Dynamics of quantum hyperbolic Ruijsenaars-Schneider particles via diagonalization of analytic difference operators

Steven William Haworth

Submitted in accordance with the requirements for the degree of
Doctor of Philosophy

The University of Leeds
School of Mathematics

September 2016

The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

Work from the following jointly-authored pre-publication is included in this thesis:

Haworth, S. and Ruijsenaars, S.N.M. (2016), Hilbert space theory for relativistic dynamics with reflection. Special cases, arXiv:1607.06675 (accepted for publication by the Journal of Integrable Systems).

Section 2.1. of this thesis is based on the above paper. The material therein was the result of joint effort between the authors. It has been written anew for the present work.

With orthogonal complements

Acknowledgements

I would like to thank my supervisor Simon without whose insights the present work would not have been possible. I would also like to thank my parents and family for their unending support. Thanks are also due to the EPSRC for their generous funding throughout the research period.

Abstract

We present a new generalised eigenfunction of the reduced two-particle, mixed-charge, hyperbolic Ruijsenaars-Schneider (or, relativistic A_1-Calogero-Moser) Hamiltonian. The asymptotics of this function displays transmission and reflection in a way that generalizes the familiar non-relativistic picture. Using this function we construct integral transforms diagonalizing the Hamiltonian (an analytic difference operator, or AΔO). When the three parametric dependences of the Hamiltonian are restricted to a certain polytope, these transforms can be used for a functional-analytic Hilbert space theory with all the desired quantum mechanical features (self-adjointness, spectrum, S-matrix etc.). As a final consideration we see how such a theory can be constructed in a different way for a special choice of the coupling parameter, with accompanying special features.

Some conventions.

✠ Throughout Chapter 1 we make use of three fixed parameters or, simply, parameters or constants for short,

$$(a_+, a_-, b) \in (0, \infty)^2 \times \mathbb{R}$$

On a handful of occasions we allude to complex b, but never in our main results, i.e. in our lemmas, theorems or corollaries. In Chapter 2, b is fixed as a positive integer multiple of a_+, with the designation $(a_+, a_-) \in (0, \infty)^2$ still in force.

✠ All the objects (functions, AΔOs, integral transforms etc.) considered in Chapters 1 and 2 have a dependence on at least one of these parameters. Whether or not we include this dependence in the symbol of the object depends on the context (often we include b but not a_+, a_-). If the symbol for the object also includes a dependence on the position variable x and/or the dual position variable y, we separate them from the parameters by writing the latter to the left of a semi-colon. E.g. $c(a_+, a_-, b; z)$.

Sometimes when defining objects we do not include the parametric dependences in the symbol; in such cases one can implement the convention just described, combined with a fixed ordering a_+, a_-, b to the left of the semi-colon, to attain an unambiguous definition. Most of the time this should not be necessary.

✠ Almost all equation references in Chapter 1 are internal to that chapter, and likewise for Chapter 2. We therefore suppress chapter number in our equation and section references, and likewise for proposition, lemma, corollary and theorem references. Instances of inter-chapter referencing are clearly indicated. These are most common in Chapter 0, the introduction.

✠ Throughout the thesis we make use of the following shorthands

$$c_\alpha(z) \equiv \cosh(\pi z/a_\alpha), \;\; s_\alpha(z) \equiv \sinh(\pi z/a_\alpha), \;\; e_\alpha(z) \equiv \exp(\pi z/a_\alpha), \;\; \alpha = +, -$$

Often without reference we make use of the following identities

$$s_\alpha(z + ia_\alpha/2) = ic_\alpha(z), \quad \alpha = +, -$$

$$s_\alpha(z + ia_\alpha) = -s_\alpha(z), \quad c_\alpha(z + ia_\alpha) = -c_\alpha(z), \quad \alpha = +, -$$

✠ The two Hilbert spaces used throughout are

$$\mathcal{H} \equiv L^2(\mathbb{R}, dx), \quad \hat{\mathcal{H}} \equiv L^2(\mathbb{R}^+, dy) \otimes \mathbb{C}^2$$

where $\mathbb{R}^+$ denotes the strictly positive real numbers, $\mathbb{R}^+ \equiv (0, \infty)$. One could equivalently use $\hat{\mathcal{H}} = L^2(\mathbb{R}^+, dy)^2$. We write elements of $\hat{\mathcal{H}}$ in the form $\hat{f} = \langle f_+, f_- \rangle$.

We use $\mathcal{C}$ to denote the functions in $C_0^\infty(\mathbb{R})$ with support away from the origin, as well as $\hat{\mathcal{C}} \equiv C_0^\infty(\mathbb{R}^+) \times C_0^\infty(\mathbb{R}^+)$. These are dense subspaces, respectively, of $\mathcal{H}$ and $\hat{\mathcal{H}}$.

✠ For us, a unitary map J between Hilbert spaces is one that strictly satisfies both $JJ^* = \mathbf{1}$ and $J^*J = \mathbf{1}$, and thus amounts to a surjective isometry.

- ✠ We always take the counting numbers $\mathbb{N}$ to include 0, i.e. $\mathbb{N} \equiv \{0, 1, \ldots\}$. If we want to remove 0 from $\mathbb{N}$, $\mathbb{Q}$ or $\mathbb{R}$, we indicate this by affixing an asterisk to the symbol, e.g. $\mathbb{N}^*$.
- ✠ The shift on functions used throughout is defined by $T^r_{ic}F(r) \equiv F(r - ic)$.
- ✠ The main analytic difference operators (AΔOs) used in this thesis consist of two shifts, one into the upper-half plane and one into the lower, each with a different, not necessarily meromorphic, coefficient (or "potential"). The presence of two shifts is said to make the AΔO second order.
- ✠ Formally, we can consider an AΔO action on any function, but this does not imply analyticity properties of the resulting function, which must be studied separately. For example the real-x restriction of (1.75) in Chapter 1 takes on a different light once the analyticity properties in Lemmas 1.1 and 3.1 are known.
- ✠ A convention in itself: the word 'formal' is used to mean 'pertaining to form', and is often a useful way to momentarily disregard any underlying analysis. Thus, colloquially, it is closer to the meaning of 'informal'.
- ✠ We always write AΔOs with an explicit argument, e.g. $A(x)$ (where x is the variable being shifted). By contrast we never use an explicit argument for an operator defined in Hilbert space, even if it happens to have an AΔO action. Thus it makes sense to ask, for example, whether some Hilbert space operator A has an AΔO action, $A(x)$.
- ✠ The word 'eigenfunction' is always meant in an algebraic, or formal, sense. Likewise for 'generalised eigenfunction'. If we want to indicate that an eigenfunction of this kind is also in some L^2-space we will refer to it as an L^2-eigenfunction. If we want to indicate that it is in the domain of a particular Hilbert space operator A, we will refer to it as an A-eigenfunction. (The latter two usages are in fact quite rare in this thesis.)
- ✠ A function $F(x, y)$ is said to be a 'generalised eigenfunction' of an AΔO, $A(x)$, when there exists a function $B(\cdot)$ such that $A(x)F(x, y) = B(y)F(x, y)$.
- ✠ In §1.2 in Chapter 1 we invoke the notion of O-asymptotics. We clarify here the meaning of this in the most general context from which other variations in the main text can be deduced. Consider the statement:

  $$f(x, y) = f_{as}(x, y) + O(g(x)), \quad \operatorname{Re} x \to \infty$$

  where the bound represented by O is uniform for $\operatorname{Im} x, y$ varying,respectively, over fixed $K_1, K_2 \subseteq \mathbb{R}$. This means: there exist positive constants $R, \mathcal{C}$ such that

  $$|f(x, y) - f_{as}(x, y)| \leq \mathcal{C}|g(x)|$$

  for all $y \in K_2$ and $x \in \mathbb{C}$ satisfying $\operatorname{Im} x \in K_1$ and $\operatorname{Re} x \geq R$.

✠ Most of the results that we prove are specific to objects arising in the Ruijsenaars-Schneider system. However, there are others that are proven for abstract objects satisfying some set of assumptions. To keep the distinction clear, any instance of the latter is given as a Proposition. Thus any result specific to a Ruijsenaars-Schneider object is either a Lemma, Theorem or Corollary (of which there are far more). This is a somewhat arbitrary, but we hope useful, convention.

✠ Many equations have symmetries that we express using indices equalling $+$ or $-$. Most of these indices arise in one of five ways, and so it helps to always use the same symbol for each of the five (equations involving several indices are thus easier to understand when they have a fixed meaning). We review what they are here. The reader may wish to refer back to them later.

- ► $\delta = +, -$ enumerates the components of a function $\hat{f} = \langle f_+, f_- \rangle$ in the Hilbert space $\hat{\mathcal{H}} \equiv L^2(\mathbb{R}^+, dy) \otimes \mathbb{C}^2$. For example, we might define one such function by $f_\delta(x) = \delta \exp(-(x + \delta\pi)^2)$.
- ► $\epsilon = +, -$ is used when describing asymptotics of the kind $x \to \epsilon\infty$. So for example, $\sinh x \sim \epsilon e^{\epsilon x}/2$ as $x \to \epsilon\infty$.
- ► $\tau = +, -$ is used to enumerate the two plane wave components of $\psi(b; x, y)$, most commonly in Chapter 2, cf. (2.14) for example.
- ► $\nu = +, -$ is used to enumerate the two shifted versions of $\mathcal{R}_r(b; x, y)$ that feature in $\psi(b; x, y)$. In Chapter 2, this passes on to the two additive components of the functions $\ell_N^{\pm}$, cf. (2.15).
- ► $\sigma = +, -$ arises when looking at the transforms $\mathcal{F}$ and $\mathcal{F}_N$. The analysis of these transforms reduces to an analysis of residues and boundary integrals of two distinct kinds that can characterised by a sign choice. The index σ arises as a product of two δ-type indices, usually occurring as $\delta\delta'$ in the main text.
- ► $\alpha = +, -$ is used in miscellaneous cases not covered by any of the above.

Contents

Chapter 0

Introduction.

1 Overview

In its mathematical content, this thesis provides an elaborate Hilbert space theory for a particular one-variable analytic difference operator (AΔO) with three fixed parametric dependences. Thus it contributes, by way of example, to the theory of unbounded self-adjoint operators. In its physical aspect, significance derives from the fact this AΔO is the Hamiltonian for the reduced two-particle, mixed-charge, quantum hyperbolic Ruijsenaars-Schneider (or, equivalently, relativistic A_1-Calogero-Moser) system, and thus also contributes to the field of quantum integrable systems. The pertinent features of this system are reviewed in §3 of this introduction. At the end of this section we look at some basic, relevant aspects of AΔO theory.

The three parameters alluded to are denoted a_+, a_-, b. The first two can be viewed respectively as the interaction and Compton wave lengths. They are always taken to be positive, whereas b represents a coupling parameter which we may on occasion treat as complex (though in all our results, a restriction to the real line is always in force). The physicist may view each as having dimension position.

One of the notable features of AΔOs which we mention straight away is the enormous degree of freedom which exists in the determination of eigenfunctions. For say we are considering an AΔO of the form $U_1(z)\exp(-i\partial_z) + U_2(z)\exp(i\partial_z)$. Any eigenfunction of this may be multiplied by any meromorphic function periodic in i to yield another eigenfunction with the same eigenvalue, thus giving rise to infinite-dimensional solution spaces. This is what we call the 'multiplicity' issue, and it is why we often speak of eigenfunctions as being 'suitable' or as meeting certain 'expectations', notions which make less sense in the relatively more rigid setting of differential operators.

Although there is presently no general theory for the Hilbert space aspects of AΔOs, various concrete cases studied by Ruijsenaars in [3], [20]-[23] and [25]-[26] all suggest a departure from the approach to differential operators (reviewed in texts such as [7], [24] and [29], as well as the notes [19]). In fact the papers just cited all share a common approach, which we successfully extend to the AΔO studied in this thesis. This turns on the central role given to generalised eigenfunctions in constructing the Hilbert space theory and is one reason why the multiplicity issue is important. The role consists roughly of this: one looks for a generalised eigenfunction of the AΔO that can be used to define an isometric integral transform from some spectral representation space into position space. The AΔO is thus diagonalised in this space as a self-adjoint operator. (We provide further motivation for this approach in a variety of places including: the toy model below; §2 of this introduction; and §4 in Chapter 1.)

In addition to the works by Ruijsenaars already cited, the literature provides other examples of AΔOs that have been studied in detail, some going back several decades. These focus on bases of orthogonal polynomial eigenfunctions, and so they are less relevant to the kind of theory we want to construct (involving continuous spectrum). Notable examples include the following: Askey-Wilson [9]; Macdonald [16]; T.H. Kooornwinder [13]; and J.F. van Diejen [15]. In fact a small subclass of the polynomials considered in [9] reappear in the present theory, cf. §2 in Chapter 1.

As the reduced Hamiltonian for a system of opposite charges we have certain expectations for the physics of the theory. The construction of the generalised eigenfunction in §1 of Chapter 1, which has also been described in the author's joint paper [32], is motivated foremost by the desire for asymptotics of the transmission and reflection kind, familiar from non-relativistic quantum mechanics. Moreover, an explicit form for the transmission and reflection coefficients is already suggested by [4]. In line with integrable system lore, we would like these coefficients to satisfy the Yang-Baxter equations, and indeed they do. And later, they should end up in the expression for the S-operator. (A suitable, easy-to-read reference for the notions invoked in this paragraph and the one below is [24]; for more in-depth accounts of spectral theory, cf. [7], [29] or [12].)

Before reaching this stage, the first task is to render the AΔO (specified in the next section) as a Hilbert space operator and prove existence of self-adjoint dynamics in $\mathcal{H}$ (which we do in §5.1, Chapter 1). As suggested already, we define a Hilbert space operator by diagonalising the AΔO using an integral transform with the generalised eigenfunction as its kernel (§4, Chapter 1). Once this same integral transform is proved to be isometric, we have a direct realisation of the spectral theorem and an explicit contribution to the absolutely continuous spectrum. We also expect a non-empty discrete spectrum which, in quantum mechanical terms, means a non-empty bound state subspace. This follows because of the orthogonality results in §3.4 of Chapter 1. (For the discrete part of the spectrum, the operator defined by diagonalisation must be supplemented with an extension, cf. §5.3, Chapter 1.) The final exercise in Chapter 1 is to prove that the L^2-eigenfunctions enumerated in §2 span this subspace, thus giving us a complete description of Hilbert space, and of the spectrum. This proof, inspired by [3], is very indirect and involves focusing on the so-called dual AΔO. (The two figures in the next section provide a detailed picture of how these results tie together.)

In Chapter 2 we see how the same results for self-adjointness and spectrum can be proved in an entirely different way when the parameter b is fixed as an integer multiple of a_+. Under this specialisation all the special functions in Chapter 1 become elementary. In addition, various novel phenomena can be seen explicitly such as isometry breakdown of the integral transform.

For the analogous same-charge system, all angles of Hilbert space theory are covered by [3]. Indistinguishability of particles entails a focus on the half-line (in contrast to the mixed-charge case). Using these results, the same-charge Hamiltonian is found to give rise to self-adjoint dynamics with purely absolutely continuous spectrum. The phase function which describes the scattering also appears in our transmission and reflection coefficients (as expected, cf. [4]).

The work in [3] involves a Hilbert space theory based on an integral transform which takes as its kernel the 8-variable BC_1 'relativistic' R-function generalising the hypergeometric ${}_2F_1$ function, cf. [1]. In fact, the very wide scope of this parameter domain encompasses two cases that correspond to an even and odd channel description of the system under consideration in this thesis. The details of these specialisations can be found in [4] (and we say more on this in §2). One consequence of this is that we are not in the dark with respect to the problem approached. Far from it, the paper [3] presents many valuable insights that guide our proofs.

The present work represents a useful addition to what is already known from [3] and [4] for at least one important reason. Our eigenfunction is constructed using the A_1 specialisation of the aforementioned R-function, recently surveyed in [5]. In recent work by Ruijsenaars and Hallnäs in [30], multi-particle (A_{N-1}) joint eigenfunctions have been constructed. By contrast there are no multi-particle versions of the BC_1 function. Accordingly, by basing our work on the A_1 function we hold out a greater hope of facilitating a >2 particle extension of the Hilbert space theory. This is not attempted in the present work, but it is the most plausible avenue for future research.

A note on sine-Gordon. One of the phenomena that makes this system special and worth considering is its connection to sine-Gordon quantum field theory. At the classical level a correspondence between the N-body point particle dynamics of the hyperbolic Ruijsenaars-Schneider system and the factorised scattering of the N-soliton sine-Gordon field theory was established by Ruijsenaars and Schneider in [10]. The extension of the N-particle correspondence to the quantum level is still an open conjecture. But it can be answered affirmatively for the $N = 2$ case. Indeed there are now two perspectives on this: the one presented in thesis and the one deriving from the even and odd channel specialisations of the R-function (a combination of [3] and [4]).

This is not the place for a detailed account of what the correspondence involves (we refer the reader to [4] for more details and other relevant references). But we can be a little more precise; when we set the coupling parameter b equal to $a_+/2$, the bound state spectrum and S-matrix for our system reproduce those of sine-Gordon quantum field theory (namely, the DHN formula and Zamolodchikov's S-matrix, respectively). Indeed, other choices of the coupling parameter have connections to other soliton systems. Since a rigorous functional-analytic treatment eludes the integrable quantum field theories, it is curious that the same physics can be reproduced here on a functional-analytic level by means of a relativistic, non-field-theoretic system.

♦ ♦ ♦

A toy model for AΔOs. To set the scene for our investigation, let us look more closely at some of the formal and Hilbert space aspects of some basic, general AΔOs. (The reader looking for more in this vein may consult [22] which inspires the sketch below. Note that we do not say anything about general AΔE theory in this thesis; a detailed account is given in [17].) First, let us think about the shift out of which all our AΔOs are built,

$$T_{ic}^{x} F(x) \equiv F(x - ic), \quad x, c \in \mathbb{R} \tag{1.1}$$

Provided $F(x)$ is analytic in a suitable strip it is well-known that $T_{ic}^{x} = \exp(-ic\partial_x)$, and so we see that the shift inherits formal self-adjointness from that of $-i\partial_x$ when c is real. Now suppose we introduce a potential term into this picture,

$$V(x) T_{ic}^{x} \tag{1.2}$$

where $V(x)$ is some meromorphic function in x. Then, one finds that, formally,

$$[V(x) T_{ic}^{x}]^* = \overline{V(x + ic)} T_{ic}^{x}, \quad x \in \mathbb{R} \tag{1.3}$$

And so we have a clear iff condition for formal self-adjointness of this AΔO, namely $V(x) = \overline{V(x + ic)}$. (Adapting this condition appropriately, one easily sees that the AΔO considered in this thesis is also formally self-adjoint.)

At the Hilbert space level there is an obvious way to render the up and down shifts as unbounded self-adjoint operators (setting $c = 1$ for simplicity). This involves noting their unitary equivalence to real-valued multiplication,

$$T_{\pm i}^{x} = J e^{\mp y} J^* \tag{1.4}$$

where J is Fourier transform,

$$J : L^2(\mathbb{R}, dy) \to \mathcal{H} \equiv L^2(\mathbb{R}, dx), \quad \hat{f} \mapsto (2\pi)^{-1/2} \int_{\mathbb{R}} dy\, e^{ixy} \hat{f}(y), \quad x \in \mathbb{R} \tag{1.5}$$

Denoting by $\mathcal{D}(e^{\pm y})$ the maximal domains of multiplication on $L^2(\mathbb{R}, dy)$ by $e^{\pm y}$ respectively, we see explicitly that unitarity of J entails the two shifts

$$T_{\pm i}^{x} : J(\mathcal{D}(e^{\mp y}) \to \mathcal{H} \tag{1.6}$$

are unbounded self-adjoint operators in $\mathcal{H}$.

To motivate the work in §4 of Chapter 1 it will be illustrative to look at this another way. Let us take the classic schoolboy approach to symmetry and write out

$$(T_i^x f, g) = \int_{\mathbb{R}} dx\, \overline{f(x - i)} g(x) = \int_{\mathbb{R}+i} dx\, \overline{f(\overline{x})} g(x - i) \tag{1.7}$$

where the last equality involves a variable change (readily adaptable for T_{-i}^{x}). At this point we ask what would be required to shift this contour down to $\mathbb{R}$, and thereby exhibit symmetry. By Cauchy's theorem it would clearly suffice if f, g were holomorphic in the strip $i[-1, 0] \times \mathbb{R}$, with uniform L^2-asymptotics. At this point, our mind jumps to the

Paley-Wiener theorems, relating analyticity properties to decay properties under Fourier transform. Indeed, Theorem IX.13 in [8], tells us precisely what we want to know: that if a function $\hat{f}$ has the necessary properties for membership of $\mathcal{D}(e^{-y})$, then $(J\hat{f})(x)$ has analytic continuation to the above strip with uniform L^2-asymptotics. In other words, we have a rigorous symmetry result

$$(T_i^x f, g) = (f, T_i^x g), \quad f, g \in J(\mathcal{D}(e^{-y})) \tag{1.8}$$

There are two things to take away this: the active role of analyticity properties of domain functions for a property like AΔO symmetry; and the possibility of grasping these properties by means of an integral transform on some spectral representation space. This is precisely the approach we take in Chapter 1 of the main text. In this regard one may think of our integral transform as a generalised Fourier transform (indeed for certain explicit values of the parameters one can see the reduction of the transform kernel to a plane wave explicitly).

As a final consideration, let us see what happens if we try to adapt (1.7)-(1.8) directly to accommodate a potential term. Writing (1.2) as $T_i^x V(x+i)$ it is clear we could reproduce the same argument, but the result would only hold for functions in

$$\mathcal{D}(V(x+i)) \cap J(\mathcal{D}(e^{-y})) \tag{1.9}$$

where $\mathcal{D}(V(x+i))$ is the maximal domain of multiplication on $L^2(\mathbb{R}, dx)$ by the function $V(x+i)$. This is a highly unusual space and we are not aware of a procedure by which it could be analysed, and proved to be dense in $\mathcal{H}$. Moreover, even if there were such a procedure, it is not clear that the symmetry property could then be strengthened to self-adjointness (something which turns out to be fairly straightforward for our generalised Fourier transform).

2 Summary of results.

In this section we give a summary of the results proved in this thesis. Their combined effect is to render the following analytic difference operator (AΔO) as an unbounded self-adjoint operator in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, dx)$ with explicit spectral properties,

$$\tilde{H}(a_+, a_-, b; x) = \left[\frac{c_+(x - ib)}{c_+(x)}\frac{c_+(x + ib - ia_-)}{c_+(x - ia_-)}\right]^{1/2} T^x_{ia_-} + (i \to -i) \tag{2.1}$$

$$c_+(x) = \cosh(\pi x/a_+), \quad T^x_{ia_-}F(x) = F(x - ia_-) \tag{2.2}$$

As relevant to quantum mechanics, we obtain an explicit description of the corresponding bound states and S-operator, and prove orthogonality and completeness (we recommend [24] as a concise summary of these notions). In the next section we see exactly how $\tilde{H}(x)$ arises as the center-of-mass-reduced two-particle, mixed-charge, hyperbolic, Ruijsenaars-Schneider Hamiltonian.

Proving these kind of facts for an AΔO requires a novel approach. As an operator in $\mathcal{H}$, its domain will be defined using an integral transform with kernel given by a particular generalised eigenfunction of $\tilde{H}(x)$. This gives us a grip on the analyticity properties of the domain functions when meromorphically-continued to a given strip (something which is clearly of interest for an AΔO). This choice of generalised eigenfunction is key to the successful construction of a self-adjoint operator with desirable spectral properties. The issue of choice is non-trivial given the highly non-unique nature of AΔO eigenfunctions. Moreover, an AΔO like $\exp(-ia_-\partial_x) + \exp(ia_-\partial_x)$, which is seemingly free, may have non-trivial (i.e. non-plane-wave) eigenfunctions and thus describe non-trivial physics. Indeed the whole of Chapter 2 is premised on this fact.

As we suggested in the introduction, this is not the first time a Hilbert space theory has been presented which enables $\tilde{H}(x)$ to be rendered as a self-adjoint operator. One such theory comes from the paper [3] by Ruijsenaars (which deals with a much larger, four-coupling, class of AΔOs). Both theories are based upon the method of generalised eigenfunctions with corresponding isometric integral transforms. However they are also different in at least two important ways.

First, [3] deals with integral transforms on the half-line, and so the Hilbert space version of $\tilde{H}(x)$ arising there acts only in $L^2(\mathbb{R}^+, dx)$. Nevertheless, it connects closely to the present work because what it gives us are *two* generalised eigenfunctions for $\tilde{H}(x)$ (and so two different transforms) which yield an even and odd channel description of the same physics below (i.e. transmission-reflection asymptotics and bound state subspace. Cf. [4]). The rigorous results for half-line transforms in [3], combined with the two specialisations described in [4], can be adapted fairly easily to make the physical equivalence rigorous on $\mathcal{H}$ (the adaptation is not presently in the literature).

Second, the two generalised eigenfunctions in question are, to be more precise, one-coupling specialisations of the four-coupling (BC_1) R-function. As we have said already, one of the achievements of this thesis is to prove everything in terms of the one-coupling (A_1) $\mathcal{R}$-function. Because of recent work by Hallnäs and Ruijsenaars in [30], this has a multi-particle (A_{n-1}) extension, unlike the R-function.

The work in this thesis involves three fixed parameters or, simply, parameters or constants for short, a_+, a_-, b, which have the physical interpretation sketched at the start of §1 (and elucidated further in §3. In fact the latter uses a different set of "physical" parameters - the only place to do so). Except for a handful of occasions when we allude to complex b, the most general domain for these parameters considered in this thesis is

$$(a_+, a_-, b) \in (0, \infty)^2 \times \mathbb{R} \tag{2.3}$$

Indeed (2.3) is always in effect for our lemmas, corollaries and theorems. The latter typically require additional restrictions on the parameters which are taken to hold in addition to (2.3). These restrictions fall under two kinds. There are those that prohibit the parameters from equalling certain linear combinations of each other (over $\mathbb{Q}$), and those that involve a size restriction. The former would disappear if we were to employ generic parameters, i.e. a_+, a_-, b taken to be linearly independent over $\mathbb{Q}$. However, we do not impose the requirement of genericity in this work. Our preference is to express both kinds as b restricted in terms of a_+, a_- (which reflects the fact that physically, b is like an interaction coupling parameter and thus more "controllable").

With respect to the first kind of restriction, the most important is given below, cf. (2.11). The Hilbert space results in this thesis require that we combine the latter with a size restriction on b. This combination amounts to the following,

$$b \in (0, a_- + a_+/2) \setminus A_-, \quad (a_+, a_-) \in (0, \infty)^2 \tag{2.4}$$

The effect of A_- here is to exclude positive integer multiples of a_- from the interval (cf. (3.1) in Chapter 1). The latter are the exceptional b-values for which the ψ-asymptotics becomes reflectionless, and for which a different Hilbert space theory applies (namely that in [20]. Cf. (3.47) in [32] for the connection). Again, we stress that (2.4) is consistent with/stronger than/sufficient for (2.11).

(Because of the symmetry (2.14), the results below are readily adaptable to the b-interval, $b \in (-a_+/2, a_-)$ with non-positive integer multiples of a_- excluded. These results mirror what we get for (2.4) in a way which is not worth detailing.)

We proceed to summarise the main results of this thesis. We stress that these are all taken from Chapter 1. The reason this chapter is said to deal with general b becomes clear when contrasted to Chapter 2 which looks at the special case when b is fixed as a positive integer multiple of a_+. The latter chapter proceeds by an entirely different chain of arguments, but since it does not yield any results bearing on self-adjointness and spectrum that do not also follow from Chapter 1, we do not summarise it here.

In almost all cases below, the defining expressions for objects are identical to those in the main text. We consider these to be their proper place, insofar as they arise there as part of the story. Nevertheless we stress that the reader does not have to read this summary to follow the main text, and vice versa. We will not re-use the equivalence symbol in the defining expressions below.

The fundamental building blocks for this work are the hyperbolic gamma function $G(z)$ [17] and the relativistic conical function $\mathcal{R}(a_+, a_-, b; x, y)$ [5] as well as its renormalised counterpart denoted by a subscript r. An overview of these functions is given at the start of Chapter 1.

Let us now specify the function central to the construction of our Hilbert space theory for $\tilde{H}(x)$, which introduces a dual variable y,

$$\psi(a_+, a_-, b; x, y) = \tilde{w}(b; x)^{1/2} \left(2s_-(ib - y)c(b; -y)\right)^{-1} \times \sum_{\nu=+,-} \nu e_-(\nu(ib - y)/2)\mathcal{R}_r(b; x + i\nu a_+/2, y) \quad (2.5)$$

This features the following meromorphic functions, whose a_+, a_--dependence we suppress,

$$c(b; y) = G(y + ia - ib)/G(y + ia) \quad (2.6)$$

$$\tilde{w}(b; x) = 1/\tilde{c}(b; x)\tilde{c}(b; -x) \quad (2.7)$$

$$\tilde{c}(b; x) = c(b; x - ia_+/2) = G(x + ia_-/2 - ib)/G(x + ia_-/2) \quad (2.8)$$

$$a = a_+/2 + a_-/2 \quad (2.9)$$

and the two entire functions

$$s_-(z) = \sinh(\pi z/a_-), \quad e_-(z) = \exp(\pi z/a_-) \quad (2.10)$$

Algebraic, asymptotic and analyticity properties of $\psi(x, y)$. We now detail certain properties which all hold when the parameters satisfy

$$b \in \mathbb{R} \setminus \mathcal{Y} \quad (2.11)$$

$$\mathcal{Y} = \{\pm [ka_+/2 + (l + 1)a_-] \,|\, k, l \in \mathbb{N}\} \cup \{-ka_+/2 \,|\, k \in \mathbb{N}\} \quad (2.12)$$

(in addition to $a_\pm \in (0, \infty)$).

The function $\psi(x, y)$ has two important symmetries

$$\overline{\psi(\overline{x}, \overline{y})} = \psi(x, -y), \quad (2.13)$$

$$\psi(b; x, y) = \psi(a_- - b; x, y) \quad (2.14)$$

Moreover, it is meromorphic in y and its square is meromorphic in x; it has no branch points on the real line (Lemma 1.1). It is smooth in both x and y (statements about smoothness involve the restriction of the variables to the real line; Corollary 3.2). It satisfies the following generalised eigenvalue equation (Lemma 1.1),

$$\tilde{H}(x)\psi(x, y) = 2\cosh(\pi y/a_+)\psi(x, y) \quad (2.15)$$

And, finally, it has the following asymptotics (Lemma 1.3),

$$\psi(x, y) = \begin{cases} t(y)\exp(i\pi xy/a_+a_-) + O(e^{-\rho \operatorname{Re} x}), & \operatorname{Re} x \to \infty \\ \exp(i\pi xy/a_+a_-) - r(y)\exp(-i\pi xy/a_+a_-) + O(e^{\rho \operatorname{Re} x}), & \operatorname{Re} x \to -\infty \end{cases} \quad (2.16)$$

where $\rho > 0$ is a constant fixed by a_+, a_-, and where the bound represented by O is uniform for $\operatorname{Im} x$ and y varying over any compact subset of $\mathbb{R}$ (the precise meaning of this statement follows from our conventions). The functions $t(y)$ and $r(y)$ are defined as follows

$$t(b;y) = \frac{s_-(y)}{s_-(ib-y)} u(y), \quad r(b;y) = \frac{s_-(ib)}{s_-(ib-y)} u(y) \tag{2.17}$$

and where $u(y)$ is a phase,

$$u(b;y) = -c(y)/c(-y) \tag{2.18}$$

From well-known trigonometric identities we thus have

$$|t(y)|^2 + |r(y)|^2 = 1, \tag{2.19}$$

as well as consistency with the Yang-Baxter equations (cf. Chapter 1, §1.1). Using (1.3) in Chapter 1 we also have the following integral representation for $\operatorname{Im} y$ in a sufficiently small interval,

$$-u(y) = \exp\left(2i \int_0^\infty \frac{du}{u} \sin(2yu) \frac{\sinh((b-2a)u)\sinh(bu)}{\sinh(a_+u)\sinh(a_-u)}\right) \tag{2.20}$$

Continuous spectrum of $\tilde{H}$. The following claims hold when the parameter restriction (2.4) is in force (a stronger version of (2.11)). It is notable that these results require a size restriction on b which is not required for the properties above, or for formal self-adjointness of $\tilde{H}(x)$. This is one of the novel phenomena of AΔOs; namely, that heuristics are a poor guide to functional-analytic results.

Out of the function $\psi(x,y)$ we construct two transforms

$$(\mathcal{F}_\pm f)(x) = \int_{\mathbb{R}^+} dy\, \psi(\pm x, y) f(y) \tag{2.21}$$

and glue these together to create a third which acts on pairs $\hat{f} = \langle f_+, f_- \rangle$,

$$\mathcal{F}\hat{f} = c\,(\mathcal{F}_+ f_+ + \mathcal{F}_- f_-), \quad c = 1/\sqrt{2a_+a_-} \tag{2.22}$$

This is an isometry on the Hilbert space $\hat{\mathcal{H}} = L^2(\mathbb{R}^+, dy) \otimes \mathbb{C}^2$ (Corollary 5.3),[1]

$$\mathcal{F} : \hat{\mathcal{H}} \to \mathcal{H} \tag{2.23}$$

We note that $\mathcal{F}$ can alternatively be viewed as an eigenfunction transform on $L^2(\mathbb{R}, dy)$ (a trivial isomorphism links the two perspectives). This may seem preferable but there

[1]It is also true that $c\mathcal{F}_\pm$ are isometries on $L^2(\mathbb{R}^+, dy)$. This is not in the main text but follows by adapting results in §4-5 in Chapter 1. This approach is not obviously preferable because it does not give us isometry of $\mathcal{F}$ without an independent proof of $\operatorname{Ran}\mathcal{F}_+ \perp \operatorname{Ran}\mathcal{F}_-$. For the adaptation one must define new operators, analogous to $\tilde{H}_{\text{ac}}$, by the intertwining $\tilde{H}_\pm \mathcal{F}_\pm = \mathcal{F}_\pm \hat{m}$. Symmetry of these operators is implied by vanishing of $d_{+,+}(f,g)$. An analogue of Lemma 5.2 can be written down for $\tilde{H}_\pm$ and $c\mathcal{F}_\pm$ which is true in light of (5.48) in Chapter 1. This analogue implies the isometry claimed.

is a catch: the integrand has a piecewise dependence on y,

$$f \mapsto c \int_{\mathbb{R}} dy\, \psi_-(x,y) f(y), \quad x \in \mathbb{R}, \quad \psi_-(x,y) \equiv \begin{cases} \psi(x,y), & y > 0 \\ \psi(-x,-y), & y < 0 \end{cases} \tag{2.24}$$

Plus, there are other issues. For instance, the S-operator below would not be neatly expressible in terms of the matrix $S(y)$ in a formalism based on (2.24). In sum, we consider $\mathcal{F}$ (2.22) to be preferable and do not use the alternative anywhere in the thesis. We note it here simply because the reader may find it more intuitive.

This isometry result has a natural application to the Hilbert space theory of $\tilde{H}(x)$. To see this we need the space of functions

$$\hat{\mathcal{C}} = C_0^\infty(\mathbb{R}^+) \times C_0^\infty(\mathbb{R}^+) \tag{2.25}$$

which is dense in the Hilbert space $\hat{\mathcal{H}}$. We also need multiplication on pairs by the unbounded function $2\cosh(\pi y/a_+)$, denoted $\hat{M}$. Clearly, $\hat{\mathcal{C}}$ is invariant under $\hat{M}$. It then follows that the operator

$$\tilde{H}_{\mathrm{ac}} = \mathcal{F}\hat{M}\mathcal{F}^{-1} \tag{2.26}$$

$$\tilde{H}_{\mathrm{ac}} : \mathcal{F}(\hat{\mathcal{C}}) \to \mathcal{F}(\hat{\mathcal{C}}) \subset \mathcal{H} \tag{2.27}$$

is a priori symmetric in the Hilbert space

$$\overline{\mathcal{F}(\hat{\mathcal{C}})} = \mathcal{F}(\hat{\mathcal{H}}) \tag{2.28}$$

where this equality follows from $\mathcal{F}$'s isometry (all the same, we often prefer to use the lhs expression for reasons of illustration; the bar denotes closure). In fact a simple argument using Nelson's Theorem allows us to strengthen this symmetry to essential self-adjointness (i.e. existence of a unique self-adjoint extension; Theorem 5.1). Because of the generalised eigenvalue equation above it is clear we have constructed a diagonalisation of $\tilde{H}(x)$. In other words, $\tilde{H}_{\mathrm{ac}}$ reproduces the action of the AΔO $\tilde{H}(x)$ on $\mathcal{F}(\hat{\mathcal{C}})$.

The wider problem is to render the AΔO $\tilde{H}(x)$ as a self-adjoint operator in the generally larger Hilbert space, $\mathcal{H}$. The result just given for $\tilde{H}_{\mathrm{ac}}$ obviously plays a major part in this. In light of it, we introduce an operator $\tilde{H}$ with the following dense domain in $\mathcal{H}$,[2]

$$\mathcal{D}(\tilde{H}) = \mathcal{F}(\hat{\mathcal{C}}) \oplus \overline{\mathcal{F}(\hat{\mathcal{C}})}^{\perp} \tag{2.29}$$

The action of $\tilde{H}$ is defined separately on the two spaces in this orthogonal sum. On $\mathcal{F}(\hat{\mathcal{C}})$ we set it to be $\tilde{H}_{\mathrm{ac}}$, unsurprisingly. However, to define a suitable action on the orthocomplement requires knowledge from the next set of results. For the present set, it is enough to define $\tilde{H}$ as an arbitrary, bounded self-adjoint operator on this space. I.e. the results (2.36)-(2.38) hold independently of this arbitrary choice.

The operator $\tilde{H}$ clearly inherits essential self-adjointness from $\tilde{H}_{\mathrm{ac}}$. As a result, it has a contribution to its absolutely continuous spectrum of multiplicity two given by the

[2] Recall our convention wherein AΔOs are written with an explicit variable, unlike operators defined in Hilbert space.

closure of the set $\{2\cosh(\pi y/a_+) \,|\, y \in \mathbb{R}^+\}$, i.e. $[2, \infty)$. When the action of $\tilde{H}$ on the orthocomplement is fixed as below, this becomes the sole contribution.

Finally we have results for time-dependent scattering theory. Essential self-adjointness entails existence of a one parameter unitary group,[3]

$$\exp(it\tilde{H}), \quad t \in \mathbb{R} \tag{2.30}$$

We compare this to a free motion defined using Fourier transform on pairs $\hat{f} = \langle f_+, f_- \rangle$,

$$\mathcal{J} : \hat{\mathcal{H}} \to \mathcal{H} \tag{2.31}$$

$$(\mathcal{J}\hat{f})(x) \equiv c \sum_{\delta=+,-} \int_{\mathbb{R}^+} dy\, \exp(i\pi\delta xy/a_+a_-) f_\delta(y), \quad c \equiv 1/\sqrt{2a_+a_-} \tag{2.32}$$

We then have an a priori self-adjoint operator

$$H_0 = \mathcal{J}\hat{M}\mathcal{J}^{-1} \tag{2.33}$$

with dense domain $\mathcal{J}(\mathcal{D}(\hat{M}))$ in $\mathcal{H}$ (here, $\mathcal{D}(\hat{M})$ is the maximal domain of all functions $f \in \hat{\mathcal{H}}$ such that $\hat{M}\hat{f} \in \hat{\mathcal{H}}$) and a corresponding one-parameter unitary group,

$$\exp(itH_0), \quad t \in \mathbb{R} \tag{2.34}$$

The operator H_0 has the free action $H_0(x) = \exp(-ia_-\partial_x) + \exp(ia_-\partial_x)$. (We note that to understand the limiting case $\tilde{H} \to H_0$, it is not enough to ask when $\tilde{H}(x) \to H_0(x)$. Rather we must ask when $\mathcal{F} \to \mathcal{J}$, which is a much more non-trivial question. One way to achieve this is to fix $b = b_N$ and $a_-/a_+ \in \mathbb{N}^*$ simultaneously, cf. Chapter 2 and [32].)

We may now consider the following wave operators on $\mathcal{H}$,

$$W_\pm = \operatorname*{s\cdot lim}_{t\to\infty} \exp(\pm it\tilde{H}) \exp(\mp itH_0) \tag{2.35}$$

(where s·lim means the limit is defined in the strong operator topology). Then, we have (Theorem 5.4),

$$W_- = \mathcal{F}\mathcal{J}^* \tag{2.36}$$

$$W_+ = \mathcal{F}\hat{S}^*\mathcal{J}^* \tag{2.37}$$

where $\hat{S}$ denotes matrix multiplication on functions in $\hat{\mathcal{H}}$ by the unitary matrix

$$S(b; y) = \begin{pmatrix} t(y) & -r(y) \\ -r(y) & t(y) \end{pmatrix} \tag{2.38}$$

Moreover, the S-operator $W_+^* W_-$ is equal to $\mathcal{J}\hat{S}\mathcal{J}^*$ and the scattering states (intersection of the ranges of the wave operators) are given by (2.28).[4]

[3] The $\tilde{H}$ in the exponent is understood to stand for its own unique self-adjoint extension.

[4] To attain positive signs for $r(y)$ in this matrix, one may redefine $\mathcal{F}$ (2.22) with subtraction in place of addition on the rhs.

Discrete spectrum of $\tilde{H}$ (plus conclusion). The following claims hold whenever (2.4) is in force. In addition to the facts about $\mathcal{F}$ above, we have the following completeness result (Theorems 7.4 and 7.7),

$$\overline{\mathcal{F}(\hat{\mathcal{C}})}^{\perp} = \mathrm{span}\{\Psi^{(0)}, \dots, \Psi^{(m_b-1)}\} \tag{2.39}$$

$$\Psi^{(m)}(b;x) = 2c_+(x)\tilde{w}(x)^{1/2}Q_m(2s_+(x)), \quad m \in \mathbb{N} \tag{2.40}$$

where m_b is the largest integer such that $m_b a_- < b$; vanishing when $b < a_-$. In the latter case, the span is empty, corresponding to unitarity of $\mathcal{F}$. The function $Q_m(b;\cdot)$ is a polynomial of degree m and parity $(-)^m$. It has a close relation to the q-ultraspherical polynomials in Askey and Wilson's survey [9] and is discussed in more detail below. For our discussion of $\tilde{H}$ what is important is that the functions $\Psi^{(m)}(x)$ in (2.39) are mutually orthogonal and engage in the following eigenvalue equations with the AΔO $\tilde{H}(x)$,

$$\tilde{H}(x)\Psi^{(m)}(x) = E_m\Psi^{(m)}(x), \quad m = 0, \dots, m_b - 1 \tag{2.41}$$

$$E_m(b) = 2\cos\left(\pi[b-(m+1)a_-]/a_+\right) \tag{2.42}$$

The parameter restrictions entail

$$0 < E_0 < E_1 < \ldots < E_{m_b-1} < 2 \tag{2.43}$$

We now fix the action of $\tilde{H}$ on the orthocomplement in (2.29) by defining it to act as real-valued multiplication on the explicit (and orthogonal) basis in (2.39),

$$\tilde{H}\Psi^{(m)} = E_m\Psi^{(m)}, \quad m = 0, \dots, m_b - 1 \tag{2.44}$$

Main Theorem. *For parameters satisfying* (2.4)*, the densely-defined operator in $\mathcal{H}$,*

$$\tilde{H} : \mathcal{F}(\hat{\mathcal{C}}) \oplus \overline{\mathcal{F}(\hat{\mathcal{C}})}^{\perp} \to \mathcal{H} \tag{2.45}$$

with action $\tilde{H}_{ac}$ (2.26) *on $\mathcal{F}(\hat{\mathcal{C}})$ and action* (2.44) *on the orthocomplement is essentially self-adjoint with absolutely continuous spectrum* $[2,\infty)$ *of multiplicity two, and point spectrum* (2.42)-(2.43) *of multiplicity one. In both cases, this action equals that of the AΔO $\tilde{H}(x)$. The map $\mathcal{F}$* (2.22) *is an isometry, and thus the closure in* (2.45) *equals $\mathcal{F}(\hat{\mathcal{H}})$.*

Let us now say more about the functions $Q_m(b;\cdot)$ and $\Psi^{(m)}(x)$ (with (2.4) still in force). The former is uniquely defined by $Q_{-1} = 0$, $Q_0 = 1$ and the recursion,[5]

$$Q_{m+1}(u) + u\sigma_m Q_m(u) + \rho_m Q_{m-1}(u) = 0, \quad m \geq 0 \tag{2.46}$$

where σ_m, ρ_m are b-dependent constants,

$$\sigma_m = \sin(\pi(b-(m+1)a_-)/a_+)/\sin(\pi(m+1)a_-/a_+) \tag{2.47}$$

$$\rho_m = \sin(\pi(2b-(m+1)a_-)/a_+)/\sin(\pi(m+1)a_-/a_+) \tag{2.48}$$

[5] In fact, for the degree and parity properties of $Q_m(b;\cdot)$ to follow from this recursion for parameters (2.3), certain restrictions on the latter are required, cf. Lemma 2.1. These are implied by (2.4), as discussed in the proof of Theorem 7.7.

The eigenvalue equation (2.41) can be seen to hold as a result of a relationship that exists between $\Psi^{(m)}(x)$ and $\psi(x,y)$ (Lemma 2.2),

$$\operatorname*{Res}_{y=y_m} \psi(x,y) \propto \Psi^{(m-1)}(x), \quad m \in \mathbb{N} \tag{2.49}$$

$$y_m = ib - ima_-, \quad \Psi^{(-1)} = 0 \tag{2.50}$$

The m-dependent proportionality constant here is equal to

$$-\, s_-(y_1)(a_-/\pi)c(y_1)/c(-y_m) \tag{2.51}$$

Thus the eigenvalues E_m arise via the spectral values y_m,

$$E_m = 2c_+(y_{m+1}) \tag{2.52}$$

In fact, as one might expect, the transmission coefficient $t(\cdot)$ in the asymptotics of $\psi(x,y)$ is singular for y_m. We have the following ordering, for $m_b \geq 1$,

$$a_+/2 > \operatorname{Im} y_1 > \operatorname{Im} y_2 > \ldots > \operatorname{Im} y_{m_b} > 0 \tag{2.53}$$

(We note that without the upper bound, this ordering is true for general parameters. The upper bound arises from the size restriction on b and, in light of (2.52), we can see why (2.53) translates into (2.43).)

We also have an expression for the norms of the functions $\Psi^{(m)}(x)$ (Theorem 7.7),[6]

$$\|\Psi^{(k)}\|^2_{\mathcal{H}} = \mathcal{N}\sigma_k \prod_{l=0}^{k}(1/\rho_l), \quad k = 0, \ldots, m_b - 1 \tag{2.54}$$

$$-\,\mathcal{N}(b) = 2(a_+a_-)^{-1/2}\sin(\pi b/a_-)\sin(\pi b/a_+)\frac{G(2ib-ia)}{G(ib+ia)G(ib-ia)} \tag{2.55}$$

In light of (2.54), the number $\mathcal{N}\sigma_0/\rho_0$ is non-obviously positive given our parameter restrictions and $m_b \geq 1$ (which entails $b > a_-$).

♦ ♦ ♦

[6] This expression for $\mathcal{N}(b)$ is a rewriting of the definition in the main text which uses (1.1) in Chapter 1, and the explicit evaluation $G(ia_+/2 - ia_-/2) = (a_+/a_-)^{1/2}$ from e.g. [5].

A more explicit view. Here we flesh out some of these restrictions and results in a (slightly) more concrete way. We focus on the case when $b \in (a_-, a_- + a_+/2)/A_-$ such that the bound state subspace is non-empty. With this restriction on b, the constant m_b is fixed by the values of a_+, a_- as follows

$a_+/a_- \in$	m_b
$(0, 2)$	1
$(2, 4)$	2
$(4, 6)$	3
$(6, 8)$	4
$\vdots$	$\vdots$

For the purposes of illustration, let us look at the results above explicitly for the case when $a_+/a_- \in (6, 8)$. Here, the reduced b-interval becomes $b \in (a_-, a_- + a_+/2) \setminus \{2a_-, 3a_-\}$ and we may say the following:

The operator $\tilde{H}$ (2.45) with action $\tilde{H}(x)$ (4.7) on its domain is essentially self-adjoint and has absolutely continuous spectrum $[2, \infty)$ of multiplicity two, and point spectrum of multiplicity one given by

$$0 < E_0 < E_1 < E_2 < E_3 < 2 \tag{2.56}$$

$$E_0 = 2\cos(\pi(b - a_-)/a_+), \quad E_1 = 2\cos(\pi(b - 2a_-)/a_+) \tag{2.57}$$

$$E_1 = 2\cos(\pi(b - 3a_-)/a_+), \quad E_3 = 2\cos(\pi(b - 4a_-)/a_+) \tag{2.58}$$

These points arise as $E_m = \cosh(\pi y_{m+1}/a_+)$ for the spectral values $y_m = ib - ima_-$. The latter are poles of the S-matrix satisfying

$$a_+/2 > \operatorname{Im} y_1 > \operatorname{Im} y_2 > \operatorname{Im} y_3 > \operatorname{Im} y_4 > 0 \tag{2.59}$$

We have the completeness result

$$\overline{\mathcal{F}(\hat{\mathcal{C}})}^{\perp} = \operatorname{span}\{\Psi^{(0)}, \Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)}\} \tag{2.60}$$

and polynomials

$$Q_0 = 1 \tag{2.61}$$

$$Q_1(u) = -\sigma_0 u \tag{2.62}$$

$$Q_2(u) = \sigma_0\sigma_1 u^2 - \rho_1 \tag{2.63}$$

$$Q_3(u) = -\sigma_0\sigma_1\sigma_2 u^3 + (\rho_1\sigma_2 + \rho_2\sigma_0)u \tag{2.64}$$

We also have, e.g.,

$$\|\Psi^{(3)}\|^2_{\mathcal{H}} = \mathcal{N}\sigma_3/\rho_0\rho_1\rho_3 \tag{2.65}$$

On the attainment of results. The summary above was designed to be as succinct as possible. Now, with a little more leg room, we discuss some subtleties concerning the proofs of the results (which, we recall, are all from Chapter 1).

First, we want to stress one way in which the summary could lead the reader astray. Above, we stated that the transform $\mathcal{F}$ was isometric and used this to conclude that the operator $\tilde{H}_{\mathrm{ac}}$ was symmetric. In reality, these two properties are proved in reverse order (something illustrated in Figure 1 below). Moreover, when we define $\tilde{H}_{\mathrm{ac}}$ and prove its symmetry, this is without knowledge that $\mathcal{F}$ is even bounded, let alone isometric. Indeed all that we possess is a well-defined map $\mathcal{F} : \hat{\mathcal{C}} \to \mathcal{H}$ (Lemma 3.3). This is why we *must* use the intertwining definition in the main text, as opposed to (2.26) above.

This also explains why we did not consider rhs of (2.26) on the maximal space $\mathcal{D}(\hat{M})$ when summarising $\tilde{H}_{\mathrm{ac}}$, something which may seem more natural since the corresponding operator would be a priori self-adjoint in light of explicit unitary equivalence to a self-adjoint multiplication operator. If we had done this, the departure from how things are actually proved in the main text would be even greater. (In any case, the rhs of (2.26) on the maximal space $\mathcal{D}(\hat{M})$ is of course nothing but the unique self-adjoint extension of $\tilde{H}_{\mathrm{ac}}$ whose existence we prove in §5.1 of Chapter 1.)[7]

If this procedure for $\mathcal{F}$ and $\tilde{H}_{\mathrm{ac}}$ seems unusual, that by which we prove the completeness result is more unusual still (both have a precedent in [3]). It rests on our ability to prove a so-called symmetry formula for the dual operator $\overline{\mathcal{S}}$, defined by intertwining the adjoint transform $\mathcal{F}^*$ and multiplication by $2s_+(x)$. This formula can be tied to an explicit expression for the orthocomplement of $\mathcal{F}(\hat{\mathcal{H}})$ when $\mathcal{F}$ is isometric. Because of the dual-variable eigenvalue equation in Lemma 1.1, the operator $\overline{\mathcal{S}}$ amounts to a Hilbert space version of the dual AΔO $\mathcal{S}(b; y)$.

These facts, and more, are illustrated in the following two figures which show how the results in Chapter 1 of this thesis are tied together. We focus on those results that reveal some fact about $\psi(x, y)$ or its close relatives. Thus we do not include our various abstract Propositions even though, for example, the symmetry result for $\tilde{H}_{\mathrm{ac}}$ hinges on three of them (recall our conventions for the meaning of 'abstract' here).

Once the first row of results is in place, i.e. Lemmas 1.1, 1.3 and 1.4, the procedure by which we arrive at the others in the picture is self-contained. These three lemmas draw upon facts about the hyperbolic gamma and relativistic conical functions from the literature.

Figure 1 begins on the next page.

[7] As it happens, one of the curious facts about the special case studied in Chapter 2 is that isometry of the eigenfunction transform *does* in fact come first, as we explain at the start of that chapter.

$$
\begin{tikzcd}
\begin{array}{c}\textbf{Lemmas 1.3 and 3.1}\\ \text{(x-asymptotics and ψ-analyticity)}\end{array} \arrow[d] & \\
\begin{array}{c}\textbf{Lemma 3.3}\\ \text{(Definedness of $\mathcal{F}_{\pm} : C_0^{\infty}(\mathbb{R}^+) \to \mathcal{H}$)}\end{array} \arrow[d] & \begin{array}{c}\textbf{Lemma 1.1}\\ \text{($\psi(x,y)$ as g.eigenfunction of $\tilde{H}(x)$)}\end{array} \arrow[dl] \\
\begin{array}{c}\textbf{Theorem 4.5}\\ \text{(Symmetry of $\tilde{H}_{\mathrm{ac}}$)}\end{array} \arrow[d] & \begin{array}{c}\textbf{Lemma 4.4}\\ \text{(x-holomorphy of $J_{\pm}(x,y,y')$)}\end{array} \arrow[l] \\
\begin{array}{c}\textbf{Theorem 5.1}\\ \text{(E.s.a. of $\tilde{H}_{\mathrm{ac}}$)}\end{array} \arrow[d] & \\
\begin{array}{c}\textbf{Lemma 5.2}\\ \text{(Preliminary result for W_{-})}\end{array} \arrow[d] \arrow[r] & \begin{array}{c}\textbf{Theorem 5.4}\\ \text{(Explicit wave operators $W_{\pm}$)}\end{array} \\
\begin{array}{c}\textbf{Corollary 5.3}\\ \text{(Isometry of $\mathcal{F}$)}\end{array} &
\end{tikzcd}
$$

Figure 1: Proof of $\mathcal{F}$'s isometry.
(All references to Chapter 1.)

$$
\begin{tikzcd}
\begin{array}{c}\textbf{Lemmas 1.4 and 3.1}\\ (y\text{-asymptotics and }\psi\text{-analyticity})\end{array} \arrow[d] & \\
\begin{array}{c}\textbf{Lemma 3.4}\\ (\text{Definedness of }\mathcal{F}_\pm^* \restriction \mathcal{C})\end{array} \arrow[d] & \begin{array}{c}\textbf{Lemma 1.1}\\ (\psi(x,y)\text{ as g.eigenfunction of }\mathcal{S}(y))\end{array} \arrow[dl] \\
\begin{array}{c}\textbf{Theorem 6.3}\\ (\text{Symmetry formula for }\overline{\mathcal{S}})\end{array} \arrow[d] & \begin{array}{c}\textbf{Lemma 6.2}\\ (y\text{-poles of }\hat{J}_\pm(y,x,x'))\end{array} \arrow[l] \\
\begin{array}{c}\textbf{Corollaries 7.1 and 7.2}\\ (\text{Symmetry/symmetry breakdown of }\overline{\mathcal{S}})\end{array} \arrow[dd] & \begin{array}{c}\textbf{Lemmas 2.2 and 2.3}\\ (y_m\text{-specialisations of }\psi(x,y))\end{array} \arrow[l] \\
 & \begin{array}{c}\textbf{Lemma 2.1}\\ (\text{Well-definedness of }Q_m)\end{array} \arrow[dl] \\
\begin{array}{c}\textbf{Theorems 7.4 and 7.7}\\ (\text{Unitarity/unitarity breakdown of }\mathcal{F})\end{array} & \begin{array}{c}\textbf{Lemma 3.5}\\ (\text{Orthogonality: }\mathcal{F}_\pm f \perp \Psi^{(m)})\end{array} \arrow[l] \\
 & \textbf{Corollary 5.3}\text{ (Isometry of }\mathcal{F}) \arrow[ul]
\end{tikzcd}
$$

Figure 2: Proof of completeness. (All references to Chapter 1.)
Note the appearance of Corollary 5.3 here, which makes the second chain dependent on the first.

3 The system at issue

We now sketch the details of the two-particle hyperbolic Ruijsenaars-Schneider system [10], and its quantisation, in order to exhibit the connection to the AΔO studied in this work. We will do this using the conventional quantities of classical and quantum mechanics. These relate to those in the main text of the thesis as described in the re-parameterisation at the end. The position variable is unchanged, but to avoid confusion with later work we nonetheless denote it with a different symbol, r.

Classically, the system is defined by the Hamiltonian,

$$H_{rel} \equiv mc^2\ (\cosh(p_1/mc) + \cosh(p_2/mc))U(r_1 - r_2), \quad m \in (0, \infty) \tag{3.1}$$

where p_i/mc is a rapidity variable and r_i its canonically conjugate partner. The potential $U : \mathbb{R} \to \mathbb{R}$ is assumed to be smooth and even. To reproduce the Lie algebra of the Poincaré group,

$$\{H_{rel}, P\} = 0, \quad \{H_{rel}, B\} = P, \quad \{P, B\} = H_{rel}/c^2 \tag{3.2}$$

we can take as space translation and boost generators,

$$P \equiv mc\ (\sinh(p_1/mc) + \sinh(p_2/mc))U(r_1 - r_2), \quad B \equiv -m(r_1 + r_2) \tag{3.3}$$

which are seen to satisfy (3.2) fairly easy; the first reduces to the identity $\cosh^2 A - \cosh^2 B = \sinh^2 A - \sinh^2 B$. (This is quite different to the general n-particle case where insistence on the first, i.e. on space-time translation invariance, leads to a set of functional equations which is known to be solved iff U is the Weierstrass elliptic function, cf. [11]).

Similar calculations show that (3.1) Poisson commutes with the four involutive (mutually commuting in the Poisson sense) Hamiltonians

$$\begin{cases} S_{\pm 1} \equiv (\exp(\pm p_1/mc) + \exp(\pm p_2/mc))U(r_1 - r_2) & (3.4) \\ S_{\pm 2} \equiv \exp(\pm(p_1/mc + p_2/mc)) & (3.5) \end{cases}$$

and thus defines a classically integrable system (we note, once again, that it is not so straightforward in the $n > 2$ particle case where one has to consider the involutivity of $2n$ Hamiltonians. For ansätze generalising (3.4),(3.5), restrictions on U arise that *encompass* those for the space-time translation invariance problem described above; that these restrictions are satisfied by the Weierstrass function can in turn be encompassed by the AΔO commutativity problem described below). We also note, importantly,

$$H_{rel} = mc^2(S_1 + S_{-1})/2 \tag{3.6}$$

The specialisations of U that interests us are the hyperbolic ones, corresponding to a same-charge potential and its mixed-charge counterpart obtained by analytic continuation $r \to r + i\pi/2\nu$. Respectively,

$$U(r) = \begin{cases} (1 + \sin^2\tau/\sinh^2(\nu r))^{1/2} \\ (1 - \sin^2\tau/\cosh^2(\nu r))^{1/2} \end{cases} \quad \tau \equiv \nu g/mc \tag{3.7}$$

(This sees the introduction of two further parameters, $g, \nu \in (0, \infty)$, akin to a coupling parameter and an inverse interaction length, respectively.) We also note that each of these two $U(r)$ functions can be written in the form $[V(r)V(-r)]^{1/2}$ where, respectively,

$$V(r) = \begin{cases} \sinh(\nu r - i\tau)/\sinh(\nu r) \\ \cosh(\nu r - i\tau)/\cosh(\nu r) \end{cases} \tag{3.8}$$

Accordingly we now add to (3.1)-(3.6) the assumption that $U(r)$ may always be written in this form, with $V(\cdot)$ smooth. To define the corresponding quantum systems we must consider how $S_{\pm k}$ and H_{rel} are to be quantised. In the general n-particle case, there is an ambiguity that arises from the canonical prescription

$$p_j \to \hat{p}_j \equiv -i\hbar\,\partial/\partial r_j \tag{3.9}$$

which is resolved by an insistence on mutual commutativity (of the $2n$ Hamiltonians). The quantisations that we are going to use are the $n = 2$ specialisations of the general solution to this commutativity problem, even though there is in fact no ambiguity in the $n = 2$ case, even for general V (these general solutions were first given in [11]). In anticipation of their promotion to formal operators, the ordering of the symbols on the rhs of the following definitions is understood to be fixed (until such a time that we set $p_i = \hat{p}_i$, whereupon the resulting operators may be manipulated),

$$\begin{aligned} \hat{S}_{\pm 1}(r_1, r_2, p_1, p_2) \equiv{}& V(\pm r)^{1/2} \exp(\pm p_1/mc) V(\mp r)^{1/2} \\ &+ V(\mp r)^{1/2} \exp(\pm p_2/mc) V(\pm r)^{1/2}, \quad r \equiv r_1 - r_2 \end{aligned} \tag{3.10}$$

$$\hat{S}_{\pm 2}(p_1, p_2) \equiv \exp\left(\pm(p_1/mc)\right) \exp\left(\pm(p_2/mc)\right) \tag{3.11}$$

where, as required, we have $\hat{S}_{\pm k} = S_{\pm k}$. The quantisation of H_{rel} is then defined by setting $p_i = \hat{p}_i$ in

$$\hat{H}_{rel}(r_1, r_2, p_1, p_2) \equiv mc^2(\hat{S}_1 + \hat{S}_{-1})(r_1, r_2, p_1, p_2)/2 \tag{3.12}$$

where a routine Taylor series argument convinces us that the action of the exponential operators is a shift,

$$\exp\left(\pm\hat{p}_j/mc\right) = T^{r_j}_{\pm i\beta\hbar}, \quad \beta \equiv 1/mc\,,\ j = 1, 2 \tag{3.13}$$

We now define the problem of finding a joint, generalised eigenfunction of the quantised $S_{\pm k}$ (and so, by implication, of the quantised H_{rel}) as finding a $\mathcal{W}$ such that

$$\begin{aligned} \hat{S}_{\pm k}(r_1, r_2, \hat{p}_1, \hat{p}_2)\mathcal{W}(r_1, r_2, p_1, p_2) &= M_{\pm k}(p_1, p_2)\mathcal{W}(r_1, r_2, p_1, p_2), \\ p_1, p_2 \in \mathbb{R}, \quad M_{\pm k} &: \mathbb{R}^2 \to \mathbb{R}^+, \quad k = 1, 2 \end{aligned} \tag{3.14}$$

The nature of AΔOs is such that we do not expect (3.14) to have unique solutions. We also note that p_j now have the role of spectral parameters (with dimension momentum).

By introducing

$$R \equiv (r_1 + r_2)/2 \tag{3.15}$$

(in addition to r above) the following separation of variables obtains in our AΔOs,

$$\hat{S}_{\pm 1}(r_1, r_2, \hat{p}_1, \hat{p}_2) = T^R_{\pm i\beta\hbar/2} H_s(r) \tag{3.16}$$

$$H_s(r) \equiv [V(r)V(-r + i\beta\hbar)]^{1/2} T^r_{i\beta\hbar} + [V(-r)V(r + i\beta\hbar)]^{1/2} T^r_{-i\beta\hbar} \tag{3.17}$$

$$\hat{S}_{\pm 2}(r_1, r_2, \hat{p}_1, \hat{p}_2) = T^R_{\pm i\beta\hbar} \tag{3.18}$$

(where one should explicitly write out the action on functions to see this).

Introducing

$$p \equiv (p_1 - p_2)/2, \quad P \equiv p_1 + p_2 \tag{3.19}$$

the ansatz

$$\mathcal{W}(r_1, r_2, p_1, p_2) = \exp(iRP/\hbar) W(r, p) \tag{3.20}$$

is then seen to satisfy the $\pm k = 2$ equations trivially with $M_{\pm 2}(p_1, p_2) = \exp(\pm\beta P)$ and so, provided we confine our attention to $M_{\pm 1}(p_1, p_2) = M(p_1 - p_2)$, (3.14) reduces to the 1-dimensional problem of finding a $W(r, p)$ such that

$$H_s(r)W(r, p) = M(p)W(r, p), \quad p \in \mathbb{R}, \quad M : \mathbb{R} \to \mathbb{R}^+ \tag{3.21}$$

which in itself can be viewed as the time-independent Schrödinger equation for some 1-particle system. For the first type of V in (3.8) the same-charge function $F(b; x, y)$ we see later is one such $W(r, p)$, when suitably parameterised. Likewise for the second type of V and the mixed-charge function $\psi(b; x, y)$. In both cases we have $M(p) = 2\cosh(\beta p)$. These claims rest on the fact the AΔOs $H(a_+, a_-, b; x)$ and $\tilde{H}(a_+, a_-, b; x)$ are nothing but disguised versions of the two types of $H_s(r)$. We can recover the latter explicitly by re-parameterising as follows

$$a_+ = \pi/\nu, \quad a_- = \beta\hbar, \quad b = \beta g, \quad x = r \tag{3.22}$$

We also use $y = \beta p/2$ when required. (These choices are fixed in accordance with dimension requirements given the scale invariance of the two AΔOs under $(a_+, a_-, b; x) \to (\lambda a_+, \lambda a_-, \lambda b; \lambda x)$).

As a final remark we note how a solution to (3.21) yields the following AΔE (the combined effect of (3.12), (3.18), (3.20) and (3.21)),

$$\begin{aligned} \hat{H}_{rel}(r_1, r_2, \hat{p}_1, \hat{p}_2) \exp(iRP/\hbar) W(r, p) = mc^2 \cosh(\beta(p_1 + p_2)/2) M(p_1 - p_2) \\ \times \exp(iRP/\hbar) W(r, p) \end{aligned} \tag{3.23}$$

where we note,

$$\begin{aligned} \frac{2}{mc^2} \hat{H}_{rel}(r_1, r_2, \hat{p}_1, \hat{p}_2) = V(r)^{1/2} \big(T^{r_1}_{i\beta\hbar} + T^{r_2}_{-i\beta\hbar}\big) V(-r)^{1/2} \\ + V(-r)^{1/2} \big(T^{r_1}_{-i\beta\hbar} + T^{r_2}_{i\beta\hbar}\big) V(r)^{1/2}, \quad r \equiv r_1 - r_2 \end{aligned} \tag{3.24}$$

Chapter 1

The case of general b.

1 The generalised eigenfunction $\psi(b; x, y)$.

1.1 Motivation and definition.

In §3 of Chapter 0 we saw how two one-variable analytic difference operators (AΔOs) arise from the quantisation and reduction of the Hamiltonian of a particular two-particle system; the two AΔOs correspond to the same- and opposite-charge potentials, and they are given again below in a re-paramterised form (so that knowledge of these origins is not required to understand the present chapter). We now face the question of how to construct suitable generalised eigenfunctions for these AΔOs, i.e. ones with desirable quantum-mechanical features. In the first instance, this means ones whose asymptotics display a familiar time-independent scattering picture. Ultimately, it means ones that serve our method of proving self-adjoint dynamics, and other phenomena, by use of an eigenfunction transform (for an overview of the goals, see the introduction; also note the idea of 'suitable' construction alludes to the multiplicity issue described earlier.)

For the same-charge case the series of papers [1]-[3] provides a complete answer to this question. This thesis is concerned with the mixed-charge case, but it is natural to approach this by building on the material from the same-charge case, as we will see.

Central to this investigation is the relativistic conical function, $\mathcal{R}(a_+, a_-, b; x, y)$. This is a specialisation of the 8-variable 'relativistic' R-function [1] which generalises the hypergeometric ${}_2F_1$ function. More recently, the relativistic conical function was the subject of [5] where many essential features were re-derived using new integral representations. These representations are defined entirely in terms of the hyperbolic gamma function which is the fundamental building block of all the special functions considered in this work. We begin with a review of its pertinent features.

Hyperbolic gamma function, $G(a_+, a_-; z)$. This function was first introduced in [17]. It satisfies two first order analytic difference equations (AΔEs),

$$\frac{G(z + ia_\alpha/2)}{G(z - ia_\alpha/2)} = 2c_{-\alpha}(z), \quad \alpha = +, - \tag{1.1}$$

$$c_\pm(z) \equiv \cosh(\pi z/a_\pm) \tag{1.2}$$

In fact it can be uniquely defined as the solution to one of these AΔEs having a particular minimality property and normalisation $G(0)=1$; the second AΔE is then satisfied as well. (These claims draw upon our standing assumption, $a_+,a_->0$, and the minimality property amounts to requiring absence of zeros and poles in a certain $|\operatorname{Im} z|$-strip, and optimal asymptotics for $|\operatorname{Re} z|\to\infty$, cf. [17].) In the strip $|\operatorname{Im} z|<a$ it has the integral representation

$$G(a_+,a_-;z)=\exp\left(i\int_0^\infty\frac{du}{u}\left(\frac{\sin 2uz}{2\sinh(a_+u)\sinh(a_-u)}-\frac{z}{a_+a_-u}\right)\right) \tag{1.3}$$

from which one reads off absence of zeros and poles in this strip, as well as the following properties, all valid for $a_+,a_->0$,

$$G(-z)=1/G(z)\quad\text{(reflection)} \tag{1.4}$$

$$\overline{G(z)}=G(-\overline{z})\quad\text{(conjugacy)} \tag{1.5}$$

$$G(a_+,a_-;z)=G(a_-,a_+;z)\quad\text{(modular invariance)} \tag{1.6}$$

$$v\in\mathbb{R}\Rightarrow G(iv)\in\mathbb{R}\quad\text{(real-valuedness)} \tag{1.7}$$

(There is also scale invariance under multiplication of all arguments by $\lambda\in\mathbb{R}^+$, but we do not have cause to use this.) The function $G(\cdot)$ has its poles at

$$-ia-ika_+-ila_-,\quad k,l\in\mathbb{N},\quad (G\text{-poles}) \tag{1.8}$$

and its zeros at

$$ia+ika_++ila_-,\quad k,l\in\mathbb{N},\quad (G\text{-zeros}) \tag{1.9}$$

The pole at $-ia$ is simple, and so is the zero at ia. We also note the dominant asymptotics, as cited for example in Appendix A of [5],

$$G(a_+,a_-;z)\sim\exp(\mp i(\chi+\pi z^2/2a_+a_-)),\quad |\operatorname{Re} z|\to\infty \tag{1.10}$$

where

$$\chi\equiv\frac{\pi}{24}\left(\frac{a_+}{a_-}+\frac{a_-}{a_+}\right) \tag{1.11}$$

Relativistic conical function. With this in place, let us now return to the relativistic conical function [5], $\mathcal{R}(a_+,a_-,b;x,y)$. Again, the parameters a_+,a_- are taken to be real and positive throughout. This function is meromorphic in b,x,y and has the following useful properties,

$$\mathcal{R}(x,y)=\mathcal{R}(y,x),\quad\text{(self-duality)} \tag{1.12}$$

$$\mathcal{R}(x,y)=\mathcal{R}(\alpha x,\alpha' y),\quad \alpha,\alpha'=+,-\quad\text{(evenness)} \tag{1.13}$$

$$\mathcal{R}(a_+,a_-;x,y)=\mathcal{R}(a_-,a_+;x,y),\quad\text{(modular invariance)} \tag{1.14}$$

$$\mathcal{R}(b;x,y)=\overline{\mathcal{R}(b;\overline{x},\overline{y})},\quad b\in\mathbb{R}\quad\text{(real-valuedness)} \tag{1.15}$$

where we have suppressed any dependence on a_+, a_-, b not relevant to the property at hand. (The same note about scale invariance also applies.) Furthermore, there are two known choices of the variables x, y for which the function reduces to a constant in the other variable, as expressed in

$$\mathcal{R}(b; x, \pm ib) = 1 \tag{1.16}$$

In all of the work that follows, we will use a renormalised version,

$$\mathcal{R}_r(a_+, a_-, b; x, y) \equiv \frac{G(ib - ia)}{G(2ib - ia)} \mathcal{R}(a_+, a_-, b; x, y) \tag{1.17}$$

which inherits the properties just listed. In addition, it has the property that all of its poles are dependent on x or y. Specifically, they are given at (or, more accurately, can only occur at),

$$\pm z = 2ia - ib + ika_+ + ila_-, \quad z = x, y, \quad k, l \in \mathbb{N} \tag{1.18}$$

$$a \equiv (a_+ + a_-)/2 \tag{1.19}$$

We will also have cause to use its asymptotics. To specify these we need a function not yet encountered, so we return to this later (cf. (1.55) below). Meanwhile, let us look at an integral representation of $\mathcal{R}_r(b; x, y)$. Of the five presented in [5], the most useful for us comes from combining [5](1.3) and [5](1.6),

$$\mathcal{R}_r(b; x, y) = \frac{G(ib - ia)}{\sqrt{a_+ a_-}} \int_{\mathbb{R}} dz \, \frac{G(z + (x-y)/2 - ib/2) G(z - (x-y)/2 - ib/2)}{G(z + (x+y)/2 + ib/2) G(z - (x+y)/2 + ib/2)} \tag{1.20}$$

which is valid in particular for $(b, x, y) \in (0, 2a) \times \mathbb{R}^2$. With this representation, the properties of self-duality and invariance under simultaneous negation of x and y can be read off straight away. Modular invariance follows from that of $G(a_+, a_-; z)$, while evenness in x and y follows from (1.4), and real-valuedness from (1.5) and (1.7).

The same-charge generalised eigenfunction, $F(b; x, y)$**.** Having reviewed these two functions, we return to the central issue of finding suitable generalised eigenfunctions for the two physical AΔOs we saw earlier ((3.17) with (3.8) in the Chapter 0, or simply (1.30) and (1.44) below). We look first at the same charge case, this having already been treated in the literature. The key fact is the following generalised eigenvalue AΔE (proved in [5]),

$$A(a_+, a_-, b; x) \mathcal{R}_r(a_+, a_-, b; x, y) = 2c_+(y) \mathcal{R}_r(a_+, a_-, b; x, y) \tag{1.21}$$

where

$$A(a_+, a_-, b; z) \equiv V(a_+, b; z) \, T^z_{ia_-} + (z \to -z) \tag{1.22}$$

and

$$V(a_+, b; z) \equiv \frac{s_+(z - ib)}{s_+(z)} \tag{1.23}$$

$$s_\pm(z) \equiv \sinh(\pi z / a_\pm) \tag{1.24}$$

Note that because of the invariance of $\mathcal{R}_r(a_+,a_-,b;x,y)$ under $a_+ \leftrightarrow a_-$ and $x \leftrightarrow y$ we immediately get three more AΔEs for this function by effecting these swaps in (1.21).

