i \leq k+1$ take distinct chimneys $H_i \subseteq \mathcal{U}_{\bowtie}$ each of height 2 and with base vertices $\{a_i, b_i\}$ such that $a_1Ea_2 \wedge a_2Ea_3 \wedge \ldots \wedge a_kEa_{k+1} \wedge a_{k+1}Ea_1$ and $b_1Eb_2 \wedge b_2Eb_3 \wedge \ldots \wedge b_kEb_{k+1} \wedge b_{k+1}Eb_1$. Let t_i be a tip vertex of H_i, so $t_ia_ib_i$ is a triangle in H_i. Let $\hat{H}_{k+1} \subseteq \mathcal{U}_{\bowtie}$ be a new distinct chimney of height 2 and with base vertices $\{\hat{a}_{k+1}, \hat{b}_{k+1}\}$ and a tip vertex $\hat{t}_{k+1}$ such that $a_kE\hat{a}_{k+1} \wedge \hat{a}_{k+1}Eb_1$ and $b_kE\hat{b}_{k+1} \wedge \hat{b}_{k+1}Ea_1$. See the figure below.

Consider the two $(k+1)$-tuples $\bar{u} = (t_1, t_2, \ldots, t_k, t_{k+1})$ and $\bar{v} = (t_1, t_2, \ldots, t_k, \hat{t}_{k+1})$. For every $I \subseteq \{1, 2, \ldots, k\}$ with $|I| = k-1$, one can see that there is a finite partial $\mathcal{L}$-isomorphism $f : \bigcup_{i \in I} H_i \cup H_{k+1} \to \bigcup_{i \in I} H_i \cup \hat{H}_{k+1}$ such that $f(t_i) = t_i$

and $f(t_{k+1}) = \hat{t}_{k+1}$. As the domain and range of f are special bowtie-free graphs, by Theorem 4.2.7(iii), there is $\tilde{f} \in \mathrm{Aut}(\mathcal{U}_{\bowtie}) = \mathrm{Aut}(\tilde{\mathcal{U}}_{\bowtie})$ extending f. Thus, every k-subtuple of $\bar{u}$ is $\tilde{\mathcal{L}}$-isomorphic to its corresponding subtuple of v. By the remark above, $\bar{u}, \bar{v}$ are $\tilde{\mathcal{L}}$-isomorphic. By homogeneity of $\tilde{\mathcal{U}}_{\bowtie}$ there is some $h \in \mathrm{Aut}(\tilde{\mathcal{U}}_{\bowtie})$ such that $h(\bar{u}) = \bar{v}$. Suppose without loss of generality that h fixes pointwise the bases of each H_i for $1 \leq i \leq k$. As $h(t_{k+1}) = \hat{t}_{k+1}$, we have that h sends the base of H_{k+1} to the base of $\hat{H}_{k+1}$, but both options $h(a_{k+1}, b_{k+1}) = (\hat{a}_{k+1}, \hat{b}_{k+1})$ and $h(a_{k+1}, b_{k+1}) = (\hat{b}_{k+1}, \hat{a}_{k+1})$ give rise to a contradiction. ■

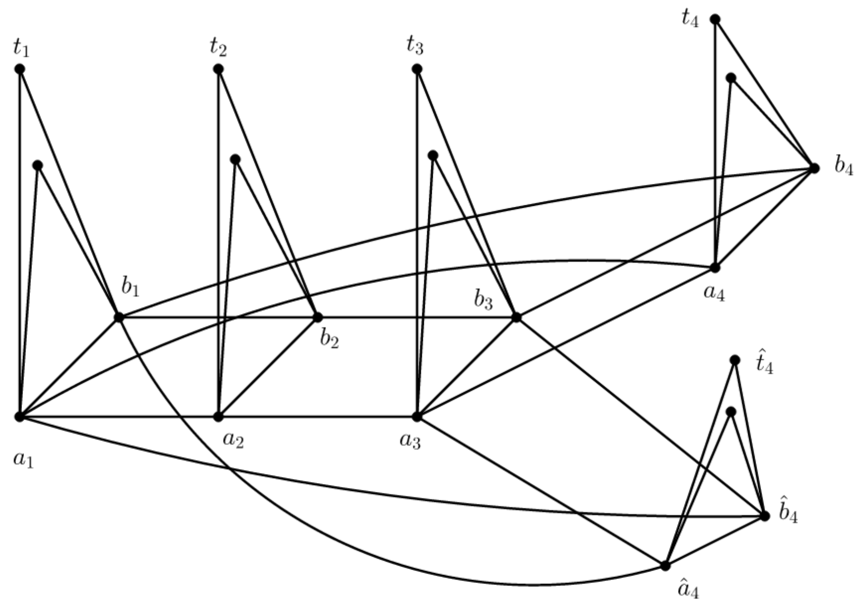


Figure 4.3: The chimneys as in the proof above for $k = 3$.

Chapter 5

Philip Hall’s Universal Locally Finite Group

The mathematical object concerned in this chapter is Philip Hall’s universal locally finite group; it is the Fraïssé limit of the class of finite groups. Our main result is to show that the class of finite groups has the amalgamation property with automorphisms (Definition 2.1.1) and consequently we obtain the following result which was also proved independently by Song [72].

Theorem. *Philip Hall’s universal locally finite group admits ample generics.*

5.1 The class of finite groups

We view groups as $\mathcal{L}$-structures in the language $\mathcal{L} = \{1, \cdot, {}^{-1}\}$. In this language, the notions of a subgroup and a substructure of a group coincide. The class of finite groups contains countably many groups up to isomorphism, has the hereditary property, and the joint embedding property (by taking direct products). Moreover, B. H. Neumann proved that the class of finite groups has the amalgamation property. He called his amalgam of groups a *permutational product with an amalgamated subgroup*.

Proposition 5.1.1. (B. H. Neumann [62, Section 3]). *The class of all finite groups has the amalgamation property.*

Proof. Suppose that A, G, H are finite groups such that $A \leq G$ and $A \leq H$. Let $S = \{s_1, s_2, \ldots, s_n\}$ be a system of representatives of right cosets of A in G, and let $T = \{t_1, t_2, \ldots, t_m\}$ be a system of representatives of right cosets of A in H. Then every $g \in G$ can be written uniquely as $g = as_i$ for $a \in A$, and $s_i \in S$ and in this case we define $[g]^A = a$ and $[g]^S = s_i$. Similarly, every $h \in H$ can be written uniquely as $h = at_i$ for $a \in A$ and $t_i \in T$, and here we define $[h]^A = a$ and $[h]^T = t_i$.

The amalgam of G and H over A is the group $K = \mathrm{Sym}(A \times S \times T)$. To justify this claim we first show that both groups G, H embed in K. We define an embedding $\phi : G \to K$ as $\phi(g) = \phi_g$ where $\phi_g(a, s, t) = ([gas]^A, [gas]^S, t)$. It is not difficult to check that ϕ is injective. We now show that ϕ is a homomorphism. Let $g, l \in G$. Then,

$$\begin{aligned} \phi_g \phi_l(a, s, t) &= \phi_g([las]^A, [las]^S, t) = ([g[las]^A[las]^S]^A, [g[las]^A[las]^S]^S, t) \\ &= ([glas]^A, [glas]^S, t) = \phi_{gl}(a, s, t). \end{aligned}$$

So $\phi(gl) = \phi(g)\phi(l)$.

Similarly, to embed H in K we define an embedding $\psi : H \to K$ as $\psi(h) = \psi_h$ where $\psi_h(a, s, t) = ([hat]^A, s, [hat]^T)$. It remains to show that $\phi(a) = \psi(a)$ for all $a \in A$. So suppose that $\hat{a} \in A$, then $\phi_{\hat{a}}(a, s, t) = ([\hat{a}as]^A, [\hat{a}as]^S, t) = (\hat{a}a, s, t)$, and $\psi_{\hat{a}}(a, s, t) = ([\hat{a}at]^A, s, [\hat{a}at]^T) = (\hat{a}a, s, t)$. Note that we may also take the subgroup $P \leq K$ generated by $\phi(G)$ and $\psi(H)$ as the amalgam of G and H over A. ∎

So the class of finite groups is an amalgamation class. Therefore it has a Fraïssé limit called *Philip Hall's universal locally finite group*.

Theorem 5.1.2 (Hall [33]). *There exists a unique countably infinite locally finite group $\mathbb{H}$ satisfying the following two properties:*

(i) Every finite group can be embedded in $\mathbb{H}$.

(ii) Every isomorphism between finite subgroups of $\mathbb{H}$ extends to an inner automorphism of $\mathbb{H}$.

