MONEY, TRANSACTIONS AND THE BUSINESS CYCLE: INSPECTING THE MECHANISM

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March 2011

Abstract

This thesis analyses and compares two monetary models - the cash-in-advance (CIA) model and the real-resource-cost (RRC) model - using a Dynamic Stochastic General Equilibrium (DSGE) framework, with perfectly competitive markets and flexible prices. The CIA model is built on the quantitative assessment developed by Cooley and Hansen (1989, 1995), while the RRC model is an implementation of the analytical work by Feenstra (1986) and Wang and Yip (1992) on transaction costs. In order to inspect the mechanisms implicit in the monetary models at hand, this thesis analyses also some extensions, building on the seminal contributions by Stockman (1981) and Abel (1985).

The main results emerging from the impulse-response functions are that the CIA and the RRC models respond in the same way to a shock in total factor productivity, while they differ in the propagation mechanism of the monetary shock, where the differences depend on the mix between cash- and credit-goods in the model economies. Instead, the impact of a transaction cost shock, in the case of the RRC model, remains weak.

When compared with the stylised facts characterising the U.S. business cycle data, the CIA and RRC approaches exhibit the 'dichotomy' typical of the standard RBC literature: the volatility of real expenditure and working hours (and the respective correlation with output) are essentially driven by the technology shock, while nominal variables are mainly affected by the monetary shock. However, when it comes to the correlations of the endogenous variables with money growth, the CIA and the RRC models fail along many dimensions, when only consumption is linked with money (Chapter 3). By contrast, when transaction technologies are extended to investment and the money supply process is modified (Chapters 4 and 5), the empirical performance of the extended CIA model is superior with respect to the extended RRC model.

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Acknowledgements

I would like to thank my supervisors, Prof. Peter N. Smith and Dr. Gulcin F. Ozkan, for their advice and guidance during these 'intense' years. I would also like to express my gratitude to my advisor, Prof. Neil Rankin, for the stimulating conversations and his endless patience in front of my pressing questions.

For helpful comments and suggestions during my postgraduate studies, I would like to thank the following professors: Roberto Tamborini, Mike Wickens, Yves Balasko, Axel Leijonhufvud, Subir Chattopadhyay, David Lopez-Salido, Tjias van Rens, Jordi Gali, Lawrence Christiano and Nobuhiro Kiyotaki.

A special mention goes to Jo Hall, Emma Fairclough, Lynn Osborne, Kathy Crocker for their courtesy and kindness during my postgraduate studies at York.

Moreover, I would like to thank Dr. Catherine P. Winnet and Prof. Christopher Martin for their support during my Teaching Fellowship at the University of Bath.

For a true friendship I am immensely grateful to Amos, Gianluca, Jorge, Anna and Dionino, while I am particularly indebt with Paolo Gelain, Mai Farid and Carlo Reggiani for their concrete help and encouragement in the 'difficult' moments.

A special 'thanks' goes to Francesco Mariotti, whose peculiar humanity represented for me a real companionship, during all these years.

Finally, my deepest gratitude goes to my family: to my mum Rosanna, my dad Renato and my grandpa Francesco, who have always encouraged me to pursue the path I have started, walking together with me.

Author's declaration

This thesis is the result of the research I have produced as a Ph.D student at the Department of Economics and Related Studies (DERS) at the University of York (UK) from 2006 to 2011. In this period, my research activity was funded by the Foundation "Luigi Einaudi" (*Young Researchers*, 2006-2007), the special funds of the Research Group "*Monetary Policy in Imperfect Capital Markets*" (Department of Economics, Universita degli Studi di Trento, 2008), and the Economic and Social Research Council (ESRC, *Open Studentship*, 2008-2009).

An earlier version of Chapter 3 entitled "*Transaction costs and endogenous velocity of money*" has been presented at the *Macroeconomic Breakfast seminars* (14th May 2009) during my visiting period at the Universidad Pompeu Fabra (Barcelona, Spain).

I am the sole author of this work having developed the original idea, prepared the data, carried out the analytical and empirical analysis and written the thesis and the paper format drafts.

The findings and the views reported in this work, however, are those of the author and should not be attributed either to the Department of Economic and Related Studies (DERS) or the University of York.

Chapter 1

Introduction

1.1 A DSGE framework for analysing business cycles

In his article '*Inspecting the mechanism: An analytical approach to the stochastic growth model*' Campbell (1994) conducts a detailed analysis of the baseline real business cycle (RBC) model. His analytical solution allows him to investigate the impact and the transmission mechanism of productivity shocks on a calibrated, artificial economy. In that case, the relative simplicity of the model - i.e., a consumer-producer representative agent, the absence of nominal frictions and perfectly competitive markets - offered the opportunity of such detailed exploration. Since then, in an attempt to include more real world features, the developments of Dynamic Stochastic General Equilibrium (DSGE) models building on the RBC workhorse has become increasingly popular and, at the same time, inevitably more complex. Whilst, this evolution has allowed the profession to explore whether additional factors other than productivity shocks can provide convincing explanations of the sources of business cycle fluctuations, in some cases the modification of the original RBC paradigm has made the inspection of the 'mechanisms' increasingly difficult.

Given the state of the art of DSGE models, one might wonder whether John Campbell's approach is still meaningful. The view put forward in this thesis is that it is. The proposed object of investigation here is not the same baseline RBC model, and the way of inspecting its mechanisms is also different. But the motivation is driven by the same fundamental need of understanding of which forces drive the model and if these forces are well addressed with respect to the real facts.