The idea is to use (1.21) to crack the issue described above. We look first at the same-charge case. Clearly $A(b;x)$ does not have the form of the pertinent Hamiltonian. Nevertheless, by using the defining AΔE for $G(a_+,a_-;z)$, (1.1), we can construct a similarity transform which connects the two. The key here is the Harish-Chandra function,

$$c(b;z) \equiv G(z+ia-ib)/G(z+ia) \tag{1.25}$$

Using (1.5) and (1.1) respectively, we find that this satisfies

$$\overline{c(b;z)} = c(b;-\overline{z}), \quad b \in \mathbb{R} \tag{1.26}$$

$$c(b;z) = \frac{s_{-\alpha}(z-ib)}{s_{-\alpha}(z)}\, c(b;z-ia_\alpha), \quad \alpha = +,- \tag{1.27}$$

Thus the following weight function will serve as such a similarity transform,

$$w(b;z) \equiv 1/c(b;z)c(b;-z) \tag{1.28}$$

(the relation (1.26) entails manifest reality and non-negativity for real z). Also note that all the same-charge functions defined here have a manifest invariance under interchange of the parameters a_+,a_-, which is not true of their mixed-charge counterparts (this is also why there is no ambiguity in the definition (1.25)).

To see how the similarity transform works, let us define

$$H(a_+,a_-,b;x) \equiv w(b;x)^{1/2}A(a_+,a_-,b;x)w(b;x)^{-1/2} \tag{1.29}$$

and later come back to the question of how these square roots behave. Using (1.27) with $\alpha = -$, we see that

$$H(a_+,a_-,b;x) = [V(a_+,b;x)V(a_+,b;-x+ia_-)]^{1/2}\, T^x_{ia_-} + (x \to -x) \tag{1.30}$$

where the term in square brackets equals

$$\frac{s_+(x-ib)}{s_+(x)}\frac{s_+(x+ib-ia_-)}{s_+(x-ia_-)} \tag{1.31}$$

Thus $H(b;x)$ has the same form as the same-charge Hamiltonian from earlier (for explicit equality we just have to specialise the variables appropriately, recall (3.22) in Chapter 0).

Invoking (1.21), we may therefore conclude that the following is a generalised eigenfunction of $H(a_+,a_-,b;x)$ (1.29),

$$F(b;x,y) \equiv w(b;x)^{1/2}w(b;y)^{1/2}\mathcal{R}_r(b;x,y) \tag{1.32}$$

with eigenvalue $2c_+(y)$. This is the function that solves the same-charge problem. From its constituent functions it inherits the properties of modular invariance and evenness, and by construction is self-dual, given (1.12). As with (1.21), we may use these symmetries to obtain three more AΔEs by effecting the swaps $a_+ \leftrightarrow a_-$ and $x \leftrightarrow y$ in the generalised eigenvalue equation involving $H(a_+, a_-, b; x)$.

From [5](2.65), or by using (1.10) and (1.55) below, we have dominant asymptotics

$$F(b; x, y) \sim (-u(b; y))^{1/2} e^{i\pi xy/a_+a_-} + (-u(b; y))^{-1/2} e^{i\pi xy/a_+a_-}, \quad x \to \infty \tag{1.33}$$

where the function

$$u(b; y) \equiv -c(b; y)/c(b; -y) \tag{1.34}$$

defined in terms of $c(b; z)$ (1.25) is manifestly a phase for $y \in \mathbb{R}$ given (1.26). Moreover, from results going back to [3], it is known that the eigenfunction transform

$$(\mathcal{E}f)(x) \equiv (2a_+a_-)^{-1/2} \int_{\mathbb{R}^+} dy\, F(b; x, y) f(y) \tag{1.35}$$

defines a unitary map $\mathcal{E} : L^2(\mathbb{R}^+, dy) \to L^2(\mathbb{R}^+, dx)$ provided the coupling parameter b satisfies $b \in [0, a_+ + a_-]$. Quantum mechanically, this corresponds to a system without bound states and scattering encoded by $u(b; y)$ (1.34). (See (2.20) in Chapter 0 for an explicit integral representation of this phase.)

Analysis of $w(b; x)$. When we introduced the AΔO $H(b; x)$ we overlooked analytic complications arising from the square roots. To address this we need to look more closely at $w(b; x)$ (1.28). We present a detailed picture of this function's behaviour in the variable x when b is real (though the claims can be adapted to complex b fairly easily).

From repeated use of (1.1) it follows that

$$G(x + ia)/G(x - ia) = 4s_+(x)s_-(x) \tag{1.36}$$

and so we may write, recalling (1.4),

$$w(b; x) = 4s_+(x)s_-(x)w_r(b; x) \tag{1.37}$$

$$w_r(b; x) \equiv \prod_{\alpha=+,-} G(\alpha x - ia + ib) \tag{1.38}$$

The property (1.5) entails the two terms in this product are conjugates of each other and so $w_r(b; x)$ is manifestly real and non-negative for $x \in \mathbb{R}$. Indeed since the zeros of $G(\cdot)$ are purely imaginary for real b this implies positivity for $x \in \mathbb{R}^*$. This can be extended to $\mathbb{R}$ iff all the x-zeros of $w_r(b; x)$ that derive from (1.9) are away from the origin. In general the latter property is neither true nor false but depends on the values of the parameters b, a_+, a_-. With our standing assumption that $a_+, a_- > 0$, we claim it holds for all real b except for certain integer combinations of a_+ and a_-. More specifically, we need

$$b \in \mathbb{R} \setminus S^+, \quad S^+ \equiv \{(k+1)a_+ + (l+1)a_- \,|\, k, l \in \mathbb{N}\} \tag{1.39}$$

(this just follows from careful use of (1.9)).

A similar consideration of the x-poles of $w_r(b; x)$, which are all on the imaginary axis provided $b \in \mathbb{R}$, reveals that they are away from the origin provided

$$b \in \mathbb{R} \setminus S^-, \quad S^- \equiv \{-ka_+ - la_- \,|\, k, l \in \mathbb{N}\} \tag{1.40}$$

Thus provided these two restrictions hold, $w_r(b;\cdot)$ is a real-analytic, even function which is positive on $\mathbb{R}$. The other term $s_+(x)s_-(x)$ is a non-negative, real-analytic, even function with a single, immutable zero at $x=0$. Thus the product with $w_r(b;x)$ has all of these properties too.

We conclude that the following restriction is sufficient to ensure $w(b;\cdot)^{1/2}$ is a real-analytic, odd function which is real-valued on $\mathbb{R}$ and whose only zero is at $x=0$,

$$b \in \mathbb{R} \setminus S, \quad S \equiv S^+ \cup S^- \tag{1.41}$$

A notable special case of this is

$$b \in (0, 2a) \tag{1.42}$$

The mixed-charged generalised eigenfunction, $\psi(b;x,y)$**.** Almost all that we have detailed so far is known from [3] and [5]. The question we now address is how to build a generalised eigenfunction for the other one-variable AΔO from §3 of Chapter 0 (i.e. (1.44) below). We aim for x-asymptotics displaying a suitable transmission-reflection picture (time-*in*dependent scattering). This function will then be taken forward and used in our construction of a complete Hilbert space theory (part of which involves time-*de*pendent scattering). The answer to the asymptotics question was presented in the author's joint paper [32]. We proceed to summarise the pertinent results.

As already claimed, the same-charge system provides the best inroad into the mixed-charge problem. The most important observation is that we can transmute the potential (1.31) into its mixed-charge analogue by applying either of the shifts $x \to x \pm ia_+/2$. Thus straight away we can assert, using (1.21), (1.29) and (1.31), that both

$$w(b;x \pm ia_+/2)^{1/2}\mathcal{R}_r(b;x \pm ia_+/2, y) \tag{1.43}$$

are generalised eigenfunctions of

$$\tilde{H}(a_+,a_-,b;x) = \left[\frac{c_+(x-ib)}{c_+(x)}\frac{c_+(x+ib-ia_-)}{c_+(x-ia_-)}\right]^{1/2} T^x_{ia_-} + (x \to -x) \tag{1.44}$$

with eigenvalue $2c_+(y)$ (we give our formal definition of $\tilde{H}(a_+,a_-,b;x)$ below, analogising (1.29), and we analyse the square roots in due course). For later use, we note that the term in square brackets here can also be written as

$$\tilde{V}(a_+,b;x)\tilde{V}(a_+,b;-x+ia_-) \tag{1.45}$$

which involves a mixed-charge analogue of (1.23),

$$\tilde{V}(a_+,b;x) \equiv \frac{c_+(x-ib)}{c_+(x)} \tag{1.46}$$

Note that by making the x-shift in $\mathcal{R}_r(b;x,y)$, we rob the object of certain symmetries - namely, self-duality (1.12) and modular invariance (1.14)) - because of the preference expressed for x and a_+. Thus these properties are *not* inherited by $\psi(b;x,y)$ below.

We can improve on the functions in (1.43), in the sense of finding ones which are simpler, by observing that the two shifted weight functions are equal to the same function multiplied by different ia_--periodic functions. More specifically, using (1.27) with $\alpha = +$ we find

$$w(b; x \pm ia_+/2) = \frac{s_-(x \pm ia_+/2)}{s_-(x \pm i(a_+/2 - b))}\, \tilde{w}(b; x) \tag{1.47}$$

where

$$\tilde{w}(b; x) \equiv 1/\tilde{c}(b; x)\tilde{c}(b; -x) \tag{1.48}$$

and

$$\tilde{c}(b; x) \equiv c(b; x - ia_+/2) = G(x + ia_-/2 - ib)/G(x + ia_-/2) \tag{1.49}$$

Given the ia_--periodicity in x of the quotients in (1.47), we can assert straight away that both

$$\tilde{w}(b; x)^{1/2}\mathcal{R}_r(b; x \pm ia_+/2, y) \tag{1.50}$$

are generalised eigenfunctions of (1.44) with eigenvalue $2c_+(y)$. Moreover, $\tilde{w}(b; x)$ is necessarily real and non-negative for real x and b, unlike the two shifted weight functions in (1.43) (to see this, just note that $\tilde{c}(b; x)$ retains the property (1.26)). Thus we may think of it as the mixed-charge counterpart of the weight function in (1.28). The square root in $\tilde{w}(b; x)^{1/2}$ is always assumed to be positive. Its properties are analysed later in this section.

We note that (1.44) can be re-interpreted in terms of $\tilde{w}(b; x)$ as follows. Using (1.1), or adapting (1.27), we have

$$\tilde{c}(b; x) = \frac{c_+(x - ib)}{c_+(x)}\, \tilde{c}(b; x - ia_-) \tag{1.51}$$

Thus, to analogise (1.29) we may write (1.44) as

$$\tilde{H}(a_+, a_-, b; x) \equiv \tilde{w}(b; x)^{1/2}\tilde{A}(a_+, a_-, b; x)\tilde{w}(b; x)^{-1/2} \tag{1.52}$$

where

$$\tilde{A}(a_+, a_-, b; x) \equiv \tilde{V}(a_+, b; x)\, T^x_{ia_-} + (i \to -i) \tag{1.53}$$

The function which will realise our aims is neither of those in (1.50), but rather a special combination of them. From what we have said so far we know that any function of the form

$$\tilde{w}(b; x)^{1/2}\big[\varphi_1(y)\mathcal{R}_r(b; x + ia_+/2, y) + \varphi_2(y)\mathcal{R}_r(b; x - ia_+/2, y)\big] \tag{1.54}$$

is a generalised eigenfunction of (1.44) with eigenvalue $2c_+(y)$. The question is how to choose $\varphi_1(y)$ and $\varphi_2(y)$ such that the x-asymptotics of this function reproduce a familiar reflection-transmission picture. At the same time, we may hope that these choices also ensure (1.54) is a generalised eigenfunction of a sensible A∆O in the dual variable y.

To pursue this we first need the dominant asymptotics of the relativistic conical function. We extract this from [5](2.71),

$$\mathcal{R}_r(b;x,y) \sim \exp(-\pi bx/a_+a_-)\big(c(b;y)e^{i\pi xy/a_+a_-} + c(b;-y)e^{-i\pi xy/a_+a_-}\big),$$
$$(b,y) \in \mathbb{R} \times (0,\infty), \quad \mathrm{Re}\, x \to \infty \quad (1.55)$$

where we recall $c(b;y)$ is defined in (1.25) (also, the limited range of y will not present any problems for us).

We also need the dominant asymptotics of the weight function (1.48). Using (1.10) and (1.25) we calculate

$$\tilde{c}(b;x) \sim \tilde{\phi}(b)^{\pm 1} \exp(\mp \pi bx/a_+a_-), \quad \mathrm{Re}\, x \to \pm\infty \quad (1.56)$$

where $\tilde{\phi}(b) \equiv \exp(i\pi b(b-a_-)/2a_+a_-)$. Thus we have

$$\tilde{w}(b;x) \sim \exp(\pm 2\pi bx/a_+a_-), \quad \mathrm{Re}\, x \to \pm\infty \quad (1.57)$$

Then, adapting (1.55), we have

$$\mathcal{R}_r(b;x \pm ia_+/2,y) \sim e^{-\pi bx/a_+a_-} e_-^{\mp ib/2}$$
$$\times \left(c(b;y)e_-^{\mp y/2} e^{i\pi xy/a_+a_-} + c(b;-y)e_-^{\pm y/2} e^{-i\pi xy/a_+a_-}\right), \quad \mathrm{Re}\, x \to \infty \quad (1.58)$$

The $\mathrm{Re}\, x \to -\infty$ asymptotics can be obtained from this by invoking the evenness (1.13),

$$\mathcal{R}_r(b;x \pm ia_+/2,y) \sim e^{\pi bx/a_+a_-} e_-^{\pm ib/2}$$
$$\times \left(c(b;y)e_-^{\pm y/2} e^{-i\pi xy/a_+a_-} + c(b;-y)e_-^{\mp y/2} e^{i\pi xy/a_+a_-}\right), \quad \mathrm{Re}\, x \to -\infty \quad (1.59)$$

The square root of the exponential in (1.57) will cancel with those in (1.58) and (1.59), leaving a product with $O(1)$ asymptotics. Even so, it is still far from obvious that they can be put together in (1.54) to achieve transmission-reflection asymptotics. However, we claim that this can be done and that the solution is contained in the following,

$$\psi(a_+,a_-,b;x,y) \equiv \tilde{w}(b;x)^{1/2} \left(2s_-(ib-y)c(b;-y)\right)^{-1}$$
$$\times \sum_{\nu=+,-} \nu e_-(\nu(ib-y)/2)\mathcal{R}_r(b;x+i\nu a_+/2,y) \quad (1.60)$$

($c(b;y)$ and $\tilde{w}(b;x)$ are defined respectively in (1.48) and (1.25)). The asymptotics of this can be calculated straightforwardly using (1.57)-(1.59). The end result is given in Lemma 1.2 below. For now, we are going to show that $\psi(b;x,y)$ is also a generalised eigenfunction of an AΔO in the dual variable y. By construction, we know that it is a generalised eigenfunction of $\tilde{H}(b;x)$, but since it no longer has self-duality we do not automatically get a dual counterpart (cf. the discussions below (1.23) and (1.46)).

Above we have used the notation from our conventions page,

$$e_\pm^z \equiv e_\pm(z) \equiv \exp(\pi z/a_\pm) \quad (1.61)$$

Dual variable properties of $\psi(b;x,y)$. Because $\mathcal{R}_r(b;x,y)$ is a generalised eigenfunction of $A(b;y)$ (1.22) with eigenvalue $2c_+(x)$, the two additive components in $\psi(b;x,y)$, corresponding to the two choices of ν, are generalised eigenfunctions respectively of

$$i\nu \frac{e_-(-\nu y/2)}{s_-(ib-y)c(b;-y)} A(b;y) \frac{s_-(ib-y)c(b;-y)}{e_-(-\nu y/2)}, \quad \nu = +,- \tag{1.62}$$

with eigenvalue $2s_+(x)$ (this is just similarity transform combined with the effect of the x-shift on $2c_+(x)$). Thus if we are to get a dual, generalised eigenvalue equation for $\psi(b;x,y)$, then (1.62) must be non-manifestly independent of ν. This is indeed the case; the claim reduces to the easily-verified operator equation,

$$e_-(-\nu y/2)\, T^y_{\pm ia_-} e_-(\nu y/2) = \mp i\nu\, T^y_{\pm ia_-}, \quad \nu = +,- \tag{1.63}$$

Because of its ia_--antiperiodicity, $s_-(ib-y)$ will pass through $A(b;y)$ in (1.62), depositing an overall minus sign. This fact, combined with (1.63), allows us to conclude that (1.62) equals

$$\frac{1}{c(b;-y)} \left[V(y) T^y_{ia_-} - V(-y) T^y_{-ia_-} \right] c(b;-y) \tag{1.64}$$

and that $\psi(b;x,y)$ is a generalised eigenfunction with eigenvalue $2s_+(x)$. To simplify this, we use (1.27) with $\alpha = -$ to write down the two AΔEs,

$$c(b;y+i\tilde{\alpha}a_-)/c(b;y) = \begin{cases} V(y+ia_-), & \tilde{\alpha} = + \quad (1.65) \\ 1/V(y), & \tilde{\alpha} = - \quad (1.66) \end{cases}$$

(which we can obviously adapt for $y \to -y$ as necessary). Thus (1.64) is the AΔO

$$\mathcal{S}(b;y) \equiv V(a_+,b;y)V(a_+,b;-y+ia_-)T^y_{ia_-} - T^y_{-ia_-} \tag{1.67}$$

where $V(a_+,b;y) = s_+(y-ib)/s_+(y)$ was first seen in (1.23). With this, the coefficient of $T^y_{ia_-}$ may also be written as

$$U(y) \equiv \frac{s_+(y-ib)}{s_+(y)} \frac{s_+(y+ib-ia_-)}{s_+(y-ia_-)} \tag{1.68}$$

There is another useful property of $\psi(b;x,y)$ (1.60) that links conjugation to negation of the dual variable. Using real-valuedness of $\tilde{w}(b;x)$ (1.48) and the conjugacy properties (1.15) and (1.26) we have

$$\begin{aligned} \overline{\psi(b;x,y)} &= \tilde{w}(b;x)^{1/2}(-2s_-(ib+y)c(b;y))^{-1} \\ &\quad \times \sum_{\nu=+,-} \nu e_-(-\nu(ib+y)/2)\mathcal{R}_r(b;x-i\nu a_+/2,y) \\ &= \psi(b;x,-y), \quad (b,x,y) \in \mathbb{R}^3 \end{aligned} \tag{1.69}$$

The final equality is manifest from (1.60) once we recall evenness of $\mathcal{R}_r(b;x,y)$ in y.

Analysis of $\tilde{w}(b;x)$. Both $\tilde{H}(a_+,a_-,b;x)$ (1.52) and $\psi(b;x,y)$ (1.60) feature the square-root function $\tilde{w}(b;x)^{1/2}$ (1.48) which we must study more closely. We present a detailed picture of its behaviour in the variable x for real b. We will impose conditions on b to ensure it has no poles or zeros for real x.

Using (1.1) it is clear that we have

$$\tilde{w}(b;x) = 2c_+(x) \prod_{\alpha=+,-} G(\alpha x - ia_-/2 + ib) \tag{1.70}$$

The property (1.5) entails the two terms in this product are conjugates of each other and so $\tilde{w}(b;x)$ is manifestly real and non-negative for $x \in \mathbb{R}$. In fact since the zeros of $G(\cdot)$ are purely imaginary when $b \in \mathbb{R}$, cf. (1.9), we have positivity for $x \in \mathbb{R}^*$. This can be extended to $\mathbb{R}$ iff all the x-zeros of $\tilde{w}(b;x)$ that derive from (1.9) are away from the origin ($c_+(x)$ has no real zeros). We claim this property is assured provided

$$b \in \mathbb{R} \setminus \tilde{S}^+, \quad \tilde{S}^+ \equiv \{(k+1/2)a_+ + (l+1)a_- \mid k,l \in \mathbb{N}\} \tag{1.71}$$

(this just follows from careful use of (1.9). For an explicit description of the zeros, cf. (3.4)).

A similar consideration of the x-poles of $\tilde{w}(b;x)$, which are all on the imaginary axis when $b \in \mathbb{R}$, reveals that they are away from the origin provided

$$b \in \mathbb{R} \setminus \tilde{S}^-, \quad \tilde{S}^- \equiv \{-(k+1/2)a_+ - la_- \mid k,l \in \mathbb{N}\} \tag{1.72}$$

(For an explicit description of the poles, cf. (3.5).) Thus $\tilde{w}(b;\cdot)$ (1.48) is a real-analytic, even function which is positive on the real line. Its positive square-root inherits these features provided

$$b \in \mathbb{R} \setminus \tilde{S}, \quad \tilde{S} \equiv \tilde{S}^+ \cup \tilde{S}^- \tag{1.73}$$

A notable special case of this is

$$b \in (0, a_- + a_+/2) \tag{1.74}$$

Generalised eigenfunction properties of $\psi(b;x,y)$. Let us summarise some of the main findings so far. (Further analyticity properties are given later in Lemma 3.1.)

Lemma 1.1. *The function $\psi(b;x,y)$ (1.60) is meromorphic in y and its square is meromorphic in x. Its branch points in x are away from the real line provided $b \in \mathbb{R}\setminus\tilde{S}$, where the point set $\tilde{S}$ is defined in (1.73). The positive square root function $\tilde{w}(b;x)^{1/2}$ has the properties described above (1.73) when this b-restriction is in force.*

The function $\psi(b;x,y)$ satisfies the following generalised eigenvalue equations with the $A\Delta Os$ $\tilde{H}(b;x)$ (1.44) and $\mathcal{S}(b;y)$ (1.67),

$$\tilde{H}(b;x)\psi(b;x,y) = 2c_+(y)\psi(b;x,y) \tag{1.75}$$

$$\mathcal{S}(b;y)\psi(b;x,y) = 2s_+(x)\psi(b;x,y) \tag{1.76}$$

Moreover it has the conjugacy property

$$\overline{\psi(b;\overline{x},\overline{y})} = \psi(b;x,-y), \quad b \in \mathbb{R} \tag{1.77}$$

and the symmetry

$$\psi(b;x,y) = \psi(a_- - b;x,y) \tag{1.78}$$

Proof. Most of these claims follow from our discussions above. Cf. in particular: (1.50) and (1.64) for the eigenvalue equations; (1.69) for the conjugacy property; and (1.73) for the claims about $\tilde{w}(b;x)^{1/2}$.

The proof of (1.78) is fairly long, and we put it at the end of this subsection. □

Asymptotics of $\psi(b;x,y)$. In preparation for the transmission and reflection asymptotics which we claim our function exhibits, we introduce the following

$$t(b;y) \equiv \frac{s_-(y)}{s_-(ib-y)} u(b;y) \tag{1.79}$$

$$r(b;y) \equiv \frac{s_-(ib)}{s_-(ib-y)} u(b;y) \tag{1.80}$$

where $u(b;y)$ (1.34) is a phase for $y \in \mathbb{R}$ which we have seen already in the asymptotics of the same-charge eigenfunction, (1.32). The property

$$|t(b;y)|^2 + |r(b;y)|^2 = 1, \tag{1.81}$$

just reduces to well-known hyperbolic trigonometric identities. The same is true of the Yang-Baxter equations,

$$r_{12}t_{13}u_{23} = t_{23}u_{13}r_{12} + r_{23}r_{13}t_{12} \tag{1.82}$$

and

$$u_{12}r_{13}u_{23} = t_{23}r_{13}t_{12} + r_{23}u_{13}r_{13} \tag{1.83}$$

with

$$s_{jk} \equiv s(y_j - y_k), \quad s = u,t,r, \quad 1 \le j < k \le 3 \tag{1.84}$$

To verify this claim in just one of the cases above, let us look more closely at (1.82). Here, the product $u(y_1-y_2)u(y_1-y_3)u(y_2-y_3)$ drops out of the equation, and what remains can be rearranged as

$$s_-(y_1-y_3)s_-(ib-(y_2-y_3)) - s_-(y_2-y_3)s_-(ib-(y_1-y_3)) = s_-(ib)s_-(y_1-y_2) \tag{1.85}$$

Using the mixed-angle identity for sinh, one can write out the lhs of this as a multiple of $c_-(ib)$ plus a multiple of $s_-(ib)$. The former is seen to vanish, and the latter is found to equal

$$s_-(y_1-y_3)c_-(y_2-y_3) - s_-(y_2-y_3)c_-(y_2-y_3) \tag{1.86}$$

Using the aforementioned identity in reverse, this equals $s_-(y_1-y_2)$ as required for manifest equality with the rhs.

We are now ready for the following lemma. (The restriction on y is needed because we have not ruled out the possibility of a pole at the origin.)

Lemma 1.2. *The dominant large-$|\operatorname{Re} x|$ asymptotics of $\psi(b;x,y)$ (1.60) are given by*

$$\psi(b;x,y) \sim \begin{cases} t(b;y)\exp(i\pi xy/a_+a_-), & \operatorname{Re} x \to \infty \\ \exp(i\pi xy/a_+a_-) - r(b;y)\exp(-i\pi xy/a_+a_-), & \operatorname{Re} x \to -\infty \end{cases} \tag{1.87}$$

provided $y \in \mathbb{R}^*$. We also have an identity

$$\psi(b;x,y) = t(b;y)\psi(b;-x,-y) - r(b;y)\psi(b;x,-y) \tag{1.88}$$

Proof. The asymptotics is just a matter of expanding $\psi(b;x,y)$ using (1.57)-(1.59). The brute force approach is somewhat messy and unilluminating, so we calculate in a more systematic way (using the conventions for indices we laid out at the start). First, we note that (1.58) and (1.59) can be combined as

$$\mathcal{R}_r(x+i\nu a_+/2,y) \sim \exp(-\epsilon\pi bx/a_+a_-)e_-(-i\epsilon\nu b/2)\sum_{\tau=+,-} c(\tau y)\exp(i\tau\epsilon\pi xy/a_+a_-) \times e_-(-\epsilon\nu\tau y/2), \quad \operatorname{Re} x \to \epsilon\infty, \quad \nu,\epsilon = +,- \tag{1.89}$$

(where we now suppress implicit b-dependence of functions). Then we substitute this and (1.57) into $\psi(x,y)$ (1.60) to get

$$\psi(x,y) \sim (2s_-(ib-y)c(-y))^{-1}\sum_{\tau=+,-} c(\tau y)\exp(i\epsilon\tau\pi xy/a_+a_-)\,u_{\epsilon,\tau}, \quad \operatorname{Re} x \to \epsilon\infty, \quad \epsilon = +,- \tag{1.90}$$

$$u_{\epsilon,\tau} \equiv \sum_{\nu=+,-} \nu e_-(\nu(ib-y)/2)e_-(-i\epsilon\nu b/2)e_-(-\epsilon\nu\tau y/2), \quad \epsilon,\tau = +,- \tag{1.91}$$

Introducing the object,

$$U_{\epsilon,\tau} \equiv (2s_-(ib-y)c(-y))^{-1}c(\tau y)u_{\epsilon,\tau} \tag{1.92}$$

we thus have the picture

$$\psi(x,y) \sim \begin{cases} U_{+,+}\exp(i\pi xy/a_+a_-) + U_{+,-}\exp(-i\pi xy/a_+a_-), & \operatorname{Re} x \to \infty \\ U_{-,+}\exp(-i\pi xy/a_+a_-) + U_{-,-}\exp(i\pi xy/a_+a_-), & \operatorname{Re} x \to -\infty \end{cases} \tag{1.93}$$

It is then trivial to compute

(ϵ,τ)	$u_{\epsilon,\tau}$
$(+,+)$	$-2s_-(y)$
$(+,-)$	0
$(-,+)$	$2s_-(ib)$
$(-,-)$	$2s_-(ib-y)$

Recombining this with (1.93) and (1.92), the claim follows.

For the identity (1.88), we first use evenness of $\tilde{w}(b;x)$ (1.48) and evenness of $\mathcal{R}_r(b;x,y)$ in both arguments to write out (using (1.60)),

$$\psi(\pm x,-y) \;=\; \tilde{w}(x)^{1/2}(2s_-(ib\,+\,y)c(y))^{-1}\sum_{\nu=+,-}\nu e_-(\nu(ib\,+\,y)/2)\mathcal{R}_r(x\,\pm\,i\nu a_+/2,y) \tag{1.94}$$

With this we find (sending $\nu\to-\nu$ in the expression for $\psi(-x,-y)$),

$$\begin{aligned} t(y)\psi(-x,-y)-r(y)\psi(x,-y) &= -\tilde{w}(x)^{1/2}(2s_-(ib+y)c(y))^{-1} \\ \times \sum_{\nu=+,-}\nu\mathcal{R}_r(x+i\nu a_+/2)&\big[e_-(-\nu(ib+y)/2)t(y)+e_-(\nu(ib+y)/2)r(y)\big] \end{aligned} \tag{1.95}$$

From the definitions of t and r, the term in square brackets equals $u(y)/s_-(ib-y)$ multiplied by

$$e_-(-\nu(ib+y)/2)s_-(y)+e_-(\nu(ib+y)/2)s_-(ib) \quad \nu=+,- \tag{1.96}$$

Expanding the $s_-(\cdot)$ terms in terms of exponentials and recombining, one finds this can be written as

$$e_-(\nu(ib-y)/2)s_-(ib+y), \quad \nu=+,- \tag{1.97}$$

Thus the rhs of (1.95) equals

$$-\,\tilde{w}(x)^{1/2}(2s_-(ib-y)c(y))^{-1}u(y)\sum_{\nu=+,-}\nu e_-(\nu(ib-y)/2)\mathcal{R}_r(x\pm i\nu a_+/2,y) \tag{1.98}$$

Recalling the definition $u(y)\equiv -c(y)/c(-y)$, we thus get manifest equality with $\psi(x,y)$ (1.60).

□

1.2 O-asymptotics.

In this section we present a more detailed account of the asymptotics in terms of order notation, what we will call O-asymptotics (this allows us to handily distinguish between dominant asymptotics). The discussion is rather technical, but the additional information is needed for some of our later proofs. In addition we will look at the y-asymptotics of $\psi(b;x,y)$ which we did not consider previously.

To calculate the asymptotics in the previous subsection we needed those of $G(a_+,a_-;z)$ (1.1) and $\mathcal{R}_r(a_+,a_-,b;x,y)$ (1.20). We took these from elsewhere, as opposed to deriving them anew. The same is true here. For example, from [5](A.13) we are going to use

$$G(a_+,a_-;z)=\exp(\mp i(\chi+\pi z^2/2a_+a_-))\big(1+O(\exp(-r|\operatorname{Re}z|))\big), \quad \operatorname{Re}z\to\pm\infty \tag{1.99}$$

where the decay constant r can be any positive number satisfying $r < 2\pi \min(1/a_+, 1/a_-)$, and where the bound represented by O is uniform for $\operatorname{Im} z$ varying over any compact subset of $\mathbb{R}$. For illustrative purposes, we can see explicitly how the imaginary part of z affects the dominant asymptotics by writing out

$$G(a_+,a_-;z) = \exp(\pi|u|v/a_+a_-)\big(1+O(\exp(-r|u|))\big), \quad u\to\pm\infty, \qquad u=\operatorname{Re} z, \quad v=\operatorname{Im} z \tag{1.100}$$

Using (1.99) we now calculate the following for $c(b;z)$ (1.25),

$$c(b;z) = \frac{G(z+ia-ib)}{G(z+ia)} = \phi(b)^{\pm 1}\exp(\mp\pi b z/a_+a_-)\big(1+O(e^{-r|\operatorname{Re} z|})\big), \quad \operatorname{Re} z\to\pm\infty \tag{1.101}$$

where

$$\phi(b) \equiv \exp(i\pi b(b-2a)/2a_+a_-), \quad a\equiv (a_-+a_+)/2 \tag{1.102}$$

What we said below (1.99) concerning the decay, holds here also. We can thus use these asymptotics to get those for $\tilde{c}(b;x)$ also,

$$\tilde{c}(b;x) = c(b;x-ia_+/2) = \tilde{\phi}(b)^{\pm 1}\exp(\mp\pi b z/a_+a_-)\big(1+O(e^{-r|\operatorname{Re} z|})\big), \quad \operatorname{Re} z\to\pm\infty \tag{1.103}$$

where $\tilde{\phi}(b)\equiv \exp(i\pi b(b-a_-)/2a_+a_-)$. And, again, what we said below (1.99) concerning the decay, holds here also

To specify the O-asymptotics for the relativistic conical function we must combine results from [5] and [2]. The end result is that[1]

$$\mathcal{R}_r(b;x,y) = \exp(-\pi b x/a_+a_-)\big[\mathcal{R}_{as}(x,y)+O(\exp(-\rho\operatorname{Re} x))\big], \qquad (b,y)\in\mathbb{R}\times(0,\infty), \quad \operatorname{Re} x\to\infty \tag{1.104}$$

$$\mathcal{R}_{as}(x,y) \equiv \sum_{\tau=+,-} c(b;\tau y)\exp(i\tau\pi x y/a_+a_-) \tag{1.105}$$

[1]Specifically, one starts from [5](2.39),

$$\mathcal{R}_r(x,y) = \phi_r^{-1}c(y)c(x)E(x,y)$$

where this function E is a specialisation of the the function $\mathcal{E}(\gamma;v,\hat{v})$ in [2], obtained by setting $(\gamma,v,\hat{v})\equiv(\gamma_r,x,y/2)$ and $\gamma_r\equiv(b-a,-a_-/2,-a_+/2,0)$, cf. [5](2.38). From Theorem 1.2. in [2] we get the O-asymptotics,

$$E(x,y) = \mathcal{R}_{as}(x,y)/c(y) + O(\exp(-\rho\operatorname{Re} x), \quad \operatorname{Re} x\to\infty$$

where $\rho>0$ is a constant fixed by a_+,a_-, and the bound is uniform for $\operatorname{Im} x$ varying over any compact subset of $\mathbb{R}$. With the O-asymptotics established for $\operatorname{Re} x\to\infty$, by combining the two equations in this footnote with our asymptotics calculation for $c(b;z)$, those for $-\infty$ follow straight away from evenness of $\mathcal{R}_r(\cdot,y)$. We note that we thus bypass the need to use [5](2.52).

Update: this same result can now be attained independently of [2] in light of [31].

where $\rho > 0$ is a constant fixed by a_+, a_-, and where the bound represented by O is uniform for $\operatorname{Im} x$ and y varying respectively over any compact subset of $\mathbb{R}$ and $\mathbb{R}^+$. Because of its plane wave structure, $\mathcal{R}_{as}(x, y)$ will be dominant in (1.104) under $\operatorname{Re} x \to \infty$. We note the lack of information about ρ turns out not to be a problem; positivity alone will suffice in the relevant proofs.

The results just given for $G(a_+, a_-; z)$ and $\mathcal{R}_r(a_+, a_-, b; x, y)$ enable us to prove the following stronger version of Lemma 1.2,

Lemma 1.3. *For $y \in \mathbb{R}^*$, the function $\psi(b; x, y)$ (1.60) satisfies the following*

$$\psi(b; x, y) = \begin{cases} t(b; y) \exp(i\pi xy/a_+a_-) + O(e^{-\rho \operatorname{Re} x}), & \operatorname{Re} x \to \infty \\ \exp(i\pi xy/a_+a_-) - r(b; y) \exp(-i\pi xy/a_+a_-) + O(e^{\rho \operatorname{Re} x}), & \operatorname{Re} x \to -\infty \end{cases} \tag{1.106}$$

where $\rho > 0$ is a constant fixed by a_+, a_-, and where the bound represented by O is uniform for $\operatorname{Im} x$ and y varying respectively over any compact subset of $\mathbb{R}$ and $\mathbb{R}^+$. (This ρ is the same that arises in (1.104).)

Proof. The preceding discussion gives us all the ingredients we need to prove this. The O-asymptotics for $\tilde{w}(b; x)^{1/2}$ (1.48) follow from (1.103),

$$\tilde{w}(b; x)^{1/2} = \exp(\epsilon\pi bx/a_+a_-)\big(1 + O(e^{-r|\operatorname{Re} x|})\big), \quad \operatorname{Re} x \to \epsilon\infty, \quad \epsilon = +, - \tag{1.107}$$

where the same statements about r and $O(\cdot)$ apply as below (1.99).

Using the asymptotics (1.104) and the evenness property (1.13) we have

$$\begin{aligned} \mathcal{R}_r(x + i\nu a_+/2, y) = \exp(-\epsilon\pi bx/a_+a_-)e_-(-i\epsilon\nu b/2)\big[&\mathcal{R}_{as}(\epsilon(x + i\nu a_+/2), y) \\ &+ O(e^{-\rho|\operatorname{Re} x|})\big], \quad \operatorname{Re} x \to \epsilon\infty, \quad \epsilon, \nu = +, - \end{aligned} \tag{1.108}$$

(where we now suppress implicit b-dependence of functions). Plugging these into $\psi(x, y)$ (1.60) we find

$$\begin{aligned} \psi(x, y) = (2s_-(ib - y)c(-y))^{-1} \sum_{\nu=+,-} &\nu e_-(\nu(ib - y)/2)e_-(-i\epsilon\nu b/2)\mathcal{R}_{as}(\epsilon(x + i\nu a_+/2), y) \\ &+ O(e^{-\rho|\operatorname{Re} x|}), \quad \operatorname{Re} x \to \epsilon\infty, \quad \epsilon = +, - \end{aligned} \tag{1.109}$$

The dominant term here must be the same as the one we saw earlier in the proof of Lemma 1.2, and so the result follows. Nonetheless, the anxious reader can use (1.105) to check explicitly that the first line of the rhs of (1.109) equals

$$\sum_{\tau=+,-} c(\tau y) \exp(i\epsilon\tau\pi xy/a_+a_-)\, U_{\epsilon,\tau}, \quad \epsilon = +, - \tag{1.110}$$

where this is the $U_{\epsilon,\tau}$ from the proof of Lemma 1.2.

□

We are now going to consider the y-asymptotics of $\psi(b;x,y)$ (1.60). We need this primarily for use in later proofs (for example, in Theorem 6.3). However, one can also view the asymptotics from the perspective of dynamics associated to the variable y (this variable, as we explained earlier, has dimension position rather than the more standard momentum). We return to this perspective again in Appendix B.

First we need the large-$|\operatorname{Re} y|$ asymptotics of $\mathcal{R}_r(b;x,y)$. Given the self-duality of this function, this follows straight away from (1.104),

$$\mathcal{R}_r(b;x,y) = \exp(-\pi by/a_+a_-)\big[\mathcal{R}_{as}(y,x) + O(\exp(-\rho \operatorname{Re} y))\big],$$
$$(b,x) \in \mathbb{R}\times\mathbb{R}^*, \quad \operatorname{Re} y \to \infty \quad (1.111)$$

where $\rho > 0$ is a constant fixed by a_+, a_-, and where the bound represented by O is uniform for $\operatorname{Im} y$ and x varying over any compact subset of $\mathbb{R}$. (Here we have used evenness in x to replace $(0,\infty)$ with $\mathbb{R}^*$.)

The substitution of (1.111) into (1.60) raises extra complications compared with the analogous situation before, because the x-shift in $\mathcal{R}_{as}(y, x+i\nu a_+/2)$ implies the latter is no longer necessarily $O(1)$ as $\operatorname{Re} y\to\infty$. We can see this by writing out

$$\mathcal{R}_{as}(y,x+i\nu a_+/2) = \sum_{\tau=+,-} c(b;\tau(x+i\nu a_+/2))e_-(-\tau\nu y/2)\exp(i\tau\pi xy/a_+a_-), \quad \nu=+,- \quad (1.112)$$

The two τ-summands now have substantially different asymptotics. The summand which dominates is $\tau=-\nu$. Thus we have

$$\mathcal{R}_{as}(y,x+i\nu a_+/2) = c(b;-(\nu x+ia_+/2))e_-(y/2)\exp(-i\nu\pi xy/a_+a_-)$$
$$+ O(e_-(-\operatorname{Re} y/2)), \quad \operatorname{Re} y\to\infty, \quad \nu=+,- \quad (1.113)$$

In preparation for the next lemma let us substitute this back into (1.111) (which we must provisionally extend for x in a suitable strip of the complex plane) to get

$$\mathcal{R}_r(b;x+i\nu a_+/2,y) = \exp(-\pi by/a_+a_-)\big[c(b;-(\nu x+ia_+/2))e_-(y/2)\exp(-i\nu\pi xy/a_+a_-)$$
$$+ O(\exp(-\tilde\rho \operatorname{Re} y))\big], \quad (b,x)\in\mathbb{R}\times\mathbb{R}^*, \quad \operatorname{Re} y\to\infty \quad (1.114)$$

where $\tilde\rho \equiv \min(\pi/2a_-, \rho)$ for ρ arising in (1.111). The same statement about the bound applies as below (1.111).

The final two ingredients we need are

$$(2s_-(ib-y))^{-1} = -e_-((ib-y))\big(1+O(e_-^{-2\operatorname{Re} y})\big), \quad \operatorname{Re} y\to\infty \quad (1.115)$$

which follows from elementary calculation, and

$$c(b;-y)^{-1} = \phi(b)\exp(\pi by/a_+a_-)\big(1+O(e^{-r\operatorname{Re} y})\big), \quad \operatorname{Re} y\to\infty \quad (1.116)$$

which follows by adapting (1.101). Here, the decay constant r is again any positive number satisfying $r < 2\pi/\max(a_+, a_-)$.

We are now ready to prove the following lemma. We note that the term in square brackets is a phase, as follows from (1.26) and (1.49).

Lemma 1.4. *Provided $x \in \mathbb{R}^*$, the function $\psi(b; x, y)$ (1.60) satisfies the following*

$$\psi(b; x, y) = \tilde{\phi}(b)^{\pm 1}\left[\frac{\tilde{c}(b; x)}{\tilde{c}(b; -x)}\right]^{\pm 1/2} \exp(i\pi xy/a_+a_-) + O(e^{-\rho|\operatorname{Re} y|}), \quad \operatorname{Re} y \to \pm\infty \quad (1.117)$$

where $\tilde{\phi}(b) \equiv \exp(i\pi b(b - a_-)/2a_+a_-)$ and the function $\tilde{c}(b; x)$ is defined in (1.49). *Also, $\rho > 0$ is a constant fixed by a_+, a_-, and the bound represented by O is uniform for* $\operatorname{Im} y$ *and x varying over any compact subset of* $\mathbb{R}$.

Proof. First we note that because of the property (1.77), we only need to prove the asymptotics for $\operatorname{Re} y \to \infty$. It then suffices to send $y \to -\overline{y}$ and conjugate the whole expression to get the asymptotics for $\operatorname{Re} y \to -\infty$.

We substitute (1.114)-(1.116) into $\psi(x, y)$ (1.60) to get

$$\begin{aligned}\psi(x, y) = -e_-(ib - y)\tilde{w}(x)^{1/2}\,\phi(b) \sum_{\nu=+,-} \nu e_-(\nu(ib - y)/2)\big[c(-(\nu x + ia_+/2))\\ \times e_-(y/2)\exp(-i\nu\pi xy/a_+a_-) + O(\exp(-\tilde{\rho}\operatorname{Re} y))\big], \quad \operatorname{Re} y \to \infty \quad (1.118)\end{aligned}$$

where the same statements about $\tilde{\rho}$ and O apply as below (1.114). The factor $e_-(\nu(ib - y)/2)$ has substantially different asymptotic for the two choices of $\nu = +, -$. That which dominates corresponds to $\nu = -$. Thus we have

$$\begin{aligned}\psi(x, y) = e_-(ib)\tilde{w}(x)^{1/2}\phi(b)e_-(-ib/2)c(x - ia_+/2)\exp(i\pi xy/a_+a_-)\\ + O(\exp(-\tilde{\rho}\operatorname{Re} y)), \quad \operatorname{Re} y \to \infty \quad (1.119)\end{aligned}$$

where the same statements about $\tilde{\rho}$ and O apply as below (1.114). It remains to conduct some algebraic simplification. Recalling (1.102), we can see $e_-(ib/2)\phi(b) = \tilde{\phi}(b)$. Recalling the definitions $\tilde{c}(x) \equiv c(x - ia_+/2)$ and $\tilde{w}(x) \equiv 1/\tilde{c}(x)\tilde{c}(-x)$ we have

$$\tilde{w}(x)^{1/2}c(x - ia_+/2) = \big[\tilde{c}(x)/\tilde{c}(-x)\big]^{1/2} \quad (1.120)$$

The lemma is thus proved. ∎

♦ ♦ ♦

Proof of (1.78)**.** This hinges on a symmetry of the relativistic conical function which we will extract from [5] without proof. Expressed in terms of

$$\mathcal{Z}(b;x,y) \equiv \mathcal{R}_r(b;x,y)/c(b;-y), \tag{1.121}$$

the symmetry is

$$\frac{\mathcal{Z}(a_- - b;x,y)}{c(a_- - b;x)} = \frac{\mathcal{Z}(b + a_+;x,y)}{c(b + a_+;x)} \tag{1.122}$$

(cf. (2.39), (2.40) and (2.48) in [5], and recall evenness, (1.13)).

Writing $\psi(b;x,y)$ (1.60) in terms of $\mathcal{Z}(b;x,y)$ (1.121) we have

$$\psi(b;x,y) = \tilde{w}(b;x)^{1/2}\,(2s_-(ib-y))^{-1}\sum_{\nu=+,-}\nu e_-(\nu(ib-y)/2)\mathcal{Z}(b;x+i\nu a_+/2,y) \tag{1.123}$$

The task is to consider how the terms on the rhs are affected by $b \to a_- - b$. It is straightforward to see that for $\nu = +,-$,

$$(2s_-(ib-y))^{-1}\,\nu e_-(\nu(ib-y)/2) \;\to\; (2s_-(ib+y))^{-1}\,ie_-(-\nu(ib+y)/2) \tag{1.124}$$

And so,

$$\psi(a_- - b;x,y) = \tilde{w}(a_- - b;x)^{1/2}\,(2s_-(ib+y))^{-1}\sum_{\nu=+,-} ie_-(-\nu(ib+y)/2) \times \mathcal{Z}(a_- - b;x+i\nu a_+/2,y) \tag{1.125}$$

The aim is to show that the rhs of this is equal to the rhs of (1.123). The symmetry (1.122) is useful to this end because of a known AΔE which describes the effect on the relativistic conical function of a shift in the parameter b by the parameter a_+. Specifically, from [5](2.88)[2] with $\delta = -$ we have

$$\mathcal{R}_r(b+a_+;x,y) = (8is_-(x)s_-(y+ib)s_-(y-ib))^{-1} \times \left[\mathcal{R}_r(b;x-ia_+,y) - \mathcal{R}_r(b;x+ia_+,y)\right] \tag{1.126}$$

(For the more general 8-variable R-function, shift relations of this kind are summarised in [28], §8.)

Considering what happens when (1.122) is substituted into (1.125) we can see the usefulness of the following formula, verifiable using (1.1), (1.25), (1.28) and (1.47),

$$\tilde{w}(a_- - b;x)^{1/2}\,\frac{c(a_- - b;x+i\nu a_+/2)}{c(b+a_+;x+i\nu a_+/2)} = -4i\nu c_+(x)\tilde{w}(a_- - b;x)^{-1/2} \times s_-(x+i\nu(a_+/2+b)), \quad \nu = +,- \tag{1.127}$$

[2]We note this equation contains a typo; the explicit a_δ on the rhs should be $a_{-\delta}$.

We also note separately, using (1.1) and (1.48), that

$$\tilde{w}(a_- - b; x)^{1/2} = 2c_+(x)\tilde{w}(b; x)^{-1/2} \tag{1.128}$$

Substituting (1.122) and the latter two equations into (1.125) we thus get

$$\begin{aligned}\psi(a_- - b; x, y) = \tilde{w}(b; x)^{1/2} s_-(ib + y)^{-1} \sum_{\nu=+,-} \nu e_-(-\nu(ib + y)/2) \\ \times s_-(x + i\nu(a_+/2 + b))\mathcal{Z}(b + a_+; x + i\nu a_+/2, y)\end{aligned} \tag{1.129}$$

We now want to apply (1.126). First we note, using (1.1) and (1.25), that

$$1/c(b + a_+; -y) = -2is_-(ib + y)/c(b; -y) \tag{1.130}$$

Substituting this and (1.126) into (1.129) we thus get

$$\begin{aligned}\psi(a_- - b; x, y) = [1/2s_-(ib+y)]\tilde{w}(b; x)^{1/2}\big(2s_-(ib-y)c(b; -y)\big)^{-1} \sum_{\nu=+,-} e_-(-\nu(ib+y)/2) \\ \times \frac{s_-(x + i\nu(a_+/2 + b))}{s_-(x + i\nu a_+/2)} \nu \left[\mathcal{R}_r(b; x + i\nu a_+/2 - ia_+, y) - \mathcal{R}_r(b; x + i\nu a_+/2 + ia_+, y)\right]\end{aligned} \tag{1.131}$$

(where the first line has been written with our target (1.60) in mind). We proceed to look at the second line more closely. It can be written as

$$\frac{s_-(x + i\nu(a_+/2 + b))}{s_-(x + i\nu a_+/2)} \big[\mathcal{R}_r(b; x - i\nu a_+/2, y) - \mathcal{R}_r(x + 3i\nu a_+/2, y)\big], \quad \nu = +, - \tag{1.132}$$

This allows us to use the fact that $\mathcal{R}_r(b; x, y)$ is a generalised eigenfunction of $A(b; x)$ (1.22) with eigenvalue $2c_-(y)$. In other words,

$$\begin{aligned}\frac{s_-(x + i\nu(a_+/2 + b))}{s_-(x + i\nu a_+/2)} \mathcal{R}_r(x + 3i\nu a_+/2, y) = 2c_-(y)\mathcal{R}_r(b; x + i\nu a_+/2, y) \\ - \frac{s_-(x + i\nu(a_+/2 - b))}{s_-(x + i\nu a_+/2)} \mathcal{R}_r(b; x - i\nu a_+/2, y), \quad \nu = +, -\end{aligned} \tag{1.133}$$

Plugging this into (1.132) we find the latter equals[3]

$$2c_-(ib)\mathcal{R}_r(b; x - i\nu a_+/2, y) - 2c_-(y)\mathcal{R}_r(b; x + i\nu a_+/2, y) \tag{1.134}$$

[3]Using $s_-(x + i\nu(a_+/2 + b)) + s_-(x + i\nu(a_+/2 - b)) = 2s_-(x + i\nu a_+/2)c_-(ib), \quad \nu = +, -.$

Thus the ν-sum in (1.131) is equal to

$$\sum_{\nu=+,-} \big\{2c_-(ib)e_-(\nu(ib+y)/2) - 2c_-(y)e_-(-\nu(ib+y)/2)\big\}\mathcal{R}_r(b;x+i\nu a_+/2,y) \quad (1.135)$$

It is now straightforward to check

$$2c_-(ib)e_-(\nu(ib+y)/2)-2c_-(y)e_-(-\nu(ib+y)/2) = 2s_-(ib+y)\nu e_-(\nu(ib-y)/2), \quad \nu=+,- \quad (1.136)$$

Equations (1.135) and (1.136) give an expression for the ν-sum in (1.131) whose effect is to make the rhs of (1.131) identical to the that of (1.60), the defining expression for $\psi(b;x,y)$.

2 Anticipating the bound states (I).

The functions $\Psi^{(m)}(b;x)$ as specialisations of $\psi(b;x,y)$.

In this section we see how for certain values of the spectral parameter y, the function $\psi(b;x,y)$ (1.60) reduces to a polynomial of the elementary function $s_+(x)$ multiplied by the positive square root weight function $2c_+(x)\tilde{w}(b;x)^{-1/2}$ (for analysis of the latter, cf. (1.73)). These polynomials are what we call $Q_m(b;u)$ below, and the product with the weight function is denoted $\Psi^{(m)}(b;x)$. The main results in this section (Lemmas 2.2 and 2.3 below) show exactly how this reduction works.

For any given choice of the parameters (a_-, b), a fixed number of the functions $\Psi^{(m)}(b;x)$ are square-integrable. Indeed, the question of integrability recalls how we can motivate the extraordinary spectral values in an a posteriori way. Following the lore of non-relativistic systems we can look for values of y for which the transmission coefficient $t(b;y)$ (1.79) is singular and of the form $y = iv$ with $v > 0$. For any such value, the (modified) asymptotics (1.87) are manifestly square-integrable, in the sense that

$$\left[\frac{\psi(b;x,y)}{t(b;y)}\right]_{y=iv} \sim \begin{cases} \exp(-\pi x v/a_+a_-), & x \to \infty \\ (s_-(ib)/s_-(iv))\exp(\pi x v/a_+a_-), & x \to -\infty \end{cases} \tag{2.1}$$

Looking at $t(b;y)$ (1.79) we can see very easily that the following values serve this purpose,

$$y_m \equiv ib - ima_-, \quad m \in \mathbb{N} \tag{2.2}$$

where positivity of the imaginary part obtains iff $ma_- < b$. Moreover, the quotient in (2.1) reduces to $(-)^m$ for these values.

In what follows we introduce functions $Q_m(b;x)$ defined by a recursion relation. Under generic conditions this produces well-defined polynomials of degree m and parity $(-)^m$. From these properties certain basic facts about $\Psi^{(m)}(b;x)$ follow straight away. After sketching these we include a subsection which looks at the polynomials more closely. Specifically, it looks at the precise conditions needed for the recursion to produce the stated properties of $Q_m(b;x)$. It also details the relation that exists between the latter and the q-ultraspherical polynomials.(In our later applications we work with $Q_m(b;x)$ rather than the q-ultrasphericals because the former has a simpler relation to $\psi(b;x,y)$, as well as the useful property of real-valuedness.)

In the final subsection we show how any given $\Psi^{(m)}(b;x)$ is related to $\psi(b;x,y)$ by the spectral values (2.2). From this it follows that $\Psi^{(m)}(b;x)$ are eigenfunctions of $\tilde{H}(b;x)$ (1.44), and thus the integrable subfamily are bound states in the heuristic sense. Later, once we have introduced the eigenfunction transforms $\mathcal{F}_\pm$, we take the first step in proving they are the bound states proper, with some orthogonality results. (The proof that they comprise *all* the bound states, i.e. the completeness problem, is altogether a different matter and concerns us in §6 and §7.)

We define

$$\Psi^{(-1)} \equiv 0 \tag{2.3}$$

$$\Psi^{(m)}(b;x) \equiv 2c_+(x)\tilde{w}(b;x)^{-1/2}Q_m(b;2s_+(x)), \quad m \in \mathbb{N} \tag{2.4}$$

where $Q_m(b;u)$ is a polynomial in u uniquely defined by $Q_{-1} \equiv 0$, $Q_0(u) \equiv 1$ and the recursion

$$Q_{m+1}(u) + u\sigma_m Q_m(u) + \rho_m Q_{m-1}(u) = 0, \quad m \geq 0 \tag{2.5}$$

where σ_m, ρ_m are b-dependent constants,

$$\sigma_m \equiv s_+(ib - i(m+1)a_-)/s_+(i(m+1)a_-) \tag{2.6}$$

$$\rho_m \equiv s_+(2ib - i(m+1)a_-)/s_+(i(m+1)a_-) \tag{2.7}$$

Or, more intuitively,

$$\sigma_m = \sin\Big(\frac{\pi}{a_+}[b-(m+1)a_-]\Big)/\sin\Big(\frac{\pi}{a_+}[m+1]a_-\Big) \tag{2.8}$$

$$\rho_m = \sin\Big(\frac{\pi}{a_+}[2b-(m+1)a_-]\Big)/\sin\Big(\frac{\pi}{a_+}[m+1]a_-\Big) \tag{2.9}$$

It is clear from this setup that $Q_m(b;u)$ is a polynomial in u of degree m and parity $(-)^m$ (ignoring for the moment any pathologies arising for non-generic parameters). The function $\Psi^{(m)}(b;x)$ inherits this parity property given evenness in x of the weight function $\tilde{w}(b;x)$ (1.48). Moreover, it is manifestly real-valued for $(b,x) \in \mathbb{R}^2$.

For later use we note an alternative writing of the ground state, which follows from (1.128),

$$\Psi^{(0)}(x) = 2c_+(x)\tilde{w}(b;x)^{-1/2} = \tilde{w}(a_- - b;x)^{1/2} \tag{2.10}$$

Using (1.57) we find dominant asymptotics,

$$\Psi^{(m)}(b;x) \sim \exp(-\pi x[b-(m+1)a_-]/a_+a_-), \quad x \to \infty \tag{2.11}$$

And so square-integrability of the function obtains iff $(m+1)a_- < b$.

2.1 A closer look at $Q_m(b;u)$.

We now turn to the question of when the recursion for $Q_m(b;u)$ (2.5) *fails* to define a polynomial $Q_m(b;u)$ with the stated properties (i.e. the question of the pathologies). One way this can happen is if one of the coefficients $\sigma_0,\ldots,\sigma_{m-1}$,$\rho_0,\ldots,\rho_{m-1}$ is singular. To exclude this possibility we must ensure that the positive ratio a_+/a_- must not take certain values. To this end we introduce the following point set (empty when $m=-1,0$),

$$\mathcal{A}_m \equiv \{(l+1)/n \,|\, l = 0,\ldots,m-1,\ n \in \mathbb{N}^*\} \tag{2.12}$$

As $m \to \infty$ this set begins to look like the positive rationals $\mathbb{Q}^+$. We define accordingly $\mathcal{A}_\infty \equiv \mathbb{Q}^+$.

We would also like to know exactly when $Q_m(u)$ has degree m, which means ensuring all $\sigma_{0\le l<m}$ are non-zero. (Assuming $Q_l(u)$ has degree l and parity $(-)^l$, then a vanishing ρ_l will return a $Q_{l+1}(u)$ as desired, so we do not need to concern ourselves about this. In the extreme case when all $\rho_0, \ldots, \rho_{m-1}$ vanish, $Q_m(u)$ is just a multiple of u^m.) This requirement on σ_m places more restrictive conditions on our parameters. We need two point sets (empty when $m=-1,0$),

$$\Pi_m^{(+)} \equiv \{(l+1)a_- + ka_+ \,|\, l=0,\ldots,m-1,\ k\in\mathbb{N}\}, \tag{2.13}$$

$$\Pi_m^{(-)} \equiv \{(l+1)a_- - (k+1)a_+ \,|\, l=0,\ldots,m-1,\ k\in\mathbb{N}\}, \tag{2.14}$$

and their union

$$\Pi_m \equiv \Pi_m^{(+)} \cup \Pi_m^{(-)} \tag{2.15}$$

We note that $\Pi_m^{(+)}$ has a relation to point sets elsewhere,

$$\Pi_m^{(+)} \subset S^+ \cup A_- \subset \mathcal{Y} \tag{2.16}$$

(cf. (1.39),(3.1),(3.2) respectively). In fact, it accords with our intuition to define $\Pi_\infty^{(+)} \equiv S^+ \cup A_-$. By contrast, $\Pi_m^{(-)}$ is unlike any others we have seen.

The point about defining these sets with an index m, is that for given b, only a finite number of the functions $\Psi^{(m)}(b;x)$ are ever integrable (as we can see from (2.11)). Accordingly, when we construct our Hilbert space theory it would be overkill to always work with the $m=\infty$ analogues of the above sets.

Lemma 2.1. *Assume the ratio a_+/a_- does not take values from $\mathcal{A}_m$* (2.12)*, and b not from Π_m* (2.15)*. Then the recursion* (2.5) *gives rise to well-defined polynomials $Q_1,\ldots,Q_l,\ldots,Q_m$ in u of degree l and parity $(-)^l$.*

Relaxing the restriction on b, (2.5) *still gives rise to well-defined polynomials $Q_1,\ldots,Q_l$, $\ldots,Q_m$ in u, but now of degree l or $l-2$, and parity $(-)^l$. Moreover, we can say that at most, two of these polynomials are of the same degree. In fact, in this case, they are multiples of each other.*

Proof. It is not hard to see that all elements in the set $\{\sigma_0,\ldots,\sigma_{m-1}\}$ are non-singular iff $a_+/a_- \notin \mathcal{A}_m$. The same is also true of $\{\rho_0,\ldots,\rho_{m-1}\}$. Thus all $Q_1,\ldots,Q_l,\ldots Q_m$ are well-defined polynomials in u.

These polynomials will have degree l iff all elements in the set $\{\sigma_0,\ldots,\sigma_{m-1}\}$ are non-zero. This is equivalent to $b\notin\Pi_m$. For instance, take a particular σ_l in this set; it will vanish iff there exists $k_0\in\mathbb{Z}$ such that $b=(l+1)a_- + k_0a_+$. Thus we see how any particular l-contribution arises in Π_m.

Let us address the second claim in the lemma. When $b\in\Pi_m$ we know that at least one element in $\{\sigma_0,\ldots,\sigma_{m-1}\}$ is zero. The point is that our restriction on a_+/a_- entails that, in this scenario, *only one* of the elements is zero. To see this, suppose σ_{l_0} is the zero element. Then we must have $b=(l_0+1)a_- + k_0a_+$ for some $k_0\in\mathbb{Z}$. Now consider $l\neq l_0$ (for $l=0,\ldots,m-1$). $\sigma_l=0$ reduces to existence of an $n\in\mathbb{Z}$ such that

$$\frac{a_+}{a_-} = \frac{l_0-l}{n-n_0} \tag{2.17}$$

But letting l range over $0, \ldots, m-1$ and recalling the positivity of a_+, a_- assumed throughout, existence of such an n would contradict $a_+/a_- \notin \mathcal{A}_m$.

Let us now look at how this $\sigma_{l_0} = 0$ affects the recursion. For all $l \leq l_0$, $Q_l(u)$ will be well-defined as a a polynomial of degree l and parity $(-)^l$. (2.5) entails

$$Q_{l_0+1} = -\rho_{l_0} Q_{l_0-1} \tag{2.18}$$

Accordingly these two polynomials have the same degree and parity. For $l \geq l_0 + 2$, the recursion will proceed as normal but with degree shifted down to $l-2$. (We note the pathological situation in which ρ_{l_0} would also vanish in (2.18) is not possible given $a_+/a_- \notin \mathcal{A}_m$). This proves the lemma.

□

In their monograph [9], Askey and Wilson describe a one-coupling specialisation of their four-coupling polynomials known as the q-ultraspherical polynomials (which go back to Rogers [6]). We claim that our polynomials $Q_m(b; u)$ are effectively the same as these (the relation is by an m-dependent constant phase multiple). To see this, we first define functions $\check{Q}_m(b; u)$ by writing

$$Q_m(b; u) = i^m e_+^{imb} \check{Q}_m(b; u), \quad m \in \mathbb{N} \tag{2.19}$$

$$\check{Q}_{-1} \equiv 0 \tag{2.20}$$

Plugging this into (2.5) we find that $\check{Q}_m(b; u)$ satisfies

$$iu\sigma_m \check{Q}_m(b; u) = e_+^{ib} \check{Q}_{m+1}(b; u) - \rho_m e_+^{-ib} \check{Q}_{m-1}(b; u) \tag{2.21}$$

Writing this out using σ_m (2.6) and ρ_m (2.7) and then rearranging we get

$$\begin{aligned} -iu(1 - e_+^{-2ib} e_+^{2i(m+1)a_-}) \check{Q}_m(b; u) &= (1 - e_+^{2i(m+1)a_-}) \check{Q}_{m+1}(b; u) \\ &= (1 - e_+^{-4ib} e_+^{2i(m+1)a_-}) \check{Q}_{m-1}(b; u) \end{aligned} \tag{2.22}$$

In [9], the q-ultraspherical polynomials are functions $C_m(z; \beta|q)$ defined by the recursion

$$\begin{aligned} 2z(1 - \beta q^m) C_m(z; \beta|q) &= (1 - q^{m+1}) C_{m+1}(z; \beta|q) \\ &\quad + (1 - \beta^2 q^{m-1}) C_{m-1}(z; \beta|q), \quad m \in \mathbb{N} \end{aligned} \tag{2.23}$$

$$C_{-1}(z; \beta|q) \equiv 0, \quad C_0(z; \beta|q) \equiv 1 \tag{2.24}$$

For the choices

$$z = -iu/2, \quad \beta = e_+^{-2ib} e_+^{2ia_-}, \quad q = e_+^{2ia_-} \tag{2.25}$$

we thus see that the recursions (2.22) and (2.23) are equivalent, implying

$$C_m(-iu/2; e_+^{-2ib} e_+^{2ia_-} | e_+^{2ia_-}) = \check{Q}_m(b; u), \quad m \in \mathbb{N} \cup \{-1\} \tag{2.26}$$

2.2 Residue lemmas.

We now return to the main question of how $\psi(b;x,y)$ (1.60) can be connected to these functions by the spectral values y_m (2.2). For reasons that will become clear it is best to work with a similarity-transformed version of $\psi(b;x,y)$ defined by

$$\hat{\psi}(b;x,y) \equiv c(b;-y)\psi(b;x,y) \tag{2.27}$$

where $c(b;z)$ is defined in (1.25). In fact, from (1.60) we see that $\hat{\psi}$ is actually a reduced version of ψ,

$$\hat{\psi}(b;x,y) = \tilde{w}(b;x)^{1/2}(2s_-(ib-y))^{-1}\sum_{\nu=+,-}\nu e_-(\nu(ib-y)/2)\mathcal{R}_r(b;x+i\nu a_+/2,y) \tag{2.28}$$

This function inherits all of the properties in Lemma 1.1 except (1.76) and (1.78). To make better sense of the lemmas below we wager some facts about its poles. The factor of $s_-(ib-y)^{-1}$ is manifestly singular at the points $y=y_m$ (2.2). But we find that $\hat{\psi}(b;x,y)$ inherits this singularity iff $m \geq 1$. In other words, $y = y_0$ turns out to be a *regular* value.

We are also interested in the points $y = -y_m$. These are regular values of both $s_-(ib-y)^{-1}$ and $\mathcal{R}_r(x \pm ia_+/2,y)$ (cf. §3.1), and consequently of $\hat{\psi}(b;x,y)$. We do not know a way to explicitly evaluate $\hat{\psi}(b;x,y)$ for these values (by which we mean writing it in a simpler form that does not involve the relativistic conical function, $\mathcal{R}_r(x,y)$). But we do know how to evaluate a particular linear combination of $\hat{\psi}(b;\pm x,-y_m)$. This is what we present in Lemma 2.3. (The combination arises in a natural way later on, and plays a role in solving the completeness problem in §7.)

When interpreting these lemmas, recall that $\Psi^{(-1)}$ is simply zero. The restriction on a_+, a_- ensures $\Psi^{(m-1)}(b;x)$ is well-defined by the recursion (2.5).

Lemma 2.2. *Using $\hat{\psi}(b;x,y)$ (2.27) and the spectral values y_m (2.2), define functions*

$$\hat{\psi}_m(b;x) \equiv \operatorname*{Res}_{y=y_m}\hat{\psi}(b;x,y) \tag{2.29}$$

Then, provided the positive parameters a_+, a_- satisfy $a_+/a_- \notin \mathcal{A}_{m-1}$ (2.12), we have

$$\hat{\psi}_m(b;x) = \eta(b)\Psi^{(m-1)}(b;x), \quad m \in \mathbb{N} \tag{2.30}$$

where

$$\eta(b) \equiv -s_-(y_1)(a_-/\pi)c(b;y_1) \tag{2.31}$$

and $c(b;y)$ is defined in (1.25).

Lemma 2.3. *Using $\hat{\psi}(b;x,y)$ (2.27) and the spectral values y_m (2.2), define functions*

$$\hat{\phi}_m(b;x) \equiv (-)^{m-1}\hat{\psi}(b;x,-y_m) + \hat{\psi}(b;-x,-y_m) \tag{2.32}$$

Then, provided the positive parameters a_+, a_- satisfy $a_+/a_- \notin \mathcal{A}_{m-1}$ (2.12), we have

$$\hat{\phi}_m(b;x) = c(b;y_1)\Psi^{(m-1)}(b;x), \quad m \in \mathbb{N} \tag{2.33}$$

Before proving these lemmas we say a word on how to reconstruct the corresponding results for $\psi(b;x,y)$. This is fairly straightforward. We just have to pay close attention to the poles of

$$1/c(b;y) = 1/G(-y-ia)G(y+ia-ib) \tag{2.34}$$

In particular we want to know if and when these overlap with $\pm y_m$ (2.2). The function $G(y)$ (1.3) has its zeros at $y = ia + i(ka_+ + la_-)$ for $k,l \in \mathbb{N}$. From this we deduce that none of the points y_m is a pole of $1/c(b;-y)$. And so, we can use (2.27) and $\hat{\psi}_m(b;x)$ (2.29) to write,

$$\operatorname*{Res}_{y=y_m} \psi(b;x,y) = \hat{\psi}_m(b;x)/c(b;-y_m), \quad m \in \mathbb{N} \tag{2.35}$$

We also deduce the point $-y_0$ is a pole of $1/c(b;-y)$, but the remaining $-y_{m\geq 1}$ are not. As a result we can use (2.27) and $\hat{\phi}_m(b;x)$ (2.32) to write

$$\operatorname*{Res}_{y=-y_0} [\psi(b;x,y) - \psi(b;-x,y)] = \hat{\phi}_0(x) \operatorname*{Res}_{y=y_0} c(b;y)^{-1} \tag{2.36}$$

and

$$(-)^{m-1}\psi(b;x,-y_m) + \psi(b;-x,-y_m) = \hat{\phi}_m(b;x)/c(b;y_m), \quad m \in \mathbb{N}^* \tag{2.37}$$

Proofs of Lemmas 2.2 and 2.3. We will employ the same strategy for both of these lemmas. This involves first showing that $\hat{\psi}_m(b;x)$ and $\hat{\phi}_m(b;x)$ both satisfy the same recursion as $\Psi^{(m-1)}(x)$. Then, we exhibit (2.30) and (2.33) explicitly for the cases $m = 0, 1$. This suffices for the desired results. Until further notice we will suppress all implicit dependences of functions on b.

The recursion satisfied by $\Psi^{(m-1)}(x)$ is of course the same as that for $Q_{m-1}(2s_+(x))$, cf. (2.5). We write it as follows

$$\frac{s_+(-ima_-)}{s_+(ib-ima_-)}C_{m+1}(x) - \frac{s_+(2ib-ima_-)}{s_+(ib-ima_-)}C_{m-1}(x) = 2s_+(x)C_m(x), \quad m \geq 1 \tag{2.38}$$

To show this is satisfied by both $\hat{\psi}_m(x)$ (2.29) and $\hat{\phi}_m(x)$ (2.32), we are going to use the dual-variable eigenvalue equation satisfied by $\psi(x,y)$, cf. Lemma 1.1. Since (2.27) just defines a similarity transform we can consider the analogous eigenvalue equation for $\hat{\psi}(x,y)$. The pleasing fact is that this has a very simple form. To see this we recall that the AΔO $\mathcal{S}(y)$ (1.67) arose as

$$\frac{1}{c(-y)}\Big[\frac{s_+(y-ib)}{s_+(y)}T^y_{ia_-} - \frac{s_+(y+ib)}{s_+(y)}T^y_{-ia_-}\Big]c(-y) \tag{2.39}$$

(cf. (1.64)). Thus we can write down straight away

$$\frac{s_+(y-ib)}{s_+(y)}\hat{\psi}(x,y-ia_-) - \frac{s_+(y+ib)}{s_+(y)}\hat{\psi}(x,y+ia_-) = 2s_+(x)\hat{\psi}(x,y) \tag{2.40}$$

The three $\hat{\psi}$-terms that feature here are all singular when $y = y_m$ (cf. our remarks below (2.28)). When we take the residue of the terms in the equation at these points, and restrict $m \geq 1$, we recover (2.38) immediately with $C_m(x) = \hat{\psi}_m(x)$ as desired.

We now show this is similarly satisfied for $\hat{\phi}_m(x)$; in other words, that (2.38) holds with $C_m(x) = \hat{\phi}_m(x)$. To achieve this we can just show that it holds for $C_m(x) = (-)^{m-1}\hat{\psi}(x, -y_m)$, since this entails that it also holds for $C_m(x) = \hat{\psi}(-x, -y_m)$. To this end we exploit the dual AΔE (2.40) again. For the three $\hat{\psi}$-terms in this equation, $y = -y_m$ are regular values (cf. our remarks above below (2.28)). Upon setting $y = -y_m$ and restricting $m \geq 1$, (2.40) becomes

$$\frac{s_+(-2ib + ima_-)}{s_+(-ib + ima_-)}\hat{\psi}(x, -y_{m-1}) - \frac{s_+(ima_-)}{s_+(-ib + ima_-)}\hat{\psi}(x, -y_{m+1}) = 2s_+(x)\hat{\psi}(x, -y_m), \quad m \geq 1 \quad (2.41)$$

which is indeed just a rearranged version of (2.38) with $C_m(x) = (-)^{m-1}\hat{\psi}(x, -y_m)$.

We now prove the two lemmas explicitly for the two cases $m = 0, 1$, looking first at Lemma 2.2. This means proving the following two equations,

$$\hat{\psi}_0 \equiv \operatorname*{Res}_{y=y_0} \hat{\psi}(x, y) = 0 \quad (2.42)$$

and

$$\hat{\psi}_1(x) \equiv \operatorname*{Res}_{y=y_1} \hat{\psi}(x, y) = \eta(b)\Psi^{(0)}(x) \quad (2.43)$$

The key to these explicit evaluations is (1.16) which originates in [5]. Adapting it for $\mathcal{R}_r(x, y)$ and using (1.25) we have

$$\mathcal{R}_r(b; x, \pm ib) = c(b; -ib), \quad x \in \mathbb{C} \quad (2.44)$$

It is clear how this will help us in the $m = 0$ case, since there we are dealing with $y = ib$ in (2.28). For the $m = 1$ case we have $y = ib - ia_-$, for which we have no analogue to (2.44). Nevertheless the latter can still be exploited in a roundabout way, as we will see.

Looking at the expression (2.28) for $\hat{\psi}(x, y)$ we can see that the $y = y_0 = ib$ pole comes from the $s_-(ib - y)^{-1}$ term, and is simple. When $y = y_0$, the other pertinent term in (2.28), namely the ν-sum, equals

$$\sum_{\nu=+,-} \nu e_-(\nu(ib - y_0)/2)\mathcal{R}_r(x + i\nu a_+/2, y_0) \quad (2.45)$$

But because of (2.44) this is just

$$c(-ib) \sum_{\nu=+,-} \nu = 0 \quad (2.46)$$

As a result, the simple pole is removed and $\hat{\psi}(x, \cdot)$ is regular at $y_0 = ib$. This proves (2.42).