Moreover, the group $\mathbb{H}$ is a simple group, and contains as subgroups $2^{\aleph_0}$ distinct copies of each countably infinite locally finite group.

By (i) above, the group $\mathbb{H}$ embeds every finite cyclic group. So for every $0 < n < \omega$ there is $h_n \in \mathbb{H}$ such that the order of h_n is n. This implies that there are infinitely many distinct 1-types of $\mathrm{Th}(\mathbb{H})$, namely $\mathrm{tp}(h_1), \mathrm{tp}(h_2), \ldots$. So by Ryll-Nardzewski's Theorem we have that $\mathrm{Th}(\mathbb{H})$ is not ω-categorical. Accordingly, Hall's group is a homogeneous structure over a finite language which is not ω-categorical.

The strategy to show that Hall's group has ample generics is by showing the its age, the class of all finite groups, has EPPA and APA. This is sufficient by Theorem 2.1.5. EPPA for finite groups was established by Hall [33, Lemma 1]. He showed that if G is a finite group, then $\mathrm{Sym}(G)$ is an EPPA-extension of G, where G is embedded in $\mathrm{Sym}(G)$ by its regular representation. Below, we build on Hall's proof to obtain coherent EPPA for finite groups.

Theorem 5.1.3. *The class of all finite groups has coherent EPPA.*

Proof. Let G be a finite group. Then the group $\mathrm{Sym}(G)$ is a coherent EPPA-extension of G. Here G is seen as a subgroup of $\mathrm{Sym}(G)$ through its regular representation by left multiplication. *Fix for every subgroup H of G an ordered system S_H of right coset representatives.*

Now suppose that H and K are subgroups of G and that $f : H \to K$ is an isomorphism. Let $m = |G : H| = |G : K|$, and let $S_H = (h_1, \ldots, h_m)$ be the fixed representatives of right cosets of H in G, and $S_K = (k_1, \ldots, k_m)$ be the fixed representatives of right cosets of K in G. Then $G = \bigsqcup_{i=1}^{m} Hh_i = \bigsqcup_{i=1}^{m} Kk_i$. Now we can define a permutation $\phi : G \to G$ of the underlying set of G as follows. For $g \in G$ write it uniquely as $g = xh_i$ for some $x \in H$ and $h_i \in S_H$, and define,

$$\phi(g) = \phi(xh_i) := f(x)k_i.$$

Clearly, $\phi \in \mathrm{Sym}(G)$. We now show that the automorphism induced by conjugation by ϕ in $\mathrm{Sym}(G)$ extends $f \in \mathrm{Part}(\mathrm{Sym}(G))$. So we need to show that for any $x \in H$ we have that $\phi x \phi^{-1} = f(x)$ in $\mathrm{Sym}(G)$. So let $u \in G$, and suppose that $u = yk_i$ for some $y \in K$ and representative $k_i \in S_K$, then

$$\phi x \phi^{-1}(u) = \phi x \phi^{-1}(yk_i) = \phi x(f^{-1}(y)h_i) = \phi\big((xf^{-1}(y))h_i\big)$$

$$= f(xf^{-1}(y))k_i = f(x)(yk_i) = f(x)u.$$

Now we show that when we always use the ordered systems of coset representatives we have fixed at the very beginning, the map from $\mathrm{Part}(G) \to \mathrm{Aut}(\mathrm{Sym}(G))$ as defined above which sends f to the inner automorphism induced by ϕ is a coherent map. So suppose that $H, K, L \leq G$, and $S_H = (h_1, \ldots, h_m)$, $S_K = (k_1, \ldots, k_m)$, and $S_L = (l_1, \ldots, l_m)$ are the fixed systems of right coset representatives of H, K, L, respectively, in G.

Let $f : H \to K$ be an isomorphism, and using f, S_H, S_K define as above the permutation $\phi \in \mathrm{Sym}(G)$ such that conjugation by ϕ in $\mathrm{Sym}(G)$ extends $f \in \mathrm{Part}(\mathrm{Sym}(G))$. Let $j : K \to L$ be another isomorphism. Similarly, using j, S_K, S_L define the permutation $\psi \in \mathrm{Sym}(G)$ such that conjugation by ψ in $\mathrm{Sym}(G)$ extends $j \in \mathrm{Part}(\mathrm{Sym}(G))$. Now consider the composition $p = jf : H \to L$. Again, using p, S_H, S_L define $\gamma \in \mathrm{Sym}(G)$ such that conjugation by γ extends $p \in \mathrm{Part}(\mathrm{Sym}(G))$.

We need to show that the inner automorphism of $\mathrm{Sym}(G)$ induced by $\psi\phi$ is the same as the one induced by γ. So take any $\delta \in \mathrm{Sym}(G)$. We will show that $\psi\phi\delta\phi^{-1}\psi^{-1} = \gamma\delta\gamma^{-1}$ in $\mathrm{Sym}(G)$. Pick any $u \in G$ and suppose that $u = zl_i$ for some $z \in L$ and representative $l_i \in S_L$. Find some $x \in H$ and representative $h_r \in S_H$ such that $\delta(p^{-1}(z))h_i) = xh_r$. Then,

$$\psi\phi\delta\phi^{-1}\psi^{-1}(u) = \psi\phi\delta\phi^{-1}\psi^{-1}(zl_i) = \psi\phi\delta\phi^{-1}(j^{-1}(z)k_i) = \psi\phi\delta(f^{-1}(j^{-1}(z))h_i)$$

$$= \psi\phi\delta(p^{-1}(z))h_i) = \psi\phi(xh_r) = \psi(f(x)k_r) = j(f(x))l_r = p(x)l_r.$$

On the other hand,

$$\gamma\delta\gamma^{-1}(u) = \gamma\delta\gamma^{-1}(zl_i) = \gamma\delta(p^{-1}(z)h_i) = \gamma(xh_r) = p(x)l_r.$$

■

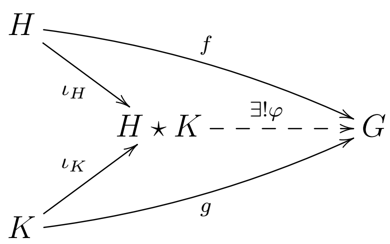
Remark 5.1.4. One can see that $\mathbb{H}$ is centreless, and so $\mathbb{H} \cong \mathrm{Inn}(\mathbb{H}) \leq \mathrm{Aut}(\mathbb{H})$. Together with Theorem 5.1.2 we get that $\mathrm{Inn}(\mathbb{H})$ is a dense locally finite simple normal subgroup of $\mathrm{Aut}(\mathbb{H})$. Compare this observation with Theorem 3.3.4.

5.2 Free Products with Amalgamation

The material in this section is based on the following books Lyndon-Schupp [52], Magnus-Karrass-Solitar [58], and Massey [61].

Our approach of establishing APA for finite groups is based on two important mathematical objects in combinatorial group theory, namely the free product and the free product with amalgamation in the category of groups. They are defined as groups which satisfy certain universal properties, and thus they are unique up to isomorphism. Their existence is established by writing down their presentations as groups. The aim of this section is merely to present some details of such objects for the sake of using them in the following section.

Definition 5.2.1 ([61])**.** Let H and K be groups. The *free product* of H and K is a group $H \star K$ together with homomorphisms $\iota_H : H \to H \star K$ and $\iota_K : K \to H \star K$ satisfying the following universal property: for any group G and homomorphisms $f : H \to G$ and $g : K \to G$, there is a unique homomorphism $\varphi : H \star K \to G$ such that $f = \varphi\iota_H$ and $g = \varphi\iota_K$. That is, the following diagram commutes.



The free product plays the role of the *coproduct* in the category of groups. The free product $H \star K$ exists and it is unique up to isomorphism. Suppose the group H has the presentation $H = \langle E_H \mid R_H \rangle$, and K has the presentation $K = \langle E_K \mid R_K \rangle$, where we also assume that E_H and E_K are disjoint. Then the free product is given by $H \star K = \langle E_H \cup E_K \mid R_H \cup R_K \rangle$, and ι_H, ι_K are the inclusion maps, which turn out to be monomorphisms. So we think of H and K as subgroups of $H \star K$.

To see this, consider the homomorphism $\iota_H : H \to H \star K$ given by $\iota_H(h) = h$ for all $h \in E_H$, and the projection homomorphism $\pi_H : H \star K \to H$ given by $\pi_H(h) = h$ for all $h \in E_H$, and $\pi_H(k) = 1_H$ for all $k \in E_K$. As $\pi_H \circ \iota_H$ is the identity map on H, we get that ι_H is an injective homomorphism. Similarly, ι_K is an injective homomorphism as well. Furthermore, $H \cap K = \{1\}$ in $H \star K$, where 1 is the empty word which is the identity element of $H \star K$. For if $x \in H \cap K$, then $\pi_H(x) = 1_H$ as $x \in K$. Since π_H is injective on $H \leq H \star K$ and $x \in H$, we must have that $x = 1$.

