Topological C*-Categories

David O'Sullivan

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Supervisor: Dr. Paul Mitchener

Abstract

Tensor C^*-categories are the result of work to recast the fundamental theory of operator algebras in the setting of category theory, in order to facilitate the study of higher-dimensional algebras that are expected to play an important role in a unified model of physics. Indeed, the application of category theory to mathematical physics is itself a highly active field of research. C^*-categories are the analogue of C^*-algebras in this context. They are defined as norm-closed self-adjoint subcategories of the category of Hilbert spaces and bounded linear operators between them. Much of the theory of C^*-algebras and their invariants generalises to C^*-categories. Often, when a C^*-algebra is associated to a particular structure it is not completely natural because certain choices are involved in its definition. Using C^*-categories instead can avoid such choices since the construction of the relevant C^*-category amounts to choosing all suitable C^*-algebras at once.

In this thesis we introduce and study C^*-categories for which the set of objects carries topological data, extending the present body of work, which exclusively considers C^*-categories with discrete object sets. We provide a construction of K-theory for topological C^*-categories, which will have applications in widening the scope of the Baum-Connes conjecture, in index theory, and in geometric quantisation. As examples of such applications, we construct the C^*-categories of topological groupoids, extending the familiar constructions of Renault.

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Contents

Introduction

A C*-category is the natural generalisation of a C*-algebra where instead of considering the bounded linear operators on a fixed Hilbert space we consider all bounded linear operators between all Hilbert spaces simultaneously. A C*-category is thus a *horizontal categorification* of a C*-algebra, and therefore the theory of C*-categories fits into the framework of the groupoidification program led by Baez, Yetter *et al* [5], [14]. Much of the elementary theory of C*-algebras carries over to C*-categories, as shown originally in [28] and then developed further in [43]. The category of all (small) C*-categories is in some sense nicer than the category of C*-algebras. For example, the collection of structure preserving homomorphisms between any two C*-categories is itself a C*-category. This observation leads to two important lines of research in which C*-categories may be used: Firstly, if A and B are C*-algebras, which we may think of as C*-categories each having just a single object, then the collection of C*-algebra homomorphisms $\{f : A \to B\}$ forms a C*-category $C^*(A, B)$ (this is almost never a C*-algebra). Since C*-algebra homomorphisms define Kasparov cycles in bivariant K-theory, such C*-categories provide a natural setting in which to characterise KK-theory using the ostensibly simpler invariant K-theory, as in [34]. Secondly, we may consider the C*-categories $\mathbf{Rep}(A)$ with all *-representations of a fixed C*-algebra A on separable Hilbert spaces as objects and all unitary intertwiners of these representations as morphisms. This and related applications to von Neumann algebras are the theme of the original paper concerning C*-categories [28]. A similar line of research has also been instigated by Bos to study continuous representations of groupoids via the representations of certain C*-categories [9].

C*-categories are also relevant to the study of quantum mechanics, where they are used to describe various aspects of duality related to algebraic quantum field theories. An early application of C*-categories in this line of research was to what is now known as Doplicher-Roberts duality, in which Tannaka-Krein duality is generalised. This led to the celebrated Doplicher-Roberts Reconstruction Theorem that allows one to reconstruct a group G from its C*-category of finite-dimensional unitary representations [20].

Finally, C^*-categories play an increasingly important role in the study of isomorphism conjectures. In [16] Davis and Lück use C^*-categories to give a unified approach to both the Farrell-Jones Isomorphism Conjecture on the algebraic K- and L-theory of integral group rings and the Baum-Connes conjecture on the topological K-theory of reduced group algebras. In doing so they introduce the notion of the reduced groupoid C^*-category, analogous to the reduced group C^*-algebra.

In the body of research described above, all of the C^*-categories considered have a discrete, countable (often finite) set of objects. Informally, a topological C^*-category is a C^*-category for which the collection of objects is no longer a discrete set, but instead is a topological space. Furthermore, the collection of all morphisms should be a topological space with a topology that is compatible with the norm-topology that is defined on each hom-set, and the structure maps of the category should all be continuous. In the formal language of internal categories, a topological C^*-category is therefore an internal category in the category **Top** of topological spaces and continuous maps. This is analogous to the definition of a topological groupoid as introduced by Ehresmann [22].

Topological C^*-categories provide a convenient language for studying topological groupoids, Fell bundles and dynamical systems that is not offered by existing algebraic methods. If one wishes to study the representations of a topological group, a well established technique is to form a convolution algebra of continuous functions on that group and complete it with respect to a C^*-norm. The *-representations of this group C^*-algebra are then in one-to-one correspondence with the continuous representations of the underlying group. By associating to a topological groupoid a topological C^*-category one has a number of analytic tools available with which to further study the groupoid and its representations. The development of the theory in this direction is mostly due to Bos [9], who has developed a theory of continuous representations of a topological groupoid (as opposed to the measurable representations studied by Renault [49]) such that these representations are in one-to-one correspondence with those of a particular C^*-category constructed in a similar fashion to the group C^*-algebra. In this thesis, we will show that this is in fact a topological C^*-category and specify the topologies involved.

The main aim of this thesis is to develop a formal theory for *topological* C^*-*categories* in order to provide a framework for the examples already present in the wider literature and to facilitate the development of further applications. Central to our considerations is the question of norm continuity: It is well established that the correct notion of C*-bundle in the decomposition theory of C*-algebras is that of either a continuous or an upper semicontinuous bundle of C*-algebras. Indeed, the very definition of a bundle in the sense of Fell [25] and Hofmann [31] forces the norm function to be at least upper semicontinuous. Furthermore, it is also known that upper semicontinuous bundles of C*-algebras correspond precisely to both $C_0(X)$-algebras and to the sheaves of C*-algebras as developed by Ara and Mathieu [3]. Conversely, there are important examples of bundle-type objects that are lower semicontinuous. Consider, for example, the fibrewise minimal tensor product of two non-exact continuous C*-bundles [4], [37], or a strongly continuous representation of a topological groupoid on a continuous field of Hilbert spaces [9].

In general, and as noted in [51], upper semicontinuous bundles tend to arise from universal constructions, whereas lower semicontinuous bundles tend to arise from families of (faithful) representations. In the latter case, this is largely due to the presence of the *-strong topology, with respect to which the norm function is lower semicontinuous. This has important consequences in the general theory. For example, our generalisation of the Gelfand-Naimark Representation Theorem is only defined for continuous C*-categories. From the category theorist's point of view, this dichotomy serves to emphasise the important role played by universal constructions.

In Chapter 1 we present the preliminary material on internal categories and groupoids that we shall use in the sequel. The basic idea of an internal category is to replace the sets of objects and morphisms and the structure maps of a small category with objects and morphisms from some other ambient category. The ambient category of interest in this thesis is of course the category **Top** of topological spaces and continuous maps. Our approach in Section 1.1 differs from classical treatments of internal categories to take into account both the ostensibly non-unital nature of C*-algebras and the additional operations that occur when considering enriched internal categories. In Section 1.2 we give a summary of some of the main results concerning topological groupoids, including the generalisation of the Haar measure from the topological group setting. We also consider an important example of a smooth groupoid introduced by Connes [13]. In contrast with the preceding section, our treatment of topological groupoids is entirely classical.

Chapter 2 contains an overview of the main definitions and results relating to bundles of C^*-algebras, Banach spaces and Hilbert spaces that we will make use of later in this thesis.

In Chapter 3 we introduce the main object of study in this thesis. We provide a formal definition of a topological C^*-category and develop the basic theory of topological C^*-categories, continuous C^*-functors and continuous bounded natural transformations. Significant results include a method for constructing a suitable topology for the morphism set of a C^*-category given that the collection of objects has a preordained topology, and that the category of small unital topological C^*-categories is a fibred category over **Top** in the sense of Grothendieck and later Benabou [7].

In Chapter 4 we then introduce the notion of a concrete topological C^*-category. The natural topology for these categories is a version of the *-strong-topology, for which the norm-function is only lower-semicontinuous. These C^*-categories are therefore distinct from the topological C^*-categories that we study in Chapter 3, but are still essential for later constructions. In the second half of the chapter we generalise the famous Gelfand-Naimark Representation Theorem to the setting of topological C^*-categories, via a construction that strongly resembles the Yoneda embedding for ordinary categories.

Chapter 5 contains a number of further constructions to develop our theory of topological C^*-categories. We construct both the full and reduced topological C^*-categories associated to a topological groupoid and show that these constructions are functorial with respect to the class of continuous C^*-categories generated by open embeddings and quotients with compact kernels. We also construct the maximal and minimal tensor products of two continuous topological C^*-categories and generalise the notion of Hilbert-module to our setting.

In Chapter 6 we define the K-theory for continuous topological C^*-categories using a classifying algebra approach derived from Hilbert-modules for topological C^*-categories. We then prove that our definition produces a family of functors satisfying the defining features of C^*-algebraic K-theory: stability, homotopy invariance, half-exactness and Bott periodicity.

Chapter 1

Topological Categories and Groupoids

In this chapter we establish the language of topological categories, in the sense of internal categories in Set. In particular, we provide a number of preliminary results that we shall require in the sequel. Whilst the content is largely expository, our initial definition of category differs from the classical one in that we don't require the existence of identity arrows. We will instead call a category equipped with identity arrows a *unital* category. Since our constructions in Chapter 3 are to be thought of as the horizontal categorifications of C^*-algebras, this approach seems more natural from the point of view of an operator algebraicist. We also we review some of the basic theory of locally compact groupoids that we require for Section 5.1.

A good general reference for the first part of this chapter is [39]. Our discussion of topological groupoids is based on ideas coming from [9], [46] and [49].

§1.1 Topological Categories

Let $\mathcal{S}$ be a fixed ambient category admitting finite products.

Definition 1.1.1. An internal category $\mathcal{C}$ in $\mathcal{S}$ consists of an object of objects $\mathcal{C}_0$ in $\mathcal{S}$, and an object of morphisms $\mathcal{C}_1$ in $\mathcal{S}$, together with source and target morphisms $s_{\mathcal{C}}, t_{\mathcal{C}} \colon \mathcal{C}_1 \rightrightarrows \mathcal{C}_0$ in $\mathcal{S}$ and a morphism $m_{\mathcal{C}} \colon \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \longrightarrow \mathcal{C}_1$ in $\mathcal{S}$ where

$$\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 := \{\, (b,a) \in \mathcal{C}_1 \times \mathcal{C}_1 \;\mid\; s(b) = t(a) \,\}$$

is the pullback of $s_{\mathcal{C}}$ along $t_{\mathcal{C}}$ in $\mathcal{S}$, such that $m_{\mathcal{C}}$ defines an associative composition on $\mathcal{C}$.

In particular, the structure maps $s_{\mathcal{C}}, t_{\mathcal{C}}$ and $m_{\mathcal{C}}$ are such that the following diagram expressing the associativity law for $m_{\mathcal{C}}$ commutes in $\mathcal{S}$:

$$\begin{array}{ccc}
\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 & \xrightarrow{m_{\mathcal{C}} \times \mathrm{id}_{\mathcal{C}_0}} & \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \\
\Big\downarrow {\scriptstyle \mathrm{id}_{\mathcal{C}_0} \times m_{\mathcal{C}}} & & \Big\downarrow {\scriptstyle m_{\mathcal{C}}} \\
\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 & \xrightarrow[m_{\mathcal{C}}]{} & \mathcal{C}_1
\end{array}$$

Definition 1.1.2. A *unital* category in $\mathcal{S}$ is a category in $\mathcal{S}$ together with an injection $\imath_{\mathcal{C}} \colon \mathcal{C}_0 \longrightarrow \mathcal{C}_1$ in $\mathcal{S}$ that assigns to each object $x \in \mathcal{C}_0$ an identity morphism $1_x \in \mathcal{C}(x,x)$ such that the following diagram commutes in $\mathcal{S}$:

$$\begin{array}{ccccc}
\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_0 & \xrightarrow{1 \times \imath_{\mathcal{C}}} & \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 & \xleftarrow{\imath_{\mathcal{C}} \times 1} & \mathcal{C}_0 \times_{\mathcal{C}_0} \mathcal{C}_1 \\
& {\scriptstyle p_1} \searrow & \Big\downarrow {\scriptstyle m} & \swarrow {\scriptstyle p_2} & \\
& & \mathcal{C}_1 & &
\end{array}$$

In the sequel we will suppress the composition morphism $m_{\mathcal{C}}$ and write either ba or $b \circ a$ for $m_{\mathcal{C}}(b,a)$. Where no confusion is likely to occur we will also omit the subscripts from the structure maps $s_{\mathcal{C}}$, $t_{\mathcal{C}}$ and $\imath_{\mathcal{C}}$.

Example 1.1.3. *A small (unital) category is a (unital) internal category in* **Set***.*

Definition 1.1.4. A *topological category* consists of a locally compact, second countable Hausdorff space of objects $\mathcal{C}_0$ and a topological space of morphisms $\mathcal{C}_1$ together with continuous maps s, t and m. That is, a topological category is an internal category in **Top**.

The notion of a structure preserving morphism between categories can also be expressed internally within **Top**:

Definition 1.1.5. Let $\mathcal{C}$ and $\mathcal{D}$ be topological categories. A *continuous functor* $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ consists of a pair of continuous maps $F_0 \colon \mathcal{C}_0 \longrightarrow \mathcal{D}_0$ and $F_1 \colon \mathcal{C}_1 \longrightarrow \mathcal{D}_1$ such that the F_i commute with the structure maps of $\mathcal{C}$ and $\mathcal{D}$ in **Top**:

$$\begin{array}{ccc}
\mathcal{C}_1 & \xrightarrow{F_1} & \mathcal{D}_1 \\
\Big\downarrow {\scriptstyle s_{\mathcal{C}}} & & \Big\downarrow {\scriptstyle s_{\mathcal{D}}} \\
\mathcal{C}_0 & \xrightarrow[F_0]{} & \mathcal{D}_0
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{C}_1 & \xrightarrow{F_1} & \mathcal{D}_1 \\
\Big\downarrow {\scriptstyle t_{\mathcal{C}}} & & \Big\downarrow {\scriptstyle t_{\mathcal{D}}} \\
\mathcal{C}_0 & \xrightarrow[F_0]{} & \mathcal{D}_0
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 & \xrightarrow{F_1 \times F_1} & \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \\
\Big\downarrow {\scriptstyle m_{\mathcal{C}}} & & \Big\downarrow {\scriptstyle m_{\mathcal{D}}} \\
\mathcal{C}_1 & \xrightarrow[F_1]{} & \mathcal{D}_1
\end{array}$$

We will frequently identify the pair (F_0, F_1) with F itself. It follows from the preceding definition that the well established properties of functors in ordinary category theory (that is, for internal categories in **Set**) hold for topological categories. In particular, if $\mathcal{C}$ and $\mathcal{D}$ are unital topological categories then the commutativity of the above diagrams ensures that $F_1(1_x) = 1_{F_0(x)}$.

Remark 1.1.6. Continuous functors are sometimes also called *strict homomorphisms* to distinguish them from weaker notions of morphism such as Morita morphisms and bimodules that are often used in the topological setting.

Definition 1.1.7. Let $F, F' \colon \mathcal{C} \longrightarrow \mathcal{D}$ be continuous functors between topological categories. A *continuous natural transformation* $F \Longrightarrow F'$ consists of a continuous map $\theta\colon \mathcal{C}_0 \longrightarrow \mathcal{D}_1$ such that $s \circ \theta = F_0$, $t \circ \theta = F'_0$, and such that the following diagram expressing naturality commutes in **Top**:

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{(F'_1, \theta\circ s)} & \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \\ \big\downarrow{\scriptstyle (\theta\circ t, F_1)} & & \big\downarrow{\scriptstyle m} \\ \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 & \xrightarrow[m]{} & \mathcal{D}_1 \end{array} \tag{1.1}$$

If $\mathcal{C}, \mathcal{D}$ are topological categories and F, F' are continuous functors then it follows from the commutativity of the above diagrams that the map θ defines a family of morphisms

$$\{\theta_x \colon F(x) \longrightarrow F'(x)\}_{x \in \mathcal{C}_0}$$

that varies continuously over $\mathcal{C}_0$, and such that for every morphism $b \in \mathcal{C}(x, y)$ the familiar naturality square commutes in **Top**:

$$\begin{array}{ccc} F(x) & \xrightarrow{\theta_x} & F'(x) \\ \big\downarrow{\scriptstyle F(b)} & & \big\downarrow{\scriptstyle F'(b)} \\ F(y) & \xrightarrow[\theta_y]{} & F'(y) \end{array} \tag{1.2}$$

If $\mathcal{D}$ is a unital topological category then denote by $\mathcal{D}_1^{iso}$ the object of **Top** consisting of the invertible morphisms in $\mathcal{D}_1$. A continuous natural transformation $\theta\colon F \Longrightarrow F'$ of continuous functors between unital topological C*-categories is called a *continuous natural isomorphism* if the map θ factors through the object

$\mathcal{D}_1^{iso}$ as follows:

$$\mathcal{C}_0 \xrightarrow{\theta} \mathcal{D}_1^{iso} \xrightarrow{(-)^{-1}} \mathcal{D}_1^{iso} \hookrightarrow \mathcal{D}_1$$

Write $\widetilde{\theta}$ for the composition of the maps above. Then the diagram (1.1) commuting is equivalent to the following diagram commuting:

$$\begin{array}{ccc} \mathcal{C}_0 \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_0 & \xrightarrow{\widetilde{\theta} \times F_1 \times \theta} & \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \\ \downarrow \cong & & \downarrow m_{\mathcal{D}} \\ \mathcal{C}_1 & \xrightarrow[F_1']{} & \mathcal{D}_1 \end{array}$$

It follows that if θ is a continuous natural isomorphism then each of the component maps θ_x is an isomorphism.

Definition 1.1.8. A *continuous strong equivalence* between topological categories $\mathcal{C}$ and $\mathcal{D}$ consists of a pair of continuous functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $F' : \mathcal{D} \longrightarrow \mathcal{C}$ such that there exist continuous natural isomorphisms $F' \circ F \Longrightarrow \mathrm{Id}_{\mathcal{C}}$ and $F \circ F' \Longrightarrow \mathrm{Id}_{\mathcal{D}}$.

Topological categories, distinguished up to continuous strong equivalence, together with continuous functors and continuous natural transformations form a 2-category denoted $\mathbf{Top}(\mathbf{Cat})$. The subcategory of unital topological categories is denoted $\mathbf{Top}(\mathbf{Cat}_1)$.

§1.2 Topological Groupoids

A groupoid is a category in which every morphism is invertible. They are useful for encoding local symmetry and describing noncommutative quotient spaces, as in [13]. A groupoid with just one object is (equivalent to) a group.

Definition 1.2.1. A topological groupoid is a unital topological category $\mathcal{G}$ equipped with a continuous involutive map

$$\mathrm{inv} : \mathcal{G}_1 \longrightarrow \mathcal{G}_1, \quad g \longmapsto g^{-1}$$

such that $m\left(g^{-1}, g\right) = 1_{s(g)}$ and $m\left(g, g^{-1}\right) = 1_{t(g)}$ for all $g \in \mathcal{C}_1$.

A topological category is called a dagger category if there exists a continuous map $\dagger : \mathcal{C}_1 \longrightarrow \mathcal{C}_1$ such that $\dagger \circ \dagger$ is the identity map on $\mathcal{C}_1$ and such that for every

$b \in \mathcal{C}(x, y)$ there exists a morphism $b^\dagger \in \mathcal{C}(y, x)$. Dagger categories include topological groupoids, where the dagger structure is the inverse map, and the category of Hilbert spaces, where the dagger structure is derived by taking adjoints.

Example 1.2.2. *To every topological dagger category $\mathcal{C}$ we can associate a topological groupoid called the* orbit equivalence groupoid *of $\mathcal{C}$, denoted $\mathbf{Orb}(\mathcal{C})$. This has object space $\mathcal{C}_0$ and morphism space $\big(\mathbf{Orb}(\mathcal{C})\big)_1$, the image of $\mathcal{C}_1$ under the continuous map (t, s). This groupoid has precisely one morphism $x \longrightarrow y$ if and only if $\mathcal{C}(y, x)$ is non-empty.*

A special case of the preceding example is the following:

Example 1.2.3. *Let X be a topological space. Then the* pair groupoid on *X is the topological groupoid $\mathcal{P}(X)$ with object space X and morphism space $X \times X$, where the source and target maps are the projections onto either factor.*

If $\mathcal{C}$ is a topological dagger category and $\mathcal{C}$ is transitive — that is, $\mathcal{C}(x, y)$ is non-empty for every pair of objects x, y — then $\mathbf{Orb}(\mathcal{C})$ is isomorphic to the pair groupoid $\mathcal{P}(\mathcal{C}_0)$.

Example 1.2.4. *Let G be a topological group acting continuously on a locally compact Hausdorff space X via α. The* action groupoid *of α is the locally compact groupoid $\mathcal{G}$ that has object space $\mathcal{G}_0 = X$ and morphism space $\mathcal{G}_1 = G \times X$. The structure maps are given by $s(g, x) = x$, $t(g, x) = g.x$ and $\imath(x) = (e_G, x)$, where e_G is the identity element of the group.*

The next example allows us to treat topological spaces as a type of topological groupoid. We will make use of this extensively later in the thesis.

Example 1.2.5. *Let X be a locally compact topological space. Then X is a topological groupoid with object and morphism space X, where all the structure maps are the identity.*

Definition 1.2.6. If $\mathcal{G}$ is a topological groupoid then a *topological subgroupoid* $\mathcal{H}$ is a subcategory of $\mathcal{G}$ such that $\mathcal{H}_0 \subseteq \mathcal{G}_0$ and $\mathcal{H}_1 \subseteq \mathcal{G}_1$ are subspaces. A subgroupoid $\mathcal{H} \leq \mathcal{G}$ is called *wide* if $\mathcal{H}_0 = \mathcal{G}_0$ and is called *full* if $\mathcal{H}(x, y) = \mathcal{G}(x, y)$ for all $x, y \in \mathcal{H}_0$.

Associated to every topological groupoid $\mathcal{G}$ are two canonical topological subgroupoids: The *identity subgroupoid* of $\mathcal{G}$ is the topological groupoid $\mathbf{Id}(\mathcal{G})$ with the same object space as $\mathcal{G}$ consisting of only the identity morphisms of $\mathcal{G}$. The structure maps for $\mathbf{Id}(\mathcal{G})$ are all the identity map. The *isotropy subgroupoid* of $\mathcal{G}$ is the topological groupoid $\mathbf{Iso}(\mathcal{G})$ with the same object space as $\mathcal{G}$ and morphism space

$$\big(\mathbf{Iso}(\mathcal{G})\big)_1 := \bigcup_{x\in\mathcal{G}_0} \mathcal{G}(x,x).$$

Definition 1.2.7. Let $\mathcal{G}$ be a topological groupoid, and $\mathcal{N}$ a wide topological subgroupoid. Then $\mathcal{N}$ is called *normal* if for all $x, y \in \mathcal{G}_0$ and given any $g \in \mathcal{G}(x,y)$ and any $k \in \mathbf{Iso}(\mathcal{G})(x,x)$ then $gkg^{-1} \in \mathcal{N}_1$.

From the preceding definition we observe that a wide topological subgroupoid is normal if and only if $\mathcal{N}(x,x)$ is a normal subgroup of $\mathcal{G}(x,x)$ for every $x \in \mathcal{G}_0$.

Definition 1.2.8. A homomorphism of topological groupoids is a continuous functor $F\colon \mathcal{G} \longrightarrow \mathcal{G}'$.

A continuous functor $\mathcal{G} \longrightarrow \mathcal{G}'$ is called an *embedding* if it is both faithful and injective on objects. If $F\colon \mathcal{G} \longrightarrow \mathcal{G}'$ is a continuous functor between topological groupoids then the *kernel* of F is the wide topological subgroupoid $\mathbf{ker}\, F \leq \mathcal{G}$ with

$$\mathbf{ker}\, F_1 := \{\, g \in \mathcal{G} \mid F(g) \in \imath(\mathcal{G}'_0) \,\}\,.$$

Clearly $\mathbf{ker}\, F$ is a normal subgroupoid of $\mathcal{G}$.

Proposition 1.2.9. *Let $F\colon \mathcal{G} \longrightarrow \mathcal{G}'$ be a continuous functor that is full and surjective on objects. Then*

$$\overline{F}\colon\ \mathcal{G}/_{\mathbf{ker}\, F} \longrightarrow \mathcal{G}' \qquad [g] \mapsto F(g)$$

defines an isomorphism of topological groupoids such that $F = Q \circ \overline{F}$ where Q is the canonical quotient functor.

Recall that a continuous map $f\colon X \longrightarrow Y$ is called *perfect* if it is closed, surjective and such that the preimage $f^{-1}(y)$ of every point y is compact.

Lemma 1.2.10. *Let $\mathcal{G}$ be a topological groupoid, and $\mathcal{N} \trianglelefteq \mathcal{G}$ a normal topological subgroupoid such that $\mathcal{N}_1$ is compact. Then the quotient functor*

$$Q\colon \mathcal{G} \longrightarrow \mathcal{G}/_{\mathcal{N}}$$

is perfect.

In particular, if $F\colon \mathcal{G} \longrightarrow \mathcal{G}'$ is a continuous quotient functor with compact kernel then

$$Q\colon \mathcal{G} \longrightarrow \mathcal{G}/_{\mathbf{ker}\, F} \cong \mathcal{G}'$$

is perfect.

§1.3 Haar Systems and Topological Amenability

For analysis on a locally compact group, it is essential to have a left Haar measure. The groupoid version of a left Haar measure is a continuous family of measures called a left Haar system. Let X, Y be locally compact, Hausdorff spaces and $p\colon X \longrightarrow Y$ a continuous map. A *continuous field of measures* on Y over X with momentum map p is a family of positive Radon measures $\{\mu^x\}_{x\in X}$ such that $\operatorname{supp}(\mu^x) \subseteq Y_x := p^{-1}(\{x\})$ and such that for all $\varphi \in C_c(Y)$ the function

$$\mu(\varphi)\colon X \longrightarrow \mathbb{C},\ x \mapsto \int_{y\in Y_x} \varphi(y)\ d\mu^x(y) \tag{1.3}$$

belongs to $C_c(X)$. A continuous field of measures is called *faithful* if $\operatorname{supp}(\mu^x) = Y_x$ for each $x \in X$.

Definition 1.3.1. Let $\mathcal{G}$ be a locally compact groupoid. A *left Haar system* for $\mathcal{G}$ is a faithful continuous field of measures $\{\lambda^x\}_{x\in\mathcal{G}_0}$ with momentum map $t_{\mathcal{G}}$ such that if $g \in \mathcal{G}(x, y)$ and $\varphi \in C_c(\mathcal{G}_1)$ then

$$\int_{\mathcal{G}^x} \varphi(gh)\ d\lambda^x(h) = \int_{\mathcal{G}^y} \varphi(h)\ d\lambda^y(h) \tag{1.4}$$

The expression in (1.4) characterises left-invariance in the groupoid setting. Unlike the left Haar measure on a locally compact group, a left Haar system need not exist, and when it does exist it is not necessarily unique.

Example 1.3.2. *If $\mathcal{G}$ is a discrete groupoid then the counting measure on each $\mathcal{G}^x$ defines a left Haar system for $\mathcal{G}$.*

Example 1.3.3. *Let X be a locally compact Hausdorff space and μ a positive Radon measure on X with full support. For $x \in X$, let δ_x denote the Dirac measure given by*

$$\delta_x(A) = \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases}$$

Then there is a left-Haar system defined by

$$\lambda^x\left(A \times \{x\}\right) = \mu(A)$$

for every Borel subset $A \subset X$. Furthermore, every left Haar system on $\mathcal{P}(X)$ is of this form for some positive Radon measure μ on X.