We now consider (2.43). As explained, (2.44) is of no use when evaluating the residue of $\hat{\psi}(x,\cdot)$ at $y_1 = ib - ia_-$. However we can cleverly exploit the property (1.78) which tells us $\psi(x,y)$ is known to be invariant under $b \to a_- - b$. In other words,

$$\psi(b;x,y) = \psi(\tilde{b};x,y), \quad \tilde{b} \equiv a_- - b \tag{2.47}$$

(note that we now show the implicit dependence on b for obvious reasons!). In terms of $\hat{\psi}(b;x,y)$ (2.27) this symmetry translates as

$$\hat{\psi}(b;x,y) = \frac{c(b;-y)}{c(\tilde{b};-y)}\hat{\psi}(\tilde{b};x,y) \tag{2.48}$$

The point is that we have $y_1 = -i\tilde{b}$, and so (2.44) *does* come into play when we take the residue of the rhs of this equation at $y = y_1$. Moreover, the factor responsible for the (simple) pole is not $\hat{\psi}(\tilde{b};x,y)$ but rather the function in the denominator in (2.48) (cf. our remarks below (2.28)). In other words,

$$\hat{\psi}_1(b;x) = \operatorname*{Res}_{y=y_1}\hat{\psi}(b;x,y) = c(b;i\tilde{b})\hat{\psi}(\tilde{b};x,-i\tilde{b})\operatorname*{Res}_{y=-i\tilde{b}} 1/c(\tilde{b};-y) \tag{2.49}$$

Using (2.28) and (2.44) we have explicitly,

$$\begin{aligned}\hat{\psi}(\tilde{b};x,-i\tilde{b}) &= \tilde{w}(\tilde{b};x)^{1/2}c(\tilde{b};-i\tilde{b}) \\ &\qquad \times [2s_-(2i\tilde{b})]^{-1}\sum_{\nu=+,-}\nu e_-(i\nu\tilde{b}) = \tilde{w}(\tilde{b};x)^{1/2}c(\tilde{b};-i\tilde{b})\end{aligned} \tag{2.50}$$

This proves that $\psi_1(b;x)$ (2.29) is proportional to $\Psi^{(0)}(b;x)$ (2.10). To check that the proportionality constant is just $\eta(b)$, we substitute (2.50) back into (2.49) and write the term multiplying $\tilde{w}(\tilde{b};x)^{1/2}$ as

$$s_-(y_1)\operatorname*{Res}_{y=y_1}\Big[\frac{c(b;-y)}{s_-(y_1+y)}\,c(\tilde{b};y)/c(\tilde{b};-y)\Big] \tag{2.51}$$

This writing allows us to use the following formula, which just follows from the AΔE (1.1) with a little work,

$$c(\tilde{b};y)/c(\tilde{b};-y) = \frac{s_-(y_1+y)}{s_-(y_1-y)}c(b;y)/c(b;-y) \tag{2.52}$$

Recalling $y_1 = ib - ia_-$, (2.51) straight away becomes

$$s_-(y_1)\operatorname*{Res}_{y=y_1}\Big[\frac{c(b;y)}{s_-(y_1-y)}\Big] \tag{2.53}$$

which is manifestly equal to $\eta(b)$ (2.31) because

$$\operatorname*{Res}_{y=y_1} 1/s_-(y_1-y) = -a_-/\pi, \tag{2.54}$$

To prove Lemma 2.3 explicitly for $m = 0, 1$ means showing

$$\hat{\phi}_0 \equiv -\hat{\psi}(x, -y_0) + \hat{\psi}(-x, -y_0) = 0 \tag{2.55}$$

and

$$\hat{\phi}_1(b; x) \equiv \hat{\psi}(x, -y_1) + \hat{\psi}(-x, -y_1) = c(b; y_1)\Psi^{(0)}(b; x) \tag{2.56}$$

Again this will hinge on (2.44). In the case of $\hat{\phi}_0$ we are dealing with $y = -ib$ in $\hat{\psi}(b; x, y)$, which we know from our discussion below (2.28) is a regular value. The vanishing comes about as a result of the special linear combination of $\hat{\psi}$ in (2.32). In the case of $\hat{\phi}_1$ we are dealing with $y = i\tilde{b}$ in $\hat{\psi}(\tilde{b}; x, y)$ which is also a regular value (this is a non-obvious fact proved earlier, cf. (2.46)). To handle the two cases, there is a useful way to write the two pertinent linear combinations of $\hat{\psi}$. From the expression (2.28) and evenness of $\tilde{w}(b; \cdot)$ we get straight away

$$\begin{aligned}\hat{\psi}(b; x, y) \mp \hat{\psi}(b; -x, y) &= \tilde{w}(b; x)^{1/2}[2s_-(ib - y)]^{-1} \\ &\sum_{\nu=+,-} \nu e_-(\nu(ib - y/2))[\mathcal{R}_r(b; x + i\nu a_+/2, y) \mp \mathcal{R}_r(b; -x + i\nu a_+/2, y)]\end{aligned} \tag{2.57}$$

Because $\mathcal{R}_r(x, y)$ is even in x, the ν-sum here can be rewritten as

$$\sum_{\nu=+,-} \mathcal{R}_r(b; x + i\nu a_+/2, y)\nu[e_-(\nu(ib - y/2)) \pm e_-(-\nu(ib - y/2))] \tag{2.58}$$

(consider sending $\nu \to -\nu$ in the second term in the square brackets). With the plus sign, the square brackets are equal to $2c_-((ib-y)/2)$; and with the minus sign, $2\nu s_-((ib-y)/2)$. Thus we have

$$\hat{\psi}(b; x, y) - \hat{\psi}(b; -x, y) = \tilde{w}(b; x)^{1/2}[2s_-((ib-y)/2))]^{-1} \sum_{\nu=+,-} \nu\mathcal{R}_r(b; x + i\nu a_+/2, y) \tag{2.59}$$

and

$$\hat{\psi}(b; x, y) + \hat{\psi}(b; -x, y) = \tilde{w}(b; x)^{1/2}[2c_-((ib - y)/2)]^{-1} \sum_{\nu=+,-} \mathcal{R}_r(b; x + i\nu a_+/2, y) \tag{2.60}$$

The first of these combined with (2.44) and (2.55) gives

$$\hat{\phi}_0(b; x) = -\hat{\psi}(b; x, -ib) + \hat{\psi}(b; -x, -ib) = -\tilde{w}(b; x)^{1/2}[2s_-(ib)]^{-1}c(b; -ib) \sum_{\nu=+,-} \nu = 0 \tag{2.61}$$

This proves (2.55). Similarly, if we take $b \to \tilde{b}$ in the second of these and combine it with (2.48) and (2.56) we get

$$\hat{\phi}_1(b; x) = \frac{c(b; y_1)}{c(\tilde{b}; y_1)}[\hat{\psi}(\tilde{b}; x, i\tilde{b}) + \hat{\psi}(\tilde{b}; -x, i\tilde{b})] = c(b; y_1)\tilde{w}(\tilde{b}; x)^{1/2} \tag{2.62}$$

Recalling (2.10), this is exactly (2.56).

□

3 Associated transforms.

This section will be somewhat technical. It provides a precise account of two eigenfunction transforms, $\mathcal{F}_\pm$, whose primary purpose is ultimately to build a third transform, $\mathcal{F}$ (4.1). The latter transform and its adjoint are central to our Hilbert space theories for the AΔOs $\tilde{H}(b;x)$ (1.44) and $\mathcal{S}(b;y)$ (1.67) respectively. Since these theories constitute the more interesting parts of our story, we help them attain a smoother flow by putting many technicalities here, and drawing upon them as required.

The final subsection, which looks at the relation between these transforms and the functions from §2, is more interesting.

3.1 Analyticity properties of $\psi(b;x,y)$.

It is important we have a better grasp of these in order to make precise statements about the eigenfunction transforms. The casual reader can skip this discussion without losing the main thread.

In a related context we have already looked at conditions for the square-roots of $w(b;x)$ (1.28) and $\tilde{w}(b;x)$ (1.48) to have certain analyticity properties for $(b,x) \in \mathbb{R}^2$. We found it was sufficient to restrict b to the real line whilst omitting certain sets of discrete points, S (1.41) and $\tilde{S}$ (1.73), in order to ensure no poles or branch points on the real line. These two sets reappear in the lemma below, along with

$$A_- \equiv \{(l+1)a_- \,|\, l \in \mathbb{N}\}, \tag{3.1}$$

In future sections we will have little explicit need for the three parameter sets $S, \tilde{S}, A_-$. More important is their union, $\tilde{S} \cup S \cup A_-$. We give this its own symbol and rewrite it in a more succinct way as follows,

$$\mathcal{Y} \equiv \{\pm [ka_+/2 + (l+1)a_-] \,|\, k,l \in \mathbb{N}\} \cup \{-ka_+/2 \,|\, k \in \mathbb{N}\} \tag{3.2}$$

We note that a special case of the restriction $b \in \mathbb{R} \setminus \mathcal{Y}$ is

$$b \in (0, a_- + a_+/2) \setminus A_- \tag{3.3}$$

The exclusion of A_- here is particularly pertinent when $a_+ \gg a_-$; if $a_+/2 < a_-$ then it only has the effect of excluding $\{a_-\}$ from the interval.

(The fact that our Hilbert space theory for systems with reflection will not apply for the values $b \in A_-$ is neither unproblematic nor surprising, since these are the values for which the asymptotics (1.87) become *reflectionless*.)

Below we focus on the case when b is real, though we note that all the results can be adapted for complex b by simply modifying exclusions of the form $b \in \mathbb{R} \setminus X$ by replacing $b \to \operatorname{Re} b$.

The following lemma builds on the meromorphy properties of $\psi(b;x,y)$ (1.60) in Lemma 1.1 (naturally, any statement about smoothness implicitly involves the restriction of the variables to the real line). The corollary follows immediately. Its importance underlies the importance of the point set $\mathcal{Y}$ (3.2).

Lemma 3.1. *The x-poles of the function $\psi(b;x,y)$ (1.60) do not depend on y, and vice versa. It is smooth in x provided $b \in \mathbb{R} \setminus \tilde{S}$ where the point set $\tilde{S}$ is defined in (1.73). It is smooth in y provided $b \in \mathbb{R} \setminus (S \cup A_-)$ where the point sets S and A_- are defined in (1.41) and (3.1) respectively.*

Corollary 3.2. *The function $\psi(b;x,y)$ (1.60) is smooth in x, y provided $b \in \mathbb{R} \setminus \mathcal{Y}$.*

Proof of Lemma 3.1. The claim about independence of x and y poles follows from the structure of $\psi(x,y)$ (1.60), and the fact this same property is known to hold for the relativistic conical function, cf. (1.18) in this regard.

Lemma 1.1 tells us $\psi(b;x,y)$ has no x-branch points when $b \in \mathbb{R} \setminus \tilde{S}$. Thus the smoothness claim in x follows once we show its x-poles are away from the real line given this restriction. This means studying the x-pole locations of the functions that comprise $\psi(x,y)$ (1.60). In line with our convention for proofs, we suppress implicit dependence of functions on b.

We have already alluded to the poles of $\tilde{w}(x)$ (1.48) in the previous section. When considering these, we recall that (1.70) is a more useful writing. Using this and the G-zeros and G-poles, (1.9) and (1.8) respectively, we have

$$\alpha x = \begin{cases} -ib + ia_+/2 + z_{k,l+1}, \quad k,l \in \mathbb{N}, \quad (\tilde{w}\text{-zeros}) & (3.4) \\ ib + ia_+/2 + z_{k,l}, \quad k,l \in \mathbb{N}, \quad (\tilde{w}\text{-poles}) & (3.5) \end{cases}$$

$$\alpha = +, -$$

$$z_{k,l} \equiv ika_+ + ila_- \tag{3.6}$$

Let us now consider the x-poles of the two functions $\mathcal{R}_r(x \pm ia_+/2, y)$. Using (1.18), we can write these as

$$\alpha x = -ib + ia_+/2 + z_{k,l+1}, \quad k,l \in \mathbb{N} \tag{3.7}$$

$$\alpha = +, -$$

To be clear, this describes the aggregated poles of the two functions, as opposed to the shared. To describe their behaviour separately we just need to supplement (3.7) by saying $\mathcal{R}_r(x + i\nu a_+/2, y)$ does *not* have poles at $\nu(-ib + ia_+/2 + z_{0,l})$, where $\nu = +, -$.

We can now use these sequences to establish some claims in Lemma 3.1. As in §1, we need $b \notin \tilde{S}^-$ (1.72) to ensure the poles (3.5) are away from the real line. To ensure (3.7) are away from the real line we need $b \notin \tilde{S}^+$ (1.71). Recalling $\tilde{S} \equiv \tilde{S}^+ \cup \tilde{S}^-$, this proves the claim about the x-poles.

(We also note here that the poles (3.7) are identical to the zeros (3.4). Without more information this does not allow us to conclude that the former are removed by the latter, however.)

To prove the claim about smoothness in y we study the y-pole locations of the functions that comprise $\psi(x,y)$ and show that $b \in \mathbb{R} \setminus (S \cup A_-)$ is sufficient to keep them away from the real line.

To study the poles for $1/c(-y)$ we first use (1.25) and (1.4) to write

$$1/c(-y) = G(-y + ia)G(y - ia + ib) \tag{3.8}$$

The poles of the first term in this product miss the origin by positivity of a_+, a_- alone. The poles of the second are at

$$y = -ib - z_{k,l} \tag{3.9}$$

The y-poles of $\mathcal{R}_r(x \pm ia_+/2, y)$ are of course unaffected by the shift in x,

$$\alpha y = i(2a - b) + z_{k,l}, \quad k, l \in \mathbb{N}, \quad \alpha = +, - \tag{3.10}$$

Finally the poles of $1/s_-(ib - y)$ are at

$$y = ib \pm ina_-, \quad n \in \mathbb{N} \tag{3.11}$$

The restriction needed to keep (3.9) away from $\operatorname{Im} y = 0$ is thus $b \notin S^-$ (1.40). For the points (3.10) we need $b \notin S^+$ (1.39). And for (3.11), $b \notin A_- \cup \{-la_- \mid l \in \mathbb{N}\}$. Recalling $S \equiv S^+ \cup S^-$, this proves the desired claim. □

3.2 The eigenfunction transforms $\mathcal{F}_\pm$

Let us now introduce the two transforms which will be central to everything that follows (see the note at start of this section). Their action is defined on functions $f : \mathbb{R}^+ \to \mathbb{C}$ by

$$(\mathcal{F}_\delta f)(x) \equiv \int_{\mathbb{R}^+} dy\, \psi(b; \delta x, y) f(y), \quad \delta = +, - \tag{3.12}$$

We find these two transforms to be the same in all essential features. Thus we talk about $\mathcal{F}_\delta$ as a single object most of the time (the corresponding statements being understood to hold for both choices of $\delta = +, -$). The reason for this sharing of features is because, plainly,

$$(\mathcal{F}_+ f)(x) = (\mathcal{F}_- f)(-x) \tag{3.13}$$

Our first task is to understand the behaviour of $\mathcal{F}_\delta$ on some suitable space of functions. A good starting point is the space of smooth, complex-valued functions on the positive half-line with compact support, denoted $C_0^\infty(\mathbb{R}^+)$. For any such function, definedness of $(\mathcal{F}_\delta f)(\cdot)$ as a meromorphic function whose only poles are the x-poles of $\psi(b; \delta x, y)$ follows provided the latter function has no y-poles on $[0, \infty)$ (routine convergence argument). As we have just seen, this latter property holds provided b is real with certain discrete values omitted. A similar restriction ensures $\psi(b; \delta x, y)$ has no x-poles on the real line and so in this case $(\mathcal{F}_\delta f)(\cdot)$ is a bounded function on $\mathbb{R}$ (recall the $O(1)$ asymptotics in Lemma 1.87).

For the purposes of quantum mechanics we would like to know when this transform maps into a suitable Hilbert space, such as $\mathcal{H} \equiv L^2(\mathbb{R}, dx)$. For considerations of this kind we need to know more about the asymptotic behaviour of $(\mathcal{F}_\delta f)(x)$ (the boundedness proclaimed above obviously does not suffice). To this end the tools are already in place, viz. the analyticity and asymptotics properties of $\psi(b; \delta x, y)$. With these the next lemma follows fairly easily.

The lemma involves the space of functions $C_0^\infty(\mathbb{R}^+)$ defined below (3.13). We find that $\mathcal{F}_\delta$ maps these functions into $\mathcal{H}$, however we do not know for sure whether this mapping is bounded (with respect to L^2-norm). So in particular we do not know whether $\mathcal{F}_\delta$ can be extended to the Hilbert space $L^2(\mathbb{R}^+, dy)$ whilst preserving the mapping into $\mathcal{H}$.

Lemma 3.3. *Assume $b \in \mathbb{R} \setminus \mathcal{Y}$ where the point set $\mathcal{Y}$ is defined in* (3.2)*. Then, for any $f \in C_0^\infty(\mathbb{R}^+)$, the functions $(\mathcal{F}_\pm f)(x)$* (3.12) *are in $\mathcal{H} \equiv L^2(\mathbb{R}, dx)$.*

More generally, with these same restrictions on b and f, the functions $(\mathcal{F}_\pm f)(x + iv)$, $v \in \mathbb{R}$ are in $\mathcal{H}$ whenever iv is a regular value of $\psi(b; \cdot, y)$.

Proof. In light of what we said below (3.12), $b \in \mathbb{R} \setminus \mathcal{Y}$ ensures $(\mathcal{F}_\pm f)(x)$ are functions in x whose poles are identical to the x-poles of $\psi(\pm x, y)$ respectively (note, we are suppressing implicit dependence of functions on b). With this in mind, let us explain how the two statements in the lemma relate to each other. The point is that we are able to bound $(\mathcal{F}_\pm f)(x + iv)$ from above by an L^2-integrable function as $x \to \pm\infty$. Thus the only obstacle to entry into $\mathcal{H}$ is the possibility that $(\mathcal{F} f_\pm)(\cdot)$ has a pole on the line $\mathbb{R} + iv$. Indeed since the x-poles of $\psi(x, y)$ are purely imaginary for real b, this amounts to the possibility that $\psi(\cdot, y)$ has a pole at the point iv. From Lemma 3.1 we see that the restriction $b \in \mathbb{R} \setminus \mathcal{Y}$ is sufficient for the special case of no pole at the origin. Thus both parts of the lemma are proved by the aforementioned bound.

(In general, we can ensure regularity of any point iv by introducing restrictions on b generalising those we saw in §3.1. They will of course depend on v, as opposed to just a_+, a_-.)

Because of (3.13), it suffices to prove the bound for just one of the transforms. We will focus on $\mathcal{F}_+$. Central to this is Lemma 1.3 which gives us O-asymptotics for $\psi(x, y)$. This lemma is valid for the b values at hand, and can be encoded as follows,

$$\psi(x, y) = \psi_{as}(x, y) + O(e_-^{-\rho|\operatorname{Re} x|}), \quad \operatorname{Re} x \to \pm\infty \tag{3.14}$$

$$\psi_{as}(x, y) \equiv \begin{cases} t(y) e^{i\pi x y / a_+ a_-}, & \operatorname{Re} x > 0 \\ e^{i\pi x y / a_+ a_-} - r(y) e^{-i\pi x y / a_+ a_-}, & \operatorname{Re} x < 0 \end{cases} \tag{3.15}$$

where $\rho > 0$ and where the bound represented by O is uniform for $\operatorname{Im} x$ and y varying respectively over any compact subset of $\mathbb{R}$ and $\mathbb{R}^+$.

To make use of (3.14) we first write

$$\begin{aligned} \frac{1}{2} \Big| \int_{\mathbb{R}^+} dy\, \psi(x, y) f(y) \Big|^2 \;\le\; & \Big| \int_{\mathbb{R}^+} dy\, \psi_{as}(x, y) f(y) \Big|^2 \\ & + \int_{\mathbb{R}^+} dy\, |\psi(x, y) - \psi_{as}(x, y)|^2 |f(y)|^2, \quad f \in C_0^\infty(\mathbb{R}^+) \end{aligned} \tag{3.16}$$

(this just uses the elementary identities $|A|^2/2 \le |B|^2 + |A - B|^2$ and $|\int dy\, X| \le \int dy |X|$). We will now argue that the two integrals on the rhs have L^2-asymptotics, first for $x \in \mathbb{R}$ and then for $x \to x + iv$ with $x, v \in \mathbb{R}$.

Given $x \in \mathbb{R}$, the plane wave structure of $\psi_{as}(x, y)$ entails that the first integral on the rhs is just a Fourier transform (or a sum of two Fourier transforms in the case of $x < 0$).

Thus the x-tail of this term is L^2 because well-known results about Fourier transforms imply we will get a map from $L^2(\mathbb{R}^+, dy)$ into $\mathcal{H}$ (multiplication by t and r preserves integrability because of their boundedness on $\mathbb{R}$).

For the second integral we can use (3.14). This tells us that for $|x|$ sufficiently large, there exists $\mathcal{C} > 0$ such that for all y in the compact set $\mathrm{supp}(f_\delta)$,

$$|\psi(x, y) - \psi_{as}(x, y)| \leq \mathcal{C}e_-^{-\rho|x|} \tag{3.17}$$

Thus the second integral is bounded from above by $(\mathcal{C}e_-^{-\rho|x|}\|f\|)^2$, and therefore has L^2-asymptotics. (Note, the fact that Lemma 1.3 only gives us a uniformity claim for compacts of $\mathbb{R}$, as opposed to the whole of $\mathbb{R}$, is what prevents us from repeating this argument for all functions in $L^2(\mathbb{R}^+, dy)$, where there is no compact support in general.)

It is straightforward to adapt these arguments to $(\mathcal{F}_+ f)(x + iv)$ for $x, v \in \mathbb{R}$. For the first integral on the rhs of (3.16), we just have to note that f's compact support entails that the functions $\exp(\pm\pi vy/a_+a_-)f(y)$, which arise from $x \to x + iv$ in (3.15), will still be in $L^2(\mathbb{R}^+, dy)$. For the second, we just have to note that the bound in (3.17) still holds for $\mathrm{Im}\, x \in [0, v]$, as implied by the uniformity statement that accompanies (3.14).

For later reference we draw some of these arguments together to assert the following. If $f \in C_0^\infty(\mathbb{R}^+)$, then the following bound holds for $|\,\mathrm{Re}\, x|$ sufficiently large,

$$\frac{1}{2}\Big|\int_{\mathbb{R}^+} dy\, \psi(x, y)f(y)\Big|^2 \;\leq\; \Big|\int_{\mathbb{R}^+} dy\, \psi_{as}(x, y)f(y)\Big|^2 + \mathcal{C}\, e_-^{-2\rho|\,\mathrm{Re}\, x|}\|f\|^2, \quad x \in \mathbb{C} \tag{3.18}$$

where $\mathcal{C} > 0$ is some constant fixed by $\mathrm{Im}\, x$ and $\mathrm{supp}(f)$, and the function $\psi_{as}(x, y)$ is defined in (3.15). The constant $\rho > 0$ derives from Lemma 1.3.

□

3.3 The adjoint transforms $\mathcal{F}_\pm^*$

In the previous subsection we saw that the transform $\mathcal{F}_\delta$ (3.12) defined a map from $C_0^\infty(\mathbb{R}^+)$ into the Hilbert space $\mathcal{H} \equiv L^2(\mathbb{R}, dx)$ (provided the coupling parameter b is real with certain discrete values omitted; namely, those in $\mathcal{Y}$ (3.2)). This claim was proved without any reference to a Hilbert space domain for $\mathcal{F}_\delta$. However, we must now bring this into our discussion in order to consider the adjoint of this transform. This can be done very easily since it is well known that $C_0^\infty(\mathbb{R}^+)$ is a dense subspace of $L^2(\mathbb{R}^+, dy)$.

I.e., we may reinterpret $\mathcal{F}_\delta$ (3.12) as a densely-defined map between Hilbert spaces,

$$\mathcal{F}_\pm : C_0^\infty(\mathbb{R}^+) \subset L^2(\mathbb{R}^+, dy) \to \mathcal{H}, \tag{3.19}$$

As such we may now consider notions of adjoint. To ensure no ambiguity in what follows, we note that the inner product on any Hilbert space $L^2(\Omega)$ is given by

$$(f, g) \equiv \int_\Omega dz\, \overline{f(z)}g(z) \tag{3.20}$$

For $\mathcal{H} \equiv L^2(\mathbb{R}, dx)$ we give this its own symbol, $(\cdot, \cdot)_{\mathcal{H}}$.

For the maps $\mathcal{F}_\pm$ (3.19), we say the adjoint element of any $f \in \mathcal{H}$ exists in $L^2(\mathbb{R}^+, dy)$ and denote it by $\mathcal{F}^*_\pm f$ iff the latter solves the following equation

$$(\mathcal{F}^*_\pm f, g) = (f, \mathcal{F}_\pm g)_{\mathcal{H}}\,, \quad g \in C_0^\infty(\mathbb{R}^+) \tag{3.21}$$

The density of $C_0^\infty(\mathbb{R}^+)$ in $L^2(\mathbb{R}^+, dy)$ ensures any solution is unique. Clearly we will have

$$(\mathcal{F}^*_\delta f)(y) = \int_{\mathbb{R}} dx\, \overline{\psi(b; \delta x, y)} f(x), \quad \delta = +, - \tag{3.22}$$

whenever the integral functions on the rhs are in $L^2(\mathbb{R}^+, dy)$. We now ask when might this be so (analogising the discussion below (3.12)). In general, if f is any smooth function with compact support then the integral will converge provided y is a regular value of $\psi(b; x, y)$, and the latter has no x-poles on the real line. As we saw earlier, this latter property is assured by $b \in \mathbb{R} \setminus \tilde{S}$. This entails that the integral functions in (3.22) are meromorphic in y with poles equal to the y-poles of $\psi(b; x, y)$. Thus they will be smooth if we strengthen to $b \in \mathbb{R} \setminus \mathcal{Y}$. Combined with the $O(1)$ y-asymptotics of $\psi(b; x, y)$, we can then assert this integral function is bounded for $y \in \mathbb{R}^+$. With the additional information in Lemma 1.4 we can get a better view on its asymptotics and prove the following lemma. It involves the space of smooth, complex-valued functions on the real line with compact support, $C_0^\infty(\mathbb{R})$. For later use we need the subset with support away from the origin, which we denote $\mathcal{C} \subset C_0^\infty(\mathbb{R})$.

Lemma 3.4. *Assume $b \in \mathbb{R} \setminus \mathcal{Y}$ where the point set $\mathcal{Y}$ is defined in (3.2). Then, for any $f \in \mathcal{C}$, the adjoint elements $\mathcal{F}^*_\pm f$ exist in $L^2(\mathbb{R}^+, dy)$ and are given by (3.22).*

More generally, with these same restrictions on b and f, the meromorphically-continued functions $(\mathcal{F}^*_\pm f)(y+iv)$, $v \in \mathbb{R}$ are in $L^2(\mathbb{R}^+, dy)$ whenever iv is a regular value of $\psi(b; x, \cdot)$.

Proof. To make our statements about the integral functions on the rhs of (3.22) more concise we will assign them their own symbol, $(\mathcal{F}^\dagger_\delta f)(y)$ where again, $\delta = +, -$. As we have noted, the restriction $b \in \mathbb{R} \setminus \mathcal{Y}$ implies $(\mathcal{F}^\dagger_\delta f)(y)$ is meromorphic in y with poles equal to the y-poles of $\psi(x, y)$. This is the same restriction that ensures well-definedness of the map $\mathcal{F}_\delta$ (3.19) and thus allows us to consider the notion of adjoint elements for functions in $\mathcal{H}$. For any $f \in \mathcal{H}$ we have $\mathcal{F}^*_\delta f = \mathcal{F}^\dagger_\delta f$ iff $\mathcal{F}^\dagger_\delta f \in L^2(\mathbb{R}^+, dy)$. (Note, we are suppressing implicit dependence of $\psi(x, y)$ on b.)

With these facts in mind let us explain how the two statements in the lemma relate to each other. The point is that we are able to bound $(\mathcal{F}^\dagger_\delta f)(y + iv)$ from above by an L^2-integrable function as $y \to \infty$. Thus the only obstacle to entry into $L^2(\mathbb{R}^+, dy)$ is the possibility that $(\mathcal{F}^\dagger_\delta f)(\cdot)$ has a pole on the line $\mathbb{R} + iv$. Indeed since the y-poles of $\psi(x, y)$ are purely imaginary for real b, this amounts to the possibility that $\psi(b; x, \cdot)$ has a pole at the point iv. From Lemma 3.1 we see that the restriction $b \in \mathbb{R} \setminus \mathcal{Y}$ is needed for the special case of no pole at the origin. Thus both parts of the lemma will be proved by the aforementioned bound.

Say that for a particular choice of $\delta = +, -$ we can prove $(\mathcal{F}^\dagger_\delta f)(y + iv)$ is asymptotically bounded from above by a function in $L^2(\mathbb{R}^+, dy)$ for all $f \in \mathcal{C}$, then this is also true for the other δ (consider invariance of $\mathcal{C}$ under $f(x) \mapsto f(-x)$). We will prove the claim for $\delta = +$. Central to this is the O-asymptotics for $\psi(x, y)$ given in Lemma 1.4. This is valid for $b \in \mathbb{R} \setminus \mathcal{Y}$ and $x \in \mathbb{R}^*$ and gives us the following

$$\overline{\psi(x,y)} = \hat{\psi}_{as}(x,y) + O(e^{-\rho\,\mathrm{Re}\,y}), \quad \mathrm{Re}\,y \to \infty \tag{3.23}$$

$$\hat{\psi}_{as}(x,y) \equiv \tilde{\phi}^{-1}\big[\tilde{c}(x)/\tilde{c}(-x)\big]^{-1/2} e^{-i\pi xy/a_+a_-} \tag{3.24}$$

where $\rho > 0$ and where the bound represented by O is uniform for $\mathrm{Im}\,y$ and x varying over any compact subset of $\mathbb{R}$.

The remainder of the proof is analogous to that of Lemma 3.3. In the same way as before, we write

$$\begin{aligned}\frac{1}{2}\Big|\int_{\mathbb{R}} dx\, \overline{\psi(x,y)} f(x)\Big|^2 \ \leq\ & \Big|\int_{\mathbb{R}} dx\, \hat{\psi}_{as}(x,y) f(x)\Big|^2 \\ & + \int_{\mathbb{R}} dx\, |\overline{\psi(x,y)} - \hat{\psi}_{as}(x,y)|^2 |f(x)|^2, \quad f \in \mathcal{C}\end{aligned} \tag{3.25}$$

Well-known facts about Fourier transforms entail the first integral on the rhs will have L^2-asymptotics for $\mathrm{Re}\,y \to \infty$ (we recall $\tilde{\phi}\big[\tilde{c}(x)/\tilde{c}(-x)\big]^{1/2}$ is a phase). The same is true of the second integral, in light of the decay described in (3.23). □

3.4 Anticipating the bound states (II). The functions $\Psi^{(m)}$ as members of $\mathcal{F}_\pm(C_0^\infty(\mathbb{R}^+))^\perp$.

Now that we have defined and studied the eigenfunction transforms $\mathcal{F}_\pm$, we look at the functions $\Psi^{(m)}(b;x)$ introduced in §2 from a different perspective. We have already seen these functions are bound states in the heuristic sense of being integrable specialisations (more accurately, residues) of the eigenfunction $\psi(b;x,y)$. In this subsection we take a step closer to proving they are bound states in the proper time-dependent scattering theory sense. Of course at this stage, these results cannot be said to establish this because we have not yet even defined any dynamics, let alone proved existence of a wave operator (both of which first require the Hamiltonian operator to be introduced in §4). The point is that once we do have a wave operator, and find that it is related to $\mathcal{F}_+$ and $\mathcal{F}_-$, these results instantly take on a new meaning.

The eigenfunction claim is a straightforward consequence of Lemmas 1.1 and 2.2. We include it here because of its central role in proving the other two claims. The integer m_b plays a central role in later sections.

Lemma 3.5. *For generic values of a_+, a_-, b, the functions $\Psi^{(m)}(b;x)$ (2.4) are well-defined in terms of $Q_m(b;x)$ (2.5) and satisfy the following eigenvalue equations,*

$$\tilde{H}(b;x)\Psi^{(m)}(b;x) = E_m\Psi^{(m)}(b;x), \quad m \in \mathbb{N}, \tag{3.26}$$

$$E_m(b) \equiv 2\cos\Big(\frac{\pi}{a_+}[b-(m+1)a_-]\Big) \tag{3.27}$$

Moreover, $\Psi^{(m)}$ is in $\mathcal{H} \equiv L^2(\mathbb{R}, dx)$ iff $m \le m_b - 1$ where m_b is the largest integer such that $m_b a_- < b$.

We can give a more precise condition than genericity: the above statements are true for $\Psi^{(0)}, \ldots, \Psi^{(m)}$ provided $a_+/a_- \notin \mathcal{A}_m$ where $\mathcal{A}_m$ is the point set defined in (2.12); empty when $m = 0$. This condition is also needed for the following claims.

We have

$$\mathcal{F}_\pm f \perp \Psi^{(m)}, \quad f \in C_0^\infty(\mathbb{R}^+) \tag{3.28}$$

provided b does not take values from the point set $\mathcal{Y}$ (3.2). We note that $b > a_-$ is needed for at least one of the functions $\Psi^{(m)}$ to be in $\mathcal{H}$.

We have

$$\Psi^{(m)} \perp \Psi^{(n)}, \quad m \neq n \tag{3.29}$$

provided the eigenvalues E_m,E_n are distinct. Such distinction holds for generic values of the parameters. More precisely, we can say a given set $\{E_0, \ldots, E_m\}$ has all distinct elements iff b does not take values from the point set Π_m (2.15); empty when $m = 0$. We note that $b > 2a_-$ is needed for at least two of the functions $\Psi^{(m)}$ to be in $\mathcal{H}$.

Proof. We note first that the eigenvalue equation (1.75) implies the following for $\hat{\psi}(x,y)$ (2.27),

$$\tilde{H}(x)\hat{\psi}(x,y) = 2c_+(y)\hat{\psi}(x,y) \tag{3.30}$$

There was nothing about the proof of Lemma 2.2 that required x to be real. Thus we may take the residue of the shifted functions in the above equation at $y = y_{m+1}$ and invoke the lemma to deduce (3.26). (There is nothing substantial about the ability to interchange the residue and the AΔO; just envisage writing out the lhs of (3.30) explicitly using $\tilde{H}(x)$ (1.44) and then taking the residue.)

The claim about membership of $\mathcal{H}$ follows immediately from (2.11).

The proofs for the two orthogonality results anticipate some of the ideas in §4. In fact we will draw upon two of the propositions in that section (where we think they are better placed overall).[4]

For the second orthogonality result we will prove direct vanishing of

$$I_{m,n} \equiv (E_n - E_m)(\Psi^{(m)}, \Psi^{(n)})_{\mathcal{H}} \tag{3.31}$$

for $m \neq n \in \mathbb{N}$. Assuming E_m, E_n are distinct for $m \neq n$ (an assumption we return to at the end), vanishing of $I_{m,n}$ will imply vanishing of the inner product.

By involving the eigenvalues in this way, we will be able to expand the integrand using the AΔO $\tilde{H}(x)$ (1.44). This may seem like a strange step but with more terms to work with, we can reveal otherwise-hidden features. (This analogises the method in [3].)

Using reality of $\Psi^{(m)}$ for $(b, x) \in \mathbb{R}^2$ and the eigenvalue equation (3.26) we write out

$$I_{m,n} = (E_n - E_m) \int_{\mathbb{R}} dx\, \Psi^{(m)}(x)\Psi^{(n)}(x) = \int_{\mathbb{R}} dx \big([\tilde{H}(x)\Psi^{(n)}(x)]\Psi^{(m)}(x) - (n \leftrightarrow m)\big) \tag{3.32}$$

To rearrange this in a more useful form we draw upon Proposition 4.1. The AΔO $\tilde{H}(x)$ (1.44) satisfies the requirements with

$$U_1(x) = [\tilde{V}(x)\tilde{V}(-x + ia_-)]^{1/2} \tag{3.33}$$

and $U_2(x) = U_1(x + ia_-)$. Moreover the integrand in (3.32) has the form (4.26) with

$$\Phi_1(x) = \Psi^{(n)}(x), \quad \Phi_2(x) = \Psi^{(m)}(x) \tag{3.34}$$

Using the proposition we therefore have

$$I_{m,n} = \lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} dx\, \big(J_{m,n}(x) - J_{m,n}(x + ia_-)\big) \tag{3.35}$$

$$J_{m,n}(x) \equiv [\tilde{V}(x)\tilde{V}(-x + ia_-)]^{1/2} \big[\Psi^{(n)}(x - ia_-)\Psi^{(m)}(x) - (n \leftrightarrow m)\big], \quad m, n \in \mathbb{N} \tag{3.36}$$

By breaking up the integral this way, (3.35) is manifestly of the form in Proposition 4.2. Provided we can argue the residue sum there is empty, we thus have

$$- I_{m,n} = \lim_{\Lambda\to\infty} \int_{\Lambda}^{\Lambda+ia_-} dx\, \big(J_{m,n}(x) - J_{m,n}(-x + ia_-)\big) \tag{3.37}$$

[4] The proofs below involve two objects which form a triad with the analogous object in the symmetry proof of §4. We are talking here about: $I_{m,n}$, $I_{m,\delta}(y)$ and $I_{\delta,\delta'}(y, y')$. In this order they form a hierarchy of complexity, which is also true of the three correlates: $J_{m,n}(x)$, $J_m(x, y)$, $J_{\delta\delta'}(x, y, y')$.

Invoking the fact $\Psi^{(m)}(\cdot)$ has parity $(-)^m$, this integrand equals $J_{m,n}(x)(1+(-)^{n+m})$. And so, vanishing when $n+m$ is odd is immediate. For the even case we have to look more closely at asymptotics.

We know from (1.107) that the decay of $\tilde{w}(x)$ is uniform for $\operatorname{Im} x \in [0, a_-]$. Since this is also clearly true for the blow-up of $2c_+(x)$ and $Q_m(2s_+(x))$ we can revamp the dominant asymptotics (2.11),

$$\Psi^{(m)}(b;x) \sim \exp(\mp\pi x[b-(m+1)a_-]/a_+a_-), \quad \operatorname{Re} x \to \pm\infty \tag{3.38}$$

and assert uniform decay for $\operatorname{Im} x \in [0, a_-]$. As a result, the limit in (3.37) is vanishing.

It remains to prove that $J_{m,n}(x)$ (3.36) is holomorphic in the strip $\operatorname{Im} x \in [0, a_-]$ given the b restriction in the lemma. Using $\Psi^{(m)}(x)$ (2.4) we write out

$$\begin{aligned} J_{m,n}(x) = {} & [\tilde{V}(x)\tilde{V}(-x+ia_-)]^{1/2}\, \tilde{w}(x-ia_-)^{1/2}\tilde{w}(x)^{1/2} \\ & \times (2c_+(x))^2\big[Q_n(2s_+(x-ia_-))Q_m(2s_+(x)) - (m \leftrightarrow n)\big] \end{aligned} \tag{3.39}$$

Since the second line is entire, we need only focus on the first. The means to an easy simplification is already in place, viz. the similarity transform in (1.52). This tells us precisely that

$$\frac{\tilde{w}(x)^{1/2}\,\tilde{V}(x)}{\tilde{w}(x-ia_-)^{1/2}} = [\tilde{V}(x)\tilde{V}(-x+ia_-)]^{1/2} \tag{3.40}$$

And so the first line of the rhs in (3.39) equals $\tilde{w}(x)\tilde{V}(x)$. Recalling $\tilde{w}(x) \equiv 1/\tilde{c}(x)\tilde{c}(-x)$ and $\tilde{V}(x)/\tilde{c}(x) = 1/\tilde{c}(x-ia_-)$ (just (1.51)) we can write it as $1/\tilde{c}(x-ia_-)\tilde{c}(-x)$. With the definition $\tilde{c}(x)$ (1.49) and the reflection property of $G(\cdot)$, it simplifies further to

$$G(x-ia_-/2+ib)G(-x+ia_-/2+ib) \tag{3.41}$$

Thus $J_{m,n}(x)$ (3.39) is holomorphic in the desired strip if the same is true of this function. As noted in §1, the poles of $G(\cdot)$ (1.3) occur at the points

$$-ia - z_{k,l}, \quad z_{k,l} \equiv ika_+ + ila_-, \quad k,l \in \mathbb{N} \tag{3.42}$$

$$a \equiv (a_+ + a_-)/2 \tag{3.43}$$

And so the x-poles of the two G-functions in (3.41) occur respectively at

$$\begin{cases} -ia_+/2 - ib - z_{k,l} \\ ia_+/2 + ib + z_{k,l+1} \end{cases} \quad k,l \in \mathbb{N} \tag{3.44}$$

With our standing assumption that $a_+, a_- > 0$, we see that all these poles lie outside $i[0, a_-]$ provided $b > -a_+/2$.

Let us now prove the first orthogonality result, based on the strategy for the second (we note that the added b-restriction is needed for $\mathcal{F}_\pm f \in \mathcal{H}$, cf. Lemma 3.3).

Given

$$(\Psi^{(m)}, \mathcal{F}_\delta f)_{\mathcal{H}} = \int_{\mathbb{R}} dx\, \Psi^{(m)}(x) \int_{\mathbb{R}^+} dy\, \psi(\delta x, y) f(y), \quad \delta = +, - \tag{3.45}$$

it is clear that

$$(2c_+(y) - E_m)(\Psi^{(m)}, \mathcal{F}_\delta f)_{\mathcal{H}} = \int_{\mathbb{R}^+} dy\, f(y) I_{m,\delta}(y), \quad \delta = +, - \tag{3.46}$$

$$I_{m,\delta}(y) \equiv (2c_+(y) - E_m) \int_{\mathbb{R}} dx\, \Psi^{(m)}(x) \psi(\delta x, y) \tag{3.47}$$

(where Fubini's theorem can be used to interchange the integration order in (3.45), given f's compact support and $\Psi^{(m)}$'s exponential decay).

Given that $E_m \leq 1$, $(2c_+(y) - E_m)$ is positive on the integration region $\mathbb{R}^+$. Hence vanishing of $I_{m,\delta}(y)$ will prove vanishing of the inner product. Because $\Psi^{(m)}(\cdot)$ has parity $(-)^m$ it follows that $I_{m,-}(y) = (-)^m I_{m,+}(y)$. Thus it suffices to prove vanishing of $I_{m,+}(y)$.

Using the eigenvalue equation (3.26) we write out

$$I_{m,+}(y) = \int_{\mathbb{R}} dx \big([\tilde{H}(x)\psi(x, y)]\Psi^{(m)}(x) - \psi(x, y)[\tilde{H}(x)\Psi^{(m)}(x)] \big) \tag{3.48}$$

We can now use Proposition 4.1 to rewrite this integrand in a more useful way. It has the form (4.26) with

$$\Phi_1(x) = \psi(x, y), \quad \Phi_2(x) = \Psi^{(m)}(x) \tag{3.49}$$

And, as noted above, the AΔO $\tilde{H}(x)$ (1.44) satisfies the necessary requirements. Thus,

$$I_{m,+}(y) = \lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} dx\, \big(J_m(x, y) - J_m(x + ia_-, y) \big) \tag{3.50}$$

$$J_m(x, y) \equiv [\tilde{V}(x)\tilde{V}(-x + ia_-)]^{1/2} \big[\psi(x - ia_-, y)\Psi^{(m)}(x) - \psi(x, y)\Psi^{(m)}(x - ia_-) \big] \tag{3.51}$$

By breaking up the integral this way, (3.35) is manifestly of the form in Proposition 4.2. Provided we can argue the residue sum there is empty, we thus have

$$-I_{m,+}(y) = \lim_{\Lambda\to\infty} \int_{\Lambda}^{\Lambda + ia_-} dx\, \big(J_m(x, y) - J_m(-x + ia_-, y) \big) \tag{3.52}$$

Given the $O(1)$ asymptotics of $\psi(x, y)$, the function $J_m(x, y)$ (3.51) inherits the exponential decay (3.38), and so $I_{m,+}(y)$ vanishes.

It remains to prove that $J_m(x, y)$ (3.51) is holomorphic in the strip $\mathrm{Im}\, x \in [0, a_-]$ given the b restriction in the lemma. Using $\psi(x, y)$ (1.60) and $\Psi^{(m)}(x)$ (2.4) we write out

$$\begin{aligned} J_m(x, y) \propto{} & [\tilde{V}(x)\tilde{V}(-x + ia_-)]^{1/2}\, \tilde{w}(x - ia_-)^{1/2} \tilde{w}(x)^{1/2} \\ & \times 2c_+(x) \sum_{\nu=+,-} \nu e_-(\nu(ib - y)/2) \big[Q_m(2s_+(x)) \mathcal{R}_r(x - ia_- + i\nu a_+/2, y) \\ & \qquad - Q_m(2s_+(x - ia_-)) \mathcal{R}_r(x + i\nu a_+/2, y) \big] \end{aligned} \tag{3.53}$$

where we are of course dealing with proportionality with respect to x (the unimportant proportionality constant is $1/c(-y)s_-(ib - y)$).

The first line of the rhs of (3.53) is exactly the same as for (3.39). We argued this was equal to $\tilde{w}(x)\tilde{V}(x)$ which has no poles in the critical strip provided $b > -a_+/2$. We must therefore focus on the poles of the remaining terms in (3.53); specifically, on the two $\mathcal{R}_r$-functions, since all the others are entire. The surprise is that the poles of these two functions are removed by $\tilde{w}(x)\tilde{V}(x)$, and so $b > -a_+/2$ suffices for the desired holomorphy. We can reveal this at the algebraic level in the following way (thus avoiding the multiplicity question that arises by comparing pole sequences).

It is known that the hyperbolic gamma function $G(z)$ can be written as a ratio of entire functions, $E(z)/E(-z)$ where $E(a_+, a_-; z)$ is defined in Appendix A of [1] and summarised in Appendix A of [5]. Thus recalling (1.70) we have

$$\tilde{w}(x)\tilde{V}(x) = 2c_+(x - ib) \prod_{\alpha=+,-} \frac{E(\alpha x - ia_-/2 + ib)}{E(\alpha x + ia_-/2 - ib)} \tag{3.54}$$

The renormalised relativistic conical function $\mathcal{R}_r(x, y)$ also has an important relation to $E(z)$. For our purposes we write this as

$$\mathcal{R}_r(x+i\nu a_+/2, y) = \hat{P}(x, y)/ \prod_{\alpha=+,-} E(\alpha x - ia_-/2 + ib - i(1-\alpha\nu)a_+/2), \quad \nu = +, - \tag{3.55}$$

where $\hat{P}(x, y)$ is an entire function in x which depends on ν (as well as, of course, (a_+, a_-, b). Cf. [5](1.12),(3.33)). We thus see explicitly how the poles of (3.55) arise as zeros of the two denominator functions. Moreover we see straight away that one of these (corresponding to $\alpha = \nu$) will cancel with one of the the numerator terms in (3.54) (also $\alpha = \nu$). A second cancellation also obtains, but to see this we must invoke the AΔE for $E(z)$,

$$E(z - ia_\kappa) = E(z) \exp(K_\kappa(a_\kappa - iz))\Gamma(iz/a_{-\kappa} + 1/2)/\sqrt{2\pi}, \quad \kappa = +, - \tag{3.56}$$

where $K_\kappa \equiv \ln(a_\kappa/a_{-\kappa})/2a_{-\kappa}$ (cf. [5](A.23)). Applying the $\kappa = +$ AΔE to the $\alpha = -\nu$ term in (3.55) we get cancellation with the numerator term in (3.54). Moreover the AΔE will not lead to any more poles because the gamma function is famously free of zeros.

The reason for also citing the $\kappa = -$ AΔE is that we can use this for the other $\mathcal{R}_r$-function in (3.53) to assert in an analogous way that its poles are removed by $\tilde{w}(x)\tilde{V}(x)$ (3.54).

Finally, we prove the claim about $E_0, \ldots E_m$. Elementary identities entail that the difference $E_l - E_{l'}$ can be written in terms of the constant σ_m (2.6) in the following way,

$$E_l - E_{l'} = \begin{cases} \sigma_{(l+l')/2}, & (l + l' \text{ even}) \\ \sigma_{(l+l'-1)/2}, & (l + l' \text{ odd}) \end{cases} \qquad l \neq l' = 0, \ldots, m \tag{3.57}$$

Thus we see how distinctness of all E_l,$E_{l'}$ is equivalent to non-vanishing of $\sigma_0, \ldots, \sigma_{m-1}$. This is exactly the same condition needed for the polynomials $Q_1(u), \ldots, Q_l(u), \ldots, Q_m(u)$ defined by (2.5) to have degree l, and we proved it was equivalent to $b \notin \Pi_m$, cf. Lemma 2.1.

□

4 The operator $\tilde{H}_{\mathrm{ac}}$

Definition and symmetry.

In this section we begin looking at the AΔO $\tilde{H}(b;x)$ (1.52) from the perspective of Hilbert space quantum mechanics. The very first task is simply to understand it *as* a Hilbert space operator, since this cannot be done without some deliberation. To promote it as such in its present form would require specifying a dense domain of L^2-functions which yield other L^2-functions under the action of the AΔO. This is an unusual problem which we can avoid by defining our operator in a different way (i.e. not via the familiar action/domain specification). In line with previous papers, this involves using an eigenfunction transform to diagonalize the action of the AΔO. This way, issues like meromorphic continuation of the domain functions can all be studied at the level of the transform (this explains our attention to $\mathcal{F}_\pm$ in the previous section).

As we will see, the diagonalization process is really just a roundabout way of specifying a domain for $\tilde{H}(b;x)$ (cf. (4.10)), but it confers other advantages. By giving a central role to the eigenfunction transform, we are able to reduce questions like those concerning operator symmetry to a series of questions about the transform kernel. The generalised eigenfunction $\psi(b;x,y)$ is sufficiently special to see each of these through successfully. We note that domain considerations are no trivial matter because the same AΔO can give rise to vastly different physics depending on how a domain is chosen.

Our association of a Hilbert space operator to the AΔO $\tilde{H}(b;x)$ involves the two eigenfunction transforms $\mathcal{F}_\pm$ introduced in §3.2. For a successful account of the dynamics, we must glue these together in a particular way to yield a new eigenfunction transform. This transform acts on function pairs $\hat{f} = \langle f_+, f_- \rangle$ as follows,

$$
\begin{aligned}
(\mathcal{F}\hat{f})(x) &\equiv c \sum_{\delta=+,-} (\mathcal{F}_\delta f_\delta)(x) \\
&= c \int_{\mathbb{R}^+} dy\, \psi(b;x,y) f_+(y) + c \int_{\mathbb{R}^+} dy\, \psi(b;-x,y) f_-(y), \quad c \equiv 1/\sqrt{2a_+a_-}
\end{aligned} \tag{4.1}
$$

From the knowledge we have of $\mathcal{F}_\pm$ (3.12), various properties of this transform follow straight away. These all hold provided the coupling parameter b is real and does not take values from a certain discrete set, $\mathcal{Y}$ (3.2). These properties also involve the space of functions

$$\hat{\mathcal{C}} \equiv C_0^\infty(\mathbb{R}^+) \times C_0^\infty(\mathbb{R}^+) \tag{4.2}$$

whose elements are always written in the form $\hat{f} = \langle f_+, f_- \rangle$. So if, for example, $\hat{f} \in \hat{\mathcal{C}}$, we can assert that $(\mathcal{F}\hat{f})(x)$ is a function whose only poles are the x-poles of $\psi(b;\pm x,y)$ (all of which are purely imaginary). Furthermore, enough is known about the asymptotics of these integral functions to yield

$$\mathcal{F} : \hat{\mathcal{C}} \to \mathcal{H} \equiv L^2(\mathbb{R}, dx) \tag{4.3}$$

(a direct consequence of Lemma 3.3).

Assuming from now on that b has values of the kind described above (4.2), we claim that we can define a Hilbert space operator $\tilde{H}_{\mathrm{ac}}$ by the intertwining relation

$$\tilde{H}_{\mathrm{ac}}\mathcal{F} = \mathcal{F}\hat{M} \tag{4.4}$$

where $\hat{M}$ denotes multiplication on pairs by the generalised eigenvalue $2c_+(y)$, i.e.

$$(\hat{M}\hat{f})(y) \equiv 2c_+(y)\langle f_+(y), f_-(y)\rangle \tag{4.5}$$

(This requires not injectivity of $\mathcal{F}$ but the weaker condition that if $\mathcal{F}\hat{f}_1 = \mathcal{F}\hat{f}_2$ for distinct $\hat{f}_1, \hat{f}_2$, then $\mathcal{F}\hat{M}\hat{f}_1 = \mathcal{F}\hat{M}\hat{f}_2$, and this follows by considering (4.10) below for $\hat{f} = \hat{f}_1 - \hat{f}_2$.)

It is not hard to see that $\hat{M}$ maps $\hat{\mathcal{C}}$-functions to $\hat{\mathcal{C}}$-functions. Thus considering (4.4) on this space of functions we indeed get a Hilbert space operator,

$$\tilde{H}_{\mathrm{ac}} : \mathcal{F}(\hat{\mathcal{C}}) \to \mathcal{F}(\hat{\mathcal{C}}) \subseteq \mathcal{H} \tag{4.6}$$

To be sure, the Hilbert space in which this acts as a densely-defined operator is the closure of $\mathcal{F}(\hat{\mathcal{C}})$. (The precise nature of the subset relation in (4.6) will be a matter of later concern, cf. §7.)

Given that $2c_+(y)$ and $\psi(b; \pm x, y)$ are generalised eigenvalue-eigenfunction pairs of $\tilde{H}(b; x)$ (1.44), it is clear we have constructed is a diagonalization of this AΔO. We proceed to show explicitly that $\tilde{H}_{\mathrm{ac}}$ (4.6) reproduces the action of

$$\tilde{H}(b; x) = \tilde{U}(x)T^x_{ia_-} + (x \to -x) \tag{4.7}$$

$$\tilde{U}(x) \equiv \left[\tilde{V}(a_+, b; x)\tilde{V}(a_+, b; -x + ia_-)\right]^{1/2} = \left[\frac{c_+(x + ib - ia_-)}{c_+(x - ia_-)}\frac{c_+(x - ib)}{c_+(x)}\right]^{1/2} \tag{4.8}$$

First we write out

$$(\tilde{H}_{\mathrm{ac}}\mathcal{F}\hat{f})(x) = (\mathcal{F}\hat{M}\hat{f})(x) = c\int_{\mathbb{R}^+} dy \sum_{\delta=+,-} 2c_+(y)\psi(\delta x, y)f_\delta(y) \tag{4.9}$$

Then we replace $2c_+(y)$ with $\tilde{H}(b; x)$ for the following equation,[5]

$$(\mathcal{F}\hat{M}\hat{f})(x) = \tilde{U}(x)(\mathcal{F}f)(x - ia_-) + \tilde{U}(-x)(\mathcal{F}f)(x + ia_-), \quad \hat{f} \in \hat{\mathcal{C}}, \quad x \in \mathbb{R} \tag{4.10}$$

Or, simply,

$$(\tilde{H}_{\mathrm{ac}}F)(x) = \tilde{H}(x)F(x), \quad F \in \mathcal{F}(\hat{\mathcal{C}}) \subseteq \mathcal{H} \tag{4.11}$$

[5] There is a subtlety to note here; our standing assumption about b ensures $(\mathcal{F}f)(\cdot)$ has no pole at the origin, but it does not necessarily ensure no poles at $\pm ia_-$. Indeed we claim it is *not* strong enough to guarantee this. Thus for certain b values it may be the case that the functions on the rhs of (4.10) are singular at $x = 0$, even though the lhs is regular. When this happens there *must* be a residue cancellation between the two functions that ensures convergence of the rhs.

For the remainder of this section we address the question of whether $\tilde{H}_{\mathrm{ac}}$ (4.4)/(4.6) is symmetric. We do this by extending the method used in [20] and [3]. Our discussion culminates in Theorem 4.5, which gives us conditions on the parameters for which the operator $\tilde{H}_{\mathrm{ac}}$ is symmetric. The method involves a direct computation of the integrals implicit in

$$D(\hat{f}, \hat{g}) \equiv (\tilde{H}_{\mathrm{ac}}\mathcal{F}\hat{f}, \mathcal{F}\hat{g})_{\mathcal{H}} - (\mathcal{F}\hat{f}, \tilde{H}_{\mathrm{ac}}\mathcal{F}\hat{g})_{\mathcal{H}}, \tag{4.12}$$

If we can show that this vanishes for all $\hat{f}, \hat{g} \in \hat{\mathcal{C}}$, then this clearly implies symmetry of $\tilde{H}_{\mathrm{ac}}$.

From the definitions of $\tilde{H}_{\mathrm{ac}}$ (4.4) and $\mathcal{F}$ (4.1) and linearity of the inner product, it follows straight away that for $\hat{f}, \hat{g} \in \hat{\mathcal{C}}$,

$$(\tilde{H}_{\mathrm{ac}}\mathcal{F}\hat{f}, \mathcal{F}\hat{g})_{\mathcal{H}} = c^2 \sum_{\delta,\delta'=+,-} (\mathcal{F}_\delta \hat{m} f_\delta, \mathcal{F}_{\delta'} g_{\delta'})_{\mathcal{H}}, \tag{4.13}$$

$$(\mathcal{F}\hat{f}, \tilde{H}_{\mathrm{ac}}\mathcal{F}\hat{g})_{\mathcal{H}} = c^2 \sum_{\delta,\delta'=+,-} (\mathcal{F}_\delta f_\delta, \mathcal{F}_{\delta'} \hat{m} g_{\delta'})_{\mathcal{H}} \tag{4.14}$$

where $\hat{m}$ denotes multiplication by the generalised eigenvalue $2c_+(y)$ (for well-definedness of the inner products on the rhs, i.e. $\mathcal{F}_\delta f_\delta \in \mathcal{H}$, recall Lemma 3.3). To consider the difference of the two sums on the rhs, we will focus on

$$d_{\delta,\delta'}(f,g) \equiv (\mathcal{F}_\delta \hat{m} f, \mathcal{F}_{\delta'} g)_{\mathcal{H}} - (\mathcal{F}_\delta f, \mathcal{F}_{\delta'} \hat{m} g)_{\mathcal{H}}, \quad \delta, \delta' = +,- \tag{4.15}$$

In fact vanishing of $D(\hat{f}, \hat{g})$ (4.12) will follow because we can prove vanishing of these paired differences for all $f, g \in C_0^\infty(\mathbb{R}^+)$. We now look at $d_{\delta,\delta'}(f,g)$ more closely. Our first round of manipulation takes us to (4.21). We begin by writing out,

$$(\mathcal{F}_\delta \hat{m} f, \mathcal{F}_{\delta'} g)_{\mathcal{H}} = \int_{\mathbb{R}} dx \int_{\mathbb{R}^+} dy\, \overline{\psi(\delta x, y)} 2c_+(y) \overline{f(y)} \int_{\mathbb{R}^+} dy'\, \psi(\delta' x, y') g(y') \tag{4.16}$$

Our first step is to push $\int dx$ through the other two integrals in order to isolate an integral which is independent of f, g. Fubini's theorem is key here. In order to use it we must first replace $\int_{\mathbb{R}} dx$ with $\lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} dx$. With this change, we obtain a bounded integration region in the variables (x, y, y') on which the integrand is bounded (note, $\int_{\mathbb{R}^+} dy$ is really just $\int_{\mathrm{supp}(f)} dy$ here; we also invoke absence of real poles for $\psi(x,y)$ from Corollary 3.2). Hence the rhs of (4.16) equals

$$\lim_{\Lambda\to\infty} \int_{\mathbb{R}^+} dy\, 2c_+(y) \overline{f(y)} \int_{\mathbb{R}^+} dy'\, g(y') \int_{-\Lambda}^{\Lambda} dx\, \overline{\psi(\delta x, y)} \psi(\delta' x, y') \tag{4.17}$$

(by a use of Fubini that takes place entirely under the limit). We can manipulate $(\mathcal{F}_\delta f, \mathcal{F}_{\delta'} \hat{m} g)_{\mathcal{H}}$ in a completely analogous way and combine it with (4.17) to get

$$d_{\delta,\delta'}(f,g) = \lim_{\Lambda\to\infty} \int_{\mathbb{R}^+} dy\, \overline{f(y)} \int_{\mathbb{R}^+} dy'\, g(y') \int_{-\Lambda}^{\Lambda} dx\, \overline{\psi(\delta x, y)} \psi(\delta' x, y') \times \big(2c_+(y) - 2c_+(y')\big), \quad f, g \in C_0^\infty(\mathbb{R}^+), \quad \delta, \delta' = +,- \tag{4.18}$$

(the ability to combine these limits is automatic from their well-definedness). A simplification of the x-integral is possible here due to the "even" integral range $[-\Lambda, \Lambda]$. It means that the $(+,+)$-integral will equal the $(-,-)$-integral, and likewise for $(+,-)$ and $(-,+)$. Thus we have

$$d_{+,+}(f,g) = d_{-,-}(f,g), \tag{4.19}$$

$$d_{+,-}(f,g) = d_{-,+}(f,g), \tag{4.20}$$

and we can write

$$d_{\delta,\delta'}(f,g) = \lim_{\Lambda\to\infty} \int_{\mathbb{R}+} dy\, \overline{f(y)} \int_{\mathbb{R}+} dy' g(y') I_{\delta\delta'}(\Lambda; y, y') \tag{4.21}$$

$$I_\sigma(\Lambda; y, y') \equiv \int_{-\Lambda}^{\Lambda} dx\, \overline{\psi(x,y)}\psi(\sigma x, y')\big(2c_+(y) - 2c_+(y')\big), \quad \sigma = +, - \tag{4.22}$$

This latter integral is independent of f, g, giving us a useful point of study. Motivated by [3], the idea is to first *expand* the integrand in (4.22) using the x-AΔE for $\psi(x,y)$,

$$\tilde{H}(x)\psi(\pm x, y) = 2c_+(y)\psi(\pm x, y) \tag{4.23}$$

(recall (1.75) and the "evenness" of $\tilde{H}(x)$ (4.7)). In other words, we write

$$I_\sigma(\Lambda; y, y') = \int_{-\Lambda}^{\Lambda} dx\, \Big(\overline{[\tilde{H}(x)\psi(x,y)]}\psi(\sigma x, y') - \overline{\psi(x,y)}[\tilde{H}(x)\psi(\sigma x, y')]\Big) \tag{4.24}$$

Such an expansion may seem counterintuitive, but it allows us to implement our main strategy for vanishing of $d_{\delta,\delta'}(f,g)$. This breaks down into three main steps, each of which we present as a proposition below. Their combined effect is to express $d_{\delta,\delta'}(f,g)$ as a sum of residues in a particular strip of the complex plane. Finally, by restricting b in a particular way we can drive the corresponding poles out of the strip and thereby prove vanishing of $d_{\delta,\delta'}(f,g)$, and thus symmetry of $\tilde{H}_{\mathrm{ac}}$. (Because of this last step, we avoid seeing the residues explicitly, so the reader will search in vain below.)

In line with our convention, the propositions below are presented in terms of minimal assumptions (which are obviously designed with our objects in mind). The first proposition shows how features of $\tilde{H}(x)$ allow the integrand in (4.24) to be rearranged in particular way. The second proposition shows why this rearrangement is useful; it allows us to rewrite the integral in terms of a residue sum and boundary integrals. The third proposition presents a condition on the integrand which secures vanishing of these boundary integrals under the large-Λ limit when recombined with (4.21).

[5] We cannot say for sure if the components that arise from writing out the action of $\tilde{H}(x)$ in (4.24) are necessarily integrable when considered separately, because we have not ruled out the possibility that $\psi(\cdot, y)$ could have poles at $\pm ia_-$. Knowledge of such integrability (on $[-\Lambda, \Lambda]$) is not needed for our argument; what we need is comparable knowledge about the function (4.37), which is established later.

Proposition 4.1. *Suppose we have an $A\Delta O$,*

$$h(x) \equiv U_1(x)T^x_{ia_-} + U_2(x)T^x_{-ia_-} \tag{4.25}$$

where $U_j(\cdot)$ are two meromorphic functions. And suppose $\Phi_j(x)$ are two more meromorphic functions. Then, the object

$$[h(x)\Phi_1(x)]\Phi_2(x) - \Phi_1(x)[h(x)\Phi_2(x)] \tag{4.26}$$

may be written as

$$J(x) - J(x+ia_-) \tag{4.27}$$

$$J(x) \equiv U_1(x)\big(\Phi_1(x-ia_-)\Phi_2(x) - \Phi_1(x)\Phi_2(x-ia_-)\big) \tag{4.28}$$

provided

$$U_2(x) = U_1(x+ia_-) \tag{4.29}$$

Proof. Writing out (4.26) we have

$$\begin{aligned}\big[U_1(x)\Phi_1(x-ia_-) + U_2(x)\Phi_1(x+ia_-)\big]&\Phi_2(x)\\ &- \Phi_1(x)\big[U_1(x)\Phi_2(x-ia_-) + U_2(x)\Phi_2(x+ia_-)\big]\end{aligned} \tag{4.30}$$

Introducing

$$L^-(x) \equiv \Phi_1(x-ia_-)\Phi_2(x) - \Phi_1(x)\Phi_2(x-ia_-), \tag{4.31}$$

it is clear that (4.30) can be written as

$$U_1(x)L^-(x) - U_2(x)L^-(x+ia_-) \tag{4.32}$$

Thus we can absorb $U_1(x)$ into $L^-(x)$ such that (4.32) equals (4.27) iff $U_2(x) = U_1(x+ia_-)$. □

To see how this proposition connects to $I_\sigma(\Lambda; y, y')$ (4.24), we first note that the structure of $\tilde{H}(x)$ (4.7) entails the "reality" property,

$$[\overline{\tilde{H}(x)\psi(x,y)}] = \tilde{H}(x)\overline{\psi(\overline{x},y)}, \quad x \in \mathbb{R} \tag{4.33}$$

(the conjugation $\overline{x}$ is not superfluous because $\tilde{H}(x)$ entails any x to the right is shifted). Furthermore, $\tilde{H}(x)$ (4.7) plainly satisfies the the conditions of Proposition 4.1 with

$$U_1(x) = \tilde{U}(x) = [\tilde{V}(x)\tilde{V}(-x+ia_-)]^{1/2} \tag{4.34}$$

and $U_2(x) = U_1(x+ia_-)$. The integrand in (4.24) then has the form (4.26) with

$$\Phi_1(x) = \overline{\psi(\overline{x},y)}, \quad \Phi_2(x) = \psi(\sigma x, y') \tag{4.35}$$

(note we implicitly deal with two choices of Φ_2 here since σ equals $+$ or $-$). In fact this Φ_1 simplifies when when we apply (1.77),

$$\Phi_1(x) = \psi(x, -y), \quad y \in \mathbb{R} \tag{4.36}$$

The two corresponding specialisation of J (4.28) are thus

$$\begin{aligned} J_\sigma(b; x, y, y') \equiv {} & [\tilde{V}(x)\tilde{V}(-x+ia_-)]^{1/2} \\ & \times \left[\psi(x-ia_-, -y)\psi(\sigma x, y') - \psi(x, -y)\psi(\sigma(x-ia_-), y')\right], \quad \sigma = +, - \end{aligned} \tag{4.37}$$

$$\tilde{V}(x) \equiv c_+(x-ib)/c_+(x) \tag{4.38}$$

(recall that the eigenfunction $\psi(b; x, y)$ is defined in (1.60)). We have therefore shown, as relevant to $I_\sigma(\Lambda; y, y')$ (4.24), that

$$\begin{aligned} & \overline{[\tilde{H}(x)\psi(x, y)]}\psi(\sigma x, y') - \overline{\psi(x, y)}[\tilde{H}(x)\psi(\sigma x, y')] \\ & \qquad = J_\sigma(x, y, y') - J_\sigma(x+ia_-, y, y'), \quad \sigma = +, -, \quad x \in \mathbb{R} \end{aligned} \tag{4.39}$$

In the next proposition we see explicitly why this rearrangement is useful with regard to the overall aim of proving vanishing of $d_{\delta,\delta'}(f, g)$ (4.15). We recall our intentions sketched below (4.24). (Note, one can envisage attaching a (y, y')-dependence to the object in this proposition, as well as the previous one. This does not change anything important.)

Proposition 4.2. *Suppose $J(\cdot)$ is a meromorphic function in the strip $i[0, a_-] \times \mathbb{R}$ with a finite number of poles which are, furthermore, away from the boundary.*

Then, denoting these poles as $x_1, \ldots, x_L$, we have for Λ sufficiently large,

$$\int_{-\Lambda}^{\Lambda} dx \left(J(x) - J(x+ia_-)\right) = 2\pi i \sum_{j=1}^{L} \operatorname*{Res}_{x=x_j} J(x) - \left(\int_{\Lambda}^{\Lambda+ia_-} + \int_{-\Lambda+ia_-}^{-\Lambda}\right) dx\, J(x) \tag{4.40}$$

Proof. This follows almost straight away from Cauchy's theorem. If we take $\Lambda > \max\{\operatorname{Re} x_j\}_{j=1}^{L}$ then the theorem entails

$$\left(\int_{-\Lambda}^{\Lambda} + \int_{\Lambda}^{\Lambda+ia_-} + \int_{\Lambda+ia_-}^{-\Lambda+ia_-} + \int_{-\Lambda+ia_-}^{-\Lambda}\right) dx\, J(x) = 2\pi i \sum_{j=1}^{L} \operatorname*{Res}_{x=x_j} J(x) \tag{4.41}$$

Applying the variable change $x \to x + ia_-$ to the third integral, we can combine it with the first to write both of them as

$$\int_{-\Lambda}^{\Lambda} dx \big(J(x) - J(x+ia_-)\big) \tag{4.42}$$

So the result follows. □

Proposition 4.3. *Suppose $J(x,y,y')$ is a function smooth in y,y' and meromorphic in x which has asymptotics satisfying*

$$J(x,y,y') = \sum_{\tau,\tau'=+,-} L_{\tau,\tau'}(y,y') \exp(i\pi x(\tau y+\tau' y')/a_+a_-) + O(e_-(-\mu \operatorname{Re} x)), \quad \operatorname{Re} x \to \infty \tag{4.43}$$

where: $L_{\pm,\pm}$ are smooth functions; μ is a positive constant; and where the bound represented by O is uniform for $(y,y',\operatorname{Im} x)$ varying over any compact subset of $\mathbb{R}^* \times \mathbb{R}^* \times \mathbb{R}$.

Furthermore, suppose $\phi \in C_0^\infty(\Omega)$ where Ω is equal to $\mathbb{R}$ or $\mathbb{R}^+$. Then,

$$\lim_{\Lambda\to\infty} \int_{\Omega^2} dydy'\, \phi(y,y') \int_{\Lambda}^{\Lambda+ia_-} ds\, J(x,y,y') = 0 \tag{4.44}$$

Proof. Vanishing of the $O(e_-(-\mu \operatorname{Re} x))$ contribution is fairly immediate from uniformity of the bound. To see this define the compact set $K \equiv \operatorname{supp}(\phi)$. Then the uniformity statement entails that for $\operatorname{Re} x$ sufficiently large, there exists $\mathcal{C} > 0$ such that for all $y,y' \in K$ and $\operatorname{Im} x \in [0,a_-]$ we have$|J(x,y,y')| \leq \mathcal{C} e_-(-\mu \operatorname{Re} x)$. Thus the modulus of the contribution to (4.44) that comes from $O(e_-(-\mu \operatorname{Re} x))$ is bounded above by

$$\mathcal{C} e_-(-\mu\Lambda) a_- \int_K dydy'\, |\phi(y,y')| \tag{4.45}$$

and subsequently vanishes under $\Lambda \to \infty$.

Vanishing of the plane wave contribution is a little less immediate but still straightforward. Unlike in (4.45), the y,y' integral is now central. The x-integrand clearly has a primitive and, with this, we get for each (τ,τ') a contribution to the integral in (4.44) equalling

$$\int_K dydy'\, \phi(y,y') L_{\tau,\tau'}(y,y') \frac{e_+(-\tau y+\tau' y')}{\tau y+\tau' y'} \exp(i\pi(\tau y+\tau' y')/a_+a_-)), \quad \tau,\tau' = +,- \tag{4.46}$$

From our assumptions, this integrand is bounded on the compact set $K \subset \Omega^2$ (true when $\tau' y' = -\tau y$ because of the basic fact $\lim_{u\to 0}(e^{-u}-1)/u = 1$). Thus the integrand is in $L^1(\Omega^2)$ and we can invoke the Riemann-Lebesgue lemma to procure vanishing under $\Lambda \to \infty$.

□

As explained below (4.24), the idea is to apply Proposition 4.2 to $I_\sigma(\Lambda; y, y')$ (4.24). The boundary integrals produced by this will then vanish under recombination with $d_{\delta,\delta'}(f,g)$ (4.21) because of Proposition 4.3, provided we can show $J_\pm(x,y,y')$ (4.37) have the necessary asymptotics. The overall result of these steps is to reduce $d_{\delta,\delta'}(f,g)$ to a sum of x-residues of $J_\sigma(x,y,y')$ in the strip $\operatorname{Im} x \in [0,a_-]$. However, from the next lemma we learn that by restricting b in a particular way, we can drive the corresponding poles out of the strip. With no poles in the strip, vanishing of $d_{\delta,\delta'}(f,g)$, and thus $D(\hat{f},\hat{g})$ (4.12), follows immediately. The holomorphy claim is not at all trivial because there is a large number of poles to consider, and we relegate the proof to the appendices.

We note that the restriction on b in the lemma below is a special case of $b \in \mathbb{R} \backslash \mathcal{Y}$ (3.2) from earlier. Indeed when proving the lemma we begin with the latter, but to achieve the desired holomorphy we have to pinch the endpoints from infinity and limit them to the interval below.

Lemma 4.4. *The functions* $J_{\pm}(b; x, y, y')$ (4.37) *are smooth in* y, y' *and holomorphic in* x *in the strip* $\operatorname{Im} x \in [0, a_-]$ *provided* b *satisfies* $b \in (-a_+/2, a_- + a_+/2)$ *and is not a* $\mathbb{Z}$*-integer multiple of* a_-.

