The connective K-theory of elementary abelian p-groups for odd primes

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To my family:
to my Mom and to the memories of my Dad

Abstract

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by
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For an odd prime p, we aim to do some calculations of connective K-theory of elementary abelian groups $V(r)$, where $V(r)$ denotes an elementary abelian p-group of rank r. The methods involve a combination of Adams spectral sequence (ASS) calculations together with local cohomology calculations. The overall plan builds on and takes its inspiration from work of Prof. J. P. C. Greenlees and Prof. R. R. Bruner.

As a step towards the Gromov-Lawson-Rosenberg (GLR) conjecture for $V(r)$, the thesis calculates the complex connective K-cohomology, $ku^*(BV(r))$, for $r \leq 3$, and the complex connective K-homology, $ku_*(BV(r))$ for $p = 3$ and $r \leq 2$.

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Chapter 0

§ 0.1 Introduction

The connective K-theory of finite groups is of central importance in the Gromov-Lawson-Rosenberg (GLR) conjecture; see the relevant description in Greenlees-Bruner [10]. In particular, Greenlees and Bruner have given a detailed analysis of the connective K-cohomology, $ku^*(BV(r))$, and connective K-homology, $ku_*(BV(r))$, of elementary abelian groups $V(r)$ for only $p = 2$.

The main aim of this thesis is to treat the corresponding problem at odd primes, and to emphasize commutative algebra in this work. This can be obtained by extending the main results from $p = 2$ to the case of an odd prime.

The calculation at the prime 2 is arranged as follows: First, use the Adams spectral sequence to calculate the cohomology ring $ku^*(BV(r))$ for $V(r)$, by studying the mod p cohomology ring of the classifying space of $V(r)$ over the exterior algebra $E(1)$ and periodic K-theory. Quillen [34] proved that the mod p cohomology ring has Krull dimension equal to the rank of $V(r)$, whilst periodic K-theory is 1-dimensional. It follows that $ku^*(BV(r))$ has dimension equal to r if $V(r)$ is nontrivial. Ossa [32] proved that the Adams spectral sequence for $ku^*(BV(r))$ collapses. Next, there is a short exact sequence [10, Chapter 4]

$$0 \longrightarrow TU \longrightarrow ku^*(BV(r)) \longrightarrow QU \longrightarrow 0$$

of $ku^*(BV(r))$-modules, where TU is the p-torsion, and QU is the image of $ku^*(BV(r))$ in $K^*(BV(r))$, and therefore has no v-torsion.

Furthermore [10], the ring homomorphism

$$ku^*(BV(r)) \longrightarrow K^*(BV(r)) \times H^*(BV(r); \mathbb{F}_p)$$

is injective. This allows us to obtain by calculation the multiplicative structure of $ku^*(BV(r))$.

To calculate $ku_*(BV(r))$ for $p = 3$ and $r \leq 2$, we use the argument above and apply the local cohomology Theorem [10].

§ 0.2 The structure of the thesis

The thesis is structured as follows: In Chapter 1, we introduce some fundamental materials in periodic K-theory and connective K-theory. We use the Adams spectral sequence to calculate the complex connective K-cohomology, $ku^*(BG)$, for a finite group G [10]. We also give some brief background on the Adams splitting of connective K-theory at odd primes p. We also collect some information about a splitting of BC_p.

In Chapter 2, we begin to calculate $lu^*(BV(r))$ for $V(r)$, $r \leq 3$, as a module over the exterior algebra $E(1)$, where lu is the Adams summand of $ku_{(p)}$. The main tool for calculating $lu^*(BV(r))$ is the Adams spectral sequence [10]. This is done by taking periodic K-theory and mod p cohomology. In all calculations we fix the odd prime p, and we give explicit calculations for $lu^*(BV(r))$, $r \leq 3$. In other words, we first calculate $H^*(BV(r);\mathbb{F}_p)$ as an $E(1)$-module and then apply the Adams spectral sequence to calculate $lu^*(BV(r))$.

This work involves considering the following short exact sequence [10]:

$$0 \longrightarrow \overline{TU} \longrightarrow lu^*(BV(r)) \longrightarrow \overline{QU} \longrightarrow 0,$$

where $\overline{TU}$ is the v-power torsion module of $lu^*(BV(r))$ and $\overline{QU}$ is the image of $lu^*(BV(r))$ in $LU^*(BV(r))$. We are interested in calculating $\overline{TU}$ as it is annihilated by (p,v), and the natural transformation $lu^*(BV(r)) \longrightarrow H^*(BV(r);\mathbb{F}_p)$ embeds $\overline{TU}$ in $H^*(BV(r);\mathbb{F}_p)$, (see Section 2.7). This means it is appropriate to introduce, for any rank r, the PC-module structure on $\overline{TU}$, where $PC = \mathbb{F}_p[y_1, y_2, \ldots, y_r]$ is the polynomial ring on the generators $y_i = c_1^{H\mathbb{F}_p}(\alpha_i)$ for the generating representations $\alpha_1, \alpha_2, \ldots, \alpha_r$. Here the generator $y_i \in H^2(BV(r);\mathbb{F}_p)$ is the image of $c_1^{ku}(\alpha_i) \in ku^2(BV(r))$. All calculations of $\overline{TU}$ focus completely on the PC-module structure. We will also consider the subring $PP = \mathbb{F}_p[Y_1, Y_2, \ldots, Y_r]$, where $Y_i = y_i^{p-1}$ and the PP-module structure obtained by restriction.

In more detail, we aim to identify explicit $PP \otimes E(1)$-submodules of $H^*(BV(r);\mathbb{F}_p)$ so that one of them is a free $E(1)$-submodule and the other has no free $E(1)$-summand. Through the Adams spectral sequence, the non free summand gives the module $\overline{QU}$. We then find an $E(1)$-basis of the free summand which is a PC-submodule of the cohomology $H^*(BV(r);\mathbb{F}_p)$.

After that, we give a summary of our calculations of $\overline{TU}$ for $V(r)$, $r \leq 3$, and our expectations for $V(r)$, $r \geq 4$. To do that, we introduce a PC-submodule $\overline{TU}^r \subseteq H^*(BV(r);\mathbb{F}_p)$ which agrees with $\overline{TU}$ for $r \leq 3$, and Conjecture 2.10.2 is that it is isomorphic to $\overline{TU}$ in general.

The additive structure of $lu^*(BV(r))$ can be read from an Adams spectral sequence and the multiplicative structure can be obtained from the mod p cohomology together with representation theory. We finish the Chapter by displaying a table of the $ku^*(BV(r))$-calculations for $p = 3, 5, 7$ for $V(r)$, $r \leq 3$ by considering the relation between the modules TU and QU, where TU is the kernel of $ku^*(BV(r))$ in $K^*(BV(r))$ so that

$$TU = \overline{TU} \oplus \Sigma^{-2}\overline{TU} \oplus \Sigma^{-4}\overline{TU} \oplus \cdots \oplus \Sigma^{-2p+4}\overline{TU},$$

and QU is the image of $ku^*(BV(r))$ in $K^*(BV(r))$.

In Chapter 3, we define certain PC-submodules $\overline{TU}^r_s \subseteq H^*(BV(r);\mathbb{F}_p)$. We then construct a free resolution for the submodule $\overline{TU}^r_n$ over the polynomial ring $PC = \mathbb{F}_p[y_1, y_2, \ldots, y_r]$. We observe that $\overline{TU}^r$ can be expressed as a sum

$$\overline{TU}^r = \overline{TU}^r_2 \oplus \overline{TU}^r_3 \oplus \cdots \oplus \overline{TU}^r_r,$$

as PC-modules [10, page 95]. We start to give some pictures for a free resolution of $\overline{TU}^r_n$ when $r = 4$. To start with,

$$\overline{TU}^4 = \overline{TU}^4_2 \oplus \overline{TU}^4_3 \oplus \overline{TU}^4_4.$$

First, $\overline{TU}^r_r$ is a free PC-module. Second, we construct a length one free resolution for $\overline{TU}^r_{r-1}$ for any odd prime p. We introduce the main result in general to establish a free resolution for $\overline{TU}^r_n$ over PC and prove it, (Section 3.4). The key idea in constructing the resolution is a truncation of the double Koszul complex for the regular sequences vertically and horizontally. Indeed, we construct the resolution as a truncation of an exact sequence.

Chapter 4 deals with the calculation of the local cohomology of the PC-module $\overline{TU}^r_i$ in degrees r and i. The main tool for calculating the local cohomology of $\overline{TU}^r_i$ is local duality [25]. At the beginning, we define a stable Koszul complex to calculate the local cohomology of $\overline{TU}^r$ and give some examples to describe Koszul complexes. After that, we use the free resolution from Chapter 3 to give precise calculations of the local cohomology of the module $\overline{TU}^r$ by using local duality.

We display a description for the general behaviour of the local cohomology of $\overline{TU}^r$. The main result is in Section 4.3 (see Theorem 4.3.1), which gives in Part(2) the dual of the top local cohomology and in Part(3) the Hilbert series of the other nonzero cohomology. It turns out that $\overline{TU}^r_i$ is extremely close to being a Cohen–Macaulay module, and $\overline{TU}^r$ is very close to being Gorenstein.

We close this Chapter by discussing the Hilbert series of Noetherian modules over the polynomial ring PC and calculating the dual local cohomology modules $H^i_{\mathfrak{m}}(\overline{TU}^r_i)^\vee$.

In Chapter 5, we introduce some examples to explain our results from the previous Chapter to calculate the local cohomology of the PC-modules $\overline{TU}^r_n$ for $r \leq 5$, which is defined using $H^*(BV(r);\mathbb{F}_p)$ as an $E(1)$-module in Chapter 3. To do that, we start to explain the organization of our calculations and give a general pattern with the consequences for $\overline{TU}^r_n$.

We consider directly the main result, Theorem 4.3.1 in Chapter 4, the calculation of the local cohomology of the PC-modules $\overline{TU}^r_n$ for $r \leq 5$.

In Chapter 6, we calculate the complex connective K-homology, $ku_*(BV(r))$ for $p = 3$ and $r \leq 2$, as a module over $ku^*(BV(r))$. The main tool for calculating $ku_*(BV(r))$ is the local cohomology Theorem. This is done by calculating the local cohomology of QU and TU and then using the short exact sequence

$$0 \longrightarrow TU \longrightarrow ku^*(BV(r)) \longrightarrow QU \longrightarrow 0,$$

where TU and QU have already been defined in Chapter 2. First, we introduce the strategy to calculate $H^*_{JU}(QU)$, which is to replace the augmentation ideal JU, the radical of the principal ideal (y^*) (i.e., $JU = \sqrt{(y^*)}$). After that, we need to calculate ideals JU_k for $r \leq 2$, and then display our calculations for $H^*_{(y^*)}(QU)$ on the $E_{1\frac{1}{2}}$-term, which is formed by taking homology of TU and QU, by Proposition 6.1.4, [10]. To obtain the E_2-term, there is only one differential coming from the long exact sequence which follows from the fact that $ku_*(BV(r))$ is connective (i.e., $ku_t(BV(r)) = 0$, for $t < 0$), together with the module structure, (see Section 6.3.4). It turns out that $E_2 = E_\infty$-term without any extension problems. We finish the Chapter by displaying our result for $ku_*(BV(r))$ as a table for $V(2)$ depending on Conjecture 6.3.3 of rank 2.

§ 0.3 Main results

The main results from our calculation in this thesis are as follows:

- $ku^*(BV(r))$, for an elementary abelian p-group $V(r)$, for $r \leq 3$;
- construction of a free resolution in general of $\overline{TU}^r_n$ over the polynomial ring PC (Proposition 3.4.1);
- the local cohomology of $\overline{TU}^r_n$ and a calculation of the Hilbert series of the dual local cohomology modules (Theorem 4.3.1);
- some examples of the local cohomology of $\overline{TU}^r_n$ for $V(r)$, $r \leq 5$;
- detailed information about $ku_*(BV(r))$ for $p = 3$ and $r \leq 2$.

§ 0.4 The conclusions:

The Bruner-Greenlees method for the calculation of connective K-theory for finite groups is a powerful gadget. We have used this machine for calculations of $ku^*(BV(r))$ and $ku_*(BV(r))$ in low rank for odd primes. These calculations show that the methods applied for $p = 2$ continue to apply for odd primes when combined with the splitting results. Our calculations provide the basis for verifying the GLR conjecture in the cases for which we have calculated $ku_*(BV(r))$.

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Chapter 1

Preliminaries

The main purpose of this Chapter is to introduce some useful elementary materials about equivariant K-theory. For our purposes, G is a compact Lie group and we consider equivariant cohomology theories on G-CW complexes.

§ 1.1 Periodic K-theory

Convenient references for this section are [4, 6, 14, 37] and the reader may refer to [5] for further details.

The aim of this section is to introduce some ideas of periodic K-theory. It is important for our purposes (connective K-theory of elementary abelian p-groups) to study G-equivariant periodic K-theory, $K_G(X)$, and group cohomology. Given any compact group G and a finite G-CW complex X, we note that by definition [37],

$$K_G(X) := \frac{\mathbb{Z}\{[\eta] \mid \eta \text{ } a \text{ } G - \text{vector bundle on } X\}}{([\eta_1 \oplus \eta_2] = [\eta_1] + [\eta_2])},$$

where η_1, η_2 are two G-vector bundles on X. A G-vector bundle is a G-map $\eta \xrightarrow{p} X$ so that for each $x \in X$, the fibre $\eta_x := p^{-1}(x)$ is a complex vector space and for $g \in G$, the translation $g : \eta_x \longrightarrow \eta_{gx}$ is linear, and p is locally trivial [14]. This theory is representable, and hence extends to arbitrary G-CW complexes.

The tensor product of G-vector bundles induces the structure of a commutative ring on $K_G(X)$. If X is a locally compact based G-space, then

$$\widetilde{K}^0_G(X) = \ker(K_G(X) \longrightarrow K_G(pt)), \quad \widetilde{K}^1_G(X) = \widetilde{K}^0_G(S^1 \wedge X),$$

and

$$K^0_G(X) \cong \widetilde{K}^0_G(X) \oplus \mathbb{Z}, \quad K^1_G(X) \cong \widetilde{K}^1_G(X).$$

If X is a point, then $K_G(X) \cong RU(G)$, the complex representation ring of G, and we apply Bott Periodicity Theorem, to find [37]

$$K^*_G(X) \cong K^G_*(X) = RU(G)[v, v^{-1}],$$

where v is the Bott element of degree 2. Now if G acts freely on X, then

$$K_G^*(X) \cong K^*(X/G).$$

It will be useful to state the Atiyah–Segal completion Theorem [4].

Theorem 1.1.1. *The equivariant K-theory of EG is given by*

$$K_G^0(EG) = K^0(BG) = RU(G)^\wedge_{JU} \quad \textit{and} \quad K_G^1(EG) = K^1(BG) = 0,$$

where $RU(G)^\wedge_{JU}$ is the JU-adic completion of $RU(G)$ with $JU = \ker(RU(G) \longrightarrow \mathbb{Z})$.

Proof. Details can be found in [4]. □

Remark 1.1.2. In homotopy theory, we notice that EG is a terminal free G-space. This means for any free G-space X, there is a G-map $\psi_X : X \longrightarrow EG$, unique up to homotopy. Indeed, EG is a free contractible G-space with $BG = (EG)/G$. From this, and our argument above, we find

$$K^*(BG) = K^*(EG/G) = K_G^*(EG) = RU(G)^\wedge_{JU}[v, v^{-1}].$$

Generally, if G is a finite p-group, then we have that the JU-adic and (p)-adic topology coincide on JU so that [6]

$$RU(G)^\wedge_{JU} \cong \mathbb{Z} \oplus JU(G)^\wedge_{JU} \cong \mathbb{Z} \oplus (\mathbb{Z}_p^\wedge \otimes JU(G)).$$

Furthermore, if G is a finite group, then the JU-adic completion $RU(G) \longrightarrow RU(G)^\wedge_{JU}$ is injective if and only if G is a p-group [14].

§ 1.2 Connective K-theory

Our sources here are [4, 10, 17] and the reader may refer to [16] for further details.

In this section, we need only consider the complex connective K-theory ku for an arbitrary compact Lie group G [17]. In fact, ku is a commutative and associative ring spectrum up to homotopy, and it is complex orientable. The connective K-cohomology ring, $ku^*(BG)$, is a Noetherian ring for a finite group G [10], and

$$ku^* = ku^*(pt) = ku_*(pt) = \mathbb{Z}[v].$$

The following cofibre sequence gives the relationship between connective K-theory ku and ordinary integral cohomology $H\mathbb{Z}$ [10],

$$\Sigma^2 ku \xrightarrow{v} ku \longrightarrow H\mathbb{Z},$$

and there is an equivalence

$$K \simeq ku[1/v].$$

Lemma 1.2.1. *For any space X we have $(ku^*X)[1/v] \xrightarrow{\cong} K^*(X)$.*

Now, let $I = \ker(ku_G^* \longrightarrow ku^*)$ be the augmentation ideal. For any compact Lie group G, the completion Theorem holds for ku [17],

$$ku^*(BG) = ku_G^*(EG) = (ku_G^*)_I^\wedge.$$

In this thesis, the Adams spectral sequence (see Section 1.5) is used to calculate the cohomology ring $ku^*(BV(r))_p^\wedge$ [10],

$$\mathrm{Ext}_{E(1)}^{*,*}(H^*(ku), H^*(BV(r); \mathbb{F}_p)) \Longrightarrow ku^*(BV(r))_p^\wedge.$$

More precisely, our calculations of $ku^*(BV(r))_p^\wedge$ start from the ordinary complex representation ring of $V(r)$ and the mod p cohomology of $BV(r)$. The representation ring $RU(V(r))$ gives the periodic K-theory

$$K^*(BV(r)) = RU(V(r))_{JU}^\wedge[v, v^{-1}],$$

where $RU(V(r))_{JU}^\wedge$ denotes the JU-adic completion of $RU(V(r))$.

The additive structure and partial multiplicative information can be obtained from an Adams spectral sequence, and to determine additive and multiplicative extensions we apply representation theory. In fact, the Atiyah–Segal completion Theorem [4] states $K^*(BG) = RU(G)_{JU}^\wedge[v, v^{-1}]$, so that the map $ku^*(BG) \longrightarrow K^*(BG)$ gives good details about additive and multiplicative structure.

§ 1.3 The splitting of $ku_{(p)}$

Our source here is [10] and the reader may refer to [32] for further details.

In this section, to calculate the Adams spectral sequence, we need to introduce the Adams splitting of $ku_{(p)}$. At odd primes p, if lu is the principal Adams summand, we have the Adams splitting [1],

$$ku_{(p)} \simeq lu \vee \Sigma^2 lu \vee \cdots \vee \Sigma^{2p-4} lu. \tag{1.1}$$

Indeed, lu is a ring spectrum and the map $lu \longrightarrow ku_{(p)}$ is a map of ring spectra. The direct sum formula for $H^*(ku)$ is given by

$$H^*(ku) = H^*(lu) \oplus H^*(\Sigma^2 lu) \oplus \cdots \oplus H^*(\Sigma^{2p-4} lu). \tag{1.2}$$

Remark 1.3.1. It is necessary to refer to lower indices as degrees and upper indices as codegrees; $v \in K_2 = K^{-2}$ has degree 2 and codegree -2.

Corollary 1.3.2. *[10, page 84] The natural maps induce an injective ring homomorphism*

$$ku^*(BV(r)) \longrightarrow K^*(BV(r)) \times H^*(BV(r); \mathbb{F}_p).$$

§ 1.4 The Steenrod algebra

A good reference for this material is [41, Chapter VI] and the reader may refer to [23, page 496] for further details.

One of the main aims in homotopy theory is to compute the homotopy groups of the sphere $\pi_*(S^0)$. An important tool is the mod p Steenrod algebra $\mathcal{A}$ and its cohomology $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$, which forms the E_2-term of the Adams spectral sequence converging to the p component of $\pi_*^s(S^0)$.

In this section, we define the mod p Steenrod algebra $\mathcal{A}$ for odd p, to be the graded associative algebra over $\mathbb{F}_p$ formed by polynomials in the noncommuting variables $\beta, \mathcal{P}^1, \mathcal{P}^2, \ldots$ modulo the Adem relations and the relation $\beta^2 = 0$, where β is of codegree 1 and $\mathcal{P}^i$ is of codegree $2i(p-1)$.

Thus for every space X, $H^*(X;\mathbb{F}_p)$ is a module over $\mathcal{A}$ for all primes p. Furthermore, the dual of $\mathcal{A}$, $\mathcal{A}_*$, is a commutative Hopf algebra over $\mathbb{F}_p$. From the structure theorems for Hopf algebras proved in [8] and [31] and the known action of $\mathcal{A}$ on certain test spaces, [30] computed the structure of $\mathcal{A}_*$.

Remark 1.4.1. We denote the monomial in $\mathcal{A}$ by

$$\mathcal{P}^I = \beta^{\epsilon_0}\mathcal{P}^{s_1}\beta^{\epsilon_1}\ldots\mathcal{P}^{s_k}\beta^{\epsilon_k},$$

where $I = (\epsilon_0, s_1, \epsilon_1, s_2, \ldots, s_k, \epsilon_k, 0, 0\ldots)$, with $\epsilon_i \in \{0,1\}$ and $s_i \in \mathbb{N}$. Now if I is a finite sequence of non-negative integers, $(\epsilon_0, s_1, \epsilon_1, s_2, \ldots, s_k, \epsilon_k)$, then I is admissible if $s_i \geq ps_{i+1} + \epsilon_i$, for $k > i \geq 1$, which means that no Adem relation can be applied to the monomial. Of course, if I is admissible, we call $\mathcal{P}^I$ admissible.

Theorem 1.4.2. *The admissible monomials form a basis for $\mathcal{A}$ as a vector space over $\mathbb{F}_p$.*

Proof. Details are given in [41]. □

The theorem implies that $\{\mathcal{P}^I : I\,\text{admissible}\}$ forms a spanning set for $\mathcal{A}$.

Theorem 1.4.3. *The Steenrod algebra $\mathcal{A}$ is a Hopf algebra for each p.*

The theorem entails that $\mathcal{A}$ has a comultiplication map $\Delta\colon \mathcal{A} \longrightarrow \mathcal{A}\otimes\mathcal{A}$. It has the following effect on the operations

$$\Delta(\mathcal{P}^k) = \sum_{i=0}^{k} \mathcal{P}^i \otimes \mathcal{P}^{k-i} \quad \text{and} \quad \Delta(\beta) = 1\otimes\beta + \beta\otimes 1.$$

One can calculate $H^*(K(C_p, n);\mathbb{F}_p)$ as a graded module over $E(1) = \Lambda(Q_0, Q_1)$, where $Q_0 = \beta$ is the Bockstein homomorphism of codegree 1 and $Q_1 = \mathcal{P}^1\beta - \beta\mathcal{P}^1$ is the first higher Milnor Bockstein of codegree $2p-1$. This information can then be utilized in determining the relations among cohomology operations. Since $\mathcal{P}^i$ raises dimension by $2i(p-1)$, the Bockstein β associated with the exact coefficient sequence

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{\times p} \mathbb{Z}_{p^2} \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

is not in the algebra generated by the operations $\mathcal{P}^i$.

Theorem 1.4.4. *([38]) As an algebra, $\mathcal{A}_*$ is isomorphic to the tensor product of an exterior and a polynomial algebra*

$$\Lambda(\tau_0, \tau_1, \dots) \otimes \mathbb{F}_p[\xi_1, \xi_2, \dots],$$

where $|\tau_i| = 2p^i - 1$ and $|\xi_j| = 2(p^j - 1)$. As a coalgebra, the comultiplication on $\mathcal{A}_*$ is given by

$$\Delta(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{p^i} \otimes \xi_i \quad \text{and} \quad \Delta(\tau_k) = \tau_k \otimes 1 + \sum_{i=0}^{k} \xi_{k-i}^{p^i} \otimes \tau_i.$$

Remark 1.4.5. In our case, the mod p cohomology of $V(r)$ is given by $Q_0 = \beta$ (of codegree 1) and $Q_1 = \mathcal{P}^1\beta - \beta\mathcal{P}^1$ (of codegree $2p-1$), where Q_0 and Q_1 act as derivations (i.e., $Q_k(xy) = Q_k(x)y + xQ_k(y)$).

Now, we define the exterior algebra $E(1) = \Lambda(Q_0, Q_1)$ as described on the preceding page, generated by Q_0 and Q_1 to be the subalgebra of $\mathcal{A}$, and $\mathcal{A}_1 = \langle \beta, \mathcal{P}^1 \rangle$ the subalgebra of $\mathcal{A}$ generated by Q_0 and the first Steenrod power $\mathcal{P}^1$. From this, we find some relations in $\mathcal{A}_1$

$$Q_0Q_0 = 0,\ Q_1Q_1 = 0,\ Q_0Q_1 = -Q_1Q_0,\ (\mathcal{P}^1)^p = 0,\ \mathcal{P}^1Q_1 = Q_1\mathcal{P}^1.$$

The mod p Steenrod algebra $\mathcal{A}$ is a free $E(1)$-module [31]. By a change of rings, we obtain

$$\operatorname{Ext}_{\mathcal{A}}^{*,*}(H^*(ku), H^*(BV(r);\mathbb{F}_p)) = \operatorname{Ext}_{E(1)}^{*,*}(\mathbb{F}_p, H^*(BV(r);\mathbb{F}_p)) \Longrightarrow ku^*(BV(r))_p^{\wedge}.$$

§ 1.5 The Adams spectral sequence

Our sources here are [29, Chapter 9] and the reader may refer to [28] for further details.

The Adams spectral sequence is an important tool for computing stable homotopy groups of spheres, and more generally the stable homotopy groups of any space.

In this section, we need to give some information about the Adams spectral sequence in detail as it is the main tool to calculate the complex connective K-cohomology in this thesis.

First, to compute the set $\{X, Y\}$ of stable homotopy classes of maps $X \longrightarrow Y$ one could consider induced homomorphisms on homology. This gives a map $\{X, Y\} \xrightarrow{f}$

$\mathrm{Hom}(H_*(X), H_*(Y))$. The first interesting instance of this is the notion of degree for maps $S^n \longrightarrow S^n$, and the degree then computes $\{S^n, S^n\}$. For maps between spheres of different dimension we obtain no information this way, however, so it is useful to look for more progressing structure. For a start we can replace homology by cohomology since this has cup products and their stable counterparts, Steenrod squares and powers. Since we are now applying a contravariant functor of X and Y, we then have a map $\{X, Y\} \longrightarrow \mathrm{Hom}_{\mathcal{A}}(H^*(Y), H^*(X))$, where $\mathcal{A}$ is the mod p Steenrod algebra and cohomology is taken with $\mathbb{F}_p$ coefficients.

Since cohomology and Steenrod operations are stable under suspension, it will help to change our viewpoint and let $\{X, Y\}_t = \varinjlim [\Sigma^{k+t} X, \Sigma^k Y]$, the direct limit under suspension of the set of maps $\Sigma^k X \longrightarrow \Sigma^k Y$. This has the advantage that the map $\{X, Y\} \longrightarrow \mathrm{Hom}_{\mathcal{A}}(H^*(Y), H^*(X))$ is a homomorphism of abelian groups, where cohomology is now to be interpreted as reduced cohomology since we want it to be stable under suspension.

Note that $\mathrm{Hom}_{\mathcal{A}}(H^*(Y), H^*(X))$ is a subgroup of $\mathrm{Hom}(H^*(Y), H^*(X))$. Now to use the $\mathcal{A}$-module structure, recall that $\mathrm{Hom}_{\mathcal{A}}$ is the $n = 0$ case of a whole sequence of functors $\mathrm{Ext}^n_{\mathcal{A}}$. Since $\mathcal{A}$ has such a complicated multiplicative structure, these higher $\mathrm{Ext}^n_{\mathcal{A}}$ groups could be nontrivial and might have more information than $\mathrm{Hom}_{\mathcal{A}}$ by itself. Consider the functor $\mathrm{Ext}^1_{\mathcal{A}}$.

This measures whether short exact sequences of $\mathcal{A}$-modules split. For a map $S^k \xrightarrow{h} S^\ell$ with $k > \ell$ one can form the mapping cone C_h, and then associated to the pair (C_h, S^ℓ) there is a short exact sequence of $\mathcal{A}$-modules

$$0 \longrightarrow H^*(S^{k+1}) \longrightarrow H^*(C_h) \longrightarrow H^*(S^\ell) \longrightarrow 0.$$

Additively this splits, but whether it splits over $\mathcal{A}$ is equivalent to whether $\mathcal{A}$ acts trivially in $H^*(C_h)$ since it acts trivially on the two adjacent terms in the short exact sequence. Since $\mathcal{A}$ is generated by the squares or powers as in the previous Section, therefore, we are asking whether some Sq^i or $\mathcal{P}^i$ is nontrivial in $H^*(C_h)$.

For $p = 2$ this is the mod 2 Hopf invariant question, and for an odd prime it is the mod p analog. The answer for $p = 2$ is the theorem of Adams that Sq^i can be nontrivial only for $i = 1, 2, 4, 8$. For odd p the corresponding statement is that only $\mathcal{P}^1$ can be nontrivial.

Thus $\mathrm{Ext}^1_{\mathcal{A}}$ does indeed detect some small but nontrivial part of the stable homotopy groups of spheres. One could hardly expect the higher $\mathrm{Ext}^n_{\mathcal{A}}$ functors to give a full description of stable homotopy groups, but the Adams spectral sequence says that they give a reasonable first approximation. In the case that X is a sphere, the Adams spectral sequence states

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(Y; \mathbb{F}_p), \mathbb{F}_p) \Longrightarrow \{S^0, Y\}^\wedge_p = \pi^s_*(Y)^\wedge_p.$$

Note that the second index t in $\mathrm{Ext}^{s,t}_{\mathcal{A}}$ denotes just a grading of $\mathrm{Ext}^s_{\mathcal{A}}$ arising from the usual grading of $H^*(Y)$.

Theorem 1.5.1. *([28])For X and Y bounded below of finite type, there is a spectral sequence, converging to $\{X,Y\}_p^\wedge$, with E_2-term given by*

$$E_2^{s,t} \cong \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(Y;\mathbb{F}_p), H^*(X;\mathbb{F}_p)),$$

and differentials $d_r\colon E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$ of bidegree $(r, r-1)$. If $\{X,Y\}$ is finitely generated, then $\{X,Y\}_p^\wedge = \{X,Y\} \otimes \mathbb{Z}_p^\wedge$.

Remark 1.5.2. The Adams spectral sequence is compatible with composition and hence multiplicative, but we will not make use of this.

Taking $X = S^0$ gives the earlier case, which suffices for the more common applications, but the general case illuminates the formal machinery, and is really no more difficult to set up than the special case. For a space X, the hypothesis needed is that X be a CW-complex with finitely many cells in each dimension.

Note that the Adams spectral sequence breaks the problem of computing stable homotopy groups of spheres up into three steps. First, there is the purely algebraic problem of computing $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_p, \mathbb{F}_p)$. After this has been done through some range of values for s and t there remain the two problems one usually has with a spectral sequence, computing differentials and resolving ambiguous extensions.

The Adams spectral sequence for a point reads

$$\begin{aligned}\mathbb{F}_p[a_0, u] &= \mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_p, \mathbb{F}_p)\\ &= \mathrm{Ext}_{\mathcal{A}}^{*,*}(H^*(lu), H^*(S^0)) \Longrightarrow lu^*{}_p^\wedge = \mathbb{Z}_p^\wedge[u],\end{aligned}$$

where $a_0 \in \mathrm{Ext}_{E(1)}^{1,1}(\mathbb{F}_p, \mathbb{F}_p)$ detects the map of degree p and $u \in \mathrm{Ext}_{E(1)}^{1,2p-1}(\mathbb{F}_p, \mathbb{F}_p)$ detects $u = v^{p-1}$.

§ 1.6 The splitting of BC_p

Our sources here are [10, 42] and the reader may refer to [32] for further details.

When $p > 2$, we have a stable splitting of BC_p. The aim of this section is to display a splitting of BC_p, and in order to treat the rank r case we shall work in the category of the p-local spectra.

Theorem 1.6.1. *There is a stable splitting of BC_p described by the formula*

$$BC_p \simeq B_1 \vee B_2 \vee \cdots \vee B_{p-1},$$

where $H^*(B_i;\mathbb{F}_p)$ is nonzero only in degrees congruent to $2i$ and $2i-1$ modulo $2p-2$.

The following Lemma will be used in connective K-theory.

Lemma 1.6.2 (Ossa [10])**.** *There is a homotopy equivalence of spectra $ku \wedge B_{i+1} \simeq ku \wedge \Sigma^2 B_i$ for $1 \le i \le p-1$, and, as an $E(1)$-module, $H^*(B_i;\mathbb{F}_p) = \Sigma^{2i} L$,* where L is a certain module starting in degree 0 (we will explain that clearly in the next Chapter).

In fact, we shall focus primarily on B_{p-1}.

Lemma 1.6.3 ([42]). *Suppose*

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

is a short exact sequence with $N \triangleleft G$ and $p \nmid |Q|$. Then $H^*(BG) = (H^*(BN))^Q$,

where the coefficients are p-local.

We are going to explain this in detail. Let $\Gamma = C_p \rtimes C_{p-1} = \mathbb{F}_p \rtimes \mathbb{F}_p^\times$, where $C_{p-1} = \langle g \rangle$ so that $\mathbb{F}_p^\times = \langle \delta \rangle \cong C_{p-1}$. We apply Lemma 1.6.3, to obtain

$$H^*(B\Gamma) = H^*(BC_p)^{C_{p-1}}.$$

We show how the Adams spectral sequence computation goes for cyclic groups. We have $H^*(BC_p) = \mathbb{F}_p[y] \otimes \Lambda(\tau)$ with $|\tau| = 1$ and $|y| = 2$, and suppose the group $\mathrm{Aut}(C_p) = \langle g \rangle$ acts via $g \cdot \tau = \delta\tau$ and $g \cdot y = \delta y$.

Then we obtain

$$g \cdot (y^i) = (g \cdot y)^i = (\delta y)^i = \delta^i y^i$$

and

$$g \cdot (\tau y^i) = (g\tau)(g \cdot y)^i = (\delta\tau)(\delta y)^i = \delta^{i+1}\tau y^i.$$

Hence

$$H^*(B\Gamma) = H^*(BC_p)^{C_{p-1}} = \mathbb{F}_p[y^{p-1}] \otimes \Lambda(\tau y^{p-2}) = \mathbb{F}_p[Y] \otimes \Lambda(T),$$

where $Y = y^{p-1}$ (of degree $2p-2$) and $T = \tau y^{p-2}$ (of degree $2p-3$).

Now, one may check that the calculation of the Adams spectral sequence [10, page 35]

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_p, H^*(B_1)) \Longrightarrow [B_1, lu]^* = lu^*(B_1)$$

works exactly as in the 2-primary case.

Chapter 2

Complex connective K-cohomology

The main aim of this Chapter is to calculate the cohomology of elementary abelian p-groups $V(r)$, $H^*(BV(r))$, for $r \leq 3$ explicitly for input to the Adams spectral sequence. In all of the calculations and diagrams in this Chapter, we will fix the odd prime p . The tool for calculating $lu^*(BV(r))$ is the Adams spectral sequence

$$\mathrm{Ext}^{*,*}_{\mathcal{A}}(H^*(lu), H^*(BV(r);\mathbb{F}_p)) \Longrightarrow lu^*(BV(r))^{\wedge}_p.$$

At odd primes p we have already seen that as in (1.1),

$$ku_{(p)} \simeq lu \vee \Sigma^2 lu \vee \Sigma^4 lu \vee \cdots \vee \Sigma^{2p-4} lu,$$

where $lu = BP\langle 1\rangle$ can be realized as a homotopy ring spectrum, with coefficient ring $\mathbb{Z}_{(p)}[v_1]$ and a map $lu \longrightarrow ku_{(p)}$ taking v_1 to v^{p-1}. Accordingly, we will focus on

$$\overline{TU} = \ker(lu^*(BV(r)) \longrightarrow LU^*(BV(r))),$$

so that

$$TU = \overline{TU} \oplus \Sigma^{-2}\overline{TU} \oplus \cdots \oplus \Sigma^{-2p+4}\overline{TU}.$$

We note that if G is a p-group, then $\widetilde{lu^*}(BG)$ is already p-complete by Atiyah-Segal completion Theorem, so the Adams spectral sequence gives the precise calculation we need. We find [1],

$$H^*(lu) = \mathcal{A} \otimes_{E(1)} \mathbb{F}_p,$$

where $E(1)$ is the exterior algebra generated by Milnor elements $Q_0 = \beta$ (of codegree 1) and $Q_1 = \mathcal{P}^1\beta - \beta\mathcal{P}^1$ (of codegree $2p-1$). With change of rings the Adams spectral sequence becomes

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_p, H^*(BV(r);\mathbb{F}_p)) \Longrightarrow lu^*(BV(r))^{\wedge}_p.$$

§ 2.1 The modules QU and $\overline{QU}$

The target of this section is to define the modules QU and $\overline{QU}$ of elementary abelian p-groups.

Definition 2.1.1. The module QU is the image of connective K-theory $ku^*(BV(r))$ in periodic K-theory $K^*(BV(r))$ [10, Chapter 4] (i.e., $QU = \mathrm{im}(ku^*(BV(r)) \longrightarrow K^*(BV(r)) = RU^{\wedge}_{JU}[v, v^{-1}])$). It is proved in [10] that for abelian groups this is the Rees ring of the completed representation ring for the augmentation ideal $JU = \ker(ku^*(BV(r)) \longrightarrow ku^*)$, where the Rees ring is defined as follows.

$$\mathrm{Rees}(\widehat{RU}, \widehat{JU}) = \bigoplus_{k \geq 1} JU^k v^{-k} \oplus \bigoplus_{k \geq 0} RU v^k,$$

where v is the Bott element of degree 2.

Definition 2.1.2. The module $\overline{QU}$ is the image of $lu^*(BV(r))$ in $LU^*(BV(r))$ (i.e., $\overline{QU} = \mathrm{im}(lu^*(BV(r)) \longrightarrow LU^*(BV(r))))$.

Remark 2.1.3. It will help to have the following notes.

1. In concrete terms QU is

$$\ldots, \widehat{JU}_3, 0, \widehat{JU}_2, 0, \widehat{JU}_1, 0, \widehat{RU}, 0, \widehat{RU}, 0, \widehat{RU}, \ldots,$$

   with $|\widehat{JU}_k| = -2k$, where $\widehat{JU}_k$ is the completion of JU^k for all k and $\widehat{RU}$ is the completed complex representation ring. In fact, QU has no v-torsion and is equal to the completed Rees ring $\mathrm{Rees}(\widehat{RU}, \widehat{JU})$.

2. QU is generated as an algebra over the coefficient ring $\mathbb{Z}_p^{\wedge}$ by the first Chern classes $c_1(\alpha) \in QU = ku^*\langle c_1(\alpha) \mid \alpha \in \mathrm{Rep}_1(V)\rangle \subseteq RU[v, v^{-1}]$, where α is nontrivial simple (α runs through 1-dimensional representations).

§ 2.2 Elementary abelian p-groups

Recall that

$$V(r) \cong (C_p)^r = \underbrace{C_p \times C_p \times \cdots \times C_p}_{r-times}$$

is the elementary abelian group of rank $r \geq 1$. It is useful preparation to study some properties of linear representations of finite groups.

Proposition 2.2.1. *Let V be a complex vector space of dimension n and G be a finite group. If χ is the character of a representation $(\rho : G \longrightarrow GL(V))$ of degree n i.e., $\chi_\rho(g) = \mathrm{tr}(\rho(g))$ for each $g \in G$, we then have*

1. *$\chi_\rho(1) = n$ if ρ has degree n.*

2. *The number of irreducible representations of G (up to isomorphism) is equal to the number of conjugacy classes of G.*

3. *The degree of an irreducible representation of G divides the order of G. Moreover, it also divides $|G\colon Z(G)|$, where $Z(G)$ is the centre of G.*

4. *Two representations are equivalent if and only if they have the same character.*

5. *The character r_G of the regular representation is given by $r_G(1) = |G|$ and $r_G(g) = 0$ for $g \neq 1$.*

Proof. Details can be found in [39]. □

§ 2.3 The character of $V(r)$

In this section, we need only display the character of $V(1)$ and $V(2)$.

Let $V(1) = \{1, x, x^2, \dots, x^{p-1}\}$ be the elementary abelian p-group of rank 1. The natural representation α of this group is

$$1 \longmapsto 1,\ x \longmapsto \omega,\ x^2 \longmapsto \omega^2,\ \dots,\ x^{p-1} \longmapsto \omega^{p-1},$$

where $\omega = e^{2\pi i/p}$. The nontrivial irreducible representations of $V(1)$ are the tensor powers of α. The jth tensor power α^j is

$$1 \longmapsto 1,\ x \longmapsto \omega^j,\ \dots,\ x^i \longmapsto \omega^{ij}.$$

We denote their characters by $\chi_1, \chi_2, \dots, \chi_{p-1}$.

Example 2.3.1. If we have $V(1)$, and choose a faithful 1-dimensional representation α, then we may write the complex representation ring as $RU(V(1)) = \mathbb{Z}[\alpha]/(\alpha^p - 1)$.

Example 2.3.2. If we have $V(2) = V(1) \times V(1)$, since $V(1)$ and $V(1)$ are represented by ρ_1 and ρ_2 with characters χ_1 and χ_2, then the direct product $V(1) \times V(1)$ is represented by representation ρ given by the tensor product of matrices, and the character of the product $V(1) \times V(1)$ is $\chi_1 \cdot \chi_2$. This product is irreducible if and only if both χ_1 and χ_2 are irreducible. We may thus describe the nontrivial irreducible representations of

$$V(2) = \{e, x, \dots, x^{p-1}, y, \dots, y^{p-1}, xy, \dots, xy^{p-1}, x^2y, \dots, x^2y^{p-1}, x^{p-1}y, \dots, (xy)^{p-1}\}.$$

The complex representation ring for this group is $RU(V(2)) = \mathbb{Z}[\alpha_1, \alpha_2]/(\alpha_1^p - 1, \alpha_2^p - 1)$.

§ 2.4 The cohomology of $BV(r)$ and L

The purpose of this section is to calculate $H^*(BV(r))$ as an $E(1)$-module. The key idea is that $H^*(BV(r))$ is more than just a graded abelian group, and even more than a graded ring (via the cup product). The representability of the cohomology functor makes $H^*(BV(r))$ a module over the algebra of its stable cohomology operations, the Steenrod algebra $\mathcal{A}$. Now, by the Künneth formula we find

$$H^*(BV(r)) = \underbrace{H^*(BC_p) \otimes H^*(BC_p) \otimes \cdots \otimes H^*(BC_p)}_{r-times}.$$

We begin by describing the answer for the cyclic group of order p. When p is odd we have a ring isomorphism

$$H^*(BC_p) = \mathbb{F}_p[y] \otimes \Lambda(\tau),$$

where $|\tau| = 1$ and $|y| = 2$. In this case y is the Bockstein $Q_0(\tau)$. This cohomology is determined as an $E(1)$-module by the fact $Q_0(\tau) = y$ and $Q_1(\tau) = y^p$, and we see that we have a direct sum decomposition

$$H^*(BC_p) = \mathbb{F}_p \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_{p-2},$$

where

$$L_i = \mathbb{F}_p\langle \tau y^a, y^{a+1} \mid a \equiv i \mod p-1\rangle = \mathbb{F}_p\langle \tau y^i, y^{i+1}\rangle[Y],$$

where $Y = y^{p-1}$ and $\langle\,\rangle$ denotes vector spaces on specified bases. Thus we have $L_i \cong \Sigma^{2i} L_0$. From this we can give the definition of the module L as follows.

Definition 2.4.1. The string module L, is a type of 'lightning flash' module as in [1, page 341] and is defined by desuspending $L_0 = \mathbb{F}_p\langle \tau y^i, y^{i+1} \mid i \equiv 0 \mod p-1\rangle$, i.e., if we introduce formal basis elements $\imath$ (of codegree 0) and Q_0 (of codegree 1), and consider the vector space $\mathbb{F}_p\langle \imath, Q_0\rangle$ with this basis, then

$$L = \Sigma^{-1} L_0 = \mathbb{F}_p\langle \imath, Q_0\rangle[Y],$$

so that it starts in degree 0. Pictorially L may be illustrated as in Figure 2.1.

Remark 2.4.2. Since $\imath\tau = \tau$ and $Q_0(\tau) = y$, it is reasonable to write $L\tau = L_0$. Note that L is not an unstable algebra, and the Q_i do not act as derivations on it.

Now the reduced mod p cohomology $\widetilde{H}^*(BC_p)$ for any odd prime is given by

$$\widetilde{H}^*(BC_p) = L_* \cong L_0 \oplus L_1 \oplus \cdots \oplus L_{p-2} = \left(\frac{L[\overline{y}]}{(\overline{y}^{p-1})} \cdot \tau\right),$$

where $\overline{y}$ is a placeholder of codegree 2. This expression is very useful for us when we take tensor powers in the next Section.

§ 2.5 The tensor powers of L

The aim of this section is to find the tensor powers of L. In detail, let $E(1)$ be the exterior algebra over $\mathbb{F}_p$ on odd degree generators Q_0 and Q_1. Since $V(r)$ is the

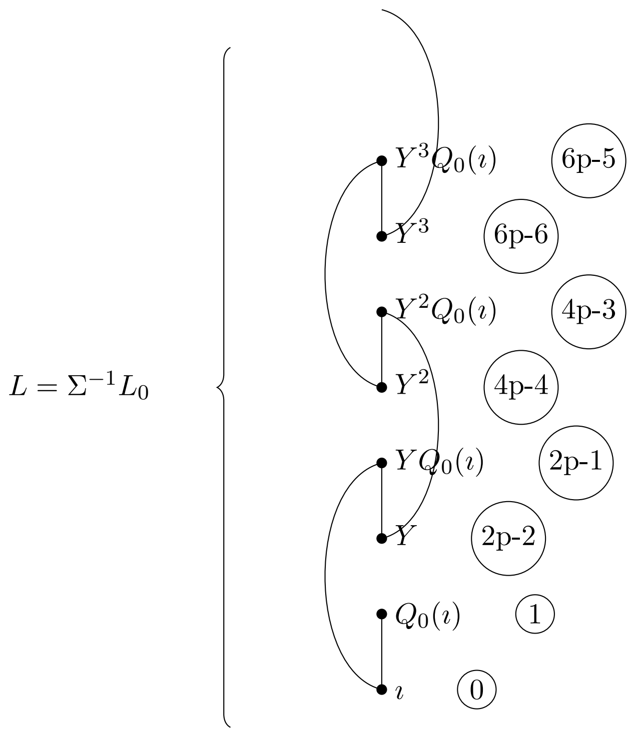


Figure 2.1: The string module L.

product of r copies of C_p, we find

$$\begin{aligned}
H^*(BV(r)) &= H^*(BC_p) \otimes \cdots \otimes H^*(BC_p) \\
&= (\mathbb{F}_p \oplus L_0 \oplus \cdots \oplus L_{p-2}) \otimes \cdots \otimes (\mathbb{F}_p \oplus L_0 \oplus \cdots \oplus L_{p-2}) \\
&= (\mathbb{F}_p \oplus L_*) \otimes \cdots \otimes (\mathbb{F}_p \oplus L_*) \\
&= (\mathbb{F}_p \oplus \frac{L[\overline{y}]}{(\overline{y}^{p-1})} \cdot \tau) \otimes \cdots \otimes (\mathbb{F}_p \oplus \frac{L[\overline{y}]}{(\overline{y}^{p-1})} \cdot \tau), \quad (L\tau = L_0 \cong \Sigma L) \\
&= (\mathbb{F}_p \oplus \frac{\Sigma L[\overline{y}]}{(\overline{y}^{p-1})}) \otimes \cdots \otimes (\mathbb{F}_p \oplus \frac{\Sigma L[\overline{y}]}{(\overline{y}^{p-1})}) \\
&= (\mathbb{F}_p \oplus \frac{\Sigma L[\overline{y}]}{(\overline{y}^{p-1})})^{\otimes r}.
\end{aligned}$$

This expresses $H^*(BV(r))$ as the sum of p^r terms. Each of the p^r terms in the tensor product is determined by choosing a subset $S \subseteq \{1, 2, \ldots, r\}$ of non-basepoint terms and then choosing one of the $(p-1)^n$ terms, where $n = |S|$ by choosing numbers $0 \leq i_s \leq p-2$ for $s \in S$.

Now let us write $\tau_S = \prod_{s \in S} \tau_s$ and $\overline{y}^I = \prod_{s \in S} \overline{y}_s^{i_s}$. In fact, this choice gives a term of the form

$$L^{\otimes |S|} \tau_S \overline{y}^I.$$

In order to determine the tensor powers of L as an $E(1)$-module, where $E(1)$ acts by the Cartan formula on a tensor product here, we prove the following important Lemma.

Lemma 2.5.1. *There is an isomorphism of $E(1)$-modules*

$$L \otimes L \cong \Sigma L \oplus E(1)[Y_1, Y_2].$$

More precisely, $L \otimes L$ is the internal direct sum of the $E(1)$-module $L \otimes Q_0(\imath)$ (isomorphic to ΣL) and the $E(1)$- module generated by $\mathbb{F}_p[Y_1, Y_2]$ (which is $E(1)$-free with basis given by the monomials in Y_1 and Y_2), with $|Y_1| = 2p-2, |Y_2| = 2p-2$.

Proof. First of all, we need to show that $L \otimes Q_0(\imath) \cong \Sigma L$. The essential fact is that Q_i vanishes on $Q_0(\imath)$ for $i = 0, 1$.

Since $L_0 \subseteq_{E(1)} H^*(BC_p)$ and $L = \Sigma^{-1} L_0$, $\imath = \Sigma^{-1}\tau$ and

$$Q_0(\imath) = \Sigma^{-1} y = \Sigma^{-1} Q_0(\tau)$$

i.e., $Q_0(\imath) \in L$. Let $\ell \in L$, then we see

$$Q_i(\ell \otimes Q_0(\imath)) = (Q_i \ell) \otimes Q_0(\imath) + (-1)^{|\ell|} \ell \otimes Q_i Q_0(\imath).$$

Since $Q_i Q_0(\imath) = 0$, then we get $Q_i(\ell \otimes Q_0(\imath)) = (Q_i \ell) \otimes Q_0(\imath)$.

Therefore $\Sigma L \cong_{E(1)} L \otimes Q_0(\imath) \subseteq L \otimes L$.

Next, we prove that for any monomial $Y_1^i Y_2^j$, $Y_1^i Y_2^j(\imath \otimes \imath)$ generates a free $E(1)$-module

$$\begin{aligned} Q_0(Y_1^i \imath \otimes Y_2^j \imath) &= (Q_0 Y_1^i \imath) \otimes Y_2^j \imath + Y_1^i \imath \otimes (Q_0 Y_2^j \imath) \\ &= Y_1^i \cdot Q_0(\imath) \otimes Y_2^j \imath + Y_1^i \imath \otimes Y_2^j \cdot Q_0(\imath) \\ &= Y_1^i Y_2^j \cdot [Q_0(\imath) \otimes \imath + \imath \otimes Q_0(\imath)]. \end{aligned}$$

Similarly,

$$\begin{aligned} Q_1(Y_1^i \imath \otimes Y_2^j \imath) &= (Q_1 Y_1^i \imath) \otimes Y_2^j \imath + Y_1^i \imath \otimes (Q_1 Y_2^j \imath) \\ &= Y_1^i \cdot Q_1(\imath) \otimes Y_2^j \imath + Y_1^i \imath \otimes Y_2^j \cdot Q_1(\imath) \\ &= Y_1^i Y_2^j \cdot [Q_1(\imath) \otimes \imath + \imath \otimes Q_1(\imath)]. \end{aligned}$$

$$\begin{aligned} Q_0 Q_1(Y_1^i \imath \otimes Y_2^j \imath) &= (Q_0 Q_1 Y_1^i \imath) \otimes Y_2^j \imath + Y_1^i \imath \otimes (Q_0 Q_1 Y_2^j \imath) \\ &= Y_1^i \cdot Q_0 Q_1(\imath) \otimes Y_2^j \imath + Y_1^i \imath \otimes Y_2^j \cdot Q_0 Q_1(\imath) \\ &= Y_1^i Y_2^j \cdot [Q_0 Q_1(\imath) \otimes \imath + \imath \otimes Q_0 Q_1(\imath)]. \end{aligned}$$

Since every one of these elements is a multiple of $Y_1^i Y_2^j$, we see any pair of the modules $F_{ij} := E(1) \cdot Y_1^i Y_2^j$ intersect trivially, and $\sum_{i,j} F_{ij} = \bigoplus_{i,j} F_{ij}$ intersects $L \otimes Q_0$ trivially. Then

$$L \otimes Q_0 + \sum_{i,j} F_{ij} = L \otimes Q_0 \oplus \bigoplus_{i,j} F_{ij}.$$

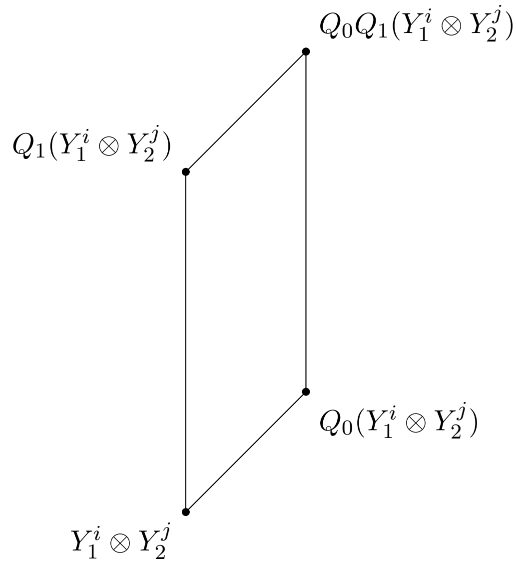


Figure 2.2: $E(1) = \Lambda(Q_0 = \beta, Q_1 = \mathcal{P}^1\beta - \beta\mathcal{P}^1)$.

To show that we get all of $L \otimes L$ we use a Hilbert series calculation.

The Hilbert series of L is $[L] = \frac{1+t}{1-T}$, and $[E(1)] = (1+t)(1+Tt)$, where $T = t^{2p-2}$.

Now, we find $[L \otimes L] = [L] \cdot [L] = (\frac{1+t}{1-T})^2$. On the other hand,

$$\begin{aligned}
[\Sigma L \oplus \bigoplus_{i,j \geq 0} Y_1^i Y_2^j E(1) \cdot (\imath \otimes \imath)] &= \frac{t(1+t)}{1-T} + \sum_{i,j \geq 0} T^{i+j}(1+t)(1+Tt) \\
&= (1+t)\big[\frac{t}{1-T} + \sum_{i,j \geq 0} T^{i+j}(1+Tt)\big] \\
&= (1+t)\big[\frac{t(1-T)}{(1-T)^2} + \frac{(1+Tt)}{(1-T)^2}\big] \\
&= \frac{1+t}{(1-T)^2}[t - Tt + 1 + Tt] \\
&= (\frac{1+t}{1-T})^2.
\end{aligned}$$

Since these two Hilbert series agree, $L \otimes Q_0 \oplus \bigoplus_{i,j} F_{ij} = L \otimes L$ as required. □

Definition 2.5.2. Let PC be the polynomial ring $\mathbb{F}_p[y_1, y_2, \ldots, y_r]$ with generators in codegree 2, and PP be the polynomial subring $\mathbb{F}_p[Y_1, Y_2, \ldots, Y_r]$ with generators in codegree $2p-2$, where $Y_i = y_i^{p-1}$ so that

$$PC = \mathbb{F}_p[Y_1, Y_2, \ldots, Y_r] \otimes \frac{\mathbb{F}_p[\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_r]}{(\overline{y}_1^{p-1}, \overline{y}_2^{p-1}, \ldots, \overline{y}_r^{p-1})},$$

as a PP-module, where $\overline{y}_i$ is of codegree 2 for $1 \leq i \leq r$. In fact, PC is the subring of $H^*(BV(r);\mathbb{F}_p)$ generated by the Chern classes $y_i = c_1^{H\mathbb{F}_p}(\alpha_i)$, which are the images of $c_1^{ku}(\alpha_i) \in ku^2(BV(r))$.