To see the universal property as stated in the definition above, note that $H \star K$ is generated by the generators of H and K. So putting $\varphi(h) = f(h)$ for every $h \in E_H$ and $\varphi(k) = g(k)$ for every $k \in E_K$ defines the unique homomorphism $\varphi : H \star K \to G$, and we have that $f = \varphi\iota_H$ and $g = \varphi\iota_K$ as required.

To see vividly what the elements of $H \star K$ look like, we recall the concept of a reduced word. A *word* on the set $H \cup K$ is just a finite sequence of elements of $H \cup K$. A word on $H \cup K$ is called a *reduced word* if it is of the form $h_1k_1h_2k_2 \ldots h_nk_n$ where $h_i \in H \setminus \{1_H\}$, $k_i \in K \setminus \{1_K\}$, and where h_1, k_n may or may not be present. Moreover, when $n = 0$, this is understood to be the empty word 1. The following theorem [52, Chapter IV] makes reduced words extremely useful.

Theorem 5.2.2 (The Normal Form Theorem for free products)**.** *Every element of $H \star K$ is equal to a unique reduced word. Moreover, if $w = h_1k_1h_2k_2 \ldots h_nk_n$ where $n \geq 1$ is a reduced word, then w is not equal to the identity element of $H \star K$.*

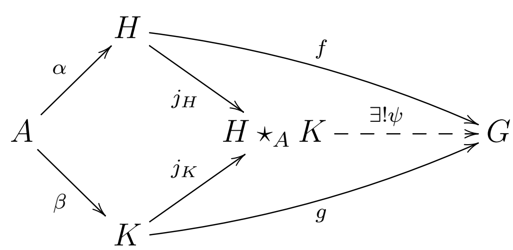
By the Normal Form Theorem of free products we have a concrete description of the group $H \star K$. Its underlying set is the set of all reduced words on $H \cup K$ and the group multiplication is concatenation of reduced words followed by reduction. In this case, the unique homomorphism $\varphi : H \star K \to G$ of the universal property is given by,

$$\varphi(h_1k_1 \dots h_nk_n) = f(h_1)g(k_1) \dots f(h_n)g(k_n),$$

where $h_1k_1 \dots h_nk_n \in H \star K$ is a reduced word.

Now we introduce the other combinatorial group-theoretic object, which is a generalisation of the free product of groups.

Definition 5.2.3. Let H and K be groups, and suppose that there is a group A together with homomorphisms $\alpha : A \to H$ and $\beta : A \to K$. The *free product of H and K with group A amalgamated* is a group $H \star_A K$ together with homomorphisms $j_H : H \to H \star_A K$ and $j_K : K \to H \star_A K$ such that $j_H\alpha = j_K\beta$, satisfying the following universal property: for any group G and homomorphisms $f : H \to G$ and $g : K \to G$ with $f\alpha = g\beta$, there is a unique homomorphism $\psi : H \star_A K \to G$ such that $f = \psi j_H$ and $g = \psi j_K$. That is, the following diagram commutes.



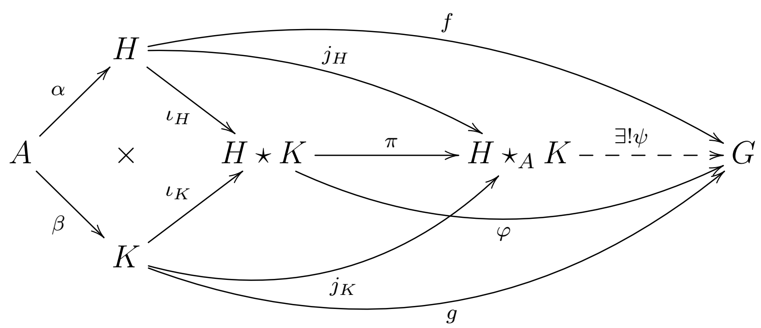
The amalgamated free product is the *pushout* in the category of groups, and it is the same as the free product when the group A is the trivial group. The amalgamated free product $H \star_A K$ exists and it is unique up to isomorphism. We will next describe how to obtain $H \star_A K$ from $H \star K$.

Suppose we are given groups A, H, K together with homomorphisms $\alpha : A \to H$ and $\beta : A \to K$. Let $H \star K$ be the free product H and K and $\iota_H : H \to H \star K$ and $\iota_K : K \to H \star K$ be the inclusion maps. Denote by N the normal subgroup of $H \star K$ generated by the set $\{\alpha(a)\beta(a^{-1}) : a \in A\} \subseteq H \star K$. Then define,

$$H \star_A K := (H \star K)/N.$$

Therefore, $H \star_A K = \langle H \star K \mid \{\alpha(a)\beta(a^{-1}) : a \in A\}\rangle$. Let $\pi : H \star K \to H \star_A K$ be the canonical projection map and let $j_H = \pi\iota_H$ and $j_K = \pi\iota_K$ (see the diagram below). If $a \in A$, then we have that $j_H\alpha(a) = \pi\iota_H\alpha(a) = \pi\alpha(a) = \alpha(a)N$ and $j_K\beta(a) = \pi\iota_K\beta(a) = \pi\beta(a) = \beta(a)N$. But since $\beta(a^{-1})\alpha(a) \in N$, we have that $\alpha(a)N = \beta(a)N$, and so $j_H\alpha = j_K\beta$.

Now we show that $H \star_A K$ together with homomorphisms j_H and j_K satisfy the universal property as stated above in the definition of the amalgamated free product. To see this, let G be any group, and consider any homomorphisms $f : H \to G$ and $g : K \to G$ such that $f\alpha = g\beta$. Using the universal property of the free product, we get a unique homomorphism $\varphi : H \star K \to G$ such that $f = \varphi\iota_H$ and $g = \varphi\iota_K$.



Now we check that $N \subseteq \ker\varphi$. For this choose an arbitrary element $\alpha(a)\beta(a^{-1})$ of the generating set of N, where $a \in A$. Now,

$$\begin{aligned}\varphi\big(\alpha(a)\beta(a^{-1})\big) &= \varphi\big(\alpha(a)\big)\ \varphi\big(\beta(a^{-1})\big) = \varphi\iota_H\big(\alpha(a)\big)\ \varphi\iota_K\big(\beta(a^{-1})\big) \\ &= f\alpha(a)\ g\beta(a^{-1}) = f\alpha(a)\ f\alpha(a^{-1}) = f\alpha(aa^{-1}) = f\alpha(1_A) = 1_G.\end{aligned}$$

Thus $\alpha(a)\beta(a^{-1}) \in \ker\varphi$, and so the generating set of N is contained in $\ker\varphi$. As $\ker\varphi$ is a normal subgroup, the whole of N is contained in $\ker\varphi$. Consequently, we can use the universal property of the projection map $\pi : H \star K \to H \star_A K$ to obtain a unique homomorphism $\psi : H \star_A K \to G$ such that $\psi\pi = \varphi$. More precisely, for a reduced word $h_1k_1 \ldots h_nk_n \in H \star K$ we have that:

$$\psi(h_1k_1 \ldots h_nk_nN) = \psi\pi(h_1k_1 \ldots h_nk_n) = \varphi(h_1k_1 \ldots h_nk_n) = f(h_1)g(k_1) \ldots f(h_n)g(k_n).$$

One can check that ψ is well-defined, and so it remains to check that the diagram commutes. But it does, for $\psi j_H = \psi\pi\iota_H = \varphi\iota_H = f$ and $\psi j_K = \psi\pi\iota_K = \varphi\iota_K = g$.

Note 5.2.4. It is the above explicit description of $\psi : H \star_A K \to G$ that we need for the next section.

5.3 The Amalgamation Property with Automorphisms

In this section we show that Philip Hall's universal locally finite group admits ample generics. Recall that our strategy is based on Theorem 2.1.5. So our interest hovers around the combinatorial properties EPPA and APA. We have seen in Theorem 5.1.3 that the age of Hall's group, the class of all finite groups, has EPPA. It is the time to establish APA for finite groups.

Recall that to say that the class of finite groups has the amalgamation property with automorphisms (APA) means the following: whenever A, H, K are finite groups with embeddings (monomorphisms) $\alpha_1 : A \to H$ and $\alpha_2 : A \to K$, then there is a finite group L with embeddings $\beta_1 : H \to L$ and $\beta_2 : K \to L$ such that $\beta_1\alpha_1 = \beta_2\alpha_2$, and whenever $f \in \mathrm{Aut}(H)$ and $g \in \mathrm{Aut}(K)$ such that $f\alpha_1(A) = \alpha_1(A)$, $g\alpha_2(A) = \alpha_2(A)$, and for every $a \in A$ we have that $\alpha_1^{-1} f\alpha_1(a) = \alpha_2^{-1} g\alpha_2(a)$, then there exists $h \in \mathrm{Aut}(L)$ which extends $\beta_1 f\beta_1^{-1} \cup \beta_2 g\beta_2^{-1}$.