The existence of a continuous field of positive measures has topological implications for the momentum map. An immediate consequence of the preceding result is that if $\mathcal{G}$ admits a left Haar system then the structure maps $t_{\mathcal{G}}, s_{\mathcal{G}}$ are necessarily continuous.

Chapter 2

Preliminaries on Bundles of C^*-Algebras and Banach Spaces

In this chapter we introduce elements of the theory of Banach and C^*-algebra bundles that will underpin constructions later in this thesis. There are numerous different definitions of Banach and C^*-algebra bundle appearing throughout the literature, from the early notion of a continuous field as introduced by Dixmier and Douady [19] to the more recent bundle-type decomposition of Kasparov that is $C_0(X)$-algebras [36]. Under fairly modest conditions these various definitions are essentially equivalent.

Arguably the most modern approach is the one used by Ara and Mathieu to relate upper semicontinuous C^*-algebra bundles to sheaves of C^*-algebras [3]. The distinctive feature of this definition is that the topology on the total space of the bundle is specified as part of the definition. We adopt this approach and take it a step further by dropping the analytical properties of the classical definition by Fell [25] and characterising a bundle entirely in terms of the topological data. This enables us to give a single, unified definition of both upper and lower semicontinuous bundles.

Most of this chapter is concerned with with giving an account of how the standard properties of Banach and C^*-algebra bundles can be derived from this definition, in order to conveniently reference established results in the sequel. The content is therefore mostly expository in nature. Details of the material presented can be found in [26] and [27] in the case of *continuous* Banach and C^*-algebra bundles. A good account of *continuous* C^*-algebra bundles and their relationship to $C_0(X)$-algebras can be found in Appendix C of [58].

§2.1 Banach and C^*-Algebra Bundles

Throughout this section let X be a locally compact Hausdorff space. We first recall some basic definitions about semicontinuity: A real-valued function $f\colon X \longrightarrow \mathbb{R}$ is called *upper semicontinuous* if the set

$$\{\, x \in X \ \mid\ f(x) < \epsilon \,\}$$

is an open set in X for every $\epsilon > 0$. It is called *lower-semicontinuous* if the set

$$\{\, x \in X \ \mid\ f(x) > \epsilon \,\}$$

is an open set (equivalently if the set $\{\, x \in X \ \mid\ f(x) \leq \epsilon \,\}$ is closed) for every $\epsilon > 0$. A function that is both upper semicontinuous and lower semicontinuous is continuous. In what follows, we suppose that that $\mathcal{A}$ is either the category $\mathbf{Ban}$ or the category $\mathbf{C^*}$-$\mathbf{Alg}$.

Definition 2.1.1. An $\mathcal{A}$-bundle over X is a triple (A, p, X) consisting of a topological space A and a continuous surjection $p\colon \mathsf{A} \longrightarrow X$ with each fibre $\mathsf{A}_x := p^{-1}(x)$ an object of $\mathcal{A}$ and such that:

i. All algebraic operations are continuous functions on A.

ii. If $\Gamma_b(\mathsf{A})$ denotes the set of all bounded continuous sections $\alpha\colon X \longrightarrow \mathsf{A}$ of p then for each $x \in X$ we have

$$\mathsf{A}_x = \overline{\{\, \alpha(x) \ \mid\ \alpha \in \Gamma_b \mathsf{A} \,\}}.$$

iii. For all $U \subset X$ open , $\alpha \in \Gamma_b(\mathsf{A})$ and $\epsilon > 0$ the set

$$\Omega(U, \alpha, \epsilon) := \{\, a \in \mathsf{A} \ \mid\ p(a) \in U \text{ and } \|a - \alpha\,(p(a))\| < \epsilon \,\}$$

is an open subset of A and these sets form a sub-base for the topology on A.

We call an $\mathcal{A}$-bundle *continuous* if the norm function $\|\cdot\| : \mathsf{A} \longrightarrow \mathbb{R}$ defined by $a \longmapsto \|a\|_{p(a)}$ is continuous. We call it upper/lower semicontinuous if the function $a \longmapsto \|a\|_{p(a)}$ is upper/lower semicontinuous.

If $\mathcal{A} = \mathbf{Ban}$ then by property (i) the operation $+$ is a continuous function $\mathsf{A} \times_p \mathsf{A} \longrightarrow \mathsf{A}$, where $\mathsf{A} \times_p \mathsf{A} = \{\, (a_1, a_2) \in \mathsf{A} \times \mathsf{A} \ \mid\ p(a_1) = p(a_2) \,\}$ and fibrewise scalar multiplication is a continuous function $\cdot_{\mathbb{C}} : \mathbb{C} \times \mathsf{A} \longrightarrow \mathsf{A}$. If $\mathcal{A} = \mathbf{C^*\text{-}Alg}$ then we also have that multiplication is a continuous function $\cdot : \mathsf{A} \times_p \mathsf{A} \longrightarrow \mathsf{A}$ and involution is a continuous function $* : \mathsf{A} \longrightarrow \mathsf{A}$.

The justification for specifying the topology in the definition of an $\mathcal{A}$-bundle comes from the following result of Fell ([26], Proposition II.13.18. See also [58], Theorem C.20).

Proposition 2.1.2. *Suppose that (A, p, X) consists of a locally compact Hausdorff space X, an untopologised set A and a surjection $p : \mathsf{A} \longrightarrow X$, such that each fibre $\mathsf{A}_x := p^{-1}(x)$ is an object in $\mathcal{A}$. Let Δ be the *-algebra of all bounded sections (not a priori continuous) of p such that:*

1. *For each $\alpha \in \Delta$ the map $x \longmapsto \|\alpha(x)\|$ is upper semicontinuous.*
2. *For each $x \in X$ the set $\{ \alpha(x) \mid \alpha \in \Delta \}$ is dense in A_x.*

Then there exists a unique topology on A making (A, p, X) an upper semicontinuous $\mathcal{A}$-bundle such that $\Delta = \Gamma_b(\mathsf{A})$.

We wish to relate our definition of an upper semicontinuous $\mathcal{A}$-bundle with the classical definitions of **Ban**-bundle and **C*-Alg**-bundle of Fell. Recall the following:

Definition 2.1.3. An upper semicontinuous *Fell-Hofmann* **Ban**-*bundle* over X consists of a continuous open surjection $p : \mathsf{A} \longrightarrow X$, together with linear operations and norms defined fibrewise so that each A_x is a Banach space, such that:

i. The map $a \longmapsto \|a\|$ is upper semicontinuous from A to $\mathbb{R}$.

ii. The map $(a, b) \longmapsto a + b$ is continuous from $\mathsf{A} \times_p \mathsf{A}$ to A.

iii. The map $(\lambda, a) \longmapsto \lambda a$ is continuous from A to A.

iv. If $\{a_i\}$ is a net in A such that $\|a_i\| \longrightarrow 0$ and $p(a_i) \longrightarrow x$ in X then $a_i \longrightarrow 0_x$, where 0_x is the zero element in A_x.

An upper semicontinous Fell-Hofmann **C*-Alg**-bundle is an upper semicontinuous Fell-Hofmann **Ban**-bundle such that each fibre A_x is a C^*-algebra and such that in addition to (i.) to (iv.) above, the following also hold:

v. The map $(a, b) \longmapsto ab$ is continuous from A to A.

vi. The involution $a \longmapsto a^*$ is continuous from A to A.

If (A, p, X) is an upper semicontinuous $\mathcal{A}$-bundle then we recover the properties of an upper semicontinuous Fell-Hofmann bundle.

Lemma 2.1.4. *Suppose that (A, p, X) is an upper semicontinuous $\mathcal{A}$-bundle.*

1. *The sets of the form $\Omega(U, \alpha, \epsilon)$ for $U \subset X$ open, $\alpha \in \Gamma_b(\mathsf{A})$ and $\epsilon > 0$ form a* basis *for the topology on A such that $p \colon \mathsf{A} \longrightarrow X$ is a continuous* open *surjection.*
2. *If $\{a_i\}$ is a net in A such that $\|a_i\| \longrightarrow 0$ and $p(a_i) \longrightarrow x$ in X then $a_i \longrightarrow 0_x$, where 0_x is the zero element in A_x.*

Proof. The proofs of these two statements are contained within the proofs of [26], Proposition II.13.18 in the continuous **Ban**-bundle case and [58], Theorem C.20 in the upper semicontinuous $\mathbf{C^*}$-**Alg**-bundle case. □

The following is an easy consequence of the preceding lemma:

Proposition 2.1.5. *Every upper semicontinuous $\mathcal{A}$-bundle is an upper semicontinuous Fell-Hofmann $\mathcal{A}$-bundle.* □

Given an upper semicontinuous $\mathcal{A}$-bundle (A, p, X), the set $\Gamma(\mathsf{A})$ of *all* continuous sections of p has the following local closure property:

Lemma 2.1.6. *Let (A, p, X) be an upper semicontinuous* **Ban**-*bundle, and let ξ be a (not a priori continuous) section. Suppose that for each $x \in X$ and each $\epsilon > 0$ there exists a continuous section $\alpha \in \Gamma(\mathsf{A})$ and a neighbourhood U of x such that $\|\alpha(u) - \xi(u)\| < \epsilon$ for all $u \in U$. Then $\xi \in \Gamma(\mathsf{A})$.*

Proof. Let ξ be as stated in the hypothesis and let $\{x_i\}$ be a net converging to $x \in X$. We show that $\xi(x_i) \longrightarrow \xi(x)$. Let $\alpha \in \Gamma(\mathsf{A})$. Since α is continuous it follows that $\{\alpha(x_i)\}$ is a net in A converging to $\alpha(x) \in \mathsf{A}$. Moreover for each index i we have $p(\xi(x_i)) = p(\alpha(x_i)) = x_i$. By assumption, for each $\epsilon > 0$ there exists a neighbourhood U such that $\|\alpha(u) - \xi(u)\| < \epsilon$ on U, hence for sufficiently large i we have $x_i \in U$. Therefore

$$\|\alpha(x_i) - \xi(x_i)\| < \epsilon \quad \text{and} \quad \|\alpha(x) - \xi(x)\| < \epsilon$$

and so by Lemma 2.1.4 we have $\xi(x_i) \longrightarrow \xi(x)$. Therefore ξ is continuous. □

§2.2 Continuous Hilbert Bundles

In this section let $\mathcal{A} = \mathbf{Hilb}$, the category of Hilbert spaces and bounded linear maps.

Definition 2.2.1. A continuous $\mathbf{Hilb}$-bundle over X is a triple (H, p, X) consisting of a topological space H and a continuous surjection $p\colon \mathsf{H} \longrightarrow X$ with each fibre $\mathsf{H}_x := p^{-1}(x)$ a Hilbert space and such that:

i All algebraic operations are continuous functions on A.

ii The function $(a, b) \longmapsto \langle\, a, b \,\rangle$ is a continuous function from H to $\mathbb{C}$.

iii If $\Gamma_b(\mathsf{H})$ denotes the set of all bounded continuous sections $\alpha\colon X \longrightarrow \mathsf{H}$ of p then for each $x \in X$ we have

$$\mathsf{H}_x = \overline{\{\, \alpha(x) \;\mid\; \alpha \in \Gamma_b\mathsf{H} \,\}}.$$

iv For all $U \subset X$ open , $\alpha \in \Gamma_b(\mathsf{H})$ and $\epsilon > 0$ the set

$$\Omega(U, \alpha, \epsilon) := \{\, a \in \mathsf{H} \;\mid\; p(a) \in U \text{ and } \|a - \alpha\,(p(a))\| < \epsilon \,\},$$

where $\|\cdot\|$ is the inner-product norm, is an open subset of H and these sets form a base for the topology on H.

A simple polarisation argument shows that a continuous $\mathbf{Hilb}$-bundle is necessarily a continuous $\mathbf{Ban}$-bundle in the sense of Definition 2.1.1.

Now let (H, p, X) and (K, q, Y) be continuous $\mathbf{Hilb}$-bundles. For each $x \in X$ and $y \in Y$ form the Hilbert space tensor product $\mathsf{H}_x \otimes \mathsf{K}_y$. Define a topological space $\mathsf{H} \otimes \mathsf{K}$ to be the disjoint union of all such $\mathsf{H}_x \otimes \mathsf{K}_y$.

Definition 2.2.2. Define $(\mathsf{H} \otimes \mathsf{K}, p \times q, X \times Y)$ to be the *tensor product* $\mathbf{Hilb}$-*bundle* of (H, p, X) and (K, q, Y), with

$$\Gamma_b(\mathsf{H} \otimes \mathsf{K}) = \mathrm{Span}\,\{\, (x, y) \longmapsto \alpha(x) \otimes \beta(y) \;\mid\; \alpha \in \Gamma_b(\mathsf{H}) \text{ and } \beta \in \Gamma_b(\mathsf{K}) \,\}.$$

We shall utilise this construction in Chapter 4.

Chapter 3

Topological C^*-Categories

In this chapter we introduce the main objects of study in this thesis: Topological C^*-categories. Our approach is to mimic as closely as possible the constructions of Fell [24], [25] and Hofmann [31] in forming continuous and upper semicontinuous $\mathbf{Ban}$- and $\mathbf{C^*\text{-}Alg}$-bundles, using our generalised definition of bundle from Chapter 2. This approach allows us to simultaneously consider both upper and lower semicontinuous topological C^*-categories, which is necessary since a fundamentally important class of examples — the *concrete* topological C^*-categories — are naturally lower semicontinuous. They therefore fall outside of traditional bundle theory for Banach spaces and C^*-algebras.

We start with a review of C^*-categories in $\mathbf{Set}$, summarising the main results about C^*-categories developed by Ghez *et al* [28] and Mitchener [43], and about multiplier C^*-categories as detailed by Vasselli [56]. The content of this section is largely expository in nature, and so we omit a number of details and refer the reader to the articles referenced above. One exception to this practice is Proposition 3.1.19 — in [56] some important details are omitted from the proof of this result, which we rectify by including them here.

In Section 3.2 we give an abstract characterisation of a topological C^*-category, and develop some of the basic theory of topological C^*-categories and their associated homomorphisms. We prove that our definitions are consistent with the established notion of topological category in both the non-unital (Proposition 3.2.3) and unital case (Proposition 3.2.5). We also give criteria under which a *-functor between topological C^*-categories is in fact an internal functor in $\mathbf{Top}$ (Proposition 3.2.8).

We conclude the chapter by studying the various categories of topological C^*-categories. This section is structured towards proving that the category of all unital topological C^*-categories is a fibred category over **Top** in the sense of Grothendieck [29] and Bénabou [7].

§3.1 Review of C^*-Categories in Set

Let $\mathcal{C}$ be a small category with countable set of objects. We call $\mathcal{C}$ a $\mathbb{C}$-linear category if each hom-object $\mathcal{C}(x,y)$ is a complex vector space and composition is given by a family of bilinear maps

$$m_{xyz}\colon \mathcal{C}(y,z)\times\mathcal{C}(x,y)\longrightarrow\mathcal{C}(x,z).$$

If A,B are Banach spaces we write $A\odot B$ for their tensor product as vector spaces, and refer to this as the *algebraic tensor product* of A and B. An element of $A\odot B$ may be written (not necessarily uniquely) as a formal sum of elementary tensors,

$$x=\sum_i \alpha_i\; a_i\otimes b_i,\qquad a_i\in A,\ b_i\in B,\ \alpha_i\in\mathbb{C}.$$

Definition 3.1.1. The *projective tensor product* $A\otimes_{pr}B$ of A and B is the completion of $A\odot B$ with respect to the *projective cross norm*

$$\|x\|=\inf\left\{\sum_i|\alpha_i|\,.\,\|a_i\|_A\,.\,\|b_i\|_B\ \mid\ x=\sum_i\alpha_i\;a_i\otimes b_i\right\}.$$

A $\mathbb{C}$-linear category $\mathcal{C}$ is called a *Banach category* if each $\mathcal{C}(x,y)$ is a Banach space with composition given by a family of linear maps

$$m_{xyz}\colon \mathcal{C}(y,z)\otimes_{pr}\mathcal{C}(x,y)\longrightarrow\mathcal{C}(x,z)$$

satisfying $\|b\circ a\|\le\|b\|\,.\,\|a\|$.

In the formal language of enriched category theory we say that $\mathcal{C}$ is a small category enriched over the closed monoidal category $(\mathbf{Ban},\otimes_{pr},\mathbb{C})$. If $\mathcal{C}$ is a $\mathbb{C}$-linear category then an involution on $\mathcal{C}$ is a functor $*:\ \mathcal{C}^{op}\longrightarrow\mathcal{C}$ given by a family of antilinear isometries

$$*_{x,y}\colon\mathcal{C}(x,y)\longrightarrow\mathcal{C}(y,x)$$

such that $*_0\colon\mathcal{C}_0\longrightarrow\mathcal{C}_0$ is the identity and $*\circ*=\mathrm{id}_{\mathcal{C}}$, the identity functor on $\mathcal{C}$. We write b^* for the image of b under $*$, and call this morphism the *adjoint* of b. A Banach category equipped with an involution is called a *Banach *-category*.

Definition 3.1.2. A C^*-category in **Set** is a Banach *-category in **Set** such that:

i For every morphism $b \in \mathcal{C}_1$ the C^*-identity $\|b^*b\| = \|b\|^2$ is satisfied;

ii For every morphism $b \in \mathcal{C}(x, y)$ there exists $a \in \mathcal{C}(x, x)$ satisfying $b^*b = a^*a$.

A C^*-category in **Set** is called *unital* if each $\mathcal{C}(x, x)$ is a unital C^*-algebra.

It follows from axiom (ii) that for every morphism $b \in \mathcal{C}(x, y)$ the morphism b^*b is a positive element of the C^*-algebra $\mathcal{C}(x, x)$. Whilst this is automatically true for C^*-algebras it is not so for C^*-categories, as shown by the following example by Schick (seen in [43]):

Example 3.1.3. *Let $\mathcal{C}$ be a unital C^*-category with set of objects $\mathcal{C}_0 = \{x, y\}$ and $\mathcal{C}(x, x) = \mathcal{C}(x, y) = \mathcal{C}(y, x) = \mathcal{C}(y, y) = \mathbb{C}$. Composition is given by multiplication of complex numbers and the norm is given by $\|b\| = |b|$. Define an involution on $\mathcal{C}$ by*

$$b^* = \begin{cases} \bar{b} & \text{if } b \in \mathcal{C}(x, y) \text{ for } x = y \\ -\bar{b} & \text{if } b \in \mathcal{C}(x, y) \text{ for } x \neq y. \end{cases}$$

It follows that for non-zero $b \in \mathcal{C}(x, y)$ with $x \neq y$ we have

$$\mathrm{Spec}(b^*b) = \{\, z \in \mathbb{C} \ \mid \ (b^*b - zI) \notin GL(\mathbb{C}) \,\} = \{-b\bar{b}\} \not\subset \mathbb{R}^{\geq 0}.$$

*Thus b^*b is not a positive element of $\mathcal{C}(x, x)$, and $\mathcal{C}$ is unital Banach *-category that does not satisfy axiom* (ii).

Definition 3.1.4. A *-functor between C^*-categories is a functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ such that each

$$F_{xy}\colon \mathcal{C}(x, y) \longrightarrow \mathcal{D}(F(x), F(y))$$

is a $\mathbb{C}$-linear map and such that $F(b)^* = F(b^*)$ for all $b \in \mathcal{C}_1$. If $\mathcal{C}$ and $\mathcal{D}$ are both unital C^*-categories then F is a unital *-functor if in addition we have $F(1_x) = 1_{F(x)}$ for each object $x \in \mathcal{C}_0$.

In [43] it is shown that *-functors exhibit a number of properties possessed by *-homomorphisms between C^*-algebras:

Proposition 3.1.5. *Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a *-functor. Then $\|F(b)\| \leq \|b\|$ for all $b \in \mathcal{C}_1$ and F is therefore continuous with respect to the norm topology. Furthermore if F is faithful then it is isometric.*

Proof. We give the proof of the first statement only, the second statement is proved similarly: Let $b \in \mathcal{C}(x,y)$. Then by definition there exists $a \in \mathcal{C}(x,x)$ such that $b^*b = a^*a$, and it follows that

$$\begin{aligned} \|F(a^*a)\| \leq \|a^*a\| &\Longrightarrow \|F(b^*b)\| \leq \|b^*b\| \\ &\Longrightarrow \|F(b)^*F(b)\| \leq \|b^*b\| \\ &\Longrightarrow \|F(b)\|^2 \leq \|b\|^2 \end{aligned}$$

by the C^*-identity. □

From the above Proposition we deduce that the norm on a C^*-category is unique: Suppose that $\|\cdot\|, \|\cdot\|'$ are two norms making $\mathcal{C}$ a C^*-category. Then the identity functor $\mathrm{Id}\colon (\mathcal{C}, \|\cdot\|) \longrightarrow (\mathcal{C}, \|\cdot\|')$ is faithful, and hence isometric.

Example 3.1.6. *Every C^*-algebra A can be thought of as a C^*-category $\mathcal{A}$ with $\mathcal{A}_0 = \{pt\}$ and $\mathcal{A} \cong A$.*

Example 3.1.7. *Let A be a C^*-algebra and let $\mathbf{Rep}(A)$ denote the category consisting of non-degenerate representations of A on Hilbert spaces as objects and intertwining operators between these representations as morphisms. Then $\mathbf{Rep}(A)$ is a C^*-category.*

Example 3.1.8. *The category $\mathbf{Hilb}$ of Hilbert spaces and bounded linear operators is a C^*-category, with composition given by composition of linear operators and involution defined by taking adjoints. More generally, if A is a C^*-algebra then the category $\mathbf{Hilb}_A$ of Hilbert A-modules and bounded A-linear operators forms a C^*-category.*

Example 3.1.8 above gives rise to an important and distinguished class of *-functors:

Definition 3.1.9. Let $\mathcal{C}$ be a C^*-category. A *representation* of $\mathcal{C}$ is a *-functor $\mathcal{C} \longrightarrow \mathbf{Hilb}$.

We obtain further examples of C^*-categories by associating to an algebraic object a $\mathbb{C}$-linear category with involution, and then constructing a representation of this $\mathbb{C}$-linear category into the category $\mathbf{Hilb}$. The closure of the image of this representation is then a C^*-category. This approach to constructing C^*-categories is demonstrated in the examples, where the algebraic object in question is a groupoid in $\mathbf{Set}$.

Let $\mathcal{G}$ be a groupoid in $\mathbf{Set}$ and denote by $\mathbb{C}\mathcal{G}$ the unital $\mathbb{C}$-linear category with set of objects $\mathcal{G}_0$ and with each hom-object $\mathbb{C}\mathcal{G}(x, y)$ the free complex vector space on $\mathcal{G}(x, y)$. Composition is defined on generators by

$$g_2 g_1 = \begin{cases} g_2 \circ g_1 & \text{if } g_2 \circ g_1 \text{ is defined in } \mathcal{G} \\ 0 & \text{otherwise} \end{cases}$$

and involution by $\alpha_g.g \longmapsto \overline{\alpha_g}.g^{-1}$. There are two C^*-categories that can be constructed from $\mathbb{C}\mathcal{G}$, the first of which is due to Davis and Lück [16] and the second is due to Mitchener [43].

Example 3.1.10. *The* reduced C^*-category of $\mathcal{G}$ *is the unital* C^**-category* $\mathcal{C}^*_r(\mathcal{G})$ *obtained by completing each hom-object of* $\mathbb{C}\mathcal{G}$ *with respect to the norm*

$$\|-\|_r := \sup \left\{ \left\|\ell^2(w, -)\right\| \;\middle|\; w \in \mathcal{G}_0 \right\}$$

where $\ell^2(x, y)$ *denotes the Hilbert space of square summable sequences in* $\mathcal{G}(x, y)$ *and the bounded linear map* $\ell^2(w, -)\colon \ell^2(w, x) \longrightarrow \ell^2(w, y)$ *is left-composition.*

Example 3.1.11. *The* full C^*-category of $\mathcal{G}$ *is the unital* C^**-category* $\mathcal{C}^*(\mathcal{G})$ *obtained by completing each hom-object of* $\mathbb{C}\mathcal{G}$ *with respect to the norm*

$$\|-\|_{\max} := \sup \left\{ \|\rho(-)\| \;\middle|\; \rho \textit{ is a representation of } \mathbb{C}\mathcal{G} \right\}.$$

The defining feature of C^*-categories is that they can be represented faithfully on the concrete category **Hilb** via a functor that is similar to the G.N.S. construction for C^*-algebras. Details of this construction for C^*-categories in $\mathbf{Set}$ can be found in [43].

Proposition 3.1.12. *Let* $\mathcal{C}$ *be a* C^**-category. Then there exists a faithful representation* $\rho\colon \mathcal{C} \longrightarrow$ **Hilb***, where* ρ *is given by the direct sum* $\bigoplus \rho_\sigma$ *over all states* σ *of all endomophism sets* $\mathcal{C}(x, x)$*.*

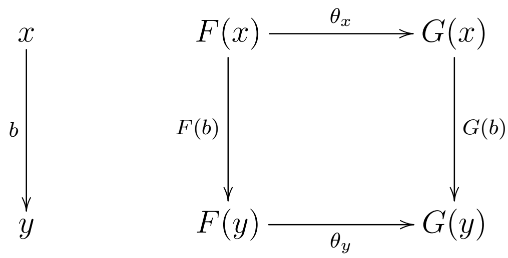
Now let $\mathcal{C}, \mathcal{D}$ be C^*-categories and let $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$ be *-functors. There are two notions of transformation $F \Longrightarrow G$ that we consider. First we recall the notion of a unitary morphism in $\mathcal{C}$.

Definition 3.1.13. If $\mathcal{C}$ is a unital C^*-category then a morphism $u \in \mathcal{C}(x, y)$ is called *unitary* if $u^*u = 1_x$ and $uu^* = 1_y$.

Lemma 3.1.14. *Let* $b \in \mathcal{C}(x, y)$ *be an isomorphism. Then there exists a unitary* $u \in \mathcal{C}(x, y)$ *such that* $u = br$ *for some self-adjoint morphism* $r \in \mathcal{C}(x, x)$*.*

Proof. Let $b \in \mathcal{C}(x, y)$ be an isomorphism in $\mathcal{C}$. Define using the functional calculus a morphism $r = (b^*b)^{-1/2}$ and let $u = br$. Then u is the desired unitary morphism. □

Definition 3.1.15. Let $\mathcal{C}, \mathcal{D}$ be C*-categories and let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ be *-functors between them. A *bounded natural transformation* $\theta : F \implies G$ is a family of bounded linear maps $\{\, \theta_x : F(x) \longrightarrow G(x) \mid x \in \mathcal{C}_0 \,\}$ in $\mathcal{D}_1$ such that for every $b \in \mathcal{C}(x, y)$ the following diagram commutes,



and such that the value $\|\theta\| := \sup \{\, \|\theta_x\| \mid x \in \mathcal{C}_0 \,\}$ is finite. If $\mathcal{C}$ and $\mathcal{D}$ are unital C*-categories then a *unitary transformation* is a bounded natural transformation such that each component map θ_x is a unitary morphism in $\mathcal{D}$.

Definition 3.1.16. Two unital C*-categories $\mathcal{C}$, $\mathcal{D}$ are said to be *unitarily equivalent* if there exists a pair of *-functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $F' : \mathcal{D} \longrightarrow \mathcal{C}$ together with unitary transformations $F' \circ F \implies \mathrm{id}_{\mathcal{C}}$ and $F \circ F' \implies \mathrm{id}_{\mathcal{D}}$.

The collection of all (small) C*-categories, *-functors between them and bounded natural transformations as 2-morphisms form a 2-category denoted C*-**Cat**. The 2-category with the same objects and *-functors but with unitary transformations as 2-morphisms is denoted C*-$\mathbf{Cat}_{\mathcal{U}}$.

