Endogenous Growth and Fiscal Policy with Productive Government Expenditure

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For Sirisakdi and his Dream

Abstract

Fiscal policy can be considered as a key driver of economic growth. The government can either provide for the public good used in firm's production process, or invest in public education that improves the abilities and skills of workers. In term of the public good and public education, the growth rate of economy is undoubtedly enhanced. However, the magnitude of growth also depends on how the government levies the distortionary tax to finance its spending. Since different kinds of taxes contribute to the different states of economy, as well as the long-run growth, the main purpose of this thesis is to examine the effect of the fiscal policy on economic growth in various theoretical frameworks, particularly the representative-agent approach and the three-period overlapping generations model.

The theoretical analysis reveals a number of interesting findings. Firstly, the productive externalities from public investment and procyclical endogenous consumption tax creates the multiplicity of balanced growth paths (BGPs) in the representative agent framework. Two BGPs arise due to the existence of the Laffer curve. In addition, local indeterminacy may occur around the lowest balanced growth path if consumption tax is mildly procyclical. As a result, there is no trade-off between growth and volatility. Secondly, the growth-maximising tax rate is investigated in the three-period overlapping generations economy in which altruistic parents provide private tuition for their children and the government subsidises public education. When the government misconceives of the existence of private tuition, public education is over-provisioned. This leads the economy to the growth-reducing area of the Armey curve. Finally, in the presence of stochastic productive government expenditure, the economy experiences a higher growth rate than it does in the perfect foresight economy when households are risk-averse agent. The inverted-U shape relationship between economic growth and permanent income tax disappears in the stochastic growth context; nevertheless, the condition of growth-enhancing tax rate remains valid. Furthermore, the first-best fiscal instruments are explored based on the difference between centralised and decentralised economies in the case of proportional congestion. To prevent the welfare loss, the initial capital should be sacrificed in the interest of the higher level of consumption. For this reason, the trade-off between growth and welfare is unavoidable in a decentralised economy.

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Declaration

I hereby declare that this thesis entitled "Endogenous Growth and Fiscal Policy with Productive Government Expenditure" is purely my own work and effort, except the second chapter that is jointly written with my supervisor; Dr.Mauro Bambi. This thesis contains the original idea which is the extension of endogenous growth theory. It has never been submitted for any other degrees in University of York and other institutions. Any errors and omissions are my sole responsibility. All sources are acknowledged as References.

Chapter 1

Introduction

With the reference to the neoclassical growth theory, Solow (1956) and Swan (1956) illustrated that economic growth is completely independent of the capital accumulation in an economy in which the diminishing marginal product of capital (MPK) takes place. With the diminishing MPK, the capital will stop accumulating when the MPK is equal to the inverse of discount factor (Jones and Manuelli, 1990). As a result, the consumption is constant over time and the long-run growth is equal to zero. To avoid a zero growth rate, the neoclassical growth theorists add some exogenous factors such as population growth and technological progress as the sources of economic growth. However, the concept of exogenous growth is challenged by endogenous-growth economists who argued that economic growth should be a result of the interactions or decision making of the economic agents. By allowing endogenous growth to exist, the assumption of diminishing MPK is adjusted to ensure that the economy will experience a positive long-run growth (Barro and Sala-i Martin, 2004).

In the absence of population growth and technological progress, the literature on endogenous growth can be classified into two groups. The first group is the endogenous growth model that incorporates convex production technology in which the diminishing MPK still exists, but with a bound. This bound prevents the MPK from approaching zero when the capital is hugely abundant. In one-sector endogenous growth model[1], Barro and Sala-I-Martin (1992) addressed three possibilities that could lead to convex production technology and a constant rate of return on aggregate capital, namely learning

[1] Rebelo (1991) and Jones et al. (1993) also confirmed that the diminishing marginal product of capital with bound can be obtained in the two-sector model.

by doing, public services with taxes, and a variety of capital goods in an imperfect market.

The other group is the endogenous growth model with non-convex production technology where the diminishing return on capital disappears. The main factor contributing to the non-convexity of production technology is the externalities that occur on the individual level but have an impact on the aggregate level. For instance, the externalities created by new knowledge may cause an increasing return on aggregate capital (Romer, 1986). Similarly, the externalities generated by human capital could potentially result in an increasing return in the production of output (Lucas, 1988). Thus, these two groups can be differentiated by production technology. While the former group applies the constant return on aggregate capital, the latter group provides the opportunity for an economy to experience an increasing return on aggregate capital. Despite the challenges and interesting implications of non-convex technology, this thesis employs the convex technology as a central analysis throughout all the chapters because it is more likely to satisfy the standard welfare theorem and to ensure a positive growth rate in the economy[2].

Although there are many sources of endogenous growth, this thesis emphasises on public investment, productive government expenditure and taxation as the drivers of economic growth. Despite the existence of different forms of productive government spending, it raises the rate of return on private capital, and thus stimulates private investment and economic growth in the long run (Aschauer, 1989; Easterly and Rebelo, 1993; Agénor, 2007). According to the paper by Barro (1990), the government invests a flow of productive government spending in the public good that is used in the production process. As a result of an inelastic labour supply, a balanced budget rule and a constant income tax rate, the impact of fiscal policy on endogenous growth can be analysed by focusing on the Euler equation. Given a sufficient level of technology, a unique balanced growth path exists and the economy experiences a positive growth rate.

Unfortunately, the model proposed by Barro (1990) has no transitional dynamics because productive government spending is a flow variable. To understand the transitional dynamics together with the long-run growth, public investment should be considered in the way that public capital stocks

[2]As explained in the work of Jones and Manuelli (1997), an economy with non-convex technology is more likely to have a negative growth rate.

can be used as the inputs in the firms' production process (Futagami et al., 1993; Irmen and Kuehnel, 2009). In addition, the method of financing public expenditure is also important to analyse the local stability of the long-run growth path. The government can levy various kinds of distortionary tax rates such as labour income tax, capital tax and time-varying consumption tax to finance government spending and to balance its budget in each period[3]. For example, Schmitt-Grohe and Uribe (1997) illustrated the conditions of global and local indeterminacy when a government levied labour income tax to finance exogenous government spending. Park and Philippopoulos (2004) confirmed that the combination of public services and capital tax generated two balanced growth paths, and the lowest balanced growth path was locally indeterminate under certain conditions. Therefore, it seems that labour income tax and capital tax may not be a preferable choice since they undoubtedly create the indeterminate equilibrium in the economy. By contrast, the effect of time-varying consumption tax on the aggregate economy is still a puzzle to economists. Giannitsarou (2007) found a unique steady state and its transitional dynamics was locally determinate when exogenous government expenditure was entirely financed by time-varying consumption tax. On the other hand, Lloyd-Braga et al. (2008) and Nourry et al. (2013) proposed the condition of the multiplicity of steady states and the condition of aggregate instability for economies in which the countercyclical consumption tax was an important financial instrument and the different types of households' preference are considered.

The role of countercyclical consumption tax is even more robust in the paper of Bambi and Venditti (2016). When productive government spending is considered together with endogenous consumption tax, the authors found the condition of global indeterminacy if consumption tax were countercyclical. Moreover, local indeterminacy arises if purely extrinsic uncertainty is introduced to the economy. Since procyclical consumption tax may prevent aggregate instability, it is then recommended to policymakers.

Should procyclical consumption tax policy be implemented? The answer to this question is much more complicated than it seems because procyclical taxation has been criticised by economists in several ways. For instance, procyclical fiscal policy can lead to the misuse of fiscal resources during period

[3]The possibility of having public debt is ruled out in this thesis for reasons of simplicity. For papers that allow public debt, please see Minea and Villieu (2009) and Groneck (2010) for their model setup.

of financial upswing in favour of the political partisans (Talvi and Vegh, 2005). Furthermore, following the standard Keynesian wisdom, procyclical fiscal policy may amplify the business cycles and the economy may suffer from a long period of recession. Thus, the scepticism about procyclical consumption tax provides the opportunity for the second chapter to explore the effect of public investment and procyclical consumption tax on the aggregate economy in the endogenous growth model. Accordingly, there are two contributions in this chapter. Firstly, public capital is considered as the input for the firms' production function. An inelastic labour supply, private capital and public capital result in the convexity of production technology and a constant return on aggregate capital. Secondly, the role of government is constrained by two rules, namely the balanced budget rule and the fiscal policy rule. In term of the fiscal policy rule, endogenous consumption tax is determined by the state-contingent variables and the time-invariant fiscal instruments, as described in Persson and Tabellini (2002). This fiscal rule makes this paper a significant departure from the work of Bambi and Venditti (2016) in which consumption tax depends on detrended control variable.

In the continuous-time framework, the Hamiltonian objective function is set and the equilibrium conditions satisfy the definition of balanced growth path (BGP). As it is common in the literature on endogenous growth, a unique BGP is found when a level of technology is sufficiently high. However, the sufficient condition for the existence of two BGPs, namely the lowest BGP and the highest BGP, is discovered. The inverse elasticity of intertemporal substitution, as well as a degree of procyclicality should be sufficiently large in order for the detrended public investment line and a non-monotonic function of detrended tax revenue (a Laffer curve) to be intersected twice. For the lowest BGP, it is locally indeterminate when purely extrinsic uncertainty is introduced and consumption tax is mildly procyclical. As a result, there is no trade-off between output growth and output volatility due to the existence of aggregate instability around the lowest BGP. For this reason, policymakers should implement the procyclical consumption tax policy with caution when the economy has a low growth rate.

Despite the fact that the effect of public investment and endogenous consumption tax in the representative agent framework potentially leads to the multiplicity of balanced growth paths and aggregate instability, the impact of productive government spending and distortionary taxation on economic growth remains unclear in the context of the heterogenous-agent model. Therefore, in the third chapter, the three-period overlapping generations model is employed to study the impact of fiscal policy on the intergenerational transfers and economic growth.

The first purpose of the third chapter is to find the growth-maximising tax rate in the economy where public education and private tuition have an impact on human capital accumulation and intergenerational transfers. In fact, there are many theoretical and empirical papers that have investigated the value of the growth-maximising tax rate. For the theoretical literature, Glomm and Ravikumar (1997) argued that the growth-maximising tax rate was equal to the elasticity of output with respect to public eduction when there was an absence of private tuition in an economy. Blankenau and Simpson (2004) attempted to incorporate both private and public education systems; however, they could not prove the existence of growth-maximising tax rate due to the complexity of the model. Furthermore, the evidence of Armey curve indicates the relationship between economic growth and the size of a government across countries. The empirical evidence shows that the growth-maximising tax rate is around 20%-30% according to the work of Rezk (2005) and Facchini and Melki (2011). To differentiate our model from other papers, the impure altruistic motive and the Cobb-Douglas human capital function are employed to explore the growth-maximising tax rate. Consequently, it departs from the work of Glomm and Ravikumar (1997) and of Blankenau and Simpson (2004) because altruistic parents will obtain the direct utility from the act of giving[4]. Based on this model, at least one growth-maximising tax rate is found within a feasible range between zero and one in which the sufficient conditions for the local maximum are also satisfied.

The second purpose of this chapter is to observe whether the public education is over-provisioned. When the government misperceives the ability of impurely altruistic parents to provide private tuition for their children, it is likely that the government may exaggerate the benefits of public education. Therefore, the growth-maximising tax rate is reinvestigated in an economy in which the misperception of private tuition take places. In such an economy, the analytical proof shows that there is a corner solution that allows the

[4]Andreoni (1990) labelled this direct utility as impure altruism in the sense that a warm-glow giver will experience a positive feeling from the act of giving. The interpretation of warm-glow giving is also similar to the 'paternalistic altruism' proposed by Michel and Pestieau (2004), and the 'joy of children receiving income' addressed by Grossman and Poutvaara (2009).

government to levy labour income tax rate by 100% to finance the public spending on education. In other words, the over-provision of public education does exist in the economy where the government misperceives the existence of private tuition. Subsequently, over-spending on public education may lead the economy into the growth-reducing area of the Armey curve.

In the second and third chapters, the impact of public investment, public education and distortionary taxation on economic growth is clearly demonstrated via the perfect foresight model. However, the perfect foresight assumption does not reflect the real economy, which is usually affected by many random factors. Such random shocks will risk one economy, and will have an impact on the decision-making processes of economic agents. Once the economy is subject to uncertainty, the household decisions regarding consumption and saving depend on the degree of risk aversion (Carroll, 2001). In the stochastic environment, the deterministic endogenous growth model is therefore incapable of analysing the effect of productive government spending on economic growth and social welfare. Consequently, the stochastic endogenous growth framework is employed in the fourth chapter.

Although the sources of uncertainty may vary across countries, the substantial amount of literature on the one-sector stochastic endogenous growth framework exploits the fact that the production process is affected by random shocks[5] (Gokan, 2002; Clemens, 2004; Clemens and Soretz, 2004). However, none of them considers productive government expenditure as the input in the production function, with the exception of the paper by Turnovsky (1999). By considering productive government spending that is devoted to the public good, Turnovsky (1999) found that economic growth and social welfare were determined by congestion and a degree of risk aversion when output was affected by technological shocks. In addition, the difference between centralised and decentralised economies was also determined by a degree of congestion. Nevertheless, such a difference disappeared in the case of proportional congestion.

The finding of Turnovsky (1999) that the solutions for a centralised economy and for a decentralised economy are identical in the case of proportional congestion is indeed limited in some respects. For example, the capital risk is the only source of uncertainty when the stochastic output is a

[5]In neoclassical growth theory, Merton (1975) was the first to introduce uncertainty in population growth, while Eaton (1981) allows a technological shock to affect the output.

linear function in capital. Subsequently, the solution of decentralised economy can replicate the social planner's outcomes. Moreover, the role of government transfer has not been studied. In fact, it is an important factor that has an impact on the volatility in households' income paths, precautionary saving and the growth rate. To overcome these limitations, the fourth chapter provides the alternative source of uncertainty by introducing a random shock to productive government expenditure to enable it to follow the stochastic process[6], and allows government transfer to have an impact on households' income paths. The impact of stochastic productive government expenditure on economic growth and social welfare in decentralised economy is also discussed in order to achieve the first-best outcomes by employing fiscal instruments.

The equilibrium of a decentralised economy is analysed by using the dynamic programming method that leads to the stochastic Bellman equation. The first finding is that an economy that has a stochastic environment experiences a higher growth rate than it does in a perfect foresight economy in which households are risk-averse agents and government transfer is allowed. This is because the intertemporal substitution effect is completely dominated by the income effect. Risk-averse households will increase precautionary saving against the risk of the uncertain income flow caused by the volatility of government transfer. Secondly, the numerical example suggests that the inverted U-shaped relationship between economic growth and permanent income tax disappears when the degree of risk aversion is sufficiently high. Nonetheless, the condition of the growth-maximising permanent income tax rate that the marginal benefit of providing for the public good should be greater than the marginal cost thereof is still valid.

Thirdly, the saving and growth rates of a decentralised economy are too high when compared to a centralised economy. Thus, our result contradicts the findings by Turnovsky (1999), which indicates that both economies have the same resource allocations when the public good is subject to proportional congestion. In fact, in our case, the resource allocations in both economies are different due to the capital risk and the income risk. Furthermore, the difference between centralised and decentralised economies provides an opportunity for the decentralised government to achieve the social planner's outcomes by employing the first-best fiscal instruments. Finally, the welfare loss is calculated based

[6] The stochastic productive government expenditure is not a new concept. Instead, it has been employed in the Neoclassical work of Baxter and King (1993), and in the New Keynesian paper of Linnemann and Schabert (2006).

on the excess amount of the initial capital in the decentralised economy when compared to the centralised economy. For this reason, the trade-off between economic growth and welfare is undoubtedly unavoidable[7]. In addition, the comparison of welfare in a risky and in a riskless economy is also characterised in term of the initial capital variation. A risky economy will experience greater welfare loss when the degree of risk aversion and the volatility of stochastic productive government expenditure are extremely high.

This thesis is organised as follows. The second chapter explains how the combination of public investment and procyclical consumption tax can be a source of global and local indeterminacy. The third chapter illustrates the existence of growth-maximising tax rate in an impurely altruistic economy that has both public education and private tuition. In addition, the over-provision of public education is highlighted when the government misperceives private tuition as the input for human capital accumulation process. With regard to the stochastic setting, the impact of productive government expenditure on economic growth and social welfare is measured in the fourth chapter. The last chapter contains the conclusion and possible avenues for future research.

[7] Unlike Barro (1990), the growth-maximising problem is not the same as welfare-maximising problem in our case.

Chapter 2

Procyclical Endogenous Taxation and Aggregate Instability

Co-authored with Dr.Mauro Bambi

2.1 Introduction

Several contributions in the literature have shown that the balanced-budget rule, together with endogenous distortionary taxes may lead to aggregate instability once embedded in a neoclassical growth model.[1] Endogenous labour income taxes and capital income taxes are responsible of aggregate instability if they are sufficiently countercyclical with respect to output growth (Schmitt-Grohe and Uribe, 1997). Although endogenous consumption taxes are preferred in such a setting because they reduce the range of parameters leading to an indeterminate steady state and therefore to sunspot equilibria (Giannitsarou, 2007), this specific result cannot be extended to more general utility functions such as those proposed by Jaimovich (2008) and by Jaimovich and Rebelo (2009). In fact, local indeterminacy is guaranteed only if the consumption tax is assumed to be countercyclical and the elasticity of intertemporal substitution in consumption is sufficiently large (Nourry et al., 2013). A common characteristic of these models is that government spending is never productive. Investigating the same issue in the endogenous growth model *à la* Barro (1990) in which government spending is productive leads to a global form of indeterminacy when the consumption taxes are endogenous and countercyclical (Bambi and Venditti, 2016). In their paper, global indeterminacy is characterised by a unique stationary equilibrium and a continuum of non-stationary equilibria. This depends on the consumers' belief in the value of countercyclical consumption tax. Once extrinsic uncertainty is introduced, sunspot equilibria and aggregate instability emerge in this context.

In general, the existing literature points out that procyclical endogenous taxes are the appropriate policy to rule out aggregate instability in models where the government balanced its budget in each period. Such a policy suggestion is relevant for at least two reasons. Firstly, the balanced budget rules have been advocated and implemented as constitutional requirements in several European countries after the 2008 crisis (for example, Article 81 of Italian Constitutional Law 1/2012). However, from 2009 to 2013, 16 European countries had a budget deficit of less than 3% as a percentage of GDP. Secondly, a great number of OECD countries have adopted the countercyclical taxes with respect to output growth (Lane, 2003)[2]. For example, Figure 2.1 shows how consumption taxes

[1]Aggregate instability emerges due to the existence of (stationary) sunspot equilibria.

[2]Consistently with the previous literature, we state that taxes are countercyclical if the tax rate expands when output shrinks, and vice versa. See footnote 5.

were adjusted countercyclically in EU countries during 2009-2013, specifically in those indicated by red and grey circles. It is interesting that eight of them had a budget deficit of less than 3% as a percentage of their GDP[3] in 2013. For this reason, these countries could benefit from switching to procyclical fiscal policy as the existing literature predicts that the stabilising role of procyclical taxation should reduce the output volatility.

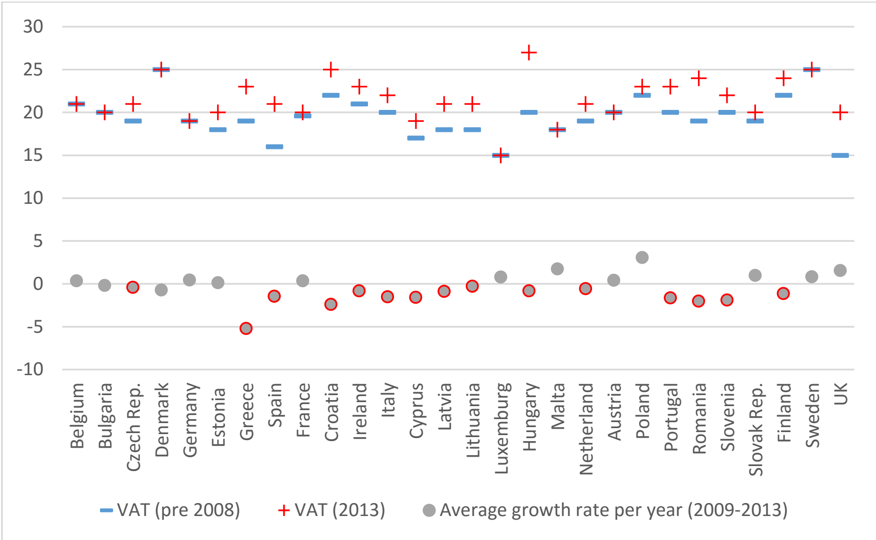


Figure 2.1: Evidence on Countercyclical VAT

The aim of this paper is to investigate the robustness of this prediction in the endogenous growth model. To make the contribution, our setting differs from that in the previous literature according to two dimensions. Firstly, the government finances 'public investment' by levying endogenous consumption taxes. The term public investment reflects the nature of public infrastructure, which is the accumulated *stock* rather than the current *flow* (Turnovsky, 1997; Fisher and Turnovsky, 1998). Using this term is also a significant departure from the work of Barro (1990) and of Bambi and Venditti(2016) since it justifies how public capital stock can be a productive input to private production. In addition, consumption taxes are considered instead of other types of taxation because it is more difficult to generate local indeterminacy even if it has been countercyclically adjusted with respect to output growth[4].

[3]According to Eurostat, these countries are the Czech Republic, Italy, Latvia, Lithuania, Hungary, the Netherlands, Romania and Finland. Moreover, the countries gradually adjusted the consumption tax during 2009-2013.

[4]Endogenous growth as a result of public investment financed by flat income taxes was originally investigated by Futagami et al. (1993).

Secondly, the government sector is characterised by two equations: a balanced budget rule *and* a fiscal policy rule. The reason for the latter is that, in our context, both public investment and the consumption tax rate are endogenous. Therefore, we need to specify a fiscal policy rule along with the balanced budget rule to avoid a trivial form of global indeterminacy. This setup is another departure from what usually done in the exogenous or no-growth models. In such models, the government spending is exogenously given and the tax rate is endogenous because it has to adjust period by period to balance the government budget[5]. By contrast, the aggregate instability does not exist in the economy if only government spending is endogenous while keeping a constant tax rate[6].

To design the fiscal policy, we follow Persson and Tabellini (2002) that a state-contingent and time-invariant fiscal policy are considered and specified by a functional form that guarantees the existence of a balanced growth path. The different fiscal policy rule literally differentiate our paper from the work of Bambi and Venditti (2016).

The main finding of this article is that our economy may admit two balanced growth paths; the lowest and highest growth. The lowest balanced growth path can be locally indeterminate even though the consumption tax is procyclical with respect to output growth. Consequently, procyclical taxation may lead the economy into a poverty trap characterised by all aggregate variables that fluctuate around the lowest balanced growth path. This finding warns policymakers that the procyclical taxation policy should be implemented with caution. Furthermore, in the presence of multiple balanced growth paths, there is no trade-off between output growth and output volatility in our setting since the aggregate instability may emerge around the balanced growth path with the lowest growth rate.

In this model, the existence of multiple BGPs depends on the existence of a Laffer curve type of relationship between the tax rate and the (detrended) tax revenue. More precisely, two balanced growth paths exist regarding the fact

[5]For example, Schmitt-Grohe and Uribe (1997, p. 80) considered the balanced budget rule $G = \tau_t w_t \ell_t$ with G exogenously given; the tax rate is endogenous because it has to adjust in each period to compensate for the changes in labour income tax and to balance the government budget. Similarly, in a model without growth, Giannitsarou (2007) considered the balanced budget rule $G = \tau_t c_t$; the tax rate is again endogenous because G is exogenously given and the budget is balanced by varying the consumption tax.

[6]Guo and Harrison (2004) could not find the aggregate instability when maintaining a flat-tax rate but adjusting the endogenous government expenditure.

that there might be two intersection of the detrended public investment and the Laffer curve. Although the existence of the Laffer curve was recently found by Nourry et al. (2013), the reason for its existence is very different from ours. In their case, the existence of the Laffer curve depends on the specification of preferences while, in our case, it depends on the shape of the fiscal policy rule and a standard CES utility function. Hence, multiple BGPs exist in this context due to a sufficiently large income effect, together with a sufficiently large procyclical taxation[7].

Similarly, our result on local indeterminacy has been proved for a sufficiently large income effect, together with mild procyclical taxation. It is indeed shown in the numerical examples that the economy can be characterised by both global indeterminacy and local indeterminacy around the lowest balanced growth path for reasonable values of the parameters.

The paper is organised as follows. In Section 2.2, the economy is fully described by the two key equations which determine the intertemporal equilibrium. Section 2.3 focuses on the existence of a balanced growth path and the sufficient conditions for global indeterminacy are found in Proposition 2. The existence of a Laffer curve is discussed and explained via figures and numerical examples. In the case of global indeterminacy, the transitional dynamics around the balanced growth path at the lowest balanced growth path are investigated in Section 2.4. Particularly, in this section, the sufficient conditions for local indeterminacy are found in Proposition 3. The numerical examples are also proposed to support that this dynamic behaviour is not only analytically possible, but also reasonable from a quantitative perspective. Section 2.5 emphasises the role of procyclical taxation in our setting and a comparison with existing results is proposed. Finally, Section 2.6 is a conclusion for this paper. The logical steps of the proofs appear in the main text while the details and the rigorous version of the proofs can be found in the Appendix A.

[7] It is worth remembering that there is no Laffer curve in flat-rate consumption taxes.

2.2 The Model Setup

In this section, we present the decision-making problem faced by the households, by the firms, and we also describe the role of government in the economy through its budget constraint and fiscal policy rule. The model setup is similar to the one proposed by Futagami et al. (1993) with the exception that the fiscal policy rule consists of endogenous and time-varying consumption tax.

Households – The economy is populated by a continuum of identical households distributed on the interval [0,1], and they are endowed with private capital stock (k) and a unit of labour in each period ($\bar{l} = 1$). They will inelastically supply this unit of labour to firms' production process and receive wage income (w) as a return. Since the identical households are an infinitely-lived agent, there is no population growth, and the population size is normalised to one ($N = 1$)[8]. Households will then choose the consumption level that maximises the intertemporal utility function,

$$\max_{c} \int_{0}^{\infty} \mathrm{e}^{-\rho t} \cdot \frac{c^{1-\sigma} - 1}{1 - \sigma} \, \mathrm{d}t$$

subject to the budget constraint,

$$\dot{k} = w + Rk - \delta k - (1 + \tau)c \tag{2.1}$$

$$c \geq 0, \qquad k \geq 0 \tag{2.2}$$

where the initial condition of capital (k_0) is exogenously given. All variables are per-capita term. Gross income per capita is the sum of the return on capital and a wage income, $y \equiv (R - \delta)k + w$, while net income is $y - \tau c$ where τ indicates the consumption tax rate. Net income is allocated between consumption (c) and gross investment $i \equiv \dot{k} + \delta k$. The intertemporal preference discount rate (ρ) and the depreciation rate of capital (δ) are assumed, as usual, to be between zero and one, while the inverse of the elasticity of intertemporal substitution in consumption (σ) is strictly greater than one[9].

The present-value Hamiltonian of this problem is

$$\mathcal{H} \equiv \frac{c^{1-\sigma} - 1}{1 - \sigma} \cdot e^{-\rho t} + \lambda[(w + (R - \delta)k - (1 + \tau)c]$$

[8]The size of labour force in this economy (L) is equal to one since $L = N\bar{l} = 1$

[9]Following the seminal paper by Barro (1990), $\sigma < 1$ may lead to the unbounded utility. For $\sigma = 1$, Giannitsarou (2007) showed that the global and local indeterminacy cannot occur. Thus, these two cases are eliminated from our study.

whose first-order conditions are:

$$\frac{\partial \mathcal{H}}{\partial c} = 0 \qquad \Leftrightarrow \qquad c^{-\sigma}e^{-\rho t} = \lambda(\tau + 1) \tag{2.3}$$

$$\frac{\partial \mathcal{H}}{\partial k} = -\dot{\lambda} \qquad \Leftrightarrow \qquad \lambda(R - \delta) = -\dot{\lambda} \tag{2.4}$$

Differentiating (2.3) and substituting it into (2.4) leads to the Euler equation;

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left(R - \delta - \rho - \frac{\dot{\tau}}{1+\tau}\right) \tag{2.5}$$

and a standard transversality condition must also hold.

$$\lim_{t\to\infty} \lambda k = 0 \tag{2.6}$$

Firms – There is a continuum of identical firms distributed on the interval [0,1]. Each firm produces a homogeneous product that can be consumed or invested in the economy. A unit of labour supply ($\bar{l}$), private capital (k) and public capital (g) are used as the inputs in the production process. With regard to the production technology, the production function is characterised by the Cobb-Douglas specification:

$$y = Ak^{\alpha}(\bar{l}g)^{1-\alpha} \tag{2.7}$$

where a constant level of technology is always greater than zero ($A > 0$). Indeed, public capital can be measured by the ratio of aggregate public capital to the size of labour force ($g = \frac{G}{L}$)[10]. Since $\bar{l} = 1$, the production function can be rewritten in the following form:

$$y = Ak^{\alpha}g^{1-\alpha} \tag{2.8}$$

Firms will then maximise their profit subject to the production technology, the total revenue and the total cost. By choosing the amount of capital, the rental rate (R) and the wage rate (w) are determined by the following two conditions:

$$R = \alpha Ak^{\alpha-1}g^{1-\alpha} \tag{2.9}$$

$$w = (1-\alpha)Ak^{\alpha}g^{1-\alpha} \tag{2.10}$$

where the condition (2.9) is directly obtained from the first-order condition. The condition (2.10) can be derived from the perfectly competitive market assumption in which firms receive zero profit.

[10]Due to the fact that $L = 1$, $g = G$ in our case.

Government – The government devotes public investment in the public good, such as a public infrastructure, that is used to enhance private productivity. This public investment consists of a change in public capital ($\dot{g}$) and a depreciation of public capital (δg) at each point in time[11]. Public capital is a productive input to individual firm's production function while the depreciation of public capital refers to the cost of maintaining the public good in each period.

To finance public investment, consumption tax is preferred to other types of taxation. Since income tax or capital tax may affect the decisions on consumption and savings indirectly via income streams and the rate of capital return, it is difficult to observe the direct impact of taxation on households' decisions. Therefore, consumption tax is imposed in order to study how households adjust their consumption/savings plans in response to taxation[12]. Assuming that the government balances its budget in each period, the public investment ($\mathcal{I}_g$) should be equal to the consumption tax revenue ($\mathcal{T}$):

$$\mathcal{I}_g \equiv \dot{g} + \delta g = \tau c \equiv \mathcal{T} \tag{2.11}$$

where the initial public capital stock (g_0) is an exogenously given and positive constant. The depreciation rate of private capital and of public capital (δ) are also assumed to be the same for the sake of simplicity.[13]

Since we want to allow both public investment and the consumption tax revenue to be endogenous and (possibly) time-varying, we need to specify not only the government balance budget constraint (2.11), but also a fiscal policy rule to avoid a trivial form of global indeterminacy. Following Persson and Tabellini (2002) (see Chapter 11, p.279), we assume a state-contingent and time-invariant fiscal policy rule:

$$\tau = \Psi(k, g) \equiv \tau_c \left(\frac{g}{k}\right)^{\eta} \tag{2.12}$$

where $\tau_c > 0$. The elasticity of the consumption tax with respect to the public-private capital ratio (η) can be a positive or negative constant.[14] The chosen

[11]The definition of public investment is indeed consistent with the paper by Futagami et al. (1993), despite a zero depreciation rate ($\delta = 0$) in their case.

[12]Park and Philipopoulos (2004) mentioned that the consumption/saving decision is driven by the rate of capital return which determines the income stream, and is not directly dependent on capital taxation itself.

[13]In this assumption, we depart from the setting presented by Futagami et al. (1993) in which public investment does not depreciate over time.

[14]A time-invariant policy rule means that its functional form does not change over time.

functional form for the fiscal policy rule implies two important characteristics of the tax rate: i) it will be constant along any BGP since $\frac{g}{k}$ will be constant; and ii) it is predetermined since it is a function of two state variables, k and g. This is indeed consistent with the fact that taxes are typically set in advance, as discussed by Schmitt-Grohe and Uribe (1997). This specific functional form differs from the case studied by Bambi and Venditti (2016) because the tax rate does not depend on a control variable, but rather on a state variable, namely the public-private capital ratio. Accordingly, the different fiscal policy rule may result in a different policy suggestion.

Furthermore, our specific fiscal policy rule provides a new characteristic of consumption tax rate. Using the fact from (2.8) and (2.12), the relationship between the tax rate and the output can be easily established:

$$\tau = \tau_c A^{\frac{\eta}{\alpha-1}} \left(\frac{y}{k}\right)^{\frac{\eta}{1-\alpha}}$$

which implies

$$\frac{\dot{y}}{y} = \frac{\dot{k}}{k} + \frac{1-\alpha}{\eta}\frac{\dot{\tau}}{\tau}$$

where the tax growth rate ($\frac{\dot{\tau}}{\tau}$) is procyclical (countercyclical) with respect to output growth ($\frac{\dot{y}}{y}$) if $\eta > 0$ ($\eta < 0$). Therefore, the elasticity of the consumption tax with respect to the public-private capital ratio is a key parameter that determines the procyclicality or countercyclicality of consumption tax rate.

The definition of an intertemporal equilibrium for this economy is subsequently described as follow:

Definition 1 – Intertemporal Equilibrium: Given the initial condition of private capital ($k_0 > 0$) and of public capital ($g_0 > 0$), an intertemporal equilibrium is any path $(c(t); k(t); \tau(t); g(t))$ that satisfies the system of equations (2.1), (2.5), (2.9), (2.10), (2.11) and (2.12), with respect to the inequality constraints; $k > 0$ and $c > 0$, and the transversality condition (2.6).