Theorem 4.5. *Assume the coupling parameter* b *has the range of the previous lemma, then the Hilbert space operator* $\tilde{H}_{ac} : \mathcal{F}(\hat{\mathcal{C}}) \to \mathcal{F}(\hat{\mathcal{C}}) \subseteq \mathcal{H}$ *defined by* (4.4) *is symmetric.*

Proof of 4.5. We recall that symmetry of $\tilde{H}_{\rm ac}$ is manifestly equivalent to vanishing of $D(\hat{f}, \hat{g})$ (4.12) for all $\hat{f}, \hat{g} \in \hat{\mathcal{C}}$ (4.2). From (4.12)-(4.15) we have

$$D(\hat{f}, \hat{g}) = c^2 \sum_{\delta, \delta' = +, -} d_{\delta, \delta'}(f_\delta, g_{\delta'}) \tag{4.47}$$

The claim is that each $d_{\delta,\delta'}(f, g)$ vanishes for all $f, g \in C_0^\infty(\mathbb{R}^+)$. We recall that the relationship between $d_{\delta,\delta'}(f, g)$ and the intermediary object $I_\sigma(\Lambda; y, y')$ (4.22) is given by

$$d_{\delta,\delta'}(f, g) = \lim_{\Lambda \to \infty} \int_{\mathbb{R}^+} dy\, \overline{f(y)} \int_{\mathbb{R}^+} dy' g(y') I_{\delta\delta'}(\Lambda; y, y'), \quad f, g \in C_0^\infty(\mathbb{R}^+) \tag{4.48}$$

(this is just (4.21) again). Plugging our result (4.39) into (4.24) we have

$$I_{\delta\delta'}(\Lambda; y, y') = \int_{-\Lambda}^{\Lambda} dx\, [J_{\delta\delta'}(x, y, y') - J_{\delta\delta'}(x + ia_-, y, y')], \quad \delta, \delta' = +, - \tag{4.49}$$

We are now in a position to apply Proposition 4.2 to this integral. The resulting residue sum will be empty because of the holomorphy of $J_{\pm}(x, y, y')$ described in Lemma 4.4 (moreover this clearly encompasses the meromorphy requirement in the proposition). Thus we get

$$I_{\delta\delta'}(\Lambda; y, y') = -\Big(\int_{\Lambda}^{\Lambda + ia_-} + \int_{-\Lambda + ia_-}^{-\Lambda} \Big) J_{\delta\delta'}(x, y, y'), \quad \delta, \delta' = +, - \tag{4.50}$$

The final step is to use Proposition 4.3 to argue for vanishing of these boundary integrals under recombination with (4.48). Applying the variable change $x \to -x + ia_-$ to the second integral, the rhs becomes

$$-\int_{\Lambda}^{\Lambda + ia_-} dy\, \big(J_{\delta\delta'}(x, y, y') - J_{\delta\delta'}(-x + ia_-, y, y') \big) \tag{4.51}$$

Thus we see that the task is to prove the four functions $J_{\pm}(x, y, y')$ and $J_{\pm}(-x + ia_-, y, y')$ (4.37) have asymptotics of the form (4.44). This concerns us for the remainder of the proof. The key here is the O-asymptotics for $\psi(x, y)$ in Lemma 1.3 Specifically we can extract from the latter the following

$$\psi(\alpha x, y) = \begin{cases} t(y)e^{i\pi xy/a_+a_-} + O(e_-^{-\rho \operatorname{Re} x}), & \alpha = + \\ e^{-i\pi xy/a_+a_-} - r(y)e^{i\pi xy/a_+a_-} + O(e_-^{-\rho \operatorname{Re} x}), & \alpha = - \end{cases} \qquad \operatorname{Re} x \to \infty \tag{4.52}$$

where $\rho > 0$ is a constant fixed by a_+, a_-, b and the bound represented by O is uniform for $\operatorname{Im} x$ varying over any compact subset of $\mathbb{R}$. Because of this latter fact we can shift x to yield

$$\psi(\alpha(x - ia_-), y) = \begin{cases} t(y)e_+^{-y}e^{i\pi xy/a_+a_-} + O(e_-^{-\rho \operatorname{Re} x}), & \alpha = + \\ e_+^{y}e^{-i\pi xy/a_+a_-} - r(y)e_+^{-y}e^{i\pi xy/a_+a_-} + O(e_-^{-\rho \operatorname{Re} x}), & \alpha = - \end{cases} \qquad \operatorname{Re} x \to \infty \tag{4.53}$$

where the same statements about ρ and O hold.

Before applying these to the problem at hand we also need the the large-$\operatorname{Re} x$ asymptotics of $\tilde{V}(x)$ (4.34). We can use the easily-verified

$$(2c_+(x + i\varphi))^{\pm 1} = e_+^{\pm x}e_+^{\pm i\varphi} + O(e_+^{-\operatorname{Re} x}), \quad \operatorname{Re} x \to \infty, \quad \varphi \in \mathbb{R} \tag{4.54}$$

to see straight away that

$$\tilde{V}(x) = 1 + O(e_+^{-\operatorname{Re} x}), \quad \operatorname{Re} x \to \infty \tag{4.55}$$

where, as with (4.54), the bound represented by O is uniform for $\operatorname{Im} x \in \mathbb{R}$.

With (4.52), (4.53) and (4.55) in place, we substitute them into (4.37) to get straight away

$$J_+(x, y, y') = t(-y)t(y')e^{i\pi x(y'-y)/a_+a_-}\big(e_+^{y} - e_+^{-y'}\big) + O(e^{-\kappa \operatorname{Re} x}), \quad \operatorname{Re} x \to \infty \tag{4.56}$$

$$\begin{aligned} J_-(x, y, y') = {}& t(-y)e^{-i\pi x(y+y')/a_+a_-}\big(e_+^{y} - e_+^{y'}\big) \\ & - e^{i\pi x(y'-y)/a_+a_-}t(-y)r(y')\big(e_+^{y} - e_+^{-y'}\big) + O(e^{-\kappa \operatorname{Re} x}), \quad \operatorname{Re} x \to \infty \end{aligned} \tag{4.57}$$

where $\kappa \equiv \pi \min(\rho/a_-, 1/a_+) > 0$, and where the bound represented by O is uniform for $(y, y', \operatorname{Im} x)$ varying over any compact subset of $\mathbb{R}^* \times \mathbb{R}^* \times \mathbb{R}$.

For the other two functions specified above (4.52), we use the fact $\tilde{V}(x)$ is invariant under $x \to -x + ia_-$ to write

$$\begin{aligned} J_\sigma(-x + ia_-, y, y'y) = {}& \tilde{V}(x)\big[\psi(-x, -y)\psi(-\sigma(x - ia_-), y') \\ & - \psi(-(x - ia_-), -y)\psi(-\sigma x, y')\big], \quad \sigma = +, - \end{aligned} \tag{4.58}$$

Using (4.52), (4.53) and (4.55), we get equations analogous to (4.56) and (4.57) in exactly the same way, with κ unchanged.

□

5 Self-adjointness and dynamics.

5.1 Nelson's theorem.

(Continuous spectrum of $\tilde{H}_{\mathbf{ac}}$.)

In the previous section we introduced an operator $\tilde{H}_{\mathrm{ac}}$ (4.4) in the Hilbert space $\overline{\mathcal{F}(\hat{\mathcal{C}})}$ whose action on its dense domain $\mathcal{F}(\hat{\mathcal{C}})$ is that of the AΔO $\tilde{H}(b;x)$ (closure here is taken with respect to the ambient space $\mathcal{H}$). We found this operator was symmetric provided the coupling parameter b satisfied certain conditions. The symmetry result can be supplemented fairly easily to establish that $\tilde{H}_{\mathrm{ac}}$ is essentially self-adjoint (i.e. has a unique self-adjoint extension). The argument hinges on Nelson's analytic vector theorem. The statement of this theorem (which can be found in its full glory in, e.g., [8]) simplifies considerably because of certain properties enjoyed by the objects at hand.

As noted already, multiplication $\hat{M}$ (4.5) maps the set of functions $\hat{\mathcal{C}}$ (4.2) into itself. But it is also onto, as can easily be seen. As a result we can iterate the intertwining (4.4) (which produces a well-defined operator provided $b \in \mathbb{R} \setminus \mathcal{Y}$ (3.2)) to yield $\tilde{H}^n_{\mathrm{ac}} \mathcal{F} = \mathcal{F} \hat{M}^n$ with $\tilde{H}^n_{\mathrm{ac}}$ having the same domain as $\tilde{H}_{\mathrm{ac}}$, namely $\mathcal{F}(\hat{\mathcal{C}})$. Because of this, Nelson's theorem gives us a sufficient condition for symmetry of $\tilde{H}_{\mathrm{ac}}$ to imply essential self-adjointness,

$$\sum_{n=0}^{\infty} \|\tilde{H}^n_{\mathrm{ac}} \mathcal{F} \hat{f}\|_{\mathcal{H}}/n! < \infty, \quad \hat{f} \in \hat{\mathcal{C}} \tag{5.1}$$

(in other words, that all $F \in \mathcal{F}(\hat{\mathcal{C}})$ are analytic vectors). This holds whenever $\tilde{H}_{\mathrm{ac}}$ is defined because for each $\hat{f} \in \hat{\mathcal{C}}$ we have a bound

$$\|\mathcal{F} \hat{M}^n \hat{f}\|_{\mathcal{H}} \leq \hat{c}^n \|\mathcal{F} \hat{f}\|_{\mathcal{H}}, \quad n \in \mathbb{N} \tag{5.2}$$

where $\hat{c}$ depends only on a_+ and $\mathrm{supp}(\hat{f}) \subset \mathbb{R}^+ \times \mathbb{R}^+$. To see this we recall $\mathcal{F}\hat{f} = c(\mathcal{F}_+ f_+ + \mathcal{F}_- f_-)$ and use the definition $\mathcal{F}_\delta$ (4.1) to assert

$$\|\mathcal{F}_\delta \hat{m} f\|_{\mathcal{H}} \leq 2c_+(y_{\max}) \|\mathcal{F}_\delta f\|_{\mathcal{H}}, \quad f \in C_0^\infty(\mathbb{R}^+), \quad \delta = +, - \tag{5.3}$$

where $y_{\max} \equiv \max(\mathrm{supp}(f))$.

Theorem 4.5 gives conditions on b for which $\tilde{H}_{\mathrm{ac}}$ is symmetric. Thus in light of the above, this theorem can be immediately strengthened to the following.

Theorem 5.1. *Assume the coupling parameter b satisfies $b \in (-a_+/2, a_- + a_+/2)$ and is not a $\mathbb{Z}$-integer multiple of a_-. Then, the Hilbert space operator $\tilde{H}_{ac} : \mathcal{F}(\hat{\mathcal{C}}) \to \mathcal{F}(\hat{\mathcal{C}})$ defined by (4.4) is essentially self-adjoint in the closure of $\mathcal{F}(\hat{\mathcal{C}})$. It has absolutely continuous spectrum $[2, \infty)$ of multiplicity two.*

5.2 Wave operators for $\tilde{H}$.

Having found conditions on the coupling parameter b for which the operator $\tilde{H}_{\mathrm{ac}} : \mathcal{F}(\hat{\mathcal{C}}) \to \mathcal{F}(\hat{\mathcal{C}})$ defined by (4.4) is essentially self-adjoint, we may consider the one-parameter unitary group on the closure of $\mathcal{F}(\hat{\mathcal{C}})$ encoding the associated dynamics. We *could* do this, but we are not going to. Instead, we are going to define a new operator which extends $\tilde{H}_{\mathrm{ac}}$ to the orthocomplement of this closure. On this new space we take it to have (initially) the action of an arbitrary bounded self-adjoint operator. The corresponding operator is denoted $\tilde{H}$ and clearly it inherits essential self-adjointness under the same conditions on b, i.e. those in Theorem 5.1. From this definition it is densely-defined in $\mathcal{H}$,

$$\tilde{H} : \mathcal{F}(\hat{\mathcal{C}}) \oplus \overline{\mathcal{F}(\hat{\mathcal{C}})}^{\perp} \to \mathcal{H} \tag{5.4}$$

From Lemma 3.5 we know that this orthocomplement is at least m_b-dimensional. In fact, because of this lemma there is a natural way to fix the action of $\tilde{H}$ on this space, but we do not discuss this until our reappraisal in the next subsection. This is because we wish to stress that the results in this subsection are *independent* of the choice of extension (we find that it drops out of the proofs as irrelevant). The point of introducing it here is that it enables us to consider a one-parameter unitary group on the physical space $\mathcal{H}$,[6]

$$\exp(it\tilde{H}), \quad t \in \mathbb{R} \tag{5.5}$$

(which still encodes the dynamics of the pre-extended system in the sense that, for any $f \in \mathcal{F}(\hat{\mathcal{C}})$, the vector $\exp(-it\tilde{H})f$ solves the time-dependent Schrödinger equation for $\tilde{H}_{\mathrm{ac}}$, and remains in the closure of this set). Whenever $\exp(it\tilde{H})$ is present, the parameter restrictions in Theorem 5.1 are understood to be in effect.

As usual, we need some free dynamics with which to compare this interacting motion. This involves the operator H_0 for which we need Fourier transform on pairs $\hat{f} = \langle f_+, f_- \rangle$,

$$\mathcal{J} : \hat{\mathcal{H}} \to \mathcal{H} \tag{5.6}$$

$$(\mathcal{J}\hat{f})(x) \equiv c \sum_{\delta=+,-} \int_{\mathbb{R}^+} dy\, \exp(i\pi\delta xy/a_+a_-) f_\delta(y), \quad c \equiv 1/\sqrt{2a_+a_-} \tag{5.7}$$

We then define

$$H_0 = \mathcal{J} \hat{M} \mathcal{J}^{-1} \tag{5.8}$$

where, again, $\hat{M}$ (4.5) denotes unbounded multiplication on pairs by $2c_+(y)$. Given $\mathcal{J}$ is unitary, we can consider (5.8) on any space up to $\mathcal{J}(\mathcal{D}(\hat{M}))$ where $\mathcal{D}(\hat{M})$ is the maximal domain of all functions $f \in \hat{\mathcal{H}}$ such that $\hat{M}\hat{f} \in \hat{\mathcal{H}}$. Thus we have a densely-defined operator in $\mathcal{H}$,

$$H_0 : \mathcal{J}(\mathcal{D}(\hat{M})) \to \mathcal{J}(\hat{\mathcal{H}}) \tag{5.9}$$

[6] Wherever this evolution operator is present, implicitly or explicitly, it is understood that the $\tilde{H}$ in the exponent stands for its own closure, i.e. the unique self-adjoint extension that follows from essential self-adjointness. This should not cause any confusion because in all applications our interest lies with the core in (5.4).

Unitarity of $\mathcal{J}$ and manifest self-adjointness of $\hat{M}$ on $\mathcal{D}(\hat{M})$ entail H_0 is self-adjoint. Thus we may consider the following one-parameter unitary group on $\mathcal{H}$,

$$\exp(itH_0), \quad t \in \mathbb{R} \tag{5.10}$$

Finally, we note that H_0 reproduces the free, relativistic action because we have

$$2c_+(y)\exp(i\delta\pi xy/a_+a_-) = H_0(x)\exp(i\delta\pi xy/a_+a_-), \quad \delta = +, - \tag{5.11}$$

$$H_0(x) \equiv T^x_{ia_-} + T^x_{-ia_-} \tag{5.12}$$

(recall our convention wherein formal AΔOs are written with an explicit variable; this helps to distinguish them from their Hilbert space counterparts).

We are now in a position to consider the following wave operators on $\mathcal{H}$,

$$W_\pm \equiv \operatorname*{s\cdot lim}_{t\to\infty} \exp(\pm it\tilde{H})\exp(\mp itH_0) \tag{5.13}$$

These limits are defined in the strong operator topology on $\mathcal{H}$. If they exist, they are necessarily isometries. We will see not only that they exist, but that they are expressible in terms of the eigenfunction transform $\mathcal{F}$ (4.3). (Keep in mind that $W_\pm$ have a dependence on b that enters through $\tilde{H}$.)

In practice, we determine a wave operator by positing an ansatz and proving vanishing of the corresponding limit. For instance, we expect one of the wave operators to be very closely related to $\mathcal{F}$. Straightforward equality is not possible, however, because this transform has a domain in $\hat{\mathcal{H}}$ rather than $\mathcal{H}$. The next closest relation would involve $\mathcal{F}$ composed with some kind of identification between the two Hilbert spaces. The adjoint Fourier transform $\mathcal{J}^* : \mathcal{H} \to \hat{\mathcal{H}}$ seems like a suitable candidate. Indeed, we are led to the following lemma (whose proof is given at the end of this subsection).

(We also note here one way in which the extension $\tilde{H}$ actually simplifies matters. Say $\exp(it\tilde{H})$ was only defined on the closure of $\mathcal{F}(\hat{\mathcal{C}})$ then we could still prove an analogue to the lemma below, and moreover in exactly the same way. However it would only hold for functions $f \in \mathcal{J}(\hat{\mathcal{C}}) \cap \mathcal{F}(\hat{\mathcal{C}})$.)

Lemma 5.2. *Suppose b satisfies $b \in (-a_+/2, a_- + a_+/2)$ and is not a $\mathbb{Z}$-integer multiple of a_-. Then, the following holds for any $f \in \mathcal{J}(\hat{\mathcal{C}}) \subset \mathcal{D}(H_0)$,*

$$\lim_{t\to\infty} \exp(-it\tilde{H})\exp(itH_0)f = \mathcal{F}\mathcal{J}^* f \tag{5.14}$$

where $\mathcal{F}$ is the eigenfunction transform defined in (4.1) *and $\mathcal{J}$ is Fourier transform* (5.7).

Corollary 5.3. *Suppose the coupling parameter b satisfies the conditions of the previous lemma. Then the map $\mathcal{F} : \hat{\mathcal{C}} \subset \hat{\mathcal{H}} \to \mathcal{H}$ with action* (4.1) *extends to an isometry on $\hat{\mathcal{H}}$. Moreover, we have existence of the wave operator W_- defined in* (5.13) *and $W_- = \mathcal{F}\mathcal{J}^*$.*

Proof of Corollary 5.3. Given Lemma 5.2, this proof is fairly routine. We introduce

$$\omega_- f \equiv \lim_{t\to\infty} e^{-it\tilde{H}} e^{itH_0} f \tag{5.15}$$

such that the result in the previous lemma may be expressed as

$$\omega_- f = \mathcal{F}\mathcal{J}^* f, \quad f \in \mathcal{J}(\hat{\mathcal{C}}) \tag{5.16}$$

It follows from one-parameter unitary group properties that

$$\|\omega_- f\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}, \quad f \in \mathcal{J}(\hat{\mathcal{C}}) \tag{5.17}$$

Writing $f = \mathcal{J}\hat{f}$ and using unitarity of $\mathcal{J}$, (5.17) becomes

$$\|\mathcal{F}\hat{f}\|_{\mathcal{H}} = \|\hat{f}\|_{\hat{\mathcal{H}}}, \quad \hat{f} \in \hat{\mathcal{C}} \tag{5.18}$$

With this we learn that $\mathcal{F}$ is bounded on the dense subspace $\hat{\mathcal{C}} \subset \hat{\mathcal{H}}$ and so has a unique linear extension to $\hat{\mathcal{H}}$. Moreover, since such an extension preserves operator norm, $\mathcal{F} : \hat{\mathcal{H}} \to \mathcal{H}$ will be an isometry. Further still, since (5.16) expresses equality of bounded operators on a dense subspace, their extensions are obviously equal too. This proves existence of W_- and equality with $\mathcal{F}\mathcal{J}^*$ as claimed.

□

We claim that this result for W_- allows us to deduce the analogue for W_+ with relative ease. This involves first getting a good idea of what we *want* W_+ to be.

With the transmission and reflection coefficients from earlier, (1.79) and (1.80), we define

$$S(b; y) \equiv \begin{pmatrix} t(b; y) & -r(b; y) \\ -r(b; y) & t(b; y) \end{pmatrix} \tag{5.19}$$

Multiplication by this matrix on functions in $\hat{\mathcal{H}}$ gives rise to a well-defined operator which we denote $\hat{S}$. It is plainly unitary given probability conservation, (1.81).

The hope is that this $\hat{S}$ will feature in the S-operator for the system, $W_+^* W_-$. With Fourier transform as the likely identification map between $\hat{\mathcal{H}}$ and $\mathcal{H}$, we speculate that $W_+^* W_-$ will be given by $\mathcal{J}\hat{S}\mathcal{J}^*$. Since we know $W_- = \mathcal{F}\mathcal{J}^*$, this gives us an expectation for W_+. It should equal $\mathcal{F}\hat{S}^*\mathcal{J}^*$.

Let us now link this expectation to what we can glean from the time-reversal operator

$$(\mathcal{T} f)(x) \equiv \overline{f(x)} \tag{5.20}$$

The role that this has in reversing time for non-relativistic systems can be analogised here. To see this we need the notion of meromorphic conjugation $\mathcal{T}'$,

$$(\mathcal{T}' f)(x) \equiv \overline{f(\overline{x})} \tag{5.21}$$

Using the AΔOs $\tilde{H}(b; x)$ (4.7) and $H_0(x)$ (5.12) one can easily verify the following operator equations on $D(\tilde{H})$ and $D(H_0)$ respectively,

$$\mathcal{T}\tilde{H} = \tilde{H}\mathcal{T}' \tag{5.22}$$

$$\mathcal{T} H_0 = H_0 \mathcal{T}' \tag{5.23}$$

(It may seem superfluous to involve $\mathcal{T}'$ here given these identities are considered on spaces of functions in the real variable x. The point is that the AΔO shifts this variable into the complex plane.)

Using one-parameter unitary group properties the upshot is that, on $\mathcal{H}$,

$$\mathcal{T} \exp(itA) = \exp(-itA)\mathcal{T}', \quad t \in \mathbb{R}, \quad A = \tilde{H}, H_0 \tag{5.24}$$

Since $\mathcal{T}$ and $\mathcal{T}'$ are involutive, this implies a priori

$$W_+ = \mathcal{T} W_- \mathcal{T}' \tag{5.25}$$

This relation is pleasing but there is in general no reason for thinking it will be of any practical use in arriving at a more explicit expression for W_+. However in our case, an identity satisfied by the transform kernel $\psi(b; x, y)$ will ensure this is so.

From Lemma 5.2 and (5.25) we have that

$$W_+ = \mathcal{T}\mathcal{F}\mathcal{J}^*\mathcal{T} \tag{5.26}$$

($\mathcal{T}'$ can now be discarded because there is no AΔO to the left of it). Let us now combine (5.26) with our expectation for W_+ above. First we should intertwine the map $\mathcal{J}^*$ and the operator $\mathcal{T}$. Writing out

$$(\mathcal{J}^*\mathcal{T} f)_\delta(y) = c \int_{\mathbb{R}} dx \, \exp(-i\delta\pi xy/a_+a_-)\overline{f(x)}, \quad f \in \mathcal{H} \tag{5.27}$$

we see that this can be written as $(\hat{\mathcal{T}}\mathcal{J}^* f)_\delta(y)$ where

$$(\hat{\mathcal{T}}\hat{f})_\delta(y) \equiv \overline{f_{-\delta}(y)}, \quad \hat{f} \in \hat{\mathcal{H}} \tag{5.28}$$

or, equivalently,

$$(\hat{\mathcal{T}}\hat{f})(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{\hat{f}(y)} \tag{5.29}$$

In other words, (5.26) is equivalent to

$$W_+ = \mathcal{T}\mathcal{F}\hat{\mathcal{T}}\mathcal{J}^* \tag{5.30}$$

We can now see what must be true of the transform $\mathcal{F}$ if W_+ is to equal $\mathcal{F}\hat{S}^*\mathcal{J}^*$,

$$\mathcal{T}\mathcal{F}\hat{\mathcal{T}} = \mathcal{F}\hat{S}^* \tag{5.31}$$

This equation can be proved explicitly using algebraic properties of $\psi(b; x, y)$, as we will see in the proof of the next theorem.[7]

We note that our discussion of W_+ could have proceeded without motivating an expected result. From (5.26) one could write down and prove (5.31). The problem

[7] Note how this shows $\mathcal{F}$ is not intertwined by $\mathcal{T}$ and $\hat{\mathcal{T}}$ in the same was as $\mathcal{J}$. This contrasts to the analogous situation in §B where $\mathcal{F}^*$ and $\mathcal{J}^*$ do intertwine the parity operators $\mathcal{P}$ and $\hat{\mathcal{P}}$ in the same way.

is that: it is not at all clear how one would conjure up such an identity without an expectation for the S-operator based on physics and scattering.

There is another angle here: with an expected result for W_+, one could try to confirm it explicitly with an analogue to Lemma 5.2. Assuming this can be done, the role of the time-reversal discussion would be to establish that (5.31) *must* be true. We see how our method, rooted in an independent proof of (5.31), essentially cuts out the need for a W_+-analogue of Lemma 5.2.

Theorem 5.4. *Suppose the coupling parameter b satisfies $b \in (-a_+/2, a_- + a_+/2)$ and is not a $\mathbb{Z}$-integer multiple of a_-. Then, the wave operators $W_\pm$ (5.13) exist on $\mathcal{H}$ and are given by*

$$W_- = \mathcal{F}\mathcal{J}^* \tag{5.32}$$

$$W_+ = \mathcal{F}\hat{S}^*\mathcal{J}^* \tag{5.33}$$

This entails that the S-operator $W_+^*W_-$ is equal to $\mathcal{J}\hat{S}\mathcal{J}^*$, where $\mathcal{J}$ is Fourier transform (5.7), and $\hat{S}$ is the operator on $\hat{\mathcal{H}} \equiv L^2(\mathbb{R}^+, dy) \otimes \mathbb{C}^2$ defined by matrix multiplication by $S(b; y)$ (5.19).

Proof of Theorem 5.4. Given Lemma 5.3 and all we have said, it remains to prove the identity (5.31). First we write out

$$(\mathcal{F}\hat{S}^*\hat{f})(x) = c \sum_{\delta=+,-} \int_{\mathbb{R}^+} dy\, \psi(\delta x, y)(\hat{S}^*\hat{f})_\delta(y) \tag{5.34}$$

where $\hat{S}^*$ acts on $\hat{f}$ as the Hermitian conjugate of the matrix (5.19),

$$(\hat{S}^*\hat{f})_\delta(y) = \overline{t(y)} f_\delta(y) - \overline{r(y)} f_{-\delta}(y), \quad \delta = +, - \tag{5.35}$$

We focus on

$$\sum_{\delta=+,-} \psi(\delta x, y)\big(\overline{t(y)} f_\delta(y) - \overline{r(y)} f_{-\delta}(y)\big) \tag{5.36}$$

Using the property $\overline{\psi(x,y)} = \psi(x, -y)$ (which holds for real x, y) we may write this as

$$\sum_{\delta=+,-} f_\delta(y)\big[\psi(\delta x, -y)t(y) - r(y)\psi(-\delta x, -y)\big]^- \tag{5.37}$$

(where superscript '$-$' denotes conjugation of the brackets). The term in the square brackets simplifies immediately to $\psi(-\delta x, y)$ because of the identity in Lemma 1.2. We therefore have

$$(\mathcal{F}\hat{S}^*\hat{f})(x) = c \sum_{\delta=+,-} \int_{\mathbb{R}^+} dy\, \overline{\psi(\delta x, y)} \hat{f}_{-\delta}(y) \tag{5.38}$$

which we can write using $\mathcal{T}$ (5.20) and $\hat{\mathcal{T}}$ (5.28) as

$$(\mathcal{T}\mathcal{F}\hat{\mathcal{T}}\hat{f})(x) \tag{5.39}$$

□

Proof of Lemma 5.2. It is more natural to tackle directly the equivalent claim,

$$\lim_{t\to\infty} e^{-it\tilde{H}} e^{itH_0} \mathcal{J}\hat{f} = \mathcal{F}\hat{f}, \quad \hat{f} \in \hat{\mathcal{C}} \tag{5.40}$$

(we now suppress implicit dependences on b). In concrete terms this means

$$\lim_{t\to\infty} \|\mathcal{F}\hat{f} - e^{-it\tilde{H}} e^{itH_0} \mathcal{J}\hat{f}\|_{\mathcal{H}} = 0, \quad \hat{f} \in \hat{\mathcal{C}} \tag{5.41}$$

Using one-parameter unitary group properties of $e^{it\tilde{H}}$ we may write the norm here as

$$\|\big(e^{it\tilde{H}}\mathcal{F} - e^{itH_0}\mathcal{J}\big)\hat{f}\|_{\mathcal{H}} \tag{5.42}$$

Given (4.4) and (5.8), the functional calculus for self-adjoint operators gives us two intertwining relations on $\hat{\mathcal{C}}$,

$$e^{it\tilde{H}}\mathcal{F} = \mathcal{F}e^{it\hat{M}}, \quad t \in \mathbb{R} \tag{5.43}$$

$$e^{itH_0}\mathcal{J} = \mathcal{J}e^{it\hat{M}}, \quad t \in \mathbb{R} \tag{5.44}$$

(in fact this is where the arbitrary extension drops out as irrelevant: $\tilde{H}\mathcal{F} = \tilde{H}_{\mathrm{ac}}\mathcal{F}$ on $\hat{\mathcal{C}}$). Accordingly, (5.41) amounts to

$$\lim_{t\to\infty} \|(\mathcal{F} - \mathcal{J})e^{it\hat{M}}\hat{f}\|_{\mathcal{H}} = 0, \quad \hat{f} \in \hat{\mathcal{C}} \tag{5.45}$$

To prove this we first write out

$$\|(\mathcal{F} - \mathcal{J})e^{it\hat{M}}\hat{f}\|_{\mathcal{H}}^2 = \int_{\mathbb{R}} dx\, |((\mathcal{F} - \mathcal{J})e^{i\hat{M}f}\hat{f})(x)|^2 \tag{5.46}$$

From $\mathcal{F}$ (4.1) and $\mathcal{J}$ (5.7) we have

$$((\mathcal{F} - \mathcal{J})\hat{f})(x) = c \sum_{\delta=+,-} \int_{\mathbb{R}^+} dy\, \big(\psi(\delta x, y) - e^{i\delta\pi xy/a_+a_-}\big) f_\delta(y) \tag{5.47}$$

The vanishing in (5.45) is therefore implied by

$$\lim_{t\to\infty} \int_{\mathbb{R}} dx\, |(I_\delta e^{i\hat{m}t} f)(x)|^2 = 0, \quad f \in C_0^\infty(\mathbb{R}^+), \quad \delta = +, - \tag{5.48}$$

where $\hat{m}$ denotes multiplication by $2c_+(y)$, and

$$(I_\delta f)(x) \equiv \int_{\mathbb{R}^+} dy\, \big(\psi(\delta x, y) - e^{i\delta\pi xy/a_+a_-}\big) f(y), \quad f : \mathbb{R}^+ \to \mathbb{C} \tag{5.49}$$

In fact, because $(I_- f)(x) = (I_+ f)(-x)$ we only have to prove (5.48) for one choice of δ. We will focus on $\delta = +$.

From the Riemann-Lebesgue lemma[8]we already have a pointwise vanishing

$$\lim_{t\to\infty} (I_+ e^{i\hat{m}t} f)(x) = 0, \quad x \in \mathbb{R}, \quad f \in C_0^\infty(\mathbb{R}^+) \tag{5.50}$$

Clearly, then, it is a question of dominated convergence; to interchange the limit and integral in (5.48) we need a dominating function $F(x) \in L^1(\mathbb{R}, dx)$ (which may depend on f) and a $t_0 \in [0, \infty)$ such that

$$|(I_+ e^{i\hat{m}t} f)(x)|^2 \leq F(x), \quad x \in \mathbb{R}, \quad t \in [t_0, \infty) \tag{5.51}$$

This dominating function will be a piecewise construction on three different x-intervals, $(-\infty, -R)$, $[-R, R]$ and (R, ∞), where $R > 0$ is to be fixed in due course.

On the middle interval it is enough to write

$$|(I_+ e^{i\hat{m}t} f)(x)| \leq \int_{\mathbb{R}^+} dy\, |\psi(x,y) - e^{i\pi xy/a_+a_-}| \cdot |f(y)| \tag{5.52}$$

and observe that the smoothness of $\psi(x, y)$ in both variables (ensured by the b-interval under consideration; recall (3.3)) implies the integral is defined and bounded on $[-R, R]$ as a function of x. The function on the rhs of the inequality can thus be used as the dominating function on this interval.

In the far reaches of $\mathbb{R}$, this argument is of no use because we need decay in x rather than boundedness. The O-asymptotics in Lemma 1.3 are crucial here. For real x they may be encoded as follows,

$$\psi(x, y) = \psi_{as}(x, y) + O(e_-^{-\rho|x|}), \quad x \to \pm\infty \tag{5.53}$$

$$\psi_{as}(x, y) \equiv \begin{cases} t(y) e^{i\pi xy/a_+a_-}, & x > 0 \\ e^{i\pi xy/a_+a_-} - r(y) e^{-i\pi xy/a_+a_-}, & x < 0 \end{cases} \tag{5.54}$$

where $\rho > 0$ and where the bound represented by O is uniform for y varying respectively over any compact subset of $\mathbb{R}^+$.

To make use of this we first write

$$\frac{1}{2}|(I_+ e^{i\hat{m}t} f)(x)|^2 \leq \left| \int_{\mathbb{R}^+} dy\, \left(\psi_{as}(x,y) - e^{i\pi xy/a_+a_-}\right) e^{2itc_+(y)} f(y) \right|^2 + \int_{\mathbb{R}^+} |\psi(x,y) - \psi_{as}(x,y)|^2 \cdot |f(y)|^2 \tag{5.55}$$

(this is just a trivial rearrangement based on a judicious choice of B in the elementary identity $|A|^2/2 \leq |B|^2 + |A - B|^2$).

The asymptotics (5.53) tells us that there exist $\mathcal{C}, x_0 > 0$ such that for all $y \in \text{supp}(f)$,

$$|\psi(x,y) - \psi_{as}(x,y)| \leq \mathcal{C} e_-^{-\rho|x|}, \quad |x| > x_0 \tag{5.56}$$

Thus the second integral in (5.55) is bounded from above by $(\mathcal{C} e_-^{-\rho|x|} \|f\|_{L^2})^2$, a function which is plainly L^1. Setting $R = x_0$ we are therefore "halfway" to finding dominating functions on the intervals $(-\infty, -R)$ and (R, ∞).

[8] To be precise, we are applying Riemann-Lebesgue to an integral of the form $\int_{\mathbb{R}^+} dy\, e^{-i2c_+(y)t} F(x,y)$ This works because we can consider a variable change $y \to u \equiv 2c_+(y)$ which is well-defined because the increasing function $2c_-(\cdot) : [2, \infty) \to \mathbb{R}^+$ is a bijection.

It remains to analyse the first integral in (5.55). This is the only case where t plays a vital role, and we cannot afford to erase it by taking the modulus into the integral. For $x > R$ we are dealing with

$$\int_{\mathbb{R}^+} dy\,\big(t(y)-1\big)e^{i\pi xy/a_+a_-}e^{2itc_+(y)}f(y) \tag{5.57}$$

and $x < -R$,

$$-\int_{\mathbb{R}^+} dy\,r(y)e^{-i\pi xy/a_+a_-}e^{2itc_+(y)}f(y) \tag{5.58}$$

The task of finding a dominating function for (the square-modulus of) these can be generalised to finding one for

$$\int_{\mathbb{R}^+} dy\,e^{i\pi wy/a_+a_-}e^{2itc_+(y)}Y(y) \tag{5.59}$$

for $w \in [R,\infty)$ where $Y \in C_0^\infty(\mathbb{R}^+)$.[9] To achieve this we write

$$e^{i\pi wy/a_+a_-}e^{2itc_+(y)} = (a_+a_-/\pi i)[w+2a_-ts_+(y)]^{-1}\partial_y\big(e^{i\pi wy/a_+a_-}e^{2itc_+(y)}\big) \tag{5.60}$$

Plugging this into (5.59) and performing integration by parts we get

$$(-a_+a_-/\pi i)\int_{\mathbb{R}^+} dy\,\partial_y\big(Y(y)[w+2a_-ts_+(y)]^{-1}\big)e^{i\pi wy/a_+a_-}e^{2itc_+(y)} \tag{5.61}$$

where the intermediate term in the integration vanishes because of Y's compact support on $(0,\infty)$ (crucially, the square-bracketed term is positive because $w, a_-, t, y > 0$). At this point we can safely take the modulus into the integral, bounding (5.61) and thus (5.59) from above by

$$(a_+a_-/\pi)\int_{\mathbb{R}^+} dy\,|\partial_y\big(Y(y)[w+2a_-ts_+(y)]^{-1}\big)| \tag{5.62}$$

This can in turn be bounded from above by

$$\begin{aligned}2ta_-^2[w+2a_-ts_+(y_0)]^{-2}&\int_{\mathbb{R}^+} dy\,c_+(y)|Y(y)|\\ &+(a_+a_-/\pi)[w+2a_-ts_+(y_0)]^{-1}\int_{\mathbb{R}^+} dy\,|Y'(y)|,\quad y_0\equiv\min(\mathrm{supp}(Y))\end{aligned} \tag{5.63}$$

which makes unambiguously clear that, for any $t > 0$, we can find a Y-dependent, L^1-integrable function in w that bounds (5.59) from above, and indeed one whose square is L^1-integrable (the integration region here can be replaced by the compact set $\mathrm{supp}(Y)$). □

[9] The functions $t(y)$ and $r(y)$ are smooth and bounded on $[0,\infty)$ provided b is not an integer multiple of a_-, and so multiplication of $f(y)$ by $(t(y)-1)$ and $r(y)$ preserves membership of $C_0^\infty(\mathbb{R}^+)$.

5.3 Reappraisal.

Continuous and discrete spectrum of $\tilde{H}$.

We are going to think more carefully about the extension of $\tilde{H}_{\rm ac}$ which we introduced in the last subsection, paving the way for the next two sections. We recall that initially (in §4) this operator was defined on $\mathcal{F}(\hat{\mathcal{C}})$ by the intertwining (4.4). We showed it was symmetric and, later, essentially self-adjoint given certain restrictions on b. In §5.2 we saw that we could extend $\tilde{H}_{\rm ac}$ to an arbitrary bounded self-adjoint operator on the orthocomplement of the closure of $\mathcal{F}(\hat{\mathcal{C}})$ in $\mathcal{H}$ without changing anything substantial about the proof of the existence and form of the wave operators. We called this extension $\tilde{H}$, and we are now concerned with the question of how to fix its action on the orthocomplement.

Most of this discussion is valid whenever $\tilde{H}_{\rm ac}$ is defined. Only (5.68) requires essential self-adjointness of $\tilde{H}_{\rm ac}$, which in turn requires we take $b \in (-a_+/2, a_- + a_+/2) \setminus A_-$, cf. Theorem 5.1. In light of Corollary 5.3 we know that $\tilde{H}$ has a contribution to its absolutely continuous spectrum of multiplicity two given by $[2, \infty)$ for these parameters.

In symbols, the above description of $\tilde{H}$ gives us

$$\mathcal{D}(\tilde{H}) = \mathcal{F}(\hat{\mathcal{C}}) \oplus \overline{\mathcal{F}(\hat{\mathcal{C}})}^{\perp} \tag{5.64}$$

which is dense in $\mathcal{H} \equiv L^2(\mathbb{R}, dx)$ by construction. We know that on $\mathcal{F}(\hat{\mathcal{C}})$ it has the unbounded AΔO action

$$(\tilde{H}\mathcal{F}\hat{f})(x) = \tilde{H}(x)(\mathcal{F}\hat{f})(x), \quad \hat{f} \in \hat{\mathcal{C}} \tag{5.65}$$

(recall (4.11)). But on the orthocomplement in (5.64), its action is yet to be fixed. Our choice is based on the desire for a Hilbert space theory for $\tilde{H}(b; x)$ motivated by physics.

The orthocomplement in (5.64) is not completely mysterious to us. In §3.4, we proved that all L^2-integrable members of the family of functions $\Psi^{(m)}(x)$ (2.4) are orthogonal to $\mathcal{F}(\hat{\mathcal{C}})$ (as well as to each other), and so to its closure too. These members are described by the count $0, \ldots m_b - 1$ where m_b is the largest integer such that $m_b a_- < b$ (vanishing if $b < a_-$). As a result,

$$\operatorname{span}\{\Psi^{(0)}, \ldots, \Psi^{(m_b-1)}\} \subseteq \overline{\mathcal{F}(\hat{\mathcal{C}})}^{\perp} \tag{5.66}$$

Given that $\Psi^{(m)}(x)$ are eigenfunctions of the AΔO $\tilde{H}(x)$ (1.44) with eigenvalue E_m (3.27), it is clear how we should fix the action of $\tilde{H}$ on this span (given $\tilde{H}$ is intended as our Hilbert space version of $\tilde{H}(b; x)$),

$$\tilde{H}\Psi^{(m)} \equiv E_m \Psi^{(m)}, \quad m = 0, \ldots, m_b - 1 \tag{5.67}$$

with m_b defined as above.

As it stands, there is still a part of Hilbert space we have not accounted for. This is the gap that potentially exists between the two sides of the subset relation in (5.66). Closing this gap is the main challenge of the next two sections. Indeed this is the problem of completeness as it manifests in our construction. In this regard we already know from Theorem 5.4 that, independently of the choice of extension, the scattering states (intersection of the ranges of the wave operators) are given by

$$\overline{\mathcal{F}(\hat{\mathcal{C}})} = \mathcal{F}(\hat{\mathcal{H}}) \tag{5.68}$$

where this equality follows because of $\mathcal{F}$'s isometry. With this in mind, we can appreciate the significance of Theorem 7.7 which §6,7 are dedicated to proving.

6 The dual operator $\overline{\mathcal{S}}$.

Definition and symmetry formula.

The primary aim of this section is to help complete our account of the dynamics associated to the operator $\tilde{H}$ (4.4). This means showing that the orthocomplement of the space $\mathcal{F}(\hat{\mathcal{C}})$ is spanned by (a particular subfamily of) the mutually-orthogonal functions $\Psi^{(m)}$ of §2 and §3.4. Following the precedent in [20] and [3], this will be done in a very indirect way. It involves focusing on the AΔO $\mathcal{S}(b;y)$, for which we know $\psi(b;x,y)$ (1.60) is also a generalised eigenfunction (and whose b-dependence we frequently suppress). This generalised eigenvalue equation in the dual variable allows us to naturally associate a Hilbert space operator to $\mathcal{S}(y)$ by intertwining the adjoint of $\mathcal{F}$ (4.1) with the eigenvalue, $2s_+(x)$. The surprise is that symmetry of this operator, and indeed symmetry breakdown, are closely related to $\mathcal{F}(\hat{\mathcal{C}})$. The precise details of this relation are the concern of the next section.

For now we focus on associating a Hilbert space operator to the AΔO $\mathcal{S}(y)$ and considering when it is symmetric. We do this by mimicking the procedure in §4. In practice this means that (from (6.17) onwards) the present section is an analogue of §4 but with $\tilde{H}(x)$ replaced by $\mathcal{S}(y)$, and the roles of x and y reversed. We also note that the results here can be alternatively viewed from the perspective of dynamics associated to $\overline{\mathcal{S}}$ below, the so-called *dual* dynamics.

We recall how the Hilbert space operator $\tilde{H}_{\text{ac}}$ (4.4) was defined using the eigenfunction transform $\mathcal{F}$ (4.1). As part of this procedure we used the fact that functions in $\hat{\mathcal{C}} \equiv C_0^\infty(\mathbb{R}^+)^2$ get mapped by $\mathcal{F}$ into $\mathcal{H} \equiv L^2(\mathbb{R}, dx)$ (provided $b \in \mathbb{R} \setminus \mathcal{Y}$ (3.2) which we assume throughout this section). At no point did we need to talk about a Hilbert space domain for $\mathcal{F}$. But now we must, because we wish to consider the adjoint of this map. A natural choice presents itself, namely

$$\hat{\mathcal{H}} \equiv L^2(\mathbb{R}^+, dy) \otimes \mathbb{C}^2 \tag{6.1}$$

The point is that $\hat{\mathcal{C}}$ is dense in this Hilbert space, allowing us to reinterpret $\mathcal{F}$ (4.1) as a densely-defined map between Hilbert spaces,

$$\mathcal{F} : \hat{\mathcal{C}} \equiv C_0^\infty(\mathbb{R}^+)^2 \subset \hat{\mathcal{H}} \to \mathcal{H} \tag{6.2}$$

(Note, since this aims to be a self-contained account of the operator $\overline{\mathcal{S}}$, we *forget* the results from §5.2 where we established isometry of $\mathcal{F}$ for very restricted b). Consistent with our earlier convention, functions in $\hat{\mathcal{H}}$ will always be written in the form $\hat{f} = \langle f_+, f_- \rangle$. To ensure no ambiguity in what follows, let us note that the inner products in these Hilbert spaces are given by

$$(d,e)_{\mathcal{H}} \equiv \int_{\mathbb{R}} dx\, \overline{d(x)} e(x) \tag{6.3}$$

$$(f,g)_{\hat{\mathcal{H}}} \equiv \sum_{\delta=+,-} \int_{\mathbb{R}^+} dy\, \overline{f_\delta(y)} g_\delta(y) \tag{6.4}$$

For the map $\mathcal{F}$ (6.2), we will say the adjoint element of $f \in \mathcal{H}$ exists in $\hat{\mathcal{H}}$, and denote it by $\mathcal{F}^* f$, iff the latter solves the following equation

$$(\mathcal{F}^* f, g)_{\hat{\mathcal{H}}} = (f, \mathcal{F} g)_{\mathcal{H}}, \quad g \in \hat{\mathcal{C}} \tag{6.5}$$

The density of $\hat{\mathcal{C}}$ in $\hat{\mathcal{H}}$ ensures any solution is unique. The collection of all adjoint elements defines the adjoint map. Clearly we will have

$$(\mathcal{F}^* f)_\delta(y) = c \int_{\mathbb{R}} dx\, \overline{\psi(b; \delta x, y)} f(x), \quad \delta = +, - \tag{6.6}$$

$$c \equiv 1/\sqrt{2a_+ a_-} \tag{6.7}$$

whenever these integral functions are in $L^2(\mathbb{R}^+, dy)$. The question raised by this is exactly the same as that for the transforms in §3.3 - which is unsurprising given we have $\mathcal{F}\hat{f} = c(\mathcal{F}_+ f_+ + \mathcal{F}_- f_-)$. Thus we can use the answer presented in Lemma 3.4 to assert that the adjoint element exists for all $f \in \mathcal{C} \subset C_0^\infty(\mathbb{R})$ and is given by (6.6). In other words, we have (a restriction of) the adjoint map

$$\mathcal{F}^* : \mathcal{C} \subset \mathcal{H} \to \hat{\mathcal{H}} \tag{6.8}$$

with action given by (6.6) and $\mathcal{C}$ representing the functions in $C_0^\infty(\mathbb{R})$ with support away from the origin. (This is restricted in the sense we are not claiming $\mathcal{C}$ constitutes the natural adjoint domain of *all* functions in $\mathcal{H}$ that have adjoint elements in $\hat{\mathcal{H}}$).

Now that we have $\mathcal{F}^*$ (6.8) we claim we can define a Hilbert space operator $\overline{\mathcal{S}}$ by the intertwining relation

$$\overline{\mathcal{S}} \mathcal{F}^* = \mathcal{F}^* M \tag{6.9}$$

where M denotes multiplication by the generalised eigenvalue $2s_+(x)$. It is not hard to see that M maps $\mathcal{C}$-functions to $\mathcal{C}$-functions. Thus (6.9) does indeed define a Hilbert space operator,

$$\overline{\mathcal{S}} : \mathcal{F}^*(\mathcal{C}) \to \mathcal{F}^*(\mathcal{C}) \subset \hat{\mathcal{H}} \tag{6.10}$$

To be sure, the Hilbert space in which this acts as a densely-defined operator is the closure of $\mathcal{F}^*(\mathcal{C})$ (moreover, (6.10) does not require injectivity of $\mathcal{F}^*$ for reasons analogous to to those below (4.5)).

We know that $2s_+(x)$ and $\psi(b; x, y)$ are a generalised eigenvalue-eigenfunction pair of the AΔO $\mathcal{S}(y)$ (1.67), thus it is clear from (6.6) and (6.9) that the action $\overline{\mathcal{S}}$ should be closely related to $\mathcal{S}(y)$. To find it explicitly we use (6.6) and the intertwining (6.9) to write out the two components of $\overline{\mathcal{S}} \mathcal{F}^* f \in \hat{\mathcal{H}}$ as follows,

$$(\overline{\mathcal{S}} \mathcal{F}^* f)_\delta(y) = (\mathcal{F}^* M f)_\delta(y) = c \int_{\mathbb{R}} dx\, \overline{\psi(b; \delta x, y)} 2s_+(x) f(x) \tag{6.11}$$

We may now take $2s_+(x)$ into the conjugation and replace it with $\delta \mathcal{S}(y)$ (the extra δ accounts for oddness of $2s_+(\cdot)$). Invoking the fact $(\mathcal{F}^* f)_\delta(y)$ are meromorphic functions in y (which follows from what we said below (3.22)), we therefore have

$$(\overline{\mathcal{S}} \mathcal{F}^* f)_\delta(y) = \delta \overline{\mathcal{S}(b; y)} (\mathcal{F}^* f)_\delta(y), \quad f \in \mathcal{C} \tag{6.12}$$

Or, simply,

$$(\overline{\mathcal{S}}\hat{F})_\delta(y) = \delta\overline{\mathcal{S}(b;y)}F_\delta(y), \quad \hat{F} \in \mathcal{F}^*(\mathcal{C}) \tag{6.13}$$

Here we are dealing with the conjugated AΔO,[10]

$$\overline{\mathcal{S}(b;y)} = \overline{U(b;y)}\, T^y_{-ia_-} - T^y_{ia_-} \tag{6.14}$$

$$U(b;y) \equiv V(y)V(-y+ia_-) \tag{6.15}$$

where from earlier, $V(a_+, b; y) = s_+(y-ib)/s_+(y)$. Thus,

$$\overline{U(y)} = \frac{s_+(y-ib+ia_-)}{s_+(y+ia_-)}\frac{s_+(y+ib)}{s_+(y)} \tag{6.16}$$

With this in place, the present section begins to analogise §4. The main result of this chapter is the formula in Theorem 6.3 which allows us to determine if and when $\overline{\mathcal{S}}$ (6.9) is symmetric. The formula arises from a direct manipulation of the integrals implicit in the following object

$$\hat{D}(f,g) \equiv (\overline{\mathcal{S}}\mathcal{F}^*f, \mathcal{F}^*g)_{\hat{\mathcal{H}}} - (\mathcal{F}^*f, \overline{\mathcal{S}}\mathcal{F}^*g)_{\hat{\mathcal{H}}} \tag{6.17}$$

Clearly, vanishing of this for all $f, g \in \mathcal{C}$ is equivalent to symmetry of $\overline{\mathcal{S}}$. However, as we have already suggested, the cases when this *fails* to vanish for all $f, g \in \mathcal{C}$ - what we call symmetry breakdown - will be of interest too. In general, (6.17) is found to equal a sum of residues in a particular strip of the complex plane. Unlike the analogous case for $\tilde{H}_{\mathrm{ac}}$, these cannot be driven out of the strip by imposing a restriction on the coupling parameter b. This is why we take up the issue of residue analysis in the next section. Symmetry is eventually established for certain parameter values in Corollary 7.1.

Let us now look at one of the two terms in (6.17) more closely (from now on we suppress implicit dependence of functions on b),

$$\begin{aligned}(\overline{\mathcal{S}}\mathcal{F}^*f, \mathcal{F}^*g)_{\hat{\mathcal{H}}} &= \sum_{\delta=+,-}\int_{\mathbb{R}^+} dy\, \overline{(\mathcal{F}^*Mf)_\delta(y)}(\mathcal{F}^*g)_\delta(y) \\ &= c^2\sum_{\delta=+,-}\int_{\mathbb{R}^+} dy \int_{\mathbb{R}} dx\, \psi(\delta x, y)2s_+(x)\overline{f(x)}\int_{\mathbb{R}} dx'\overline{\psi(\delta x', y)}g(x'), \quad f,g \in \mathcal{C}\end{aligned} \tag{6.18}$$

Our first step is to push $\int_{\mathbb{R}^+} dy$ through the other two integrals in order to isolate an integral which is independent of f, g. Fubini's theorem is key here. In order to use it we must first replace $\int_{\mathbb{R}^+} dy$ with $\lim_{\Lambda\to\infty}\int_0^\Lambda dy$. With this change, we obtain a bounded integration region in the variables (y, x, x') on which the integrand is bounded (note, $\int_{\mathbb{R}} dx$ is really just $\int_{\mathrm{supp}(f)} dx$ here; we also invoke absence of real poles for $\psi(x,y)$ from Corollary 3.2). Hence the rhs of (6.18) equals

[10] For a precise notion of conjugated AΔO we should define $\overline{\mathcal{S}(y)}$ as the AΔO that solves $\overline{\mathcal{S}(y)}\,\overline{\varphi(y)} = \overline{\mathcal{S}(y)\varphi(\overline{y})}$ where is $\varphi(y)$ is an arbitrary meromorphic function. This is a straightforward algebraic exercise, the solution of which agrees with our "naive" (6.14).

$$c^2 \lim_{\Lambda\to\infty} \int_{\mathbb{R}} dx\, 2s_+(x)\overline{f(x)} \int_{\mathbb{R}} dx'\, g(x') \int_0^{\Lambda} dy \sum_{\delta=+,-} \psi(\delta x, y)\overline{\psi(\delta x', y')} \tag{6.19}$$

We can manipulate $(\mathcal{F}^* f, \overline{\mathcal{S}}\mathcal{F}^* g)_{\hat{\mathcal{H}}}$ in a completely analogous way and combine it with (6.19) to get

$$\hat{D}(f,g) = c^2 \lim_{\Lambda\to\infty} \int_{\mathbb{R}} dx\, \overline{f(x)} \int_{\mathbb{R}} dx'\, g(x') \sum_{\delta=+,-} \hat{I}_\delta(\Lambda; x, x'), \quad f, g \in \mathcal{C} \tag{6.20}$$

$$\hat{I}_\delta(\Lambda; x, x') \equiv \int_0^{\Lambda} dy\, \psi(\delta x, y)\overline{\psi(\delta x', y')}\big(2s_+(x) - 2s_+(x')\big), \quad \delta = +, - \tag{6.21}$$

At this point we analogise the strategy pursued in §4. In other words, we are going to *expand* this integrand using the AΔEs,

$$\delta\mathcal{S}(y)\psi(\delta x, y) = 2s_+(x)\psi(\delta x, y), \quad \delta = +, - \tag{6.22}$$

(this is just the eigenvalue equation (1.76) with oddness of $2s_+(\cdot)$). We get

$$\hat{I}_\delta(\Lambda; x, x') = \delta \int_0^{\Lambda} dy \Big([\mathcal{S}(y)\psi(\delta x, y)]\overline{\psi(\delta x', y)} - \psi(\delta x, y)[\overline{\mathcal{S}(y)\psi(\delta x', y)}] \Big) \tag{6.23}$$

The point of this expansion is that it will allow us to rearrange the integrand in a way that makes it (almost) amenable to Proposition 4.2 with $x \to y$. This proposition will re-express the integral in terms of residues and boundary integrals. The catch is that the proposition requires an integral range $[-\Lambda, \Lambda]$ which we do not have at present. We will get around this because of a non-obvious evenness in y, which will allow us to replace $\int_0^{\Lambda} dy$ with $\int_{-\Lambda}^{\Lambda} dy/2$ (this step is detailed in the proof of Theorem 6.3).

To rearrange the integrand in (6.23) such that it has the form $J(y) - J(y + ia_-)$, Proposition 4.1 will not be of any use (not least because $\mathcal{S}(y)$ does not have the "reality" property analogous to (4.33); we can see this explicitly in (6.14)). However if we switch from $\psi(x, y)$ to the reduced version

$$\hat{\psi}(x, y) = c(y)\psi(x, y) \tag{6.24}$$

(to see why this is reduced, recall (2.28)) we can make some progress.

Introducing a similarity-transformed AΔO

$$\hat{\mathcal{S}}(b; y) \equiv c(b; -y)\mathcal{S}(b; y)\frac{1}{c(b; -y)} \tag{6.25}$$

we can use the property $\overline{c(-y)} = c(y)$, $y \in \mathbb{R}$ to write (6.23) as

$$\hat{I}_\delta(\Lambda; x, x') = \delta \int_0^{\Lambda} dy \Big(\big[\hat{\mathcal{S}}(y)\hat{\psi}(\delta x, y)\big]\overline{\hat{\psi}(\delta x', y)} - \hat{\psi}(\delta x, y)\big[\overline{\hat{\mathcal{S}}(y)\hat{\psi}(\delta x', y)}\big] \Big) w(y) \tag{6.26}$$

where, recall, $w(y) = 1/c(y)c(-y)$, cf. (1.28). At this point we recall that the AΔO $\mathcal{S}(y)$ arose as

$$\frac{1}{c(-y)}\big[V(y)T^y_{ia_-} - V(-y)T^y_{-ia_-}\big]c(-y) \tag{6.27}$$

where $V(y) = s_+(y-ib)/s_+(y)$, cf. (1.64). Thus we can write down straight away

$$\hat{\mathcal{S}}(y) = \frac{s_+(y-ib)}{s_+(y)}T^y_{ia_-} - \frac{s_+(y+ib)}{s_+(y)}T^y_{-ia_-} \tag{6.28}$$

(in fact we have seen this already in (2.40)). Unlike $\mathcal{S}(y)$, this now has a definite real/imaginary parity, since

$$\overline{\hat{\mathcal{S}}(y)\varphi(y)} = -\hat{\mathcal{S}}(y)\overline{\varphi(\overline{y})}, \quad y \in \mathbb{R}^+ \tag{6.29}$$

where this holds for an arbitrary meromorphic function $\varphi(y)$ (the conjugation $\overline{y}$ is not superfluous here because $\hat{\mathcal{S}}(y)$ entails any y to the right is shifted).

Thus we have

$$\hat{I}_\delta(\Lambda; x, x') = \delta \int_0^\Lambda dy\Big(\big[\hat{\mathcal{S}}(y)\hat{\psi}(\delta x, y)\big]\overline{\hat{\psi}(\delta x', y)} + \hat{\psi}(\delta x, y)\big[\hat{\mathcal{S}}(y)\overline{\hat{\psi}(\delta x', \overline{y})}\big]\Big)w(y), \qquad \delta = +,- \tag{6.30}$$

Now we propose

Proposition 6.1. *Suppose we have an AΔO,*

$$S(y) \equiv V_1(y)T^y_{ia_-} - V_2(y)T^y_{-ia_-} \tag{6.31}$$

where $V_j(\cdot)$ are two meromorphic functions.

And suppose $\Phi_j(y)$ are two more meromorphic functions. Then the object

$$[S(y)\Phi_1(y)]\Phi_2(y) + \Phi_1(y)[S(y)\Phi_2(y)] \tag{6.32}$$

may be written as

$$J(y) - J(y + ia_-) \tag{6.33}$$

$$J(y) \equiv V_1(y)\big[\Phi_1(y-ia_-)\Phi_2(y) + \Phi_1(y)\Phi_2(y-ia_-)\big] \tag{6.34}$$

provided

$$V_2(y) = V_1(y + ia_-) \tag{6.35}$$

Proof. Writing out (6.32) we have

$$\big[V_1(y)\Phi_1(y-ia_-) - V_2(y)\Phi_1(y+ia_-)\big]\Phi_2(y) + \Phi_1(y)\big[V_1(y)\Phi_2(y-ia_-) - V_2(y)\Phi_2(y+ia_-)\big] \tag{6.36}$$

Introducing

$$L^+(y) \equiv \Phi_1(y-ia_-)\Phi_2(y) + \Phi_1(y)\Phi_2(y-ia_-), \tag{6.37}$$

it is clear that (6.36) can be written as

$$V_1(y)L^+(y) - V_2(y)L^+(y+ia_-) \tag{6.38}$$

Thus we can absorb $V_1(y)$ into $L^+(y)$ such that (6.38) equals (6.33) iff $V_2(y) = V_1(y+ia_-)$. □

It is fairly clear how this will apply to the integrand in (6.30) (however, as we will see, the requirement on $V_j(y)$ requires us to invoke the AΔEs (1.65)-(1.66)).[11] First, we view the product $w(y)\hat{\mathcal{S}}(y)$ in (6.30) as an AΔO in itself (i.e. the function $w(y)$ just multiplies the coefficients in $\hat{\mathcal{S}}(y)$ (6.28)). Then, the integrand in in (6.30) has the form (6.31) with

$$V_1(y) = w(y)s_+(y-ib)/s_+(y), \quad V_2(y) = w(y)s_+(y+ib)/s_+(y) \tag{6.39}$$

and

$$\Phi_1(y) = \hat{\psi}(\delta x, y), \quad \Phi_2(y) = \overline{\hat{\psi}(\delta x', \overline{y})} \tag{6.40}$$

(note we implicitly deal with two choices of Φ_1 and Φ_2 here since δ equals $+$ or $-$). In fact because of (1.26), the conjugacy property (1.77) passes on to $\hat{\psi}(x,y)$, meaning this choice of Φ_2 simplifies to

$$\Phi_2(y) = \hat{\psi}(\delta x', -y) \tag{6.41}$$

Finally, to see that these choices of $V_1(y)$ and $V_2(y)$ satisfy (6.35), we recall $w(y) = 1/c(y)c(-y)$ and use the AΔEs for $c(y)$, (1.65)-(1.66), to write

$$w(y+ia_-)\frac{s_+(y+ia_- - ib)}{s_+(y+ia_-)} = w(y)\frac{s_+(y+ib)}{s_+(y)} \tag{6.42}$$

Thus we have shown the two integrands in (6.30), corresponding to the two choices of $\delta = +, -$, are amenable to Proposition 6.1. The two corresponding specialisations of $J(y)$ (6.34) are

$$\hat{J}_\delta(b; y, x, x') \equiv w(b;y)\frac{s_+(y-ib)}{s_+(y)}\left[\hat{\psi}(\delta x, y-ia_-)\hat{\psi}(\delta x', -y) + \hat{\psi}(\delta x, y)\hat{\psi}(\delta x', -y+ia_-)\right], \quad \delta = +,- \tag{6.43}$$

where $\hat{\psi}(x,y) = c(y)\psi(x,y)$ is the reduced version of $\psi(x,y)$ (to see why this is reduced, recall (2.28)). The conclusion is that

$$\begin{aligned}\Big(\big[\hat{\mathcal{S}}(y)\hat{\psi}(\delta x, y)\big]\overline{\hat{\psi}(\delta x', y)} + \hat{\psi}(\delta x, y)\big[\hat{\mathcal{S}}(y)\overline{\hat{\psi}(\delta x', \overline{y})}\big]\Big)w(y) \\ = \hat{J}_\delta(y, x, x') - \hat{J}_\delta(y+ia_-, x, x'), \quad \delta = +,-\end{aligned} \tag{6.44}$$

[11] This is one argument in favour of *not* switching to $\hat{\psi}(x,y)$ after (6.23). One can still arrive at (6.53), but then in place of Proposition 6.1 we would get a proposition which loses its manifest (anti)symmetry with Proposition 4.1. Moreover, $\hat{\psi}(x,y)$ is more conducive to residues and eases the analysis in §A.2.

As sketched below (6.23), the purpose of this rewriting is that when substituted into (6.30) it will give us a way to apply Proposition 4.2 (with $x \to y$) to the resulting integral (recalling our remarks about integrand evenness below (6.23)). In other words, it allows us to re-express $\hat{I}_\delta(\Lambda; x, x')$ (6.21) in terms of boundary integrals and residues. The former will vanish provided we can meet the conditions of Proposition 4.3 (with $(x, y, y') \to (y, x, x')$). For the residues we need to know about the pole structure of $\hat{J}_\delta(b; y, x, x')$ (6.43). The relevant facts are given in the following lemma whose proof we relegate to an appendix. With the lemma in place we can launch straight into the theorem.

Lemma 6.2. *Suppose b satisfies $b \in (0, a_+ + a_-)$ and is not: a positive integer multiple of a_-; or $a_+/2$ plus a positive integer multiple of a_-. Then, the functions $\hat{J}_\pm(b; y, x, x')$ (6.43) are smooth in x, x' and have two y-poles in the strip $\operatorname{Im} y \in [0, a_-]$. These poles are: simple; away from the boundary; and given at y_{m_b} and $-y_{m_b} + ia_-$ where*

$$y_m \equiv ib - ima_-, \quad m \in \mathbb{N} \tag{6.45}$$

and where m_b is the uniquely defined integer ensuring $\operatorname{Im} y_{m_b} \in (0, a_-)$.

Theorem 6.3. *Suppose the coupling parameter b is given as in the previous lemma. Then, for functions $f, g \in \mathcal{C}$ (6.8), the object $\hat{D}(f, g)$ (6.17), which measures symmetry violation of the operator $\overline{\mathcal{S}}$ (6.9), is equal to*

$$\hat{D}(f, g) = \frac{\pi i}{a_+ a_-} \int_{\mathbb{R}} dx\, \overline{f(x)} \int_{\mathbb{R}} dx'\, g(x') \operatorname*{Res}_{y = y_{m_b}} \sum_{\delta = +,-} \delta \hat{J}_\delta(b; y, x, x') \tag{6.46}$$

where the functions $\hat{J}_\pm(b; y, x, x')$ (6.43) are defined in terms of $\hat{\psi}(b; x, y)$ (6.24) and where y_{m_b} is as described in the previous lemma.

Proof. This is mostly just a matter of bringing together arguments we have already made. We recall that the relationship between $\hat{D}(f, g)$ (6.17) and the intermediary object $\hat{I}_\delta(\Lambda; x, x')$ (6.21) is given by

$$\hat{D}(f, g) = c^2 \lim_{\Lambda \to \infty} \int_{\mathbb{R}} dx\, \overline{f(x)} \int_{\mathbb{R}} dx'\, g(x') \sum_{\delta = +,-} \hat{I}_\delta(\Lambda; x, x'), \quad f, g, \in \mathcal{C} \tag{6.47}$$

(this is just (6.20) again). Plugging our result (6.44) into (6.30) we have

$$\sum_{\delta = +,-} \hat{I}_\delta(\Lambda; x, x') = \int_{-\Lambda}^{\Lambda} \frac{dy}{2} \sum_{\delta = +,-} \delta \big[\hat{J}_\delta(y, x, x') - \hat{J}_\delta(y + ia_-, x, x') \big] \tag{6.48}$$

Here we are making the claim that this integrand is non-obviously even in y, allowing us to replace $\int_0^\Lambda dy$ with $\int_{-\Lambda}^\Lambda dy/2$. We will prove this shortly. For now we use Proposition 4.2 (with $x \to y$) and Lemma 6.2 to get

$$\sum_{\delta=+,-} \hat{I}_\delta(\Lambda; x, x')$$
$$= \left[\pi i \Big(\operatorname*{Res}_{y=y_{m_b}} + \operatorname*{Res}_{y=-y_{m_b}+ia_-} \Big) - \Big(\int_\Lambda^{\Lambda+ia_-} + \int_{-\Lambda+ia_-}^{-\Lambda} \Big) \frac{dy}{2} \right] \sum_{\delta=+,-} \delta \hat{J}_\delta(y, x, x') \quad (6.49)$$

At this point we make a further claim, that

$$\sum_{\delta=+,-} \delta \hat{J}_\delta(y, x, x') = - \sum_{\delta=+,-} \delta \hat{J}_\delta(-y + ia_-, x, x') \quad (6.50)$$

This subsumes the evenness assumption above but, more importantly, it grants an immediate simplification to the residues and boundary integrals in (6.49) (consider the variable change $y \to -y + ia_-$),

$$\sum_{\delta=+,-} \hat{I}_\delta(\Lambda; x, x') = \left[2\pi i \operatorname*{Res}_{y=y_{m_b}} - \int_\Lambda^{\Lambda+ia_-} dy \right] \sum_{\delta=+,-} \delta \hat{J}_\delta(y, x, x') \quad (6.51)$$

Accordingly, if we can argue that this remaining boundary integral vanishes when recombined with (6.47), the theorem follows (recall, $c^2 = 1/a_+a_-$).

We establish such vanishing by using Proposition 4.3 with $(x, y, y') \to (y, x, x')$ and $\Omega = \mathbb{R}$. As required, we will show that $\hat{J}_\pm(y, x, x')$ (6.43) have O-asymptotics in y of the form (4.43). To this end it obviously helps to use Lemma 1.4, but first we need to rewrite $\hat{J}_\pm(y, x, x')$ (6.43) in terms of $\psi(x, y)$.

Two of the $\hat{\psi}$-terms in (6.43) can be written in terms of ψ straight away (recall that $w(y) = 1/c(y)c(-y)$ and $\hat{\psi}(x, y)/c(y) = \psi(x, y)$), leaving

$$\hat{J}_\delta(y, x, x') = \frac{s_+(y - ib)}{s_+(y)} \left[\frac{\hat{\psi}(\delta x, y - ia_-)}{c(y)} \psi(\delta x', -y) + \psi(\delta x, y) \frac{\hat{\psi}(\delta x', -y + ia_-)}{c(-y)} \right], \quad \delta = +, - \quad (6.52)$$

For the other two $\hat{\psi}$-terms here we can rewrite $c(y)$ as $c(y - ia_-)$ using (1.65); and for the second, $c(-y)$ as $c(-y - ia_-)$ using (1.66). The end result it

$$\hat{J}_\delta(y, x, x') = \psi(\delta x, y - ia_-)\psi(\delta x', -y) + U(y)\psi(\delta x, y)\psi(\delta x', -y + ia_-), \quad \delta = +, - \quad (6.53)$$

where $U(y) = V(y)V(-y + ia_-)$, cf. (6.15).

To consider the large-$\operatorname{Re} y$ O-asymptotics of this function we first extract the following from Lemma 1.4,

$$\psi(x, \alpha y) = \begin{cases} \tilde{\phi} \big[\tilde{c}(x)/\tilde{c}(-x)\big]^{1/2} e^{i\pi xy/a_+a_-} + O(e_-^{-\rho \operatorname{Re} y}), & \alpha = + \\ \tilde{\phi}^{-1} \big[\tilde{c}(-x)/\tilde{c}(x)\big]^{1/2} e^{-i\pi xy/a_+a_-} + O(e_-^{-\rho \operatorname{Re} y}), & \alpha = - \end{cases} \quad \operatorname{Re} y \to \infty \quad (6.54)$$

Since the bound represented by O is uniform for $\mathrm{Im}\, y$ varying over any compact subset of $\mathbb{R}$, cf. the lemma, we can shift y to yield

$$\psi(x,\alpha(y-ia_-)) = \begin{cases} \tilde{\phi}\big[\tilde{c}(x)/\tilde{c}(-x)\big]^{1/2}\, e_+^{x}\, e^{i\pi xy/a_+a_-} + O(e_-^{-\rho\,\mathrm{Re}\, y}), & \alpha = + \\ \tilde{\phi}^{-1}\big[\tilde{c}(-x)/\tilde{c}(x)\big]^{1/2}\, e_+^{-x}\, e^{-i\pi xy/a_+a_-} + O(e_-^{-\rho\,\mathrm{Re}\, y}), & \alpha = - \end{cases} \quad \mathrm{Re}\, y \to \infty \tag{6.55}$$

Moreover, the statement about O still holds here.

We also need the asymptotics for $U(y)$ in (6.15). From the easily-verified

$$(2s_+(y+i\varphi))^{\pm 1} = e_+^{\pm y} e_+^{\pm i\varphi} + O(e_+^{-\,\mathrm{Re}\, y}), \quad \mathrm{Re}\, y \to \infty, \quad \varphi \in \mathbb{R} \tag{6.56}$$

we get straight away that

$$U(y) = \frac{s_+(y-ia_-+ib)}{s_+(y-ia_-)}\frac{s_+(y-ib)}{s_+(y)} = 1 + O(e_+^{-\,\mathrm{Re}\, y}), \quad \mathrm{Re}\, y \to \infty \tag{6.57}$$

where the bound represented by O is uniform for $\mathrm{Im}\, y \in \mathbb{R}$.

Putting this all together we have

$$\hat{J}_\delta(y,x,x') = \left[\frac{\tilde{c}(\delta x)}{\tilde{c}(-\delta x)}\,\frac{\tilde{c}(-\delta x')}{\tilde{c}(\delta x')}\right]^{1/2} e^{i\delta\pi y(x-x')/a_+a_-}\big(e_+^{\delta x} + e_+^{-\delta x'}\big) \\ + O(e^{-\tilde{\rho}\,\mathrm{Re}\, y}), \quad \mathrm{Re} \to \infty, \quad \delta = +,- \tag{6.58}$$

where $\tilde{\rho} > 0$ and with a uniform bound, as required by (4.43).

It now remains to prove (6.50). We can make this property seem less baroque if we write it in terms of

$$\ell(y) \equiv \sum_{\delta=+,-} \delta \hat{J}_\delta(y+ia_-/2, x, x') \tag{6.59}$$

since, now, (6.50) is equivalent to oddness of the function $\ell(\cdot)$. We also have

$$\big(2s_+(x) - 2s_+(x')\big)\sum_{\delta=+,-}\psi(\delta x, y)\overline{\psi(\delta x', y)} = \ell(y-ia_-/2) - \ell(y+ia_-/2) \tag{6.60}$$

where this links up our starting point in (6.20) with (6.48). Moreover this makes it clear how oddness of $\ell(\cdot)$ implies evenness of the lhs. It also suggests a possible alternative way of proving this oddness. This would involve finding some minimal assumptions for when evenness of F implies oddness of φ in the following

$$F(y) = \varphi(y-ia_-/2) - \varphi(y+ia_-/2) \tag{6.61}$$

(of course these should be simple assumptions satisfied by the objects at hand). This approach would be preferable because proving evenness of the lhs function in (6.60) is a simpler task than proving oddness of $\ell(\cdot)$. However at the time of writing we do not know whether this is possible, and so instead we provide a brute force approach to (6.50). To this end we help ourselves to the expression for $\hat{J}_\delta(y,x,x')$ in the proof of Lemma A.2, namely (A.26). The only δ-dependent terms in this expression are the two exponentials in the ν,ν'-sum. Under $\sum_\delta \delta$ these equal $\mu_{\nu\nu'}(y)$ where

$$\mu_\alpha(y) = \begin{cases} 2c_-(ib), & \alpha = + \\ 2c_-(y), & \alpha = - \end{cases} \tag{6.62}$$

And so

$$\sum_{\delta=+,-} \delta \hat{J}_\delta(y,x,x') \propto \hat{w}(y)(s_-(ib-y)s_-(ib+y))^{-1} \sum_{\nu,\nu'=+,-} \mu_{\nu\nu'}(y)\big[\nu'\mathcal{R}_r(x+i\nu a_+/2, y-ia_-) \\ \times \mathcal{R}_r(x'+i\nu' a_+/2, y) - (x,x',\nu,\nu') \to (x',x,\nu',\nu)\big] \tag{6.63}$$

where

$$\hat{w}(y) \equiv \frac{s_+(y-ib)}{s_+(y)} w(y) \tag{6.64}$$

(the unimportant proportionality constant is given by (A.27)). We now argue that the expression on the rhs yields a sign flip under $y \to ia_- - y$. The function (6.64) is invariant under this variable change because of (6.42), and likewise for the reciprocal term in (6.63) because of ia_--antiperiodicity of $s_-(\cdot)$. Applying $y \to ia_- - y$ to the square-bracketed term gives

$$\nu'\mathcal{R}_r(x+i\nu a_+/2, -y)\mathcal{R}_r(x'+i\nu' a_+/2, ia_- - y) - (x,x',\nu,\nu') \to (x',x,\nu',\nu) \tag{6.65}$$

for fixed $\nu,\nu' = +,-$. Because of evenness of $\mathcal{R}_r(x,\cdot)$ this equals the original square-bracketed term multiplied by $-\nu\nu'$. The claim follows then follows because we have $\mu_{\nu\nu'}(-y+ia_-) = \nu\nu'\mu_{\nu\nu'}(y)$. □

7 Residue analysis.

Completing the spectrum of $\tilde{H}$ using symmetry formula for $\overline{\mathcal{S}}$.