The largest embedding of a non-unital C*-algebra A as an essential ideal of a unital one is the multiplier algebra of A. Following Vaselli [56] we review the analogous construction for C*-categories.

Let $\mathcal{C}$ be a C*-category. Each $\mathcal{C}(x, y)$ is a Hilbert-$\mathcal{C}(y, y)$-$\mathcal{C}(x, x)$-bimodule. If $a \in \mathcal{C}(x, x)$, $c \in \mathcal{C}(y, y)$ and $b, b' \in \mathcal{C}(x, y)$ then the left and right actions are respectively given by

$$(c, b) \longmapsto c \circ b \quad \text{and} \quad (b, a) \longmapsto b \circ a$$

and the inner products are given by

$$\langle\, b', b \,\rangle_y := b' \circ b^* \quad \text{and} \quad \langle\, b, b' \,\rangle_x := b^* \circ b'.$$

We observe that each $\mathcal{C}(x,y)$ is complete with respect to the norm

$$\|b\|^2 := \|< b,b >_x\| = \|< b,b >_y\|.$$

A bounded linear map $m\colon \mathcal{C}(w,x) \longrightarrow \mathcal{C}(w,y)$ is called a right-$\mathcal{C}(w,w)$-module operator if $m(b \circ a) = m(b) \circ a$. Similarly, a bounded linear map $m\colon \mathcal{C}(x,z) \longrightarrow \mathcal{C}(y,z)$ is called a left-$\mathcal{C}(z,z)$-module operator if $m(c \circ b) = c \circ m(b)$.

Definition 3.1.17. Let $x, y \in \mathcal{C}_0$. A multiplier from x to y is a pair $m = (m^L, m^R)$ such that:

1. $m^L\colon \mathcal{C}(x,x) \longrightarrow \mathcal{C}(x,y)$ is a right $\mathcal{C}(x,x)$-module operator;
2. $m^R\colon \mathcal{C}(y,y) \longrightarrow \mathcal{C}(x,y)$ is a left-$\mathcal{C}(y,y)$-module operator;
3. $c \circ m^L(a) = m^R(c) \circ a \in \mathcal{C}(x,y)$, for all $a \in \mathcal{C}(x,x)$, $c \in \mathcal{C}(y,y)$.

We write $M\mathcal{C}(x,y)$ for the collection of all multipliers from x to y.

Remark 3.1.18. We will write $cm = m^R(c)$ and $ma = m^L(a)$, and hence the compatibility axiom above can be thought of as the associativity condition $c(ma) = (cm)a$.

We define the norm of a multiplier m to be $\|m\| := \|m^L\|$. Note that we could have equivalently declared that $\|m\| := \|m^R\|$ since for each a such that $\|a\| \leq 1$ we have

$$\begin{aligned}
\|m^L\| &= \sup \left\{ \|m^L(a)\| \;\mid\; \|a\| \leq 1 \right\} \\
&= \sup \left\{ \|c \circ m^L(a)\| \;\mid\; \|a\| \leq 1,\ \|c\| \leq 1 \right\} \\
&= \sup \left\{ \|m^R(c) \circ a\| \;\mid\; \|a\| \leq 1,\ \|c\| \leq 1 \right\} \\
&\leq \sup \left\{ \|m^R(c)\| \;\mid\; \|c\| \leq 1 \right\} \\
&= \|m^R\|.
\end{aligned}$$

The converse is proved similarly, and therefore $\|m^L\| = \|m^R\|$. If $l \in M\mathcal{C}(x,y)$ and $m \in M\mathcal{C}(y,z)$ are multipliers we define the composition

$$M\mathcal{C}(y,z) \times M\mathcal{C}(x,y) \longrightarrow M\mathcal{C}(x,z), \qquad (m,l) \longmapsto m \circ l$$

by

$$(m \circ l)^L(a) = \lim_i \left((me_i^y) \circ la \right) \qquad \text{and} \qquad (m \circ l)^R(c) = \lim_i \left(cm \circ (e_i^y l) \right),$$

where $(e_i^y)_{i\in I}$ is an approximate unit in $\mathcal{C}(y,y)$. If $m \in M\mathcal{C}(x,y)$ then we define $m^* \in M(y,x)$ by the formulae $m^*c := (c^*m)^*$ and $am^* := (ma^*)^*$.

We write $\mathbb{M}(\mathcal{C})$ for the unital category of all multipliers between the objects of $\mathcal{C}$. This has set of objects $\mathbb{M}(\mathcal{C})_0 = \mathcal{C}_0$ and for $x, y \in \mathcal{C}_0$ a hom-object

$$\mathbb{M}(\mathcal{C})(x,y) = M\mathcal{C}(x,y).$$

Composition and involution are defined as above, which (from the compatibility axiom for multipliers) makes composition necessarily associative. The identity morphism in each $\mathbb{M}(\mathcal{C})(x,x)$ is given by the identity multiplier $m_1 = (m_1^L, m_1^R)$ which is defined by the formulae $m_1^L(a) = a$ and $m_1^R(c) = c$. We claim that this is a C^*-category.

Proposition 3.1.19. *Let $\mathcal{C}$ be a C^*-category. Then $\mathbb{M}(\mathcal{C})$ is a C^*-category.*

Proof. For each $m = (m^L, m^R) \in \mathbb{M}(\mathcal{C})(x,y)$ we have $m^L \in \boldsymbol{\mathcal{L}}\big(\mathcal{C}(x,x), \mathcal{C}(x,y)\big)$ and $m^R \in \boldsymbol{\mathcal{L}}\big(\mathcal{C}(y,y), \mathcal{C}(x,y)\big)$, hence each $\mathbb{M}(\mathcal{C})(x,y)$ is a $\mathbb{C}$-vector space under pointwise operations. Now if $l \in \mathbb{M}(\mathcal{C})(x,y)$ and $m \in \mathbb{M}(\mathcal{C})(y,z)$ then

$$\begin{aligned}\|m\circ l\| = \left\|(m\circ l)^R\right\| &= \sup\left\{ \left\|(m\circ l)^R a\right\| \;\mid\; \|a\|\le 1 \right\}\\ &= \sup\left\{ \left\|\lim_i (me_i^y)\circ la\right\| \;\mid\; \|a\|\le 1 \right\}\\ &\le \left\|\lim_i (me_i^y)\right\| . \sup\left\{ la \;\mid\; \|a] \le 1\| \right\}\\ &\le \|m\| . \|l\|\end{aligned}$$

and therefore composition is submultiplicative. Thus, $\mathbb{M}(\mathcal{C})$ is a $\mathbb{C}$-linear category.

For completeness of hom-objects, let (m_n) be a Cauchy sequence in $\mathbb{M}(\mathcal{C})(x,y)$. Then (m_n^L) and (m_n^R) are Cauchy sequences in the Banach spaces $\boldsymbol{\mathcal{L}}\big(\mathcal{C}(x,x), \mathcal{C}(x,y)\big)$ and $\boldsymbol{\mathcal{L}}\big(\mathcal{C}(y,y), \mathcal{C}(x,y)\big)$ respectively, and hence converge to uniformly to limits m^L and m^R respectively. If $a \in \mathcal{C}(x,x)$ and $c \in \mathcal{C}(y,y)$ then

$$c\circ m^L(a) = \lim_{n\longrightarrow\infty} c\circ m_n^L(a) = \lim_{n\longrightarrow\infty} m_n^R(c)\circ a = m^R(c)\circ a.$$

Therefore $m = (m^L, m^R) = \lim_{n\longrightarrow\infty} m_n \in \mathbb{M}\mathcal{C}(x,y)$, hence $\mathbb{M}(\mathcal{C})(x,y)$ is complete. Thus, $\mathbb{M}(\mathcal{C})$ is a Banach *-category.

It remains to prove that the C^*-identity and positivity axioms hold. For every

$m \in \mathbb{M}(\mathcal{C})_1$ we have

$$\begin{aligned}\|m\|^2 &= \sup \{ \|ma\|^2 \mid \|a\| \leq 1 \} \\ &= \sup \{ (ma)^*(ma) \mid \|a\| \leq 1 \} \\ &= \sup \{ \|a^* m^* ma\| \mid \|a\| \leq 1 \} \\ &\leq \sup \{ \|c^* m^* ma\| \mid \|a\| \leq 1,\ \|c\| \leq 1 \} \\ &= \|m^* ma\| .\end{aligned}$$

It follows that $\|m\|^2 \leq \|m^* m\|$, and hence by submultiplicativity of composition the C*-identity is satisfied. Finally, let $a \in \mathcal{C}(x, x)$. Then $a^* a$ is a positive element of the C*-algebra $\mathcal{C}(x, x)$, and

$$\begin{aligned}(m^* m)^L(a^* a) &= m^*(\lim_i m e_i^y \circ a^* a) \\ &= \lim_i (m e_i^y \circ a^* a \circ (e_i^y)^* m^*) \\ &= \lim_i (m e_i^y \circ a)(a \circ (e_i^y)^* m^*) \\ &= \lim_i (a \circ (e_i^y)^* m^*)^* (a \circ (e_i^y)^* m^*),\end{aligned}$$

which is a positive element of $\mathcal{C}(x, x)$. A similar calculation shows that $(m^* m)^R(a^* a)$ is a positive element of $\mathcal{C}(x, x)$, and so $m^* m$ sends positive elements to positive elements. It follows that there exists $l \in \mathbb{M}(\mathcal{C})(x, x)$ such that $m^* m = l^* l$. □

Definition 3.1.20. Let $\mathcal{C}$ be a C*-category in **Set**. The *multiplier* C*-*category of* $\mathcal{C}$ is the unital category $\mathbb{M}(\mathcal{C})$.

Example 3.1.21. *Let $\mathcal{C}$ be the subcategory of* **Hilb** *such that for every $H, H' \in \mathcal{C}_0$ we have $\mathcal{C}(H, H') = \mathfrak{K}(H, H')$, the set of all compact operators from H to H'. Then $M\mathcal{C}(H, H') = \mathfrak{B}(H, H')$, and $\mathbb{M}(\mathcal{C}) =$* **Hilb**.

§3.2 Topological C*-Categories

Let $\mathcal{C}$ be a C*-category such that the collection of objects $\mathcal{C}_0$ is a locally compact, Hausdorff space.

Definition 3.2.1. A *vector field* for $\mathcal{C}$ is a function $\alpha \colon \mathcal{C}_0 \times \mathcal{C}_0 \longrightarrow \mathcal{C}_1$ such that $\alpha(x, y) \in \mathcal{C}(x, y)$ for every $x, y \in \mathcal{C}_0$. Call a vector field α *norm-continuous* (resp. norm- upper/lower semicontinuous) if the numerical function $(x, y) \longmapsto \|\alpha(x, y)\|$ is continuous (resp. upper/lower semicontinuous) with respect to the product topology on $\mathcal{C}_0 \times \mathcal{C}_0$ and the standard topology on $\mathbb{R}$.

We identify vector fields with sections of the *anchor map* $(s,t)\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_0 \times \mathcal{C}_0$. The *support* of a vector field α is the set

$$\operatorname{supp}(\alpha) = \overline{\{\, (x,y) \in \mathcal{C}_0 \times \mathcal{C}_0 \;\mid\; \alpha(x,y) \neq 0 \,\}}$$

which we equip with the subspace topology relative to $\mathcal{C}_0 \times \mathcal{C}_0$. We say that a vector field α is *compactly supported* if $\operatorname{supp}(\alpha)$ is a compact subset. Let $A_{\mathcal{C}}$ be a *-algebra of compactly supported vector fields with product given by convolution and involution by the formula $\alpha^*(x,y) := \alpha(y,x)^*$. Since α, β are compactly supported vector fields it follows that $\operatorname{supp}(\beta \star \alpha) \subseteq \operatorname{supp}(\beta)\operatorname{supp}(\alpha)$ is compact, and hence this product is well-defined on $A_{\mathcal{C}}$. We say that $A_{\mathcal{C}}$ is *dense in each fibre* if for every $x, y \in \mathcal{C}_0$ the set $\{\, \alpha(x,y) \;\mid\; \alpha \in A_{\mathcal{C}} \,\}$ is dense in the hom-object $\mathcal{C}(x,y)$.

Definition 3.2.2. An *upper semicontinuous (resp. lower semicontinuous) topological* C^**-category* is a pair $(\mathcal{C}, A_{\mathcal{C}})$ where $\mathcal{C}$ is a C^*-category with $\mathcal{C}_0$ a locally compact, Hausdorff space and $A_{\mathcal{C}}$ is a *-algebra of norm-upper semicontinuous (resp. norm-lower semicontinuous) compactly supported vector fields for $\mathcal{C}$ that is dense in each fibre, such that the collection of morphisms $\mathcal{C}_1$ is equipped with the topology generated by sets of the form

$$\Omega(U,\alpha,\epsilon) := \big\{\, b \in \mathcal{C}_1 \;\mid\; (s,t)(b) \in U,\; \big\| b - \alpha\big((s,t)(b)\big) \big\| < \epsilon \,\big\}, \tag{3.1}$$

where $U \subseteq \mathcal{C}_0 \times \mathcal{C}_0$ is an open subset, $\epsilon > 0$ and $\alpha \in A_{\mathcal{C}}$. A *continuous topological* C^**-category* is a pair $(\mathcal{C}, A_{\mathcal{C}})$ that is both an upper semicontinuous topological C^*-category and a lower semicontinuous topological C^*-category.

Where no confusion is likely to occur we will often omit reference to $A_{\mathcal{C}}$ and identify a topological C^*-category $(\mathcal{C}, A_{\mathcal{C}})$ by its underlying C^*-category $\mathcal{C}$.

Let $\mathcal{C}$ be an upper semicontinuous topological C^*-category. It follows from Proposition 2.1.4 that the collection of sets of the form $\Omega(U,\alpha,\epsilon)$ as given in (3.1) forms a basis for the topology on $\mathcal{C}_1$, and this topology is the unique topology for $\mathcal{C}_1$ making $A_{\mathcal{C}}$ the set of compactly supported *continuous* sections of the anchor map $(s,t)\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_0 \times \mathcal{C}_0$. Moreover, we claim that Definition 3.2.2 is consistent with the notion of a topological category as described in Chapter 1. An essentially equivalent result for Fell bundles over étale groupoids was proven independently by Takeishi [55]. We first consider the case of non-unital topological C^*-categories:

Proposition 3.2.3. *Let $\mathcal{C}$ be a non-unital upper semicontinuous topological* C^**-category. Then $\mathcal{C}$ is an internal category in* **Top**.

Proof. The *-algebra $A_{\mathcal{C}}$ satisfies the hypotheses of Proposition 2.1.2 and therefore the topology on $\mathcal{C}_1$ generated by the sets $\Omega(U, \alpha, \epsilon)$ as given in (3.1) makes the triple $(\mathcal{C}_1, (s,t), \mathcal{C}_0 \times \mathcal{C}_0)$ an upper semicontinuous $\mathbf{Ban}$-bundle. In particular, the anchor map (s,t) is a continuous open surjection with respect to this topology, and hence the structure maps s and t are continuous. Furthermore, addition on each hom-object defines a continuous map

$$+\colon \mathcal{C}_1 \times_{\mathcal{C}_0 \times \mathcal{C}_0} \mathcal{C}_1 \longrightarrow \mathcal{C}_1$$

and scalar multiplication on each hom-object defines a continuous map

$$\cdot\colon \mathbb{C} \times \mathcal{C}_1 \longrightarrow \mathcal{C}_1.$$

For composition in $\mathcal{C}$, let $\{a_i\}$ and $\{b_i\}$ be nets in $\mathcal{C}_1$ such that for each index i we have $a_i \in \mathcal{C}(x_i, y_i)$ and $b_i \in \mathcal{C}(y_i, z_i)$ and such that $a_i \longrightarrow a$ and $b_i \longrightarrow b$ where $a \in \mathcal{C}(x,y)$ and $b \in \mathcal{C}(y,z)$. We prove that $b_i a_i \longrightarrow ba$. If either of a, b is the zero morphism then $\|b_i a_i\| \leq \|b_i\| . \|a_i\| \longrightarrow 0$ and hence $b_i a_i$ converges to $0 = ba$. Now assume that neither a nor b are the zero morphism and let $\epsilon > 0$. Since $A_{\mathcal{C}}$ is dense in each fibre there exist $\alpha, \beta \in A_{\mathcal{C}}$ such that

$$\|a - \alpha(x,y)\| < \frac{\epsilon}{2\,\|\beta(y,z)\|} \quad \text{and} \quad \|b - \beta(y,z)\| < \frac{\epsilon}{2\,\|a\|}.$$

By Proposition 2.1.4, the vector fields α, β are continuous, and hence for sufficiently large i we have

$$\|a_i - \alpha(x_i,y_i)\| < \frac{\epsilon}{2\,\|\beta(y_i,z_i)\|} \quad \text{and} \quad \|b_i - \beta(y_i,z_i)\| < \frac{\epsilon}{2\,\|a_i\|}.$$

Let $U_1 \subseteq \mathcal{C}_0 \times \mathcal{C}_0$ be an open neighbourhood of (x,y) and $U_2 \subseteq \mathcal{C}_0 \times \mathcal{C}_0$ be an open neighbourhood of (y,z). Let $K_i \subseteq U_i$, $i = 1, 2$ be compact subsets, and define $U := U_2 U_1 \subseteq \mathcal{C}_0 \times \mathcal{C}_0$ and $K := K_2 K_1 \subseteq U$. Let $f \in C_0(\mathcal{C}_0 \times \mathcal{C}_0)$ be such that $f \equiv 1$ on K and $f \equiv 0$ outside of U. Since $A_{\mathcal{C}}$ is closed under the pointwise action of $C_0(\mathcal{C}_0 \times \mathcal{C}_0)$ it follows that $\eta := f \cdot (\beta * \alpha)$ is a continuous vector field for $\mathcal{C}$. Therefore, for sufficiently large i,

$$\eta\big((y,z) \cdot (x,y)\big) = \beta(y,z)\alpha(x,y)$$

and

$$\eta\big((y_i,z_i).(x_i,y_i)\big) = \beta(y_i,z_i)\alpha(x_i,y_i),$$

and hence on K we have

$$\begin{aligned}\big\|ba - \eta\big((y,z).(x,y)\big)\big\| &\leq \|ba - \beta(y,z)\alpha(x,y)\| \\ &\leq \|b - \beta(y,z)\| \cdot \|a\| + \|\beta(y,z)\| \cdot \|a - \alpha(x,y)\| \\ &\leq \frac{\epsilon}{2\,\|a\|} \cdot \|a\| + \|\beta(y,z)\| \cdot \frac{\epsilon}{2\,\|\beta(y,z)\|} \\ &= \epsilon.\end{aligned}$$

Similarly, for sufficiently large i,

$$\begin{aligned}\big\|b_i a_i - \eta\big((y_i,z_i).(x_i,y_i)\big)\big\| &\leq \|b_i a_i - \beta(y_i,z_i)\alpha(x_i,y_i)\| \\ &\leq \|b_i - \beta(y_i,z_i)\| \,.\, \|a_i\| + \|\beta(y_i,z_i)\| \,.\, \|a_i - \alpha(x_i,y_i)\| \\ &\leq \frac{\epsilon}{2\,\|a_i\|} . \|a_i\| + \|\beta(y_i,z_i)\| \,. \frac{\epsilon}{2\,\|\beta(y_i,z_i)\|} \\ &= \epsilon\end{aligned}$$

on K. Therefore $b_i a_i \longrightarrow ba$ by Lemma 2.1.4 and hence composition for $\mathcal{C}$ defines a continuous map

$$m \colon \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \longrightarrow \mathcal{C}_1.$$

Finally, for involution, let $\{b_i\}$ be a net in $\mathcal{C}_1$ such that for each index i we have $b_i \in \mathcal{C}(x_i, y_i)$, and such that $b_i \longrightarrow b$ where $b \in \mathcal{C}(x,y)$. Since $A_{\mathcal{C}}$ is dense in each fibre it follows that for each $\epsilon > 0$ there exists a vector field $\alpha \in A_{\mathcal{C}}$ such that for sufficiently large i

$$\|b - \alpha(x,y)\| < \epsilon \quad \text{and} \quad \|b_i - \alpha(x_i,y_i)\| < \epsilon.$$

Involution is isometric, and therefore

$$\begin{aligned}\|b^* - \alpha^*(y,x)\| &= \|b^* - \alpha(x,y)^*\| \\ &= \big\|\big(b - \alpha(x,y)\big)^*\big\| \\ &= \|b - \alpha(x,y)\| \\ &\leq \epsilon.\end{aligned}$$

Similarly, for sufficiently large i,

$$\begin{aligned}\|b_i^* - \alpha^*(y_i, x_i)\| &= \|b_i^* - \alpha(x_i, y_i)^*\| \\ &= \big\|\big(b_i - \alpha(x_i, y_i)\big)^*\big\| \\ &= \|b_i - \alpha(x_i, y_i)\| \\ &\leq \epsilon.\end{aligned}$$

Therefore $b_i^* \longrightarrow b^*$ by Lemma 2.1.4, and hence $*\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_1$ is continuous. ☐

To extend the preceding result to the unital case we first need the following closure property of the space of continuous vector fields:

Lemma 3.2.4. *Let $\mathcal{C}$ be an upper semicontinuous topological C^*-category and let η be a norm-upper semicontinuous vector field. Suppose that for every $x, y \in \mathcal{C}_0$ and each $\epsilon > 0$ there exists a compactly supported continuous vector field $\alpha \in A_{\mathcal{C}}$ and a neighbourhood U of (x, y) such that $\|\beta(u, v) - \eta(u, v)\| < \epsilon$ for all $(u, v) \in U$. Then the vector field η is continuous.*

Proof. Let η be as stated in the hypothesis and let $\{(x_i, y_i)\}$ be a net in $\mathcal{C}_0 \times \mathcal{C}_0$ converging to objects (x, y). We show that $\eta(x_i, y_i) \longrightarrow \eta(x, y)$. Let $\alpha \in A_{\mathcal{C}}$. Then since α is continuous $\{\alpha(x_i, y_i)\}$ is a net of morphisms converging to $\alpha(x, y)$ in $\mathcal{C}_1$, and for each i we have

$$(s, t)(\eta(x_i, y_i)) = (s, t)(\alpha(x_i, y_i)) = (x_i, y_i).$$

By assumption, for each $\epsilon > 0$ there exists a neighbourhood U such that

$$\|\alpha(u, v) - \eta(u, v)\| < \epsilon$$

on U, and hence for large enough i we have $(x_i, y_i) \in U$. Therefore,

$$\|\alpha(x_i, y_i) - \eta(x_i, y_i)\| < \epsilon \quad \text{and} \quad \|\alpha(x, y) - \eta(x, y)\| < \epsilon,$$

and hence by Lemma 2.1.4 we have $\eta(x_i, y_i) \longrightarrow \eta(x, y)$ and thus η is continuous. ☐

Proposition 3.2.5. *Let $\mathcal{C}$ be a unital upper semicontinuous topological C^*-category. Then $\mathcal{C}$ is an internal category in* **Top**.

Proof. We prove that the identity map $\imath\colon \mathcal{C}_0 \longrightarrow \mathcal{C}_1$ is continuous. Let η be a vector field for $\mathcal{C}$ such that

$$\eta(x,y) = \begin{cases} 1_x & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

We claim that η is continuous at every point (x,x). Denote by D the diagonal space of $\mathcal{C}_0$,

$$D = \{ (x,x) \mid x \in \mathcal{C}_0 \}$$

and let $U \subset D$ be an open neighbourhood of (x,x), and $\alpha \in A_{\mathcal{C}}$ a continuous vector field for $\mathcal{C}$. For each $\epsilon > 0$ consider the set

$$W_\epsilon := (s,t)\big(\Omega(U,\alpha,\epsilon)\big).$$

By Proposition 2.1.2, the map (s,t) is open, whence it follows that W_ϵ is an open neighbourhood of (x,x), and on W_ϵ we have $\|\eta(w,w) - \alpha(w,w)\| < \epsilon$. Hence by Lemma 3.2.4, η is continuous at each (x,x). The diagonal defines a continuous map $D\colon \mathcal{C}_0 \longrightarrow D(\mathcal{C}_0)$ and therefore the unit map $\imath = \eta \circ D$ is continuous. □

Now let $\mathcal{C}$ be a unital topological C*-category and let $\mathbf{End}_{\mathcal{C}}$ denote the subcategory of $\mathcal{C}$ with the same space of objects as $\mathcal{C}$ and morphisms given by

$$\mathbf{End}_{\mathcal{C}}(x,y) = \begin{cases} \mathcal{C}(x,x) & \text{if } x = y \\ \{0\} & \text{otherwise.} \end{cases}$$

From $\mathbf{End}_{\mathcal{C}}$ we obtain a $\mathbf{C^*}$-$\mathbf{Alg}$-bundle $(\mathsf{E},(s,t),D)$ where D is the diagonal space of $\mathcal{C}_0$ and

$$\mathsf{E} := \coprod_{x \in \mathcal{C}_0} \mathcal{C}(x,x).$$

Lemma 3.2.6. *Let $\mathcal{C}$ be a topological* C*-*category. Then $\mathcal{C}$ is a continuous (resp. upper/lower semicontinuous) topological* C*-*category if and only if the triple $(\mathsf{E},(s,t),D)$ is continuous (resp. upper/lower semicontinuous) $\mathbf{C^*}$-$\mathbf{Alg}$-bundle.*

Proof. Let $b \in \mathcal{C}_1$ be a morphism in $\mathcal{C}(x,y)$. The norm function $b \longmapsto \|b\|$ factorises into maps

$$b \longmapsto (b^*, b) \longmapsto b^*b \longmapsto \|b^*b\|^{1/2} = \|b\|\,. \tag{3.2}$$

By Definition 3.1.2, there exists an endomorphism $a \in \mathcal{C}(x,x)$ such that $a^*a = b^*b$, and therefore $b \longmapsto \|b\|$ is continuous (resp. upper/lower semicontinuous) if and only if $a \longmapsto \|a\|$ is continuous (resp. upper/lower semicontinuous) on $(\mathsf{E},(s,t),D)$. □

Our development of topological C*-categories thus far only considers them as isolated objects. In order to develop a meaningful theory of topological C*-categories we need to consider them as objects of some category, and investigate the structure-preserving morphisms between them.

Definition 3.2.7. A *continuous *-functor* between topological C*-categories $\mathcal{C}$ and $\mathcal{D}$ is a *-functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ on the underlying categories such that the object component $F_0\colon \mathcal{C}_0 \longrightarrow \mathcal{D}_0$ and the morphism component $F_1\colon \mathcal{C}_1 \longrightarrow \mathcal{D}_1$ are both continuous maps. If $\mathcal{C}$ and $\mathcal{D}$ are unital topological C*-categories then F must also satisfy $\imath_{\mathcal{D}} \circ F_1 = F_0 \circ \imath_{\mathcal{C}}$. We write $[\mathcal{C}, \mathcal{D}]$ for the collection of all continuous *-functors between topological C*-categories.