As a common feature in the literature on endogenous growth, the dynamics associated with such an equilibrium can be described by combining these equations to obtain a system of two ordinary differential equations (ODEs):

$$\frac{\dot{x}}{x} = (\tau_c x^{\eta-1} + 1 + \tau_c x^{\eta})y - Ax^{1-\alpha} \tag{2.13}$$

$$\frac{\dot{y}}{y} = \frac{1}{\sigma}\left[\alpha A x^{1-\alpha} - \delta - \rho - \frac{\tau_c \eta x^{\eta}}{1+\tau_c x^{\eta}}\frac{\dot{x}}{x}\right] - \left[Ax^{1-\alpha} - \delta - (1+\tau_c x^{\eta})y\right] \tag{2.14}$$

in the state-like variable $x \equiv \frac{g}{k}$ and the control-like variable $y \equiv \frac{c}{k}$. The interested reader may refer to Appendix A.1 for further details regarding the derivations leading to system (2.13)-(2.14).

2.3 Balanced Growth Paths

In this section, the existence and uniqueness of a balanced growth path is investigated in this economy. Two main results are proved in the followings. Firstly, a unique balanced growth path always exists within reasonable parameter choices, particularly the level of technology and the tax rate. Second, multiple balanced growth paths possibly occur in the economy under the plausible choices of parameters.

A balanced growth path (hereafter BGP) is a particular intertemporal equilibrium in which consumption, public capital and private capital grow exponentially at the same positive rate, γ:

$$c = c_0 e^{\gamma t}, \qquad g = g_0 e^{\gamma t} \qquad \textit{and} \qquad k = k_0 e^{\gamma t}$$

Along a BGP, the public-private capital ratio and the consumption-capital ratio are constant and their value x^*, y^* is a steady state of the system (2.13)-(2.14). In particular, along a BGP, these two equations are rewritten as follows:

$$y^* = \frac{Ax^{*1-\alpha}}{\tau_c x^{*\eta-1} + 1 + \tau_c x^{*\eta}} \tag{2.15}$$

$$x^* = \left(\frac{\sigma\gamma + \delta + \rho}{\alpha A}\right)^{\frac{1}{1-\alpha}} \quad \textit{with} \tag{2.16}$$

$$\gamma = Ax^{*1-\alpha} - \delta - (1 + \tau_c x^{*\eta})y^*. \tag{2.17}$$

The existence and uniqueness of a BGP can be explored by examining at the roots of the following equation when $\gamma \in (0, +\infty)$:

$$\tilde{\mathcal{T}}(\gamma) \equiv \underbrace{\frac{A\tau_c}{\tau_c x^{*\alpha-1} + x^{*\alpha-\eta} + \tau_c x^{*\alpha}}}_{\textit{detrended tax revenue}} = \underbrace{\gamma + \delta}_{\textit{detrended public investment}} \equiv \tilde{\mathcal{I}}_g(\gamma) \tag{2.18}$$

This equation can be obtained by solving (2.17) for y^* and substituting it into (2.15). It is worth mentioning that x^* is a one-to-one function of γ taken from (2.17). Alternatively, (2.18) can be obtained by combining equation (2.15) with the government budget constraint (2.11), and evaluating it along BGP. Without loss of generality, we can assume $g_0 = 1$. The left hand side of (2.18) is then known as the detrended tax revenues, $\tilde{\mathcal{T}} \equiv \mathcal{T}e^{-\gamma t}$, while the right hand side is called the detrended public investment, $\tilde{\mathcal{I}_g} \equiv \mathcal{I}_g e^{-\gamma t}$. Consequently, on the BGP, equation (2.18) is employed for our analytical illustration as it represents the government's balanced budget rule.

All the preliminaries are now ready to prove the conditions under which a unique balanced growth path would exist in this economy.

Proposition 1: A unique balanced growth path exists if

$$A > \underline{A} \qquad and \qquad \tau_c > \underline{\tau}_c \tag{2.19}$$

where $\Gamma \equiv \left(\frac{\delta+\rho}{\alpha A}\right)^{\frac{1}{1-\alpha}}$, $\underline{A} \equiv \frac{\delta^{1-\alpha}(\delta+\rho)}{[(1-\alpha)\delta+\rho]^{1-\alpha}\alpha^{\alpha}} > 0$ and $\underline{\tau}_c \equiv \frac{\delta\Gamma^{\alpha-\eta}}{A-(\Gamma^{-1}+1)\Gamma^{\alpha}\delta} > 0$.

Proof: A unique BGP exists as long as $\tilde{\mathcal{T}}(\gamma)$ intersects *only* once on the straight line $\gamma+\delta$. If $\tilde{\mathcal{T}}(0) \geq \delta$ then there is always *at least* one intersection since $\lim_{\gamma\to\infty} \tilde{\mathcal{T}}(\gamma) = \lim_{x\to\infty} \tilde{\mathcal{T}}(\gamma) = 0^+$ and $\tilde{\mathcal{T}}(\gamma)$ is continuous and differentiable in its domain. The rigorous proof is shown in Appendix A.2.1:

$$\tilde{\mathcal{T}}(0) \geq \delta \qquad \Leftrightarrow \qquad A > \underline{A} \qquad and \qquad \tau_c \geq \underline{\tau}_c \tag{2.20}$$

Finally, there is only one intersection when $\tilde{\mathcal{T}}(0) > \delta$ because the function $\tilde{\mathcal{T}}(\gamma)$ has at most a unique critical point when $\hat{\gamma} > 0$ as shown in Appendix A.2.2. Consequently, equation (2.19) implies the existence of a unique BGP in this economy. Q.E.D

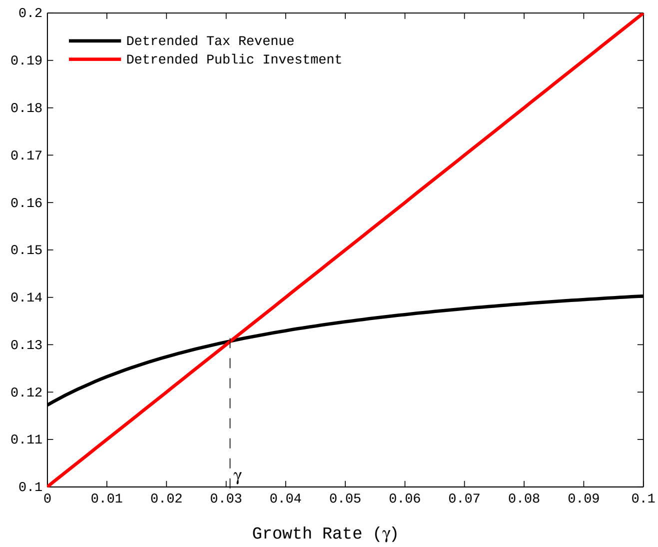


Figure 2.2: Existence and Uniqueness of the BGP

A discussion of these conditions is in order. To have a positive growth rate of the economy, the level of technology (A) and the tax rate (τ_c) should be sufficiently large. In particular, the condition on A is similar to the one required in the AK model. Besides, the condition on τ_c tells us that economic growth can be sustained only if the government provides a sufficient amount of public investment in the public good. These conditions are indeed fairly similar to those found in Bambi and Venditti (2016).

Note that, although the conditions for the parameters identified in this Proposition 1 are sufficient but not necessary for the existence and uniqueness of a BGP, the set of parameters that has *a measurement of zero* is excluded in our model. In fact, it is the set of parameters for which the function $\tilde{\mathcal{T}}(\gamma)$ is tangential to the straight line $\gamma + \delta$. Accordingly, for any positive value of γ both x^* and y^* are positive and all the inequality constraints are respected. The transversality condition along the BGP is always satisfied as long as $(1-\sigma)\gamma - \rho < 0$, since we have assumed $\sigma > 1$. Indeed, $\sigma > 1$ ensures that the attainable utility is bounded and the economy can start from a positive value of the initial consumption ($c_0 > 0$)[15].

A numerical example is now proposed to show that the parameter values to create a unique BGP are plausible. Considering the standard value of parameters in on a yearly basis, an intertemporal discount rate (ρ) is equal to 0.0101 and the elasticity of output with respect to private capital (α) is 0.33[16]. Suppose that a depreciation rate (δ) is 0.1, a unique BGP exists in this economy when $(A, \tau_c) = (0.94, 0.2)$. In fact, the two conditions in (2.19) are both respected given that $(\underline{A}, \underline{\tau}_c) = (0.19, 0.1477)$. Assuming $\sigma = 3$, the resulting growth rate is 3.28%, i.e. $\gamma = 0.0328$, and the public-private capital ratio is $x^* = 0.5527$ while the tax rate is 17.24%, i.e. $\tau^* = \tau_c x^{*\eta} = 0.1724$. The numerical example is clearly illustrated in Figure 2.2. The growth rate of the economy has been found at the intersection of the detrended tax revenue curve and the detrended public investment line.

The next step is to find sufficient conditions, if any, that could lead to global indeterminacy.

[15]For the case of $\sigma < 1$, there are two additional conditions in the Proposition 1 for the bounded utility and a positive value of the initial consumption. See Appendix A.2.3 for the further explanation.

[16]With regard to our production function (2.7), public capital enhances the productivity of labour as it is a labour-augmenting process. Thus, the capital share of income is approximately 1/3 while the labour share of income is around 2/3.

Proposition 2 – Global Indeterminacy: Two balanced growth paths exist if the following parametrical conditions hold:

$$A > \underline{A}, \qquad \underline{\tau}_c - \epsilon < \tau_c \leq \underline{\tau}_c, \qquad \eta > \underline{\eta} \qquad \textit{and} \;\; \sigma > \underline{\sigma} \tag{2.21}$$

with $\epsilon > 0$ sufficiently small real number,

$$\underline{\eta} \equiv \frac{\rho\alpha A}{A - (\Gamma^{-1}+1)\Gamma^{\alpha}\delta} > 0 \qquad \textit{and} \qquad \underline{\sigma} \equiv \frac{\alpha(1-\alpha)\left[(\Gamma^{-1}+1)\tau_c + \Gamma^{-\eta}\right]^2 \Gamma}{\tau_c\{\tau_c[(1-\alpha)\Gamma^{-1}-\alpha] + (\eta-\alpha)\Gamma^{-\eta}\}}.$$

Proof: Given the properties of the function $\tilde{\mathcal{T}}(\gamma)$ found in Proposition 1, two BGPs exist as long as the following two conditions hold:

a) $\delta - \varepsilon \leq \tilde{\mathcal{T}}(0) < \delta$, for any $\varepsilon > 0$ sufficiently small real number;

b) $\left.\frac{d\tilde{\mathcal{T}}(\gamma)}{d\gamma}\right|_{\gamma=0} > 1$

In fact, condition *a)* means that the curve $\tilde{\mathcal{T}}(\gamma)$ is slightly below the straight line $\gamma + \delta$ at $\gamma = 0$, but it is steeper than it is in condition *b)*. Therefore, the curve must intersect the straight line twice since $\tilde{\mathcal{T}}(\gamma)$ is continuous, has a unique critical point and $\lim_{\gamma\to\infty} \tilde{\mathcal{T}}(\gamma) = 0^+$. The proof for the subset of parameters that makes these two conditions hold can be found in the Appendices A.3 and A.4. Q.E.D

In the case of $\tau_c = \underline{\tau}_c$, the BGP with lowest growth rate, the so-called BGP_ℓ, is characterised by a zero growth rate ($\gamma_\ell = 0$) while the other, called BGP_h, is characterised by a strictly positive rate ($\gamma_h > 0$). For the condition; $\eta > \underline{\eta}$ and $\sigma > \underline{\sigma}$, the Proposition 2 suggests that global indeterminacy possibly arises in an economy in which the consumption tax rate is sufficiently procyclical and the inverse elasticity of intertemporal substitution is sufficiently high. The latter condition also implies that the income effect should be sufficiently large to allow a multiplicity of BGPs to exist. In fact, the condition $\sigma > \underline{\sigma}$ can be written as $\sigma > \max\{\underline{\sigma}, 1\}$ to satisfy the transversality condition[17].

Remark 1: Proposition 2 uses a continuity argument to prove that two BGPs may exist for an open set of parameters. The set of parameters found in Proposition 2 is clearly not the largest set that allows global indeterminacy to emerge. In particular, the lower bound for $\underline{\tau}_c$, namely $\underline{\tau}_c - \epsilon$, can be computationally enlarged.

[17]The case of $\sigma \leq 1$ does not alter the sufficient conditions of Proposition 2.

To illustrate the last point in more details, the numerical example is proposed to show that global indeterminacy may arise for plausible values of the parameters.

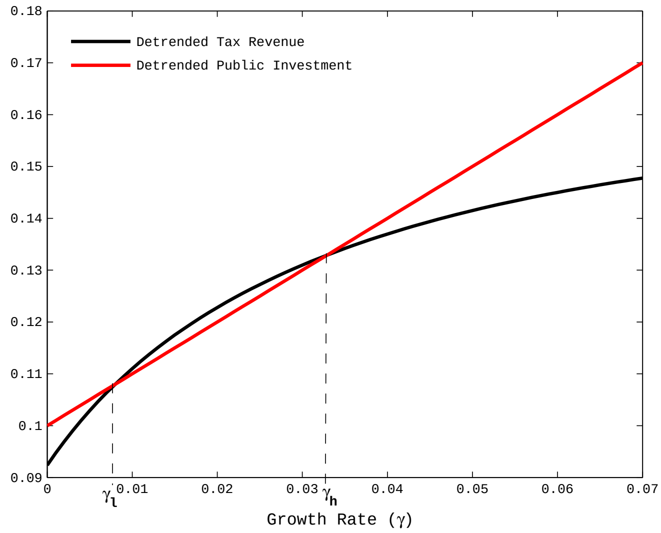


Figure 2.3: Existence of two BGPs when conditions (2.21) hold.

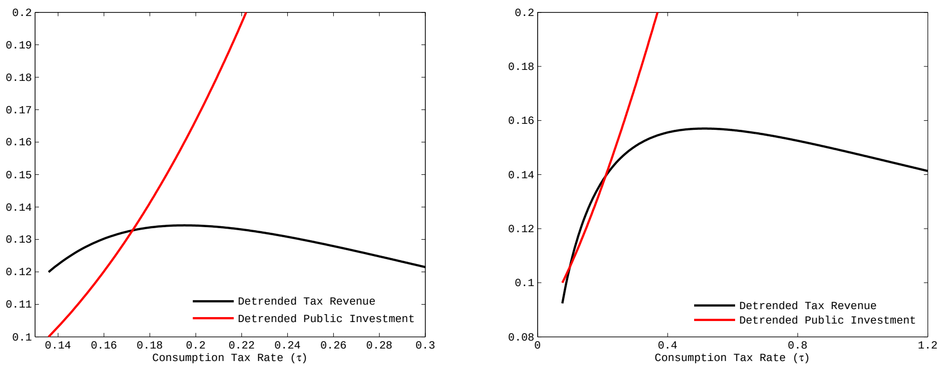
As in the previous numerical exercise, we assume a depreciation rate $\delta = 0.1$, an intertemporal discount factor $\rho = 0.0101$ and the elasticity of output with respect to private capital $\alpha = 0.33$. Global indeterminacy is then revealed for $(A, \tau_c, \eta, \sigma) = (0.9, 0.2, 0.5, 7)$ when all the conditions in (2.21) are satisfied. In particular, we have obtained the constrained values which are $(\underline{A}, \underline{\tau}_c, \underline{\eta}, \underline{\sigma}) = (0.20, 0.22, 0.004, 1.72)$. As shown in Figure 2.3, the economy has two BGPs due to two points of intersection. The lowest BGP is characterised by the growth rate $\gamma_\ell = 0.008$, implying the public-private capital ratio $x_\ell^* = 0.4201$ and the tax rate $\tau_\ell^* = \tau_c x_\ell^{*\eta} = 0.13$. The highest BGP is characterised by the growth rate $\gamma_h = 0.033$, implying the public-private capital ratio $x_h^* = 1.22$ and a tax rate $\tau_h^* = \tau_c x_h^{*\eta} = 0.22$. Therefore, the reasonable growth rates of 0.8% and 3.3% are associated with admissible values of the consumption tax rates of 13% and 22%, respectively.

Remark 2 – Futagami et al. (1993) case: Consider the case of acyclical taxation ($\eta = 0$) which corresponds to the economy described by Futagami et al. (1993), public investment is financed by levying a constant consumption tax, $\tau = \tau_c$, instead of an income tax. By employing the same parameters' values as in the previous exercise, we found a unique BGP in Futagami et al.'s (1993) economy. The growth rate of their economy was in between γ_ℓ and γ_h, precisely equal to 2.67%, i.e. $\gamma = 0.0267$.

We now conclude this section with some considerations regarding the existence of a Laffer curve in this economy. The existence of a Laffer curve can be observed and investigated from equation (2.18) after rewriting it as a function of τ. This can be done easily by using equations (2.15)-(2.17) to write x, y and γ as functions of τ. Equation (2.18) can then be rewritten as:

$$\tilde{\mathcal{I}}_g(\tau) \equiv \gamma(\tau) + \delta = \tau \cdot y(\tau) \equiv \tilde{\mathcal{T}}(\tau).$$

As can be seen in Figure 2.4, the detrended tax revenue always has an inverted U-shape. This means that the Laffer curve exists, both under the assumption of a unique BGP and under the assumption of two BGPs. Remarkably, Figure 2.4 has been obtained using the numerical choices for the parameters suggested previously, but allowing the tax rate (and therefore the growth rate of the economy) to change. Considering the result of Proposition 1, Figure 2.4a shows that a unique BGP emerges when the detrended public investment, in other words the red curve, has only one positive intersection with the detrended tax revenue, namely the black curve. With regard to the numerical exercise proposed following Proposition 1, the consumption tax rate at the intersection point in Figure 2.4 is slightly higher than 17%, which implies a 3.28% growth rate of the economy.



(a) A Unique BGP

(b) Multiple BGPs

Figure 2.4: Existence of a Laffer Curve

On the right side of Figure 2.4b, two BGPs emerge when the detrended public investment, the red curve, has two positive intersections with the detrended tax revenue, the black curve. This indicates the case of the global indeterminacy that is a result of Proposition 2. The numerical example demonstrates that, at these intersections, the consumption tax rates are 13% and 22%, implying a growth rate of the economy equal to 0.8% and 3.3% respectively.

Considering Figure 2.2 2.3 and 2.4, and taking into account the proof of Proposition 2, it can be observed that the consumption tax rate depends on the parameter τ_c, the elasticity of the consumption tax with respect to the public-private capital ratio (η), and the inverse of the elasticity of intertemporal substitution in consumption (σ). These parameters play a fundamental role to generate global indeterminacy. Particularly, global indeterminacy emerges when the detrended tax revenue is initially lower than the detrended public investment at a low lever of tax rate, while η and σ are sufficiently high. This is because the variation in the detrended tax revenue within a small range of the tax rate, namely from τ_ℓ to τ_h[18], should be sufficiently larger than the detrended public investment to allow global indeterminacy to exist. Under these circumstances, the detrended tax revenue curve intersects the detrended public investment curve twice; once when the tax rate is very low and once when tax rate is very high (see Figure 2.4b), with regard to the existence of a Laffer curve.

2.4 Transitional Dynamics

In this section, the transitional dynamics around the steady state(s) are analysed. Linearising the system of ODEs (2.13) and (2.14) around a generic steady state (x^*, y^*)[19] gives us:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \approx \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \tag{2.22}$$

where

$$\begin{aligned}
a &= \left[\tau_c(\eta-1)x^{*\eta-1} + \tau_c\eta x^{*\eta}\right] y^* - (1-\alpha)Ax^{*1-\alpha} && (2.23)\\
b &= \tau_c x^{*\eta} + x^* + \tau_c x^{*\eta+1} > 0 && (2.24)\\
c &= y^*\left[-\frac{1}{\sigma}\left((1-\alpha)(\sigma-\alpha)Ax^{*-\alpha} + \frac{\tau_c\eta}{x^{*1-\eta}+\tau_c x^*}\cdot a\right) + \tau_c\eta x^{*\eta-1}y^*\right] && (2.25)\\
d &= y^*\left(1+\tau_c x^{*\eta} - \frac{1}{\sigma}\cdot\frac{\tau_c\eta}{x^{*1-\eta}+\tau_c x^*}\cdot b\right) && (2.26)
\end{aligned}$$

[18] τ_ℓ and τ_h are corresponding to γ_ℓ and γ_h respectively.

[19] Both equations have a form $\dot{z} = f(z,w)z$ whose first-order Taylor approximation around the steady state (z^*, w^*) is $\dot{z} \approx z^*\left(\frac{\partial f}{\partial z}(z^*,w^*)\cdot\tilde{z} + \frac{\partial f}{\partial w}(z^*,w^*)\cdot\tilde{w}\right)$. Note that, the tilde sign indicates the deviation from the steady state.

In general, the stability of a steady state (x^*, y^*) can be revealed by observing the sign of the determinant and trace of the Jacobian matrix, $J \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Long and tedious computations, summarised in the Appendix A.5, lead to the following the determinant and the trace of the Jacobian matrix:

$$\det(J) \equiv y^{*2} x^{*2\eta} \left\{ \tau_c x^{*-1} \left[(1-\alpha)\tau_c x^{*-1} - \tau_c \alpha + (\eta - \alpha) x^{*-\eta} \right] - \frac{(1-\alpha)\alpha}{\sigma} \left(\tau_c x^{*-1} + x^{*-\eta} + \tau_c \right)^2 \right\}$$

$$\mathrm{tr}(J) \equiv y^* \left\{ x^{*\eta} \tau_c \left[(\eta + \alpha - 2) x^{*-1} + \eta + \alpha \right] + \alpha - \frac{1}{\sigma} \cdot \frac{\tau_c \eta \left(\tau_c x^{*\eta - 1} + 1 + \tau_c x^{*\eta} \right)}{x^{*-\eta} + \tau_c} \right\}$$

The focus of this section is on studying the local stability properties of the BGP with the lowest growth rate in the case of global indeterminacy. In other words, the trade-off between growth and volatility is investigated in the presence of endogenous consumption tax. If the consumption tax induces the aggregate instability (i.e. one of the steady state is locally indeterminate) around the lowest BGP (BGP_ℓ), a poverty trap characterised by low growth and high volatility may exist in our model. On the other hand, one may argue that the aggregate instability is the price to pay for a high growth rate if the endogenous fluctuation emerges around the highest BGP (BGP_h).

To avoid the confusion of terminology, the steady state that corresponds to BGP_ℓ is denoted by (x_ℓ^*, y_ℓ^*). Using a continuity argument, we simplify our analysis by studying the aggregate instability around BGP_ℓ that is sufficiently close to zero ($\gamma_\ell = 0$) where the steady state is (x_ℓ^*, y_ℓ^*)[20]. We begin presenting an intermediary result that is crucial for explaining the transitional dynamics around (x_ℓ^*, y_ℓ^*). The next Lemma finds some sufficient conditions for the parameters in order for the Jacobian matrix evaluated around (x_ℓ^*, y_ℓ^*) to have a positive determinant and a negative trace.

Lemma 1: Considering the case of (x_ℓ^*, y_ℓ^*) with $\gamma_\ell = 0$, the following results then hold:

i) if $A > \bar{A}$, $\eta > \alpha$ and $\sigma > \underline{\sigma}$ then $\det(J) > 0$;

ii) if $-\alpha < \eta < 2 - \alpha$, $A > \hat{A}$ and $\tau_c > \hat{\tau}_c$ then $\mathrm{tr}(J) < 0$;

where $\hat{A} \equiv \frac{\delta+\rho}{\alpha} \left(\frac{\eta+\alpha}{2-\eta-\alpha} \right)^{1-\alpha}$ and $\hat{\tau}_c \equiv \frac{\alpha}{\Gamma^\eta [\Gamma^{-1}(2-\eta-\alpha) - \eta - \alpha]}$.

Proof: See Appendix A.6.

[20]Based on a continuity argument, such results still hold for any growth rate that is sufficiently close to zero.

Next, the two conditions found in this Lemma are combined with the conditions on parameters found in Proposition 2 in order to explore the set of parameters that leads to global and local indeterminacy. The next proposition shows that the intersection of these different sets of parameters is non-empty; therefore, we may have global indeterminacy when the lowest steady state is locally indeterminate.

Proposition 3 – Local Indeterminacy: The steady state (x_ℓ^*, y_ℓ^*) with $\gamma_\ell = 0$ is locally indeterminate if

$$A > \hat{A}, \qquad 0 < \rho < \epsilon, \qquad \alpha < \eta < \eta^\circ, \qquad \tau_c = \underline{\tau}_c, \qquad \textit{and} \;\; \sigma > \underline{\sigma} \qquad (2.27)$$

with $\epsilon > 0$ sufficiently small real number and $\eta^\circ \equiv \frac{\delta - \rho}{\delta(1+\Gamma)}$.

Proof: The proof consists of the open set of parameters that satisfies the conditions of global indeterminacy and of local indeterminacy, $\det(J) \geq 0$ and $\mathrm{tr}(J) < 0$, where the Jacobian Matrix is evaluated at the lowest BGP. The completed proof can be found in Appendix A.7. Q.E.D

For the existence of global indeterminacy and local indeterminacy around the lowest BGP, the elasticity of consumption tax with respect to public-private capital ratio as known as a degree of cyclicality (η) is the important parameter that should be crucially discussed. In the global indeterminacy case, there is the lower bound ($\underline{\eta}$) but no upper bound value for η. This means that two BGPs exists in the economy when the consumption tax is sufficiently procyclical ($\eta > \underline{\eta}$). However, it is not necessarily true that such a procyclical consumption tax can generate the aggregate instability around the lowest BGP. In fact, both global and local indeterminacy can emerge only in the case of a mild procyclical taxation ($\alpha < \eta < \eta^\circ$)[21].

Remark 3: Based on continuity reason, the results stated in Proposition 3 hold for any τ_c lower than, but still sufficiently close to $\underline{\tau}_c$. This is indeed shown computationally in the following numerical exercise.

The numerical exercises is now examined to show that the lowest BGP can be indeterminate for reasonable choices of the parameters. Suppose that the parameters are set exactly as in the numerical exercise proposed to show the possibility of global indeterminacy. The only difference is that now $\tau_c = 0.208$. Then, the growth rate on the lowest BGP is 0.43% and the corresponding steady

[21] The condition $\eta < \eta^\circ$ can be rewritten as $\eta(1+\Gamma) = 1 - \frac{\rho}{\delta}$. Since $\frac{\rho}{\delta}$ is small, we can observe that $\eta < \frac{1}{1+\Gamma}$. Thus, in our case, local indeterminacy arises when the degree of procyclicality is not too strong, i.e $\alpha < \eta < 1$.

state is locally indeterminate. Specifically, there are two negative eigenvalues because $\det(J) = 0.0055$ and $\mathrm{tr}(J) = -0.0027$. Furthermore, the conditions in Proposition 3 are also respected, since $(\hat{A}, \eta^{\circ}, \underline{\sigma}) = (0.265, 0.678, 3.55)$.

Comparing the numerical value of parameters between Proposition 2 and Proposition 3, a greater value for the level of technology (A) and for the inverse elasticity of intertemporal substitution (σ) are required to allow this economy to experience the aggregate instability at the lowest BGP. Once again, it implies that the income effect should be sufficiently large in this economy[22]. In contrast to the case of global indeterminacy, local indeterminacy will arise if and only if the consumption tax is mildly procyclical ($0.330 < \eta < 0.678$). Further details regarding the degree of cyclicality for both cases will be substantially discussed in the next section.

2.5 Procyclical versus Countercyclical Taxation

In the introduction, we observed that the existing literature on the time varying endogenous taxation has often argued in favour of procyclical taxation or procyclical government spending. It was indeed shown in the different settings that procyclical taxes should be preferred to countercyclical taxes because they guarantee the local determinacy of the steady state. Examples of this result include Nourry et al.'s (2013, p. 1989 bullet point v) work where it could be observed that consumption tax has to be countercyclical with respect to output growth to have a locally indeterminate steady state. Schmitt-Grohe and Uribe (1997, p. 977) explained that "the rational expectations equilibrium is more likely to be indeterminate...the less procyclical government expenditure." More recently, Bambi and Venditti (2016) confirmed that procyclical taxation should be implemented to stabilise the economy in which productive government spending and endogenous time-varying consumption taxes play a crucial role in determining balanced growth paths.

In our framework, this issue has been raised and re-addressed in order to create the awareness of procyclical taxation policy. According to Propositions 2 and 3, as well as the numerical exercises presented throughout the paper, there is evidence that multiple BGPs and aggregate instability around the lowest BGP may arise when consumption tax is procyclical. In particular, one of the sufficient conditions for having two BGPs is that the consumption tax growth

[22]For the case of $0 < \sigma < 1$, the numerical example shows that the aggregate instability does not exist since the condition $\sigma > \underline{\sigma}$ is violated.

rate has to be sufficiently procyclical, as the elasticity of consumption tax should be greater than its lower bound ($\eta > \underline{\eta} \equiv 0.004$). While it is true that this is not a necessary but only a sufficient condition, our numerical exercises based on the conditions (2.21) show that multiple BGPs emerge as a result of procyclical taxation, specifically $\eta = 0.5$. However, global indeterminacy is not always generated by procyclical taxation alone. By reducing from 0.5 to 0.4 but keeping other parameters unchanged, we found a unique BGP in this economy. This is because one of the conditions (2.21) is violated, namely $\tau_c > \underline{\tau}_c = -0.97$.

With regard to the local stability, the numerical example shows that both BGPs are locally determinate when the parametrical values from the global indeterminacy case are chosen. As mentioned in the previous section, this implies that the conditions for global indeterminacy may not necessarily lead to the indeterminate equilibria since the sufficient conditions (2.27) of local indeterminacy for the lowest BGP might be violated. On the other hand, a slight increase in τ_c from 0.2 to 0.208 changes the stability properties of the lowest BGP, and local indeterminacy arises. Therefore, the aggregate instability around the lowest BGP possibly occurs once extrinsic uncertainty is introduced into the model and all conditions of Proposition 3 are satisfied.

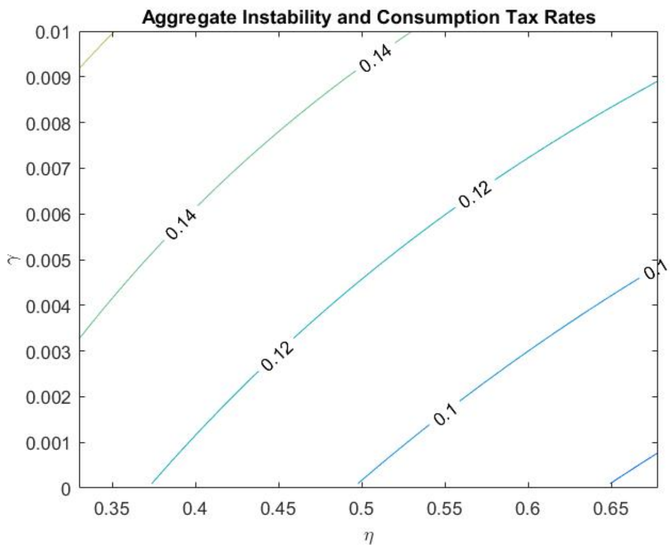


Figure 2.5: Aggregate Instability and Consumption Tax Rates

Although the role of procyclical consumption tax is required to generate local indeterminacy around the lowest BGP, our analytical proof recommends that the degree of procyclicality should not be too strong. In fact, the numerical example suggests that the value range of η should be between 0.330 and 0.678

based on the conditions of Proposition 3 and the feasible set of parametrical values. Considering this value range of η together with the low growth rate ($\gamma < 1\%$), Figure 2.5 demonstrates the value of consumption tax rates that can lead to aggregate instability. These consumption tax rates (τ) are characterised by contour curves with the corresponding points (η, γ), whereby both global and local indeterminacy conditions are satisfied. More precisely, if we choose $\eta = 0.5$ to have τ at around 10%-14% in the low-growth economy and assume the parameter choices of the public-private capital ratio are sufficiently close to x_ℓ^*, this economy will converge towards the lowest BGP with an infinite number of trajectories. Hence, procyclical taxation may lead the economy into a poverty trap characterised by the volatility of aggregate variables and the low growth rate. A fiscal-led growth policy is therefore not recommended for a low-growth country in which consumption tax is mildly procyclical.

2.6 Conclusion

In this paper, public investment and endogenous consumption tax were proposed as sources of endogenous growth in the economy. According to the state-contingent and time-invariant fiscal policy rule, we can observe that the combination of procyclical consumption tax and public investment has a potential to generate the multiplicity of balanced growth paths. This finding is also numerically supported by the existence of a Laffer curve. Two balanced growth paths, namely the lowest BGP and the highest BGP, were derived from the two intersections of the detrended public investment line to the detrended tax revenue curve. When the inverse intertemporal elasticity of substitution in consumption is sufficiently greater than its threshold and the consumption tax is mildly procyclical, there is a possibility for the aggregate instability to emerge around the lowest BGP. Once the purely extrinsic uncertainty is introduced into the model, our model predicts that the economy will experience a poverty trap characterised by a high volatility of aggregate variables and a low growth rate. Therefore, the procyclical taxation policy should be considered carefully before implementation.

Chapter 3

Growth-Maximising Spending on Public Education and the Role of Private Tuition

3.1 Introduction

Since 1990, macroeconomists have debated whether economic growth is generated by internal economic activities instead of by external factors. Human capital has been mentioned as one of the key engine of endogenous growth (Lucas, 1988), and it can be accumulated via the education system. When people have a high level of education, a country will prosper and will have a healthy growth rate. Consequently, the quality of the education system for young generation should be focused, not only for economic growth, but also for reasons pertaining to social welfare.