Our method to show APA for finite groups relies upon the universal property of the amalgamated free product of groups discussed in the previous section.

Theorem 5.3.1. *The class of finite groups has the amalgamation property with automorphisms.*

Proof. Suppose we are given finite groups A, H, and K, together with embeddings $\alpha : A \to H$ and $\beta : A \to K$. So H has a subgroup isomorphic to a subgroup of K. Without loss of generality we may assume that H and K are disjoint sets. Suppose also that we have $f \in \operatorname{Aut}(H)$ with $f\alpha(A) = \alpha(A)$ and $g \in \operatorname{Aut}(K)$ with $g\beta(A) = \beta(A)$ such that $\alpha^{-1}f\alpha = \beta^{-1}g\beta$. This just says that f and g induce the same automorphism on the common subgroup A. Before we proceed recall that $H \star_A K = (H \star K)/N$ where $N \trianglelefteq H \star K$ is the normal subgroup generated by the set $\{\alpha(a)\beta(a^{-1}) : a \in A\}$.

STEP 1. We apply the universal property of the amalgamated free product as illustrated in its diagram above with $G = H \star_A K$, and the embeddings $\hat{f}, \hat{g}$ into G induced by f and g, respectively. That is, $\hat{f} : H \to H \star_A K$ where $\hat{f}(h) = f(h)N$ and $\hat{g} : K \to H \star_A K$ where $\hat{g}(k) = g(k)N$. By the universal property of $H \star_A K$ and Note 5.2.4 we get a unique homomorphism $\psi : H \star_A K \to H \star_A K$ where for every reduced word $h_1k_1 \dots h_nk_n \in H \star K$ we have that:

$$\psi(h_1k_1 \dots h_nk_nN) = \hat{f}(h_1)\hat{g}(k_1) \dots \hat{f}(h_n)\hat{g}(k_n) = f(h_1)g(k_1) \dots f(h_n)g(k_n)N.$$

We claim that $\psi \in \operatorname{Aut}(H \star_A K)$. To this end, let $\chi : H \star K \to H \star K$ be the unique homomorphism determined by $f \in \operatorname{Aut}(H)$ and $g \in \operatorname{Aut}(K)$ using the universal property of $H \star K$. By the previous section, $\chi(h_1k_1 \dots h_nk_n) = f(h_1)g(k_1) \dots f(h_n)g(k_n)$. Notice that $\psi(h_1k_1 \dots h_nk_nN) = \chi(h_1k_1 \dots h_nk_n)N$. We will first show that $\chi \in \operatorname{Aut}(H \star K)$. Take any nonempty reduced word $h_1k_1 \dots h_nk_n \in H \star K$ where $n \geq 1$. It is easy to see that χ is a surjective map, for $\chi\big(f^{-1}(h_1)g^{-1}(k_1) \dots f^{-1}(h_n)g^{-1}(k_n)\big) = h_1k_1 \dots h_nk_n$ as required.

To see that χ is injective, we will show that $\ker\chi = \{1\}$ where 1 is the empty word. Let $h_1k_1 \dots h_nk_n \in H \star K$ be a nonempty reduced word. As $f \in \operatorname{Aut}(H)$ and $g \in \operatorname{Aut}(K)$, and $h_i, k_i \neq 1$, it follows that $f(h_i), g(k_i) \neq 1$, so $\chi(h_1k_1 \dots h_nk_n) = f(h_1)g(k_1) \dots f(h_n)g(k_n)$ is also a nonempty reduced word.

Now we show that $\psi \in \operatorname{Aut}(H \star_A K)$. Take any element $wN \in H \star_A K$ where $w \in H \star K$ is a nonempty reduced word. Consider the element $\chi^{-1}(w)N \in H \star_A K$, and observe that $\psi(\chi^{-1}(w)N) = \chi\big(\chi^{-1}(w)\big)N = wN$, showing that ψ is surjective.

To show that ψ is injective, we will show first that for every nonempty reduced word $w \in H \star K$ we have that $w \in N$ if and only if $\chi(w) \in N$. By the definition of the normal subgroup $N \trianglelefteq H \star K$ we have that $w \in N$ if and only if $w = \prod_{i=1}^{k} w_i \alpha(a_i)\beta(a_i^{-1})w_i^{-1}$ for some $k \in \mathbb{N}$, $w_i \in H \star K$, $a_i \in A$. In what follows $b_i := \alpha^{-1} f \alpha(a_i)$ which is also an element of A. We also have by our assumptions on f and g that $b_i = \beta^{-1} g \beta(a_i)$ as well. So,

$$\begin{aligned}
w \in N \Leftrightarrow w &= \prod_{i=1}^{k} w_i \alpha(a_i)\beta(a_i^{-1})w_i^{-1} \\
\Leftrightarrow \chi(w) &= \prod_{i=1}^{k} \chi(w_i)\chi(\alpha(a_i))\chi(\beta(a_i^{-1}))\chi(w_i^{-1}) \\
\Leftrightarrow \chi(w) &= \prod_{i=1}^{k} \chi(w_i) f(\alpha(a_i)) g(\beta(a_i^{-1}))\chi(w_i^{-1}) \\
\Leftrightarrow \chi(w) &= \prod_{i=1}^{k} \chi(w_i)\alpha\big(\alpha^{-1} f \alpha(a_i)\big)\beta\big(\beta^{-1} g \beta(a_i^{-1})\big)\chi(w_i^{-1}) \\
\Leftrightarrow \chi(w) &= \prod_{i=1}^{k} \chi(w_i)\alpha(b_i)\beta(b_i^{-1})\chi(w_i^{-1}) \\
\Leftrightarrow \chi(w) &\in N.
\end{aligned}$$

Notice that the injectivity of χ is needed to establish the second equivalence above.

Now suppose that $wN \in \ker \psi$. Thus, $\psi(wN) = \chi(w)N = N$, which implies that $\chi(w) \in N$. By the above, $w \in N$, and so $wN = N$. This shows that $\ker \psi = \{N\}$, and so ψ is injective.

STEP 2. Let $\Sigma = \big\{(f_i, g_i, \psi_i) : 1 \leq i \leq r\big\}$ be the set of *all* triples as (f, g, ψ) above. That is, all triples (f_i, g_i, ψ_i) where $f_i \in \mathrm{Aut}(H)$ with $f_i\alpha(A) = \alpha(A)$ and $g_i \in \mathrm{Aut}(K)$ with $g_i\beta(A) = \beta(A)$ such that $\alpha^{-1} f_i \alpha = \beta^{-1} g_i \beta$. Moreover, ψ_i is the automorphism of $H \star_A K$ that is obtained as above; it extends both f_i and g_i. Note that $H \star_A K$ almost satisfies APA for H, K, and the common subgroup A. The only obstacle left is that in general $H \star_A K$ is not a finite group.

STEP 3. Let $\Psi := \langle \psi_i : 1 \leq i \leq r \rangle$. We claim that $\Psi \leq \mathrm{Aut}(H \star_A K)$ is a finite subgroup. To see this, note that every $\psi \in \Psi$ is determined by a pair (f, g) where

$f \in \langle f_i : 1 \leq i \leq r \rangle \leq \operatorname{Aut}(H)$ and $g \in \langle g_i : 1 \leq i \leq r \rangle \leq \operatorname{Aut}(K)$. As both H and K are finite, the claim follows.

STEP 4. By Proposition 5.1.1, the class of finite groups has the amalgamation property, and so there is a finite group P which embeds H and K such that $H \cap K = A$ inside P, and we may assume harmlessly that P is generated by H and K. Thus, we get a surjective homomorphism $j : H \star_A K \to P$. So we may think of the amalgam P as $(H \star_A K)/M$ for some normal subgroup $M \trianglelefteq H \star_A K$ of finite index. Note that j is injective on $H \cup K$ viewed as a subset of $H \star_A K$.