In this Chapter, we want to introduce for any rank r, the PC-module structure on $\overline{TU}$. All calculations of $\overline{TU}$ focus entirely on the PC-module structure.

§ 2.6 Calculation of $lu^*(BV(1))$

The target of this section is to calculate $lu^*(BV(1))$ by using the Adams spectral sequence. We find for rank 1 group, $lu^*(BV(1))$ is a subring of $LU^*(BV(1))$, so that once we have identified the generators of $lu^*(BV(1))$ using the Adams spectral sequence, we can use RU (via equivariant K-theory) to determine the relations.

2.6.1 E_2-term

In this section, we want to calculate $lu^*(BV(1))$ by using the E_2-term of the Adams spectral sequence. First, we have

$$H^*(BV(1)) \cong \mathbb{F}_p \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_{p-2} = \mathbb{F}_p \oplus \frac{\Sigma L[\overline{y}]}{(\overline{y}^{p-1})}.$$

The Adams spectral sequence for $lu^*(BV(1))$ reads

$$\operatorname{Ext}^{*,*}_{E(1)}(\mathbb{F}_p, H^*(BV(1);\mathbb{F}_p)) \Longrightarrow lu^*(BV(1))^\wedge_p. \tag{2.1}$$

Next, we find

$$\begin{aligned}\operatorname{Ext}^{*,*}_{E(1)}(\mathbb{F}_p, H^*(BV(1);\mathbb{F}_p)) &= \operatorname{Ext}^{*,*}_{E(1)}\left(\mathbb{F}_p, \mathbb{F}_p \oplus \frac{\Sigma L[\overline{y}]}{(\overline{y}^{p-1})}\right)\\ &= \operatorname{Ext}^{*,*}_{E(1)}(\mathbb{F}_p, \mathbb{F}_p) \oplus \operatorname{Ext}^{*,*}_{E(1)}\left(\mathbb{F}_p, \frac{\Sigma L[\overline{y}]}{(\overline{y}^{p-1})}\right)\\ &= \mathbb{F}_p[a_0, u] \oplus \bigoplus_{k=0}^{p-2} \Sigma^{2k+1} \operatorname{Ext}^{*,*}_{E(1)}(\mathbb{F}_p, L), \qquad (2.2)\end{aligned}$$

where a_0 is of bidegree $(1,1)$, and u is of bidegree $(1,2p-1)$. Now, we want to calculate $\operatorname{Ext}^{*,*}_{E(1)}(\mathbb{F}_p, L)$ by taking a projective resolution of $\mathbb{F}_p$ over $E(1)$ and calculating $H^*(\operatorname{Hom}_{E(1)}(P_\bullet, L))$ as follows.

Consider the projective resolution of $\mathbb{F}_p$

$$\cdots \longrightarrow P_4 \xrightarrow{d_4} P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{F}_p \longrightarrow 0,$$

where the P_s are projective modules over $\mathbb{F}_p$ and the differentials d_s are defined as follows. $d_s\colon P_s \longrightarrow P_{s-1}$, for $s \geq 0$, where

$$P_s = \bigoplus_{k=0}^{k=s} \Sigma^{s+(2p-2)k} E(1), \quad s \geq 0. \tag{2.3}$$

If we name the generators of P_s, $a_0^s, a_0^{s-1}u, a_0^{s-2}u^2, \ldots, a_0^1 u^{s-1}, u^s$, then

$$d_s(a_0^i u^j) = Q_0(a_0^{i-1}u^j) - Q_1(a_0^i u^{j-1}),$$

where the term is interpreted as zero if $i-1$ or $j-1$ is negative.

Then, we have a long exact sequence

$$\begin{aligned}
\cdots \longrightarrow & \underbrace{\Sigma^4 E(1) \oplus \Sigma^{2p+2}E(1) \oplus \Sigma^{4p}E(1) \oplus \Sigma^{6p-2}E(1) \oplus \Sigma^{8p-4}E(1)}_{P_4} \\
& \longrightarrow \underbrace{\Sigma^3 E(1) \oplus \Sigma^{2p+1}E(1) \oplus \Sigma^{4p-1}E(1) \oplus \Sigma^{6p-3}E(1)}_{P_3} \\
& \longrightarrow \underbrace{\Sigma^2 E(1) \oplus \Sigma^{2p}E(1) \oplus \Sigma^{4p-2}E(1)}_{P_2} \longrightarrow \underbrace{\Sigma^1 E(1) \oplus \Sigma^{2p-1}E(1)}_{P_1} \\
& \longrightarrow \underbrace{E(1)}_{P_0} \longrightarrow \mathbb{F}_p \longrightarrow 0.
\end{aligned}$$

Now, we apply $\mathrm{Hom}_{E(1)}^{s,t}(P_s, L)$, for $s \geq 0$, we get

$$\begin{aligned}
0 \longrightarrow \mathrm{Hom}_{E(1)}^{0,t}(E(1), L) \xrightarrow{d_0} \mathrm{Hom}_{E(1)}^{1,t}(\Sigma^1 E(1) \oplus \Sigma^{2p-1}E(1), L) \xrightarrow{d_1} \\
\mathrm{Hom}_{E(1)}^{2,t}(\Sigma^2 E(1) \oplus \Sigma^{2p}E(1) \oplus \Sigma^{4p-2}E(1), L) \xrightarrow{d_2} \\
\mathrm{Hom}_{E(1)}^{3,t}(\Sigma^3 E(1) \oplus \Sigma^{2p+1}E(1) \oplus \Sigma^{4p-1}E(1) \oplus \Sigma^{6p-3}E(1), L) \xrightarrow{d_3} \\
\mathrm{Hom}_{E(1)}^{4,t}(\Sigma^4 E(1) \oplus \Sigma^{2p+2}E(1) \oplus \Sigma^{4p}E(1) \oplus \Sigma^{6p-2}E(1) \oplus \Sigma^{8p-4}E(1), L) \xrightarrow{d_4} \cdots
\end{aligned}$$

Taking homology of this chain complex at stage s gives us the Ext groups.

Since $\mathrm{Hom}_{E(1)}(E(1), L) = L$,

$$\mathrm{Hom}_{E(1)}(P_s, L) = \Sigma^{-s}L \oplus \Sigma^{-s-(2p-2)}L \oplus \Sigma^{-s-(4p-4)}L \oplus \cdots \oplus \Sigma^{-s-s(2p-2)}L.$$

Then

$$\begin{aligned}
\mathrm{Hom}_{E(1)}(P_{s+1}, L) = \Sigma^{-(s+1)}L \oplus \Sigma^{-(s+1)-(2p-2)}L \oplus \Sigma^{-(s+1)-(4p-4)}L \\
\oplus \cdots \oplus \Sigma^{-(s+1)-(s+1)(2p-2)}L.
\end{aligned}$$

- If $s = 0$, then

$$\begin{aligned}
\mathrm{Ext}_{E(1)}^{0,t}(\mathbb{F}_p, L) &= H^0(\mathrm{Hom}_{E(1)}(P_\bullet, L)) \\
&= \ker(\mathrm{Hom}_{E(1)}^{0,t}(E(1), L) \longrightarrow \mathrm{Hom}_{E(1)}^{1,t}(\Sigma^1 E(1) \oplus \Sigma^{2p-1}E(1), L)) \\
&= \ker(L \xrightarrow{\begin{pmatrix} Q_0 & Q_1 \end{pmatrix}} \Sigma^{-1}L \oplus \Sigma^{-2p+1}L).
\end{aligned}$$

- If $s = 1$, then

$$
\begin{aligned}
\operatorname{Ext}^{1,t}_{E(1)}(\mathbb{F}_p, L) &= H^1(\operatorname{Hom}_{E(1)}(P_\bullet, L)) \\
&= \frac{\ker(d_1)}{\operatorname{im}(d_0)} \\
&= \frac{\ker(\Sigma^{-1}L \oplus \Sigma^{-2p+1}L \longrightarrow \Sigma^{-2}L \oplus \Sigma^{-2p}L \oplus \Sigma^{-4p+2}L)}{\operatorname{im}(L \longrightarrow \Sigma^{-1}L \oplus \Sigma^{-2p+1}L))} \\
&= \frac{\ker \begin{pmatrix} Q_0 & 0 \\ Q_1 & -Q_0 \\ 0 & Q_1 \end{pmatrix}}{\operatorname{im} \begin{pmatrix} Q_0 & Q_1 \end{pmatrix}}.
\end{aligned}
$$

- If $s = 2$, then

$$
\begin{aligned}
\operatorname{Ext}^{2,t}_{E(1)}(\mathbb{F}_p, L) &= H^2(\operatorname{Hom}_{E(1)}(P_\bullet, L)) \\
&= \frac{\ker(d_2)}{\operatorname{im}(d_1)} \\
&= \frac{\ker(\Sigma^{-2}L \oplus \Sigma^{-2p}L \oplus \Sigma^{-4p+2}L \longrightarrow \Sigma^{-3}L \oplus \Sigma^{-2p-1}L \oplus \Sigma^{-4p+1}L \oplus \Sigma^{-6p+3}L)}{\operatorname{im}(\Sigma^{-1}L \oplus \Sigma^{-2p+1}L \longrightarrow \Sigma^{-2}L \oplus \Sigma^{-2p}L \oplus \Sigma^{-4p+2}L)} \\
&= \frac{\ker \begin{pmatrix} Q_0 & 0 \\ Q_1 & -Q_0 \\ Q_1 & -Q_0 \\ 0 & Q_1 \end{pmatrix}}{\operatorname{im} \begin{pmatrix} Q_0 & 0 \\ Q_1 & -Q_0 \\ 0 & Q_1 \end{pmatrix}}.
\end{aligned}
$$

- If $s = 3$, then

$$
\begin{aligned}
\operatorname{Ext}^{3,t}_{E(1)}(\mathbb{F}_p, L) &= H^3(\operatorname{Hom}_{E(1)}(P_\bullet, L)) \\
&= \frac{\ker(d_3)}{\operatorname{im}(d_2)} \\
&= \frac{\ker \begin{pmatrix} Q_0 & 0 \\ Q_1 & -Q_0 \\ Q_1 & -Q_0 \\ Q_1 & -Q_0 \\ 0 & Q_1 \end{pmatrix}}{\operatorname{im} \begin{pmatrix} Q_0 & 0 \\ Q_1 & -Q_0 \\ Q_1 & -Q_0 \\ 0 & Q_1 \end{pmatrix}}.
\end{aligned}
$$

- If $s = 4$, then

$$\begin{aligned}
\operatorname{Ext}^{4,t}_{E(1)}(\mathbb{F}_p, L) &= H^4(\operatorname{Hom}_{E(1)}(P_\bullet, L)) \\
&= \frac{\ker(d_4)}{\operatorname{im}(d_3)} \\
&= \frac{\ker\begin{pmatrix} Q_0 & 0 \\ Q_1 & -Q_0 \\ Q_1 & -Q_0 \\ Q_1 & -Q_0 \\ Q_1 & -Q_0 \\ 0 & Q_1 \end{pmatrix}}{\operatorname{im}\begin{pmatrix} Q_0 & 0 \\ Q_1 & -Q_0 \\ Q_1 & -Q_0 \\ Q_1 & -Q_0 \\ 0 & Q_1 \end{pmatrix}}.
\end{aligned}$$

Note that (Figure 2.3) on the next page gives us the E_2-term of the Adams spectral sequence for $lu^*(BV(1))$, where each bullet in this Figure denotes an $\mathbb{F}_p$. This is the sum of

(1) $\operatorname{Ext}^{*,*}_{E(1)}(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p[a_0, u]$ (in red).

(2) $\operatorname{Ext}^{*,*}_{E(1)}(\mathbb{F}_p, \Sigma^{2p-3}L)$ (in black).

(3) $\operatorname{Ext}^{*,*}_{E(1)}(\mathbb{F}_p, \Sigma^{2k+1}L)$, for $k = 0, 1, \ldots, p-3$ (in green).

Note that

(1) gives the E_2-term of the Adams spectral sequence for $lu^*(pt)$.

(2) gives the E_2-term of the Adams spectral sequence for $\widetilde{lu}^*(B_{p-1})$, and (1) and (2) give the E_2-term of the Adams spectral sequence for $lu^*(B_{p-1})$.

2.6.2 $\overline{QU}$

In this section, we want to calculate the module $\overline{QU}$. We focus on the part of the E_2-term of (2.2) given by

$$\bigoplus_{k=0}^{p-2} \Sigma^{2k+1} \operatorname{Ext}^{*,*}_{E(1)}(\mathbb{F}_p, L).$$

Note that $\mathbb{F}_p[a_0, u]$ also contributes to $\overline{QU}$.

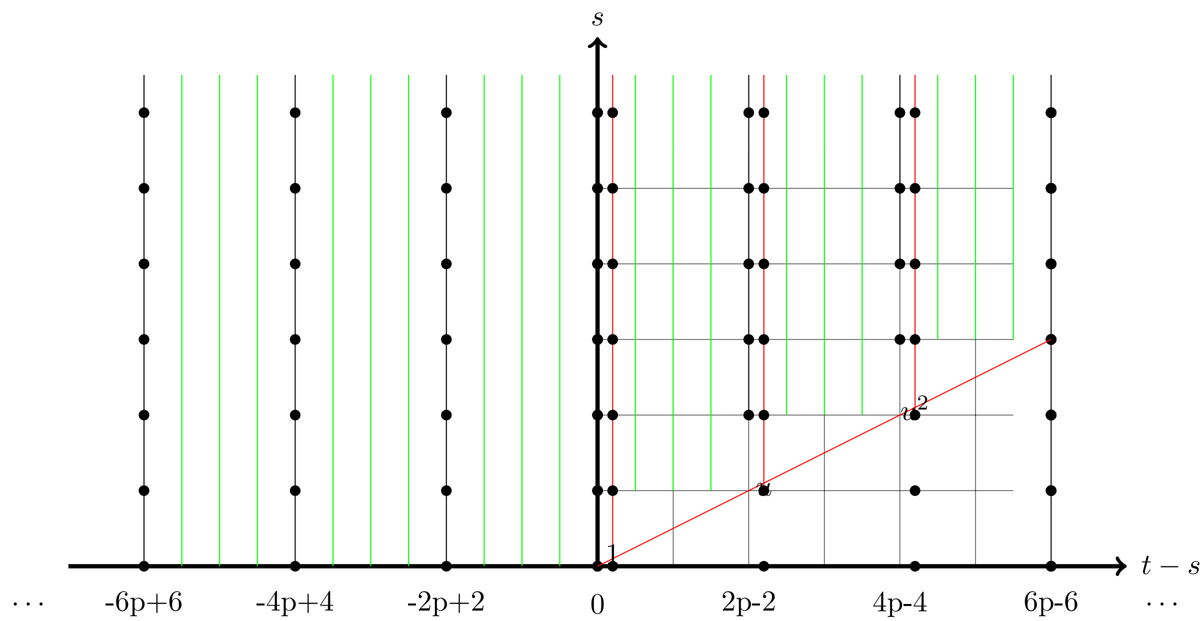


Figure 2.3: The Adams spectral sequence for $lu^*(BV(1))$.

By definition 2.1.2, $\overline{QU} \supseteq lu^*\langle c_1^{lu}(\alpha) \mid \alpha \in \mathrm{Rep}_1(V)\rangle$. One may check from the general theory of complex oriented cohomology theories that

$$\overline{QU} = \frac{lu^*[[c_1^{lu}(\alpha)]]}{c_1^{lu}(\alpha^p)} = \frac{lu^*[[c_1^{lu}(\alpha)]]}{[p]_{lu}\big(c_1^{lu}(\alpha)\big)},$$

where $c_1^{lu}(\alpha^n) = [n]_{lu}(\alpha)$ is the power series p-typification of $[n]_{ku}$ and α is the natural representation of $V(1)$.

2.6.3 $\overline{TU}$

If we have $V(1)$, then v_1 acts injectively on $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_p, L)$ [10, page 32], and $\overline{TU} = 0$.

Consider the short exact sequence of $lu^*(BV(1))$-modules

$$0 \longrightarrow \overline{TU} \longrightarrow lu^*(BV(1)) \longrightarrow \overline{QU} \longrightarrow 0. \tag{2.4}$$

Since $\overline{TU} = 0$, $lu^*(BV(1)) \cong \overline{QU}$.

§ 2.7 PP-module $\overline{TU}$.

We are now considering a general r again. In this section, we want to look at the module $\overline{TU}$ as a PP-module. We need only assemble the pieces.

The following Lemma, which is a consequence of Corollary 1.3.2, will be used.

Lemma 2.7.1. *The v-power torsion module $\overline{TU}$ is annihilated by (p, v), and the natural transformation $lu^*(BV(r)) \longrightarrow H^*(BV(r); \mathbb{F}_p)$ embeds $\overline{TU}$ in $H^*(BV(r); \mathbb{F}_p)$.*

In order to see $\overline{TU}$ is a PP-module, we discuss as follows.

Consider the following diagram

$$\begin{tikzcd}
 & & \overline{QU} \arrow[dr, hook] & \\
0 \arrow[r] & \overline{TU} \arrow[r, hook] \arrow[dr, hook, "d"'] & lu^*(BV(r)) \arrow[u, two heads] \arrow[r] \arrow[d] & LU^*(BV(r)) \\
 & & H^*(BV(r); \mathbb{F}_p) &
\end{tikzcd}$$

First, $\overline{TU}$ is a module over $\mathbb{Z}[c_1^{lu}(\alpha_1), c_1^{lu}(\alpha_2), \ldots, c_1^{lu}(\alpha_r)] = PC_{\mathbb{Z}}[y_1, y_2, \ldots, y_r]$. Since $p = 0$ on $\overline{TU}$, the $PC_{\mathbb{Z}}$- module structure comes from $PC_{\mathbb{Z}/p} = PC = \mathbb{Z}/p[y_1, y_2, \ldots, y_r]$. Second, we have a map $PP = \mathbb{Z}/p[Y_1, Y_2, \ldots, Y_r] \longrightarrow \mathbb{Z}/p[y_1, y_2, \ldots, y_r] = PC$, $(Y_i \longmapsto y_i^{p-1})$, and hence $\overline{TU}$ is a module over PP by restriction.

To see $d \colon \overline{TU} \longrightarrow H^*(BV(r); \mathbb{F}_p)$ is a PP-module map, note that there are compatible orientations on $lu, ku, H\mathbb{F}_p$. Note that d is a monomorphism by the Adams spectral sequence [10]. Therefore, PP-module structure on $\overline{TU}$ follows from the image of d.

Similarly, we can do the same argument as above for PC-module structure on TU with $PC = \mathbb{F}_p[y_1, y_2, \ldots, y_r]$, where $y_i = c_1^{H\mathbb{F}_p}(\alpha_i)$ is the image of $c_1^{ku}(\alpha_i) \in ku^2(BV(r))$ in codegree 2 of the representations α_i in $H^*(BV(r); \mathbb{F}_p)$.

Suppose g_j are given and define the PP-submodule M by

$$M := PP\{g_j \mid j \in J\} \subseteq H^*(BV(r)).$$

Our aim is to show that M is a PC-submodule.

Remark 2.7.2. Suppose $PP = S \subseteq R = PC$ and V is an R-module, $W \subseteq V$ is an S-submodule. If $R = S\{\rho_1, \ldots, \rho_n\}$ as S-module and $\rho_i W \subseteq W$ for all $i = 1, 2, \ldots, n$, then W is an R-submodule.

Lemma 2.7.3. *Let $I = \{(a_1, a_2, \ldots, a_r) \mid 0 \leq a_i \leq p-2\}$ for $M = PP\{g_j \mid j \in J\} \subseteq H^*(BV(r))$ as above, if $y^{\underline{a}} g_j \in M$ for all $\underline{a} \in I, j \in J$ then M is a PC-submodule of $H^*(BV(r))$, where PC and PP are given as in definition 2.5.2.*

Proof. Since PC is a ring and V is a PC-module, and $W \subseteq V$, then W is a PC-submodule of V if

(1) $w_1, w_2 \in W$, then $\quad w_1 + w_2 \in W$.

(2) $w \in W, \lambda \in PC,$ then $\quad \lambda w \in W$.

In fact, in our case we may take $V = H^*(BV(r))$, and $W = PP\{g_j \mid j \in J\} \subseteq V$. Now, we verify (1). Given $w_1, w_2 \in W$ i.e., $w_1 = \sum_j \nu_j^1 g_j$ and $w_2 = \sum_j \nu_j^2 g_j$, $\quad (\nu_j \in PP)$. Then

$$w_1 + w_2 = \sum_j \nu_j^1 g_j + \sum_j \nu_j^2 g_j = \sum_j (\nu_j^1 + \nu_j^2) g_j \in W, \quad (\text{since} \quad \nu_j^1 + \nu_j^2 \in PP).$$

For (2), given $w \in W$ and $\lambda \in PC = PP \otimes \frac{\mathbb{F}_p[\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_r]}{(\overline{y}_1^{p-1}, \overline{y}_2^{p-1}, \ldots, \overline{y}_r^{p-1})}$.
Let us say $w = \sum_j \nu_j g_j$ and $\lambda = \sum_{T \subseteq \{1,2,\ldots,r\}} \nu_T \, \tau_T$. Then

$$\lambda w = \sum_T \nu_T \, \tau_T \sum_j \nu_j g_j = \sum_{T,j} \nu_T \nu_j \, \tau_T g_j \in M, \quad (\text{since} \quad \nu_j^1 + \nu_j^2 \in PP).$$

By Remark 2.7.2, we find $\tau_T g_j \in M$ as required. □

§ 2.8 Calculation of $lu^*(BV(2))$

The purpose of this section is to calculate $lu^*(BV(2))$ by the Adams spectral sequence. If we have $V(2)$, then we will find $\overline{TU} \cong \overline{TU}^2 = \overline{TU}_2^2$ is a free module of rank 1 over $PC = \mathbb{F}_p[y_1, y_2]$ with $|y_i| = 2$ on a generator of degree $-2p-2$.

2.8.1 E_2-term.

In this section, we apply the Adams spectral sequence to calculate $lu^*(BV(2))$. The calculation is complicated by the presence of (p, v)-torsion; although the Adams spectral sequence does not give the complete answer, it shows that there is no v-torsion in positive Adams filtration. Accordingly, ordinary mod p cohomology together with representation theory determines the multiplicative structure.

First, we need to calculate $H^*(BV(2))$

$$\begin{aligned}
H^*(BV(2)) &\cong (H^* BV(1))^{\otimes 2} \\
&= (\mathbb{F}_p \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_{p-2})^{\otimes 2} \\
&= \big(\mathbb{F}_p \oplus \frac{\Sigma L[\overline{y}]}{(\overline{y}^{p-1})}\big)^{\otimes 2} \\
&= \big(\mathbb{F}_p \oplus \frac{\Sigma L[\overline{y_1}]}{(\overline{y_1}^{p-1})}\big) \otimes \big(\mathbb{F}_p \oplus \frac{\Sigma L[\overline{y_2}]}{(\overline{y_2}^{p-1})}\big) \\
&= \mathbb{F}_p \oplus \frac{\Sigma L[\overline{y_1}]}{(\overline{y_1}^{p-1})} \oplus \frac{\Sigma L[\overline{y_2}]}{(\overline{y_2}^{p-1})} \oplus \frac{\Sigma^2 L \otimes L[\overline{y_1}, \overline{y_2}]}{(\overline{y_1}^{p-1}, \overline{y_2}^{p-1})}.
\end{aligned}$$

We apply Lemma 2.5.1 to the last expression above. Then $H^*(BV(2))$ becomes

$$H^*(BV(2)) \cong \mathbb{F}_p \oplus \frac{\Sigma L[\overline{y_1}]}{(\overline{y_1}^{p-1})} \oplus \frac{\Sigma L[\overline{y_2}]}{(\overline{y_2}^{p-1})} \oplus \frac{\Sigma^3 L[\overline{y_1}, \overline{y_2}]}{(\overline{y_1}^{p-1}, \overline{y_2}^{p-1})} \oplus \frac{\Sigma^2 E(1)[Y_1, Y_2][\overline{y_1}, \overline{y_2}]}{(\overline{y_1}^{p-1}, \overline{y_2}^{p-1})}.$$

Since the spectral sequence collapses at E_2-term, see [10, Th. 4.2.4], $\mathrm{Gr}(lu^*(BV(2))) = \mathrm{Ext}$. As in rank 1, we calculate

$$\begin{aligned}
\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_p, H^*(BV(2);\mathbb{F}_p)) &= \mathrm{Ext}^{*,*}_{E(1)}\Big(\mathbb{F}_p, \mathbb{F}_p \oplus \frac{\Sigma L[\overline{y_1}]}{(\overline{y_1}^{p-1})} \oplus \frac{\Sigma L[\overline{y_2}]}{(\overline{y_2}^{p-1})} \oplus \frac{\Sigma^3 L[\overline{y_1}, \overline{y_2}]}{(\overline{y_1}^{p-1}, \overline{y_2}^{p-1})} \\
&\quad \oplus \frac{\Sigma^2 E(1)[Y_1, Y_2][\overline{y_1}, \overline{y_2}]}{(\overline{y_1}^{p-1}, \overline{y_2}^{p-1})}\Big) \\
&= \mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_p, \mathbb{F}_p) \oplus \mathrm{Ext}^{*,*}_{E(1)}\Big(\mathbb{F}_p, \frac{\Sigma L[\overline{y_1}]}{(\overline{y_1}^{p-1})} \oplus \frac{\Sigma L[\overline{y_2}]}{(\overline{y_2}^{p-1})} \\
&\quad \oplus \frac{\Sigma^3 L[\overline{y_1}, \overline{y_2}]}{(\overline{y_1}^{p-1}, \overline{y_2}^{p-1})}\Big) \oplus \mathrm{Ext}^{*,*}_{E(1)}\Big(\mathbb{F}_p, \frac{\Sigma^2 E(1)[Y_1, Y_2][\overline{y_1}, \overline{y_2}]}{(\overline{y_1}^{p-1}, \overline{y_2}^{p-1})}\Big).
\end{aligned} \tag{2.5}$$

2.8.2 $\overline{QU}$.

In this section, we remark that $\overline{QU}$ consists of the coefficient ring, coming from $\mathbb{F}_p[a_0, u]$ together with the part of the E_2-term of (2.5),

$$\mathrm{Ext}^{*,*}_{E(1)}\Big(\mathbb{F}_p, \frac{\Sigma L[\overline{y_1}]}{(\overline{y_1}^{p-1})} \oplus \frac{\Sigma L[\overline{y_2}]}{(\overline{y_2}^{p-1})} \oplus \frac{\Sigma^3 L[\overline{y_1}, \overline{y_2}]}{(\overline{y_1}^{p-1}, \overline{y_2}^{p-1})}\Big).$$

Since we are aiming to calculate the module QU, which is known already (see Remark 2.1.3), we will not make $\overline{QU}$ explicit here.

2.8.3 PP-module $\overline{TU}$.

In this section, we want to explain $\overline{TU}$ as a PP-module and find submodules A_{ij} and B_{ij} in $H^*(BV(2))$ as $PP \otimes E(1)$-modules, where B_{ij} is a free $E(1)$-module.

Now, we have $L_i \subseteq H^*(BV(1))$ for all $0 \le i \le p-2$, then $L_i \otimes L_j \subseteq H^*(BV(1)) \otimes H^*(BV(1)) \cong H^*(BV(2))$, for all $0 \le i, j \le p-2$.

Our aim at rank 2 is to show for all $0 \le i, j \le p-2$, that we can realize the abstract isomorphism

$$L_i \otimes L_j \cong \Sigma^{2i+2j+3} L \oplus \Sigma^{2i+2j+2} E(1)[Y_1, Y_2]$$

by actual submodules. Thus we aim to find submodules $A_{ij}, B_{ij} \subseteq L_i \otimes L_j$ so that

$$L_i \otimes L_j = A_{ij} \oplus B_{ij}$$

(internal direct sum), and $A_{ij} \cong \Sigma^{2i+2j+3} L$, $B_{ij} \cong \Sigma^{2i+2j+2} E(1)[Y_1, Y_2]$.

We must choose $A = A_{ij}$, $B = B_{ij} \subseteq H^*(BV(2))$ so that $A \cap B = 0$ and $A + B = L_i \otimes L_j$.

Lemma 2.8.1. *We may take*

$$A = A_{ij} := L_i \otimes y_2^{j+1},$$

and

$$B = B_{ij} := PP \otimes E(1) \cdot y_1^i \tau_1 \otimes y_2^j \tau_2,$$

where A_{ij} is an $E(1)$-submodule and B_{ij} is a free PP-module.

Proof. We need only prove the following things.

(1) $A \cap B = 0$.

(2) $A + B = L_i \otimes L_j$.

First, for all $0 \leq i, j \leq p-2$, we define a PP-module generator $g_{ij} := y_1^i \tau_1 \otimes y_2^j \tau_2$, with $|\tau_1| = 1$ and $|\tau_2| = 1$.

For (1) we obtain explicit vector space basis for B. In fact, B is a free PP-module on 4-elements.

$$\begin{aligned} g_{ij} &= y_1^i \tau_1 \otimes y_2^j \tau_2. \\ Q_0(g_{ij}) &= y_1^{i+1} \otimes y_2^j \tau_2 - y_1^i \tau_1 \otimes y_2^{j+1}. \\ Q_1(g_{ij}) &= y_1^{i+1} Y_1 \otimes y_2^j \tau_2 - y_1^i \tau_1 \otimes y_2^{j+1} Y_2. \\ Q_0 Q_1(g_{ij}) &= y_1^{i+1} \otimes y_2^{j+1} Y_2 - y_1^{i+1} Y_1 \otimes y_2^{j+1}. \end{aligned}$$

A general element of B is a linear combination of $Y_1^a \otimes Y_2^b$ times these generators. We observe that no such combination lies in A.

For (2), by definition, $A \subseteq L_i \otimes L_j$ is true since $y_2^{j+1} \in L_j$, and $B \subseteq L_i \otimes L_j$ is also true since $y_1^{i+1}, y_1^i \tau_1 \in L_i$, $y_2^{j+1}, y_2^j \tau_2 \in L_j$.

Since $L_i \otimes L_j$ is finite dimensional in each degree, it is suffice to show that $\dim(L_i \otimes L_j)$ is correct. We can do this by Hilbert series calculation.

The Hilbert series of L_i, L_j is given by

$[L_i] = \frac{1+t}{1-T} \cdot t^{2i+1}$ and $[L_j] = \frac{1+t}{1-T} \cdot t^{2j+1}$, where $T = t^{2p-2}$.

Then

$$[L_i \otimes L_j] = (\frac{1+t}{1-T})^2 \cdot t^{2i+2j+2}.$$

The Hilbert series of A, B is given by

$$[A] = [L_i \otimes y_2^{j+1}] = \frac{1+t}{1-T} \cdot t^{2i+1+2j+2}$$

and

$$[B] = [E(1) \otimes PP \cdot y_1^i \tau_1 \otimes y_2^j \tau_2] = \frac{(1+t)(1+Tt)}{(1-T)^2} \cdot t^{2i+1+2j+1}.$$

Since $A \cap B = 0$ by (1), then

$$\begin{aligned}
[A+B] = [A] + [B] &= \frac{1+t}{1-T} \cdot t^{2i+2j+3} + \frac{(1+t)(1+Tt)}{(1-T)^2} \cdot t^{2i+2j+2} \\
&= \frac{t^{2i+2j+2}}{(1-T)^2}[(1+t)(1-T)t + (1+t)(1+Tt)] \\
&= \frac{t^{2i+2j+2}(1+t)}{(1-T)^2}[t - Tt + 1 + Tt] \\
&= (\frac{1+t}{1-T})^2 \cdot t^{2i+2j+2} \\
&= [L_i \otimes L_j].
\end{aligned}$$

Since these two Hilbert series are equal, $A + B = L_i \otimes L_j$ as required. □

2.8.4 PC-module structure of $\overline{TU}$.

The purpose of this section is to introduce PC-module structure of $\overline{TU}$ for rank 2.

Define $B_{**} := \sum_{0 \le i,j \le p-2} B_{ij}$. We want now to show that B_{**} is a PC-submodule of $H^*(BV(2))$. As above, we know that B_{ij} is a free PP-module on 4-generators g_{ij}, $Q_0(g_{ij})$, $Q_1(g_{ij})$, and $Q_0Q_1(g_{ij})$ and given i, j.

We note

$$y_1 g_{ij} = \begin{cases} g_{i+1j}, & i+1 \le p-2 \\ Y_1 g_{0j}, & i+1 = p-1, \end{cases}$$

and

$$y_2 g_{ij} = \begin{cases} g_{ij+1}, & j+1 \le p-2 \\ Y_2 g_{0j}, & j+1 = p-1. \end{cases}$$

Applying Lemma 2.7.3, we obtain B_{**} is a PC-submodule of $H^*(BV(2))$. As an $E(1) \otimes PC$-module it is generated by $g_{00} = \tau_1 \otimes \tau_2$, and furthermore it is a free module on g_{00}. Finally, we calculate

$$\begin{aligned}
\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_p, B_{**}) &= \mathrm{Hom}_{E(1)}(\mathbb{F}_p, B_{**}) \\
&= Q_0Q_1(B_{**}) \\
&= PC \cdot [Q_0Q_1(g_{00})] \\
&= \overline{TU}^2_2,
\end{aligned}$$

where

$$\begin{aligned}
Q_0Q_1(g_{00}) &= Q_0Q_1(\tau_1 \otimes \tau_2) \\
&= y_1 \otimes y_2Y_2 - y_1Y_1 \otimes y_2.
\end{aligned}$$

This calculation gives $\overline{TU} = \overline{TU}_2^2 \cong PC(-2p-2)$ is a free module of rank 1 generated by g_{00} over $PC = \mathbb{F}_p[y_1, y_2]$ shifted up $-2p-2$.

Therefore, the v-power torsion part of $lu^*(BV(2))$ is $\overline{TU}$ (i.e., $lu^*(BV(2))_{tors} = \overline{TU} \cong \overline{TU}^2 = \overline{TU}_2^2$), where the new notation tors as a subscript here denotes the v-power torsion part.

Using the short exact sequence of (2.4), our calculation at rank 2 records

$$
\begin{aligned}
lu^{-2p-2}(BV(2))_{tors} &= p^1 \\
lu^{-2p-3}(BV(2))_{tors} &= 0 \\
lu^{-2p-4}(BV(2))_{tors} &= p^2 \\
lu^{-2p-5}(BV(2))_{tors} &= 0 \\
lu^{-2p-6}(BV(2))_{tors} &= p^3 \\
lu^{-2p-7}(BV(2))_{tors} &= 0 \\
lu^{-2p-8}(BV(2))_{tors} &= p^4 \\
lu^{-2p-9}(BV(2))_{tors} &= 0 \\
lu^{-2p-10}(BV(2))_{tors} &= p^5 \\
lu^{-2p-11}(BV(2))_{tors} &= 0 \\
lu^{-2p-12}(BV(2))_{tors} &= p^6 \\
lu^{-2p-13}(BV(2))_{tors} &= 0 \\
lu^{-2p-14}(BV(2))_{tors} &= p^7, \\
&\vdots
\end{aligned}
$$

where p^k on the right hand side denotes an elementary abelian p-group of rank k.

§ 2.9 Calculation of $lu^*(BV(3))$

The aim of this section is to calculate $lu^*(BV(3))$. If we have $V(3)$, then $\overline{TU} \cong \overline{TU}_2^3 \oplus \overline{TU}_3^3$, and we will see $\overline{TU}_3^3$ is a free module of rank 1 over $PC = \mathbb{F}_p[y_1, y_2, y_3]$ with $|y_i| = 2$ on a generator of degree $-2p-3$, whilst $\overline{TU}_2^3$ is not free.

2.9.1 E_2-term

In this section, we use the Adams spectral sequence to calculate $lu^*(BV(3))$. First, we calculate $H^*(BV(3))$ as follows.

$$
\begin{aligned}
H^*(BV(3)) &\cong H^*(BV(1))^{\otimes 3}\\
&= (\mathbb{F}_p \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_{p-2})^{\otimes 3}\\
&= \big(\mathbb{F}_p \oplus \frac{\Sigma L[\overline{y}]}{(\overline{y}^{p-1})}\big)^{\otimes 3}\\
&= \big(\mathbb{F}_p \oplus \frac{\Sigma L[\overline{y_1}]}{(\overline{y_1}^{p-1})}\big) \otimes \big(\mathbb{F}_p \oplus \frac{\Sigma L[\overline{y_2}]}{(\overline{y_2}^{p-1})}\big) \otimes \big(\mathbb{F}_p \oplus \frac{\Sigma L[\overline{y_3}]}{(\overline{y_3}^{p-1})}\big)\\
&= \mathbb{F}_p \otimes \Big[\big(\mathbb{F}_p \oplus \frac{\Sigma L[\overline{y_2}]}{(\overline{y_2}^{p-1})}\big) \otimes \big(\mathbb{F}_p \oplus \frac{\Sigma L[\overline{y_3}]}{(\overline{y_3}^{p-1})}\big)\Big] \oplus \big(\frac{\Sigma L[\overline{y_1}]}{(\overline{y_1}^{p-1})}\big)\\
&\quad \otimes \Big[\big(\mathbb{F}_p \oplus \frac{\Sigma L[\overline{y_2}]}{(\overline{y_2}^{p-1})}\big) \otimes \big(\mathbb{F}_p \oplus \frac{\Sigma L[\overline{y_3}]}{(\overline{y_3}^{p-1})}\big)\Big]\\
&= \mathbb{F}_p \otimes \Big(\mathbb{F}_p \oplus \frac{\Sigma L[\overline{y_2}]}{(\overline{y_2}^{p-1})} \oplus \frac{\Sigma L[\overline{y_3}]}{(\overline{y_3}^{p-1})} \oplus \frac{\Sigma^2 L \otimes L[\overline{y_2}, \overline{y_3}]}{(\overline{y_2}^{p-1}, \overline{y_3}^{p-1})}\Big) \oplus \big(\frac{\Sigma L[\overline{y_1}]}{(\overline{y_1}^{p-1})}\big)\\
&\quad \otimes \Big(\mathbb{F}_p \oplus \frac{\Sigma L[\overline{y_2}]}{(\overline{y_2}^{p-1})} \oplus \frac{\Sigma L[\overline{y_3}]}{(\overline{y_3}^{p-1})} \oplus \frac{\Sigma^2 L \otimes L[\overline{y_2}, \overline{y_3}]}{(\overline{y_2}^{p-1}, \overline{y_3}^{p-1})}\Big)\\
&= \mathbb{F}_p \oplus \frac{\Sigma L[\overline{y_1}]}{(\overline{y_1}^{p-1})} \oplus \frac{\Sigma L[\overline{y_2}]}{(\overline{y_2}^{p-1})} \oplus \frac{\Sigma L[\overline{y_3}]}{(\overline{y_3}^{p-1})} \oplus \frac{\Sigma^2 L \otimes L[\overline{y_1}, \overline{y_2}]}{(\overline{y_1}^{p-1}, \overline{y_2}^{p-1})} \oplus \frac{\Sigma^2 L \otimes L[\overline{y_1}, \overline{y_3}]}{(\overline{y_1}^{p-1}, \overline{y_3}^{p-1})}\\
&\quad \oplus \frac{\Sigma^2 L \otimes L[\overline{y_2}, \overline{y_3}]}{(\overline{y_2}^{p-1}, \overline{y_3}^{p-1})} \oplus \frac{\Sigma^3 L \otimes L \otimes L[\overline{y_1}, \overline{y_2}, \overline{y_3}]}{(\overline{y_1}^{p-1}, \overline{y_2}^{p-1}, \overline{y_3}^{p-1})}.
\end{aligned}
$$

As we can see from the last expression above, we need to apply Lemma 2.5.1, and use the definition of $L \cong_{\mathbb{F}_p} \mathbb{F}_p[Y]\{\gamma_0, \gamma_1\}$ with $|\gamma_0| = 0$ and $|\gamma_1| = 1$ to calculate $L \otimes L \otimes L$.

Now, we do the tensor product of $L \otimes L \otimes L$.

$$
\begin{aligned}
L \otimes L \otimes L &\cong (\Sigma L \oplus E(1)[Y_1, Y_2]) \otimes L\\
&= (\Sigma L \otimes L) \oplus (E(1)[Y_1, Y_2] \otimes \mathbb{F}_p[Y_3]\{\gamma_0, \gamma_1\})\\
&= \Sigma^2 L \oplus \Sigma E(1)[Y_1, Y_3] \oplus E(1)[Y_1, Y_2, Y_3]\{\gamma_0, \gamma_1\}, \qquad (2.6)
\end{aligned}
$$

where this isomorphism comes from rank 2. Using the terms as in (2.6), then $H^*(BV(3))$ becomes

$$
\begin{aligned}
H^*(BV(3)) \cong \mathbb{F}_p &\oplus \frac{\Sigma L[\overline{y_1}]}{(\overline{y_1}^{p-1})} \oplus \frac{\Sigma L[\overline{y_2}]}{(\overline{y_2}^{p-1})} \oplus \frac{\Sigma L[\overline{y_3}]}{(\overline{y_3}^{p-1})} \oplus \frac{\Sigma^3 L[\overline{y_1}, \overline{y_2}]}{(\overline{y_1}^{p-1}, \overline{y_2}^{p-1})} \oplus \frac{\Sigma^3 L[\overline{y_1}, \overline{y_3}]}{(\overline{y_1}^{p-1}, \overline{y_3}^{p-1})}\\
&\oplus \frac{\Sigma^3 L[\overline{y_2}, \overline{y_3}]}{(\overline{y_2}^{p-1}, \overline{y_3}^{p-1})} \oplus \frac{\Sigma^5 L[\overline{y_1}, \overline{y_2}, \overline{y_3}]}{(\overline{y_1}^{p-1}, \overline{y_2}^{p-1}, \overline{y_3}^{p-1})} \oplus \frac{\Sigma^2 E(1)[Y_1, Y_2][\overline{y_1}, \overline{y_2}]}{(\overline{y_1}^{p-1}, \overline{y_2}^{p-1})}\\
&\oplus \frac{\Sigma^2 E(1)[Y_1, Y_2][\overline{y_1}, \overline{y_3}]}{(\overline{y_1}^{p-1}, \overline{y_3}^{p-1})} \oplus \frac{\Sigma^2 E(1)[Y_1, Y_2][\overline{y_2}, \overline{y_3}]}{(\overline{y_2}^{p-1}, \overline{y_3}^{p-1})}\\
&\oplus \frac{\Sigma^4 E(1)[Y_1, Y_3][\overline{y_1}, \overline{y_2}, \overline{y_3}]}{(\overline{y_1}^{p-1}, \overline{y_2}^{p-1}, \overline{y_3}^{p-1})}\\
&\oplus \frac{\Sigma^3 E(1)[Y_1, Y_2, Y_3][\overline{y_1}, \overline{y_2}, \overline{y_3}]\{\gamma_0, \gamma_1\}}{(\overline{y_1}^{p-1}, \overline{y_2}^{p-1}, \overline{y_3}^{p-1})}.
\end{aligned}
$$

Now, we apply the same formula (2.1) as before to calculate

$$\begin{aligned}\mathrm{Ext}^{*,*}_{E(1)}((\mathbb{F}_p, H^*(BV(3);\mathbb{F}_p)) = \mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_p,\mathbb{F}_p) \oplus \mathrm{Ext}^{*,*}_{E(1)}\Big(\mathbb{F}_p, \frac{\Sigma L[\overline{y_1}]}{(\overline{y_1}^{p-1})} \oplus \frac{\Sigma L[\overline{y_2}]}{(\overline{y_2}^{p-1})} \oplus \frac{\Sigma L[\overline{y_3}]}{(\overline{y_3}^{p-1})} \\ \oplus \frac{\Sigma^3 L[\overline{y_1},\overline{y_2}]}{(\overline{y_1}^{p-1},\overline{y_2}^{p-1})} \oplus \frac{\Sigma^3 L[\overline{y_1},\overline{y_3}]}{(\overline{y_1}^{p-1},\overline{y_3}^{p-1})} \oplus \frac{\Sigma^3 L[\overline{y_2},\overline{y_3}]}{(\overline{y_2}^{p-1},\overline{y_3}^{p-1})} \\ \oplus \frac{\Sigma^5 L[\overline{y_1},\overline{y_2},\overline{y_3}]}{(\overline{y_1}^{p-1},\overline{y_2}^{p-1},\overline{y_3}^{p-1})} \oplus \frac{\Sigma^2 E(1)[Y_1,Y_2][\overline{y_1},\overline{y_2}]}{(\overline{y_1}^{p-1},\overline{y_2}^{p-1})} \\ \oplus \frac{\Sigma^2 E(1)[Y_1,Y_2][\overline{y_1},\overline{y_3}]}{(\overline{y_1}^{p-1},\overline{y_3}^{p-1})} \oplus \frac{\Sigma^2 E(1)[Y_1,Y_2][\overline{y_2},\overline{y_3}]}{(\overline{y_2}^{p-1},\overline{y_3}^{p-1})} \\ \oplus \frac{\Sigma^4 E(1)[Y_1,Y_3][\overline{y_1},\overline{y_2},\overline{y_3}]}{(\overline{y_1}^{p-1},\overline{y_2}^{p-1},\overline{y_3}^{p-1})} \\ \oplus \frac{\Sigma^3 E(1)[Y_1,Y_2,Y_3][\overline{y_1},\overline{y_2},\overline{y_3}]\{\gamma_0,\gamma_1\}}{(\overline{y_1}^{p-1},\overline{y_2}^{p-1},\overline{y_3}^{p-1})}\Big).\end{aligned} \tag{2.7}$$

2.9.2 $\overline{QU}$.

In this section, we remark that $\overline{QU}$ consists of the coefficient ring, coming from $\mathbb{F}_p[a_0,u]$ together with the part of the E_2-term of (2.7),

$$\begin{aligned}\mathrm{Ext}^{*,*}_{E(1)}\Big(\mathbb{F}_p, \frac{\Sigma L[\overline{y_1}]}{(\overline{y_1}^{p-1})} \oplus \frac{\Sigma L[\overline{y_2}]}{(\overline{y_2}^{p-1})} \oplus \frac{\Sigma L[\overline{y_3}]}{(\overline{y_3}^{p-1})} \oplus \frac{\Sigma^3 L[\overline{y_1},\overline{y_2}]}{(\overline{y_1}^{p-1},\overline{y_2}^{p-1})} \oplus \frac{\Sigma^3 L[\overline{y_1},\overline{y_3}]}{(\overline{y_1}^{p-1},\overline{y_3}^{p-1})} \\ \oplus \frac{\Sigma^3 L[\overline{y_2},\overline{y_3}]}{(\overline{y_2}^{p-1},\overline{y_3}^{p-1})} \oplus \frac{\Sigma^5 L[\overline{y_1},\overline{y_2},\overline{y_3}]}{(\overline{y_1}^{p-1},\overline{y_2}^{p-1},\overline{y_3}^{p-1})}\Big).\end{aligned}$$

Since we are aiming to calculate the module QU, which is known already (see Remark 2.1.3), we will not make $\overline{QU}$ explicit here.

2.9.3 PP-module $\overline{TU}$.

In this section, we need to explain $\overline{TU}$ as a PP-module. In fact, we aim to obtain explicit submodules A_{ijk} and B_{ijk} in $H^*(BV(3))$ as $PP \otimes E(1)$-modules, where B_{ijk} is a free $E(1)$-module.

We begin first to describe the abstract isomorphism by the following Lemma so that $L_i \otimes L_j \otimes L_k \cong A_{ijk} \oplus B_{ijk}$.

Lemma 2.9.1. *There is an abstract isomorphism for all* $0 \le i,j,k \le p-2$

$$\begin{aligned}L_i \otimes L_j \otimes L_k \cong \Sigma^{2i+2j+2k+5} L \oplus \Big(\Sigma^{2i+2j+2k+4} E(1)[Y_1,Y_3] \\ \oplus \Sigma^{2i+2j+2k+3} E(1)[Y_1,Y_2,Y_3]\{\gamma_0,\gamma_1\}\Big),\end{aligned}$$

so that

$$A_{ijk} \cong \Sigma^{2i+2j+2k+5} L,$$

and

$$B_{ijk} \cong \Big(\Sigma^{2i+2j+2k+4}E(1)[Y_1, Y_3] \oplus \Sigma^{2i+2j+2k+3}E(1)[Y_1, Y_2, Y_3]\{\gamma_0, \gamma_1\}\Big),$$

where B_{ijk} is a free $E(1)$-module and A_{ijk} is an $E(1)$-submodule.

Proof.

$$\begin{aligned}
L_i \otimes L_j \otimes L_k &\cong \Big(\Sigma^{2i+2j+3}L \oplus \Sigma^{2i+2j+2}E(1)[Y_1, Y_2]\Big) \otimes L_k \\
&= (\Sigma^{2i+2j+3}L \otimes L_k) \oplus (\Sigma^{2i+2j+2}E(1)[Y_1, Y_2] \otimes L_k) \\
&= (\Sigma^{2i+2j+2k+4}L \otimes L) \oplus (\Sigma^{2i+2j+2k+3}E(1)[Y_1, Y_2] \otimes L) \\
&= (\Sigma^{2i+2j+2k+4}L \otimes L) \oplus (\Sigma^{2i+2j+2k+3}E(1)[Y_1, Y_2] \otimes \mathbb{F}_p[Y_3]\{\gamma_0, \gamma_1\}) \\
&= \Sigma^{2i+2j+2k+5}L \oplus \Big(\Sigma^{2i+2j+2k+4}E(1)[Y_1, Y_3] \\
&\quad \oplus \Sigma^{2i+2j+2k+3}E(1)[Y_1, Y_2, Y_3]\{\gamma_0, \gamma_1\}\Big).
\end{aligned}$$

□

It is important to realise the abstract isomorphism by actual submodules as follows. We have $H^*(BV(1)) = \mathbb{F}_p \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_{p-2}$, then

$$H^*(BV(3)) \cong H^*(BV(1)) \otimes H^*(BV(1)) \otimes H^*(BV(1)) = \bigoplus_{P_i \in \{\mathbb{F}_p, L_0, \ldots, L_{p-2}\}} P_1 \otimes P_2 \otimes P_3$$

i.e., we must choose explicit submodules $A(P_1, P_2, P_3), B(P_1, P_2, P_3) \subseteq H^*(BV(3))$ so that

$$P_1 \otimes P_2 \otimes P_3 = A(P_1, P_2, P_3) \oplus B(P_1, P_2, P_3),$$

as $PP \otimes E(1)$-modules with $B(P_1, P_2, P_3)$ a free $E(1)$-module, and $A(P_1, P_2, P_3)$ having no free summand.

It is sufficient to analyse $P_1 \otimes P_2 \otimes P_3$ for $P_i \in \{\mathbb{F}_p, L_0, \ldots, L_{p-2}\}$ as follows.

1. If 3 of P_1, P_2, P_3 are $\mathbb{F}_p$, then $\mathbb{F}_p \otimes \mathbb{F}_p \otimes \mathbb{F}_p = \mathbb{F}_p$.
2. If 2 of P_1, P_2, P_3 are $\mathbb{F}_p$, then $\mathbb{F}_p \otimes \mathbb{F}_p \otimes L_k \cong \Sigma^{2k+1}L$, $\mathbb{F}_p \otimes L_j \otimes \mathbb{F}_p \cong \Sigma^{2j+1}L$, and $L_i \otimes \mathbb{F}_p \otimes \mathbb{F}_p \cong \Sigma^{2i+1}L$.
3. If 1 of P_1, P_2, P_3 is $\mathbb{F}_p$, then
   a. $\mathbb{F}_p \otimes L_j \otimes L_k \cong \Sigma^{2j+2k+3}L \oplus \Sigma^{2j+2k+2}E(1)[Y_2, Y_3]$,
   b. $L_i \otimes \mathbb{F}_p \otimes L_k \cong \Sigma^{2i+2k+3}L \oplus \Sigma^{2i+2k+2}E(1)[Y_1, Y_3]$,
   c. $L_i \otimes L_j \otimes \mathbb{F}_p \cong \Sigma^{2i+2j+3}L \oplus \Sigma^{2i+2j+2}E(1)[Y_1, Y_2]$.
4. If 0 of P_1, P_2, P_3 is $\mathbb{F}_p$, then $L_i \otimes L_j \otimes L_k$ becomes

$$L_i \otimes L_j \otimes L_k \cong \Sigma^{2i+2j+2k+5} L \oplus \Big(\Sigma^{2i+2j+2k+4} E(1)[Y_1, Y_3] \\ \oplus \Sigma^{2i+2j+2k+3} E(1)[Y_1, Y_2, Y_3]\{\gamma_0, \gamma_1\} \Big).$$

Now, we must choose $A = A_{ijk} = A(L_i, L_j, L_k)$, $B = B_{ijk} = B(L_i, L_j, L_k) \subseteq H^*(BV(3))$, so that $A \cap B = 0$ and $A + B = L_i \otimes L_j \otimes L_k$.

Lemma 2.9.2. *We may take*

$$A = A_{ijk} := L_i \otimes y_2^{j+1} \otimes y_3^{k+1},$$

and

$$B = B_{ijk} := B^{od}(L_i, L_j, L_k) \oplus B^{ev_1}(L_i, L_j, L_k) \oplus B^{ev_2}(L_i, L_j, L_k),$$

where $B^{od}(L_i, L_j, L_k) := E(1) \otimes PP \cdot y_1^i \tau_1 \otimes y_2^j \tau_2 \otimes y_3^k \tau_3$ with generators in odd degree, and

$$B^{ev_1}(L_i, L_j, L_k) := E(1) \otimes PP \cdot y_1^i \tau_1 \otimes y_2^j \tau_2 \otimes y_3^{k+1},$$
$$B^{ev_2}(L_i, L_j, L_k) := E(1) \otimes PP \cdot y_1^i \tau_1 \otimes y_2^{j+1} \otimes y_3^k \tau_3,$$

together with

$$B_{*jk} := E(1) \otimes PP \cdot 1 \otimes y_2^j \tau_2 \otimes y_3^k \tau_3,$$
$$B_{i*k} := E(1) \otimes PP \cdot y_1^i \tau_1 \otimes 1 \otimes y_3^k \tau_3,$$

and

$$B_{ij*} := E(1) \otimes PP \cdot y_1^i \tau_1 \otimes y_2^j \tau_2 \otimes 1$$

with generators in even degrees.

Proof. We want to prove the following things.

(1) $A \cap B = 0$.

(2) $A + B = L_i \otimes L_j \otimes L_k$.

First, for all $0 \le i, j, k \le p-2$. Define a PP-module generators $g_{ijk} := y_1^i \tau_1 \otimes y_2^j \tau_2 \otimes y_3^k \tau_3$ in odd degrees, and $g^1_{ijk+1} := y_1^i \tau_1 \otimes y_2^j \tau_2 \otimes y_3^{k+1}$, $g^2_{ij+1k} := y_1^i \tau_1 \otimes y_2^{j+1} \otimes y_3^k \tau_3$, $g_{*jk} := 1 \otimes y_2^j \tau_2 \otimes y_3^k \tau_3$, $g_{i*k} := y_1^i \tau_1 \otimes 1 \otimes y_3^k \tau_3$, and $g_{ij*} := y_1^i \tau_1 \otimes y_2^j \tau_2 \otimes 1$ in even degrees with $|\tau_1| = 1$, $|\tau_2| = 1$ and $|\tau_3| = 1$.