In the literature on endogenous growth, two separate education systems are currently being analysed using the overlapping-generation model, namely public education[1] and private education[2]. Public education plays a role when parents are unable to provide private education for their children. Since the young has limited access to the credit market, the result is a shortage of human capital stock and a low growth rate (Boldrin and Montes, 2005). Without doubt, public schooling, which is financed by income tax from the worker generation, can be used as a fiscal instrument that solves the inadequate intergenerational income distribution (Fender and Wang, 2003). Furthermore, for a poor country that is experiencing the poverty trap, Boldrin (2005) found that the investment in private education was nearly zero despite a zero tax rate. One possible reason is that parents were unable to appropriate their contribution to human capital in the aggregate economy; thus, they might enjoy being a free rider. Therefore, the provision of public education can solve a free rider problem and can help a poor country start to its growth process.

In contrast to public education, private education allows the altruistic parents to choose the optimal level of education for their children (Glomm, 1997). According to the empirical work of Luis Bernal (2005), Spanish parents in Zaragoza believed that the quality of private education was higher than the quality of free-public schooling since the massive enrolment in public schools might result in a lower quality of education. Moreover, the performance of students in private schools is statistically greater than it is in public schools, particularly in Belgium and Brazil (Vandenberghe and Robin, 2004). Accordingly, altruistic parents may choose to pay for private tuition rather

[1]Public education is covered by the government expenditure on education. No tuition fee is paid by parents.

[2]Private education refers to any kinds of education that is paid by parents.

than to send their children to public schools.

Altruistic motivation can be divided into two categories: pure altruism and impure altruism. Pure altruism is the type described by Barro (1974) in which parents take the utility of their children into account. Nevertheless, pure altruism cannot explain the direct utility that parents obtain from their giving. Such a direct utility is considered to be an impure altruism, which is represented by different labels such as warm-glow altruism (Andreoni, 1990), paternalistic altruism (Michel and Pestieau, 2004) and joy of children receiving income (Grossmann and Poutvaara, 2009). In addition, the direct utility obtained from the act of giving can also create the externalities among intergenerational transfers that have an impact on economic growth in the long run.

Although both public and private education offer different benefits to the economy, most of the previous literature has treated them as perfect substitutes. In reality, such a substitutability may not be perfect. For example, Benabou (1996) emphasised the complementarity between public and private education since both are inputs for the human capital accumulation process. Blankenau and Simpson (2004) also proposed that the human capital of the young generation is a combination of general skills gained from public schooling and specific skills provided privately by universities and by firms that offer on-the-job training. Similarly, Glomm and Kaganovich (2003) claimed that an increase in public education expenditure increased the private input in education when there was a public pension scheme.

The evidence of complementarity between public schooling and private education can be observed across countries. For instance, 90 % of British parents said they were likely to send their children to state schools[3] during the day (Department for Education, United Kingdom, 2014). At the end of the day, a quarter of parents send their children to private tutors (Garner, 2013). There are two reasons for this. Firstly, parents aim to provide more knowledge or activities for their children in addition to the subjects taught by state schools. The other reason is that the insufficient knowledge is obtained at state school; thus, children may need additional help to understand the materials. That's why there are 130 private tuition institutions across the UK (Tanner et al., 2009).

[3]The definition of state school includes direct grant nurseries, non-maintained schools, academies and free-schools with alternative provisions.

By considering both public and private education regimes, this paper aims to study the effect of fiscal policy on economic growth and macroeconomic variables when parents who exhibit impure altruism can provide private tuition and bequests across generations. Due to its advantages, the three-period overlapping generations model is employed to analyse the impact of the labour income tax, which is considered as a proxy for the size of the educational provision, on aggregate economy[4]. Our model replicates the stylised facts from the paper by Samuelson (1958) and extends it in a similar way to Grossman and Poutvaara (2009).

There are two main contributions of this paper. Our first contribution is to find the existence of a growth maximising tax rate in the economy that is driven by both public education and private tuition. This finding also fills the gap in the work of Blankenau and Simpson (2004) because they could not provide proof for the existence of a growth maximising tax rate when human capital is accumulated via both types of education systems. Despite the complexity of solutions, the growth-maximising tax rate will prevent the economy from embarking on the over-provision of public education. The second contribution is to create the policy awareness for the government that misconceives of the existence of private tuition. In such an economy, the over-provision of public education may occur because the government overestimates the benefit of public education. Our analytical result illustrates that the value of a growth-maximising tax rate is always greater than one, which indicates the over-provision of public education.

This paper is organised as follows. The existing literature on a growth-maximising tax is reviewed in Section 3.2. Section 3.3 is a basic model of the three-period overlapping generations, while the effect of the labour income tax on key macroeconomic variables is analysed in Section 3.4. In Section 3.5, the steady-state equilibrium is studied. The growth maximising-tax rate and the argument about over-provision of public education are also discussed in this section. Section 3.6 emphasises the importance of public education, and the last section, Section 3.7, presents the conclusion.

[4]Wicken (2008) mentioned many advantages of the overlapping generations model in order to analyse the impact of fiscal policy on an economy, such as there being no Ricardian equivalence, and allowing for heterogeneous agents and intergenerational transfers.

3.2 Literature Reviews

The relationship between economic growth and the optimal size of government expenditure has been studied crucially for the last two decades. When the government levies a distortionary taxation to finance its expenditure on the public good, there are two opposite impacts on the economy. On the negative side, levying a distortionary tax will discourage saving and investment decisions. Consequently, the capital accumulation will decrease and the economy will experience a low growth rate. By contrast, an increase in (productive) government spending can raise the marginal product of capital and enhance the growth rate in the long run. Hence, the overall impact of fiscal policy depends on the compensation between the negative effect of distortionary taxation and the positive effect of government expenditure.

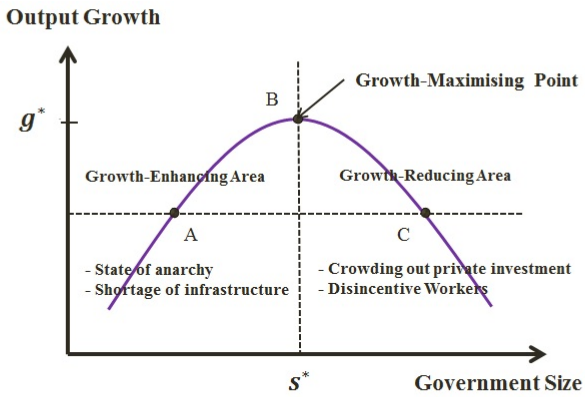


Figure 3.1: The Armey Curve

The relationship between economic growth and the optimal government size can be explained by using the Armey curve. Armey (1995) explained that changing in government size should be associated with changing in output (or in the growth rate of output). Once the economy begins to experience a state of anarchy, output per capita is very low and there is no incentive for people to invest. Consequently, the government is required to provide for the public good, to build the economic infrastructure, and to enforce the rules of law. As a result of the combination of government expenditure and private investment, the output will be larger and will become the largest when the size of the government reaches a certain point. After the government size exceeds this threshold, the output will decrease because government expenditure will crowd out private investment due to the distortionary taxation. For this reason, the Armey curve is a bell curve that illustrates a non-linear relationship between output and the size of the government, as presented in Figure 3.1.

The remaining task for economists is to find the appropriate size of a government to maximise the output growth. The growth-maximising size of a government is indicated at the peak of Armey curve, where the marginal benefit of government expenditure is equal to zero. According to the previous literature, two economic variables are considered as proxies of government size. One is the government expenditure and the other is the distortionary tax rate. Empirically, there is a lot of evidence across countries showing that the government should levy the average tax rate at around 20-30 % in order to maximise the economic growth. For instance, in the U.S economy, Scully (1994) estimated the growth-maximising tax rate and concluded that it should be around 21.5-22.9 %. Rezk (2005) found an average tax rate of 30 % would maximise the growth rate in Argentina, while Facchini and Melki (2011) suggested the same figure of the growth-maximising tax rate in France.

The theoretical literature on endogenous growth with the representative agent framework also supports the existence of Armey curve. Barro (1990) explored the inverted U-shaped relationship between economic growth and the income tax rate, in which the entire flow of income tax revenue is devoted to the provision of the public good. The literature suggests that the growth-maximising tax rate should be equal to the elasticity of output with respect to productive government spending. This value of growth-maximising tax rate is also supported by the work of Futagami et al. (1993), Turnovsky (1997) and Tsoukis and Miller (2003), despite changing from a flow of government spending to a stock variable.

Although the general conclusion for the growth-maximising tax rate is to equate the tax rate with the government elasticity of output, this should be done with awareness of how public capital contributes to the aggregate output. Different forms of public capital formulation result in different values for the growth-maximising tax rates. For instance, Baier and Glomm (2001) introduced a degree of substitutability between public and private capital by using the constant elasticity of substitution (CES) production function. They found that the growth-maximising tax rate depended on the degree of substitutability, the public capital elasticity of output and the elasticity of output with respect to aggregate capital. In the work of Agénor (2007), the production function consisted of three economic inputs, namely private capital, public infrastructure and labourers' education level. Accordingly, the public infrastructure elasticity of output and labourers' education elasticity of output significantly determines the growth-maximising tax rate in this economy. Therefore, it can be concluded that the value of growth-maximising tax rate depends on how public capital contributes to the aggregate economy.

In the context of overlapping-generation framework with public education, the literature on growth-maximising taxation remains scarce and insufficient. Using two-period overlapping generations model, Glomm and Ravikumar (1997) found that the endogenous growth can be generated by human capital via public education. They reached the same conclusion as Barro (1990), in that the growth-maximising tax rate should be equal to the elasticity of output with respect to human capital. However, the misperception of the existence of private education may cause the over-provision of public education. Levying a high labour income tax rate to finance public education may lead the economy into the growth-reducing area of the Armey curve.

For the case in which both public education and private education play roles on human capital in three-period overlapping generations, Blankenau and Simpson (2004) investigated the determinants of growth-maximising tax rate that varied along with tax schemes. In their model, the young generation can borrow money to finance education at the tertiary level and thus acquire the specific skills, while general skills are obtained at public schools at the primary and secondary levels. Due to the complexity of the solutions, the growth-maximising tax rate has not been explored in Blankenau and Simpson's (2004) paper. Instead, they considered the growth-maximising tax rate in special cases, such as the case of unproductive spending on public education, and of unproductive private investment in human capital. Furthermore, the growth-maximising tax rate in the case of unproductive private investment in human capital implies that the over-provision of public education still remains unclear in term of the value range of the growth-maximising tax rate. Thus, there are many unanswered research questions in their work that should be discussed.

In addition, the impure altruistic framework with public education spending and private tuition is another avenue that has not been investigated. There are two important reasons that impure altruistic intergenerational transfers should be considered. Firstly, it is very difficult for children to access to the credit market. Secondly, private institutions to which parents can send their children to obtain additional knowledge do exist. For these reasons, it is reasonable to estimate the growth-maximising tax rate and to re-address the issue of the over-provision of public education in the economy in which impure altruism determines the level of private tuition.

3.3 The Basic Model

The three-period overlapping generations model is employed to study the impact of labour income tax on economic growth and macroeconomic variables in an impurely altruistic economy with both private tuition and public education systems. There are three economic agents in a competitive market economy, namely households, firms and the government. Given wage and rental rates, households will decide on optimal amounts for saving, investing in private tuition and bequests for their offspring based on their after-tax income. The government provides public education and levies labour income tax to finance its expenditure on such education. This discrete-time model is built based on a perfect foresight assumption.

3.3.1 Households

In one generation, individuals are homogeneous in term of preference and the capacity to learn. There is no population growth due to the fact that the population is the same across generations. Thus, the life-time utility of individuals who born in period t-1 will be

$$U_{t-1} = U(c_{2t}) + \beta U(c_{3t+1}) + \beta V(I_{t+1}) \tag{3.1}$$

where β is a discount factor and its value is between zero and one. The quantity of consumption on the part of adult and old generations are represented by c_{2t} and c_{3t+1}, respectively. In the first period of life, the consumption by the young generation (c_{t-1}) does not take into account in utility function because they live with their parents (Samuelson, 1958). Therefore, only the consumption on the part of adult and old generations are incorporated into utility function. In addition, altruistic parents show that they care about the economic situation of their children by leaving them bequests in term of income, and gain the direct utility from this action. This motivation is called 'joy-of-giving', which is similar to the motivation of 'joy-of-children-receiving-income' addressed by Grossman and Poutvaara (2009, p. 651). Similarly, this motive is closely related to the model by Michel and Pestieau (2004), in which bequests are considered as consumption. Therefore, the direct utility derived from the joy-of-giving is prevailed into the life-time utility in term of $V(I_{t+1})$, and its future value is discounted by the same discount factor, β[5].

It is worth mentioning that the joy-of-giving motivation is different from the dynastic altruism asserted by Barro (1974). In fact, it is a kind of impure altruism that parents are motivated by the direct utility from giving an income (for education or as a bequest) to their children. Such an income creates externalities among intergenerational transfers that lead the economy into the second-best world.

Considering the altruistic parents who were born in the first generation at period -1, the initial value of private tuition (e_{-1}) is given, as it is innate. As they become working age in the next period, the initial value of disposable income (I_0), together with the initial value of a bequest (b_0), are also given. Assuming that the preference is additive separable and has a natural logarithmic form of consumption, the life-time utility becomes

$$U_{t-1} = ln(c_{2t}) + \beta ln(c_{3t+1}) + \beta ln(I_{t+1}) \tag{3.2}$$

where t indicates the period of being adult, or of parenthood.

With regard to the constraint on each generation, children will devote all their time to study at state schools during the day and will study with private tutor in the evening (or at weekends). Adults (parents) supply an inelastic labour for work and receive disposable income. They then choose the optimal amount to save (s_t) for retirement and will invest in private tuition (e_t) for their offspring. When the adults become old, they will consume part of the return from their savings as living expense and will leave part as bequests (b_{t+1}) for their descendants,

$$e_{t-1} \geq 0 \tag{3.3}$$

$$c_{2t} + s_t + e_t = I_t \tag{3.4}$$

$$c_{3t+1} + b_{t+1} = (1 + r_{t+1})s_t \tag{3.5}$$

$$c_{2t}, c_{3t+1}, e_t, b_{t+1} \geq 0 \tag{3.6}$$

where r_t is denoted as the real interest rate. The net income of individuals who were born in period t-1 at time t (I_t) is equal to an after-tax income plus

[5] In fact, a degree of altruism can be different from β. However, we continue to use β because parents will obtain the utility derived from bequests as future consumption.

bequests given by the old generation. That is,

$$I_t = (1 - \tau_w)w_t h_t(e_{t-1}, E_{t-1}, h_{t-1}) + b_t \tag{3.7}$$

where $h_t(e_{t-1}, E_{t-1}, h_{t-1})$ is human capital, which is a function of private tuition (e_{t-1}), public education (E_{t-1}) and parental education (h_{t-1}). The bequests left by the old generation, those who born in the period t-2, is indicated by b_t.

3.3.2 Representative Firms

Assuming that the output is a homogeneous commodity and that the production technology is linearly homogeneous, the per capita output (y_t) can be characterised as a function of physical capital (k_t) and human capital (h_t).

$$y_t = Ak_t^{\alpha} h_t^{1-\alpha} \tag{3.8}$$

Representative firms will maximise their profit, and the first-order conditions will be

$$R_t = \alpha A k_t^{\alpha-1} h_t^{1-\alpha} \tag{3.9}$$

$$w_t = (1-\alpha) A k_t^{\alpha} h_t^{-\alpha} \tag{3.10}$$

where R_t and w_t are rental and wage rates respectively. Note that the depreciation rate is assumed to be zero, and thus the rental rate (R_t) is equal to one plus the real interest rate ($1 + r_t$). The elasticity of output with respect to physical capital (α) and the elasticity of output with respect to human capital ($1 - \alpha$) are between zero and one.

3.3.3 Government

With regard to the balanced budget rule, the government levies labour income tax and devotes its expenditure to public education. Public education (E_t) is an endogenous variable determined by labour income tax (τ_w), wage rate (w_t) and human capital (h_t).

$$E_t = \tau_w w_t h_t \tag{3.11}$$

where $0 < \tau_w < 1$. It is worth mentioning that public education can be considered as a pure public good as explained by Samuelson (1954), and it has two important properties: non-excludability and non-rivalry. According to non-excludability, the price mechanism in competitive market is incapable of controlling the access of public education efficiently[6]. Thus, the market has failed and the government is justified in providing public education because it is a social benefit[7].

The relationship between public education and private tuition is a crucial issue in this paper. Based on the work of Glomm and Kaganovich (2003) and of Benabou (1996), the substitutability of public education and private tuition might not be perfect, although they have access to the same educational technology[8]. Consequently, the constant elasticity of substitution (CES) of the human capital function is employed to study how human capital will accumulate over time. Following Glomm and Kaganovich (2003), human capital accumulation is determined by three components: private tuition (e_t), public investment in education (E_t) and parental education[9] (h_t),

$$h_{t+1} = B(ae_t^{\rho} + bE_t^{\rho})^{\frac{z}{\rho}} \cdot h_t^{1-z} \tag{3.12}$$

where $h_0 > 0$ is given, as it is the basic knowledge humans have when they are born. The private tuition elasticity of human capital and the public education elasticity of human capital are denoted by a and b respectively, and their values are between zero and one ($a, b \in (0, 1)$). To allow for a constant growth rate, the human capital technology is assumed to be a constant return to scale. Thus, the value of scale parameter (z) is also between zero and one ($z \in (0, 1)$). In addition, a degree of substitutability between public education and private tuition (ρ) can vary from $-\infty$ to 1.

As in the work of Glomm and Ravikumar (1992) and of Blankenau and Simpson (2004), the human capital accumulation function can be given a Cobb-Douglas form, which is a special case of CES human capital function in which an elasticity of substitution between private tuition and public education (σ) is

[6]According to the Samuelson's rule, the sum of marginal rate of substitution between the public good and individual private good of all households should be equal to the marginal rate of transformation between the public good and individual private good to achieve the Pareto-efficient allocation. This rule contrasts with the rule of efficient for two private goods in a competitive market. See Myles (1995) for further details.

[7]One may argue that the competitive market equilibrium is efficient under Lindahl's (1919) equilibrium. However, there is no incentive for households to reveal their preferences. Thus, no personalised price could be applied.

[8]Glomm (1997) explained that the accessibility of education technology for both public and private education should be the same so that the input, which is essentially financial expenditure, is transformed into output as human capital.

[9]Coleman et al. (1966) found a positive correlation between parental education and students' standardised test.

equal to one. Thus, equation (3.12) can be rewritten as

$$h_{t+1} = Be_t^{\varphi} E_t^{\mu} h_t^{\phi} \tag{3.13}$$

where $\varphi = \frac{za}{a+b}$, $\mu = \frac{zb}{a+b}$, $z = \varphi + \mu$, and $\phi = 1 - \varphi - \mu$. The elasticity of human capital with respect to private tuition (φ), to public education (μ), and to parental education (ϕ) is between zero and one ($\varphi, \mu, \phi \in (0, 1)$). Notably, B can be interpreted as the ability of each individual and it is the same in one generation. This ability is a complementary input to private tuition and to public education[10]; $\frac{\partial^2 h_{t+1}}{\partial e_t \partial B} = e_t^{\varphi-1} E_t^{\mu} h_t^{\phi} > 0$ and $\frac{\partial^2 h_{t+1}}{\partial E_t \partial B} = e_t^{\varphi} E_t^{\mu-1} h_t^{\phi} > 0$.

Substituting government budget constraint (3.11) into (3.13), the function of human capital accumulation depends partly on the labour income tax rate (τ_w) as it refers to the level of public spending on education.

$$h_{t+1} = Be_t^{\varphi} (\tau_w w_t)^{\mu} h_t^{\mu+\phi} \tag{3.14}$$

3.3.4 Equilibrium

Households' decisions regarding savings, education and bequests

The optimal decisions of households for savings (s_t), private tuition (e_t) and bequests (b_{t+1}) can be derived from the indirect utility function. By substituting c_{2t} and c_{3t+1} using adult (3.4) and old (3.5) budget constraints into the utility function (3.2), the indirect utility function will be

$$\begin{aligned} U_{t-1} = ln[(1-\tau_w)w_t h_t + b_t - s_t - e_t] + \beta ln[(1+r_{t+1})s_t - b_{t+1}] \\ + \beta ln[(1-\tau_w)w_{t+1}h_{t+1} + b_{t+1})] \end{aligned} \tag{3.15}$$

where h_{t+1} is given by equation (3.13) or (3.14).

Given the wage rate and the rental rate, the first-order conditions are:

$$\frac{c_{3t+1}}{c_2 t} = \beta(1 + r_{t+1}) \tag{3.16}$$

$$\frac{I_{t+1}}{c_2 t} = \beta(1-\tau_w)w_{t+1} \cdot \frac{\varphi h_{t+1}}{e_t} \tag{3.17}$$

$$c_{3t+1} = I_{t+1} \tag{3.18}$$

Definition 1: Given the initial conditions of physical capital (k_0), human capital (h_0) and bequests (b_1), the equilibrium of the economy is fully described by sequences of resource allocation $\{k_{t+1}, h_{t+1}, b_{t+1}, e_t\}_{t=0}^{\infty}$, sequences of price

[10]See Jacobs and Bovenberg (2010).

$\{w_t, r_t\}_{t=0}^{\infty}$ and sequence of government expenditure on public education $\{E_t\}_{t=0}^{\infty}$ that solve equations (3.9), (3.10), (3.16), (3.17) and (3.18).

Equation (3.16) is a standard Euler equation. Individuals will smooth their consumption paths by making decisions concerning how much they consume today against tomorrow. Thus, the marginal rate of substitution between current consumption and future consumption is equal to a discounted rate of return on savings. With regard to equation (3.17), the marginal rate of substitution between current consumption and the future income of children is equated by the marginal rate of return on an investment in private tuition for their children. Finally, in equation (3.18), the marginal rate of substitution between the future consumption and future bequests is one. This means that the old generation will sacrifice one unit of their consumption to give one unit of bequest to their offspring.

Since parents can decide whether to invest money in their offspring's education or to save it for retirement period, their decisions will not be optimal unless the rate of return on private tuition is equal to the market rate of return on savings. That is,

$$(1-\tau_w)w_{t+1}\frac{\varphi h_{t+1}}{e_t} = 1 + r_{t+1} \tag{3.19}$$

Equation (3.19) is known as a no-arbitrage condition. This condition makes altruistic parents feel indifferent about savings for retirement versus an investment in private tuition. This no-arbitrage condition can also be exploited to determine the dynamic movement of investment in private tuition, bequests, physical capital accumulation and human capital accumulation.

Investment in private tuition and bequest functions

Investment in private tuition and bequests are two alternative ways for parents to give an income to their offspring. Given the rental rate (3.9), the wage rate (3.10) and the no-arbitrage condition (3.19), investment in private tuition (e_t) can be rewritten as a function of future physical capital (k_{t+1}) and other exogenous parameters (α, φ, τ_w).

$$e_t = \frac{(1-\tau_w)(1-\alpha)\varphi}{\alpha} \cdot k_{t+1} \tag{3.20}$$

According to equation (3.20), parents have to consider a trade-off between the amount of future physical capital that they will consume tomorrow and a joy-of-giving motive in terms of investment in private tuition.

Considering another form of joy-of-giving motive, a bequest function can be constructed from the optimal choice of households concerning bequests (3.18), the old budget constraint (3.5) and the adult's disposable income (3.7). After substituting w_{t+1} and R_{t+1} using firms' first-order conditions (3.9) and (3.10), the future bequest will be determined by future physical capital (k_{t+1}), future human capital (h_{t+1}) and exogenous parameters (α, φ, τ_w).

$$b_{t+1} = \frac{1}{2} \cdot [\alpha - (1-\tau_w)(1-\alpha)]\, Ak_{t+1}^{\alpha} h_{t+1}^{1-\alpha} \tag{3.21}$$

$$b_t = \frac{1}{2} \cdot [\alpha - (1-\tau_w)(1-\alpha)]\, Ak_t^{\alpha} h_t^{1-\alpha} \tag{3.22}$$

Based on equation (3.21) and (3.22), a bequest is a function of capital and human capital in each period. This means that the old generation will consider future income and decide on the amount of bequests to forward to the adult generation.

Physical capital and human capital accumulation

As physical capital and human capital are predetermined variables, it is important to characterise the dynamic movement of these variables. The dynamic movement of physical capital can be found from the Euler equation (3.16). After substituting c_{2t} and c_{3t+1} using households' budget constraints, the Euler equation will be

$$(1+\beta)(1+r_{t+1}) = \beta(1+r_{t+1})[(1-\tau_w)w_t h_t + b_t - e_t] + b_{t+1}$$

Using the firms' first-order conditions (3.9)-(3.10), bequest functions (3.21)-(3.22) and the market clearing condition ($s_t = k_{t+1}$), the physical capital accumulation function is a function of two predetermined variables: k_t and h_t, as well as exogenous parameters.

$$k_{t+1} = \gamma y_t = \gamma A k_t^{\alpha} h_t^{1-\alpha} \tag{3.23}$$

where $\gamma = \frac{[(1-\tau_w)(1-\alpha)+\alpha]\beta\alpha}{\alpha(1+2\beta)+(1-\tau_w)(1-\alpha)(1+2\beta\varphi)} > 0$.

Turning to the human capital accumulation function (3.14), it can be rewritten as a function of future physical capital (k_{t+1}), current human capital (h_t) and other exogenous parameters by substituting for private tuition (e_t) using (3.20).

$$h_{t+1} = B\left[\frac{(1-\tau_w)(1-\alpha)\varphi}{\alpha}\right]^{\varphi} k_{t+1}^{\varphi} (\tau_w w_t)^{\mu} h_t^{\mu+\phi} \tag{3.24}$$

Due to the fact that k_{t+1} is known from equation (3.23) and the real wage (w_t) is given by (3.10), the human capital accumulation equation is also driven by two predetermined variables: k_t and h_t.

$$h_{t+1} = B\left[\frac{(1-\tau_w)(1-\alpha)\varphi}{\alpha}\right]^{\varphi}\gamma^{\varphi}((1-\alpha)\tau_w)^{\mu}(Ak_t^{\alpha})^{\varphi+\mu}h_t^{1-\alpha(\varphi+\mu)} \tag{3.25}$$

Since both future physical capital and future human capital are functions of two predetermined variables, namely current physical capital (k_t) and current human capital (h_t), the dynamic movement of this model is characterised by a pair of first-order simultaneous difference equations, (3.23) and (3.25). This dynamic movement is similar to the model proposed by Boldrin and Montes (2005), which allows children to borrow from adult generation for their education. Intuitively, the way that the children optimally choose their level of education is the same as is the optimal choice of education selected by impurely altruistic parents, except that the children do not need to pay to their altruistic parents back. Thus, both optimal consumption and private tuition choices that contribute to physical capital accumulation (3.23) are the same.

3.4 Macroeconomic Variables and Taxation

The effect of labour income tax on the key macroeconomic variables in equilibrium is studied in this section. Physical capital accumulation, human capital accumulation, investment in private tuition and future bequest are the main variables that have been focused because they can provide a broad picture and economic intuition before analysing the steady-state equilibrium.

3.4.1 Physical Capital and Labour Income Tax

Since labour income tax distorts a decision on savings, it will unambiguously reduce the physical capital accumulation. This statement is algebraically confirmed in this impure altruistic economy with public education and private tuition. By differentiating (3.23) with respect to labour income tax, it is obvious that there is a negative relationship between the accumulation of physical capital and the labour income tax rate, as indicated in equation (3.26).

$$\frac{\partial k_{t+1}}{\partial \tau_w} = \frac{-2\alpha^2\beta^2(1-\alpha)(1-\varphi)}{[\alpha(1+2\beta)+(1-\tau_w)(1-\alpha)(1+2\beta\varphi)]^2}\cdot Ak_t^{\alpha}h_t^{1-\alpha} < 0 \tag{3.26}$$

The economic intuition is that levying a high level of labour income tax will move the resource away from physical capital sector into human capital sector via public education system. Subsequently, physical capital will be accumulated slowly.

3.4.2 Human Capital Accumulation and Labour Income Tax

On the condition that government expenditure is exogenous, levying labour income tax definitely decreases the investment in human capital (Grossmann and Poutvaara, 2009; Jacobs and Bovenberg, 2010). However, in this model, government spending on education is an endogenous variable. While the revenue received from labour income tax that provides public schooling will enhance the human capital of children, the labour income tax can distort private tuition investment. Hence, the effect of labour income tax on human capital accumulation depends on the balance of these two effects.

To clarify the effect of labour income tax, the first-order derivative of h_{t+1} in equation (3.25) with respect to τ_w is as follow:

$$\frac{\partial h_{t+1}}{\partial \tau_w} = h_{t+1} \cdot \left[\frac{\mu}{\tau_w} + \frac{\varphi}{\gamma} \cdot \frac{\partial \gamma}{\partial \tau_w} - \frac{\varphi}{(1-\tau_w)} \right] \tag{3.27}$$

where

$$\frac{\partial \gamma}{\partial \tau_w} = \frac{-2\alpha^2\beta^2(1-\alpha)(1-\varphi)}{[\alpha(1+2\beta)+(1-\tau_w)(1-\alpha)(1+2\beta\varphi)]^2} < 0$$

The sign depends on the value in the square brackets, which can be either positive or negative depending on the parameters' value. Thus, an increase in labour income tax has an ambiguous impact on human capital accumulation, which is in contrast to the result found by Grossman and Poutvaara (2009), as well as the result indicated by Jacobs and Bovenberg (2010).

With regard to the term in the square brackets, the first term reflects the benefits of public education. If the human capital of children relies heavily on state schools (high μ), it is worthwhile for the government to devote its tax revenue for public education. The middle term refers to the indirect effect of how labour income tax distorts physical capital accumulation. According to the analysis in 3.4.1, an increase in labour income tax contributes to a lower level of saving and less physical capital accumulation. Consequently, the partial derivative of γ with respect to τ_w is negative because γ characterises how physical capital evolves over time. Finally, the last term demonstrates the direct effect of private tuition's contribution on human capital accumulation.

When the government increases the rate of labour income tax, parents will reduce investment in private tuition, which affects the human capital of children through the elasticity of human capital with respect to private tuition (φ).

3.4.3 Private Tuition and Labour Income Tax

In the previous literature, an increase in labour income tax discourages parents from investing in private education for two reasons. The first reason is that collecting labour income tax wastes economic resources and distorts investment in private education when government expenditure is an exogenous variable (Grossmann and Poutvaara, 2009). The second reason is related to the relationship between public and private education. If public and private education were perfect substitutes, an increase in labour income tax would surely dampen the investment in private education (Benabou, 1996).

In this paper, a new idea is suggested. Not only is government spending considered to be an endogenous variable that is devoted to public education, but public education and private tuition are not perfect substitutes in human capital accumulation function. To see the relationship between private tuition and labour income tax, the partial derivative of private tuition (3.20) with respect to the labour income tax rate is taken.

$$\begin{aligned}\frac{\partial e_t}{\partial \tau_w} &= -\frac{(1-\alpha)\beta\varphi}{\kappa^2}\left[\alpha(1+2\beta)(2(1-\tau_w)(1-\alpha)+\alpha)\right]\cdot y_t \\ &\quad -\frac{(1-\alpha)\beta\varphi}{\kappa^2}\left[(1-\tau_w)^2(1-\alpha)^2(1+2\beta\varphi)\right]\cdot y_t < 0\end{aligned} \tag{3.28}$$

where

$$\kappa = \alpha(1+2\beta)(1-\tau_w)(1-\alpha)(1+2\beta\varphi)$$

As the partial derivative is negative, it indicates that an increase in labour income tax will discourage parents from investing in private education. The economic intuition is the same way that labour income tax distorts physical capital accumulation (k_{t+1}). When the government increases the labour income tax rate, this will decrease the after-tax income of households. Therefore, the amount of savings and private tuition are reduced. This is commonly known as the trade-off between public education and private tuition.

3.4.4 Bequest and Labour Income Tax

Parents can give bequests to their offspring in term of income that can be used for consumption. Such bequests work similarly to the model suggested by Michel and and Pestieau (2004), which considered bequests as consumption. In general, the source of bequests is the parental income, which depends on both physical and human capital.

According to Section 3.4.1 and 3.4.2, an increase in labour income tax will reduce the physical capital accumulation, while having an ambiguous impact on human capital accumulation. Thus, one could expect that the effect of labour income tax on bequest would also be equivocal. Taking the derivative of bequest function (3.21) with respect to the labour income tax rate into account produces

$$\begin{aligned}\frac{\partial b_{t+1}}{\partial \tau_w} &= \frac{1}{2}(1-\alpha)y_{t+1} \\ &+ \frac{1}{2}(\alpha-(1-\tau_w)(1-\alpha))y_{t+1}\left[\frac{\alpha}{k_{t+1}}\cdot\frac{\partial k_{t+1}}{\partial \tau_w}+\frac{(1-\alpha)}{h_{t+1}}\cdot\frac{\partial h_{t+1}}{\partial \tau_w}\right]\end{aligned} \tag{3.29}$$

where the relationship between future bequest and labour income tax depends on the parameters' values in the square bracket, which can be either positive or negative.

3.5 Steady-state Analysis

The balanced growth path can be explored by a pair of first-order simultaneous difference equations, (3.23) and (3.25). Following the analysis by Boldrin and Montes (2005), the two predetermined variables, physical capital (k_t) and human capital (h_t), are firstly transformed into a physical-human capital ratio ($\frac{k_t}{h_t}$) to define a balanced growth path.

Definition 2: A balanced growth path is a particular solution of physical capital accumulation (3.23) and human capital accumulation (3.25) such that all economic variables have the same constant growth rate in the long run.