The following result provides a criterion for a *-functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ between upper semicontinuous topological C*-categories to be continuous:

Proposition 3.2.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be upper semicontinuous topological* C**-categories and let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a (not necessarily continuous) *-functor such that the object map $F_0\colon \mathcal{C}_0 \longrightarrow \mathcal{D}_0$ is continuous. If the map $F_1 \circ \alpha\colon \mathcal{C}_0 \times \mathcal{C}_0 \longrightarrow \mathcal{D}_1$ is continuous for every $\alpha \in A_{\mathcal{C}}$ then F is a continuous *-functor.*

Proof. Let $\{b_i\}$ be a net in $\mathcal{C}_1$ such that for each index i we have $b_i \in \mathcal{C}(x_i, y_i)$ and such that $b_i \longrightarrow b$ for some $b \in \mathcal{C}(x, y)$. Let $\epsilon > 0$ and choose $\alpha \in A_{\mathcal{C}}$ such that $\|\alpha(x, y) - b\| \leq \epsilon$. By continuity it follows that for sufficiently large i we have $\|\alpha(x_i, y_i) - b_i\| \leq \epsilon$. Since F is norm-decreasing we have

$$\|(F \circ \alpha)(x, y) - F(b)\| = \|F(\alpha(x, y) - b)\| \leq \|\alpha(x, y) - b\| \leq \epsilon$$

and for sufficiently large i we also have

$$\|(F \circ \alpha)(x_i, y_i) - F(b_i)\| = \|F(\alpha(x_i, y_i) - b_i)\| \leq \|\alpha(x_i, y_i) - b_i\| \leq \epsilon.$$

By assumption the composite $F_1 \circ \alpha$ is a continuous map $\mathcal{C}_0 \times \mathcal{C}_0 \longrightarrow \mathcal{D}_1$ and therefore $(F \circ \alpha)(x_i, y_i) \longrightarrow (F \circ \alpha)(x, y)$ in $\mathcal{D}_1$. Then $F(b_i) \longrightarrow F(b)$ by Lemma 2.1.4. □

If $\mathcal{C}$ is a unital topological C*-category we write $\mathcal{U}(\mathcal{C}_1)$ for the subspace of $\mathcal{C}_1$ consisting of unitary morphisms of $\mathcal{C}$. Let $\mathcal{C}$ and $\mathcal{D}$ be topological C*-categories and $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$ continuous *-functors.

Definition 3.2.9. A *continuous natural transformation* $F \Longrightarrow G$ consists of a continuous map $\theta\colon \mathcal{C}_0 \longrightarrow \mathcal{D}_1$ such that $s_{\mathcal{D}} \circ \theta = F_0$ and $t_{\mathcal{D}} \circ \theta = G_0$ and such that the following diagram commutes in $\mathbf{Top}$:

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{(G_1, \theta\circ s_{\mathcal{C}})} & \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 \\ {\scriptstyle (\theta\circ t_{\mathcal{C}}, F_1)}\downarrow & & \downarrow{\scriptstyle m} \\ \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1 & \xrightarrow[m]{} & \mathcal{D}_1 \end{array} \tag{3.3}$$

A continuous natural transformation is *bounded* if $\|\theta\| := \sup\{\, \|\theta(x)\| \mid x \in \mathcal{C}_0 \,\}$ is finite. It is called a *continuous unitary transformation* if $\theta(x) \in \mathcal{U}(\mathcal{C}_1)$ for all $x \in \mathcal{C}_0$.

If $b \in \mathcal{C}(x, y)$ then the commutativity of the diagram in (3.3) implies that

$$\Big[G(b) \circ \theta(x) \colon F(x) \longrightarrow G(x) \longrightarrow G(y)\Big] = \Big[\theta(y) \circ F(b) \colon F(x) \longrightarrow F(y) \longrightarrow G(y)\Big]$$

and therefore we recover a the usual commutative diagram of a natural transformation:

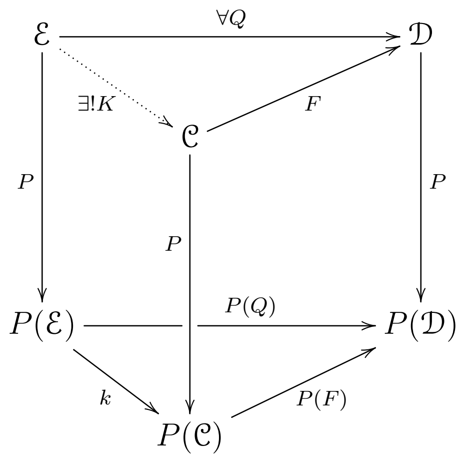
$$\begin{array}{ccccc} x & \qquad & F(x) & \xrightarrow{\theta_x} & G(x) \\ {\scriptstyle b}\downarrow & & {\scriptstyle F(b)}\downarrow & & \downarrow{\scriptstyle G(b)} \\ y & & F(y) & \xrightarrow[\theta_y]{} & G(y) \end{array}$$

The collection of all topological C*-categories forms a 2-category C*-$\mathbf{Cat}(\mathbf{Top})$ with continuous *-functors as 1-morphisms and continuous bounded natural transformations as 2-morphisms. We write C*-$\mathbf{Cat}_{\mathcal{U}}(\mathbf{Top})$ for the 2-category with the same objects and 1-morphisms but with continuous unitary transformations as 2-morphisms.

In the remainder of this section, we prove that the category C*-$\mathbf{Cat_1}(\mathbf{Top})$ of unital topological C*-categories and continuous *-functors is fibred over the category of topological spaces. First we must translate the concept of a cartesian morphism into the language of continuous *-functors between topological C*-categories.

Definition 3.2.10. Let $P\colon$ C*-$\mathbf{Cat_1}(\mathbf{Top}) \longrightarrow \mathbf{Top}$ be a functor. A continuous unital *-functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ between topological C*-categories is called *strongly cartesian* if for every topological C*-category $\mathcal{E} \in$ C*-$\mathbf{Cat_1}(\mathbf{Top})$, for every continuous unital *-functor $Q\colon \mathcal{E} \longrightarrow \mathcal{D}$ and for every continuous map $k\colon P(\mathcal{E}) \longrightarrow P(\mathcal{C})$ such that $P(Q) = P(F) \circ k$ there exists a unique continuous unital *-functor $K\colon \mathcal{E} \longrightarrow \mathcal{C}$

such that $Q = F \circ K$ and $k = P(K)$:



Write $C(\ ,\ )$ for a hom-object in $\mathbf{Top}$. Then we may equivalently characterise a continuous unital *-functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ as strongly cartesian if for every unital topological C*-category $\mathcal{E} \in \mathrm{C}^*\text{-}\mathbf{Cat_1}(\mathbf{Top})$ the map

$$\mathrm{C}^*\text{-}\mathbf{Cat_1}(\mathbf{Top})[\mathcal{E}, \mathcal{C}] \longrightarrow \mathrm{C}^*\text{-}\mathbf{Cat_1}(\mathbf{Top})[\mathcal{E}, \mathcal{D}] \times_{(P(\mathcal{E}), P(\mathcal{D}))} C(P(\mathcal{E}), P(\mathcal{C}))$$

given by $K \longmapsto (F \circ K, P(F))$ is a bijection.

Now let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a continuous *-functor between topological C*-categories. Define a category $F^*\mathcal{D}$ with the same space of objects as $\mathcal{C}$ and with the space of morphisms

$$F^*\mathcal{D}_1 := \{\, ((x, y), b) \in \mathcal{C}_0 \times \mathcal{C}_0 \times \mathcal{D}_1 \;\mid\; s(b) = F(x),\ t(b) = F(y) \,\},$$

which we equip with the subspace topology relative to $\mathcal{C}_0 \times \mathcal{C}_0 \times \mathcal{D}_1$. The hom-objects of $F^*\mathcal{D}$ are defined as $F^*\mathcal{D}(x, y) := \mathcal{D}(F(x), F(y))$ from which it follows immediately that $F^*\mathcal{D}$ is a topological C*-category. The category $F^*\mathcal{D}$ is unital, with unit map

$$\imath_{(F^*\mathcal{D})}\colon \mathcal{C}_0 \longrightarrow F^*\mathcal{D}_1, \qquad x \longmapsto \big((x, x), \mathrm{id}_{F(x)}\big)$$

if and only if the topological C*-category $\mathcal{D}$ is unital. Furthermore, it is a continuous (resp. upper/lower semicontinuous) topological C*-category if and only if $\mathcal{D}$ is. There is a canonical continuous *-functor $F^*\colon F^*\mathcal{D} \longrightarrow \mathcal{D}$ with object component $F^*_0\colon F^*\mathcal{D}_0 \longrightarrow \mathcal{D}_0$ given by F_0 and morphism component $F^*_1\colon F^*\mathcal{D}_1 \longrightarrow \mathcal{D}_1$ given by projection onto the final factor.

Now if $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a fully faithful continuous *-functor between unital topological C*-categories and $F^*\colon F^*\mathcal{D} \longrightarrow \mathcal{D}$ is the canonical projection functor then there exists a unique (up to natural isomorphism) continuous *-functor $K\colon \mathcal{C} \longrightarrow F^*\mathcal{D}$ such that $F = F^* \circ K$, implementing an isomorphism $\mathcal{C} \cong F^*\mathcal{D}$. Let

$$\mathrm{Obj}\colon \mathrm{C}^*\text{-}\mathbf{Cat_1}(\mathbf{Top}) \longrightarrow \mathbf{Top}$$

be the functor that sends a unital topological C*-category to its space of objects $\mathcal{C}_0$ and sends a continuous *-functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ to the object component of F, $F_0\colon \mathcal{C}_0 \longrightarrow \mathcal{D}_0$.

Lemma 3.2.11. *Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a fully faithful continuous *-functor between unital topological* C**-categories. Then F is strongly cartesian with respect to* Obj.

Proof. Let $\mathcal{E}$ be a topological C*-category and $k_0\colon \mathcal{E}_0 \longrightarrow \mathcal{C}_0$ a continuous map. Then we have a continuous *-functor $k_0^*\colon k_0^*\mathcal{C} \longrightarrow \mathcal{C}$ with object component k_0 and hence a continuous *-functor

$$F \circ k_0^* \,:\, k_0^*\mathcal{C} \longrightarrow \mathcal{D}$$

with object component $k_0 \circ F_0$. Since F is fully-faithful it follows that $k_0^* \circ F$ is also fully-faithful, and therefore there exists a unique (up to natural isomorphism) continuous *-functor

$$K'\colon \mathcal{E} \longrightarrow k_0^*\mathcal{C}$$

such that $F \circ k_0^* \circ K' = Q$. Define a continuous *-functor K by $k_0^* \circ K'$, then $\mathrm{Obj}(K) = k_0$ and $F \circ K = Q$, thus proving existence.

Now suppose that $K, H\colon \mathcal{E} \longrightarrow \mathcal{C}$ are two continuous unital *-functors such that $K_0 = H_0 = k_0$ and such that $F \circ K = Q = F \circ H$. Then for every $x \in \mathcal{E}$ we have $K(x) = H(x)$, and therefore the map $x \longmapsto \mathrm{id}_{K(x)}$ defines a continuous natural isomorphism $K \Longrightarrow H$. □

We conclude with the main result of this section.

Definition 3.2.12. A functor $P\colon \mathcal{C} \longrightarrow \mathcal{S}$ is called a *fibration* if given any $x \in \mathcal{C}_0$ and $(f\colon a \longrightarrow P(x)) \in \mathcal{S}_1$ there exists a strongly cartesian morphism $\phi\colon x \longrightarrow y$ such that $P(\phi) = f$. The morphism ϕ is called a *lifting of f* and we say that $\mathcal{C}$ is *fibred over* $\mathcal{S}$.

Proposition 3.2.13. *The category* $\mathrm{C}^*\text{-}\mathbf{Cat_1}(\mathbf{Top})$ *of topological* C**-categories is fibred over* $\mathbf{Top}$.

Proof. We prove that functor the $\mathrm{Obj}\colon \mathrm{C}^*\text{-}\mathbf{Cat_1}(\mathbf{Top}) \longrightarrow \mathbf{Top}$ is a fibration. Let $\mathcal{C}$ be a topological C^*-category, and let $f\colon X \longrightarrow \mathcal{C}_0$ be a continuous map from X, a locally compact Hausdorff space. Then there exists a continuous *-functor

$$f^*\colon f^*\mathcal{C} \longrightarrow \mathcal{C}$$

such that $\mathrm{Obj}(f^*) = f$. By construction, f^* is fully-faithful, and hence by Lemma 3.2.11 f^* is cartesian. Therefore, Obj is a fibration. □

Chapter 4

Concrete Topological C^*-Categories

In Chapter 3 we introduced topological C^*-categories by defining them axiomatically. In this chapter we construct a concrete characterisation of a topological C^*-category as a category with a locally compact Hausdorff space of objects and whose hom-objects are spaces of bounded linear maps between Hilbert spaces. This is precisely the topological C^*-category analogue of a concrete C^*-algebra. Crucially, however, the collection of all Hilbert spaces (even all separable Hilbert spaces) forms a proper class as opposed to a set, and so a characterisation of **Hilb** as an internal category in **Top** is not possible. Our approach is therefore to construct (small) topological categories that are in some sense equivalent to **Hilb**. These are C^*-categories associated to continuous **Hilb**-bundles.

The content of this chapter falls naturally into two sections. In the first section we provide a construction of a C^*-category from a continuous **Hilb**-bundle by considering the topological multiplier C^*-category of the category of compact operators (Definition 4.1.5). We consider two topologies on the set of morphisms of such categories. For the *strict topology* we regard the construction simply as a category of multipliers, whereas for the **-strong operator topology* we acknowledge that the hom-objects are in fact spaces of operators. These are both analogous to the corresponding topologies on operator algebras. The C^*-category obtained by equipping the morphism set with the *-strong operator topology is essentially the same as the lower semicontinuous Fell bundle constructed independently by Bos [9], although we approach the construction of the topology in a different (but equivalent) way.

In Section 4.2 we are concerned with proving a generalisation of the Gelfand-Naimark Theorem for topological C^*-categories. This section contains two of the main results in this thesis: Firstly, we assert that every continuous topological C^*-

category can be embedded into a functor category in a manner analogous to the Yoneda embedding (Theorem 4.2.5). We then use this result and reason as in [43] to prove that every topological C^*-category can be faithfully and continuously represented as a concrete topological C^*-category associated to some continuous **Hilb**-bundle over its object space. Furthermore, we show that if $\mathcal{C}$ is a topological C^*-category with faithful representation then $\mathcal{C}$ is necessarily continuous (Theorem 4.2.8).

§4.1 Concrete Topological C^*-Categories

Throughout this section let X be a locally compact, Hausdorff space and let (H, p, X) be a continuous **Hilb**-bundle. We start by constructing the analogue of the algebra of compact operators in the context of topological C^*-categories.

Definition 4.1.1. Define a category $\mathcal{K}(\mathsf{H})$ with space of objects X and for each $x, y \in X$ a hom-object $\mathcal{K}(\mathsf{H})(x,y) := \mathcal{K}(H_x, H_y)$, where H_x and H_y are fibres of H. Composition in $\mathcal{K}(\mathsf{H})$ is defined as composition of operators and involution is given by the usual adjoint of linear operators.

For each pair of continuous bounded sections $\alpha, \beta \in \Gamma_b(\mathsf{H})$ define a vector field $\theta_{\alpha,\beta}$ for $\mathcal{K}(\mathsf{H})$ by the formula

$$\Theta_{\alpha,\beta}(x,y)(-) := \langle\, \beta(x), - \,\rangle \alpha(y) \qquad \in \mathcal{K}(H_x, H_y).$$

Denote by $A_{\mathcal{K}(\mathsf{H})}$ the set $\{\, \sum_i \theta_{\alpha_i,\beta_i} \;\mid\; \alpha_i, \beta_i \in \Gamma_b(\mathsf{H}) \,\}$ of all finite sums of such $\theta_{\alpha,\beta}$.

Proposition 4.1.2. *The pair $\big(\mathcal{K}(\mathsf{H}), A_{\mathcal{K}(\mathsf{H})}\big)$ form a continuous topological C^*-category.*

Proof. The set $\mathcal{K}(H_x, H_y)$ is defined as the closure of the set of all finite rank operators $H_x \longrightarrow H_y$. Therefore each $\mathcal{K}(\mathsf{H})(H_x, H_y)$ is a closed linear subspace of $\mathbf{Hilb}(H_x, H_y)$ and hence the underlying category $\mathcal{K}(\mathsf{H})$ is a C^*-category. Moreover, $A_{\mathcal{K}(\mathsf{H})}$ is dense in each fibre and for each $\alpha, \beta \in \Gamma_b(\mathsf{H})$ the map

$$(x,y) \longmapsto \|\langle\, \beta(x), h \,\rangle \alpha(y)\|$$

is continuous. It follows that $\big(\mathcal{K}(\mathsf{H}), A_{\mathcal{K}(\mathsf{H})}\big)$ forms a continuous topological C^*-category. □

We now extend this construction to a category with hom-objects consisting of *all* bounded linear maps between fibres of a continuous **Hilb**-bundle (H, p, X).

Recall that if H, H' are Hilbert spaces then $\mathcal{B}(H, H') = M\mathcal{K}(H, H')$.

Definition 4.1.3. Let (H, p, X) be a continuous **Hilb**-bundle. Define the category $\mathcal{B}(\mathsf{H})$ to be the multiplier C*-category $\mathbb{M}(\mathcal{K}(\mathsf{H}))$, which has space of objects X, and for every $x, y \in X$ a hom-object $\mathcal{B}(\mathsf{H})(H_x, H_y) := M(\mathcal{K}(\mathsf{H}))$ consisting of all bounded linear maps from H_x to H_y.

We consider two distinct topologies on the set of morphisms $\mathcal{B}(\mathsf{H})_1$. The first is a generalisation of a natural topology on multipliers.

Definition 4.1.4. Let $\mathcal{C}$ be a topological C*-category and let $\mathbb{M}(\mathcal{C})$ be the multiplier C*-category of $\mathcal{C}$. The *strict topology* on $\mathbb{M}(\mathcal{C})_1$ is the topology generated by the family of seminorms $\{\, \|-\|_\alpha^s \mid \alpha \in A_{\mathcal{C}} \,\}$ where

$$\|m\|_\alpha^s := \left\|m\alpha\big(s(m), s(m)\big)\right\| + \left\|\alpha\big(t(m), t(m)\big)m\right\|.$$

It follows that a net of morphisms $\{m_i\}$ in $\mathbb{M}(\mathcal{C})$, where $m_i \in \mathbb{M}(\mathcal{C})(x_i, y_i)$ for each index i, converges strictly to $m \in \mathbb{M}(\mathcal{C})(x, y)$ if and only if

$$m_i\alpha\,(x_i, x_i) \longrightarrow m\alpha\,(x, x) \quad \text{and} \quad \alpha\,(y_i, y_i)\, m_i \longrightarrow \alpha\,(y, y)\, m$$

for all $\alpha \in A_{\mathcal{C}}$. In particular, a net $\{T_i\}$ of operators such that $T_i \in \mathcal{B}(H_{x_i}, H_{y_i})$ converges strictly to $T \in \mathcal{B}(H_x, H_y)$ if and only if

$$T_i\theta_{\alpha,\beta}\,(x_i, y_i) \longrightarrow T\theta_{\alpha,\beta}(x, y) \quad \text{and} \quad \theta_{\alpha,\beta}(x_i, y_i)T_i \longrightarrow \theta_{\alpha,\beta}(x, y)T$$

in $\mathcal{B}(\mathsf{H})_1$ for all $\alpha, \beta \in \Gamma_b(\mathsf{H})$.

A vector field μ for $\mathbb{M}(\mathcal{C})$ is called *strictly continuous* if the maps

$$(x, y) \longmapsto \mu(x, y)\alpha(x, x) \quad \text{and} \quad (x, y) \longmapsto \alpha(y, y)\mu(x, y)$$

are continuous on $X \times X \longrightarrow \mathcal{C}_1$ for every $\alpha \in A_{\mathcal{C}}$. Write $A^s_{\mathbb{M}(\mathcal{C})}$ for the set of strictly continuous vector fields for $\mathbb{M}(\mathcal{C})$. Then a sub-base for the strict topology on $\mathbb{M}(\mathcal{C})_1$ is given by the collection of sets of the form

$$\Omega(U, \mu, \alpha, \epsilon) = \left\{\, m \in \mathbb{M}(\mathcal{C}) \;\middle|\; (s,t)(m) \in U,\ \left\|(s,t)(m) - \mu\big((s,t)(m)\big)\right\|_\alpha^s < \epsilon \,\right\},$$

for $U \subseteq \mathcal{C}_0 \times \mathcal{C}_0$ an open subset, $\alpha \in A_{\mathcal{C}}$, $\mu \in A^s_{\mathbb{M}(\mathcal{C})}$ and $\epsilon > 0$.

Definition 4.1.5. Let $\mathcal{C}$ be a topological C*-category. Then $\left(\mathbb{M}(\mathcal{C}), A^s_{\mathbb{M}(\mathcal{C})}\right)$ is called the *topological multiplier* C*-*category* of $\mathcal{C}$.

Example 4.1.6. *The category* $\left(\mathfrak{B}(\mathsf{H}), A^s_{\mathbb{M}(\mathfrak{K}(\mathsf{H}))}\right)$ *is the topological multiplier* C^*-*category of* $(\mathfrak{K}(\mathsf{H}), A_{\mathfrak{K}})$.

Since the morphisms of $\mathfrak{B}(\mathsf{H})$ are linear maps we can also equip $\mathfrak{B}(\mathsf{H})_1$ with an operator topology. For each $\alpha \in \Gamma_b(\mathsf{H})$ define a seminorm

$$\|T\|^{\omega}_{\alpha} := \left\|T\big(\alpha\left(s(T)\right)\big)\right\| + \left\|T^*\big(\alpha\left(t(T)\right)\big)\right\|,$$

where the norms $\left\|T\big(\alpha\left(s(T)\right)\big)\right\|$ and $\left\|T^*\big(\alpha\left(t(T)\right)\big)\right\|$ are the inner-product norms on $H_{s(T)}$ and $H_{t(T)}$ respectively.

Definition 4.1.7. The $*$-*strong operator topology* on $\mathfrak{B}(\mathsf{H})_1$ is the topology generated by the family of seminorms $\{\, \|-\|^{\omega}_{\alpha} \mid \alpha \in \Gamma_b(\mathsf{H}) \,\}$. A net of morphisms $\{T_i\}$ in $\mathfrak{B}(\mathsf{H})$ such that $T_i \in \mathfrak{B}(H_{x_i}, H_{y_i})$ converges to $T \in \mathfrak{B}(\mathsf{H})$ if and only if

$$T_i\left(\alpha(x_i)\right) \longrightarrow T\left(\alpha(x)\right) \quad \text{and} \quad T_i^*\left(\alpha(y_i)\right) \longrightarrow T^*\left(\alpha(y)\right).$$

A vector field Θ for $\mathfrak{B}(\mathsf{H})$ is called $*$-*strongly continuous* if the maps

$$(x,y) \longmapsto \Theta(x,y)\alpha(x) \quad \text{and} \quad (x,y) \longmapsto \Theta(x,y)^*\alpha(y)$$

are continuous on $X \times X \longrightarrow \mathfrak{B}(\mathsf{H})_1$ for every $\alpha \in \Gamma_b(\mathsf{H})$. Let $A_{\mathfrak{B}(\mathsf{H})}$ be the set of all strictly continuous vector fields for $\mathfrak{B}(\mathsf{H})$.

Lemma 4.1.8. *For each pair of objects* $x, y \in X$ *the set* $\left\{\, \Theta(x,y) \mid \Theta \in A_{\mathfrak{B}(\mathsf{H})} \,\right\}$ *is dense in* $\mathfrak{B}(\mathsf{H})(x,y)$.

Proof. Let $x_0, y_0 \in X$ be objects, $T \in \mathfrak{B}(\mathsf{H})(x_0,y_0)$ be a morphism and let $\epsilon > 0$. Let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$ be such that

$$\left\| h - \sum_{i=1}^{n} \langle\, \alpha_i(x_0), h \,\rangle \alpha_i(x_0) \right\| < \epsilon \quad \text{and} \quad \left\| h' - \sum_{j=1}^{m} \langle\, \beta_j(y_0), h' \,\rangle \alpha_j(y_0) \right\| < \epsilon$$

for any $h \in H_{x_0}$ and $h' \in H_{y_0}$. Define a vector field Θ by the formula

$$\Theta(x,y)(h) := \sum_{i=1}^{n} \sum_{j=1}^{m} \langle\, \alpha_i(x), h \,\rangle \langle\, \beta_j(y_0), T\left(\alpha_i(x_0)\right) \,\rangle \beta_j(y)$$

for $h \in H_{x_0}$. By continuity of the inner-product function it follows that $\Theta \in A_{\mathfrak{B}(\mathsf{H})}$.

Furthermore

$$\begin{aligned}
\|(T-\Theta(x,y))\,h\| &= \|T(h)-\Theta(x,y)h\| \\
&\le \left\| T(h) - \sum_{i=1}^{n} \langle\, \alpha_i(x), h \,\rangle T\left(\alpha_i(x)\right) \right\| \\
&\quad + \left\| \sum_{i=1}^{n} \langle\, \alpha_i(x), h \,\rangle T\left(\alpha_i(x)\right) - \sum_{i=1}^{n}\sum_{j=1}^{m} \langle\, \alpha_i(x), h \,\rangle \langle\, \beta_j(y_0), T\left(\alpha_i(x_0)\right) \rangle \beta_j(y) \right\| \\
&\le \left\| T(h) - \sum_{i=1}^{n} \langle\, \alpha_i(x), h \,\rangle T\left(\alpha_i(x)\right) \right\| + \|\epsilon\|
\end{aligned}$$

and

$$\left\| T(h) - \sum_{i=1}^{n} \langle\, \alpha_i(x), h \,\rangle T\left(\alpha_i(x)\right) \right\| \le \left\| T\left(h - \sum_{i=1}^{n} \langle\, \alpha_i(x), h \,\rangle \alpha_i(x) \right) \right\| \le \|T\|\,\epsilon.$$

Therefore $\|(T-\Theta(x,y))\,h\| \le (\|T\|+1)\,\epsilon$, and hence $\left\{\, \Theta(x,y) \;\mid\; S \in A_{\mathfrak{B}(\mathsf{H})} \,\right\}$ is dense in $\mathfrak{B}(\mathsf{H})(x,y)$. □

A sub-base for the $*$-strong operator topology on $\mathfrak{B}(\mathsf{H})_1$ is given by the collection of all sets of the form $\Omega(U,\Theta,\alpha,\epsilon)$ where the set $\Omega(U,\Theta,\alpha,\epsilon)$ is defined as

$$\Omega(U,\Theta,\alpha,\epsilon) = \left\{\, T \in \mathfrak{B}(\mathsf{H})_1 \;\mid\; (s,t)(T) \in U,\ \left\| T - \Theta\big((s,t)(T)\big) \right\|_{\alpha}^{\omega} < \epsilon \,\right\},$$

for $U \subseteq X \times X$ an open subset, $\Theta \in A_{\mathfrak{B}(\mathsf{H})}$, $\alpha \in \Gamma_b(\mathsf{H})$ and $\epsilon > 0$. A base for a topology is the finite intersection of all such sets, and hence a base for the $*$-strong operator topology on $\mathfrak{B}(\mathsf{H})_1$ is given by the collection of all sets of the form

$$\Omega(U,\Theta,\boldsymbol{\alpha},\epsilon) = \left\{ T \in \mathfrak{B}(\mathsf{H})_1 \;\mid\; (s,t)(T) \in U,\ \sum_{i=1}^{k} \left\| T - \Theta\big((s,t)(T)\big) \right\|_{\alpha_i}^{\omega} < \epsilon,\ k \in \mathbb{N} \right\},$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$ such that $\alpha_i \in \Gamma_b(\mathsf{H})$ for each index i, and where U, Θ and ϵ are as above.

We claim that the category $\big(\mathfrak{B}(\mathsf{H}), A_{\mathfrak{B}(\mathsf{H})}\big)$ is a lower semicontinuous C^*-category. We use a similar line of argument to that used in Section 5.2 of [9].