For (1) we obtain explicit vector space basis for B. In fact, B is a free module over $PP \otimes E(1)$ on 4-elements respectively,

$$g_{ijk} = y_1^i \tau_1 \otimes y_2^j \tau_2 \otimes y_3^k \tau_3.$$
$$Q_0(g_{ijk}) = y_1^{i+1} \otimes y_2^j \tau_2 \otimes y_3^k \tau_3 \;-\; y_1^i \tau_1 \otimes y_2^{j+1} \otimes y_3^k \tau_3 \;+\; y_1^i \tau_1 \otimes y_2^j \tau_2 \otimes y_3^{k+1}.$$
$$Q_1(g_{ijk}) = y_1^{i+1} Y_1 \otimes y_2^j \tau_2 \otimes y_3^k \tau_3 \;-\; y_1^i \tau_1 \otimes y_2^{j+1} Y_2 \otimes y_3^k \tau_3 \;+\; y_1^i \tau_1 \otimes y_2^j \tau_2 \otimes y_3^{k+1} Y_3.$$
$$Q_0 Q_1(g_{ijk}) = y_1^{i+1} \otimes y_2^{j+1} \otimes y_3^{k+1} Y_3 \;-\; y_1^{i+1} \otimes y_2^{j+1} Y_2 \otimes y_3^{k+1} \;+\; y_1^{i+1} Y_1 \otimes y_2^{j+1} \otimes y_3^{k+1},$$

and

$$\begin{aligned}
g^1_{ijk+1} &= y_1^i\tau_1 \otimes y_2^j\tau_2 \otimes y_3^{k+1}. \\
Q_0(g^1_{ijk+1}) &= y_1^{i+1}\tau_1 \otimes y_2^j\tau_2 \otimes y_3^{k+1} \;-\; y_1^i\tau_1 \otimes y_2^{j+1} \otimes y_3^{k+1}. \\
Q_1(g^1_{ijk+1}) &= y_1^{i+1}Y_1 \otimes y_2^j\tau_2 \otimes y_3^{k+1} \;-\; y_1^i\tau_1 \otimes y_2^{j+1}Y_2 \otimes y_3^{k+1}. \\
Q_0Q_1(g^1_{ijk+1}) &= y_1^{i+1} \otimes y_2^{j+1}Y_2 \otimes y_3^{k+1} \;-\; y_1^{i+1}Y_1 \otimes y_2^{j+1} \otimes y_3^{k+1},
\end{aligned}$$

and

$$\begin{aligned}
g^2_{ij+1k} &= y_1^i\tau_1 \otimes y_2^{j+1} \otimes y_3^k\tau_3. \\
Q_0(g^2_{ij+1k}) &= y_1^{i+1} \otimes y_2^{j+1} \otimes y_3^k\tau_3 \;-\; y_1^i\tau_1 \otimes y_2^{j+1} \otimes y_3^{k+1}. \\
Q_1(g^2_{ij+1k}) &= y_1^{i+1}Y_1 \otimes y_2^{j+1} \otimes y_3^k\tau_3 \;-\; y_1^i\tau_1 \otimes y_2^{j+1} \otimes y_3^{k+1}Y_3. \\
Q_0Q_1(g^2_{ij+1k}) &= y_1^{i+1}Y_1 \otimes y_2^{j+1} \otimes y_3^{k+1} \;-\; y_1^{i+1} \otimes y_2^{j+1} \otimes y_3^{k+1}Y_3.
\end{aligned}$$

A general element of B is a linear combination of $Y_1^a \otimes Y_2^b \otimes Y_3^c$ times these generators. We observe that no such combination lies in A.

For (2), by definition, $A \subseteq L_i \otimes L_j \otimes L_k$ is true since $y_2^{j+1} \otimes y_3^{k+1} \subseteq L_j \otimes L_k$, and $B \subseteq L_i \otimes L_j \otimes L_k$ is also true since $y_1^{i+1}, y^i\tau_1 \in L_i$, $y_2^{j+1}, y_2^j\tau_2 \in L_j$, and $y_3^{k+1}, y_3^k\tau_3 \in L_k$.

Since $L_i \otimes L_j \otimes L_k$ is finite dimensional in each degree, it is suffice to show that $\dim(L_i \otimes L_j \otimes L_k)$ is correct. We can do this by Hilbert series calculation.

The Hilbert series of L_i, L_j, L_k is given by

$[L_i] = \frac{1+t}{1-T} \cdot t^{2i+1}$, $[L_j] = \frac{1+t}{1-T} \cdot t^{2j+1}$, and $[L_k] = \frac{1+t}{1-T} \cdot t^{2k+1}$, where $T = t^{2p-2}$.

Then

$$[L_i \otimes L_j \otimes L_k] = \left(\frac{1+t}{1-T}\right)^3 \cdot t^{2i+2j+2k+3}.$$

The Hilbert series of A is given by

$$[A] = [L_i \otimes y_2^{j+1} \otimes y_3^{k+1}] = (\frac{1+t}{1-T}) \cdot t^{2i+2j+2k+5}.$$

On the other hand, the Hilbert series of $B^{od}(L_i, L_j, L_k)$, $B^{ev_1}(L_i, L_j, L_k)$, and $B^{ev_2}(L_i, L_j, L_k)$ are given by

$$\begin{aligned}
[B^{od}(L_i, L_j, L_k)] &= [E(1) \otimes PP \cdot y_1^i\tau_1 \otimes y_2^j\tau_2 \otimes y_3^k\tau_3] \\
&= \frac{(1+t)(1+Tt)}{(1-T)^3} \cdot t^{2i+2j+2k+3}, \\
[B^{ev_1}(L_i, L_j, L_k)] &= [E(1) \otimes PP \cdot y_1^i\tau_1 \otimes y_2^j\tau_2 \otimes y_3^{k+1}] \\
&= \frac{(1+t)(1+Tt)}{(1-T)^3} \cdot t^{2i+2j+2k+4}, \\
[B^{ev_2}(L_i, L_j, L_k)] &= [E(1) \otimes PP \cdot y_1^i\tau_1 \otimes y_2^{j+1} \otimes y_3^k\tau_3] \\
&= \frac{(1+t)(1+Tt)}{(1-T)^2} \cdot t^{2i+2j+2k+4}.
\end{aligned}$$

Since $A \cap B = 0$ by (1), then

$$\begin{aligned}
[A+B] &= [A + (B^{od} + B^{ev_1} + B^{ev_2})] \\
&= [A] + [B^{od}] + [B^{ev_1}] + [B^{ev_2}] \\
&= (\frac{1+t}{1-T}) \cdot t^{2i+2j+2k+5} + \frac{(1+t)(1+Tt)}{(1-T)^3} \cdot t^{2i+2j+2k+3} \\
&\quad + \frac{(1+t)(1+Tt)}{(1-T)^3} \cdot t^{2i+2j+2k+4} + \frac{(1+t)(1+Tt)}{(1-T)^2} \cdot t^{2i+2j+2k+4} \\
&= \Big(\frac{1+t}{(1-T)^3}\Big) \cdot t^{2i+2j+2k+3}[(1-T)^2t^2 + (1+Tt) + (1+Tt)t + (1-T)(1+Tt)t] \\
&= \Big(\frac{1+t}{(1-T)^3}\Big) \cdot t^{2i+2j+2k+3}[t^2 - 2Tt^2 + T^2t^2 + 1 + Tt + t + Tt^2 + t + Tt^2 - Tt - T^2t^2] \\
&= \Big(\frac{1+t}{(1-T)^3}\Big) \cdot t^{2i+2j+2k+3}[t^2 + 2t + 1] \\
&= \Big(\frac{1+t}{1-T}\Big)^3 \cdot t^{2i+2j+2k+3} \\
&= [L_i \otimes L_j \otimes L_k].
\end{aligned}$$

Since these two Hilbert series agree, $A + B = L_i \otimes L_j \otimes L_k$ as required. □

2.9.4 PC-module structure of $\overline{TU}$.

In this section, we want to deduce the PC-module structure over $\overline{TU}$ from the PP-module structure.

It is reasonable to write

$$\begin{aligned}
B_{***} &= \sum_{0 \le i,j,k \le p-2} B_{ijk} + \sum_{0 \le j,k \le p-2} B_{*jk} + \sum_{0 \le i,k \le p-2} B_{i*k} + \sum_{0 \le i,j \le p-2} B_{ij*} \\
&= \sum_{0 \le i,j,k \le p-2} B^{od}(L_i, L_j, L_k) + \sum_{0 \le i,j,k \le p-2} B^{ev_1}(L_i, L_j, L_k) + \sum_{0 \le i,j,k \le p-2} B^{ev_2}(L_i, L_j, L_k) \\
&\quad + \sum_{0 \le j,k \le p-2} B_{*jk} + \sum_{0 \le i,k \le p-2} B_{i*k} + \sum_{0 \le i,j \le p-2} B_{ij*}.
\end{aligned}$$

Our aim here is to show that B_{***} is a PC-submodule of $H^*(BV(3))$, and has PC-generators $g_{000} := \tau_{123} = \tau_1 \otimes \tau_2 \otimes \tau_3$, $g_{*00} := \tau_{23} = 1 \otimes \tau_2 \otimes \tau_3$, $g_{0*0} := \tau_{13} = \tau_1 \otimes 1 \otimes \tau_3$, and $g_{00*} := \tau_{12} = \tau_1 \otimes \tau_2 \otimes 1$.

Lemma 2.9.3. *For all $0 \le i,j,k \le p-2$. We note that*

$$y_1 g_{ijk} = \begin{cases} g_{i+1jk}, & i+1 \le p-2 \\ Y_1 g_{i-p+1jk}, & i+1 = p-1 \end{cases}$$

$$y_2 g_{ijk} = \begin{cases} g_{ij+1k}, & j+1 \le p-2 \\ Y_2 g_{i-p+1jk}, & j+1 = p-1 \end{cases}$$

$$y_3 g_{ijk} = \begin{cases} g_{ijk+1}, & k{+}1 \leq p-2 \\ Y_3 g_{i-p+1jk}, & k{+}1 = p-1, \end{cases}$$

and

$$y_1 g^1_{ijk+1} = \begin{cases} g^1_{i+1jk+1}, & i{+}1 \leq p-2 \\ Y_1 g_{i-p+1jk}, & i{+}1 = p-1 \end{cases}$$

$$y_2 g^1_{ijk+1} = \begin{cases} g^1_{ij+1k+1}, & j{+}1 \leq p-2 \\ Y_2 g_{i-p+1jk}, & j{+}1 = p-1 \end{cases}$$

$$y_3 g^1_{ijk+1} = \begin{cases} g^1_{ijk+2}, & k{+}1 \leq p-2 \\ Y_3 g_{i-p+1jk}, & k{+}1 = p-1, \end{cases}$$

and

$$y_1 g^2_{ij+1k} = \begin{cases} g^2_{i+1j+1k}, & i{+}1 \leq p-2 \\ Y_1 g_{i-p+1jk}, & i{+}1 = p-1 \end{cases}$$

$$y_2 g^2_{ij+1k} = \begin{cases} g^2_{ij+2k}, & j{+}1 \leq p-2 \\ Y_2 g_{i-p+1jk}, & j{+}1 = p-1 \end{cases}$$

$$y_3 g^2_{ij+1k} = \begin{cases} g^2_{ij+1k+1}, & k{+}1 \leq p-2 \\ Y_3 g_{i-p+1jk}, & k{+}1 = p-1, \end{cases}$$

and

$$y_1 g_{*jk} = Q_0(g_{0jk}) + g_{0j+1k} - g_{0jk+1},$$

where

$$\begin{aligned} Q_0(g_{0jk}) &= Q_0(\tau_1 \otimes y_2^j \tau_2 \otimes y_3^k \tau_3) \\ &= y_1 \otimes y_2^j \tau_2 \otimes y_3^k \tau_3 - \tau_1 \otimes y_2^{j+1} \otimes y_3^k \tau_3 + \tau_1 \otimes y_2^j \tau_2 \otimes y_3^{k+1} \\ &= y_1 g_{*jk} - g_{0j+1k} + g_{0jk+1}, \end{aligned}$$

$$y_2 g_{*jk} = \begin{cases} g_{*j+1k}, & j{+}1 \leq p-2 \\ Y_2 g_{*0k}, & j{+}1 = p-1, \end{cases}$$

$$y_3 g_{*jk} = \begin{cases} g_{*jk+1}, & k{+}1 \leq p-2 \\ Y_3 g_{*0k}, & k{+}1 = p-1, \end{cases}$$

and

$$y_1 g_{i*k} = \begin{cases} g_{i+1*k}, & i{+}1 \leq p-2 \\ Y_1 g_{0*k}, & i{+}1 = p-1, \end{cases}$$

$$y_2 g_{i*k} = g_{i+10k} + g_{i0k+1} - Q_0(g_{i0k}),$$

where

$$\begin{aligned} Q_0(g_{i0k}) &= Q_0(y_1^i\tau_1 \otimes \tau_2 \otimes y_3^k\tau_3) \\ &= y_1^{i+1} \otimes \tau_2 \otimes y_3^k\tau_3 - y_1^i\tau_1 \otimes y_2 \otimes y_3^k\tau_3 + y_1^i\tau_1 \otimes \tau_2 \otimes y_3^{k+1} \\ &= g_{i+10k} - y_2 g_{i*k} + g_{i0k+1}, \end{aligned}$$

$$y_3 g_{i*k} = \begin{cases} g_{i*k+1}, & k{+}1 \le p-2 \\ Y_3 g_{0*k}, & k{+}1 = p-1, \end{cases}$$

and

$$y_1 g_{ij*} = \begin{cases} g_{i+1j*}, & i{+}1 \le p-2 \\ Y_1 g_{0k*}, & i{+}1 = p-1, \end{cases}$$

$$y_2 g_{ij*} = \begin{cases} g_{ij+1*}, & j{+}1 \le p-2 \\ Y_2 g_{0k*}, & j{+}1 = p-1, \end{cases}$$

$$y_3 g_{ij*} = Q_0(g_{ij0}) - g_{i+1j0} + g_{ij+10},$$

where

$$\begin{aligned} Q_0(g_{ij0}) &= Q_0(y_1^i\tau_1 \otimes y_2^j\tau_2 \otimes \tau_3) \\ &= y_1^{i+1} \otimes y_2^j\tau_2 \otimes \tau_3 - y_1^i\tau_1 \otimes y_2^{j+1} \otimes \tau_3 + y_1^i\tau_1 \otimes y_2^j\tau_2 \otimes y_3 \\ &= g_{i+1j0} - g_{ij+10} + y_3 g_{ij*}. \end{aligned}$$

Now, we discuss to describe the PC-submodule in odd and even degrees of B_{***} by the appropriate sums as follows. By definition, $B^{od}(L_i, L_j, L_k)$, the odd part of B_{***}, is a free PP-module on 4-generators g_{ijk}, $Q_0(g_{ijk})$, $Q_1(g_{ijk})$, and $Q_0Q_1(g_{ijk})$. We may thus describe the PC-submodule in odd degrees

$$\overline{TU}_3^3 = \sum_{0 \le i,j,k \le p-2} B^{od}(L_i, L_j, L_k).$$

For the even part of B_{***}, $B^{ev_1}(L_i, L_j, L_k), B^{ev_2}(L_i, L_j, L_k)$, B_{*jk}, B_{i*k}, and B_{ij*}, we note that the first generator $y_1 g_{*jk} \in B_{***}$ (since $PP \cdot E(1) \cdot g_{0jk} \subseteq B_{***}$), and similarly for the other two generators $y_2 g_{i*k}$ and $y_3 g_{ij*}$. Accordingly, there are only three even generators of B_{***}. We may now describe the PC-submodule in even degrees by the appropriate sums

$$\begin{aligned} \overline{TU}_2^3 = &\sum_{0 \le i,j,k \le p-2} B^{ev_1}(L_i, L_j, L_k) + \sum_{0 \le i,j,k \le p-2} B^{ev_2}(L_i, L_j, L_k) + \sum_{0 \le j,k \le p-2} B_{*jk} \\ &+ \sum_{0 \le i,k \le p-2} B_{i*k} + \sum_{0 \le i,j \le p-2} B_{ij*}. \end{aligned}$$

We use Lemma 2.9.3, to obtain B_{***} is a PC-submodule of $H^*(BV(3))$. As an $E(1) \otimes PC$-module it is generated by $g_{000} = \tau_{123} = \tau_1 \otimes \tau_2 \otimes \tau_3$ for the odd part, and by 3 types of generator $g_{*00} = \tau_{23} = 1 \otimes \tau_2 \otimes \tau_3$, $g_{0*0} = \tau_{13} = \tau_1 \otimes 1 \otimes \tau_3$, $g_{00*} = \tau_{12} = \tau_1 \otimes \tau_2 \otimes 1$

for the even part, and furthermore it is a free module on generators g_{000}, g_{*00}, g_{0*0}, and g_{00*}.

Now we calculate

$$\begin{aligned}
\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_p, B_{***}) &= \mathrm{Hom}_{E(1)}(\mathbb{F}_p, B_{***}) \\
&= Q_0Q_1(B_{***}) \\
&= [PC\cdot Q_0Q_1(g_{000})] \oplus \big[PC\cdot Q_0Q_1(g_{*00}) + PC\cdot Q_0Q_1(g_{0*0}) \\
&\quad + PC\cdot Q_0Q_1(g_{00*})\big] \\
&= \overline{TU}^3_3 \oplus \overline{TU}^3_2,
\end{aligned}$$

where

$$\begin{aligned}
Q_0Q_1(g_{000}) &= Q_0Q_1(\tau_1\otimes\tau_2\otimes\tau_3) \\
&= (y_1Y_1\otimes y_2 - y_2Y_2\otimes y_1)\otimes\tau_3 - (y_1Y_1\otimes y_3 - y_3Y_3\otimes y_1)\otimes\tau_2 \\
&\quad - (y_3Y_3\otimes y_2 - y_2Y_2\otimes y_3)\otimes\tau_1,
\end{aligned}$$

and

$$\begin{aligned}
Q_0Q_1(g_{*00}) &= Q_0Q_1(1\otimes\tau_2\otimes\tau_3) \\
&= y_2\otimes y_3Y_3 - y_2Y_2\otimes y_3. \\
Q_0Q_1(g_{0*0}) &= Q_0Q_1(\tau_1\otimes 1\otimes\tau_3) \\
&= y_1\otimes y_3Y_3 - y_1Y_1\otimes y_3. \\
Q_0Q_1(g_{00*}) &= Q_0Q_1(\tau_1\otimes\tau_2\otimes 1) \\
&= y_1\otimes y_2Y_2 - y_1Y_1\otimes y_2.
\end{aligned}$$

This calculation gives us $\overline{TU} \cong \overline{TU}^3 = \overline{TU}^3_3 \oplus \overline{TU}^3_2$ so that $\overline{TU}^3_3 \cong PC(-2p-3)$ is a free module of rank 1 over $PC = \mathbb{F}_p[y_1, y_2, y_3]$ on a generator g_{000} of degree $-2p-3$, whilst $\overline{TU}^3_2$ has 3 types of generator g_{*00}, g_{0*0}, g_{00*} over PC of degree $-2p-2$ (we will see that clearly in Chapter 5).

$$0 \longleftarrow \overline{TU}^3_2 \longleftarrow \binom{3}{2} PC(-(2p+2)) \longleftarrow \binom{3}{3}[PC(-(2p+4)) \oplus PC(-(4p+2))] \longleftarrow 0.$$

Therefore, the odd part $lu^{od}(BV(3))$ of $lu^*(BV(3))$ is $\overline{TU}^3_3$ (i.e., $lu^{od}(BV(3))) = \overline{TU}^3_3 \cong PC(-2p-3)$), and

$$\begin{aligned}
lu^{-2p-3}(BV(3)) &= p^1. \\
lu^{-2p-5}(BV(3)) &= p^3. \\
lu^{-2p-7}(BV(3)) &= p^6. \\
lu^{-2p-9}(BV(3)) &= p^{10}. \\
lu^{-2p-11}(BV(3)) &= p^{15}. \\
lu^{-2p-13}(BV(3)) &= p^{21}. \\
&\vdots
\end{aligned}$$

The v-power torsion even part $lu^{ev}(BV(3))$ of $lu^*(BV(3))$ is $\overline{TU}_2^3$ (i.e., $lu^{ev}(BV(3))_{tors} = \overline{TU}_2^3$), and is calculated by the Hilbert series

$$\begin{aligned}[\overline{TU}_2^3] &= 3[PC(-(2p+2))] - [PC(-(2p+4)) + PC(-(4p+2))] \\ &= \frac{3t^{2p+2}}{(1-t^2)^3} - \frac{t^{2p+4}}{(1-t^2)^3} - \frac{t^{4p+2}}{(1-t^2)^3} \\ &= \frac{3t^{2p+2} - t^{2p+4} - t^{4p+2}}{(1-t^2)^3} \\ &= \frac{t^{2p+2}(3 - t^2 - t^{2p})}{(1-t^2)^3}.\end{aligned}$$

If we have $p = 3, 5, 7$, then $\overline{TU}_2^3$ is calculated at even degree torsion by

Degree	$\overline{TU}_2^3$ (p =3)	$\overline{TU}_2^3$ (p=5)	$\overline{TU}_2^3$ (p=7)
$-2p-2$	p^3	p^3	p^3
$-2p-4$	p^8	p^8	p^8
$-2p-6$	p^{15}	p^{15}	p^{15}
$-2p-8$	p^{23}	p^{24}	p^{24}
$-2p-10$	p^{32}	p^{35}	p^{35}
$-2p-12$	p^{42}	p^{47}	p^{48}
$-2p-14$	p^{53}	p^{60}	p^{63}
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 2.4: $\overline{TU}_2^3$

§ 2.10 Summary of rank ≤ 3 and expectations for rank ≥ 4

The target of this section is to give a summary of our calculations for $V(r)$, $r \leq 3$ and expectations for rank $r \geq 4$. If we have $V(r)$, $r \leq 3$, then $\overline{TU}_r^r$ is a free PC-module, and $\overline{TU}_{r-1}^r$ is not a free module, but this module will admit a free resolution of length one, and we will prove that as a result in the next Chapter.

If we have $V(r)$, $r \geq 4$, then we expect $\overline{TU}_r^r \cong PC(-2p-r)$ is a free PC-module of rank 1 over $PC = \mathbb{F}_p[y_1, y_2, \ldots, y_r]$ on a generator of degree $-2p-r$, and in general $\overline{TU}_i^r$ has a free resolution of length $r-i$.

Definition 2.10.1. $\overline{TU}$ was defined as the v-power torsion module in $lu^*(BV(r))$. Analogous to [10, page 95], we define

$$\overline{TU}^r := PC\{q_S \mid |S| \geq 2\} \subseteq H^*(BV(r); \mathbb{F}_p),$$

to be the PC-module generated by the elements q_S,

where

$$q_S = Q_0Q_1(\tau_S) = \sum_{s<t} (-1)^{t-s}(y_s y_t^p - y_s^p y_t)\tau_{S\setminus\{s,t\}},$$

for subsets $S \subseteq \{1, 2, \ldots, r\}$ with $|S| \geq 2$, and $\tau_S = \prod_{s \in S} \tau_s$. It was proved in 2.8.4 and 2.9.4 that $\overline{TU} \cong \overline{TU}^r$ for $r \leq 3$.

Conjecture 2.10.2. *The v-power torsion $\overline{TU}$ is the PC-module generated by elements q_S, for $|S| \geq 2$. Accordingly*

$$\overline{TU}^r = \overline{TU}^r_2 \oplus \overline{TU}^r_3 \oplus \cdots \oplus \overline{TU}^r_r,$$

where $\overline{TU}^r_s$ is generated by the elements q_S with $|S| = s$.

The idea of the proof depends on the complementary part of $H^*(BV(r); \mathbb{F}_p)$ which is in the $E(1)$-free submodule, and has basis the τ_S with $|S| \geq 2$. Each factor $E(1)$ in the sum gives rise to $\Sigma^{2p}\mathbb{F}_p = \mathrm{Hom}_{E(1)}(\mathbb{F}_p, E(1))$, and is generated by Q_0Q_1 times the generator of the free module, giving $Q_0Q_1(\tau_S)$ from the submodule generated by τ_S.

The directness of the sum follows since the number of elements τ_s in every monomial in q_S is precisely $|S| - 2$, so that every term in every element of $\overline{TU}^r$ is a monomial involving precisely the product of $n - 2$ of the τ_s.

§ 2.11 Calculation of $ku^*(BV(r))$ for $r \leq 3$

In this section, we want to calculate $ku^*(BV(r))$, for $r \leq 3$. This can be done by calculating the QU-module together with TU-module.

- If we have $V(1)$, then $\overline{TU} = 0$, and also $TU = 0$.

Consider the short exact sequence

$$0 \longrightarrow TU \longrightarrow ku^*(BV(1)) \longrightarrow QU \longrightarrow 0. \tag{2.8}$$

Since $TU = 0$,

$$\begin{aligned} ku^*(BV(1)) &= QU \\ &= \overline{QU} \oplus \Sigma^2\overline{QU} \oplus \Sigma^4\overline{QU} \oplus \cdots \oplus \Sigma^{2p-4}\overline{QU} \\ &= lu^*(BV(1)) \oplus \Sigma^2 lu^*(BV(1)) \oplus \Sigma^4 lu^*(BV(1)) \oplus \cdots \oplus \Sigma^{2p-4} lu^*(BV(1)). \end{aligned}$$

- If we have $V(2)$, then $\overline{TU} \cong \overline{TU}^2 = \overline{TU}^2_2 \cong PC(-2p-2)$, with $PC = \mathbb{F}_p[y_1, y_2]$, and

$$\begin{aligned} TU &= \overline{TU}^2_2 \oplus \Sigma^{-2}\overline{TU}^2_2 \oplus \Sigma^{-4}\overline{TU}^2_2 \oplus \cdots \oplus \Sigma^{-2p+4}\overline{TU}^2_2 \\ &= PC(-2p-2) \oplus PC(-2p-4) \oplus PC(-2p-6) \oplus \cdots \oplus PC(-4p+2). \end{aligned}$$

By definition 2.1.1, $QU := \mathrm{im}(ku^*(BV(2)) \longrightarrow K^*(BV(2)) = RU^{\wedge}_{JU}[v, v^{-1}])$, and $QU = ku^*\langle c_1(\alpha_i) \mid \alpha \in \mathrm{Rep}_1(V)\rangle$. Then

$$\begin{aligned} K^{-2n}(BV(2)) &= \mathbb{Z} \oplus JU^{\wedge}_p \\ &= \mathbb{Z} \oplus \langle c_1(\alpha_i) \,|\, i = 1, 2\rangle^{\wedge}_p, \text{ for } n \geq 0. \end{aligned}$$

Degree	$PC(-2p-2)$	$PC(-2p-4)$	$PC(-2p-6)$	$\cdots$	$PC(-4p+2)$	TU
$-2p-2$	p^1	0	0	$\cdots$	0	p^1
$-2p-4$	p^2	p^1	0	$\cdots$	0	p^3
$-2p-6$	p^3	p^2	p^1	$\cdots$	0	p^6
$-2p-8$	p^4	p^3	p^2	$\cdots$	0	p^{10}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-4p+2$	p^{p-1}	p^{p-2}	p^{p-3}	$\cdots$	p^1	$p^{\binom{p}{2}}$
$-4p$	p^p	p^{p-1}	p^{p-2}	$\cdots$	p^2	$p^{\binom{p+1}{2}-1}$
$-4p-2$	p^{p+1}	p^p	p^{p-1}	$\cdots$	p^3	$p^{\binom{p+2}{2}-3}$

Figure 2.5: TU for rank 2

and for $n \geq 1$

$$QU^{-2n}(BV(2)) = (JU_p^\wedge)^n.$$

Note that the final column as in Figure 2.5 only gives the torsion part of $ku^*(BV(2))$ which is TU. Now, the whole calculation of QU and TU together gives $ku^*(BV(2))$ as follows.

If $p = 3$, we then have

$$ku^{2p+2}(BV(2)) = JU^{p+1} \oplus p^1.$$
$$ku^{2p+4}(BV(2)) = JU^{p+2} \oplus p^3.$$

If $p = 5$, we then have

$$ku^{2p+2}(BV(2)) = JU^{p+1} \oplus p^1.$$
$$ku^{2p+4}(BV(2)) = JU^{p+2} \oplus p^3.$$
$$ku^{2p+6}(BV(2)) = JU^{p+3} \oplus p^6.$$
$$ku^{2p+8}(BV(2)) = JU^{p+4} \oplus p^{10}.$$

If p=7, we then have

$$ku^{2p+2}(BV(2)) = JU^{p+1} \oplus p^1.$$
$$ku^{2p+4}(BV(2)) = JU^{p+2} \oplus p^3.$$
$$ku^{2p+6}(BV(2)) = JU^{p+3} \oplus p^6.$$
$$ku^{2p+8}(BV(2)) = JU^{p+4} \oplus p^{10}.$$
$$ku^{2p+10}(BV(2)) = JU^{p+5} \oplus p^{15}.$$
$$ku^{2p+12}(BV(2)) = JU^{p+6} \oplus p^{21}.$$

- If we have $V(3)$, then $\overline{TU} \cong \overline{TU}^3 = \overline{TU}_2^3 \oplus \overline{TU}_3^3$, where $\overline{TU}_3^3 \cong PC(-2p-3)$ with $PC = \mathbb{F}_p[y_1, y_2, y_3]$, and

$$\begin{aligned} TU_3 &= \overline{TU}_3^3 \oplus \Sigma^{-2}\overline{TU}_3^3 \oplus \Sigma^{-4}\overline{TU}_3^3 \oplus \cdots \oplus \Sigma^{-2p+4}\overline{TU}_3^3 \\ &= PC(-2p-3) \oplus PC(-2p-5) \oplus PC(-2p-7) \oplus \cdots \oplus PC(-4p+1). \end{aligned}$$

Degree	$PC(-2p-3)$	$PC(-2p-5)$	$PC(-2p-7)$	$\cdots$	$PC(-4p+1)$	TU_3
$-2p-3$	p^1	0	0	$\cdots$	0	p^1
$-2p-5$	p^3	p^1	0	$\cdots$	0	p^4
$-2p-7$	p^6	p^3	p^1	$\cdots$	0	p^{10}
$-2p-9$	p^{10}	p^6	p^3	$\cdots$	0	p^{20}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-4p+1$	$p^{\binom{p}{2}}$	$p^{\binom{p-1}{2}}$	$p^{\binom{p-2}{2}}$	$\cdots$	p^1	$p^{\binom{p+1}{3}}$
$-4p-1$	$p^{\binom{p+1}{2}-1}$	$p^{\binom{p}{2}-1}$	$p^{\binom{p-1}{2}-1}$	$\cdots$	p^2	$p^{\binom{p+2}{3}-p}$
$-4p-3$	$p^{\binom{p+2}{2}-3}$	$p^{\binom{p+1}{2}-3}$	$p^{\binom{p}{2}-3}$	$\cdots$	p^3	$p^{\binom{p+3}{3}-(3p+1)}$

Figure 2.6: TU_3 for rank 3

Note that the final column as in Figure 2.6 only gives the odd degree part of $ku^*(BV(3))$ which is TU_3, and

$$\begin{aligned} ku^{2p+3}(BV(3)) &= p^1. \\ ku^{2p+5}(BV(3)) &= p^4. \\ ku^{2p+7}(BV(3)) &= p^{10}. \\ ku^{2p+9}(BV(3)) &= p^{20}. \\ ku^{2p+11}(BV(3)) &= p^{35}. \\ ku^{2p+13}(BV(3)) &= p^{56}. \\ ku^{2p+15}(BV(3)) &= p^{84}. \\ &\vdots \end{aligned}$$

As in rank 2, the module QU can be done by calculating

$$\begin{aligned} K^{-2n}(BV(3)) &= \mathbb{Z} \oplus JU_p^\wedge \\ &= \mathbb{Z} \oplus \langle c_1(\alpha_i) \,|\, i = 1, 2, 3\rangle_p^\wedge, \text{ for } n \geq 0. \end{aligned}$$

and for $n \geq 1$

$$QU^{-2n}(BV(3)) = (JU_p^\wedge)^n.$$

We can now describe the whole of $ku^*(BV(3))$ as follows.

The odd degree part is TU_3 as described above, while the even degree part is the sum of QU, also described above, and TU_2. Combining the splitting of ku in (1.1) with the values given in Figure 2.4 for $lu^{ev}(BV(3))_{tors} = \overline{TU}_2^3$, gives the values of TU below by summing $p-1$ shifts of $\overline{TU}$. For small values of p, for example, we have the following :

If $p=3$, we then have

$$\begin{aligned}
ku^{2p+2}(BV(3)) &= JU^{p+1} \oplus p^{3}.\\
ku^{2p+4}(BV(3)) &= JU^{p+2} \oplus p^{11}.\\
ku^{2p+6}(BV(3)) &= JU^{p+3} \oplus p^{23}.\\
ku^{2p+8}(BV(3)) &= JU^{p+4} \oplus p^{38}.
\end{aligned}$$

If $p=5$, we then have

$$\begin{aligned}
ku^{2p+2}(BV(3)) &= JU^{p+1} \oplus p^{3}.\\
ku^{2p+4}(BV(3)) &= JU^{p+2} \oplus p^{11}.\\
ku^{2p+6}(BV(3)) &= JU^{p+3} \oplus p^{26}.\\
ku^{2p+8}(BV(3)) &= JU^{p+4} \oplus p^{50}.\\
ku^{2p+10}(BV(3)) &= JU^{p+5} \oplus p^{82}.\\
ku^{2p+12}(BV(3)) &= JU^{p+6} \oplus p^{121}.
\end{aligned}$$

If $p=7$, we then have

$$\begin{aligned}
ku^{2p+2}(BV(3)) &= JU^{p+1} \oplus p^{3}.\\
ku^{2p+4}(BV(3)) &= JU^{p+2} \oplus p^{11}.\\
ku^{2p+6}(BV(3)) &= JU^{p+3} \oplus p^{26}.\\
ku^{2p+8}(BV(3)) &= JU^{p+4} \oplus p^{50}.\\
ku^{2p+10}(BV(3)) &= JU^{p+5} \oplus p^{85}.\\
ku^{2p+12}(BV(3)) &= JU^{p+6} \oplus p^{133}.\\
ku^{2p+14}(BV(3)) &= JU^{p+7} \oplus p^{193}.\\
ku^{2p+16}(BV(3)) &= JU^{p+8} \oplus p^{264}.
\end{aligned}$$

Degree	Rank 1	Rank 2	Rank 3 (p=3)	Rank 3 (p=5)	Rank 3 (p=7)
4	RU	RU	RU	RU	RU
3	0	0	0	0	0
2	RU	RU	RU	RU	RU
1	0	0	0	0	0
0	RU	RU	RU	RU	RU
-1	0	0	0	0	0
-2	JU	JU	JU	JU	JU
-3	0	0	0	0	0
-4	JU^2	JU^2	JU^2	JU^2	JU^2
-5	0	0	0	0	0
-6	JU^3	JU^3	JU^3	JU^3	JU^3
-7	0	0	0	0	0
-8	JU^4	JU^4	JU^4	JU^4	JU^4
-9	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-2p$	JU^p	JU^p	JU^p	JU^p	JU^p
$-2p-2$	JU^{p+1}	$JU^{p+1} \oplus p^1$	$JU^{p+1} \oplus p^3$	$JU^{p+1} \oplus p^3$	$JU^{p+1} \oplus p^3$
$-2p-3$	0	0	p^1	p^1	p^1
$-2p-4$	JU^{p+2}	$JU^{p+2} \oplus p^3$	$JU^{p+2} \oplus p^{11}$	$JU^{p+2} \oplus p^{11}$	$JU^{p+2} \oplus p^{11}$
$-2p-5$	0	0	p^4	p^4	p^4
$-2p-6$	JU^{p+3}	$JU^{p+3} \oplus p^6$	$JU^{p+3} \oplus p^{23}$	$JU^{p+3} \oplus p^{26}$	$JU^{p+3} \oplus p^{26}$
$-2p-7$	0	0	p^{10}	p^{10}	p^{10}
$-2p-8$	JU^{p+4}	$JU^{p+4} \oplus p^{10}$	$JU^{p+4} \oplus p^{38}$	$JU^{p+4} \oplus p^{50}$	$JU^{p+4} \oplus p^{50}$
$-2p-9$	0	0	p^{20}	p^{20}	p^{20}
$-2p-10$	JU^{p+5}	$JU^{p+5} \oplus p^{15}$	$\vdots$	$JU^{p+5} \oplus p^{82}$	$JU^{p+5} \oplus p^{85}$
$-2p-11$	0	0	$\vdots$	p^{35}	p^{35}
$-2p-12$	JU^{p+6}	$JU^{p+6} \oplus p^{21}$	$\vdots$	$JU^{p+6} \oplus p^{121}$	$JU^{p+6} \oplus p^{133}$
$-2p-13$	0	0	$\vdots$	p^{56}	p^{56}
$-2p-14$	JU^{p+7}	$JU^{p+7} \oplus p^{28}$	$\vdots$	$\vdots$	$JU^{p+7} \oplus p^{193}$
$-2p-15$	0	0	$\vdots$	p^{84}	p^{84}
$-2p-16$	JU^{p+8}	$JU^{p+8} \oplus p^{36}$	$\vdots$	$\vdots$	$JU^{p+8} \oplus p^{264}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 2.7: $ku^*(BV(r))$ for $r = 1, 2, 3$.
The symbol p^k denotes an elementary abelian p-group of rank k.

Chapter 3

Free resolution of $\overline{TU}_n^r$

First, recall that $\overline{TU}^r$ is the PC-submodule of $H^*(BV(r);\mathbb{F}_p)$ generated by

$$q_S = Q_0Q_1(\tau_S) = \sum_{s<t}(-1)^{t-s}(y_s y_t^p - y_s^p y_t)\tau_{S\setminus\{s,t\}},$$

for subsets $S \subseteq \{1,2,\ldots,r\}$ with $|S| \geq 2$, and $\tau_S = \prod_{s\in S}\tau_s$, and that this is isomorphic to the v-power torsion module $\overline{TU}$ in $lu^*(BV(r))$ for $r \leq 3$ (and conjecturally for all r).

In this Chapter, we will work entirely with the purely algebraic object $\overline{TU}^r$. Next, recall the PC-submodules $\overline{TU}_s^r$.

The purpose of this Chapter is to create a free resolution for the submodule $\overline{TU}_s^r$ over the polynomial ring PC. This will in turn feed in to the calculation of the local cohomology of $\overline{TU}^r$ and show that it is very close to being Gorenstein.

Definition 3.0.1. The PC-submodule $\overline{TU}_s^r$ is defined as [10, page 95]

$$\overline{TU}_s^r := PC\{q_S \mid |S| = s\} \subseteq H^*(BV(r);\mathbb{F}_p),$$

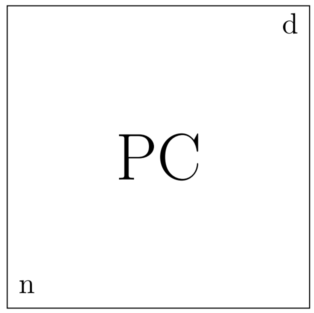
and is generated by the elements q_S with $|S| = s$.

Since q_S is a sum of monomials each involving exactly $s-2$ exterior generators, the same is true for every monomial in a term of $\overline{TU}_s^r$. Hence, in particular, the submodules $\overline{TU}_s^r$ have trivial intersection, and

$$\overline{TU}^r = \bigoplus_{s=2}^{r}\overline{TU}_s^r.$$

We will need to consider modules $\binom{r}{n}PC(-d)$, where $\binom{r}{n}$ is the binomial coefficient counting n-element subsets of a set with r elements. The notation $\binom{r}{n}PC(-d)$

indicates the free PC-module of rank $\binom{r}{n}$ on generators of degree $-d$. The principle is that a direct sum of n copies of $PC(-d)$ should be represented by a square:



§ 3.1 Pictures for resolutions

In this section, we want to display some pictures for free resolutions of $\overline{TU}_s^r$ using the following example.

Example 3.1.1. Considering rank 4 for definiteness, the resolutions of $\overline{TU}_4^4, \overline{TU}_3^4$, and $\overline{TU}_2^4$ are as follows, where we have indicated the names of the generators in the appropriate box. The letters Q, X, Y, and Z are the names of the generators of the free modules in the boxes defined as follows. $Q = Q_{ij}, X = X_{1234}, Y = Y_{1234}$, and $Z = Z_{1234}$ are as given in Figure 3.3.

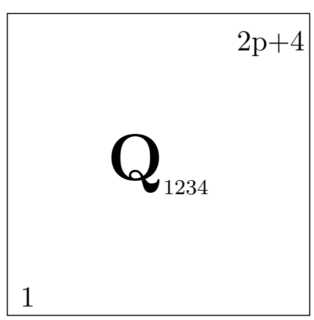


Figure 3.1: $\overline{TU}_4^4$

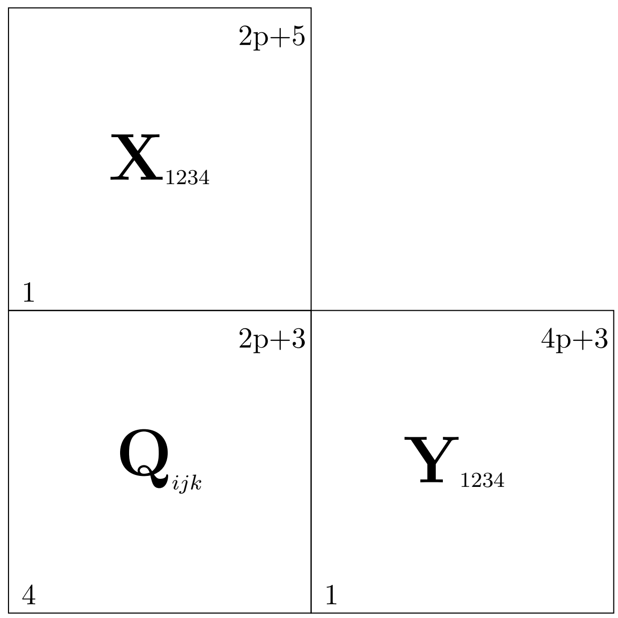


Figure 3.2: $\overline{TU}_3^4$

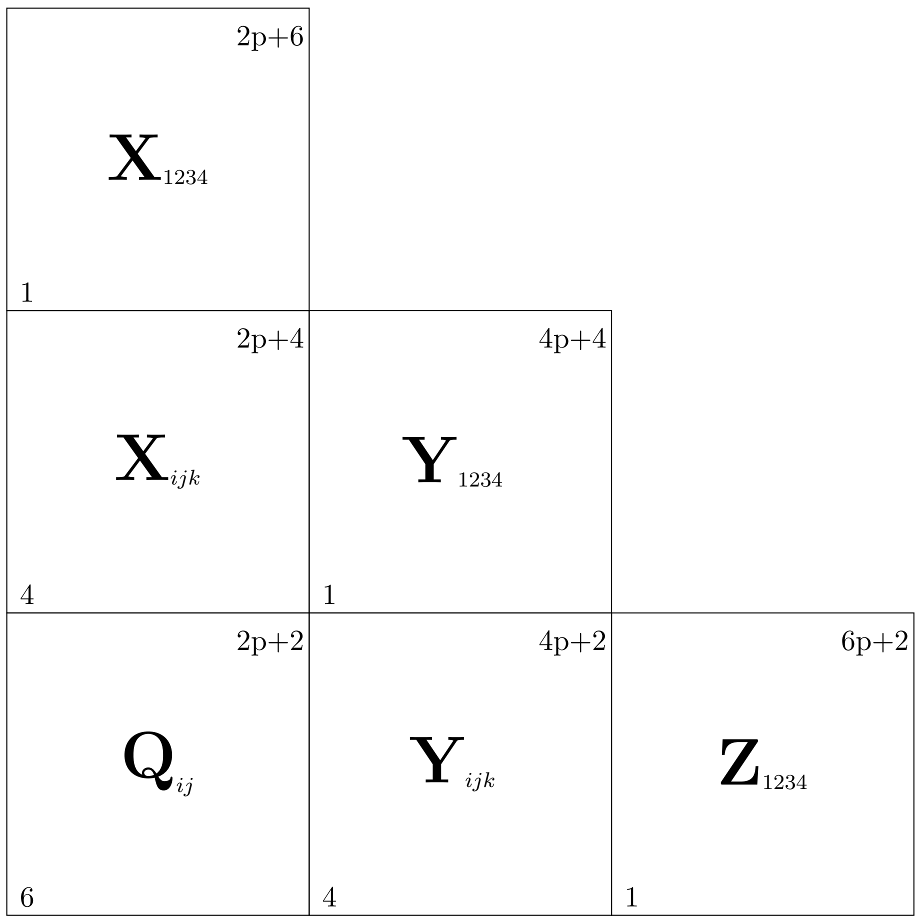


Figure 3.3: $\overline{TU}_2^4$

§ 3.2 The structure of $\overline{TU}$

In this section, we need only recall the structure of $\overline{TU}$, and, by Definition 3.0.1, that $\overline{TU}_s^r$ is the PC-submodule generated by elements $q_S = Q_0Q_1(\tau_S)$ with $|S| = s$. In fact, $\overline{TU}_r^r$ is a free module of rank 1 over PC on a generator of degree $-2p-r$. All other modules require more detailed analysis.

As we proved in Chapter 2, we obtain the following results.

Rank 1: If we have $V(1)$, then $\overline{TU} = 0$.

Rank 2: If we have $V(2)$, then $\overline{TU} \cong \overline{TU}^2 = \overline{TU}^2_2 \cong PC(-2p-2)$ is a free module of rank 1 over PC on a generator of degree $-2p-2$.

Rank 3: If we have $V(3)$, then $\overline{TU} \cong \overline{TU}^3 = \overline{TU}^3_2 \oplus \overline{TU}^3_3$, where $\overline{TU}^3_3 \cong PC(-2p-3)$ is a free module of rank 1 over PC on a generator of degree $-2p-3$.

Rank r: More generally, if we have $V(r)$, $r \geq 4$, then

$$\overline{TU}^r = \overline{TU}^r_2 \oplus \overline{TU}^r_3 \oplus \cdots \oplus \overline{TU}^r_r,$$

where $\overline{TU}^r_r \cong PC(-2p-r)$ is a free module of rank 1 over PC on a generator of degree $-2p-r$.

§ 3.3 A length one free resolution for $\overline{TU}^r_{r-1}$

The aim of this section is to introduce a free resolution for $\overline{TU}^r_{r-1}$ for any odd prime p. We do this separately so as to give full details in the first nontrivial case.

The following Proposition will be used.

Proposition 3.3.1.

$$(Q_1Q_0) - ann(H^*(BV(r))) = \operatorname{im}(Q_0) + \operatorname{im}(Q_1) + PC + \bigoplus_{i=1}^{r} PC\tau_i,$$

where for any $E(1)$-module M,

$$(Q_0Q_1) - ann(M) = \{x \in M |\ Q_0Q_1x = 0\}.$$

Proof. Details are given in [10, Proposition 4.2.11, page 86]. □

Proposition 3.3.2. *The PC-module $\overline{TU}^r_{r-1}$ has a free resolution of the form*

$$0 \longleftarrow \overline{TU}^r_{r-1} \xleftarrow{\pi} F_0 \xleftarrow{d} F_1 \longleftarrow 0, \tag{3.1}$$

where

$$F_0 = \binom{r}{r-1} PC(-(2p+r-1)),$$

with generators $Q_{\widehat{i}}$ for $1 \leq i \leq r$ where $\widehat{i} = \{1, 2, \ldots, r\} \setminus \{i\}$, and

$$F_1 = \binom{r}{r} [PC(-(2p+r+1)) \oplus PC(-(4p+r-1))]$$

with generators X and Y, and π is defined by

$$\pi(Q_{\widehat{i}}) = q_{\widehat{i}},$$

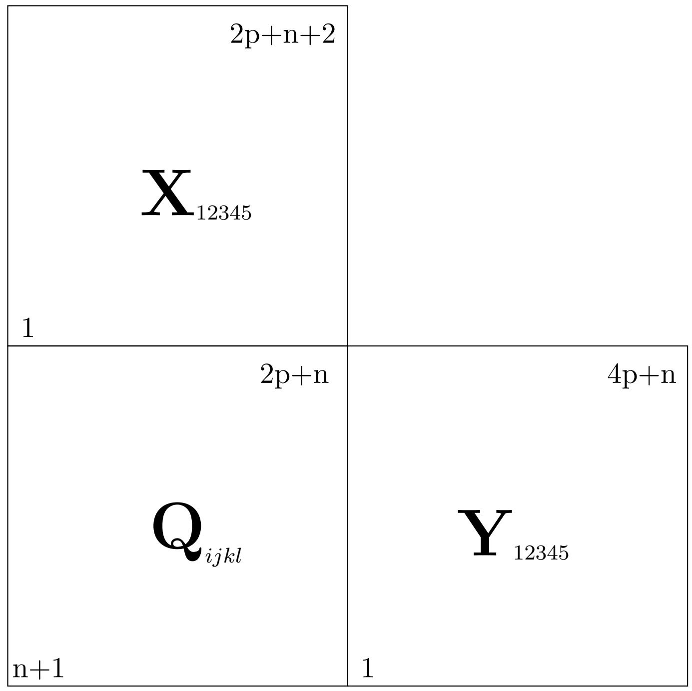


Figure 3.4: $\overline{TU}_{r-1}^r$

and d is defined by

$$d(X) = \sum_{i \in I} (-1)^{i|I} y_i Q_{\widehat{i}} \quad \textit{and} \quad d(Y) = \sum_{i \in I} (-1)^{i|I} y_i^p Q_{\widehat{i}}.$$

We write $i|I$ for the position α of $i \in I$, where $I = i_0 i_1 \dots i \dots i_n$ with $i = i_\alpha$.

Proof. First of all, by Definition 3.0.1, $\overline{TU}_{r-1}^r$ is the PC-module generated by elements $q_S = Q_0 Q_1(\tau_S)$ with $|S| = r-1$. This gives exactness at $\overline{TU}_{r-1}^r$. Here, we write $q_S = q_{\widehat{i}}$ when $S \cup \{i\} = \{1, 2, \dots, r\}$. Then we have $q_{\widehat{i}} = Q_0 Q_1(\tau_{\widehat{i}})$.

For exactness at F_0, we first notice that $\pi \circ d = 0$. We calculate

$$\begin{aligned}
\pi \circ d(X) &= \sum_{i \in I} (-1)^{i|I} Q_0(\tau_i) q_{\widehat{i}} \\
&= \sum_{i \in I} (-1)^{i|I} y_i Q_0 Q_1(\tau_{\widehat{i}}) \\
&= Q_0 Q_1 \left(\sum_{i \in I} (-1)^{i|I} y_i \tau_{\widehat{i}} \right) \\
&= Q_0 Q_1 \left(Q_0(\tau_{all}) \right) \\
&= 0.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\pi \circ d(Y) &= \sum_{i \in I} (-1)^{i|I} Q_1(\tau_i) q_{\widehat{i}} \\
&= \sum_{i \in I} (-1)^{i|I} y_i^p Q_0 Q_1(\tau_{\widehat{i}}) \\
&= Q_0 Q_1 \left(\sum_{i \in I} (-1)^{i|I} y_i^p \tau_{\widehat{i}} \right) \\
&= Q_0 Q_1 \left(Q_1(\tau_{all}) \right) \\
&= 0.
\end{aligned}$$

Suppose

$$\begin{aligned}
\pi\Big(\sum_S \beta_S Q_S \Big) &= \sum_S \beta_S \pi(Q_S) \\
&= \sum_S \beta_S q_S \\
&= 0
\end{aligned}$$

for suitable elements $\beta_S \in PC$. By PC-linearity of $Q_1 Q_0$, the last equality is of the form

$$\begin{aligned}
0 &= \sum_S \beta_S q_S \\
&= \sum_S \beta_S Q_0 Q_1(\tau_S) \\
&= Q_0 Q_1 (\sum_S \beta_S \tau_S).
\end{aligned}$$

Since the action of $Q_1 Q_0$ is PC-linear, the exactness at the next step comes from this action together with Proposition 3.3.1, that

$$\ker(Q_1 Q_0) = \mathrm{im}(Q_1) + \mathrm{im}(Q_0) + PC + \bigoplus_{i=1}^{r} PC\tau_i.$$

Since $\sum_S \beta_S \tau_S$ lies in the part of $H^*(BV(r); \mathbb{F}_p)$ spanned as a PC-module by τ_S with $|S| = i \geq 2$, we see that

$$\sum_S \beta_S \tau_S = Q_0(t_0) + Q_1(t_1),$$

for some t_0, t_1. Let us take $T=$ all. Then we find

$$\begin{aligned}
Q_0(\tau_{all}) &= Q_0(\tau_T) \\
&= \sum_{t \in T} (-1)^{t|T} y_t \tau_{T \setminus \{t\}}.
\end{aligned}$$

Next, we write t_0, t_1 in terms of the PC basis $\{\tau_T\}$, so that $t_0 = \sum_T \nu_T^0 \tau_T,\ t_1 = \sum_T \nu_T^1 \tau_T$ with the sums over T with $|T| = i+1$, and $\nu_T^0, \nu_T^1 \in PC$.

Then

$$\beta_S = \sum_{T=S\cup\{t\}} (-1)^{t|T}(y_t\nu_T^0 + y_t^p\nu_T^1),$$

and

$$\begin{aligned}
d\left(\sum_T \left(\nu_T^0 X + \nu_T^1 Y\right)\right) &= \sum_T \left(\nu_T^0 d(X) + \nu_T^1 d(Y)\right) \\
&= \sum_T \left(\nu_T^0 (\sum_{i\in I}(-1)^{i|I} y_t Q_{\widehat{i}}) + \nu_T^1(\sum_{i\in I}(-1)^{i|I} y_t^p Q_{\widehat{i}})\right) \\
&= \sum_T (-1)^{t|T}\Big(\sum_{T=S\cup\{t\}} y_t\nu_T^0 + y_t^p\nu_T^1\Big) Q_{\widehat{i}} \\
&= \sum_S \beta_S Q_S.
\end{aligned}$$

Finally, we prove that the first map of (3.1) is a monomorphism.

Suppose $d(fX+gY) = 0$, where $fX+gY$ is the smallest codegree for which this happens. Then

$$\begin{aligned}
0 &= d(fX+gY) \\
&= f(\sum_{i\in I}(-1)^{i|I} Q_0(\tau_i) Q_{\widehat{i}}) + g(\sum_{i\in I}(-1)^{i|I} Q_1(\tau_i) Q_{\widehat{i}}) \\
&= \sum_{i\in I}(-1)^{i|I} Q_{\widehat{i}}[y_i f + y_i^p g],
\end{aligned}$$

i.e., $y_i f + y_i^p g = 0$ for all i, so $f + y_i^{p-1} g = 0$, and $f \in \cap_i (y_i^{p-1})$. Now $\cap_i(y_i^{p-1}) = (y_1^{p-1}\dots y_r^{p-1})$, since y_i and y_j are coprime.

Therefore,

$$f = (y_1 \dots y_r)^{p-1}\bar{f}.$$

Now we obtain

$$(y_1\dots y_r)^{p-1}\bar{f} + y_i^{p-1} g = 0$$

and

$$(y_{\widehat{i}})^{p-1}\bar{f} + g = 0.$$

Hence, $g \in \cap_i (y_{\widehat{i}})^{p-1} = (y_1^{p-1}\dots y_r^{p-1})$, so $g = (y_1\dots y_r)^{p-1}\bar{g}$.

Let us define $(y_1\dots y_r)^{p-1} = \lambda$. Then we obtain

$$0 = d(\lambda\bar{f}X + \lambda\bar{g}Y) = \lambda d(\bar{f}X + \bar{g}Y)$$

Since λ is regular, we get $0 = d(\bar{f}X + \bar{g}Y)$. But $\operatorname{codeg}(\bar{f}X + \bar{g}Y) = \operatorname{codeg}(fX + gY) - 2(p-1)$. This is a contradiction.

The proposition is completely proved. $\square$

Now we have the start of a resolution visibly related to the Koszul complexes for the regular sequences $y_1, y_2, \ldots, y_r$ and $y_1^p, y_2^p, \ldots, y_r^p$.

§ 3.4 A free resolution for $\overline{TU}_n^r$ in general

In this section, we are going to construct a free resolution of $\overline{TU}_n^r$ over the polynomial ring $PC = \mathbb{F}_p[y_1, y_2, \ldots, y_r]$.

The key result in this Chapter is given as follows.