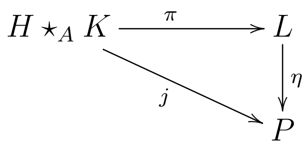
STEP 5. Let $\Gamma := (H \star_A K) \rtimes \Psi$, their semidirect product. As $\Gamma/(H \star_A K) \cong \Psi$, we have that $H \star_A K \leq \Gamma$ has finite index. As $M \leq H \star_A K$ has finite index, $M \leq \Gamma$ is of finite index. Define,

$$\hat{M} := \bigcap_{\gamma \in \Gamma} \gamma M \gamma^{-1}.$$

Let $\mu : \Gamma \to \operatorname{Sym}(\Gamma/M)$ be the induced homomorphism by left multiplication, and note that $\hat{M} = \ker \mu$. Therefore $\hat{M} \leq M \leq H \star_A K \leq \Gamma = (H \star_A K) \rtimes \Psi$, and $\hat{M} \trianglelefteq \Gamma$ is a normal subgroup of Γ of finite index. Moreover, as $\hat{M} \trianglelefteq \Gamma$ we have that $\hat{M}$ is ψ-invariant, that is, $\psi(\hat{M}) = \hat{M}$, for every $\psi \in \Psi$. In particular, $\hat{M}$ is ψ_i-invariant for all $1 \leq i \leq r$.

STEP 6. Let $L := (H \star_A K)/\hat{M}$, and note that L is a finite group. Let $\pi : H \star_A K \to L$ be the projection homomorphism. Moreover, as $\hat{M}$ is ψ_i-invariant, for every $1 \leq i \leq r$ the automorphism $\psi_i \in \operatorname{Aut}(H \star_A K)$ induces an automorphism $\hat{\psi}_i \in \operatorname{Aut}(L)$ given by $\hat{\psi}_i(x\hat{M}) = \psi_i(x)\hat{M}$, for any $x \in H \star_A K$.

STEP 7. We now argue that the group L is the desired finite group. We will show that L amalgamates H and K over A via the map π. As $\hat{M} = \ker \pi \subseteq \ker j = M$, there is a unique homomorphism $\eta : L \to P$ such that $j = \eta\pi$. Here η is given by $\eta(x\hat{M}) = xM$, for any $x \in H \star_A K$.



Recall that the amalgamated free product $H \star_A K$ embeds H and K over A. As j is injective on each of H and K, we have that π embeds both groups H and K in L.

It remains to show that every pair of automorphisms of H and K agreeing on A extends simultaneously to an automorphism of L. So let $(f_i, g_i, \psi_i) \in \Sigma$, and $h \in H \leq H \star_A K$, and $k \in K \leq H \star_A K$. Then

$$\hat{\psi}_i(\pi(h)) = \hat{\psi}_i(h\hat{M}) = \psi_i(h)\hat{M} = f_i(h)\hat{M} = \pi(f_i(h)).$$

Similarly, we have that $\hat{\psi}_i(\pi(k)) = \pi(g_i(k))$. So for every $1 \leq i \leq r$ we have that $\hat{\psi}_i \in \mathrm{Aut}(L)$ extends the copy of $f_i \cup g_i$ in L. ■

We have arrived now to the main result of this chapter.

Theorem 5.3.2. *Philip Hall's universal locally finite group admits ample generics.*

Proof. By Theorem 5.1.3, the class of all finite groups has the extension property for partial automorphisms, and by Theorem 5.3.1, it has the amalgamation property with automorphisms. So applying Theorem 2.1.5 to Hall's universal group $\mathbb{H}$ we obtain that $\mathbb{H}$ admits ample generics. ■

Chapter 6

On the Weak Amalgamation Property

In this chapter we explore the significance of the weak amalgamation property (Definition 2.2.4) as a powerful tool to investigate the existence of generic automorphisms for homogeneous structures. We consider three cases: linear orders, ordered graphs, and tournaments. Recall that if $\mathcal{C}$ is an amalgamation class of finite $\mathcal{L}$-structures and $n \in \omega$, then an n-system over $\mathcal{C}$ is a tuple $\langle A, p_1, \ldots, p_n \rangle$ where $A \in \mathcal{C}$ and each $p_i \in \mathrm{Part}(A)$. The class of all n-systems over $\mathcal{C}$ is denoted by $\mathcal{C}^n$.

6.1 Linear Orders

We use the weak amalgamation property to prove Hodkinson's result (see Section 1.4) that there is no generic pair of automorphisms of the countable dense linear ordering. We hope that this method will adapt easily to other cases such as the universal homogeneous partial order, and possibly give an answer to Question 10 in Chapter 7.

Lemma 6.1.1. *Let $\mathcal{C}$ be the class of all finite linear orders. The class $\mathcal{C}^2$ of all 2-systems over $\mathcal{C}$ does not have the weak amalgamation property. Consequently, $(\mathbb{Q}, <)$ does not have 2-generic automorphisms.*

Proof. Suppose that $\mathcal{C}^2$ has the weak amalgamation property. Let $S = \langle A, p_1, p_2 \rangle \in \mathcal{C}^2$

where $A = \{a_1, a_2\}$, and $a_1 < a_2$, and $p_1(a_1) = a_2$, and $p_2(a_1) = a_2$. Then S has an extension $T = \langle B, f_1, f_2\rangle$ such that any two extensions of T amalgamate over S.

Let $b := \max\{\eta(a_1) : \eta \in \mathrm{Word}(f_1^{\pm 1}, f_2^{\pm 1})\}$. Choose $k \in \omega$ minimal such that $b = g_k g_{k-1} \ldots g_2 g_1(a_1)$ where each $g_i \in \{f_1, f_1^{-1}, f_2, f_2^{-1}\}$. By maximality of b and minimality of k, we have that $g_k(b)$ is undefined. Moreover, there is no $b' \in \mathrm{dom}(g_k)$ such that $g_k(b') < b < b'$. So it is possible to define g_k on $\{b\}$ such that $b < g_k(b)$.

We build an extension $\bar{B} = B \cup \{b_1, \ldots, b_n\}$ such that $b = b_0 < b_1 < \ldots < b_{n-1} < b_n$, and simultaneously extending f_1, f_2 by adding each element in $\{b_i : 0 \leq i < n\}$ to either $\mathrm{dom}(f_1) \cup \mathrm{dom}(f_1^{-1})$ or $\mathrm{dom}(f_2) \cup \mathrm{dom}(f_2^{-1})$, with $n \in \omega$ big enough such that if $f_1(b_{n-1}) = b_n$, say, then there is no $b' \in B$ such that $b' < b_n < f_2(b')$ or $f_2(b') < b_n < b'$. That is, we have the freedom of choice to define f_2 on $\{b_n\}$ such that $b_n < f_2(b_n)$, or $f_2(b_n) = b_n$, or $f_2(b_n) < b_n$. Strictly speaking there are four possibilities, namely $f_1^{\pm 1}, f_2^{\pm 1}$, for the map sending b_{n-1} to b_n. Without loss of generality, assume that $f_1(b_{n-1}) = b_n$.

Now extend $\bar{B}$ to $C = \bar{B} \cup \{c\}$ where $b_n < c$. Let $h_2 := f_2 \cup \{(b_n, c)\}$. So $\langle C, f_1, h_2\rangle \in \mathcal{C}^2$. Next, extend $\bar{B}$ to $D = \bar{B} \cup \{d\}$ where $d < b_n$. Let $h_2' = f_2 \cup \{(b_n, d)\}$. So $\langle D, f_1, h_2'\rangle \in \mathcal{C}^2$. By the weak amalgamation property there is $\langle G, \phi_1, \phi_2\rangle \in \mathcal{C}^2$ amalgamating $\langle C, f_1, h_2\rangle$ and $\langle D, f_1, h_2'\rangle$ over $\langle A, p_1, p_2\rangle$. Since there is a single word $w \in \mathrm{Word}(f_1^{\pm 1}, f_2^{\pm 1})$ which takes a_1 to b_n, the corresponding two copies of b_n in C and D are identified in G. Therefore, $c = h_2(b_n) = \phi_2(b_n) = h_2'(b_n) = d$, contradicting that $d < b_n < c$ in G. ■

6.2 Ordered Graphs

The language here is $\mathcal{L} = \{E, <\}$ where $E, <$ are binary relation symbols. An *ordered graph* G is an $\mathcal{L}$-structure where E is interpreted as an irreflexive symmetric binary relation on G, and $<$ is interpreted as a total order on G. Note that neither E nor $<$ depends on the other. We also note that the class of all finite ordered graphs is an amalgamation class, and its Fraïssé limit is called the *universal homogeneous ordered graph*. By $[a_1, a_2 \ldots, a_n]$ we mean an incomplete n-cycle mapping a_i to a_{i+1} for $1 \leq i < n$.