Let $x_t = \frac{k_t}{h_t}$ and dividing (3.23) by (3.25), the dynamic movement of physical-human capital ratio (x_t) can be demonstrated as follow:

$$x_{t+1} = \left[\frac{\gamma^{1-\varphi}A^{1-\varphi-\mu}}{B\eta}\right]\cdot x_t^{\alpha\phi} \tag{3.30}$$

where $\eta = \left[\frac{(1-\tau_w)(1-\alpha)\varphi}{\alpha}\right]^{\varphi}\cdot\tau_w^{\mu}(1-\alpha)^{\mu} > 0$

At the steady state, the economic variables remain constant over time ($x_{t+1} = x_t = x^*$). For all positive initial conditions of (k_0, h_0), the steady-state value of physical-human capital ratio (x^*) will define the ray of balanced growth path. That is,

$$x^* = \left[\frac{\gamma^{1-\varphi}A^{1-\varphi-\mu}}{B\eta}\right]^{\frac{1}{1-\alpha\phi}} \tag{3.31}$$

Let us assume that g^* is a net long-run growth rate, the value of g^* can be found from the law of motion of physical capital accumulation (3.23) and a steady-state value of capital ratio (3.31). According to the definition 2, two capital stocks will grow in the same rate at $1 + g$ along the balanced growth path.

$$1 + g^* = \gamma A\left[\frac{\gamma^{1-\varphi}A^{1-\varphi-\mu}}{B\eta}\right]^{\frac{\alpha-1}{1-\alpha\phi}} \tag{3.32}$$

To have a positive net growth rate ($g^* > 0$), the value on the right hand side should be greater than one. Hence, the necessary condition for a positive growth rate will be

$$\gamma > \left[\frac{1}{(B\eta)^{1-\alpha}A^{\mu+\varphi}}\right]^{\frac{1}{\alpha\mu+\varphi}} \tag{3.33}$$

where all parameters are greater than zero.

3.5.1 Growth-Maximising Tax

In the previous section, endogenous growth and its necessary conditions were determined by the labour income tax rate and other exogenous parameters. Obviously, the different labour income tax rates chosen by policymakers contribute to the different levels of growth rate. Choosing a high tax rate may distort the saving and private tuition decisions, while the human capital may rapidly accumulated via an increase in public education expenditure. Thus, the effect of public education on economic growth may be ambiguous in the sense that it can be either in the growth-enhancing area, or in growth-reducing area of the Armey Curve. To avoid the growth-reducing area, it is important to find the growth-maximising tax rate that indicates the appropriate level of public education.

Proposition 1: According to the balanced growth path (3.32), there must be at least one value of the labour income tax rate (τ_w) that locally maximises the long-run growth rate given a specific set of parameters.

Proof: From (3.32), the derivative of gross growth rate $(1+g^*)$ with respect to labour income tax (τ_w) is calculated and is set as equal to zero,

$$\frac{\partial(1+g^*)}{\partial\tau_w} = \frac{(1+g^*)(1-\alpha)}{1-\alpha\phi} \cdot [\Omega + \Lambda] = 0 \tag{3.34}$$

where

$$\begin{aligned}
\Omega &= \frac{\mu - \tau_w(\mu+\varphi)}{\tau_w(1-\tau_w)} \\
\Lambda &= \frac{-2\alpha\beta(\alpha\mu+\varphi)(1-\varphi)}{[(1-\tau_w)(1-\alpha)+\alpha][\alpha(1+2\beta)+(1-\tau_w)(1-\alpha)(1+2\beta\varphi)]}.
\end{aligned}$$

When the necessary condition (3.33) for a positive growth rate holds, the solutions of (3.34) could imply the growth-maximising tax rate of this economy. In fact, equation (3.34) is equal to zero if, and only if, the sum of Ω and Λ is zero. Rearranging the term in $[\Omega + \Lambda]$, the value of labour income tax rate that leads to either the possible highest or lowest growth rate can be derived from roots of a polynomial of degree three, namely a cubic function.

$$f(\tau_w) = a\tau_w^3 + b\tau_w^2 + c\tau_w + d = 0 \tag{3.35}$$

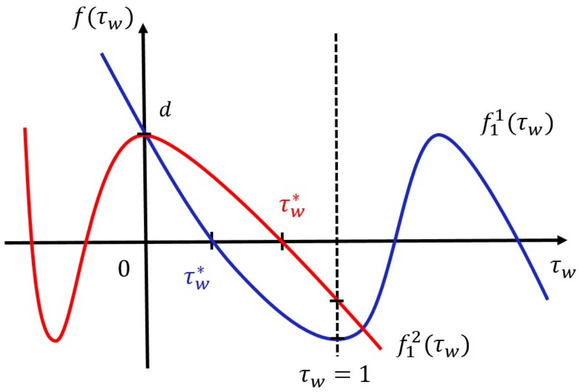
where,

$$\begin{aligned}
a &= -(\mu+\varphi)(1-\alpha)^2(1+2\beta\varphi) \\
b &= (1+2\beta\varphi)(1-\alpha)\cdot[(1-\alpha)(3\mu+2\varphi)+\alpha(\mu+\varphi)] \\
&\qquad + \alpha\cdot[(\mu+\varphi)(1-\alpha)(1+2\beta)+2\beta(\alpha\mu+\varphi)(1-\varphi)] \\
c &= -2\alpha\beta(\alpha\mu+\varphi)(1-\varphi)-(1+2\beta\varphi)(1-\alpha)[3\mu+\varphi-\alpha\mu] \\
&\quad - \alpha(1+2\beta)[2\mu+\varphi-\alpha\mu] \\
d &= \mu\cdot[(1-\alpha)(1+2\beta\varphi)+\alpha(1+2\beta)]
\end{aligned}$$

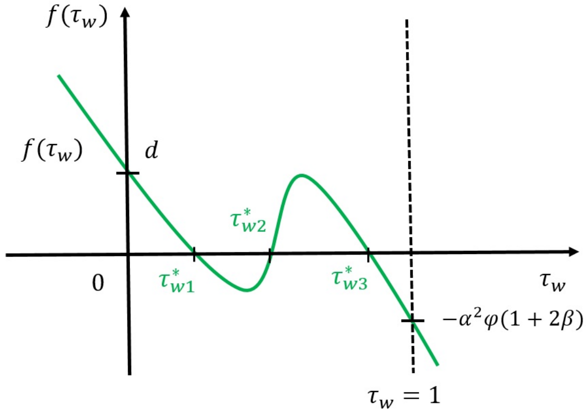
Unlike a quadratic equation that may have no real solution, a cubic function has at least one real root. This means that policymakers can find a value of the labour income tax rate that maximises (or minimise) the economic growth in the long run. Since a feasible range of values for labour income tax is between zero and one, it is interesting to find whether the possible roots of the cubic function lie within that range or not.

Considering the sign of coefficients, the positive value of d indicates the intercept term on the y-axis when τ_w is set to be zero. The negative coefficient a refers to the starting point of the graph that should begin from the second quadrant and descend to the fourth quadrant. In addition, the positive coefficient b determines the curvature of the parabolic element, whilst the negative coefficient c alters the slope of the cubic function. Using the properties of coefficients and evaluating $\tau_w = 1$, the sum of the coefficient values becomes negative. That is,

$$a + b + c + d = -\alpha^2\varphi(1 + 2\beta) < 0 \tag{3.36}$$



(a) Only one root of labour income tax lies within zero and one range



(b) Three roots of labour income tax lie within zero and one range

Figure 3.2: Possible Roots of Labour Income Tax Rate within the Range between Zero and One for the Cubic Function

From equation (3.36), together with a negative value of a and a positive value of d, the value of the cubic function will lie in the fourth quadrant when τ_w is equal to one. This implies that there are only two possibilities of a growth-maximising (minimising) tax rate for this economy, as illustrated in Figure 3.2.

In Figure 3.2a, there is only one real root of labour income tax that lies within the range between zero and one. However, other two roots that lie outside $(0,1)$ may be positive (see f_1^1) or negative (see f_1^2). By contrast, in Figure 3.2b, it is possible that three roots of the labour income tax rate are between zero and one. Accordingly, the economy can have either one growth-maximising (minimising) tax rate, or three growth-maximising (minimising) tax rates regarding a specific set of parameters in one economy.

To clarify whether it is a local maximum or minimum rate, the second order-derivative of gross growth rate $(1+g^*)$ with respect to labour income tax (τ_w) is taken into account[11],

$$\frac{\partial^2(1+g^*)}{\partial \tau_w^2} = \frac{(1-\alpha)(1+g^*)}{1-\alpha\phi} \cdot \left[\frac{\partial \Omega}{\partial \tau_w} + \frac{\partial \Lambda}{\partial \tau_w} + \frac{(\Omega+\Lambda)^2(1-\alpha)}{1-\alpha\phi}\right] \tag{3.37}$$

where

$$\begin{aligned}
\frac{\partial \Omega}{\partial \tau_w} &= \frac{-\tau_w^2(\mu+\varphi)-\mu(1-2\tau_w)}{[\tau_w(1-\tau_w)]^2} \\
\frac{\partial \Lambda}{\partial \tau_w} &= \frac{1}{(\Delta \cdot \Pi)^2} \cdot \{-2\alpha\beta(\alpha\mu+\varphi)(1-\varphi)(1-\alpha)[(1+2\beta\varphi)\Delta+\Pi]\} \\
\Delta &= (1-\tau_w)(1-\alpha)+\alpha \\
\Pi &= \alpha(1+2\beta)+(1-\tau_w)(1-\alpha)(1+2\beta\varphi)
\end{aligned}$$

The sign of second-order derivative depends on the value in the square brackets in equation (3.37). In the square brackets, the third component is eliminated due to the growth-maximisation condition (3.34), $\Omega+\Lambda=0$. Since the second component is always negative, the value of first component will determine the sign of the second-order condition. To have a local maximum, the first component should be negative. Thus, the sufficient condition will be demonstrated in equation (3.38).

$$(\mu+\varphi)\tau_w^2 - 2\mu\tau_w + \mu > 0 \tag{3.38}$$

According to the sufficient condition for the local maximum (3.38), the positive coefficients, $\mu+\varphi$ and μ, indicate that this quadratic equation is a U-shaped curve that crosses the y-axis at μ when $\tau_w = 0$. To determine whether the value of labour income tax rates lying in the a range between zero and one satisfy the sufficient condition for the local maximum, the minimum

[11]See Appendix B.1 for the derivation

point of the quadratic equation is firstly explored by differentiating (3.38) with respect to τ_w and equating it to zero. The minimum value will exist at

$$\tau_w = \frac{\mu}{\mu + \varphi} \tag{3.39}$$

which lies in the range between zero and one. Substituting (3.39) into (3.38), it can be observed that the minimum value of the quadratic equation, $\frac{\varphi\mu}{\mu+\varphi}$ is greater than zero.

Finally, the value of the quadratic equation at $\tau_w = 1$ is evaluated. One could easily find that its value will be φ when substituting $\tau_w = 1$ into (3.38). The possible coordinated points for different values of labour income tax rate are depicted in Figure 3.3. Accordingly, the U-shaped curve implies that the possible values of labour income tax that are between zero and one will satisfy the sufficient condition for the local maximum. With regard to the fact that the value of labour income tax satisfies the necessary condition (3.34) and the sufficient condition (3.38) for the local maximum, there must be at least one positive value of the labour income tax rate that maximises economic growth in the long run. Therefore, Proposition 1 is proved. Q.E.D.

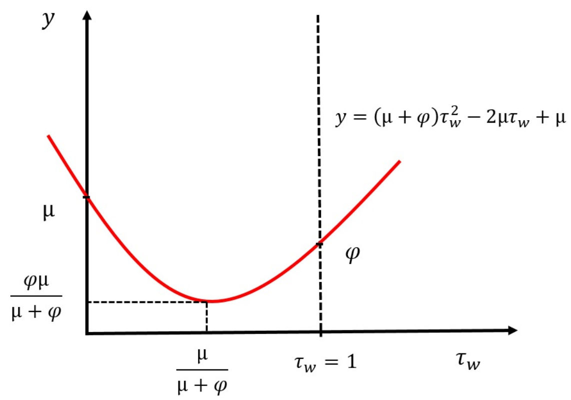


Figure 3.3: Quadratic Equation for the Local Maximum Condition

Despite the complexity of the roots of the cubic function of labour income tax (3.35), the growth-maximising tax rate is a function of exogenous parameters such as the discount factor (β), the human capital elasticity of output ($1-\alpha$), the physical capital elasticity of output (α), the private tuition elasticity of human capital (φ) and the public education elasticity of human capital (μ). Overlooking one of these variables may cause an overestimation or underestimation of the benefits of public education.

3.5.2 Is the Effect of Public Education Overestimated?

The conjecture of over-provision of public education can potentially be observed in countries in which the public education programme is inefficient. Levying a high income tax to finance an inefficient public education programme may distort households' saving decisions, discourage private investment and deteriorate the growth rate of economy.

The conjecture reagrding such over-provision may arise for two reasons which are "the misperception of the government about the households' preference" and "the misperception of the government about the private tuition option" that opens to altruistic parents. Due to the first reason, the government does not realise that parents are altruistic. Therefore, the government does not take both private tuition and bequests into account in the households' preference[12]. With regard to the second reason concerned in this paper, the government knows that the parents are altruistic and willing to give bequests to their children. However, let us suppose that the government does not realise that private tuition is also possible in a form of a bequest. In other words, the government misperceives the existence of private tuition as a technological option for human capital accumulation. We now ask whether the government will spend more of its budget for public education than the necessary amount that maximises economic growth.

Although one can easily find that the economy described by Glomm and Ravikumar (1997) coincides with our model when parents are not willing to provide private education and bequests to their children, a comparison between our roots of cubic function (τ_w^*) and the growth-maximising tax rate in their economy (τ^{GR}) is extremely difficult due to the complexity of the roots. Instead, the hypothesis that the government's misperception concerning the existence of private tuition will generate the over-provision of public education is explored in this section. The growth-maximising tax rate of an economy with only public education (τ_w^E) is then compared to the growth-maximising tax rate of an economy with both public education and private tuition (τ_w^*) in order to prove the over-provision analytically.

Proposition 2: The government's misperception concerning the existence of private tuition will lead to the overestimation of benefits of public education. The economy becomes situated in the growth-reducing area of the Armey curve

[12] An economy in which the government misperceives the households' preference could be compared to an economy without altruism; see Glomm and Ravikumar (1997).

since $\tau_w^E > \tau_w^*$.

Proof: Considering an economy with only public education denoted by superscript E, such an economy can be obtained from the general model where the elasticity of human capital with respect to private tuition is equal to zero ($\varphi = 0$). The physical capital accumulation and the human capital accumulation equations can then be written as follows[13]:

$$k_{t+1}^E = \gamma^E A k_t^\alpha h_t^{1-\alpha} \tag{3.40}$$

$$h_{t+1}^E = B\eta^E A^\mu k_t^{\alpha\mu} h_t^{1-\alpha\mu} \tag{3.41}$$

where

$$\gamma^E = \frac{[(1-\tau_w)(1-\alpha)+\alpha]\beta\alpha}{\alpha(1+2\beta)+(1-\tau_w)(1-\alpha)} \tag{3.42}$$

$$\eta^E = \tau_w^\mu (1-\alpha)^\mu \tag{3.43}$$

Using a similar process of derivation as in the economy with public education and private tuition, the growth rate of an economy with only public education can be deduced from a pair of first-order simultaneous difference equations (3.40) and (3.41). That is,

$$(1+g^E) = \gamma^E A \left[\frac{\gamma^E A^{1-\mu}}{B\eta^E}\right]^{\frac{\alpha-1}{1-\alpha\phi}} \tag{3.44}$$

where $\phi = 1-\mu$ due to the misperception that private tuition can be a part of human capital accumulation process. To find the growth-maximising tax rate, the derivative of a gross growth rate $(1+g^E)$ is taken with respect to the labour income tax rate,

$$\frac{\partial(1+g^E)}{\partial\tau_w} = \frac{\mu(1+g^E)(1-\alpha)}{1-\alpha\phi}\left[\frac{1}{\tau_w} - \frac{2\alpha^2\beta}{\Psi}\right] \tag{3.45}$$

where

$$\Psi = [(1-\tau_w)(1-\alpha)+\alpha][\alpha(1+2\beta)+(1-\tau_w)(1-\alpha)] \tag{3.46}$$

By setting the value of the square brackets of equation (3.44) equal to zero, the condition for the growth-maximising (minimising) tax rate can be found from the roots of the quadratic equation,

$$f(\tau_w^E) = (1-\alpha)^2\tau_w^2 - [2\alpha\beta + 2(1-\alpha)]\tau_w + 1 + 2\alpha\beta = 0 \tag{3.47}$$

[13] The derivation of the difference equations is explained in Appendix B.2.1

which are

$$\tau_{w1}^{E} = \frac{\alpha\beta + 1 - \alpha - \alpha\sqrt{\beta(\beta + 2(1-\alpha))}}{(\alpha - 1)^2} > 0 \tag{3.48}$$

$$\tau_{w2}^{E} = \frac{\alpha\beta + 1 - \alpha + \alpha\sqrt{\beta(\beta + 2(1-\alpha))}}{(\alpha - 1)^2} > 0 \tag{3.49}$$

$$\tau_{w1}^{E} < \tau_{w2}^{E} \tag{3.50}$$

To ensure that the roots of the quadratic equation indicate the local maximum, the second-order derivative of the gross growth rate with respect to labour income tax is calculated[14],

$$\frac{\partial^2(1+g^E)}{\partial \tau_w^2} = \frac{\mu(1-\alpha)(1+g^E)}{1-\alpha\phi}\left[-\frac{1}{\tau_w^2} + \frac{2\alpha^2\beta}{\Psi}\frac{\partial \Psi}{\partial \tau_w}\right] < 0 \tag{3.51}$$

where

$$\frac{\partial \Psi}{\partial \tau_w} = -\{\alpha(1-\alpha)(1+2\beta) + 2(1-\tau_w)(1-\alpha)^2 + \alpha(1-\alpha)\} < 0 \tag{3.52}$$

Using the information from (3.51) and (3.52), the roots of quadratic equation, τ_{w1}^{E} and τ_{w2}^{E}, indicates the growth-maximising tax rates for this economy. Considering the two roots of quadratic equations, the elasticity of human capital with respect to public education (μ) has no effect on either of the growth-maximising tax rates. This implies that there is no trade-off between public education and savings (or with private tuition), as explained by Blankenau and Simpson (2004). Thus, the growth-maximising tax rates depend on only the elasticity of output with respect to physical capital (α) and the discount factor (β).

Unfortunately, the roots of quadratic equation (3.47) cannot be compared to the roots of cubic function (3.35) due to the complexity of the roots. Therefore, the shape of the quadratic equation is investigated in the feasible range of labour income tax between zero and one. The intercept term is equal to $1 + 2\alpha\beta$ when the labour income tax rate is evaluated at zero ($\tau_w = 0$). At $\tau_w = 1$, the value of the quadratic equation is positive and it is equal to α^2. Since the coefficient of τ_w^2 is always positive and both τ_{w1}^{E} and τ_{w2}^{E} are greater than zero, all the properties of the quadratic equation indicate that there are two possible scenarios for the value of the growth-maximising tax rates.

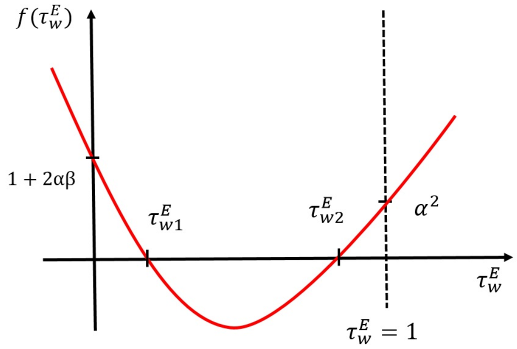
Figure 3.4a demonstrates the first scenario in which both τ_{w1}^{E} and τ_{w2}^{E} are within the feasible range of zero and one, while Figure 3.4b shows the second

[14]See Appendix B.2.2 for further calculation.

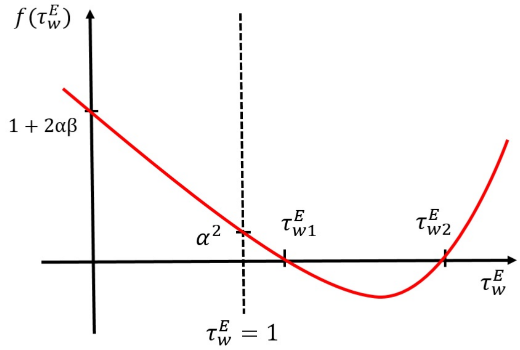
scenario that both τ_{w1}^E and τ_{w2}^E lie outside the feasible range. For $\tau_{w1}^E < 0$, we need

$$\tau_{w1}^E = \frac{\alpha\beta + 1 - \alpha - \alpha\sqrt{\beta(\beta + 2(1-\alpha))}}{(\alpha - 1)^2} < 1 \tag{3.53}$$

However, this inequality (3.53) cannot hold[15]. Therefore, it can be concluded that $\tau_{w1}^E > 1$ and $\tau_{w2}^E > 1$ since $\tau_{w1}^E < \tau_{w2}^E$. In other words, this analytical proof confirms that both growth-maximising tax rates in an economy that the government misperconceives of private tuition will be greater than one, as demonstrated in Figure 3.4b.



(a) Two roots of labour income tax lies within zero and one range



(b) Two roots of labour income tax lie outside zero and one range

Figure 3.4: Two Possible cases for the Quadratic Equation

Once the government realised that the growth-maximising tax rates exceed one, the best response of the government would be to set the labour income tax rate (τ_w^E) at 1 or at 100% regarding the feasibility constraint. This corner

[15] See Appendix B.2.3 for the proof.

solution can be confirmed by the Kuhn-Tucker conditions for the local maximum point. These conditions are

$$\frac{\partial(1+g^E)}{\partial\tau_w} \leq 0, \qquad \tau_w \geq 0 \qquad and \qquad \tau_w\frac{\partial(1+g^E)}{\partial\tau_w} = 0$$

where either the first or the second conditions can be slack. However, the third condition prevents both $\frac{\partial(1+g^E)}{\partial\tau_w}$ and τ_w from not being equal to zero. Evaluating (3.45) at $\tau_w = 1$ gives:

$$\left.\frac{\partial(1+g^E)}{\partial\tau_w}\right|_{\tau_w=1} = \frac{\mu}{1-\alpha\phi}\left\{\left(\frac{\alpha\beta}{1+2\beta}\right)^{\alpha\mu} A^{\mu}B^{1-\alpha}(1-\alpha)^{1-\alpha+\mu}\right\}^{\frac{1}{1-\alpha\phi}}\left[\frac{1}{1+2\beta}\right] > 0$$

which confirms that, at $\tau_w^E = 1$, the equation does not satisfy the growth-maximum conditions regarding the Kuhn-Tucker conditions. As a result, the government is likely to set the tax rate at 100% due to the misperception that the economy is in the growth-enhancing area. In conclusion, the growth-maximising tax rate in an economy with only public education is always greater than the growth-maximising tax rate in an economy with both public education and private tuition ($\tau_w^E > \tau_w^*$). The overestimation of the benefits of public education will lead the economy into the growth-reducing area of the Armey curve. Q.E.D

In an impure altruistic economy with only public education, the corner solution at $\tau_w^E = 1$ can be explained by reasons of resource allocation. The misperception of the government concerning the existence of private tuition leads to the exaggeration of the benefits of public education, as it is the only driver of economic growth. The government will then attempt to draw resources from unproductive activities, such as bequests, to finance public spending on education due to a lack of awareness of the option of private tuition[16]. With a shortage of funding for private tuition, the aggregate human capital accumulation may decrease in conjunction with a decline in economic growth.

The corner solution for the misperception concerning the existence of private tuition in this model is similar to that in the work of Blankenau and Simpson (2004), in which private investment in human capital is unproductive[17]. They showed that the growth-maximising labour income tax rate can be greater than one. Although Blankenau and Simpson (2004) did not mention about the over-provision of public education, their result suggests

[16]In the economy of Glomm and Ravikumar (1997), parents are selfish; consequently, no resources are wasted on bequests. This might explain why they have an interior solution for the growth-maximising tax rate.

[17]See their Proposition 6 on page 599.

the possibility that the government may overspend on public education. Thus, the possibility of the over-provision of public education creates the awareness for policymakers not to overestimate the benefits of public education.

3.5.3 A Numerical Example

The value of labour income tax that maximises the growth rate depends on the value of parameters for a particular economy. Therefore, the numerical calibration is required to provide the growth-maximising tax rate for the economy in the realistic way.

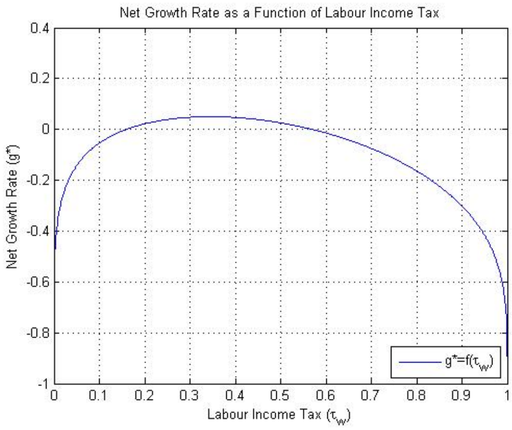


Figure 3.5: Relationship between Net Growth and Labour Income Tax

The numerical example is constructed based on the value of parameters in the UK's economy. The discount factor (β) is commonly assumed to be 0.99 and the elasticity of output with respect to physical capital (α) is 0.3 (Meeks et al., 2014). The elasticity of human capital with respect to public education (μ) is chosen by folllowing the empirical work of Blankenau et al. (2007). In their paper, the OLS estimation shows that μ is equal to 0.2 when the government revenue and other fixed effects are taken into account[18]. According to the work of Kindermann (2009), the fraction of income spent by parents on their children's education is used as a proxy of the elasticity of human capital with respect to private tuition (φ). Since there is a direct utility for altruistic parents from 'act of giving' private tuition, φ is set to be 0.3 which is approximately a two-fold

[18]Blankenau et al. (2007) found that the elasticity of human capital with respect to public education expenditure varies from 0.128 to 0.201. However, only 0.201 is statistically significant when considering all fixed effects.

increase in this paper[19]. Given this specific set of parameters, the relationship between the net growth rate (g) and the labour income tax rate (τ_w) can be calibrated by MATLAB programme, as demonstrated in Figure 3.5[20].

The hump-shape curve in Figure 3.5 indicates that the relationship between the net growth rate and labour income tax rate is a non-monotonic function. This confirms that the Armey curve does exist and it shares similar features of the inverted U-shape, as described in Barro (1990). In addition, one may expect that the growth-maximising tax rate should be between $0.2 - 0.4$. To provide an exact number for the growth-maximising tax rate, a cubic function of labour income tax is calculated in order to obtain the possible roots.

$$-0.3905\tau_w^3 + 1.5672\tau_w^2 - 1.6591\tau_w + 0.4020 = 0 \tag{3.54}$$

With respect to these parametric values, the roots of equation (3.54) are 0.3451, 1.2168 and 2.4512, as can be seen in Figure 3.6. According to the feasible range of labour income tax ($0 < \tau_w < 1$), policymakers should set labour income tax at rate 34.51% to maximise the long-run growth.

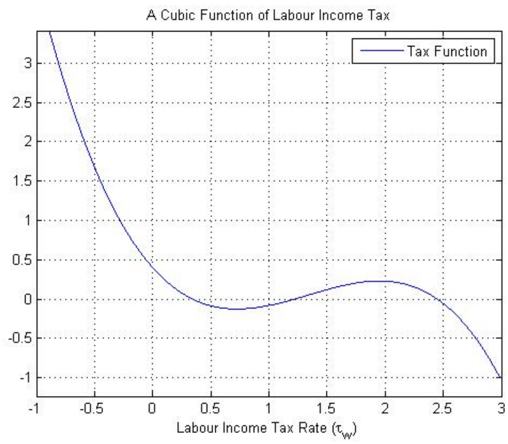


Figure 3.6: Cubic Function of Labour Income Tax

Substituting labour income tax at 34.51% and other parametric values into the growth equation (3.32), the numerical result confirms that the economy would reach the highest growth rate at 5.02% if the government were to levy the labour income tax at 34.51%. Note that the elasticity of human capital with respect to public education and to private tuition may vary across

[19]With regard to the parameterisation, Kindermann (2009) set the fraction of income spending from parents to children to be 0.16.

[20]The value of growth-maximising tax rates in the wider range of values of μ and φ will be explored in Figure 3.7.

countries. Different sets of parameters will definitely produce the different growth-maximising tax rates.

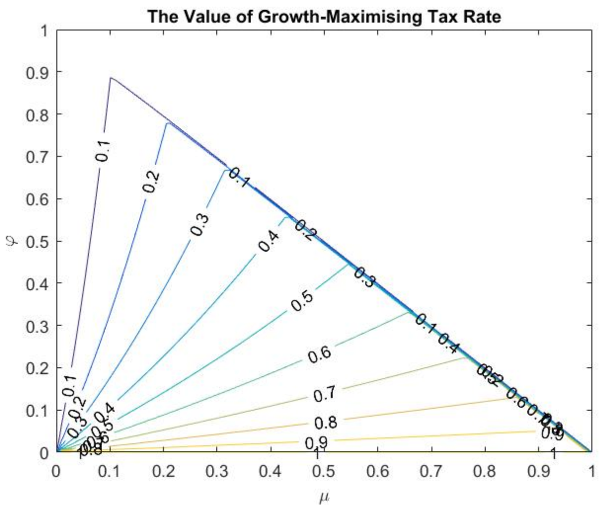


Figure 3.7: Value of the Growth-Maximising Tax Rates

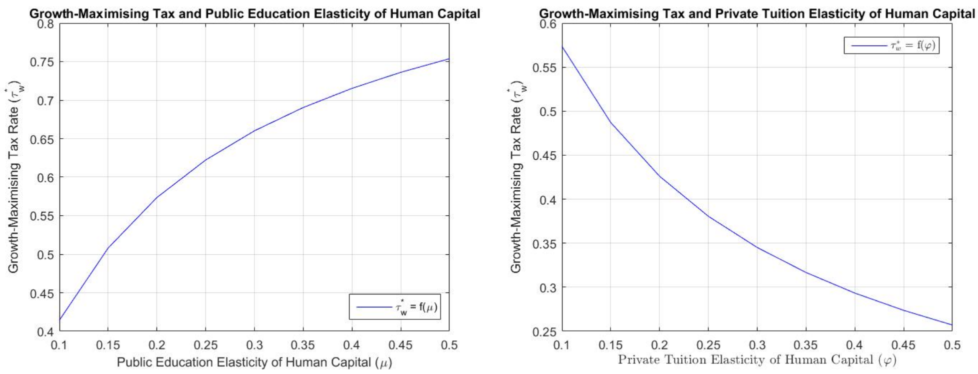
Contour Figure 3.7 illustrates the value of the growth-maximising tax rate ($\tau_w *$) when the elasticity of human capital with respect to public education (μ) and the elasticity of human capital with respect to private tuition (φ) are restricted to be less than one ($\mu + \varphi < 1$). Thus, the feasible values of τ_w^* is indicated by contour lines located in a lower triangle area only[21]. By decreasing the efficiency of private tuition ($\varphi \downarrow$), the growth-maximising tax rate is increased dramatically if the human capital is highly affected by public education ($\mu \uparrow$). Otherwise, τ_w^* will be low. In addition, the relationship between μ and φ can be interpreted from this graph. An increase in the efficiency of public education should be offset by an increase in the efficiency of private tuition if a country wants to retain the growth-maximising tax rate. This is a reason why the contour curve is an upward sloping line[22].

Accordingly, the relationship between the growth-maximising tax rate and the efficiency of public education can be easily observed in Figure 3.8a. Given a constant value for the private tuition elasticity of human capital ($\varphi = 0.2$), the growth-maximising tax rate is an increasing function of the elasticity of human

[21]The growth-maximising tax rate cannot be obtained in the upper triangle area due to the fact that $\mu + \varphi > 1$ violates the condition of human capital technology, which is assumed to be a constant return to scale.

[22]The magnitude of slope depends on the diminishing marginal return of each input to human capital.

capital with respect to public education. The economic intuition is simple. As the human capital is a driver of growth, it is reasonable to increase the tax rate to finance public education which contributes to the human capital accumulation process. Conversely, the government should reduce the tax rate when private tuition is more efficient than public education in terms of increasing human capital. This argument is also supported by Figure 3.8b, in that the growth-maximising tax rate is a decreasing function of the elasticity of human capital with respect to private tuition when the public education elasticity of human capital remains constant ($\mu = 0.2$).



(a) τ_w^* and μ when $\varphi = 0.2$ (b) τ_w^* and φ when $\mu = 0.2$

Figure 3.8: Relationship between Growth-Maximising Tax and the Elasticity of Human Capital with respect to Public Education and to Private Tuition

The over-provision of public education is also explored using the numerical calibration. All the parameters are the same as when we discover the value of the growth-maximising tax rates in Figure 3.7, except that the sum of the elasticity of human capital with respect to public education and to private education can vary within a range of 0 to 0.8 ($\mu + \varphi \leq 0.8$)[23]. This is because the over-provision of public education takes place at the boundary ($\mu = 0.8$ and $\varphi = 0$) at which the misperception concerning the existence of private tuition occurs[24]. The over-provision of public education can then be observed in Figure 3.9.

Once again, Figure 3.9 shows that the growth-maximising tax rate is an increasing function of the public education elasticity of human capital, which

[23]If we were to set the constraint as $\mu+\varphi \leq 1$, this may violate the over-provision condition. At $\mu = 1$, it implies that both φ and ϕ are equal to zero. In this case, the government completely ignores both private tuition and parental education.