Lemma 4.1.9. *For every $\Theta \in A_{\mathfrak{B}(\mathsf{H})}$ the map $(x,y,h) \longmapsto \|\Theta(x,y)h\|$ is continuous on $\{\, X \times X \times \mathsf{H} \mid h \in H_x \,\} \longrightarrow \mathbb{R}^{\ge 0}$.*

Proof. Let $(x,y,h) \in \{\, X \times X \times \mathsf{H} \mid h \in H_x \,\}$ and let $\alpha \in \Gamma(\mathsf{H})$ be such that

$\alpha(x) = h$. Let (x_i, y_i, h_i) be a net in $\{ X \times X \times \mathsf{H} \mid h \in H_x \}$ such that (x_i, y_i, h_i) converges to (x, y, h). By continuity we have, for sufficiently large i,

$$\Big| \|\Theta(x_i, y_i)h_i\| - \|\Theta(x_i, y)h_i\| \Big| \leq \frac{\epsilon}{2}$$

and

$$\Big| \|\Theta(x_i, y_i)\alpha(x_i)\| - \|\Theta(x, y)h\| \Big| \leq \frac{\epsilon}{2}.$$

It follows that

$$\begin{aligned}\Big| \|\Theta(x_i, y_i)h_i\| - \|\Theta(x, y)h\| \Big| &\leq \epsilon + \Big| \|\Theta(x_i, y)h_i\| - \|\Theta(x_i, y)\alpha(x_i)\| \Big| + \epsilon \\ &\leq (2 + \|\Theta(x_i, y)\|)\, \epsilon\end{aligned}$$

and therefore $(x, y, h) \longmapsto \|\Theta(x, y)h\|$ is continuous. ☐

Lemma 4.1.10. *For every $\Theta \in A_{\mathfrak{B}(\mathsf{H})}$ the map $(x, y) \longmapsto \|\Theta(x, y)\|$ is lower semicontinuous.*

Proof. Fix $\epsilon > 0$. Let (x_i, y_i, h_i) be a net in $\{ X \times X \times \mathsf{H} \mid h \in H_x \}$ such that (x_i, y_i, h_i) converges to (x, y, h) where h is such that

$$\big| \|\Theta(x, y)h\| - \|\Theta(x, y)\| \big| \leq \frac{\epsilon}{2}.$$

By continuity, for sufficiently large i we have

$$\big| \|\Theta(x_i, y_i)h_i\| - \|\Theta(x, y)h\| \big| \leq \frac{\epsilon}{2},$$

and therefore

$$\|\Theta(x_i, y_i)\| \;\geq\; \|\Theta(x_i, y_i)h_i\| \;\geq\; \|\Theta(x, y)h\| - \frac{\epsilon}{2} \;\geq\; \|\Theta(x, y)\| - \epsilon,$$

hence the map $(x, y) \longmapsto \|\Theta(x, y)\|$ is lower semicontinuous for every $\Theta \in A_{\mathfrak{B}(\mathsf{H})}$. ☐

Proposition 4.1.11. *With respect to the strong*-bundle topology on $\mathfrak{B}(\mathsf{H})_1$, the norm function $\|-\| : \mathfrak{B}(\mathsf{H})_1 \longrightarrow \mathbb{R}^{\geq 0}$, $T \longmapsto \|T\|$ is lower semicontinuous.*

Proof. We prove that the level sets $Q_\delta := \{ T \in \mathfrak{B}(\mathsf{H})_1 \mid \|T\| > \delta \}$ are open. Let $\delta > 0$ and take $T_0 \in Q_\delta$ with $(s, t)(T_0) = (x_0, y_0)$. Choose $\Theta \in A_{\mathfrak{B}(\mathsf{H})}$ such that $\Theta(x_0, y_0) = T_0$. and pick $\delta' > 0$ such that $\delta < \delta' < \|T_0\|$. Define a subset of $X \times X$

by

$$U := \{\, (x,y) \in X \times X \;\mid\; \|\Theta(x,y)\| > \delta' \,\}\,.$$

By Lemma 4.1.10 the function $(x,y) \longmapsto \|\Theta(x,y)\|$ is lower semicontinuous, and therefore U is open. Let $\epsilon > 0$ be such that $\epsilon < \delta' - \delta$, then for $T \in \Omega(U,\Theta,\alpha,\epsilon)$ we have

$$\|T\| \geq \big\|\Theta\big((s,t)(T)\alpha(s(T))\big)\big\| - \big\|\Theta\big((s,t)(T)\alpha(s(T))\big) - T\big\| > \delta' - \epsilon > \delta.$$

Therefore $T_0 \in \Omega(U,\Theta,\alpha,\epsilon) \subseteq Q_\delta$, and hence Q_δ is open. □

The $*$-strong operator topology does not, in general, make $\mathcal{B}(\mathsf{H})$ an internal category in **Top**, since composition of morphisms $m\colon \mathcal{B}(\mathsf{H})_1 \times_X \mathcal{B}(\mathsf{H})_1 \longrightarrow \mathcal{B}(\mathsf{H})_1$ is only separately $*$-strongly continuous. We do, however, have the following:

Proposition 4.1.12. *Consider the category $\big(\mathcal{B}(\mathsf{H}), A_{\mathcal{B}(\mathsf{H})}\big)$ equipped with the $*$-strong operator topology. The involution map $*\colon \mathcal{B}(\mathsf{H})_1 \longrightarrow \mathcal{B}(\mathsf{H})_1$ is $*$-strongly continuous.*

Proof. Let (T_i) be a net of morphisms in $\mathcal{B}(\mathsf{H})$ converging to T. Then for every $\alpha \in \Gamma_b(\mathsf{H})$ we have

$$T_i\big(\alpha\big(s(T_i)\big)\big) \longrightarrow T\big(\alpha\big(s(T)\big)\big) \quad \text{and} \quad T_i^*\big(\alpha\big(t(T_i)\big)\big) \longrightarrow T^*\big(\alpha\big(t(T)\big)\big).$$

Now

$$T_i\big(\alpha\big(s(T_i)\big)\big) \longrightarrow T\big(\alpha\big(s(T)\big)\big) \iff (T_i^*)^*\big(\alpha\big(t(T_i^*)\big)\big) \longrightarrow (T^*)^*\big(\alpha\big(t(T^*)\big)\big)$$

and

$$T_i^*\big(\alpha\big(t(T_i)\big)\big) \longrightarrow T^*\big(\alpha\big(t(T)\big)\big) \iff T_i^*\big(\alpha\big(s(T_i^*)\big)\big) \longrightarrow T^*\big(\alpha\big(s(T^*)\big)\big)$$

and therefore $T_i^* \longrightarrow T^*$. □

Definition 4.1.13. A *concrete topological* C^**-category* is a *-category $\mathcal{C}$ with a locally compact, Hausdorff space of objects $\mathcal{C}_0$ such that each $\mathcal{C}(x,y)$ is a norm-closed linear subspace of $\mathcal{B}(\mathsf{H})(x,y)$ for some continuous **Hilb**-bundle $(\mathsf{H}, p, \mathcal{C}_0)$.

§4.2 Representations

An extension of the GNS construction to C^*-categories appeared in the original article by Ghez, Lima and Roberts [28], and was used to prove that every C^*-

category can be faithfully represented as a norm-closed subcategory of **Hilb**. The details of this construction were then explored in greater detail in [43]. We now look to prove an analogue of this result for topological C*-categories.

Definition 4.2.1. Let $\mathcal{C}$ be a topological C*-category. A *representation* of $\mathcal{C}$ is a *-functor $R\colon \mathcal{C} \longrightarrow \mathfrak{B}(\mathsf{H})$ where:

i $(\mathsf{H}, p, \mathcal{C}_0)$ is a continuous **Hilb**-bundle,

ii The object map $R_0\colon \mathcal{C}_0 \longrightarrow \mathfrak{B}(\mathsf{H})_0$ is continuous,

iii The map $b \longmapsto R(b)\Big(\alpha\big(s(b)\big)\Big)$ is continuous on $\mathcal{C}_1 \longrightarrow \mathsf{H}$ for every $\alpha \in \Gamma_b(\mathsf{H})$.

Recall that a state on a unital C*-algebra A is a linear functional $\sigma\colon A \longrightarrow \mathbb{C}$ such that $\sigma(1) = 1$ and $\sigma(a^*a) \geq 0$ for all $a \in A$. Given a unital topological C*-category $\mathcal{C}$ and a state σ on $\mathcal{C}(x,x)$ we define the *σ-null space* of $\mathcal{C}(x,y)$ to be the set

$$N(x,y;\sigma) = \{\, a \in \mathcal{C}(x,y) \;\mid\; \sigma(a^*a) = 0 \,\}\,.$$

This is a vector space and hence we may form the quotient

$$H^0(x,y;\sigma) := \mathcal{C}(x,y)\Big/N(x,y;\sigma)\,.$$

We write $[a]$ for the class of the morphism $a \in \mathcal{C}(x,y)$ under the quotient mapping. From the properties of σ the map $\langle -,- \rangle\colon \mathcal{C}(x,y) \times \mathcal{C}(x,y) \longrightarrow \mathbb{C}$ given by the formula $\langle a,b \rangle = \sigma(a^*b)$ is positive and sesquilinear, and therefore the induced map on the quotient space $H^0(x,y;\sigma)$ is a well-defined inner product.

Definition 4.2.2. For each $x,y \in \mathcal{C}_0$ and each σ a state on the C*-algebra $\mathcal{C}(x,x)$ define $H(x,y;\sigma)$ to the be Hilbert space obtained by completing $H^0(x,y;\sigma)$ with respect to the inner-product norm. Define

$$H(x,y) := \bigoplus_{\sigma} H(x,y;\sigma),$$

taking the Hilbert space direct-sum over all states on $\mathcal{C}(x,x)$.

In order to proceed we require some technical lemmas. The first is a standard result about C*-algebras that we state without proof.

Lemma 4.2.3. *Let A be a unital C*-algebra, and let $a \in A$. Then there exists a state σ on A such that $\sigma(a^*a) = \|a\|^2$.*

We also require the following result by Mitchener [43], the proof of which we reproduce here.

Lemma 4.2.4. *Let $\mathcal{C}$ be a unital topological C^*-category, $x \in \mathcal{C}_0$ and σ a state on $\mathcal{C}(x,x)$. Let $a \in \mathcal{C}(x,y)$ and $b \in \mathcal{C}(y,z)$. Then $\|[ba]\| \leq \|b\| \, \|[a]\|$ where $\|[ba]\|$ denotes the norm of the class $[ba]$ in $H(x,z)$ and $\|[a]\|$ denotes the norm of the class $[a]$ in $H(x,y)$.*

Proof. The result is immediate when b is the zero morphism, so assume that $b \neq 0$. Define a morphism $d \in \mathcal{C}(y,y)$ by

$$d := \frac{b^*b}{\|b\|^2}.$$

Since $\mathcal{C}$ is a C^*-category the endomorphism d is positive, and $\|d\| = 1$. Therefore the endomorphism $1_y - d$ is positive, and hence by the functional calculus there exists a positive element $e \in \mathcal{C}(y,y)$ such that $e^2 = 1_y - d$.

Observe that $(ea)^*(ea) = a^*(1-e)a$, and therefore the morphism $a^*(a-d)a$ is positive. Moreover, $\sigma\left(a^*(1_y - d)a\right) \geq 0$. Therefore

$$\begin{aligned}
\sigma(a^*da) \leq \sigma(a^*a) &\implies \sigma\left(\frac{a^*b^*ba}{\|b\|^2}\right) \leq \sigma(a^*a) \\
&\implies \sigma(a^*b^*ba) \leq \|b\|^2 \sigma(a^*a) \\
&\implies \|[ba]\| \leq \|b\| \, \|[a]\| .
\end{aligned}$$

□

For each $y \in \mathcal{C}_0$ define a map on objects by $H_y(u) := H(u,y)$ and for each $b \in \mathcal{C}(u,v)$ define a map $H_y(b) \colon H_y(v) \longrightarrow H_y(u)$ by the composition $[a] \longmapsto [ab]$. Then H_y defines a *-functor $H_y \colon \mathcal{C}^{op} \longrightarrow \mathbf{Hilb}$.

Theorem 4.2.5. *The assignment $y \longmapsto H_y$ extends to a faithful *-functor*

$$H_\bullet \colon \mathcal{C} \hookrightarrow [\mathcal{C}^{op}, \mathbf{Hilb}],$$

*where $[\mathcal{C}^{op}, \mathbf{Hilb}]$ denotes the category of contravariant *-functors (in* **Set***) from $\mathcal{C}$ to* **Hilb**.

Proof. On objects we write $H_\bullet(y) = H_y$. Let $b \in \mathcal{C}(y,z)$, then b determines a transformation $H_\bullet(b)\colon H_y \implies H_z$ with components

$$(H_\bullet(b))_x : H_y(x) \longrightarrow H_z(x), \qquad [a] \longmapsto [ba].$$

If $d\colon x \longrightarrow w$ is a morphism in $\mathcal{C}(x,w)$ then for each object $y \in \mathcal{C}_0$ there is a contravariantly induced linear map $H_y(w) \longrightarrow H_y(x)$ given by the assignment $[a] \longmapsto [ad]$, hence by the associativity of composition in $\mathcal{C}$ the diagram

$$\begin{array}{ccc} H_y(w) & \xrightarrow{(H_\bullet(b))_w} & H_z(w) \\ \downarrow{\scriptstyle [-\circ d]} & & \downarrow{\scriptstyle [-\circ d]} \\ H_y(x) & \xrightarrow[(H_\bullet(b))_x]{} & H_z(x) \end{array}$$

commutes for every $d \in \mathcal{C}(x,w)$. The transformation $H_\bullet(b)$ is therefore natural. Moreover, by Lemma 4.2.4 we have

$$\|(H_\bullet(b))_x\| = \sup_{\|[a]\|=1} \|[ba]\| \leq \|b\| \, \|[a]\|$$

for each $x \in \mathcal{C}_0$. Therefore $\|H_\bullet(b)\| = \sup_{x\in\mathcal{C}_0} \|(H_\bullet(b))_x\|$ is finite, and hence $H_\bullet(b)$ is a bounded natural transformation – that is, a morphism in $[\mathcal{C}^{op}, \mathbf{Hilb}]$.

Now we prove that $H_\bullet$ is a faithful *-functor. If $b \in \mathcal{C}(y,z)$ and $c \in \mathcal{C}(z,w)$ then for each $x \in \mathcal{C}_0$ we have

$$H_\bullet(c)_x \, (H_\bullet(b))_x \, [a] = H_\bullet(c)_x[ba] = [cba] = H_\bullet(cb)[a].$$

Therefore $H_\bullet$ is a functor. Define

$$H_\bullet(b)^*_x \colon H(x,z) \longrightarrow H(x,y), \qquad [a] \longmapsto [b^*a],$$

then $H_\bullet(b)^*_x = H_\bullet(b^*)_x$, making $H_\bullet$ a *-functor. To see that $H_\bullet$ is faithful, observe that if $x \in \mathcal{C}_0$ is such that $\mathcal{C}(x,y) = \{0\}$ then $(H_\bullet(b))_x$ must be the zero map, hence $\|(H_\bullet(b))_x\| = 0$. Now suppose that $x \in \mathcal{C}_0$ such that $\mathcal{C}(x,y) \neq \{0\}$, and choose $a \in \mathcal{C}(x,y)$ such that $\|a\| = 1$. Then by the positivity axiom of Definition 3.1.2 there exists an endomorphism $e \in \mathcal{C}(x,x)$ such that $(ba)^*ba = e^*e$, and hence $\|e\|^2 = \|ba\|^2$. It follows that by Lemma 4.2.3 there exists a state σ_x on $\mathcal{C}(x,x)$ such that

$$\sigma_x\left((ba)^*ba\right) = \|ba\|^2 .$$

Therefore, letting $\|[ba]\|_\sigma$ denote the norm of the image of ba in $H(x,z;\sigma)$, we have,

$$\begin{aligned}
\|(H_\bullet(b))_x\| &= \sup\{\, \|[ba]\|_\sigma \ \mid \ \|a\|=1,\ \sigma \text{ a state on } \mathcal{C}(x,x)\,\} \\
&\geq \sup\{\, \|[ba]_{\sigma_x}\| \ \mid \ \|a\|=1\,\} \\
&= \sup\{\, ba \ \mid \ \|a\|=1\,\} \\
&= \|b\|\,.
\end{aligned}$$

Therefore, $H_\bullet$ is faithful. □

To complete the construction of a faithful *-functor we require a topological analogue of the direct-sum of Hilbert spaces. Let X be a locally compact Hausdorff space equipped with Borel structure and positive Radon measure μ, and let (H, p, X) be a continuous **Hilb**-bundle. Denote by $L^2(X,\mathsf{H})$ the set of all continuous sections $\alpha \in \Gamma_b(\mathsf{H})$ such that the integral

$$\int_{x\in X} \|\alpha(x)\|^2 \; d\mu(x)$$

is finite. The set $L^2(X,\mathsf{H})$ is an inner-product space under pointwise linear operations with inner-product given by

$$\langle\, \alpha, \beta \,\rangle := \int_{x\in X} \langle\, \alpha(x), \beta(x) \,\rangle \; d\mu(x).$$

Since each continuous section $\alpha,\beta \in \Gamma_b(\mathsf{H})$ is square integrable this inner-product is well-defined.

Definition 4.2.6. If (H, p, X) is a continuous **Hilb**-bundle with fibres H_x then the *topological direct integral*

$$\int_{x\in X}^{\oplus} H_x \; d\mu(x)$$

is the completion of $L^2(X,\mathsf{H})$ under the inner-product norm.

Now for each $y \in \mathcal{C}_0$ define

$$H(y) := \int_{\mathcal{C}_0}^{\oplus} H(x,y) \; d\mu(x)$$

where $H(x,y) = \bigoplus_\sigma H(x,y;\sigma)$ as above. The collection $\{H(y)\}_{y\in\mathcal{C}_0}$ form the fibres of a **Hilb**-bundle $(\mathsf{H}_{\mathcal{C}}, t, \mathcal{C}_0)$ where t is the target map of $\mathcal{C}$. Furthermore, each compactly supported continuous section $\alpha \in \Gamma_c(\mathcal{C})$ determines a section

$$\widetilde{\alpha} \colon \mathcal{C}_0 \longrightarrow \mathsf{H}_{\mathcal{C}}, \qquad \widetilde{\alpha}(y) = \big[\alpha|_{X\times\{y\}}\big]$$

where $\big[\alpha|_{X\times\{y\}}\big]$ denotes the image of $\alpha(-,y)$ in $H_y(-)$. From the vector-valued Tietze Extension Theorem, the topology on $\mathsf{H}_{\mathcal{C}}$ is then determined by the set $\{\, \widetilde{\alpha} \mid \alpha \in \Gamma_c(\mathcal{C})\,\}$, making $\mathsf{H}_{\mathcal{C}}$ a continuous **Hilb**-bundle. Denote by $H_\bullet(\mathcal{C})$ the image of $\mathcal{C}$ under the faithful *-functor $H_\bullet \colon \mathcal{C} \hookrightarrow [\mathcal{C}^{op}, \mathbf{Hilb}]$ and write $\big(\mathfrak{B}(\mathsf{H}_{\mathcal{C}}), A_{\mathfrak{B}(\mathsf{H}_{\mathcal{C}})}\big)$ for the concrete topological C*-category associated to $\mathsf{H}_{\mathcal{C}}$.

Proposition 4.2.7. *Let $\mathcal{C}$ be a continuous, unital topological* C**-category. Then there exists an isometric *-functor $R \colon H_\bullet(\mathcal{C}) \longrightarrow \mathfrak{B}(\mathsf{H}_{\mathcal{C}})$.*

Proof. Define a map $R_0 \colon H_\bullet(\mathcal{C})_0 \longrightarrow \mathfrak{B}(\mathsf{H}_{\mathcal{C}})$ by

$$H_y \longmapsto \int_{x\in\mathcal{C}_0}^{\oplus} H(x,y)\; d\mu(x).$$

On morphisms, let $H_\bullet(b)$ be the bounded natural transformation defined above. Then each $x \in \mathcal{C}_0$ determines a bounded linear operator

$$H_\bullet(b)_x \colon H_y(x) \longrightarrow H_z(x)$$

defined by composition $[a] \longmapsto [ba]$. Since $H_\bullet(b)$ is bounded there exists a constant $M > 0$ such that $\|H_\bullet(b)_x\| \leq M$ for all $x \in \mathcal{C}_0$. Define a map

$$R_1 \colon H_\bullet(\mathcal{C})_1 \longrightarrow \mathfrak{B}(\mathsf{H}_{\mathcal{C}})_1$$

on morphisms by sending $H_\bullet(b)$ to the bounded linear map

$$\int_{x\in\mathcal{C}_0}^{\oplus} H_\bullet(b)_x\; d\mu(x) \colon \int_{x\in\mathcal{C}_0}^{\oplus} H(x,y)\; d\mu(x) \longrightarrow \int_{x\in\mathcal{C}_0}^{\oplus} H(x,z)\; d\mu(x).$$

We claim that $R = (R_0, R_1)$ is a faithful *-functor: First let

$$H_\bullet(b) \in [\mathcal{C}^{op}, \mathbf{Hilb}]\,\big(H_y, H_z\big) \quad \text{and} \quad H_\bullet(c) \in [\mathcal{C}^{op}, \mathbf{Hilb}]\,\big(H_z, H_w\big)$$

be bounded natural transformations. Then $R_1\,(H_\bullet(c) \circ H_\bullet(b))$ is defined on fibres by $H_\bullet(c)_x \circ H_\bullet(b)_x$ which, by the *-functoriality of $H_\bullet$, is equal to $H_\bullet(cb)_x$. The family $\{H_\bullet(b)_x\}$ is uniformly bounded by M, therefore

$$\int_{x\in\mathcal{C}_0} \|H_\bullet(b)_x\widetilde{\alpha}(y)\|^2 \; d\mu(x) \leq M^2 \int_{x\in\mathcal{C}_0} \|\widetilde{\alpha}(y)\|^2 \; d\mu(x) < \infty$$

and hence $R\left(H_\bullet(c) \circ H_\bullet(b)\right) = R\left(H_\bullet(c)\right) \circ R\left(H_\bullet(b)\right)$. Now define $R(H_\bullet(b))^*$ to be the linear map

$$\int_{x\in\mathcal{C}_0}^{\oplus} H(x,z)\; d\mu(x) \longrightarrow \int_{x\in\mathcal{C}_0}^{\oplus} H(x,y)\; d\mu(x)$$

given fibrewise by linear operators $H_\bullet(b^*)_x \colon H_z(x) \longrightarrow H_y(x)$, $[a] \longmapsto [b^*a]$. By *-functoriality of $H_\bullet$ we have $R(H_\bullet(b))^* = R(H_\bullet(b))^*$, and hence R is a *-functor. Finally,

$$\|R(H_\bullet(b)\| = \left\| \int_{x\in\mathcal{C}_0}^{\oplus} H_\bullet(b)_x \; d\mu(x) \right\| = \sup_{x\in\mathcal{C}_0} \|H_\bullet(b)_x\| = \|H_\bullet(b)\| \,,$$

and therefore R is isometric. □

Theorem 4.2.8. *Let $\mathcal{C}$ be a continuous, unital topological C^*-category. Then there exists a faithful representation $\rho\colon \mathcal{C} \longrightarrow \mathfrak{B}(\mathsf{H}_{\mathcal{C}})$. Moreover if $\mathcal{C}$ is a topological C^*-category with any faithful representation $\mathcal{C} \longrightarrow \mathfrak{B}(\mathsf{H})$ then $\mathcal{C}$ is necessarily (at least) lower semicontinuous.*

Proof. Define ρ to be the composition $R \circ H_\bullet$. By Propositions 4.2.5 and 4.2.7 both R and $H_\bullet$ are faithful *-functors, whence ρ is also. For the final claim, let $\mathcal{C}$ be a topological C^*-category with faithful representation $\rho\colon \mathcal{C} \longrightarrow \mathfrak{B}(\mathsf{H})$ for some continuous **Hilb**-bundle $(\mathsf{H}, p, \mathcal{C}_0)$. Finally, since ρ is faithful we have

$$b \longmapsto \|b\| = \|\rho(b)\| \,,$$

and $\rho(b) \longmapsto \|\rho(b)\|$ is lower semicontinuous. □

Corollary 4.2.9. *Let $\mathcal{C}$ be a non-unital, continuous topological C^*-category. Then there exists a faithful representation $\rho\colon \mathcal{C} \longrightarrow \mathfrak{B}(\mathsf{H}_{\mathcal{C}^+})$ where $\mathcal{C}^+$ is the unitisation of $\mathcal{C}$.*

Proof. There exists an isometric embedding of $\mathcal{C}$ into its unitisation $\mathcal{C}^+$. The result then follows by applying Theorem 4.2.8 to the unital continuous topological C^*-category $\mathcal{C}^+$. □

Chapter 5

Further Constructions

In this chapter, we introduce three further constructions associated with topological C^*-categories.

The first construction is an extension of the notion of a groupoid C^*-category to the setting of topological groupoids. This combines the work of Davis and Lück [16] and Mitchener [43] with the earlier work on groupoid C^*-algebras by Renault [49]. We construct both a reduced topological groupoid C^*-category (Definition 5.1.8) and a full topological groupoid C^*-category (Definition 5.1.10) by taking appropriate completions of a category whose hom-objects are spaces of compactly supported continuous functions on the underlying groupoid hom-set. Composition in this category is given by convolution of continuous functions, giving rise to the general term *convolution category*. A closely related convolution category $L^1(\mathcal{G})$ was introduced independently by Bos [9]. We then show that these constructions are functorial with respect to a certain class of groupoid morphisms (Proposition 5.1.12).

In Section 5.2 we define tensor products of two continuous topological C^*-categories. Our constructions rely on the observation from Chapter 3 that the family of endomorphism sets $\{\mathcal{C}(x,x)\}_{x\in\mathcal{C}_0}$ of a continuous topological C^*-category form a continuous $\mathbf{C^*}$-$\mathbf{Alg}$-bundle. We therefore extend the definitions contained in [4], [37] and [41] of tensor-product $\mathbf{C^*}$-$\mathbf{Alg}$-bundles to form the minimal tensor product (Definition 5.2.2) and maximal tensor product (Definition 5.2.7) of two continuous topological C^*-catgories. We also prove that these tensor-product topological C^*-categories possess the same continuity properties exhibited by continuous $\mathbf{C^*}$-$\mathbf{Alg}$-bundles.

The final construction studied in this chapter is that of a Hilbert module over a topological C^*-category. Hilbert modules over C^*-categories in $\mathbf{Set}$ have been considered by both Joachim [33] and Mitchener [43], in the study of K-theory and KK-theory respectively. The definitions used by the two authors are not equivalent. We adopt a definition similar to that of Joachim and define a Hilbert module over a topological C^*-category as a contravariant functor satisfying certain properties. The main observation in this section is that the set of all bounded natural transformations from a Hilbert module to itself forms a C^*-algebra (Proposition 5.3.3)

§5.1 Groupoid C^*-Categories

Let $\mathcal{G}$ be a locally compact, second-countable Hausdorff groupoid endowed with a left Haar system $\{\lambda^x\}_{\mathcal{G}_0}$, such that there exist continuous families of measures $\{\lambda_x^y\}_{(x,y)\in\mathcal{G}_0\times\mathcal{G}_0}$ and $\{\mu_x\}_{x\in\mathcal{G}_0}$ satisfying

$$\lambda^y = \int_{x\in s(\mathcal{G}(-,y))} \lambda_x^y \; d\mu_y(x)$$

(c.f. Chapter 1). Define a $\mathbb{C}$-linear category $\mathcal{C}_c(\mathcal{G})$ with space of objects $\mathcal{G}_0$ and for each pair of objects $x, y \in \mathcal{G}_0$ a hom-object

$$\mathcal{C}_c(\mathcal{G})(x,y) := C_c(\mathcal{G}(x,y)),$$

the space of compactly supported continuous functions on $\mathcal{G}(x,y)$. Define composition by

$$(\psi \star \varphi)(g) := \int_{h\in\mathcal{G}(x,y)} \psi(gh^{-1})\varphi(h)\; d\lambda_x^y(h) \tag{5.1}$$

for $\psi \in C_c(\mathcal{G}(y,z))$, $\varphi \in C_c(\mathcal{G}(x,y))$ and $g \in \mathcal{G}(x,z)$, and involution by

$$\varphi^*(g) := \overline{\varphi(g^{-1})}$$

for $\varphi \in C_c(\mathcal{G}(x,y))$ and $g \in \mathcal{G}(y,x)$.