Proposition 6.2.1. *Let $\mathcal{C}$ be the class of all finite ordered graphs. Then the class of 1-systems over $\mathcal{C}$ does not have the weak amalgamation property. Consequently, the universal homogeneous ordered graph does not have generic automorphisms.*

Proof. Suppose that $\mathcal{C}^1$ has the weak amalgamation property. Let $S = \langle A, p\rangle \in \mathcal{C}^1$ where $A = \{a_0, a_1\}$ such that $A \models \neg(a_0 E a_1) \wedge \neg(a_1 E a_0) \wedge (a_0 < a_1)$, and $p \in \mathrm{Part}(A)$ is given by $p(a_0) = a_1$. Let $T = \langle B, f\rangle \in \mathcal{C}^1$ be the extension of $\langle A, p\rangle$ guaranteed by the weak amalgamation property such that any two extensions of T amalgamate over S.

Let $[a_{-m}, \ldots, a_{-1}, a_0, a_1, \ldots, a_k]$ be the maximal incomplete cycle of f extending p. So $a_{-m} \notin \mathrm{range}(f)$ and $a_k \notin \mathrm{dom}(f)$. Define $C := B \cup \{c\}$ where c is a new point, and define $E^C := E^B \cup \{(a_{-m}, c), (c, a_{-m})\}$. Put $g := f \cup \{(a_k, c)\}$. Extend the total order of B to a total order on C by locating the new point c in the right interval such that $\langle C, g\rangle \in \mathcal{C}^1$. Similarly, define $D := B \cup \{d\}$ where d is a new point, and define $E^D := E^B$. Put $h := f \cup \{(a_k, d)\}$. Extend the total order of B to a total order on D such that $\langle D, h\rangle \in \mathcal{C}^1$. Now, by the weak amalgamation property there is $\langle G, \phi\rangle \in \mathcal{C}^1$ which amalgamates $\langle C, g\rangle$ and $\langle D, h\rangle$ over $\langle A, p\rangle$. We may assume that $C, D \subseteq G$. Thus, $p \subseteq [a_{-m}, \ldots, a_{-1}, a_0, a_1, \ldots, a_k, c] \subseteq g \subseteq \phi$ and $p \subseteq [a_{-m}, \ldots, a_{-1}, a_0, a_1, \ldots, a_k, d] \subseteq h \subseteq \phi$. Therefore $c = \phi(a_k) = d$, which leads us to identify the vertices $c \in C$ and $d \in D$ in G. As C is a substructure of G we have that $G \models E(a_{-m}, c)$, and as D is a substructure of G we have that $G \models \neg E(a_{-m}, d)$, contradicting $c = d$ in G. ■

Corollary 6.2.2. *The universal homogeneous ordered graph Γ does not have locally generic automorphisms.*

Proof. Let $\mathcal{C}$ be the class of all finite ordered graphs. By Theorem 2.2.9, Γ has a locally generic automorphism if and only if $\mathcal{C}^1$ satisfies the local weak amalgamation property. Suppose for the sake of contradiction that $\mathcal{C}^1$ satisfies the local weak amalgamation property. Then there is a 1-system $S = \langle A, p \rangle \in \mathcal{C}^1$ such that the subclass $\mathcal{K} := \{T \in \mathcal{C}^1 : S \text{ embeds in } T\}$ of $\mathcal{C}^1$ has the weak amalgamation property. Let $B := A \cup \{b_0, b_1\}$ and set $a < b_0 < b_1$ where $a := \max(A)$, and there are no edges between A and the b_i and b_0, b_1 are non-adjacent. Moreover, define a partial automorphism $f \in \mathrm{Part}(B)$ by setting $f{\restriction}_A = p$ and $f(b_0) = b_1$. Now apply the argument in the proof above to $T = \langle B, f \rangle \in \mathcal{K}$ to deduce a contradiction. ■

So the universal homogeneous ordered graph Γ is an example of a homogeneous structure over a finite relational language without locally generic automorphisms. To the best of our knowledge it is the first such example. However, as the class of all n-systems over $\mathrm{Age}(\Gamma)$ has the joint embedding property for each $n \in \omega$ we have by Theorem 2.2.3 that $G = \mathrm{Aut}(\Gamma)$ does have a dense diagonal conjugacy class in G^n for each $n \in \omega$.

6.3 Tournaments

We work in the language of directed graphs, $\mathcal{L} = \{\rightarrow\}$ where $\rightarrow$ is a binary relation symbol. A *directed graph* or *digraph* D is an $\mathcal{L}$-structure where $\rightarrow$ is interpreted as an irreflexive anti-symmetric ($x \rightarrow y$ implies $y \not\rightarrow x$) binary relation on D. A *tournament* is a digraph $(T, \rightarrow)$ such that for any distinct vertices $u, v \in T$ we have that either $u \rightarrow v$ or $v \rightarrow u$. The class of all finite tournaments is an amalgamation class, and its Fraïssé limit is called the *universal homogeneous tournament*.

The fact that EPPA for tournaments is an open problem (see Chapter 7) forms an obstacle to proving the existence or non-existence of ample generics for the universal homogeneous tournament. Below we show that EPPA is a sufficient and necessary condition for the universal tournament to have ample generics. Thus, we cannot bypass the EPPA obstacle to prove or disprove the existence of ample generics for the universal homogeneous tournament via the weak amalgamation property of the Kechris-Rosendal characterisation.

Proposition 6.3.1. *The universal homogeneous tournament has ample generics if and only if the class of finite tournaments has EPPA.*

Proof. Let $\mathbb{T}$ be the universal homogeneous tournament. Suppose that $\mathbb{T}$ has ample generics. Let $\mathcal{C} = \mathrm{Age}(\mathbb{T})$, the class of all finite tournaments. Then by Kechris-Rosendal characterisation we have that for each $n \in \omega$, the class $\mathcal{C}^n$ of n-systems has the weak amalgamation property. Suppose we are given a finite tournament $A \in \mathcal{C}$. Let $\mathrm{Part}(A) = \{p_1, p_2, \ldots, p_n\}$. Then $S := \langle A, p_1, \ldots, p_n \rangle \in \mathcal{C}^n$. By the weak amalgamation property, there is $T = \langle B, f_1, \ldots, f_n \rangle \in \mathcal{C}^n$ extending S such that any two extensions of T amalgamate over S. Define:

$$B_0 := \{\eta(a) : a \in A \text{ and } \eta \in \text{Word}(f_1^{\pm 1}, \ldots, f_n^{\pm 1})\}.$$

Let $V = \langle B_0, f_1{\restriction}_{B_0}, \ldots, f_n{\restriction}_{B_0} \rangle \in \mathcal{C}^n$, and note that any two extensions of T which are amalgamated over S are actually amalgamated over V. So B_0 will always be contained in the amalgam.

Claim. The substructure $B_0 \subseteq B$ is an EPPA-extension of A.

Proof of claim. By definition of B_0, if $b \in B_0 \cap \mathrm{dom}(f_i)$ where $1 \leq i \leq n$, then $f_i(b) \in B_0$. So it remains to show that $B_0 \subseteq \mathrm{dom}(f_i)$ for all i such that $1 \leq i \leq n$. It follows then that $f_i{\restriction}_{B_0} \in \mathrm{Aut}(B_0)$ extending $p_i \in \mathrm{Part}(A)$, and so B_0 is an EPPA-extension of A. Fix some $f \in \{f_1, \ldots, f_n\}$, and let p be the corresponding p_i. Suppose that there is some $\hat{b} \in B_0$ and $\hat{b} \notin \mathrm{dom}(f)$.

Let $l = [b_1, b_2, \ldots, b_k := \hat{b}]$ be a maximal incomplete cycle of f (allowing the possibility of $k = 1$), and note that each $b_j \in B_0$. By maximality, $b_1 \notin \mathrm{range}(f)$. We now define two extensions of $\langle B, f \rangle$. Define $C := B \cup \{c\}$ where c is a new point. Make C into a tournament as follows. For each $u \in B$, define:

$$\begin{cases} u \to c, & \text{if } u \notin \mathrm{range}(f) \\ u \to c, & \text{if } u \in \mathrm{range}(f) \text{ and } f^{-1}(u) \to b_k \\ c \to u, & \text{if } u \in \mathrm{range}(f) \text{ and } b_k \to f^{-1}(u) \end{cases}$$

Define $g := f \cup \{(b_k, c)\}$, then $\langle C, g\rangle \in \mathcal{C}^1$, that is, $g \in \mathrm{Part}(C)$, because the structure on C was defined such that $v \to b_k$ if and only if $f(v) \to c$ for every $v \in \mathrm{dom}(f)$. Next, define $D := B \cup \{d\}$ where d is a new point. Make D into a tournament as follows. For each $u \in B$, define:

$$\begin{cases} d \to u, & \text{if } u \notin \mathrm{range}(f) \\ u \to d, & \text{if } u \in \mathrm{range}(f) \text{ and } f^{-1}(u) \to b_k \\ d \to u, & \text{if } u \in \mathrm{range}(f) \text{ and } b_k \to f^{-1}(u) \end{cases}$$

Define $h := f \cup \{(b_k, d)\}$, and so $\langle D, h\rangle \in \mathcal{C}^1$.