[24]This means that the elasticity of human capital with respect to parental education (ϕ) is assumed to be 0.2.

is consistent to Figure 3.8a. Considering the point that the government misperceives the existence of private tuition ($\mu = 0.8$ and $\varphi = 0$), the government will collect the labour income tax at 1 or 100% which is a corner solution. Therefore, Figure 3.9 confirms that the over-provision of public education does exist in this economy.

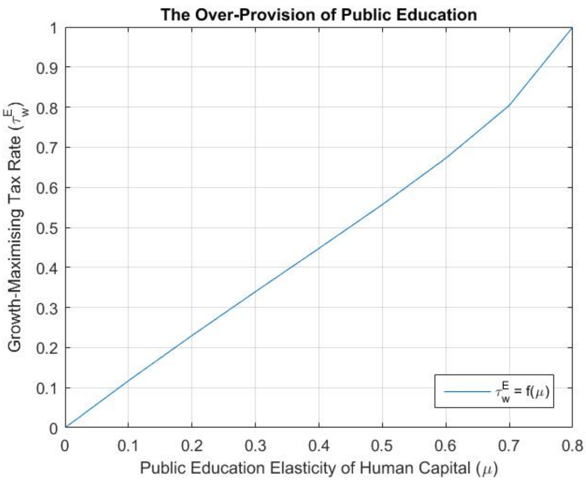


Figure 3.9: The Over-provision of Public Education

3.5.4 Degree of Substitutability

In this paper, the value of the parameter that refers to the degree of substitutability between public education and private tuition is assumed be zero ($\rho = 0$) since the human capital accumulation has a Cobb-Douglas form[25]. However, ρ can vary within a certain range $(-\infty, 1]$, which reflects the substitutability between these two types of education systems (Glomm and Kaganovich, 2003). Thus, it is interesting to observe how endogenous growth responds to tax rate when the degree of substitutability is altered, given other parametric values remain unchanged.

Suppose that the value of the degree of substitutability increases from zero to one ($\rho = 1$), this mean that public education and private tuition are perfect substitutes and the elasticity of substitution approaches infinity ($\sigma \to +\infty$). In this case, the function of human capital accumulation (3.12) can be rewritten as

$$h_{t+1} = B[ae_t + bE_t]^z h_t^{1-z} \tag{3.55}$$

[25] At $\rho = 0$, there is still substitutability between public education and private tuition, as the elasticity of substitution is equal to one.

Following the same process for finding the endogenous growth rate, the gross growth rate of an economy with a perfect substitution between public education and private tuition (g_p) is

$$(1 + g_p) = \gamma_p A \left[\frac{(\gamma_p A)^{1-z}}{B\eta_p} \right]^{\frac{\alpha-1}{1-\alpha(1-z)}} \tag{3.56}$$

where

$$\begin{aligned}
\gamma_p &= \frac{[(1 + \tau_w(1-\alpha)(\frac{2b}{a} - 1)]\beta\alpha}{\alpha(1+2\beta) + (1-\tau_w)(1-\alpha)(1+2\beta z)} \\
\eta_p &= \left[\frac{(1-\tau_w)(1-\alpha)z \cdot a}{\alpha} \right]^z \\
z &= \varphi + \mu
\end{aligned}$$

For simplicity, the share of private tuition and public education on human capital is assumed to be the same ($\varphi = a = 0.3$ and $\mu = b = 0.2$), in order for the economies to be compared. Given the same parameters from UK's economy ($\beta = 0.99$ and $\alpha = 0.3$), the relationship between the net growth rate and the labour income tax rate for both cases are compared in Figure 3.10.

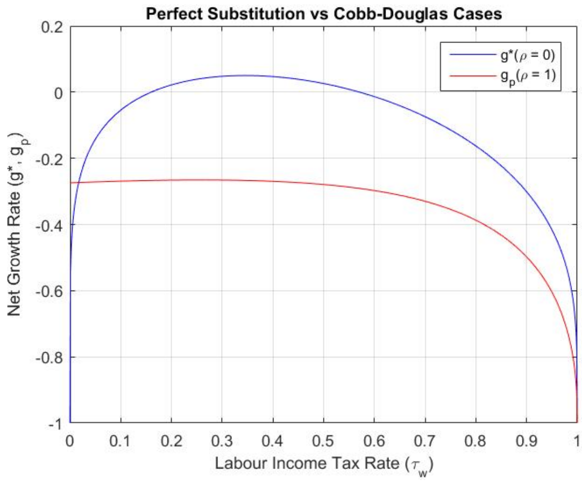


Figure 3.10: Perfect Substitution (g_p) vs Cobb-Douglas (g^*)

Although the Armey curve exists in both economies, Figure 3.10 shows that only the economy with perfect substitution experiences a negative growth rate at any level of tax rates. One possible explanation is that the economy with perfect substitution between public education and private tuition might suffer from inferior resource allocation. Due to the perfect substitutability, children may receive the same level of education, but parents are wasting resources by making double payment, both directly to private tuition and indirectly to public education via taxation. While the human capital in this economy remain unchanged, the process of physical capital accumulation is distorted by labour income tax. As a result, the growth rate of the economy declines.

Once the government realises that the quality of education provided by public and private institutions is identical, our numerical example suggests that the government should decrease the labour income tax rate and reduce the subsidy for public education. For this reason, the growth-maximising tax rate in the economy with perfect substitutability case (25.72%)[26] is smaller than the growth-maximising tax rate in the economy with complementarity between public education and private tuition (34.51%).

3.6 Public Education: A Necessity

The role of public education was clarified in the previous section. The government should increase the public education expenditure if it can enhance the human capital differently from private tuition. If not, public spending on education should be reduced because labour income tax will distort the physical capital accumulation process. Nevertheless, the existence of public education is still required in the economy even though there is a perfect substitution between public education and private tuition.

In the overlapping generation model, agents are heterogeneous in terms of income. While the middle-age generation receives income by working, the young generation has a zero income. Consequently, the inequality in income distribution between generations is a potential problem in this economy. This problem could be solved by the intergenerational transfers from parents to children, in the form of private tuition or bequests (Zilcha, 2003). However, parents may not willing to offer private tuition to their children since they may consider their own consumption to be the highest priority (Glomm and Kaganovich, 2008). According to the empirical evidence in US economy, the self-interest issue was raised in the work of Ladd and Murray (2001). They used the econometric model to study the intergenerational conflict in K-12 education and found that the middle-age and the old generations voted overwhelmingly for a reduction in per-child education spending in 16 states. In

[26]The growth-maximising tax rate cannot be zero in the perfect substitutability since both public education and private tuition operate together.

addition, Poterba (1997) used the panel data to analyse the demographic structure and political economy of public education across all states in the US from 1960 to 1990. His findings suggest that an increase in local property taxes and income taxes to finance K-12 education created tensions for the old generation. As a result, the elderly were less willing to support elementary and secondary education.

The income distribution gap and a decline in the investment in children's education lead to a reduction in human capital stock and in the growth rate. To prevent the income distribution gap across generations, the government should provide public education in order to secure the minimum level of human capital stock, as well as the growth rate of economy (Eckstein and Zilcha, 1994; Fernandez and Rogerson, 1996; Benabou, 2002). Furthermore, public education also alleviates the financial burden on households that experience a free-rider problem (Boldrin, 2005). Once the financial burden is mitigated, the intergenerational transfers from parents to children improve. This statement is also supported by the theoretical work of Holter (2015), which studied the impact of income taxation and public education expenditure on the intergenerational earning persistence across countries. In his calibrated model, the number of credit-constrained parents was reduced when the government subsidises public education. Hence, public education expenditure and income taxation will have a positive impact on intergenerational earning persistence because parents can increase the intergenerational transfer to their children. Without doubt, maintaining the growth rate and providing the equitable income distribution are justified reasons for the existence of public education.

3.7 Conclusion

The three-period overlapping generations model was employed to characterise the intergenerational transfers, such as private tuition and bequests from impure altruistic parents to their offspring. Private tuition is considered as the complementary input, together with the public education, in the human capital function. To observe how fiscal policy affect the aggregate economy, the effect of the labour income tax on macroeconomic variables was studied first. Despite the disincentive to save or spend on private tuition, the impacts of labour income tax on the accumulation of human capital and bequests are ambiguous. Due to this ambiguity, it is suggested that policymakers should levy an appropriate level of labour income tax to stimulate economic growth in the long run.

As indicated in the Armey curve, levying a high labour income tax rate may lead the economy into the growth-reducing area. Therefore, the growth-maximising tax rate was investigated to find the appropriate size of the government in an impure altruistic economy that is driven by both public education and private tuition. According to the cubic function of labour income tax, the necessary and sufficient conditions ensure that at least one value of growth-maximising tax rate lies within the feasible range between zero and one. The growth-maximising tax rate in an economy with both public education and private tuition were then compared with the growth-maximising tax rate in an economy in which the government may misconceive of the existence of private tuition. The analytical solution illustrates that the growth-maximising tax rate in the economy with only public education is equal to 100%, as it is the corner solution for this economy. This corner solution implies that there is the over-provision of public education when there is a misperception concerning private tuition. This cautions policymakers not to overestimate the benefits of public education. In addition, the numerical example illustrates that an increase in the degree of substitutability between public education and private tuition will reduce both economic growth and the growth-maximising tax rate. Thus, the government should minimise the amount spent on public education. However, this amount should not be zero since public education maintains the growth rate and enables the equitable income distribution.

Chapter 4

Stochastic Productive Government Expenditure, Congestion and Economic Growth

4.1 Introduction

For the past thirty years, productive government spending has been considered to be an important factor that enhances long-term growth. Governments can invest in productive spending for the public good, such as the creation of public roads, bridges and power plants in order to stimulate economic activities among economic agents[1].

According to the seminal work by Barro (1990), purely public good is introduced as a productive input into a firm's production function. With the non-rivalry and non-excludability properties of the public good[2], it can undoubtedly enhance economic growth when the size of government is sufficiently small. However, the positive magnitude of growth is reduced significantly if the non-rivalry property does not hold[3]. In other words, the public good may be subject to a congestion.

Although congestion generally refers to a reduction in the capacity of the public good (G) to be exploited by firms and households, the terminology and formulae for congestion may vary in the existing literature. For example, Glomm and Ravikumar (1994) applied congestion to both capital and labour, $G = \tilde{G}\left(\frac{1}{K^{\rho}L^{\phi}}\right)$, since households may use public infrastructure in an unproductive way. Turnovsky (1996) defined the congestion of public input as a ratio that was proportionate to output, $G = \frac{\tilde{G}}{Y^{1-\delta}}$, because an increase in income encourages households using a great number of public infrastructures. To avoid confusion, this paper defines congestion by following the papers by Fisher and Turnovsky (1998) and by Eicher and Turnovsky (2000), which indicated that it is a ratio describing an individual firm's use of private capital to aggregate use of private capital, $G = \tilde{G}\left(\frac{k}{K}\right)^{1-\delta}$. Despite the different types of congestion, a high degree of congestion will significantly reduce the positive impact of the public good on economic growth in the deterministic endogenous growth framework.

In the stochastic environment, Turnovsky (1999) took a degree of congestion and a degree of risk aversion into account and discovered two remarkable points. Firstly, in the presence of a stochastic process of output, the long-run growth could be enhanced if a degree of risk aversion were sufficiently high. This means

[1] Unlike chapter 2 in which public capital is a stock variable, productive government spending in this chapter is defined as a flow variable.

[2] See Samuelson (1954) for further explanation.

[3] This paper only considers the rivalry property of the public good. For the non-excludability property, please see Ott and Turnovsky (2006).

that the effect of precautionary saving, together with a degree of congestion, plays a central role in determining the magnitude of growth. Secondly, the economic growth and the social welfare of a decentralised economy is equivalent to those of a centrally planned economy when the public good is subject to proportional congestion. Therefore, the first-best solution can be achieved in a decentralised economy.

Unfortunately, the model of Turnovsky (1999) is limited in some respects. One limitation is that the role of government transfer has been ignored. Since the effect of precautionary saving requires a high income effect, the fluctuation of government transfer may increase the volatility in households' income paths. Consequently, risk-averse households are likely to increase their savings regardless of the degree of risk aversion. Other aspects are the source of uncertainty and a number of risks concerned in the stochastic AK model. According to Turnovsky (1999), a productivity shock causes outputs to follow the stochastic process and it is a linear function in capital. Thus, households will only experience capital risks that affect their decision making (Clemens and Soretz, 2003). Turning a blind eye to the income risk may cause a high volatility of income. Furthermore, in the stochastic AK setting, the effect of proportional congestion disappears when the market clearing condition is applied. This is because decentralised households enjoy an individual share of the public good in the same way that the planner does. Hence, the solutions for a decentralised economy and for a centralised economy are identical.

The main purpose of this paper is to overcome the limitations of Turnovsky's (1999) work, as described above. In this chapter, the stochastic AK model is constructed by introducing a random shock to productive government expenditure that is devoted to the public good subject to proportional congestion[4]. In this scenario, the government will decide to use the deterministic part as public input into firms' production function and to use the diffusion part for government consumption and government transfer. Thus, government transfer will definitely have an impact on households' income paths and will allow the proportional congestion to determine the difference between centralised and decentralised economies in a significant way.

The impact of stochastic productive government expenditure on economic growth is first explored in a decentralised economy. By allowing the

[4]In fact, a fiscal shock on productive government expenditure is not a new perspective. It has been used widely in the neoclassical growth framework (Baxter and King, 1993; Turnovsky and Fisher, 1995), as well as in the new Keynesian literature (Linnemann and Schabert, 2006).

government transfer, economic growth becomes greater when households are risk-averse agents. This is because the uncertain income effect induces precautionary saving. Although the interpretation of precautionary saving is similar to that in other stochastic endogenous growth models (Clemens, 2004; Ott and Soretz, 2002), it can be observed that the income effect strictly dominates the intertemporal substitution effect regardless of the degree of risk aversion in our case.

In addition, an increase in the growth rate in the presence of stochastic setting leads to scepticism concerning the existence of an inverted U- shaped relationship between economic growth and permanent income tax in the deterministic growth model. If the inverted U-shape does not exist, the condition for the growth enhancing tax rate may be altered. After investigating this point, the numerical example shows that such a relationship disappears when a degree of risk aversion is sufficiently large. Nevertheless, the condition for growth-enhancing income tax that the marginal benefit of providing for the public good is greater than the marginal cost remains valid.

With regard to the debate concerning the similarity between centralised and decentralised economies raised by Turnovsky (1999) in his proportional congestion case, it has been proved in our model that the solutions of a decentralised economy will not be the same as the social planner's outcomes. Therefore, a decentralised government should employ the first-best fiscal policy instruments in order to reduce saving and to achieve the first-best welfare level. Furthermore, the welfare loss in a decentralised economy can be calculated, and it can be compensated by the variation of the initial capital to reach the welfare target. Lastly, the trade-off between growth and welfare is unavoidable in a risky economy. The planner in a risky economy could either sacrifice the amount of initial capital, or apply the certainty equivalent interest rate to level off the welfare, as in a riskless economy.

The remainder of this paper is organised as follows. Section 4.2 is the basic model that characterises the equilibrium of a decentralised economy and the impact of stochastic productive government expenditure on economic growth. Section 4.3 discusses endogenous growth and the fiscal policy while the comparison between centralised and decentralised economies is presented in Section 4.4. Section 4.5 demonstrates the welfare loss measured by the variation of the initial capital, and Section 4.6 contains the conclusion.

4.2 The Basic Model

The impact of stochastic productive government expenditure on economic growth is analysed through a decentralised economy that has three economic agents, namely households, firms and the government, in a stochastic endogenous growth model. The perfect foresight assumption cannot be applied since the market economy is subject to uncertainty.

4.2.1 Households

The economy is inhabited by a fixed number of N infinitely lived households and there is no population growth. Each household will supply an inelastic unit of labour in a period of time ($\bar{l} = 1$) and will maximise the expected utility by smoothing the optimal consumption path. They are identical in terms of preference, the income received and the initial capital per person. Therefore, the representative-agent framework can be used to determine the optimal decision of households. Assuming that the utility function is a constant relative risk aversion (CRRA), each household will maximise the expected utility

$$U(c(t)) = E_0 \int_0^{\infty} \mathrm{e}^{-\rho t} \cdot \frac{c(t)^{1-\eta} - 1}{1-\eta} \, \mathrm{d}t \tag{4.1}$$

subject to the budget constraint (4.2)

$$dk(t) = [(1 - \tau_y)R(t)k(t) - c(t)] \cdot dt + \frac{T(t)}{N} \tag{4.2}$$

$$c(t) \geq 0, \quad k(t) \geq 0 \tag{4.3}$$

where equation (4.3) is a non-negativity constraint for consumption (c) and capital (k).

The value range of the parameters are chosen based on the literature on endogenous growth. The value of the discount rate is between zero and one ($0 < \rho < 1$), while E_0 is an expectation operator at period zero. An inverse intertemporal elasticity of substitution, which is known as the Arrow-Pratt relative risk aversion, is greater than zero ($\eta > 0$) since risk-averse households want to smooth their consumption path. Note that each household receives a government transfer ($\frac{T}{N}$) and treats it as a lump-sum amount. This lump-sum transfer will have an impact on the fluctuation of households' income paths. After that, they will optimally choose the consumption and saving plans for their futures.

By applying the dynamic programming method and Ito's Lemma, the value function of individual households can be constructed in the form of a stochastic Bellman equation[5]

$$\rho V(k) = \frac{c^{1-\eta} - 1}{1-\eta} - \rho V(k) + V'(k)((1-\tau_y)Rk - c) + \frac{1}{2}V''(k)\left(\frac{T}{N}\right)^2 \quad (4.4)$$

where $V(k)$ is a value function that depends only on the predetermined variable which is private capital (k)[6]. Given the rental rate, the optimal consumption and saving of households are

$$c^{-\eta} = V'(k) \quad (4.5)$$

$$V'(k)((1-\tau_y)R - \rho) + V''(k)((1-\tau_y)Rk - c) + \frac{1}{2}V'''(k)\left(\frac{T}{N}\right)^2 = 0 \quad (4.6)$$

In addition, the solution to the stochastic Bellman equation will be represented as a function form $(V(k))$ that satisfies the first-order conditions (4.5) and (4.6) and the transversality condition for the bounded utility (4.7).

$$\lim_{t\to\infty} E_t[\mathrm{e}^{-\rho t}V(k)] = 0 \quad (4.7)$$

4.2.2 Representative Firms

The representative firms will maximise their profit subject to revenue and cost. The production function consists of three inputs, namely the inelastic labour $(\bar{l})$, private capital (k) and a capacity of individual firm to use the public good (g). Since the public good may enhance the productivity of labour, the production function can be written as a labour-augmenting process:

$$y(t) = Ak(t)^{\alpha}(\bar{l}g(t))^{1-\alpha} \quad (4.8)$$

where A is a positive constant technology $(A > 0)$ and the elasticity of output with respect to private capital is between zero and one $(0 < \alpha < 1)$. Since $\bar{l} = 1$, an individual firm's production function can be rewritten as

$$y(t) = Ak(t)^{\alpha}g(t)^{1-\alpha}. \quad (4.9)$$

which is a Cobb-Douglas specification.

[5]Instead of using a stochastic Bellman equation, one may alternatively apply the Hamiltinian-Jacobi-Bellman (HJB) equation to solve the problem. The results of both methods are the same.

[6]The derivation of value function $V(k)$ is shown in Appendix C.1.1

The capacity of individual firm to use the public good can also imply that the public good is subject to proportional congestion. Assuming that the government invests aggregate productive government expenditure (G_k) in the public good, each individual firm will receive only a proportionate share of the public good. Therefore, in this paper, proportional congestion is defined by the ratio of an individual firm's usage (k) to the total usage of the public good (K). Thus, the capacity of an individual firm using the public good (g) will be:

$$g(t) = G_k(t) \left(\frac{k(t)}{K(t)} \right) \tag{4.10}$$

Substituting (4.10) into (4.9), the individual output can be written as a linear function in the individual capital.

$$y(t) = A \left(\frac{G_k(t)}{K(t)} \right)^{1-\alpha} k(t) \tag{4.11}$$

Although an individual firm realises the existence of proportional congestion, the effect of externalities from proportional congestion has never been taken into account. Consequently, the first-order condition for the profit maximisation is

$$R(t) = A \left(\frac{G_k(t)}{K(t)} \right)^{1-\alpha} \tag{4.12}$$

where the wage rate is equal to zero ($w = 0$) and the rental rate (R) is equal to the real interest rate (r), as the depreciation rate of capital is assumed to be zero ($\delta = 0$) for reasons of simplicity.

4.2.3 Government

The government invests aggregate productive government expenditure (G_k) in the public good and views the use of the public good by individual firms as productive per capita government expenditure ($g = \frac{G_k}{N}$). However, in the stochastic setting, this productive per capita government expenditure may be affected by many random factors. For example, in democratic countries, the population may vote for more public roads to mitigate the traffic jams[7]. By contrast, protesters may take actions against the deforestation required to build a new power plant. These factors can be considered as fiscal policy

[7]Park and Philippopulos (2003) revealed that rational voters can alter the reallocation of resources for public consumption, public investment and redistributive transfer. In addition, Kaas (2003) argued that sequential voting may generate a larger government size than the growth-maximising government size.

shocks that could have a significant impact on a long-run growth[8]. Thus, a random shock is introduced into productive per capita government spending that follows the stochastic process, which is a Brownian motion

$$dg(t) = \gamma_g g(t)dt + g(t)\sigma du \tag{4.13}$$

that is similar to

$$dg(t) = \gamma_g \frac{G_k(t)}{N} \cdot dt + \frac{G_k(t)}{N}\sigma \cdot du \tag{4.14}$$

where γ_g is a rate of change in g when a diffusion process is equal to zero ($\sigma = 0$). Note that, $\gamma_g g$ and $g^2\sigma^2$ are the instantaneous mean and variance per unit of time, respectively.

Based on equation (4.13), the generalised theory of stochastic differential equations developed by Ito and McKean (1964) is employed to explain the dynamics of productive government expenditure corresponding to the discrete-time model[9]. It consists of two parts. The first part refers to the adjustment of deterministic productive government spending over time, while the second part is a stochastic process depending on the standard deviation of productive spending (σ) and a random shock (u). Assuming that du follows a Wiener process, its disturbances are serially uncorrelated, and are normally distributed with zero mean and $\sqrt{dt}$ standard deviation ($du \sim N(0, dt)$)[10].

After realising the stochastic process, the government instantaneously decides to invest the deterministic part into the public good and to divide the diffusion part into two components. The first component is the government consumption (G_c) and the second component is rebated to households as a government transfer (T). Assuming the government budget is balanced each period, the stochastic aggregate government expenditure can be written as follows:

$$dG(t) = \gamma_g G_k(t)dt + G_c(t) + T(t) \tag{4.15}$$

[8]For the empirical testing, please see Gemmell et al. (2011).

[9]The diffusion of stochastic productive per capita government expenditure is continuous but not differentiable. Consequently, a differential equation with standard time derivatives is not applicable when describing the dynamics (Merton, 1990).

[10]The poisson process is an alternative stochastic process for a random shock. Besides, the Wiener process was chosen due to the linear relationship between risk premium and a degree of risk aversion. See Steger (2005) for the comparison.

where

$$G_k(t) = \tau_y Y(t) \tag{4.16}$$

$$G_c(t) = \theta G_k(t)\sigma du \tag{4.17}$$

$$T(t) = (1-\theta)G_k(t)\sigma du \tag{4.18}$$

when the permanent income tax rate (τ_y) and the percentage of shock absorption (θ) are between zero and one.

With regard to equations (4.16) and (4.17), the aggregate productive government expenditure (G_k) is financed by the permanent income tax revenue, while the aggregate government consumption (G_c) is subsidised by the percentage of shock absorption from the diffusion process. This scenario is consistent with the empirical evidence provided by Ramey (2011) in that there is a smaller fluctuation in productive government spending than there is in non-productive government spending. Government consumption will act like a sponge that absorbs the fiscal shock from the stochastic productive per capita government expenditure.

4.2.4 Decentralised Economy Equilibrium

In the decentralised economy, the demand for goods is equal to the supply when the market clearing conditions are imposed.

$$K(t) = Nk(t), \qquad C(t) = Nc(t), \qquad Y(t) = Ny(t) \tag{4.19}$$

Given the initial value of capital ($k(0) = k_0$) and the initial stochastic process ($u(0) = u_0$), the equilibrium of a decentralised economy is fully described by the resource allocation path $\{c(t), k(t)\}_{t\geq 0}$, the path of price $\{R(t)\}_{t\geq 0}$ and the path of government policy $\{G_k(t), G_c(t), T(t)\}_{t\geq 0}$ that solve equations (4.15), (4.16), (4.17), (4.18), (4.12), (4.5), (4.6), (4.7) and the market clearing conditions (4.19).

By substituting the aggregate productive government expenditure (4.16), the rental rate (4.12) and the market clearing conditions (4.19) into the households' optimal choice of saving (4.6), the equilibrium of capital becomes

$$\begin{aligned} V'(k)\left((1-\tau_y)\tau_y^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} - \rho\right) + \frac{1}{2}V'''(k)\left[(1-\theta)(\tau_y A)^{\frac{1}{\alpha}} k\sigma\right]^2 \\ V''(k)\left((1-\tau_y)\tau_y^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} k(t) - c(t)\right) = 0 \end{aligned} \tag{4.20}$$

where $V'(k) = \frac{\partial V}{\partial k}$, $V''(k) = \frac{\partial^2 V}{\partial k^2}$ and $V'''(k) = \frac{\partial^3 V}{\partial k^3}$ respectively.

To obtain the explicit function of $V(k)$, the relationship between consumption and private capital is needed to be clarified. In fact, such a relationship can be drawn from the fact that households have a constant relative risk aversion, while the utility function depends on time-invariant parameters. Therefore, consumption can be written as a linear function of private capital[11],

$$c(t) = \lambda k(t) \tag{4.21}$$

where λ is a constant propensity to consume capital. The constant propensity to consume capital (4.21) enables us to derive a closed-form solution for the optimal consumption as a function of private capital[12].

$$\begin{aligned} \lambda = \frac{c(t)}{k(t)} = \frac{\rho}{\eta} - \frac{1}{\eta}(1-\tau_y)\tau_y^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} \\ + (1-\tau_y)\tau_y^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} - (1-\theta)^2 (\tau_y A)^{\frac{2}{\alpha}} \sigma^2 \left(\frac{\eta+1}{2}\right) \end{aligned} \tag{4.22}$$

From Equation (4.22), the ratio of consumption to private capital depends on the variance of productive per capita government expenditure (σ^2). This means that households will take the uncertainty resulting from government transfer into account in order to smooth their consumption paths. With regard to the fiscal instruments, the permanent income tax rate will also distort a consumption-saving decisions. However, a percentage of shock absorption chosen by the government has a positive impact on consumption. An increase in the percentage of shock absorption will reduce the fluctuation in government transfer, as well as the uncertainty. This will allow households to increase their consumption.

4.2.5 Stochastic Productive Government Expenditure and Expected Growth Rate

The impact of stochastic productive government expenditure on the long-run growth rate is explored in this section. The expected growth rate of capital can be derived from productive government spending (4.16), households' budget constraint (4.2) and the constant propensity to consume capital (4.22). Using the fact that a random shock has a zero mean value ($E(du) = 0$), the expected growth rate of capital (γ_k) can be characterised as

$$\gamma_k = \frac{E(dK)}{Kdt} = (1-\tau_y)\tau_y^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} - \lambda \tag{4.23}$$

[11]Clemens and Soretz (2003) explained that the consumption-capital ratio is a constant when the utility of households is CRRA.

[12]See Appendix C.1.2 and C.1.3 for derivations.

where γ_k depends on the fiscal variable and other exogenous parameters. With regard to the production function (4.11) and the constant propensity to consume capital (4.22), the expected growth rate of capital can be considered to be the expected growth rate of this economy (γ).

$$\gamma = \gamma_k = \gamma_c = \gamma_y$$

Proposition 1: In the presence of stochastic setting, the economy will experience a higher growth rate than it will in the perfect foresight economy when the government transfer is allowed and households are risk-averse agents.

Proof: By substituting equation (4.22) into (4.23), the expected growth rate can be rewritten as a function of time-invariant parameters and fiscal policy instruments.

$$\gamma = \frac{1}{\eta}\left[(1-\tau_y)\tau_y^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} - \rho\right] + (1-\theta)^2(\tau_y A)^{\frac{2}{\alpha}}\sigma^2\left(\frac{\eta-1}{2}+1\right) \tag{4.24}$$

According to equation (4.24), the expected growth rate consists of two components. The first component is the deterministic growth rate in the literature on perfect foresight endogenous growth, while the second component is the additional growth rate generated by a diffusion process. The second component indicates the degree to which the economic growth rate deviates from its deterministic trend. This deviation depends on the degree of risk aversion (η). Nevertheless, the negative value of η is ruled out since the aim of households is to smooth their consumption paths[13]. Due to the stochastic process of productive government spending, the economic growth becomes larger than the growth rate in the perfect foresight endogenous growth model when the government transfer contributes to households' income paths. Q.E.D

The economic intuition behind this is the way in which households' concern about the risk resulting from stochastic productive government expenditure that might affect their income paths via government transfer. To prepare for an uncertain future income, risk-averse households may increase the amount they save in order to smooth their consumption paths. This kind of saving is known as precautionary saving, or emergency saving. It is also worth mentioning that the level of precautionary saving crucially depends on the degree of risk aversion of households.

The effect of precautionary saving on the long-run growth rate is interpreted in the same way as it is in the works by Levhari and Srinivasan (1969) and by

[13]Due to the concavity of the utility function, the marginal utility of consumption should be decreasing in consumption. Thus, the degree of risk aversion is strictly greater than zero.

Sandmo (1970). When impatient households are sufficiently risk averse, they will increase their precautionary saving because they are aware of the uncertain future income that might affect their future consumption. In fact, there is a tug-of-war between impatience and uncertainty that can be explained by the intertemporal substitution and the income effects. On one hand, impatient households may increase their current consumption against future consumption in order to attain the maximum amount of happiness in the current period. This effect is so called the **intertemporal substitution effect**. On the other hand, they may decrease the current consumption and increase their precautionary saving since they are aware of the uncertain future income. This precautionary saving a result of the **income effect**. By allowing government transfer to have an impact on households' income paths, it can be observed that the income effect completely dominates the intertemporal substitution effect in this model.

Although the relationship between the degree of risk aversion and economic growth is indeed similar to that described in other stochastic endogenous growth models, there are three different features that differentiates this model from others. Firstly, our source of uncertainty is the stochastic productive government expenditure that distinguishes this work from the paper by Turnovsky (1999) and by Clemens and Soretz (2004), which applied the stochastic process to production technology, and from the paper by de Hek (1999) that required the stochastic property for knowledge creation. Secondly, the possibility that the economic growth in the stochastic setting is smaller than the deterministic growth is ruled out from this model. As the government transfer creates a large income effect, it dominates the intertemporal substitution effect. This result is in contrast to the results in the work by de Hek (1999) and by Clemens and Soretz (2003). In their models, the substitution effect has a potential to dominate the income effect when the degree of risk aversion is sufficiently low. Lastly, the concept of risk neutral in stochastic endogenous growth models has been challenged. For instance, de Hek (1999) and Clemens and Soretz (2003) defined the terminology of risk neutral that the degree of risk aversion is equal to one. As $\eta = 1$, it can be argued that households still possess the degree of risk aversion since there is a decreasing of marginal utility in consumption. Thus, this chapter defines the risk neutral at $\eta = 0$, which is consistent with microeconomic textbooks.

Assuming that households are risk neutral ($\eta = 0$), one may suspect that the expected growth rate (4.24) is still affected by the diffusion process of productive government expenditure. The economic explanation for this is a role of government transfer. Although households are risk neutral, the impact of government transfer on households' income paths still exists. Consequently, the consumption-saving decision is unavoidably affected and the expected growth rate is changed. This explanation also supports why the income effect in this model is particularly strong when government transfer and the degree of risk aversion play a combined role.

4.3 Endogenous Growth and Fiscal Policy

The government levies permanent income tax to finance its productive spending, whilst government consumption is subsidised by the percentage of shock absorption from the diffusion process. In this section, the impact of both fiscal instruments on economic growth is analysed carefully in order to determine the conditions of growth-enhancing fiscal instruments for this stochastic economy.

4.3.1 Permanent Income Tax

In the deterministic growth model with productive government expenditure, the relationship between the long-run growth rate and permanent income tax is illustrated by the inverted U-shape (Barro, 1990). An increase in permanent income tax will enhance economic growth only if the marginal benefit of providing for the public good is greater than the marginal cost. After the marginal cost exceeds the marginal benefit, the growth rate reduces (Glomm and Ravikumar, 1997). Nevertheless, comparing only the marginal benefit and the marginal cost of the public good may not be sufficient to explain the relationship between economic growth and permanent income tax in the stochastic environment. Since the level of government transfer is affected by the diffusion process of productive government expenditure, permanent income tax that is used to finance such spending will play both direct and indirect roles in determining economic growth in the long run.

The argument regarding the inverted U-shaped relationship between the expected growth rate and permanent income tax is firstly explored by a numerical example. Given the standard set of parameters[14], the net growth

[14] The parametrical values for this calibration are $A = 0.8$, $\rho = 0.0101$, $\alpha = 0.3$, $\sigma = 0.4$ and $\theta = 0.1$, which are the standard numbers in macroeconomic literature (Meeks et al., 2014).

rate can be plotted as a function of permanent income tax when degrees of risk aversion vary arbitrarily within a range [1,4][15]. Using the MATLAB programme, Figure 4.1 confirms the existence of the inverted U-shape relationship when the degree of risk aversion is equal to one ($\eta = 1$). Nonetheless, such an inverted U-shaped relationship will disappear if the degree of risk aversion is sufficiently large. In the case where $\eta > 1$, the growth rate is strictly increasing in permanent income tax. This means that the government can arbitrarily increase permanent income tax within (0,1) to maximise the growth rate.