In this section we roll out the consequences of Theorem 6.3, seeking to determine if and when the operator $\overline{\mathcal{S}} : \mathcal{F}^*(\mathcal{C}) \to \mathcal{F}^*(\mathcal{C})$ (6.9) is symmetric. The work in §2 will be pivotal here.

The main reason for doing this is to shed light on $\mathcal{F}$ (6.2), which we know has the role of wave operator for the dynamics described by $\tilde{H}$ (4.4). More specifically, symmetry and symmetry-breakdown of $\overline{\mathcal{S}}$ can be tied to the range of $\mathcal{F}$, and thus to the bound states. In the simplest case, symmetry of $\overline{\mathcal{S}}$ implies surjectivity and thus unitarity of $\mathcal{F}$. Before exploring this connection, the first task is to get a better grip on the issue of symmetry and symmetry breakdown of $\overline{\mathcal{S}}$. This means returning to the expression for $\hat{D}(f, g)$ (6.17) in Theorem 6.3 and studying the residue featuring therein. The aim is to get a more useful expression for $\hat{D}(f, g)$ whose vanishing, we recall, is manifestly equivalent to symmetry of $\overline{\mathcal{S}}$. This first discussion culminates in the two corollaries below. At issue is

$$\operatorname*{Res}_{y=y_m} \sum_{\delta=+,-} \delta \hat{J}_\delta(b; y, x, x'), \tag{7.1}$$

$$y_m \equiv ib - ima_-, \quad m \in \mathbb{N} \tag{7.2}$$

where the function $\hat{J}_\delta(y, x, x')$ (6.43) is given by

$$\hat{J}_\delta(b; y, x, x') = \hat{w}(b; y)\big[\hat{\psi}(\delta x, y - ia_-)\hat{\psi}(\delta x', -y) + \hat{\psi}(\delta x, y)\hat{\psi}(\delta x', -y + ia_-)\big], \quad \delta = +, - \tag{7.3}$$

$$\hat{w}(b; y) \equiv \frac{s_+(y - ib)}{s_+(y)} w(b; y), \quad w(b; y) \equiv 1/c(b; y)c(b; -y) \tag{7.4}$$

Here, $c(b; y)$ is defined in (1.25) and $\hat{\psi}(x, y)$ is a reduced version of the eigenfunction $\psi(x, y)$, cf. (2.28) (we suppress the implicit dependence of these latter two functions on b for easier viewing). Well-definedness of (7.3) as a function meromorphic in y and smooth in x, x' follows for $b \in \mathbb{R} \setminus \tilde{S}$ (1.73). Thus the latter is the only assumption we need to analyse the properties of (7.3) and is assumed from now on. (If we want well-definedness of $\mathcal{F}^*$ (6.8) we require the stronger $b \in \mathbb{R} \setminus \mathcal{Y}$ (3.2) which entails smoothness of $\psi(x, y)$ in y. This does not imply smoothness of (7.3) because of the y-shifts in $\hat{\psi}(x, y)$.)

To express (7.1) in a more useful form we are going to draw heavily upon the results in §2 which showed that for the spectral values $\pm y_m$, the functions $\psi(x, y)$ could be linked to $\Psi^{(m)}(x)$ (2.4).

Before processing the residue of $\hat{J}_\pm(b; \cdot, x, x')$ (7.3) at y_m we note a couple of things. First, the the y-shift present in (7.3) will entail $y = y_m \to y_{m+1}$. Second, from §2 we know that $y_{m\geq 1}$ are simple poles of $\hat{\psi}(x, \cdot)$, whereas y_0 and $-y_m$ are regular values. Thus we may write

$$\operatorname*{Res}_{y=y_m} \hat{J}_\delta(b;y,x,x') = \hat{w}(b;y_m)\Big[\Big(\operatorname*{Res}_{y=y_{m+1}} \hat{\psi}(\delta x,y)\Big)\hat{\psi}(\delta x',-y_m) + \Big(\operatorname*{Res}_{y=y_m} \hat{\psi}(\delta x,y)\Big)\hat{\psi}(\delta x',-y_{m+1})\Big], \quad \delta=+,-, \quad m\geq 0 \tag{7.5}$$

And thus we encounter the functions $\hat{\psi}_m(x)$ from Lemma 2.2. Specifically, the two residues here are equal to $\hat{\psi}_{m+1}(x)$ and $\hat{\psi}_m(x)$, respectively. With the restriction $a_+/a_- \notin \mathcal{A}_m$ (2.12) in force, the same lemma tells us that $\hat{\psi}_m(x)$ is proportional to $\Psi^{(m-1)}(x)$ (with proportionality constant $\eta(b)$ (2.31)). The rhs of (7.5) therefore equals

$$\eta(b)\,\hat{w}(b;y_m)\big[\Psi^{(m)}(\delta x)\hat{\psi}(\delta x',-y_m) + \Psi^{(m-1)}(\delta x)\hat{\psi}(\delta x',-y_{m+1})\big] \tag{7.6}$$

When we reconsider this under $\sum_{\delta=+,-}\delta$, the remaining $\hat{\psi}$-functions can be written in terms of $\hat{\phi}_m(x)$ which we introduced in Lemma 2.3. To see why, we recall that $\Psi^{(m)}(x)$ has parity $(-)^m$ in x. Thus the δ's can be pulled out of the Ψ-terms in (7.6) to give

$$\eta(b)\,\hat{w}(b;y_m)\big[\Psi^{(m)}(x)\delta^m\hat{\psi}(\delta x',-y_m) + \Psi^{(m-1)}(x)\delta^{m-1}\hat{\psi}(\delta x',-y_{m+1})\big] \tag{7.7}$$

The connection to $\hat{\phi}_m(x)$ is now manifest because the definition (2.32) can be rearranged as

$$\sum_{\delta=+,-}\delta^{m+1}\hat{\psi}(\delta x,-y_m) = (-)^{m-1}\hat{\phi}_m(x), \quad m\in\mathbb{N} \tag{7.8}$$

And so

$$\sum_{\delta=+,-}\delta \operatorname*{Res}_{y=y_m} \hat{J}_\delta(b;y,x,x') = \eta(b)\hat{w}(b;y_m)(-)^{m-1}\big[\Psi^{(m)}(x)\hat{\phi}_m(x') - \Psi^{(m-1)}(x)\hat{\phi}_{m+1}(x')\big] \tag{7.9}$$

Lemma 2.3 tells us $\hat{\phi}_m(x)$ are similarly proportional to $\Psi^{(m)}(x)$ (now with proportionality constant $c(b;y_1)$). The rhs of (7.9) therefore equals

$$\eta(b)\,\hat{w}(b;y_m)c(b;y_1)(-)^{m-1}\big[\Psi^{(m)}(x)\Psi^{(m-1)}(x') - \Psi^{(m-1)}(x)\Psi^{(m)}(x')\big], \quad m\in\mathbb{N} \tag{7.10}$$

For one thing, this expression for the residue (7.1) exhibits vanishing when $m=0$ (recall, $\Psi^{-1}\equiv 0$). Thus we have a sufficient condition for symmetry of $\overline{\mathcal{S}}$ (6.9), namely $m_b=0$ (recall, $\mathcal{A}_0$ is empty). Recalling how the constant m_b was defined in Theorem 6.3, this condition amounts to $b\in(0,a_-)$. (We note that $b\in(0,a_-)$ is compatible with all the conditions in Lemma 6.2.)

We draw these observations together into the following two corollaries. The first is immediate given what we have just said. The second, which gives us a new expression for symmetry breakdown of $\overline{\mathcal{S}}$, is almost immediate and just requires us to handle some constants.

Corollary 7.1. *Suppose the coupling parameter b satisfies $b \in (0, a_-)$. Then the operator $\overline{\mathcal{S}} : \mathcal{F}^*(\mathcal{C}) \to \mathcal{F}^*(\mathcal{C})$ (6.9) is symmetric.*

Corollary 7.2. *Suppose b is as described in Lemma 6.2 but with lower limit $b > a_-$. And suppose a_+, a_- satisfy $a_+/a_- \notin \mathcal{A}_{m_b}$ (2.12). Then, for functions $f, g \in \mathcal{C}$ (4.2), the object $\hat{D}(f, g)$ (6.17), which measures symmetry violation of $\overline{\mathcal{S}}$ (6.9), satisfies*

$$\hat{D}(f,g) = \mathcal{N}(b) P_{m_b} \int_{\mathbb{R}} dx\, \overline{f(x)} \int_{\mathbb{R}} dx'\, g(x') \\ \times \left[\Psi^{(m)}(x)\Psi^{(m-1)}(x') - \Psi^{(m-1)}(x)\Psi^{(m)}(x')\right]\Big|_{m=m_b} \quad (7.11)$$

where $\Psi^{(m)}(b; x)$ (2.4) are the functions studied in §2 and where $m_b \geq 1$ is the largest integer satisfying $m_b a_- < b$. We have also introduced the constants

$$\mathcal{N}(b) \equiv \frac{i}{a_+} s_-(y_1) \frac{c(b; y_1)}{c(b; -y_0)} \quad (7.12)$$

and

$$P_m \equiv \prod_{l=0}^{m-1} (1/\rho_l) \quad (7.13)$$

which feature the function $c(b; z)$ (1.25), the spectral values y_m (7.2) and the b-dependent constant, ρ_l (2.9).

Proof of Corollary 7.2. First we note the description of m_b in this corollary is equivalent to the definition in Lemma 6.2 given the b-interval under consideration.

Plugging (7.9)\(7.10) into Theorem 6.3, we see straight away how the x, x'-dependent part of (7.11) arises. It just remains to consider the constants. Specifically, we have to compute

$$\frac{\pi i}{a_+ a_-} \eta(b) \hat{w}(y_m) c(y_1) (-)^{m-1}, \quad m \in \mathbb{N}^* \quad (7.14)$$

From (7.4) we have

$$\hat{w}(y_m) = \frac{s_+(y_m - ib)}{s_+(y_m)} w(y_m) \quad (7.15)$$

where $w(y_m) = 1/c(y_m)c(-y_m)$. Unless otherwise stated, equations in this proof hold for $m \in \mathbb{N}^*$ (necessary because $c(y_0) = \infty$ generically). Recalling $y_m \equiv ib - ima_-$, we can treat $c(\pm y_m)$ recursively using the AΔE for $c(y)$, (1.66). Setting $y = y_m$ in this we get

$$\frac{c(y_m)}{c(y_{m+1})} = \frac{s_+(y_m - ib)}{s_+(y_m)} = -1/\sigma_{m-1} \quad (7.16)$$

and $y = -y_{m+1}$,

$$\frac{c(-y_m)}{c(-y_{m+1})} = \frac{s_+(-y_{m+1})}{s_+(-y_{m+1} - ib)} = \sigma_m/\rho_m, \quad m \in \mathbb{N} \quad (7.17)$$

where $\sigma_m \backslash \rho_m$ are the constants defined in (2.6)\(2.7) (and which feature implicitly in $\Psi^{(m)}(x)$). Thus

$$w(y_{m+1})/w(y_m) = -\sigma_m/\rho_m\sigma_{m-1} \tag{7.18}$$

Since the quotient in (7.15) may be written as $(-1/\sigma_{m-1})$ this implies

$$\hat{w}(y_m) = (-)^m w(y_1)\frac{1}{\sigma_0}\prod_{l=1}^{m-1}(1/\rho_l) \tag{7.19}$$

By using (7.17) with $m = 0$ we have $w(y_1) = \sigma_0/\rho_0 c(y_1)c(-y_0)$. Putting this together with (7.19), and recalling $\eta(b) = -s_-(y_1)(a_-/\pi)c(y_1)$ from (2.31), we see that (7.14) equals

$$\frac{i}{a_+}s_-(y_1)(c(y_1)/c(-y_0))P_m \tag{7.20}$$

This proves the claim. □

We now begin our discussion of the connection between the range of the transform $\mathcal{F}$ (6.2), which has the the role of wave operator for the dynamics described by $\tilde{H}$ (5.4), and the symmetry analysis of $\overline{\mathcal{S}} : \mathcal{F}^*(\mathcal{C}) \to \mathcal{F}^*(\mathcal{C})$ (6.9). Isometry of $\mathcal{F}$ (6.2) is critical to establishing this connection. From Corollary 5.3 we have an explicit description of certain conditions under which this property obtains.

As usual we present our propositions in terms of minimal assumptions. We have designed these so that the relationship to our concrete objects should be obvious (all the same, it can be found explicitly in, e.g., the proof of Theorem 7.4).

Proposition 7.3. *Let $J : \hat{\mathcal{H}} \to \mathcal{H}$ be an isometry and $\lambda(x)$ a real-valued function which, as a multiplication operator on $\mathcal{H}$, maps some dense subspace $\mathcal{D}$ into itself. With these objects we may define an operator $S : J^*(\mathcal{D}) \to J^*(\mathcal{D})$ by the intertwining relation*

$$SJ^* = J^*\lambda \tag{7.21}$$

Furthermore, suppose that for some $\kappa \geq 0$, $1/(\lambda + i\kappa)$ is a bounded function. We also need the assumption that at least one function in $J(\hat{\mathcal{H}})$ has support on $\mathbb{R}$. Then, if S is symmetric, J is surjective (and thus unitary).

Proof. We first note that isometry of J implies J^* exists and is bounded on $\mathcal{H}$. Thus the operator S is well-defined.

Symmetry of S (7.21) is equivalent to vanishing of

$$\hat{D}_S(f, g) \equiv (SJ^*f, J^*g)_{\hat{\mathcal{H}}} - (J^*f, SJ^*g)_{\hat{\mathcal{H}}} \tag{7.22}$$

for all $f, g \in \mathcal{D}$. Using (7.21) we can write this as

$$(J^*\lambda f, J^*g)_{\hat{\mathcal{H}}} - (J^*f, J^*\lambda g)_{\hat{\mathcal{H}}} = (J^*(\lambda - i\kappa)f, J^*g)_{\hat{\mathcal{H}}} - (J^*f, J^*(\lambda + i\kappa)g)_{\hat{\mathcal{H}}} \tag{7.23}$$

where the functions $\tilde{f} \equiv (\lambda - i\kappa)f$ and $\tilde{g} \equiv (\lambda + i\kappa)g$ are in $\mathcal{H}$ provided $f, g, \in \mathcal{D}$. Writing (7.23) in terms of these two functions gives

$$\begin{aligned}(J^*\tilde{f}, J^*(\lambda + i\kappa)^{-1}g)_{\hat{\mathcal{H}}} - (J^*(\lambda - i\kappa)^{-1}\tilde{f}, J^*\tilde{g})_{\hat{\mathcal{H}}} \\ = (\tilde{f}, JJ^*(\lambda + i\kappa)^{-1}\tilde{g})_{\mathcal{H}} - (\tilde{f}, (\lambda + i\kappa)^{-1}JJ^*\tilde{g})_{\mathcal{H}} = (\tilde{f}, [JJ^*, (\lambda + i\kappa)^{-1}]\,\tilde{g})_{\mathcal{H}}\end{aligned} \quad (7.24)$$

Since J is an isometry we have automatically that

$$JJ^* = \mathbf{1}_{\mathcal{H}} - \mathrm{proj}_{J(\hat{\mathcal{H}})^\perp} \quad (7.25)$$

where proj_X denotes orthogonal projection onto $X \subseteq \mathcal{H}$, and $X^\perp$ denotes the orthocomplement of X in $\mathcal{H}$. Consequently,

$$\hat{D}_S(f, g) = (\tilde{f}, [(\lambda + i\kappa)^{-1}, \mathrm{proj}_{J(\hat{\mathcal{H}})^\perp}]\,\tilde{g})_{\mathcal{H}} \quad (7.26)$$

Since the functions $(\lambda(x) \pm i\kappa)$ are non-zero everywhere, the subspaces $(\lambda \pm i\kappa)\mathcal{D}$ arising for $\tilde{f}, \tilde{g}$ are clearly dense in $\mathcal{H}$. Thus if we now let $\hat{D}_S(f, g)$ vanish, (7.26) is strong enough for us to conclude

$$[(\lambda + i\kappa)^{-1}, \mathrm{proj}_{J(\hat{\mathcal{H}})^\perp}] = 0 \quad (7.27)$$

We now invoke the well-known result that the algebra of the bounded multiplication operators is equal to its own commutant. Since these operators are generated by $\mathbf{1}$ and $(\lambda + i\kappa)^{-1}$, there must exist a function $\mu(x) \in L^\infty(\mathbb{R}, dx)$ such that

$$\mathrm{proj}_{J(\hat{\mathcal{H}})^\perp} = \mu(\cdot) \quad (7.28)$$

We need only consider this operator equation on a function $f \in J(\hat{\mathcal{H}})$ supported on $\mathbb{R}$ to conclude that $\mu = 0$, and so the projection is just the zero operator for all functions in $\mathcal{H}$.

We note that without the assumption involving κ, we can still show $[\lambda, \mathrm{proj}_{J(\mathcal{H})^\perp}] = 0$ (and moreover we do not need to use $\tilde{f}, \tilde{g}$). But without assuming boundedness of $\lambda(x)$ we could not invoke the commutant result cited above.

$\square$

Theorem 7.4. *If the coupling parameter b satisfies $b \in (0, a_-)$ then the map $\mathcal{F} : \hat{\mathcal{H}} \to \mathcal{H}$, which extends* (4.3) *with action* (4.1)*, is unitary.*

Proof. For b in this interval we know that $\mathcal{F} : \hat{\mathcal{H}} \to \mathcal{H}$ (6.2) is an isometry, cf. Corollary 5.3. Thus we can apply Proposition 7.3 with

$$J = \mathcal{F}, \quad \lambda(x) = 2s_+(x), \quad \mathcal{D} = \mathcal{C}, \quad \kappa = 1 \quad (7.29)$$

The operator S (7.21) corresponding to these choices is clearly just $\overline{\mathcal{S}} : \mathcal{F}^*(\mathcal{C}) \to \mathcal{F}^*(\mathcal{C})$ (6.9). We know from Corollary 7.1 that this is symmetric given $b \in (0, a_-)$, and so the theorem follows.

$\square$

We now look at the far less obvious relationship that exists between $\mathcal{F}(\hat{\mathcal{H}})$ and symmetry breakdown of $\overline{\mathcal{S}}$, building on the precedent in [3]. In Corollary 7.2 we presented an expression for $\hat{D}(f,g)$ (6.17) - the object measuring symmetry violation - which was more illuminating than the original in Theorem 6.3. To cement the relationship, we must re-express it again.

This can be loosely motivated by the observation that two of the $\Psi^{(m)}$-functions in (7.11) are not in themselves integrable (recall from §2 that $\Psi^{(m)} \in \mathcal{H}$ iff $m = 0, \dots, m_b - 1$).[12] Thus we might consider taking the index of these two problem terms down a notch by using the recursion for $\Psi^{(m)}$, (2.5). When we do this we realise the whole square-bracketed term in (7.11) has its own recursion which is *first* order and allows for a closed-form solution. The result is a Christoffel-Darboux identity which we present in the following lemma.

Lemma 7.5. *As relevant to the integrand in* (7.11)*, we have*

$$\begin{aligned} &P_m\big[\Psi^{(m)}(x)\Psi^{(m-1)}(x') - \Psi^{(m-1)}(x)\Psi^{(m)}(x')\big] \\ &\qquad = \big(2s_+(x') - 2s_+(x)\big)\sum_{k=0}^{m-1} P_{k+1}\sigma_k\Psi^{(k)}(x)\Psi^{(k)}(x'), \quad m \in \mathbb{N}^* \end{aligned} \tag{7.30}$$

which sees the reappearance of the constant (2.6)*,*

$$\sigma_k \equiv s_+(ib - i(k+1)a_-)/s_+(i(k+1)a_-) \tag{7.31}$$

As suggested, we may also weaken the $a_\pm$-restriction in Corollary 7.2 to $a_+/a_- \notin \mathcal{A}_{m_b-1}$.

Proof. We start by writing the object on the lhs of (7.30) as $D_m(x,x')$ and allow $m \in \mathbb{N}$ such that $D_0(x,x') = 0$ (recall $\Psi^{(-1)} \equiv 0$). From (2.5) we know that $\Psi^{(m)}(x)$ satisfies the recursion

$$\Psi^{(m+1)}(x) = -2s_+(x)\sigma_m\Psi^{(m)}(x) - \rho_m\Psi^{(m-1)}(x), \quad m \geq 0 \tag{7.32}$$

so that

$$D_{m+1}(x,x') = -P_{m+1}\big[\big(2s_+(x)\sigma_m\Psi^{(m)}(x) + \rho_m\Psi^{(m-1)}(x)\big)\Psi^{(m)}(x') - \ (x \leftrightarrow x')\big] \tag{7.33}$$

If we rewrite the rhs as

$$P_{m+1}\big[\big(2s_+(x')\sigma_m\Psi^{(m)}(x)\Psi^{(m)}(x') + \rho_m\Psi^{(m-1)}(x')\big)\Psi^{(m)}(x) - \ (x \leftrightarrow x')\big] \tag{7.34}$$

then we can see straight away

$$\begin{aligned} &D_{m+1}(x,x') - D_m(x,x') \\ &= \big[2s_+(x')P_{m+1}\sigma_m\Psi^{(m)}(x)\Psi^{(m)}(x') + \big\{P_{m+1}\rho_m - P_m\big\}\Psi^{(m-1)}(x')\big)\Psi^{(m)}(x) - \ (x \leftrightarrow x')\big] \end{aligned} \tag{7.35}$$

[12] Though of course, the integrand as a whole still is.

From the form of P_m (7.13) alone, this curly-bracketed term is manifestly vanishing. Thus we have the first order recursion,

$$D_{m+1}(x,x') - D_m(x,x') = P_{m+1}\sigma_m \Psi^{(m)}(x)\Psi^{(m)}(x')\big(2s_+(x') - 2s_+(x)\big), \quad m \in \mathbb{N} \tag{7.36}$$

and the claim follows. □

Envisioning how this lemma recombines with Corollary 7.2 we can understand the significance of the following proposition. Straight away it leads to our completeness theorem, which is the final result of this section (in our summary in introduction, we saw how it looks under more concrete conditions).

(The assumption in the proposition that γ_k are merely 'constants' may sound a bit wishy washy. The point is that depending on how a formula like (7.37) arises, it may not be obvious that γ_k are positive, or even real. In this case, the positivity implied by (7.39) is a result in itself.)

Proposition 7.6. *Let $S : J^*(\mathcal{D}) \to J^*(\mathcal{D})$ be defined as in Proposition 7.3, and $\hat{D}_S(f,g)$ as in* (7.22)*. Then, if the following formula holds for $f, g \in \mathcal{D}$,*

$$\hat{D}_S(f,g) = \int_{\mathbb{R}} dx\, \overline{f(x)} \int_{\mathbb{R}} dx'\, g(x')[\lambda(x') - \lambda(x)] \sum_{k=0}^{L} \gamma_k\, \varphi_k(x)\overline{\varphi_k(x')}, \tag{7.37}$$

where $L \geq 0$ is some integer, γ_k are constants, and $\varphi_k \in J(\hat{\mathcal{H}})^\perp$ are mutually orthogonal functions at least one of which has support on $\mathbb{R}$, we have

$$J(\hat{\mathcal{H}})^\perp = \operatorname{span}\{\varphi_0, \dots \varphi_L\} \subset \mathcal{H} \tag{7.38}$$

and

$$1/\gamma_k = \|\varphi_k\|^2_{\mathcal{H}}, \quad k = 0, \dots L \tag{7.39}$$

Theorem 7.7. *Suppose the coupling parameter b satisfies $b \in (a_-, a_- + a_+/2)$ and is not a positive integer multiple of a_-. Then, the map $\mathcal{F} : \hat{\mathcal{H}} \to \mathcal{H}$, which extends* (4.3) *with action* (4.1)*, is an isometry which partakes in the following orthogonal decomposition of Hilbert space,*

$$\mathcal{H} = \mathcal{F}(\hat{\mathcal{H}}) \oplus \operatorname{span}\{\Psi^{(0)}, \dots, \Psi^{(m_b - 1)}\} \tag{7.40}$$

where $\Psi^{(m)}(b;x)$ are the mutually orthogonal functions defined in (2.4)*, and m_b is the largest integer satisfying $m_b a_- < b$. The polynomial $Q_m(b;\cdot)$* (2.5) *on which a given $\Psi^{(m)}$ depends has degree m and parity $(-)^m$.*

Furthermore, we have function norms given by

$$1/\mathcal{N}(b) P_{k+1}\sigma_k = \|\Psi^{(k)}\|^2_{\mathcal{H}}, \quad k = 0, \dots, m_b - 1 \tag{7.41}$$

where the b-dependent constants on the lhs are defined respectively by (7.12), (7.13) *and* (7.31).

Proof of Theorem 7.7. We consider Proposition 7.6 with

$$J = \mathcal{F}, \quad \lambda(x) = 2s_+(x), \quad \mathcal{D} = \mathcal{C}, \quad \kappa = 1 \tag{7.42}$$

(noting the restriction on b in the theorem ensures $\mathcal{F}$ is an isometry, cf. Corollary 5.3). The operator S (7.21) arising for these choices is clearly $\overline{\mathcal{S}} : \mathcal{F}^*(\mathcal{C}) \to \mathcal{F}^*(\mathcal{C})$ (6.9) and we have accordingly,

$$\hat{D}_{\bar{\mathcal{S}}}(f, g) = \hat{D}(f, g) \tag{7.43}$$

By combining Corollary 7.2 and Lemma 7.5 we thus realise the form of (7.37) for the choices

$$L = m_b - 1, \quad \gamma_k = \mathcal{N}(b)\sigma_k P_{k+1}, \quad \varphi_k = \Psi^{(k)} \tag{7.44}$$

where we also have that the functions $\Psi^{(k)}(b; x)$ are real-valued. From Lemma 3.5 we know they satisfy $\Psi^{(k)} \in \mathcal{F}(\hat{\mathcal{C}})^{\perp}$ (recall (3.3)) and are mutually orthogonal provided $b \notin \Pi_{m_b-1}$. It automatically follows that $\Psi^{(k)}$ are orthogonal to

$$\overline{\mathcal{F}(\hat{\mathcal{C}})} = \mathcal{F}(\hat{\mathcal{H}}) \tag{7.45}$$

where this equality holds when $\mathcal{F}$ is an isometry. Accordingly, all the conditions are met for us to deduce the statements in the theorem via Proposition 7.6. It just remains to reconcile various claims about restrictions on the parameters.

First, it is easy to see that the restrictions on b in the theorem are compatible with those in Corollary 7.2. Second, it is less obvious, but true, that the former are compatible with $b \notin \Pi_{m_b-1}$ (2.15), $m_b \geq 1$. Because of (2.16) and (3.3) we already know the b-restriction in the lemma implies $b \notin \Pi^{(+)}_{m_b-1}$. However, dealing with $\Pi^{(-)}_m$ is a little more complicated. The issue is seeing that, for $m_b \geq 2$,

$$(l + 1)a_- - k'a_+ \notin (a_-, a_- + a_+/2), \quad l = 0, \ldots, m_b - 2, \quad k' \in \mathbb{N}^* \tag{7.46}$$

which is not obvious at first glance. A sufficient condition for (7.46) is

$$a_+/a_- > m_b - 1 \tag{7.47}$$

(since this implies all the numbers on the lhs of (7.46) are negative). When $b \in (a_-, a_- + a_+/2)$ there is a manifest relation between the ratio a_+/a_- and the integer m_b (by definition the largest such that $m_b a_- < b$); namely,

$$a_+/a_- > 2m_b - 2 \tag{7.48}$$

Thus (7.47) clearly holds for $m_b \geq 2$ as required.

As regards the restriction on the ratio a_+/a_-, the point is that

$$a_+/a_- \notin \mathcal{A}_{m_b-1}, \quad m_b \geq 1 \tag{7.49}$$

holds in general for the b-interval at issue. To see this we note from (2.12) that $\max \mathcal{A}_m = m$ when $m \geq 1$. Then, it is a matter of invoking (7.48) again to see that (7.49) is plainly true.

□

Proof. Proof of Proposition 7.6 The aim is to establish equality in the subset relation

$$\mathrm{span}\{\varphi_0, \dots \varphi_L\} \subseteq J(\hat{\mathcal{H}})^\perp \tag{7.50}$$

(which itself follows immediately from the assumptions on φ_k). The argument we use, adapted from [3], throws out the function norms (7.39) as a highly desirable and non-trivial by-product.

From (7.26) we know that $\hat{D}_S(f, g)$ (7.22) can be written as

$$\hat{D}_S(f, g) = (\tilde{f}, [(\lambda + i\kappa)^{-1}, \mathrm{proj}_{J(\hat{\mathcal{H}})^\perp}]\, \tilde{g})_{\mathcal{H}} \tag{7.51}$$

where $\tilde{f} \equiv (\lambda - i\kappa)f$ and $\tilde{g} \equiv (\lambda + i\kappa)g$. The aim is to re-express the rhs of (7.37) in such a way that it can be fruitfully combined with (7.51). First we write it as

$$\sum_{k=0}^{L} \gamma_k \left[(f, \varphi_k)(\varphi_k, \lambda g) - (\lambda f, \varphi_k)(\varphi_k, g) \right], \tag{7.52}$$

meanwhile suppressing the $\mathcal{H}$-subscript on the inner product $(\cdot, \cdot)_{\mathcal{H}}$. The term in square brackets may then be rewritten as

$$(f, \varphi_k)(\varphi_k, (\lambda + i\kappa)g) - ((\lambda - i\kappa)f, \varphi_k)(\varphi_k, g) \tag{7.53}$$

Using $\tilde{f}, \tilde{g}$ above, and the fact the bounded adjoint of $(\lambda - i\kappa)^{-1}$ is $(\lambda + i\kappa)^{-1}$, this equals

$$\begin{aligned} &(\tilde{f}, (\lambda + i\kappa)^{-1}\varphi_k)(\varphi_k, \tilde{g}) - (\tilde{f}, \varphi_k)(\varphi_k, (\lambda + i\kappa)^{-1}\tilde{g}) \\ &\quad = (\tilde{f}, (\varphi_k, \tilde{g})(\lambda + i\kappa)^{-1}\varphi_k - (\varphi_k, (\lambda + i\kappa)^{-1}\tilde{g})\varphi_k) = (\tilde{f}, [(\lambda + i\kappa)^{-1}, \varphi_k \otimes \overline{\varphi}_k]\tilde{g}) \end{aligned} \tag{7.54}$$

where $(F \otimes G)H \equiv (\overline{G}, H)F$. And so, introducing

$$Q \equiv \sum_{k=0}^{L} \gamma_k \, \varphi_k \otimes \overline{\varphi_k}, \tag{7.55}$$

we may write (7.52) as

$$(\tilde{f}, [(\lambda + i\kappa)^{-1}, Q\,]\tilde{g}) \tag{7.56}$$

Combining this with (7.26), our assumption (7.37) may be written as

$$(\tilde{f}, [(\lambda + i\kappa)^{-1}, Q - \mathrm{proj}_{J(\hat{\mathcal{H}})^\perp}]\, \tilde{g})_{\mathcal{H}} = 0, \quad f, g \in \mathcal{D} \tag{7.57}$$

As noted earlier, the subspaces $(\lambda \pm i\kappa)\mathcal{D}$ arising for $\tilde{f}, \tilde{g}$ are clearly dense in $\mathcal{H}$. Thus (7.57) is strong enough for us to conclude

$$[(\lambda + i\kappa)^{-1}, Q - \mathrm{proj}_{J(\hat{\mathcal{H}})^\perp}] = 0 \tag{7.58}$$

Repeating the argument below (7.27), there must consequently exist a function $\mu(x) \in L^\infty(\mathbb{R}, dx)$ such that

$$Q - \mathrm{proj}_{J(\hat{\mathcal{H}})^\perp} = \mu(\cdot) \tag{7.59}$$

Considering this operator equation on φ_l, $l = 0, \ldots, L$, and using mutual orthogonality, we get

$$[\gamma_l(\varphi_l, \varphi_l) - 1]\varphi_l(x) = \mu(x)\varphi_l(x), \quad x \in \mathbb{R} \tag{7.60}$$

Now if we choose φ_l with support on $\mathbb{R}$, it follows that μ must be a constant function. Now reconsider (7.59) on any non-zero function in $J(\hat{\mathcal{H}})$ (unproblematic since J is an isometry) then the lhs vanishes and so μ must be zero. Consequently, (7.59) becomes

$$\mathrm{proj}_{J(\hat{\mathcal{H}})^\perp} = Q \tag{7.61}$$

and (7.60),

$$(\varphi_l, \varphi_l) = 1/\gamma_l \tag{7.62}$$

With the former, we get a complete expression of the orthocomplement $J(\hat{\mathcal{H}})^\perp$ as claimed. $\square$

A Appendix. Proof of Lemmas 4.4 and 6.2.

A.1 Proof of Lemma 4.4.

At issue is the function

$$J_\sigma(x,y,y') = \tilde{U}(x)\big[\psi(x-ia_-,-y)\psi(\sigma x,y') - \psi(x,-y)\psi(\sigma(x-ia_-),y')\big], \quad \sigma = +,- \tag{A.1}$$

where all the objects here have an implicit dependence on the parameters (a_+, a_-, b), and

$$\tilde{U}(x) = \big[\tilde{V}(x)\tilde{V}(-x+ia_-)\big]^{1/2} = \left[\frac{c_+(x+ib-ia_-)}{c_+(x-ia_-)}\frac{c_+(x-ib)}{c_+(x)}\right]^{1/2} \tag{A.2}$$

which derives from the A∆O $\tilde{H}(x)$ (1.52). Throughout this appendix we will suppress implicit dependence on these parameters.

From Lemma 3.1 it is clear that (A.1) is smooth in y, y' for both choices of σ given the restriction $b \in \mathbb{R} \setminus \mathcal{Y}$ (3.2) (this also ensures well-definedness of the square root in (A.1)). With this restriction in force we proceed to consider the analyticity properties in x of (A.1). We claim that for both choices of σ, the function (A.1) is holomorphic in x in the strip $\mathrm{Im}\, x \in [0, a_-]$ provided $b \in (-a_+/2, a_- + a_+/2)$. If we extract the $\mathbb{Z}$-integer multiples of a_- from this interval, then it lies inside $\mathbb{R} \setminus \mathcal{Y}$. Hence the condition on b in Lemma 4.4.

We break up the holomorphy claim into the following two lemmas. The first isolates the algebraic aspect, the second the analytic.

Lemma A.1. *Any x-pole of $J_\pm(x,y,y')$* (A.1) *is an x-pole of one of the four products*

$$\frac{G(x-ia_-/2+ib)}{G(x-ia_-/2-ib)}\mathcal{R}_r(x-ia_-+i\nu a_+/2,y)\mathcal{R}_r(x+i\nu' a_+/2,y'), \quad \nu,\nu' = +,- \tag{A.3}$$

Lemma A.2. *For y, y' any non-singular values, the four products* (A.3) *are holomorphic in x in the strip* $\mathrm{Im}\, x \in [0, a_-]$ *provided* $b \in (-a_+/2, a_- + a_+/2)$.

Proof of Lemma A.1. The first task is to write out (A.1) using $\psi(x,y)$ (1.60). Recalling evenness of $\tilde{w}(x)$ and $\mathcal{R}_r(x,y)$ in both variables we get, for $\sigma = +,-$,

$$\begin{aligned}\psi(\sigma x, \pm y) &= \tilde{w}(x)^{1/2}c(\mp y)^{-1}(2s_-(ib\mp y))^{-1}\sum_{\nu=+,-}\nu e_-(\nu(ib\mp y)/2)\mathcal{R}_r(\sigma x+i\nu a_+/2,y)\\ &= \tilde{w}(x)^{1/2}c(\mp y)^{-1}(2s_-(ib\mp y))^{-1}\sum_{\nu=+,-}\sigma\nu e_-(\sigma\nu(ib\mp y)/2)\mathcal{R}_r(x+i\nu a_+/2,y)\end{aligned} \tag{A.4}$$

By shifting x in this we also get

$$\begin{aligned}\psi(\sigma(x-ia_-),\pm y) &= \tilde{w}(x-ia_-)^{1/2}c(\mp y)^{-1}(2s_-(ib\mp y))^{-1}\\ &\quad\times\sum_{\nu=+,-}\sigma\nu e_-(\sigma\nu(ib\mp y)/2)\mathcal{R}_r(x-ia_-+i\nu a_+/2,y), \quad \sigma=+,-\end{aligned} \tag{A.5}$$

Substituting these last two equations into (A.1) we get straight away

$$J_\sigma(x,y,y') \propto \tilde{U}(x)\tilde{w}(x)^{1/2}\tilde{w}(x-ia_-)^{1/2} \sum_{\nu,\nu'=+,-} \nu\nu' e_-(\nu(ib+y)/2)e_-(\sigma\nu'(ib-y')/2)$$
$$\times \left[\mathcal{R}_r(x-ia_-+i\nu a_+/2,y)\mathcal{R}_r(x+i\nu' a_+/2,y') - (y,y',\nu,\nu') \to (y',y,\nu',\nu)\right] \quad \text{(A.6)}$$

where we are of course dealing with proportionality with respect to x. We note that the unimportant proportionality constant is

$$\sigma/4c(y)c(-y')s_-(ib+y)s_-(ib-y') \quad \text{(A.7)}$$

We thus see how any x-pole of (A.6) must come from one of the four functions

$$\tilde{U}(x)\tilde{w}(x)^{1/2}\tilde{w}(x-ia_-)^{1/2}\mathcal{R}_r(x-ia_-+i\nu a_+/2,y)\mathcal{R}_r(x+i\nu' a_+/2,y'), \quad \nu,\nu'=+,- \quad \text{(A.8)}$$

This combination of $\tilde{U}$ and $\tilde{w}$ has been seen before. It arose in a related context in §3.4. There we showed it was equal to $\tilde{w}(x)\tilde{V}(x)$ and thence to $G(x-ia_-/2+ib)$ $G(-x+ia_-/2+ib)$, cf. (3.41). And the latter is just a rewriting of the quotient in the lemma (recall (1.4)).

□

Proof of Lemma A.2. The x-poles of the quotient in (A.3) have already been analysed in the proof of Lemma 3.5. There we saw they could all be banished from $i[0,a_-]\times\mathbb{R}$ by taking $b > -a_+/2$.

As a result, we need only study the x-poles of the $\mathcal{R}_r$-functions in (A.3). To banish these from the strip would require inequalities on b more restrictive than the ones proposed. Instead we proceed on the basis of a crucial observation, namely that the poles of these $\mathcal{R}_r$-functions are removed by x-zeros of the quotient term in (A.3). In fact we know already this must be true from what we saw in the proof of Lemma 3.5 (where the same functions appeared in a related context). It is corroborated again by the pole analysis below.

The bottom line is that we need only be concerned with the double poles that arise when those of the two $\mathcal{R}_r$-functions in the product in (A.3) overlap - with no corresponding overlap of the zeros of the quotient term in (A.3). Such poles will *not* be removed by the zeros. It is these that can be banished from the strip by imposing $b \in (-a_+/2, a_-+a_+/2)$.

As noted in §1, the zeros of $G(\cdot)$ (1.3) occur at the points

$$ia + z_{k,l}, \quad z_{k,l} \equiv ika_+ + ila_-, \quad k,l \in \mathbb{N} \quad \text{(A.9)}$$

And so the x-zeros of the quotient term in (A.3) (recalling $1/G(z) = G(-z)$ are given by

$$\begin{cases} x = -ib + ia_+/2 + z_{k,l+1} \\ -x = -ib + ia_+/2 + z_{k,l} \end{cases} \quad k,l \in \mathbb{N} \quad \text{(A.10)}$$

To consider the poles of the $\mathcal{R}_r$-functions in (A.3) it helps to recall (3.7) which tells us the aggregated poles of the two functions $\mathcal{R}_r(x + i\nu' a_+/2, y')$ (corresponding to the two choices $\nu' = +, -$) are given by

$$\pm x = -ib + ia_+/2 + z_{k,l+1}, \quad k, l \in \mathbb{N} \tag{A.11}$$

We can adapt this straight away to assert that the aggregated poles of the other two functions $\mathcal{R}_r(x + i\nu a_+/2 - ia_-, y)$ are given by

$$\begin{cases} x = -ib + ia_+/2 + z_{k,l+2} \\ -x = -ib + ia_+/2 + z_{k,l} \end{cases} \quad k, l \in \mathbb{N} \tag{A.12}$$

Thus we can see the pole sequences (A.11)-(A.12) are manifestly encompassed by the zeros (A.10). This in itself does not establish that the poles are removed (because we have not addressed multiplicity) but we know this is true at the algebraic level from (3.54)-(3.56).

With these sequences in place, we now ask for a restriction on b which will ensure that the following four products have no double poles in $i[0, a_-]$ (with no corresponding double zeros from (A.10)),

$$\mathcal{R}_r(x - ia_- + i\nu a_+/2, y)\mathcal{R}_r(x + i\nu' a_+/2, y'), \quad \nu, \nu' = +, - \tag{A.13}$$

This amounts to ensuring no overlap of the upwards poles of the two constituent functions, and similarly no overlap of the downwards. To simplify this problem we can make a "worst case scenario" assumption wherein we assume that, regardless of the choice of ν' in $\mathcal{R}_r(x + i\nu' a_+/2, y')$, the latter has poles at *all* the points (A.11), even though we know this is not the case (cf. the text below (3.7)). We make the analogous assumption for $\mathcal{R}_r(x - ia_- + i\nu a_+/2, y)$ and (A.12). Under these assumptions we will see that the upper bound $b < a_- + a_+/2$ is sufficient for no overlap.[13]

Writing down all the upwards poles that are shared between (A.11) and (A.12) we get a sequence,

$$-ib + ia_+/2 + z_{k,l+2}, \quad k, l \in \mathbb{N} \tag{A.14}$$

and for the downwards,

$$ib - ia_+/2 - z_{k,l+1}, \quad k, l \in \mathbb{N} \tag{A.15}$$

From these sequences we pick out the poles,

$$-ib + ia_+/2 + z_{0,2} \tag{A.16}$$

$$ib - ia_+/2 - z_{0,1} \tag{A.17}$$

The first of these has the property that when its imaginary part is $> a_-$, the same is true of all the other poles in (A.14). Analogously, when the second has imaginary part < 0, the

[13]We claim that by studying each of the cases (ν, ν') separately without this assumption - a task which is somewhat tedious - one does not improve on this restriction, so there is no shrinking of the b interval because of it.

same is true of all the other poles in (A.15). Thus by ensuring these two restrictions we drive out all the shared poles from $i[0, a_-]$. It is not hard to see they are both equivalent to $b < a_- + a_+/2$. $\square$

A.2 Proof of Lemma 4.4.

At issue is the function

$$\hat{J}_\delta(y, x, x') = \hat{w}(y)\big[\hat{\psi}(\delta x, y - ia_-)\hat{\psi}(\delta x', -y) + \hat{\psi}(\delta x, y)\hat{\psi}(\delta x', -y + ia_-)\big], \quad \delta = +, - \tag{A.18}$$

where all the objects here have an implicit dependence on the parameters (a_+, a_-, b), and where

$$\hat{w}(y) \equiv \frac{s_+(y - ib)}{s_+(y)} w(y) \tag{A.19}$$

Here, $w(y)$ is the weight function (1.28); $\hat{\psi}(x, y)$ (6.24) is the reduced version of $\psi(x, y)$ (1.60); and the quotient term derives from the AΔO $\mathcal{S}(y)$ (1.67).

From Lemma 3.1 it is clear that (A.18) is smooth in x, x' for both choices of δ given the restriction $b \in \mathbb{R} \setminus \tilde{S}$ (1.73), (which is implied by the stronger $b \in \mathbb{R} \setminus \mathcal{Y}$ we were assuming throughout §6 for definedness of the maps $\mathcal{F}$ (6.2) and $\mathcal{F}^*$ (6.8)). The only points in $\tilde{S}$ that can lie in $b \in (0, a_+ + a_-)$ are $(l + 1)a_- + a_+/2, l \in \mathbb{N}$ which is why they are excluded in the lemma. With $b \in \mathbb{R} \setminus \tilde{S}$ in force we proceed to consider the analyticity properties in y of (A.18).

We claim that for both choices of δ, the only y-poles of (A.18) in the strip $\operatorname{Im} y \in [0, a_-]$ arise from the two functions

$$s_-(ib \pm y)^{-1}, \tag{A.20}$$

provided $b \in (0, a_+ + a_-)$. The poles of these functions are simple and given, respectively, at

$$\pm\, y = ib + ina_-, \quad n \in \mathbb{Z} \tag{A.21}$$

We can see there are precisely two points in these sequences that lie in the strip $\operatorname{Im} y \in [0, a_-]$. Provided b is not a positive integer multiple of a_- these are away from the boundary, and are exactly those described in the lemma.

The claim described above can be broken up into two further lemmas. The first isolates the algebraic aspect, the second the analytic.

Lemma A.3. *Any y-pole of $\hat{J}_\pm(y, x, x')$ (A.18) is a y-pole of one of the three functions,*

$$s_-(ib \pm y)^{-1}, \tag{A.22}$$

$$\hat{w}(y)\mathcal{R}_r(x, y - ia_-)\mathcal{R}_r(x', y) \tag{A.23}$$

Lemma A.4. *For x, x' any non-singular values, the function (A.23) is holomorphic in y in the strip $\operatorname{Im} y \in [0, a_-]$ provided $b \in (0, 2a)$.*

Proof of Lemma A.3. Adapting (A.4) for $\hat{\psi}(x,y)$ (6.24) we have, for $\delta = +,-$,

$$\hat{\psi}(\delta x, \pm y) = \tilde{w}(x)^{1/2}(2s_-(ib \mp y))^{-1} \sum_{\nu=+,-} \delta\nu e_-(\delta\nu(ib \mp y)/2)\mathcal{R}_r(x + i\nu a_+/2, y) \quad \text{(A.24)}$$

By applying a shift in y we then get, using antiperiodicity of $s_-(\cdot)$ in ia_- and $e_-(\pm i\delta\nu a_-/2) = \pm i\delta\nu$,

$$\hat{\psi}(\delta x, \pm(y - ia_-)) = \tilde{w}(x)^{1/2}(-2s_-(ib \mp y))^{-1} \sum_{\nu=+,-} (\pm i)e_-(\delta\nu(ib \mp y)/2)\mathcal{R}_r(x + i\nu a_+/2, y - ia_-) \quad \text{(A.25)}$$

Substituting these last two equations into (A.18) we get straight away, for $\delta = +,-$,

$$\hat{J}_\delta(y, x, x') \propto \hat{w}(y)(s_-(ib - y)s_-(ib + y))^{-1} \sum_{\nu,\nu'=+,-} \delta e_-(\delta\nu(ib - y)/2)e_-(\delta\nu'(ib + y)/2)$$
$$\times \left[\nu'\mathcal{R}_r(x + i\nu a_+/2, y - ia_-)\mathcal{R}_r(x' + i\nu' a_+/2, y) - (x, x', \nu, \nu') \to (x', x, \nu', \nu)\right] \quad \text{(A.26)}$$

where we are of course dealing with proportionality with respect to y. The unimportant proportionality constant is

$$-i\tilde{w}(x)^{1/2}\tilde{w}(x')^{1/2}/4 \quad \text{(A.27)}$$

The shifts on x, x' in the square brackets in (A.26) do not affect the structure of the y-poles. Since the exponentials in (A.26) are entire, we see that the y-poles can only come from the functions claimed (which we note are ν, ν'-independent).

□

Proof of Lemma A.4. It is natural to analyse the poles of $\hat{w}(y)$ (A.19) and those of the product of $\mathcal{R}_r$-functions separately. As regards the former, we use (1.37) to write

$$\hat{w}(y) = 4s_+(y - ib)s_-(y) \prod_{\alpha=+,-} G(\alpha y - ia + ib) \quad \text{(A.28)}$$

Using (3.42) we can see the y-poles of the G-product in (A.28) are given by

$$\pm y = ib + z_{k,l} \quad \text{(A.29)}$$

However, the $l = 0$ terms in the upwards sequence $(+y)$ will be removed by the zeros of $s_+(y - ib)$ at $ib + ina_+, n \in \mathbb{N}$, such that those of $\hat{w}(y)$ (A.28) are given by

$$\begin{cases} +y = ib + z_{k,l+1} \\ -y = ib + z_{k,l} \end{cases} \quad k, l \in \mathbb{N} \quad \text{(A.30)}$$

With our standing assumption that $a_+, a_- > 0$, we see that all these poles lie outside $i[0, a_-]$ provided $b > 0$.

Dealing with the product of $\mathcal{R}_r$-functions in (A.23) is not so straightforward. Like in the previous subappendix, if we tried to banish all of its y-poles directly, this would lead to an inequality on b more restrictive than the one proposed. We get around this by observing that all these poles are y-zeros of $\hat{w}(y)$. This does not in itself imply these poles are removed, but one can see this algebraically level by analogising (3.54)-(3.56). As a result we need only be concerned with the double poles that arise when those in the product overlap - with no corresponding overlap of the zeros of $\hat{w}(y)$. We find that these can be banished from the strip by imposing $b < 2a$.

We first prove the claim that the y-poles of the two $\mathcal{R}_r$-functions in (A.23) are y-zeros of $\hat{w}(y)$ (A.28). Using (A.9) we can see that the b-dependent y-zeros of the G-product in $\hat{w}(y)$ are given by

$$\pm\, y = i(2a-b) + z_{k,l} \tag{A.31}$$

As noted in §1, the y-poles of the function $\mathcal{R}_r(x,y)$ are given at

$$\pm\, y = i(2a-b) + z_{k,l}, \quad k,l \in \mathbb{N} \tag{A.32}$$

And so those of $\mathcal{R}_r(x, y - ia_-)$ by

$$\begin{cases} +y = i(2a-b) + z_{k,l+1} & \text{(A.33)} \\ -y = i(2a-b) + z_{k,l} & \text{(A.34)} \end{cases}$$

$$-\, y = -ib + ia_+ + z_{k,0} \tag{A.35}$$

for $k, l \in \mathbb{N}$. The pole sequences (A.32)-(A.34) are all encompassed by the zeros (A.31). Those in (A.35) are removed by the zeros of $s_+(y - ib)$ at $y = ib - ina_+, n \in \mathbb{N}^*$. Thus all of these poles are removable in (A.23).

The upwards poles that are shared by $\mathcal{R}_r(x', y)$ and $\mathcal{R}_r(x, y - ia_-)$ give rise to double poles of the product of $\mathcal{R}_r$-functions in (A.23) which will not be removed by the zeros, and similarly for the downwards. To establish the claim in the lemma, we have to show that these can be banished from the strip $\operatorname{Im} y \in [0, a_-]$ by imposing $b < 2a$.

We can see that (A.33) is contained in (A.32) and thus the former describes all the shared upwards poles. We can also see that (A.34) is contained in (A.32), whereas the a_--independent (A.35) has no overlap with (A.32). Thus (A.34) describes all the shared downwards poles. From amongst these shared poles we pick out for special attention

$$+\, y = i(2a-b) + z_{0,1} \tag{A.36}$$

$$-\, y = i(2a-b) \tag{A.37}$$

The significance of the first is that whenever its imaginary part is $> a_-$, the same is true of all the other poles in (A.33). Analogously for the second, when its imaginary part is < 0, this is also true for all the other poles in (A.34). It is not hard to see that both of these conditions are equivalent to $b < 2a$.

□

B Appendix. Dual dynamics.

A different perspective on $\overline{\mathcal{S}}$.

The main motivation for a Hilbert space theory of the AΔO $S(b;y)$ (1.67) was to complete our account of the dynamics associated to $\tilde{H}$, the Hilbert space version of the AΔO $\tilde{H}(b;x)$ (1.67). At the same time, the results we proved for $\overline{\mathcal{S}}$ (6.9) in §6 and §7 portend a quantum mechanical interpretation of their own. This is what we call the dual dynamics, and it can be viewed completely independently of the dynamics in the main text. Indeed in an alien world with no knowledge of the Ruijsenaars-Schneider system, the function $\psi(b;x,y)$ could be of interest primarily for its role in the dynamics outlined here.

We recall that Corollary 7.1 tells us $\overline{\mathcal{S}}$ is symmetric on its domain $\mathcal{F}^*(\mathcal{C})$, given a restriction on the coupling parameter: $b \in (0, a_-)$. It is not hard to see from (6.6) and (6.9) that the necessary bound exists for us to strengthen this using Nelson's theorem; for any $f \in \mathcal{C}$ there is a constant c such that

$$\|\overline{\mathcal{S}}^n \mathcal{F}^* f\|_{\hat{\mathcal{H}}} \leq c^n \|\mathcal{F}^* f\|_{\hat{\mathcal{H}}}, \quad n \in \mathbb{N} \tag{B.1}$$

(cf. §5.1 for details). Thus we have straight away

Theorem B.1. *Assume the coupling parameter b satisfies $b \in (0, a_-)$. Then, the operator $\overline{\mathcal{S}} : \mathcal{F}^*(\mathcal{C}) \to \mathcal{F}^*(\mathcal{C}) \subset \hat{\mathcal{H}}$ defined by* (6.9) *is essentially self-adjoint in the closure of $\mathcal{F}^*(\mathcal{C})$.*

By mimicking the procedure in §5.2, we can prove the existence and form of the wave operators associated to this system fairly easily. In doing so we find the system to be reflectionless and without bound states, i.e. $\hat{\mathcal{H}}$ consists entirely of scattering states. With the wave operators in place, there are two routes to the latter claim; one of them uses the results in §5.2 (and in doing so, makes the dual dynamics dependent on the dynamics for $\tilde{H}$) whilst the other is self-contained and analogises the argument in §7 (making it dependent on symmetry of $\tilde{H}$ but not its dynamics).[14]

Before using Theorem B.1 to define a one-parameter unitary group for $\overline{\mathcal{S}}$, we should make a provisional extension of the latter to the orthocomplement of the closure of $\mathcal{F}^*(\mathcal{C})$, where it is formally undefined. With the extension defined as an arbitrary, bounded self-adjoint operator (and denoted by the same symbol) we thus have a unitary one-parameter group on $\hat{\mathcal{H}}$,

$$\exp(it\overline{\mathcal{S}}), \quad t \in \mathbb{R} \tag{B.2}$$

We will later prove that $\mathcal{F}^*$ is isometric and onto for this range of b.[15] It follows that $\mathcal{F}^*(\mathcal{C})$ is dense in $\hat{\mathcal{H}}$, and so $\overline{\mathcal{S}}$ is densely-defined in this Hilbert space and, as an essentially self-adjoint operator, has absolutely continuous spectrum $(-\infty, \infty)$ of multiplicity one. Thus the extension happens to be unnecessary, but we cannot claim to know this yet.

As usual we need a free motion with which to compare this interacting motion. This involves a dual Hamiltonian $\hat{H}_0$ analogous to H_0 (5.8) for which we need the adjoint Fourier transform $\mathcal{J}^* : \mathcal{H} \to \hat{\mathcal{H}}$,

[14] We omit the details of this. The point is that Proposition 7.1 can easily be adapted for $\mathcal{H} \leftrightarrow \hat{\mathcal{H}}$ and applied to $S = \tilde{H}$.

[15] In fact, from Corollary 5.3 we "know" this already. However, this knowledge rests on the dynamics in §5.2 which we are forgetting here.

$$(\mathcal{J}^* f)_\delta(x) = c \sum_{\delta=+,-} \int_{\mathbb{R}} dx\, \exp(-i\pi\delta xy/a_+a_-) f(x), \quad c \equiv 1/\sqrt{2a_+a_-} \tag{B.3}$$

The operator $\hat{H}_0$ is defined by

$$\hat{H}_0 = \mathcal{J}^* M \mathcal{J} \tag{B.4}$$

where, again, M (6.9) denotes unbounded multiplication on $\mathcal{H}$ by $2s_+(x)$. We can consider (B.4) on any space up to $\mathcal{D}(M)$ where $\mathcal{D}(M)$ is the the maximal domain of all functions $f \in \mathcal{H}$ such that $Mf \in \mathcal{H}$. Thus we have a densely-defined operator in $\hat{\mathcal{H}}$,

$$\hat{H}_0 : \mathcal{J}^*(\mathcal{D}(M)) \to \mathcal{J}^*(\mathcal{H}) \tag{B.5}$$

Unitarity of $\mathcal{J}^*$ entails $\hat{H}_0$ is self-adjoint and so we may consider the unitary one-parameter group,

$$\exp(it\hat{H}_0), \quad t \in \mathbb{R} \tag{B.6}$$

Finally, $\hat{H}_0$ (B.5) has an AΔO action,

$$(\hat{H}_0 \hat{f})(y) = \begin{pmatrix} \hat{H}_0(y) f_+(y) \\ -\hat{H}_0(y) f_-(y) \end{pmatrix} \tag{B.7}$$

$$\hat{H}_0(y) \equiv T^y_{ia_-} - T^y_{-ia_-} \tag{B.8}$$

which follows because

$$2s_+(x) e^{-i\pi\delta xy/a_+a_-} = \delta \hat{H}_0(y) e^{-i\pi\delta xy/a_+a_-}, \quad \delta = +, - \tag{B.9}$$

We are now in a position to define wave operators on $\hat{\mathcal{H}}$,

$$\hat{W}_\pm \equiv \operatorname*{s\text{-}lim}_{t\to\infty} \exp(\pm it\overline{\mathcal{S}}) \exp(\mp it\hat{H}_0) \tag{B.10}$$

Our expectation for what these will be involves the following square-root phase function

$$\theta(b;x) \equiv \tilde{\phi}(b)\, [\tilde{c}(b;x)/\tilde{c}(b;-x)]^{1/2} \tag{B.11}$$

where we recall $\tilde{c}(b;x)$ is defined in (1.49), and $\tilde{\phi}(b) \equiv \exp(i\pi b(b-a_-)/2a_+a_-)$ has been seen already in Lemma 1.4. Because of (1.26), we have $|\theta(b;x)|^2 = 1$ for $(b,x) \in \mathbb{R}^2$, as well as

$$\overline{\theta(b;-x)} = \tilde{\phi}(b)^{-2}\, \theta(b;x) \tag{B.12}$$

The significance of $\theta(b;x)$ derives from its role in the large-y asymptotics of the function $\psi(b;x,y)$. This asymptotics, given in Lemma 1.4, may be written as

$$\overline{\psi(b;x,y)} = \theta(b;x)^{-1} e^{-i\pi xy/a_+a_-} + O(e^{-\rho y}), \quad y \to \infty \tag{B.13}$$

where $\rho > 0$ is a constant fixed by a_+, a_-, and the bound represented by O is uniform for x varying over any compact subset of $\mathbb{R}$. (We are interested in the conjugated function here because of its appearance in the transform kernel of $\mathcal{F}^*$, (6.6)).

We are now ready for the first result and its corollary. Having chosen the above definitions carefully, we are able to use exactly the same ideas as in Lemma 5.2 (with the roles of x and y reversed). The corollary is fairly intuitive and we omit the details (they are virtually the same as those in the proof of Corollary 5.3 but with relevant objects interchanged).

Lemma B.2. *Suppose $b \in (0, a_-)$. Then, the following holds for any $\hat{f} \in \mathcal{J}^*(\mathcal{C}) \subset \mathcal{D}(\hat{H}_0)$,*

$$\lim_{t\to\infty} \exp(-it\overline{\mathcal{S}}) \exp(it\hat{H}_0)\hat{f} = \mathcal{F}^*\theta^+\mathcal{J}\hat{f} \tag{B.14}$$

where: θ^+ denotes multiplication by $\theta(b; x)$ (B.11) on functions in $\mathcal{H} \equiv L^2(\mathbb{R}, dx)$; the map $\mathcal{F}^* : \mathcal{C} \to \hat{\mathcal{H}}$ is a restriction of the adjoint of $\mathcal{F}$ (4.1) with action (6.6); and $\mathcal{J}$ is Fourier transform (5.7).

Corollary B.3. *Suppose the coupling parameter b satisfies the condition of the previous lemma. Then the adjoint map $\mathcal{F}^*$ (6.5) is bounded on $\mathcal{H}$ with action (6.6) and is, furthermore, an isometry. Moreover, we have existence of the wave operator $\hat{W}_-$ defined in (B.10) and $\hat{W}_- = \mathcal{F}^*\theta^+\mathcal{J}$.*

Let us now discuss the a priori relationship that exists between $\hat{W}_-$ and $\hat{W}_+$. With this in place, we will be able to obtain the latter from the former by means of an identity satisfied by the transform $\mathcal{F}^*$.

To express this relationship we need the notion of multiplication on function 2-tuples in $\hat{\mathcal{H}}$ by

$$\hat{\mathcal{P}} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{B.15}$$

In other words,

$$(\hat{\mathcal{P}}\hat{f})_\delta(y) = \hat{f}_{-\delta}(y), \quad \hat{f} \in \hat{\mathcal{H}} \tag{B.16}$$

Recalling that the action of $\overline{\mathcal{S}}$ is given by (6.13) we have straight away that

$$(\hat{\mathcal{P}}\overline{\mathcal{S}}\hat{f})_\delta(y) = (\overline{\mathcal{S}}\hat{f})_{-\delta}(y) = -\delta\overline{\mathcal{S}(b; y)}f_{-\delta}(y) = -(\overline{\mathcal{S}}\hat{\mathcal{P}}\hat{f})_\delta(y), \quad f \in \mathcal{F}^*(\mathcal{C}) \tag{B.17}$$

I.e.,

$$\hat{\mathcal{P}}\overline{\mathcal{S}} = -\overline{\mathcal{S}}\hat{\mathcal{P}} \tag{B.18}$$

(In fact this property follows more generally from the intertwining (6.9) whenever $\mathcal{F}^*$ is an isometry; this also involves (B.23) and (B.24) below.) Thus whenever $\overline{\mathcal{S}}$ is essentially self-adjoint it follows that

$$\hat{\mathcal{P}}e^{it\overline{\mathcal{S}}} = e^{-it\overline{\mathcal{S}}}\hat{\mathcal{P}}, \quad t \in \mathbb{R} \tag{B.19}$$

Thus we see that an intimate connection exists between $\hat{\mathcal{P}}$ and time reversal. Adapting this argument we also have

$$\hat{\mathcal{P}}e^{it\hat{H}_0} = e^{-it\hat{H}_0}\hat{\mathcal{P}}, \quad t \in \mathbb{R} \tag{B.20}$$

Before exploring the consequences of this for the wave operators, let us look at $\hat{\mathcal{P}}$ from a different perspective. This is as the dual-level counterpart of the familiar parity operator on $\mathcal{H}$,

$$(\mathcal{P}f)(x) \equiv f(-x), \quad f \in \mathcal{H}, \tag{B.21}$$

$\mathcal{P}$ and $\hat{\mathcal{P}}$ engage in the following intertwining relations,

$$\mathcal{P}\mathcal{J} = \mathcal{J}\hat{\mathcal{P}} \tag{B.22}$$

$$\mathcal{F}^*\mathcal{P} = \hat{\mathcal{P}}\mathcal{F}^* \tag{B.23}$$

(which follow from the forms of the transforms alone; i.e. (B.22) still holds with $\mathcal{J} \to \mathcal{F}$ and vice versa for (B.23).[16]We have presented these specific versions because they are needed in various places below.)

Oddness of $s_-(\cdot)$ entails that on $\mathcal{H}$,

$$\mathcal{P}M = -M\mathcal{P} \tag{B.24}$$

and so 9

$$\mathcal{P}e^{itM} = e^{-itM}\mathcal{P}, \quad t \in \mathbb{R} \tag{B.25}$$

In fact whenever $\mathcal{F}^*$ is an isometry, this property is equivalent to (B.19) (this involves (B.23) and (B.39) below).

As a result of (B.19) and (B.20) it follows straight away that

$$\hat{W}_+ = \hat{\mathcal{P}}\hat{W}_-\hat{\mathcal{P}} \tag{B.26}$$

Thus using Corollary B.3 and (B.22) we have

$$\hat{W}_+ = \hat{\mathcal{P}}\mathcal{F}^*\theta^+\mathcal{P}\mathcal{J} \tag{B.27}$$

A more explicit expression for $\hat{W}_+$ follows because of a transform identity that we can prove explicitly for the term on the rhs of (B.27) (cf. the proof of Theorem B.4 below). This gives a desirable form to the S-operator.

[16]The form we are talking about here is that seen in

$$(\mathcal{K}\hat{f})(x) = \sum_{\delta=+,-}\int_{\mathbb{R}^+} dy\, \Psi(x,y) f_\delta(y)$$

and its formal adjoint,

$$(\mathcal{K}^*f)_\delta(y) = \int_{\mathbb{R}} dx\, \overline{\Psi(\delta x, y)} f(x), \quad \delta = +,-$$

Theorem B.4. *Suppose the coupling parameter b satisfies $b \in (0, a_-)$. Then, the wave operators $\hat{W}_\pm$ (B.10) exist on $\hat{\mathcal{H}} \equiv L^2(\mathbb{R}^+, dy) \otimes \mathbb{C}^2$ and are given by*

$$\hat{W}_- = \mathcal{F}^* \theta^+ \mathcal{J} \tag{B.28}$$

$$\hat{W}_+ = \mathcal{F}^* \theta^- \mathcal{J} \tag{B.29}$$

where $\theta^\pm$ denotes multiplication by $\theta(b; \pm x)$ (B.11) on functions in $\mathcal{H} \equiv L^2(\mathbb{R}, dx)$ This entails an S-operator,

$$\hat{W}_+^* \hat{W}_- = \mathcal{J}^*(-\tilde{u}(b; \cdot))\mathcal{J} \tag{B.30}$$

$$\tilde{u}(b; x) \equiv -\tilde{c}(b; x)/\tilde{c}(b; -x) \tag{B.31}$$

where: $\mathcal{J}$ is Fourier transform (5.7); $\tilde{c}(b; x)$ is the function in (1.49); and $\tilde{u}(b; x)$ has been defined to analogise $u(b; y)$ (1.34). $\tilde{u}(b; x)$ differs from $\theta(b; x)^2$ by a constant phase multiple and in (B.30) it is used for multiplication on $\mathcal{H} \equiv L^2(\mathbb{R}, dx)$.

Proof of Theorem B.4. (B.28) is already given in Corollary B.3. Because of what we said above, circa (B.27), the result for $\hat{W}_+$ follows because of the following transform identity on $\mathcal{C}$,

$$\hat{\mathcal{P}} \mathcal{F}^* \theta^+ \mathcal{P} = \mathcal{F}^* \theta^- \tag{B.32}$$

This follows straightforwardly from the form of $\mathcal{F}^*$ alone, (6.6); we write out

$$\begin{aligned}(\mathcal{F}^* \theta^+ \mathcal{P} f)_\delta(y) &= \int_{\mathbb{R}} dx\, \overline{\psi(\delta x, y)}\, \theta(x) f(-x) \\ &= \int_{\mathbb{R}} dx\, \overline{\psi(-\delta x, y)}\, \theta(-x) f(x) = (\mathcal{F}^* \theta^- f)_{-\delta}(y), \quad f \in \mathcal{C}\end{aligned} \tag{B.33}$$

and so (B.32) follows as claimed (note we have suppressed implicit dependences on b).

Now for the S-operator. From (B.28) and (B.29) we have

$$\hat{W}_+^* \hat{W}_- = \mathcal{J}^* \overline{\theta^-} \mathcal{F} \mathcal{F}^* \theta^+ \mathcal{J} \tag{B.34}$$

From Corollary (B.3) we know that $\mathcal{F}^*$ is an isometry for the b-interval under consideration. This implies $\mathcal{F}\mathcal{F}^* = \mathbf{1}_{\mathcal{H}}$ and so (B.30) follows because of (B.12). To see this with explicit variables we can write out (B.34) as follows,

$$(\hat{W}_+^* \hat{W}_- \hat{f})_\delta(y) = \int_{\mathbb{R}} dx\, e^{-i\delta\pi xy/a_+a_-} \left[\, \overline{\theta(-x)}\, \theta(x)\right](\mathcal{J}\hat{f})(x), \quad \delta = +, - \tag{B.35}$$

Then, from $\theta(x)$ (B.11),

$$\left[\, \overline{\theta(-x)}\, \theta(x)\right] = \tilde{\phi}^{-2}\, \theta(x)^2 = \tilde{c}(b; x)/\tilde{c}(b; -x) \tag{B.36}$$

□

Proof of Lemma B.2. This is mostly just a matter of adapting the proof of Lemma 5.2.

We aim to prove the following equivalent statement to (B.14),

$$\lim_{t\to\infty} \exp(-it\overline{\mathcal{S}})\ \exp(it\hat{H}_0)\mathcal{J}^* f = \mathcal{F}^*\theta^+ f, \quad f \in \mathcal{C} \tag{B.37}$$

In concrete terms this means

$$\lim_{t\to\infty} \|(\mathcal{F}^*\theta^+ - e^{-it\overline{\mathcal{S}}} e^{it\hat{H}_0}\mathcal{J}^*)f\|_{\hat{\mathcal{H}}} = 0, \quad f \in \mathcal{C} \tag{B.38}$$

(we are now suppressing implicit dependences on b). Unitary one-parameter group properties entail the following two intertwining relations on $\hat{\mathcal{H}}$,

$$e^{it\overline{\mathcal{S}}}\mathcal{F}^* = \mathcal{F}^* e^{itM} \tag{B.39}$$

$$e^{it\hat{H}_0}\mathcal{J}^* = \mathcal{J}^* e^{itM} \tag{B.40}$$

Using these we may write the norm in (B.38) as

$$\|(\mathcal{F}^*\theta^+ - \mathcal{J}^*)e^{itM} f\|_{\hat{\mathcal{H}}} \tag{B.41}$$

Since

$$\|\hat{f}\|^2_{\hat{\mathcal{H}}} = \sum_{\delta=+,-} \int_{\mathbb{R}^+} dy\, |f_\delta(y)|^2 \tag{B.42}$$

(recall (6.4)) we see how (B.38) is equivalent to

$$\lim_{t\to\infty} \int_{\mathbb{R}^+} dy\, |(\hat{I}_\delta e^{itM} f)(y)|^2 = 0, \quad f \in \mathcal{C}, \quad \delta = +,- \tag{B.43}$$

$$(\hat{I}_\delta f)(y) \equiv \int_{\mathbb{R}} dx, \left(\overline{\psi(\delta x, y)}\theta(x) - e^{-i\delta\pi xy/a_+a_-}\right) f(x), \quad f : \mathbb{R} \to \mathbb{C} \tag{B.44}$$

(which is such that $c\hat{I}_\pm f = ((\mathcal{F}^*\theta^+ - \mathcal{J}^*)f)_\pm$).

From the Riemann-Lebesgue we already have a pointwise vanishing

$$\lim_{t\to\infty} (\hat{I}_\pm e^{iMt} f)(y) = 0, \quad y \in \mathbb{R}^+, \quad f \in \mathcal{C} \tag{B.45}$$

Thus it is a question of dominated convergence; to interchange the limit and integral in (B.45) we need two dominating functions $F_\pm(y) \in L^1(\mathbb{R}^+, dy)$ and a $t_0 \in [0,\infty)$ such that

$$|(\hat{I}_\delta e^{iMt} f)(y)|^2 \le F_\delta(y), \quad y \in \mathbb{R}^+, \quad t \in [t_0, \infty), \quad \delta = +,- \tag{B.46}$$

These dominating functions will be piecewise constructions on two different intervals, $(0,R)$ and (R,∞), where $R > 0$ is to be fixed in due course. (Unlike in the proof of Lemma 5.2 these two functions are qualitatively different for different choices of $\delta = +,-$.)

As a result of smoothness of $\psi(x,y)$ in both variables (recall (3.3)) the integral functions in (B.44) are defined and bounded on $(0,R)$, and so a dominating functions is trivial to write down.

For the second interval we require decay rather than boundedness. For this the asymptotics (B.13) are central. They entail that

$$\overline{\psi(\pm x, y)}\,\theta(x) = \psi^{\pm}_{as}(x,y) + O(e^{-\rho y}), \quad y \to \infty \tag{B.47}$$

$$\psi^{\delta}_{as}(x,y) \equiv \begin{cases} e^{-i\pi xy/a_+a_-}, & \delta = + \qquad \text{(B.48)} \\ (\theta(x)/\theta(-x))\, e^{i\pi xy/a_+a_-}, & \delta = - \qquad \text{(B.49)} \end{cases}$$

where the bound represented by O has the properties described below (B.13). Telescoping as in (5.55) we thus have

$$\frac{1}{2}|(\hat{I}_\delta e^{iMt} f)(y)|^2 \leq \left| \int_{\mathbb{R}} dx\, \big(\psi^{\delta}_{as}(x,y) - e^{-i\delta\pi xy/a_+a_-}\big) e^{2its_+(x)} f(x) \right|^2 + \int_{\mathbb{R}} dx\, |\psi(\delta x, y)\theta(x) - \psi^{\delta}_{as}(x,y)|^2 \cdot |f(x)|^2, \quad \delta = +,- \tag{B.50}$$

From (B.47) it is immediately clear that the second of these integrals can be bounded from above by a function integrable on (y_0, ∞) for some $y_0 \geq 0$. Thus setting $R = y_0$ we are "halfway" to finding a dominating function on (R, ∞) for both choices of δ; it remains to analyse the first integral on the rhs.

For $\delta = +$, the first integral in (B.50) has a vanishing integrand, and so we are in fact "the whole way" to finding a dominating function for this choice of δ.

For $\delta = -$, we must look at the first integral in (B.50) more closely,

$$\int_{\mathbb{R}} dx\, \big((\theta(x)/\theta(-x)) - 1\big) e^{i\pi xy/a_+a_- + 2its_+(x)} f(x) \tag{B.51}$$

We can bound (the modulus squared of) this from above by an integrable function on (R, ∞) for any $t_0 > 0$ by employing an argument analogous to (5.59)-(5.62) (with the roles of x and y reversed). This hinges on the fact we can rewrite the exponential in (B.51) as

$$(\pi i/a_+a_-)^{-1}[y + 2a_- t c_+(x)]^{-1} \partial_x \big(e^{i\pi xy/a_+a_+ + 2its_+(x)}\big) \tag{B.52}$$

The term in square brackets is non-zero because we have $y, a_-, t, c_+(x) > 0$. Thus performing integration by parts (with an intermediate term which vanishes because of f's compact support away from the origin) we can readily find a bound as described.