Proposition 5.1.1. *The category $\mathcal{C}_c(\mathcal{G})$ is a well-defined $\mathbb{C}$-linear involutive category. It is non-unital unless $\mathcal{G}$ is compact.*

Proof. The function $(\psi \star \varphi)$ is non-zero at $g \in \mathcal{G}(x,z)$ if and only if there exists $h \in \mathcal{G}(x,y)$ such that $\psi(gh^{-1})$ and $\varphi(h)$ are both non-zero. Therefore,

$$\operatorname{supp}(\psi\star\varphi) \subseteq (\operatorname{supp}\psi)\,(\operatorname{supp}\varphi)$$

which is compact. Moreover, for $g \in C_c(\mathcal{G}(x,z))$ we have

$$|\psi \star \varphi(g)| \leq \int_{h\in\mathcal{G}(x,y)} \left|\psi(gh^{-1})\varphi(h)\right| \, d\lambda_x^y(h)$$
$$\leq \left(\int_{h\in\mathcal{G}(x,y)} \left|\psi(gh^{-1})\right| \, d\lambda_x^y(h)\right)^{\frac{1}{2}} \left(\int_{h\in\mathcal{G}(x,y)} |\varphi(h)| \, d\lambda_x^y(h)\right)^{\frac{1}{2}}$$

by Hölder's inequality, and therefore the function $(\psi \star \varphi)$ is continuous on

$$\mathcal{G}(y,z) \cdot \mathcal{G}(x,y) \subseteq \mathcal{G}(x,z),$$

hence $(\psi \star \varphi) \in C_c(\mathcal{G}(x,z))$.

Now let $\phi \in \mathcal{C}_c(\mathcal{G})(w,x)$, $\varphi \in \mathcal{C}_c(\mathcal{G})(x,y)$ and $\psi \in \mathcal{C}_c(\mathcal{G})(y,z)$. Then

$$(\psi \star \varphi) \star \phi(g) = \int_{h\in\mathcal{G}(w,x)} (\psi \star \varphi)(gh^{-1})\phi(h) \, d\lambda_w^x(h)$$
$$= \int_{h\in\mathcal{G}(w,x)} \int_{k\in\mathcal{G}(x,y)} \psi(gh^{-1}k^{-1})\varphi(k)\phi(h) \, d\lambda_x^y(k) \, d\lambda_w^x(h). \qquad (5.2)$$

By left-invariance of the Haar system, we have

$$\int_{k\in\mathcal{G}(x,y)} \psi(gh^{-1}k^{-1}) \, d\lambda_x^y(k) = \int_{\gamma\in\mathcal{G}(w,y)} \psi(g\gamma^{-1} \, d\lambda_w^y(\gamma)$$

and

$$\int_{k\in\mathcal{G}(x,y)} \varphi(k) \, d\lambda_x^y(k) = \int_{\gamma\in\mathcal{G}(w,y)} \varphi(\gamma h^{-1}) \, d\lambda_w^y(\gamma)$$

where $\gamma = kh$. Therefore (5.2) may be written as

$$\int_{\gamma\in\mathcal{G}(w,y)} \psi(g\gamma^{-1}) \int_{h\in\mathcal{G}(w,x)} \varphi(\gamma h^{-1})\phi(h) \, d\lambda_w^x(\gamma) \, d\lambda_w^y(\gamma)$$
$$= \int_{\gamma\in\mathcal{G}(w,y)} \psi(g\gamma^{-1})(\varphi \star \phi)(\gamma) \, d\lambda_w^y(\gamma)$$
$$= \left(\psi \star (\varphi \star \phi)\right)(g)$$

and hence composition is associative. Furthermore, composition is bilinear and is compatible with involution, since for $g \in \mathcal{G}(y,x)$ we have

$$\begin{aligned}(\psi \star \varphi)^*(g) &= \overline{(\psi \star \varphi)(g^{-1})} \\ &= \overline{\int_{h \in \mathcal{G}(x,y)} \psi(g^{-1}h^{-1})\varphi(h)\, d\lambda_x^y(h)} \\ &= \int_{h \in \mathcal{G}(x,y)} \overline{\psi(h^{-1})\varphi(gh)}\, d\lambda_x^y(h) \\ &= \int_{h \in \mathcal{G}(x,y)} \psi^*(h)\varphi^*(g^{-1}h^{-1})\, d\lambda_x^y(h) \\ &= (\varphi^* \star \psi^*)(g).\end{aligned}$$

by left-invariance of the Haar system.

For the final statement, let $\imath$ denote the unit map for $\mathcal{C}_c(\mathcal{G})$. Clearly $\operatorname{supp}(\imath)$ is compact if and only if $\mathcal{G}$ is. □

The category $\mathcal{C}_c(\mathcal{G})$ is in fact a category enriched in normed vector spaces, with topology given by the family of norms $\|\varphi\|_{x,y} := \sup\{\,|\varphi(g)| \mid g \in \mathcal{G}(x,y)\,\}$. Since this supremum is 0 if and only if $\varphi(g) = 0$ for all $g \in \mathcal{G}(x,y)$, this is a well defined, non-degenerate norm.

Lemma 5.1.2. *The fibrewise operations of composition and involution are continuous with respect to the norm topology.*

Proof. Let $\{\varphi_i\}$ be a net of morphisms in $\mathcal{C}_c(\mathcal{G})$ such that $\varphi_i \in \mathcal{C}_c(\mathcal{G})(x_i, y_i)$ for each index i, and such that $\varphi_i \longrightarrow \varphi$ in $\mathcal{C}_c(\mathcal{G})(x,y)$. Similarly, let $\{\psi_j\}$ be a net of morphisms in $\mathcal{C}_c(\mathcal{G})$ such that $\psi_j \in \mathcal{C}_c(\mathcal{G})(y_j, z_i)$ for each index j, and such that $\psi_j \longrightarrow \psi$ in $\mathcal{C}_c(\mathcal{G})(y,z)$. Then there exist compact sets K_1, K_2 such that for sufficiently large i, j we have $\operatorname{supp}(\psi_j \star \varphi_i) \subseteq K_2 \cdot K_1$, which is compact. Then

$$\begin{aligned}|(\psi_j \star \varphi_i)(g) - (\psi \star \varphi)(g)| &\le \int_{h \in \mathcal{G}(x,y)} \left|\psi_j(gh^{-1})\varphi_i(h) - \psi(gh^{-1})\varphi(h)\right|\, d\lambda_x^y(h) \\ &\le \int_{h \in \mathcal{G}(x,y)} \left|\psi_j(gh^{-1}) - \psi(gh^{-1})\right| \cdot |\varphi_i(h)|\, d\lambda_x^y(h) \\ &\quad + \int_{h \in \mathcal{G}(x,y)} \left|\psi(gh^{-1})\right| \cdot |\varphi_i(h) - \varphi(h)|\, d\lambda_x^y(h),\end{aligned}$$

and therefore $(\psi_j \star \varphi_i) \longrightarrow (\psi \star \varphi)$ uniformly on $K_2 \cdot K_1$. It follows that composition is continuous. Similarly, $|\varphi_i^*(g) - \varphi^*(g)| = \left|\overline{\varphi_i(g^{-1})} - \overline{\varphi(g^{-1})}\right|$, hence $\varphi_i^* \longrightarrow \varphi^*$ uniformly and therefore involution is continuous. □

For each $f \in C_c(\mathcal{G}_1)$ define a vector field $\widetilde{f}$ for $\mathcal{C}_c(\mathcal{G})$ by $\widetilde{f}(x,y) := f|_{\mathcal{G}(x,y)}$. Let $A_{\mathcal{C}_c(\mathcal{G})}$ be the set $\left\{ \widetilde{f} \mid f \in C_c(\mathcal{G}_1) \right\}$ of all such vector fields.

Proposition 5.1.3. *There exists a topology on the set of morphisms $\mathcal{C}_c(\mathcal{G})_1$ generated by basic open sets of the form*

$$\Omega(U, \widetilde{f}, \epsilon) := \left\{ \varphi \in \mathcal{C}_c(\mathcal{G})_1 \mid (s,t)(\varphi) \in U, \left\| \varphi - \widetilde{f}((s,t)(\varphi)) \right\| \leq \epsilon \right\},$$

*where $U \subseteq \mathcal{G}_0 \times \mathcal{G}_0$ is an open set, $\widetilde{f} \in A_{\mathcal{C}_c(\mathcal{G})}$ and $\epsilon > 0$, making $\mathcal{C}_c(\mathcal{G})$ an internal $\mathbb{C}$-linear *-category in* **Top**.

Proof. By Lemma 5.1.2 composition and involution on $\mathcal{C}_c(\mathcal{G})$ are fibrewise continuous with respect to the norm topology. Moreover, for every continuous function $f \in C_c(\mathcal{G}_1)$ the map $f \longmapsto \sup_g \left| f|_{\mathcal{G}(x,y)}(g) \right|$ is continuous, and since $\mathcal{G}(x,y)$ is a closed subspace of $\mathcal{G}_1$ the set

$$\left\{ \widetilde{f}(x,y) \mid f \in C_c(\mathcal{G}_1) \right\}$$

is dense in $\mathcal{C}_c(\mathcal{G})(x,y)$ for each $x, y \in \mathcal{G}_0$. The result then follows from arguing as in Proposition 3.2.3, noting that the set $A_{\mathcal{C}_c(\mathcal{G})}$ satisfies the hypotheses required there and that the proof of that result does not rely on the completeness of the hom-objects. □

The assignment $\mathcal{G} \longmapsto \mathcal{C}_c(\mathcal{G})$ is not a functor, for if such a functor F were to exist then we would require, for each $g \in \mathcal{G}_1$, that $F(g)$ be a continuous function (with compact support) such that

$$F(g)(h) = \delta_g(h) = \begin{cases} 1 & \text{if } g = h, \\ 0 & \text{otherwise} \end{cases}$$

and no such continuous function exists. We are, however, able to functorially induce morphisms for certain classes of groupoid morphism.

Proposition 5.1.4. *Let $F : \mathcal{G} \longrightarrow \mathcal{G}'$ be an embedding. Then there exists a functorially induced continuous *-functor $\widetilde{F} : \mathcal{C}_c(\mathcal{G}) \longrightarrow \mathcal{C}_c(\mathcal{G}')$ given by the (trivial) pushforward $\widetilde{F}(\varphi)(g') := \varphi(F^{-1}(g'))$.*

Proof. We first note that since F is an embedding the maps

$$F_{xy} : \mathcal{C}_c(\mathcal{G})(x,y) \longrightarrow \mathcal{C}_c(\mathcal{G}')(F(x), F(y))$$

are all injective, and hence $F^{-1}(g')$ consists of just a single morphism in $\mathcal{G}_1$, g say. Therefore $\widetilde{F}$ is well-defined. Furthermore it is clearly continuous, and the support

$$\operatorname{supp}(\widetilde{F}(\varphi)) = F(\operatorname{supp}(\varphi))$$

is compact. Now if $\varphi \in \mathcal{C}_c(\mathcal{G})(x,y)$ and $\psi \in \mathcal{C}_c(\mathcal{G})(y,z)$ then

$$\begin{aligned}
\widetilde{F}(\psi \star \varphi)(g') &= (\psi \star \varphi)(F^{-1}(g')) \\
&= \int_{h \in \mathcal{G}(x,y)} \psi(F^{-1}(g')h^{-1})\varphi(h)\ d\lambda_x^y(h) \\
&= \int_{k \in \mathcal{G}(F(x),F(y))} \psi(F^{-1}(g'k^{-1}))\varphi(F^{-1}(k))\ d\lambda_{F(x)}^{F(y)}(k) \\
&= \int_{k \in \mathcal{G}(F(x),F(y))} \widetilde{F}(\psi)(g'k^{-1})\widetilde{F}(\varphi)(k)\ d\lambda_{F(x)}^{F(y)}(k) \\
&= (\widetilde{F}(\psi) \star \widetilde{F}(\varphi))(g').
\end{aligned}$$

Therefore, $\widetilde{F}$ is a functor. Moreover,

$$\widetilde{F}(\varphi^*)(g') = \varphi^*(F^{-1}(g')) = \overline{\varphi(F^{-1}((g')^{-1}))} = \widetilde{F}(\varphi)^*(g')$$

and for all $x, y \in \mathcal{G}_0$ and all $f \in C_c(\mathcal{G}_1)$, the map $(x,y) \longmapsto \widetilde{F} \circ \widetilde{f}$ is continuous. Therefore, by Proposition 3.2.8, the functor $\widetilde{F}$ is a continuous *-functor.

Finally, we prove that the induced continuous functor is functorial with respect to embeddings of topological groupoids. Let $F\colon \mathcal{G} \longrightarrow \mathcal{G}'$ and $F'\colon \mathcal{G}' \longrightarrow \mathcal{G}''$ be embeddings. Then

$$\begin{aligned}
(\widetilde{F' \circ F})(\varphi)(g'') &= \varphi((F' \circ F)^{-1}(g'')) \\
&= \varphi((F^{-1} \circ (F')^{-1})(g'') \\
&= (\widetilde{F'} \circ \widetilde{F})(\varphi)(g'')
\end{aligned}$$

thus proving functoriality. □

We also have functoriality with respect to quotient functors with compact kernels.

Proposition 5.1.5. *Let $F\colon \mathcal{G} \longrightarrow \mathcal{G}'$ be a continuous quotient functor with compact kernel. There exists a functorially induced continuous *-functor $\widetilde{F}\colon \mathcal{C}_c(\mathcal{G}) \longrightarrow \mathcal{C}_c(\mathcal{G}')$ given by the pushforward*

$$\widetilde{F}(\varphi)(g') := \int_{k \in F^{-1}(g')} \varphi(k)\ d\lambda_x^y(k).$$

Proof. By Lemma 1.2.10 the map $F_1 : \mathcal{C}_c(\mathcal{G})_1 \longrightarrow \mathcal{C}_c(\mathcal{G}')_1$ is perfect, and therefore the set $\{ k \in \mathcal{G} \mid k \in [g] \}$ is compact, hence $\widetilde{F}(\varphi)$ is well-defined. Furthermore,

$$\operatorname{supp}(\widetilde{F}(\varphi)) \subseteq \operatorname{supp}(\varphi)$$

and therefore $\widetilde{F}(\varphi) \in C_c(F(x), F(y))$. Let $\varphi \in \mathcal{C}_c(\mathcal{G})(x,y)$ and $\psi \in \mathcal{C}_c(\mathcal{G})(y,z)$, then

$$\begin{aligned}
(\widetilde{F}(\psi) \star \widetilde{F}(\varphi))(g') &= \int_{h' \in \mathcal{G}(F(x),F(y))} \widetilde{F}(\psi)(g(h')^{-1})\widetilde{F}(\varphi)(h')\, d\lambda_{F(x)}^{F(y)}(h') \\
&= \int_{h \in \mathcal{G}(x,y)} \int_{k \in F^{-1}(g')} \psi(kh^{-1})\varphi(h) d\lambda_x^z(k)\, d\lambda_x^y(h) \\
&= \int_{k \in F^{-1}(g')} \int_{h \in \mathcal{G}(x,y)} \psi(kh^{-1})\varphi(h)\, d\lambda_x^y(h)\, d\lambda_x^z(k) \\
&= \int_{k \in F^{-1}(g')} (\psi \star \varphi)(k)\, d\lambda_x^z(k) \\
&= \widetilde{F}(\psi \star \varphi)(g')
\end{aligned}$$

and so $\widetilde{F}$ is a functor. Furthermore

$$\widetilde{F}(\varphi^*)(g') = \int_{k \in F^{-1}(g')} \varphi^*(k)\, d\lambda_x^z(k) = \int_{k \in F^{-1}(g')} \overline{\varphi(k^{-1})}\, d\lambda_x^z(k) = \widetilde{F}(\varphi)^*$$

for all $x, y \in \mathcal{G}_0$, and therefore $\widetilde{F}$ is a *-functor. Moreover, for all $f \in C_c(\mathcal{G}_1)$ the map

$$(x,y) \longmapsto \left(\widetilde{F} \circ \widetilde{f}\right)(x,y)$$

is continuous and hence by Propositoin 3.2.8 the functor $\widetilde{F}$ is continuous. The fact that $\widetilde{F}$ is functorial with respect to groupoid homomorphisms follows from the standard properties of functors. □

For each $x, y \in \mathcal{G}_0$ denote by $L^1(\mathcal{G}(x,y))$ the space of continuous functions $\varphi : \mathcal{G}(x,y) \longrightarrow \mathbb{C}$ such that the integral

$$\int_{g \in \mathcal{G}(x,y)} |\varphi(g)|\, d\lambda_x^y(g)$$

is finite. Construct a category $L^1(\mathcal{G})$ with object space $\mathcal{G}_0$ and for each $x, y \in \mathcal{G}_0$ a hom-object

$$L^1(\mathcal{G})(x,y) := L^1(\mathcal{G}(x,y))$$

with composition as given by the continuous extension of (5.1). We claim that $L^1(\mathcal{G})$ is a Banach *-category. Indeed, by construction each $L^1(\mathcal{G})(x,y)$ is a complex

Banach space. For $\varphi \in C_c(\mathcal{G}(x,y))$ and $\psi \in C_c(\mathcal{G}(y,z))$ we have

$$\begin{aligned}
\|\psi \circ \varphi\|_{L^1(\mathcal{G}(x,y))} &= \int_{g\in\mathcal{G}(x,y)} |(\psi\circ\varphi)(g)| \; d\lambda_x^z(g) \\
&= \int_{g\in\mathcal{G}(x,y)} \int_{k\in\mathcal{G}(y,z)} |\psi(k)| \,.\, \left|\varphi(k^{-1}g)\right| \; d\lambda_y^z(k)\; d\lambda_x^y(g) \\
&\leq \left(\int_{k\in\mathcal{G}(y,z)} |\psi(k)| \; d\lambda_y^z(k)\right)\left(\int_{\gamma\in\mathcal{G}(x,y)} |\varphi(\gamma)| \; d\lambda_x^y(\gamma)\right) \\
&= \|\psi\|_{L^1(\mathcal{G}(y,z))} \, \|\varphi\|_{L^1(\mathcal{G}(x,y))} \,.
\end{aligned}$$

It follows that the continuous extension of (5.1) exists and that composition satisfies the triangle inequality. Therefore, $L^1(\mathcal{G})$ is a Banach category. Furthermore, it is a Banach *-category when equipped with involution defined by the continuous extension of $\varphi^*(g) = \overline{\varphi(g^{-1})}$ for $\varphi \in C_c(\mathcal{G}(x,y))$ and $g \in \mathcal{G}(y,x)$.

Now write $L^2(\mathcal{G}(x,y))$ for the space of continuous functions $\varphi\colon \mathcal{G}(x,y) \longrightarrow \mathbb{C}$ such that the integral

$$\int_{g\in\mathcal{G}(x,y)} |\varphi(g)|^2 \; d\lambda_x^y(g)$$

is finite. For each $x, y \in \mathcal{G}_0$ the set $L^2(\mathcal{G}(x,y))$ is a Hilbert space with inner product given by

$$\langle\, \varphi_1, \varphi_2 \,\rangle \coloneqq \int_{g\in\mathcal{G}(x,y)} \overline{\varphi_1(g)}\varphi_2(g) \; d\lambda_x^y(g).$$

Fix an object $w \in \mathcal{G}_0$ and define a map $I_w\colon \mathcal{G}_0 \longrightarrow \mathbf{Hilb}_0$ by $I_w(x) = L^2(\mathcal{G}(w,x))$.

Proposition 5.1.6. *There exists a continuous* **Hilb***-bundle* $(\mathsf{H}_{\mathcal{G}}, t, \mathcal{G}_0)$ *where the total space is* $\mathsf{H}_{\mathcal{G}} = \coprod_{x\in\mathcal{G}_0} I_w(x)$ *and* t *is the target map of the topological groupoid* $\mathcal{G}$.

Proof. We have an untopologised surjection with fibres in **Hilb**, so it suffices to show that there exists a set of norm-continuous vector fields for $(\mathsf{H}_{\mathcal{G}}, t, \mathcal{G}_0)$. Consider the *-algebra of compactly supported functions on $\mathcal{G}$. Define a norm on $C_c(\mathcal{G})$ by

$$\|f\| \coloneqq \sup_{x\in\mathcal{G}_0} \left\| f|_{\mathcal{G}(w,x)} \right\|_{L^2(\mathcal{G}(w,x))}$$

where $\|-\|_{L^2(\mathcal{G}(w,x))}$ is the usual L^2-norm. By construction, this supremum is finite. Then for every $x \in \mathcal{G}_0$ the map $f \longmapsto \left\| f|_{\mathcal{G}(w,x)} \right\|$ is continuous. Moreover the set $\left\{ f|_{\mathcal{G}(w,x)} \;\middle|\; f \in C_c(\mathcal{G}) \right\}$ is dense in $L^2(\mathcal{G}(w,x))$. $\square$

For the same fixed object $w \in \mathcal{G}_0$ and $\varphi \in C_c(\mathcal{G}(x,y))$, define $I_w(\varphi)$ to be the linear map

$$I(w)(\varphi)\colon L^2(\mathcal{G}(w,x)) \longrightarrow L^2(\mathcal{G}(w,y)), \quad \phi \longmapsto (\varphi \star \phi).$$

Lemma 5.1.7. *The linear map $I(w)(\varphi)$ is bounded.*

Proof. By the construction of $L^1(\mathcal{G})$, we have $\|\varphi \star \phi\| \leq M \|\phi\|$ where

$$M = \int_{g \in \mathcal{G}(x,y)} |\varphi(g)| \; d\lambda_x^y(g) < \infty.$$

□

It follows that I_w defines a $\mathbb{C}$-linear map

$$L^1(\mathcal{G})(x,y) \longrightarrow \mathbf{Hilb}\left(L^2(\mathcal{G}(w,x)), L^2(\mathcal{G}(w,y))\right).$$

We define $(I(w)(\varphi))^*$ to be the linear map $L^2(\mathcal{G}(w,y)) \longrightarrow L^2(\mathcal{G}(w,x))$ defined by the composition $\phi \longmapsto \varphi^* \circ \phi$, which is necessarily bounded. Then

$$(I(w)(\varphi))^* = I(w)(\varphi^*)$$

and therefore I_w defines a *-functor $\mathcal{C}_c(\mathcal{G}) \longrightarrow \mathfrak{B}(\mathsf{H}_{\mathcal{G}})$. Furthermore,

$$\|I_w\| = \sup\{\, \|I_w(\varphi)\| \;\mid\; \|\varphi\| = 1\,\} = 1$$

in the operator norm, hence each *-functor I_w is faithful. We can therefore define a C*-norm on $\mathcal{C}_c(\mathcal{G})$ by writing

$$\|\varphi\|_{\mathrm{r}} := \sup_{w \in \mathcal{G}_0} \|I_w(\varphi)\|, \qquad \varphi \in \mathcal{C}_c(\mathcal{G})(x,y).$$

Definition 5.1.8. Define the *reduced topological* C**-category* of $\mathcal{G}$ to be the continuous topological C*-category $\left(\mathcal{C}_r^*(\mathcal{G}), A_{\mathcal{C}_c(\mathcal{G})}\right)$ obtained by completing each $\mathcal{C}_c(\mathcal{G})$ with respect to $\|-\|_{\mathrm{r}}$.

We define the maximal norm on $C_c(\mathcal{G})$ by defining $\|\varphi\|_{\max}$ to be the supremum $\sup\{\|\rho(\varphi)\|\}$ taken across all continuous bounded representations ρ of $\mathcal{C}_c(\mathcal{G})$.

Lemma 5.1.9. *The function $\|-\|_{\max}$ is a well-defined* C**-norm.*

Proof. For each continuous bounded representation ρ we have

$$\|\rho(\varphi)\|_{\mathrm{op}} \leq \|\rho\| \,.\, |\varphi| \leq |\varphi| < \infty$$

since ρ is a (necessarily norm non-increasing) *-functor. Therefore the supremum $\sup\{\|\rho(\varphi)\|\}$ is well-defined. Note also that $\|\rho(\varphi)\|_{op}$ is the operator norm. It therefore follows from the properties of the operator norm and of suprema that $\|-\|_{\max}$ is indeed a C*-norm. □

Definition 5.1.10. Define the *full* C*-*category of* $\mathcal{G}$ to be the continuous topological C*-category $\big(\mathcal{C}^*(\mathcal{G}), A_{\mathcal{C}_c(\mathcal{G})}\big)$ obtained by completing each $\mathcal{C}_c(\mathcal{G})(x,y)$ with respect to the norm $\|-\|_{\max}$.

Proposition 5.1.11. *The topological* C*-*categories* $\big(\mathcal{C}_r^*(\mathcal{G}), A_{\mathcal{C}_c(\mathcal{G})}\big)$ *and* $\big(\mathcal{C}^*(\mathcal{G}), A_{\mathcal{C}_c(\mathcal{G})}\big)$ *are internal* C*-*categories in* **Top**.

Proof. By Proposition 5.1.3, the category $\big(\mathcal{C}_c(\mathcal{G}), A_{\mathcal{C}_c(\mathcal{G})}\big)$ is an internal category in **Top**. The result then follows by extending continuously to $\big(\mathcal{C}_r^*(\mathcal{G}), A_{\mathcal{C}_c(\mathcal{G})}\big)$ and $\big(\mathcal{C}^*(\mathcal{G}), A_{\mathcal{C}_c(\mathcal{G})}\big)$. □

Proposition 5.1.12. *The assignments* $\mathcal{G} \longmapsto \mathcal{C}_r^*(\mathcal{G})$ *and* $\mathcal{G} \longmapsto \mathcal{C}^*(\mathcal{G})$ *are functorial with respect to the class of continuous functors between groupoids generated by all embeddings and all quotient functors with compact kernels.*

Proof. Let $F\colon \mathcal{G} \longrightarrow \mathcal{G}'$ belong to the stated class of continuous functors. By Propositions 5.1.4 and 5.1.5, there exists a functorially induced continuous *-functor $\widetilde{F}\colon \mathcal{C}_r^*(\mathcal{G}) \longrightarrow \mathcal{C}^*(\mathcal{G}')$. The result follows by extending continuously to the completions. □

§5.2 Tensor Products

Let (H, p, X) and (K, q, Y) be continuous **Hilb**-bundles, and let $(\mathsf{H} \otimes \mathsf{K}, (p,q), X \times Y)$ be the associated tensor product bundle. Let $\mathcal{C}$ and $\mathcal{D}$ be continuous topological C*-categories and construct a $\mathbb{C}$-linear *-category $\mathcal{C} \odot \mathcal{D}$ with space of objects $\mathcal{C}_0 \times \mathcal{D}_0$ and for each $x, x' \in \mathcal{C}_0$ and $y, y' \in \mathcal{D}_0$ a hom-object

$$\big(\mathcal{C} \odot \mathcal{D}\big)((x,y),(x',y')) := \mathcal{C}(x,x') \odot \mathcal{D}(y,y')$$

where $\odot$ denotes the algebraic tensor product of the underlying vector spaces. Define composition on elementary tensors by

$$(c_2 \otimes d_2) \circ (c_1 \otimes d_1) := c_2 c_1 \otimes d_2 d_1 \tag{5.3}$$

and involution by

$$(c \otimes d)^* := c^* \otimes d^* \tag{5.4}$$

and extend these by linearity. Define $A_{\mathcal{C}\odot\mathcal{D}}$ to be the set of vector fields

$$\big((x,x'),(y,y')\big) \longmapsto \sum_{i=1}^{n} \alpha_i(x,x') \otimes \beta_i(y,y')$$

where $\alpha_i \in A_{\mathcal{C}}$ and $\beta_i \in A_{\mathcal{D}}$ for each index i.