By the weak amalgamation property, there is $\langle G, \phi\rangle \in \mathcal{C}^1$ which amalgamates $\langle C, g\rangle$ and $\langle D, h\rangle$ over $\langle A, p\rangle$. We have arrived at a situation where the incomplete cycle $l \subseteq B_0 \subseteq G$, and $l \subseteq g, h \subseteq \phi$. This forces $c \in C$ to be identified with $d \in D$ in G because $c = \phi(b_k) = d$. Such identification gives a contradiction as $b_1 \to c$ and $d \to b_1$. Therefore, $B_0 \subseteq \mathrm{dom}(f)$ for all $f \in \{f_1, \ldots, f_n\}$, proving the claim.

Conversely, suppose that the class of all finite tournaments $\mathcal{C}$ has EPPA. We want to show that $\mathcal{C}$ has APA as well. It follows then that the universal homogeneous tournament has ample generics by Theorem 2.1.5. So let $A, B, C \in \mathcal{C}$ be such that $A \subseteq B$ and $A \subseteq C$. Let D be the disjoint union of B and C over A, and make D into a tournament by setting $b \to c$ for all $b \in B \setminus A$ and $c \in C \setminus A$. Then D amalgamates B and C over A. Now suppose that $f \in \mathrm{Aut}(B)$ and $g \in \mathrm{Aut}(C)$ such that $f{\restriction}_A = g{\restriction}_A$. Then their union $h := f \cup g \in \mathrm{Aut}(D)$. ■

Remark. The proof above yields the following. The universal homogeneous tournament has n-generic automorphisms if and only if for every finite tournament A and $f_1, \ldots f_n \in \mathrm{Part}(A)$ there is a tournament B such that $A \subseteq B$ and each f_i extends to an automorphism of B.

Chapter 7

Open Questions

We now present some future research questions related to the work in this thesis. We have seen in Section 1.4 that the countable dense linear order $(\mathbb{Q}, <)$ admits generic automorphisms, but it does *not* have 2-generic automorphisms. This observation stimulates the following question.

Question 1. Is there for every $n \in \omega$, a countably infinite structure which has n-generic automorphisms, but not $(n+1)$-generic automorphisms?

In Section 3.2 we showed that any free amalgamation class over a finite relational language has coherent EPPA. The proof uses a work of Herwig-Lascar, Hodkinson-Otto, and Solecki.

Question 2. Can we prove (coherent) EPPA for free amalgamation classes using and extending the necklace argument of Ivanov? See Proposition 4.2.6.

Question 3. Is there a class of finite structures which has EPPA, but not coherent EPPA?

The following is a strengthening of Proposition 3.3.5.

Question 4. If M is ω-categorical and $\mathrm{Aut}(M)$ has a dense locally finite subgroup, must $\mathrm{Age}(M)$ have *coherent* EPPA?

Question 5. Does the class of all finite two-graphs have EPPA? Does its Fraïssé limit, the universal homogeneous two-graph, have ample generics? See Example 3.2.4.

Question 6. Let M be the universal homogeneous directed graph not containing an independent set of size 3. Does $\mathrm{Age}(M)$ have EPPA?

Question 7. Does the universal bowtie-free graph $\mathcal{U}_{\bowtie}$ of Chapter 4 have ample generics?

A *tournament* is a directed graph such that any pair of distinct vertices is connected by exactly one directed edge. The class of all finite tournaments is an amalgamation class, and so it has a Fraïssé limit, called the *universal homogeneous tournament*. We note that if T is a finite tournament, then $\mathrm{Aut}(T)$ has an odd order, and so is a soluble group by the Feit-Thompson Theorem. See [55, Remark 5.3.11] for more details on $\mathrm{Aut}(T)$. A long-standing open problem about finite tournaments is the following:

Question 8. Does the class of finite tournaments have EPPA?

Another question related to tournaments arose in a visit from Phillip Wesolek to Dugald Macpherson in Leeds in February 2017. A group is called *locally soluble* if every finitely generated subgroup is soluble.

Question 9. Is there a locally soluble oligomorphic permutation group?

If the class of finite tournaments has coherent EPPA, then the answer of the previous question is yes. For by Theorem 3.3.4 the automorphism group of the universal homogeneous tournament has a dense locally finite subgroup H. The group H is constructed as a union of a chain of subgroups of automorphism groups of finite tournaments (soluble groups), and so H would be an example of a locally soluble oligomorphic group.

Remarks. **(i)** Macpherson [53] proved that if Ω is an infinite set and $G \leq \mathrm{Sym}(\Omega)$ is soluble, then G has infinitely many orbits on Ω^4. So we cannot hope for a soluble oligomorphic group.

(ii) Suppose that M is an ω-categorical structure. If there is a subset $A \subseteq M$ of size 5 such that $\mathrm{Alt}(A)$ or $\mathrm{Sym}(A)$ is induced by $\{g \in \mathrm{Aut}(M) : g(A) = A\}$, then $\mathrm{Aut}(M)$ has no dense locally soluble subgroup as $\mathrm{Alt}(A)$ is insoluble.

We are interested in finding out whether EPPA is a necessary condition for the existence of ample generics. We have seen that if M is a homogeneous ω-categorical structure such that $\mathrm{Th}(M)$ has the strict order property, then $\mathrm{Age}(M)$ does not have EPPA.

Question 10. Is there a homogeneous ω-categorical structure with the strict order property which admits ample generics?

Recall that the cofinality of a group which is not finitely generated is the least cardinality of a chain of its proper subgroups whose union is the whole group. Macpherson-Neumann [56, Section 5] considered covering a group with an arbitrary family of proper subgroups, not necessarily a chain of proper subgroups, and they defined the *subgroup covering number* of G to be the least cardinal λ such that there is a family of size λ of proper subgroups whose union is G. So the subgroup covering number of any group is less than or equal to its cofinality. One of their theorems shows that the subgroup covering number of $\mathrm{Sym}(\mathbb{N})$ is uncountable. Motivated by their result, we ask the following:

Question 11. Does the automorphism group of an ω-categorical structure with ample generics have an uncountable subgroup covering number?

Another line of research related to Fraïssé Theory is the connection with structural Ramsey theory and topological dynamics of automorphism groups of countable structures. Following [55, Section 6.5], suppose that $\mathcal{C}$ is an amalgamation class of finite $\mathcal{L}$-structures, and let $<$ be a new binary relation symbol added to the original language. Define $\mathcal{C}_< := \{(A, <) \text{ linear order } : A \in \mathcal{C}\}$, and suppose that $\mathcal{C}_<$ is an amalgamation class. Then the Kechris-Pestov-Todorcevic correspondence ([46]) states that the automorphism group of the Fraïssé limit of $\mathcal{C}_<$ is *'extremely amenable'* if and only if $\mathcal{C}_<$ has the *'Ramsey property'*.

Question 12. Is there a formal connection between EPPA and the Ramsey property?

We elaborate more on this question. On the one hand, if $\mathcal{C}$ is a free amalgamation class, then $\mathcal{C}_<$ has the Ramsey property—see [55, Theorem 6.5.3], and $\mathcal{C}$ has EPPA by Chapter 3. On the other hand, Böttcher-Foniok [6] proved that each of Cameron's five amalgamation classes of permutations [9] is a Ramsey class. Here permutations are viewed as $\mathcal{L}$-structures where $\mathcal{L} = \{<_1, <_2\}$ and both symbols are interpreted as linear orders. So an amalgamation class of permutations will have the form $\mathcal{C}_{<_2}$ where $\mathcal{C} = \{(A, <_1) : <_1 \text{ is a linear order on } A\}$. In this case, $\mathcal{C}_{<_2}$ has the Ramsey property, but $\mathcal{C}$ does not have EPPA. So suppose we have an amalgamation class $\mathcal{C}$ with further assumptions. If $\mathcal{C}_<$ has the Ramsey property, is it possible to derive EPPA for $\mathcal{C}$? Conversely, is it possible to show that $\mathcal{C}_<$ has the Ramsey property if $\mathcal{C}$ has EPPA?

Appendix

A Proof of Fraïssé's Theorem

The material in this section stems from [5, Section 14.4] and [80, Proposition 2.3]. For simplicity we work with a relational language $\mathcal{L}$. Recall that a countable $\mathcal{L}$-structure is homogeneous if every finite partial automorphism extends to an automorphism.