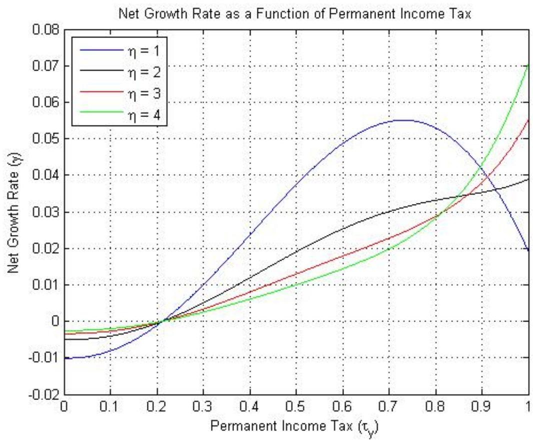


Figure 4.1: Net Growth Rate as a Function of Permanent Income Tax

As the growth rate is an increasing function of permanent income tax, it could be explained through the domination of the income effect over the intertemporal substitution effect. When the degree of risk aversion is high, households feel insecure about the uncertain future income that might affect their future consumption and expected utility[16]. To prepare for this, households may decide to increase an amount of precautionary saving. Once the government knows that precautionary saving is positively adjusted by risk-averse households, permanent income tax can be increased. On one hand, an increase in precautionary saving will enhance private investment and economic growth. On the other hand, the extra amount of permanent income tax revenue can be used to supply greater public good, and thus boost up the growth rate of economy. With these two effects, the long-run growth rate increases in the permanent income tax rate if the degree of risk aversion is

[15] As rational households will smooth their consumption paths, the degree of risk aversion should be greater than zero.

[16] With regard to the CRRA utility, the marginal utility is convex in consumption.

sufficiently high. This is why the inverted U-shaped relationship between economic growth and permanent income tax disappears[17].

The point of intersection of these four curves indicates that the minimum rate of permanent income tax that should be greater than 0.2127 or 21.27% to ensure a positive growth rate for this economy. Therefore, the economy will experience a negative growth rate if the government imposes an extremely low permanent income tax rate.

The extinction of a famous inverted U-shape creates doubt about the conditions for the growth-enhancing permanent income tax rate. Does a condition of a marginal benefit of providing for the public good greater than the marginal cost thereof still hold in the stochastic growth context? This doubt leads to Proposition 2.

Proposition 2: The condition for growth-enhancing permanent income tax rate that the marginal benefit of providing for the public good is greater than its marginal cost thereof remains valid in the stochastic growth model in which households are risk-averse agent.

Proof: Differentiating the stochastic growth rate (4.24) with respect to the permanent income tax (τ_y) gives us:

$$\frac{\partial\gamma}{\partial\tau_y} = \frac{1}{\alpha\eta}\tau_y^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}\left[\frac{1-\alpha}{\tau_y} - 1\right] + (1-\theta)^2\tau_y^{\frac{2-\alpha}{\alpha}}A^{\frac{2}{\alpha}}\sigma^2\left[\frac{\eta+1}{\alpha}\right] \tag{4.25}$$

where its sign depends on the value of two square brackets. In order to have a growth-enhancing permanent income tax rate, both square brackets should be strictly positive. Hence, the necessary conditions are

$$\eta > -1 \tag{4.26}$$

$$1-\alpha > \tau_y \tag{4.27}$$

where the necessary condition (4.26) always holds in an economy with rise-averse households. The second necessary condition (4.27) implies that the growth-enhancing permanent income tax rate can be achieved when the marginal benefit of providing for the public good $(1-\alpha)$ is greater than the marginal cost (τ_y). This condition is also consistent with the implication of the growth-enhancing tax rate in the perfect foresight growth model. Given the presence of these two necessary conditions, Proposition 2 is proved. Q.E.D

[17] Although the inverted-U shape does not disappear, there is empirical evidence from the OECD countries that have a high average growth-enhancing tax rate because of the size of their family are small (De Witte and Moesen, 2010) and they have efficient public policies (Afonso et al., 2005).

In the case that the marginal benefit is lower than is the marginal cost, policymakers may experience a trade-off between the deterministic growth and its diffusion part while increasing permanent income tax. This is because raising permanent income tax allows the government to provide more public good, which will generate a positive diffusion for the stochastic growth rate. Thus, the government can levy a higher permanent income tax rate as long as the benefits of government transfer can compensate for the loss of permanent income.

4.3.2 Percentage of Shock Absorption

A percentage of shock absorption is another fiscal policy instrument that can determine the impact of the stochastic productive government expenditure on the volatility of income path. To choose an appropriate level for the percentage of shock absorption, policymakers should first consider how it affects the long-run growth rate.

Proposition 3: The growth-enhancing condition for the percentage of shock absorption does not exist when households are risk-averse agents. An increase in the percentage of shock absorption will reduce economic growth definitely.

Proof: Taking derivative of the stochastic growth rate (4.24) with respect to the percentage of shock absorption (θ) gives:

$$\frac{\partial \gamma}{\partial \theta} = -(1-\theta)(\tau_y A)^{\frac{2}{\alpha}} \sigma^2 (\eta + 1) \tag{4.28}$$

Based on equation (4.28), the percentage of shock absorption will boost the stochastic growth rate if, and only if, the degree of risk aversion is below minus one ($\eta < -1$). The low degree of risk aversion implies that households are risk-lover. They have a motivation to increase consumption today against tomorrow since they know that the government will rebate the government transfer to them. Unfortunately, it is impossible to incorporate risk-lover households into this model because their behaviour violates the assumption of the concavity of the utility function. For the risk-averse households ($\eta > 0$), the stochastic growth rate will decrease when the government increases the percentage of shock absorption. Consequently, the growth-enhancing condition for the percentage of shock absorption does not exist in this economy. Q.E.D

In Figure 4.2, the numerical example confirms that an increase in the percentage of shock absorption will reduce the long-run growth rate slightly given the specific set of parameters[18]. The economic intuition is that an

[18] $\alpha = 0.3$, $\rho = 0.0101$, $\sigma = 0.5$, $A = 1.5$, $\eta = 4$ and $\tau_y = 0.3$

increase in the percentage of shock absorption will reduce the volatility of government transfer, as well as of precautionary saving. Therefore, the expected growth rate will decrease.

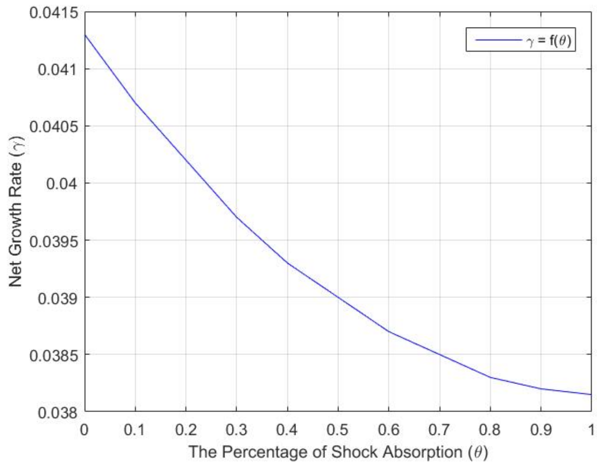


Figure 4.2: Net Growth Rate (γ) as a Function of the Percentage of Shock Absorption (θ)

With regard to Proposition 2 and 3, the policy recommendation is to levy the permanent income tax rate because it can enhance economic growth in the long run. By contrast, the percentage of shock absorption should be applied carefully since it is harmful for growth and the government consumption is purely waste. Nonetheless, the percentage of shock absorption is a useful fiscal instrument. The government can use it to determine the impact of stochastic productive government expenditure via government transfer on the volatility of households' income path. This will help a decentralised government achieve its desired level of the growth rate in the long run.

4.4 Centralised vs Decentralised Economy

In the previous section, it was suggested that policymakers should increase permanent income tax and reduce the percentage of shock absorption for growth-enhancing purpose. However, it did not suggest the ways of implementing both fiscal instruments to achieve the first-best outcomes that can be obtained as a result of having the benevolent social planner. Consequently, in this section, the first-best fiscal policy is investigated to eliminate the wedge between centralised and the decentralised economies.

4.4.1 Centralised Economy

The social planner will provide the first-best solution for the economy by internalising households' degree of risk aversion and the externalities from stochastic productive government expenditure. Thus, proportional congestion and government transfer are taken into account. The objective of the social planner is to maximise a representative household's utility

$$U(C/N) = E_0 \int_0^\infty \mathrm{e}^{-\rho t} \cdot \frac{(C/N)^{1-\eta} - 1}{1-\eta} \, \mathrm{d}t \tag{4.29}$$

subject to the aggregate resource constraint,

$$dK(t) = Y(t) - C(t) - dG(t) \tag{4.30}$$

where the aggregate production function (Y), stochastic aggregate government expenditure (dG), productive government spending (G_k), government consumption (G_c) and government transfer (T) are

$$Y(t) = AK(t)^\alpha G_k(t)^{1-\alpha} \cdot dt \tag{4.31}$$

$$dG(t) = G_k(t)dt + G_c(t) + T(t) \tag{4.32}$$

$$G_k(t) = \phi Y(t) \tag{4.33}$$

$$G_c(t) = \phi' G_k(t)\sigma du \tag{4.34}$$

$$T(t) = (1-\phi')G_k(t)\sigma du \tag{4.35}$$

in which ϕ and ϕ' are fractions that indicate the level of productive government expenditure and government consumption, respectively. The value function of centralised economy can be constructed by applying the method of dynamic programming and Ito's Lemma:

$$\begin{aligned}\rho V(K) = \frac{(C(t)/N)^{1-\eta} - 1}{1-\eta} &- \rho V(K) + V'(K)[(1-\phi)AK^\alpha G_k^{1-\alpha} - C] \\ &+ \frac{1}{2}V''(K)[(1-\phi')\sigma\phi AK^\alpha G_k^{1-\alpha}]^2\end{aligned} \tag{4.36}$$

After choosing the optimal amount of consumption and saving, the optimal consumption path and the expected growth rate for a centralised economy can be written as follows[19]:

[19]The derivation is provided in Appendix C.2

$$\lambda_s = \frac{C}{K} = \frac{\rho}{\eta} - \frac{1}{\eta}(1-\phi)\alpha\phi^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}$$
$$+ (1-\phi)\phi^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}} + \alpha\left((1-\phi')\sigma(\phi A)^{\frac{1}{\alpha}}\right)^2 \tag{4.37}$$
$$- \frac{1}{2}(\eta+1)\left((1-\phi')\sigma(\phi A)^{\frac{1}{\alpha}}\right)^2$$

$$\gamma_s = \frac{1}{\eta}\left[\alpha(1-\phi)\phi^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}} - \rho\right]$$
$$+ \left((1-\phi')\sigma(\phi A)^{\frac{1}{\alpha}}\right)^2 \cdot \left(\frac{\eta+1}{2} - \alpha\right) \tag{4.38}$$

where λ_s is the constant propensity to consume capital and γ_s is expected growth rate in the centralised economy. Then, centralised and decentralised economies can be compared.

Proposition 4: In contrast to Turnovsky (1999), the stochastic growth rate in a centralised economy is smaller than it is in a decentralised economy when productive government expenditure follows the stochastic process and the public good is subject to proportional congestion.

Proof: To compare a decentralised economy with a centralised economy, the flow of government expenditure should be the same for both economies. This implies that the fraction of productive government expenditure is set to be equal to the permanent income tax rate ($\phi = \tau_y$), and that the fraction of government consumption is the same rate as the percentage of shock absorption ($\phi' = \theta$). Therefore, the growth rate of a centralised economy (4.38) can be rewritten as

$$\gamma_s = \frac{1}{\eta}\left[\alpha(1-\tau_y)\tau_y^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}} - \rho\right]$$
$$+ \left((1-\theta)\sigma(\tau_y A)^{\frac{1}{\alpha}}\right)^2 \cdot \left(\frac{\eta-1}{2} + 1 - \alpha\right) \tag{4.39}$$

where γ_s is the growth rate of a centralised economy.

From equation (4.39) the difference between the stochastic growth rate of centralised and decentralised economies can be found by subtracting equation (4.24) with (4.39).

$$\gamma - \gamma_s = \frac{1}{\eta}(1-\alpha)(1-\tau_y)\tau_y^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}} + \alpha[(1-\theta)\sigma(\tau_y A)^{\frac{1}{\alpha}}]^2 > 0 \tag{4.40}$$

According to equation (4.40), the expected growth rate of a centralised economy is smaller than the growth rate of a decentralised economy for two reasons. Firstly, the deterministic part of the growth rate in a centralised economy is smaller than it is in a decentralised economy due to the elasticity of income with respect to private capital (α). Since the social planner internalises the externalities from the public good and proportional congestion, the private capital elasticity of income (α) is taken into account for the marginal product of private capital. Hence, the rate of return on capital in a centralised economy is lower than it is in a decentralised economy. Secondly, the social planner also internalises the effect of the externalities from the public good and proportional congestion on government transfer. By considering the public good elasticity of income ($1 - \alpha$), the social planner observes a lower fluctuation of income flow. Subsequently, the amount of precautionary saving in a centralised economy is not too high. This contrasts with the individual households who treat government transfer as a lump-sum amount and allow it to have a significant effect on their income paths. As a result, the diffusion part of the growth rate in a centralised economy is always smaller than it is in the decentralised one[20]. Q.E.D

Intuitively, the overestimation of the return on private capital (R) and the exaggeration of government transfer (T) contribute to a high volatility of households' income in a decentralised economy. With the high volatility of income, risk-averse households may increase their precautionary saving as protection against uncertainty. This increases the long-run growth rate tremendously. By contrast, the social planner has a precise knowledge of the return on capital and the fluctuation of households' income by considering all possible factors. The planner then chooses low levels of saving and growth rates with a low volatility of income paths to obtain the greatest social welfare. Furthermore, the numerical result also supports our findings. Given that all parameters are the same for both economies[21], Figure 4.3 confirms that the growth rate of a decentralised economy is too high compared to the growth rate in a centralised economy.

The theoretical justification is that a centralised economy can be considered to be the Arrow-Romer economy that is exposed to both capital risk and to income risk (Clemens and Soretz, 2003). The (public) capital risk is similar to a decentralised economy in which the (public) capital risk creates an uncertain income path for households. However, a centralised economy also experiences

[20]This result is also consistent to the previous literature in which the public good is subject to congestion that a degree of congestion reduces the long-run growth rate (Eicher and Turnovsky, 2000).

[21]Again, we used the standard parametric values in the macroeconomic literature; $\alpha = 0.3$, $\rho = 0.0101$, $\sigma = 0.4$, $A = 0.8$, $\eta = 0.6$ and $\theta = \phi' = 0.3$.

the income risk generated by the factor of income distribution from the public good $(1-\alpha)$. In other words, income risk is another factor that contributes to the uncertain amount of income. Due to the risk sharing between both capital and income risks, the stream of income may be less volatile in a centralised economy.

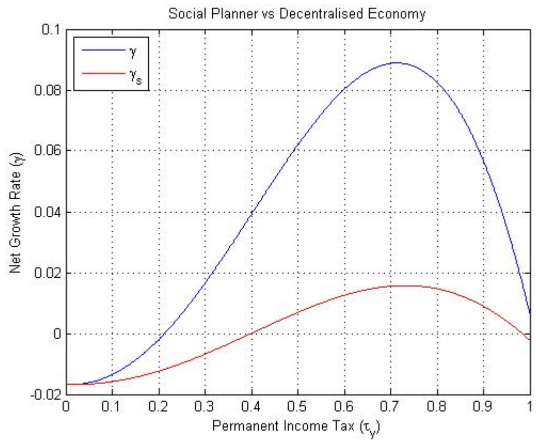


Figure 4.3: Comparison of Growth and Permanent Income Tax between Centralised (γ_s) and Decentralised (γ) Economies

In our proportional congestion case, the difference between centralised and decentralised economies is in stark contrast to the work of Turnovsky (1999). In his work, the marginal product of capital is equal to the average product of capital when the externalities derived from the public good and proportional congestion are taken into account. This implies that output is a linear function of capital on both individual and on aggregate levels. Subsequently, the effect of proportional congestion on output is vanished completely due to the market clearing condition[22]. There is no doubt that, in Turnovsky's (1999) model, the decentralised economy simply replicates the centralised economy.

In conclusion, Turnovsky's (1999) result cannot be generally applied to the stochastic growth model when the stochastic process is not generated by output itself and government transfer is not allowed. In the case of stochastic productive government expenditure, individual firms and the social planner experience different production technology even though the aggregate production technology is the same. The social planner encounters the Cobb-Douglas production technology after internalising the externalities derived from the public good and proportional congestion. On the contrary,

[22]See Turnovsky (1999), equation (22) p.558 where $\delta = 0$.

individual firms observe that output is a linear function of capital regardless of the externalities. Thus, the outcomes for each economy will never be the same.

4.4.2 The First-Best Fiscal Instruments

In the last section, it was proved that the saving and growth rates in a decentralised economy is too high when compared to the social planner's outcomes. Therefore, fiscal instruments should be used to reduce the amount of savings and to enhance a level of aggregate consumption, together with social welfare. To maximise social welfare, the government should apply fiscal instruments that can imitate the first-best solutions. Due to the fact that there are two fiscal instruments in a decentralised economy, namely permanent income tax and the percentage of shock absorption, the first-best fiscal policy instrument is then explored in this section.

To mimic the first-best world, all the economic variables in the decentralised economy are adjusted to be the same as those in the social planner's economy. Accordingly, the consumption-capital ratio (C/K) and the economic growth rate (γ) should be the same in both economies. By equating (4.24) and (4.39), the permanent income tax rate can be written as the implicit function of the percentage of shock absorption, as follow:

$$F(\tau_y, \theta) = (1-\tau_y)\tau_y^{-\frac{1+\alpha}{\alpha}} + \left(\frac{\alpha}{1-\alpha}\right)(1-\theta)^2\eta\sigma^2 A^{\frac{1}{\alpha}} = 0 \tag{4.41}$$

which can be exploited to determine the relationship between the fiscal instruments. Taking the total differentiation of equation (4.41) with respect to permanent income tax and the percentage of shock absorption, it can be observed that permanent income tax has a negative relationship with the percentage of shock absorption when households have a positive degree of risk aversion ($\eta > 0$).

$$\frac{\mathrm{d}\tau_y}{\mathrm{d}\theta} = -\frac{2\alpha^2\eta(1-\theta)\tau_y^{\frac{1+2\alpha}{\alpha}}\sigma^2 A^{\frac{1}{\alpha}}}{(1-\alpha)\cdot[(1+\alpha)(1-\tau_y)+\alpha\tau_y]} < 0 \tag{4.42}$$

Equation (4.42) could be explained by economic intuition. To achieve the first-best solution, the government is required to raise private consumption and social welfare levels. An increase in the percentage of shock absorption (θ) will definitely reduce the volatility (risk) of government transfer (T). Consequently, risk-averse households will decrease their precautionary saving and enjoy consumption. Similarly, the government can encourage private consumption by decreasing the permanent income tax rate. A low level of the permanent income tax rate (τ_y) is less likely to distort the households' consumption. Therefore, the direction of permanent income tax should be the opposite of the percentage of shock absorption in order to maximise social welfare[23].

Although the implication of the fiscal instruments is quite clear, there is a special case in which the government chooses the percentage of shock absorption to be the same as the permanent income tax rate. By setting $\tau_y = \theta = \tau$ in equation (4.41), the first-best income tax rate (τ) will take the following form:

$$(1-\tau)\tau^{\frac{1+\alpha}{\alpha}} = -\frac{1-\alpha}{\eta\alpha\sigma^2 A^{\frac{1}{\alpha}}} \tag{4.43}$$

which is a function of exogenous parameters (η, α, σ and A). Hence, the government should consider these parametrical values in the economy before adjusting the income tax rate to achieve the first-best outcomes.

4.5 Welfare Analysis

4.5.1 Trade-off between Growth and Welfare

In a decentralised economy, the amount of savings and the growth rate are too high; therefore, social welfare that relies on a level of consumption will be low. To achieve the first-best social welfare, the optimal amount of consumption in a decentralised economy should be the same as it is in a centralised economy. This optimal amount of consumption can be found from the first-order conditions obtained in both economies

$$V'(k) = c^{-\eta}$$
$$V_s'(K_s) = \left(\frac{C}{N}\right)^{-\eta}\frac{1}{N}$$

where the subscript 's' refers to the social planner's economic variable.

Using the fact that the consumption is a proportion of capital ($c = \lambda k$) and taking the integral to the first-order conditions, it is clear that the optimal amount of consumption is determined by the value function of capital in each economy.

[23]The opposite will be true if households are risk-seeking ($\eta < 0$). An increase in the percentage of shock absorption leads to the overconsumption. Thus, the government will reduce the extra amount of consumption by increasing the permanent income tax rate.

$$V(k) = c^{-\eta} \cdot \frac{k}{1-\eta} \tag{4.44}$$

$$V_s(K) = \left(\frac{C}{N}\right)^{-\eta} \cdot \frac{k_s}{1-\eta} \tag{4.45}$$

To obtain the same level of consumption and social welfare, the value function in the decentralised economy should be equal to the value function from the centralised economy. That is,

$$V_s(k_s) = V(k) \tag{4.46}$$

which also indicates the same welfare level in both economies.

Substituting equations (4.44) and (4.45) into (4.46) and rewrite them in terms of the consumption-capital ratio, the condition (4.46) for reaching the first-best welfare target is illustrated in equation (4.47).

$$k^{1-\eta} \cdot \left(\frac{c}{k}\right)^{-\eta} = k_s^{1-\eta} \cdot \left(\frac{C}{K}\right)_s^{-\eta} \tag{4.47}$$

Applying the market clearing conditions ($K = Nk$ and $C = Nc$) and the definition of a constant propensity to consume (λ) to equation (4.47), the condition for the first-best social welfare can be specified by a relative amount of capital ratio.

$$\frac{k(t)}{k_s(t)} = \left[\frac{(C(t)/K(t))_s}{(C(t)/K(t))}\right]^{\frac{\eta}{\eta-1}} = \left[\frac{\lambda_s}{\lambda}\right]^{\frac{\eta}{\eta-1}} \tag{4.48}$$

Evaluating equation (4.48) at period 0, the excess amount of capital in the decentralised economy relative to the centralised economy is alternatively calculated by the ratio of the constant propensity to consume capital (λ), as presented in equation (4.49).

$$\frac{k_0 - k_{s,0}}{k_{s,0}} = \left[\frac{\lambda_s}{\lambda}\right]^{\frac{\eta}{\eta-1}} - 1 \tag{4.49}$$

When the excess amount of initial capital is equal to zero, it means that the decentralised economy achieves the first-best welfare target. Nevertheless, the decentralised economy will experience the welfare loss if the excess amount of initial capital is positive[24].

[24]The idea of welfare loss expressed in term of variation in capital is consistent with the work of Turnovsky (1999).

Intuitively, this excess amount of initial capital is caused by a lack of consideration of the externalities from proportional congestion and the volatility of government transfer. As a result, households will increase their savings and reduce consumption because of the uncertain income paths. Equation (4.49) demonstrates the trade-off between economic growth and social welfare. To increase the social welfare, the government should implement a fiscal policy to encourage households to sacrifice the excess amount of capital and to consume more. A lower savings will result in a reduction in the growth rate.

This trade-off between growth and welfare contradicts the underlying result suggested by Barro (1990), in which the same conditions for growth maximisation and welfare maximisation may be obtained. In the deterministic growth model, Barro (1990) argued that the utility maximisation problem coincides the welfare maximisation problem if the following three conditions hold. Firstly, the households' utility depends solely on consumption. Secondly, the utility is bounded. Thirdly, the output elasticity of the public good is constant. Despite the fact that all three conditions are satisfied in our growth model with the stochastic process, the fear of uncertainty increases economic growth but reduces social welfare. In fact, the existence of the trade-off between growth and welfare is consistent with the paper by Futagami et al. (1993), which included the stock of productive government expenditure into the production process in the deterministic growth model, and the paper by Chatterjee et al. (2003), in which the trade-off occurred in a small open economy with a pure transfer.

4.5.2 Trade-off between Risky and Riskless Economy

Although the social planner provides the highest social welfare level that could be possibly obtained, a centralised economy will still experience a welfare loss due to the volatility of productive government expenditure (σ). By equating the value function of a riskless economy and the value function of a risky economy ($V_f(K_{f,0}) = V_r(K_{r,0})$), the welfare loss in a centralised economy can be also measured by the excess amount of initial capital in a risky economy,

$$\frac{K_{r,0} - K_{f,0}}{K_{f,0}} = \left[\frac{\lambda_f}{\lambda_r}\right]^{\frac{\eta}{\eta-1}} - 1 \tag{4.50}$$

where subscript 'r' and 'f' refer to a risky and a riskless economies, respectively. λ_f is obtained from equation (4.37) when $\sigma = 0$ and $\lambda_r = \lambda_s$.

From equation (4.50), it can be concluded that the social planner has to sacrifice the amount of initial capital in a risky economy to reach the highest social welfare at the same level as it is in a riskless economy[25]. When the economy experiences a high volatility of productive government spending, it will suffer from a substantial loss of welfare. This conclusion is also supported by the numerical example, as represented in Figure 4.4.

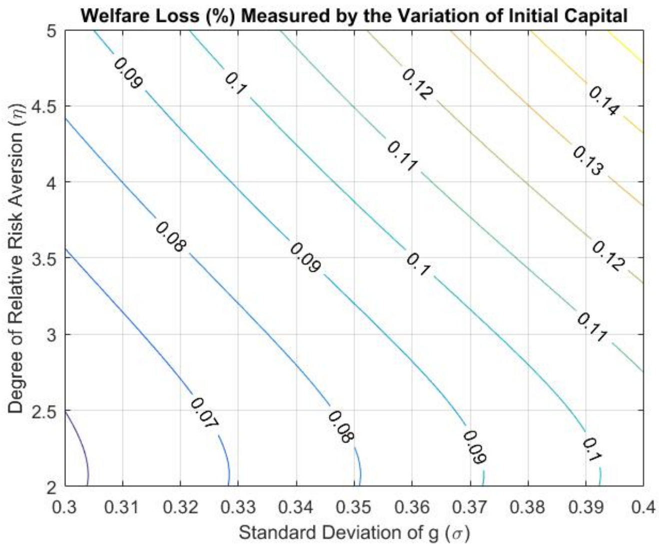


Figure 4.4: Welfare Loss (%) Measured by the Variation of Initial Capital

Given the specific set of parameters in standard macroeconomic literature, the discount rate (ρ) is equal to 0.0101 while the elasticity of output with respect to capital (α) is 0.3. A constant technology parameter (A) is equal to 0.8. With regard to the policy instruments, the fractions that indicate the level of productive government spending ($\phi = \tau_y$) and government consumption ($\phi' = \theta$) are set to be 0.3 and 0.1, respectively. The selected value range of degree of risk aversion (η) is chosen from 2 to 5, based on the work of Brown and Gibbons (1985)[26]. In addition, the standard deviation of government expenditure on economic affairs is used as a proxy for the standard deviation of productive government spending (σ). It varies from 0.3-0.4 across European countries when

[25]This is similar to the work of Epaulard and Pommeret (2003), which measured the cost of welfare from the volatility by the percentage of capital that the representative agent has to surrender at period 0.

[26]Brown and Gibbons (1985) applied a method moment of estimator (MME) to measure the value of relative risk aversion from the aggregate consumption and NYSE data from 1926 to 1981.

considering at least 12 European countries during 2006-2014[27]. With respect to all corresponding parameters, Figure 4.4 illustrates the welfare loss when households' degree of risk aversion (η) and the standard deviation of productive spending (σ) vary. The contour curves indicate the welfare loss from the lowest (south-west) to the highest levels (north-east). This means that an increase in the volatility of productive spending, together with the degree of risk aversion will lead to a significant loss of social welfare.

4.5.3 Certainty Equivalent Interest Rate

In a centralised economy, the welfare loss occurs when the economy is subject to uncertainty caused by the stochastic process of productive government expenditure. However, the welfare loss can be eliminated in the following three cases. Firstly, the government spends all the amount of the diffusion parts of productive government expenditure on the government consumption that is completely waste ($\phi' = 1$). Thus, households' income is not affected by the volatility of government transfer ($T = 0$). Second, the volatility of productive spending will have no impact on growth and welfare when households' degree of risk aversion (η) is equal to $2\alpha - 1$. Lastly, the social planner may suggest a certainty equivalent interest rate that covers the volatility of the aggregate wealth of households from the government transfer. In these three cases, economic growth and social welfare in a risky economy will be the same as in a riskless economy.

The certainty equivalent interest rate (r_c) can be derived from the optimal aggregate consumption, the aggregate private capital and the aggregate wealth chosen by the social planner[28]. After internalising externalities from the public good and government transfer, the certainty equivalent interest rate (r_c) is:

$$r_c = (1-\phi)\phi^{\frac{1-\alpha}{\alpha}}\alpha A^{\frac{1}{\alpha}} - \alpha\eta(1-\phi')^2(\phi A)^{\frac{2}{\alpha}}\sigma^2 \tag{4.51}$$

where r_c is also known as the social risk-adjusted after tax return on aggregate capital accumulation, and $\alpha\eta(1-\phi')^2(\phi A)^{\frac{2}{\alpha}}\sigma^2$ is the risk premium in this stochastic environment. If the certainty equivalent interest rate is applied to the economy, it is more likely that households will reduce their precautionary saving. The welfare loss would then never be incurred in a risky economy.

[27]According to Eurostat data, the standard deviation of government spending on economic affairs is smaller when the number of European countries increases. See Appendix C.3 for the calculation.

[28]See Appendix C.4 for the derivation.

4.6 Conclusion

The impact of fiscal policy on economic growth and social welfare was analysed through the stochastic endogenous growth model. In a decentralised economy, risk-averse households receive the government transfer that is generated by the stochastic process of productive government expenditure. Due to the volatility of government transfer, households will increase precautionary saving against the uncertain future income. This amount of precautionary saving can be explained by the domination of the income effect over the intertemporal substitution effect. As a result, the expected growth rate becomes larger than the growth rate in the perfect foresight economy.

According to the numerical result, the inverted U-shaped relationship between economic growth and permanent income tax disappears if the degree of risk aversion is sufficiently high. The argument that the growth rate is an increasing function of permanent income tax is justified by a high level of precautionary saving. However, the necessary condition for the growth enhancing permanent income tax rate that the marginal benefit of providing for the public good should be greater than the marginal cost remains valid in the stochastic context. On the other hand, the growth-enhancing condition for the percentage of shock absorption does not exist in this economy.

In contrast to Turnovsky's (1999) findings, the growth rates in centralised and decentralised economies are not the same. After internalising the volatility of government transfer and the externalities from the public good subject to proportional congestion, the social planner experiences the Cobb-Douglas production technology while individual firms observe the linear production technology in capital. Due to the exaggeration of capital return and the volatility of income, the saving and growth rates in a decentralised economy are too high. Therefore, the first-best fiscal instruments can be used to obtain the first-best outcomes. The government should decrease permanent income tax to encourage private consumption, but should increase the percentage of shock absorption to reduce the volatility of government transfer. To achieve the same welfare target, a decentralised economy should compensate for welfare loss by sacrificing the initial capital. Nevertheless, the welfare target in a risky economy will be lower than the welfare level in a riskless economy. Accordingly, the social planner can either sacrifice the initial capital to cover the welfare loss from risks, or apply the certainty equivalent interest rate.

Chapter 5

Conclusion

Productive government spending can be considered as a source of endogenous growth in one economy. This spending may take various forms, such as public capital, public education and the public good that incorporates with private capital in the production function. Even though the diminishing marginal return of each type of capital still exists, the combination of private capital and public capital will prevent the marginal product of aggregate capital from having zero rate of return. Despite the fact that an increase in productive government expenditure undoubtedly raises the output and economic growth, the magnitude of growth depends on how the government levies the distortionary taxation to finance its expenditure. A high level of distortionary tax rate will create a heavy tax burden and distort the households' decisions. Therefore, the compensation between the positive effect of productive government expenditure and the negative effect from distortionary taxation leads to the different states of economy that have been clarified throughout this thesis.

As discussed in the second chapter, the government provides public capital stock that is used in private production function, and levies procyclical consumption tax to finance public investment in each period regarding the balanced budget rule. The contribution of this chapter can be characterised by the feature of consumption tax that depends on the state-contingent variables and the time-invariant fiscal policy rule. Under certain conditions, it can be observed that public investment and procyclical consumption tax have a potential to generate the multiplicity of balanced growth paths (BGPs). Two BGPs were found from the evidence of a Laffer curve since there were two intersections of the detrended tax revenue curve and the detrended public investment line. Consequently, the economy can converge either to the lowest BGP or to the highest BGP when the inverse elasticity of intertemporal substitution and the degree of procyclicality are sufficiently large. As we are aware of the poverty trap and the aggregate instability, the local stability properties of the lowest BGP were studied. Assuming that the lowest BGP was sufficiently close to zero, the sufficient conditions for the local indeterminacy were carefully derived. Unlike the global indeterminacy case, the lowest BGP is locally indeterminate if, and only if, the consumption tax is mildly procyclical. This analytical derivation is also supported by the numerical example. By choosing a value of exogenous parameters that is consistent with the sufficient conditions, the numerical example shows that aggregate instability occurs around the lowest BGPs when a purely extrinsic uncertainty is introduced and the degree of procyclicality is not too large. Accordingly, policymakers should implement a procyclical consumption tax policy with caution. A possible extension of this chapter would be to check the robustness of the results when relaxing the balanced budget rule and introducing the public debt as another fiscal policy instrument.