Proposition 3.4.1. *There is a resolution of $\overline{TU}_n^r$ by free PC-modules as follows:*

$$0 \longleftarrow \overline{TU}_n^r \longleftarrow \binom{r}{n} PC(-(2p+n)) \longleftarrow \binom{r}{n+1} PC(-(2p+n+2)) \oplus PC(-(4p+n))$$
$$\longleftarrow \binom{r}{n+2} PC(-(2p+n+4)) \oplus PC(-(4p+n+2)) \oplus PC(-(6p+n)) \longleftarrow \cdots$$

Thus the hth syzygy is

$$F_h = \binom{r}{n+h} [PC(-(2p+n+2h)) \oplus PC(-(4p+n+2h-2)) \oplus \cdots \oplus PC(-(2p+n+2ph))].$$

We name generators as follows. In homological degree h, for each subset $R \subseteq \{1, 2, \ldots, r\}$ with $(n+h)$ elements, there are generators $Q_R^0, Q_R^1, \ldots, Q_R^h$, with Q_R^i in degree $-(2p(i+1) + n + 2(h-i))$.

In homological degree 0, the generator $Q_S = Q_S^0$ maps to q_S, and for higher syzygies we think of Q_R^i as the ith letter in a sequence.

In positive homological degree, there are two differentials, d_0 and d_1, defined as

$$d_0(Q_R^i) = \sum_{r \in R} (-1)^{r|R} Q_0(\tau_r) Q_{R \setminus \{r\}}^i$$

and

$$d_1(Q_R^i) = \sum_{r \in R} (-1)^{r|R} Q_1(\tau_r) Q_{R \setminus \{r\}}^{i-1},$$

and the differential d is defined as the sum of the two differentials d_0 and d_1:

$$d(Q_R^i) = d_0(Q_R^i) + d_1(Q_R^i),$$

where $r|R$ is the position of r in R, counting from 0.

$$
\begin{tikzcd}
Q_V^0 \arrow[d, "d_0"'] & & & \\
Q_U^0 \arrow[d, "d_0"'] & Q_V^1 & & \\
Q_T^0 \arrow[d, "d_0"'] & Q_U^1 & Q_V^2 & \\
Q_S^0 & Q_T^1 \arrow[l, "d_1"] & Q_U^2 \arrow[l, "d_1"] & Q_V^3 \arrow[l, "d_1"]
\end{tikzcd}
$$

Figure 3.5: The double complex resolution for $\overline{TU}_{r-3}^r$ in rank r.

Now we have the ingredients to construct a diagram (Figure 3.5) exactly analogous to the one for the stable Koszul complex. To explain this diagram, let us take the following free resolution of $\overline{TU}_{r-3}^r$:

$$0 \longleftarrow \overline{TU}_{r-3}^r \xleftarrow{d} F_0 \xleftarrow{d} F_1 \xleftarrow{d} F_2 \xleftarrow{d} F_3 \longleftarrow 0.$$

We note that $F_0 = PC\{Q_S^0 \mid S \subseteq \{1,2,\dots,r\},\ |S| = r-3\}$ with $\deg(Q_S^0) = 2p+n$, $F_1 = PC\{Q_T^0, Q_T^1 \mid T \subseteq \{1,2,\dots,r\},\ |T| = r-2\}$ with $\deg(Q_T^0) = 2p+n+2$, and $\deg(Q_T^1) = 4p+n$.

Similarly for F_2, F_3, we find that $F_2 = PC\{Q_U^0, Q_U^1, Q_U^2 \mid U \subseteq \{1,2,\dots,r\},\ |U| = r-1\}$ with $\deg(Q_U^0) = 2p+n+4$, $\deg(Q_U^1) = 4p+n+2$, and $\deg(Q_U^2) = 6p+n$, and $F_3 = PC\{Q_V^0, Q_V^1, Q_V^2, Q_V^3 \mid V \subseteq \{1,2,\dots,r\},\ |V| = r\}$ with $\deg(Q_V^0) = 2p+n+6$, $\deg(Q_V^1) = 4p+n+4$, $\deg(Q_V^2) = 6p+n+2$, and $\deg(Q_V^3) = 8p+n$.

For the differential, with, for example, $Q_U^1 \in F_2$, we have

$$\begin{aligned} d(Q_U^1) &= d_0(Q_U^1) + d_1(Q_U^1) \\ &= \sum_{u\in U} (-1)^{u|U} y_u Q_{U\setminus\{u\}}^1 + \sum_{u\in U} (-1)^{u|U} y_u^p Q_{U\setminus\{u\}}^0. \end{aligned}$$

Similarly, with $Q_V^2 \in F_3$ we have

$$\begin{aligned} d(Q_V^2) &= d_0(Q_V^2) + d_1(Q_V^2) \\ &= \sum_{v\in V} (-1)^{v|V} y_v Q_{V\setminus\{v\}}^2 + \sum_{v\in V} (-1)^{v|V} y_v^p Q_{V\setminus\{v\}}^1. \end{aligned}$$

Now we return to the proof of Proposition 3.4.1.

Proof. The proof is based on the idea that the resolution is a truncation of the double Koszul complex based on the sequences $y_1, y_2, \ldots, y_r$ (vertically) and $y_1^p, y_2^p, \ldots, y_r^p$ (horizontally). In other words, we construct the above resolution as a truncation of an exact complex. We form the double Koszul complex K as the free PC-module on generators $\{Q_R^i\}$ of homological degree $0 \leq i \leq h$. Here, h indicates the homological degree.

First of all, we have to show that $d_0^2 = 0$, $d_1^2 = 0$, and $d_0 d_1 = -d_1 d_0$.

$$\begin{aligned}
d_0^2(Q_R^i) = d_0\left(d_0(Q_R^i)\right) &= d_0\Big(\sum_{r\in R}(-1)^{r|R}Q_0(\tau_r)Q_{R\setminus\{r\}}^i\Big)\\
&= \sum_{r\in R}(-1)^{r|R}y_r d_0(Q_{R\setminus\{r\}}^i)\\
&= \sum_{r\in R}(-1)^{r|R}y_r\sum_{s\neq r}(-1)^{s|R\setminus\{r\}}Q_0(\tau_s)(Q_{R\setminus\{r,s\}}^{i-1})\\
&= \sum_{r\in R}\sum_{s\neq r}y_r y_s(-1)^{r|R}(-1)^{s|R\setminus\{r\}}(Q_{R\setminus\{r,s\}}^{i-1})\\
&= \sum_{a\neq b}(y_a y_b - y_b y_a)(Q_{R\setminus\{a,b\}}^{i-1})\\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
d_1^2(Q_R^i) = d_1\left(d_1(Q_R^i)\right) &= d_1\Big(\sum_{r\in R}(-1)^{r|R}Q_1(\tau_r)Q_{R\setminus\{r\}}^{i-1}\Big)\\
&= \sum_{r\in R}(-1)^{r|R}y_r^p d_1(Q_{R\setminus\{r\}}^{i-1})\\
&= \sum_{r\in R}(-1)^{r|R}y_r^p\sum_{s\neq r}(-1)^{s|R\setminus\{r\}}Q_1(\tau_s)(Q_{R\setminus\{r,s\}}^{i-2})\\
&= \sum_{r\in R}\sum_{s\neq r}y_r^p y_s^p(-1)^{r|R}(-1)^{s|R\setminus\{r\}}(Q_{R\setminus\{r,s\}}^{i-2}).
\end{aligned}$$

We notice that $y_r^p y_s^p$ occurs twice, for $r = i$, $s = j$ and $r = j$, $s = i$, and that these have opposite signs. Then

$$\begin{aligned}
d_1^2(Q_R^i) &= \sum_{r\neq s}\Big((-1)^{r|R+s|R\setminus\{r\}} + (-1)^{s|R+r|R\setminus\{s\}}\Big)y_r^p y_s^p(Q_{R\setminus\{r,s\}}^{i-2})\\
&= \sum_{a\neq b}(y_a^p y_b^p - y_b^p y_a^p)(Q_{R\setminus\{a,b\}}^{i-2})\\
&= 0.
\end{aligned}$$

Finally, we calculate

$$\begin{aligned}
d_0 d_1(Q_R^i) &= d_0\Big(\sum_{r\in R}(-1)^{r|R} Q_1(\tau_r) Q^{i-1}_{R\setminus\{r\}}\Big)\\
&= \sum_{r\in R}(-1)^{r|R} y_r^p d_0(Q^{i-1}_{R\setminus\{r\}})\\
&= \sum_{r\in R}(-1)^{r|R} y_r^p \sum_{s\neq r}(-1)^{s|R\setminus\{r\}} Q_0(\tau_s)(Q^{i-2}_{R\setminus\{r,s\}})\\
&= \sum_{r\in R}\sum_{s\neq r} y_r^p y_s (-1)^{r|R}(-1)^{s|R\setminus\{r\}}(Q^{i-2}_{R\setminus\{r,s\}})\\
&= \sum_{r\neq s}\Big((-1)^{r|R+s|R\setminus\{r\}} + (-1)^{s|R+r|R\setminus\{s\}}\Big) y_r^p y_s (Q^{i-2}_{R\setminus\{r,s\}})\\
&= \sum_{a\neq b}(y_a^p y_b - y_a y_b^p)(Q^{i-2}_{R\setminus\{a,b\}})\\
&= -d_1 d_0(Q_R^i).
\end{aligned}$$

Since $d_0 d_1 = -d_1 d_0$, $d = d_0 + d_1$ is a differential. Now, the homological degree of Q_R^i is $|R| - i$, and it is convenient to display K as a double complex with Q_R^i at $(2i - |R|, 2|R| - 3i)$. This means that d_0 moves down one step and d_1 moves left one step as in Figure 3.5.

For definiteness, we give the rest of the argument for s even. The modifications for s odd are as given in [10, Proposition 4.6.3, page 96-98].

This suggests introducing a filtration by left half-planes :

$$\ldots \subseteq K_t \subseteq K_{t+1} \subseteq K_{t+2} \subseteq \ldots \subseteq K,$$

where

$$K_t = \langle Q_R^i \mid 2i - |R| \leq t\rangle.$$

This gives rise to a homological type spectral sequence of modules $(E^r_{p,q}, d^r)$ converging to $H_*(K_t)$, i.e.,

$$E^0_{p,q} = H_{p,q}(K_t, K_{t-1}),$$

standard in the homological grading, so that the differentials d_0 and d_1 defined above are named so as to fit the standard spectral sequence notation.

Note that by construction K_t/K_{t-1} is the Koszul complex for the sequence $y_1, \ldots, y_r$. Accordingly, since $y_1, y_2, \ldots, y_r$ is a regular sequence in PC, it follows that d_0 is exact except in the bottom nonzero degree in each column. Since this is in homological degree 0, there are no other differentials. We conclude that (K, d) is exact everywhere except in the 0th row.

Now, the proposed resolution $S = S(\overline{TU}_s^r)$ of $\overline{TU}_s^r$ is the quotient complex of K represented in the plane by the first quadrant with bottom corner generated by Q_R^i with $|R| = s$ (i.e., at $(2-s, 2s-2)$). By Proposition 3.3.2, we know that the bottom homology of S is $\overline{TU}^r_s$, and it remains to show that S is exact except at the bottom. We deduce this from the acyclicity of K.

We claim that $H_*(S) = H_0(S) = \overline{TU}^r_s$.

Now, consider the filtration of S, which is coming from the one on K,

$$\ldots \subseteq S_s \subseteq S_{s+1} \subseteq S_{s+2} \subseteq \ldots \subseteq S_r = S,$$

where

$$S_s = \langle Q^i_R \mid 2i - |R| \leq s \rangle.$$

Then we find

$$H_*\Big(H_*(\mathrm{Gr}(S_\bullet), Q_0), Q_1\Big) = H_*\Big(\operatorname{coker}(K_{t-1} \longrightarrow K_t), Q_1\Big).$$

Since S_t/S_{t-1} is a truncation of a Koszul complex, $E_1(S)$ is a chain complex C concentrated at the bottom edge, and a diagram chase establishes that d_1 is exact on C except at the bottom.

Suppose $t \in C$ is a d_1-cycle not in the bottom degree, we show that t is a d_1-boundary. By definition of C, $t = [\hat{t}]$ for some $\hat{t}$, where $[.]$ denotes d_0-homology classes. Since t is a cycle, there is $\hat{q}$ so that $d_1(\hat{t}) = \hat{r}$ and $d_0(\hat{q}) = \hat{r}$.

As we proved above, $d = d_0 + d_1$; we then obtain

$$d(\hat{t} + \hat{q}) = (d_0 + d_1)(\hat{t} + \hat{q}) = d_0(\hat{t}) + d_1(\hat{q}).$$

Since $d_0(\hat{t})$ and $d_1(\hat{q})$ are d-cycles, and we are above homological degree 0, there are two elements s, u with $ds = \hat{t}$ and $du = \hat{q}$. By the definition above and previous discussion, we find

$$\begin{aligned}
d(\hat{t} - \hat{q} + s - u) &= (d_0 + d_1)(\hat{t} - \hat{q} + s - u)\\
&= d_0(\hat{t} - \hat{q} + s - u) + d_1(\hat{t} - \hat{q} + s - u)\\
&= d_0(\hat{t}) - d_0(\hat{q}) + d_0 s - d_0 u + d_1(\hat{t}) - d_1(\hat{q}) + d_1 s - d_1 u\\
&= 0 - \hat{r} + d_0 s - d_0 u + \hat{r} - 0 + d_1 s - d_1 u\\
&= (d_0 + d_1)(s) - (d_0 + d_1)(u)\\
&= ds - du\\
&= \hat{t} - \hat{q}\\
&= 0.
\end{aligned}$$

Hence there is an element z with $dz = \hat{t} - \hat{q} + s - u$. Resolving z into its components we find $\hat{t} = d_0(\acute{z}) + d_1(\acute{\acute{z}})$, and so

$$t = [\hat{t}] = [\hat{t} + d_0(\acute{z})] = [d_1(\acute{\acute{z}})] = d_1[\acute{\acute{z}}]$$

as required. $\square$

We are going to investigate the local cohomology of $\overline{TU}^r_n$ in the next Chapter.

Chapter 4

Local Cohomology of $\overline{TU}_n^r$

This Chapter discusses the local cohomology of $\overline{TU}_i^r$. We will introduce the main theorem as an important result and prove it in Section 4.3, which describes that the PC-module $\overline{TU}_i^r$ has depth i and only has local cohomology in degrees r and i. Furthermore, the dual of $H^i_{\mathfrak{m}}(\overline{TU}_i^r)$ is only 1-dimensional rather than i-dimensional.

Therefore for any odd p, $\overline{TU}_i^r$ is extremely close to being Cohen-Macaulay, and it turns out that $\overline{TU}^r$ is very close to being Gorenstein. Indeed, we are working over a commutative local ring $(R, \mathfrak{m}, k)$, where $R = PC = \mathbb{F}_p[y_1, y_2, \ldots, y_r]$, $\mathfrak{m}$ is the maximal ideal of PC, and $k = R/\mathfrak{m}$ which is the field $\mathbb{F}_p$.

The principal tool we use in this calculation is local duality [25], which states that for any finitely generated R-module M,

$$H^i_{\mathfrak{m}}(M) = \mathrm{Ext}^{r-i}_{PC}(M, PC)^{\vee}(-d),$$

for some d independent of M.

Working with the dual of local cohomology allows us to measure the significance of local cohomology modules by their dimension.

§ 4.1 Koszul complexes and Local cohomology

For further details of this section see [25, page 6]. In this section, we define and use a stable Koszul complex to calculate the local cohomology of $\overline{TU}^r$. In fact, it calculates the right derived functors of the J-power torsion functor on graded ku^*-modules $M \longmapsto \Gamma_J(M)$, where

$$\Gamma_J(M) = \{m \in M \mid \text{for some } s > 0, J^s m = 0\}$$

on ku^*-modules M.

Before dealing with the local cohomology of $\overline{TU}^r$, we need to equip ourselves with Koszul complexes.

Definition 4.1.1. Let R be a commutative ring with unity and let $J = (x_1, \dots, x_n)$ be an ideal of R. The stable Koszul complex K of R at J is

$$K^{\infty}(x_1, x_2, \dots, x_n; R) = K^{\infty}(x_1; R) \otimes_R K^{\infty}(x_2; R) \otimes_R \cdots \otimes_R K^{\infty}(x_n; R),$$

the tensor product of cochain complexes, where $K^{\infty}(x_r; R)$ is the cochain complex $(R \longrightarrow R[\frac{1}{x_r}]), (t \longmapsto \frac{t}{1})$, for each $r \in \{1, 2, \dots, n\}$. The local cohomology of a module M over the ring R at J is as follows.

$$H^*_J(R; M) := H^*(K^{\infty}(x_1, x_2, \dots, x_n; R) \otimes_R M),$$

where $H^*(C)$ is the homology of a chain complex C.

In particular, we define

$$H^*_J(R) := H^*_J(R; R).$$

Observe that from the definition that $H^r_J(R; M) = 0$, for $r > n$.

Remark 4.1.2. Let R be a ring and (a) be an R-ideal. The cochain complex $K^{\infty}(a) = (R \longrightarrow R[\frac{1}{a}])$ has a natural map $\sigma : K^{\infty}(a) \longrightarrow R$. More precisely, there is a commutative diagram;

$$\begin{array}{ccccc}
K^{\infty} = & (R & \longrightarrow & R[\frac{1}{a}]) \\
\downarrow{\scriptstyle \sigma} & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
R = & (R & \longrightarrow & 0).
\end{array}$$

Hence, for any ideal $J = (x_1, x_2, \dots, x_i)$ and $I = (y_1, y_2, \dots, y_j)$ of R, there exists a map of chain complexes

$$1 \otimes \sigma^J : K^{\infty}(J + I) = K^{\infty}(J) \otimes_R K^{\infty}(I) \longrightarrow K^{\infty}(J) = K^{\infty}(J) \otimes_R R.$$

When we are applying $\otimes_R M$, M is an R-module, and taking homology, we obtain the map

$$\mu : H^s_{J+I}(R; M) \longrightarrow H^s_J(R; M).$$

We now give some examples to describe a Koszul complexes.

Example 4.1.3. If $R = \mathbb{Z}$ and $J = (2)$, we then have $K^{\infty}(2; \mathbb{Z}) = (\mathbb{Z} \longrightarrow \mathbb{Z}[\frac{1}{2}])$. It is clear that the map in this cochain complex is monomorphism and also the cokernel is not hard to calculate. That is,

$$H^r_{(2)}(\mathbb{Z}) = \begin{cases} \mathbb{Z}/2^{\infty}, & if \quad r = 1 \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathbb{Z}/2^{\infty} = \mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$.

Example 4.1.4. Let $R = k[x]$ be a polynomial ring over a field k with indeterminate x of degree r and $J = (x)$, we have $K^{\infty}(x; R) = K^{\infty}(x; k[x]) = (k[x] \longrightarrow k[x][\frac{1}{x}])$.

The calculation of $H^i_{(x)}(k[x])$ is easier if we look at the picture on the next page.

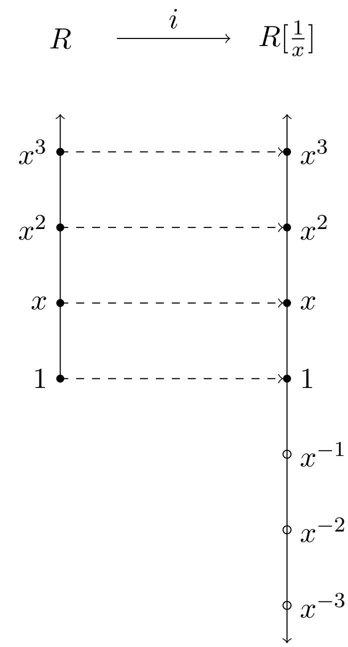


Figure 4.1: Koszul complex of $k[x]$ at (x).

This means the kernel of i is zero and the cokernel of i is $H^1_J(R) \cong R[x^{-1}]/R = k[x,x^{-1}]/k[x]$ which is $\Sigma^{-r}(k[x]^\vee)$, dual vector space of $k[x]$ shifted up by degree $-r$, where $k[x]^\vee := \mathrm{Hom}_k(k[x],k)$. It follows that

$$H^i_{(x)}(k[x]) = \begin{cases} \Sigma^{-r}(k[x]^\vee) = k[x,x^{-1}]/k[x], & if \quad \text{i=1} \\ 0, & \text{otherwise.} \end{cases}$$

Example 4.1.5. Let $R = k[x,y]$ be a polynomial ring over a field k with indeterminates x, y of degrees r, s and $J = (x,y)$, we have

$$K^\infty(J;R) = K^\infty(x;R) \otimes_R K^\infty(y;R) = (R \longrightarrow R[\frac{1}{x}] \oplus R[\frac{1}{y}] \longrightarrow R[\frac{1}{xy}]).$$

As in an Example 4.1.4, we illustrate the picture of Koszul complex for this ring as below.

$$R \xrightarrow{\{i,i\}} R[\tfrac{1}{x}] \oplus R[\tfrac{1}{y}] \xrightarrow{(i,-i)} R[\tfrac{1}{xy}].$$

From the Figure 4.3, as given in page 67, it is easy to see that this cochain complex is exact at the first and second term. Hence, $H^0_J(R)$ and $H^1_J(R)$ are zero. For the third term, the cokernel of $(i,-i)$ map is all the circle points in the third quadrant, which is isomorphic to $\Sigma^{-(r+s)}(k[x,y]^\vee)$. Therefore

$$H^i_J(R) = \begin{cases} 0, & if \quad \text{i=0;} \\ 0, & if \quad \text{i=1;} \\ \Sigma^{-(r+s)}(k[x,y]^\vee), & if \quad \text{i=2;} \\ 0, & otherwise. \end{cases}$$

The other definition of the local cohomology for a module M over a commutative ring R (with unity) with ideal J, is given by using functor $\Gamma_J(-)$, [25].

Definition 4.1.6. Let R be a Noetherian ring and let $J \subseteq R$ be an ideal. For an R-module M and a submodule $N \subseteq M$. Let

$$(N :_M J) := \{m \in M |\ Jm \in N, \forall\, J\}$$

Observe that, $(N :_M J)$ is a submodule of M and that $N \subseteq (N :_M J)$.

Definition 4.1.7. The J-torsion submodule of an R-module M is defined by

$$\Gamma_J(M) := \bigcup_{s\in\mathbb{N}} (0 :_M J^s) = \{m \in M |\, \exists\, s \in \mathbb{N} : J^s m = 0\},$$

where

$$(0 :_M J^s) \cong \mathrm{Hom}_R(R/J^s, M).$$

In fact, $H^r_J(.)$ is defined to be the rth right derived functor of Γ_J (i.e., it can be calculated by taking an injective resolution of M, applying Γ_J and taking cohomology). It is simple to show that $\Gamma_J(-)$ is an additive left exact covariant functor and thus,

$$R\Gamma^0_J(M) = \Gamma_J(M).$$

One can show that this definition and the previous definition coincide for a module over Noetherian ring (see, e.g., [25, page 7]).

Remark 4.1.8. If M is an R- module, then $H^*_J(M) = H^*_J(R; M)$ and $H^*_J(R) = H^*_J(R; R)$.

This applies in our case, since $ku^*(BG)$ is a Noetherian ring for a finite group G [10]. Elementary properties of this construction are the following:

Proposition 4.1.9. *Let R be a commutative Noetherian ring with unity, $J \lhd R$, and M is an R-module. The following holds.*

1. If K and N are R-modules such that $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ is a short exact sequence, then we have an induced long exact sequence

$$0 \longrightarrow H^0_J(K) \longrightarrow H^0_J(M) \longrightarrow H^0_J(N) \longrightarrow H^1_J(K) \longrightarrow H^1_J(M) \longrightarrow H^1_J(N) \longrightarrow \cdots.$$

2. For I an R-ideal, if $\sqrt{I} = \sqrt{J}$, then $H_J^r(M) = H_I^r(M)$ for all $r \geq 0$ and for all R-modules M.

3. For a Noetherian ring F, $\Phi : R \longrightarrow F$ a ring homomorphism and N an F-module, $H_J^r(N) \cong H_{JF}^r(N)$ for each r as F-modules.

4. Let Λ be a directed set and $\{M_\lambda\}_{\lambda \in \Lambda}$ a direct system of R-modules. Then
$$\lim_{\to\lambda} H_J^r(M_\lambda) \cong H_J^r(\lim_{\to\lambda} M_\lambda).$$

5. If F is an R-flat module, then $H_J^r(M) \otimes_R F = H_{JF}^r(M \otimes_R F)$.

6. If $(R, \mathfrak{m})$ is local, then $H_\mathfrak{m}^r(M) \cong H_{\mathfrak{m}\widehat{R}}^r(\widehat{R} \otimes_R M)$ which is isomorphic to $H_{\mathfrak{m}\widehat{R}}^r(\widehat{M})$ if M is finitely generated.

Proof. Details can be found, for example, in [25]. □

§ 4.2 Local duality

The aim of this section is to use the resolution in Proposition 3.4.1 to give an exact calculation of the local cohomology of the PC-module $\overline{TU}^r$, using local duality [25]. Since PC is a polynomial ring,
$$H_\mathfrak{m}^*(PC) = H_\mathfrak{m}^r(PC) = PC^\vee(2r),$$
where $(.)^\vee$ denotes graded vector space duality.

This immediately gives the answer for any finitely generated free module F, and noting the functoriality and behaviour of suspensions, this is
$$H_\mathfrak{m}^*(F) = H_\mathfrak{m}^r(F) = \mathrm{Hom}_{PC}(F, PC)^\vee(2r).$$

This means that a free resolution gives rise to a complex for calculating local cohomology.

Lemma 4.2.1. *(Local duality)If we have a free resolution*
$$0 \longleftarrow M \longleftarrow F_0 \longleftarrow \cdots \longleftarrow F_{r-1} \longleftarrow F_r \longleftarrow 0,$$
of the PC-module M, we obtain a complex
$$0 \longleftarrow H_\mathfrak{m}^r(F_0) \longleftarrow \cdots \longleftarrow H_\mathfrak{m}^r(F_{r-1}) \longleftarrow H_\mathfrak{m}^r(F_r) \longleftarrow 0$$
whose cohomology is the local cohomology of M:
$$H_\mathfrak{m}^i(M) = \mathrm{Ext}_{PC}^{r-i}(M, PC)^\vee(-2r).$$

Proof. Let $R = k[x_1, \ldots, x_r]$ be a polynomial ring over a field k with the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_r)$. Then

$$H^i_{\mathfrak{m}}(R) = \begin{cases} R^\vee(-2r), & \text{if } r = i \\ 0 & \text{if } r \neq i. \end{cases}$$

Now we have a free resolution of M

$$0 \longleftarrow M \longleftarrow F_0 \longleftarrow \cdots \longleftarrow F_{r-1} \longleftarrow F_r \longleftarrow 0.$$

Taking the local cohomology of this sequence gives us

$$\begin{aligned} H^i_{\mathfrak{m}}(M) &= H_{r-i}\Big(H^r_{\mathfrak{m}}(F_0) \longleftarrow H^r_{\mathfrak{m}}(F_1) \longleftarrow \cdots \longleftarrow H^r_{\mathfrak{m}}(F_r)\Big) \\ &= H_{r-i}\Big(\operatorname{Hom}_{PC}(F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_r, PC)^\vee\Big)(-2r) \\ &= \operatorname{Ext}^{r-i}_{PC}(M, PC)^\vee(-2r) \end{aligned}$$

as required. □

Since we are working with Noetherian graded-commutative local ring of dimension r, there are two types of Noetherian rings which are important in understanding, and using local cohomology. These are Cohen-Macaulay and Gorenstein rings. Goresnstein rings are Cohen-Macaulay, and both properties are defined using local cohomology.

We need to deal with modules M that are zero above a certain degree; the dual $M^\vee$ of such a module will therefore be zero below a certain degree. It is reasonable to write

$$Start(i)M^\vee$$

for the suspension of $M^\vee$ whose lowest nonzero degree is i.

The following remark helps to recognize the dual Koszul complex for our calculations in the next Chapter.

Remark 4.2.2. If F and F' are finitely generated free PC-modules and $\Theta : F \longrightarrow F'$ is represented by the matrix Θ (of elements of PC), then

$$H^r_{\mathfrak{m}}(\Theta)^\vee : H^r_{\mathfrak{m}}(F')^\vee \longrightarrow H^r_{\mathfrak{m}}(F)^\vee,$$

is represented by Θ^t (the transpose of Θ).

More precisely, we know that the Koszul complex is exact except in homological degree 0, and the original resolution which is proved as in Proposition 3.4.1, was a truncation of a double Koszul complex is represented by the map Θ. By local duality, we find the map $H^r_{\mathfrak{m}}(\Theta)^\vee$ occurs in the dual of a double Koszul complex which is again a double Koszul complex. Thus we find the matrix Θ^t also occurs in a double Koszul complex.

§ 4.3 General behaviour of the local cohomology of $\overline{TU}^r$

Further details are given in [10, page 87]. In this section we will describe the general behaviour of the J-local cohomology $H_J^*(M)$ of an R-module M, and then impose the conditions giving the best behaviour. The local cohomology modules vanish above the dimension d of the module M (i.e., above the Krull dimension of the ring $R/ann(M)$). On the other hand, the local cohomology $H_J^*(M)$ vanishes up to the J-depth of M, so that if there is an M-regular sequence of length ℓ in J we find $H_J^i(M) = 0$, for $i < \ell$, and the potentially non-zero modules are

$$H_J^\ell(M), H_J^{\ell+1}(M), \dots, H_J^d(M).$$

We are going to give a description for the general behaviour of the local cohomology of $\overline{TU}^r$. We recall that

$$\overline{TU}^r = \overline{TU}_2^r \oplus \overline{TU}_3^r \oplus \cdots \oplus \overline{TU}_r^r.$$

The following theorem is the main result of this section. In fact, this result gives us detailed and remarkable behaviour about the PC-module $\overline{TU}^r$.

Theorem 4.3.1. *The local cohomology of the modules $\overline{TU}_i^r$ is as follows.*

(1) $\overline{TU}_i^r$ only has local cohomology in degrees r and i.

(2) For $i = 2, 3, \dots, r-1$

$$H_{\mathfrak{m}}^r(\overline{TU}_i^r) = (\overline{TU}_{r-i+2}^r)^\vee(r-2p),$$

and

$$H_{\mathfrak{m}}^r(\overline{TU}_r^r) = PC^\vee(r-2p).$$

(3) The module $H_{\mathfrak{m}}^i(\overline{TU}_i^r)^\vee$ is 1-dimensional if $i < r$, and has Hilbert series

$$[H_{\mathfrak{m}}^i(\overline{TU}_i^r)^\vee] = t^{i-2}Y^{i-r-1}\frac{[(1+y+y^2+\cdots+y^{p-1})^r - Y^{r-i+1}]}{(1-Y)},$$

where $y = t^2, Y = y^{p-1} = t^{2p-2}$ and t is of degree -1.

Here, we wish to explain the dual of the Figure 3.5, (see Chapter 3). This explanation is useful for our Figures in the next Chapter. Its dual is given by

$$\begin{aligned}0 \longrightarrow H_{\mathfrak{m}}^r(\overline{TU}_{r-3}^r)^\vee \longrightarrow H_{\mathfrak{m}}^r(F_0)^\vee \longrightarrow H_{\mathfrak{m}}^r(F_1)^\vee \longrightarrow H_{\mathfrak{m}}^r(F_2)^\vee \longrightarrow H_{\mathfrak{m}}^r(F_3)^\vee \\ \longrightarrow H_{\mathfrak{m}}^{r-3}(\overline{TU}_{r-3}^r)^\vee \longrightarrow 0.\end{aligned}$$

We note that

$$H_{\mathfrak{m}}^r(F_0)^\vee = PC\{(Q_S^0)^* \mid S \subseteq \{1, 2, \dots, r\},\ |S| = r-3\},$$

with $\deg((Q_S^0)^*) = 2p - n - 2$, where $\{(Q_S^0)^*\}$ is dual to the basis $\{Q_S^0\}$ of F_0, and

$$H_{\mathfrak{m}}^r(F_1)^\vee = PC\{(Q_T^0)^*, (Q_T^1)^* \mid T \subseteq \{1, 2, \dots, r\},\ |T| = r-2\},$$

with $\deg((Q_T^0)^*) = 2p-n$, and $\deg((Q_T^1)^*) = 4p-n-2$, where $\{(Q_T^0)^*\}$, $\{(Q_T^1)^*\}$ are dual to the basis $\{Q_T^0\}$, $\{Q_T^1\}$ of F_1.

Similarly for $H^r_{\mathfrak{m}}(F_2)^\vee, H^r_{\mathfrak{m}}(F_3)^\vee$, we find that

$$H^r_{\mathfrak{m}}(F_2)^\vee = PC\{(Q_U^0)^*, (Q_U^1)^*, (Q_U^2)^* \mid U \subseteq \{1,2,\ldots,r\},\ |U| = r-1\},$$

with $\deg((Q_U^0)^*) = 2p-n$, $\deg((Q_U^1)^*) = 4p-n-2$, and $\deg((Q_U^2)^*) = 6p-n-4$, where $\{(Q_U^0)^*\}$, $\{(Q_U^1)^*\}$, $\{(Q_U^2)^*\}$ are dual to the basis $\{Q_U^0\}$, $\{Q_U^1\}$, $\{Q_U^2\}$ of F_2, and

$$H^r_{\mathfrak{m}}(F_3)^\vee = PC\{(Q_V^0)^*, (Q_V^1)^*, (Q_V^2)^*, (Q_V^3)^* \mid V \subseteq \{1,2,\ldots,r\},\ |V| = r\},$$

with $\deg((Q_V^0)^*) = 2p-n$, $\deg((Q_V^1)^*) = 4p-n-2$, $\deg((Q_V^2)^*) = 6p-n-4$, and $\deg((Q_V^3)^*) = 8p-n-6$, where $\{(Q_V^0)^*\}$, $\{(Q_V^1)^*\}$, $\{(Q_V^2)^*\}$, and $\{(Q_V^3)^*\}$ are dual to the basis $\{Q_V^0\}$, $\{Q_V^1\}$, $\{Q_V^2\}$, and $\{Q_V^3\}$ of F_3 respectively.

$$\begin{array}{cccccc}
 & \xleftarrow{d_1} & \xleftarrow{d_1} & \xleftarrow{d_1} & & \\
(Q_V^0)^* & (Q_U^0)^* & (Q_T^0)^* & (Q_S^0)^* & & \\
 & & & & \Big\downarrow d_0 & \\
 & (Q_V^1)^* & (Q_U^1)^* & (Q_T^1)^* & & \\
 & & & & \Big\downarrow d_0 & \\
 & & (Q_V^2)^* & (Q_U^2)^* & & \\
 & & & & \Big\downarrow d_0 & \\
 & & & (Q_V^3)^* & &
\end{array}$$

Figure 4.2: The double complex for the local cohomology of $\overline{TU}^r_{r-3}$ in rank r.

Remark 4.3.2. Since the dual of the local cohomology below the top degree is of dimension 1, $\overline{TU}^r_s$ is extremely close to being Cohen-Macaulay. Furthermore, if we take $\overline{TU}' = \bigoplus_{s=3}^{r-1} \overline{TU}^{r-1}_s$, then

$$H^r_{\mathfrak{m}}(\overline{TU}') = (\overline{TU}')^\vee(r-2p),$$

so that $\overline{TU}'$ is quasi-Gorenstein. Since the difference between $\overline{TU}^r_2$ and PC (as measured by $PC/\overline{TU}^r_2$) is similarly very small, $\overline{TU}^r$ is very close to being Gorenstein.

Proof. The strategy to calculate the local cohomology of $\overline{TU}^r_i$ is as follows. We consider the resolution $S = S(\overline{TU}^r_s)$ from the proof of 3.4.1. By Lemma 4.2.1, we need only reverse the direction of the arrows and change the suspensions to obtain a complex $S^\vee = (S(\overline{TU}^r_s))^\vee$ calculating the local cohomology. This will give an algorithm for calculating it.

Lemma 4.3.3. *The extreme local cohomology modules of $\overline{TU}^r_i$ are described as follows.*

(i) The bottom local cohomology $H^i_{\mathfrak{m}}(\overline{TU}^r_i)^\vee$ has a presentation as follows (where we write $q = 2p-2$):

$$\begin{aligned} &rPC(2p-i-2) \oplus rPC(2p-i-2+q) \oplus \cdots \oplus rPC(2r-i-2+(r-i-1)q) \longrightarrow \\ &PC(2p-i) \oplus PC(2p-i+q) \oplus \cdots \oplus PC(2p-i+(r-i)q) \longrightarrow H^i_{\mathfrak{m}}(\overline{TU}^r_i)^\vee \longrightarrow 0; \end{aligned}$$

In particular, it is generated by elements in degrees $2p-i, 2p-i+q, \ldots, 2p-i+2(r-i)q$.

(ii) The top local cohomology lies in an exact sequence

$$\begin{aligned} 0 \longrightarrow H^r_{\mathfrak{m}}(\overline{TU}^r_i)^\vee \xrightarrow{i} \binom{r}{i} PC(2p-2r+i) \longrightarrow &\binom{r}{i+1} PC(2p-2r+2+i) \\ \oplus &\binom{r}{i+1} PC(4p-2r+i), \end{aligned}$$

where the map i is described by

$$Q^*_S \longmapsto \sum_{r \notin S} (-1)^{r|S} \Big(Q_0(\tau_r) X^*_S + Q_1(\tau_r) Y^*_S \Big),$$

where Q^*_S are the generators in degree $2p-2r+i$ with $(r-i)$-elements and X^*_S, Y^*_S are the generators in degrees $2p-2r+2+i$, $4p-2r+i$ with $r-(i+1)$-elements.

To see that $\overline{TU}^r_i$ only has local cohomology in degrees r and i, we have seen that $\overline{TU}^r_i$ has a free resolution given by a truncation of the double Koszul complex in the previous chapter. Applying $\mathrm{Hom}_{PC}(\cdot, PC)$ we obtain another truncated double Koszul complex, since the dual of a Koszul complex is again a Koszul complex, see the terms of module as in Figure 4.2 The subquotients are truncated Koszul complexes for the regular sequence $y_1, y_2, \ldots, y_r$, and so they have homology only at the truncation point, giving a chain complex for the dual local cohomology which only has homology at the top and bottom [10, 4.7.3]. This completes the proof of Part (1).

Since $TU^r_r \cong PC(-r-2p)$, the statement about its local cohomology is immediate from $H^r_{\mathfrak{m}}(PC) = PC^\vee(2r)$.

Part (2) is therefore a consequence of the following Lemma.

Lemma 4.3.4. *There is a short exact sequence, for $2 \le i < r-1$,*

$$0 \longrightarrow \overline{TU}^r_{r-i+2} \xrightarrow{j} \binom{r}{r-i} PC(i-r) \longrightarrow \binom{r}{r-i-1} [PC(2+i-r) \oplus PC(2p+i-r)],$$

where the map j is described by

$$\tau(T) \longmapsto \sum_{t\in T}(-1)^{t|T\setminus\{t\}}\Big(Q_0(\tau_t)\xi_{T\setminus\{t\}} + Q_1(\tau_t)\eta_{T\setminus\{t\}}\Big),$$

where $\binom{r}{r-i}PC(i-r)$ *has basis* $\{\tau(T) \mid |T| = r-i\}$*, and* $\binom{r}{r-i-1}PC(2+i-r)$*,* $\binom{r}{r-i-1}PC(2p+i-r)$ *are generated by the elements* $\xi_{T\setminus\{t\}}, \eta_{T\setminus\{t\}}$ *in degrees* $2+i-r, 2p+i-r$ *respectively.*

Proof. First, observe that (for p odd) $H^*(BV(r);\mathbb{F}_p) = PC \otimes \Lambda(\tau_i \mid 1 \le i \le r)$, and then consider the PC-submodule Λ_{r-i} of $H^*(BV(r);\mathbb{F}_p)$ generated by the elements τ_T, with $|T| = r-i$, and note that it is a free PC-module isomorphic to $\binom{r}{r-i}PC(i-r)$. This explains the naming of the generators $\tau(T)$ in the statement.

Now, $\overline{TU}^r_{r-i+2}$ is the PC-submodule of $H^*(BV(r);\mathbb{F}_p)$ generated by the elements $q_S = Q_0Q_1(\tau_S)$ with $|S| = r-i+2$. Since

$$Q_0Q_1(\tau_S) = \sum_{s<t}(-1)^{t-s}(y_s y_t^p - y_s^p y_t)\tau_{S\setminus\{s,t\}} \in \Lambda_{r-i},$$

we see that $\overline{TU}^r_{r-i+2} \subseteq \Lambda_{r-i}$. We claim that $\overline{TU}^r_{r-i+2}$ is precisely the kernel of the above map. Indeed, under the correspondence $\tau(T) = \tau_T$, the first summand is Q_0 and the second is Q_1. The exactness of the sequence now follows exactly as in Proposition 3.3.2. It is a 0-sequence since $Q_0(q_S) = Q_0(Q_1Q_0(\tau_S)) = 0$ and $Q_1(q_S) = Q_1(Q_1Q_0(\tau_S)) = 0$. Thus, the exactness states that $\overline{TU}^r_{r-i+2} = \ker(Q_0) \cap \ker(Q_1)$. We find containment from the definition of the generators q_S. Our calculation of the $E(1)$-module structure showed $\ker(Q_0) \cap \ker(Q_1) = \mathrm{im}(Q_0Q_1) + PC$, from which the desired result follows. □

It remains to calculate the Hilbert series of $H^i_{\mathfrak{m}}(\overline{TU}^r_i)$, which we do in the next Section. □

§ 4.4 Hilbert series

Our goal in this section is to calculate the Hilbert series for $H^i_{\mathfrak{m}}(\overline{TU}^r_i)$. We shall be discussing Hilbert series of Noetherian modules over the polynomial ring PC. For a finitely generated $[PC]$-module M, the Hilbert series of M is written [10],

$$[M] = \sum_k t^k dim(M_{-k}),$$

where t is of degree -1. We let $y = t^2$, and $Y = y^{p-1}$, so that $yY = y^p$.

In any case, $[PC] = 1/(1-t^2)^r = 1/(1-y)^r$, and it is then immediate by taking a resolution of M that the Hilbert series takes the form $[M] = t^s p(t)/(1-y)^r$ for some integer s and polynomial $p(t)$.

Definition 4.4.1. For $0 \leq i \leq r$, we define truncations of the polynomial $(1-y)^r$ by

$$(1-y)^r_{[i]} = \binom{r}{i}(-y)^i + \binom{r}{i+1}(-y)^{i+1} + \cdots + \binom{r}{r}(-y)^r.$$

The identity $(-y)^r(1-(1/y))^r = (1-y)^r$ gives a duality property we need.

Lemma 4.4.2. *The truncated polynomials have the following duality property.*

$$(1-y)^r = (-y)^r(1-(1/y))^r_{[i]} + (1-y)^r_{[r-i+1]}. \quad \square$$

It is now rather straightforward to translate our structure results into statements about Hilbert series.

Lemma 4.4.3. *The Hilbert series of $\overline{TU}_i^r$ is given by*

$$[\overline{TU}_i^r] = (-t)^{2p-i}\frac{(1-y)^r_{[i]} - Y^{1-i}(1-yY)^r_{[i]}}{(1-y)^r(1-Y)}.$$

Proof. The resolution of Proposition 3.4.1, gives

$$\begin{aligned}(1-y)^r[\overline{TU}_i^r] &= \tbinom{r}{i}t^{2p+i} - \tbinom{r}{i+1}(t^{2p+2+i}+t^{4p+i}) + \tbinom{r}{i+2}(t^{2p+4+i}+t^{4p+2+i}+t^{6p+i}) - \cdots \\ &= (-t)^{2p-i}\Big[\tbinom{r}{i}(-y)^i - \tbinom{r}{i+1}(-y)^{i+1}(1+Y) + \tbinom{r}{i+2}(-y)^{i+2}(1+Y+Y^2) - \cdots\Big] \\ &= \frac{(-t)^{2p-i}}{1-Y}\Big[\tbinom{r}{i}(-y)^i(1-Y) - \tbinom{r}{i+1}(-y)^{i+1}(1-Y^2) + \tbinom{r}{i+2}(-y)^{i+2} \\ &\quad (1-Y^3) - \cdots\Big]\end{aligned}$$

as required. $\square$

From this, local duality and Theorem 4.3.1, we may deduce the Hilbert series of the dual local cohomology modules.

Lemma 4.4.4. *For $2 \leq i \leq r-1$, we have*

$$[H^r_{\mathfrak{m}}(\overline{TU}_i^r)^\vee] = (-1)^r[\overline{TU}_i^r](1/t) - (-1)^{-i+r}[H^i_{\mathfrak{m}}(\overline{TU}_i^r)^\vee].$$

Proof. We take a resolution F_* of $\overline{TU}_i^r$ (for example that given in Proposition 3.4.1). As in the proof of Lemma 4.4.3, we have $[\overline{TU}_i^r] = \chi([F_*])$. By local duality, the cohomology of the $2r$-th desuspension of the PC-dual of F_* is the dual local cohomology. Theorem 4.3.1, states that the local cohomology of $\overline{TU}$ is only in degrees r and i, so

$$\begin{aligned}[H^r_{\mathfrak{m}}(\overline{TU}_i^r)^\vee] + (-1)^{-i+r}[H^i_{\mathfrak{m}}(\overline{TU}_i^r)^\vee] &= y^r\chi([\mathrm{Hom}(F_*, PC)]) \\ &= (-1)^r\chi([F_*])(1/t) \\ &= (-1)^r[\overline{TU}_i^r](1/t)\end{aligned}$$

as required. $\square$

As a consequence, the Hilbert series of $H^i_{\mathfrak{m}}(\overline{TU}^r_i)^\vee$ now follows from Lemma 4.4.3.

Corollary 4.4.5. *For* $2 \leq i \leq r-1$, *we have*

$$\begin{aligned}
[H^i_{\mathfrak{m}}(\overline{TU}^r_i)^\vee] &= t^{i-2}Y^{i-r-1}\frac{\Big[(1-yY)^r - Y^{r-i+1}(1-y)^r\Big]}{(1-Y)(1-y)^r} \\
&= t^{i-2}Y^{i-r-1}\frac{\Big[(1+y+y^2+\cdots+y^{p-1})^r - Y^{r-i+1}\Big]}{(1-Y)}.
\end{aligned}$$

Remark 4.4.6. We find that the first form is more compact, but the second shows the pattern more clearly. Most importantly, it is apparent that the dimensions are bounded.

Proof. By combining Part(2) of Theorem 4.3.1, with Lemma 4.4.4, we find

$$(-1)^{r-i}[H^i_{\mathfrak{m}}(\overline{TU}^r_i)^\vee] = (-1)^r[\overline{TU}^r_i](1/t) - t^{r-2p}[\overline{TU}^r_{r-i+2}].$$

By Lemma 4.4.3,

$$\begin{aligned}
t^{r-2p}[\overline{TU}^r_{r-i+2}] &= (-1)^{r-i}t^{i-2}\frac{\Big[(1-y)^r_{[r-i+2]} - Y^{i-r-1}(1-yY)^r_{[r-i+2]}\Big]}{(1-y)^r(1-Y)} \\
&= (-1)^{r-i}t^{i-2}\frac{\Big[(1-y)^r_{[r-i+1]} - Y^{i-r-1}(1-yY)^r_{[r-i+1]}\Big]}{(1-y)^r(1-Y)}.
\end{aligned}$$

Now we use Lemma 4.4.3, and Lemma 4.4.2, to deduce

$$\begin{aligned}
(-1)^r[\overline{TU}^r_i](1/t) &= (-1)^r(-t)^{i-2p}\Big[(1-1/y)^r_{[i]} - Y^{i-1}(1-1/(yY))^r_{[i]}\Big]\Big/(1-1/y)^r(1-1/Y) \\
&= -Y(-t)^{i-2p}\Big[y^r(1-1/y)^r_{[i]} - y^rY^{i-1}(1-1/(yY))^r_{[i]}\Big]\Big/(1-y)^r(1-Y) \\
&= (-1)^{r+1}(-t)^{i-2}\Big[[(1-y)^r - (1-y)^r_{[r-i+1]}] - Y^{i-r-1} \\
&\qquad [(1-yY)^r - (1-yY)^r_{[r-i+1]}]\Big]\Big/(1-y)^r(1-Y) \\
&= (-1)^{r+1}(-t)^{i-2}\Big[[(1-y)^r - Y^{i-r-1}(1-yY)^r] \\
&\qquad - [(1-y)^r_{[r-i+1]} - Y^{i-r-1}(1-yY)^r_{[r-i+1]}]\Big]\Big/(1-y)^r(1-Y)
\end{aligned}$$

By subtracting $t^{r-2p}[\overline{TU}^r_{r-i+2}]$ we obtain the required result.

∎

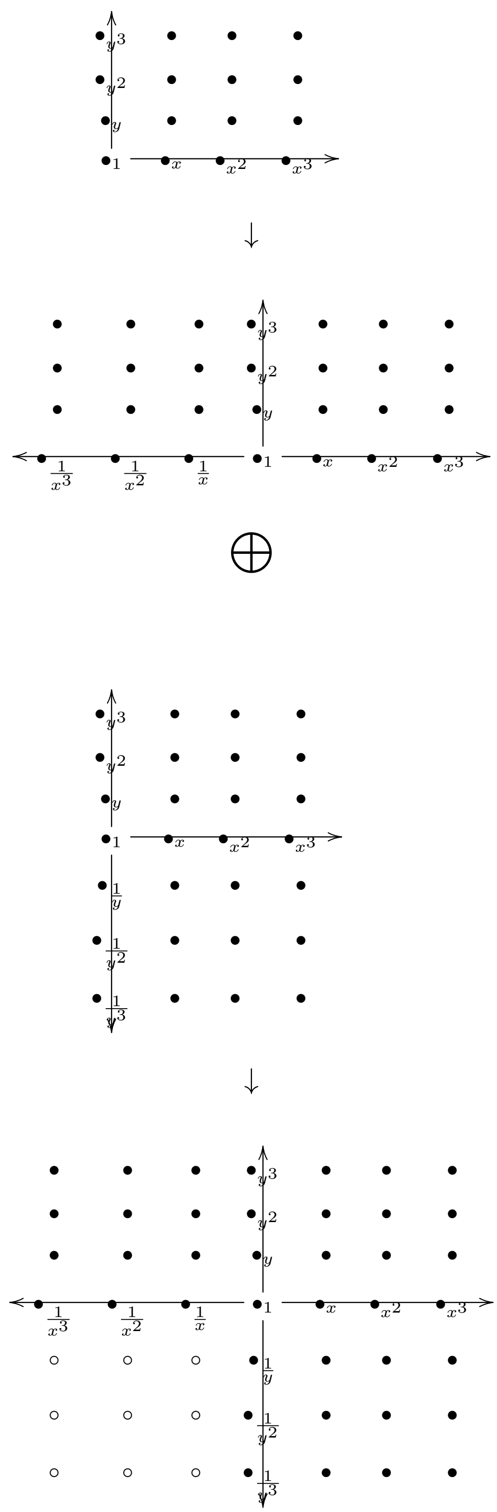


Figure 4.3: Koszul complex of $k[x, y]$ at (x, y).

Chapter 5

Examples

The main aim of this Chapter is to illustrate our results on the PC-submodule $\overline{TU}^r_n$ as defined, purely algebraically using $H^*(BV(r);\mathbb{F}_p)$ as an $E(1)$-module, in 3.0.1, and its local cohomology in ranks $r = 2, 3, 4, 5$ for all odd primes. First, by Proposition 3.4.1, we recall there is a resolution of $\overline{TU}^r_n$ by free PC-modules in general.

Second, by Theorem 4.3.1, $\overline{TU}^r_i$ only has local cohomology in degrees r and i with

$$H^r_{\mathfrak{m}}(\overline{TU}^r_r) = PC^\vee(-2p + r)$$

and

$$H^r_{\mathfrak{m}}(\overline{TU}^r_i) = (\overline{TU}^r_{r-i+2})^\vee(-2p + r)$$

for $i = 2, 3, \ldots, r-1$.

§ 5.1 The general pattern

The purpose of this section is to explain the organization of our calculations and give a general pattern with the consequences for $\overline{TU}^r_n$. We begin with the PC-module $\overline{TU}^r_r$, we have $\overline{TU}^r_r \cong PC(-2p - r)$, as in Section 2.10, is a free module of rank 1 over PC on a generator of degree $-2p - r$.

We apply Theorem 4.3.1, Part(2). Its local cohomology is given by

$$H^r_{\mathfrak{m}}(\overline{TU}^r_r) = PC^\vee(-2p + r).$$

For the PC-module $\overline{TU}^r_{r-1}$. First, by Proposition 3.3.2, the module $\overline{TU}^r_{r-1}$ has a free resolution of the form

$$0 \longleftarrow \overline{TU}^r_{r-1} \longleftarrow \tbinom{r}{r-1}\underbrace{PC(-(2p+r-1))}_{Q_{\widehat{i}}} \xleftarrow{\langle d_0, d_1\rangle} \tbinom{r}{r}[\underbrace{PC(-(2p+r+1))}_{X} \oplus$$
$$\underbrace{PC(-(4p+r-1))}_{Y}] \longleftarrow 0.$$

The generators $Q_{\widehat{i}}$ are of $\binom{r}{r-1}PC(-(2p+r-1))$. The differential is given by

$$d_0(X) = \sum_{i\in I}(-1)^{i|I}y_iQ_{\widehat{i}} \quad \text{and} \quad d_1(Y) = \sum_{i\in I}(-1)^{i|I}y_i^pQ_{\widehat{i}},$$

for $1 \le i \le r$, where $\widehat{i} = \{1,2,\dots,r\} \setminus \{i\}$.

By Remark 4.2.2, we can represent the map $\langle d_0, d_1\rangle \colon F_1 \longrightarrow F_0$ with respect to the chosen generators by a matrix Θ.

Next, for local cohomology of $\overline{TU}^r_{r-1}$, we obtain an exact sequence

$$\begin{aligned} 0 \longleftarrow H^r_{\mathfrak{m}}(\overline{TU}^r_{r-1}) \longleftarrow \tbinom{r}{r-1}PC^\vee(-2p+r+1) \longleftarrow \tbinom{r}{r}[PC^\vee(-2p+r-1)\oplus \\ PC^\vee(-4p+r+1)] \longleftarrow H^{r-1}_{\mathfrak{m}}(\overline{TU}^r_{r-1}) \longleftarrow 0. \end{aligned}$$

Dualizing, we obtain an exact sequence

$$\begin{aligned} 0 \longrightarrow \big(H^r_{\mathfrak{m}}(\overline{TU}^r_{r-1})\big)^\vee \longrightarrow \tbinom{r}{r-1}\overbrace{PC(2p-r-1)}^{(Q_{\widehat{i}})^*} \xrightarrow{\Theta^t} \tbinom{r}{r}\overbrace{[PC(2p-r+1)}^{X^*}\oplus\overbrace{PC(4p-r-1)]}^{Y^*} \\ \longrightarrow \big(H^{r-1}_{\mathfrak{m}}(\overline{TU}^r_{r-1})\big)^\vee \longrightarrow 0. \end{aligned}$$

Here, the indicated generators $(Q_{\widehat{i}})^*$ of $\binom{r}{r-1}PC(2p-r-1)$, where $\widehat{i} = \{1,2,\dots,r\} \setminus \{i\}$, are dual to the generators $Q_{\widehat{i}}$ of Proposition 3.3.2 under local duality and similarly for X^*, Y^*.

By using local duality, the dual map $H^r_{\mathfrak{m}}(F_0)^\vee \longleftarrow H^r_{\mathfrak{m}}(F_1)^\vee$ is represented with respect to the dual generators by a matrix Θ^t.

Finally, in order to get the results in this calculation, we know that the Koszul complex is exact except in homological degree 0, and since the dual of a Koszul complex is again a Koszul complex, then we find that the matrix Θ^t also occurs in the dual Koszul complex.