Definition A.1. An $\mathcal{L}$-structure M is *weakly homogeneous* if whenever $A, B \in \mathrm{Age}(M)$ with $A \subseteq B$ and $f : A \to M$ is an embedding, then there is an embedding $g : B \to M$ which extends f.

Lemma A.2. *A countable $\mathcal{L}$-structure M is homogeneous if and only if it is weakly homogeneous.*

Proof. Suppose that M is a homogeneous structure. Let $A, B \in \mathrm{Age}(M)$ with $A \subseteq B$ and let $f : A \to M$ be an embedding. Let $B' \subseteq M$ be such that there is an isomorphism $\alpha : B \cong B'$. By homogeneity, the finite partial automorphism $f\alpha^{-1} : \alpha(A) \to f(A)$ extends to an automorphism $h \in \mathrm{Aut}(M)$. Then the embedding $h\alpha : B \to M$ extends f. So M is weakly homogeneous.

Now for the other direction suppose that M is weakly homogeneous. Let $f : U \to V$ be an isomorphism between finite substructures of M. We will construct an automorphism $\hat{f} \in \mathrm{Aut}(M)$ extending f. Write M as a countable union of a chain of finite substructures in two ways: $M = \bigcup_{n\in\mathbb{N}} U_n$ where $U_0 = U$ and $M = \bigcup_{n\in\mathbb{N}} V_n$ where $V_0 = V$.

We proceed by alternating between even and odd steps to define an increasing chain of embeddings $\big(f_n : n \in \omega\big)$ as follows. Start with $f_0 := f$.

At the even step $(2n)$, we ensure that $U_n \subseteq \operatorname{dom}(\hat{f})$. At this point, we have that $U_{n-1} \subseteq \operatorname{dom}(f_{2n-1})$. We can find $k \in \mathbb{N}$ large enough so that $(\operatorname{dom}(f_{2n-1}) \cup U_n) \subseteq U_k$. Since M is weakly homogeneous, there is an embedding $f_{2n} : U_k \to M$ which extends f_{2n-1}. Clearly $U_n \subseteq \operatorname{dom}(f_{2n})$ and $f_{2n-1} \subseteq f_{2n}$.

At the odd step $(2n+1)$, we ensure that $V_n \subseteq \operatorname{range}(\hat{f})$. At this point we have that $V_{n-1} \subseteq \operatorname{range}(f_{2n})$. We can find $k \in \mathbb{N}$ large enough so that $(\operatorname{range}(f_{2n}) \cup V_n) \subseteq V_k$. Since M is weakly homogeneous, there is an embedding $g : V_k \to M$ which extends f_{2n}^{-1}. Now define $f_{2n+1} := g^{-1}$. Clearly $V_n \subseteq \operatorname{range}(f_{2n+1})$ and $f_{2n} \subseteq f_{2n+1}$.

Finally $\hat{f} := \bigcup_{n \in \mathbb{N}} f_n$ is an automorphism of M extending f. So M is homogeneous. ■

Theorem A.3 (Fraïssé's Theorem [28])**.** *Suppose that $\mathcal{C}$ is an amalgamation class of finite $\mathcal{L}$-structures. Then there is a unique, up to isomorphism, homogeneous $\mathcal{L}$-structure M such that* $\operatorname{Age}(M) = \mathcal{C}$*. Conversely, if N is a homogeneous $\mathcal{L}$-structure then* $\operatorname{Age}(N)$ *is an amalgamation class.*

Proof. Enumerate representatives of all the isomorphism types in $\mathcal{C}$ as $A_0, A_1, A_2, \ldots$. The aim is to construct a chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ of structures $M_n \in \mathcal{C}$ such that for all $n \in \omega$ we have that M_n has a substructure isomorphic to A_n, and whenever $i, j \leq n$, and $\alpha : A_i \to A_j$ and $f : A_i \to M_n$ are embeddings, then there is an embedding $g : A_j \to M_{n+1}$ such that $f = g\alpha$. Start by taking $M_0 = A_0$. Suppose that some finite chain $M_0 \subseteq \ldots \subseteq M_n$ has been constructed as required. Enumerate the finitely many, say r many, pairs of embeddings $\big(\alpha_k : A_{k_i} \to A_{k_j},\ f_k : A_{k_i} \to M_n\big)_{k=1}^{r}$ with $k_i, k_j \leq n$. Using the amalgamation property, we build inductively a chain $M_n = B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots \subseteq B_r =: M_{n+1}$ where for $1 \leq k \leq r$, we have that B_k is the amalgam of B_{k-1} and A_{k_j} over A_{k_i} via the embeddings α_k and f_k. Using the joint embedding property we may assume that M_{n+1} contains a copy of A_{n+1}. It is not difficult to see that M_{n+1} satisfies the desired requirements. Finally, put $M = \bigcup_{n\in\omega} M_n$. By construction and the hereditary property we get that $\mathrm{Age}(M) = \mathcal{C}$. Moreover M is weakly homogeneous, and so by the Lemma A.2 above, the structure M is homogeneous.

We now show the uniqueness of M. Suppose that N is homogeneous $\mathcal{L}$-structure such that $\mathrm{Age}(M) = \mathrm{Age}(N)$. Enumerate the elements of M as $M = \{m_i : i \in \omega\}$ and the elements of N as $N = \{n_i : i \in \omega\}$. We build up an isomorphism $h : M \to N$ by using a back-and-forth argument.

Step 0: Let $i \in \omega$ be the least i such that $\{m_0\} \cong \{n_i\}$. Define $h_0(m_0) := n_i$.

Step $2i + 1$: To ensure that $\mathrm{dom}(h) = M$. Suppose that the partial isomorphism h_{2i} has been defined with $\mathrm{dom}(h_{2i}) = \{m'_1, m'_2 \ldots, m'_k\}$. Let $j \in \omega$ be the least j such that $m_j \notin \mathrm{dom}(h_{2i})$. We want to add m_j to the domain. Let $A = \{m'_1, m'_2 \ldots, m'_k, m_j\} \subseteq M$. As $\mathrm{Age}(M) = \mathrm{Age}(N)$, we can find an isomorphic copy $\{n'_1, n'_2 \ldots, n'_k, n'_{k+1}\}$ of A in N. Thus, $\{n'_1, n'_2 \ldots, n'_k\} \cong \{m'_1, m'_2 \ldots, m'_k\} \cong \{h_{2i}(m'_1), h_{2i}(m'_2) \ldots, h_{2i}(m'_k)\}$. Let $g : \{n'_1, n'_2 \ldots, n'_k\} \cong \{h_{2i}(m'_1), h_{2i}(m'_2) \ldots, h_{2i}(m'_k)\}$ be the isomorphism above. As N is homogeneous, there is $\hat{g} \in \mathrm{Aut}(N)$ extending g. Hence,

$$\{m'_1, \ldots, m'_k, m_j\} \cong \{n'_1, \ldots, n'_k, n'_{k+1}\} \cong \{h_{2i}(m'_1), \ldots, h_{2i}(m'_k), \hat{g}(n'_{k+1})\}.$$

Define $h_{2i+1} := h_{2i} \cup \{(m_j, \hat{g}(n'_{k+1}))\}$.

Step $2i + 2$: To ensure that $\mathrm{range}(h) = N$. Suppose that the partial isomorphism h_{2i+1} has been defined. Let $j \in \omega$ be the least j such that $n_j \notin \mathrm{range}(h_{2i+1})$. Using the same strategy above, we can find a suitable preimage of n_j and extend h_{2i+1} to h_{2i+2} with $n_j \in \mathrm{range}(h_{2i+2})$.

Finally, define $h := \bigcup_{i\in\omega} h_i$. By construction $\mathrm{dom}(h) = M$ and $\mathrm{range}(h) = N$, and so $h : M \cong N$.

It remains to show that last part of the theorem. So suppose that N is a homogeneous $\mathcal{L}$-structure. We show that $\mathrm{Age}(N)$ satisfies the amalgamation property. Suppose that $A, B_1, B_2 \in \mathrm{Age}(N)$ with embeddings $f_1 : A \to B_1$ and $f_2 : A \to B_2$. We can assume without loss of generality that A, B_1, B_2 are substructures of N. Since N is homogeneous, there are $\alpha, \beta \in \mathrm{Aut}(N)$ which extend f_1, f_2 respectively.

Define $C := \alpha^{-1}(B_1) \cup \beta^{-1}(B_2)$, and let $g_1 = \alpha^{-1}|_{B_1}$ and $g_2 = \beta^{-1}|_{B_2}$. For any $a \in A$ we have that $g_1 \circ f_1(a) = \alpha^{-1} \circ \alpha(a) = a = \beta^{-1} \circ \beta(a) = g_2 \circ f_2(a)$. Thus C amalgamates B_1 and B_2 over A. ■

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