The third chapter demonstrated how public education and labour income tax affected economic growth in the three-period overlapping generations model where impure altruistic parents provided private tuition to their children. The growth-maximising tax rate was explored first, since it could prevent the economy from entering into the growth-reducing area of the Armey curve. To determine the growth-maximising tax rate, the steady-state growth rate was investigated and written as a function of labour income tax. The first-order derivative of growth with respect to labour income tax is then calculated to prove the existence of growth-maximising tax rate. By studying the lower bound and the upper bound of the labour income tax rate within a feasible range between zero and one, at least one growth-maximising tax rate was found from the real roots of a cubic function of labour income tax. Furthermore, the second-order derivative confirmed that the roots of cubic function satisfied the sufficient condition for the local maximum. Nevertheless, it is extremely difficult to draw economic intuition from the formula for the growth-maximising tax rate due to the complexity of the roots.

The second finding of the third chapter was related to the over-provision of public education when the government misconceived of the ability of parents to provide private tuition for their children. Once again, the growth-maximising tax rate was evaluated in the economy that only had public education system. When there is a misperception concerning the existence of private tuition, the growth-maximising tax rate lies outside of its feasible range of value between zero and one. The corner solution exists, and it implies that the over-provision of public education arises if the government overestimates the benefits of public education. Therefore, policymakers should reallocate the funding of public education carefully by taking private tuition into account. Not only can this prevent the government from wasting resources on public education, it can also reduce the distortion created by labour income tax. Finally, it is worth mentioning that, in this paper, the discount factor and the degree of impure altruism are assumed to be the same for reasons of simplicity. Thus, the most direct extension of the second chapter would be to distinguish the degree of impure altruism from the discount factor in order to study its impact on the intergenerational transfers and the provision of public education.

In the fourth chapter, the stochastic endogenous growth model was employed to study the impact of productive government expenditure and fiscal instruments on economic growth and social welfare. A random shock was introduced into productive per-capita government expenditure, as it is a source of uncertainty in this kind of economy. In this scenario, the government would decide to devote the deterministic parts of productive spending to the public good that is subject to proportional congestion, and partially transfers the stochastic part to households. This type of government transfer creates the volatility in households' income paths. Given a stochastic setting, the economy will experience a higher growth rate than it will in the perfect foresight economy. Since households are aware of the uncertain future income flow, they will increase their precautionary saving to smooth their consumption paths. Thus, in our case, the income effect strictly dominates the intertemporal substitution effect. Moreover, the difference in the growth rates also challenges the concept of the inverted U-shaped relationship between economic growth and permanent income tax. Although the numerical example shows that the inverted U-shaped relationship disappears when the degree of risk aversion is sufficiently large, the condition of growth-enhancing permanent income tax remains valid. However, there is no growth-enhancing condition for the percentage of shock absorption regarding the concavity of the utility function.

In contrast to Turnovsky's (1999) work, the existence of the difference between centralised and decentralised economies can be demonstrated via the difference in growth rates. In the case of proportional congestion, the growth rate of a centralised economy is always smaller than the growth rate of a decentralised economy. The theoretical justification for this difference is the way that the social planner internalises the externalities from the public good that is subject to proportional congestion, but individual households and firms do not. Therefore, the social planner will choose a small growth rate that reduces the volatility of households' income streams in order to obtain the highest level of social welfare. Due to the difference between the economies, a decentralised government has an opportunity to employ the first-best fiscal instruments to achieve the first-best welfare target. To increase private consumption as well as social welfare, the government should reduce the permanent income tax rate, but increase the percentage of shock absorption. With regard to the welfare analysis, the magnitude of the welfare loss in a decentralised economy can be measured as an excess amount of the initial capital when compared to a centralised economy. This excess amount of the initial capital also indicates that the trade-off between economic growth and social welfare is unavoidable. In addition, the trade-off between risky and riskless economies in terms of social welfare can be illustrated by the variation in the initial capital. When the degree of risk aversion and the volatility of productive spending are large, a risky economy will suffer greatly from a loss of social welfare.

These three chapters justified the role of productive government expenditure and taxation as sources of endogenous growth in an economy. However, policymakers should apply the fiscal policy to enhance economic growth with caution, since it could lead to the multiplicity of balanced growth paths, aggregate instability, the over-provision of public education and the reduction of social welfare. Furthermore, the policy implications in the entire thesis has been suggested by considering the assumptions of the new classical school of thought, such as the rational expectation, and the flexibility of price and wage. Thus, our policy recommendations may not be applicable to an economy in which nominal rigidity occurs.

The limitation of this thesis can be categorised according to three main factors. The first factor is the assumption of a constant return to scale technology (CRTS) which is a knife-edge in this study. The property of a constant return on aggregate capital ensures that one economy will have a constant endogenous growth rate over time. However, the production technology may exhibit the increasing-return-to-scale (IRTS), or non-convex technology, as described by Romer (1986) and by Lucas (1988). Thus, the effect of productive government expenditure on economic growth in an economy with an increasing return on aggregate capital has been left for future work. The second factor is the lack of empirical evidence testing for the policy recommendations. As Solow (1994) criticised the endogenous growth model for being very 'un-robust', it may not be applicable to the empirical evidences across countries in which the constant return on aggregate capital may not occur. Therefore, another possible extension of this research is to have a robustness check of our policy suggestions. The final factor is the government's balanced budget rule, which is employed throughout all the chapters in this thesis. The criticism of the balanced budget rule is that a government may not balance its budget in each period since the economy is more painful during the economic downturn. For this reason, the constraint of the government balanced budget rule should be relaxed, or applied with caution.

Appendix A

A.1 Further details on how to obtain the system (2.13)-(2.14)

Equation (2.13) can be obtained as follows. Combining the firms' FOC with the intertemporal budget constraint of households and rewriting it in a form of new variables, $x \equiv \frac{g}{k}$ and $y \equiv \frac{c}{k}$, leads to:

$$\frac{\dot{k}}{k} = Ax^{1-\alpha} - \delta - (1+\tau)y = Ax^{1-\alpha} - \delta - (1+\tau_c x^{\eta})y$$

where the last equality is based on the fact that $\tau = \tau_c x^{\eta}$. Similarly, the government budget constraint can be rewritten as follows:

$$\frac{\dot{g}}{g} = \tau y x^{-1} - \delta = \tau_c y x^{\eta-1} - \delta$$

Since $\frac{\dot{x}}{x} = \frac{\dot{g}}{g} - \frac{\dot{k}}{k}$, it must be that

$$\frac{\dot{x}}{x} = (\tau_c x^{\eta-1} + 1 + \tau_c x^{\eta})y - Ax^{1-\alpha}$$

With regard to equation (2.14), the steps to obtain it are the following. Combining the Euler equation with the firms' FOCs and rewriting it in a form of new variables leads to

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left(\alpha A x^{1-\alpha} - \delta - \rho - \frac{\dot{\tau}}{1+\tau}\right)$$

From the fiscal policy (2.12), we have that

$$\log(1+\tau) = \log(1+\tau_c x^{\eta}) \qquad \Rightarrow \qquad \frac{\dot{\tau}}{1+\tau} = \frac{\tau_c \eta x^{\eta}}{1+\tau_c x^{\eta}}\frac{\dot{x}}{x}$$

Substituting this last expression into the Euler equation and considering that $\frac{\dot{y}}{y} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k}$, it gives

$$\frac{\dot{y}}{y} = \frac{1}{\sigma}\left[\alpha A x^{1-\alpha} - \delta - \rho - \frac{\tau_c \eta x^{\eta}}{1+\tau_c x^{\eta}}\frac{\dot{x}}{x}\right] - \left[Ax^{1-\alpha} - \delta - (1+\tau_c x^{\eta})y\right].$$

A.2 Further details on the proof of Proposition 1

A.2.1 Conditions for $\tilde{\mathcal{T}}(0) \geq \delta$

With regard to the conditions which emerge from imposing $\tilde{\mathcal{T}}(0) \geq \delta$, it can be observed that

$$\tilde{\mathcal{T}}(0) = \frac{\tau_c A}{\tau_c \Gamma^{\alpha-1} + \Gamma^{\alpha-\eta} + \tau_c \Gamma^{\alpha}} \geq \delta$$

Note that, the denominator is always strictly positive so we may rewrite the inequality as

$$\tau_c[A - (\Gamma^{\alpha-1} + \Gamma^{\alpha})\delta] \geq \delta \Gamma^{\alpha-\eta}$$

Since $\Gamma = \left(\frac{\delta+\rho}{\alpha A}\right)^{\frac{1}{1-\alpha}}$, it follows immediately that

$$A - (\Gamma^{\alpha-1} + \Gamma^{\alpha})\delta > 0 \qquad \Leftrightarrow \qquad A > \frac{\delta^{1-\alpha}(\delta+\rho)}{[(1-\alpha)\delta+\rho]^{1-\alpha}\alpha^{\alpha}} \equiv \underline{A}$$

Assuming $A > \underline{A}$, then $\tilde{\mathcal{T}}(0) \geq \delta$ if and only if

$$\tau_c \geq \frac{\delta \Gamma^{\alpha-\eta}}{A - (\Gamma^{\alpha-1} + \Gamma^{\alpha})\delta} \equiv \underline{\tau}_c$$

where $\underline{\tau}_c > 0$. On the other hand, the case $0 < A \leq \underline{A}$ is not admissible since it implies that $\tilde{\mathcal{T}}(0) \geq \delta$ only if Γ or δ are negative for any positive values of τ_c.

A.2.2 Existence of (at most) a critical point for $\tilde{\mathcal{T}}(\gamma)$

Second, $\tilde{\mathcal{T}}(\gamma)$ has a unique critical point. In fact, it can be shown that

$$\frac{d\tilde{\mathcal{T}}}{d\gamma} = \frac{d\tilde{\mathcal{T}}}{dx^*} \cdot \frac{dx^*}{d\gamma} = \underbrace{\frac{-A\tau_c \frac{dx^*}{d\gamma} x^{*\alpha-1}}{[\tau_c x^{*\alpha-1} + x^{*\alpha-\eta} + \tau_c x^{*\alpha}]^2}}_{<0} [\tau_c(\alpha-1)x^{*-1} + (\alpha-\eta)x^{*-\eta} + \tau_c \alpha]$$

and therefore

$$\frac{d\tilde{\mathcal{T}}}{d\gamma} = 0 \qquad \Leftrightarrow \qquad \underbrace{\tau_c(\alpha-1)x^{*-1} + (\alpha-\eta)x^{*-\eta} + \tau_c\alpha}_{\equiv g(x^*)} = 0$$

Now, there exists always a unique positive root $\hat{x}$ of $g(x^*) = 0$. This is a direct consequences of the following arguments. First, the function $g(x^*)$ is not continuous in $x^* = 0$ and has a critical point at $x^{\bullet} = \left(\frac{\eta(\alpha-\eta)}{\tau_c(1-\alpha)}\right)^{\frac{1}{\eta-1}}$. Second,

$$\lim_{x^* \to +\infty} g(x^*) = \begin{cases} (\tau_c\alpha)^- & if \;\; \eta > \alpha \\ (\tau_c\alpha)^+ & if \;\; 0 < \eta < \alpha \\ +\infty & if \;\; \eta < 0 \end{cases} \qquad and \qquad \lim_{x^* \to 0^+} g(x^*) = -\infty \;\; always$$

Combining this information, there is always the case that $g(x^*)$ intersects the x^*-axis only once. Since x^* is a one-to-one function of γ, we can confirm that a unique critical point $\hat{\gamma}$ of $\tilde{\mathcal{T}}(\gamma)$ can be found in the extended domain $\gamma > -\frac{\delta+\rho}{\alpha A\sigma}$.[1] Therefore, $\tilde{\mathcal{T}}(\gamma)$ has *at most* a critical point in the domain $\gamma \geq 0$.

A.2.3 Additional conditions for the case of $\sigma < 1$

Proposition 1 has been proved for the case that the inverse elasticity of intertemporal substitution is strictly greater than one ($\sigma > 1$). However, in the case of $\sigma < 1$, it is possible to show that a unique BGP can arise by adding two following conditions[2]:

a) $\sigma > 1 - \frac{\rho}{\gamma}$ for a bounded utility

b) $\sigma > \alpha$ for $c_0 > 0$

Proof of condition *a)***:** From the CRRA utility function and the definition of BGP, the utility function can be written as:

$$U = \int_0^\infty \mathrm{e}^{-\rho t} \cdot \frac{(c_0 \mathrm{e}^{\gamma t})^{1-\sigma} - 1}{1-\sigma} \, \mathrm{d}t \tag{A.1}$$

Executing the integral and omitting the constant part, the attainable utility is obtained as follow:

$$U = \frac{c_0^{1-\sigma}}{(1-\sigma)(\rho - \gamma(1-\sigma))} \cdot \mathrm{e}^{-(\rho-\gamma(1-\sigma))t} \tag{A.2}$$

To have a bounded utility, $\rho - \gamma(1-\sigma)$ should be positive which means that the value of σ is:

$$\sigma > 1 - \frac{\rho}{\gamma} \tag{A.3}$$

which is consistent to the condition *a)*[3].

Proof of condition *b)***:** According to the intertemporal equilibrium, the households' budget constraint can be written as:

$$\frac{\dot{k}}{k} = A\left(\frac{g}{k}\right)^{1-\alpha} - \delta - (1+\tau)\left(\frac{c}{k}\right) \tag{A.4}$$

[1] In fact, looking at equation (2.16) we have that $\gamma \to -\frac{\delta+\rho}{\alpha A\sigma}$ as $x^* \to 0^+$

[2] The case of $\sigma = 1$ was proved in the paper by Giannitsarou (2007) in which a unique steady state was found.

[3] This condition also satisfies the transversality condition:

$$\lim_{t\to+\infty} \frac{k_0}{c_0^{-\sigma}(1+\tau)} \cdot \mathrm{e}^{-(\rho-\gamma(1-\sigma))t} = 0$$

Using the definition of BGP and the fact from the consumption tax function, equation (A.4) will be

$$\gamma = A\left(\frac{g}{k}\right)^{1-\alpha} - \delta - \left(1 + \tau_c\left(\frac{g}{k}\right)^{\eta}\right)\left(\frac{c}{k}\right) \tag{A.5}$$

Evaluating the Euler equation; $\frac{\dot{c}}{c} = \frac{1}{\sigma}\left[\alpha A\left(\frac{g}{k}\right)^{1-\alpha} - \delta - \rho\right]$, at the BGP, we get

$$\frac{g}{k} = \left[\frac{\sigma\gamma + \delta + \rho}{\alpha A}\right]^{\frac{1}{1-\alpha}} \tag{A.6}$$

Substituting (A.6) into (A.5), the ratio of consumption to capital is

$$\frac{c}{k} = \frac{(\sigma - \alpha)\gamma + (1-\alpha)\delta + \rho}{\alpha(1 + \tau^{\bullet})} \tag{A.7}$$

where $\tau^{\bullet} = \tau_c\left[\frac{\sigma\gamma+\delta+\rho}{\alpha A}\right]^{\frac{\eta}{1-\alpha}}$. Suppose that this economy initially starts on the BGP, the initial value of consumption and the initial value of capital should satisfy the equation (A.7) always. Given a positive value of the initial capital ($k_0 > 0$), the initial value of consumption (c_0) will be

$$c_0 = \left[\frac{(\sigma - \alpha)\gamma + (1-\alpha)\delta + \rho}{\alpha(1 + \tau^{\bullet})}\right]k_0 \tag{A.8}$$

From equation (A.8), the initial value of consumption is positive ($c_0 > 0$) if, and only if, $\sigma > \alpha$. Therefore, the condition *b*) is proved.

Combining conditions *a*) and *b*) with the conditions from Proposition 1, a unique BGP exists in the case of $0 < \sigma < 1$ when all the following conditions are satisfied:

$$A > \underline{A}, \qquad \tau_c > \underline{\tau}_c, \qquad \sigma > 1 - \frac{\rho}{\gamma} \qquad \textit{and} \qquad \sigma > \alpha \tag{A.9}$$

where the additional conditions; $\sigma > 1 - \frac{\rho}{\gamma}$ and $\sigma > \alpha$, are always respected since we assume that $\sigma > 1$.

A.3 Proof of Proposition 2

Given the properties of the function $\tilde{\mathcal{T}}(\gamma)$ found in Proposition 1, two BGPs exist as long as the following two conditions hold:

a) $\delta - \varepsilon \leq \tilde{\mathcal{T}}(0) < \delta$, for any $\varepsilon > 0$ sufficiently small real number;

b) $\left.\frac{d\tilde{\mathcal{T}}(\gamma)}{d\gamma}\right|_{\gamma=0} > 1$

In fact, condition *a)* means that the curve $\tilde{\mathcal{T}}(\gamma)$ is slightly below the straight line $\gamma+\delta$ at $\gamma=0$ but it is steeper for condition *b)*. Therefore, the curve must intersect the straight line twice since $\tilde{\mathcal{T}}(\gamma)$ is continuous, has a unique critical point and $\lim_{\gamma\to\infty}\tilde{\mathcal{T}}(\gamma)=0^+$.

Step 1 – Parameters Conditions for a) to hold. Using the same argument of the proof of Proposition 1, it can be proved that for any given ε sufficiently small positive constant we have that $\tilde{\mathcal{T}}(0)\geq\delta-\varepsilon$ if $A>\underline{A}(\varepsilon)$ and $\tau_c\geq\underline{\tau}_c(\varepsilon)$ where $\underline{\tau}_c(\varepsilon)\equiv\frac{(\delta-\varepsilon)\Gamma^{\alpha-\eta}}{A-(\Gamma^{-1}+1)\Gamma^{\alpha}(\delta-\varepsilon)}$, $\underline{A}(\varepsilon)\equiv\frac{(\delta-\varepsilon)^{1-\alpha}(\delta+\rho)}{[(1-\alpha)\delta+\rho+\alpha\varepsilon]^{1-\alpha}\alpha^{\alpha}}$, $\underline{\tau}_c(\varepsilon)\leq\underline{\tau}_c$ and $\underline{A}(\varepsilon)\leq\underline{A}$ with equality when $\varepsilon=0$, as shown in Appendix A.4.1. In the same Appendix, we also show that $\epsilon\equiv\underline{\tau}_c-\underline{\tau}_c(\varepsilon)$ and ε are infinitesimals of the same order.[4]

Based on previous results, it is also the case that $\tilde{\mathcal{T}}(0)<\delta$ if $A>\underline{A}$ and $\tau_c<\underline{\tau}_c$. Summing up, condition *a)* always holds if

$$A>\underline{A}\qquad \textit{and}\qquad \underline{\tau}_c(\varepsilon)=\underline{\tau}_c-\epsilon<\tau_c<\underline{\tau}_c.$$

Step 2 – Parameters Conditions for b) to hold. Taking into account Appendix A.2.2, we have that

$$\left.\frac{d\tilde{\mathcal{T}}(\gamma)}{d\gamma}\right|_{\gamma=0}>1\;\Leftrightarrow\;\frac{-A\tau_c\left.\frac{dx^*}{d\gamma}\right|_{\gamma=0}\Gamma^{\alpha-1}}{[\tau_c\Gamma^{\alpha-1}+\Gamma^{\alpha-\eta}+\tau_c\Gamma^{\alpha}]^2}[\tau_c(\alpha-1)\Gamma^{-1}+(\alpha-\eta)\Gamma^{-\eta}+\tau_c\alpha]>1$$

Given that $\left.\frac{dx^*}{d\gamma}\right|_{\gamma=0}=\frac{\sigma\Gamma^{\alpha}}{\alpha(1-\alpha)A}$, the last inequality can be rewritten as follows:

$$\frac{\tau_c\frac{\sigma}{\alpha(1-\alpha)}\Gamma^{-1}}{[\tau_c\Gamma^{-1}+\Gamma^{-\eta}+\tau_c]^2}\{\tau_c[(1-\alpha)\Gamma^{-1}-\alpha]+(\eta-\alpha)\Gamma^{-\eta}\}>1\tag{A.10}$$

Then, condition *b)* holds as long as this inequality is satisfied. Clearly, the inequality is never satisfied if the term inside the curly brackets is negative. To avoid that, we look for condition on τ_c such that

$$\eta-\alpha+\tau_c[(1-\alpha)\Gamma^{-1}-\alpha]\Gamma^{\eta}>0\tag{A.11}$$

We need to distinguish two cases.

Case 1: $(1-\alpha)\Gamma^{-1}-\alpha>0$ which is indeed the case when $A>\bar{A}\equiv\frac{\delta+\rho}{(1-\alpha)^{1-\alpha}\alpha^{\alpha}}$.[5] In this case, (A.11) implies

$$\tau_c>\frac{\alpha-\eta}{[(1-\alpha)\Gamma^{-1}-\alpha]\Gamma^{\eta}}\equiv\bar{\tau}_c$$

[4]The infinitesimals have the same order if their speed of convergence toward zero is the same. This is indeed shown in Appendix A.4.1.

[5]This last inequality and the value of $\bar{A}$ can be found easily by combining $(1-\alpha)\Gamma^{-1}-\alpha>0$ with the definition of Γ given in Proposition 1.

Once this inequality is imposed, it follows that (A.10) holds as long as $\sigma > \underline{\sigma}$. Subsequently, condition *b)* in case 1 is satisfied when

$$A > \bar{A}, \qquad \tau_c > \bar{\tau}_c \qquad \textit{and} \qquad \sigma > \underline{\sigma}.$$

Case 2: $(1-\alpha)\Gamma^{-1} - \alpha < 0$ which is indeed the case when $A < \bar{A}$. In this case, (A.11) implies

$$\tau_c < \frac{\alpha - \eta}{[(1-\alpha)\Gamma^{-1} - \alpha]\Gamma^{\eta}} \equiv \bar{\tau}_c$$

Once this inequality is imposed, it follows that (A.10) holds as long as $\sigma > \underline{\sigma}$. Then, condition *b)* in case 2 is satisfied when

$$A < \bar{A}, \qquad \tau_c < \bar{\tau}_c \qquad \textit{and} \qquad \sigma > \underline{\sigma}.$$

Step 3 – Combining Steps 1 and 2. The following inequalities are proved in Appendix A.4.2:

- $\underline{A} < \bar{A}$ always;
- if $A > \bar{A}$ and $\eta > \underline{\eta}$ then $\bar{\tau}_c < \underline{\tau}_c$;
- if $A < \bar{A}$ and $\eta > \underline{\eta}$ then $\bar{\tau}_c > \underline{\tau}_c$;

Taking into account these results, both conditions *a)* and *b)* - case 1 holds if

$$A > \bar{A}, \qquad \eta > \underline{\eta}, \qquad \underline{\tau}_c - \epsilon < \tau_c < \underline{\tau}_c, \qquad \textit{and} \;\; \sigma > \underline{\sigma}. \tag{A.12}$$

On the other hand, both conditions *a)* and *b)* - case 2 holds if

$$\underline{A} < A < \bar{A}, \qquad \eta > \underline{\eta}, \qquad \underline{\tau}_c - \epsilon < \tau_c < \underline{\tau}_c, \qquad \textit{and} \;\; \sigma > \underline{\sigma}. \tag{A.13}$$

But then, it follows immediately that conditions *a)* and *b)* hold when (2.21) is satisfied. Once again, if the condition $\eta > \underline{\eta}$ is replaced by $\eta > \alpha$, the result of case 1 remains unchanged since equation (A.11) is respected for any choices of τ_c.

A.4 Further details on the proof of Proposition 2

A.4.1 Details for step 1

We prove that $\underline{A} \geq \underline{A}(\varepsilon)$ by contradiction. Suppose that $\underline{A} < \underline{A}(\varepsilon)$ then it follows that

$$\underline{A} \equiv \frac{(\delta+\rho)\delta^{1-\alpha}}{[\rho + (1-\alpha)\delta]^{1-\alpha}\alpha^{\alpha}} < \frac{(\delta+\rho)(\delta-\varepsilon)^{1-\alpha}}{[\rho + (1-\alpha)\delta + \alpha\varepsilon]^{1-\alpha}\alpha^{\alpha}} \equiv \underline{A}(\varepsilon).$$

Simplifying these expressions, the inequality boils down to $-\varepsilon(\rho+\delta) > 0$ which is clearly impossible since $\varepsilon, \delta, \rho$ are positive.

We prove now by contradiction that $\underline{\tau}_c \geq \underline{\tau}_c(\varepsilon)$. Suppose that $\underline{\tau}_c < \underline{\tau}_c(\varepsilon)$, it then follows that

$$\underline{\tau}_c(\varepsilon) \equiv \frac{\delta - \varepsilon}{A - (\Gamma^{\alpha-1}) + \Gamma^{\alpha}\delta} > \frac{\delta}{A - (\Gamma^{\alpha-1}) + \Gamma^{\alpha}\delta} \equiv \underline{\tau}_c$$

After the simplification, the inequality boils down to $-\varepsilon A > 0$ which is clearly impossible due to the positive value of ε and A.

Finally, we want to show that $\epsilon \equiv \underline{\tau}_c - \underline{\tau}_c(\varepsilon)$ and ε are infinitesimals of the same order. To do so, we need to show that $\lim_{\varepsilon\to 0} \frac{\underline{\tau}_c - \underline{\tau}_c(\varepsilon)}{\varepsilon}$ is a positive constant. Since the argument of the limit has the indeterminate form $\frac{0}{0}$, the Hopital's rule is then applied and we find that

$$\lim_{\varepsilon\to 0} \frac{\underline{\tau}_c - \underline{\tau}_c(\varepsilon)}{\varepsilon} = \lim_{\varepsilon\to 0} -\tau_c'(\varepsilon) = \frac{\Gamma^{\alpha-\eta}A}{[A - (\Gamma^{-1}+1)\delta\Gamma^{\alpha}]^2} > 0.$$

A.4.2 Details for step 3

We want here to show that $\underline{A} < \bar{A}$. In fact,

$$\underline{A} < \bar{A} \qquad \Leftrightarrow \qquad \frac{\delta^{1-\alpha}(\delta+\rho)}{[(1-\alpha)\delta+\rho]^{1-\alpha}\alpha^{\alpha}} < \frac{\delta+\rho}{(1-\alpha)^{1-\alpha}\alpha^{\alpha}} \qquad \Leftrightarrow \qquad \rho > 0.$$

Moreover, we want to find conditions such that $\underline{\tau}_c < \bar{\tau}_c$. Taking into account the parameters' restrictions, we observe that,

$$\underline{\tau}_c \equiv \frac{\delta\Gamma^{\alpha-\eta}}{A - (\Gamma^{-1}+1)\Gamma^{\alpha}\delta} < \frac{\alpha-\eta}{[(1-\alpha)\Gamma^{-1}-\alpha]\,\Gamma^{\eta}} \equiv \bar{\tau}_c \tag{A.14}$$

From Appendix A.2.1, we know that $A > \underline{A}$ implies $A - (\Gamma^{-1}+1)\Gamma^{\alpha}\delta > 0$; let us first focus on case 1, i.e. $A > \bar{A}$ and then $(1-\alpha)\Gamma^{-1} - \alpha > 0$, condition (A.14) holds if and only if

$$\eta < \underline{\eta} \equiv \frac{\rho\alpha A}{A - (\Gamma^{-1}+1)\Gamma^{*\alpha}\delta} \equiv \underline{\eta}$$

On the other hand, on case 2, i.e. $A < \bar{A}$ and then $(1-\alpha)\Gamma^{-1} - \alpha < 0$, condition (A.14) holds if and only if $\eta > \underline{\eta}$. Combining these results leads to the last two inequalities listed in step 3 of Proposition 2.

A.5 Further details on det(J) and tr(J)

A.5.1 Determinant of Jacobian matrix (det(J))

From the system of differential equations, the determinant of Jacobian matrix can be found from its components: a, b, c and d.

$$det(J) = ad - bc$$

Recall the value of a, b, c and d, the determinant of Jacobian matrix (det(J)) is written as

$$\begin{aligned} det(J) = a \cdot y^* &\left(1 + \tau_c x^{*\eta} - \frac{b}{\sigma} \cdot \frac{\tau_c \eta}{x^* + \tau_c x^*}\right) \\ - b \cdot y^* &\left[-\left(\frac{1}{\sigma}(1-\alpha)(\sigma - \alpha)Ax^{*-\alpha} + \frac{\tau_c \eta}{x^{*1-\eta} + \tau_c x^*} \cdot a\right) + \tau_c \eta x^{*\eta-1} y^*\right] \end{aligned}$$

Rewriting component a as a function of b gives:

$$a = [\eta(b - x) - \tau_c x^{*\eta}]\, y^* x^{*-1} - (1-\alpha)Ax^{*1-\alpha}$$

Substituting a into the equation, the determinant of Jacobian matrix can be rewritten in term of b.

$$\begin{aligned} det(J) = y^* &\left(\tau_c x^{*\eta-1} y^* (\eta - 1 - \tau_c x^{*\eta})\right) \\ &- (1-\alpha)Ax^{*1-\alpha} y^* \left(1 + \tau_c x^{*\eta} - bx^{*-1} - \frac{b\alpha}{\sigma} x^{*-1}\right) \end{aligned}$$

Extracting b out, det(J) will be

$$\begin{aligned} det(J) = y^* &\left(\tau_c x^{*\eta-1} y^* (\eta - 1 - \tau_c x^{*\eta})\right) \\ &- (1-\alpha)Ax^{*1-\alpha} y^* \left(-\tau_c x^{*\eta-1} + \frac{\alpha}{\sigma}(\tau_c x^{*\eta-1} + 1 + \tau_c x^{*\eta})\right) \end{aligned}$$

Using the fact from BGP, it must be that

$$y^* = \frac{Ax^{*1-\alpha}}{\tau_c x^{*\eta-1} + 1 + \tau_c x^{*\eta}}$$

Fractioning y^* out and do some algebraic manipulations, the Jacobian determinant can be obtained as follows:

$$\begin{aligned} det(J) = y^{*2} x^{*2\eta} &\left\{\tau_c x^{*-1}[(1-\alpha)\tau_c x^{*-1} - \alpha\tau_c + (\eta - \alpha)x^{*-\eta}]\right\} \\ &- y^{*2} x^{*2\eta} \left\{\frac{(1-\alpha)\alpha}{\sigma}(\tau_c x^{*-1} + x^{*-\eta} + \tau_c)^2\right\} \end{aligned}$$

A.5.2 Trace of Jacobian matrix (tr(J))

The Trace of Jacobian Matrix (tr(J)) can be calculated from the summation of the diagonal components of Jacobian Matrix.

$$tr(J) = a + d$$

That is,

$$\begin{aligned} tr(J) = {} & (\tau_c(\eta-1)x^{*\eta-1} + \tau_c\eta x^{*\eta}) \cdot y^* - (1-\alpha)Ax^{*1-\alpha} \\ & + y^* \left[1 + \tau_c x^{*\eta} - \frac{1}{\sigma} \cdot \frac{\tau_c\eta}{x^{*1-\eta} + \tau_c x^*} \cdot b\right] \end{aligned}$$

Extracting b out gives:

$$\begin{aligned} tr(J) = {} & (\tau_c(\eta-1)x^{*\eta-1} + \tau_c\eta x^{*\eta}) \cdot y^* - (1-\alpha)Ax^{*1-\alpha} \\ & + y^* \left[1 + \tau_c x^{*\eta} - \frac{1}{\sigma} \cdot \frac{\tau_c\eta}{x^{*1-\eta} + \tau_c x^*}(\tau_c x^{*\eta} + x^* + \tau_c x^{*\eta+1})\right] \end{aligned}$$

Adding and Subtracting $(1-\alpha)[\tau_c x^{*\eta-1} + 1 + \tau_c x^{*\eta}]y^*$ on the right hand side, the value of tr(J) will be

$$\begin{aligned} tr(J) = {} & y^* \left[\tau_c(\eta-1)x^{*\eta-1} + \tau_c\eta x^{*\eta} - \frac{1}{\sigma}\frac{\tau_c\eta}{x^{*1-\eta} + \tau_c x^*}(\tau_c x^{*\eta} + x^* + \tau_c x^{*\eta+1})\right] \\ & + (1 + \tau_c x^{*\eta})y^* - (1-\alpha)Ax^{*1-\alpha} + (1-\alpha)(\tau_c x^{*\eta-1} + x^* + \tau_c x^{*\eta}) \\ & - (1-\alpha)(\tau_c x^{*\eta-1} + x^* + \tau_c x^{*\eta}) \end{aligned}$$

Using the fact that $\frac{\dot{x}}{x} = 0$ on the BGP, the trace of Jacobian Matrix can be rewritten as follows:

$$\begin{aligned} tr(J) = {} & y^*\{\tau_c x^{*\eta}[(\eta - 2 + \alpha)x^{*-1} + \eta + \alpha] + \alpha\} \\ & - y^* \left\{\frac{1}{\sigma} \cdot \frac{\tau_c\eta(\tau_c x^{*\eta-1} + 1 + \tau_c x^{*\eta})}{x^{*-\eta} + \tau_c}\right\} \end{aligned}$$

A.6 Proof of Lemma 1

From step 2 of Proposition 2, we know that $(1-\alpha)\Gamma^{-1} - \alpha > 0$ if, and only if, $A > \bar{A}$. Based on that, condition i) and ii) hold immediately.