□

Chapter 2

The special case $b = (N+1)a_+$.

1 Introduction

In this part of the thesis we look at how a Hilbert space theory can be constructed independently of the previous results for a special case of the coupling parameter b. This is the case when b equals an integer multiple of a_+,

$$b = b_N \equiv (N+1)a_+, \quad N \in \mathbb{N} \equiv \{0, 1, \ldots\} \tag{1.1}$$

The theory that follows will still be based on the function $\psi(b; x, y)$ that was central to our earlier work. But now, we can exploit some very different properties that it has under the specialisation (1.1). For one thing, its component parts become elementary. In addition, it can be broken up into two components which have quasiperiodicity in x and y. This may seem like an obscure property, but it enables a totally different approach to the construction of a Hilbert space theory. (The idea to look for quasiperiodicity comes from [20], where it exists for reflectionless eigenfunctions and plays a crucial role in the isometry proof of the corresponding eigenfunction transforms.)

Our goal is once again to prove self-adjoint dynamics, complete with a description of Hilbert space in terms of scattering and bound states (i.e. continuous and discrete spectrum). But now, the process by which we arrive at these things is reversed. Let us expand on what we mean by this. In Chapter 1, the eigenfunction transform $\mathcal{F}$ played a central role in the Hilbert space theory. It was used to define the Hilbert space version of $\tilde{H}(b; x)$, namely $\tilde{H}_{\mathrm{ac}}$, which we proved was symmetric and, later, essentially self-adjoint. This was done without knowledge that $\mathcal{F}$ was isometric, or even bounded. These properties were proved eventually, but only by exhibiting $\mathcal{F}$'s relationship to the wave operators (any consideration of which *requires* self-adjointness).

For the b_N-specialised versions of $\mathcal{F}$, which we call $\mathcal{F}_N$, we are able to prove isometry directly (with no mention of an AΔO). This is the main preoccupation of Section 3.1. For the b_N-specialised versions of $\tilde{H}_{\mathrm{ac}}$, which we call $\tilde{H}_{\mathrm{ac},N}$, symmetry and self-adjointness follow easily once we have isometry. This is because the intertwining satisfied by $\tilde{H}_{\mathrm{ac},N}$ can then be read as an explicit rendering of unitary equivalence to the unbounded self-adjoint multiplication operator $\hat{M}$ (cf. §3.3 for details).

The proof of orthogonality and completeness is also different in the specialised setting. It exploits the fact we can also tackle directly the isometry properties of the *adjoint* eigenfunction transform. More specifically, isometry *breakdown* of the adjoint is connected to the presence of bound states (we find there can be at most one bound state in this regime). In fact we can give an illuminating description of how isometry breaks down for both $\mathcal{F}_N$ and $\mathcal{F}_N^*$, cf. §5 (something which we cannot do in the general b case).

We note that isometry of the specialised eigenfunction transforms $\mathcal{F}_N$ is not proved for all values of the parameters a_+, a_-. These must be restricted in a way that corresponds exactly with the parameter restrictions in Chapter 1, and summarised in Chapter 0. Thus, the physics that derives from the new approach is no more than what we get from Part I.

An important feature about AΔOs, which we have alluded to already, can be seen explicitly in the special case. If we look at how $\tilde{H}(b;x)$ specialises for (1.1) we see that

$$\tilde{H}_N(x) \equiv \tilde{H}(b_N;x) = T^x_{ia_-} + T^x_{-ia_-} \tag{1.2}$$

In other words, this appears to be a *free* AΔO. Accordingly, the claim that we are going to build a Hilbert space theory with non-trivial dynamics, bound states, scattering and so on, may seem surprising. The point is that these things all flow from the special choice of operator *domain* (which comes by way of the intertwining definition of $\tilde{H}_{\mathrm{ac}}$)).

In §3.1 and §4.1 (which address isometry of $\mathcal{F}_N$ and $\mathcal{F}_N^*$ respectively) we break up the critical steps into a series of propositions. As per our convention, these are presented in terms of minimal assumptions (which are satisfied by the objects at hand). This has the practical advantage of isolating the conceptual core of any given step (which is useful not least because the eigenfunctions $\psi_N(x,y)$ are somewhat unwieldy). But beyond this, it opens up the possibility that other eigenfunction transforms might be found that also satisfy the assumptions. Indeed this was realised in the author's joint paper [32].

This part of the thesis is intended to be as self-contained as possible, which is why we have reset the section numbers. Any references to Chapter 1 will be clearly indicated.

Let us make one technical remark about certain constants that feature in this chapter (though not explicitly until §5). In Chapter 1, an important role was played by the constant m_b defined to the largest integer such that $m_b a_- < b$. This reappears in the present chapter as m_{N+1}. In addition there are other constants $m_1, \ldots, m_N$ and $n_0, \ldots, n_N$, for any given N. These are all fixed by the value of the ratio a_+/a_- in a particular way, and they do not have an analogue in Chapter 1. The n_j-kind arise here because the x-integrand of the following object, which is the focus of study in §3.1,

$$(\mathcal{F}_N \hat{f}, \mathcal{F}_N \hat{g})_{\mathcal{H}} \tag{1.3}$$

is found to have $2(N+1)$ intractable poles in the strip $\operatorname{Im} x \in [0, a_-]$ at points x_j^+ depending on n_j, cf. (5.1). This contrasts to the situation for

$$(\tilde{H}_{\mathrm{ac},N}\mathcal{F}_N \hat{f}, \mathcal{F}_N \hat{g})_{\mathcal{H}} - (\mathcal{F}_N \hat{f}, \tilde{H}_{\mathrm{ac},N}\mathcal{F}_N \hat{g})_{\mathcal{H}} \tag{1.4}$$

(manifestly relevant to symmetry of $\tilde{H}_{\mathrm{ac},N}$) which was studied for general b in §4 of Chapter 1. The x-integrand implicit in this object was found to have no poles in the strip for certain parameter restrictions (corresponding here to $n_N = 0$). Thus we see that the task of showing (1.3) equals $(\hat{f}, \hat{g})_{\hat{\mathcal{H}}}$ involves residue concerns that are not present in showing that (1.4) vanishes. This is one way in which the present chapter is more difficult than the previous one; the extra difficulty is captured in the comparison of Proposition 3.1 in this chapter to Proposition 4.2 in Chapter 1.

The constants $m_1, \ldots, m_{N+1}$, defined circa (5.51), arise analogously in the calculation of

$$(\mathcal{F}_N^* f, \mathcal{F}_N^* g)_{\hat{\mathcal{H}}} \tag{1.5}$$

In short, $\{0, \ldots, 0\}$ corresponds to unitarity of $\mathcal{F}_N$, and $\{0, \ldots, 0, 1\}$ to a one-dimensional bound state subspace, i.e. $\mathcal{F}_N(\hat{\mathcal{H}})^\perp$ as one dimensional. Isometry of $\mathcal{F}_N$ hinges on $\{n_j\}$ and requires $\{0, \ldots 0\}$. (The fact that 0 is the most important value of these constants is why we can suppress their explicit appearance until §5.)

2 The specialised functions $\psi_N(x,y)$.

Definition and evaluation.

We write the $b = b_N \equiv (N+1)a_+$ specialisations of $\psi(b;x,y)$ from Chapter 1 as follows

$$\psi_N(x,y) \equiv \psi(b_N;x,y), \quad N \in \mathbb{N} \equiv \{0,1,\ldots\} \tag{2.1}$$

To compute the rhs of this we need various formulas from Chapter 1, as well as a formula for the corresponding specialisation of the relativistic conical function (which we take directly from [5]). Once we have obtained our working expression for $\psi_N(x,y)$, references to Chapter 1 are minimal. Indeed, one does not need to know how this family of functions derives from $\psi(b;x,y)$ in order to follow the story. Accordingly, the reader can skip to (2.6).

As claimed already, the component parts of $\psi(b;x,y)$ become elementary under this specialisation. Central to this is the AΔE for the hyperbolic gamma function, (1.1) in Chapter 1, which gives us the following

$$G(z+ib_N) = G(z)\prod_{j=0}^{N} 2c_-(x+i(j+1/2)a_+), \quad N \in \mathbb{N} \tag{2.2}$$

With this we can see (recalling (1.25), (1.48) and (1.49) from Chapter 1),

$$1/c(b_N;-y) = \frac{G(y-ia+ib_N)}{G(y-ia)} = \prod_{j=0}^{N}(-2i)s_-(y+ija_+) \tag{2.3}$$

$$\begin{aligned}\tilde{w}(b_N;x) &= \frac{G(x+ia_-/2)}{G(x-ia_-/2)}\,\frac{G(x-ia_-/2+i(N+1)a_+)}{G(x+ia_-/2-i(N+1)a_+)} \\ &= \prod_{j=0}^{N} 4s_-(x+i(j+1/2)a_+)s_-(x-i(j+1/2)a_+)\end{aligned} \tag{2.4}$$

From [5](4.8)-(4.12) we then have

$$\begin{aligned}\mathcal{R}_r(b_N;x,y) = (-i)^{N+1}\big(&\Sigma_N(x,y)\exp(i\pi xy/a_+a_-) \\ &- \Sigma_N(x,-y)\exp(-i\pi xy/a_+a_-)\big)\Big/\prod_{j=-N}^{N} 4s_-(x+ija_+)s_-(y+ija_+)\end{aligned} \tag{2.5}$$

where $\Sigma_N(x,y)$ is an entire function defined and discussed below.

Bringing (2.1)-(2.5) together we have

$$\begin{aligned}\psi_N(x,y) = (-)^N i^{N+1} w_N(x)^{1/2} v_N(y) \\ \times \sum_{\nu=+,-} 2\nu s_-(x - i\nu(N+1/2)a_+) e_-(\nu(ib_N - y)/2) \\ \times \Big(\exp(i\pi xy/a_+a_-) e_-(-\nu y/2)\Sigma_N(x + i\nu a_+/2, y) - (y \to -y)\Big)\end{aligned} \tag{2.6}$$

where

$$w_N(x) \equiv 1/\prod_{j=0}^{N} 4s_-(x + i(j+1/2)a_+)s_-(x - i(j+1/2)a_+) \tag{2.7}$$

$$v_N(y) \equiv 1/\prod_{j=1}^{N+1} 2is_-(y - ija_+) \tag{2.8}$$

We note these are the only non-entire x, y-functions on the rhs of (2.6). We also have

$$\Sigma_N(x,y) \equiv \sum_{k,l=0}^{N} c_{k,l}^{(N)} e_-((N-2k)x)e_-((N-2l)y) \tag{2.9}$$

where the coefficients $c_{k,l}^{(N)}$ are Laurent polynomials in $e_-(ia_+)$ with integer coefficients (cf. [18] for a full definition). Relevant to us is the fact they satisfy $c_{k,l}^{(N)} = c_{l,k}^{(N)} = c_{N-k,N-l}^{(N)} = (-)^N \overline{c}_{k,N-l}^{(N)}$, which entail respectively

$$\Sigma_N(x,y) = \begin{cases} \Sigma_N(y,x) & (2.10) \\ \Sigma_N(-x,-y) & (2.11) \\ (-)^N\,\overline{\Sigma_N(x,-y)} & (2.12) \end{cases}$$

These coefficients are also such that

$$\sum_{l=0}^{N} c_{0,l}^{(N)} e_-((N-2l)z) = \prod_{l=1}^{N} 2s_-(z + ila_+) \tag{2.13}$$

(The lhs of this appears in the lead asymptotics of Σ_N, cf. §2.3.)

When studying $\psi_N(x,y)$ it is useful to write it in a form which isolates the plane wave structure,

$$\psi_N(x,y) = w_N(x)^{1/2} v_N(y) \sum_{\tau=+,-} \exp(i\tau\pi xy/a_+a_-)\ell_N^\tau(x,y) \tag{2.14}$$

where

$$\begin{aligned}\ell_N^\tau(x,y) \equiv (-)^N i^{N+1}\,\tau \sum_{\nu=+,-} 2\nu s_-(x - i\nu(N+1/2)a_+) \\ \times\, e_-(\nu(ib_N - y)/2)e_-(-\nu\tau y/2)\Sigma_N(x + i\nu a_+/2, \tau y), \quad \tau = +,-\end{aligned} \tag{2.15}$$

This entire function has several important properties which can all be verified from the definitions with a little work,

$$\overline{\ell_N^\tau(x,y)} = \ell_N^\tau(x,-y) \tag{2.16}$$

$$\ell_N^\tau(x+ia_-,y) = (-)^{N+1}\ell_N^\tau(x,y) \tag{2.17}$$

$$\ell_N^\tau(x,y+ia_-) = \tau(-)^{N+1}\ell_N^\tau(x,y) \tag{2.18}$$

Indeed it these quasiperiodicity properties combined with (2.14) which are key to the direct isometry proof.

In the course of the proofs it will be useful to write $\psi_N(x,y)$ in terms of

$$K_N(x,y) \equiv \exp(i\pi xy/a_+a_-)\Sigma_N(x,y) \tag{2.19}$$

This function inherits the properties (2.10) and (2.11),

$$K_N(x,y) = K_N(y,x) \tag{2.20}$$

$$K_N(x,y) = K_N(-x,-y) \tag{2.21}$$

In addition we know from [18] that it satisfies the AΔE,

$$\begin{aligned} s_-(x-iNa_+)K_N(x+ia_+,y) + s_-(x+iNa_+)K_N(x-ia_+,y) \\ = 2s_-(x)c_-(y)K_N(x,y) \end{aligned} \tag{2.22}$$

From [18] we also get the very important specialisations,

$$K_N(\pm ija_+,y) = i^N B_{N-j}^{(N)}(c_-(y)), \quad j=0,\ldots,N \tag{2.23}$$

$$K_N(x,\pm ija_+) = i^N B_{N-j}^{(N)}(c_-(x)), \quad j=0,\ldots,N \tag{2.24}$$

Here $B_k^{(N)}(\cdot)$ is a polynomial of degree k and parity $(-)^k$ with real coefficients. (For completeness we note that the degree-k property is in fact only valid if

$$ja_+/a_- \neq \{1,2,\ldots\}, \quad \forall j=1,\ldots,2N \tag{2.25}$$

We also note that with just one of the four formulas in (2.23) and (2.24), the other three follow from (2.20) and (2.21).) We note the special case

$$i^N B_0^{(N)}(\cdot) = \prod_{j=N+1}^{2N} 2s_-(ija_+) \tag{2.26}$$

We proceed to illustrate why it can be advantageous to work with K_N instead of Σ_N. From the definition (2.19) it easily follows that

$$K_N(x+i\nu a_+/2,\tau y) = \exp(i\tau\pi xy/a_+a_-)e_-(-\nu\tau y/2)\Sigma_N(x+i\nu a_+/2,\tau y), \quad \nu,\tau=+,- \tag{2.27}$$

This means that if we define

$$\lambda_N^\tau(x,y) \equiv \exp(i\pi\tau xy/a_+a_-)\ell_N^\tau(x,y), \quad \tau=+,- \tag{2.28}$$

then comparison of (2.27) with (2.15) makes it clear that

$$\lambda_N^\tau(x,y) = (-)^N i^{N+1}\,\tau \sum_{\nu=+,-} 2\nu s_-(x-i\nu(N+1/2)a_+) \\ \times e_-(\nu(ib_N-y)/2)K_N(x+i\nu a_+/2,\tau y) \tag{2.29}$$

which is a little simpler than the corresponding expression for ℓ_N. All the same, there are occasions when we choose to work with Σ_N rather than K_N, because the formulas involved are more intuitive when the plane waves are visible rather buried in K_N. We note here the $\lambda_N^\pm$-analogue of (2.14),

$$\psi_N(x,y) = w_N(x)^{1/2}v_N(y)\sum_{\tau=+,-}\lambda_N^\tau(x,y) \tag{2.30}$$

2.1 The functions M_N and Λ_N

The reader is advised to skip this subsection on a first reading. It introduces and studies an entire function M_N, and its close friend Λ_N. First,

$$M_N^{\tau,\tau'}(x,x';y,y') \equiv \sum_{\alpha=+,-}\ell_N^{\alpha\tau}(\alpha x,y)\ell_N^{\alpha\tau'}(\alpha x',y'), \quad \tau,\tau'=+,- \tag{2.31}$$

This function has several "surface" symmetries that can be read off straight away, as well as several "deep" symmetries that derive from properties of the components of $\ell_N^\tau(x,y)$. Two examples of the former are

$$M_N^{\tau,\tau'}(x,x';y,y') = \begin{cases} M_N^{\tau',\tau}(x',x;y',y) & (2.32) \\ M_N^{-\tau,-\tau'}(-x,-x';y,y') & (2.33)\end{cases}$$

To find the "deep" symmetries there is a way to express (2.31) using (2.15) which is particularly useful. We begin with the following expression, which is just (2.31) with (2.15) substituted in,

$$(-)^{N+1}\tau\tau'\sum_{\alpha,\nu,\nu'=+,-}4\nu\nu' s_-(\alpha x-\nu\eta_N)s_-(\alpha x'-\nu'\eta_N)e_-(\nu(\xi-y)/2)e_-(\nu'(\xi-y')/2) \\ \times e_-(-\nu\alpha\tau y/2)e_-(-\nu'\alpha\tau'y'/2)\Sigma_N(\alpha x+i\nu a_+/2,\alpha\tau y)\Sigma_N(\alpha x'+i\nu' a_+/2,\alpha\tau'y'), \quad \tau,\tau'=+,- \tag{2.34}$$

$$\eta_N \equiv i(N+1/2)a_+, \quad \xi\equiv i(N+1)a_+ \tag{2.35}$$

If we now relabel $\nu\to\alpha\nu$, $\nu'\to\alpha\nu'$, as permitted under $\sum_{\nu,\nu'}$, and use oddness of $s_-(\cdot)$ and the symmetry (2.11) we find

$$M_N^{\tau,\tau'}(x, x'; y, y') = (-)^{N+1}\tau\tau' \sum_{\nu,\nu'=+,-} 4\nu\nu' s_-(x - \nu\eta_N)s_-(x' - \nu'\eta_N)\mu_{\nu\nu'}(y, y')$$
$$\times\, e_-(-\nu\tau y/2)e_-(-\nu'\tau' y'/2)\Sigma_N(x + i\nu a_+/2, \tau y)\Sigma_N(x' + i\nu' a_+/2, \tau' y'), \quad \tau, \tau' = +, - \tag{2.36}$$

where

$$\mu_{\nu\nu'}(y, y') \equiv \begin{cases} \sum_{\alpha=+,-} e_-(\alpha(y + y')/2)e_-(-i\alpha b_N), & \nu\nu' = + \\ \sum_{\alpha=+,-} e_-(\alpha(y - y')/2), & \nu\nu' = - \end{cases} \tag{2.37}$$

The only part of this rewriting which is not manifest is

$$\sum_{\alpha=+,-} e_-(\alpha\nu(\xi - y)/2)e_-(\alpha\nu'(\xi - y')/2) = \mu_{\nu\nu'}(y, y'), \quad \nu, \nu' = +, - \tag{2.38}$$

but this is seen to be true with a little thought .

Since the above is a little dense, let us we expand on what has happened in the passage from (2.31) to (2.36) (the indices τ, τ' are considered fixed throughout). We learn from our starting expression (2.34) that each of the α-summands in (2.31) consists of four additive components, corresponding to the four choices of $\nu, \nu' = +, -$. What is not clear in (2.34), but revealed by $s_-(\cdot)$'s oddness and the Σ_N-symmetry, (2.11), is that for the two choices of $\alpha = +, -$ these four components are actually very similar. By relabelling, we bring the similar terms together and factorise; these similar terms differ only in the α-summand in (2.38).

The point of (2.36) is to find symmetries of M_N not manifest from its definition, (2.31). The first of these is

$$M_N^{\tau,\tau'}(x, x'; y, -y) = M_N^{-\tau,-\tau'}(x, x'; -y, y), \quad \tau, \tau' = +, - \tag{2.39}$$

This follows because with $y' = -y$ in (2.36), the y variable occurs only in the pairs τy, $\tau' y$, as well as in $\mu_{\nu\nu'}(y, -y)$. The latter equals

$$\mu_{\nu\nu'}(y, -y) = \begin{cases} 2c_-(i(N + 1)a_+), & \nu\nu' = +, \\ 2c_-(y), & \nu\nu' = - \end{cases} \tag{2.40}$$

which is manifestly even as a function of y.

As a consequence of (2.39), we get a further two,

$$M_N^{\tau,\tau}(x, x; -y, y) = M_N^{\tau,\tau}(-x, -x; -y, y), \quad \tau = +, - \tag{2.41}$$
$$M_N^{\tau,-\tau}(x, -x; -y, y) = M_N^{\tau,-\tau}(-x, x; -y, y), \quad \tau = +, - \tag{2.42}$$

The first of these arises by applying (2.32), (2.33) and (2.39) sequentially. And the second by applying (2.32) and (2.39). They will be significant later on.

In our later analysis of the "residue functions", the proofs are aided by using the $\ell_N \to \lambda_N$ modification of M_N,

$$\Lambda_N^{\tau,\tau'}(x,x';y,y') \equiv \sum_{\alpha=+,-} \lambda_N^{\alpha\tau}(\alpha x,y)\lambda_N^{\alpha\tau'}(\alpha x',y'), \quad \tau,\tau'=+,- \tag{2.43}$$

It readily follows from the definition of λ_N (2.28) that

$$\begin{aligned}&\lambda_N^{\alpha\tau}(\alpha x,y)\lambda_N^{\alpha\tau'}(\alpha x',y')\\ &= \exp(i\pi\tau xy/a_+a_-)\exp(i\pi\tau' x'y'/a_+a_-)\ell_N^{\alpha\tau}(\alpha x,y)\ell_N^{\alpha\tau'}(\alpha x',y'), \quad \alpha,\tau,\tau'=+,-\end{aligned} \tag{2.44}$$

And so

$$\Lambda_N^{\tau,\tau'}(x,x';y,y') = \exp(i\pi\tau xy/a_+a_-)\exp(i\pi\tau' x'y'/a_+a_-)M_N^{\tau,\tau'}(x,x';y,y') \tag{2.45}$$

From this we can see that Λ_N inherits all the symmetries of M_N listed above. Furthermore, we can adapt (2.36) to yield an analogous expression for Λ_N. Specifically, by combining (2.27), (2.36) and (2.45) we get straight away

$$\begin{aligned}\Lambda_N^{\tau,\tau'}(x,x';y,y') = (-)^{N+1}\tau\tau' \sum_{\nu,\nu'=+,-} 4\nu\nu' s_-(x-\nu\eta_N)s_-(x'-\nu'\eta_N)\mu_{\nu\nu'}(y,y')\\ \times K_N(x+i\nu a_+/2,\tau y)K_N(x'+i\nu' a_+/2,\tau' y'), \quad \tau,\tau'=+,-\end{aligned} \tag{2.46}$$

2.2 The functions $w_N(x)$ and $\hat{w}_N(y)$

We first take a closer look at the function $w_N(x)$ (2.7). It immediately follows from oddness and ia_--antiperiodicity of $s_-(\cdot)$ that it is even and ia_--periodic in x. We can also show that it is real-valued and positive for real x by writing it as

$$w_N(x) \equiv 1/\prod_{j=0}^{N} |2s_-(x-i(j+1/2)a_+)|^2 \tag{2.47}$$

If $\tilde{x}$ is a pole of this function, then so are $-\tilde{x}$ and $\tilde{x} \pm ia_-$ by evenness and periodicity respectively. Clearly all poles lie on the imaginary axis.

To examine the poles more closely let us zone in on $s_-(x-i(j+1/2)a_+)^{-1}$ and define x_j^+ to be **the unique x-pole of this function in the period strip** $\mathrm{Im}\, x \in [0,a_-)$. Here, j runs from 0 to N. Since the x-poles of this function are at

$$x = i(j+1/2)a_+ + ina_-, \quad n \in \mathbb{Z}, \tag{2.48}$$

we know x_j^+ will be one of these. In fact, since a_+, a_- are positive, it will correspond to a particular $n \leq 0$.

Defining x_j^- to be **the unique x-pole of** $s_-(x+i(j+1/2)a_+)^{-1}$ **in** $\mathrm{Im}\, x \in [0,a_-)$, we can now express all the poles of $w_N(x)$ in the period strip as $\{x_j^+, x_j^-\}_{j=0}^N$. Provided

x_j^+ is away from the origin, then clearly $x_j^- = ia_- - x_j^+$. This is the case iff the number $(j+1/2)a_+/a_-$ is non-integer.

We now address the question of pole order. Specifically, we want an iff condition for all the poles x_j^+, x_j^- to be simple. Given that the poles of $\sinh(\cdot)^{-1}$ are themselves simple, non-simplicity can arise for $w_N(\cdot)$ only by the overlap of poles from different product terms. Hence it is this possibility that we want to exclude. From (2.48), we see that the pole x_j^+ is going to be simple iff the only solution to the following equation is the trivial one, $\pm = +$, $j' = j$ and $n' = n$,

$$i(j+1/2)a_+ + ina_- = \pm i(j'+1/2) + in'a_- \tag{2.49}$$

where we take $j' = 0, \dots, N$ and $n' \in \mathbb{Z}$ on the rhs in order to express all the poles of $w_N(\cdot)$. This is solved by a fixed j' iff

$$[(j+1/2) \mp (j'+1/2))]a_+/a_- \in \mathbb{Z} \tag{2.50}$$

Since

$$\{(j+1/2) \mp (j'+1/2) \,\big|\, j,j' = 0,\dots,N\} = \{-2N-1,\dots,0,\dots,2N+1\}, \tag{2.51}$$

we conclude that the poles $\{x_j^+\}$ are simple iff the numbers ka_+/a_- are non-integer for all $k = 1,\dots,2N+1$. This also ensures $\{x_j^+\}$ are away from the origin (this is clear from what we said above but also a priori, since if $x_j^+ = 0$ then $x_j^- = 0$. But this would be a case of overlap, of the kind we have just excluded).

We summarise and expand on this in the following lemma.

Lemma 2.1. *The function $w_N(x)$ (2.7) has all simple poles iff the ratio a_-/a_+ is not in the following point set,*

$$\mathcal{E}_N \equiv \left\{ \frac{k}{n} \,\middle|\, k = 1,2,\dots,2N+1,\ n \in \mathbb{N} \right\} \tag{2.52}$$

This condition also ensures that all poles are away from the lines $\operatorname{Im} x/a_- \in \mathbb{Z}$.

Proof. This follows largely from what was said above. We just have to note that the argument above can be easily adapted for x_j^- and that simplicity of $\{x_j^\pm\}$ is clearly equivalent to simplicity of all poles (because of $w_N(\cdot)$'s ia_--periodicity). Likewise, $\{x_j^\pm\}$ away from the origin is equivalent to all poles away from the lines $\operatorname{Im} x/a_- \in \mathbb{Z}$. □

We now introduce the dual weight function $\hat{w}_N(y)$. Although this is yet to appear in our story, we define and study it here because of the obvious similarities to $w_N(x)$. It is defined in terms of $v_N(y)$ (2.8) as follows

$$\hat{w}_N(y) \equiv v_N(y)v_N(-y) = |v_N(y)|^2 = 1/\prod_{j=1}^{N+1} 4s_-(y+ija_+)s_-(y-ija_+) \tag{2.53}$$

To understand the second equality here, note from the definition of v_N that

$$\overline{v_N(y)} = v_N(-y) \tag{2.54}$$

The properties of evenness and real-valuedness and positivity for real y are all manifest from (2.53). As before, ia_--periodicity in y follows from ia_--antiperiodicity of $s_-(\cdot)$.

All the properties we listed above for a generic pole of $w_N(\cdot)$ also hold for a generic pole of $\hat{w}_N(\cdot)$. Furthering the analogy, we formalise our study of the poles by defining y_j^+ to be **the unique y-pole of $s_-(y - ija_+)^{-1}$ in the strip** $\operatorname{Im} y \in [0, a_-)$. Clearly y_j^+ will be given by

$$y = ija_+ + ima_-, \quad m \in \mathbb{Z}, \tag{2.55}$$

for a particular $m \leq 0$. We also **define y_j^- similary for** $s_-(y - ija_+)^{-1}$. Here, j runs from 1 to $N+1$. We have $y_j^- = ia_- - y_j^+$ whenever y_j^+ is away from the origin. This is the case iff ja_+/a_- is non-integer. On the matter of pole order, the argument above can be easily adapted, arriving at the following analogue to (2.51),

$$\{j \mp j' \mid j, j' = 1, \ldots, N+1\} = \{-2N-2, \ldots, 0, \ldots, 2N+2\} \tag{2.56}$$

With this we conclude that the poles $\{y_j^+\}$ are simple iff the numbers ka_+/a_- are non-integer for all $k = 1, \ldots, 2N+2$. This also ensures that $\{y_j^+\}$ are away from the origin.

We summarise and expand on this in the following lemma.

Lemma 2.2. *The function $\hat{w}_N(y)$ (2.53) has all simple poles iff the ratio a_-/a_+ is not in the following point set,*

$$\hat{\mathcal{E}}_N \equiv \left\{ \frac{k}{n} \,\middle|\, k = 1, 2, \ldots, 2N+2,\ n \in \mathbb{N} \right\} \tag{2.57}$$

This condition also ensures that all poles are away from the lines $\operatorname{Im} y/a_- \in \mathbb{Z}$.

Proof. This follows largely from what was said above. We just have to note that the argument there can be easily adapted for y_j^- and that simplicity of $\{y_j^\pm\}$ is clearly equivalent to simplicity of all poles. Likewise, $\{y_j^\pm\}$ away from the origin is equivalent to all poles away from the lines $\operatorname{Im} y/a_- \in \mathbb{Z}$. □

We note one phenomenon here whereby the poles of the weight functions might unexpectedly turn out to be regular values of $\psi_N(x, y)$. This is evident in the case of $\psi_0(x, y)$ in §6 which becomes equal to a plane wave for the exceptional values $a_-/a_+ \in \mathbb{N}^*$ (cf. [32] for more on this).

2.3 Asymptotics.

Here we catalogue the x and y asymptotics of the functions $w_N(x)$, $w_N(y)$, $\Sigma_N(x,y)$ and $\ell_N^{\pm}(x,y)$ that were introduced in §2. In each case, the dominant and subdominant terms are powers of $e_-(\ldots)$. To keep the formulas a little more compact, we will write these terms as $e_-^{\cdots}$. All the results here follow by elementary calculations from the definitions in §2.

We look first at the x asymptotics. The results below assume $x \in \mathbb{R}$. For $x \in \mathbb{C}$ and $\mathrm{Re}\, x \to \pm\infty$, they can all be adapted by just replacing x with $\mathrm{Re}\, x$ when this occurs inside $O(\cdot)$. The implied bound is then uniform for $\mathrm{Im}\, x \in \mathbb{R}$.

$$w_N(x) = e_-^{-2(N+1)|x|} + O(e_-^{-2(N+3)|x|}), \quad x \to \pm\infty \tag{2.58}$$

$$\Sigma_N(x,y) = e_-^{N|x|} s_N(\pm y) + O(e_-^{(N-2)|x|}), \quad x \to \pm\infty \tag{2.59}$$

Here

$$s_N(y) \equiv \prod_{j=1}^{N} 2s_-(y + ija_+) \tag{2.60}$$

arises due to (2.13). It is closely related to the reciprocal of $v_N(-y)$. In fact,

$$2(-)^N i^{N+1} s_N(y) v_N(\alpha y) = \begin{cases} u_N(y)/s_-(y - i(N+1)a_+), & \alpha = + \quad (2.61) \\ 1/s_-(y + i(N+1)a_+), & \alpha = - \quad (2.62) \end{cases}$$

With (2.59) we can deduce

$$\ell_N^\tau(x,y) = (-)^N i^{N+1} \tau s_N(\tau y) e_-^{(N+1)x} \sum_{\nu=+,-} \nu e_-(-\nu(1+\tau)y/2) + O(e_-^{(N-1)x}), \quad x \to \infty, \quad \tau = +,- \tag{2.63}$$

and

$$\ell_N^\tau(x,y) = (-i)^{N+1} \tau s_N(-\tau y) e_-^{-(N+1)x} \sum_{\nu=+,-} \nu e_-(i\nu(N+1)a_+) e_-(-\nu(1+\tau)y/2) + O(e_-^{-(N-1)x}), \quad x \to -\infty, \quad \tau = +,- \tag{2.64}$$

We now consider the y asymptotics, The results below assume $y \in \mathbb{R}$. For $y \in \mathbb{C}$ and $\mathrm{Re}\, y \to \pm\infty$, they can all be adapted by just replacing y with $\mathrm{Re}\, y$ when this occurs inside $O(\cdot)$. The implied bound is then uniform for $\mathrm{Im}\, y \in \mathbb{R}$.

$$\hat{w}_N(y) = e_-^{-2(N+1)|y|} + O(e_-^{-2(N+3)|y|}), \quad y \to \pm\infty \tag{2.65}$$

Because of the symmetry (2.10), the y-asymptotics for Σ_N are just those in (2.59) with $x \leftrightarrow y$. In preparation for the $\ell_N^{\pm}$ asymptotics we note the adaptation,

$$\Sigma_N(x + i\nu a_+/2, \tau y) = e_-^{N|y|} s_N(\epsilon\tau(x + i\nu a_+/2)) + O(e_-^{(N-2)|y|}),$$
$$y \to \epsilon\infty, \quad \epsilon, \nu, \tau = +, - \quad (2.66)$$

The $\ell_N^{\pm}$ asymptotics is best expressed for $\tau = +$ and $\tau = -$ separately. For $\tau = +$, the $\nu = -\epsilon$ term in the sum in (2.15) gives the dominant $y \to \epsilon\infty$ asymptotics,

$$\ell_N^{+}(x, y) = (-i)^{N+1} 2\epsilon s_-(x + i\epsilon(N + 1/2)a_+) e_-(-i\epsilon(N + 1)a_+/2) e_-^{(N+1)|y|}$$
$$\times\, s_N(\epsilon x - ia_+/2) + O(e_-^{(N-1)|y|}), \quad y \to \epsilon\infty, \quad \epsilon = +, - \quad (2.67)$$

For $\tau = -$, it comes from $\nu = \epsilon$. All we need to know about these asymptotics is

$$\ell_N^{-}(x, y) = O(e_-^{N|y|}), \quad y \to \epsilon\infty, \quad \epsilon = +, - \quad (2.68)$$

2.4 Defining the eigenfunction transform $\mathcal{F}_N$

We now use the functions ψ_N (2.6) to define a special class of integral transforms. These act on function pairs of the form $\hat{f} = \langle f_+, f_- \rangle$, according to

$$(\mathcal{F}_N \hat{f})(x) \equiv (2a_+a_-)^{-1/2} \int_0^\infty dy \sum_{\delta=+,-} \psi_N(\delta x, y) f_\delta(y) \quad (2.69)$$

or equivalently, in terms of $\mathcal{F}$ from (4.1) in Chapter 1,

$$\mathcal{F}_N f = \mathcal{F}((N + 1)a_+) f \quad (2.70)$$

The coefficient upfront in (2.69) is necessary to secure certain properties of this transform such as isometry. In general, properties of the transform may depend on the parameters a_+, a_- implicit in ψ_N, in the sense that a property may only hold when the parameters are restricted in some way. Indeed this is what we find in the case of isometry.

Next, we want to consider how this transform acts on functions in the Hilbert space $\hat{\mathcal{H}} \equiv L^2(\mathbb{R}^+, dy) \otimes \mathbb{C}^2$. Provided $\psi_N(\cdot, \cdot)$ has no poles on $\mathbb{R}^2$, the integral function in (2.69) is manifestly convergent if we restrict f to $\hat{\mathcal{C}} \equiv C_0^\infty(\mathbb{R}^+)^2 \subset \hat{\mathcal{H}}$. However, if we want to render $\mathcal{F}_N$ as a map from $\hat{\mathcal{C}}$ into $L^2(\mathbb{R}, dx)$, more is required. Concretely, we need to know that $(\mathcal{F}_N \hat{f})(\cdot)$ has L^2-asymptotics. This, and more, follows because $\mathcal{F}_N$ is essentially a weighted sum of Fourier transforms, in the sense that we now explain.

From the definitions (2.9) and (2.15) we can see that separation of variables obtains in $\ell_N^{\pm}$. In other words, it has the form $\sum_k \ell_k^{\tau}(x) \hat{\ell}_k^{\tau}(y)$ for $\tau = +, -$ for k in some suitable subset of $\mathbb{N}$. By writing out ψ_N using (2.14), $(\mathcal{F}_N \hat{f})(x)$ thus equals

$$w_N(x)^{1/2} \sum_k \sum_{\delta,\tau=+,-} \ell_k^{\tau}(\delta x) \big(F \hat{\ell}_k^{\tau}(\cdot) v_N(\cdot) f_\delta\big)(\tau x) \quad (2.71)$$

where $F : L^2(\mathbb{R}^+) \to L^2(\mathbb{R})$ is the Fourier transform,

$$(Fg)(x) \equiv c \int_0^\infty dy\, e^{i\pi xy/a_+a_-} g(y), \quad c \equiv (2a_+a_-)^{-1/2} \tag{2.72}$$

Whenever the poles of $w_N(\cdot)$ and $v_N(\cdot)$ are off the real axis, the $O(1)$ asymptotics of ψ_N and entirety of $\ell_N^{\pm}$ imply that $w_N(\cdot)^{1/2}\ell_k^{\tau}(\cdot)$ and $v_N(\cdot)\hat{\ell}_k^{\tau}(\cdot)$ are bounded functions on $\mathbb{R}$. Consequently, (2.71) represents a sum of products of three bounded maps, and is thus bounded.

We will formalise this result in the theorem below. But first we note an important fact about ψ_N which has implications for how we think about $\mathcal{F}_N$. Suppose we introduce some dimensionless variables r, k. Then, by writing out $\psi_N(a_-r, a_-k)$ using (2.6), one sees that the parameters a_+ and a_- always occur together in the ratio a_+/a_-. Thus if we consider

$$(\mathcal{F}_N\hat{f})(a_-r) = c \int_0^\infty dk\, a_- \sum_{\delta=+,-} \psi_N(\delta r a_-, k a_-) f_\delta(k a_-) \tag{2.73}$$

we can see that there are grounds for viewing $\mathcal{F}_N$ as a one-parameter family in the space of bounded maps from $\hat{\mathcal{H}}$ to $\mathcal{H}$. In the symbols of functional analysis: $\mathcal{F}_N(\cdot) \in \mathcal{L}(\hat{\mathcal{H}}, \mathcal{H})$.

Lemma 2.3. *Provided the positive parameters a_+, a_- satisfy $a_-/a_+ \notin \mathcal{E}_N$ (2.52), the transform $\mathcal{F}_N$ defined by (2.69) is a bounded map from $\hat{\mathcal{H}}$ into $\mathcal{H}$.*

Proof. The crux of this argument has been given already. Excluding the $\mathcal{E}_N$ values ensures $w_N(\cdot)$ has no pole at the origin, cf. §2.2.

□

3 Isometry of $\mathcal{F}_N$

3.1 Isometry formula

The eigenfunction transform $\mathcal{F}_N$, (2.69), defines a bounded map from $\hat{\mathcal{H}}$ into $\mathcal{H}$. In this subsection we present a formula, Theorem 3.3, that allows us to determine if and when $\mathcal{F}_N$ is an isometry. The question of isometry reduces to an analysis of residues, which is what we take up in the next subsection. We first sketch some basic steps on the way to the theorem and then prove two propositions that carry this forward.

Using the definition (2.69) we write out

$$(\mathcal{F}_N \hat{f}, \mathcal{F}_N \hat{g})_{\mathcal{H}} = c^2 \int_{\mathbb{R}} dx \int_{\mathbb{R}^+} dy \sum_{\delta=+,-} \overline{\psi_N(\delta x, y)}\, \overline{f_\delta(y)} \times \int_{\mathbb{R}^+} dy' \sum_{\delta'=+,-} \psi_N(\delta' x, y') g_{\delta'}(y'), \quad c \equiv (2a_+a_-)^{-1/2} \quad (3.1)$$

The aim is to directly compute the rhs of this and find conditions under which it equals $(f, g)_{\hat{\mathcal{H}}}$. Invoking Fubini's theorem we may rewrite it as[1]

$$c^2 \sum_{\delta,\delta'=+,-} \lim_{\Lambda\to\infty} \int_{\mathbb{R}^+} dy\, \overline{f_\delta(y)} \int_{\mathbb{R}^+} dy'\, g_{\delta'}(y') \int_{-\Lambda}^{\Lambda} dx\, \overline{\psi_N(\delta x, y)} \psi_N(\delta' x, y'). \quad (3.2)$$

The rightmost integral here is independent of f and g, giving us a useful point of study. The "even" integral range $[-\Lambda, \Lambda]$ means we can write it as

$$\int_{-\Lambda}^{\Lambda} dx\, \overline{\psi_N(x, y)} \psi_N(\delta\delta' x, y'), \quad \delta, \delta' = +, - \quad (3.3)$$

and so the four x-integrals that arise in (3.2) (corresponding to the four choices of δ, δ') reduce to two essential kinds,

$$\int_{-\Lambda}^{\Lambda} dx\, \overline{\psi_N(x, y)} \psi_N(\sigma x, y'), \quad \sigma = +, -. \quad (3.4)$$

Now we recall from §2 that ψ_N can be written as

$$\psi_N(x, y) = w_N(x)^{1/2} v_N(y) \sum_{\tau=+,-} e^{i\tau\pi xy/a_+a_-} \ell_N^\tau(x, y). \quad (3.5)$$

And so, invoking evenness of $w_N(\cdot)$ and the conjugacy relations (2.54),(2.16), we get

$$\overline{\psi_N(x, y)} \psi_N(\sigma x, y') = w_N(x) v_N(-y) v_N(y') \times \sum_{\tau,\tau'=+,-} e^{-i\tau\pi xy/a_+a_-} e^{i\tau'\pi xy'/a_+a_-} \ell_N^\tau(x, -y) \ell_N^{\sigma\tau'}(\sigma x, y'), \quad \sigma = +, -, \quad x, y, y' \in \mathbb{R}. \quad (3.6)$$

A primitive for this is obviously out of the question but it turns out we can integrate it over $[-\Lambda, \Lambda]$ using contour methods. Since $w_N(\cdot)$ has poles that extend along the imaginary

[1] Cf. §4 in Chapter 1 for more on this.

axis in both directions, a semi-circular contour is unsuitable. However, particular features of the integrand, revealed by (3.6), make it amenable to a rectangular contour of fixed height, a_-. These are: periodicity of $w_N(\cdot)$ and quasiperiodicity of the τ, τ'-summand. The latter picks up a multiplier $e_+(\tau y)e_+(-\tau' y')$ when shifted by ia_- due to its plane wave structure and (2.17). We abstract these properties in the following proposition, which will enable us to rewrite (3.4) in terms of a residue sum and boundary integrals. (The proposition is phrased in terms of a variable s in order to avoid ambiguity when we later apply it to an integral in y. This way we can set $s = x$ or $s = y$ at leisure.)

Proposition 3.1. *Suppose $W(\cdot)$ is an even and ia_--periodic function which has simple and finitely many poles in the period strip $i[0, a_-] \times \mathbb{R}$ which are away from the boundary; and suppose we have four entire functions $J^{\pm,\pm}(\cdot)$ satisfying*

$$J^{\tau,\tau'}(s + ia_-) = M^{\tau,\tau'} J^{\tau,\tau'}(s), \quad \tau, \tau' = +, - \tag{3.7}$$

where $M^{\tau,\tau'} \in \mathbb{R} \setminus \{0, 1\}$ satisfies $M^{-\tau,-\tau'} = 1/M^{\tau,\tau'}$.

Then, $W(\cdot)$ has an even number of poles in $i[0, a_-] \times \mathbb{R}$ which can be ordered as $s_1, s_2, \ldots, s_{2L}$ such that $s_{L+j} = ia_- - s_j$, $j = 1, \ldots, L$. For any such ordering, define W_j to be the residue of $W(\cdot)$ at s_j. Then, for $\Lambda > 0$ sufficiently large,

$$\begin{aligned}\int_{-\Lambda}^{\Lambda} ds\, W(s) \sum_{\tau,\tau'=+,-} J^{\tau,\tau'}(s) = \sum_{\tau,\tau'=+,-} (1 - M^{\tau,\tau'})^{-1} \Big[2\pi i \sum_{j=1}^{L} W_j [J^{\tau,\tau'}(s_j) + J^{-\tau,-\tau'}(-s_j)] \\ - \Big(\int_{\Lambda}^{\Lambda+ia_-} + \int_{-\Lambda+ia_-}^{-\Lambda} \Big) ds\, W(s) J^{\tau,\tau'}(s) \Big]\end{aligned} \tag{3.8}$$

Proof. We observe that if $\tilde{s}$ is any pole of $W(\cdot)$ then so is $ia_- - \tilde{s}$ by evenness and ia_--periodicity. Furthermore by these same properties, the corresponding residue is minus that at $\tilde{s}$. Thus we have an even number of poles which we may order as the proposition suggests. With this ordering we obtain the useful relation

$$W_{j+L} = -W_j, \quad j = 1, \ldots, L. \tag{3.9}$$

Next, we take $\Lambda > \max\{\mathrm{Re}\, s_j\}_{j=1}^{L}$ and apply Cauchy's theorem for the rectangular contour with vertices at $\Lambda, \Lambda + ia_-, -\Lambda + ia_-, -\Lambda$ in order to deduce

$$\begin{aligned}(1 - M^{\tau,\tau'}) \int_{-\Lambda}^{\Lambda} ds\, W(s) J^{\tau,\tau'}(s) = 2\pi i \sum_{j=1}^{2L} W_j J^{\tau,\tau'}(s_j) \\ - \Big(\int_{\Lambda}^{\Lambda+ia_-} + \int_{-\Lambda+ia_-}^{-\Lambda} \Big) ds\, W(s) J^{\tau,\tau'}(s).\end{aligned} \tag{3.10}$$

It remains to focus on the residue sum. Using $s_{L+j} = ia_- - s_j$, (3.7) and (3.9) we have

$$\sum_{j=1}^{2L} W_j J^{\tau,\tau'}(s_j) = \sum_{j=1}^{L} W_j [J^{\tau,\tau'}(s_j) - M^{\tau,\tau'} J^{\tau,\tau'}(-s_j)]. \tag{3.11}$$

The reflection property of $M^{\tau,\tau'}$ entails $-(1-M^{\tau,\tau'})^{-1}M^{\tau,\tau'} = (1-M^{-\tau,-\tau'})^{-1}$ and so

$$-\sum_{\tau,\tau'=+,-}(1-M^{\tau,\tau'})^{-1}M^{\tau,\tau'}J^{\tau,\tau'}(-s_j) = \sum_{\tau,\tau'=+,-}(1-M^{\tau,\tau'})^{-1}J^{-\tau,-\tau'}(-s_j) \tag{3.12}$$

Thus dividing (3.10) through by $(1-M^{\tau,\tau'})$ and applying $\sum_{\tau,\tau'}$, the claim follows. □

We need one more proposition before we can prove the first theorem of this section. This tells us how the boundary integrals that arise from Proposition 3.1 when it is applied to (3.4) will behave when substituted into (3.2). Like the previous proposition, it is expressed in terms of some minimal assumptions. These may seem baroque but they are all satisfied for the choices $W(x) = w_N(x)$, $m_1^\tau(x,y) = v_N(-y)\ell_N^\tau(x,-y)$ and $m_2^{\tau'}(x,y') = v_N(y')\ell_N^{\sigma\tau'}(\sigma x, y')$ arising from (3.6). (The $J^{\tau,\tau'}$ in this proposition is of the same type as in the previous one.)

Proposition 3.2. *Suppose we have a function*

$$J^{\tau,\tau'}(x,y,y') = \exp(-i\pi\tau xy/a_+a_-)\exp(i\pi\tau' xy'/a_+a_-)m_1^\tau(x,y)m_2^{\tau'}(x,y'), \quad \tau,\tau' = +,- \tag{3.13}$$

where: $m_j^\pm(x,y)$ are entire in x and smooth in $y\in\mathbb{R}$; the product $m_1^\tau(x,y)m_2^{\tau'}(x,y')$ is ia_--periodic in x for both $\tau = +,-$; and the following function is even in x,

$$\sum_{\tau=+,-} m_1^\tau(\tau x,y)m_2^\tau(\tau x,y), \quad x\in\mathbb{C}. \tag{3.14}$$

Suppose, furthermore, that $W(\cdot)$ satisfies the conditions of the previous proposition and, in addition, satisfies the following asymptotics relation with $m_1^\pm$ and $m_2^\pm$ for $\operatorname{Re} x\to\epsilon\infty$,

$$W(x)m_1^\tau(x,y)m_2^{\tau'}(x,y') = \mathcal{M}_{(\epsilon)}^{\tau,\tau'}(y,y') + O(e_-(-\eta|\operatorname{Re} x|)), \quad \epsilon,\tau,\tau' = +,- \tag{3.15}$$

where $\eta > 0$ is some constant, $\mathcal{M}_{(\epsilon)}^{\tau,\tau'}$ is smooth in $y,y'\in\mathbb{R}$. Also, the bound represented by $O(\cdot)$ is uniform for $\operatorname{Im} x, y, y'$ in compact subsets of $\mathbb{R}$. The function represented by $O(\cdot)$ is smooth in $y,y'\in\mathbb{R}$ and may depend on ϵ,τ,τ'. The y'-partial derivative of this function is assumed to inherit these same asymptotics properties.

Then,

$$\sum_{\alpha,\tau=+,-}\alpha\mathcal{M}_{(\alpha\tau)}^{\tau,\tau}(y,y) = 0 \tag{3.16}$$

And, for functions $f,g\in C_0^\infty(\mathbb{R}^+)$,

$$\begin{aligned}(2a_+a_-)^{-1}\lim_{\Lambda\to\infty}\int_{\mathbb{R}^+}dy\,\overline{f(y)}\int_{\mathbb{R}^+}dy'\,g(y')\sum_{\tau,\tau'=+,-}(1-e_+(\tau y)e_+(-\tau' y'))^{-1}\\ \times\left(\int_\Lambda^{\Lambda+ia_-}+\int_{-\Lambda+ia_-}^{-\Lambda}\right)dx\,W(x)J^{\tau,\tau'}(x,y,y') = -\int_{\mathbb{R}^+}dy\,\overline{f(y)}g(y)\sum_{\tau=+,-}\mathcal{M}_{(\tau)}^{\tau,\tau}(y,y)\end{aligned} \tag{3.17}$$

Proof. The first claim follows straightforwardly from (3.15) and the evenness of (3.14) in x. (Note that it means we could equally use $\sum_\tau \mathcal{M}^{\tau,\tau}_{(-\tau)}(y,y)$ on the rhs of (3.17).)

For the main claim, we will study separately the $\tau'=\tau$ and $\tau'=-\tau$ terms in the sum. Those for which $\tau'=-\tau$ are by far the simplest since for this choice, the reciprocal term in (3.17) equals $(1-e_+(\tau(y+y')))^{-1}$, which is non-singular on the pertinent integration region (recall, f,g have compact support away from the origin). At issue is

$$\int_K dydy' \frac{\overline{f(y)}g(y')}{1-e_+(\tau(y+y'))}\left(\int_\Lambda^{\Lambda+ia_-} + \int_{-\Lambda+ia_-}^{-\Lambda}\right) dx\, W(x) J^{\tau,-\tau}(x,y,y'), \quad \tau=+,- \tag{3.18}$$

where we have introduced the compact set $K \equiv \mathrm{supp}(f)\times\mathrm{supp}(g) \subset \mathbb{R}^+\times\mathbb{R}^+$ away from the origin. We proceed to break up the x-integrand using (3.15), where the function represented by $O(\cdot)$ is now given the symbol $\rho^{\tau,\tau'}_{(\epsilon)}(x,y,y')$. For the $\int_\Lambda^{\Lambda+ia_-}dx$ integral, we are interested in $\epsilon=+$. The $O(1)$ x-integrand term can be integrated directly, leading to a contribution proportional to

$$\int_K dydy'\, \overline{f(y)}g(y')\mathcal{M}^{\tau,-\tau}_{(+)}(y,y')\frac{\exp(-i\pi\tau\Lambda(y+y')/a_+a_-)}{-i\pi\tau\Lambda(y+y')/a_+a_-} \tag{3.19}$$

Under our assumptions, the y,y'-integrand is bounded on K and so (3.19) vanishes under $\Lambda\to\infty$ by the Riemann-Lebesgue lemma. The $O(e_-(-\eta\,\mathrm{Re}\,x))$ term cannot be integrated directly but its contribution to (3.18) has the form

$$\int_K dydy' \int_\Lambda^{\Lambda+ia_-} dx\, \phi(x,y,y')\rho^{\tau,-\tau}_{(+)}(x,y,y') \tag{3.20}$$

where $\phi(x,y,y')$ is some function bounded for $(y,y')\in K$ as well as for $\mathrm{Re}\,x\in\mathbb{R}$ and $\mathrm{Im}\,x$ in compacts of $\mathbb{R}$. (These properties readily follow from the assumptions.) Estimating in the obvious way, the modulus of (3.20) is, for Λ sufficiently large, found to be less than or equal to

$$e_-(-2\Lambda)\,\mathcal{C}|K|a_- \sup_{(x,y,y')\in\, i(0,a_-)\times K} |\phi(x+\Lambda,y,y')| \tag{3.21}$$

where $\mathcal{C}>0$ is some fixed number deriving from the uniform x-asymptotics of $\rho^{\tau,-\tau}_{(+)}(x,y,y')$. The three properties of $\phi(x,y,y')$ stated below (3.20) ensure that this supremum is finite under $\Lambda\to\infty$ and so, overall, the limit procures vanishing. We have thus shown the contribution to (3.18) deriving from $\int_\Lambda^{\Lambda+ia_-}dx$ vanishes. The analogous argument can easily be made for $\int_{-\Lambda+ia_-}^{-\Lambda}dx$, using $\epsilon=-$.

We now consider the $\tau'=\tau$ terms. For this choice, the reciprocal term in (3.17) equals $(1-e_+(\tau(y-y')))^{-1}$ which is singular right along $y=y'$. To deal with this, we proceed to write these terms in a way which makes them amenable to Proposition A.1 (this isolates the tricky part of the argument). With vanishing established for $\tau'=-\tau$, we know the lhs of (3.17) equals

$$(2a_+a_-)^{-1}\lim_{\Lambda\to\infty}\int_K dydy'\,\overline{f(y)}g(y')\sum_{\tau=+,-}(1-e_+(\tau(y-y')))^{-1}$$
$$\times\left(\int_\Lambda^{\Lambda+ia_-}+\int_{-\Lambda+ia_-}^{-\Lambda}\right)dx\,W(x)J^{\tau,\tau}(x,y,y') \quad (3.22)$$

To prepare the way for the proposition we first note that for any objects B^+ and B^-,

$$\sum_{\tau=+,-}(1-e_+(\tau(y-y')))^{-1}B^\tau=(2s_+((y-y')/2))^{-1}\sum_{\tau=+,-}(-\tau)e_+(-\tau(y-y')/2)B^\tau \quad (3.23)$$

We also note that by changing variables $x\to x+ia_-/2$ and $x\to -x+ia_-/2$, and using $W(\cdot)$'s evenness and periodicity, we can write the second line in (3.22) in terms of just one contour,

$$\int_{\Lambda-ia_-/2}^{\Lambda+ia_-/2}dx\,W(x+ia_-/2)\sum_{\alpha=+,-}\alpha J^{\tau,\tau}(\alpha x+ia_-/2,y,y') \quad (3.24)$$

Recalling (3.13), we can see that the two $J^{\tau,\tau}$ terms corresponding to $\alpha=+,-$, both contain a factor of $e_+(\tau(y-y')/2)$. Thus combining (3.23) and (3.24) we can write (3.22) as

$$(2a_+a_-)^{-1}\lim_{\Lambda\to\infty}\int_K dydy'\,\overline{f(y)}g(y')\,(2s_+((y-y')/2))^{-1}\int_{\Lambda-ia_-/2}^{\Lambda+ia_-/2}dx\,W(x+ia_-/2)$$
$$\times\sum_{\alpha=+,-}(-\alpha)\exp(-i\pi\alpha x(y-y')/a_+a_-)\sum_{\tau=+,-}m_1^\tau(\tau\alpha x+ia_-/2,y)m_2^\tau(\tau\alpha x+ia_-/2,y') \quad (3.25)$$

(where we have taken the τ-sum into the integral and relabelled $\alpha\to\tau\alpha$, as permitted under $\sum_\alpha$). We now show this connects to Proposition A.1 for the obvious choice

$$G_{-\alpha}(x,y,y')=W(x+ia_-/2)\sum_{\tau=+,-}m_1^\tau(\tau\alpha(x+ia_-/2),y)m_2^\tau(\tau\alpha(x+ia_-/2),y'),\quad \alpha=+,- \quad (3.26)$$

The equality $G_+(x,y,y)=G_-(x,y,y)$ (A.4) is a straightforward consequence of the evenness assumption on (3.14) and the ia_--periodicity in x of $m_1^\tau(x,y)m_2^{\tau'}(x,y')$. For the asymptotics, assumption (3.15) and evenness of $W(\cdot)$ means that

$$W(x)m_1^\tau(\tau\alpha x,y)m_2^\tau(\tau\alpha x,y')=\mathcal{M}^{\tau,\tau}_{(\tau\alpha)}(y,y')+O(e_-(-\eta\,\mathrm{Re}\,x)),\quad \mathrm{Re}\,x\to\infty,\quad \tau,\alpha=+,- \quad (3.27)$$

and so, exploiting the ia_--periodcity properties of the functions involved,

$$G_{-\alpha}(x,y,y') = \sum_{\tau=+,-} \mathcal{M}^{\tau,\tau}_{(\tau\alpha)}(y,y') + O(e_-(-\eta \operatorname{Re} x)), \quad \operatorname{Re} x \to \infty \tag{3.28}$$

(given the convention introduced below (3.18), the $O(\cdot)$ term here can be explicitly rendered as $\sum_\tau \rho^{\tau,\tau}_{(\tau\alpha)}(\tau\alpha(x+ia_-/2), y, y')$ for $\alpha = +,-$). Thus the connection to Proposition A.1 is complete because of assumption (3.16). Explicitly, we must set $(s,t,t') = (x,y,y')$, $\Omega = \mathbb{R}^+$ and $\phi(y,y') = \overline{f(y)}g(y')$ in (A.7). Because of (3.28), we have $A_\pm(y,y) = \sum_{\tau=+,-} \mathcal{M}^{\tau,\tau}_{(\tau)}(y,y)$ and so by Proposition (A.1), (3.25) equals

$$-\int_{\mathbb{R}^+} dy\, \overline{f(y)}g(y) \sum_{\tau=+,-} \mathcal{M}^{\tau,\tau}_{(\tau)}(y,y) \tag{3.29}$$

□

We are now ready for the theorem that expresses (3.1) in a form manifestly conducive to questions of isometry. It involves residues of the function w_N analysed in §2.2,

$$w_j \equiv \operatorname*{Res}_{x=x_j^+} w_N(x), \quad j = 0,\ldots,N \tag{3.30}$$

as well as the following function, which we call a **residue function** because of its dependence on the pole locations x_j^+ (2.48),

$$R^{(j)}_\sigma(y,y') \equiv \sum_{\tau,\tau'=+,-} (1 - e_+(\tau y)e_+(-\tau' y'))^{-1} \exp(-i\tau\pi x_j^+ y/a_+a_-) \exp(i\tau'\pi x_j^+ y'/a_+a_-)$$
$$\times \left[\ell^\tau_N(x_j^+, -y)\ell^{\sigma\tau'}_N(\sigma x_j^+, y') + \ell^{-\tau}_N(-x_j^+, -y)\ell^{-\sigma\tau'}_N(-\sigma x_j^+, y')\right], \quad j = 0,\ldots,N, \quad \sigma = +,- \tag{3.31}$$

The formula we present is valid only when all the poles of $w_N(\cdot)$ are simple, a property that hinges on the values of the underlying parameters a_+, a_-. More precisely, we know from Lemma 2.1 that the ratio a_-/a_+ cannot take values from the point set $\mathcal{E}_N$ if this property is to hold. The theorem applies to functions in $\hat{\mathcal{C}} \equiv C_0^\infty(\mathbb{R}^+)^2$, a set which is dense in $\hat{\mathcal{H}}$.

Theorem 3.3. *For the map $\mathcal{F}_N : \hat{\mathcal{H}} \to \mathcal{H}$ defined by (2.69), the following is true for functions $f, g \in \hat{\mathcal{C}}$ whenever $a_-/a_+ \notin \mathcal{E}_N$,*

$$(\mathcal{F}_N \hat{f}, \mathcal{F}_N \hat{g})_{\mathcal{H}} = (\hat{f}, \hat{g})_{\hat{\mathcal{H}}}$$
$$+ \frac{\pi i}{a_+a_-} \sum_{\delta,\delta'=+,-} \int_{\mathbb{R}^+} dy\, \overline{f_\delta(y)} \int_{\mathbb{R}^+} dy'\, g_{\delta'}(y') v_N(-y) v_N(y') \sum_{j=0}^N w_j R^{(j)}_{\delta\delta'}(y,y') \tag{3.32}$$

Moreover, this integral is absolutely convergent.

Proof. Denoting the integral function in (3.4) as $I_\sigma(\Lambda, y, y')$, we already know from (3.1)-(3.4) that

$$(\mathcal{F}_N\hat{f}, \mathcal{F}_N\hat{g})_{\mathcal{H}} = (2a_+a_-)^{-1} \sum_{\delta,\delta'=+,-} \lim_{\Lambda\to\infty} \int_{\mathbb{R}+} dy\, \overline{f_\delta(y)} \int_{\mathbb{R}+} dy'\, g_{\delta'}(y') I_{\delta\delta'}(\Lambda, y, y') \quad (3.33)$$

The idea is to apply Proposition 3.1 to $I_\sigma(\Lambda, y, y')$ for $\sigma = +$ and $\sigma = -$, and then let the resulting boundary integrals be handled by Proposition 3.2. We need the choices $m_1^\tau(x, y) = v_N(-y)\ell_N^\tau(x, -y)$ and $m_2^{\tau'}(x, y') = v_N(y')\ell_N^{\sigma\tau'}(\sigma x, y')$ for which the function (3.13) in Proposition 3.2 equals

$$\begin{aligned} J^{\tau,\tau'}(\sigma; x, y, y') \equiv v_N(-y)v_N(y') \exp(-i\tau\pi xy/a_+a_-) \exp(i\tau'\pi xy'/a_+a_-) \\ \times \ell_N^\tau(x, -y)\ell_N^{\sigma\tau'}(\sigma x, y'), \quad \sigma, \tau, \tau' = +, - \quad (3.34) \end{aligned}$$

(where we have made the σ dependence explicit). This function connects with $I_\sigma(\Lambda, y, y')$ because, multiplied by $w_N(x)$, it is just the rhs of (3.6) and, moreover, because of the quasiperiodicity relation (2.17), it satisfies

$$J^{\tau,\tau'}(\sigma; x + ia_-, y, y') = e_+(\tau y)e_+(-\tau' y')J^{\tau,\tau'}(\sigma; x, y, y'), \quad \sigma, \tau, \tau' = +, - \quad (3.35)$$

This means it is a candidate for Proposition 3.1 with $s = x$, $W(x) = w_N(x)$ and $M^{\tau,\tau'} = e_+(\tau y)e_+(-\tau' y')$ (for the required properties of $w_N(\cdot)$, cf. §2.2). The support of $\hat{f}, \hat{g}$ in (3.33) means $y, y' \in (0, \infty)$, and so a problem arises for this multiplier when $y = y'$ and $\tau = \tau'$. This ceases to be a problem, however, when (3.8) is considered under $\lim_{\Lambda\to\infty} \int_K dydy'$ for K compact and away from the origin, like we have in (3.33). The divergence affecting the residue term is handled by our absolute convergence claim (to be proved shortly), and that affecting the boundary integrals is handled by Proposition 3.2, assuming that the $m_1^\pm$, $m_2^\pm$ introduced above satisfy the various criteria of the proposition. Applying Proposition (3.1) we thus have

$$\begin{aligned} (\mathcal{F}_N f, \mathcal{F}_N g)_{\mathcal{H}} = \frac{\pi i}{a_+a_-} \sum_{\delta,\delta'=+,-} \int_{\mathbb{R}+} dy\, \overline{f_\delta(y)} \int_{\mathbb{R}+} dy' g_{\delta'}(y') v_N(-y)v_N(y') \sum_{j=0}^N w_j R_{\delta\delta'}^{(j)}(y, y') \\ - (2a_+a_-)^{-1} \lim_{\Lambda\to\infty} \sum_{\delta,\delta'=+,-} \int_{\mathbb{R}+} dy\, \overline{f_\delta(y)} \int_{\mathbb{R}+} dy' g_{\delta'}(y') \Big(\int_\Lambda^{\Lambda+ia_-} + \int_{-\Lambda+ia_-}^{-\Lambda} \Big) dx \\ \times \sum_{\tau,\tau'=+,-} (1 - e_+(\tau y)e_+(-\tau' y'))^{-1} w_N(x) J^{\tau,\tau'}(\delta\delta'; x + ia_-, y, y') \quad (3.36) \end{aligned}$$

(concerning the poles, we have made the obvious choice $s_{j+1} = x_j^+$, $j = 0, \ldots, N$). For the first line of the rhs we have used

$$\begin{aligned} J^{\tau,\tau'}(\sigma; x_j^+, y, y') + J^{-\tau,-\tau'}(\sigma; -x_j^+, y, y') = v_N(-y)v_N(y') \exp(-i\tau\pi x_j^+ y/a_+a_-) \\ \times \exp(i\tau'\pi x_j^+ y'/a_+a_-) \big[\ell_N^\tau(x_j^+, -y)\ell_N^{\sigma\tau'}(\sigma x_j^+, y') + \ell_N^{-\tau}(-x_j^+, -y)\ell_N^{-\sigma\tau'}(-\sigma x_j^+, y')\big] \quad (3.37) \end{aligned}$$

(which follows straight away from the definition (3.34)). If we now apply Proposition 3.2 to (3.36), it follows that the term constituting the second and third lines of (3.36) equals

$$+\sum_{\delta,\delta'=+,-}\int_{\mathbb{R}^+}dy\,\overline{f_\delta(y)}\int_{\mathbb{R}^+}dy' g_{\delta'}(y')\sum_{\tau=+,-}\mathcal{M}^{\tau,\tau}_{(\tau)}(\delta\delta';y,y) \tag{3.38}$$

(where we have made explicit the $\sigma=\delta\delta'$ dependence of $\mathcal{M}^{\tau,\tau'}_{(\epsilon)}(y,y')$ that enters through our choice of $m_2^\pm$). Let us now confirm that the various criteria of Proposition 3.2 are met, and compute this τ-sum.

Entirety of $\ell^\tau_N(x,y)$ in both variables means that $m^\tau_j(\cdot,y)$ is entire in x and $m^\tau_j(x,\cdot)$ is smooth on $\mathbb{R}$ when the poles of $v_N(\cdot)$ are off the real axis, which we know from §2.2 is secured by $a_-/a_+\notin\mathcal{E}_N$. Periodicity follows from (2.17). We can prove the evenness property by writing (3.14) in terms of the function M_N in §2.1, which we have studied in detail. For this choice of $m_j^\pm$, (3.14) equals

$$v_N(-y)v_N(y)\sum_{\alpha=+,-}\ell^\tau_N(\tau x,-y)\ell^{\sigma\tau}_N(\sigma\tau x,y),\quad \sigma=+,- \tag{3.39}$$

Comparing with (2.31) we see this is just

$$v_N(-y)v_N(y)M_N^{+,\sigma}(x,\sigma x;-y,y),\quad \sigma=+,- \tag{3.40}$$

For $\sigma=+$, evenness in x is thus immediately given by the symmetry (2.41), and for $\sigma=-$ by (2.42).

To realise the asymptotic form (3.15) for these choices of m_1^τ and m_2^τ, we need the asymptotics (2.58),(2.63) and (2.64). The ν-sum in the latter two equations can be rewritten using

$$\phi^\tau_{(+)}(y)\equiv\begin{cases}2s_-(y), & \tau=+\\ 0, & \tau=-\end{cases}\qquad \phi^\tau_{(-)}(y)\equiv\begin{cases}2s_-(i(N+1)a_+-y), & \tau=+\\ 2s_-(i(N+1)a_+), & \tau=-\end{cases} \tag{3.41}$$

With these functions the two equations in question become

$$\ell^\tau_N(x,y)=(-i)^{N+1}\tau s_N(\pm\tau y)e_-^{(N+1)|\operatorname{Re}x|}\phi^\tau_{(\pm)}(y)+O(e_-^{(N-1)|\operatorname{Re}x|}),\quad \operatorname{Re}x\to\pm\infty \tag{3.42}$$

Recalling, (2.58),

$$w_N(x)=e_-^{-2(N+1)|\operatorname{Re}x|}+O(e_-^{-2(N+3)|\operatorname{Re}x|}),\quad \operatorname{Re}x\to\pm\infty \tag{3.43}$$

we then get the following, for fixed $\epsilon,\sigma=+,-$,

$$\begin{aligned}&w_N(x)\ell^\tau_N(x,-y)\ell^{\sigma\tau'}_N(\sigma x,y')\\ &=(-)^{N+1}\sigma\tau\tau's_N(-\epsilon\tau y)s_N(\epsilon\tau'y')\phi^\tau_{(\epsilon)}(-y)\phi^{\sigma\tau'}_{(\epsilon\sigma)}(y')+O(e_-^{-2|\operatorname{Re}x|}),\quad \operatorname{Re}x\to\epsilon\infty\end{aligned} \tag{3.44}$$

With this we recall our choices of $m_j^\pm$ above (3.34) and read off,

$$\begin{aligned}\mathcal{M}^{\tau,\tau'}_{(\epsilon)}(\sigma;y,y')=v_N(-y)v_N(y')(-)^{N+1}\sigma\tau\tau's_N(-\epsilon\tau y)s_N(\epsilon\tau'y')\phi^\tau_{(\epsilon)}(-y)\phi^{\sigma\tau'}_{(\epsilon\sigma)}(y'),&\\ \epsilon,\sigma,\tau,\tau'=+,-&\end{aligned} \tag{3.45}$$

As a check, we note that for this $\mathcal{M}^{\tau,\tau'}_{(\epsilon)}$, the vanishing property (3.16) reduces to

$$\sum_{\tau=+,-} \phi^{\tau}_{(\tau)}(-y)\phi^{\sigma\tau}_{(\sigma\tau)}(y) = \sum_{\tau=+,-} \phi^{\tau}_{(-\tau)}(-y)\phi^{\sigma\tau}_{(-\sigma\tau)}(y), \quad \sigma = +,- \tag{3.46}$$

which should be satisfied for both $\sigma = +$ and $\sigma = -$. Indeed in the second case both sides of the equation vanish, meaning $\sum_{\tau=+,-} \mathcal{M}^{\tau,\tau}_{(\tau)}(-; y, y) = 0$, and in the first case both sides are found to equal

$$4s_-(i(N+1)a_+ - y)s_-(i(N+1)a_+ + y) \tag{3.47}$$

Because of (2.62), and its $y \to -y$ counterpart, this means

$$\sum_{\tau=+,-} \mathcal{M}^{\tau,\tau}_{(\tau)}(+; y, y) = 1 \tag{3.48}$$

And so, with regard to (3.36),

$$\sum_{\delta,\delta'=+,-} \int_{\mathbb{R}^+} dy\, \overline{f_\delta(y)} g_{\delta'}(y) \sum_{\tau=+,-} \mathcal{M}^{\tau,\tau}_{(\tau)}(\delta\delta'; y, y) = \sum_{\delta=+,-} \int_{\mathbb{R}^+} dy\, \overline{f_\delta(y)} g_\delta(y) \tag{3.49}$$

We now deal with the absolute convergence claim. First we note that the poles of $w_N(\cdot)$ are off the real axis when $a_-/a_+ \notin \mathcal{E}_N$, cf. §2.2. Recalling that $\ell^\tau_N(\cdot,\cdot)$ is entire, it is therefore clear that the only threat can come from the $\tau' = \tau$ terms in (3.31); specifically, from the reciprocal term which is singular right along $y = y'$ (there is no threat at the origin for $\tau' = -\tau$ because f, g have compact support away from the origin). Suppose we can argue that for both choices of $\sigma = +,-$, the square bracketed term in (3.31) is in fact independent of the value of τ whenever $\tau' = \tau$ and $y' = y$. Then, finitude of $R^{(j)}_\pm(y, y')$ follows because of the limit

$$\lim_{y'\to y} \sum_{\tau=+,-} \frac{L^\tau(y, y')}{1 - e_+(\tau(y - y'))} = L^+(y, y) + (\partial_1 L^+ - \partial_1 L^-)(y, y) \tag{3.50}$$

which can be established by a routine Taylor series argument for generic smooth functions $L^\pm(y, y')$ satisfying $L^+(y, y) = L^-(y, y)$ (∂_1 denotes partial derivative with respect to the first argument).

In fact, this supposition holds as a consequence of the evenness in x of (3.39) that we have just exhibited (in other words, convergence of the residue function integral and vanishing of the boundary integrals are secured by the same fact). On a more abstract level this is due to the fact that evenness in x of (3.14) is automatically equivalent to τ-independence of

$$\sum_{\alpha=+,-} m^{\alpha\tau}_1(\alpha x, y) m^{\alpha\tau}_2(\alpha x, y), \quad \tau = +,- \tag{3.51}$$

□

3.2 Residue analysis.

We first restate the definition of our residue function (3.31),

$$R_\sigma^{(j)}(y,y') \equiv \sum_{\tau,\tau'=+,-} (1-e_+(\tau y)e_+(-\tau' y'))^{-1}\exp(-i\pi\tau x_j^+ y/a_+a_-)\exp(i\pi\tau' x_j^+ y'/a_+a_-)$$
$$\times\left[\ell_N^\tau(x_j^+,-y)\ell_N^{\sigma\tau'}(\sigma x_j^+,y')+\ell_N^{-\tau}(-x_j^+,-y)\ell_N^{-\sigma\tau'}(-\sigma x_j^+,y')\right],\quad j=0,\ldots,N,\quad \sigma=+,- \tag{3.52}$$

and recall that: $\ell_N^\tau(\cdot,\cdot)$ is an entire function defined in (2.15); x_j^+ is one of the $2(N+1)$ poles of the weight function $w_N(x)$ in the strip $\mathrm{Im}\, x\in i(0,a_-)$, cf. §2.2; and the residue of this function at $x=x_j^+$ is denoted by w_j.