By using Theorem 4.3.1, our calculations give two things at any rank r:

$$(H^r_{\mathfrak{m}}(\overline{TU}^r_{r-1}))^\vee = \ker(\Theta^t) = (\overline{TU}^r_{r-i+2})(2p-r), \tag{5.1}$$

and

$$(H^{r-1}_{\mathfrak{m}}(\overline{TU}^r_{r-1}))^\vee = \operatorname{coker}(\Theta^t) \tag{5.2}$$

is generated in degrees $2p-r+1$ and $4p-r-1$. From the above exact sequence, its Hilbert series satisfies

$$\begin{aligned} [(H^{r-1}_{\mathfrak{m}}(\overline{TU}^r_{r-1}))^\vee] = [(H^r_{\mathfrak{m}}(\overline{TU}^r_{r-1}))^\vee] + [PC(2p-r+1)] + [PC(4p-r-1)] \\ - r[PC(2p-r-1)]. \end{aligned} \tag{5.3}$$

Similarly, we do the same calculation as above for the PC-modules $\overline{TU}^r_{r-2}, \overline{TU}^r_{r-3}$.

For the PC-module $\overline{TU}^r_{r-2}$, we find by Proposition 3.4.1 that it admits a resolution

$$0 \longleftarrow \overline{TU}^r_{r-2} \longleftarrow \binom{r}{r-2}\underbrace{PC(-(2p+r-2))}_{Q_{\widehat{ij}}} \longleftarrow \binom{r}{r-1}[\underbrace{PC(-(2p+r))}_{X_{\widehat{i}}} \oplus \underbrace{PC(-(4p+r-2))]}_{Y_{\widehat{i}}}$$
$$\longleftarrow \binom{r}{r}[\underbrace{PC(-(2p+r+2))}_{X} \oplus \underbrace{PC(-(4p+r))}_{Y} \oplus \underbrace{PC(-(6p+r-2))]}_{Z} \longleftarrow 0.$$

The generators $Q_{\widehat{ij}}$ are of $\binom{r}{r-2}PC(-(2p+r-2))$, and $X_{\widehat{i}}$, $Y_{\widehat{i}}$ are of $\binom{r}{r-1}PC(-(2p+r))$, $\binom{r}{r-1}PC(-(4p+r-2))$ respectively.

To see that $d^2 = 0$ as given on (page 52) of Chapter 3,

$$\begin{aligned}
d^2(X) &= \sum_{j\neq i}\Big((-1)^{i|I+j|I\setminus\{i\}} + (-1)^{j|I+i|I\setminus\{j\}}\Big)y_i y_j Q_{\widehat{ij}} = 0,\\
d^2(Y) &= \sum_{j\neq i}\Big((-1)^{i|I+j|I\setminus\{i\}} + (-1)^{j|I+i|I\setminus\{j\}}\Big)y_i^p y_j Q_{\widehat{ij}} = 0,\\
d^2(Z) &= \sum_{j\neq i}\Big((-1)^{i|I+j|I\setminus\{i\}} + (-1)^{j|I+i|I\setminus\{j\}}\Big)y_i^p y_j^p Q_{\widehat{ij}} = 0.
\end{aligned}$$

Now, we use the same argument by apply Theorem 4.3.1 this calculation records

$$(H^r_{\mathfrak{m}}(\overline{TU}^r_{r-2}))^\vee = \ker(\Theta^t) = (\overline{TU}^r_{r-i+2})(2p-r), \tag{5.4}$$

and

$$(H^{r-2}_{\mathfrak{m}}(\overline{TU}^r_{r-2}))^\vee = \operatorname{coker}(\Theta^t) \tag{5.5}$$

is generated in degrees $2p-r+2$, $4p-r$, and $6p-r-2$. Its Hilbert series satisfies

$$\begin{aligned}
[(H^{r-2}_{\mathfrak{m}}(\overline{TU}^r_{r-2}))^\vee] = \frac{r(r-1)}{2}&[PC(2p-r-2)] + [PC(4p-r+2)] + [PC(4p-r)]\\
&+ [PC(6p-r-2)] - [(H^r_{\mathfrak{m}}(\overline{TU}^r_{r-2}))^\vee] - r([PC(2p-r)]\\
&+ [PC(4p-r-2)]).
\end{aligned} \tag{5.6}$$

Finally, the PC-module $\overline{TU}^r_{r-3}$ has a free resolution of the form by Proposition 3.4.1

$$0 \longleftarrow \overline{TU}^r_{r-3} \longleftarrow \binom{r}{r-3}\underbrace{PC(-(2p+r-3))}_{Q_{\widehat{ijk}}} \longleftarrow \binom{r}{r-2}[\underbrace{PC(-(2p+r-1))}_{X_{\widehat{ij}}}$$
$$\oplus \underbrace{PC(-(4p+r-3))]}_{Y_{\widehat{ij}}} \longleftarrow \binom{r}{r-1}[\underbrace{PC(-(2p+r+1))}_{X_{\widehat{i}}} \oplus \underbrace{PC(-(4p+r-1))}_{Y_{\widehat{i}}} \oplus$$
$$\underbrace{PC(-(6p+r-3))]}_{Z_{\widehat{i}}} \longleftarrow \binom{r}{r}[\underbrace{PC(-(2p+r+3))}_{X} \oplus \underbrace{PC(-(4p+r+1))}_{Y}$$
$$\oplus \underbrace{PC(-(6p+r-1))}_{Z} \oplus \underbrace{PC(-(8p+r-3))]}_{W} \longleftarrow 0.$$

The generators $Q_{\widehat{ijk}}$ are of $\binom{r}{r-3}PC(-(2p+r-3))$, $X_{\widehat{ij}}$, $Y_{\widehat{ij}}$ are of $\binom{r}{r-2}PC(-(2p+r-1))$, $\binom{r}{r-2}PC(-(4p+r-3))$, and $X_{\widehat{i}}$, $Y_{\widehat{i}}$, $Z_{\widehat{i}}$ are of $\binom{r}{r-1}[PC(-(2p+r+1)) \oplus PC(-(4p+r-1)) \oplus PC(-(6p+r-3))]$, respectively.

To see that $d^2 = 0$ as given on (page 52) of Chapter 3,

$$\begin{aligned}
d^2(X_{\widehat{i}}) &= \sum_{k\neq j} \Big((-1)^{j|J+k|J\setminus\{j\}} + (-1)^{k|J+j|J\setminus\{k\}}\Big) y_j y_k Q_{\widehat{ijk}} = 0, \text{ for all } i \\
d^2(Y_{\widehat{i}}) &= \sum_{k\neq j} \Big((-1)^{j|J+k|J\setminus\{j\}} + (-1)^{k|J+j|J\setminus\{k\}}\Big) y_j^p y_k Q_{\widehat{ijk}} = 0, \text{ for all } i \\
d^2(Z_{\widehat{i}}) &= \sum_{k\neq j} \Big((-1)^{j|J+k|J\setminus\{j\}} + (-1)^{k|J+j|J\setminus\{k\}}\Big) y_j^p y_k^p Q_{\widehat{ijk}} = 0, \text{ for all } i.
\end{aligned}$$

As before, by Theorem 4.3.1 this calculation gives us

$$(H^r_{\mathfrak{m}}(\overline{TU}^r_{r-3})^{\vee} = \ker(\Theta^t) = (\overline{TU}^r_{r-i+2})(2p-r), \tag{5.7}$$

and

$$(H^{r-3}_{\mathfrak{m}}(\overline{TU}^r_{r-3}))^{\vee} = \operatorname{coker}(\Theta^t) \tag{5.8}$$

is generated in degrees $2p-r+3$, $4p-r+1$, $6p-r-1$, and $8p-r-3$. Its Hilbert series satisfies

$$\begin{aligned}
[(H^{r-3}_{\mathfrak{m}}(\overline{TU}^r_{r-3}))^{\vee}] = {} & [(H^{r}_{\mathfrak{m}}(\overline{TU}^r_{r-3}))^{\vee}] + \frac{r(r-1)}{2}([PC(2p-r-1)] + [PC(4p-r-3)]) \\
& - r([PC(2p-r+1)] + [PC(4p-r-1)] + [PC(6p-r-3)]) \\
& + [PC(2p-r+3)] + [PC(4p-r+1)] + [PC(6p-r-1)] \\
& + [PC(8p-r-3)] - \frac{r(r-1)(r-2)}{6}[PC(2p-r-3)]. \qquad (5.9)
\end{aligned}$$

§ 5.2 Rank 2

If we have $V(2)$, then $\overline{TU}^2 = \overline{TU}^2_2 \cong PC(-2p-2)$ and $\overline{TU}^2_2$ is a free module of rank 1 over PC on a generator of degree $-2p-2$. Its local cohomology is given by

$$H^r_{\mathfrak{m}}(\overline{TU}^r_r) = H^2_{\mathfrak{m}}(\overline{TU}^2_2) = PC^{\vee}(-2p+2).$$

§ 5.3 Rank 3

If we have $V(3)$, then we want to calculate the local cohomology of the PC-modules $\overline{TU}^3_3, \overline{TU}^3_2$. We have $\overline{TU}^3 = \overline{TU}^3_2 \oplus \overline{TU}^3_3$.

5.3.1 $\overline{TU}_3^3$

The PC-module $\overline{TU}_3^3 \cong PC(-2p-3)$ is a free module of rank 1 over PC on a generator of degree $-2p-3$. Its local cohomology is given by

$$H^r_{\mathfrak{m}}(\overline{TU}_r^r) = H^3_{\mathfrak{m}}(\overline{TU}_3^3) = PC^\vee(-2p+3).$$

5.3.2 $\overline{TU}_2^3$

By Proposition 3.3.2, the PC-module $\overline{TU}_2^3$ has a free resolution

$$0 \longleftarrow \overline{TU}_2^3 \longleftarrow \binom{3}{2} \underbrace{PC(-(2p+2))}_{(Q_{12},Q_{13},Q_{23})} \xleftarrow{\langle d_0,d_1\rangle} \binom{3}{3} [\underbrace{PC(-(2p+4))}_{X=X_{123}} \oplus \underbrace{PC(-(4p+2))}_{Y=Y_{123}}] \longleftarrow 0.$$

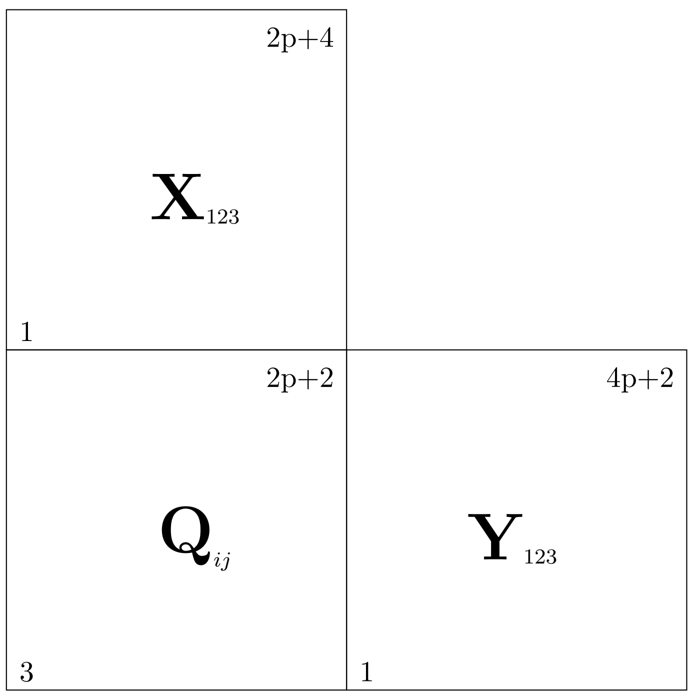


Figure 5.1: The double complex resolution for $\overline{TU}_2^3$.

When $i<j$, the generator Q_{ij} maps to $q_{ij} = y_i^p y_j - y_i y_j^p$. The generators satisfy the relations

$$y_3 q_{12} - y_2 q_{13} + y_1 q_{23} = 0 \quad \text{and} \quad d_0(X) = y_3 Q_{12} - y_2 Q_{13} + y_1 Q_{23}.$$

Similarly,

$$y_3^p q_{12} - y_2^p q_{13} + y_1^p q_{23} = 0 \quad \text{and} \quad d_1(Y) = y_3^p Q_{12} - y_2^p Q_{13} + y_1^p Q_{23}.$$

We calculate

$$\begin{aligned} q_{12} &= Q_0Q_1(\tau_{12}) = y_1^p y_2 - y_1 y_2^p. \\ q_{13} &= Q_0Q_1(\tau_{13}) = y_1^p y_3 - y_1 y_3^p. \\ q_{23} &= Q_0Q_1(\tau_{23}) = y_2^p y_3 - y_2 y_3^p. \end{aligned}$$

For calculational purposes, we observe that

$$\begin{aligned} y_3q_{12} - y_2q_{13} + y_1q_{23} &= y_3(y_1^py_2 - y_1y_2^p) - y_2(y_1^py_3 - y_1y_3^p) + y_1(y_2^py_3 - y_2y_3^p) \\ &= y_1y_2y_3\big[(y_1^{p-1} - y_2^{p-1}) - (y_1^{p-1} - y_3^{p-1}) + (y_2^{p-1} - y_3^{p-1})\big] \\ &= 0. \end{aligned}$$

Similarly, we find

$$\begin{aligned} y_3^pq_{12} - y_2^pq_{13} + y_1^pq_{23} &= y_3^p(y_1^py_2 - y_1y_2^p) - y_2^p(y_1^py_3 - y_1y_3^p) + y_1^p(y_2^py_3 - y_2y_3^p) \\ &= y_1^py_2^py_3^p\big[(\frac{1}{y_2^{p-1}} - \frac{1}{y_1^{p-1}}) - (\frac{1}{y_3^{p-1}} - \frac{1}{y_1^{p-1}}) + (\frac{1}{y_3^{p-1}} - \frac{1}{y_2^{p-1}})\big] \\ &= 0. \end{aligned}$$

We can now represent the map $\langle d_0, d_1\rangle \colon F_1 \longrightarrow F_0$ with respect to the chosen generators by a matrix

$$\Theta = \begin{array}{c} \\ Q_{12} \\ Q_{13} \\ Q_{23} \end{array} \begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{pmatrix} y_3 & y_3^p \\ -y_2 & -y_2^p \\ y_1 & y_1^p \end{pmatrix} \end{array}$$

For local cohomology of $\overline{TU}_2^3$, we obtain an exact sequence

$$\begin{aligned} 0 \longleftarrow H^3_{\mathfrak{m}}(\overline{TU}_2^3) \longleftarrow 3PC^\vee(-2p+4) \longleftarrow PC^\vee(-2p+2) \oplus PC^\vee(-4p+4) \\ \longleftarrow H^2_{\mathfrak{m}}(\overline{TU}_2^3) \longleftarrow 0. \end{aligned}$$

The original resolution was a truncation of a Koszul complex, and the same is true here.

Dualizing, we obtain an exact sequence

$$\begin{aligned} 0 \longrightarrow \big(H^3_{\mathfrak{m}}(\overline{TU}_2^3)\big)^\vee \longrightarrow \overbrace{3PC(2p-4)}^{(Q_{12}^*,Q_{13}^*,Q_{23}^*)} \xrightarrow{\Theta^t} \overbrace{PC(2p-2)}^{X^*} \oplus \overbrace{PC(4p-4)}^{Y^*} \longrightarrow \\ \big(H^2_{\mathfrak{m}}(\overline{TU}_2^3)\big)^\vee \longrightarrow 0. \end{aligned}$$

Here, the indicated generators $Q_{12}^*, Q_{13}^*, Q_{23}^*$ of $3PC(2p-4)$ are dual to the generators Q_{12}, Q_{13}, Q_{23} respectively of Proposition 3.3.2 under local duality, and similarly for X^*, Y^*.

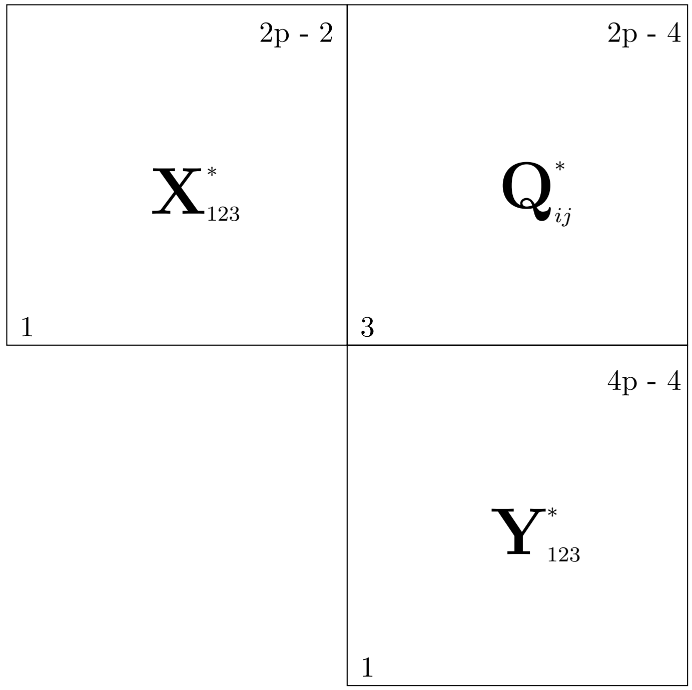


Figure 5.2: The double complex resolution for the local cohomology of $\overline{TU}^3_2$.

By local duality, the map $H^3_{\mathfrak{m}}(F_0)^\vee \longleftarrow H^3_{\mathfrak{m}}(F_1)^\vee$ is represented with respect to the dual generators by a matrix

$$\Theta^t = \begin{array}{c c} & \begin{array}{ccc} Q^*_{12} & Q^*_{13} & Q^*_{23} \end{array} \\ \begin{array}{c} X^* \\ Y^* \end{array} & \begin{pmatrix} y_3 & -y_2 & y_1 \\ y_3^p & -y_2^p & y_1^p \end{pmatrix} \end{array}$$

By Theorem 4.3.1, and (5.1), we deduce

$$\begin{aligned} (H^3_{\mathfrak{m}}(\overline{TU}^3_2))^\vee &= \ker \begin{pmatrix} y_3 & -y_2 & y_1 \\ y_3^p & -y_2^p & y_1^p \end{pmatrix} \\ &= (\overline{TU}^3_3)(2p-3) \\ &= (PC(-2p-3))(2p-3) \\ &= PC(-6), \end{aligned}$$

and by using (5.2),

$$(H^2_{\mathfrak{m}}(\overline{TU}^3_2))^\vee = \operatorname{coker} \begin{pmatrix} y_3 & -y_2 & y_1 \\ y_3^p & -y_2^p & y_1^p \end{pmatrix}$$

is generated in degrees $2p-2$ and $4p-4$. Its Hilbert series can be read off from the exact sequence (5.3),

$$\begin{aligned}
[H^2_{\mathfrak{m}}(\overline{TU}^3_2)^\vee] &= [PC(-6)] + [PC(2p-2)] + [PC(4p-4)] - 3[PC(2p-4)] \\
&= \frac{t^6}{(1-t^2)^3} + \frac{t^{-2p+2}}{(1-t^2)^3} + \frac{t^{-4p+4}}{(1-t^2)^3} - \frac{3t^{-2p+4}}{(1-t^2)^3} \\
&= \frac{t^{-4p+4}(t^{4p+2} + t^{2p-2} + 1 - 3t^{2p})}{(1-t^2)^3} \\
&= \frac{y^{-2p+2}(y^{2p+1} + y^{p-1} + 1 - 3y^p)}{(1-y)^3} \\
&= \frac{Y^{-2}(y^3Y^2 - 3yY + Y + 1)}{(1-y)^3},
\end{aligned}$$

where $y = t^2, Y = y^{p-1} = t^{2p-2}$ with t is of degree -1, and $yY = y^p$.

§ 5.4 Rank 4

If we have $V(4)$, then we calculate the local cohomology of the PC-modules $\overline{TU}^4_4, \overline{TU}^4_3, \overline{TU}^4_2$ by applying Theorem 4.3.1, Part(2). We have

$$\overline{TU}^4 = \overline{TU}^4_2 \oplus \overline{TU}^4_3 \oplus \overline{TU}^4_4.$$

5.4.1 $\overline{TU}^4_4$

The PC-module $\overline{TU}^4_4 \cong PC(-2p-4)$ is a free module of rank 1 over PC on a generator of degree $-2p-4$. Its local cohomology is given by

$$H^r_{\mathfrak{m}}(\overline{TU}^r_r) = H^4_{\mathfrak{m}}(\overline{TU}^4_4) = PC^\vee(-2p+4).$$

5.4.2 $\overline{TU}^4_3$

By Proposition 3.3.2, the resolution of PC-module $\overline{TU}^4_3$ is the short exact sequence

$$0 \longleftarrow \overline{TU}^4_3 \longleftarrow \binom{4}{3} \underbrace{PC(-(2p+3))}_{(Q_{123},Q_{124},Q_{134},Q_{234})} \xleftarrow{\langle d_0, d_1\rangle} \binom{4}{4} [\underbrace{PC(-(2p+5))}_{X=X_{1234}} \oplus \underbrace{PC(-(4p+3))}_{Y=Y_{1234}}] \longleftarrow 0.$$

The generators $Q_{123}, Q_{124}, Q_{134}, Q_{234}$ of $\binom{4}{3} PC(-(2p+3))$ satisfy the relations

$$y_4q_{123} - y_3q_{124} + y_2q_{134} - y_1q_{234} = 0 \quad \text{and} \quad d_0(X) = y_4Q_{123} - y_3Q_{124} + y_2Q_{134} - y_1Q_{234}.$$

Similarly,

$$y_4^pq_{123} - y_3^pq_{124} + y_2^pq_{134} - y_1^pq_{234} = 0 \quad \text{and} \quad d_1(Y) = y_4^pQ_{123} - y_3^pQ_{124} + y_2^pQ_{134} - y_1^pQ_{234}.$$

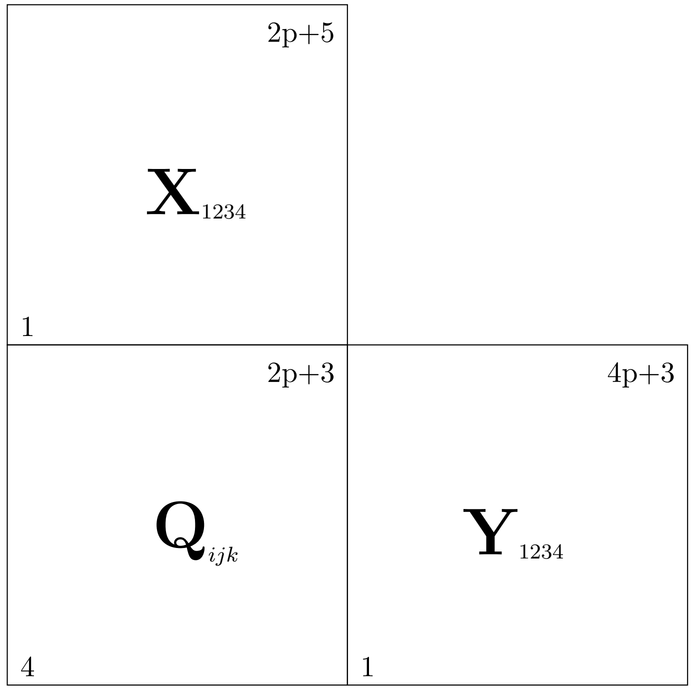


Figure 5.3: The double complex resolution for $\overline{TU}_3^4$.

We now calculate

$$
\begin{aligned}
q_{123} &= Q_0Q_1(\tau_{123}) = (y_1^py_2 - y_1y_2^p)\tau_3 - (y_1^py_3 - y_1y_3^p)\tau_2 + (y_2^py_3 - y_2y_3^p)\tau_1.\\
q_{124} &= Q_0Q_1(\tau_{124}) = (y_1^py_2 - y_1y_2^p)\tau_4 - (y_1^py_4 - y_1y_4^p)\tau_2 + (y_2^py_4 - y_2y_4^p)\tau_1.\\
q_{134} &= Q_0Q_1(\tau_{134}) = (y_1^py_3 - y_1y_3^p)\tau_4 - (y_1^py_4 - y_1y_4^p)\tau_3 + (y_3^py_4 - y_3y_4^p)\tau_1.\\
q_{234} &= Q_0Q_1(\tau_{234}) = (y_2^py_3 - y_2y_3^p)\tau_4 - (y_2^py_4 - y_2y_4^p)\tau_3 + (y_3^py_4 - y_3y_4^p)\tau_2.
\end{aligned}
$$

We can represent the map $\langle d_0, d_1\rangle\colon F_1 \longrightarrow F_0$ with respect to the chosen generators by a matrix

$$
\Theta = \begin{array}{c} \\ Q_{123} \\ Q_{124} \\ Q_{134} \\ Q_{234} \end{array}
\begin{array}{c} \begin{array}{cc} X & \;\; Y \end{array} \\ \begin{pmatrix} y_4 & y_4^p \\ -y_3 & -y_3^p \\ y_2 & y_2^p \\ -y_1 & -y_1^p \end{pmatrix} \end{array}
$$

For local cohomology of $\overline{TU}_3^4$, we obtain an exact sequence

$$
\begin{aligned}
0 \longleftarrow H^4_{\mathfrak{m}}(\overline{TU}_3^4) \longleftarrow 4PC^\vee(-2p+5) \longleftarrow PC^\vee(-2p+3) \oplus PC^\vee(-4p+5)\\
\longleftarrow H^3_{\mathfrak{m}}(\overline{TU}_3^4) \longleftarrow 0.
\end{aligned}
$$

Dualizing, we obtain an exact sequence

$$0 \longrightarrow \left(H^4_{\mathfrak{m}}(\overline{TU}^4_3)\right)^\vee \longrightarrow \overbrace{4PC(2p-5)}^{(Q^*_{123},Q^*_{124},Q^*_{134},Q^*_{234})} \xrightarrow{\Theta^t} [\overbrace{PC(2p-3)}^{X^*} \oplus \overbrace{PC(4p-5)}^{Y^*}] \longrightarrow \left(H^3_{\mathfrak{m}}(\overline{TU}^4_3)\right)^\vee \longrightarrow 0.$$

The indicated generators $Q^*_{123}, Q^*_{124}, Q^*_{134}, Q^*_{234}$ of $4PC(2p-5)$ are dual to the generators $Q_{123}, Q_{124}, Q_{134}, Q_{234}$ respectively of Proposition 3.3.2 under local duality, and similarly for X^*, Y^*.

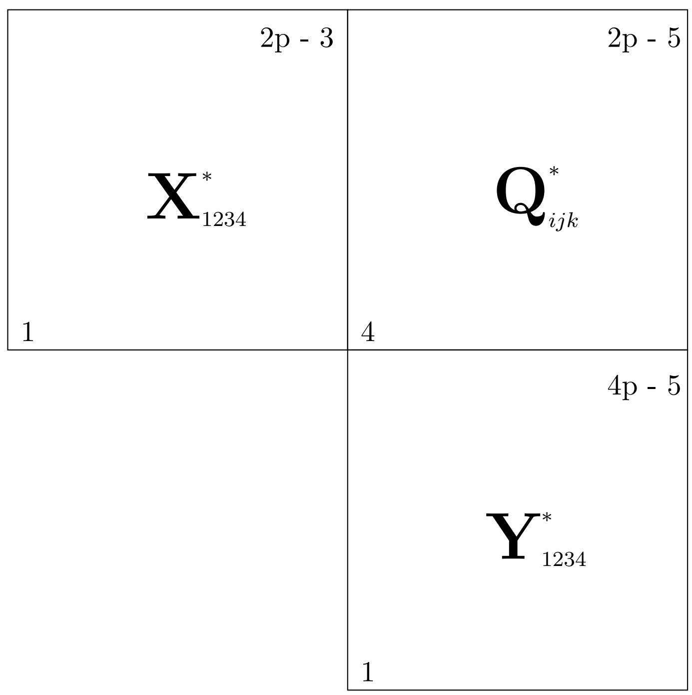


Figure 5.4: The double complex resolution for the local cohomology of $\overline{TU}^4_3$.

By local duality, the map $H^4_{\mathfrak{m}}(F_0)^\vee \longleftarrow H^4_{\mathfrak{m}}(F_1)^\vee$ is represented with respect to the dual generators by a matrix

$$\Theta^t = \begin{array}{c c} & \begin{array}{cccc} Q^*_{123} & Q^*_{124} & Q^*_{134} & Q^*_{234} \end{array} \\ \begin{array}{c} X^* \\ Y^* \end{array} & \begin{pmatrix} y_4 & -y_3 & y_2 & -y_1 \\ y_4^p & -y_3^p & y_2^p & -y_1^p \end{pmatrix} \end{array}$$

This calculation, by Theorem 4.3.1 and (5.1), records

$$\begin{aligned} (H^4_{\mathfrak{m}}(\overline{TU}^4_3))^\vee &= \ker \begin{pmatrix} y_4 & -y_3 & y_2 & -y_1 \\ y_4^p & -y_3^p & y_2^p & -y_1^p \end{pmatrix} \\ &= (\overline{TU}^4_3)(2p-4), \end{aligned}$$

and by (5.2),

$$(H^3_{\mathfrak{m}}(\overline{TU}^4_3))^\vee = \operatorname{coker}\begin{pmatrix} y_4 & -y_3 & y_2 & -y_1 \\ y_4^p & -y_3^p & y_2^p & -y_1^p \end{pmatrix}$$

is generated in degrees $2p-3$ and $4p-5$. Its Hilbert series can be read off from the exact sequence (5.3),

$$\begin{aligned}
[(H^3_{\mathfrak{m}}(\overline{TU}^4_3))^\vee] &= [\overline{TU}^4_3(2p-4)] + [PC(2p-3)] + [PC(4p-5)] - 4[PC(2p-5)] \\
&= \frac{4t^7 - t^9 - t^{2p+7} + t^{-2p+3} + t^{-4p+5} - 4t^{-2p+5}}{(1-t^2)^4} \\
&= \frac{t^{-4p+5}(4t^{4p+2} - t^{4p+4} - t^{6p+2} + t^{2p-2} + 1 - 4t^{2p})}{(1-t^2)^4} \\
&= \frac{tY^{-2}(4y^{2p+1} - y^{2p+2} - y^{3p+1} + y^{p-1} + 1 - 4y^p)}{(1-y)^4} \\
&= \frac{tY^{-2}(-y^4Y^3 - y^4Y^2 + 4y^3Y^2 - 4yY + Y + 1)}{(1-y)^4}.
\end{aligned}$$

5.4.3 $\overline{TU}^4_2$

By Proposition 3.4.1, the PC-module $\overline{TU}^4_2$ has a free resolution

$$\begin{aligned}
0 \longleftarrow \overline{TU}^4_2 \longleftarrow \binom{4}{2}\underbrace{PC(-(2p+2))}_{Q_{\widehat{ij}}} &\xleftarrow{\langle d_0, d_1\rangle} \binom{4}{3}[\underbrace{PC(-(2p+4))}_{X_{\widehat{i}}} \oplus \underbrace{PC(-(4p+2))}_{Y_{\widehat{i}}}] \\
\longleftarrow \binom{4}{4}[\underbrace{PC(-(2p+6))}_{X=X_{1234}} &\oplus \underbrace{PC(-(4p+4))}_{Y=Y_{1234}} \oplus \underbrace{PC(-(6p+2))}_{Z=Z_{1234}}] \longleftarrow 0.
\end{aligned}$$

The generators $Q_{\widehat{12}} = Q_{34}$, $Q_{\widehat{13}} = Q_{24}$, $Q_{\widehat{14}} = Q_{23}$, $Q_{\widehat{23}} = Q_{14}$, $Q_{\widehat{24}} = Q_{13}$, $Q_{\widehat{34}} = Q_{12}$ are of $\binom{4}{2} PC(-(2p+2))$, and the generators $X_{\widehat{1}} = X_{234}$, $X_{\widehat{2}} = X_{134}$, $X_{\widehat{3}} = X_{124}$, $X_{\widehat{4}} = X_{123}$, $Y_{\widehat{1}} = Y_{234}$, $Y_{\widehat{2}} = Y_{134}$, $Y_{\widehat{3}} = Y_{124}$, $Y_{\widehat{4}} = Y_{123}$ are of $\binom{4}{3} PC(-(2p+4))$, $\binom{4}{3} PC(-(4p+2))$ respectively.

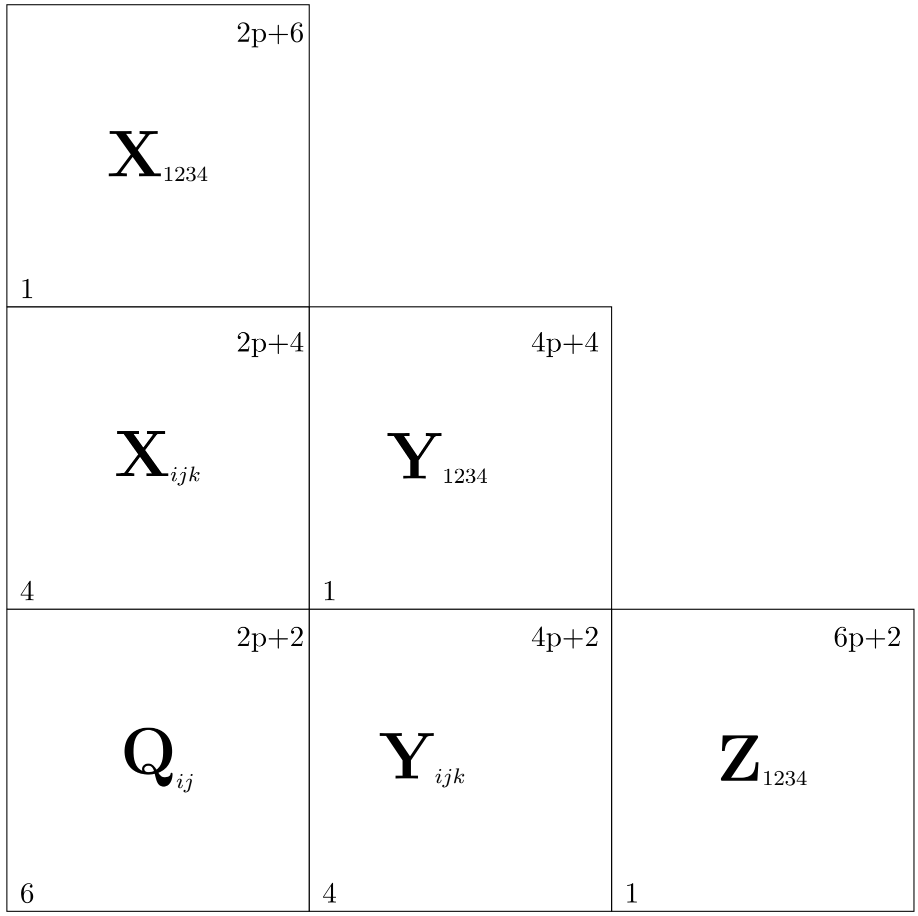


Figure 5.5: The double complex resolution for $\overline{TU}_2^4$.

We may verify $d^2 = 0$ as given on (page 52) of Chapter 3,

$$
\begin{aligned}
d^2(X) &= d^2(X_{1234}) \\
&= (y_1y_2 - y_2y_1)Q_{\widehat{12}} - (y_1y_3 - y_3y_1)Q_{\widehat{13}} + (y_1y_4 - y_4y_1)Q_{\widehat{14}} \\
&\quad - (y_2y_3 - y_3y_2)Q_{\widehat{23}} + (y_2y_4 - y_4y_2)Q_{\widehat{24}} - (y_3y_4 - y_4y_3)Q_{\widehat{34}} = 0, \\
d^2(Y) &= d^2(Y_{1234}) \\
&= (y_1^py_2 - y_1y_2^p)Q_{\widehat{12}} - (y_1^py_3 - y_1y_3^p)Q_{\widehat{13}} + (y_1^py_4 - y_1y_4^p)Q_{\widehat{14}} \\
&\quad - (y_2^py_3 - y_2y_3^p)Q_{\widehat{23}} + (y_2^py_4 - y_2y_4^p)Q_{\widehat{24}} - (y_3^py_4 - y_3y_4^p)Q_{\widehat{34}} = 0, \\
d^2(Z) &= d^2(Z_{1234}) \\
&= (y_1^py_2^p - y_2^py_1^p)Q_{\widehat{12}} - (y_1^py_3^p - y_3^py_1^p)Q_{\widehat{13}} + (y_1^py_4^p - y_4^py_1^p)Q_{\widehat{14}} \\
&\quad - (y_2^py_3^p - y_3^py_2^p)Q_{\widehat{23}} + (y_2^py_4^p - y_4^py_2^p)Q_{\widehat{24}} - (y_3^py_4^p - y_4^py_3^p)Q_{\widehat{34}} = 0.
\end{aligned}
$$

Note that the differentials $d_0(X_{\widehat{i}}) = \sum_{j \neq i} (-1)^{j|J} y_j Q_{\widehat{ij}}$ and $d_1(Y_{\widehat{i}}) = \sum_{j \neq i} (-1)^{j|J} y_j^p Q_{\widehat{ij}}$ are given by

$$
\begin{aligned}
d_0(X_{\widehat{1}}) = d_0(X_{234}) &= y_2 Q_{\widehat{12}} - y_3 Q_{\widehat{13}} + y_4 Q_{\widehat{14}}.\\
d_0(X_{\widehat{2}}) = d_0(X_{134}) &= y_1 Q_{\widehat{12}} - y_3 Q_{\widehat{23}} + y_4 Q_{\widehat{24}}.\\
d_0(X_{\widehat{3}}) = d_0(X_{124}) &= y_1 Q_{\widehat{13}} - y_2 Q_{\widehat{23}} + y_4 Q_{\widehat{34}}.\\
d_0(X_{\widehat{4}}) = d_0(X_{123}) &= y_1 Q_{\widehat{14}} - y_2 Q_{\widehat{24}} + y_3 Q_{\widehat{34}}.\\
d_1(Y_{\widehat{1}}) = d_1(Y_{234}) &= y_2^p Q_{\widehat{12}} - y_3^p Q_{\widehat{13}} + y_4^p Q_{\widehat{14}}.\\
d_1(Y_{\widehat{2}}) = d_1(Y_{134}) &= y_1^p Q_{\widehat{12}} - y_3^p Q_{\widehat{23}} + y_4^p Q_{\widehat{24}}.\\
d_1(Y_{\widehat{3}}) = d_1(Y_{124}) &= y_1^p Q_{\widehat{13}} - y_2^p Q_{\widehat{23}} + y_4^p Q_{\widehat{34}}.\\
d_1(Y_{\widehat{4}}) = d_1(Y_{123}) &= y_1^p Q_{\widehat{14}} - y_2^p Q_{\widehat{24}} + y_3^p Q_{\widehat{34}}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
q_{34} = Q_0 Q_1(\tau_{34}) &= y_3^p y_4 - y_3 y_4^p.\\
q_{24} = Q_0 Q_1(\tau_{24}) &= y_2^p y_4 - y_2 y_4^p.\\
q_{23} = Q_0 Q_1(\tau_{23}) &= y_2^p y_3 - y_2 y_3^p.\\
q_{14} = Q_0 Q_1(\tau_{14}) &= y_1^p y_4 - y_1 y_4^p.\\
q_{13} = Q_0 Q_1(\tau_{13}) &= y_1^p y_3 - y_1 y_3^p.\\
q_{12} = Q_0 Q_1(\tau_{12}) &= y_1^p y_2 - y_1 y_2^p.
\end{aligned}
$$

We can now represent the first map $F_1 \longrightarrow F_0$ with respect to the chosen generators by a matrix

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} X_{\widehat{1}} & X_{\widehat{2}} & X_{\widehat{3}} & X_{\widehat{4}} & Y_{\widehat{1}} & Y_{\widehat{2}} & Y_{\widehat{3}} & Y_{\widehat{4}} \end{array} \\
\begin{array}{c} Q_{\widehat{12}} \\ Q_{\widehat{13}} \\ Q_{\widehat{14}} \\ Q_{\widehat{23}} \\ Q_{\widehat{24}} \\ Q_{\widehat{34}} \end{array} &
\begin{pmatrix}
y_2 & y_1 & & & y_2^p & y_1^p & & \\
-y_3 & & y_1 & & -y_3^p & & y_1^p & \\
y_4 & & & y_1 & y_4^p & & & y_1^p \\
 & -y_3 & -y_2 & & & -y_3^p & -y_2^p & \\
 & y_4 & & -y_2 & & y_4^p & & -y_2^p \\
 & & y_4 & y_3 & & & y_4^p & y_3^p
\end{pmatrix}
\end{array}
$$

We can also represent the second map $F_2 \longrightarrow F_1$ with to respect to the chosen generators by a matrix

$$
\Theta = \begin{array}{cc}
 & \begin{array}{ccc} X & Y & Z \end{array} \\
\begin{array}{c} X_{\widehat{1}} \\ X_{\widehat{2}} \\ X_{\widehat{3}} \\ X_{\widehat{4}} \\ Y_{\widehat{1}} \\ Y_{\widehat{2}} \\ Y_{\widehat{3}} \\ Y_{\widehat{4}} \end{array} &
\begin{pmatrix}
y_1 & y_1^p & \\
-y_2 & -y_2^p & \\
y_3 & y_3^p & \\
-y_4 & -y_4^p & \\
 & y_1 & y_1^p \\
 & -y_2 & -y_2^p \\
 & y_3 & y_3^p \\
 & -y_4 & -y_4^p
\end{pmatrix}
\end{array}
$$

For local cohomology of $\overline{TU}^4_2$, we obtain an exact sequence,

$$0 \longleftarrow H^4_{\mathfrak{m}}(\overline{TU}^4_2) \longleftarrow \binom{4}{2} PC^\vee(-2p+6) \longleftarrow \binom{4}{3} [PC^\vee(-2p+4) \oplus PC^\vee(-4p+6)] \longleftarrow$$
$$\binom{4}{4} [PC^\vee(-2p+2) \oplus PC^\vee(-4p+4) \oplus PC^\vee(-6p+6)] \longleftarrow H^2_{\mathfrak{m}}(\overline{TU}^4_2) \longleftarrow 0.$$

Dualizing, we obtain an exact sequence,

$$0 \longrightarrow (H^4_{\mathfrak{m}}(\overline{TU}^4_2))^\vee \longrightarrow \overbrace{6PC(2p-6)}^{(Q_{\widehat{ij}})^*} \longrightarrow 4[\overbrace{PC(2p-4)}^{(X_{\widehat{i}})^*} \oplus \overbrace{PC(4p-6)}^{(Y_{\widehat{i}})^*}] \xrightarrow{\Theta^t}$$
$$[\overbrace{PC(2p-2)}^{X^*} \oplus \overbrace{PC(4p-4)}^{Y^*} \oplus \overbrace{PC(6p-6)}^{Z^*}] \longrightarrow (H^2_{\mathfrak{m}}(\overline{TU}^4_2))^\vee \longrightarrow 0.$$

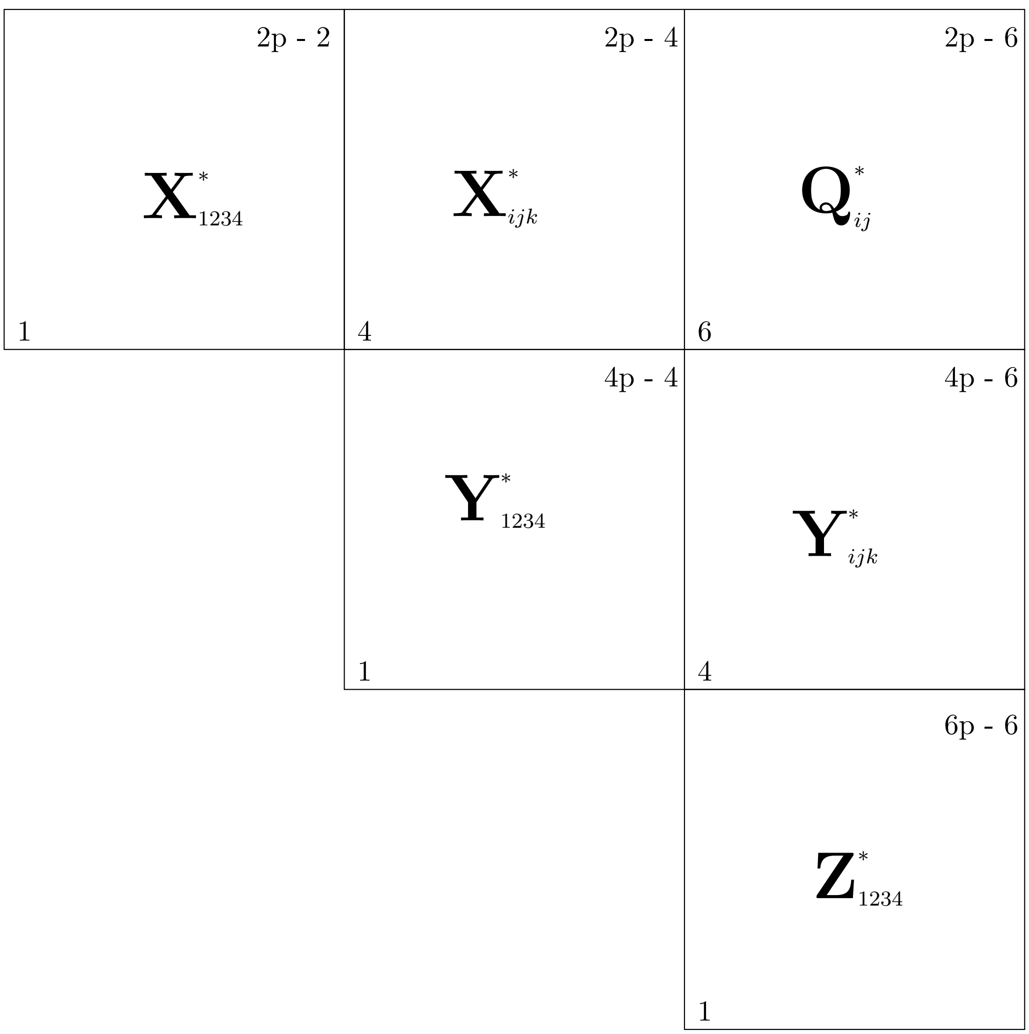


Figure 5.6: The double complex resolution for the local cohomology of $\overline{TU}^4_2$.

The indicated generators $(Q_{\widehat{ij}})^*$ of $6PC(2p-6)$ are dual to the generators $Q_{\widehat{ij}}$ so that $(Q_{\widehat{12}})^* = Q^*_{34}$, $(Q_{\widehat{13}})^* = Q^*_{24}$, $(Q_{\widehat{14}})^* = Q^*_{23}$, $(Q_{\widehat{23}})^* = Q^*_{14}$, $(Q_{\widehat{24}})^* = Q^*_{13}$, $(Q_{\widehat{34}})^* = Q^*_{12}$.

Also, the generators $(X_{\widehat{i}})^*, (Y^*_{\widehat{j}})$ of $4PC(2p-4)$, $4PC(4p-6)$ are dual to the generators $X_{\widehat{i}}, Y_{\widehat{i}}$ so that $(X_{\widehat{1}})^* = X^*_{234}, (Y_{\widehat{1}})^* = Y^*_{234}$, $(X_{\widehat{2}})^* = X^*_{134}, (Y_{\widehat{2}})^* = Y^*_{134}$, $(X_{\widehat{3}})^* = X^*_{124}, (Y_{\widehat{3}})^* = Y^*_{124}$, $(X_{\widehat{4}})^* = X^*_{123}, (Y_{\widehat{4}})^* = Y^*_{123}$ of Proposition 3.3.2 under local duality, and similarly for X^*, Y^*, Z^*.

By local duality, the map $H^4_{\mathfrak{m}}(F_1)^\vee \longleftarrow H^4_{\mathfrak{m}}(F_2)^\vee$ is represented with respect to the dual generators by a matrix

$$\Theta^t = \begin{array}{c} \\ X^* \\ Y^* \\ Z^* \end{array} \begin{array}{c} \begin{array}{cccccccc} (X_{\widehat{1}})^* & (X_{\widehat{2}})^* & (X_{\widehat{3}})^* & (X_{\widehat{4}})^* & (Y_{\widehat{1}})^* & (Y_{\widehat{2}})^* & (Y_{\widehat{3}})^* & (Y_{\widehat{4}})^* \end{array} \\ \begin{pmatrix} y_1 & -y_2 & y_3 & -y_4 & & & & \\ y_1^p & -y_2^p & y_3^p & -y_4^p & y_1 & -y_2 & y_3 & -y_4 \\ & & & & y_1^p & -y_2^p & y_3^p & -y_4^p \end{pmatrix} \end{array}$$

Hence, by Theorem 4.3.1, and (5.4), the first result is

$$\begin{aligned}(H^4_{\mathfrak{m}}(\overline{TU}^4_2))^\vee &= \ker \begin{pmatrix} y_1 & -y_2 & y_3 & -y_4 & & & & \\ y_1^p & -y_2^p & y_3^p & -y_4^p & y_1 & -y_2 & y_3 & -y_4 \\ & & & & y_1^p & -y_2^p & y_3^p & -y_4^p \end{pmatrix} \\ &= (\overline{TU}^4_4)(2p-4) \\ &= (PC(-2p-4))(2p-4) \\ &= PC(-8)\end{aligned}$$

and by (5.5), the second result

$$H^2_{\mathfrak{m}}(\overline{TU}^4_2)^\vee = \operatorname{coker} \begin{pmatrix} y_1 & -y_2 & y_3 & -y_4 & & & & \\ y_1^p & -y_2^p & y_3^p & -y_4^p & y_1 & -y_2 & y_3 & -y_4 \\ & & & & y_1^p & -y_2^p & y_3^p & -y_4^p \end{pmatrix}$$

is generated in degrees $2p-2$, $4p-4$, and $6p-6$. Its Hilbert series can be read off from the exact sequence (5.6),

$$\begin{aligned}[H^2_{\mathfrak{m}}(\overline{TU}^4_2)^\vee] &= 6[PC(2p-6)] + [PC(2p-2)] + [PC(4p-4)] + [PC(6p-6)] - [PC(-8)] \\ &\quad - 4([PC(2p-4)] + [PC(4p-6)]) \\ &= \frac{6t^{-2p+6} + t^{-2p+2} + t^{-4p+4} + t^{-6p+6} - t^8 - 4(t^{-2p+4} + t^{-4p+6})}{(1-t^2)^4} \\ &= \frac{t^{-6p+6}(6t^{4p} + t^{4p-4} + t^{2p-2} + 1 - t^{6p+2} - 4(t^{4p-2} + t^{2p}))}{(1-t^2)^4} \\ &= \frac{y^{-3p+3}(6y^{2p} + y^{2p-2} + y^{p-1} + 1 - y^{3p+1} - 4(y^{2p-1} + y^p))}{(1-y)^4} \\ &= \frac{y^{-3p+3}(-y^{3p+1} + 6y^{2p} - 4y^{2p-1} + y^{2p-2} - 4y^p + y^{p-1} + 1)}{(1-y)^4} \\ &= \frac{Y^{-3}(-y^4Y^3 + 6y^2Y^2 - 4yY^2 + Y^2 - 4yY + Y + 1)}{(1-y)^4}.\end{aligned}$$

§ 5.5 Rank 5

If we have $V(5)$, we want to calculate the local cohomology of the PC-modules $\overline{TU}^5_5, \overline{TU}^5_4, \overline{TU}^5_3, \overline{TU}^5_2$ by applying Theorem 4.3.1, Part (2). We have

$$\overline{TU}^5 = \overline{TU}^5_2 \oplus \overline{TU}^5_3 \oplus \overline{TU}^5_4 \oplus \overline{TU}^5_5.$$

5.5.1 $\overline{TU}^5_5$

The PC-module $\overline{TU}^5_5 \cong PC(-2p-5)$ is a free module of rank 1 over PC on a generator of degree $-2p-5$. Its local cohomology is given by

$$H^r_{\mathfrak{m}}(\overline{TU}^r_r) = H^5_{\mathfrak{m}}(\overline{TU}^5_5) = PC^\vee(-2p+5).$$

5.5.2 $\overline{TU}^5_4$

By Proposition 3.3.2, the PC-module $\overline{TU}^5_4$ has a free resolution

$$0 \longleftarrow \overline{TU}^5_4 \longleftarrow \binom{5}{4} \underbrace{PC(-(2p+4))}_{Q_{\widehat{i}}} \overset{\langle d_0, d_1 \rangle}{\longleftarrow} \binom{4}{4} [\underbrace{PC(-(2p+6))}_{X = X_{12345}} \oplus \underbrace{PC(-(4p+4))}_{Y = Y_{12345}}] \longleftarrow 0.$$

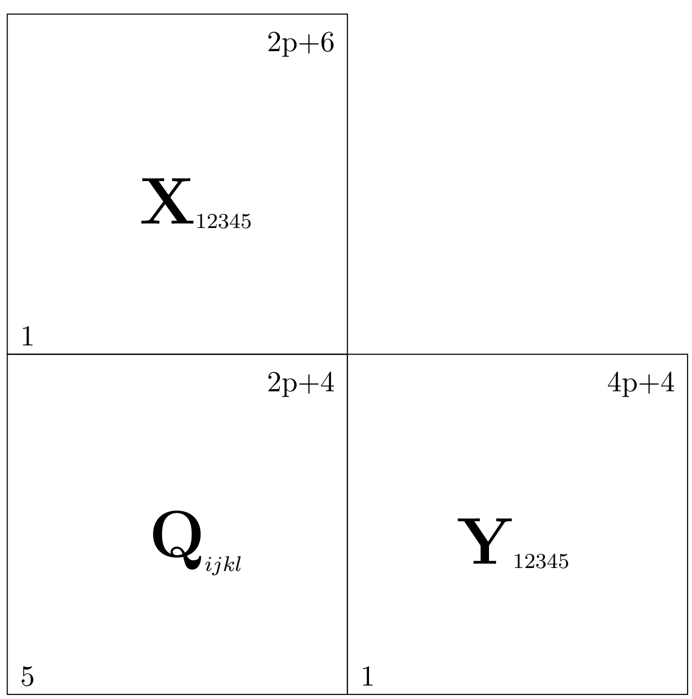


Figure 5.7: The double complex resolution for $\overline{TU}^5_4$.