On the other hand, tr$(J) < 0$ if the sum of the first two terms within the curly parenthesis is negative:

$$x^{*\eta}\tau_c \left[(\eta + \alpha - 2)x^{*-1} + \eta + \alpha\right] + \alpha < 0 \tag{A.15}$$

Clearly, this never happens if the term in the square brackets is positive. Therefore, let us consider the case when it is negative. This is indeed possible when, for example, $-\alpha < \eta < 2-\alpha$ and $\Gamma < \frac{2-\eta-\alpha}{\eta+\alpha}$. The last condition can be rewritten in terms of A by taking into account the definition of Γ and leads to $A > \hat{A}$. Under these assumptions on the parameters, the inequality (A.15) is respected when $\tau_c > \hat{\tau}_c$, and therefore $\mathrm{tr}(J) < 0$.

A.7 Proof of Proposition 3

To have local indeterminacy, it requires that $\det(J) \geq 0$. We begin combining the conditions on the parameters $(A, \eta, \tau_c, \sigma)$ which guarantee the sign of the determinant with those for multiple BGPs. From step 3 of the proof of Proposition 2, we know that $\underline{A} < \bar{A}$ and therefore the resulting condition on A for having $\det(J) \geq 0$ and multiple BGPs is $A > \bar{A}$. Accordingly, the conditions on τ_c and η must hold for having multiple BGPs since $\det(J) \geq 0$ independently on their values. Finally, we have $\sigma > \underline{\sigma}$. Considering all conditions of the parameters, we observe

$$A > \bar{A}, \qquad \eta > \alpha, \qquad \tau_c = \underline{\tau}_c, \quad \textit{and} \quad \sigma > \hat{\sigma} \tag{A.16}$$

We now need to combine these inequalities with the conditions for having $\mathrm{tr}(J) < 0$. Let us begin with the condition on A. It is obvious that $A > \max\{\bar{A}, \hat{A}\}$ where $\hat{A} > \bar{A}$ if $\eta > \alpha$. In fact, it can be shown that

$$\hat{A} \equiv \frac{\delta+\rho}{\alpha}\left(\frac{\eta+\alpha}{2-\eta-\alpha}\right)^{1-\alpha} > \frac{\delta+\rho}{\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \equiv \bar{A} \tag{A.17}$$

which is satisfied as long as $\frac{\eta+\alpha}{2-\eta-\alpha} > \frac{\alpha}{1-\alpha}$. It is indeed always the case when $\eta > \alpha$; the condition for $det(J) > 0$. Additionally, the level of technology should be positive ($\hat{A} > 0$). Therefore, $\frac{\eta+\alpha}{2-\eta-\alpha}$ will be positive if $-\alpha < \eta < 2-\alpha$.

Moving to the conditions on τ_c, we have $\tau_c \leq \underline{\tau}_c$ for global indeterminacy and $\tau_c > \hat{\tau}_c$ for local indeterminacy. Taking into account the definition of $\hat{\tau}_c$ and $\underline{\tau}_c$ we have that

$$\hat{\tau}_c \equiv \frac{\alpha}{\Gamma^{\eta}[\Gamma^{-1}(2-\eta-\alpha)-\eta-\alpha]} < \frac{\delta\Gamma^{\alpha-\eta}}{A-(\Gamma^{-1}+1)\Gamma^{\alpha}\delta} \equiv \underline{\tau}_c$$

After some simplifications, it emerges that such inequalities lead to

$$\alpha A < -\delta\Gamma^{\alpha}\left[\Gamma^{-1}(\eta-2)+\eta\right]$$

and solving for η, we obtain that

$$\eta < \frac{\alpha A(\delta-\rho)}{\delta(\alpha A+(\delta+\rho)\Gamma^{\alpha})} = \frac{\delta-\rho}{\delta(1+\Gamma)} \equiv \eta^{\circ} \tag{A.18}$$

here the last equality is obtained by dividing both sides of the right hand side of the inequality by αA and using the definition of Γ.

Regarding all conditions on η, it is restricted with respect to condition A.13, A.16 and A.18. Let us consider the case $\rho \to 0^+$; in this case $\underline{\eta} \to 0^+$ and then, by continuity, we have that $\exists \epsilon > 0 : \forall \rho \in (0, \epsilon)$. Consequently, the resulting condition on η that $\alpha < \eta < 2 - \alpha$ definitely satisfies conditions A.13 and A.16. To place η° between α and $2 - \alpha$, $\alpha < \eta^\circ$ leads to another condition for A;

$$A > \frac{(\delta + \rho)\delta^{1-\alpha}}{[(1-\alpha)\delta - \rho]^{1-\alpha}\alpha^\alpha} \equiv A^\bullet \tag{A.19}$$

Comparing $\hat{A}$ and $A^\bullet$, it can be easily shown that

$$A^\bullet \equiv \frac{(\delta + \rho)\delta^{1-\alpha}}{[(1-\alpha)\delta - \rho]^{1-\alpha}\alpha^\alpha} < \frac{\delta + \rho}{\alpha}\left(\frac{\eta + \alpha}{2 - \eta - \alpha}\right)^{1-\alpha} \equiv \hat{A} \tag{A.20}$$

Assuming $\rho < \delta$, this inequality is equivalent to

$$\eta > \alpha \cdot \frac{\delta + \rho}{\delta - \rho}$$

which clearly implies $\eta > \alpha$, as $\rho \to 0^+$. Finally, combining the resulting inequalities together with a sufficient condition ($\sigma > \underline{\sigma}$) leads to a condition (2.27).

Appendix B

B.1 Economy with both education systems

B.1.1 Local maximum condition

The local maximum condition for the labour income tax rate (τ_w) can be derived by taking the second-order derivative of the steady state growth rate ($1 + g^*$) with respect to τ_w.

$$\frac{\partial^2(1+g^*)}{\partial \tau_w^2} = \frac{(1-\alpha)}{1-\alpha\phi} \cdot \left\{ \left(\frac{\partial \Omega}{\partial \tau_w} + \frac{\partial \Lambda}{\partial \tau_w} \right) + (\Omega + \Lambda) \frac{\partial(1+g^*)}{\partial \tau_w} \right\}. \tag{B.1}$$

where $\frac{\partial(1+g^*)}{\partial \tau_w}$ is given from the first-order condition (3.34).

Simplifying equation (B.1), we will get the second-order condition that indicates whether it is a local maximum or a local minimum

$$\frac{\partial^2(1+g^*)}{\partial \tau_w^2} = \frac{(1-\alpha)(1+g^*)}{1-\alpha\phi} \cdot \left[\frac{\partial \Omega}{\partial \tau_w} + \frac{\partial \Lambda}{\partial \tau_w} + \frac{(\Omega+\Lambda)^2(1-\alpha)}{1-\alpha\phi} \right] \tag{B.2}$$

which depends on the value of the square bracket.

B.2 Economy with misperception of private tuition

B.2.1 The model setup

In this economy, the government misconceives of the existence of private tuition. Therefore, the human capital accumulation function consists of only two inputs; public education (E_t) and parental education (h_t)

$$h_{t+1}^E = B E_t^\mu h_t^{\mu+\phi} \tag{B.3}$$

where it can be compared with (3.13) when $\varphi = 0$. Despite the misperception of government, the optimal choices of households remain unchanged. Thus, equation (3.16), (3.17), (3.18) and (3.19) are still valid for this analysis.

At the equilibrium, the decision of giving bequests is affected by changing in the rental rate and the wage rate. The amount of bequests is determined by

$$b^E_{t+1} = \frac{1}{2} \cdot [\alpha - (1-\tau_w)(1-\alpha)]\, A(k^E_{t+1})^{\alpha}(h^E_{t+1})^{1-\alpha}. \tag{B.4}$$

Substituting the market clearing condition ($s_t = k_{t+1}$), the adult and old constraints into (3.16), the Euler equation can be rewritten as

$$(1+r_{t+1})(1+\beta)k_{t+1} = \beta(1+r_{t+1})[(1-\tau_w)w_t h_t + b_t] + b_{t+1}. \tag{B.5}$$

Substituting (3.9), (3.10) and (B.4) into (B.5), it can be shown that the future physical capital is a function of current income

$$k^E_{t+1} = \frac{[(1-\tau_w)(1-\alpha)+\alpha]\beta\alpha}{\alpha(1+2\beta)+(1-\tau_w)(1-\alpha)} y^E_t \tag{B.6}$$

which is represented in equation (3.40). For the human capital function, it can be found by substituting the government budget constraint (3.11) and the wage rate (3.10) into (B.3). The future human capital is also a function of current income

$$h_{t+1} = B\tau_w^{\mu}(1-\alpha)^{\mu}(y^E)^{\mu} \tag{B.7}$$

which is the same as equation (3.41).

B.2.2 Local maximum condition for the economy with only public education

Considering the first-order derivative of gross growth rate with respect to the labour income tax rate, the necessary condition for the local maximum (or the local minimum) can be obtained by setting the first-order derivative at zero

$$\frac{\partial(1+g^E)}{\partial\tau_w} = \frac{\mu(1+g^E)(1-\alpha)}{1-\alpha\phi}\left[\frac{1}{\tau_w} - \frac{2\alpha^2\beta}{\Psi}\right] = 0 \tag{B.8}$$

where $\frac{\mu(1+g^E)(1-\alpha)}{1-\alpha\phi} > 0$ due to the constraint of parameters. Therefore, the square brackets should be equal to zero to make equation (B.8) holds.

$$\frac{1}{\tau_w} - \frac{2\alpha^2\beta}{\Psi} = 0 \tag{B.9}$$

Taking the second-order derivative of the gross growth rate with respect to the labour income tax rate gives:

$$\begin{aligned}\frac{\partial^2(1+g^E)}{\partial\tau_w} &= \frac{\mu(1-\alpha)(1+g^E)}{1-\alpha\phi}\left[-\frac{1}{\tau_w^2} + \frac{2\alpha^2\beta}{\Psi}\frac{\partial\Psi}{\partial\tau_w}\right] \\ &\quad + \frac{\mu(1-\alpha)}{1-\alpha\phi}\left[\frac{1}{\tau_w} - \frac{2\alpha^2\beta}{\Psi}\right]\frac{\partial(1+g^E)}{\partial\tau_w}\end{aligned} \tag{B.10}$$

The second term on the right hand side of equation (B.10) is eliminated by using the fact from the necessary condition (B.8). Thus, the sufficient condition for the local maximum is finally obtained

$$\frac{\partial^2(1+g^E)}{\partial\tau_w^2} = \frac{\mu(1-\alpha)(1+g^E)}{1-\alpha\phi}\left[-\frac{1}{\tau_w^2} + \frac{2\alpha^2\beta}{\Psi}\frac{\partial\Psi}{\partial\tau_w}\right] \tag{B.11}$$

which is similar to equation (3.51).

B.2.3 Proof of the over-provision of public education

Considering the growth-maximising tax rate in the economy that has only public education (τ_{w1}^E), it can be proved by the contradiction that this growth-maximising tax rate lies outside the range of zero and one. Setting τ_{w1}^E from (3.48) to be less than one gives:

$$\alpha\beta + 1 - \alpha - \alpha\sqrt{\beta(\beta + 2(1-\alpha))} < (\alpha - 1)^2. \tag{B.12}$$

Expanding (B.12) and cancelling the same variables in both sides, we get

$$\beta + 1 - \alpha < \sqrt{\beta(\beta + 2(1-\alpha))} \tag{B.13}$$

where $\beta(\beta+2(1-\alpha))$ is always greater than zero due to the positive parametrical values. Since both sides are positive, squaring both sides does not alter the sign of inequality in (B.13).

$$(1-\alpha)^2 + 2(1-\alpha)\beta + \beta^2 < \beta^2 + 2(1-\alpha)\beta$$

Rearranging the terms, the inequality expression becomes

$$(1-\alpha)^2 < 0$$

which is impossible due to $\alpha \in (0, 1)$. Therefore, the value of τ_{w1}^E is proved by the contradiction such that τ_{w1}^E is greater than one ($\tau_{w1}^E > 1$).

Since $\tau_{w1}^E < \tau_{w2}^E$, it can be concluded that both growth-maximising tax rates lie outside the range (0,1). Therefore, the best feasible tax rate that the government should levy is at 1 or at 100%. In other words, the economy has a corner solution when the government aims to maximise the growth rate in this economy without considering the existence of private tuition.

Appendix C

C.1 Decentralised economy

C.1.1 Value function

Considering the value function from the Bellman equation gives:

$$V(k,t) = \max_c \left\{ U(c,t) + \frac{1}{1+\rho dt} \cdot E(t)V(k,t+1) \right\} \tag{C.1}$$

Multiplying by $1 + \rho dt$ in both sides, the Bellman equation will be

$$(1+\rho dt)V(k,t) = \max_c \{(1+\rho dt)U(c,t) + E(t)V(k,t+1)\} \tag{C.2}$$

Since the utility is CRRA function (4.1) and $(dt)^2 \approx 0$, (C.2) is rewritten as

$$\rho dt V(k,t) = \max_c \{U(c,t) + E(dV)\} \tag{C.3}$$

Using the Taylor's expansion up to second order, it implies the application of Ito's Lemma.

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial k}dk + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(dt)^2 + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(dk)^2 \tag{C.4}$$

Due to the fact that $(dt)^2 \approx 0$, it can be shown that

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial k}dk + \frac{1}{2}\frac{\partial^2 V}{\partial t^2}(dk)^2 \tag{C.5}$$

Substituting equation (4.2) into (C.5), it becomes

$$\begin{aligned} dV = \frac{\partial V}{\partial t}dt &+ V'(k)\left[\left((1-\phi)Rk - c\right)dt + \frac{T}{N}\right] \\ &+ \frac{1}{2}V''(k)\left[\left((1-\phi)Rk - c\right)^2 (dt)^2\right] \\ &+ \frac{1}{2}V''(k)\left[2\left((1-\phi)Rk - c\right)dt \cdot \frac{T}{N} + \left(\frac{T}{N}\right)^2\right] \end{aligned} \tag{C.6}$$

where $T = (1-\theta)G_k \sigma du$.

Using the fact that $(dt)^2 \approx 0$, $dt \cdot du \approx 0$ and $(du)^2 = dt$, equation (C.1.1) can be rewritten as

$$
\begin{aligned}
dV = \frac{\partial V}{\partial t} dt + V'(k) \left[((1-\phi)Rk - c)\, dt + (1-\theta)\frac{G_k}{N}\sigma du \right] \\
+ \frac{1}{2} V''(k) \left[(1-\theta)\frac{G_k}{N}\sigma \right]^2 dt.
\end{aligned}
\tag{C.7}
$$

Taking the expectation, the mean of a random shock is equal to zero ($E(du) = 0$) regarding the Wiener process.

$$
\begin{aligned}
E(dV) = \frac{\partial V}{\partial t} dt + V'(k) \left[((1-\tau_y)Rk - c)\, dt \right] \\
+ \frac{1}{2} V''(k) \left[(1-\theta)\frac{G_k}{N}\sigma \right]^2 dt
\end{aligned}
\tag{C.8}
$$

Substituting equation (C.8) and the CRRA utility function into (C.3) gives us:

$$
\begin{aligned}
\rho dt V(k,t) = \max_c \Bigg\{ \int_0^\infty \mathrm{e}^{-\rho t} \cdot \frac{c(t)^{1-\eta} - 1}{1-\eta} dt + \frac{\partial V}{\partial t} dt \\
+ V'(k) \left[((1-\tau_y)Rk - c)\, dt \right] + \frac{1}{2} V''(k) \left[(1-\theta)\frac{G_k}{N}\sigma \right]^2 dt \Bigg\}
\end{aligned}
\tag{C.9}
$$

Dividing by dt and using the fact that $V(k,t) = e^{-\rho t} V(k)$, we get

$$
\begin{aligned}
\rho V(k) = \frac{c(t)^{1-\eta} - 1}{1-\eta} - \rho V(k) + V'(k) \left[((1-\tau_y)Rk - c) \right] \\
+ \frac{1}{2} V''(k) \left[(1-\theta)\frac{G_k}{N}\sigma \right]^2
\end{aligned}
\tag{C.10}
$$

which is known as the stochastic Bellman equation.

C.1.2 Ratio of productive government expenditure to private capital

The government will devote all productive government expenditure to the public good which is financed by the permanent income tax rate (τ_y). Recall that the aggregate productive government expenditure (G_k) is proportional to the aggregate income (Y).

$$
G_k = \tau_y Y
$$

Substituting out Y by using the production function gives us:

$$
G_k = \tau_y A K^{\alpha} G_k^{1-\alpha} \tag{C.11}
$$

Dividing equation (C.11) by K, the ratio of productive government expenditure to private capital will depend on the permanent income tax rate (τ_y), the capital elasticity of income (α) and the constant technology (A).

$$\frac{G_k}{K} = (\tau_y A)^{\frac{1}{\alpha}} \tag{C.12}$$

This means that the the ratio of productive government expenditure to private capital can be treated as a constant, as described in the paper by Barro (1990).

C.1.3 Derivation of optimal consumption

According to the optimal amount of consumption (4.5) and the constant propensity to consume capital (4.21), the first-order condition of consumption can be rewritten as

$$V'(k) = (\lambda k)^{-\eta} \tag{C.13}$$

which implies

$$V''(k) = \frac{-\eta(\lambda k)^{-\eta}}{k} \tag{C.14}$$

$$V'''(k) = \frac{\eta(\eta+1)(\lambda k)^{-\eta}}{k^2} \tag{C.15}$$

Substituting equation (C.13), (C.14) and (C.15) into the first-order condition of capital (4.6) gives:

$$(1-\tau_y)R - \rho - \eta\left[(1-\tau_y)Rk - c\right] + \frac{\eta(\eta+1)}{2}\left[(1-\theta)\left(\frac{G_k}{N}\right)\sigma\right]^2 = 0 \tag{C.16}$$

Substituting rental rate (4.12), the market clearing condition and the ratio of aggregate productive government expenditure to private capital (C.12) into (C.16) and dividing by η, the closed-form solution for the equilibrium consumption is obtained, and it is determined solely by capital. Indeed, one can observe that the consumption-capital ratio is a constant in this stochastic economy.

$$\begin{aligned}\frac{c(t)}{k(t)} &= \frac{\rho}{\eta} - \frac{1}{\eta}(1-\tau_y)\tau_y^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} \\ &\quad + (1-\tau_y)\tau_y^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} - \left[\frac{\eta+1}{2}\right]\left[(1-\theta)(\tau_y A)^{\frac{1}{\alpha}}\sigma\right]^2\end{aligned} \tag{C.17}$$

where $\frac{c(t)}{k(t)} = \frac{C(t)}{K(t)}$ since $C = Nc$ and $K = Nk$.

C.2 Centralised economy

The social planner will maximise households' utility

$$U(C/N) = E_0 \int_0^\infty \mathrm{e}^{-\rho t} \cdot \frac{(C/N)^{1-\eta} - 1}{1-\eta} \, \mathrm{d}t \tag{C.18}$$

subject to the resource constraint, which can be written as a form of the stochastic aggregate capital accumulation

$$dK = [(1-\phi)AK^\alpha G_k^{1-\alpha} - C] \cdot dt + [(1-\phi')\sigma\phi AK^\alpha G_k^{1-\alpha}] \cdot du \tag{C.19}$$

where ϕ and ϕ' are the fraction of productive government expenditure and government consumption, respectively.

Applying the Ito's Lemma and following the same step of derivation that is used for analysing the decentralised economy, the stochastic Bellman equation will be

$$\begin{aligned} \rho V(K) = {} & \frac{(C(t)/N)^{1-\eta} - 1}{1-\eta} - \rho V(K) \\ & + V'(K)[(1-\phi)AK^\alpha G_k^{1-\alpha} - C] \\ & + \frac{1}{2} V''(K)[(1-\phi')\sigma\phi AK^\alpha G_k^{1-\alpha}]^2 \end{aligned} \tag{C.20}$$

The social planner will optimally choose the amount of aggregate consumption and of capital for the economy. Then, the first-order conditions for the optimal consumption and savings are

$$\frac{1}{N}\left(\frac{C}{N}\right)^{-\eta} = V'(K) \tag{C.21}$$

$$\begin{aligned} & -\rho V'(K) + V'(K)(1-\phi)\alpha A \left(\frac{G_k}{K}\right)^{1-\alpha} \\ & + V''(K)\left[(1-\phi)A\left(\frac{G_k}{K}\right)^{1-\alpha} \cdot K - C + \left((1-\phi')\sigma\phi A \left(\frac{G_k}{K}\right)^{1-\alpha}\right)^2 \alpha K\right] \\ & \qquad + \frac{1}{2} V'''(K)\left[(1-\phi')\sigma\phi AK^\alpha G_k^{1-\alpha}\right]^2 = 0. \end{aligned} \tag{C.22}$$

Assuming that the aggregate consumption is proportionate to the aggregate private capital ($C = \lambda_s K$) and using the fact from the first-order condition of consumption (C.21), the optimal ratio of consumption to capital will be

$$\begin{aligned} \lambda_s = \frac{C}{K} = \frac{\rho}{\eta} - \frac{1}{\eta}(1-\phi)\alpha A \left(\frac{G_k}{K}\right)^{1-\alpha} + (1-\phi)A\left(\frac{G_k}{K}\right)^{1-\alpha} \\ + \left[\alpha - \frac{1}{2}(\eta+1)\right] \cdot \left((1-\phi')\sigma\phi A\left(\frac{G_k}{K}\right)^{1-\alpha}\right)^2 \end{aligned}$$

Substituting out the ratio of productive government spending to private capital by using the government budget constraint ($G_k = \phi Y$), one can observe that the consumption-capital ratio (λ_s) is a constant value

$$\lambda_s = \frac{C}{K} = \frac{\rho}{\eta} - \frac{1}{\eta}(1-\phi)\alpha\phi^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}} \\ + (1-\phi)\phi^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}} + \left[\alpha - \frac{1}{2}(\eta+1)\right] \cdot \left((1-\phi')\sigma(\phi A)^{\frac{1}{\alpha}}\right)^2$$

which is similar to equation (4.37).

Taking the expectation operator into stochastic capital accumulation (C.19) and using the fact that $C = \lambda_s K$ and $G_k = \phi Y$ give us:

$$EdK = [(1-\phi)\phi^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}}K - \lambda K]dt$$

Dividing by Kdt, the expected growth rate of a centralised economy will be

$$\gamma_s = \frac{EdK}{Kdt} = \left[(1-\phi)\phi^{\frac{1-\alpha}{\alpha}}A^{\frac{1}{\alpha}} - \lambda_s\right]$$

Substituting λ_s into this equation, the growth rate of a centralised economy becomes

$$\gamma_s = \frac{1}{\eta}\left[(1-\phi)\phi^{\frac{1-\alpha}{\alpha}}\alpha A^{\frac{1}{\alpha}} - \rho\right] + \left((1-\phi')(\phi A)^{\frac{1}{\alpha}}\sigma\right)^2 \cdot \left(\frac{\eta+1}{2} - \alpha\right)$$

which is indeed represented in equation (4.38).

C.3 Volatility of productive spending

In Figure 4.4, the welfare loss is calculated when varying the degree of risk aversion (η) and the standard deviation of productive government expenditure (σ). Since it is difficult to find the data on productive government spending, we decide to use government spending on economic affairs as a proxy for productive government expenditure. This data is then collected annually between 2006 and 2014 from Eurostat. As we aware of the unit of measurement, government spending on economic affairs is calculated as a percentage of GDP.

For the calculation of the standard deviation (σ_i), we employ a simple formula in a statistic textbook;

$$\sigma_i = \sqrt{\frac{\sum_{t=2006}^{2014}(X_{it} - \bar{X})}{N}} \tag{C.23}$$

where X_{it} refers to the annual data. Each country and time period are denoted by i and t, respectively. $\bar{X}$ is the mean value of X_{it}, and N is a number of observations.

Table C.1: Standard Deviation of Productive Spending in European Union

Spending on economic affairs as a percentage of GDP										
Country	2006	2007	2008	2009	2010	2011	2012	2013	2014	σ_i
EU (28)	4.2	4.1	4.6	4.9	5.1	4.5	4.7	4.3	4.2	0.344
EU (25)	4.2	4.0	4.6	4.9	5.1	4.4	4.7	4.3	4.2	0.362
EU (19)	4.4	4.2	4.4	5.0	5.4	4.6	4.9	4.5	4.4	0.381
EU (15)	4.4	4.1	4.4	4.9	5.3	4.5	4.8	4.5	-	0.372
EU (12)	4.4	4.1	4.4	4.9	5.4	4.5	4.8	4.4	-	0.405

- EU (12) includes Belgium, Denmark, France, Germany, Greece, Ireland, Italy, Luxembourg, The Netherlands, Portugal, Spain and The United Kingdom.
- EU (15) is EU (12) plus Austria, Finland and Sweden.
- EU (25) is EU (15) plus Cyprus, The Czech Republic, Estonia, Hungary, Latvia, Lithuania, Malta, Poland, Slovakia and Slovenia.
- EU (28) is EU (25) plus Bulgaria, Romania and Croatia.

Table C.2: Standard Deviation of Productive Spending in European Countries

Spending on economic affairs as a percentage of GDP										
Country	2006	2007	2008	2009	2010	2011	2012	2013	2014	σ_i
Belgium	5.3	5.5	6.0	6.7	6.8	7.4	7.7	6.9	7.0	0.822
Denmark	3.1	2.9	2.8	3.3	3.4	3.4	3.7	3.5	3.6	0.308
France	4.5	4.3	4.5	4.9	5.1	4.8	5.0	4.9	5.1	0.289
Germany	3.5	3.3	3.6	4.0	4.8	3.7	3.7	3.5	3.3	0.462
Greece	3.8	4.3	5.7	5.4	4.5	4.1	6.7	15.0	3.7	3.549
Ireland	3.7	3.9	5.5	6.8	25.4	7.4	3.2	2.7	3.2	7.148
Italy	5.1	4.2	4.0	4.7	4.2	4.2	4.1	4.2	4.2	0.355
Luxembourg	4.8	4.1	4.0	4.7	4.5	4.7	4.7	4.5	4.5	0.278
Netherlands	4.1	4.2	4.4	5.4	5.2	4.8	4.5	3.8	4.2	0.528
Portugal	4.3	4.2	4.6	4.8	6.4	4.4	3.8	3.8	6.9	1.106
Spain	5.0	5.2	5.4	5.7	5.8	5.7	7.9	4.5	4.4	1.030
UK	3.1	3.0	5.2	4.4	3.4	3.0	3.5	3.1	3.0	0.773

From table C.1, the empirical evidence demonstrates that the standard deviation of productive government expenditure varies from 0.3-0.4 when taking the countries in European Union into account. However, it can be observed that the volatility of productive spending reduces if the number of European countries increases. In table C.2, the standard deviation of productive government spending for each European country is also calculated. It is obvious that government spending on economic affairs is very volatile in Greece, Ireland, Portugal and Spain. One possible explanation is that governments of these countries may attempt to strengthen the economy after the 2008 financial crisis by increasing government expenditure.

C.4 Derivation of certainty equivalence

The social planner has a precise knowledge that the aggregate wealth of households is affected by the volatility of government transfer. Thus, the planner will carefully choose the optimal consumption and saving paths while keeping the volatility of aggregate wealth of households less volatile.

After taking government transfer into account (T), the planner may suggest households to invest in either bond (B) or capital (K). Therefore, the constraint on accumulated wealth will be

$$dW(t) = dB(t) + dK(t) \tag{C.24}$$

where

$$\begin{aligned} dB(t) &= r_b B(t)dt && \text{(C.25)} \\ dK(t) &= [(1-\phi)r_k K(t) - C(t)]dt + T(t) && \text{(C.26)} \\ T(t) &= (1-\phi')\phi A K(t)^{\alpha} G_k(t)^{1-\alpha}\sigma du && \text{(C.27)} \end{aligned}$$

in which r_b is the return on bond and r_k is the return on capital. Due to equations (C.25), (C.26) and (C.27), the constraint on accumulated wealth can be rewritten as

$$\begin{aligned} dW(t) = {} & [r_b W(t) + ((1-\phi)r_k - r_b)K(t) - C(t)] \cdot dt \\ & + (1-\phi')\phi A K(t)^{\alpha} G_k(t)^{1-\alpha}\sigma \cdot du. \end{aligned} \tag{C.28}$$

Considering the utility function that is subject to the aggregate wealth constraint (C.28) and applying the Ito's Lemma, the stochastic Bellman equation will be

$$\begin{aligned} \rho J(W) = {} & \frac{(C/N)^{1-\eta} - 1}{1-\eta} - \rho J(W) \\ & + J'(W)[r_b W + (1-\phi)r_k K - r_b K - C] \\ & + \frac{1}{2} J''(W)\left[(1-\phi')\phi A K^{\alpha} G_k^{1-\alpha}\sigma\right]^2 \end{aligned} \tag{C.29}$$

Next, the social planner optimally chooses the amount of aggregate consumption and capital. The corresponded first-order conditions are

$$\frac{1}{N}\left(\frac{C^*}{N}\right)^{-\eta} = J'(W) \tag{C.30}$$

$$K^* = -\frac{J'(W)[(1-\phi)r_k - r_b]}{J''(W)\alpha\left[(1-\phi')\phi A\left(\frac{G_k}{K}\right)^{1-\alpha}\sigma\right]^2} \tag{C.31}$$

and the optimal allocation of wealth can be found from

$$\begin{aligned} J'(W)[r_b-\rho] + J''(W)[r_b W_t + ((1-\phi)r_k - r_b)K - C] \\ + \frac{1}{2}J'''(W)\left[(1-\phi')\phi A\left(\frac{G_k}{K}\right)^{1-\alpha}\sigma\right]^2 K^2 = 0 \end{aligned} \tag{C.32}$$

Substituting equation (C.31) into (C.32), the optimal allocation of wealth can be rewritten as

$$\begin{aligned} J'(W)(r_b-\rho) + J''(W)[r_b W + ((1-\phi)r_k - r_b)K^* - C^*] \\ + \frac{1}{2}\left(\frac{-J'(W)}{J''(W)}\right)^2\left(\frac{((1-\phi)r_k - r_b)^2}{\left[\alpha(1-\phi')\phi A\left(\frac{G_k}{K}\right)^{1-\alpha}\sigma\right]^2}\right)J'''(w) = 0. \end{aligned} \tag{C.33}$$

Assuming that the function of the optimal consumption, the optimal capital and the optimal bond are proportionate to wealth, the equilibrium allocation becomes

$$C^* = \mu W \tag{C.34}$$

$$K^* = nW \tag{C.35}$$

$$B* = (1-n)W \tag{C.36}$$

where n is a fraction of capital investment and μ is a proportional to consume capital. Substituting (C.34) and (C.35) into (C.33), the optimal condition for wealth is obtained.

$$\begin{aligned} J'(W)(r_b-\rho) + J''(W)\left[r_b W + ((1-\phi)r_k - r_b)nW - \mu W\right] \\ + \frac{1}{2}\left(\frac{-J'(W)}{J''(W)}\right)^2\left(\frac{((1-\phi)r_k - r_b)^2}{\left[\alpha(1-\phi')\tau_y A\left(\frac{G_k}{K}\right)^{1-\alpha}\sigma\right]^2}\right)J'''(w) = 0 \end{aligned} \tag{C.37}$$

With regard to the optimal condition of consumption (C.30) and (C.34), the second-order and third-order conditions of $J(W)$ can be calculated.

$$J''(W) = -\frac{\eta}{WN}(\mu W)^{-\eta} \tag{C.38}$$

$$J'''(W) = \frac{\eta(\eta+1)}{W^2 N}(\mu W)^{-\eta} \tag{C.39}$$

Substituting equation (C.30), (C.38) and (C.39) into (C.37) and dividing it by η, the optimal wealth condition (C.37) can be rewritten as

$$\frac{r_b}{\eta} - \frac{\rho}{\eta} - r_b - ((1-\phi)r_k - r_b)n + \mu$$
$$+ \frac{1}{2}\left(\frac{\eta+1}{\eta}\right)\left(\frac{((1-\phi)r_k - r_b)^2}{\eta\left[\alpha(1-\phi')\phi A\left(\frac{G_k}{K}\right)^{1-\alpha}\sigma\right]^2}\right) = 0 \tag{C.40}$$

One can find the value of proportional to consume capital (μ) by substituting equations (C.30), (C.38) and (C.39) into (C.33), and then dividing by η.

$$\mu = -\frac{r_b}{\eta} + \frac{\rho}{\eta} + r_b + \frac{((1-\phi)r_k - r_b)^2}{\eta\alpha\left[(1-\phi')\tau_y A\left(\frac{G_k}{K}\right)^{1-\alpha}\sigma\right]^2}$$
$$- \frac{1}{2}\left(\frac{\eta+1}{\eta}\right)\left(\frac{((1-\phi)r_k - r_b)^2}{\eta\left[\alpha(1-\phi')\phi A\left(\frac{G_k}{K}\right)^{1-\alpha}\sigma\right]^2}\right) \tag{C.41}$$

Substituting out μ in equation (C.40) out by using (C.41) and using the fact that $G_k = \tau_y Y$ give us:

$$[(1-\phi)r_k - r_b]n = \frac{((1-\phi)r_k - r_b)^2}{\eta\alpha[(1-\phi')(\phi A)^{\frac{1}{\alpha}}\sigma]^2}. \tag{C.42}$$

However, in this paper, there is only the investment in capital. Therefore, $K^* = W^*$ and $n = 1$. Consequently, the certainty equivalence return interest rate (r_c) becomes

$$r_c = r_b = (1-\phi)r_k - \eta\alpha[(1-\phi')(\phi A)^{\frac{1}{\alpha}}\sigma]^2. \tag{C.43}$$

Replacing r_k by the marginal rate of return to capital (4.12) and using the fact that $G_k = \phi Y$ again, the explicit function of the certainty equivalent interest rate will be

$$r_c = (1-\phi)\phi^{\frac{1-\alpha}{\alpha}} A^{\frac{1}{\alpha}} - \alpha\eta(1-\phi')^2(\phi A)^{\frac{2}{\alpha}}\sigma^2 \tag{C.44}$$

which is similar to equation (4.51).

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