Now suppose that $\mathcal{C}$ and $\mathcal{D}$ have faithful strongly continuous representations $\rho\colon \mathcal{C} \longrightarrow \boldsymbol{\mathcal{B}}(\mathsf{H}_{\mathcal{C}})$ and $\tau\colon \mathcal{D} \longrightarrow \boldsymbol{\mathcal{B}}(\mathsf{H}_{\mathcal{D}})$ as in Proposition 4.2.8. Since $\mathcal{C}$ and $\mathcal{D}$ are both continuous such representations always exist.

Lemma 5.2.1. *Let $x, x' \in \mathcal{C}_0$ and $y, y' \in \mathcal{D}_0$. Then there exists a natural inclusion*

$$\mathcal{C}(x,x') \odot \mathcal{D}(y,y') \hookrightarrow \mathbf{Hilb}(\rho(x)\otimes\tau(y), \rho(x')\otimes\tau(y')),$$

where $\odot$ denotes the algebraic tensor product of the underlying vector spaces and $\otimes$ is the usual tensor product of Hilbert spaces.

Proof. We define an injection

$$f_1\colon\ \mathcal{C}(x,x') \odot \mathcal{D}(y,y') \longrightarrow \mathbf{Hilb}(\rho(x),\rho(x')) \odot \mathbf{Hilb}(\tau(y),\tau(y'))$$

by

$$f_1\left(\sum_{i=1}^{n} c_i \otimes d_i\right) = \sum_{i=1}^{n} \rho(c_i)\otimes\tau(d_i)$$

and an injection

$$f_2\colon\ \mathbf{Hilb}\big(\rho(x),\rho(x')\big) \odot \mathbf{Hilb}\big(\tau(y),\tau(y')\big) \longrightarrow \mathbf{Hilb}\big(\rho(x)\otimes\tau(y),\rho(x')\otimes\tau(y')\big)$$

by

$$f_2\left(\sum_{i=1}^{n} (\rho(c_i)\otimes\tau(d_i))(h\otimes k)\right) = \sum_{i=1}^{n} \rho(c_i)(h)\otimes\tau(d_i)(k).$$

Set $f := f_2 \circ f_1$ and extend by linearity. □

We define $\mathcal{C}(x,x') \otimes_{min} \mathcal{D}(y,y')$ to be the closure of $\mathcal{C}(x,x') \odot \mathcal{D}(y,y')$ in the space $\mathbf{Hilb}(\rho(x)\otimes\tau(y), \rho(x')\otimes\tau(y'))$.

Definition 5.2.2. Let $\mathcal{C}$ and $\mathcal{D}$ be continuous C*-categories. Define the *minimal tensor product* of $\mathcal{C}$ and $\mathcal{D}$ to be the topological C*-category $\mathcal{C}\otimes_{min}\mathcal{D}$ with space of objects $\mathcal{C}_0 \times \mathcal{D}_0$ and for every $x, x' \in \mathcal{C}_0$ and $y, y' \in \mathcal{D}_0$ a hom-object

$$\big(\mathcal{C}\otimes\mathcal{D}\big)((x,y),(x',y')) := \mathcal{C}(x,x') \otimes_{min} \mathcal{D}(y,y')$$

defined as above. Composition and involution are defined as the continuous extensions of the corresponding operations in (5.3).

We call a continuous topological C^*-category $\mathcal{C}$ *exact* if for every short exact sequence

$$0 \longrightarrow \mathcal{A} \xrightarrow{F} \mathcal{A}' \xrightarrow{G} \mathcal{A}'' \longrightarrow 0$$

of continuous topological C^*-categories and all $x, x' \in \mathcal{C}_0$ and $w, w' \in \mathcal{A}_0$ we have a short-exact sequence:

$$\begin{array}{ccc} 0 \longrightarrow \mathcal{C}(x,x') \otimes \mathcal{A}(w,w') & \longrightarrow & \mathcal{C}(x,x') \otimes \mathcal{A}'\big(F(w), F(w')\big) \\ & & \downarrow \\ & & \mathcal{C}(x,x') \otimes \mathcal{A}''\Big(G\big(F(w)\big), G\big(F(w')\big)\Big) \longrightarrow 0 \end{array}$$

In particular, if $\mathcal{C}$ is exact then so is every endomorphism hom-object $\mathcal{C}(x,x)$. We recall the following result from [37]:

Proposition 5.2.3. *Let* (A, p, X) *be a continuous* $\mathbf{C}^*$-$\mathbf{Alg}$*-bundle, and let* B *be a* C^**-algebra, thought of as a trivial* $\mathbf{C}^*$-$\mathbf{Alg}$*-bundle* $(B, q, \{pt\})$*. Then*

1. *The minimal tensor product* $\mathsf{A} \otimes_{min} B$ *is lower semicontinuous. It is continuous if and only if* B *is exact.*

2. *The maximal tensor product* $\mathsf{A} \otimes_{max} B$ *is upper semicontinuous. It is continuous if and only if* B *is nuclear.*

Write $\mathsf{E}_{\mathcal{C}}$ and $\mathsf{E}_{\mathcal{D}}$ for the endomorphism bundles of $\mathcal{C}$ and $\mathcal{D}$ respectively. It follows from Proposition 5.2.3 that if $\mathcal{D}$ is an exact continuous topological C^*-category then the endomorphism bundle $\mathsf{E}_{\mathcal{C}} \otimes \mathsf{E}_{\mathcal{D}}$ is continuous.

Proposition 5.2.4. *Let* $\mathcal{C}, \mathcal{D}$ *be continuous topological* C^**-categories such that* $\mathcal{D}$ *is exact. Let* $\alpha_1, \ldots, \alpha_n \in A_{\mathcal{C}}$ *and* $\beta_1, \ldots, \beta_n \in A_{\mathcal{D}}$*. Then the map*

$$\big((x,x'),(y,y')\big) \longmapsto \left\| \sum_{i=1}^{n} \alpha_i(x,x') \otimes \beta_i(y,y') \right\|$$

is continuous.

Proof. For each $\alpha_i \in A_{\mathcal{C}}$ and $\beta_i \in A_{\mathcal{D}}$ the maps

$$(x,x) \longmapsto \alpha_i(x,x)^*\alpha_i(x,x)$$
$$(y,y) \longmapsto \beta_i(y,y)^*\beta_i(y,y)$$

are continuous. By assumption the categories $\mathcal{C}, \mathcal{D}$ are continuous, and hence by Lemma 3.2.6 the endomorphism bundles $\mathsf{E}_{\mathcal{C}}$ and $\mathsf{E}_{\mathcal{D}}$ are continuous $\mathbf{C^*}$-$\mathbf{Alg}$-bundles over $\mathcal{C}_0$ and $\mathcal{D}_0$ respectively. Moreover, since $\mathcal{D}$ is exact it follows that for each $y \in \mathcal{D}_0$ the C^*-algebra $\mathcal{D}(y,y)$ is exact, hence by Proposition 5.2.3 the tensor product $\mathbf{C^*}$-$\mathbf{Alg}$-bundle $\mathsf{E}_{\mathcal{C}} \otimes \mathsf{E}_{\mathcal{D}}$ is continuous. The map

$$\big((x,x),(y,y)\big) \longmapsto \left\| \sum_{i=1}^{n} \alpha_i(x,x) \otimes \beta_i(y,y) \right\|$$

is thus continuous. By axiom (ii) of Definition 3.2.2 there exists some collection $\widetilde{\alpha}_i \in A_{\mathcal{C}}$ and some collection $\widetilde{\beta}_i \in A_{\mathcal{D}}$ such that

$$\widetilde{\alpha}_i(x,x)^*\widetilde{\alpha}_i(x,x) = \alpha_i(x,x')^*\alpha_i(x,x')$$

and

$$\widetilde{\beta}_i(y,y)^*\widetilde{\beta}_i(y,y) = \beta_i(y,y')^*\beta_i(y,y')$$

and hence the map

$$\begin{aligned}\big((x,x'),(y,y')\big) \longmapsto \quad & \left\| \sum_{i=1}^{n} \alpha_i(x,x') \otimes \beta_i(y,y') \right\| \\ = & \left\| \sum_{i=1}^{n} \alpha_i(x,x')^*\alpha(x,x') \otimes \beta_i(y,y')^*\beta_i(y,y') \right\|^{1/2} \\ = & \left\| \widetilde{\alpha}_i(x,x)^*\widetilde{\alpha}_i(x,x) \otimes \widetilde{\beta}_i(y,y)^*\widetilde{\beta}_i(y,y) \right\|^{1/2}\end{aligned}$$

is continuous. □

Corollary 5.2.5. *The category $\mathcal{C} \otimes_{min} \mathcal{D}$ is a lower semicontinuous topological C^*-category. If $\mathcal{D}$ is exact then it is a continuous topological C^*-category.*

Proof. Since $\mathcal{D}$ is exact it follows from Proposition 5.2.3 that the $\mathbf{C^*}$-$\mathbf{Alg}$-bundle $\mathsf{H}_{\mathcal{C}} \otimes_{min} \mathsf{H}_{\mathcal{D}}$ is continuous, hence by Lemma 3.2.6 the tensor product $\mathcal{C} \otimes_{min} \mathcal{D}$ is a continuous topological C^*-category. □

We now construct the maximal tensor product. Let $\mathcal{A}, \mathcal{C}$ and $\mathcal{D}$ be continuous topological C*-categories, and let $[\mathcal{C} \odot \mathcal{D}, \mathcal{A}]$ be the set of continuous *-functors $\mathcal{C} \odot \mathcal{D} \longrightarrow \mathcal{A}$. Consider the norm defined by

$$\|f\|_{max} := \sup \{ \rho(f) \mid \rho \in [\mathcal{C} \odot \mathcal{D}, \mathcal{A}],\ \mathcal{A} \in \mathrm{C}^*\text{-}\mathbf{Cat}(\mathbf{Top}) \} .$$

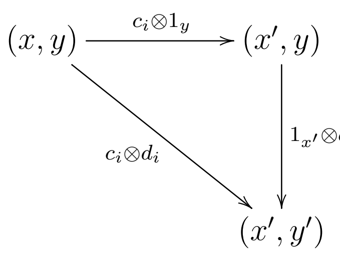
where $f \in \big(\mathcal{C} \odot \mathcal{D}\big)((x, y), (x', y'))$ and $\mathcal{A}$ varies across all continuous topological C*-categories. We call this the *supremum norm on* $\mathcal{C} \odot \mathcal{D}$.

Lemma 5.2.6. *The supremum norm on* $\mathcal{C} \odot \mathcal{D}$ *is finite.*

Proof. First assume that $\mathcal{C}$ and $\mathcal{D}$ are unital, and let $x \in \mathcal{C}_0$ and $y \in \mathcal{D}_0$. Define continuous *-functors $J_x\colon \mathcal{D} \longrightarrow \mathcal{C} \odot \mathcal{D}$ and $I_y\colon \mathcal{C} \longrightarrow \mathcal{C} \odot \mathcal{D}$ on objects by the continuous inclusions $\mathcal{C}_0 \hookrightarrow \mathcal{C}_0 \times \mathcal{D}_0$ and $\mathcal{D}_0 \hookrightarrow \mathcal{C}_0 \times \mathcal{D}_0$ and on morphisms by the assignments

$$\begin{aligned} I_y\colon \Big[c\colon x \longrightarrow x'\Big] &\longmapsto \Big[(c \otimes 1_y)\colon (x, y) \longrightarrow (x', y)\Big] \\ J_x\colon \Big[d\colon y \longrightarrow y'\Big] &\longmapsto \Big[(1_x \otimes d)\colon (x, y) \longrightarrow (x, y')\Big]. \end{aligned}$$

Let $\mathcal{A}$ be a unital topological C*-category and let $F\colon \mathcal{C} \odot \mathcal{D} \longrightarrow \mathcal{A}$ be any continuous *-functor into $\mathcal{A}$. Then $F \circ I_y$ and $F \circ J_x$ are continuous *-functors and are hence norm-decreasing. Furthermore, we know that



commutes in $\mathcal{C} \odot \mathcal{D}$ and therefore for every $f = \sum_{i=1}^{n} \alpha_i\, c_i \otimes d_i$ in $\big(\mathcal{C} \odot \mathcal{D}\big)((x, y), (x', y'))$ we have

$$\begin{aligned} \left\| F\Big(\sum_{i=1}^{n} \alpha_i\, c_i \otimes d_i \Big) \right\| &\leq \sum_{i=1}^{n} |\alpha_i|\, \|F(c_i \otimes d_i)\| \\ &= \sum_{i=1}^{n} |\alpha_i|\, \|F(1_{x'} \otimes d_i \circ c_i \otimes 1_y)\| \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} |\alpha_i| \, \|F(1_{x'} \otimes d_i) \circ F(c_i \otimes 1_y)\| \\
&\leq \sum_{i=1}^{n} |\alpha_i| \, \|F \circ J_{x'}(d_i)\| \, . \, \|F \circ I_y(c_i)\| \\
&\leq \sum_{i=1}^{n} |\alpha_i| \, \|d_i\| \, . \, \|c_i\|
\end{aligned}
$$

independently of F. Therefore $\|f\|_{max}$ is finite. Now if $\mathcal{C}$ and $\mathcal{D}$ are non-unital then there exist isometric continuous C*-functors $U_{\mathcal{C}} \colon \mathcal{C} \longrightarrow \mathcal{C}^+$ and $U_{\mathcal{D}} \colon \mathcal{D} \longrightarrow \mathcal{D}^+$ into unital topological C*-categories. The result then follows by composition of $U_{\mathcal{C}}$ and $U_{\mathcal{D}}$ with F above. □

Definition 5.2.7. If $\mathcal{C}$ and $\mathcal{D}$ are topological C*-categories then we define the *maximal tensor product* of $\mathcal{C}$ and $\mathcal{D}$ to be the topological C*-category $(\mathcal{C} \otimes_{max} \mathcal{D}, A_{\mathcal{C}\odot\mathcal{D}})$ with space of objects $\mathcal{C}_0 \times \mathcal{D}_0$, and for each $x, x' \in \mathcal{C}_0$ and $y, y' \in \mathcal{D}_0$ a hom-object

$$\big(\mathcal{C} \otimes_{max} \mathcal{D}\big)((x, y), (x', y'))$$

defined as the completion of $\big(\mathcal{C} \odot \mathcal{D}\big)((x, y), (x', y'))$ with respect to the supremum norm.

Consider the comparison functor $C \colon \mathcal{C} \otimes_{max} \mathcal{D} \longrightarrow \mathcal{C} \otimes_{min} \mathcal{D}$. A topological C*-category $\mathcal{C}$ is called *nuclear* if C is a continuous isomorphism for every topological C*-category $\mathcal{D}$.

Proposition 5.2.8. *The category $\mathcal{C} \otimes_{max} \mathcal{D}$ is a continuous topological C*-category if and only if $\mathcal{D}$ is nuclear.*

Proof. If $\mathcal{D}$ is nuclear then for every $y \in \mathcal{D}_0$ there exists an isomorphism of C*-algebras

$$B \otimes_{max} \mathcal{D}(y, y) \cong B \otimes_{min} \mathcal{D}(y, y)$$

for every C*-algebra B. By Proposition 5.2.3, the endomorphism bundle $\mathsf{H}_{\mathcal{D}}$ is therefore continuous. Hence, by Lemma 3.2.6, the C*-category $\mathcal{C} \otimes_{max} \mathcal{D}$ is continuous. The reverse implication is proved similarly. □

§5.3 Hilbert Modules over Topological C*-Categories

Let $\mathcal{C}$ be a topological C*-category. We define a (non-internal) functor from $\mathcal{C}$ into the category **Ban** as a $\mathbb{C}$-linear functor $E \colon \mathcal{C} \longrightarrow \mathbf{Ban}$ such that the family

$\{E(x)\}_{x\in\mathcal{C}_0}$ forms the fibres of a **Ban**-bundle $(\mathsf{E}, p, \mathcal{C}_0)$.

Definition 5.3.1. A *topological* **Hilb**$_{\mathcal{C}}$*-module* is a functor $E\colon \mathcal{C}^{op} \longrightarrow$ **Ban** together with a $\mathcal{C}_0 \times \mathcal{C}_0$-indexed family of sesquilinear forms

$$\langle\ ,\ \rangle_{x,y}\colon E(x) \times E(y) \longrightarrow \mathcal{C}(y,x)$$

such that for all objects $x, y, z \in \mathcal{C}_0$ and morphisms $a, b \in \mathcal{C}(x,y)$,

1. $\langle u, v\rangle^*_{x,y} = \langle v, u\rangle_{y,x}$ for all $u \in E(x)$ and $v \in E(y)$,
2. $\langle w, E(b)v\rangle_{z,x} = \langle w, v\rangle_{z,y} \circ b$ for all $v \in E(y)$ and $w \in E(z)$,
3. $\langle u, u\rangle_{x,x} \geq 0$ in $\mathcal{C}(x,x)$ for all $u \in E(x)$ and $\langle u, u\rangle_{x,x}$ if and only if $u = 0$

and such that each $E(x)$ is complete with respect to the norm given by $\|u\|_x := \|\langle u, u\rangle_{x,x}\|^{1/2}$.

If E is a **Hilb**$_{\mathcal{C}}$-module and $b \in \mathcal{C}(x,y)$ then $E(b)$ is a linear map $E(y) \longrightarrow E(x)$, $u \mapsto E(b)u$. We denote by $u.b$ the action of $E(b)$ on u. The family of sesquilinear forms induce a left pre-**Hilb**$_{A_{\mathcal{C}}}$-structure on $\Gamma_b(\mathsf{E})$ with action

$$A_{\mathcal{C}} \times \Gamma_b(\mathsf{E}) \longrightarrow \Gamma_b(\mathsf{E}),\ (\alpha, \eta) \longmapsto \eta \cdot \alpha$$

where $\eta \cdot \alpha(x) = E(\alpha(x,x))\eta(x)$, and $A_{\mathcal{C}}$-valued inner-product

$$\langle\ ,\ \rangle\colon \Gamma_b(\mathsf{E}) \times \Gamma_b(\mathsf{E}) \longrightarrow A_{\mathcal{C}},\ (\eta, \xi) \longmapsto \langle \eta, \xi\rangle$$

where $\langle \eta, \xi\rangle(x,y) = \langle \eta(x), \xi(y)\rangle_{x,y} \in \mathcal{C}(y,x)$.

Definition 5.3.2. Let $E, F\colon \mathcal{C}^{op} \longrightarrow$ **Ban** be topological **Hilb**$_{\mathcal{C}}$-modules. A **Hilb**$_{\mathcal{C}}$-*operator* is a natural transformation $T\colon E \Longrightarrow F$ for which there exists a natural transformation $T^*\colon F \Longrightarrow E$ such that

$$\langle T\eta, \xi\rangle(x,y) = \langle \eta, T^*\xi\rangle(x,y)$$

for all $x, y \in \mathcal{C}_0$ and all $\eta, \xi \in \Gamma_b(\mathsf{E})$.

For each $x \in \mathcal{C}_0$, the components of T and T^* at x are bounded linear maps $T_x\colon E(x) \longrightarrow F(x)$ and $T^*_x\colon F(x) \longrightarrow E(x)$ respectively. By applying the Riesz

Representation Theorem to each component map it follows that the natural transformation T^* is uniquely determined by T. We call T^* the *adjoint transformation*. A $\mathbf{Hilb}_{\mathcal{C}}$-operator $T\colon E \Longrightarrow F$ is called *bounded* if

$$\|T\| := \sup\left\{ \|T_x\|_{\mathfrak{B}(E,F)} \;\mid\; x \in \mathcal{C}_0 \right\} < \infty, \tag{5.5}$$

and we denote the set $\{T\colon E \Longrightarrow F\}$ of all bounded $\mathbf{Hilb}_{\mathcal{C}}$-operators by $\mathfrak{B}(E,F)$. This is a Banach space with respect to the norm (5.5) above, with addition $(T+S)_x = T_x + S_x$ and scalar multiplication $(\lambda T)_x = \lambda T_x$ defined on components T_x, S_x for each $x \in \mathcal{C}_0$.

Proposition 5.3.3. *If E and F are topological $\mathbf{Hilb}_{\mathcal{C}}$-modules such that $E = F$ then the set $\mathfrak{B}(E,E) = \mathfrak{B}(E)$ has the structure of a C^*-algebra.*

Proof. The set $\mathfrak{B}(E)$ is a Banach space with respect to the norm (5.5). For $T, S \in \mathfrak{B}(E)$ we define the product TS component-wise by

$$(TS)_x = T_x \circ S_x$$

for each $x \in \mathcal{C}_0$. Since $\|T_x \circ S_x\| \leq \|T_x\| \cdot \|S_x\|$ for each $x \in \mathcal{C}_0$ it follows that $\|TS\| \leq \|T\| \cdot \|S\|$. For each $T \in \mathfrak{B}(E)$ there is a natural adjoint T^*, so it remains to check that the C^*-identity holds. But

$$\begin{aligned} \|T\|^2 &= \sup\left\{ \|T_x\|^2 \;\mid\; x \in \mathcal{C}_0 \right\}, \\ &= \sup\left\{ \|T_x^* T_x\| \;\mid\; x \in \mathcal{C}_0 \right\}, \\ &= \sup\left\{ \|(T^*T)_x\| \;\mid\; x \in \mathcal{C}_0 \right\} = \|T^*T\| \end{aligned}$$

and therefore $\mathfrak{B}(E)$ is a C^*-algebra. □

Chapter 6

K-Theory for Topological C^*-Categories

In this final chapter, we define K-theory groups K_n for continuous topological C^*-categories. Our approach is to first construct a topological $\mathbf{Hilb}_{\mathcal{C}}$-module E and a faithful *-homomorphism $A_{\mathcal{C}} \hookrightarrow \mathfrak{B}(E)$, allowing us to take the closure of $A_{\mathcal{C}}$ in $\mathfrak{B}(E)$. By Proposition 5.3.3, this forms a C^*-algebra that we call the classifying algebra for $\mathcal{C}$. This classifying algebra is thought of as a noncommutative analogue of a classifying space. We then show that this construction is functorial with respect to a certain class of continuous *-functors (Proposition 6.1.8).

In the final section we define the K-theory group K_n of a continuous topological C^*-category to be the composition of the classifying algebra construction with the K-theory functor K_n for C^*-algebras (Definitions 6.2.1 and 6.2.7) and show that the defining properties of K-theory — namely stability, homotopy invariance and half-exactness — hold (Theorem 6.2.6). These properties follow easily from previous results on the classifying algebra. We conclude the thesis by proving Bott Periodicity for topological C^*-category K-theory (Theorem 6.2.9).

§6.1 A Classifying Algebra for K-Theory

Let $\mathcal{C}$ be a *continuous* topological C^*-category with $\mathcal{C}_0$ equipped with a Borel structure, and μ a positive Radon measure on $\mathcal{C}_0$. It follows from Proposition 3.2.3 that if $\alpha, \beta \in A_{\mathcal{C}}$ then $(\beta \star \alpha) \in A_{\mathcal{C}}$, and $A_{\mathcal{C}}$ is a topological *-algebra with respect to the inductive limit topology. For each $x \in \mathcal{C}_0$ we define $E(x)$ to be the topological direct integral

$$\int_{z \in \mathcal{C}_0}^{\oplus} \mathcal{C}(x, z)\, d\mu(z),$$

which is complete with respect to the canonical inner-product

$$\langle \alpha, \beta \rangle_{x,x} = \int_{z \in \mathcal{C}_0} \alpha(x,z)^* \beta(x,z)\, d\mu(z), \qquad \alpha, \beta \in A_{\mathcal{C}}.$$

To every morphism $b \in \mathcal{C}(x, y)$ we can associate the right-composition operator $R_b \colon E(y) \longrightarrow E(x)$ that is bounded in the L^2-norm. Therefore, the assignments $x \longmapsto E(x)$ and $b \longmapsto E(b)$, where $E(b) = R_b$, define a functor $E \colon \mathcal{C}^{op} \longrightarrow \mathbf{Ban}$.

Lemma 6.1.1. *For every $\alpha, \beta \in A_{\mathcal{C}}$ the map $x \longmapsto \langle \alpha, \beta \rangle(x)$ is continuous. Furthermore, there exists a continuous $\mathbf{Ban}$-bundle $(\mathsf{E}, \nu, \mathcal{C}_0)$ such that for each $\alpha \in A_{\mathcal{C}}$ the map $x \longmapsto \alpha(x, -)$ is a continuous section of $(\mathsf{E}, \nu, \mathcal{C}_0)$.*

Proof. The norm function $x \longmapsto \|\alpha(x, -)\|$ is defined as

$$\|\langle \alpha, \alpha \rangle_{x,x}\|^{1/2} = \left\| \int_{z \in \mathcal{C}_0} \alpha(x,z)^* \alpha(x,z)\, d\mu(z) \right\|_{\mathcal{C}}$$

and is the composition of α with the norm on $\mathcal{C}$. Both of these are continuous, and hence the norm function $x \longmapsto \|\alpha(x, -)\|$ is continuous. By the vector-valued Tietze Extension Theorem the set $\{ \alpha(x, -) \mid \alpha \in A_{\mathcal{C}} \}$ is dense in $E(x)$ for each $x \in \mathcal{C}_0$, from which the result follows. □

Proposition 6.1.2. *The functor $E \colon \mathcal{C}^{op} \longrightarrow \mathbf{Ban}$ defines a topological $\mathbf{Hilb}_{\mathcal{C}}$-module.*

Proof. For each pair of objects $x, y \in \mathcal{C}_0$ define a sesquilinear form

$$\langle -, - \rangle_{x,y} \colon E(x) \times E(y) \longrightarrow \mathcal{C}(y, x)$$

by the formula

$$\langle \alpha, \beta \rangle_{x,y} = \int_{w \in \mathcal{C}_0} \alpha(x,w)^* \beta(y,w)\, d\mu(w)$$

where $\alpha(x, -) \in E(x)$ and $\beta(y, -) \in E(y)$. Then for $\alpha \in E(x)$, $\beta \in E(y)$, $\eta \in E(z)$ and $b \in \mathcal{C}(x, y)$ we have

$$\begin{aligned} \langle \eta, E(b)\beta \rangle_{z,x} &= \int_{w \in \mathcal{C}_0} \eta(z,w)^* E(b)\beta(y,w)\, d\mu(w) \\ &= \int_{w \in \mathcal{C}_0} \eta(z,w)^* \beta(y,w) \circ b\, d\mu(w) \\ &= \langle \eta, \beta \rangle_{z,y} \circ b. \end{aligned}$$

Furthermore,

$$\begin{aligned}\langle\, \alpha, \beta \,\rangle_{x,y}^{*} &= \left(\int_{w\in\mathcal{C}_0} \alpha(x,w)^{*}\beta(y,w)\; d\mu(w)\right)^{*}\\ &= \int_{w\in\mathcal{C}_0} \beta(y,w)^{*}\left(\alpha(x,w)^{*}\right)^{*}\; d\mu(w)\\ &= \langle\, \beta, \alpha \,\rangle_{y,x}.\end{aligned}$$

Finally

$$\langle\, \alpha, \alpha \,\rangle_{x,x} = \int_{w\in\mathcal{C}_0} \alpha(x,w)^{*}\alpha(x,w)\; d\mu(w)$$

is a positive element in $\mathcal{C}(x,x)$, and this integral is 0 if and only if $\alpha(x,w) = 0$ for all $w \in \mathcal{C}_0$. The result then follows from Lemma 6.1.1. □

For each $x \in \mathcal{C}_0$, left-multiplication by elements of $A_{\mathcal{C}}$ induces a *-representation $\pi_x^l\colon A_{\mathcal{C}} \longrightarrow \mathbf{\mathcal{B}}(E)$ where $\pi_x^l(\alpha)$ is the map $\beta(x,-) \longmapsto (\alpha \star \beta)(x,-)$ for $\beta \in E(x)$.