The generators $Q_{\widehat{i}}$ of $\binom{5}{4} PC(-(2p+4))$, where $\widehat{i} = \{1,2,3,4,5\} \setminus \{i\}$ satisfy the relations

$$y_5 q_{1234} - y_4 q_{1235} + y_3 q_{1245} - y_2 q_{1345} + y_1 q_{2345} = 0,$$

and

$$\begin{aligned} d_0(X) &= d_0(X_{12345}) \\ &= y_5 Q_{1234} - y_4 Q_{1235} + y_3 Q_{1245} - y_2 Q_{1345} + y_1 Q_{2345}. \end{aligned}$$

Similarly,

$$y_5^p q_{1234} - y_4^p q_{1235} + y_3^p q_{1245} - y_2^p q_{1345} + y_1^p q_{2345} = 0,$$

and

$$\begin{aligned} d_1(Y) &= d_1(Y_{12345}) \\ &= y_5^p Q_{1234} - y_4^p Q_{1235} + y_3^p Q_{1245} - y_2^p Q_{1345} + y_1^p Q_{2345}. \end{aligned}$$

We calculate

$$\begin{aligned} q_{1234} &= Q_0 Q_1(\tau_{1234}) \\ &= (y_1^p y_2 - y_1 y_2^p)\tau_{34} - (y_1^p y_3 - y_1 y_3^p)\tau_{24} + (y_1^p y_4 - y_1 y_4^p)\tau_{23} - (y_2^p y_3 - y_2 y_3^p)\tau_{14} \\ &\quad + (y_2^p y_4 - y_2 y_4^p)\tau_{13} - (y_3^p y_4 - y_3 y_4^p)\tau_{12}. \\ q_{1235} &= Q_0 Q_1(\tau_{1235}) \\ &= (y_1^p y_2 - y_1 y_2^p)\tau_{35} - (y_1^p y_3 - y_1 y_3^p)\tau_{25} + (y_1^p y_5 - y_1 y_5^p)\tau_{23} - (y_2^p y_3 - y_2 y_3^p)\tau_{15} \\ &\quad + (y_2^p y_5 - y_2 y_5^p)\tau_{13} - (y_3^p y_5 - y_3 y_5^p)\tau_{12}. \\ q_{1245} &= Q_0 Q_1(\tau_{1245}) \\ &= (y_1^p y_2 - y_1 y_2^p)\tau_{45} - (y_1^p y_4 - y_1 y_4^p)\tau_{25} + (y_1^p y_5 - y_1 y_5^p)\tau_{24} - (y_2^p y_4 - y_2 y_4^p)\tau_{15} \\ &\quad + (y_2^p y_5 - y_2 y_5^p)\tau_{14} - (y_4^p y_5 - y_4 y_5^p)\tau_{12}. \\ q_{1345} &= Q_0 Q_1(\tau_{1345}) \\ &= (y_1^p y_3 - y_1 y_3^p)\tau_{45} - (y_1^p y_4 - y_1 y_4^p)\tau_{35} + (y_1^p y_5 - y_1 y_5^p)\tau_{34} - (y_3^p y_4 - y_3 y_4^p)\tau_{15} \\ &\quad + (y_3^p y_5 - y_3 y_5^p)\tau_{14} - (y_4^p y_5 - y_4 y_5^p)\tau_{13}. \\ q_{2345} &= Q_0 Q_1(\tau_{2345}) \\ &= (y_2^p y_3 - y_2 y_3^p)\tau_{45} - (y_2^p y_4 - y_2 y_4^p)\tau_{35} + (y_2^p y_5 - y_2 y_5^p)\tau_{34} - (y_3^p y_4 - y_3 y_4^p)\tau_{25} \\ &\quad + (y_3^p y_5 - y_3 y_5^p)\tau_{24} - (y_4^p y_5 - y_4 y_5^p)\tau_{23}. \end{aligned}$$

We can now represent the map $\langle d_0, d_1 \rangle \colon F_1 \longrightarrow F_0$ with respect to the chosen generators by a matrix

$$\Theta = \begin{array}{c} \\ Q_{1234} \\ Q_{1235} \\ Q_{1245} \\ Q_{1345} \\ Q_{2345} \end{array} \begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{pmatrix} y_5 & y_5^p \\ -y_4 & -y_4^p \\ y_3 & y_3^p \\ -y_2 & -y_2^p \\ y_1 & y_1^p \end{pmatrix} \end{array}$$

For local cohomology of $\overline{TU}_4^5$, we obtain an exact sequence

$$0 \longleftarrow H^5_{\mathfrak{m}}(\overline{TU}_4^5) \longleftarrow \binom{5}{4} PC^{\vee}(-2p+6) \longleftarrow \binom{5}{5} [PC^{\vee}(-2p+4) \oplus PC^{\vee}(-4p+6)]$$
$$\longleftarrow H^4_{\mathfrak{m}}(\overline{TU}_4^5) \longleftarrow 0.$$

Dualizing, we obtain a short exact sequence

$$0 \longrightarrow (H^5_{\mathfrak{m}}(\overline{TU}_4^5))^{\vee} \longrightarrow \overbrace{5PC(2p-6)}^{(Q_{\widehat{i}})^*} \xrightarrow{\Theta^t} [\overbrace{PC(2p-4)}^{X^*} \oplus \overbrace{PC(4p-6)}^{Y^*}] \longrightarrow (H^4_{\mathfrak{m}}(\overline{TU}_4^5))^{\vee} \longrightarrow 0.$$

The indicated generators $(Q_{\widehat{i}})^*$ of $5PC(2p-6)$ are dual to the generators $Q_{\widehat{i}}$ so that $(Q_{\widehat{1}})^* = Q^*_{2345}$, $(Q_{\widehat{2}})^* = Q^*_{1345}$, $(Q_{\widehat{3}})^* = Q^*_{1245}$, $(Q_{\widehat{4}})^* = Q^*_{1235}$, $(Q_{\widehat{5}})^* = Q^*_{1234}$ of Proposition 3.3.2 under local duality, and similarly for X^*, Y^*.

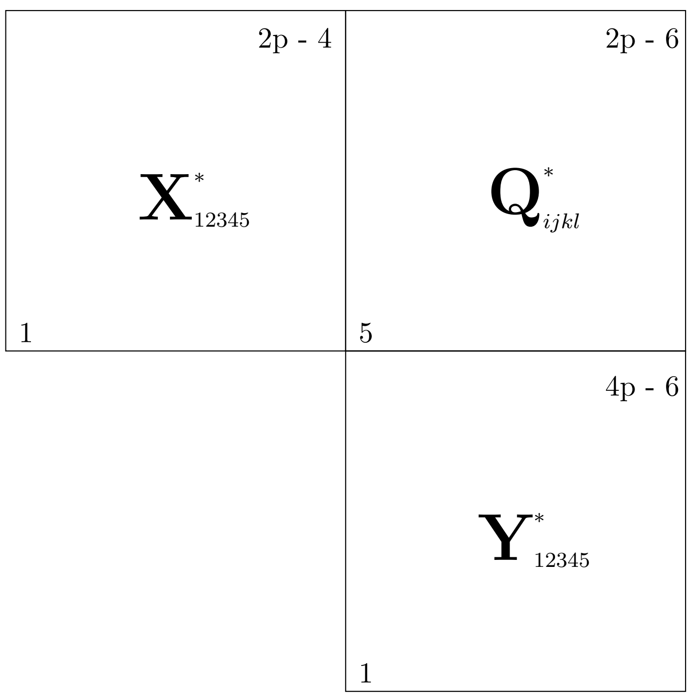


Figure 5.8: The double complex resolution for the local cohomology of $\overline{TU}_4^5$.

By local duality, the map $H^5_{\mathfrak{m}}(F_0)^{\vee} \longleftarrow H^5_{\mathfrak{m}}(F_1)^{\vee}$ is represented with respect to the dual generators by a matrix

$$\Theta^t = \begin{array}{c} \\ X^* \\ Y^* \end{array} \begin{array}{c} \begin{array}{ccccc} Q^*_{1234} & Q^*_{1235} & Q^*_{1245} & Q^*_{1345} & Q^*_{2345} \end{array} \\ \begin{pmatrix} y_5 & -y_4 & y_3 & -y_2 & y_1 \\ y_5^p & -y_4^p & y_3^p & -y_2^p & y_1^p \end{pmatrix} \end{array}$$

Therefore, by Theorem 4.3.1, and (5.1), we deduce

$$(H^5_{\mathfrak{m}}(\overline{TU}^5_4))^\vee = \ker \begin{pmatrix} y_5 & -y_4 & y_3 & -y_2 & y_1 \\ y_5^p & -y_4^p & y_3^p & -y_2^p & y_1^p \end{pmatrix} = (\overline{TU}^5_3)(2p-5),$$

and by (5.2),

$$(H^4_{\mathfrak{m}}(\overline{TU}^5_4))^\vee = \operatorname{coker} \begin{pmatrix} y_5 & -y_4 & y_3 & -y_2 & y_1 \\ y_5^p & -y_4^p & y_3^p & -y_2^p & y_1^p \end{pmatrix}$$

is generated in degrees $2p-4$ and $4p-6$. Its Hilbert series can be read off from the exact sequence (5.3),

$$\begin{aligned}
[(H^4_{\mathfrak{m}}(\overline{TU}^5_4))^\vee] &= [\overline{TU}^5_3(2p-5)] + [PC(2p-4)] + [PC(4p-6)] - 5[PC(2p-6)] \\
&= \frac{10t^8 + t^{12} + t^{2p+10} + t^{4p+8} - 5t^{10} - 5t^{2p+8} + t^{-2p+4} + t^{-4p+6} - 5t^{-2p+6}}{(1-t^2)^5} \\
&= \frac{t^{-4p+6}(10t^{4p+2} + t^{4p+6} + t^{6p+4} + t^{8p+2} - 5t^{4p+4} - 5t^{6p+2} + t^{2p-2} + 1 - 5t^{2p})}{(1-t^2)^5} \\
&= \frac{y^{-2p+3}(10y^{2p+1} + y^{2p+3} + y^{3p+2} + y^{4p+1} - 5y^{2p+2} - 5y^{3p+1} + y^{p-1} + 1 - 5y^p)}{(1-y)^5} \\
&= \frac{y^{-2p+3}(y^{4p+1} + y^{3p+2} - 5y^{3p+1} + y^{2p+3} - 5y^{2p+2} + 10y^{2p+1} - 5y^p + y^{p-1} + 1)}{(1-y)^5} \\
&= \frac{yY^{-2}(y^5Y^4 + y^5Y^3 - 5y^4Y^3 + y^5Y^2 - 5y^4Y^2 + 10y^3Y^2 - 5yY + Y + 1)}{(1-y)^5}.
\end{aligned}$$

5.5.3 $\overline{TU}^5_3$

By Proposition 3.4.1, the PC-module $\overline{TU}^5_3$ has a free resolution

$$0 \longleftarrow \overline{TU}^5_3 \longleftarrow \binom{5}{3} \underbrace{PC(-(2p+3))}_{Q_{\widehat{ij}}} \overset{\langle d_0, d_1\rangle}{\longleftarrow} \binom{5}{4} [\underbrace{PC(-(2p+5))}_{X_{\widehat{i}}} \oplus \underbrace{PC(-(4p+3))}_{Y_{\widehat{i}}}]$$

$$\longleftarrow \binom{5}{5} [\underbrace{PC(-(2p+7))}_{X = X_{12345}} \oplus \underbrace{PC(-(4p+5))}_{Y = Y_{12345}} \oplus \underbrace{PC(-(6p+3))}_{Z = Z_{12345}}] \longleftarrow 0.$$

The generators $Q_{\widehat{12}} = Q_{345}$, $Q_{\widehat{13}} = Q_{245}$, $Q_{\widehat{14}} = Q_{235}$, $Q_{\widehat{15}} = Q_{234}$, $Q_{\widehat{23}} = Q_{145}$, $Q_{\widehat{24}} = Q_{135}$, $Q_{\widehat{25}} = Q_{134}$, $Q_{\widehat{34}} = Q_{125}$, $Q_{\widehat{35}} = Q_{124}$, $Q_{\widehat{45}} = Q_{123}$ are of $\binom{5}{3} PC(-(2p+3))$, and the generators $X_{\widehat{1}} = X_{2345}$, $X_{\widehat{2}} = X_{1345}$, $X_{\widehat{3}} = X_{1245}$, $X_{\widehat{4}} = X_{1235}$, $X_{\widehat{5}} = X_{1234}$, $Y_{\widehat{1}} = Y_{2345}$, $Y_{\widehat{2}} = Y_{1345}$, $Y_{\widehat{3}} = Y_{1245}$, $Y_{\widehat{4}} = Y_{1235}$, $Y_{\widehat{5}} = Y_{1234}$ are of $\binom{5}{4} PC(-(2p+5))$, $\binom{5}{4} PC(-(4p+3))$ respectively.

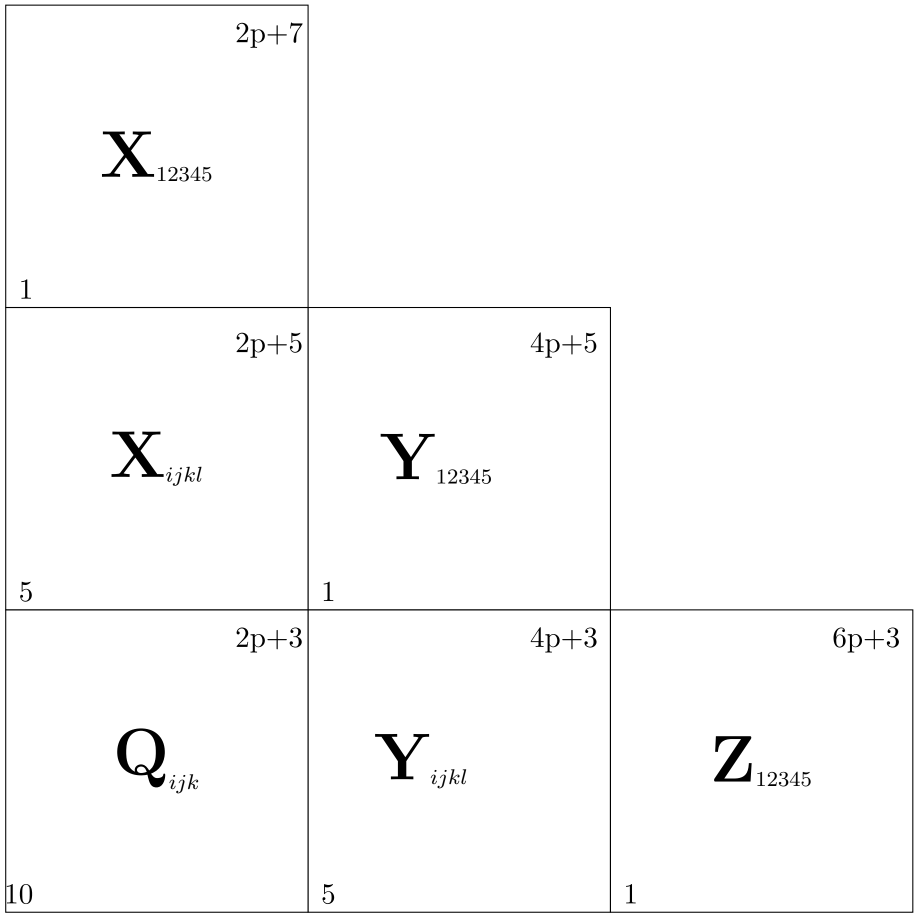


Figure 5.9: The double complex resolution for $\overline{TU}_3^5$.

We may verify $d^2 = 0$ as given on (page 52) of Chapter 3,

$$\begin{aligned}
d^2(X) &= d^2(X_{12345}) \\
&= (y_1y_2 - y_2y_1)Q_{\widehat{12}} - (y_1y_3 - y_3y_1)Q_{\widehat{13}} + (y_1y_4 - y_4y_1)Q_{\widehat{14}} - (y_1y_5 - y_5y_1)Q_{\widehat{15}} \\
&\quad + (y_2y_3 - y_3y_2)Q_{\widehat{23}} - (y_2y_4 - y_4y_2)Q_{\widehat{24}} + (y_2y_5 - y_5y_2)Q_{\widehat{25}} - (y_3y_4 - y_4y_3)Q_{\widehat{34}} \\
&\quad + (y_3y_5 - y_5y_3)Q_{\widehat{35}} - (y_4y_5 - y_5y_4)Q_{\widehat{45}} = 0, \\
d^2(Y) &= d^2(Y_{12345}) \\
&= (y_1^py_2 - y_1y_2^p)Q_{\widehat{12}} - (y_1^py_3 - y_1y_3^p)Q_{\widehat{13}} + (y_1^py_4 - y_1y_4^p)Q_{\widehat{14}} - (y_1^py_5 - y_1y_5^p)Q_{\widehat{15}} \\
&\quad + (y_2^py_3 - y_2y_3^p)Q_{\widehat{23}} - (y_2^py_4 - y_2y_4^p)Q_{\widehat{24}} + (y_2^py_5 - y_2y_5^p)Q_{\widehat{25}} - (y_3^py_4 - y_3y_4^p)Q_{\widehat{34}} \\
&\quad + (y_3^py_5 - y_3y_5^p)Q_{\widehat{35}} - (y_4^py_5 - y_4y_5^p)Q_{\widehat{45}} = 0, \\
d^2(Z) &= d^2(Z_{12345}) \\
&= (y_1^py_2^p - y_2^py_1^p)Q_{\widehat{12}} - (y_1^py_3^p - y_3^py_1^p)Q_{\widehat{13}} + (y_1^py_4^p - y_4^py_1^p)Q_{\widehat{14}} - (y_1^py_5^p - y_5^py_1^p)Q_{\widehat{15}} \\
&\quad + (y_2^py_3^p - y_3^py_2^p)Q_{\widehat{23}} - (y_2^py_4^p - y_4^py_2^p)Q_{\widehat{24}} + (y_2^py_5^p - y_5^py_2^p)Q_{\widehat{25}} - (y_3^py_4^p - y_4^py_3^p)Q_{\widehat{34}} \\
&\quad + (y_3^py_5^p - y_5^py_3^p)Q_{\widehat{35}} - (y_4^py_5^p - y_5^py_4^p)Q_{\widehat{45}} = 0.
\end{aligned}$$

The differentials $d_0(X_{\hat{i}}) = \sum\limits_{j\neq i}(-1)^{j|J} y_j Q_{\widehat{ij}}$, $d_1(Y_{\hat{i}}) = \sum\limits_{j\neq i}(-1)^{j|J} y_j^p Q_{\widehat{ij}}$ are given by

$$\begin{aligned}
d_0(X_{\hat{1}}) = d_0(X_{2345}) &= y_2 Q_{\widehat{12}} - y_3 Q_{\widehat{13}} + y_4 Q_{\widehat{14}} - y_5 Q_{\widehat{15}}.\\
d_0(X_{\hat{2}}) = d_0(X_{1345}) &= y_1 Q_{\widehat{12}} - y_3 Q_{\widehat{23}} + y_4 Q_{\widehat{24}} - y_5 Q_{\widehat{25}}.\\
d_0(X_{\hat{3}}) = d_0(X_{1245}) &= y_1 Q_{\widehat{13}} - y_2 Q_{\widehat{23}} + y_4 Q_{\widehat{34}} - y_5 Q_{\widehat{35}}.\\
d_0(X_{\hat{4}}) = d_0(X_{1235}) &= y_1 Q_{\widehat{14}} - y_2 Q_{\widehat{24}} + y_3 Q_{\widehat{34}} - y_5 Q_{\widehat{45}}.\\
d_0(X_{\hat{5}}) = d_0(X_{1234}) &= y_1 Q_{\widehat{15}} - y_2 Q_{\widehat{25}} + y_3 Q_{\widehat{35}} - y_4 Q_{\widehat{45}}.\\
d_1(Y_{\hat{1}}) = d_1(Y_{2345}) &= y_2^p Q_{\widehat{12}} - y_3^p Q_{\widehat{13}} + y_4^p Q_{\widehat{14}} - y_5^p Q_{\widehat{15}}.\\
d_1(Y_{\hat{2}}) = d_1(Y_{1345}) &= y_1^p Q_{\widehat{12}} - y_3^p Q_{\widehat{23}} + y_4^p Q_{\widehat{24}} - y_5^p Q_{\widehat{25}}.\\
d_1(Y_{\hat{3}}) = d_1(Y_{1245}) &= y_1^p Q_{\widehat{13}} - y_2^p Q_{\widehat{23}} + y_4^p Q_{\widehat{34}} - y_5^p Q_{\widehat{35}}.\\
d_1(Y_{\hat{4}}) = d_1(Y_{1235}) &= y_1^p Q_{\widehat{14}} - y_2^p Q_{\widehat{24}} + y_3^p Q_{\widehat{34}} - y_5^p Q_{\widehat{45}}.\\
d_1(Y_{\hat{5}}) = d_1(Y_{1234}) &= y_1^p Q_{\widehat{15}} - y_2^p Q_{\widehat{25}} + y_3^p Q_{\widehat{35}} - y_4^p Q_{\widehat{45}}.
\end{aligned}$$

We now calculate

$$\begin{aligned}
q_{345} = Q_0Q_1(\tau_{345}) &= (y_3^p y_4 - y_3 y_4^p)\tau_5 - (y_3^p y_5 - y_3 y_5^p)\tau_4 + (y_4^p y_5 - y_4 y_5^p)\tau_3.\\
q_{245} = Q_0Q_1(\tau_{245}) &= (y_2^p y_4 - y_2 y_4^p)\tau_5 - (y_2^p y_5 - y_2 y_5^p)\tau_4 + (y_4^p y_5 - y_4 y_5^p)\tau_2.\\
q_{235} = Q_0Q_1(\tau_{235}) &= (y_2^p y_3 - y_2 y_3^p)\tau_5 - (y_2^p y_5 - y_2 y_5^p)\tau_3 + (y_3^p y_5 - y_3 y_5^p)\tau_2.\\
q_{234} = Q_0Q_1(\tau_{234}) &= (y_2^p y_3 - y_2 y_3^p)\tau_4 - (y_2^p y_4 - y_2 y_4^p)\tau_3 + (y_3^p y_4 - y_3 y_4^p)\tau_2.\\
q_{145} = Q_0Q_1(\tau_{145}) &= (y_1^p y_4 - y_1 y_4^p)\tau_5 - (y_1^p y_5 - y_1 y_5^p)\tau_4 + (y_4^p y_5 - y_4 y_5^p)\tau_1.\\
q_{135} = Q_0Q_1(\tau_{135}) &= (y_1^p y_3 - y_1 y_3^p)\tau_5 - (y_1^p y_5 - y_1 y_5^p)\tau_3 + (y_3^p y_5 - y_3 y_5^p)\tau_1.\\
q_{134} = Q_0Q_1(\tau_{134}) &= (y_1^p y_3 - y_1 y_3^p)\tau_4 - (y_1^p y_4 - y_1 y_4^p)\tau_3 + (y_3^p y_4 - y_3 y_4^p)\tau_1.\\
q_{125} = Q_0Q_1(\tau_{125}) &= (y_1^p y_2 - y_1 y_2^p)\tau_5 - (y_1^p y_5 - y_1 y_5^p)\tau_2 + (y_2^p y_5 - y_2 y_5^p)\tau_1.\\
q_{124} = Q_0Q_1(\tau_{124}) &= (y_1^p y_2 - y_1 y_2^p)\tau_4 - (y_1^p y_4 - y_1 y_4^p)\tau_2 + (y_2^p y_4 - y_2 y_4^p)\tau_1.\\
q_{123} = Q_0Q_1(\tau_{123}) &= (y_1^p y_2 - y_1 y_2^p)\tau_3 - (y_1^p y_3 - y_1 y_3^p)\tau_2 + (y_2^p y_3 - y_2 y_3^p)\tau_1.
\end{aligned}$$

We can now represent the first map $F_1 \longrightarrow F_0$ with to respect to the chosen generators by a matrix

$$\begin{array}{cc}
 & \begin{array}{cccccccccc} X_{\hat{1}} & X_{\hat{2}} & X_{\hat{3}} & X_{\hat{4}} & X_{\hat{5}} & Y_{\hat{1}} & Y_{\hat{2}} & Y_{\hat{3}} & Y_{\hat{4}} & Y_{\hat{5}} \end{array} \\
\begin{array}{c} Q_{\widehat{12}} \\ Q_{\widehat{13}} \\ Q_{\widehat{14}} \\ Q_{\widehat{15}} \\ Q_{\widehat{23}} \\ Q_{\widehat{24}} \\ Q_{\widehat{25}} \\ Q_{\widehat{34}} \\ Q_{\widehat{35}} \\ Q_{\widehat{45}} \end{array} &
\begin{pmatrix}
y_2 & y_1 & & & & y_2^p & y_1^p & & & \\
-y_3 & & y_1 & & & -y_3^p & & y_1^p & & \\
y_4 & & & y_1 & & y_4^p & & & y_1^p & \\
-y_5 & & & & y_1 & -y_5^p & & & & y_1^p \\
 & -y_3 & -y_2 & & & & -y_3^p & -y_2^p & & \\
 & y_4 & & -y_2 & & & y_4^p & & -y_2^p & \\
 & -y_5 & & & -y_2 & & -y_5^p & & & -y_2^p \\
 & & y_4 & y_3 & & & & y_4^p & y_3^p & \\
 & & -y_5 & & y_3 & & & -y_5^p & & y_3^p \\
 & & & -y_5 & -y_4 & & & & -y_5^p & -y_4^p
\end{pmatrix}
\end{array}$$

We can also represent the second map $F_2 \longrightarrow F_1$ with respect to the chosen generators by a matrix

$$\Theta = \begin{array}{c} \\ X_{\widehat{1}} \\ X_{\widehat{2}} \\ X_{\widehat{3}} \\ X_{\widehat{4}} \\ X_{\widehat{5}} \\ Y_{\widehat{1}} \\ Y_{\widehat{2}} \\ Y_{\widehat{3}} \\ Y_{\widehat{4}} \\ Y_{\widehat{5}} \end{array} \begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{pmatrix} y_1 & y_1^p & \\ -y_2 & -y_2^p & \\ y_3 & y_3^p & \\ -y_4 & -y_4^p & \\ y_5 & y_5^p & \\ & y_1 & y_1^p \\ & -y_2 & -y_2^p \\ & y_3 & y_3^p \\ & -y_4 & -y_4^p \\ & y_5 & y_5^p \end{pmatrix} \end{array}$$

For local cohomology of $\overline{TU}_3^5$, we obtain an exact sequence

$$0 \longleftarrow H^5_{\mathfrak{m}}(\overline{TU}_3^5) \longleftarrow \binom{5}{3} PC^\vee(-2p+7) \longleftarrow \binom{5}{4} [PC^\vee(-2p+5) \oplus PC^\vee(-4p+7)] \longleftarrow$$
$$\binom{5}{5} [PC^\vee(-2p+3) \oplus PC^\vee(-4p+5) \oplus PC^\vee(-6p+7)] \longleftarrow H^3_{\mathfrak{m}}(\overline{TU}_3^5) \longleftarrow 0.$$

Dualizing, we obtain an exact sequence

$$0 \longrightarrow (H^5_{\mathfrak{m}}(\overline{TU}_3^5))^\vee \longrightarrow \overbrace{10PC(2p-7)}^{(Q_{\widehat{ij}})^*} \longrightarrow 5[\overbrace{PC(2p-5)}^{(X_{\widehat{i}})^*} \oplus \overbrace{PC(4p-7)}^{(Y_{\widehat{i}})^*}] \longrightarrow$$
$$[\overbrace{PC(2p-3)}^{X^*} \oplus \overbrace{PC(4p-5)}^{Y^*} \oplus \overbrace{PC(6p-7)}^{Z^*}] \longrightarrow (H^3_{\mathfrak{m}}(\overline{TU}_3^5))^\vee \longrightarrow 0.$$

The indicated generators $(Q_{\widehat{ij}})^*$ of $10PC(2p-7)$ are dual to the generators $Q_{\widehat{ij}}$ so that $(Q_{\widehat{12}})^* = Q^*_{345}$, $(Q_{\widehat{13}})^* = Q^*_{245}$, $(Q_{\widehat{14}})^* = Q^*_{235}$, $(Q_{\widehat{15}})^* = Q^*_{234}$, $(Q_{\widehat{23}})^* = Q^*_{145}$, $(Q_{\widehat{24}})^* = Q^*_{135}$, $(Q_{\widehat{25}})^* = Q^*_{134}$, $(Q_{\widehat{34}})^* = Q^*_{125}$, $(Q_{\widehat{35}})^* = Q^*_{124}$, $(Q_{\widehat{45}})^* = Q^*_{123}$.

Also, the generators $(X_{\widehat{i}})^*$, $(Y_{\widehat{j}})^*$ of $5PC(2p-5)$, $5PC(4p-7)$ are dual to the generators X and Y so that $(X_{\widehat{1}})^* = X^*_{2345}, (Y_{\widehat{1}})^* = Y^*_{2345}$, $(X_{\widehat{2}})^* = X^*_{1345}, (Y_{\widehat{2}})^* = Y^*_{1345}$, $(X_{\widehat{3}})^* = X^*_{1245}, (Y_{\widehat{3}})^* = Y^*_{1245}$, $(X_{\widehat{4}})^* = X^*_{1235}, (Y_{\widehat{4}})^* = Y^*_{1235}$, $(X_{\widehat{5}})^* = X^*_{1234}, (Y_{\widehat{5}})^* = Y^*_{1234}$ of Proposition 3.3.2 under local duality, and similarly for X^*, Y^*, Z^*.

By local duality, the map $H^5_{\mathfrak{m}}(F_1)^\vee \longleftarrow H^5_{\mathfrak{m}}(F_2)^\vee$ is represented with respect to the dual generators by a matrix

$$\Theta^t = \begin{array}{c} \\ X^* \\ Y^* \\ Z^* \end{array} \begin{array}{c} \begin{array}{cccccccccc} (X_{\widehat{1}})^* & (X_{\widehat{2}})^* & (X_{\widehat{3}})^* & (X_{\widehat{4}})^* & (X_{\widehat{5}})^* & (Y_{\widehat{1}})^* & (Y_{\widehat{2}})^* & (Y_{\widehat{3}})^* & (Y_{\widehat{4}})^* & (Y_{\widehat{5}})^* \end{array} \\ \begin{pmatrix} y_1 & -y_2 & y_3 & -y_4 & y_5 & & & & & \\ y_1^p & -y_2^p & y_3^p & -y_4^p & y_5^p & y_1 & -y_2 & y_3 & -y_4 & y_5 \\ & & & & & y_1^p & -y_2^p & y_3^p & -y_4^p & y_5^p \end{pmatrix} \end{array}$$

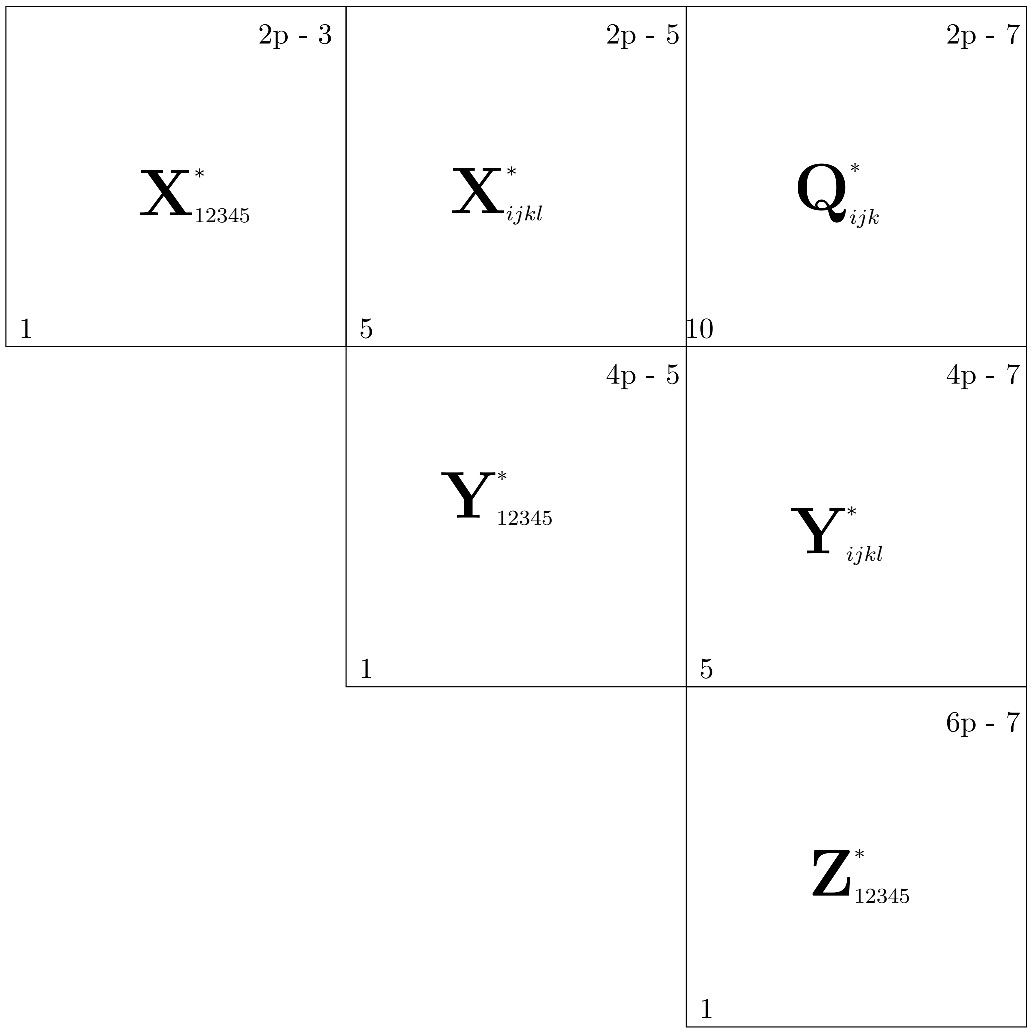


Figure 5.10: The double complex resolution for the local cohomology of $\overline{TU}_3^5$.

The calculation by Theorem 4.3.1 and (5.4) gives us

$$
\begin{aligned}
(H^5_{\mathfrak{m}}(\overline{TU}_3^5))^\vee &= \ker \begin{pmatrix} y_1 & -y_2 & y_3 & -y_4 & y_5 & & & & & \\ y_1^p & -y_2^p & y_3^p & -y_4^p & y_5^p & y_1 & -y_2 & y_3 & -y_4 & y_5 \\ & & & & & y_1^p & -y_2^p & y_3^p & -y_4^p & y_5^p \end{pmatrix} \\
&= (\overline{TU}_4^5)(2p-5),
\end{aligned}
$$

and by (5.5),

$$
(H^3_{\mathfrak{m}}(\overline{TU}_3^5))^\vee = \operatorname{coker} \begin{pmatrix} y_1 & -y_2 & y_3 & -y_4 & y_5 & & & & & \\ y_1^p & -y_2^p & y_3^p & -y_4^p & y_5^p & y_1 & -y_2 & y_3 & -y_4 & y_5 \\ & & & & & y_1^p & -y_2^p & y_3^p & -y_4^p & y_5^p \end{pmatrix}
$$

is generated in degrees $2p-3$, $4p-5$, and $6p-7$. Its Hilbert series can be read off from the exact sequence (5.6),

$$
\begin{aligned}
[(H^3_{\mathfrak{m}}(\overline{TU}^5_3))^\vee] &= 10[PC(2p-7)] + [PC(2p-3)] + [PC(4p-5)] + [PC(6p-7)] \\
&\quad - [\overline{TU}^5_4(2p-5)] - 5([PC(2p-5)] + [PC(4p-7)]) \\
&= \frac{10t^{-2p+7} + t^{-2p+3} + t^{-4p+5} + t^{-6p+7} - 5t^9 + t^{11} + t^{2p+9} - 5(t^{-2p+5} + t^{-4p+7})}{(1-t^2)^5} \\
&= \frac{t^{-6p+7}(10t^{4p} + t^{4p-4} + t^{2p-2} + 1 - 5t^{6p+2} + t^{6p+4} + t^{8p+2} - 5(t^{4p-2} + t^{2p}))}{(1-t^2)^5} \\
&= \frac{t^{-6p+7}(10y^{2p} + y^{2p-2} + y^{p-1} + 1 - 5y^{3p+1} + y^{3p+2} + y^{4p+1} - 5y^{2p-1} - 5y^p)}{(1-y)^5} \\
&= \frac{tY^{-3}(y^{4p+1} + y^{3p+2} - 5y^{3p+1} + 10y^{2p} - 5y^{2p-1} + y^{2p-2} - 5y^p + y^{p-1} + 1)}{(1-y)^5} \\
&= \frac{tY^{-3}(y^5Y^4 + y^5Y^3 - 5y^4Y^3 + 10y^2Y^2 - 5yY^2 + Y^2 - 5yY + Y + 1)}{(1-y)^5}.
\end{aligned}
$$

5.5.4 $\overline{TU}^5_2$

Finally, we calculate the local cohomology of the PC-module $\overline{TU}^5_2$. By Proposition 3.4.1, the module $\overline{TU}^5_2$ has a free resolution (see Figure 5.11)

$$
\begin{aligned}
0 \longleftarrow \overline{TU}^5_2 \longleftarrow \binom{5}{2} \underbrace{PC(-(2p+2))}_{Q_{\widehat{ijk}}} &\xleftarrow{\langle d_0, d_1\rangle} \binom{5}{3} [\underbrace{PC(-(2p+4))}_{X_{\widehat{ij}}} \oplus \underbrace{PC(-(4p+2))}_{Y_{\widehat{ij}}}] \longleftarrow \\
&\binom{5}{4} [\underbrace{PC(-(2p+6))}_{X_{\widehat{i}}} \oplus \underbrace{PC(-(4p+4))}_{Y_{\widehat{i}}} \oplus \underbrace{PC(-(6p+2))}_{Z_{\widehat{i}}}] \longleftarrow \\
\binom{5}{5} [\underbrace{PC(-(2p+8))}_{X=X_{12345}} &\oplus \underbrace{PC(-(4p+6))}_{Y=Y_{12345}} \oplus \underbrace{PC(-(6p+4))}_{Z=Z_{12345}} \oplus \underbrace{PC(-(8p+2))}_{W=W_{12345}}] \longleftarrow 0.
\end{aligned}
$$

The generators $Q_{\widehat{123}} = Q_{45}$, $Q_{\widehat{124}} = Q_{35}$, $Q_{\widehat{125}} = Q_{34}$, $Q_{\widehat{134}} = Q_{25}$, $Q_{\widehat{135}} = Q_{24}$, $Q_{\widehat{145}} = Q_{23}$, $Q_{\widehat{234}} = Q_{15}$, $Q_{\widehat{235}} = Q_{14}$, $Q_{\widehat{245}} = Q_{13}$, $Q_{\widehat{345}} = Q_{12}$ are of $\binom{5}{2} PC(-(2p+2))$, and the generators $X_{\widehat{12}} = X_{345}$, $X_{\widehat{13}} = X_{245}$, $X_{\widehat{14}} = X_{235}$, $X_{\widehat{15}} = X_{234}$, $X_{\widehat{23}} = X_{145}$, $X_{\widehat{24}} = X_{135}$, $X_{\widehat{25}} = X_{134}$, $X_{\widehat{34}} = X_{125}$, $X_{\widehat{35}} = X_{124}$, $X_{\widehat{45}} = X_{123}$, $Y_{\widehat{12}} = Y_{345}$, $Y_{\widehat{13}} = Y_{245}$, $Y_{\widehat{14}} = Y_{235}$, $Y_{\widehat{15}} = Y_{234}$, $Y_{\widehat{23}} = Y_{145}$, $Y_{\widehat{24}} = Y_{135}$, $Y_{\widehat{25}} = Y_{134}$, $Y_{\widehat{34}} = Y_{125}$, $Y_{\widehat{35}} = Y_{124}$, $Y_{\widehat{45}} = Y_{123}$ are of $\binom{5}{3} PC(-(2p+4))$, $\binom{5}{3} PC(-(4p+2))$ respectively.

We may verify $d^2 = 0$ as given on (page 52) of Chapter 3,

$$\begin{aligned}
d^2(X_{\widehat{i}}) &= (y_2y_3 - y_3y_2)Q_{\widehat{123}} - (y_2y_4 - y_4y_2)Q_{\widehat{124}} + (y_2y_5 - y_5y_2)Q_{\widehat{125}} - (y_3y_4 - y_4y_3)Q_{\widehat{134}} \\
&\quad + (y_3y_5 - y_5y_3)Q_{\widehat{135}} - (y_4y_5 - y_5y_4)Q_{\widehat{145}} + (y_3y_4 - y_4y_3)Q_{\widehat{234}} - (y_3y_5 - y_5y_3)Q_{\widehat{235}} \\
&\quad + (y_4y_5 - y_5y_4)Q_{\widehat{245}} - (y_4y_5 - y_5y_4)Q_{\widehat{345}} = 0, \quad \text{for all } i \\
d^2(Y_{\widehat{i}}) &= (y_2^py_3 - y_2y_3^p)Q_{\widehat{123}} - (y_2^py_4 - y_2y_4^p)Q_{\widehat{124}} + (y_2^py_5 - y_2y_5^p)Q_{\widehat{125}} - (y_3^py_4 - y_3y_4^p)Q_{\widehat{134}} \\
&\quad + (y_3^py_5 - y_3y_5^p)Q_{\widehat{135}} - (y_4^py_5 - y_4y_5^p)Q_{\widehat{145}} + (y_3^py_4 - y_3y_4^p)Q_{\widehat{234}} - (y_3^py_5 - y_3y_5^p)Q_{\widehat{235}} \\
&\quad + (y_4^py_5 - y_4y_5^p)Q_{\widehat{245}} - (y_4^py_5 - y_4y_5^p)Q_{\widehat{345}} = 0, \quad \text{for all } i \\
d^2(Z_{\widehat{i}}) &= (y_2^py_3^p - y_3^py_2^p)Q_{\widehat{123}} - (y_2^py_4^p - y_4^py_2^p)Q_{\widehat{124}} + (y_2^py_5^p - y_5^py_2^p)Q_{\widehat{125}} - (y_3^py_4^p - y_4^py_3^p)Q_{\widehat{134}} \\
&\quad + (y_3^py_5^p - y_5^py_3^p)Q_{\widehat{135}} - (y_4^py_5^p - y_5^py_4^p)Q_{\widehat{145}} + (y_3^py_4^p - y_4^py_3^p)Q_{\widehat{234}} - (y_3^py_5^p - y_5^py_3^p)Q_{\widehat{235}} \\
&\quad + (y_4^py_5^p - y_5^py_4^p)Q_{\widehat{245}} - (y_4^py_5^p - y_5^py_4^p)Q_{\widehat{345}} = 0, \quad \text{for all } i.
\end{aligned}$$

The differential $d_0(X_{\widehat{ij}}) = \sum_{k\neq j, j\neq i} (-1)^{k|K} y_k Q_{\widehat{ijk}}$ is given by

$$\begin{aligned}
d_0(X_{\widehat{12}}) &= d_0(X_{345}) = y_3Q_{\widehat{123}} - y_4Q_{\widehat{124}} + y_5Q_{\widehat{125}}. \\
d_0(X_{\widehat{13}}) &= d_0(X_{245}) = y_2Q_{\widehat{123}} - y_4Q_{\widehat{134}} + y_5Q_{\widehat{135}}. \\
d_0(X_{\widehat{14}}) &= d_0(X_{235}) = y_2Q_{\widehat{124}} - y_3Q_{\widehat{134}} + y_5Q_{\widehat{145}}. \\
d_0(X_{\widehat{15}}) &= d_0(X_{234}) = y_2Q_{\widehat{125}} - y_3Q_{\widehat{135}} + y_4Q_{\widehat{145}}. \\
d_0(X_{\widehat{23}}) &= d_0(X_{145}) = y_1Q_{\widehat{123}} - y_4Q_{\widehat{234}} + y_5Q_{\widehat{235}}. \\
d_0(X_{\widehat{24}}) &= d_0(X_{135}) = y_1Q_{\widehat{124}} - y_3Q_{\widehat{234}} + y_5Q_{\widehat{245}}. \\
d_0(X_{\widehat{25}}) &= d_0(X_{134}) = y_1Q_{\widehat{125}} - y_3Q_{\widehat{235}} + y_4Q_{\widehat{245}}. \\
d_0(X_{\widehat{34}}) &= d_0(X_{125}) = y_1Q_{\widehat{134}} - y_2Q_{\widehat{234}} + y_5Q_{\widehat{345}}. \\
d_0(X_{\widehat{35}}) &= d_0(X_{124}) = y_1Q_{\widehat{135}} - y_2Q_{\widehat{235}} + y_4Q_{\widehat{345}}. \\
d_0(X_{\widehat{45}}) &= d_0(X_{123}) = y_1Q_{\widehat{145}} - y_2Q_{\widehat{245}} + y_3Q_{\widehat{345}}.
\end{aligned}$$

Similarly for the differential $d_1(Y_{\widehat{ij}}) = \sum_{k\neq j, j\neq i} (-1)^{k|K} y_k^p Q_{\widehat{ijk}}$,

$$\begin{aligned}
d_1(Y_{\widehat{12}}) &= d_1(Y_{345}) = y_3^pQ_{\widehat{123}} - y_4^pQ_{\widehat{124}} + y_5^pQ_{\widehat{125}}. \\
d_1(Y_{\widehat{13}}) &= d_1(Y_{245}) = y_2^pQ_{\widehat{123}} - y_4^pQ_{\widehat{134}} + y_5^pQ_{\widehat{135}}. \\
d_1(Y_{\widehat{14}}) &= d_1(Y_{235}) = y_2^pQ_{\widehat{124}} - y_3^pQ_{\widehat{134}} + y_5^pQ_{\widehat{145}}. \\
d_1(Y_{\widehat{15}}) &= d_1(Y_{234}) = y_2^pQ_{\widehat{125}} - y_3^pQ_{\widehat{135}} + y_4^pQ_{\widehat{145}}. \\
d_1(Y_{\widehat{23}}) &= d_1(Y_{145}) = y_1^pQ_{\widehat{123}} - y_4^pQ_{\widehat{234}} + y_5^pQ_{\widehat{235}}. \\
d_1(Y_{\widehat{24}}) &= d_1(Y_{135}) = y_1^pQ_{\widehat{124}} - y_3^pQ_{\widehat{234}} + y_5^pQ_{\widehat{245}}. \\
d_1(Y_{\widehat{25}}) &= d_1(Y_{134}) = y_1^pQ_{\widehat{125}} - y_3^pQ_{\widehat{235}} + y_4^pQ_{\widehat{245}}. \\
d_1(Y_{\widehat{34}}) &= d_1(Y_{125}) = y_1^pQ_{\widehat{134}} - y_2^pQ_{\widehat{234}} + y_5^pQ_{\widehat{345}}. \\
d_1(Y_{\widehat{35}}) &= d_1(Y_{124}) = y_1^pQ_{\widehat{135}} - y_2^pQ_{\widehat{235}} + y_4^pQ_{\widehat{345}}. \\
d_1(Y_{\widehat{45}}) &= d_1(Y_{123}) = y_1^pQ_{\widehat{145}} - y_2^pQ_{\widehat{245}} + y_3^pQ_{\widehat{345}}.
\end{aligned}$$

We now calculate

$$
\begin{aligned}
q_{\widehat{123}} &= q_{45} = Q_0Q_1(\tau_{45}) = (y_4^p y_5 - y_4 y_5^p).\\
q_{\widehat{124}} &= q_{35} = Q_0Q_1(\tau_{35}) = (y_3^p y_5 - y_3 y_5^p).\\
q_{\widehat{125}} &= q_{34} = Q_0Q_1(\tau_{34}) = (y_3^p y_4 - y_3 y_4^p).\\
q_{\widehat{134}} &= q_{25} = Q_0Q_1(\tau_{25}) = (y_2^p y_5 - y_2 y_5^p).\\
q_{\widehat{135}} &= q_{24} = Q_0Q_1(\tau_{24}) = (y_2^p y_4 - y_2 y_4^p).\\
q_{\widehat{145}} &= q_{23} = Q_0Q_1(\tau_{23}) = (y_2^p y_3 - y_2 y_3^p).\\
q_{\widehat{234}} &= q_{15} = Q_0Q_1(\tau_{15}) = (y_1^p y_5 - y_1 y_5^p).\\
q_{\widehat{235}} &= q_{14} = Q_0Q_1(\tau_{14}) = (y_1^p y_4 - y_1 y_4^p).\\
q_{\widehat{245}} &= q_{13} = Q_0Q_1(\tau_{13}) = (y_1^p y_3 - y_1 y_3^p).\\
q_{\widehat{345}} &= q_{12} = Q_0Q_1(\tau_{12}) = (y_1^p y_2 - y_1 y_2^p).
\end{aligned}
$$

We can now represent the first map $\langle d_0, d_1\rangle \colon F_1 \longrightarrow F_0$ with respect to the chosen generators by the two matrices

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} X_{\widehat{12}} & X_{\widehat{13}} & X_{\widehat{14}} & X_{\widehat{15}} & X_{\widehat{23}} & X_{\widehat{24}} & X_{\widehat{25}} & X_{\widehat{34}} & X_{\widehat{35}} & X_{\widehat{45}} \end{array} \\
\begin{array}{c} Q_{\widehat{123}} \\ Q_{\widehat{124}} \\ Q_{\widehat{125}} \\ Q_{\widehat{134}} \\ Q_{\widehat{135}} \\ Q_{\widehat{145}} \\ Q_{\widehat{234}} \\ Q_{\widehat{235}} \\ Q_{\widehat{245}} \\ Q_{\widehat{345}} \end{array} &
\begin{pmatrix}
y_3 & y_2 & & & y_1 & & & & & \\
-y_4 & & y_2 & & & y_1 & & & & \\
y_5 & & & y_2 & & & y_1 & & & \\
 & -y_4 & -y_3 & & & & & y_1 & & \\
 & y_5 & & -y_3 & & & & & y_1 & \\
 & & y_5 & y_4 & & & & & & y_1 \\
 & & & & -y_4 & -y_3 & & -y_2 & & \\
 & & & & y_5 & & -y_3 & & -y_2 & \\
 & & & & & y_5 & y_4 & & & -y_2 \\
 & & & & & & & y_5 & y_4 & y_3
\end{pmatrix}
\end{array}
$$

and

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} Y_{\widehat{12}} & Y_{\widehat{13}} & Y_{\widehat{14}} & Y_{\widehat{15}} & Y_{\widehat{23}} & Y_{\widehat{24}} & Y_{\widehat{25}} & Y_{\widehat{34}} & Y_{\widehat{35}} & Y_{\widehat{45}} \end{array} \\
\begin{array}{c} Q_{\widehat{123}} \\ Q_{\widehat{124}} \\ Q_{\widehat{125}} \\ Q_{\widehat{134}} \\ Q_{\widehat{135}} \\ Q_{\widehat{145}} \\ Q_{\widehat{234}} \\ Q_{\widehat{235}} \\ Q_{\widehat{245}} \\ Q_{\widehat{345}} \end{array} &
\begin{pmatrix}
y_3^p & y_2^p & & & y_1^p & & & & & \\
-y_4^p & & y_2^p & & & y_1^p & & & & \\
y_5^p & & & y_2^p & & & y_1^p & & & \\
 & -y_4^p & -y_3^p & & & & & y_1^p & & \\
 & y_5^p & & -y_3^p & & & & & y_1^p & \\
 & & y_5^p & y_4^p & & & & & & y_1^p \\
 & & & & -y_4^p & -y_3^p & & -y_2^p & & \\
 & & & & y_5^p & & -y_3^p & & -y_2^p & \\
 & & & & & y_5^p & y_4^p & & & -y_2^p \\
 & & & & & & & y_5^p & y_4^p & y_3^p
\end{pmatrix}
\end{array}
$$

The generators $X_{\widehat{1}} = X_{2345}\,,\ X_{\widehat{2}} = X_{1345}\,,\ X_{\widehat{3}} = X_{1245}\,,\ X_{\widehat{4}} = X_{1235}\,,\ X_{\widehat{5}} = X_{1234}\,,$ $Y_{\widehat{1}} = Y_{2345}\,,\ Y_{\widehat{2}} = Y_{1345}\,,\ Y_{\widehat{3}} = Y_{1245}\,,\ Y_{\widehat{4}} = Y_{1235}\,,\ Y_{\widehat{5}} = Y_{1234}\,,\ Z_{\widehat{1}} = Z_{2345}\,,\ Z_{\widehat{2}} = Z_{1345}\,,$ $Z_{\widehat{3}} = Z_{1245}\,,\ Z_{\widehat{4}} = Z_{1235}\,,\ Z_{\widehat{5}} = Z_{1234}$ are of $\binom{5}{4}PC(-(2p+6)),\ \binom{5}{4}PC(-(4p+4)),$ $\binom{5}{4}PC(-(6p+2))$ respectively.