With this restated, we can make sense of the following corollary of Theorem 3.3, which serves as the focal point of this subsection. The restriction $a_-/a_+\notin\mathcal{E}_N$ is simply there to ensure the validity of the formula (3.32). Whenever this formula holds, the claim is immediate and there is no need for further proof,

Corollary 3.4. *Provided $a_-/a_+\notin\mathcal{E}_N$, the map $\mathcal{F}_N:\hat{\mathcal{H}}\to\mathcal{H}$ defined by* (2.69) *is an isometry iff the residue sums $\sum_{j=0}^N w_j R_\sigma^{(j)}(y,y')$ vanishes for both $\sigma=+$ and $\sigma=-$.*

In all the examples we are going to see, vanishing of the residue sum comes about because of vanishing of each individual summand. (Indeed in §5 we prove that for fixed N and σ, the residue functions are linearly independent.) To provide some guidance for the general case we will run through the $N=0$ case explicitly before returning to general N in Lemma 3.5. An important observation to make is that the square bracketed term in (3.52) is just the $x=x_j^+$ specialisation of

$$M_N^{\tau;\sigma\tau'}(x,\sigma x;-y,y'),\quad \sigma,\tau,\tau'=+,- \tag{3.53}$$

where M_N is the function we introduced in §2.1, and about which a great deal is known.

When $N=0$ there are only two residue functions to consider, corresponding to $\sigma=+$ and $\sigma=-$. Focusing first on $\sigma=+$, we use (2.36) and $\Sigma_0=1$ to write out

$$M_0^{\tau,\tau'}(x,x;-y,y') = -\tau\tau'\sum_{\nu,\nu'=+,-} 4\nu\nu' s_-(x-i\nu a_+/2)s_-(x-i\nu a_+/2)$$
$$\times\,\mu_{\nu\nu'}(-y,y')e_-(\nu\tau y/2)e_-(-\nu'\tau' y'/2),\quad \tau,\tau'=+,- \tag{3.54}$$

where

$$\mu_{\nu\nu'}(-y,y') = \begin{cases} 2c_-((y-y')/2+ia_+), & \nu\nu'=+\\ 2c_-((y+y')/2), & \nu\nu'=-\end{cases} \tag{3.55}$$

We see that if we set $x=x_0^+=ia_+/2$ in (3.54), then all terms vanish except $\nu=\nu'=-$, leaving

$$M_0^{\tau,\tau'}(x_0^+, x_0^+; -y, y') = -\tau\tau' s_-(ia_+)^2 \mu_+(-y, y') e_-(-\tau y/2) e_-(\tau' y'/2) \tag{3.56}$$

Noting that the plane wave term in (3.52) just equals $e_-(\tau y/2)e_-(-\tau' y'/2)$ when $x_0^+ = ia_+/2$, we conclude that

$$R_+^{(0)}(y, y') = -s_-(ia_+)^2 \mu_+(-y, y') \sum_{\tau,\tau'=+,-} (1 - e_+(\tau y)e_+(-\tau' y'))^{-1} \tau\tau' \tag{3.57}$$

In fact this sum vanishes outright due to the simple identity

$$\frac{1}{1 - A/A'} + \frac{1}{1 - A'/A} - \frac{1}{1 - AA'} - \frac{1}{1 - 1/AA'} = 0 \tag{3.58}$$

Rerunning the process for $\sigma = -$ we find

$$R_-^{(0)}(y, y') = s_-(ia_+)^2 \mu_-(-y, y') \sum_{\tau,\tau'=+,-} (1 - e_+(\tau y)e_+(-\tau' y'))^{-1} \tau\tau' \tag{3.59}$$

And so, for $N = j = 0$ and $x_0^+ = ia_+/2$,

$$R_+^{(0)}(y, y') = R_-^{(0)}(y, y') = 0 \tag{3.60}$$

In Lemma 3.5 we learn that this vanishing extends to all (N, j) when a_+, a_- are suitably restricted. Above we simply assumed that $x_0^+ = ia_+/2$, but in general x_j^+ (2.48) only has the form

$$x_j^+ = i(j + 1/2)a_+, \quad j = 0, \ldots, N \tag{3.61}$$

if $(j + 1/2)a_+ < a_-$, cf. §2.2. When a_-/a_+ is sufficiently small such that this latter inequality is no longer satisfied, we must subtract some multiple of ia_- from the rhs of (3.61) to obtain the correct expression for x_j^+. This change spoils the vanishing argument, as we will see in §5. The restriction in Lemma 3.5 thus ensures a given x_j^+ has the form (3.61).

Comparing to the $N = 0$ vanishing argument, the main difference in the general N case is that $\Sigma_N(\cdot, \cdot)$ is no longer unity. In fact each argument of this function corresponds to an even polynomial of exponentials of degree $2N$, cf. (2.9). Because of this increasing degree, a brute force approach like we gave for $N = 0$ is simply not possible. The proof will hinge on an important fact about these polynomials that is proved in [18]. The end-goal is again to isolate the τ, τ'-sum in (3.57)

Lemma 3.5. *If the positive parameters a_+, a_- satisfy $a_-/a_+ > j + 1/2$ for a fixed $j = 0, \ldots, N$, then the residue functions $R_\pm^{(j)}(y, y')$ (3.52) vanish.*

Theorem 3.6. *If the positive parameters a_+, a_- satisfy $a_-/a_+ > N + 1/2$, and $a_-/a_+ \notin \mathcal{E}_N$ (2.52), then the map $\mathcal{F}_N : \hat{\mathcal{H}} \to \mathcal{H}$ defined by (2.69) is an isometry.*

Proof of Theorem 3.6. Follows from Corollary 3.4 and Lemma 3.5. □

Proof of Lemma 3.5. We first rewrite $R^{(j)}_\sigma(y,y')$ in terms of

$$\lambda^\tau_N(x,y) = \exp(i\pi\tau xy/a_+a_-)\ell^\tau_N(x,y), \quad \tau = +,- \tag{3.62}$$

for which we have the explicit expression (2.29). It is best to conduct this rearrangement with $x^+_j \to x$ in (3.52), and then to set $x = x^+_j$ at the end. This is straightforward because from (3.62) we have

$$\begin{aligned} e_-(-i\pi\tau xy/a_+a_-)e_-(i\pi\tau' xy'/a_+a_-)\left[\ell^\tau_N(x,-y)\ell^{\sigma\tau'}_N(\sigma x,y')+\ell^{-\tau}_N(-x,-y)\ell^{-\sigma\tau'}_N(-\sigma x,y')\right] \\ = \left[\lambda^\tau_N(x,-y)\lambda^{\sigma\tau'}_N(\sigma x,y') + \lambda^{-\tau}_N(-x,-y)\lambda^{-\sigma\tau'}_N(-\sigma x,y')\right], \quad \sigma,\tau,\tau' = +,- \end{aligned} \tag{3.63}$$

and so

$$\begin{aligned} R^{(j)}_\sigma(y,y') = \sum_{\tau,\tau'=+,-} (1-e_+(\tau y)e_+(-\tau' y'))^{-1}\left[\lambda^\tau_N(x^+_j,-y)\lambda^{\sigma\tau'}_N(\sigma x^+_j,y')\right. \\ \left. + \lambda^{-\tau}_N(-x^+_j,-y)\lambda^{-\sigma\tau'}_N(-\sigma x^+_j,y')\right], \quad j = 0,\ldots,N, \sigma = +,- \end{aligned} \tag{3.64}$$

The four λ_N-functions that feature here are all variants of

$$\lambda^{\alpha\tau}_N(\alpha x^+_j, z), \quad \alpha,\tau = +,- \tag{3.65}$$

and so we proceed to study this function more closely. From (2.29) and (2.21) we learn that its τ-dependence hinges on

$$K_N(x^+_j + i\alpha\nu a_+/2, \tau z), \quad \alpha,\nu,\tau = +,- \tag{3.66}$$

in the sense that (3.65) equals

$$\tau\phi K_N(x^+_j + i\alpha a_+/2, \tau z) + \tau\varphi K_N(x^+_j - i\alpha a_+/2, \tau z) \tag{3.67}$$

where ϕ, φ are some j,α-dependent entire functions in z.

We now look at (3.66) more closely. Since the restriction in the lemma ensures $x^+_j = i(j+1/2)a_+$, its first argument equals $ika_+, k \le N$ whenever $j < N$, or whenever $j = N$ and $\alpha\nu = -$. Accordingly for these cases, (2.23) tells us that (3.66) equals

$$i^N B^{(N)}_{N-j-(1+\alpha\nu)/2}(c_-(z)) \tag{3.68}$$

which, crucially, is independent of τ. For the one remaining case, $j = N$ and $\alpha\nu = +$, (2.23) cannot be applied. However, for $j = N$, the function ϕ in (3.67) contains a multiplicative factor of

$$s_-(x^+_N - i\alpha\nu(N+1/2)a_+) \tag{3.69}$$

(as (2.29) makes clear), and this clearly vanishes for $\alpha\nu = +$.

We have thus shown that in all cases, (3.65) is straightforwardly proportional to τ with respect to the variable τ (i.e. $\lambda_N^{\alpha\tau}(\alpha x_j^+, z) = \tau\Psi_{j,\alpha}(z)$, $\tau = +, -$ for some function $\Psi_{j,\alpha}(z)$). As a result, (3.64) equals some j-dependent entire function in y, y' multiplied by

$$\sum_{\tau,\tau'=+,-} (1 - e_+(\tau y)e_+(-\tau' y'))^{-1}\tau\tau' \tag{3.70}$$

This is the function we saw in (3.57) and we know it vanishes due to (3.58).

□

3.3 Dynamics and wave operator. (Inc. Main Theorem)

So far we have made only a cursory reference to the AΔO which it is the purpose of the present sections to serve. Although we know the function $\psi_N(x, y)$ is a generalised eigenfunction of $\tilde{H}_N(x)$, we have not needed this fact at any time. Thus we are in the strange situation of being able to prove the existence of some non-trivial self-adjoint dynamics without any invocation of an explicit operator action. This is the dynamics for the operator defined by intertwining $\mathcal{F}_N$ and $\hat{M}$, where the latter denotes multiplication on pairs in $\hat{\mathcal{H}}$ by $2c_+(y)$. Such a definition amounts to a diagonalisation of $\tilde{H}_N(x)$.

To connect with $\tilde{H}_{\rm ac}$ in Chapter 1 we define

$$\tilde{H}_{{\rm ac},N} \equiv \tilde{H}_{\rm ac}((N + 1)a_+), \quad N \in \mathbb{N} \tag{3.71}$$

where $\tilde{H}_{\rm ac}$ is the operator on $\mathcal{F}(\hat{\mathcal{C}})$ defined by (4.4) in Chapter 1 (we have made its b-dependence manifest in (3.71)). Although this is our chosen definition of $\tilde{H}_{{\rm ac},N}$, it can be equivalently defined by the intertwining relation,

$$\tilde{H}_{{\rm ac},N}\mathcal{F}_N\hat{f} = \mathcal{F}_N\hat{M}\hat{f}, \quad \hat{f} \in \hat{\mathcal{C}} \tag{3.72}$$

which employs the eigenfunction transform $\mathcal{F}_N$ (2.69) and the set of functions $\hat{\mathcal{C}} \equiv C_0^\infty(\mathbb{R}^+)^2$. Indeed (3.72) is preferable if one wishes to view the present section as self-contained. Because $\hat{M}$ maps $\hat{\mathcal{C}}$-functions into $\hat{\mathcal{C}}$-functions, we have

$$\tilde{H}_{{\rm ac},N} : \mathcal{F}_N(\hat{\mathcal{C}}) \to \mathcal{F}_N(\hat{\mathcal{C}}) \subset \mathcal{H} \tag{3.73}$$

When we introduced $\mathcal{F}_N$ in §2.4 we employed a simple argument to establish its boundedness on $\hat{\mathcal{H}}$. (By contrast, there was no such argument for $\mathcal{F}$.) This entails that we can immediately extend the intertwining (3.72) to all functions in the maximal multiplication domain of $\hat{M}$ on $\hat{\mathcal{H}}$, denoted $\mathcal{D}(\hat{M})$. Thus whenever $\mathcal{F}_N$ is an isometry it readily follows that the extension of $\tilde{H}_{{\rm ac},N}$ to $\mathcal{F}_N(\mathcal{D}(\hat{M}))$ will be self-adjoint (because of explicit unitary equivalence to the self-adjoint $\hat{M}$). Indeed this is nothing but the unique self-adjoint extension which we know exists from Theorem 5.1 in Chapter 1.

The intertwining above combined with Theorem 3.6 gives us the following result. (We omit the argument that strengthens symmetry to essential self-adjointness since this is identical to that in §5.1 in Chapter 1.)

Theorem 3.7. *If the parameters a_+, a_- satisfy $a_-/a_+ > N + 1/2$, and $a_-/a_+ \notin \mathcal{E}_N$ (2.52), then $\tilde{H}_{ac,N}$ (3.72) is essentially self-adjoint in the closure of $\mathcal{F}_N(\hat{\mathcal{C}})$ with absolutely continuous spectrum $[2, \infty)$ of multiplicity two. This closure equals $\mathcal{F}_N(\hat{\mathcal{H}})$, and the unique self-adjoint extension of $\tilde{H}_{ac,N}$ is described below* (3.73).

In preparation for our look at dynamics, we note down the dominant asymptotics of $\psi_N(x,y)$ (which can just be understood as a specialisation of Lemma 1.2 in Chapter 1, but an explicit computation using the formulas in §2.3 is also not difficult),

$$\psi_N(x,y) \sim \begin{cases} t_N(y)\exp(i\pi xy/a_+a_-), & x \to \infty \\ \exp(i\pi xy/a_+a_-) - r_N(y)\exp(-i\pi xy/a_+a_-), & x \to -\infty \end{cases} \tag{3.74}$$

where

$$t_N(y) \equiv \frac{s_-(y)}{s_-(ib_N - y)}\, u_N(y), \quad r_N(y) \equiv \frac{s_-(ib_N)}{s_-(ib_N - y)}\, u_N(y) \tag{3.75}$$

and where we have the phase

$$u_N(y) \equiv u(b_N; y) = \prod_{j=1}^{N} \frac{s_-(ija_+ + y)}{s_-(ija_+ - y)} \tag{3.76}$$

(the latter equality just follows from (2.3)).

At this point we do not know whether $\mathcal{F}_N(\hat{\mathcal{C}})$ is dense in $\mathcal{H}$ (recall, we are "forgetting" results from Chapter 1), and so to consider dynamics on $\mathcal{H}$ we will extend $\tilde{H}_{\mathrm{ac},N}$ to an arbitrary bounded self-adjoint operator on

$$\overline{\mathcal{F}_N(\hat{\mathcal{C}})}^{\perp} \tag{3.77}$$

We call the resulting densely-defined operator $\tilde{H}_N$ (cf. (3.82) below). Clearly it inherits self-adjointness under the same conditions as $\tilde{H}_{\mathrm{ac},N}$. This extension procedure is the same as that in §5.2 of Chapter 1. Thus we are just going to apply the results of the latter directly to $\tilde{H}_N$.[2] This involves the wave operators[3]

$$W_\pm(b_N) = \operatorname*{s\text{-}lim}_{t\to\infty} \exp(\pm it\tilde{H}_N)\exp(\mp itH_0) \tag{3.78}$$

for which Theorem 5.4 in Chapter 1 tells us

$$W_-(b_N) = \mathcal{F}_N J^*, \quad W_+(b_N) = \mathcal{F}_N \hat{S}_N^* J^* \tag{3.79}$$

where $J : \hat{\mathcal{H}} \to \mathcal{H}$ is Fourier transform and $\hat{S}_N$ is matrix multiplication on $\hat{\mathcal{H}}$ by

$$\begin{pmatrix} t_N(y) & -r_N(y) \\ -r_N(y) & t_N(y) \end{pmatrix} \tag{3.80}$$

[2] We claim there isnothing novel about an independent proof of this same result in the special b case.

[3] It is understood that the $\tilde{H}_N$ in the exponent stands for its own closure, i.e. the unique self-adjoint extension which we have discussed.

It follows that the scattering states (intersection of the ranges of the wave operators) are given by

$$\overline{\mathcal{F}_N(\hat{\mathcal{C}})} = \mathcal{F}_N(\hat{\mathcal{H}}) \tag{3.81}$$

(where this fact about the closure is an immediate consequence of $\mathcal{F}_N$'s isometry).

When it comes to fixing the arbitrary action on $\overline{\mathcal{F}_N(\hat{\mathcal{C}})}^{\perp}$ in accordance with the desired physics the situation is somewhat different to before. We recall that in §3.4 of Chapter 1 we showed explicitly that a particular family of functions $\Psi^{(m)}(b;x)$ lived in the orthocomplement of $\mathcal{F}(\hat{\mathcal{C}})$. This result is of course still applicable here. However, we are going to "forget" about it, because the same fact can be obtained by different, and more satisfactory, means.

It turns out that isometry, and isometry breakdown, of the adjoint $\mathcal{F}_N^*$ has a non-obvious connection to the range of $\mathcal{F}_N$, and thus to the bound states. This connection is stronger than the one that exists between the range of $\mathcal{F}_N$ and symmetry breakdown of the dual operator $\overline{\mathcal{S}}$ (which we explored in §6-7 of Chapter 1) because it procures a complete description of the orthocomplement of this range with no prior knowledge of any of its elements.

(Let us assess our expectations from Chapter 1 with regard to the parameters. We recall that a given $\Psi^{(m)}(b;x)$ was integrable iff $(m+1)a_- < b$, cf. (2.11). And we know that isometry of $\mathcal{F}_N$ requires $a_-/a_+ > N + 1/2$. Thus we can see that $m = 0$ will procure the only integrable member of the family of $\Psi^{(m)}$-functions. Our independent exhibition of the orthocomplement of $\mathcal{F}(\hat{\mathcal{H}})$ should therefore yield something at most one-dimensional. This is indeed the case, as we will see in Theorem 4.7.)

We finish the story of $\tilde{H}_N$ by combining Theorem 3.7 above with Theorem 4.7 from the next section. What we get does not improve on the results of Chapter 1 (those summarised in Chapter 1). The pertinent difference is that the theorem below hinges on our direct isometry proof of $\mathcal{F}_N$ which does not invoke the AΔO $\tilde{H}_N(x)$ at any time.

Main Theorem of Chapter 2. *For positive parameters a_+, a_- satisfying $a_-/a_+ \in (N+1/2, N+1)$ and $a_-/a_+ \notin \mathcal{E}_N, \hat{\mathcal{E}}_N$,* (2.52) *and* (2.57)*, we may define a densely-defined operator in $\mathcal{H}$,*

$$\tilde{H}_N : \mathcal{F}_N(\hat{\mathcal{C}}) \oplus \overline{\mathcal{F}_N(\hat{\mathcal{C}})}^{\perp} \to \mathcal{H} \tag{3.82}$$

with action equal to $\tilde{H}_{ac,N}$ (3.72) *on $\mathcal{F}_N(\hat{\mathcal{C}})$ and multiplication by $2\cos(\pi a_-/a_+)$ on the one-dimensional orthocomplement in* (3.82) *which is spanned by*

$$\Psi_N(x) \equiv 2c_+(x)w_N(x)^{1/2} \tag{3.83}$$

This is the ground state defined using the weight function (2.4)*. In both cases, this action equals that of the AΔO $\tilde{H}_N(x)$. The map $\mathcal{F}_N$* (2.22) *is an isometry, and thus the closure in* (3.82) *equals $\mathcal{F}_N(\hat{\mathcal{H}})$.*

It follows that $\tilde{H}_N$ is essentially self-adjoint with absolutely continuous spectrum $[2,\infty)$ *of multiplicity two, and point spectrum $2\cos(\pi a_-/a_+)$ of multiplicity one. Its unique self-adjoint extension was described below* (3.73)*.*

We note that Theorem 4.5 of the next section tells us the extension $\tilde{H}_N$ is redundant when $a_-/a_+ > N + 1$, and so in this case there is no improvement on Theorem 3.7.

4 Isometry of $\mathcal{F}_N^*$

4.1 Isometry formula

In this subsection we apply the same treatment to the adjoint transform, $\mathcal{F}_N^* : \mathcal{H} \to \hat{\mathcal{H}}$. We know that this exists and is bounded because Lemma 2.3 established that $\mathcal{F}_N$ (2.69) was a bounded map from $\hat{\mathcal{H}}$ into $\mathcal{H}$ provided $a_-/a_+ \notin \mathcal{E}_N$ (2.52), and so these properties follow automatically. Clearly the action is given by

$$(\mathcal{F}_N^* f)_\delta(y) = c \int_{\mathbb{R}} dx\, \overline{\psi_N(\delta x, y)} f(x), \quad \delta = +, - \tag{4.1}$$

What we seek, then, is a formula analogous to (3.32) that will allow us to deduce the isometry properties of $\mathcal{F}_N^*$. Because the roles x and y are now reversed, this amounts to a wholly different problem and we stress our target theorem is not simply an adaptation of its analogue in §3.1. It is the special nature of ψ_N that makes this possible.

Although the end-goal is the same, our motives are somewhat different. The main reason for wanting such a formula for $\mathcal{F}_N^*$ is to shed light more light on $\mathcal{F}_N$. For instance, if there is a range of a_+, a_- for which $\mathcal{F}_N^*$ and $\mathcal{F}_N$ are both isometries, then clearly $\mathcal{F}_N$ is unitary for this range. More generally we find there is a connection between isometry breakdown of $\mathcal{F}_N^*$ and the range of $\mathcal{F}_N$. (We say the 'main' reason because it is also the case that $\mathcal{F}_N^*$ can be used to describe a dual dynamics, cf. B in Chapter 1.)

From (4.1) we have

$$(\mathcal{F}_N^* f, \mathcal{F}_N^* g)_{\hat{\mathcal{H}}} = c^2 \sum_{\delta=+,-} \int_{\mathbb{R}^+} dy \int_{\mathbb{R}} dx\, \psi_N(\delta x, y) \overline{f(x)} \int_{\mathbb{R}} dx'\, \overline{\psi_N(\delta x', y)} g(x') \tag{4.2}$$

The aim is to directly compute the rhs of this with the aim of finding conditions under which it equals $(f, g)_{\mathcal{H}}$. Due to Fubini's theorem we may rewrite it as

$$c^2 \lim_{\Lambda\to\infty} \int_{\mathbb{R}} dx\, \overline{f(x)} \int_{\mathbb{R}} dx'\, g(x') \int_0^\Lambda dy \sum_{\delta=+,-} \psi_N(\delta x, y) \overline{\psi_N(\delta x', y)} \tag{4.3}$$

Once again we will focus on the rightmost integral here and show that it is amenable to Proposition 3.1 with $s = y$. First we recall the expression for ψ_N that isolates its main structural features, (2.14),

$$\psi_N(x, y) = w_N(x)^{1/2} v_N(y) \sum_{\tau=+,-} e^{i\tau\pi xy/a_+a_-} \ell_N^\tau(x, y). \tag{4.4}$$

Invoking reality and evenness of $w_N(\cdot)$, as well as (2.54) and (2.16), this entails

$$\begin{aligned} &\psi_N(\delta x, y)\overline{\psi_N(\delta x', y)} = \hat{w}_N(y) w_N(x)^{1/2} w_N(x')^{1/2} \\ &\times \sum_{\tau,\tau'=+,-} e^{i\tau\pi xy/a_+a_-} e^{-i\tau'\pi x'y/a_+a_-} \ell_N^{\delta\tau}(\delta x, y) \ell_N^{\delta\tau'}(\delta x', -y), \quad \delta = +, - \quad x, x', y \in \mathbb{R} \end{aligned} \tag{4.5}$$

(to make the connection with (4.4) more transparent, relabel $\tau \to \delta\tau$ and $\tau' \to \delta\tau'$, as permitted under $\sum_{\tau,\tau'}$). When this is plugged back into (4.3), the δ-dependence gets confined to the following entire function,

$$L_N^{\tau,\tau'}(y,x,x') \equiv \sum_{\delta=+,-} \ell_N^{\delta\tau}(\delta x,y)\ell_N^{\delta\tau'}(\delta x',-y), \quad \tau,\tau'=+,- \tag{4.6}$$

This object has a non-obvious symmetry which is of critical importance; namely,

$$L_N^{\tau,\tau'}(y,x,x') = L_N^{-\tau,-\tau'}(-y,x,x'), \quad \tau,\tau'=+,- \tag{4.7}$$

Proved later, it implies that the y-integrand in (4.3) is even in y. This implies we can write the y-integral in (4.3) as

$$w_N(x)^{1/2}w_N(x')^{1/2}\int_{-\Lambda}^{\Lambda}\frac{dy}{2}\,\hat{w}_N(y)\sum_{\tau,\tau'=+,-} e^{i\tau\pi xy/a_+a_-}e^{-i\tau'\pi x'y/a_+a_-}L_N^{\tau,\tau'}(y,x,x') \tag{4.8}$$

This change in integral limits is crucial since $\hat{w}_N(\cdot)$ (2.53) has its poles along the imaginary axis which would frustrate an attempt to tackle the original integral by a rectangular contour. Now, however, (4.8) is a candidate for Proposition 3.1 with $s=y$. To see this we just have to note that the quasiperiodicity relation (2.18) means the integrand picks up a multiplier $\tau\tau' e_+(-\tau x)e_+(\tau' x)$ when y is shifted by ia_-. Like before, the use of this proposition leaves us with boundary integrals which have to be looked at more closely. This is done abstractly in the following proposition, which connects to (4.3) via the choice $m^\tau(x,y) = w_N(x)^{1/2}\ell_N^\tau(x,y)$. (We note that this proposition is very similar in form to its analogue, Proposition 3.2 - even more so if we negate the indices τ,τ' in one of them. Despite this, however, we claim they are not simply adaptations of each other.)

Proposition 4.1. *Suppose we have a function*

$$J^{\tau,\tau'}(y,x,x') = \exp(i\pi\tau xy/a_+a_-)\exp(-i\pi\tau' x'y/a_+a_-)\mathcal{J}^{\tau,\tau'}(y,x,x') \tag{4.9}$$

$$\mathcal{J}^{\tau,\tau'}(y,x,x') \equiv \sum_{\delta=+,-} m^{\delta\tau}(\delta x,y)m^{\delta\tau'}(\delta x',-y), \quad \tau,\tau'=+,- \tag{4.10}$$

where: $m^\pm(x,y)$ are entire in y and smooth in $x\in\mathbb{R}$; m^+ is ia_--periodic in y; and m^- is ia_--antiperiodic in y.

Suppose, furthermore, that $W(\cdot)$ satisfies the conditions of Proposition (3.1) *and, in addition, satisfies the following asymptotics relation with $\mathcal{J}^{\tau,\tau'}$ for* $\operatorname{Re} y \to \epsilon\infty$*,*

$$W(y)\mathcal{J}^{\tau,\tau'}(y,x,x') = \mathcal{L}_{(\epsilon)}^{\tau,\tau'}(x,x') + O(e_-(-\eta|\operatorname{Re} y|)), \quad \epsilon,\tau,\tau'=+,- \tag{4.11}$$

where $\eta>0$ is some constant and $\mathcal{L}_{(\epsilon)}^{\tau,\tau'}$ is smooth in x,x'. Also, the bound represented by $O(\cdot)$ is uniform for $\operatorname{Im} y, x, x'$ *in compact subsets of* $\mathbb{R}$. *The function represented by $O(\cdot)$ is smooth in x,x' and may depend on ϵ,τ,τ'. The x'-partial derivative of this function is assumed to inherit these same asymptotics properties.*

Then, for functions $f, g \in \mathcal{C}$,

$$(2a_+a_-)^{-1} \lim_{\Lambda\to\infty} \int_{\mathbb{R}} dx\, \overline{f(x)} \int_{\mathbb{R}} dx'\, g(x') \sum_{\tau,\tau'=+,-} (1 - \tau\tau' e_+(-\tau x)e_+(\tau' x'))^{-1}$$
$$\times \left(\int_{\Lambda}^{\Lambda+ia_-} + \int_{-\Lambda+ia_-}^{-\Lambda} \right) dy\, W(y) J^{\tau,\tau'}(y,x,x') = \int_{\mathbb{R}} dx\, \overline{f(x)} g(x) \sum_{\tau=+,-} \mathcal{L}^{\tau,\tau}_{(+)}(x,x) \quad (4.12)$$

Proof. The proof proceeds in a similar way to Proposition 3.2. Again, the $\tau' = -\tau$ terms in the sum are by far the simplest to handle since the reciprocal term in (4.12) is non-singular on the pertinent integration region. At issue is

$$\int_K dx dx' \frac{\overline{f(x)} g(x')}{1 + e_+(-\tau(x+x'))} \left(\int_{\Lambda}^{\Lambda+ia_-} + \int_{-\Lambda+ia_-}^{-\Lambda} \right) dy\, W(y) J^{\tau,-\tau}(y,x,x'), \quad \tau = +,- \quad (4.13)$$

where we have introduced the compact set $K \equiv \text{supp}(f) \times \text{supp}(g)$. With the $O(\cdot)$ term in (4.11) represented by the symbol $\hat{\rho}^{\tau,\tau'}_{(\epsilon)}(y,x,x')$, the reasoning is now identical to that below (3.18), with $(x,y,y') \to (y,x,x')$, $\mathcal{M} \to \mathcal{L}$ and $\rho \to \hat{\rho}$. We can thus be sure that the $\tau' = -\tau$ terms on the lhs of (4.12) vanish under $\Lambda \to \infty$.

Before proceeding to consider the $\tau' = \tau$ terms, we note that the function in the proposition has the following easily verified properties

$$\mathcal{J}^{\tau,\tau'}(y + ia_-, x, x') = \tau\tau' \mathcal{J}^{\tau,\tau'}(y,x,x') \quad (4.14)$$

$$\mathcal{J}^{\tau,\tau'}(y,x,x) = \mathcal{J}^{\tau',\tau}(-y,x,x) \quad (4.15)$$

The latter entails, furthermore,

$$\mathcal{L}^{\tau,\tau'}_{(+)}(x,x) = \mathcal{L}^{\tau',\tau}_{(-)}(x,x) \quad (4.16)$$

(meaning the rhs of (4.12) could equally be expressed in terms of $\mathcal{L}^{\tau,\tau}_{(-)}(x,x)$).

With the vanishing established for the $\tau' = -\tau$ terms, we know the lhs of (4.12) equals

$$(2a_+a_-)^{-1} \lim_{\Lambda\to\infty} \int_K dx dx'\, \overline{f(x)} g(x') \sum_{\tau=+,-} (1 - e_+(-\tau(x-x')))^{-1}$$
$$\times \left(\int_{\Lambda}^{\Lambda+ia_-} + \int_{-\Lambda+ia_-}^{-\Lambda} \right) dy\, W(y) J^{\tau,\tau}(y,x,x') \quad (4.17)$$

This is almost identical in form to its analogue, (3.22). Indeed we can easily adapt the formulas (3.23) and (3.24) in order to write it as

$$(2a_+a_-)^{-1} \lim_{\Lambda\to\infty} \int_K dx dx'\, \overline{f(x)} g(x')\, (2s_+((x-x')/2))^{-1} \int_{\Lambda-ia_-/2}^{\Lambda+ia_-/2} dy\, W(y + ia_-/2)$$
$$\times \sum_{\alpha,\tau=+,-} \alpha \exp(i\pi\alpha y(x-x')/a_+a_-) \mathcal{J}^{\tau,\tau}(\alpha\tau y + ia_-/2, x, x') \quad (4.18)$$

For a fixed $\tau = +, -$, the connection with Proposition A.1 is then given by

$$G_\alpha(y, x, x') = W(y + ia_-/2)\, \mathcal{J}^{\tau,\tau}(\alpha\tau y + ia_-/2, x, x'), \quad \alpha = +, - \tag{4.19}$$

To see that this satisfies the necessary assumptions we note that ia_--periodicity of $\mathcal{J}^{\tau,\tau}(\cdot, x, x')$, cf. (4.14), means we can write it as

$$W(y + ia_-/2)\, \mathcal{J}^{\tau,\tau}(\alpha\tau(y + ia_-/2), x, x') \tag{4.20}$$

The property $G_+(y, x, x) = G_-(y, x, x)$, (A.4), is then an immediate consequence of the symmetry (4.15). For the asymptotics (A.1), we use (4.11), (4.16) as well as evenness and ia_--periodicity of $W(\cdot)$ to get

$$G_\alpha(y, x, x') = \mathcal{L}^{\tau,\tau}_{(\tau\alpha)}(x, x') + O(e_-(-\eta \operatorname{Re} y)), \quad \operatorname{Re} y \to \infty \tag{4.21}$$

(As a rather technical point, note how it is not necessary for our choice of G_α to include a sum over τ, unlike in Proposition 3.2.)

To make the connection to Proposition A.1 explicit we must set $(s, t, t') = (y, x, x')$, $\Omega = \mathbb{R}$, $\phi(x, x') = \overline{f(x)}g(x')$ and $\pm = +$ in (A.7). Because of (4.21), we have $A(x) = \mathcal{L}^{\tau,\tau}_{(+)}(x, x)$ and so, summing over τ, (4.18) equals

$$\int_{\mathbb{R}^+} dx\, \overline{f(x)} g(x) \sum_{\tau=+,-} \mathcal{L}^{\tau,\tau}_{(+)}(x, x) \tag{4.22}$$

□

We are now ready for the theorem that expresses (4.2) in a form manifestly conducive to questions of isometry. It involves residues of the function $\hat{w}_N(\cdot)$ (2.53),

$$\hat{w}_j \equiv \operatorname*{Res}_{y=y_j^+} \hat{w}_N(y), \quad j = 1, \ldots, N+1 \tag{4.23}$$

as well as the following residue function, so-called because it depends on y_j^+ (2.55),

$$\hat{R}^{(j)}(x, x') \equiv \sum_{\tau,\tau'=+,-} (1 - \tau\tau' e_+(-\tau x)e_+(\tau' x'))^{-1} \exp(i\pi\tau x y_j^+/a_+a_-) \exp(-i\pi\tau' x' y_j^+/a_+a_-)$$
$$\times L_N^{\tau,\tau'}(y_j^+, x, x'), \quad j = 1, \ldots, N+1 \tag{4.24}$$

The formula we present is valid only when all the poles of $\hat{w}_N(\cdot)$ are simple, a property that hinges on the values of the underlying parameters a_+, a_-. More precisely, we know from Lemma 2.2 that the ratio a_-/a_+ cannot take values from the point set $\hat{\mathcal{E}}_N$ (2.57) if this property is to hold. The theorem applies to functions in $C_0^\infty(\mathbb{R})$ with support away from the origin. This set is dense in $\mathcal{H}$ and denoted $\mathcal{C}$.

Theorem 4.2. *For the map $\mathcal{F}_N^* : \mathcal{H} \to \hat{\mathcal{H}}$ with action* (4.1)*, the following is true for functions $f, g \in \mathcal{C}$ whenever $a_-/a_+ \notin \hat{\mathcal{E}}_N$*

$$(\mathcal{F}_N^* f, \mathcal{F}_N^* g)_{\hat{\mathcal{H}}} = (f, g)_{\mathcal{H}}$$
$$+ \frac{\pi i}{a_+a_-} \int_{\mathbb{R}} dx\, \overline{f(x)} \int_{\mathbb{R}} dx'\, g(x') w_N(x)^{1/2} w_N(x')^{1/2} \sum_{j=1}^{N+1} \hat{w}_j \hat{R}^{(j)}(x, x') \tag{4.25}$$

Moreover, this integral is absolutely convergent.

Proof. Denoting two times the integral function in (4.8) as $I(\Lambda, x, x')$ we already know from (4.2)-(4.6) that

$$(\mathcal{F}_N^* f, \mathcal{F}_N^* g)_{\hat{\mathcal{H}}} = \frac{c^2}{2} \lim_{\Lambda\to\infty} \int_{\mathbb{R}} dx\, \overline{f(x)} \int_{\mathbb{R}} dx'\, g(x') I(\Lambda, x, x') \tag{4.26}$$

The idea is to apply Proposition 3.1 to $I(\Lambda, x, x')$, and then let the resulting boundary integrals be handled by Proposition 4.1. For the choice $m^\tau(x,y) = w_N(x)^{1/2}\ell_N^\tau(x,y)$, evenness of $w_N(\cdot)$ means that the function (4.9) in Proposition 4.1 equals

$$J^{\tau,\tau'}(y,x,x') = w_N(x)^{1/2} w_N(x')^{1/2} e^{i\tau\pi xy/a_+a_-} e^{-i\tau'\pi x' y/a_+a_-} L_N^{\tau,\tau'}(y,x,x'), \quad \tau,\tau' = +,- \tag{4.27}$$

This function connects with $I(\Lambda, x, x')$ because, multiplied by $\hat{w}_N(y)/2$, it is just (4.8) and, moreover, because of the quasiperiodicity relation (2.18), it satisfies

$$J^{\tau,\tau'}(y + ia_-, x, x') = \tau\tau' e_+(-\tau x) e_+(\tau' x') J^{\tau,\tau'}(y,x,x'), \quad \tau,\tau' = +,- \tag{4.28}$$

This means it is a candidate for Proposition 3.1 with $s = y$, $W(y) = \hat{w}_N(y)$ and $M^{\tau,\tau'} = \tau\tau' e_+(-\tau x)e_+(\tau' x')$ (for the required properties of $\hat{w}_N(\cdot)$, cf. §2.2). Since we have $x, x' \in \mathbb{R}$, a problem arises for this multiplier when $x = x'$ and $\tau = \tau'$. This ceases to be a problem, however, when (3.8) is considered under $\lim_{\Lambda\to\infty}\int_K dxdx'$ for K compact and away from the origin, as we have in (4.26). The divergence affecting the residue term is handled by our absolute convergence claim (to be proved shortly), and that affecting the boundary integrals is handled by Proposition 4.1. Assuming therefore that $m^\pm$, introduced above satisfy the various criteria of Proposition 4.1, the two propositions combine to reveal that the rhs of (4.26) equals

$$\begin{aligned}\frac{1}{2}\frac{\pi i}{a_+a_-} \int_{\mathbb{R}} dx\, \overline{f(x)} \int_{\mathbb{R}} dx'\, g(x') \sum_{\tau,\tau'=+,-} (1 - \tau\tau' e_+(-\tau x)e_+(\tau' x'))^{-1} \sum_{j=1}^{N+1} \hat{w}_j \\ \times \left[J^{\tau,\tau'}(y_j^+, x, x') + J^{-\tau,-\tau'}(-y_j^+, x, x')\right] - \frac{1}{2}\int_{\mathbb{R}} dx\, \overline{f(x)} g(x) \sum_{\tau=+,-} \mathcal{L}_{(+)}^{\tau,\tau}(x,x)\end{aligned} \tag{4.29}$$

(concerning the poles, we have made the obvious choice $s_j = y_j^+$, $j = 1, \ldots, N+1$). We note two immediate simplifications of this that follow from the symmetry (4.7). In the first case, this same symmetry gets passed onto $J^{\tau,\tau'}(y,x,x')$ in the same form, simplifying the residue term. In the second, it entails via (4.11) that the τ-sum in the integral equals $2\mathcal{L}_{(+)}^{+,+}(x,x)$, as a result of

$$\mathcal{J}^{\tau,\tau'}(y,x,x') = w_N(x)^{1/2} w_N(x')^{1/2} L_N^{\tau,\tau'}(y,x,x'), \quad \tau,\tau' = +,- \tag{4.30}$$

Thus the rhs of (4.26) equals

$$\frac{\pi i}{a_+a_-}\int_{\mathbb{R}} dx\,\overline{f(x)}\int_{\mathbb{R}} dx'\,g(x')\sum_{\tau,\tau'=+,-}(1-\tau\tau' e_+(-\tau x)e_+(\tau' x'))^{-1}\sum_{j=1}^{N+1}\hat{w}_j J^{\tau,\tau'}(y_j^+,x,x') \\ -\int_{\mathbb{R}} dx\,\overline{f(x)}g(x)\mathcal{L}^{+,+}_{(+)}(x,x) \quad (4.31)$$

The second line in (4.25) then follows immediately from (4.27).

We now show that these $m^{\pm}(x,y)$ satisfy the criteria of Proposition 4.1. Entirety of $\ell_N^{\tau}(x,y)$ in both variables means that $m^{\tau}(x,\cdot)$ is entire and $m^{\tau}(\cdot,y)$ is smooth on $\mathbb{R}$ when the poles of $w_N(\cdot)$ are off the real axis, which we know is the case if $a_-/a_+\notin\hat{\mathcal{E}_N}$, cf. §2.2. Periodicity follows from (2.18).

The tricky criteria are those concerning (4.11). We first use the asymptotics (2.67), (2.68) and (2.65) to shed some light on $\hat{w}_N(y)\ell_N^{\tau}(x,y)\ell^{\tau'}(x',-y)$. These equations tell us that this object is $O(e_-(-|\operatorname{Re} y|))$ whenever τ or τ' is $-$. Indeed from this asymptotics we see that only $\tau=\tau'=+$ gives a non-decaying $O(1)$ term,

$$\hat{w}_N(y)\ell_N^{+}(x,y)\ell_N^{+}(x',-y)=(-)^N 4s_-(x+\epsilon\eta_N)s_-(x'-\epsilon\eta_N)s_N(\epsilon x-ia_+/2) \\ \times s_N(-\epsilon x'-ia_+/2)+O(e_-^{-2|\operatorname{Re} y|}),\quad y\to\epsilon\infty,\quad \epsilon=+,- \quad (4.32)$$

In order to realise (4.11) we must consider what this means for $\hat{w}_N(y)L_N^{\tau,\tau'}(y,x,x')$, (4.6). We can see that only the $\tau'=\tau$ terms will feature a specialisation of the kind (4.32), which will come from the $\delta=\tau$ term in the sum. In other words, we have

$$\hat{w}_N(y)L_N^{\tau,-\tau}(y,x,x')=O(e_-^{-|\operatorname{Re} y|}),\quad \operatorname{Re} y\to\pm\infty,\quad \tau=+,- \quad (4.33)$$

$$\hat{w}_N(y)L_N^{\tau,\tau}(y,x,x')=(-)^N 4s_-(\tau x+\epsilon\eta_N)s_-(\tau x'-\epsilon\eta_N)s_N(\epsilon\tau x-ia_+/2) \\ \times s_N(-\epsilon\tau x'-ia_+/2)+O(e_-^{-2|\operatorname{Re} y|}),\quad \operatorname{Re} y\to\epsilon\infty,\quad \epsilon,\tau=+,- \quad (4.34)$$

From this we can read off

$$\mathcal{L}^{\tau,-\tau}_{(\epsilon)}(x,x')=0,\quad \epsilon,\tau=+,- \quad (4.35)$$

and

$$\mathcal{L}^{\tau,\tau}_{(\epsilon)}(x,x')=w_N(x)^{1/2}w_N(x')^{1/2}(-)^N 4s_-(\tau x-\epsilon\eta_N)s_-(\tau x'+\epsilon\eta_N)s_N(-\epsilon\tau x-ia_+/2) \\ \times s_N(\epsilon\tau x'-ia_+/2),\quad \epsilon,\tau=+,- \quad (4.36)$$

(as a check one can see that these satisfy (4.16)). The symmetry (4.7) means that the sum on the rhs of (4.12) equals $2\mathcal{L}^{+,+}_{(+)}(x,x)$. Accordingly we compute,

$$\mathcal{L}^{+,+}_{(+)}(x,x)=w_N(x)(-)^N 4s_-(x+\eta_N)s_-(x-\eta_N)s_N(x-ia_+/2)s_N(-x-ia_+/2) \quad (4.37)$$

The result follows because from (2.7) and (2.60) we have

$$(-)^N 4s_-(x+\eta_N)s_-(x-\eta_N)s_N(x-ia_+/2)s_N(-x-ia_+/2) = 1/w_N(x) \tag{4.38}$$

We now deal with the absolute convergence claim. First we note that the poles of $\hat{w}_N(\cdot)$ are off the real axis when $a_-/a_+ \notin \hat{\mathcal{E}}_N$, cf. §2.2. Since the functions $\ell_N^\tau(\cdot,\cdot)$ that compose L_N are entire, it is thus clear that the only threat of divergence can come from the reciprocal term when $\tau' = \tau$. Suppose we can argue that the L_N term in (4.24) is in fact independent of the value of τ whenever $\tau' = \tau$ and $x' = x$. Then $\hat{R}^{(j)}(x,x)$ will be finite because of the limit (3.50) with $(y,y') \to (x,x')$. The supposition holds because the symmetries (4.7) and (4.43) together imply

$$L_N^{+,+}(y,x,x) = L_N^{-,-}(y,x,x) \tag{4.39}$$

□

We now take a closer look at the entire function L_N, which we recall is defined by

$$L_N^{\tau,\tau'}(y,x,x') \equiv \sum_{\delta=+,-} \ell_N^{\delta\tau}(\delta x, y)\ell_N^{\delta\tau'}(\delta x', -y), \quad \tau,\tau' = +,- \tag{4.40}$$

In particular we want to prove the symmetry which we have assumed throughout this section,

$$L_N^{\tau,\tau'}(y,x,x') = L_N^{-\tau,-\tau'}(-y,x,x') \tag{4.41}$$

(cf. (4.7)). To do this we will connect it to the function M_N in §2.1 which we have studied in detail. Comparison of the definitions reveals

$$L_N^{\tau,\tau'}(y,x,x') = M_N^{\tau,\tau'}(x,x';y,-y), \quad \tau,\tau' = +,- \tag{4.42}$$

Thus (4.41) follows immediately from the "deep" symmetry (2.39). In addition we have the "surface" symmetries, which can either be read off from (4.40) or viewed as special cases of (2.32) and (2.33),

$$L_N^{\tau,\tau'}(y,x,x') = \begin{cases} L_N^{\tau',\tau}(-y,x',x) & (4.43) \\ L_N^{-\tau,-\tau'}(y,-x,-x') & (4.44) \end{cases}$$

(these are also satisfied for $\mathcal{I}^{\tau,\tau'}$ (4.10) since this has the same form). There is also the quasiperiodicity property which we have been using implicitly, which follows straightforwardly from (2.18),

$$L_N^{\tau,\tau'}(y+ia_-,x,x') = \tau\tau' L_N^{\tau,\tau'}(y,x,x') \tag{4.45}$$

4.2 Residue analysis

As we explained at the start of the previous subsection, the main reason for studying $\mathcal{F}_N^*$ is to learn more about $\mathcal{F}_N$. In particular we are interested in the behaviour of $\mathcal{F}_N^*$ for the polytope of a_+, a_- described by $a_-/a_+ > N + 1/2$. These are the values for which $\mathcal{F}_N$ is isometric and thus pertinent to quantum mechanics. Suppose we can show that $\mathcal{F}_N^*$ is also an isometry for some subset of this polytope, then we learn that $\mathcal{F}_N$ is onto and thus unitary. This is a consequence of the following general equality

$$\ker(T^*) = T(\mathcal{H}_2)^\perp \tag{4.46}$$

which holds for any bounded map $T : \mathcal{H}_2 \to \mathcal{H}_1$ and its adjoint $T^* : \mathcal{H}_1 \to \mathcal{H}_2$ (which is necessarily bounded). In the quantum mechanical picture, the implication of $\operatorname{Ran} \mathcal{F}_N = \mathcal{H}$ is that the Hilbert space $\mathcal{H}$ is made up entirely of scattering states - provided of course that we can prove $\mathcal{F}_N$ is indeed the wave operator (forgetting the results from Chapter 1 this is just an expectation).

The following corollary gives us grounds for assessing isometry of $\mathcal{F}_N^*$. The restriction $a_-/a_+ \notin \hat{\mathcal{E}}_N$ is simply there to ensure the validity of the formula (4.25). Whenever this formula holds, the claim is immediate and there is no need for further proof,

Corollary 4.3. *Provided $a_-/a_+ \notin \hat{\mathcal{E}}_N$, the map $\mathcal{F}_N^* : \mathcal{H} \to \hat{\mathcal{H}}$ with action* (4.1) *is an isometry iff the residue sum $\sum_{j=1}^{N+1} \hat{w}_j \hat{R}^{(j)}(x, x')$ vanishes.*

We recall that the residue function here is defined by

$$\hat{R}^{(j)}(x, x') \equiv \sum_{\tau,\tau'=+,-} (1 - \tau\tau' e_+(-\tau x) e_+(\tau' x'))^{-1} \exp(i\pi\tau x y_j^+/a_+a_-) \exp(-i\pi\tau' x' y_j^+/a_+a_-) \times L_N^{\tau,\tau'}(y_j^+, x, x'), \quad j = 1, \ldots, N+1 \tag{4.47}$$

$$L_N^{\tau,\tau'}(y_j^+, x, x') = \ell_N^\tau(x, y_j^+) \ell_N^{\tau'}(x', -y_j^+) + \ell_N^{-\tau}(-x, y_j^+) \ell_N^{-\tau'}(-x', -y_j^+) \tag{4.48}$$

and recall further that: $\ell_N^\tau(\cdot, \cdot)$ is an entire function defined in (2.15); y_j^+ is one of the $2(N+1)$ poles of the weight function $\hat{w}_N(y)$ in the strip $\operatorname{Im} y \in i(0, a_-)$, cf. §2.2. The residue of this function at $y = y_j^+$ is denoted $\hat{w}_j$.

In all the examples we are going to see, vanishing of the residue sum comes about because of vanishing of each individual summand. (Indeed in §5 we prove that for fixed N, the residue functions (4.47) are linearly independent.) To provide some guidance, we run through the $N = 0$ case explicitly, before turning to the general case in Lemma 4.4. Exploiting the connection in (4.42), we use (2.36) and $\Sigma_0 = 1$ to deduce,

$$L_0^{\tau,\tau'}(y, x, x') = M_0^{\tau,\tau'}(x, x'; y, -y) = -\tau\tau' \sum_{\nu,\nu'=+,-} 4\nu\nu' s_-(x - i\nu a_+/2) s_-(x' - i\nu' a_+/2) \times \mu_{\nu\nu'}(y, -y) e_-(-\nu\tau y/2) e_-(\nu'\tau' y/2) \tag{4.49}$$

We have seen the function $\mu_{\nu\nu'}(y,-y)$ already in (2.40). Indeed from the latter we have that $\mu_\pm(y_1^+,-y_1^+) = 2c_-(ia_+)$. Setting $y = y_1^+ = ia_+$ in (4.49), we isolate

$$\sum_{\nu=+,-} 2\nu s_-(x - ia_+/2)e_-(-i\nu\tau a_+/2) = \sum_{\nu=+,-} \nu\big(e_-^x e_-^{-i\nu a_+/2} - e_-^{-x} e_-^{i\nu a_+/2}\big)e_-^{-i\nu\tau a_+/2}$$
$$= -e_-(\tau x)s_-(ia_+), \quad \tau = +,- \tag{4.50}$$

Accordingly,

$$L_0^{\tau,\tau'}(y_1^+, x, x') = -8\tau\tau' c_-(ia_+)s_-(ia_+)^2 e_-(\tau x)e_-(-\tau' x') \tag{4.51}$$

Noting that for $y = ia_+$, the plane wave product in (4.47) is just $e_-(-\tau x)e_-(\tau' x')$ we therefore have

$$\hat{R}^{(1)}(x, x') = -8c_-(ia_+)s_-(ia_+)^2 \sum_{\tau,\tau'=+,-} (1 - \tau\tau' e_+(-\tau x)e_+(\tau' x'))^{-1}\tau\tau' \tag{4.52}$$

In fact this sum vanishes outright because of the simple identity

$$\frac{1}{1 - A'/A} + \frac{1}{1 - A/A'} - \frac{1}{1 + 1/AA'} - \frac{1}{1 + AA'} = 0 \tag{4.53}$$

Thus $\mathcal{F}_0^*$ is an isometry when $y_1^+ = ia_+$.

In Lemma 4.4 we learn that this vanishing extends to all N, j when a_+, a_- are suitably restricted. Above we simply assumed that $y_1^+ = ia_+$, but in general y_j^+ only has the form

$$y_j^+ = ija_+, \quad j = 1, \ldots, N+1 \tag{4.54}$$

if $ja_+ < a_-$, cf. §2.2. When a_-/a_+ is sufficiently small such that this is no longer satisfied, we must subtract some multiple of ia_- from the rhs of (4.54) to obtain the correct expression for y_j^+. This change spoils the vanishing argument, as we will see in Lemma 4.6. The restriction in Lemma 4.4 thus ensures y_j^+ has the form (4.54), and that in Theorem 4.5 ensures *all* y_j^+ have the form (4.54).

With regard to how the general N argument differs from the $N = 0$ vanishing argument, the main difference in the general N case is the same as that which we described before Lemma 3.5. The end-goal is again to isolate the same τ, τ'-sum in (4.52).

Lemma 4.4. *If the positive parameters a_+, a_- satisfy $a_-/a_+ > j$ for a fixed $j = 1, \ldots, N+1$, then the residue function $\hat{R}^{(j)}(x, x')$ (4.47) vanishes.*

Theorem 4.5. *If the positive parameters a_+, a_- satisfy $a_-/a_+ > N+1$, and $a_-/a_+ \notin \hat{\mathcal{E}}_N$ (2.57), then the adjoint $\mathcal{F}_N^* : \mathcal{H} \to \hat{\mathcal{H}}$ with action (4.1) is an isometry. Equivalently, $\mathcal{F}_N : \hat{\mathcal{H}} \to \mathcal{H}$ is an isometry onto $\mathcal{H}$, i.e. unitary.*

Proof of Theorem 4.5. Because $a_-/a_+ > N + 1/2$, we know from Theorem 3.6 that $\mathcal{F}_N$ is an isometry. Corollary 4.3 and Lemma 4.4 establish that $\mathcal{F}_N^*$ is an isometry for $a_-/a_+ > N+1$ and $a_-/a_+ \notin \hat{\mathcal{E}}_N$. $\square$

Proof of Lemma 4.4. We first rewrite $\hat{R}^{(j)}(x,x')$ in terms of

$$\lambda_N^\tau(x,y) = \exp(i\pi\tau xy/a_+a_-)\ell_N^\tau(x,y), \quad \tau = +,- \tag{4.55}$$

for which we have the explicit expression (2.29). It is best to conduct this rearrangement with $y_j^+ \to y$ in (4.47), and then to set $y = y_j^+$ at the end. This is straightforward because from (4.55) it follows that

$$\begin{aligned}&\exp(i\pi\tau xy/a_+a_-)\exp(-i\pi\tau' x'y/a_+a_-)L_N^{\tau,\tau'}(y,x,x')\\ &\qquad = \lambda_N^\tau(x,y)\lambda_N^{\tau'}(x',-y) + \lambda_N^{-\tau}(-x,y)\lambda_N^{-\tau'}(-x',-y), \quad \tau,\tau' = +,-\end{aligned} \tag{4.56}$$

And so

$$\begin{aligned}\hat{R}^{(j)}(x,x') = \sum_{\tau,\tau'=+,-}&(1-\tau\tau' e_+(-\tau x)e_+(\tau' x'))^{-1}\big[\lambda_N^\tau(x,y_j^+)\lambda_N^{\tau'}(x',-y_j^+)\\ &+ \lambda_N^{-\tau}(-x,y_j^+)\lambda_N^{-\tau'}(-x',-y_j^+)\big], \quad j = 1,\ldots,N+1\end{aligned} \tag{4.57}$$

The four λ_N-functions that feature here are all variants of

$$\lambda_N^\tau(x,\alpha y_j^+), \quad \alpha,\tau = +,- \tag{4.58}$$

and so we proceed to study this function more closely. From (2.29) we learn that its τ-dependence hinges on

$$K_N(x + i\nu a_+/2, \alpha\tau y_j^+), \quad \alpha,\nu,\tau = +,- \tag{4.59}$$

in the sense that (4.58) equals

$$\tau\hat{\phi}K_N(x + ia_+/2, \alpha\tau y_j^+) + \tau\hat{\varphi}K_N(x - ia_+/2, \alpha\tau y_j^+) \tag{4.60}$$

where $\hat{\phi},\hat{\varphi}$ are some j,α-dependent entire functions in x.

We now look at (4.59) more closely. Since $y_j^+ = ija_+$, we can use (2.24) whenever $j \le N$. This tells us that (4.59) equals

$$i^N B_{N-j}^{(N)}(c_-(x + i\nu a_+/2)) \tag{4.61}$$

which is independent of α and, more crucially, of τ. It implies that when $j \le N$, (4.59) is proportional to τ (with respect to the variable τ, cf. note below (3.69)). As a result, (4.57) equals some j-dependent entire function in x, x' multiplied by

$$\sum_{\tau,\tau'=+,-}(1-\tau\tau' e_+(-\tau x)e_+(\tau' x'))^{-1}\tau\tau' \tag{4.62}$$

This is the function we saw in (4.52) and we know it vanishes due to (4.53).

Thus we have shown that $\hat{R}^{(j)}(x,x') = 0$ when $y_j^+ = ija_+$ and $j \le N$. It remains to establish vanishing when $j = N+1$. It turns out this is much trickier because (2.24) can no longer be used.

Looking first at the $\alpha = +$ term in (4.58), it follows from (2.29) that

$$\lambda_N^\tau(x, y_{N+1}^+) = 2(-)^N i^{N+1} \tau\, p_N \tag{4.63}$$

where we have isolated for special attention

$$p_N \equiv \sum_{\nu=+,-} \nu s_-(x - i\nu(N + 1/2)a_+) K_N(x + i\nu a_+/2, \tau\xi) \tag{4.64}$$

$$\xi \equiv i(N + 1)a_+ \tag{4.65}$$

We have attached neither x nor τ dependence to the symbol p_N because we claim it is in fact independent of these variables, i.e. constant. Assuming this for the moment, we turn to the $\alpha = -$ term in (4.58). The analogous sum one finds is

$$\sum_{\nu=+,-} \nu s_-(x - i\nu(N + 1/2)a_+) e_-(\nu\xi) K_N(x + i\nu a_+/2, -\tau\xi) \tag{4.66}$$

But for this, the argument that we will apply to the rhs of (4.64) does not work. To get around this we should stop studying the $\alpha = +, -$ terms separately, and consider the whole square-bracketed term in (4.57). We note that the latter is just the $y = y_{N+1}^+ = \xi$ specialisation of $\Lambda_N^{\tau,\tau'}(x, x'; y, -y)$ where Λ_N is the function we introduced and studied in §2.1. Amongst the things we have for Λ_N is the partially evaluated expression (2.46). From this we learn that

$$\begin{aligned}\Lambda_N^{\tau,\tau'}(x, x'; \xi, -\xi) = 2c_-(\xi)(-)^{N+1}\tau\tau' &\sum_{\nu,\nu'=+,-} 4\nu\nu' s_-(x - \nu\eta_N) s_-(x' - \nu'\eta_N) \\ &\times K_N(x + i\nu a_+/2, \tau\xi) K_N(x' + i\nu' a_+/2, -\tau'\xi)\end{aligned} \tag{4.67}$$

$$\eta_N \equiv i(N + 1/2) \tag{4.68}$$

The nuisance term in (4.66), namely $e_-(\nu\xi)$, has passed outside the sum as part of $\mu_{\nu\nu'}(\xi, -\xi) = 2c_-(\xi)$. In fact (4.67) is now manifestly equal to

$$8c_-(\xi)(-)^{N+1}\tau\tau' p_N^2 \tag{4.69}$$

Thus the square bracketed term in (4.57) is again straightforwardly proportional to $\tau\tau'$ (cf. note below (3.69)) and so vanishing follows due to (4.53).

It remains to prove that the rhs of (4.64) is indeed independent of x and τ (there is no need to compute p_N until Theorem 4.7). From (2.19) and $2ia_-$-periodicity of Σ_N, we deduce

$$K_N(x + 2ia_-, y) = e_+(-2y) K_N(x, y) \tag{4.70}$$

Thus $K_N(x, y)$ is $2ia_-$-periodic in x when $y = \pm i(N + 1)a_+$. It is therefore clear that the rhs of (4.64) is $2ia_-$-periodic overall. Denoting the rhs of (4.64) as $S(x)$, we compute

$$S(x+ia_+/2)-S(x-ia_+/2) = \big[s_-(x-iNa_+)K_N(x+ia_+,\xi)+s_-(x+iNa_+)K_N(x-ia_+,\xi)\big] \\ - K_N(x,\tau\xi)\big(s_-(x+\xi)+s_-(x-\xi)\big) \quad (4.71)$$

The term in square brackets is just the lhs of the AΔE, (2.22), and so we know it equals $2s_-(x)c_-(\xi)K_N(x,\xi)$. In addition, $s_-(x+\xi)+s_-(x-\xi) = 2s_-(x)c_-(\xi)$, and so (4.71) vanishes. This implies $S(\cdot)$ is both $2ia_-$ and $2ia_+$-periodic. Thus whenever a_+/a_- is irrational, $S(\cdot)$ is constant. This constancy extends to all $a_+/a_- \in (0,\infty)$ because $S(a_+,a_-;x)$ is made up from functions that are smooth in a_+,a_-,x.

□

So far we have established that $\mathcal{F}_N^*$ is an isometry when $a_-/a_+ > N+1$, and subsequently that $\operatorname{Ran}\mathcal{F}_N = \mathcal{H}$. It is now natural to ask what happens in the range $a_-/a_+ \in (N+1/2, N+1)$. Here we know that $\mathcal{F}_N$ is an isometry, but we do not yet know anything about the behaviour of $\mathcal{F}_N^*$, i.e. about the residue sum in Corollary 4.3. We will find that the sum is non-vanishing in this range, meaning that $\mathcal{F}_N^*$ is not an isometry. Furthermore we will find that its kernel is non-zero. Recalling (4.46), this entails that $\mathcal{F}_N$ is no longer onto. In the quantum mechanical picture, this corresponds to existence of bound states. To describe these explicitly we need to know $\operatorname{Ran}\mathcal{F}_N$ explicitly. It turns out that Theorem 4.2 allows us to do this, and the findings are presented in Theorem 4.7. The picture is then complete once we show that these bound states are eigenfunctions of the Hamiltonian.

Let us return to a study of the residue function, (4.47). As we discussed circa (4.54), a consequence of a_-/a_+ dropping below $N+1$ is that the pole y_{N+1}^+ no longer has the form $i(N+1)a_+$. The latter exceeds the strip $i[0,a_-)$ and so we must subtract some multiple of ia_-. In the case of $N \geq 1$ we can fix $a_-/a_+ \in (N,N+1)$ to get

$$y_{N+1}^+ = i(N+1)a_+ - ia_- \quad (4.72)$$

(which is now guaranteed to live in the pertinent strip). For this choice of interval, all the remaining $y_{j\leq N}^+$ still have the form ija_+. In the case of $N=0$ the latter consideration does not apply. Instead we fix $a_-/a_+ \in (1/2,1)$ to get

$$y_1^+ = ia_+ - ia_- \quad (4.73)$$

Let us run through the computation of the $N=0$ residue function explicitly to offer some guidance for the general case. The quasiperiodicity relation (4.45) tells us that

$$L_0^{\tau,\tau'}(y_1^+,x,x') = \tau\tau' L_0^{\tau,\tau'}(ia_+,x,x') \quad (4.74)$$

The L_0 term on the rhs here is now identical to that in (4.51) (where we were dealing with $y_1^+ = ia_+$). For $y_1^+ = ia_+$, the plane wave term in (4.47) reduces to to $e_-(-\tau x)$ $e_-(\tau' x')e_+(\tau x)e_+(-\tau' x')$ and so

$$\hat{R}^{(1)}(x,x') = -8c_-(ia_+)s_-(ia_+)^2 \sum_{\tau,\tau'=+,-} (1-\tau\tau' e_+(-\tau x)e_+(\tau' x'))^{-1}\tau\tau' \quad (4.75)$$

Unlike with (4.62), the sum here does not vanish outright. Indeed, setting $A \equiv e_+(x)$ and $A' \equiv e_+(x')$ it equals

$$\begin{aligned}\frac{A}{A'}\frac{1}{1-A'/A}+\frac{A'}{A}\frac{1}{1-A/A'}+AA'\frac{1}{1+1/A'A}+\frac{1}{AA'}\frac{1}{1+AA'}&\\ =\frac{A'}{A}+\frac{A}{A'}+A'A+\frac{1}{A'A}&\\ =2c_+(x-x')+2c_+(x+x')=4c_+(x)c_+(x')&\end{aligned} \tag{4.76}$$

(Doubtless there are many ways to get from the first line to the second, but one is to make use of the identity

$$\frac{B}{1\mp 1/B}+\frac{1/B}{1\mp B}=B+\frac{1}{B}\pm 1\,.) \tag{4.77}$$

Returning to (4.25), what this implies is that

$$(\mathcal{F}_0^* f, \mathcal{F}_0^* g)_{\hat{\mathcal{H}}} = (f,g)_{\mathcal{H}} - c_0 (f,\Psi_0)_{\mathcal{H}}(\Psi_0, g)_{\mathcal{H}} \tag{4.78}$$

where $c_0 = -\sin(\pi a_+/a_-)/a_+$ and $\Psi_0(x) = 2c_+(x)w_0(x)^{1/2}$ (in the process we have computed $\hat{w}_0 = -ia_- \sin(2\pi a_+/a_-)/4\pi$). Note that the sign of c_0 is positive for the range of this formula's validity, $a_-/a_+ \in (1/2,1)$. The implications of (4.78), and its general N analogue, are explored in Theorem 4.7 below. This theorem involves the point set $\hat{\mathcal{E}}_N$ (2.57) and the weight function $w_N(x)$ (2.53). First we give the lemma that generalises the above $N=0$ calculation.

Lemma 4.6. *If for $N>0$, the positive parameters a_+, a_- satisfy $a_-/a_+ \in (N, N+1)$ then the $j=N+1$ residue function (4.47) is given by*

$$\hat{R}^{(N+1)}(x,x') = 8\cos(\pi(N+1)a_+/a_-)c_+(x)c_+(x')\prod_{j=N+1}^{2N+1}[2\sin(\pi j a_+/a_-)]^2 \tag{4.79}$$

Theorem 4.7. *If the positive parameters a_+, a_- satisfy $a_-/a_+ \in (N+1/2, N+1)\setminus \hat{\mathcal{E}}_N$ then the map $\mathcal{F}_N : \hat{\mathcal{H}} \to \mathcal{H}$ defined by (2.69) is an isometry which partakes in the following orthogonal decomposition of Hilbert space,*

$$\mathcal{H} = \operatorname{Ran}\mathcal{F}_N \oplus \operatorname{span}\{\Psi_N\} \tag{4.80}$$

where $\Psi_N(x) \equiv 2c_+(x)w_N(x)^{1/2}$ satisfies the AΔE

$$\Psi_N(x-ia_-)+\Psi_N(x+ia_-) = 2\cos(\pi a_-/a_+)\Psi_N(x) \tag{4.81}$$

and has norm

$$\|\Psi_N\|^2 = (-)^{N+1}a_+\frac{\prod_{j=1}^{N}\sin(\pi j a_+/a_-)}{\prod_{j=N+1}^{2N+1}\sin(\pi j a_+/a_-)} \tag{4.82}$$

Proof of Lemma 4.6. As explained above (4.72), the restriction in the lemma entails $y^+_{N+1} = i(N+1)a_+ - ia_-$. In the proof of Lemma 4.4 we dealt with this residue function for the case when $y^+_{N+1} = i(N+1)a_+$. Much of the material there can be recycled due to the ia_--quasiperiodicity properties of some of the terms in the residue function, (4.47). Specifically, the plane wave product picks up a multiplier $e_+(\tau x)e_+(-\tau' x')$ when y is shifted by $-ia_-$, and the L_N term picks up $\tau\tau'$, cf. (4.45). Thus we get

$$\hat{R}^{(N+1)}(x,x') = \sum_{\tau,\tau'=+,-} (1-\tau\tau' e_+(-\tau x)e_+(\tau' x'))^{-1} e_+(\tau x)e_+(-\tau' x')\tau\tau' \\ \times \exp(i\pi\tau x\xi/a_+a_-)\exp(-i\pi\tau' x'\xi/a_+a_-) L_N^{\tau;\tau'}(\xi,x,x') \quad (4.83)$$

$$\xi \equiv i(N+1)a_+ \quad (4.84)$$

The second line here is now identical to the square-bracketed term in (4.57) for $y^+_j = y^+_{N+1} = \xi$. In other words, it equals $\Lambda_N^{\tau;\tau'}(x,x';\xi,-\xi)$, cf. (4.67). Above we learned that this object is independent of x, x' and proportional to $\tau\tau'$ (cf. note below (3.69)). Thus $\hat{R}^{(N+1)}(x,x')$ equals some constant multiplied by

$$\sum_{\tau,\tau'=+,-} (1-\tau\tau' e_+(-\tau x)e_+(\tau' x'))^{-1} e_+(\tau x)e_+(-\tau' x') \quad (4.85)$$

We have seen this sum already in (4.75) and we know it equals $4c_+(x)4c_+(x')$. Furthermore, from (4.69) it is clear that the constant just mentioned is $8c_-(\xi)(-)^{N+1}p_N^2$, and so

$$\hat{R}^{(N+1)}(x,x') = 32c_-(\xi)(-)^{N+1}p_N^2 c_+(x)c_+(x') \quad (4.86)$$

It remains to determine the constant p_N, (4.64), explicitly. We can set x and τ on the rhs of (4.64) to be anything we want. For example $x = \eta_N$ and $\tau = +$ gives

$$p_N = \sum_{\nu=+,-} \nu s_-(\eta_N - \nu\eta_N)K_N(\eta_N + \nu\eta_0, \xi) = -s_-(2\eta_N)K_N(\eta_N - \eta_0, \xi) \quad (4.87)$$

This enables us to now use (2.23), since $\eta_N - \eta_0 = iNa_+$. With this we find

$$p_N = -\frac{1}{2}\prod_{j=N+1}^{2N+1} 2s_-(ija_+) \quad (4.88)$$

□

Proof of Theorem 4.7. The point about $a_-/a_+ > N + 1/2$ is that it ensures $\mathcal{F}_N$ is an isometry, cf. Theorem 3.6. We recall the general fact that if a map $T : \mathcal{H}_2 \to \mathcal{H}_1$ is an isometry then

$$T^*T = \mathbf{1}_2, \quad TT^* = \mathbf{1}_1 - \mathrm{proj}_{\mathcal{P}} \quad (4.89)$$

where: $\mathbf{1}_1/\mathbf{1}_2$ denotes identity on $\mathcal{H}_1/\mathcal{H}_2$; proj_X denotes orthogonal projection onto $X \subseteq \mathcal{H}_1$; and $\mathcal{P} \equiv (\mathrm{Ran}\, T)^\perp$ is the orthogonal complement of $\mathrm{Ran}\, T$.

From (4.89) we therefore have

$$\mathcal{F}_N \mathcal{F}_N^* = \mathbf{1}_{\mathcal{H}} - \mathrm{proj}_{(\mathrm{Ran}\,\mathcal{F}_N)^\perp} \tag{4.90}$$

By using the residue result in Lemma 4.6 and the formula in Theorem 4.2 we will show that

$$(\mathcal{F}_N^* f, \mathcal{F}_N^* g)_{\hat{\mathcal{H}}} = (f,g)_{\mathcal{H}} - c_N (f, \Psi_N)_{\mathcal{H}} (\Psi_N, g)_{\mathcal{H}} \tag{4.91}$$

where c_N is some constant. Rewriting this as

$$(\mathcal{F}_N \mathcal{F}_N^* f, g)_{\hat{\mathcal{H}}} = (f,g)_{\mathcal{H}} - c_N ((\Psi_N, f)_{\mathcal{H}} \Psi_N, g)_{\mathcal{H}} \tag{4.92}$$

and invoking the fact f, g are arbitrary elements of a dense subspace of $\mathcal{H}$, it is clear that we get the operator equation

$$\mathrm{proj}_{[\mathrm{Ran}\,\mathcal{F}_N]^\perp} = c_N (\Psi_N, \cdot\,)_{\mathcal{H}} \Psi_N \tag{4.93}$$

From this we learn that $[\mathrm{Ran}\,\mathcal{F}_N]^\perp = \mathrm{span}\{\Psi_N\}$. Furthermore, with knowledge that the rhs is a projection, it necessarily follows that

$$c_N (\Psi_N, \Psi_N)_{\mathcal{H}} = 1 \tag{4.94}$$

We now exhibit (4.91) and find c_N. With vanishing of the N functions $\hat{R}^{(j \le N)}(x, x')$ secured by Lemma 4.4 for $a_-/a_+ > N$, formula (4.25) becomes

$$\begin{aligned}(\mathcal{F}_N^* f, \mathcal{F}_N^* g)_{\hat{\mathcal{H}}} &= (f,g)_{\mathcal{H}} \\ &+ \frac{\pi i}{a_+ a_-} \int_{\mathbb{R}} dx\, \overline{f(x)} \int_{\mathbb{R}} dx'\, g(x') w_N(x)^{1/2} w_N(x')^{1/2} \hat{w}_{N+1} \hat{R}^{(N+1)}(x, x')\end{aligned} \tag{4.95}$$

Already it is clear from Lemma 4.6 that this residue function has the separability of variables needed to realise (4.91). But to provide an explicit, elementary expression for c_N, and thus for the norm of Ψ_N, we must also calculate $\hat{w}_{N+1}$.

The restriction $a_-/a_+ \notin \hat{\mathcal{E}}_N$ ensures simplicity of the pole at y^+_{N+1}. Furthermore, because of ia_--periodicity of $\hat{w}_N(\cdot)$, the residue is equal to that at $i(N+1)a_+$. The substantial part of the calculation is

$$\operatorname*{Res}_{y = i(N+1)a_+} s_-(y - i(N+1)a_+)^{-1} = (a_-/\pi) \tag{4.96}$$

with which we find

$$\hat{w}_{N+1} = (a_-/\pi) i \sin(\pi(N+1)a_+/a_-)(-)^{N+1} \Big[\prod_{j=1}^{2N+2} 2 \sin(\pi j a_+/a_-) \Big]^{-1} \tag{4.97}$$

And so, using Lemma 4.6 for $N > 0$ and (4.75) for $N = 0$, we get

$$\frac{\pi i}{a_+ a_-} \hat{w}_{N+1} \hat{R}^{(N+1)}(x, x') = \frac{(-)^N}{a_+} 4 c_+(x) c_+(x') \frac{\prod_{j=N+1}^{2N+1} \sin(\pi j a_+/a_-)}{\prod_{j=1}^{N} \sin(\pi j a_+/a_-)} \tag{4.98}$$

Recalling that $\Psi_N(x) \equiv 2c_+(x)w_N(x)^{1/2}$ we realise (4.91) with

$$c_N \equiv \frac{(-)}{a_+}^{N+1} \frac{\prod_{j=N+1}^{2N+1} \sin(\pi j a_+/a_-)}{\prod_{j=1}^{N} \sin(\pi j a_+/a_-)} \tag{4.99}$$

We note how it is not even obvious that this number is positive for $a_-/a_+ > N + 1/2$, but we know this must be so from (4.94).