Definition 6.1.3. Define $\mathbb{A}(\mathcal{C})$ to be the completion of $A_{\mathcal{C}}$ with respect to the norm defined by

$$\|\alpha\| := \sup\left\{ \left\|\pi_x^l(\alpha)\right\|_{op} \;\mid\; x \in \mathcal{C}_0 \right\},$$

where $\left\|\pi_x^l(\alpha)\right\|_{op}$ is the operator norm on $\mathbf{\mathcal{B}}(E(x))$. We call $\mathbb{A}(\mathcal{C})$ the *classifying algebra of* $\mathcal{C}$.

We observe that if $\mathcal{C}_0$ is non-compact then the C*-algebra $\mathbb{A}(\mathcal{C})$ is necessarily non-unital, since the unit map $x \longmapsto 1_x$ is supported on all of $\mathcal{C}_0$. In fact, the classifying algebra $\mathbb{A}(\mathcal{C})$ is unital if and only if *both* of the following criteria are satisfied:

1. $\mathcal{C}$ is a unital topological C*-category, and
2. $\mathcal{C}_0$ is compact.

In the remainder of this section we explore some of the functoriality and stability properties of the classifying algebra. We start with the classifying algebra of the tensor product of a continuous topological C*-category with a commutative C*-algebra:

Let $\mathcal{C}$ be a continuous topological C*-category and A a commutative C*-algebra. Form the tensor product $\mathcal{C} \otimes_{min} A$. Since A is commutative it is nuclear, and therefore $\mathcal{C} \otimes_{min} A$ is a continuous topological C*-category. It follows that the endomorphism bundle $\mathsf{E}_{\mathcal{C}} \otimes_{min} A$ is a continuous $\mathbf{C^*}$-$\mathbf{Alg}$-bundle. The following result is due to Kirchberg and Wassermann [37].

Proposition 6.1.4. *Let* A *be a continuous* $\mathbf{C^*}$-$\mathbf{Alg}$*-bundle and* B *a nuclear* C^**-algebra. Then* $\mathsf{A} \otimes_{min} B$ *is a continuous* $\mathbf{C^*}$-$\mathbf{Alg}$*-bundle and*

$$\Gamma_c(\mathsf{A} \otimes_{min} B) = \Gamma_c(\mathsf{A}) \otimes_{min} B.$$

From the preceding Proposition it follows that $\Gamma_c\,(\mathsf{E}_{\mathcal{C}} \otimes_{min} A) = \Gamma_c(\mathsf{E}_{\mathcal{C}}) \otimes_{min} A$, and hence $\Gamma_c(\mathsf{E}_{\mathcal{C}}) \otimes_{min} A$ is a C^*-subalgebra of $\mathbb{A}(\mathcal{C} \otimes_{min} A)$.

To proceed, we require a basic result concerning conditional expectations, the proof of which can be found in [55].

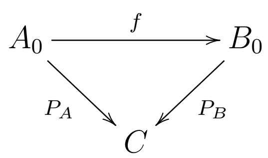
Definition 6.1.5. Let A be a C^*-algebra and $B \subseteq A$ a C^*-subalgebra. A conditional expectation is a surjective bounded linear map $P\colon A \longrightarrow B$ such that:

1. P is a projection with $\|P\| = 1$,

2. $P(a^*)P(a) < P(a^*a)$ and $P(a^*a) > 0$ for every $a \in A$,

3. $P(bac) = bP(a)c$ for every $a \in A$ and every $b, c \in B$.

Lemma 6.1.6. *Let* A *and* B *be* C^**-algebras and* C *a common* C^**-subalgebra. Let* A_0 *and* B_0 *be dense *-subalgebras of* A *and* B *respectively, and let* $f\colon A_0 \longrightarrow B_0$ *be a surjective *-homomorphism. Suppose there exist faithful conditional expectations* $P_A\colon A \longrightarrow C$ *and* $P_B\colon B \longrightarrow C$ *such that the following diagram commutes:*



Then f *extends to an isomorphism* $\tilde{f}\colon A \xrightarrow{\cong} B$.

Proposition 6.1.7. *Let* $\mathcal{C}$ *be a continuous topological* C^**-category, and* A *a commutative* C^**-algebra. Then* $\mathbb{A}(\mathcal{C} \otimes A) \cong \mathbb{A}(\mathcal{C}) \otimes A$.

Proof. Let $\Gamma_c(\mathsf{E}_{\mathcal{C}}) \otimes_{min} A$ be the common C^*-subalgebra of $\mathbb{A}(\mathcal{C} \otimes_{min} A)$ and $\mathbb{A}(\mathcal{C}) \otimes_{min} A$. Also let

$$P\colon \mathbb{A}(\mathcal{C} \otimes_{min} A) \longrightarrow \Gamma_c(\mathsf{E}_{\mathcal{C}})$$

be the faithful conditional expectation arising from the restriction of $\mathbb{A}(\mathcal{C} \otimes_{min} A)$ to $\Gamma_c(\mathsf{E}_{\mathcal{C}})$, and let

$$Q \otimes \mathrm{id} \colon \mathbb{A}(\mathcal{C}) \otimes_{min} A \longrightarrow \Gamma_c(\mathsf{E}_{\mathcal{C}})$$

be the faithful conditional expectation obtained from the restriction Q of $\mathbb{A}(\mathcal{C})$ to $\Gamma_c(\mathsf{E}_{\mathcal{C}})$. Then the following diagram commutes,

$$\begin{array}{ccc} A_{\mathcal{C}} \odot A & \xrightarrow{\qquad \mathrm{id} \qquad} & A_{\mathcal{C}} \odot A \\ & {}_{P}\searrow \qquad \swarrow_{Q} & \\ & \Gamma_c(\mathsf{E}_{\mathcal{C}}) \otimes_{min} A & \end{array}$$

and hence by Lemma 6.1.6 the identity map $A_{\mathcal{C}} \odot A \longrightarrow A_{\mathcal{C}} \odot A$ extends to an isomorphism $\mathbb{A}(\mathcal{C} \otimes_{min} A) \xrightarrow{\cong} \mathbb{A}(\mathcal{C}) \otimes_{min} A$. □

Let $\mathcal{C}, \mathcal{D}$ be continuous topological C*-categories, and let $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ be a continuous *-functor. We call F an *open inclusion* if the object component

$$F_0 \colon \mathcal{C}_0 \longrightarrow \mathcal{D}_0$$

is an injective, open continuous map. If $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ is an open inclusion then the image of F is the continuous topological C*-category $F(\mathcal{C})$ with object space

$$F(\mathcal{C})_0 = F_0\left(\mathcal{C}_0\right)$$

an open subset of $\mathcal{D}_0$, and hom-objects $F(\mathcal{C})(x, y) = \mathcal{D}(F(x), F(y))$. We identify $F(\mathcal{C})$ with the continuous topological C*-category with the same object space as $\mathcal{D}$ by declaring

$$F(\mathcal{C})(x, y) = \begin{cases} \mathcal{D}(x, y) & \text{if } x, y \in F_0(\mathcal{C}_0), \\ \{0\} & \text{otherwise,} \end{cases}$$

for $x, y \in \mathcal{D}_0$.

Proposition 6.1.8. *The assignment* $\mathcal{C} \longmapsto \mathbb{A}(\mathcal{C})$ *is functorial with respect to open inclusions.*

Proof. Let $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ be an open inclusion. Then F induces a map

$$\mathbb{A}(F) \colon A_{\mathcal{C}} \longrightarrow A_{\mathcal{D}}$$

by sending $\alpha \in A_{\mathcal{C}}$ to its image in $A_{F(\mathcal{C})}$ and extending by zero. For $x, y \in \mathcal{D}_0$ the continuous section $\mathbb{A}(F)(\alpha)$ is therefore given by the formula

$$\mathbb{A}(F)(\alpha)(x,y) = F_1 \alpha F_0^{-1}(x,y),$$

and since $\operatorname{supp}(\alpha) \subset \mathcal{C}_0 \times \mathcal{C}_0$ and F is open it follows that

$$\operatorname{supp}\big(\mathbb{A}(F)(\alpha)\big) \subset F(\mathcal{C})_0 \times F_0(\mathcal{C})_0 \subset \mathcal{D}_0 \times \mathcal{D}_0.$$

Therefore, the support of $\mathbb{A}(F)(\alpha)$ is compact in $\mathcal{D}_0 \times \mathcal{D}_0$, and $\mathbb{A}(F)(\alpha) \in A_{\mathcal{D}}$.

Now, suppose that $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ and $G\colon \mathcal{D} \longrightarrow \mathcal{E}$ are open inclusions. Then for $x, y \in \mathcal{E}_0$ we have

$$\begin{aligned}\big(\mathbb{A}(G) \circ \mathbb{A}(F)\big)(\alpha)(x,y) &= G_1\big(\mathbb{A}(F)(\alpha)\big)G_0^{-1}(x,y)\\ &= G_1\big(F_1 \alpha F_0^{-1}\big)G_0^{-1}(x,y)\\ &= G_1 F_1 \alpha G_0 F_0^{-1}(x,y) = \mathbb{A}(GF)(\alpha)(x,y)\end{aligned}$$

and therefore $\mathbb{A}$ is functorial with respect to open inclusions. □

Now let $\mathcal{C}$ be a continuous topological C^*-category and let $\mathcal{J} \trianglelefteq \mathcal{C}$ be an ideal of $\mathcal{C}$. We may form the quotient topological C^*-category $\mathcal{C}/\mathcal{J}$ with object space $\mathcal{C}_0$ and for each $x, y \in \mathcal{C}_0$ a hom-object

$$\mathcal{C}/\mathcal{J}(x,y) = \mathcal{C}(x,y) \big/ \mathcal{J}(x,y)\ .$$

Since $\mathcal{C}$ is a continuous C^*-category, so are $\mathcal{J}$ and $\mathcal{C}/\mathcal{J}$. Since $\mathcal{C}_0 = \mathcal{J}_0 = (\mathcal{C}/\mathcal{J})_0$, the canonical inclusion and quotient functors $i\colon \mathcal{J} \longrightarrow \mathcal{C}$ and $q\colon \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{J}$ respectively are open inclusions. We have a short-exact sequence of continuous topological C^*-categories

$$0 \longrightarrow \mathcal{J} \xrightarrow{\ i\ } \mathcal{C} \xrightarrow{\ q\ } \mathcal{C}/\mathcal{J} \longrightarrow 0\ .$$

By Proposition 6.1.8, there exists a functorially induced sequence of C^*-algebras

$$0 \longrightarrow \mathbb{A}(\mathcal{J}) \xrightarrow{\mathbb{A}(i)} \mathbb{A}(\mathcal{C}) \xrightarrow{\mathbb{A}(q)} \mathbb{A}(\mathcal{C}/\mathcal{J}) \longrightarrow 0\ .$$

Proposition 6.1.9. *For $\mathcal{C}$ a continuous topological C^*-category and $\mathcal{J} \trianglelefteq \mathcal{C}$ an ideal,*

$$0 \longrightarrow \mathbb{A}(\mathcal{J}) \xrightarrow{\mathbb{A}(i)} \mathbb{A}(\mathcal{C}) \xrightarrow{\mathbb{A}(q)} \mathbb{A}(\mathcal{C}/\mathcal{J}) \longrightarrow 0.$$

is a short-exact sequence of C^*-algebras.

Proof. Consider the following short-exact sequence of continuous topological C^*-categories,

$$0 \longrightarrow \mathcal{J} \xrightarrow{i} \mathcal{C} \xrightarrow{q} \mathcal{C}/\mathcal{J} \longrightarrow 0 .$$

There is a canonical injection $i_* : A_{\mathcal{J}} \longrightarrow A_{\mathcal{C}}$ from which we obtain

$$A_{\mathcal{J}} \xrightarrow{i_*} A_{\mathcal{C}} \longrightarrow \mathfrak{B}(E) ,$$

where E is the topological $\mathbf{Hilb}_{\mathcal{C}}$-module defined in Section 5.3. This embeds $A_{\mathcal{J}}$ into $\mathfrak{B}(E)$ as a *-subalgebra. Hence, by taking the closure of $A_{\mathcal{J}}$ in $\mathfrak{B}(E)$ we obtain an injective *-homomorphism $\mathbb{A}(i) : \mathbb{A}(\mathcal{J}) \longrightarrow \mathbb{A}(\mathcal{C})$. We also have a canonical surjection $q_* : A_{\mathcal{C}} \longrightarrow A_{\mathcal{C}/\mathcal{J}}$. The set $\{ q(\alpha) \mid \alpha \in A_{\mathcal{C}} \}$ is a dense subspace of $A_{\mathcal{C}/\mathcal{J}}$ in the inductive limit topology, and hence $q_* (A_{\mathcal{C}})$ is dense in $\mathbb{A}(\mathcal{C}/\mathcal{J})$. Therefore, q_* extends continuously to a surjective *-homomorphism $\mathbb{A}(q) : \mathbb{A}(\mathcal{C}) \longrightarrow \mathbb{A}(\mathcal{C}/\mathcal{J})$.

To prove exactness in the middle, observe that $\mathbb{A}(\mathcal{J})$ is a C^*-ideal of $\mathbb{A}(\mathcal{C})$, and therefore we have a short-exact sequence of C^*-algebras as follows,

$$0 \longrightarrow \mathbb{A}(\mathcal{J}) \xrightarrow{\mathbb{A}(i)} \mathbb{A}(\mathcal{C}) \xrightarrow{\widetilde{q}} \mathbb{A}(\mathcal{C}) \big/ \mathbb{A}(\mathcal{J}) \longrightarrow 0 .$$

Therefore $\ker \widetilde{q} = \operatorname{Im} \mathbb{A}(i)$. In particular, $\ker \widetilde{q} \subseteq \operatorname{Im} \mathbb{A}(i)$. Now, let $\alpha \in \ker \mathbb{A}(q)$. Then $\alpha(x, y) \in \mathcal{J}$ for all $x, y \in \mathcal{C}_0$. Therefore $\alpha = 0$ in $\mathbb{A}(\mathcal{C})/\mathbb{A}(\mathcal{J})$, and hence $\alpha \in \ker \widetilde{q}$. We therefore have

$$\ker \mathbb{A}(q) \;\subseteq\; \ker \widetilde{q} \;\subseteq\; \operatorname{Im} \mathbb{A}(i).$$

Conversely, if $\alpha \in \mathbb{A}(\mathcal{J})$ then $\alpha(x, y) \in \mathcal{J}$ for all $x, y \in \mathcal{C}_0$. Therefore $q(\alpha)(x, y) = 0$ for all $x, y \in \mathcal{C}_0$, and hence $\alpha \in \ker \mathbb{A}(q)$. We thus have $\operatorname{Im} \mathbb{A}(i) \subseteq \ker \mathbb{A}(q)$, and hence $\ker \mathbb{A}(q) = \operatorname{Im} \mathbb{A}(i)$. Therefore, the sequence

$$0 \longrightarrow \mathbb{A}(\mathcal{J}) \xrightarrow{\mathbb{A}(i)} \mathbb{A}(\mathcal{C}) \xrightarrow{\mathbb{A}(q)} \mathbb{A}(\mathcal{C}/\mathcal{J}) \longrightarrow 0$$

is exact. □

§6.2 K-Theory and Properties

Let $\mathcal{C}$ be a continuous topological C^*-category, and $\mathbb{A}(\mathcal{C})$ its classifying algebra as defined in Section 6.1.

Definition 6.2.1. Define the Abelian group $K_0(\mathcal{C})$ by

$$K_0(\mathcal{C}) := K_0(\mathbb{A}(\mathcal{C})),$$

where $K_0(\mathbb{A}(\mathcal{C}))$ is the C*-algebra K-theory group of $\mathbb{A}(\mathcal{C})$.

Let X be a locally compact, Hausdorff topological space, equipped with a Borel structure, together with a positive Radon measure μ. We treat X as a (trivial) topological groupoid $\mathcal{G} := X \rightrightarrows X$ and form its associated continuous topological C*-category $\mathcal{C}^*(X)$ as in Section 5.1. The category $\mathcal{C}^*(X)$ has object space X and for each $x, y \in X$ a hom-object

$$\mathcal{C}^*(X)(x,y) = \begin{cases} \mathbb{C} & \text{if } x = y, \\ \{0\} & \text{if } x \neq y. \end{cases}$$

Proposition 6.2.2. *Let X be a locally compact Hausdorff space as above. Then*

$$K_0(\mathbb{A}(X)) = K_0^{top}(X),$$

where $K_0^{top}(X)$ is the topological K-theory of X.

Proof. By construction, $A_{\mathcal{C}^*(X)}$ can be identified with the set of compactly supported continuous functions $C_c(X)$. Consider the topological $\mathbf{Hilb}_{\mathcal{C}^*(X)}$-module E. Then for each $x \in X$ we have $E(x) = \mathbb{C}$ and therefore the family $\{E(x)\}$ forms the fibres of a trivial **C*-Alg**-bundle over X. It follows that $\mathfrak{B}(E) \cong C_0(X)$. We thus have an injective *-homomorphism $C_c(X) \longrightarrow C_0(X)$, and from the uniqueness of C*-norms it follows that $\mathbb{A}(\mathcal{C}^*(X)) \cong C_0(X)$. Therefore,

$$K_0(\mathcal{C}^*(X)) \cong K_0(C_0(X)) \cong K_0^{top}(X).$$

□

Now consider the case of a topological C*-category $\mathcal{C}$ with a single object, $\mathcal{C}_0 = \{pt\}$. We denote the C*-algebra $\mathcal{C}(pt, pt)$ by A.

Proposition 6.2.3. *Let A be a C*-algebra, thought of as a topological C*-category as above. Then $K_0(\mathcal{C})$ is the C*-algebra K-theory group of A.*

Proof. A compactly supported section $pt \times pt \longrightarrow \mathcal{C}(x,x)$ is a choice of element $a \in A$. Furthermore, any such function is continuous. For the $\mathbf{Hilb}_{\mathcal{C}}$-module E we have $E(pt) = A$, and so we have an isometry

$$A \cong A_{\mathcal{C}} \longrightarrow \boldsymbol{\mathcal{B}}(E) \cong \boldsymbol{\mathcal{B}}(A)$$

given by sending $a \in A$ to the left-multiplication operator L_a. Therefore, $\mathbb{A}(A)$ is isomorphic to the C^*-algebra A, and hence

$$K_0(\mathcal{C}) = K_0(\mathbb{A}(\mathcal{C})) = K_0(A).$$

□

The K-theory of C^*-algebras is an invariant with respect to C^*-algebra homotopies. To fully characterise the K-theory of continuous topological C^*-categories we require a corresponding notion of homotopy between continuous *-functors.

Definition 6.2.4. Let $\mathcal{C}$ and $\mathcal{D}$ be continuous topological C^*-categories and let $F, G \colon \mathcal{C} \longrightarrow \mathcal{D}$ be continuous open inclusions such that $F_0(x) = G_0(x)$ for all $x \in \mathcal{C}_0$. A *homotopy* between F and G is a continuous *-functor

$$H \colon \mathcal{C} \longrightarrow \mathcal{D} \otimes C([0,1])$$

such that $\mathrm{ev}_0 \circ H = F$ and $\mathrm{ev}_1 \circ H = G$. We call F and G *homotopic* if there exists a homotopy from F to G and write $F \simeq G$.

Lemma 6.2.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be continuous topological C^*-categories, and let $F, G \colon \mathcal{C} \longrightarrow \mathcal{D}$ be continuous open inclusions. Suppose that there exists a homotopy*

$$H \colon \mathcal{C} \longrightarrow \mathcal{D} \otimes C([0,1])$$

from F to G. Then H induces a homotopy $\mathbb{A}(H) \colon \mathbb{A}(F) \Longrightarrow \mathbb{A}(G)$ on the classifying algebras of $\mathcal{C}$ and $\mathcal{D}$.

Proof. Since $H_0 = F_0 = G_0$ and F, G are both open inclusions it follows that H must also be an open inclusion. By Proposition 6.1.8, it follows that there exists a functorially induced *-homomorphism

$$\mathbb{A}(H) \colon \mathbb{A}(\mathcal{C}) \longrightarrow \mathbb{A}(\mathcal{D} \otimes C([0,1])),$$

which is isomorphic to $\mathbb{A}(\mathcal{D}) \otimes C([0,1])$. Furthermore, for each $t \in [0,1]$ we have

$$\mathbb{A}(\mathrm{ev}_t \circ H) = \mathrm{ev}_t \circ \mathbb{A}(H)$$

and so $\mathrm{ev}_0 \circ \mathbb{A}(H) = \mathbb{A}(F)$ and $\mathrm{ev}_1 \circ \mathbb{A}(H) = \mathbb{A}(G)$, making $\mathbb{A}(H)$ a homotopy of C^*-algebras. □

Theorem 6.2.6. *The K-theory group $K_0(\mathcal{C})$ defines a functor K_0 from the category of continuous topological C^*-categories with open inclusions as morphisms to the category of Abelian groups, such that the following properties are satisfied:*

1. (Stability): *If K is the C^*-algebra of compact operators and $\mathcal{C} \otimes K$ is the tensor product topological C^*-category then there exists an isomorphism*

$$K_0(\mathcal{C}) \cong K_0(\mathcal{C} \otimes K);$$

2. (Homotopy invariance): *If $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ are homotopic open inclusions then the induced maps on the K-theory groups,*

$$K_0(F) : K_0(\mathcal{C}) \longrightarrow K_0(\mathcal{D}) \quad \textit{and} \quad K_0(G) : K_0(\mathcal{C}) \longrightarrow K_0(\mathcal{D}),$$

are equal;

3. (Half-exactness): *If $\mathcal{J} \trianglelefteq \mathcal{C}$ is an ideal of $\mathcal{C}$ then the short-exact-sequence*

$$0 \longrightarrow \mathcal{J} \xrightarrow{\;i\;} \mathcal{C} \xrightarrow{\;q\;} \mathcal{C}/\mathcal{J} \longrightarrow 0$$

of topological C^*-categories lifts to an exact sequence

$$K_0(\mathcal{J}) \longrightarrow K_0(\mathcal{C}) \longrightarrow K_0(\mathcal{C}/\mathcal{J})$$

of Abelian groups.

Proof. By Proposition 6.1.8, we know that the assignment $\mathcal{C} \longmapsto \mathbb{A}(\mathcal{C})$ defines a functor from the category of continuous topological C^*-categories with open inclusions as morphisms to the category of C^*-algebras. This extends to a functor into the category of Abelian groups by composition with the K-theory functor $K : \mathbf{C^*\text{-}Alg} \longrightarrow \mathbf{Ab}$.

For stability we note that the algebra K of compact operators is nuclear and hence exact. We may therefore form the continuous topological C^*-category tensor product $\mathcal{C} \otimes K$. There exists an isomorphism $\mathbb{A}(\mathcal{C} \otimes K) \cong \mathbb{A}(\mathcal{C}) \otimes K$, and therefore

$$K_0(\mathcal{C} \otimes K) = K_0(\mathbb{A}(\mathcal{C}) \otimes K) \cong K_0(\mathbb{A}(\mathcal{C})) = K_0(\mathcal{C})$$

by the stability of C*-algebra K-theory.

Now, let $\mathcal{C}, \mathcal{D}$ be continuous topological C*-categories and suppose that we have homotopic open-inclusions $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$. Then by Lemma 6.2.5, the induced *-homomorphisms $\mathbb{A}(F)$ and $\mathbb{A}(G)$ are homotopic, and therefore by homotopy invariance of C*-algebra K-theory the group homomorphisms $K_0(F) = K_0\big(\mathbb{A}(F)\big)$ and $K_0(G) = K_0\big(\mathbb{A}(G)\big)$ are equal.

Finally, let $\mathcal{J} \trianglelefteq \mathcal{C}$ be an ideal of $\mathcal{C}$ and

$$0 \longrightarrow \mathcal{J} \xrightarrow{\ i\ } \mathcal{C} \xrightarrow{\ q\ } \mathcal{C}/\mathcal{J} \longrightarrow 0$$

its associated short-exact sequence. This lifts to a short-exact sequence

$$0 \longrightarrow \mathbb{A}(\mathcal{J}) \xrightarrow{\mathbb{A}(i)} \mathbb{A}(\mathcal{C}) \xrightarrow{\mathbb{A}(q)} \mathbb{A}(\mathcal{C}/\mathcal{J}) \longrightarrow 0$$

of C*-algebras and hence, by the half-exactness of C*-algebra K-theory, to an exact sequence

$$K_0(\mathcal{J}) \longrightarrow K_0(\mathcal{C}) \longrightarrow K_0(\mathcal{C}/\mathcal{J})$$

of Abelian groups. □

To complete our description of K-theory for continuous topological C*-categories, we define the higher K-theory groups $K_n(\mathcal{C})$ for $n \geq 1$. Let $\mathcal{C}$ be a continuous topological C*-category and for each $n \geq 1$ form the tensor product topological C*-category $\mathcal{C} \otimes C_0(\mathbb{R}^n)$.

Definition 6.2.7. For $n \geq 1$ define the higher K-theory groups

$$K_n(\mathcal{C}) := K_0(\mathcal{C} \otimes C_0(\mathbb{R}^n)).$$

Proposition 6.2.8. *If $\mathcal{C}$ is a continuous topological C*-category then there exists an isomorphism $K_n(\mathcal{C}) \cong K_n(\mathbb{A}(\mathcal{C}))$.*

Proof. It follows from Proposition 6.1.7 that for each $n \in \mathbb{N}$ there exists an isomorphism $\mathbb{A}(\mathcal{C} \otimes C_0(\mathbb{R}^n)) \cong \mathbb{A}(\mathcal{C}) \otimes C_0(\mathbb{R}^n)$. Therefore,

$$K_n(\mathcal{C}) = K_0(\mathcal{C} \otimes C_0(\mathbb{R}^n)) \cong K_0(\mathbb{A}(\mathcal{C})) \otimes C_0(\mathbb{R}^n) \cong K_n(\mathbb{A}(\mathcal{C}).$$

□

The K-theory of C*-algebras exhibits Bott Periodicity. To conclude this thesis we prove that the same result holds for the K-theory of continuous topological C^*-categories.

Theorem 6.2.9 (Bott Periodicity)**.** *For each $n \in \mathbb{N}$ there exists an isomorphism $K_n(\mathcal{C}) \cong K_{n+2}(\mathcal{C})$.*

Proof. It follows from Proposition 6.2.8 that for each $n \in \mathbb{N}$ there exists an isomorphism $K_n(\mathcal{C}) \cong K_n(\mathbb{A}(\mathcal{C}))$. Therefore,

$$K_n(\mathcal{C}) \cong K_n(\mathbb{A}(\mathcal{C})) \cong K_{n+2}(\mathbb{A}(\mathcal{C})) \cong K_{n+2}(\mathcal{C}),$$

where the middle isomorphism is given by the Bott map in the C^*-algebra K-theory of $\mathbb{A}(\mathcal{C})$. □

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