The differential $d_0(X_{\widehat{i}}) = \sum_{j\neq i}(-1)^{j|J}y_jX_{\widehat{ij}}$ is given by

$$\begin{aligned}
d_0(X_{\widehat{1}}) &= d_0(X_{2345}) = y_2X_{\widehat{12}} - y_3X_{\widehat{13}} + y_4X_{\widehat{14}} - y_5X_{\widehat{15}}.\\
d_0(X_{\widehat{2}}) &= d_0(X_{1345}) = y_1X_{\widehat{12}} - y_3X_{\widehat{23}} + y_4X_{\widehat{24}} - y_5X_{\widehat{25}}.\\
d_0(X_{\widehat{3}}) &= d_0(X_{1245}) = y_1X_{\widehat{13}} - y_2X_{\widehat{23}} + y_4X_{\widehat{34}} - y_5X_{\widehat{35}}.\\
d_0(X_{\widehat{4}}) &= d_0(X_{1235}) = y_1X_{\widehat{14}} - y_2X_{\widehat{24}} + y_3X_{\widehat{34}} - y_5X_{\widehat{45}}.\\
d_0(X_{\widehat{5}}) &= d_0(X_{1234}) = y_1X_{\widehat{15}} - y_2X_{\widehat{25}} + y_3X_{\widehat{35}} - y_4X_{\widehat{45}}.
\end{aligned}$$

For the differential $d(Y_{\widehat{i}})$. Since $d = d_1 + d_0$ as we proved in Chapter 3,

$$\begin{aligned}
d(Y_{\widehat{i}}) &= (d_1+d_0)(Y_{\widehat{i}})\\
&= d_1(Y_{\widehat{i}}) + d_0(Y_{\widehat{i}})\\
&= \sum_{j\neq i}(-1)^{j|J}y_j^pX_{\widehat{ij}} \pm \sum_{j\neq i}(-1)^{j|J}y_jY_{\widehat{ij}}\\
&= \sum_{j\neq i}(-1)^{j|J}(y_j^pX_{\widehat{ij}} \pm y_jY_{\widehat{ij}}).
\end{aligned}$$

We now calculate

$$\begin{aligned}
d(Y_{\widehat{1}}) &= d(Y_{2345})\\
&= (d_1+d_0)(Y_{2345})\\
&= d_1(Y_{2345}) + d_0(Y_{2345})\\
&= (y_2^pX_{\widehat{12}} + y_2Y_{\widehat{12}}) - (y_3^pX_{\widehat{13}} + y_3Y_{\widehat{13}}) + (y_4^pX_{\widehat{14}} + y_4Y_{\widehat{14}}) - (y_5^pX_{\widehat{15}} + y_5Y_{\widehat{15}}).
\end{aligned}$$

$$\begin{aligned}
d(Y_{\widehat{2}}) &= d(Y_{1345})\\
&= (d_1+d_0)(Y_{1345})\\
&= d_1(Y_{1345}) + d_0(Y_{1345})\\
&= (y_1^pX_{\widehat{12}} + y_1Y_{\widehat{12}}) - (y_3^pX_{\widehat{23}} + y_3Y_{\widehat{23}}) + (y_4^pX_{\widehat{24}} + y_4Y_{\widehat{24}}) - (y_5^pX_{\widehat{25}} + y_5Y_{\widehat{25}}).
\end{aligned}$$

$$\begin{aligned}
d(Y_{\widehat{3}}) &= d(Y_{1245})\\
&= (d_1+d_0)(Y_{1245})\\
&= d_1(Y_{1245}) + d_0(Y_{1245})\\
&= (y_1^pX_{\widehat{13}} + y_1Y_{\widehat{13}}) - (y_2^pX_{\widehat{23}} + y_2Y_{\widehat{23}}) + (y_4^pX_{\widehat{34}} + y_4Y_{\widehat{34}}) - (y_5^pX_{\widehat{35}} + y_5Y_{\widehat{35}}).
\end{aligned}$$

$$\begin{aligned} d(Y_{\widehat{4}}) &= d(Y_{1235}) \\ &= (d_1 + d_0)(Y_{1235}) \\ &= d_1(Y_{1235}) + d_0(Y_{1235}) \\ &= (y_1^p X_{\widehat{14}} + y_1 Y_{\widehat{14}}) - (y_2^p X_{\widehat{24}} + y_2 Y_{\widehat{24}}) + (y_3^p X_{\widehat{34}} + y_3 Y_{\widehat{34}}) - (y_5^p X_{\widehat{45}} + y_5 Y_{\widehat{45}}). \end{aligned}$$

$$\begin{aligned} d(Y_{\widehat{5}}) &= d(Y_{1234}) \\ &= (d_1 + d_0)(Y_{1234}) \\ &= d_1(Y_{1234}) + d_0(Y_{1234}) \\ &= (y_1^p X_{\widehat{15}} + y_1 Y_{\widehat{15}}) - (y_2^p X_{\widehat{25}} + y_2 Y_{\widehat{25}}) + (y_3^p X_{\widehat{35}} + y_3 Y_{\widehat{35}}) - (y_4^p X_{\widehat{45}} + y_4 Y_{\widehat{45}}). \end{aligned}$$

Similarly, $d_1(Z_{\widehat{i}}) = \sum_{j \neq i} (-1)^{j|J} y_j^p Y_{\widehat{ij}}$ is given by

$$\begin{aligned} d_1(Z_{\widehat{1}}) &= d_1(Z_{2345}) = y_2^p Y_{\widehat{12}} - y_3^p Y_{\widehat{13}} + y_4^p Y_{\widehat{14}} - y_5^p Y_{\widehat{15}}. \\ d_1(Z_{\widehat{2}}) &= d_1(Z_{1345}) = y_1^p Y_{\widehat{12}} - y_3^p Y_{\widehat{23}} + y_4^p Y_{\widehat{24}} - y_5^p Y_{\widehat{25}}. \\ d_1(Z_{\widehat{3}}) &= d_1(Z_{1245}) = y_1^p Y_{\widehat{13}} - y_2^p Y_{\widehat{23}} + y_4^p Y_{\widehat{34}} - y_5^p Y_{\widehat{35}}. \\ d_1(Z_{\widehat{4}}) &= d_1(Z_{1235}) = y_1^p Y_{\widehat{14}} - y_2^p Y_{\widehat{24}} + y_3^p Y_{\widehat{34}} - y_5^p Y_{\widehat{45}}. \\ d_1(Z_{\widehat{5}}) &= d_1(Z_{1234}) = y_1^p Y_{\widehat{15}} - y_2^p Y_{\widehat{25}} + y_3^p Y_{\widehat{35}} - y_4^p Y_{\widehat{45}}. \end{aligned}$$

We can also represent the second map $F_2 \longrightarrow F_1$ with respect to the chosen generators by a matrix

$$\begin{array}{cc} & \begin{array}{ccccccccccccccc} X_{\widehat{1}} & X_{\widehat{2}} & X_{\widehat{3}} & X_{\widehat{4}} & X_{\widehat{5}} & Y_{\widehat{1}} & Y_{\widehat{2}} & Y_{\widehat{3}} & Y_{\widehat{4}} & Y_{\widehat{5}} & Z_{\widehat{1}} & Z_{\widehat{2}} & Z_{\widehat{3}} & Z_{\widehat{4}} & Z_{\widehat{5}} \end{array} \\ \begin{array}{c} X_{\widehat{12}} \\ X_{\widehat{13}} \\ X_{\widehat{14}} \\ X_{\widehat{15}} \\ X_{\widehat{23}} \\ X_{\widehat{24}} \\ X_{\widehat{25}} \\ X_{\widehat{34}} \\ X_{\widehat{35}} \\ X_{\widehat{45}} \\ Y_{\widehat{12}} \\ Y_{\widehat{13}} \\ Y_{\widehat{14}} \\ Y_{\widehat{15}} \\ Y_{\widehat{23}} \\ Y_{\widehat{24}} \\ Y_{\widehat{25}} \\ Y_{\widehat{34}} \\ Y_{\widehat{35}} \\ Y_{\widehat{45}} \end{array} & \begin{pmatrix} y_2 & y_1 & & & & y_2^p & y_1^p & & & & & & & & \\ -y_3 & & y_1 & & & -y_3^p & & y_1^p & & & & & & & \\ y_4 & & & y_1 & & y_4^p & & & y_1^p & & & & & & \\ -y_5 & & & & y_1 & -y_5^p & & & & y_1^p & & & & & \\ & -y_3 & -y_2 & & & & -y_3^p & -y_2^p & & & & & & & \\ & y_4 & & -y_2 & & & y_4^p & & -y_2^p & & & & & & \\ & -y_5 & & & -y_2 & & -y_5^p & & & -y_2^p & & & & & \\ & & y_4 & y_3 & & & & y_4^p & y_3^p & & & & & & \\ & & -y_5 & & y_3 & & & -y_5^p & & y_3^p & & & & & \\ & & & -y_5 & -y_4 & & & & -y_5^p & -y_4^p & & & & & \\ & & & & & y_2 & y_1 & & & & y_2^p & y_1^p & & & \\ & & & & & -y_3 & & y_1 & & & -y_3^p & & y_1^p & & \\ & & & & & y_4 & & & y_1 & & y_4^p & & & y_1^p & \\ & & & & & -y_5 & & & & y_1 & -y_5^p & & & & y_1^p \\ & & & & & & -y_3 & -y_2 & & & & -y_3^p & -y_2^p & & \\ & & & & & & y_4 & & -y_4 & & & y_4^p & & -y_2^p & \\ & & & & & & -y_5 & & & -y_2 & & -y_5^p & & & -y_2^p \\ & & & & & & & y_4 & y_3 & & & & y_4^p & y_3^p & \\ & & & & & & & -y_5 & & y_3 & & & -y_5^p & & y_3^p \\ & & & & & & & & -y_5 & -y_4 & & & & -y_5^p & -y_4^p \end{pmatrix} \end{array}$$

We represent the third map $F_3 \longrightarrow F_2$ with respect to the chosen generators by a matrix

$$\Theta = \begin{array}{r} \\ X_{\widehat{1}} \\ X_{\widehat{2}} \\ X_{\widehat{3}} \\ X_{\widehat{4}} \\ X_{\widehat{5}} \\ Y_{\widehat{1}} \\ Y_{\widehat{2}} \\ Y_{\widehat{3}} \\ Y_{\widehat{4}} \\ Y_{\widehat{5}} \\ Z_{\widehat{1}} \\ Z_{\widehat{2}} \\ Z_{\widehat{3}} \\ Z_{\widehat{4}} \\ Z_{\widehat{5}} \end{array} \begin{array}{c} \begin{array}{cccc} X & Y & Z & W \end{array} \\ \begin{pmatrix} y_1 & y_1^p & & \\ -y_2 & -y_2^p & & \\ y_3 & y_3^p & & \\ -y_4 & -y_4^p & & \\ y_5 & y_5^p & & \\ & y_1 & y_1^p & \\ & -y_2 & -y_2^p & \\ & y_3 & y_3^p & \\ & -y_4 & -y_4^p & \\ & y_5 & y_5^p & \\ & y_1^p & y_1 & y_1^p \\ & -y_2^p & -y_2 & -y_2^p \\ & y_3^p & y_3 & y_3^p \\ & -y_4^p & -y_4 & -y_4^p \\ & y_5^p & y_5 & y_5^p \end{pmatrix} \end{array}$$

For local cohomology of $\overline{TU}_2$, we obtain an exact sequence

$$\begin{aligned} 0 \longleftarrow H^5_{\mathfrak{m}}(\overline{TU}^5_2) \longleftarrow \binom{5}{2} PC^\vee(-2p+8) \longleftarrow \binom{5}{3} [PC^\vee(-2p+6) \oplus PC^\vee(-4p+8)] \longleftarrow \\ \binom{5}{4} [PC^\vee(-2p+4) \oplus PC(-4p+6) \oplus PC^\vee(-6p+8)] \longleftarrow \\ \binom{5}{5} [PC^\vee(-2p+2) \oplus PC^\vee(-4p+4) \oplus PC^\vee(-6p+6) \oplus PC^\vee(-8p+8)] \\ \longleftarrow H^2_{\mathfrak{m}}(\overline{TU}^5_2) \longleftarrow 0. \end{aligned}$$

Dualizing, we obtain a short exact sequence (see Figure 5.12)

$$\begin{aligned} 0 \longrightarrow (H^5_{\mathfrak{m}}(\overline{TU}^5_2))^\vee \longrightarrow \overbrace{10PC(2p-8)}^{(Q_{\widehat{ijk}})^*} \longrightarrow 10[\overbrace{PC(2p-6)}^{(X_{\widehat{ij}})^*} \oplus \overbrace{PC(4p-8)}^{(Y_{\widehat{ij}})^*}] \longrightarrow \\ 5[\overbrace{PC(2p-4)}^{(X_{\widehat{i}})^*} \oplus \overbrace{PC(4p-6)}^{(Y_{\widehat{i}})^*} \oplus \overbrace{PC(6p-8)}^{(Z_{\widehat{i}})^*}] \xrightarrow{\Theta^t} [\overbrace{PC(2p-2)}^{X^*} \oplus \overbrace{PC(4p-4)}^{Y^*} \oplus \\ \overbrace{PC(6p-6)}^{Z^*} \oplus \overbrace{PC(8p-8)}^{W^*}] \longrightarrow (H^2_{\mathfrak{m}}(\overline{TU}^5_2))^\vee \longrightarrow 0. \end{aligned}$$

Here, the indicated generators $(Q_{\widehat{ijk}})^*$ of $10PC(2p-8)$ are dual to the generators $Q_{\widehat{ijk}}$, so that $(Q_{\widehat{123}})^* = Q^*_{45}$, $(Q_{\widehat{124}})^* = Q^*_{35}$, $(Q_{\widehat{125}})^* = Q^*_{34}$, $(Q_{\widehat{134}})^* = Q^*_{25}$, $(Q_{\widehat{135}})^* = Q^*_{24}$, $(Q_{\widehat{145}})^* = Q^*_{23}$, $(Q_{\widehat{234}})^* = Q^*_{15}$, $(Q_{\widehat{235}})^* = Q^*_{14}$, $(Q_{\widehat{245}})^* = Q^*_{13}$, $(Q_{\widehat{345}})^* = Q^*_{12}$ of Proposition 3.3.2 under local duality.

Also, the indicated generators $(X_{\widehat{ij}})^*$, $(Y_{\widehat{ij}})^*$ of $10PC(2p-6), 10PC(4p-8)$ are dual to the generators $X_{\widehat{ij}}$, $Y_{\widehat{ij}}$, so that $(X_{\widehat{12}})^* = X^*_{345}, (Y_{\widehat{12}})^* = Y^*_{345}$, $(X_{\widehat{13}})^* = X^*_{245}, (Y_{\widehat{13}})^* = Y^*_{245}$, $(X_{\widehat{14}})^* = X^*_{235}, (Y_{\widehat{14}})^* = Y^*_{235}$, $(X_{\widehat{15}})^* = X^*_{234}, (Y_{\widehat{15}})^* = Y^*_{234}$, $(X_{\widehat{23}})^* = X^*_{145}, (Y_{\widehat{23}})^* = Y^*_{145}$, $(X_{\widehat{24}})^* = X^*_{135}, (Y_{\widehat{24}})^* = Y^*_{135}$, $(X_{\widehat{25}})^* = X^*_{134}, (Y_{\widehat{25}})^* = Y^*_{134}$, $(X_{\widehat{34}})^* = X^*_{125}, (Y_{\widehat{34}})^* = Y^*_{125}$, $(X_{\widehat{35}})^* = X^*_{124}, (Y_{\widehat{35}})^* = Y^*_{124}$, $(X_{\widehat{45}})^* = Y^*_{123}, (Y_{\widehat{45}})^* = Y^*_{123}$.

The generators $(X_{\widehat{i}})^*, (Y_{\widehat{i}})^*, (Z_{\widehat{i}})^*$ of $5PC(2p-4)$, $5PC(4p-6)$, $5PC(6p-8)$ are dual to the generators X, Y, Z, so that $(X_{\widehat{1}})^* = X^*_{2345}, (Y_{\widehat{1}})^* = Y^*_{2345}, (Z_{\widehat{1}})^* = Z^*_{2345}$, $(X_{\widehat{2}})^* = X^*_{1345}, (Y_{\widehat{2}})^* = Y^*_{1345}, (Z_{\widehat{2}})^* = Z^*_{1345}$, $(X_{\widehat{3}})^* = X^*_{1245}, (Y_{\widehat{3}})^* = Y^*_{1245}, (Z_{\widehat{3}})^* = Z^*_{1245}$, $(X_{\widehat{4}})^* = X^*_{1235}, (Y_{\widehat{4}})^* = Y^*_{1235}, (Z_{\widehat{4}})^* = Z^*_{1235}$, $(X_{\widehat{5}})^* = X^*_{1234}, (Y_{\widehat{5}})^* = Y^*_{1234}, (Z_{\widehat{5}})^* = Z^*_{1234}$.

Similarly for X^*, Y^*, Z^*, W^*.

By local duality, the map $H^5_{\mathfrak{m}}(F_2)^\vee \longleftarrow H^5_{\mathfrak{m}}(F_3)^\vee$ is represented with respect to the dual generators by a matrix

$$\Theta^t = \begin{array}{c} \\ X^* \\ Y^* \\ Z^* \\ W^* \end{array} \begin{array}{c} \begin{array}{ccccccccccccccc} X^*_{\widehat{1}} & X^*_{\widehat{2}} & X^*_{\widehat{3}} & X^*_{\widehat{4}} & X^*_{\widehat{5}} & Y^*_{\widehat{1}} & Y^*_{\widehat{2}} & Y^*_{\widehat{3}} & Y^*_{\widehat{4}} & Y^*_{\widehat{5}} & Z^*_{\widehat{1}} & Z^*_{\widehat{2}} & Z^*_{\widehat{3}} & Z^*_{\widehat{4}} & Z^*_{\widehat{5}} \end{array} \\ \begin{pmatrix} y_1 & -y_2 & y_3 & -y_4 & y_5 & & & & & & & & & & \\ y_1^p & -y_2^p & y_3^p & -y_4^p & y_5^p & y_1 & -y_2 & y_3 & -y_4 & y_5 & & & & & \\ & & & & & y_1^p & -y_2^p & y_3^p & -y_4^p & y_5^p & y_1 & -y_2 & y_3 & -y_4 & y_5 \\ & & & & & & & & & & y_1^p & -y_2^p & y_3^p & -y_4^p & y_5^p \end{pmatrix} \end{array}$$

This calculation by Theorem 4.3.1 and (5.7) gives us

$$\begin{aligned} (H^5_{\mathfrak{m}}(\overline{TU}^5_2))^\vee &= \ker \begin{pmatrix} y_1 & -y_2 & y_3 & -y_4 & y_5 & & & & & & & & & & \\ y_1^p & -y_2^p & y_3^p & -y_4^p & y_5^p & y_1 & -y_2 & y_3 & -y_4 & y_5 & & & & & \\ & & & & & y_1^p & -y_2^p & y_3^p & -y_4^p & y_5^p & y_1 & -y_2 & y_3 & -y_4 & y_5 \\ & & & & & & & & & & y_1^p & -y_2^p & y_3^p & -y_4^p & y_5^p \end{pmatrix} \\ &= (\overline{TU}^5_5)(2p-5) \\ &= (PC(-2p-5))(2p-5) \\ &= PC(-10), \end{aligned}$$

and by (5.8),

$$(H^2_{\mathfrak{m}}(\overline{TU}^5_2))^\vee = \operatorname{coker} \begin{pmatrix} y_1 & -y_2 & y_3 & -y_4 & y_5 & & & & & & & & & & \\ y_1^p & -y_2^p & y_3^p & -y_4^p & y_5^p & y_1 & -y_2 & y_3 & -y_4 & y_5 & & & & & \\ & & & & & y_1^p & -y_2^p & y_3^p & -y_4^p & y_5^p & y_1 & -y_2 & y_3 & -y_4 & y_5 \\ & & & & & & & & & & y_1^p & -y_2^p & y_3^p & -y_4^p & y_5^p \end{pmatrix}$$

is generated in degrees $2p-2$, $4p-4$, $6p-6$, and $8p-8$. Its Hilbert series can be read off from the exact sequence (5.9),

$$
\begin{aligned}
[(H^2_{\mathfrak{m}}(\overline{TU}^5_2))^\vee] &= [PC(-10)] + 10[PC(2p-6)] + 10[PC(4p-8)] - 5([PC(2p-4)] \\
&\quad + [PC(4p-6)] + [PC(6p-8)]) + [PC(2p-2)] + [PC(4p-4)] \\
&\quad + [PC(6p-6)] + [PC(8p-8)] - 10[PC(2p-8)] \\
&= \frac{t^{10} + 10t^{-2p+6} + 10t^{-4p+8} - 5(t^{-2p+4} + t^{-4p+6} + t^{-6p+8}) + t^{-2p+2}}{(1-t^2)^5} \\
&\quad + \frac{t^{-4p+4} + t^{-6p+6} + t^{-8p+8} - 10t^{-2p+8}}{(1-t^2)^5} \\
&= \frac{t^{-8p+8}(t^{8p+2} + 10t^{6p-2} + 10t^{4p} - 5(t^{6p-4} + t^{4p-2} + t^{2p}) + t^{6p-6}}{(1-t^2)^5} \\
&\quad + \frac{t^{4p-4} + t^{2p-2} + 1 - 10t^{6p})}{(1-t^2)^5} \\
&= \frac{y^{-4p+4}(y^{4p+1} + 10y^{3p-1} + 10y^{2p} - 5(y^{3p-2} + y^{2p-1} + y^p) + y^{3p-3}}{(1-y)^5} \\
&\quad + \frac{y^{2p-2} + y^{p-1} + 1 - 10y^{3p})}{(1-y)^5} \\
&= \frac{Y^{-4}(y^{4p+1} - 10y^{3p} + 10y^{3p-1} - 5y^{3p-2} + y^{3p-3} + 10y^{2p} - 5y^{2p-1}}{(1-y)^5} \\
&\quad + \frac{y^{2p-2} - 5y^p + y^{p-1} + 1)}{(1-y)^5} \\
&= \frac{Y^{-4}(y^5Y^4 - 10y^3Y^3 + 10y^2Y^3 - 5yY^3 + Y^3 + 10y^2Y^2 - 5yY^2 + Y^2)}{(1-y)^5} \\
&\quad - \frac{5yY + Y + 1}{(1-y)^5}.
\end{aligned}
$$

2p+8 $\mathbf{X}_{12345}$ 1			
2p+6 $\mathbf{X}_{ijkl}$ 5	4p+6 $\mathbf{Y}_{12345}$ 1		
2p+4 $\mathbf{X}_{ijk}$ 10	4p+4 $\mathbf{Y}_{ijkl}$ 5	6p+4 $\mathbf{Z}_{12345}$ 1	
2p+2 $\mathbf{Q}_{ij}$ 10	4p+2 $\mathbf{Y}_{ijk}$ 10	6p+2 $\mathbf{Z}_{ijkl}$ 5	8p+2 $\mathbf{W}_{12345}$ 1

Figure 5.11: The double complex resolution for $\overline{TU}_2^5$.

2p - 2 $\mathbf{X}^*_{12345}$ 1	2p - 4 $\mathbf{X}^*_{ijkl}$ 5	2p - 6 $\mathbf{X}^*_{ijk}$ 10	2p - 8 $\mathbf{Q}^*_{ij}$ 10
	4p - 4 $\mathbf{Y}^*_{12345}$ 1	4p - 6 $\mathbf{Y}^*_{ijkl}$ 5	4p - 8 $\mathbf{Y}^*_{ijk}$ 10
		6p - 6 $\mathbf{Z}^*_{12345}$ 1	6p - 8 $\mathbf{Z}^*_{ijkl}$ 5
			8p - 8 $\mathbf{W}^*_{12345}$ 1

Figure 5.12: The double complex resolution for the local cohomology of $\overline{TU}_2^5$.

Chapter 6

Calculating connective K-homology using Local Cohomology

The local cohomology Theorem states that there is a spectral sequence

$$E_2^{p,q} = H_{JU}^{p,q}(ku^*(BG)) \Longrightarrow ku_*(BG) \quad p, q \geq 0, \tag{6.1}$$

for all finite groups G, where the differential $d_r \colon E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ is of the cohomological bidegree $(r, -r+1)$, $r \geq 1$, and $JU = \ker(ku^*(BG) \longrightarrow ku^*)$. In this Chapter, we apply it for $p = 3$ and $r \leq 2$ to calculate the connective K-homology $ku_*(BV(r))$.

§ 6.1 The strategy

The strategy to calculate $ku_*(BV(r))$ is as follows. First, we begin with calculating the local cohomology of QU, $H_{JU}^*(QU)$. The local cohomology of QU is 0 except in degrees 0 and 1 (i.e., $H_{JU}^i(QU) = 0$ for $i \geq 2$), [10]. To do this, by definition 2.1.1, we recall that QU is the image of $ku^*(BV(r))$ in $K^*(BV(r))$, and is therefore for abelian groups the Rees ring of the completed representation ring for the augmentation ideal JU. The idea here involves defining a special element y^* in the representation ring RU so that the principal ideal (y^*) is a reduction of JU. The notation y^* is chosen by analogy with [10, 4.4.3, page 89]. Since local cohomology only depends on the radical of the ideal, we may replace JU by a convenient smaller ideal $(y^*) \subseteq JU$ which is a reduction of it. The central point of calculating $H_{JU}^*(QU)$ is that if (y^*) is a reduction of JU, then $H_{JU}^*(QU) = H_{(y^*)}^*(QU)$.

Second, we then apply the local cohomology Theorem as given in (6.1).

Definition 6.1.1. Let R be a commutative ring, and let J be an ideal with the filtration by subideals

$$R \supseteq J \supseteq J^2 \supseteq J^3 \supseteq \ldots \supseteq J^k.$$

We say an ideal I is a reduction of J, if $I \subseteq J$, there exist k_0 and s such that $IJ^k = J^{k+s}$, for $k \geq k_0$.

Remark 6.1.2. In all calculations, $H^*_{JU}(QU)$ can be obtained as $H^*_{(y^*)}(QU)$ for an element $y^* \in JU$, and JU is the radical of the principal ideal (y^*) (i.e., $JU = \sqrt{(y^*)}$).

We need to give a formal definition of the element y^*.

Definition 6.1.3. For any rank r and odd primes p, we define y^* in the representation ring RU as follows.

$$y^* := \sum_{\alpha \in V(r)^\vee} e(\alpha) = \sum_{\alpha \in V(r)^\vee} (1-\alpha) = p^r - \rho \in JU,$$

where $e(\alpha) := (1-\alpha)$ is the Euler class in RU, and ρ is the regular representation of $V(r)$ given by

$$\begin{aligned} \rho &= \sum_{\alpha \in V(r)^\vee} \alpha \\ &= p^r - \sum_{\alpha \in V(r)^\vee} (1-\alpha) \\ &= p^r - \sum_{\alpha \in V(r)^\vee} v e_{ku}(\alpha), \end{aligned}$$

where $e_{ku}(\alpha) := \frac{1-\alpha}{v}$ is the Euler class in $ku^*(BV(r))$, and v is the Bott element of degree 2. This makes the element y^* take the value 0 at the identity and p^r at the other elements (i.e., $y^* = 0 \quad p^r \quad p^r \quad \cdots \quad p^r \in JU^{2r}$).

This defines an element of $K^{4r}(BV(r))$. We will see in our cases that this lies in JU^{2r} and hence here it comes from $ku^{4r}(BV(r))$.

Proposition 6.1.4. *The local cohomology of QU is calculated by*

$$H^i_{JU}(QU) = \begin{cases} ku^* \cdot \rho & if\ i=0 \\ QU[1/y^*]/QU & if\ i=1 \\ 0 & otherwise, \end{cases}$$

and $H^1_{JU}(QU)$ has an element of order p^{r-1}, so that in all cases $H^1_{JU}(QU)$ is non-zero in some negative degree.

Proof. Details can be found in [10, page 91-93]. □

From here on we specialise to the prime $p = 3$.

§ 6.2 Rank 1

The aim of this section is to calculate $ku_*(BV(1))$, and we start first to calculate $H^1_{(y^*)}(QU)$.

6.2.1 Ideals JU_k

In this section, we need only calculate the augmentation ideals $JU_k,\ k \geq 1$ for $V(1)$.

It will help to use the following Lemma for calculational purposes [16].

Lemma 6.2.1. *For each $n \in \mathbb{Z}$*

$$(QU)_n = (QU)^{-n} \cong \begin{cases} \widehat{RU} & if\ n\ is\ even\ and \geq 0 \\ \widehat{JU_k} & if\ n = -2k < 0, \end{cases}$$

where $\widehat{RU}$ is the completed complex representation ring and $\widehat{JU_k}$ is the p-adic completion of JU_k for all k, the augmentation ideal generated by the Chern classes of k-dimensional representations, so that if G is a p-group, then

$$\widehat{JU_k} \cong \mathbb{Z}_p^\wedge \otimes JU_k,$$

where $JU_k = (JU_1)^k$, for $k \geq 1$. Note that JU_1 is the augmentation ideal generated by the first Chern class $c_1(\alpha)$, where α is a 1-dimensional representation of G.

Remark 6.2.2. Here, we now consider $p = 3$ and write $\gamma_1 = 1 - \omega$ and $\gamma_2 = 1 - \omega^2$, where $\omega = e^{2\pi i/3}$.

To start with, $\widehat{JU_1}$. There are two characters u_1, u_2 generating $\widehat{JU_1}$ over $\mathbb{Z}_3^\wedge$, where $u_1 = e(\widehat{x}) = 1 - \widehat{x},\ u_2 = e((\widehat{x})^2) = 1 - (\widehat{x})^2$ so that $\widehat{x}$ indicates a natural representation of the quotient by $\langle x \rangle$. This means if $V(1) = \{e, x, x^2\}$, then

$$(QU)_{-2} \cong \widehat{JU_1} \cong \mathbb{Z}_3^\wedge \otimes JU_1 = \mathbb{Z}_3^\wedge \left\langle \begin{array}{rccc} & e & x & x^2 \\ u_1 : (& 0 & \gamma_1 & \gamma_2 \ ) \\ u_2 : (& 0 & \gamma_2 & \gamma_1 \ ) \end{array} \right\rangle.$$

Next, $\widehat{JU_2}$. By definition, $JU_2 := JU_1 {\cdot} JU_1 = (JU_1)^2$, an ideal generated by $c_1(\alpha)c_1(\beta)$ for each pair α, β of 1-dimensional representations. Using a straightforward calculation to obtain a 2×2 matrix excepting e-column. This calculation shows us $y^* \in (JU_1)^2 = ku^4(BV(1)) = ku_{-4}(BV(1))$, and it is in degree -4.

$$(QU)_{-4} \cong \widehat{JU_2} \cong \mathbb{Z}_3^\wedge \otimes JU_2 = \mathbb{Z}_3^\wedge \left\langle \begin{array}{rccc} & e & x & x^2 \\ v_1 : (& 0 & 3\omega & 3\omega^2 \ ) \\ v_2 : (& 0 & 3 & 3 \ ) \end{array} \right\rangle,$$

where $v_1 = u_1 u_1,\ v_2 = u_1 u_2$.

Now, $\widehat{JU_3}$. As above, $JU_3 := JU_1 {\cdot} (JU_1)^2 = (JU_1)^3$, an ideal generated by $c_1(\alpha)c_1(\beta)c_1(\gamma)$ for each α, β, γ of 1-dimensional representations. We find

$$(QU)_{-6} \cong \widehat{JU_3} \cong \mathbb{Z}_3^\wedge \otimes JU_3 = \mathbb{Z}_3^\wedge \left\langle \begin{array}{rccc} & e & x & x^2 \\ w_1 : (& 0 & 3\gamma_1 & 3\gamma_2 \ ) \\ w_2 : (& 0 & 3\gamma_2 & 3\gamma_1 \ ) \end{array} \right\rangle,$$

where $w_1 = y^* \cdot u_1,\ w_2 = y^* \cdot u_2$.

From this, we obtain $JU_3 = 3JU_1$, and $y^* \cdot JU_1 = JU_3$. This means our calculations for the rest of the work give us

$$y^* \cdot JU_2 = JU_4 = JU_1 \cdot (JU_1)^3 = JU_1 \cdot 3JU_1 = 3(JU_1)^2.$$

Thus, we obtain

$$(QU)_{-8} \cong \widehat{JU_4} \cong \mathbb{Z}_3^\wedge \otimes JU_4 = \mathbb{Z}_3^\wedge \left\langle \begin{array}{cccccc} & & e & x & x^2 & \\ t_1 : & (& 0 & 9\omega & 9\omega^2 &) \\ t_2 : & (& 0 & 9 & 9 &) \end{array} \right\rangle,$$

where $t_1 = y^* \cdot v_1,\ t_2 = y^* \cdot v_2$.

Similarly

$$y^* \cdot JU_3 = JU_5 = JU_1 \cdot (JU_1)^4 = JU_1 \cdot 3(JU_1)^2 = 3(JU_1)^3 = 9JU_1.$$

We find

$$(QU)_{-10} \cong \widehat{JU_5} \cong \mathbb{Z}_3^\wedge \otimes JU_5 = \mathbb{Z}_3^\wedge \left\langle \begin{array}{cccccc} & & e & x & x^2 & \\ r_1 : & (& 0 & 9\gamma_1 & 9\gamma_2 &) \\ r_2 : & (& 0 & 9\gamma_2 & 9\gamma_1 &) \end{array} \right\rangle,$$

where $r_1 = (y^*)^2 \cdot u_1,\ r_2 = (y^*)^2 \cdot u_2$.

Finally, we find that the general formula for $V(1)$ is calculated by $(y^*) \cdot (JU_1)^k = (JU_1)^{k+2}$, for $k \geq 1$.

6.2.2 $H^0_{JU}(QU)$

In this section, we want to calculate $H^0_{JU}(QU)$. To do this, we know that QU has no v-torsion, and TU has v-torsion, $QU \subseteq K^*(BV(1)) \xrightarrow{\cong} ku^*(BV(1))[1/v]$. Hence $H^0_{JU}(QU)$ is the submodule of $H^0_{JU}(K^*(BV(1))) = K^* \cdot \rho$ consisting of elements from $ku^*(BV(1))$. This proves $H^0_{JU}(QU) = ku^* \cdot \rho$.

6.2.3 $H^1_{JU}(QU)$

The purpose of this section is to calculate $H^1_{JU}(QU)$ by using the principal ideal (y^*).

By definition 6.1.3, $y^* = p^r - \rho$. Since we have $V(1)$, $\overline{TU} = 0$ and then $TU = 0$, it is not hard to calculate $H^1_{JU}(QU) = H^1_{(y^*)}(QU)$, and see that $H^1_{(y^*)}(QU) = 0$ for negative degrees.

The main point here is that $H^1_{(y^*)}(QU) = 0$ below degree $-2r$ and the order of its $\mathbb{Z}$-torsion increases with degree. To prove this, it will be helpful to use Proposition 6.2.5 [36].

Definition 6.2.3 ([36])**.** Let R be a ring. Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of R-modules. A direct system consists of modules M_n, $n \in \mathbb{N}$ and module maps $a_{m,n} : M_m \longrightarrow M_n$, for each $m, n \in \mathbb{N}$ with $m \leq n$ such that the following holds:

(1) a_{nn} is the identity mapping of M_n for all $n \in \mathbb{N}$;

(2) a_{mn} is the composite of $a_{n-1n}a_{n-2n-1}\dots a_{mm+1}$.

Remark 6.2.4. If D is the direct sum of M_n and each module M_n is identified by its canonical image in D, and E is the submodule of D generated by all elements of the form $x_n - a_{m,n}(x_n)$ for $x_n \in M_n$, then the direct limit of $(M_n, a_{m,n})$ is defined by $\varinjlim M_n := D/E$.

Proposition 6.2.5 ([36])**.** *Let $(M_n, a_{m,n})$ be a direct system of R-modules over $\mathbb{N}$. If $A \in \mathbb{N}$ such that $a_{(A+k)(A+k+1)}$ is an R-isomorphism for all $k \geq 0$, then* $\underrightarrow{\operatorname{colim}} M_n \cong M_A$.

Consider the exact sequence

$$0 \longrightarrow H^0_{JU}(QU) \longrightarrow QU \longrightarrow QU[1/y^*] \longrightarrow H^1_{(y^*)}(QU) \longrightarrow 0.$$

Since QU is a graded module, so both $QU[1/y^*]$ and $H^1_{(y^*)}(QU)$ are graded. We aim to calculate $H^1_{(y^*)}(QU)$ in degree n as

$$H^1_{(y^*)}(QU)_n = \operatorname{coker}(QU_n \longrightarrow \big(QU[1/y^*])_n\big).$$

Note that

$$\varinjlim\Big(QU_n \xrightarrow{(y^*)^a} QU_{n-4ar} \xrightarrow{(y^*)^a} QU_{n-2(4ar)} \xrightarrow{(y^*)^a} \cdots\Big) \overset{g}{\cong} (QU[1/y^*])_n,$$

where $\cong^{(g)}$ is an isomorphism given by the natural map g, and a is the least number so that $n - 4ar < 0$.

In fact, this filtered direct system is constant at $QU_{-n} = \widehat{JU}_k$, if $n = 2k$, for $2 \leq k \leq 5$. Applying Proposition 6.2.5, we see that $\widehat{JU}_k \cong (QU[1/y^*])_n$, for $k \in \{2, 3, 4, 5\}$.

Therefore

$$H^1_{(y^*)}(QU)_n = \operatorname{coker}(QU_n \xrightarrow{(y^*)^a} \widehat{JU}_k),$$

Now, it is easy to see that $H^1_{(y^*)}(QU)_n = 0$, for $n \leq -2r = -4$.

If $n \geq -4$ and $k \geq 2$. We have

$$\begin{aligned}
H^1_{(y^*)}(QU)_{-4} &= \widehat{JU}_6/y^* \cdot \widehat{JU}_2. \\
H^1_{(y^*)}(QU)_{-2} &= \widehat{JU}_5/y^* \cdot \widehat{JU}_1. \\
H^1_{(y^*)}(QU)_{0} &= \widehat{JU}_4/y^* \cdot RU. \\
H^1_{(y^*)}(QU)_{2} &= \widehat{JU}_3/y^* \cdot RU. \\
H^1_{(y^*)}(QU)_{4} &= \widehat{JU}_6/(y^*)^k \cdot RU. \\
H^1_{(y^*)}(QU)_{6} &= \widehat{JU}_5/(y^*)^k \cdot RU. \\
H^1_{(y^*)}(QU)_{8} &= \widehat{JU}_4/(y^*)^k \cdot RU. \\
H^1_{(y^*)}(QU)_{10} &= \widehat{JU}_3/(y^*)^k \cdot RU. \\
&\vdots
\end{aligned}$$

From this discussion, we find $H^1_{(y^*)}(QU)_{-2n} = 0$, for $n \geq 1$, and

$$H^1_{(y^*)}(QU)_{+2n} = \operatorname{coker}(RU \xrightarrow{(y^*)^a} QU_{n-4ar}), \text{ for } n - 4ar < 0.$$

We start to calculate $H^1_{(y^*)}(QU)$ for positive degrees.

- If $n = 0$, then $H^1_{(y^*)}(QU)_0 \cong \widehat{JU_2}/y^* \cdot RU$.

First, we need to write $y^* \cdot RU$ as a matrix so that $RU = \mathbb{Z} + JU$.

$$y^* \cdot RU = \mathbb{Z}_3^\wedge \left\langle \begin{array}{r|ccc} & e & x & x^2 \\ \hline y_0 = y^* \cdot 1 : & 0 & 3 & 3 \\ y_1 = y^* \cdot u_1 : & 0 & 3\gamma_1 & 3\gamma_2 \\ y_2 = y^* \cdot u_2 : & 0 & 3\gamma_2 & 3\gamma_1 \end{array} \right\rangle$$

Next, we can now express $y^* \cdot RU$ as a linear combination from JU_2. It is easy to see $\widehat{JU_2}/y^* \cdot RU \cong 0$, and by direct calculation, we obtain

$$H^1_{(y^*)}(QU)_0 \cong \widehat{JU_2}/y^* \cdot RU \cong 0.$$

- If $n = 2$, then $H^1_{(y^*)}(QU)_2 \cong \widehat{JU_3}/(y^*)^2 \cdot RU$.

Now, we want to represent $(y^*)^2 \cdot RU$ by a matrix

$$(y^*)^2 \cdot RU = \mathbb{Z}_3^\wedge \left\langle \begin{array}{r|ccc} & e & x & x^2 \\ \hline x_0 = (y^*)^2 \cdot 1 : & 0 & 9 & 9 \\ x_1 = (y^*)^2 \cdot u_1 : & 0 & 9\gamma_1 & 9\gamma_2 \\ x_2 = (y^*)^2 \cdot u_2 : & 0 & 9\gamma_2 & 9\gamma_1 \end{array} \right\rangle$$

As above, we write $(y^*)^2 \cdot RU$ as a linear combination from JU_3, and we see

$$\begin{array}{r|cc|} & w_1 & w_2 \\ x_0 : & 1 & 0 \\ x_1 : & 3 & 0 \\ x_2 : & 0 & 3 \end{array}$$

By direct row operations

$$\begin{array}{c|cc|} & w_1 & w_2 \\ x_0 : & 1 & 0 \\ x_2 : & 0 & 3 \end{array}$$

Therefore

$$H^1_{(y^*)}(QU)_2 \cong \widehat{JU_3}/(y^*)^2 \cdot RU \cong \mathbb{Z}/3.$$

- if $n = 4$, then $H^1_{(y^*)}(QU)_4 \cong \widehat{JU_2}/(y^*)^2 \cdot RU.$

We have obtained $(y^*)^2 \cdot RU$, as in degree 2, is represented by a matrix

$$(y^*)^2 \cdot RU = \mathbb{Z}_3^\wedge \left\langle \begin{array}{cc} & \begin{array}{ccc} e & x & x^2 \end{array} \\ x_0 : & \begin{pmatrix} 0 & 9 & 9 \end{pmatrix} \\ x_1 : & \begin{pmatrix} 0 & 9\gamma_1 & 9\gamma_2 \end{pmatrix} \\ x_2 : & \begin{pmatrix} 0 & 9\gamma_2 & 9\gamma_1 \end{pmatrix} \end{array} \right\rangle$$

Now, we write $(y^*)^2 \cdot RU$ as a linear combination from JU_2, and we see

$$\begin{array}{c|cc|} & v_1 & v_2 \\ x_0 : & 0 & 3 \\ x_1 : & -3 & 3 \\ x_2 : & -3 & 3 \end{array}$$

Again, it is not hard to see the matrix $\widehat{JU_2}/(y^*)^2 \cdot RU$, and we do direct operation to obtain

$$\begin{array}{c|cc|} & v_1 & v_2 \\ x_0 : & 3 & 0 \\ x_1 : & 0 & 3 \end{array}$$

Hence

$$H^1_{(y^*)}(QU)_4 \cong \widehat{JU_2}/(y^*)^2 \cdot RU \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3.$$

- if $n = 6$, then $H^1_{(y^*)}(QU)_6 \cong \widehat{JU_3}/(y^*)^3 \cdot RU.$

We can represent $(y^*)^3 \cdot RU$ by a matrix

$$(y^*)^3 \cdot RU = \mathbb{Z}_3^\wedge \left\langle \begin{array}{cc} & \begin{array}{ccc} e & x & x^2 \end{array} \\ z_0 = (y^*)^3 \cdot 1 : & \begin{pmatrix} 0 & 27 & 27 \end{pmatrix} \\ z_1 = (y^*)^3 \cdot u_1 : & \begin{pmatrix} 0 & 27\gamma_1 & 27\gamma_2 \end{pmatrix} \\ z_1 = (y^*)^3 \cdot u_2 : & \begin{pmatrix} 0 & 27\gamma_2 & 27\gamma_1 \end{pmatrix} \end{array} \right\rangle$$

We write $(y^*)^2 \cdot RU$ as a linear combination from JU_3, and we find

$$\begin{array}{c|cc|} & w_1 & w_2 \\ z_0 : & 3 & 0 \\ z_1 : & 9 & 0 \\ z_2 : & 0 & 9 \end{array}$$

We do row calculations to obtain $\widehat{JU_3}/(y^*)^3 \cdot RU$,

$$\begin{array}{c|cc|} & w_1 & w_2 \\ z_0: & 3 & 0 \\ z_2: & 0 & 9 \end{array}$$

Therefore,

$$H^1_{(y^*)}(QU)_6 \cong \widehat{JU_3}/(y^*)^3 \cdot RU \cong \mathbb{Z}/3 \oplus \mathbb{Z}/9.$$

Finally, we use the same argument for the rest of this work of positive degrees because all calculations rely on $(y^*)^a \cdot RU$, for $a > 0$. The results are displayed in Figure 6.1.

$H^1_{(y^*)}(QU)$	$H^0_{(y^*)}(QU)$	degrees
$\vdots$	$\vdots$	$\vdots$
$[81] \oplus [243]$	$\mathbb{Z}$	18
0	0	17
$[81]^2$	$\mathbb{Z}$	16
0	0	15
$[27] \oplus [81]$	$\mathbb{Z}$	14
0	0	13
$[27]^2$	$\mathbb{Z}$	12
0	0	11
$[9] \oplus [27]$	$\mathbb{Z}$	10
0	0	9
$[9]^2$	$\mathbb{Z}$	8
0	0	7
$[3] \oplus [9]$	$\mathbb{Z}$	6
0	0	5
$[3]^2$	$\mathbb{Z}$	4
0	0	3
$[3]$	$\mathbb{Z}$	2
0	0	1
0	$\mathbb{Z}$	0
0	0	-2

Figure 6.1: $H^0_{(y^*)}(QU)$ and $H^1_{(y^*)}(QU)$.
The symbol $[n]$ denotes a cyclic group of order n.

6.2.4 $ku_*(BV(1))$

If we have $V(1)$, then there are no differentials in negative degrees and the local cohomology spectral sequence obviously collapses.

Now, we have

$$ku_*(BV(1)) = ku_{ev}(BV(1)) \oplus ku_{od}(BV(1)),$$

where

$$ku_{ev}(BV(1)) = H^0_{JU}(ku^*BV(1)) = ku^* \cdot \rho,$$

and

$$ku_{od}(BV(1)) = \Sigma^{-1}(3^{r-1}H^1_{(y^*)}(ku^*(BV(1))) = \Sigma^{-1}H^1_{(y^*)}(QU).$$

§ 6.3 Rank 2

The aim of this section is to calculate $ku_*(BV(2))$. This can be done by using the local cohomology Theorem. Since the local cohomology vanishes above the rank of the group $V(2)$, the local cohomology spectral sequence is a finite spectral sequence.

6.3.1 Ideals JU_k

In this section, we want to calculate $JU_k,\ k \geq 1$ for $V(2)$. We have $\ker(\alpha) = \langle y \rangle$, $\ker(\beta) = \langle x \rangle$, and $(y^*) \subseteq JU \subseteq RU \subseteq \mathbb{Z}[w]^9$, where α, β are 1-dimensional representations of $V(2)$. Again, the notation $\widehat{x}, \widehat{y}$ as given in Figure 6.2, indicates a natural representations of the quotient by $\langle x \rangle$, $\langle y \rangle$ respectively.

$V(2) =$	e	x	x^2	y	y^2	xy	x^2y^2	xy^2	x^2y
$e(\widehat{x}) = 1 - \widehat{x}$	0	0	0	γ_1	γ_2	γ_1	γ_2	γ_2	γ_1
$e((\widehat{x})^2) = 1 - (\widehat{x})^2$	0	0	0	γ_2	γ_1	γ_2	γ_1	γ_1	γ_2
$e(\widehat{y}) = 1 - \widehat{y}$	0	γ_1	γ_2	0	0	γ_1	γ_2	γ_1	γ_2
$e(\widehat{x}\widehat{y}) = 1 - (\widehat{xy})$	0	γ_1	γ_2	γ_1	γ_2	γ_2	γ_1	0	0
$e((\widehat{x})^2\widehat{y}) = 1 - (\widehat{x})^2\widehat{y}$	0	γ_1	γ_2	γ_2	γ_1	0	0	γ_2	γ_1
$e((\widehat{y})^2) = 1 - (\widehat{y})^2$	0	γ_2	γ_1	0	0	γ_2	γ_1	γ_2	γ_1
$e(\widehat{x}(\widehat{y})^2) = 1 - (\widehat{x})(\widehat{y})^2$	0	γ_2	γ_1	γ_1	γ_2	0	0	γ_1	γ_2
$e((\widehat{x})^2(\widehat{y})^2) = 1 - (\widehat{xy})^2$	0	γ_2	γ_1	γ_2	γ_1	γ_1	γ_2	0	0

Figure 6.2: The augmentation ideal, JU_1.

Lemma 6.3.1. *If α, β are 1-dimensional representations of $V(2)$, then we have*

$$e(\alpha\beta) = e(\alpha) + e(\beta) - e(\alpha)e(\beta),$$

where $e(\alpha) = 1 - \alpha$, $e(\beta) = 1 - \beta$, and $e(\alpha\beta) = 1 - \alpha\beta$.

Proof. To prove this, we note that

$$\begin{aligned} e(\alpha)e(\beta) &= (1-\alpha)(1-\beta) \\ &= 1 - \alpha - \beta + \alpha\beta \\ &= (1-\alpha) + (1-\beta) - (1-\alpha\beta) \\ &= e(\alpha) + e(\beta) - e(\alpha\beta). \end{aligned}$$

□

Remark 6.3.2. From Figure 6.2, note that the first augmentation ideal JU_1 is defined by an 8×8 matrix with complex entries $\gamma_1 = 1 - \omega$ and $\gamma_2 = 1 - \omega^2$ excepting the e-column, which is 0. Since $\chi(x^{-1}) = \overline{\chi(x)}$, we need only display the values on x, y, xy, xy^2. Hence we just display an 8×4 matrix.

We start first to calculate JU_k with the augmentation ideal $JU_1 = \mathbb{Z}\Big\{e(\delta_i) \mid 1 \leq i \leq 8\Big\}$. We note that $\langle \gamma_1, \gamma_2 \rangle$ is an ideal in $\mathbb{Z}[\omega]$, but JU_1 is an ideal in RU, then the character values of elements of JU_1 all lie in the ideal $\langle \gamma_1, \gamma_2 \rangle$. In Remark 6.3.2, we find $\begin{pmatrix} e & x & y & xy & xy^2 & x^{-1} & y^{-1} & (xy)^{-1} & (xy^2)^{-1} \end{pmatrix}$ for the character u_1: $\begin{pmatrix} 0 & 0 & \gamma_1 & \gamma_1 & \gamma_2 & 0 & \overline{\gamma_1} & \overline{\gamma_1} & \overline{\gamma_2} \end{pmatrix}$. In abbreviated form we work with $\begin{pmatrix} x & y & xy & xy^2 \end{pmatrix}$, for u_1: $\begin{pmatrix} 0 & \gamma_1 & \gamma_1 & \gamma_2 \end{pmatrix}$, and similarly for the other characters.

$$(QU)_{-2} \cong \widehat{JU_1} \cong \mathbb{Z}_3^\wedge \otimes JU_1 = \mathbb{Z}_3^\wedge \left\langle \begin{array}{cccccc} & x & y & xy & xy^2 & \\ u_1 : (& 0 & \gamma_1 & \gamma_1 & \gamma_2 &) \\ u_2 : (& 0 & \gamma_2 & \gamma_2 & \gamma_1 &) \\ u_3 : (& \gamma_1 & 0 & \gamma_1 & \gamma_1 &) \\ u_4 : (& \gamma_1 & \gamma_1 & \gamma_2 & 0 &) \\ u_5 : (& \gamma_1 & \gamma_2 & 0 & \gamma_2 &) \\ u_6 : (& \gamma_2 & 0 & \gamma_2 & \gamma_2 &) \\ u_7 : (& \gamma_2 & \gamma_1 & 0 & \gamma_1 &) \\ u_8 : (& \gamma_2 & \gamma_2 & \gamma_1 & 0 &) \end{array} \right\rangle,$$

where $u_1 = e(\widehat{x}) = 1 - \widehat{x}, u_2 = e((\widehat{x})^2) = 1 - (\widehat{x})^2, u_3 = e(\widehat{y}) = 1 - \widehat{y}, u_4 = e(\widehat{xy}) = 1 - (\widehat{xy}), u_5 = e((\widehat{x})^2\widehat{y}) = 1 - (\widehat{x})^2\widehat{y}, u_6 = e((\widehat{y})^2) = 1 - (\widehat{y})^2, u_7 = e(\widehat{x}(\widehat{y})^2) = 1 - (\widehat{x})(\widehat{y})^2$, and $u_8 = e((\widehat{x})^2(\widehat{y})^2) = 1 - (\widehat{xy})^2$.

Next, $(JU_1)^2$. By definition,

$$JU_2 := JU_1 \cdot JU_1 = (JU_1)^2 = = \mathbb{Z}\Big\{e(\delta_i)e(\delta_j) \mid 1 \leq i, j \leq 8\Big\} = \mathbb{Z}\Big\{\text{row reduced}\Big\},$$

where $e(\delta_i)e(\delta_j) \in (JU_1)^2$ because by Lemma 6.3.1, $e(\sigma)e(\delta_i)e(\delta_j) = \Big(e(\sigma) + e(\delta_i) - e(\sigma\delta_i)\Big)e(\delta_j) \in \mathbb{Z}\Big\{e(\delta_i)e(\delta_j)\Big\}$. Therefore there are 31 products -$e(\delta_i)e(\delta_j)$ for $1 \leq i, j \leq 8$. This means $(JU_1)^2$ is generated as an abelian group by 31 characters.

One may then use row reduction to obtain an 8×4 matrix for JU_2 so that the character values of elements of JU_2 all lie in the ideal $\langle \gamma_1, \gamma_2 \rangle$. Thus

$$(QU)_{-4} \cong \widehat{JU_2} \cong \mathbb{Z}_3^\wedge \otimes JU_2 = \mathbb{Z}_3^\wedge \left\langle \begin{array}{cccccc} & x & y & xy & xy^2 & \\ v_1 : (& 3\omega & 0 & 3\omega & 3\omega &) \\ v_2 : (& 3 & 0 & 3 & 3 &) \\ v_3 : (& 0 & 3\omega & 3\omega & 3\omega^2 &) \\ v_4 : (& 0 & 3 & 3 & 3 &) \\ v_5 : (& 0 & 0 & 3\omega & 3 &) \\ v_6 : (& 0 & 0 & 3\omega^2 & 3 &) \\ v_7 : (& 0 & 0 & 3 & 3\omega^2 &) \\ v_8 : (& 0 & 0 & 3 & 3\omega &) \end{array} \right\rangle,$$

where the generators from v_1 to v_8 come from calculation of the 31 products in $(JU_1)^2$.

Next, $(JU_1)^3$. By definition, $JU_3 := JU_1 \cdot (JU_1)^2 = (JU_1)^3$. As a result, we find this ideal is generated as an abelian group by 64 characters since it comes from the product of JU_1 with $(JU_1)^2$. We do some row calculation to obtain an 8×4 diagonal non-singular matrix

$$(QU)_{-6} \cong \widehat{JU_3} \cong \mathbb{Z}_3^\wedge \otimes JU_3 = \mathbb{Z}_3^\wedge \left\langle \begin{array}{rcccccc} & & x & y & xy & xy^2 & \\ w_1 : & (& 3\gamma_1 & 0 & 0 & 0 &) \\ w_2 : & (& 3\gamma_2 & 0 & 0 & 0 &) \\ w_3 : & (& 0 & 3\gamma_1 & 0 & 0 &) \\ w_4 : & (& 0 & 3\gamma_2 & 0 & 0 &) \\ w_5 : & (& 0 & 0 & 3\gamma_1 & 0 &) \\ w_6 : & (& 0 & 0 & 3\gamma_2 & 0 &) \\ w_7 : & (& 0 & 0 & 0 & 3\gamma_1 &) \\ w_8 : & (& 0 & 0 & 0 & 3\gamma_2 &) \end{array} \right\rangle,$$

where the generators from w_1 to w_8 come from calculation of the 64 products of JU_1 with $(JU_1)^2$.

As before, we note that the character values of elements of JU_3 all lie in the ideal $\langle \gamma_1, \gamma_2 \rangle$. On the other hand, our calculations after this ideal gives a diagonal non-singular matrices to all the other augmentation ideals.

We carry on with $(JU_1)^4$. By definition, $JU_4 := JU_1 \cdot (JU_1)^3 = (JU_1)^4$. The character values of elements of JU_4 all lie in the ideal $\langle \gamma_1, \gamma_2 \rangle$. From this calculation, note that $y^* \in (JU_1)^4 = ku^8(BV(2)) = ku_{-8}(BV(2))$, and it is of degree -8. We do row calculation

$$(QU)_{-8} \cong \widehat{JU_4} \cong \mathbb{Z}_3^\wedge \otimes JU_4 = \mathbb{Z}_3^\wedge \left\langle \begin{array}{rcccccc} & & x & y & xy & xy^2 & \\ t_1 : & (& 9 & 0 & 0 & 0 &) \\ t_2 : & (& 9\omega & 0 & 0 & 0 &) \\ t_3 : & (& 0 & 9 & 0 & 0 &) \\ t_4 : & (& 0 & 9\omega & 0 & 0 &) \\ t_5 : & (& 0 & 0 & 9 & 0 &) \\ t_6 : & (& 0 & 0 & 9\omega & 0 &) \\ t_7 : & (& 0 & 0 & 0 & 9 &) \\ t_8 : & (& 0 & 0 & 0 & 9\omega &) \end{array} \right\rangle,$$

where $t_1 = w_1 + w_2$, $t_2 = -\frac{1}{3}w_1^2$, $t_3 = w_3 + w_4$, $t_4 = -\frac{1}{3}w_3^2$, $t_5 = w_5 + w_6$, $t_6 = -\frac{1}{3}w_5^2$, $t_7 = w_7 + w_8$, and $t_8 = -\frac{1}{3}w_7^2$.

Next, $(JU_1)^5$. This is by definition $JU_5 := JU_1 \cdot (JU_1)^4 = (JU_1)^5$ so that the character values of elements of JU_5 all lie in the ideal $\langle \gamma_1, \gamma_2 \rangle$. We see

$$(QU)_{-10} \cong \widehat{JU_5} \cong \mathbb{Z}_3^\wedge \otimes JU_5 = \mathbb{Z}_3^\wedge \left\langle \begin{array}{r} \\ r_1 : \\ r_2 : \\ r_3 : \\ r_4 : \\ r_5 : \\ r_6 : \\ r_7 : \\ r_8 : \end{array} \begin{array}{cccccc} & x & y & xy & xy^2 & \\ (& 9\gamma_1 & 0 & 0 & 0 &) \\ (& 9\gamma_2 & 0 & 0 & 0 &) \\ (& 0 & 9\gamma_1 & 0 & 0 &) \\ (& 0 & 9\gamma_2 & 0 & 0 &) \\ (& 0 & 0 & 9\gamma_1 & 0 &) \\ (& 0 & 0 & 9\gamma_2 & 0 &) \\ (& 0 & 0 & 0 & 9\gamma_1 &) \\ (& 0 & 0 & 0 & 9\gamma_2 &) \end{array} \right\rangle,$$

where $r_1 = 3w_1, r_2 = 3w_2, r_3 = 3w_3, r_4 = 3w_4, r_5 = 3w_5, r_6 = 3w_6, r_7 = 3w_7$, and $r_8 = 3w_8$.

Now, $(JU_1)^6$. By definition, $JU_6 := JU_1 \cdot (JU_1)^5 = (JU_1)^6$ so that the character values of elements of JU_6 all lie in the ideal $\langle \gamma_1, \gamma_2 \rangle$. We see

$$(QU)_{-12} \cong \widehat{JU_6} \cong \mathbb{Z}_3^\wedge \otimes JU_6 = \mathbb{Z}_3^\wedge \left\langle \begin{array}{r} \\ z_1 : \\ z_2 : \\ z_3 : \\ z_4 : \\ z_5 : \\ z_6 : \\ z_7 : \\ z_8 : \end{array} \begin{array}{cccccc} & x & y & xy & xy^2 & \\ (& 27 & 0 & 0 & 0 &) \\ (& 27\omega & 0 & 0 & 0 &) \\ (& 0 & 27 & 0 & 0 &) \\ (& 0 & 27\omega & 0 & 0 &) \\ (& 0 & 0 & 27 & 0 &) \\ (& 0 & 0 & 27\omega & 0 &) \\ (& 0 & 0 & 0 & 27 &) \\ (& 0 & 0 & 0 & 27\omega &) \end{array} \right\rangle,$$

where $z_1 = 3t_1, z_2 = 3t_2, z_3 = 3t_3, z_4 = 3t_4, z_5 = 3t_5, z_6 = 3t_6, z_7 = 3t_7$, and $z_8 = 3t_8$.