Finally, the AΔE (4.81) is a trivial consequence of ia_--periodicity of $w_N(\cdot)$ and the fact $c_+(x - ia_-) + c_+(x + ia_-) = 2c_+(x)c_+(ia_-)$.

$\square$

5 Isometry breakdown.

We now look more closely at the situation when isometry of $\mathcal{F}_N$ and $\mathcal{F}_N^*$ breaks down. We have already seen an instance of $\mathcal{F}_N^*$ isometry breakdown in the emergence of the ground state in Theorem 4.7 (and for this reason, Theorem 5.7 in this subsection is fairly immediate). Now we put aside our interest in quantum mechanics and study the matter in a purely mathematical light, looking first at $\mathcal{F}_N$ and then returning to $\mathcal{F}_N^*$.

5.1 Breakdown for $\mathcal{F}_N$

As we explained earlier, the pole x_j^+ (cf. §2.2) is no longer given by $i(j+1/2)a_+$ when the ratio a_-/a_+ drops below $j+1/2$. In the most general case, x_j^+ is given by

$$x_j^+ = i(j+1/2)a_+ - in_j a_- \tag{5.1}$$

where the precise value of the integer $n_j \geq 0$ is fixed by the requirement that x_j^+ be in $\operatorname{Im} x \in [0, a_-)$, and hence depends on a_-/a_+. We know already that when $a_-/a_+ > N+1/2$, all the poles have the simple form $i(j+1/2)a_+$ and this procures isometry of $\mathcal{F}_N$. Below this value, the relationship between a_-/a_+ and the constants $\{n_j\}$ is fairly complicated. For example, the first **breakdown interval** is given by

$$a_-/a_+ \in (N-1/2, N+1/2), \quad N \geq 1 \tag{5.2}$$

(and $a_-/a_+ \in (1/4, 1/2)$ for $N=0$). In this interval, the poles have the form

$$\begin{cases} x_N^+ = i(N+1/2)a_+ - ia_- \\ x_j^+ = i(j+1/2)a_+, \quad j = 0, \ldots, N-1 \end{cases} \tag{5.3}$$

which corresponds to the choice of constants $n_N = 1$ and $n_{j<N} = 0$. As a_-/a_+ moves further below $N-1/2$, the breakdown intervals become increasingly difficult to specify. Each fixes the constants $\{n_j\}$ in a different way (indeed this is what defines an interval as such). For any set of constants other than $\{0, \ldots 0\}$ we will see that the residue sum in (3.32) is non-vanishing. This is what gives us the notion of breakdown, and the formula (3.32) allows us to express it in precise terms. As the intervals approach the origin, the constants grow and the complexity of this breakdown increases. This increase can be encoded in terms of an operator of increasing rank, cf. Theorem 5.4.

In the following lemma we learn that when $n_j > 0$ in (5.1), the argument used for vanishing in Lemma 3.5 breaks down. This alone does not mean that the residue sum in Corollary 3.4 is necessarily non-vanishing. But this is indeed also the case as established in the second part of the lemma. The corollary to the lemma is an immediate consequence of Corollary 3.4 and needs no further proof. (We must exclude the point set $\mathcal{E}_N$, (2.52), as part of the 'only if', since we know that for some of these a_-/a_+-values, $\mathcal{F}_N$ reduces to Fourier transform and in this case is obviously an isometry.)

Lemma 5.1. *If the positive parameters a_+, a_- satisfy $a_-/a_+ < j+1/2$ for a fixed $j = 0, \ldots, N$, then the residue function $R_\sigma^{(j)}(y, y')$ (3.31) is generically non-vanishing for both $\sigma = +, -$. Moreover, when $a_-/a_+ < N+1/2$, the $N+1$ residue functions corresponding to a choice of σ are linearly independent, and so the sum in Corollary 3.4 is non-vanishing.*

Corollary 5.2. *Assume the positive parameters a_+, a_- satisfy $a_-/a_+ \notin \mathcal{E}_N$. Then the map $\mathcal{F}_N : \hat{\mathcal{H}} \to \mathcal{H}$ defined by* (2.69) *is an isometry iff $a_-/a_+ > N + 1/2$.*

Proof of Lemma 5.1. In Lemma 3.5 we established that the residue function vanishes when $x_j^+ = \eta_j \equiv i(j+1/2)a_+$. In the present case, the restriction in the lemma means that $x_j^+ = \eta_j - in_j a_-$ where $n_j > 0$, cf. (5.1). Because of the quasiperiodicity of certain terms in (3.52) it turns out the present case can be linked to the one in Lemma 3.5.

First we note that the rewriting of the residue sum, (3.64), is still valid since it is not tied to any particular form of x_j^+,

$$R_\sigma^{(j)}(y,y') = \sum_{\tau,\tau'=+,-} (1 - e_+(\tau y)e_+(-\tau' y'))^{-1} \big[\lambda_N^\tau(x_j^+, -y)\lambda_N^{\sigma\tau'}(\sigma x_j^+, y') \\ + \lambda_N^{-\tau}(-x_j^+, -y)\lambda_N^{-\sigma\tau'}(-\sigma x_j^+, y')\big], \quad j = 0, \dots, N, \sigma = +,- \tag{5.4}$$

The square-bracketed term here emerged from the function $J^{\tau,\tau'}(\sigma; x, y, y')$ in the proof of Theorem 3.3. Specifically it was equal to the $x = x_j^+$ specialisation of

$$\big[J^{\tau,\tau'}(\sigma; x, y, y') + J^{-\tau,-\tau'}(\sigma; -x, y, y')\big]/v_N(-y)v_N(y') \tag{5.5}$$

(cf. (3.37)). A fundamental requirement of that proof was that this J be ia_--quasiperiodic in x. The quasiperiodicity relation took the form

$$J^{\tau,\tau'}(x - ia_-) = e_+(-\tau y)e_+(\tau' y')J^{\tau,\tau'}(x), \quad \tau,\tau' = +,- \tag{5.6}$$

(cf. (3.7) and (3.35)), which is equivalent to

$$J^{-\tau,-\tau'}(-x + ia_-) = e_+(-\tau y)e_+(\tau' y')J^{-\tau,-\tau'}(-x), \quad \tau,\tau' = +,-, \tag{5.7}$$

and so we can conclude straight away that (5.5) picks up a multiplier $e_+(-\tau y)e_+(\tau' y')$ when x is shifted by $-ia_-$.

Accordingly when we set $x_j^+ = \eta_j - in_j a_-$ in (5.4) it becomes

$$R_\sigma^{(j)}(y,y') = \sum_{\tau,\tau'=+,-} (1 - e_+(\tau y)e_+(-\tau' y'))^{-1} e_+(-\tau n_j y)e_+(\tau' n_j y') \\ \times \big[\lambda_N^\tau(\eta_j, -y)\lambda_N^{\sigma\tau'}(\sigma\eta_j, y') + \lambda_N^{-\tau}(-\eta_j, -y)\lambda_N^{-\sigma\tau'}(-\sigma\eta_j, y')\big] \tag{5.8}$$

This square-bracketed term is now identical to the one in the proof of Lemma 3.5 which we argued was "proportional to $\tau\tau'$" (i.e. equal to $\tau\tau'$ times something τ,τ'-independent). Thus we can deduce that $R_\sigma^{(j)}(y,y')$ equals some j-dependent entire function in y, y' multiplied by the $n = n_j$ specialisation of

$$\mathcal{J}_n(y,y') \equiv \sum_{\tau,\tau'=+,-} (1 - e_+(\tau y)e_+(-\tau' y'))^{-1} e_+(-n(\tau y - \tau' y'))\tau\tau', \quad n \in \mathbb{N} \tag{5.9}$$

We have already seen this function for $n = 0$ and we know that it vanishes, cf. (3.58). More generally, if we write out the four summands and rearrange we find

$$\mathcal{J}_n(y,y') = \frac{s_+((n+1/2)(y-y'))}{s_+((y-y')/2)} - \frac{s_+((n+1/2)(y+y'))}{s_+((y+y')/2)} \tag{5.10}$$

Using the identity

$$s_+((n+1/2)z) - s_+((n-1/2)z) = 2s_+(z/2)c_+(nz) \tag{5.11}$$

we then get the following recursion relation

$$\big(\mathcal{J}_n - \mathcal{J}_{n-1}\big)(y,y') = -4s_+(ny)s_+(ny') \tag{5.12}$$

for which we can write down the closed-form solution straight away,

$$\mathcal{J}_n(y,y') = -4\sum_{k=0}^{n} s_+(ky)s_+(ky') \tag{5.13}$$

Since this is generically non-vanishing for all $n > 0$, the first claim in the lemma is established once we argue that the entire function mentioned above (5.9) is also non-vanishing. This function is just the square-bracketed term in (5.8) divided through by $\tau\tau'$. We can write it as $[\Lambda_N^{\tau,\sigma\tau'}(\eta_j,\sigma\eta_j;-y,y')/\tau\tau']$, where Λ_N is the entire function defined in §2.1. Amongst the things we have for Λ_N is the partially evaluated expression (2.46). This gives us

$$\begin{aligned}\Lambda_N^{\tau,\sigma\tau'}(\eta_j,\sigma\eta_j;-y,y') = (-)^{N+1}\sigma\tau\tau' \sum_{\nu,\nu'=+,-} 4\nu\nu' s_-(\eta_j-\nu\eta_N)s_-(\sigma\eta_j-\nu'\eta_N)\mu_{\nu\nu'}(-y,y')\\ \times K_N(\eta_j+i\nu a_+/2,-\tau y)K_N(\sigma\eta_j+i\nu' a_+/2,\sigma\tau' y'),\quad \sigma,\tau,\tau'=+,-\end{aligned} \tag{5.14}$$

When $j = N$ the situation is much simpler because the four $s_-(\cdot)$-products (corresponding to the four choices of ν,ν') all vanish except for $(\nu,\nu') = (-,-\sigma)$. In other words,

$$\begin{aligned}\Lambda_N^{\tau,\sigma\tau'}(\eta_N,\sigma\eta_N;-y,y') = (-)^{N+1}\sigma\tau\tau' 4s_-(2\eta_N)^2\mu_\sigma(-y,y')\\ \times K_N(\eta_N-ia_+/2,-\tau y)K_N(\sigma\eta_N-i\sigma a_+/2,\sigma\tau' y')\end{aligned} \tag{5.15}$$

Because of (2.21), the σ's drop out of the second K_N, meaning that in all cases we are dealing with $K_N(iNa_+,z)$. Subsequently we learn from (2.23) that (5.15) equals

$$-\sigma\tau\tau'[2s_-(2\eta_N)B_0^{(N)}(\cdot)]^2\mu_\sigma(-y,y') \tag{5.16}$$

All the terms in this product are non-vanishing for generic parameters a_+, a_-.

For $j < N$ we relabel $\nu' \to \sigma\nu'$, as permitted under $\sum_{\nu'}$, to obtain

$$\begin{aligned}\Lambda_N^{\tau,\sigma\tau'}(\eta_j,\sigma\eta_j;-y,y') = (-)^{N+1}\sigma\tau\tau' \sum_{\nu,\nu'=+,-} 4\nu\nu' s_-(\eta_j-\nu\eta_N)s_-(\eta_j-\nu'\eta_N)\mu_{\sigma\nu\nu'}(-y,y')\\ \times K_N(\eta_j+i\nu a_+/2,-\tau y)K_N(\sigma\eta_j+i\sigma\nu' a_+/2,\sigma\tau' y'),\quad \sigma,\tau,\tau'=+,-\end{aligned} \tag{5.17}$$

Again, (2.21) means the σ's drop out of the second K_N. Because of (2.23), we have

$$K_N(\eta_j + i\nu a_+/2, z) = i^N B^{(N)}_{N-j-(1+\nu)/2}(c_-(z)), \quad \nu = +, -, \quad j < N \tag{5.18}$$

And so

$$\Lambda_N^{\tau,\sigma\tau'}(\eta_j, \sigma\eta_j; -y, y') = -\sigma\tau\tau' \sum_{\nu,\nu'=+,-} 4\nu\nu' s_-(\eta_j - \nu\eta_N) s_-(\eta_j - \nu'\eta_N)\mu_{\sigma\nu\nu'}(-y, y')$$
$$\times B^{(N)}_{N-j-(1+\nu)/2}(c_-(y)) B^{(N)}_{N-j-(1+\nu')/2}(c_-(y')), \quad \sigma, \tau, \tau' = +, -, \quad j < N \tag{5.19}$$

Each ν, ν' summand is proportional to

$$\mu_{\sigma\nu\nu'}(-y, y') B^{(N)}_{N-j-(1+\nu)/2}(c_-(y)) B^{(N)}_{N-j-(1+\nu')/2}(c_-(y')) \tag{5.20}$$

where, recalling (2.37),

$$\mu_{\sigma\nu\nu'}(-y, y') = \begin{cases} e_-^{(y-y')/2} e_-^{\xi} + e_-^{-(y-y')/2} e_-^{-\xi}, & \sigma\nu\nu' = + \\ e_-^{(y+y')/2} + e_-^{-(y+y')/2}, & \sigma\nu\nu' = - \end{cases} \tag{5.21}$$

Since $B_k^{(N)}(\cdot)$, §2, is a polynomial of degree k, it follows that for a fixed j and σ, the four ν, ν' summands in (5.19) are linearly independent and non-vanishing for generic parameters a_+, a_-.

We have thus established that when $n_j > 0$, the residue function $R_\sigma^{(j)}(y, y')$ is generically non-vanishing for all $j = 0, \ldots, N$ and $\sigma = +, -$. We now turn to the second part of the lemma. For this it is useful to write

$$R_\sigma^{(j)}(y, y') = \mathcal{J}_{n_j}(y, y')[\Lambda_N^{\tau,\sigma\tau'}(\eta_j, \sigma\eta_j; -y, y')/\tau\tau'] \tag{5.22}$$

(this equation is well-defined because we have just seen that this quotient term is in fact τ, τ'-independent). We recall that the set of constants $\{n_j\}$ arising in (5.1) is fixed by the value of a_-/a_+. We will argue for linear independence of (5.22) for fixed σ and arbitrary $\{n_j\}$ (this will save us the non-trivial task of describing how the set is actually fixed by a particular a_-/a_+), The restriction $a_-/a_+ < N + 1/2$ means that at least one of the constants is non-zero, namely n_N, and so at least one of the residue functions is non-vanishing. Since $\mathcal{J}_{n_j}(y, y') = \mathcal{J}_{n_{j'}}(y, y')$ is readily attainable under our assumptions, it is necessary that (5.16) and the N functions in (5.19), corresponding to $j = 0, \ldots, N-1$, are linearly independent (the latter functions are thought of monolithically, with regards to $\sum_{\nu,\nu'}$). This linear independence is also sufficient because $\mathcal{J}_n(y, y')$ is built from powers of $e_+(y)$ and $e_+(y')$, whereas the $N+1$ functions just described are built from $e_-(y)$ and $e_-(y')$. Specifically, the max/min powers of $e_-(y)$ and $e_-(y')$ in (5.16) are ± 1, respectively. And, as follows from inspection of (5.20), the (actually attained) max/min powers of $e_-(y)$ and $e_-(y')$ in (5.19) are $\pm(N - j + 1)$ respectively. The j-dependence of the latter establishes the claim.

□

We now return to the matter of isometry breakdown and the claim that it can be encoded in terms of a finite rank operator. A suitable definition for such an operator is

$$R_N \equiv \mathcal{F}_N^* \mathcal{F}_N - \mathbf{1}_{\hat{\mathcal{H}}} \tag{5.23}$$

which we note is a priori self-adjoint on $\hat{\mathcal{H}}$ (manifestly symmetric and bounded due to $\mathcal{F}_N$'s boundedness). We can see explicitly how R_N captures the notion of isometry breakdown by writing

$$\|\mathcal{F}_N \hat{f}\|^2_{\mathcal{H}} - \|\hat{f}\|^2_{\hat{\mathcal{H}}} = (\hat{f}, R_N \hat{f})_{\hat{\mathcal{H}}} \tag{5.24}$$

Clearly then, isometry of $\mathcal{F}_N$ is equivalent to vanishing of R_N. Thus from Corollary 5.2 we can be sure that $R_N \neq 0$ when $a_-/a_+ < N + 1/2$.

Now, say we have an equation of the form

$$(\mathcal{F}_N \hat{f}, \mathcal{F}_N \hat{g})_{\mathcal{H}} = (\hat{f}, \hat{g})_{\hat{\mathcal{H}}} + \sum_{\delta,\delta'=+,-} \int_{\mathbb{R}^+} dy\, \overline{f_\delta(y)} \int_{\mathbb{R}^+} dy'\, g_{\delta'}(y') \sum_{l=1}^{L} \lambda_l (\Theta_a^l)_\delta(y) \overline{(\Theta_b^l)_{\delta'}(y')} \tag{5.25}$$

where $\Theta_a^l, \Theta_b^l \in \hat{\mathcal{H}}$, $\lambda_l \in \mathbb{C}$ and L is some positive integer, then a priori,

$$R_N = \sum_{l=1}^{L} \lambda_l\, \Theta_a^l \otimes \overline{\Theta_b^l} \tag{5.26}$$

where, formally,

$$(F \otimes G) H \equiv (\overline{G}, H) F \tag{5.27}$$

We note that the rhs of (5.26) is *manifestly* self-adjoint if $\lambda_l \in \mathbb{R}$ and $\Theta_b^l = \Theta_a^l$ (because $((\Theta_a \otimes \overline{\Theta_b})\hat{f}, \hat{g}) = (\hat{f}, (\Theta_b \otimes \overline{\Theta_a})\hat{g})$. Under these conditions the rhs of (5.24) equals

$$\sum_{l=1}^{L} \lambda_l |(f, \Theta_a^l)_{\hat{\mathcal{H}}}|^2 \tag{5.28}$$

The potential to realise (5.25) is clearly given to us by Theorem 3.3. We achieve this, and more, in the following two theorems. The first deals with the first breakdown interval (5.2) for $N > 0$ (it also applies to $N = 0$ provided we change the lower interval limit, $N - 1/2 \to 1/4$). The second encompasses all breakdown intervals for generic N. We recall that $v_N(y)/w_j$ are defined in (2.8)/(3.30) respectively, and that $B_k^{(N)}(\cdot)$ is a polynomial of degree k, equalling a constant given by (2.26) when $k = 0$. The polynomial drops out as unity when $N = 0$. We exclude the point set $\mathcal{E}_N$ (2.52) to ensure the poles of $w_N(\cdot)$ are simple.

Theorem 5.3. *Let the positive parameters a_+, a_- satisfy $a_-/a_+ \in (N-1/2, N+1/2)\backslash\mathcal{E}_N$. Then for $N > 0$, the operator R_N, (5.23), has the manifestly self-adjoint form,*

$$R_N = \lambda_N \big(\Gamma_1^{(N)} \otimes \overline{\Gamma_1^{(N)}} - \Pi_1^{(N)} \otimes \overline{\Pi_1^{(N)}}\big) \tag{5.29}$$

where

$$\lambda_N \equiv \frac{2\pi i}{a_+ a_-} w_N \in \mathbb{R}, \tag{5.30}$$

$$\Gamma_k^{(N)} \equiv \begin{pmatrix} \gamma_k^{(N)} \\ \gamma_k^{(N)} \end{pmatrix}, \quad \Pi_k^{(N)} \equiv \begin{pmatrix} \pi_k^{(N)} \\ -\pi_k^{(N)} \end{pmatrix}, \tag{5.31}$$

$$\gamma_k^{(N)}(y) \equiv 4v_N(-y)s_+(ky)s_-(2\eta_N)s_-((y+i(N+1)a_+)/2)B_0^{(N)}(\cdot), \tag{5.32}$$

$$\pi_k^{(N)}(y) \equiv 4v_N(-y)s_+(ky)s_-(2\eta_N)c_-((y+i(N+1)a_+)/2)B_0^{(N)}(\cdot), \tag{5.33}$$

and

$$\eta_j \equiv i(j+1/2)a_+ \tag{5.34}$$

Theorem 5.4. *Let the positive parameters a_+, a_- satisfy $a_-/a_+ \in (0, N+1/2) \setminus \mathcal{E}_N$. Then the operator R_N, (5.23), can be written in the manifestly self-adjoint form,*

$$R_N = \lambda_N \sum_{k=1}^{n_N} \left(\Gamma_k^{(N)} \otimes \overline{\Gamma_k^{(N)}} - \Pi_k^{(N)} \otimes \overline{\Pi_k^{(N)}}\right) + \sum_{j=0}^{N-1} \lambda_j \sum_{k=1}^{n_j} \left(\left(\Gamma_k^{(N,j)} \otimes \overline{\Gamma_k^{(N,j)}} - \Pi_k^{(N,j)} \otimes \overline{\Pi_k^{(N,j)}}\right) \tag{5.35}$$

where

$$\lambda_j \equiv \frac{2\pi i}{a_+ a_-} w_j \in \mathbb{R}, \tag{5.36}$$

$$\Gamma_k^{(N,j)} \equiv \begin{pmatrix} \gamma_k^{(N,j)} \\ \gamma_k^{(N,j)} \end{pmatrix}, \quad \Pi_k^{(N,j)} \equiv \begin{pmatrix} \pi_k^{(N,j)} \\ -\pi_k^{(N,j)} \end{pmatrix}, \quad j < N \tag{5.37}$$

$$\gamma_k^{(N,j)}(y) \equiv 4v_N(-y)s_+(ky) \times s_-((y+i(N+1)a_+)/2) \sum_{\nu=+,-} s_-(\eta_j - \nu\eta_N) B^{(N)}_{N-j-(1+\nu)/2}(c_-(y)) \tag{5.38}$$

$$\pi_k^{(N,j)}(y) \equiv 4v_N(-y)s_+(ky) \times c_-((y+i(N+1)a_+)/2) \sum_{\nu=+,-} \nu s_-(\eta_j - \nu\eta_N) B^{(N)}_{N-j-(1+\nu)/2}(c_-(y)), \tag{5.39}$$

and where $\{n_j\}_{j=0}^N$ is the set of constants fixed by the value of a_-/a_+ according to (5.1) and the definition of x_j^+.

Proofs of Theorems 5.3 and 5.4. We first note λ_j are real because w_j is the residue of a real-valued function at an imaginary pole.

Next, we catalogue the residue functions for the most general form of the poles (5.1), i.e. $x_j^+ = \eta_j - in_j a_-$ where $\eta_j \equiv i(j+1/2)a_+$. The argument in the previous proof brought us close to an explicit expression for these. Indeed from what was said below (5.13) we have

$$R_\sigma^{(j)}(y,y') = \mathcal{J}_{n_j}(y,y')[\Lambda_N^{\tau,\sigma\tau'}(\eta_j,\sigma\eta_j;-y,y')/\tau\tau'], \quad \sigma=+,-, \quad j=0,\ldots,N \tag{5.40}$$

(this equation is well-defined because the quotient term is known to be τ,τ'-independent). And so, with the computations (5.16) and (5.19),

$$R_{\delta\delta'}^{(N)}(y,y') = -\mathcal{J}_{n_N}(y,y')\delta\delta'[2s_-(2\eta_N)B_0^{(N)}(\cdot)]^2\mu_{\delta\delta'}(-y,y'), \quad \delta,\delta'=+,- \tag{5.41}$$

$$\begin{aligned} R_{\delta\delta'}^{(j)}(y,y') = -\mathcal{J}_{n_j}(y,y')\delta\delta' \sum_{\nu,\nu'=+,-} & 4\nu\nu' s_-(\eta_j-\nu\eta_N)s_-(\eta_j-\nu'\eta_N)\mu_{\delta\delta'\nu\nu'}(-y,y') \\ & \times B_{N-j-(1+\nu)/2}^{(N)}(c_-(y))B_{N-j-(1+\nu')/2}^{(N)}(c_-(y')), \quad \delta,\delta'=+,-, \quad j<N \end{aligned} \tag{5.42}$$

At this point we note a subtlety (which the reader may ignore). For almost all of this chapter, the decision to transfer the δ,δ' dependence of the integral in (3.2) onto just one of the terms in the integrand, à la (3.3), has been a good one (giving birth to the σ index, which became our stand-in for $\delta\delta'$ and thus reduced the overall number of indices, simplified formulas and so on). Now, however, in wanting to rebuild the inner product form, we are forced to undo this transference and find expressions for (5.41) and (5.42) of the form $\sum_k A_k^\delta B_k^{\delta'}$. This form is achieved with the objects in the theorem.

Using Theorem (3.3) and what was said about (5.25), the claims in the theorem follow because

$$\begin{aligned} & \frac{\pi i}{a_+a_-}v_N(-y)v_N(y')w_N R_{\delta\delta'}^{(N)}(y,y') \\ & \qquad = \sum_{k=1}^{n_N}\lambda_N\big(\gamma_k^{(N)}(y)\overline{\gamma_k^{(N)}(y')} - \delta\pi_k^{(N)}(y)\delta'\overline{\pi_k^{(N)}(y')}\big), \quad \delta,\delta'=+,- \end{aligned} \tag{5.43}$$

and

$$\begin{aligned} & \frac{\pi i}{a_+a_-}v_N(-y)v_N(y')w_j R_{\delta\delta'}^{(j)}(y,y') \\ & \qquad = \sum_{k=1}^{n_j}\lambda_j\big(\gamma_k^{(N,j)}(y)\overline{\gamma_k^{(N,j)}(y')} - \delta\pi_k^{(N,j)}(y)\delta'\overline{\pi_k^{(N,j)}(y')}\big), \quad \delta,\delta'=+,-, \quad j<N \end{aligned} \tag{5.44}$$

To see that these hold, we use reality of the coefficients of the polynomial $B_k^{(N)}(\cdot)$ and the fact $\bar{\eta}_j = -\eta_j$, $\bar{\xi} = -\xi$, to write out the k-summand in (5.43) as $(\pi i/a_+a_-)w_N$ multiplied by

$$2v_N(-y)v_N(y')[2s_-(2\eta_N)B_0^{(N)}(\cdot)]^2(-4s_+(ky)s_+(ky'))$$
$$\times \left\{s_-((y+\xi)/2)s_-((y'-\xi)/2) - \delta\delta' c_-((y+\xi)/2)c_-((y'-\xi)/2)\right\} \quad (5.45)$$

$$\xi \equiv i(N+1)a_+ \quad (5.46)$$

From elementary manipulation of, e.g., (5.21) it follows that

$$s_-((y+\xi)/2)s_-((y'-\xi)/2) = \frac{1}{4}\big(\mu_- - \mu_+\big)(-y,y') \quad (5.47)$$

and

$$c_-((y+\xi)/2)c_-((y'-\xi)/2) = \frac{1}{4}\big(\mu_- + \mu_+\big)(-y,y') \quad (5.48)$$

And so we obtain the following identity, which is exactly what we need to prove (5.43) (recall (5.41) and (5.13)),

$$s_-((y+\xi)/2)s_-((y'-\xi)/2) - \delta\delta' c_-((y+\xi)/2)c_-((y'-\xi)/2)$$
$$= -\frac{\delta\delta'}{2}\mu_{\delta\delta'}(-y,y'), \quad \delta,\delta' = +,- \quad (5.49)$$

Similarly, the k-summand in (5.44) equals $(\pi i/a_+a_-)w_j$ multiplied by

$$2v_N(-y)v_N(y')(-4s_+(ky)s_+(ky')) \sum_{\nu,\nu'=+,-} 4s_-(\eta_j - \nu\eta_N)s_-(\eta_j - \nu'\eta_N)B^{(N)}_{N-j-(1+\nu)/2}(c_-(y))$$
$$\times B^{(N)}_{N-j-(1+\nu')/2}(c_-(y'))\left\{s_-((y+\xi)/2)s_-((y'-\xi)/2) - \delta\delta'\nu\nu' c_-((y+\xi)/2)c_-((y'-\xi)/2)\right\}$$
$$(5.50)$$

And so (5.44) again reduces to the identity (5.49) (recalling (5.42) and (5.13)).

□

5.2 Breakdown for $\mathcal{F}_N^*$

As we explained earlier, the pole y_j^+ (cf. §2.2) is no longer given by ija_+ when the ratio a_-/a_+ drops below j. In the most general case, y_j^+ is given by

$$y_j^+ = ija_+ - im_ja_- \quad (5.51)$$

where the precise value of the integer $m_j \geq 0$ is fixed by the requirement that y_j^+ be in $\mathrm{Im}\, y \in [0, a_-)$, and hence depends on a_-/a_+. When $a_-/a_+ > N+1$ all the poles have the simple form ija_+, and this is synonymous with isometry of $\mathcal{F}_N^*$. Below this value, the relationship between the a_-/a_+ and the constants $\{m_j\}$ becomes fairly intricate. For example, the first breakdown interval for $\mathcal{F}_N^*$ is given by

$$a_-/a_+ \in (N, N+1) \tag{5.52}$$

(and $a_-/a_+ \in (1/2, 1)$ for $N = 0$). In this interval, the poles have the form

$$\begin{cases} y_{N+1}^+ = i(N+1)a_+ - ia_- \\ y_j^+ = ija_+, \quad j = 1, \ldots, N \end{cases} \tag{5.53}$$

which corresponds to the choice of constants $m_{N+1} = 1$ and $m_{j<N} = 0$. There is a one-to-one relationship between intervals and constants $\{m_j\}$. In the lemma below we confirm that isometry exists only for $\{0, \ldots 0\}$. For all other sets, i.e. for $a_-/a_+ < N+1$, the formula (4.25) allows us to express breakdown in precise terms. As a_-/a_+ approaches the origin, the constants get larger and complexity of the breakdown increases. This increase can be reflected in the increasing rank of an operator, cf. Theorem 5.8.

The corollary to the following lemma is an immediate consequence of Corollary 4.3 and needs no further proof. (We must exclude the point set $\mathcal{E}_N$, (2.52), as part of the 'only if', since for some of these a_-/a_+-values, $\mathcal{F}_N$ reduces to Fourier transform and so is obviously an isometry.)

Lemma 5.5. *When $a_-/a_+ < j$, the residue function $\hat{R}^{(j)}(x, x')$, (4.47), is non-vanishing for fixed $j = 1, \ldots, N+1$. Moreover, when $a_-/a_+ < N+1$, the $N+1$ residue functions are linearly independent, and so the sum in Corollary 4.3 is also non-vanishing.*

Corollary 5.6. *Assume the positive parameters a_+, a_- satisfy $a_-/a_+ \notin \hat{\mathcal{E}}_N$. Then the map $\mathcal{F}_N^* : \mathcal{H} \to \hat{\mathcal{H}}$ with action (4.1) is an isometry iff $a_-/a_+ > N+1$.*

Proof of Lemma 5.5. In Lemma 4.4 we learned that the residue function vanishes when $y_j^+ = \xi_j \equiv ija_+$. Now, the restriction in the lemma means that $y_j^+ = \xi_j - im_j a_-$ where $m_j > 0$, cf. (5.51). We can connect with the earlier work because of the quasiperiodicity of certain terms in (4.47). Specifically, the plane wave product picks up a multiplier $e_+(\tau x)e_+(-\tau' x')$ when y is shifted by $-ia_-$, and the L_N term picks up $\tau\tau'$, cf. (4.45). Thus we get

$$\begin{aligned} \hat{R}^{(j)}(x, x') = \sum_{\tau, \tau' = +,-} & (1 - \tau\tau' e_+(-\tau x)e_+(\tau' x'))^{-1} e_+(m_j \tau x)e_+(-m_j \tau' x')(\tau\tau')^{m_j} \\ & \times \exp(i\pi\tau x \xi_j / a_+ a_-) \exp(-i\pi\tau' x' \xi_j / a_+ a_-) L_N^{\tau, \tau'}(\xi_j, x, x'), \quad j = 1, \ldots, N+1 \end{aligned} \tag{5.54}$$

Recalling (4.56), the second line here is now identical to the square-bracketed term in (4.57) for $y_j^+ = \xi_j$, namely

$$\left[\lambda_N^\tau(x, \xi_j)\lambda_N^{\tau'}(x', -\xi_j) + \lambda_N^{-\tau}(-x, \xi_j)\lambda_N^{-\tau'}(-x', -\xi_j)\right] \tag{5.55}$$

In the proof of Lemma 4.4 we argued that this was "proportional" to $\tau\tau'$, and thus we may conclude from (5.54) that $\hat{R}^{(j)}(x, x')$ equals some j-dependent entire function in x, x' multiplied by the $m = m_j$ specialisation of

$$\mathcal{K}_m(x,x') \equiv \sum_{\tau,\tau'=+,-} (1-\tau\tau' e_+(-\tau x)e_+(\tau' x'))^{-1} e_+(m(\tau x - \tau' x'))(\tau\tau')^{m+1}, \quad m \in \mathbb{N} \tag{5.56}$$

We have already seen this for $m = 0$ and $m = 1$, cf. (4.53) and (4.76) respectively. More generally, if we write out the four summands and rearrange we find

$$\mathcal{K}_m(x,x') = \frac{s_+((m+1/2)(x-x'))}{s_+((x-x')/2)} + (-)^{m+1}\frac{c_+((m+1/2)(x+x'))}{c_+((x+x')/2)} \tag{5.57}$$

which, for one thing, shows that $\mathcal{K}_0(x,x') = 0$. Setting $z = x - x'$ and $z = x + x'$ respectively in the following identities

$$s_+((m+1/2)z) - s_+((m-1/2)z) = 2s_+(z/2)c_+(mz) \tag{5.58}$$

$$c_+((m+1/2)z) + c_+((m-1/2)z) = 2c_+(z/2)c_+(mz) \tag{5.59}$$

we get the recursion relation

$$\big(\mathcal{K}_m - \mathcal{K}_{m-1}\big)(x,x') = \begin{cases} -4s_+(mx)s_+(mx') & (m \text{ even}) \\ 4c_+(mx)c_+(mx') & (m \text{ odd}) \end{cases} \tag{5.60}$$

for which we can write down the closed-form solution straight away,

$$\mathcal{K}_m(x,x') = 4\sum_{k=1}^{m}(-)^{k-1}\tilde{Q}_k(x)\tilde{Q}_k(x') \tag{5.61}$$

where

$$\tilde{Q}_k(x) \equiv \begin{cases} s_+(kx), & (k \text{ even}) \\ c_+(kx), & (k \text{ odd}) \end{cases} \quad k \in \mathbb{N}^* \tag{5.62}$$

Now we must check that the entire function mentioned below (5.54) is also non-vanishing. This function is just (5.55) divided through by $\tau\tau'$. Recalling (4.56), we can write (5.55) as $\Lambda_N^{\tau,\tau'}(x,x';\xi_j,-\xi_j)$, where Λ_N is the function we introduced in §2.1. We are already familiar with this for $j = N+1$, cf. the proof of Lemma 4.4. Indeed we know from (4.69) that

$$\Lambda_N^{\tau,\tau'}(x,x';\xi_{N+1},-\xi_{N+1}) = 8c_-(i(N+1)a_+)(-)^{N+1}\tau\tau' p_N^2 \tag{5.63}$$

where p_N is a non-zero constant, cf. (4.88) (note that as elsewhere we use the shorthand $\xi \equiv i(N+1)a_+$).

We now consider the $j \leq N$ case. Amongst the things we have for Λ_N is the partially evaluated expression (2.46). From this we learn

$$\begin{aligned}
\Lambda_N^{\tau,\tau'}(x,x';\xi_j,-\xi_j) &= (-)^{N+1}\tau\tau' \sum_{\nu,\nu'=+,-} 4\nu\nu' s_-(x-\nu\eta)s_-(x'-\nu'\eta)\mu_{\nu\nu'}(\xi_j,-\xi_j) \\
&\qquad \times K_N(x+i\nu a_+/2,\tau\xi_j)K_N(x'+i\nu' a_+/2,-\tau'\xi_j) \\
&= -\tau\tau' \sum_{\nu,\nu'=+,-} 4\nu\nu' s_-(x-\nu\eta)s_-(x'-\nu'\eta)\mu_{\nu\nu'}(\xi_j,-\xi_j)B_{N-j}^{(N)}(c_-(x+i\nu a_+/2)) \\
&\qquad \times B_{N-j}^{(N)}(c_-(x'+i\nu' a_+/2)), \quad j=1,\ldots,N, \quad \eta\equiv i(N+1/2)a_+ \qquad (5.64)
\end{aligned}$$

To reach the second equality we have used (2.24). Since $B_k^{(N)}(\cdot)$, §2, is a polynomial of degree k, it follows that for a fixed j, the four ν,ν' summands in (5.64) are linearly independent and non-vanishing for generic parameters a_+, a_-.

We have established that when $m_j > 0$, the residue function $\hat{R}^{(j)}(x,x')$ is generically non-vanishing for all $j = 1,\ldots,N+1$. The second part of the lemma claims that these $N+1$ functions are also linearly independent. This follows using arguments analogous to those given at the end of the proof of Lemma 5.1. One just has to make the appropriate modifications ($n_j \to m_j$, $\mathcal{J} \to \mathcal{K}$ etc.).

□

Now we consider the matter of isometry breakdown for $\mathcal{F}_N^*$. The analogue of R_N, (5.23), is

$$\hat{R}_N \equiv \mathcal{F}_N\mathcal{F}_N^* - \mathbf{1}_{\mathcal{H}} \qquad (5.65)$$

which is a priori self-adjoint on $\mathcal{H}$. It equals the zero operator iff $\mathcal{F}_N^*$ is an isometry.

Say we have an equation of the form

$$(\mathcal{F}_N^* f, \mathcal{F}_N^* g)_{\hat{\mathcal{H}}} = (f,g)_{\mathcal{H}} + \int_{\mathbb{R}} dx\, \overline{f(x)} \int_{\mathbb{R}} dx'\, g(x') \sum_{l=1}^{\hat{L}} \hat{\lambda}_l \hat{\Theta}_a^l(x)\overline{\hat{\Theta}_b^l(x')} \qquad (5.66)$$

where $\hat{\Theta}_a^l, \hat{\Theta}_b^l \in \mathcal{H}$, $\hat{\lambda}_l \in \mathbb{C}$ and $\hat{L}$ is some positive integer, then a priori,

$$\hat{R}_N = \sum_{l=1}^{\hat{L}} \hat{\lambda}_l (\hat{\Theta}_a^l \otimes \overline{\hat{\Theta}_b^l}) \qquad (5.67)$$

Moreover, if $\hat{\lambda}_l \in \mathbb{R}$ and $\hat{\Theta}_b^l = \hat{\Theta}_a^l$ then (5.67) is *manifestly* self-adjoint and we have

$$\|\mathcal{F}_N^* f\|_{\hat{\mathcal{H}}}^2 - \|f\|_{\mathcal{H}}^2 = \sum_{l=1}^{\hat{L}} \hat{\lambda}_l |(f,\hat{\Theta}_a^l)_{\mathcal{H}}|^2 \qquad (5.68)$$

The potential to realise (5.66) is clearly given to us by Theorem 4.2. We achieve this, and more, in the following theorems. The first deals with the first breakdown interval (5.52) for $N > 0$ (it also applies to $N = 0$ provided we change the lower interval limit, $N \to 1/2$). The second encompasses all breakdown intervals for general N and thus subsumes the first theorem. We recall that $w_N(x)/\hat{w}_j$ are defined in (2.53)/(4.23) respectively, and that $B_k^{(N)}(\cdot)$ is a polynomial of degree k. We exclude the point set $\hat{\mathcal{E}}_N$ (2.57) to ensure the poles of $\hat{w}_N(\cdot)$ are simple.

Theorem 5.7. *Let the positive parameters a_+, a_- satisfy $a_-/a_+ \in (N, N+1) \setminus \hat{\mathcal{E}}_N$. Then for $N > 0$, the operator $\hat{R}_N$,* (5.65)*, can be written in the manifestly self-adjoint form,*

$$\hat{R}_N = -c_N \Psi_N \otimes \Psi_N \tag{5.69}$$

where Ψ_N is the ground state function from Theorem 4.7,

$$\Psi_N(x) \equiv 2c_+(x) w_N(x)^{1/2}, \tag{5.70}$$

and c_N is the constant (4.99) *which we know equals $1/\|\Psi_N\|^2$.*

Proof. Since $\mathcal{F}_N$ is an isometry in this range (cf. Theorem 3.6 or Corollary 5.2) we know a priori that

$$\mathcal{F}_N \mathcal{F}_N^* = \mathbf{1}_{\mathcal{H}} - \mathrm{proj}_{[\mathrm{Ran}\,\mathcal{F}_N]^\perp} \tag{5.71}$$

(cf. (4.89)), and so $\hat{R}_N = -\mathrm{proj}_{[\mathrm{Ran}\,\mathcal{F}_N]^\perp}$. We established in Theorem 4.7 that $[\mathrm{Ran}\,\mathcal{F}_N]^\perp = \mathrm{span}\{\Psi_N\}$, and so $\hat{R}_N$ is as given in the theorem. □

The next theorem involves a family of real functions that generalises Ψ_N, defined below (4.80), according to

$$\Psi_N^{(k)}(x) \equiv Q^{(k)}(x) \Psi_N(x), \quad k \in \mathbb{N} \tag{5.72}$$

where

$$Q^{(k)}(x) \equiv \begin{cases} c_+((k+1)x)/c_+(x), & (k \text{ even}) \\ s_+((k+1)x)/c_+(x), & (k \text{ odd}) \end{cases} \quad k \in \mathbb{N} \tag{5.73}$$

This function is a Chebyshev polynomial in $s_+(x)$ of degree k. This can be seen explicitly by noting that $Q^{(k)}$ satisfies the recursion relation

$$Q^{(k+1)}(x) = 2s_+(x) Q^{(k)}(x) + Q^{(k-1)}(x), \quad k \geq 1 \tag{5.74}$$

Since $Q^{(0)}(x) = 1$, we have $\Psi_N^{(0)} = \Psi_N$. (In general, these relate to the functions in §2 of Chapter 1 by a sign change, $\Psi_N^{(m)}(x) = (-)^{Nm} \Psi^{(m)}(b_N; x)$, $m \in \mathbb{N}$.)

Theorem 5.8. *Let the positive parameters a_+, a_- satisfy $a_-/a_+ \in (0, N+1) \setminus \hat{\mathcal{E}}_N$. Then the operator $\hat{R}_N$,* (5.65)*, can be written in the manifestly self-adjoint form,*

$$\hat{R}_N = c_N \sum_{k=1}^{m_{N+1}} (-)^k \Psi_N^{(k-1)} \otimes \Psi_N^{(k-1)} + \sum_{j=1}^{N} \sum_{k=1}^{m_j} (-)^k \big(\hat{\lambda}_j \, \hat{\Gamma}_k^{(N,j)} \otimes \overline{\hat{\Gamma}_k^{(N,j)}} - \hat{\lambda}'_j \, \hat{\Pi}_k^{(N,j)} \otimes \overline{\hat{\Pi}_k^{(N,j)}} \big) \tag{5.75}$$

where

$$\hat{\lambda}_j \equiv \frac{2\pi i}{a_+a_-}\hat{w}_j\, s_-(i(N+1\pm j)a_+/2) \in \mathbb{R}, \quad j \leq N, \tag{5.76}$$

$$\hat{\lambda}'_j \equiv \frac{2\pi i}{a_+a_-}\hat{w}_j\, c_-(i(N+1\pm j)a_+/2) \in \mathbb{R}, \quad j \leq N, \tag{5.77}$$

$(f(\pm u) \equiv f(u)f(-u))$,

$$\hat{\Gamma}_k^{(N,j)}(x) \equiv \Psi_N^{(k-1)}(x) \sum_{\nu=+,-} 2s_-(x-\nu\eta_N)B_{N-j}^{(N)}(c_-(x+i\nu a_+/2)) \in \mathbb{R}, \tag{5.78}$$

$$\hat{\Pi}_k^{(N,j)}(x) \equiv \Psi_N^{(k-1)}(x) \sum_{\nu=+,-} 2\nu s_-(x-\nu\eta_N)B_{N-j}^{(N)}(c_-(x+i\nu a_+/2)) \in i\mathbb{R}, \tag{5.79}$$

$$\eta_N \equiv i(N+1/2)a_+, \tag{5.80}$$

and where $\{m_j\}_{j=1}^{N+1}$ *is the set of constants fixed by the value of* a_-/a_+ *according to* (5.51) *and the definition of* y_j^+*.*

Proof. We first note $\hat{\lambda}_j$ are real because $\hat{w}_j$ is the residue of a real-valued function at an imaginary pole. (5.78)/(5.79) are real/imaginary because $\Psi_N^{(k-1)}$ is real and the coefficients of the polynomial $B_k^{(N)}(\cdot)$ are real, i.e.

$$\overline{B_k^{(N)}(u)} = B_k^{(N)}(\bar{u}) \tag{5.81}$$

Next, we catalogue the residue functions for the most general form of the poles (5.1), i.e. $y_j^+ = \xi_j - im_ja_-$ where $\xi_j \equiv ija_+$. The argument in the previous proof brought us close to an explicit expression for these. Indeed from what was said circa (5.55)-(5.56) we have

$$R^{(j)}(x,x') = \mathcal{K}_{m_j}(y,y')[\Lambda_N^{\tau,\tau'}(x,x';\xi_j,-\xi_j)/\tau\tau'], \quad j=1,\ldots,N+1 \tag{5.82}$$

(this equation balances because the quotient term is known to be τ,τ'-independent). And so, with the computations (5.63) and (5.64),

$$\hat{R}^{(N+1)}(x,x') = \mathcal{K}_{m_{N+1}}(x,x')8c_-(\xi_{N+1})(-)^{N+1}p_N^2 \tag{5.83}$$

$$\hat{R}^{(j)}(x,x') = -\mathcal{K}_{m_j}(x,x') \sum_{\nu,\nu'=+,-} 4\nu\nu' s_-(x-\nu\eta)s_-(x-\nu'\eta)\mu_{\nu\nu'}(\xi_j,-\xi_j)$$
$$\times B_{N-j}^{(N)}(c_-(x+i\nu a_+/2))B_{N-j}^{(N)}(c_-(x'+i\nu' a_+/2)), \quad j=1,\ldots,N, \quad \eta \equiv i(N+1/2)a_+ \tag{5.84}$$

Using Theorem (3.3) and what was said about (5.66), the claims in the theorem follow because

$$\frac{\pi i}{a_+a_-} w_N(x)^{1/2} w_N(x')^{1/2} \hat{w}_{N+1} \hat{R}^{(N+1)}(x,x') = c_N \sum_{k=1}^{m_{N+1}} (-)^k \Psi_N^{(k-1)}(x) \Psi_N^{(k-1)}(x') \tag{5.85}$$

$$\begin{aligned}&\frac{\pi i}{a_+a_-} w_N(x)^{1/2} w_N(x')^{1/2} \hat{w}_j \hat{R}^{(j)}(x,x') \\ &\quad = \sum_{k=1}^{m_j} (-)^k \big(\hat{\lambda}_j \hat{\Gamma}_k^{(N,j)}(x) \overline{\hat{\Gamma}_k^{(N,j)}(x')} - \hat{\lambda}'_j \hat{\Pi}_k^{(N,j)}(x) \overline{\hat{\Pi}_k^{(N,j)}(x')}\big), \quad j \le N\end{aligned} \tag{5.86}$$

The first of these holds because, writing $\mathcal{K}_m$ (5.61) in terms of $Q^{(k)}$ we have

$$\mathcal{K}_m(x,x') = 4c_+(x)c_+(x') \sum_{k=1}^{m} (-)^{k-1} Q^{(k-1)}(x) Q^{(k-1)}(x') \tag{5.87}$$

such that, as features in the lhs of (5.85) via (5.83),

$$w_N(x)^{1/2} w_N(x')^{1/2} \mathcal{K}_m(x,x') = \sum_{k=1}^{m} (-)^{k-1} \Psi_N^{(k-1)}(x) \Psi_N^{(k-1)}(x') \tag{5.88}$$

(keep in mind m is just some arbitrary integer whereas m_j is a particular integer fixed by the value of a_-/a_+). The constants balance as required because from (4.88) and (4.97) we find

$$\frac{\pi i}{a_+a_-} \hat{w}_{N+1} 8c_-(\xi_{N+1})(-)^{N+1} p_N^2 = -c_N \tag{5.89}$$

To see that (5.85) holds, we use (5.81) and the fact $\bar{\eta} = -\eta$, $\bar{\xi}_j = -\xi_j$, to write out the k-summand on the rhs as $(\pi i/a_+a_-)\hat{w}_j$ multiplied by

$$\begin{aligned}&2(-)^k \Psi_N^{(k-1)}(x) \Psi_N^{(k-1)}(x') \sum_{\nu,\nu'=+,-} 4s_-(x-\nu\eta)s_-(x'-\nu'\eta) B_{N-j}^{(N)}(c_-(x+i\nu a_+/2)) \\ &\times B_{N-j}^{(N)}(c_-(x'+i\nu' a_+/2)) \big\{ s_-(i(N+1\pm j)a_+/2) + \nu\nu' c_-(i(N+1\pm j)a_+/2) \big\}\end{aligned} \tag{5.90}$$

To connect with the lhs of (5.86) we specialise (5.49) in the obvious way to get

$$\begin{aligned}&s_-(i(N+1\pm j)a_+/2) + \nu\nu' c_-(i(N+1\pm j)a_+/2) \\ &\qquad = \frac{\nu\nu'}{2} \mu_{\nu\nu'}(\xi_j, -\xi_j), \quad \nu,\nu' = +,-\end{aligned} \tag{5.91}$$

and then invoke (5.88).

□

6 Summary theorem for $N = 0$

We now bring together various results for the special case $N = 0$. For this choice of N, the formulas in the previous two sections simplify considerably. This allows us to highlight some of the important concepts more easily. However, the reader should be aware that the $N = 0$ case is a poor guide to the general N case. The latter involves complications not found below.

The transform at issue here acts on pairs $\hat{f} = \langle f_+, f_- \rangle$,

$$(\mathcal{F}_0 \hat{f})(x) = (2a_+a_-)^{-1/2} \int_0^\infty dy \sum_{\delta=+,-} \psi_0(\delta x, y) f_\delta(y) \tag{6.1}$$

which we know from §2.4 defines a bounded map from the Hilbert space $\hat{\mathcal{H}} \equiv L^2(\mathbb{R}^+, dy) \otimes \mathbb{C}^2$ into $\mathcal{H} \equiv L^2(\mathbb{R}, dx)$. The writing of ψ_N that we have been using throughout, namely (2.14), now looks like

$$\psi_0(x, y) = w_0(x)^{1/2} v_0(y) \sum_{\tau=+,-} \exp(i\tau\pi xy/a_+a_-) \ell_0^\tau(x, y) \tag{6.2}$$

where the functions

$$w_0(x) = 1/4s_-(x + ia_+/2)s_-(x - ia_+/2), \tag{6.3}$$

$$v_0(y) = 1/2is_-(y - ia_+) \tag{6.4}$$

have poles on the imaginary axis, and $\ell_0^\pm$ are entire,

$$\ell_0^\tau(x, y) = i\tau \sum_{\nu=+,-} 2\nu s_-(x - i\nu a_+/2)e_-(\nu(ia_+ - y)/2)e_-(-\nu\tau y/2), \quad \tau = +, - \tag{6.5}$$

Written out fully, we have

$$\begin{aligned} \ell_0^+(x, y) = 2i[s_-(x - ia_+/2)e_-^{-y} e_-^{ia_+/2} - s_-(x + ia_+/2)e_-^y e_-^{-ia_+/2}] \\ = -2i[s_-(y)e_-^x + s_-(y - ia_+)e_-^{-x}] \end{aligned} \tag{6.6}$$

and

$$\ell_0^-(x, y) = -2i[s_-(x - ia_+/2)e_-^{ia_+/2} - s_-(x + ia_+/2)e_-^{-ia_+/2}] = 2ie_-^{-x}s_-(ia_+) \tag{6.7}$$

A detailed account of the poles of (6.3)-(6.4) has already been given in §2.2. Of relevance here is the complex number x_0^+, defined as the unique x-pole of $1/s_-(x - ia_+/2)$ in the strip $\operatorname{Im} x \in [0, a_-)$. Clearly,

$$x_0^+ = ia_+/2 - in_0a_- \tag{6.8}$$

for some uniquely defined, positive integer n_0. It is this integer which features explicitly below, fixed by the value of a_-/a_+. To ensure all the poles of $w_0(x)$ are simple we must restrict the positive parameters a_+, a_- according to

$$a_-/a_+ \notin \mathcal{E}_0 = \{1, 1/2, 1/3, 1/4, \dots\} \tag{6.9}$$

Similarly, y_1^+ is defined as the unique y-pole of $1/s_-(y-ia_+)$ in the strip $\operatorname{Im} y \in [0, a_-)$. Clearly

$$y_1^+ = ia_+ - im_1 a_- \tag{6.10}$$

for some uniquely defined, positive integer m_1. It is this integer which features explicitly below, fixed by the value of a_-/a_+. To ensure all the poles of the function $v_0(y)v_0(-y)$ are simple we must take

$$a_-/a_+ \notin \hat{\mathcal{E}}_0 = \mathcal{E}_0 \cup \{2, 2/3, 2/5, 2/7, \dots\} \tag{6.11}$$

Now we return to $\mathcal{F}_0 : \hat{\mathcal{H}} \to \mathcal{H}$. It has a bounded adjoint

$$\mathcal{F}_0^* : \mathcal{H} \to \hat{\mathcal{H}}, \tag{6.12}$$

$$(\mathcal{F}_0^* f)_\delta(y) = (2a_+a_-)^{-1/2} \int_{\mathbb{R}} dx\, \overline{\psi_0(\delta x, y)} f(x), \quad \delta = +, - \tag{6.13}$$

where

$$\overline{\psi_0(x,y)} = w_0(x)^{1/2} v_0(-y) \sum_{\tau=+,-} \exp(-i\tau\pi xy/a_+a_-)\ell_0^\tau(x,-y)$$
$$= \psi(x,-y), \quad x, y \in \mathbb{R} \tag{6.14}$$

This uses the easily verified properties $\overline{v_0(y)} = v_0(-y)$ and $\overline{\ell_0^\tau(x,y)} = \ell_0^\tau(x,-y)$.

The operator on $\hat{\mathcal{H}}$ given by

$$R_0 = \mathcal{F}_0^* \mathcal{F}_0 - \mathbf{1}_{\hat{\mathcal{H}}} \tag{6.15}$$

is closely linked to isometry of $\mathcal{F}_0$ in the sense that it satisfies a priori

$$\|\mathcal{F}_0 \hat{f}\|_{\mathcal{H}}^2 - \|\hat{f}\|_{\hat{\mathcal{H}}}^2 = (\hat{f}, R_0 \hat{f})_{\hat{\mathcal{H}}} \tag{6.16}$$

We can thus see that vanishing of R_0 is equivalent to isometry of $\mathcal{F}_0$. For $\mathcal{F}_0^*$ we define analogously on $\mathcal{H}$,

$$\hat{R}_0 = \mathcal{F}_0 \mathcal{F}_0^* - \mathbf{1}_{\mathcal{H}} \tag{6.17}$$

In our summary theorem below we draw upon many of our earlier results in order to present explicit expressions for R_0 and $\hat{R}_0$ for (almost) the whole range of positive parameters a_+, a_-. (We must exclude a countable number of non-intersecting lines from the a_+, a_--plane corresponding to the ratios $a_-/a_+ \in \hat{\mathcal{E}}_0$. These ratios are those for which some of our earlier theorems do not hold.)

The following table lays out how the constants n_0,m_1 that feature in the theorem are fixed by the values of a_+, a_-. We switch to the ratio a_+/a_- which is preferable in the $N = 0$ case.

$a_+/a_- \in$	m_1	n_0
$(0,1)$	0	0
$(1,2)$	1	0
$(2,3)$	2	1
$(3,4)$	3	1
$(4,5)$	4	2
$\vdots$	$\vdots$	$\vdots$

Theorem 6.1. *Assume the positive parameters a_+, a_- satisfy $a_-/a_+ \notin \hat{\mathcal{E}}_0$. Then,*

$$R_0 = \frac{\sin(\pi a_+/a_-)}{2a_+} \sum_{k=1}^{n_0} \left(\Gamma_k^{(0)} \otimes \overline{\Gamma_k^{(0)}} - \Pi_k^{(0)} \otimes \overline{\Pi_k^{(0)}} \right) \tag{6.18}$$

and

$$\hat{R}_0 = \frac{\sin(\pi a_+/a_-)}{a_+} \sum_{k=1}^{m_1} \Psi_0^{(k-1)} \otimes \Psi_0^{(k-1)} \tag{6.19}$$

where

$$\Gamma_k^{(0)} = \begin{pmatrix} \gamma_k^{(0)} \\ \gamma_k^{(0)} \end{pmatrix}, \quad \Pi_k^{(0)} = \begin{pmatrix} \pi_k^{(0)} \\ -\pi_k^{(0)} \end{pmatrix}, \tag{6.20}$$

$$\gamma_k^{(0)}(y) = s_+(ky)/c_-((y + ia_+)/2), \tag{6.21}$$

$$\pi_k^{(0)}(y) = s_+(ky)/s_-((y + ia_+)/2), \tag{6.22}$$

$$\Psi_0^{(k)}(x) = 2c_+(x)Q^{(k)}(x)[s_-(x + ia_+/2)s_-(x - ia_+/2)]^{-1/2} \tag{6.23}$$

where $Q^{(k)}(x)$, (5.73), is a Chebyshev polynomial in $s_+(x)$ of degree k and parity $(-)^k$; unity when $k = 0$. Empty sums are defined to vanish. Indeed these are the only cases when R_0 and $\hat{R}_0$ vanish.

Proof. This is virtually immediate from Corollaries 5.2 and 5.6 and Theorems 5.4 and 5.8 upon specialising to $N = 0$. We simply note the residue calculations $w_0 = a_-/4\pi s_-(ia_+)$ and $\hat{w}_1 = a_-/4\pi s_-(2ia_+)$. □

A Appendix. Boundary proposition

The proposition in this appendix addresses two problems that arise in the main text. In each, a different set of variables is at issue; either (x, y, y') or (y, x, x'). Below we opt for a neutral set of variables (s, t, t') so that the proposition can be readily adapted with minimal confusion. The variables (t, t') range over either $\mathbb{R}^2$ or $\mathbb{R}^+ \times \mathbb{R}^+$; the symbol Ω is used to denote either of these sets. Most of the ideas we use to prove the proposition originate in [20].

Suppose $G_\pm(s, t, t')$ are two $\mathbb{C}$-valued functions on $\{\operatorname{Re} s > 0\} \times \Omega$, which are analytic in s and smooth in t, t'. And suppose they satisfy

$$G_\alpha(s, t, t') = A_\alpha(t, t') + \rho_\alpha(s, t, t'), \quad \alpha = +, - \tag{A.1}$$

where

$$\rho_\pm(s, t, t') = O(\exp(-\eta s)), \quad \operatorname{Re} s \to \infty, \quad \eta > 0 \tag{A.2}$$

$$(\partial_3 \rho_\pm)(s, t, t') = O(\exp(-\eta s)), \quad \operatorname{Re} s \to \infty, \quad \eta > 0 \tag{A.3}$$

with bounds that are uniform for $(\operatorname{Im} s, t, t')$ in compacts of $\mathbb{R} \times \Omega$. Furthermore suppose that $G_\pm$ satisfy

$$G_+(s, t, t) = G_-(s, t, t) \tag{A.4}$$

and that

$$A_+(t, t) = A_-(t, t) \tag{A.5}$$

We are now prepared for the proposition.

Proposition A.1. *Let $\phi(t, t') \in C_0^\infty(\Omega)$ with Ω equal to $\mathbb{R}^2$ or $\mathbb{R}^+ \times \mathbb{R}^+$, and*

$$I_\Lambda \equiv \int_\Omega dt dt' \, \phi(t, t') \frac{B_\Lambda(t, t')}{2s_+((t - t')/2)}, \quad \Lambda > 0 \tag{A.6}$$

where

$$B_\Lambda(t, t') \equiv \int_{\Lambda - ia_-/2}^{\Lambda + ia_-/2} ds \sum_{\alpha = +, -} \alpha \exp(i\alpha\pi s(t - t')/a_+ a_-) G_\alpha(s, t, t') \tag{A.7}$$

Then,

$$\lim_{\Lambda \to \infty} I_\Lambda = -2a_+ a_- \int_\Omega dt \, \phi(t, t) A_+(t, t) \tag{A.8}$$

Proof. We will break up $B_\Lambda(t, t')$ into four subsidiary integrals using two expansions:

$$\exp(i\alpha\pi s(t - t')/a_+ a_-) = \cos(\pi s(t - t')/a_+ a_-) + i\alpha \sin(\pi s(t - t')/a_+ a_-), \quad \alpha = +, -, \tag{A.9}$$

and the asymptotics (A.1). In other words we consider,

$$B_\Lambda(t,t') = \sum_{j=1}^{4} \int_{\Lambda - ia_-/2}^{\Lambda + ia_-/2} ds\, b_j(s,t,t') \tag{A.10}$$

$$b_1 \equiv i \sin(\pi s(t-t')/a_+a_-) A^{(+)}(t,t') \tag{A.11}$$

$$b_2 \equiv i \sin(\pi s(t-t')/a_+a_-) \rho^{(+)}(s,t,t') \tag{A.12}$$

$$b_3 \equiv \cos(\pi s(t-t')/a_+a_-) A^{(-)}(t,t') \tag{A.13}$$

$$b_4 \equiv \cos(\pi s(t-t')/a_+a_-) \rho^{(-)}(s,t,t') \tag{A.14}$$

where

$$A^{(\pm)}(t,t') \equiv A_+(t,t') \pm A_-(t,t') \tag{A.15}$$

$$\rho^{(\pm)}(s,t,t') \equiv \rho_+(s,t,t') \pm \rho_-(s,t,t') \tag{A.16}$$

Clearly, b_1,b_2 vanish when $t = t'$. The same is also true of b_3,b_4 in light of our assumptions (A.4) and (A.5). It follows that each of the four summands in (A.10) is a smooth function in t, t' that vanishes when $t = t'$ (analyticity of $\rho_\pm$ in s follows from that of $G_\pm$) . This same vanishing confirms that our integral I_Λ is well-defined, and will play a crucial role in neutralising the problem denominator in (A.6).

On the basis of this break-up we have

$$I_\Lambda = \sum_{j=1}^{4} I_j(\Lambda) \tag{A.17}$$

$$I_j(\Lambda) \equiv \int_\Omega dt dt' \, \frac{\phi(t,t')}{2s_+((t-t')/2)} \int_{\Lambda - ia_-/2}^{\Lambda + ia_-/2} ds\, b_j(s,t,t'), \quad j = 1, \ldots, 4 \tag{A.18}$$

The result follows because we will show

$$\lim_{\Lambda \to \infty} I_1(\Lambda) = -2a_+a_- \int_\Omega dt\, \phi(t,t) A(t) \tag{A.19}$$

$$\lim_{\Lambda \to \infty} I_j(\Lambda) = 0, \quad j = 2, 3, 4 \tag{A.20}$$

The nature of these computations divides naturally along the lines of the $\mathrm{Re}\, s$-asymptotics of b_1-b_4. The argument we use for $j = 2, 4$ requires decay of the integrand as $\mathrm{Re}\, s \to \infty$ and so it cannot encompass $j = 1, 3$. In the latter cases however, a direct evaluation of the s-integral is readily available, unlike for $j = 2, 4$, and this turns out to be useful. To see what these are, and why they help, we note

$$\int_{\Lambda - ia_-/2}^{\Lambda + ia_-/2} ds \sin \gamma s = 2i\gamma^{-1} \sin \gamma\Lambda \, \sinh(\gamma a_-/2) \ , \ \gamma \in \mathbb{R} \tag{A.21}$$

$$\int_{\Lambda - ia_-/2}^{\Lambda + ia_-/2} ds \cos \gamma s = 2i\gamma^{-1} \cos \gamma\Lambda \, \sinh(\gamma a_-/2) \ , \ \gamma \in \mathbb{R} \tag{A.22}$$

Applying the first of these to the $j = 1$ s-integral in (A.18) we have

$$I_1(\Lambda) = -\int_\Omega dtdt'\phi(t,t')\frac{\sin(\pi\Lambda(t-t')/a_+a_-)}{\pi(t-t')/a_+a_-}A^{(+)}(t,t') \tag{A.23}$$

Thus the problem denominator $s_+((t-t')/2)$ has been replaced with $(t-t')$. To handle this in the large-Λ limit we can invoke the tempered distribution

$$\lim_{\Lambda\to\infty}\gamma^{-1}\sin\gamma\Lambda = \pi\Delta(\gamma), \quad \gamma\in\mathbb{R} \tag{A.24}$$

where Δ is the Dirac delta function, with which we find

$$I_1(\Lambda) = -a_+a_-\int_\Omega dt\,\phi(t,t)A^{(+)}(t,t) \tag{A.25}$$

We have $A^{(+)}(t,t) = 2A_+(t,t)$ by assumption, and so (A.19) follows.

By applying (A.22) to the $j=3$ s-integral in (A.18) we have

$$I_3(\Lambda) = i\int_\Omega dtdt'\phi(t,t')\cos(\pi\Lambda(t-t')/a_+a_-)\frac{A^{(-)}(t,t')}{\pi(t-t')/a_+a_-} \tag{A.26}$$

This quotient is bounded when $t=t'$, and indeed on all of Ω, because of the vanishing of $A^{(-)}(t,t)$. Since ϕ has compact support on Ω by assumption, we may thus invoke the Riemann-Lebesgue lemma to procure vanishing.

The problem for $I_2(\Lambda)$ is bounding the s-integral by a function which decays in Λ, whilst still having a t,t'-dependence that maintains convergence of the t,t'-integral (which at present is ensured by the sine term). To solve this we write

$$I_2(\Lambda) = i\int_\Omega dtdt'\phi(t,t')\,\frac{\pi(t-t')/a_+a_-}{2s_+((t-t')/2)}\int_{\Lambda-ia_-/2}^{\Lambda+ia_-/2}ds\,\frac{\sin(\pi s(t-t')/a_+a_-)}{\pi(t-t')/a_+a_-}\rho^{(+)}(s,t,t') \tag{A.27}$$

where the first quotient is once again bounded on Ω. We then use the estimate

$$\Big|\frac{\sin\gamma s}{\gamma}\Big| = \frac{1}{2}\Big|\int_{-s}^{s}du\,\exp(i\gamma u)\Big| \leq |s|\exp|\gamma\,\mathrm{Im}\,s|, \quad \gamma\in\mathbb{R} \tag{A.28}$$

and the fact that the assumption about the asymptotics of $\rho_\pm(s,t,t')$ entails that for sufficiently large Λ there exists a $\mathcal{C}>0$ such that for all $(t,t')\in\mathrm{supp}(\phi)$,

$$|\rho^{(+)}(s,t,t')| \leq \mathcal{C}\exp(-\eta\Lambda), \quad \mathrm{Re}\,s = \Lambda \tag{A.29}$$

These two equations imply that the s-integral in (A.27) can be bounded from above by, for example,

$$2\mathcal{C}a_-\Lambda\exp(-\eta\Lambda)\exp|\pi(t-t')/a_+| \tag{A.30}$$

From this fact, $I_2(\Lambda)$ plainly vanishes under the large-Λ limit, recalling the compact support of ϕ on Ω.

We face the same problem for $I_4(\Lambda)$, but now if we insert a factor of unity in the same way the quotient $\cos(\pi s(t-t')/a_+a_-)/(t-t')$ is not at all convergent along $t=t'$. However, $\rho^{(-)}(s,t,t')/(t-t')$ *is* convergent along $t=t'$, because the assumptions (A.1) and (A.4) entail $\rho^{(-)}(s,t,t) = 0$. Thus it remains to find an analogous bound for this quotient which will ensure vanishing of $I_4(\Lambda)$ under the large-Λ limit. This is where the assumptions made about $\partial_3\rho_\pm$ come in. First we construct

$$\begin{aligned}
|\rho^{(-)}(s,t,t')| &= \Big| \int_t^{t'} du\, (\partial_3\rho^{(-)})(s,t,u) \Big| \\
&\leq |t-t'| \max_{u\in[t,t']} |(\partial_3\rho^{(-)})(s,t,u)| \\
&= |t-t'| \max_{\theta\in[0,1]} |(\partial_3\rho^{(-)})(s,t,t+\theta(t'-t))|
\end{aligned} \tag{A.31}$$

such that we get the following bound on the t,t'-integration region,

$$\Big|\frac{\rho^{(-)}(s,t,t')}{(t-t')}\Big| \leq \max_{(t,t',\theta)\in \mathrm{supp}(\phi)\times[0,1]} |(\partial_3\rho^{(-)})(s,t,t+\theta(t'-t))| \tag{A.32}$$

From (A.16), the rhs inherits large-$\mathrm{Re}\, s$ exponential decay with the same uniformity properties. With this we have addressed the hard part of the problem. It is then just a matter of constructing some elementary bounds like those we gave for $I_2(\Lambda)$ above in (A.30) to prove that $I_4(\Lambda)$ vanishes under the large-Λ limit.

□

Bibliography

[1] Ruijsenaars, S.N.M. (1999) A generalized hypergeometric function satisfying four analytic difference equations of Askey-Wilson type, *Comm. Math. Phys.*

[2] Ruijsenaars, S.N.M. (2003), A generalized hypergeometric function II. Asymptotics and D_4 Symmetry, *Commun. Math. Phys.* ***243****, 389-412*

[3] Ruijsenaars, S.N.M. (2003), A generalized hypergeometric function III. Associated Hilbert space transform, *Commun. Math. Phys.* ***243****, 413-448*

[4] Ruijsenaars, S.N.M. (2001) Sine-Gordon solitons vs. relativistic Calogero-Moser particles, *NATO Science Series Vol. 35, Kluwer, Dordrecht*

[5] Ruijsenaars, S.N.M. (2011), A Relativistic Conical Function and its Whittaker Limits, *SIGMA* ***7****, 101*

[6] L. J. Rogers (1894), Second memoir on the expansion of certain infinite products, *Proc. London Math. Soc.,* ***25****, 318-343*

[7] M. Reed and B. Simon (1972), Methods of modern mathematical physics. I. Functional analysis, *Academic Press, New York*

[8] M. Reed and B. Simon (1975), Methods of modern mathematical physics. II. Fourier analysis and self-adjointness, *Academic Press, New York*

[9] R. Askey and J. Wilson (1985), Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, *Mem. Am. Math Soc.* ***319***

[10] Ruijsenaars, S.N.M. and Schneider, H. (1985), A new class of integrable systems and its relation to solitons, *Annals of physics* ***170****, 370-405*

[11] Ruijsenaars, S.N.M. (1987), Complete integrability of relativistic Calogero-Moser systems and elliptic function identities, *Comm. Math. Phys,*

[12] M.S. Birman and M.Z. Solomjak (1987), Spectral theory of self-adjoint operators in Hilbert space, *Mathematics and its applications. Soviet series. Springer; 1987 edition*

[13] T. H. Koornwinder (1992), Askey-Wilson polynomials for root systems of type BC, *Contemp. Math.* ***138****, 189-204*

[14] J. F. van Diejen (1994), Integrability of difference Calogero-Moser systems, *Journal of Mathematical Physics 35 (6), 2983-3004*

[15] J. F. van Diejen and A. N. Kirillov (1995), Formulas for q-spherical functions using inverse scattering theory of reflectionless Jacobi operators, *Commun. Math. Phys.* ***210****, 335-369*

[16] I. G. Macdonald (1995), Symmetric functions and Hall polynomials, *Oxford University Press, Oxford, 1995*

[17] Ruijsenaars, S.N.M. (1997), First-order analytic difference equations and integrable quantum systems, *J. Math. Phys.* ***38****, 1069-1146*

[18] Ruijsenaars, S.N.M. (1998), Generalized Lamé functions. II Hyperbolic and trigonometric specializations, *J. Math. Phys. 40, 1627–1663*

[19] Ruijsenaars, S.N.M. (1999), Systems of Calogero-Moser type, *in: Proceedings of the 1994 Banff summer school, CRM series in mathematical physics, (G. Semenoff, L. Vinet, Eds.), pp. 251–352, Springer, New York*

[20] Ruijsenaars, S.N.M (2000), Hilbert space theory for reflectionless relativistic potentials, *RIMS Kyoto Univ.* ***36****, 707-753*

[21] Ruijsenaars, S.N.M. (2001), Self-adjoint A∆Os with vanishing reflection, *in: Proceedings NEEDS 2000, Theor. Math. Phys.* ***128****, 933–945*

[22] Ruijsenaars, S.N.M. (2002), Reflectionless Analytic Difference Operators III. Hilbert Space Aspects, *JNMP, Vol 9, Number 2, 181208*

[23] Ruijsenaars, S.N.M. (2002), A new class of reflectionless second-order A∆Os and its relation to nonlocal solitons, *Regular and Chaotic Dynamics,* ***7*** *351–391*

[24] Gustafson, S.J. and Sigal, I.M. (2003) *Mathematical Concepts of Quantum Mechanics*, Springer

[25] Ruijsenaars, S.N.M. (2005), A unitary joint eigenfunction transform for the A∆Os $\exp(ia_\pm d/dz) + \exp(2\pi z/a_\mp)$, *in: Proceedings Helsinki SIDE VI, J. Nonlinear Math. Phys., Suppl. 2 to Vol. 12 253–294*

[26] Ruijsenaars, S.N.M. (2005), Isometric reflectionless eigenfunction transforms for higher-order A∆Os, *Calogero Festschrift issue, J. Nonlinear Math. Phys. 12 Suppl. 1, 565–598.*

[27] S.N.M. Ruijsenaars (2005), A relativistic hypergeometric function, *Journal of Computational and Applied Mathematics* ***178****, 393417*

[28] Ruijsenaars, S.N.M (2007), Quadratic transformations for a function that generalizes ${}_2F_1$ and the Askey-Wilson polynomials, *Ramanujan J, 13:339-364*

[29] Teschl, G. (2014), Mathematical Methods in Quantum Mechanics With Applications to Schrödinger operators, Second Edition, *Graduate Studies in Mathematics, Volume 157, American Mathematical Society*

[30] M. Hallnäs and S.N.M. Ruijsenaars (2014), Joint eigenfunctions for the relativistic Calogero-Moser Hamiltonians of hyperbolic type. I. First steps, *Int. Math. Res. Not, no.16, 4400-4456*

[31] M. Hallnäs and S.N.M. Ruijsenaars (2016), Joint eigenfunctions for the relativistic Calogero-Moser Hamiltonians of hyperbolic type. II. The two- and three-variable cases, *arXiv:1607.06672*

[32] Haworth, S. and Ruijsenaars, S.N.M. (2016) Hilbert space theory for relativistic dynamics with reflection. I. Special cases *Journal of Integrable Systems*

We shall not cease from exploration, and the end of all our exploring will be to arrive where we started and know the place for the first time.

T.S. Eliot