We deduce $JU_7 = y^* \cdot (JU_1)^3 = (JU_1)^7$ so that the character values of elements of JU_7 all lie in the ideal $\langle \gamma_1, \gamma_2 \rangle$. We obtain

$$(QU)_{-14} \cong \widehat{JU_7} \cong \mathbb{Z}_3^\wedge \otimes JU_7 = \mathbb{Z}_3^\wedge \left\langle \begin{array}{r} \\ p_1 : \\ p_2 : \\ p_3 : \\ p_4 : \\ p_5 : \\ p_6 : \\ p_7 : \\ p_8 : \end{array} \begin{array}{cccccc} & x & y & xy & xy^2 & \\ (& 27\gamma_1 & 0 & 0 & 0 &) \\ (& 27\gamma_2 & 0 & 0 & 0 &) \\ (& 0 & 27\gamma_1 & 0 & 0 &) \\ (& 0 & 27\gamma_2 & 0 & 0 &) \\ (& 0 & 0 & 27\gamma_1 & 0 &) \\ (& 0 & 0 & 27\gamma_2 & 0 &) \\ (& 0 & 0 & 0 & 27\gamma_1 &) \\ (& 0 & 0 & 0 & 27\gamma_2 &) \end{array} \right\rangle,$$

where $p_1 = y^* \cdot w_1, p_2 = y^* \cdot w_2, p_3 = y^* \cdot w_3, p_4 = y^* \cdot w_4, p_5 = y^* \cdot w_5, p_6 = y^* \cdot w_6, p_7 = y^* \cdot w_7$, and $p_8 = y^* \cdot w_8$.

Furthermore, $JU_8 = y^* \cdot (JU_1)^4$ so that the character values of elements of JU_8 all lie in the ideal $\langle\gamma_1,\gamma_2\rangle$. We find

$$(QU)_{-16} \cong \widehat{JU_8} \cong \mathbb{Z}_3^\wedge \otimes JU_8 = \mathbb{Z}_3^\wedge \left\langle \begin{array}{r} \\ q_1 : \\ q_2 : \\ q_3 : \\ q_4 : \\ q_5 : \\ q_6 : \\ q_7 : \\ q_8 : \end{array} \begin{array}{cccc} x & y & xy & xy^2 \\ (\; 81 & 0 & 0 & 0 \;) \\ (\; 81\omega & 0 & 0 & 0 \;) \\ (\; 0 & 81 & 0 & 0 \;) \\ (\; 0 & 81\omega & 0 & 0 \;) \\ (\; 0 & 0 & 81 & 0 \;) \\ (\; 0 & 0 & 81\omega & 0 \;) \\ (\; 0 & 0 & 0 & 81 \;) \\ (\; 0 & 0 & 0 & 81\omega \;) \end{array} \right\rangle,$$

where $q_1 = 3p_1, q_2 = 3p_2, q_3 = 3p_3, q_4 = 3p_4, q_5 = 3p_5, q_6 = 3p_6, q_7 = 3p_7$, and $q_8 = 3p_8$.

Finally, we deduce the general formula to calculate the ideals JU_k for $V(2)$ is given by $(y^*) \cdot (JU_1)^k = (JU_1)^{k+4}$, for $k \geq 3$.

6.3.2 $H^1_{(y^*)}(QU)$

In this section, we aim to calculate $H^1_{(y^*)}(QU)$ with negative and positive degrees. We begin first to calculate $H^1_{(y^*)}(QU)$ for negative degrees. By Proposition 6.1.4, we find

- If $n = -4$, then $H^1_{(y^*)}(QU)_{-4} \cong \widehat{JU_6}/y^* \cdot \widehat{JU_2}$.

First, we need to express $y^* \cdot \widehat{JU_2}$ by a matrix.

$$y^* \cdot \widehat{JU_2} = \mathbb{Z}_3^\wedge \left\langle \begin{array}{r} \\ y_1 = y^* \cdot v_1 : \\ y_2 = y^* \cdot v_2 : \\ y_3 = y^* \cdot v_3 : \\ y_4 = y^* \cdot v_4 : \\ y_5 = y^* \cdot v_5 : \\ y_6 = y^* \cdot v_6 : \\ y_7 = y^* \cdot v_7 : \\ y_8 = y^* \cdot v_8 : \end{array} \begin{array}{cccc} x & y & xy & xy^2 \\ (\; 27\omega & 0 & 27\omega & 27\omega \;) \\ (\; 27 & 0 & 27 & 27 \;) \\ (\; 0 & 27\omega & 27\omega & 27\omega^2 \;) \\ (\; 0 & 27 & 27 & 27 \;) \\ (\; 0 & 0 & 27\omega & 27 \;) \\ (\; 0 & 0 & 27\omega^2 & 27 \;) \\ (\; 0 & 0 & 27 & 27\omega^2 \;) \\ (\; 0 & 0 & 27 & 27\omega \;) \end{array} \right\rangle$$

Next, we write $y^* \cdot \widehat{JU_2}$ as a linear combination from the matrix JU_6 (see Section 6.3.1), and we can now represent $\widehat{JU_6}/y^* \cdot \widehat{JU_2}$.

$$\begin{array}{r} \\ y_1 : \\ y_2 : \\ y_3 : \\ y_4 : \\ y_5 : \\ y_6 : \\ y_7 : \\ y_8 : \end{array} \begin{array}{|cccccccc|} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}$$

We do direct row calculation

$$
\begin{array}{rc}
 & \begin{array}{cccccccc} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 \end{array} \\
\begin{array}{r} y_2 : \\ y_1 : \\ y_4 : \\ y_3 : \\ y_8 : \\ y_5 : \\ y_7^+ : \\ y_6^+ : \end{array} &
\left|\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{array}\right|,
\end{array}
$$

where $y_7^+ = y_7 - y_8$ and $y_6^+ = y_6 + y_8 + y_5 + 2y_7^+$.

Doing column operations

$$
\begin{array}{rc}
 & \begin{array}{cccccccc} z_1 & z_2 & z_3 & z_4 & z_5^* & z_6^* & z_7^* & z_8^* \end{array} \\
\begin{array}{r} y_2 : \\ y_1 : \\ y_4 : \\ y_3 : \\ y_8 : \\ y_5 : \\ y_7^+ : \\ y_6^+ : \end{array} &
\left|\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -3
\end{array}\right|,
\end{array}
$$

where $z_5^* = z_1 - z_5$, $z_6^* = z_2 - z_6$, $z_7^* = z_1 - z_7$, and $z_8^* = z_2 - z_8$.

Again, we do column operations to obtain the required matrix

$$
\begin{array}{rc}
 & \begin{array}{cccccccc} z_1 & z_2 & z_3 & z_4 & z_5^+ & z_6^+ & z_7^+ & z_8^+ \end{array} \\
\begin{array}{r} y_2 : \\ y_1 : \\ y_4 : \\ y_3 : \\ y_8 : \\ y_5 : \\ y_7^+ : \\ y_6^+ : \end{array} &
\left|\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{array}\right|,
\end{array}
$$

where $z_5^+ = z_3 + z_5^*$, $z_6^+ = z_4 + z_6^*$, $z_7^+ = z_6^+ + z_4 - (z_3 + z_7^*)$, and $z_8^+ = 2z_7^+ - (z_5^+ + z_4 - z_8^*)$.

Thus, we deduce $\widehat{JU_6}/y^* \cdot \widehat{JU_2} \cong \mathbb{Z}/3$ is generated by $z_8^+ + (y^* \cdot \widehat{JU_2})$, and therefore the result is

$$H^1_{(y^*)}(QU)_{-4} \cong \mathbb{Z}/3.$$

- If $n = -2$, then $H^1_{(y^*)}(QU)_{-2} \cong \widehat{JU_5}/y^* \cdot \widehat{JU_1}$.

We need to represent $y^* \cdot \widehat{JU_1}$ by a matrix

$$y^* \cdot \widehat{JU_1} = \mathbb{Z}_3^\wedge \left\langle \begin{array}{l} \\ x_1 = y^* \cdot u_1 : \\ x_2 = y^* \cdot u_2 : \\ x_3 = y^* \cdot u_3 : \\ x_4 = y^* \cdot u_4 : \\ x_5 = y^* \cdot u_5 : \\ x_6 = y^* \cdot u_6 : \\ x_7 = y^* \cdot u_7 : \\ x_8 = y^* \cdot u_8 : \end{array} \begin{array}{cccc} x & y & xy & xy^2 \\ (\; 0 & 9\gamma_1 & 9\gamma_1 & 9\gamma_2 \;) \\ (\; 0 & 9\gamma_2 & 9\gamma_2 & 9\gamma_1 \;) \\ (\; 9\gamma_1 & 0 & 9\gamma_1 & 9\gamma_1 \;) \\ (\; 9\gamma_1 & 9\gamma_1 & 9\gamma_2 & 0 \;) \\ (\; 9\gamma_1 & 9\gamma_2 & 0 & 9\gamma_2 \;) \\ (\; 9\gamma_2 & 0 & 9\gamma_2 & 9\gamma_2 \;) \\ (\; 9\gamma_2 & 9\gamma_1 & 0 & 9\gamma_1 \;) \\ (\; 9\gamma_2 & 9\gamma_2 & 9\gamma_1 & 0 \;) \end{array} \right\rangle$$

We use $JU_5 = \langle 9\gamma_1, 9\gamma_2 \rangle$, and express $y^* \cdot \widehat{JU_1}$ as a linear combination from JU_5, and we do $\widehat{JU_5}/y^* \cdot \widehat{JU_1}$

$$\begin{array}{r} \\ x_1 : \\ x_2 : \\ x_3 : \\ x_4 : \\ x_5 : \\ x_6 : \\ x_7 : \\ x_8 : \end{array} \begin{array}{|cccccccc|} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}$$

We do direct row calculations

$$\begin{array}{r} \\ x_5 : \\ x_7 : \\ x_1 : \\ x_2 : \\ x_6^+ : \\ x_4^+ : \\ x_3^+ : \\ x_8^+ : \end{array} \begin{array}{|cccccccc|} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{array},$$

where $x_6^+ = x_6 - x_7 + x_1$, $x_4^+ = x_4 - x_5 - x_1 + x_2 + x_6^+$, $x_3^+ = x_3 - x_5 + x_2 - x_6^+$, and $x_8^+ = x_8 - x_7 + x_1 - x_2 - 2x_6^+ + x_4^+$.

Doing column operations

$$\begin{array}{c} \\ x_5 : \\ x_7 : \\ x_1 : \\ x_2 : \\ x_6^+ : \\ x_4^+ : \\ x_3^+ : \\ x_8^+ : \end{array}\begin{array}{c|cccccccc|} & r_1 & r_2 & r_3^+ & r_4^+ & r_5^+ & r_6^+ & r_7^+ & r_8^+ \\ & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{array},$$

where $r_3^+ = r_2 - r_3$, $r_4^+ = r_1 - r_4$, $r_5^+ = r_3^+ + r_5$, $r_6^+ = r_5^+ - (r_4^+ + r_6)$, $r_7^+ = r_5^+ + r_4^+ - r_2 + r_7$, and $r_8^+ = r_7^+ - (r_2 - r_3 + 2r_5 + r_1 - r_8)$.

This calculation records

$$\widehat{JU_5}/y^* \cdot \widehat{JU_1} \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$$

is generated by $r_6^+ + (y^* \cdot \widehat{JU_1})$, $r_7^+ + (y^* \cdot \widehat{JU_1})$ and $r_8^+ + (y^* \cdot \widehat{JU_1})$ respectively, and hence the result follows

$$H^1_{(y^*)}(QU)_{-2} \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3.$$

This completes calculating of $H^1_{(y^*)}(QU)$ for negative degrees. For positive degrees, let us begin with $n = 0$, and we find $H^1_{(y^*)}(QU)_0 \cong \widehat{JU_4}/y^* \cdot RU$.

First, we need to write $y^* \cdot RU$ as a matrix so that $RU = \mathbb{Z} + JU$.

$$y^* \cdot RU = \mathbb{Z}_3^\wedge \left\langle \begin{array}{rcccccc} & & e & x & y & xy & xy^2 \\ z_0 = y^* \cdot 1 : & (& 0 & 9 & 9 & 9 & 9 &) \\ z_1 = y^* \cdot u_1 : & (& 0 & 0 & 9\gamma_1 & 9\gamma_1 & 9\gamma_2 &) \\ z_2 = y^* \cdot u_2 : & (& 0 & 0 & 9\gamma_2 & 9\gamma_2 & 9\gamma_1 &) \\ z_3 = y^* \cdot u_3 : & (& 0 & 9\gamma_1 & 0 & 9\gamma_1 & 9\gamma_1 &) \\ z_4 = y^* \cdot u_4 : & (& 0 & 9\gamma_1 & 9\gamma_1 & 9\gamma_2 & 0 &) \\ z_5 = y^* \cdot u_5 : & (& 0 & 9\gamma_1 & 9\gamma_2 & 0 & 9\gamma_2 &) \\ z_6 = y^* \cdot u_6 : & (& 0 & 9\gamma_2 & 0 & 9\gamma_2 & 9\gamma_2 &) \\ z_7 = y^* \cdot u_7 : & (& 0 & 9\gamma_2 & 9\gamma_1 & 0 & 9\gamma_1 &) \\ z_8 = y^* \cdot u_8 : & (& 0 & 9\gamma_2 & 9\gamma_2 & 9\gamma_1 & 0 &) \end{array} \right\rangle$$

Next, we can now to express $y^* \cdot RU$ as a linear combination from JU_4. We compute $\widehat{JU_4}/y^* \cdot RU$

$$
\begin{array}{r|cccccccc|}
 & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 \\
z_0: & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
z_1: & 0 & 0 & 1 & -1 & 1 & -1 & 2 & 1 \\
z_2: & 0 & 0 & 2 & 1 & 2 & 1 & 1 & -1 \\
z_3: & 1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 \\
z_4: & 1 & -1 & 1 & -1 & 2 & 1 & 0 & 0 \\
z_5: & 1 & -1 & 2 & 1 & 0 & 0 & 2 & 1 \\
z_6: & 2 & 1 & 0 & 0 & 2 & 1 & 2 & 1 \\
z_7: & 2 & 1 & 1 & -1 & 0 & 0 & 1 & -1 \\
z_8: & 2 & 1 & 2 & 1 & 1 & -1 & 0 & 0
\end{array}
$$

In this calculation, we have obtained an matrix 9×8 matrix, and in order to reduce it to an 8×8 matrix. We do some row calculations

$$
\begin{array}{r|cccccccc|}
 & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 \\
z_0: & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
z_3^+: & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
z_1: & 0 & 0 & 1 & -1 & 1 & -1 & 2 & 1 \\
z_7^+: & 0 & 0 & 0 & 3 & 0 & 3 & -3 & 0 \\
z_5^+: & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
z_8^+: & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
z_4^+: & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
z_2^+: & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{array},
$$

where $z_3^+ = z_3 - z_0$, $z_7^+ = z_7 - 2z_0 + z_3^+ + 2z_1$, $z_5^+ = z_5 - (z_0 + z_3^+ + 2z_1) + z_7^+$, $z_8^+ = z_8 - 2z_0 + z_3^+ + z_1$, $z_4^+ = z_4 - (z_0 + z_3^+ + z_1) + z_8^+$, and $z_2^+ = z_2 - 2z_1 + z_7^+$.

Doing some column operations

$$
\begin{array}{r|cccccccc|}
 & t_1 & t_2 & t_3^+ & t_4^+ & t_5^+ & t_6^+ & t_7^+ & t_8^+ \\
z_0: & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
z_3^+: & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
z_1: & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
z_7^+: & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
z_5^+: & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
z_8^+: & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
z_4^+: & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
z_2^+: & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{array},
$$

where $t_3^+ = t_2 + t_1 - t_3$, $t_4^+ = t_3^+ - t_4$, $t_5^+ = t_3^+ - t_1 + t_5$, $t_6^+ = t_4^+ - (t_3^+ + t_2 - t_6)$, $t_7^+ = t_4^+ - (2t_3^+ - t_1 + t_7)$, and $t_8^+ = t_3^+ - t_2 + t_8$.

Thus, we obtain

$$\widehat{JU_4}/y^* \cdot RU \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$$

is generated by $t_4^+ + (y^* \cdot RU)$, $t_5^+ + (y^* \cdot RU)$, $t_6^+ + (y^* \cdot RU)$, $t_7^+ + (y^* \cdot RU)$, $t_8^+ + (y^* \cdot RU)$ respectively. Therefore

$$H^1_{(y^*)}(QU)_0 \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3.$$

- If $n = 2$, then $H^1_{(y^*)}(QU)_2 \cong \widehat{JU_3}/y^* \cdot RU$,

where $JU_3 = \langle 3\gamma_1, 3\gamma_2 \rangle$, and we use the above matrix to calculate $H^1_{(y^*)}(QU)_2$.

We can now express $y^* \cdot RU$ as a linear combination from JU_3, and we compute $\widehat{JU_3}/y^* \cdot RU$ to obtain

$$\begin{array}{r|cccccccc|}
 & w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \\
z_0 : & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
z_1 : & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 3 \\
z_2 : & 0 & 0 & 0 & 3 & 0 & 3 & 3 & 0 \\
z_3 : & 3 & 0 & 0 & 0 & 3 & 0 & 3 & 0 \\
z_4 : & 3 & 0 & 3 & 0 & 0 & 3 & 0 & 0 \\
z_5 : & 3 & 0 & 0 & 3 & 0 & 0 & 0 & 3 \\
z_6 : & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 3 \\
z_7 : & 0 & 3 & 3 & 0 & 0 & 0 & 3 & 0 \\
z_8 : & 0 & 3 & 0 & 3 & 3 & 0 & 0 & 0
\end{array}$$

As before, we do row calculations to obtain an 8×8 matrix

$$\begin{array}{r|cccccccc|}
 & w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \\
z_0 : & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
z_7 : & 0 & 3 & 3 & 0 & 0 & 0 & 3 & 0 \\
z_1 : & 0 & 0 & 3 & 0 & 3 & 0 & 0 & 3 \\
z_2 : & 0 & 0 & 0 & 3 & 0 & 3 & 3 & 0 \\
z_5^+ : & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 \\
z_4^+ : & 0 & 0 & 0 & 0 & 0 & 9 & 3 & -6 \\
z_6^+ : & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -6 \\
z_8^+ : & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9
\end{array},$$

where $z_5^+ = z_5 - 3z_0 + z_7$, $z_4^+ = (z_4 - 3z_0 + z_7) - z_1 + z_2 - 2z_5^+$, $z_6^+ = z_6 - z_7 + z_1 + z_5^+$, and $z_8^+ = (z_8 - z_7 + z_1 - z_2 + 2z_5^+) + z_4^+ + z_6^+$.

We do some column operations

$$
\begin{array}{c|cccccccc|}
 & w_1 & w_2^+ & w_3^+ & w_4^+ & w_5^+ & w_6^+ & w_7^+ & w_8^+ \\
z_0 : & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
z_3^+ : & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
z_1 : & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
z_7^+ : & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
z_5^+ : & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
z_8^+ : & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 \\
z_4^+ : & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
z_2^+ : & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9
\end{array},
$$

where $w_2^+ = w_1 - w_2$, $w_3^+ = w_2^+ - (w_1 - w_3)$, $w_4^+ = w_1 - w_4$, $w_5^+ = w_3^+ + w_1 - w_5$, $w_6^+ = w_5^+ + w_4^+ - (w_1 - w_6)$, $w_7^+ = w_4^+ + w_2^+ - (w_1 - w_7) - 1/3w_6^+$, and $w_8^+ = 6/9(w_6^+ - (w_3^+ + w_1 - w_8) - w_7^+)$.

Hence, we deduce

$$\widehat{JU_3}/y^* \cdot RU \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9$$

is generated by $w_2^+ + (y^* \cdot RU)$, $w_3^+ + (y^* \cdot RU)$, $w_4^+ + (y^* \cdot RU)$, $w_5^+ + (y^* \cdot RU)$, $w_6^+ + (y^* \cdot RU)$, $w_7^+ + (y^* \cdot RU)$, $w_8^+ + (y^* \cdot RU)$ respectively.

Therefore

$$H^1_{(y^*)}(QU)_2 \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9.$$

Since we work to calculate the local cohomology of QU for positive degrees, all calculations are available in $(y^*)^a \cdot RU$ for $a > 1$. We continue to have the following results.

- If $n = 4$, then $H^1_{(y^*)}(QU)_4 \cong \widehat{JU_6}/(y^*)^2 \cdot RU$.

We can represent $(y^*)^2 \cdot RU$ by a matrix

$$
(y^*)^2.\ RU = \mathbb{Z}_3^\wedge \left\langle
\begin{array}{r}
 \\
s_0 = (y^*)^2 \cdot 1 : \\
s_1 = (y^*)^2 \cdot u_1 : \\
s_2 = (y^*)^2 \cdot u_2 : \\
s_3 = (y^*)^2 \cdot u_3 : \\
s_4 = (y^*)^2 \cdot u_4 : \\
s_5 = (y^*)^2 \cdot u_5 : \\
s_6 = (y^*)^2 \cdot u_6 : \\
s_7 = (y^*)^2 \cdot u_7 : \\
s_8 = (y^*)^2 \cdot u_8 :
\end{array}
\begin{array}{(ccccc)}
e & x & y & xy & xy^2 \\
0 & 81 & 81 & 81 & 81 \\
0 & 0 & 81\gamma_1 & 81\gamma_1 & 81\gamma_2 \\
0 & 0 & 81\gamma_2 & 81\gamma_2 & 81\gamma_1 \\
0 & 81\gamma_1 & 0 & 81\gamma_1 & 81\gamma_1 \\
0 & 81\gamma_1 & 81\gamma_1 & 81\gamma_2 & 0 \\
0 & 81\gamma_1 & 81\gamma_2 & 0 & 81\gamma_2 \\
0 & 81\gamma_2 & 0 & 81\gamma_2 & 81\gamma_2 \\
0 & 81\gamma_2 & 81\gamma_1 & 0 & 81\gamma_1 \\
0 & 81\gamma_2 & 81\gamma_2 & 81\gamma_1 & 0
\end{array}
\right\rangle
$$

As before, we can write $(y^*)^2 \cdot RU$ as a linear combination from JU_6

$$\begin{array}{r|cccccccc|}
 & z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 \\
s_0: & 3 & 0 & 3 & 0 & 3 & 0 & 3 & 0 \\
s_1: & 0 & 0 & 3 & -3 & 3 & -3 & 6 & 3 \\
s_2: & 0 & 0 & 6 & 3 & 6 & 3 & 3 & -3 \\
s_3: & 3 & -3 & 0 & 0 & 3 & -3 & 3 & -3 \\
s_4: & 3 & -3 & 3 & -3 & 6 & 3 & 0 & 0 \\
s_5: & 3 & -3 & 6 & 3 & 0 & 0 & 6 & 3 \\
s_6: & 6 & 3 & 0 & 0 & 6 & 3 & 6 & 3 \\
s_7: & 6 & 3 & 3 & -3 & 0 & 0 & 3 & -3 \\
s_8: & 6 & 3 & 6 & 3 & 3 & -3 & 0 & 0
\end{array}$$

We do some row calculations

$$\begin{array}{r|cccccccc|}
 & z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 & z_8 \\
s_0: & 3 & 0 & 3 & 0 & 3 & 0 & 3 & 0 \\
s_3^+: & 0 & -3 & -3 & 0 & 0 & -3 & 0 & -3 \\
s_1: & 0 & 0 & 3 & -3 & 3 & -3 & 6 & 3 \\
s_7^+: & 0 & 0 & 0 & 9 & 0 & 9 & -9 & 0 \\
s_5^+: & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 \\
s_8^+: & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 \\
s_4^+: & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 \\
s_6^+: & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9
\end{array},$$

where $s_3^+ = s_1 - s_3$, $s_7^+ = s_7 - 2s_0 + s_3^+ + 2s_1$, $s_5^+ = s_5 - s_0 - s_3^+ - 2s_1 + s_7^+$, $s_8^+ = s_8 - 2s_0 + s_3^+ + s_1$, $s_4^+ = s_4 - s_0 - s_3^+ - s_1 + s_8^+$, and $s_6^+ = (s_6 - 2s_0 + s_3^+ + 3s_1) - s_7^+ + s_4^+ + s_5^+$.

Doing column operations

$$\begin{array}{r|cccccccc|}
 & z_1 & z_2 & z_3^+ & z_4^+ & z_5^+ & z_6^+ & z_7^+ & z_8^+ \\
s_0: & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
s_3^+: & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
s_1: & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
s_7^+: & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 \\
s_5^+: & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 \\
s_8^+: & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 \\
s_4^+: & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 \\
s_6^+: & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9
\end{array},$$

where $z_3^+ = z_2 + z_1 - z_3$, $z_4^+ = z_3^+ - z_4$, $z_5^+ = z_3^+ - (z_1 - z_5)$, $z_6^+ = z_4^+ - (z_3^+ + z_2 - z_6)$, $z_7^+ = z_4^+ - (2z_3^+ - z_1 + z_7)$, and $z_8^+ = z_3^+ - (z_2 - z_8)$.

Thus, we deduce

$$\widehat{JU_6}/(y^*)^2 \cdot RU \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9$$

is generated by $z_1 + ((y^*)^2 \cdot RU)$, $z_2 + ((y^*)^2 \cdot RU)$, $z_3^+ + ((y^*)^2 \cdot RU)$, $z_4^+ + ((y^*)^2 \cdot RU)$, $z_5^+ + ((y^*)^2 \cdot RU)$, $z_6^+ + ((y^*)^2 \cdot RU)$, $z_7^+ + ((y^*)^2 \cdot RU)$, $z_8^+ + ((y^*)^2 \cdot RU)$ respectively.

Therefore,

$$H^1_{(y^*)}(QU)_4 \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9.$$

- If $n = 6$, then $H^1_{(y^*)}(QU)_6 \cong \widehat{JU_5}/(y^*)^2 \cdot RU$.

As in degree of 4, we have obtained $(y^*)^2 \cdot RU$. We now compute $\widehat{JU_5}/(y^*)^2 \cdot RU$ to obtain

$$\begin{array}{r|cccccccc|}
 & r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 \\
s_0 : & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
s_1 : & 0 & 0 & 9 & 0 & 9 & 0 & 0 & 9 \\
s_2 : & 0 & 0 & 0 & 9 & 0 & 9 & 9 & 0 \\
s_3 : & 9 & 0 & 0 & 0 & 9 & 0 & 9 & 0 \\
s_4 : & 9 & 0 & 9 & 0 & 0 & 9 & 0 & 0 \\
s_5 : & 9 & 0 & 0 & 9 & 0 & 0 & 0 & 9 \\
s_6 : & 0 & 9 & 0 & 0 & 0 & 9 & 0 & 9 \\
s_7 : & 0 & 9 & 9 & 0 & 0 & 0 & 9 & 0 \\
s_8 : & 0 & 9 & 0 & 9 & 9 & 0 & 0 & 0
\end{array}$$

We do row calculations to obtain an 8×8 matrix

$$\begin{array}{r|cccccccc|}
 & r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 \\
s_0 : & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
s_7 : & 0 & 9 & 9 & 0 & 0 & 0 & 9 & 0 \\
s_1 : & 0 & 0 & 9 & 0 & 9 & 0 & 0 & 9 \\
s_2 : & 0 & 0 & 0 & 9 & 0 & 9 & 9 & 0 \\
s_5^+ : & 0 & 0 & 0 & 0 & -9 & -9 & 0 & 0 \\
s_4^+ : & 0 & 0 & 0 & 0 & 0 & 27 & 9 & -18 \\
s_6^+ : & 0 & 0 & 0 & 0 & 0 & 0 & -9 & 18 \\
s_8^+ : & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -27
\end{array},$$

where $s_5^+ = s_5 - 3s_0 + s_7$, $s_4^+ = (s_4 - 3s_0 + s_7 - s_1 + s_2) - 2s_5^+$, $s_6^+ = s_6 - s_7 + s_1 + s_5^+$, and $s_8^+ = s_8 - s_7 + s_1 - s_2 + 2s_5^+ + s_4^+ - s_6^+$.

After doing column calculations, we find

$$\begin{array}{r|cccccccc|}
 & r_1 & r_2^+ & r_3^+ & r_4^+ & r_5^+ & r_6^+ & r_7^+ & r_8^+ \\
s_0 : & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
s_7 : & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\
s_1 : & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\
s_2 : & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 \\
s_5^+ : & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 \\
s_4^+ : & 0 & 0 & 0 & 0 & 0 & 27 & 0 & 0 \\
s_6^+ : & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 \\
s_8^+ : & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27
\end{array},$$

where $r_2^+ = r_1 - r_2$, $r_3^+ = r_2^+ - (r_1 - r_3)$, $r_4^+ = r_1 - r_4$, $r_5^+ = r_3^+ + r_1 - r_5$, $r_6^+ = r_5^+ + r_4^+ - (r_1 - r_6)$, $r_7^+ = r_4^+ + r_2^+ - (r_1 - r_7) - 1/3r_6^+$, and $r_8^+ = 2r_7^+ - 2/3r_6^+ - (r_3^+ + r_1 - r_8)$.

Hence,

$$\widehat{JU_5}/(y^*)^2 \cdot RU \cong \mathbb{Z}/3 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/27.$$

is generated by $r_1 + ((y^*)^2 \cdot RU)$, $r_2^+ + ((y^*)^2 \cdot RU)$, $r_3^+ + ((y^*)^2 \cdot RU)$, $r_4^+ + ((y^*)^2 \cdot RU)$, $r_5^+ + ((y^*)^2 \cdot RU)$, $r_6^+ + ((y^*)^2 \cdot RU)$, $r_7^+ + ((y^*)^2 \cdot RU)$, $r_8^+ + ((y^*)^2 \cdot RU)$ respectively.

Therefore, the result is

$$H^1_{(y^*)}(QU)_6 \cong \mathbb{Z}/3 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/27.$$

- If $n = 8$, then $H^1_{(y^*)}(QU)_8 \cong \widehat{JU_4}/(y^*)^2 \cdot RU$.

As in degree of 6, we have already calculated a matrix $(y^*)^2 \cdot RU$, and we need only compute $\widehat{JU_4}/(y^*)^2 \cdot RU$. We obtain

$$\begin{array}{c|cccccccc|}
 & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 \\
s_0 : & 9 & 0 & 9 & 0 & 9 & 0 & 9 & 0 \\
s_1 : & 0 & 0 & 9 & -9 & 9 & -9 & 18 & 9 \\
s_2 : & 0 & 0 & 18 & 9 & 18 & 9 & 9 & -9 \\
s_3 : & 9 & -9 & 0 & 0 & 9 & -9 & 9 & -9 \\
s_4 : & 9 & -9 & 9 & -9 & 18 & 9 & 0 & 0 \\
s_5 : & 9 & -9 & 18 & 9 & 0 & 0 & 18 & 9 \\
s_6 : & 18 & 9 & 0 & 0 & 18 & 9 & 18 & 9 \\
s_7 : & 18 & 9 & 9 & -9 & 0 & 0 & 9 & -9 \\
s_8 : & 18 & 9 & 18 & 9 & 9 & -9 & 0 & 0
\end{array},$$

By doing some row calculations

$$\begin{array}{c|cccccccc|}
 & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 \\
s_0 : & 9 & 0 & 9 & 0 & 9 & 0 & 9 & 0 \\
s_3^+ : & 0 & -9 & -9 & 0 & 0 & -9 & 0 & -9 \\
s_1 : & 0 & 0 & 9 & -9 & 9 & -9 & 18 & 9 \\
s_7^+ : & 0 & 0 & 0 & -27 & 0 & -27 & 27 & 0 \\
s_5^+ : & 0 & 0 & 0 & 0 & -27 & 0 & 0 & 0 \\
s_8^+ : & 0 & 0 & 0 & 0 & 0 & -27 & 0 & 0 \\
s_4^+ : & 0 & 0 & 0 & 0 & 0 & 0 & -27 & 0 \\
s_6^+ : & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27
\end{array},$$

where $s_3^+ = s_1 - s_3$, $s_7^+ = (s_7 - 2s_0) + s_3^+ + 2s_1$, $s_5^+ = (s_5 - s_0 - s_3^+) - 2s_1 + s_7^+$, $s_8^+ = (s_8 - 2s_0 + s_3^+) + s_1$, $s_4^+ = (s_4 - s_0 - s_3^+) - s_1 + s_8^+$, and $s_6^+ = (s_6 - 2s_0 + s_3^+ + 3s_1) - s_7^+ + s_4^+ + s_5^+$.

We do column operations to obtain the required matrix

$$\begin{array}{c|cccccccc|}
 & t_1 & t_2 & t_3^+ & t_4^+ & t_5^+ & t_6^+ & t_7^+ & t_8^+ \\
s_0 : & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
s_3^+ : & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\
s_1 : & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\
s_7^+ : & 0 & 0 & 0 & 27 & 0 & 0 & 0 & 0 \\
s_5^+ : & 0 & 0 & 0 & 0 & 27 & 0 & 0 & 0 \\
s_8^+ : & 0 & 0 & 0 & 0 & 0 & 27 & 0 & 0 \\
s_4^+ : & 0 & 0 & 0 & 0 & 0 & 0 & 27 & 0 \\
s_6^+ : & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27
\end{array},$$

where $t_3^+ = t_2 + t_1 - t_3$, $t_4^+ = t_3^+ - t_4$, $t_5^+ = t_3^+ - (t_1 - t_5)$, $t_6^+ = t_4^+ - (t_3^+ + t_2 - t_6)$, $t_7^+ = t_4^+ - (2t_3^+ - t_1 + t_7)$, and $t_8^+ = t_3^+ - t_2 + t_8$.

Thus

$$\widehat{JU_4}/(y^*)^2 \cdot RU \cong \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/27.$$

is generated by $t_1 + ((y^*)^2 \cdot RU)$, $t_2 + ((y^*)^2 \cdot RU)$, $t_3^+ + ((y^*)^2 \cdot RU)$, $t_4^+ + ((y^*)^2 \cdot RU)$, $t_5^+ + ((y^*)^2 \cdot RU)$, $t_6^+ + ((y^*)^2 \cdot RU)$, $t_7^+ + ((y^*)^2 \cdot RU)$, $t_8^+ + ((y^*)^2 \cdot RU)$ respectively.

Therefore, the result is

$$H^1_{(y^*)}(QU)_8 \cong \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/27.$$

- Finally, if $n = 10$, then $H^1_{(y^*)}(QU)_{10} \cong \widehat{JU_3}/(y^*)^2 \cdot RU$.

Again, we have $(y^*)^2 \cdot RU$, and we want to express it as a linear combination from $JU_3 = \langle 3\gamma_1, 3\gamma_2 \rangle$

$$\begin{array}{c|cccccccc|}
 & w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \\
s_0 : & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
s_1 : & 0 & 0 & 27 & 0 & 27 & 0 & 0 & 27 \\
s_2 : & 0 & 0 & 0 & 27 & 0 & 27 & 27 & 0 \\
s_3 : & 27 & 0 & 0 & 0 & 27 & 0 & 27 & 0 \\
s_4 : & 27 & 0 & 27 & 0 & 0 & 27 & 0 & 0 \\
s_5 : & 27 & 0 & 0 & 27 & 0 & 0 & 0 & 27 \\
s_6 : & 0 & 27 & 0 & 0 & 0 & 27 & 0 & 27 \\
s_7 : & 0 & 27 & 27 & 0 & 0 & 0 & 27 & 0 \\
s_8 : & 0 & 27 & 0 & 27 & 27 & 0 & 0 & 0
\end{array}$$

We do row calculations

$$\begin{array}{r|cccccccc|}
 & w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \\
s_0: & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
s_7: & 0 & 27 & 27 & 0 & 0 & 0 & 27 & 0 \\
s_1: & 0 & 0 & 27 & 0 & 27 & 0 & 0 & 27 \\
s_2: & 0 & 0 & 0 & 27 & 0 & 27 & 27 & 0 \\
s_5^+: & 0 & 0 & 0 & 0 & -27 & -27 & 0 & 0 \\
s_4^+: & 0 & 0 & 0 & 0 & 0 & 81 & 27 & -54 \\
s_6^+: & 0 & 0 & 0 & 0 & 0 & 0 & -27 & 54 \\
s_8^+: & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -81
\end{array},$$

where $s_5^+ = (s_5 - 3s_0) + s_7$, $s_4^+ = (s_4 - 3s_0 + s_7) - s_1 + s_2 - 2s_5^+$, $s_6^+ = (s_6 - s_7) + s_1 + s_5^+$, and $s_8^+ = (s_8 - s_7) + s_1 - s_2 + 2s_5^+ + s_4^+ - s_6^+$.

Doing column operations

$$\begin{array}{r|cccccccc|}
 & w_1 & w_2^+ & w_3^+ & w_4^+ & w_5^+ & w_6^+ & w_7^+ & w_8^+ \\
s_0: & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
s_7: & 0 & 27 & 0 & 0 & 0 & 0 & 0 & 0 \\
s_1: & 0 & 0 & 27 & 0 & 0 & 0 & 0 & 0 \\
s_2: & 0 & 0 & 0 & 27 & 0 & 0 & 0 & 0 \\
s_5^+: & 0 & 0 & 0 & 0 & 27 & 0 & 0 & 0 \\
s_4^+: & 0 & 0 & 0 & 0 & 0 & 81 & 0 & 0 \\
s_6^+: & 0 & 0 & 0 & 0 & 0 & 0 & 27 & 0 \\
s_8^+: & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 81
\end{array}$$

where $w_2^+ = w_1 - w_2$, $w_3^+ = w_2^+ - (w_1 - w_3)$, $w_4^+ = w_1 - w_4$, $w_5^+ = w_3^+ + w_1 - w_5$, $w_6^+ = w_5^+ + w_4^+ - (w_1 - w_6)$, $w_7^+ = w_4^+ + w_2^+ - (w_1 - w_7) - 1/3w_6^+$, and $w_8^+ = 2w_7^+ - 2/3w_6^+ - (w_3^+ + w_1 - w_8)$.

Therefore

$$\widehat{JU_3}/(y^*)^2 \cdot RU \cong \mathbb{Z}/9 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/81 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/81.$$

is generated by $w_1 + ((y^*)^2 \cdot RU)$, $w_2^+ + ((y^*)^2 \cdot RU)$, $w_3^+ + ((y^*)^2 \cdot RU)$, $w_4^+ + ((y^*)^2 \cdot RU)$, $w_5^+ + ((y^*)^2 \cdot RU)$, $w_6^+ + ((y^*)^2 \cdot RU)$, $w_7^+ + ((y^*)^2 \cdot RU)$, $w_8^+ + ((y^*)^2 \cdot RU)$ respectively.

Hence, we obtain

$$H^1_{(y^*)}(QU)_{10} \cong \mathbb{Z}/9 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/81 \oplus \mathbb{Z}/27 \oplus \mathbb{Z}/81.$$

One may check in a similar way for the rest of the work by apply Proportion 6.1.4, to calculate $H^1_{(y^*)}(QU)$.

6.3.3 $E_{1\frac{1}{2}}$-TERM

The E_2-term is the local cohomology of the extension of QU by TU. The $E_{1\frac{1}{2}}$-term is formed by taking local cohomology of TU and QU. The E_2-term is then obtained by including the effect of the connecting homomorphism of the short exact sequence

$$0 \longrightarrow TU \longrightarrow ku^*(BV(2)) \longrightarrow QU \longrightarrow 0, \tag{6.2}$$

which we view as a late d_1, and call it $d_{1\frac{1}{2}}$ since it is a map on the $E_{1\frac{1}{2}}$-term. Thus

$$d_{1\frac{1}{2}} : H^1_{JU}(QU) \longrightarrow H^2_{\mathfrak{m}}(TU).$$

We refer to this display as the $E_{1\frac{1}{2}}$-term. In rank 1 there is no differential in negative degrees. In rank 2 there is only a differential $d_{1\frac{1}{2}}$ and it is forced by the fact that the E_∞-term of the spectral sequence must be on the 0,1 and 2 line.

Our results of $H^1_{(y^*)}(QU)$ on the $E_{1\frac{1}{2}}$-term display as in Figure 6.3.

6.3.4 E_2 -term

The goal of this section is to calculate $H^1_{JU}(ku^*(BV(2)))$ and $H^2_{JU}(ku^*(BV(2)))$. In fact, we need only find $\ker(d_{1\frac{1}{2}})$ and $\operatorname{coker}(d_{1\frac{1}{2}})$, where

$$d_{1\frac{1}{2}} : H^1_{JU}(QU) \longrightarrow H^2_{\mathfrak{m}}(TU).$$

The differential $d_{1\frac{1}{2}}$ is of bidegree $(1,0)$, and the E_2-term consists of the $d_{1\frac{1}{2}}$ cycles on the 0-line and the $d_{1\frac{1}{2}}$ homology on each higher line. Since $d_{1\frac{1}{2}}$ preserves the torsion subgroup, the sequence (6.2) is an exact sequence of chain complexes. It is suitable to pass from $E_{1\frac{1}{2}}$ to E_2-term.

Now, by Proposition 6.1.4, QU only has local cohomology in degrees 0 and 1, and TU has local cohomology is given by

$$H^*_{\mathfrak{m}}(TU) = H^*_{\mathfrak{m}}(\overline{TU}) \oplus H^*_{\mathfrak{m}}(\Sigma^2\overline{TU}) \oplus H^*_{\mathfrak{m}}(\Sigma^4\overline{TU}) \oplus \cdots \oplus H^*_{\mathfrak{m}}(\Sigma^{2p-4}\overline{TU}).$$

This is enough to find the structure of $(H^1_{JU}(QU)/3H^1_{JU}(QU))^\vee$ as a module over $PC = \mathbb{F}_3[y_1, y_2]$ with $|y_i| = 2$. In this Chapter, we calculate $H^1_{JU}(QU)$ as an abelian group, and in order to obtain E_2-term we will show that the PC-module $(H^1_{JU}(QU)/3H^1_{JU}(QU))^\vee$ is generated by its elements in degrees ≥ 0.

The following Conjecture will be used to find $\ker(d_{1\frac{1}{2}})$ and $\operatorname{coker}(d_{1\frac{1}{2}})$ on the E_2-term.

Now, consider an exact sequence

$$0 \longrightarrow H^1_{JU}(ku^*(BV(2))) \longrightarrow H^1_{JU}(QU) \xrightarrow{d_{1\frac{1}{2}}} H^2_{\mathfrak{m}}(TU) \longrightarrow H^2_{JU}(ku^*(BV(2))) \longrightarrow 0.$$

Since the codomain of the map $d_{1\frac{1}{2}}$ is dual of a free module on a generator of degree 4, $d_{1\frac{1}{2}}$ is determined by its effect in degree -4. Since $H^1_{JU}(QU)$ is 1-dimensional in degree -4, it is enough to show it is non-trivial. Since the local cohomology spectral sequence collapses and $ku_*(BV(r))$ is connective, we will prove that $H^2_{JU}(ku^*(BV(2))) = 0$ below degree 4 (see Figure 6.4).

Conjecture 6.3.3. *The PC-module $(H^1_{JU}(QU)/3H^1_{JU}(QU))^\vee$ is generated by degrees 4 and 2 over PC.*

Remark 6.3.4. For a proof of the analogue for prime two see [10, 4.5.2, page 93-94].

6.3.5 $ku_*(BV(2))$

In this section, we aim to calculate the complex connective K-homology by using the local cohomology of $V(2)$ for $p=3$. This is done by applying (6.1), and note that differentials are all forced by the fact that $ku_*(BV(2))$ is in degrees ≥ 0.

We note that local cohomology is a covariant functor of $ku^*(BV(2))$, so our calculation of $ku_*(BV(2))$ is also covariant. Thus the local cohomology Theorem shows that $ku^*(BV(2))$ is isomorphic to a type of dual of itself, and we obtain a duality property for the commutative ring $ku^*(BV(2))$ closely related to Gorenstein duality. In fact, the appropriate commutative algebra shows that $ku^*(BV(2))$ and $ku_*(BV(2))$ contain the same information up to duality.

Again, we consider the short exact sequence

$$0 \longrightarrow TU \longrightarrow ku^*(BV(2)) \longrightarrow QU \longrightarrow 0.$$

Then we have an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0_{JU}(TU) \arrow[r] & H^0_{JU}(ku^*(BV(2))) \arrow[r] & H^0_{JU}(QU) \arrow[dll, rounded corners, "d_{1\frac{1}{2}}"] \\
 & H^1_{JU}(TU) \arrow[r] & H^1_{JU}(ku^*(BV(2))) \arrow[r] & H^1_{JU}(QU) \arrow[dll, rounded corners, "d_{1\frac{1}{2}}"] \\
 & H^2_{JU}(TU) \arrow[r] & H^2_{JU}(ku^*(BV(2))) \arrow[r] & H^2_{JU}(QU) \arrow[dll, rounded corners, "d_{1\frac{1}{2}}"] \\
 & H^3_{JU}(TU) \arrow[r] & H^3_{JU}(ku^*(BV(2))) \arrow[r] & H^3_{JU}(QU) \arrow[r] & \dots
\end{tikzcd}$$

The cohomology $H^*_{JU}(QU)$ appears as the 0th and $1st$ columns of the E_2-term of the local cohomology spectral sequence, where $H^s_{JU}(.)$ is in the sth column.

If $r=2$, then the spectral sequence also collapses, and $H^0_{JU}(ku^*BV(2)) = ku^* \cdot \rho$. For $p=3$, we have $\overline{TU} \cong \overline{TU}^2 = \overline{TU}_2^2 \cong PC(-8)$, and $TU = PC(-8) \oplus PC(-6)$. As above, we have only one differential which is $d_{1\frac{1}{2}}$. The differential $d_{1\frac{1}{2}}$ is surjective with $\ker(d_{1\frac{1}{2}}) = 3 \cdot H^1_{JU}(QU)$.

We have

$$ku_*(BV(2)) = ku_* \oplus \widetilde{ku}_*(BV(2)) = \mathbb{Z}[v] \oplus \widetilde{ku}_*(BV(2)).$$

This means there is no non-zero differential on the E_2-term, and therefore $E_2 = E_\infty$.

In our case, we have $ku_{(3)} \simeq lu \vee \Sigma^2 lu$, and

$$ku^*(BV(2)) = [BV(2), ku]^* = [BV(2), lu]^* \oplus [BV(2), \Sigma^2 lu]^* = lu^*(BV(2)) \oplus \Sigma^2 lu^*(BV(2)).$$

The Adams spectral sequence for $p = 3$ reads

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_3, H^*(BV(2); \mathbb{F}_3)) \Longrightarrow lu^*(BV(2)).$$

Since $\overline{TU} = \overline{TU}(lu) = \ker(lu^*(BV(2)) \longrightarrow LU^*(BV(2))) = PC(-8)$, its local cohomology is given by

$$H^*_{\mathfrak{m}}(\overline{TU}(lu)) = H^2_{\mathfrak{m}}(PC(-8)) = PC^\vee(-8)(4) = PC^\vee(-4),$$

and $\Sigma^2 \overline{TU}(lu) = \Sigma^2 PC(-8) = PC(-8)(2) = PC(-6)$. Its local cohomology is given by

$$H^*_{\mathfrak{m}}(\Sigma^2 \overline{TU}(lu)) = H^2_{\mathfrak{m}}(PC(-6)) = PC^\vee(-6)(4) = PC^\vee(-2).$$

Now, we have

$$TU = \ker(ku^*(BV(2)) \longrightarrow K^*(BV(2))) = PC(-8) \oplus PC(-6).$$

For local cohomology of TU, we obtain

$$H^2_{\mathfrak{m}}(TU) = H^2_{\mathfrak{m}}(\overline{TU}(lu)) \oplus H^2_{\mathfrak{m}}(\Sigma^2 \overline{TU}(lu)) = PC^\vee(-4) \oplus PC^\vee(-2).$$

Dualizing, we obtain

$$(H^2_{\mathfrak{m}}(TU))^\vee = PC(4) \oplus PC(2).$$

In fact, the codomain of each differential is annihilated by 3, and so the kernel of each differential includes all multiples of 3. We now apply 6.3.3, note that

$$d'_{1\frac{1}{2}} \colon H^1_J(QU)/3H^1_{JU}(QU) \longrightarrow H^2_{\mathfrak{m}}(TU) = PC^\vee(-4) \oplus PC^\vee(-2),$$

and the dual differential

$$(d'_{1\frac{1}{2}})^\vee \colon (H^1_{JU}(QU)/3H^1_{JU}(QU))^\vee \longleftarrow (H^2_{\mathfrak{m}}(TU))^\vee = PC(4) \oplus PC(2).$$

$$\begin{tikzcd}
H^2_{\mathfrak{m}}(TU) & H^1_{JU}(QU) \arrow[l, "d_{1\frac{1}{2}}"'] \arrow[d] \\
& H^1_{JU}(QU)/3H^1_{JU}(QU) \arrow[ul, dashed, "d'_{1\frac{1}{2}}"]
\end{tikzcd}$$

Since the local cohomology spectral sequence converges to $ku_*(BV(2))$, and from the above discussion, it is clear to identify the abelian group in each degree, and the action of $GL(V)$, and there is no additive extension problem. Therefore, $ku_*(BV(2))$ can be read from $E_\infty = E_2$, and the main results are as follows.

As a module over $ku^*(BV(2))$, we have

$$ku_*(BV(2)) = ku_{ev}(BV(2)) \oplus ku_{od}(BV(2)),$$

where

$$ku_{ev}(BV(2)) = lu_{ev}(BV(2)) \oplus \Sigma^2 lu_{ev}(BV(2)),$$

where

$$lu_{ev}(BV(2)) = lu_* \oplus PC^\vee(2)$$

and

$$\Sigma^2 lu_{ev}(BV(2)) = PC^\vee(4).$$

From this, we find $ku_{ev}(BV(2))$ satisfies

$$\begin{aligned} ku_{ev}(BV(2)) &= ku_* \oplus PC^\vee(2) \oplus PC^\vee(4) \\ &= \mathbb{Z}[v] \oplus \mathrm{coker}(d_{1\frac{1}{2}}) \\ &= \mathbb{Z}[v] \oplus H^2_{\mathfrak{m}}(ku^* BV(2)). \end{aligned}$$

We record our results below, assuming Conjecture 6.3.3.

$$\begin{aligned} ku_0(BV(2)) &\cong \mathbb{Z}. \\ ku_2(BV(2)) &\cong \mathbb{Z} \oplus 3. \\ ku_4(BV(2)) &\cong \mathbb{Z} \oplus 3^3. \\ ku_6(BV(2)) &\cong \mathbb{Z} \oplus 3^5. \\ ku_8(BV(2)) &\cong \mathbb{Z} \oplus 3^7. \\ ku_{10}(BV(2)) &\cong \mathbb{Z} \oplus 3^9. \\ ku_{12}(BV(2)) &\cong \mathbb{Z} \oplus 3^{11}. \\ ku_{14}(BV(2)) &\cong \mathbb{Z} \oplus 3^{13}. \\ ku_{16}(BV(2)) &\cong \mathbb{Z} \oplus 3^{15}. \\ ku_{18}(BV(2)) &\cong \mathbb{Z} \oplus 3^{17}. \\ ku_{2k+2}(BV(2)) &\cong \mathbb{Z} \oplus 3^{2k+1}, \text{ for } k \geq 0, \end{aligned}$$

and

$$\begin{aligned}
ku_1(BV(2)) &\cong [3]^2.\\
ku_3(BV(2)) &\cong [3]^5.\\
ku_5(BV(2)) &\cong [3]^5 \oplus [9]^2.\\
ku_7(BV(2)) &\cong [3]^3 \oplus [9]^5.\\
ku_9(BV(2)) &\cong [3] \oplus [9]^5 \oplus [27]^2.\\
ku_{11}(BV(2)) &\cong [9]^3 \oplus [27]^5.\\
ku_{13}(BV(2)) &\cong [9] \oplus [27]^5 \oplus [81]^2.\\
ku_{15}(BV(2)) &\cong [27]^3 \oplus [81]^5.\\
ku_{17}(BV(2)) &\cong [27] \oplus [81]^5 \oplus [243]^2.\\
ku_{4k-1}(BV(2)) &\cong [3^{k-1}]^3 \oplus [3^k]^5.\\
ku_{4k+1}(BV(2)) &\cong [3^{k-1}] \oplus [3^k]^5 \oplus [3^{k+1}]^2.
\end{aligned}$$

$H^3_{\mathfrak{m}}(ku^*(BV(2)))$	$H^2_{\mathfrak{m}}(TU)$	$H^1_{(y^*)}(QU)$	$H^0_{\mathfrak{m}}(ku^*(BV(2)))$	degrees
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	$3^{12} \oplus 3^{11}$	$[81] \oplus [243]^5 \oplus [729]^2$	$\mathbb{Z}$	18
0	0	0	0	17
0	$3^{11} \oplus 3^{10}$	$[81]^3 \oplus [243]^5$	$\mathbb{Z}$	16
0	0	0	0	15
0	$3^{10} \oplus 3^9$	$[27] \oplus [81]^5 \oplus [243]^2$	$\mathbb{Z}$	14
0	0	0	0	13
0	$3^9 \oplus 3^8$	$[27]^3 \oplus [81]^5$	$\mathbb{Z}$	12
0	0	0	0	11
0	$3^8 \oplus 3^7$	$[9] \oplus [27]^5 \oplus [81]^2$	$\mathbb{Z}$	10
0	0	0	0	9
0	$3^7 \oplus 3^6$	$[9]^3 \oplus [27]^5$	$\mathbb{Z}$	8
0	0	0	0	7
0	$3^6 \oplus 3^5$	$[3] \oplus [9]^5 \oplus [27]^2$	$\mathbb{Z}$	6
0	0	0	0	5
0	$3^5 \oplus 3^4$	$[3]^3 \oplus [9]^5$	$\mathbb{Z}$	4
0	0	0	0	3
0	$3^4 \oplus 3^3$	$[1] \oplus [3]^5 \oplus [9]^2$	$\mathbb{Z}$	2
0	0	0	0	1
0	$3^3 \oplus 3^2$	$[1] \oplus [3]^5$	$\mathbb{Z}$	0
0	0	0	0	-1
0	$3^2 \oplus 3$	$[3]^3$	0	-2
0	0	0	0	-3
0	3	[3]	0	-4
0	0	0	0	-5
0	0	0	0	-6

Figure 6.3: The $E_{1\frac{1}{2}}$-term of the local cohomology spectral sequence for $ku_*(BV(2))$. The symbol $[n]$ denotes a cyclic group of order n, and 3^r denotes an elementary abelian group of rank r.

$H^i_{\mathfrak{m}}(ku^*(BV(2)))_{i\geq 3}$	$H^2_{\mathfrak{m}}(ku^*(BV(2)))$	$H^1_{\mathfrak{m}}(ku^*(BV(2)))$	$H^0_{\mathfrak{m}}(ku^*(BV(2)))$	degrees
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	3^{15}	$[27] \oplus [81]^5 \oplus [243]^2$	$\mathbb{Z} \cdot \rho$	18
0	0	0	0	17
0	3^{13}	$[27]^3 \oplus [81]^5$	$\mathbb{Z} \cdot \rho$	16
0	0	0	0	15
0	3^{11}	$[9] \oplus [27]^5 \oplus [81]^2$	$\mathbb{Z} \cdot \rho$	14
0	0	0	0	13
0	3^9	$[9]^3 \oplus [27]^5$	$\mathbb{Z} \cdot \rho$	12
0	0	0	0	11
0	3^7	$[3] \oplus [9]^5 \oplus [27]^2$	$\mathbb{Z} \cdot \rho$	10
0	0	0	0	9
0	3^5	$[3]^3 \oplus [9]^5$	$\mathbb{Z} \cdot \rho$	8
0	0	0	0	7
0	3^3	$[3]^5 \oplus [9]^2$	$\mathbb{Z} \cdot \rho$	6
0	0	0	0	5
0	3	$[3]^5$	$\mathbb{Z} \cdot \rho$	4
0	0	0	0	3
0	0	$[3]^2$	$\mathbb{Z} \cdot \rho$	2
0	0	0	0	1
0	0	0	$\mathbb{Z} \cdot \rho$	0
0	0	0	0	-1
0	0	0	0	-2
0	0	0	0	-3
0	0	0	0	-4
0	0	0	0	-5
0	0	0	0	-6

Figure 6.4: The E_2-term of the local cohomology spectral sequence for $ku_*(BV(2))$. The symbol $[n]$ denotes a cyclic group of order n, and 3^r denotes an elementary abelian group of rank r.

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