The primary motivation behind Campbell's effort was "*to study shocks to technology and shocks to government spending financed by lump-sum or distortionary taxation*". In this sense his main concern was about 'how does the model work'. By taking this technical motivation seriously he was able to show that "*the persistence of shocks is an important determinant of their macroeconomic effects*". Now, one might conclude that a technical motivation led to a technical conclusion. But the value of that conclusion extends well beyond the technicalities. In fact, it actually made it possible to raise some deeper questions regarding these models, such as 'What is the nature of shocks?', 'Are all shocks alike?', 'Are shocks actually persistent?', 'How does the structure of the economy interact with these shocks?', 'What is the meaning of a policy shock?', and so on. Clearly these are interesting questions, especially for those interested in practical issues of stabilisation policy and welfare analysis. It is equally true that these questions make sense within the context of the RBC (or DSGE) paradigm, where the general equilibrium principle rules and the informational content of the shocks is somewhat crucial, given the assumption of rational expectations. But before moving to (or searching for) other paradigms, it seems reasonable to see whether this framework can give (at least some) answers. After all, in order to verify whether a working hypothesis is true or not, it is good to start by taking it seriously.

This thesis believes that one way of taking these models seriously consists in investigating how do they propagate the shocks. And one good reason lies precisely in the type of findings John Campbell's 'inspection' brought to general attention. In fact, his approach is particularly useful in revealing a potential danger, when it comes to assess the quantitative performance of calibrated DSGE models[1]. The great potentiality of the so-called 'microfoundations program' lies in the possibility for the researcher (or the policymaker) to pursue an explicit investigation of the sources of business cycle phenomena, which

[1]In the present study the focus is on calibration. For a survey of other quantitative approaches to macroeconomics see Hoover (1995).

would otherwise remain hidden below the surface of aggregate data. This is certainly one of the main motivations (and advantages) of the methodology which started with the RBC research agenda. However, one major drawback of DSGE models consists in the temptation of 'pushing' the calibration *to the limit*, or introducing (perhaps persistent) unexplained shocks, in order to match features of the data characterising business cycles in reality[2]. In other words, instead of mantaining what one of the leading promoters of this research agenda, Edward Prescott, defines as a "*theory ahead of business cycle measurement*", sometimes a dangerous inversion occurs: calibration is used to 'stretch' the *theory* in order to cover the *measurement* fit. As warned by Prescott (1986): "*The models constructed within this theoretical framework are necessarily highly abstract. Consequently, they are necessarily false, and statistical hypothesis testing will reject them. This does not imply however that nothing can be learned from such quantitative theoretical exercises.*".

The main purpose of this study is to face the challenge, asking whether *something* can be learned from a quantitative theoretical assessment of monetary versions of the RBC model, by inspecting their mechanisms.

1.2 Monetary issues in the RBC agenda

The idea that monetary factors are at the source of business cycle fluctuations is an old problem in economics, which dates back at least to Hume (1752). A modern revival of this theme was brought back on stage through the work of Friedman and Schwartz (1963) with their "*Monetary History of the United States*", where empirical evidence was used to illustrate two stylised facts: a close link exists between money and output, and - most importantly for macroeconomic policy - the *direction of causation* goes from the first to the second. Almost ten year later, Lucas (1972), in his famous 'island parable' model offered a possible theoretical explanation for the short run *non*-neutrality of money, starting what later on would have been defined as the rational expectations 'revolution' in macroeconomics. It took some time before Lucas' intuition could be translated into a self-contained, canonical modelling strategy for the

[2]See the critique by Chari, Kehoe and McGrattan (2009).

study of business cycle fluctuations. Nowadays, the model developed by Kydland and Prescott (1982) is considered to be the seminal contribution to those attempts. Couriously enough, after all this scientific trajectory, when the first RBC model appeared on the scene, money was no more part of it. However, since this new modelling paradigm was formally grounded in a general equilibrium setup, almost immediately it appeared flexible enough to encompass a great variety of extensions and to be tested on the basis of new (and old) questions concerning the business cycle. Among these, the problem of introducing money in an otherwise standard RBC model arose almost immediately after the publication of the work by Kydland and Prescott (1982). For illustrative purposes it will be useful to distinguish between two streams of literature dealing with this problem.

The first development is due to King and Plosser (1984), where (inside) money is incorporated in the RBC model through intermediaries and enters the production function of firms. The main message from their study was that the RBC model implies what is commonly known as *reverse causation*: i.e., the direction of causality runs from output to money and not - as stated by Friedman and Schwartz (1963) - vice versa. Subsequent work in this direction attempted to reinforce the argument that monetary fluctuations were due to the effects of real shocks to money demand. The thesis of the reverse causation is summarised in the title of Kydland and Prescott's (1990) article: "*Business cycles: real facts and a monetary myth*".

Almost contemporaneously, a second stream of literature tried to exploit the RBC paradigm searching for a more active role of money within this new framework. This research agenda culminated in a significant article by Cooley and Hansen (1989), where monetary shocks were modelled as an additional source of business cycle fluctuations, complementary (and orthogonal) to productivity shocks. The fact that this article received great stimulus from the theoretical developments of Lucas (1982, 1987) and Lucas and Stokey (1983, 1987), is evident for at least four reasons: the analysis is conducted in a perfectly competitive environment with fully flexible prices and wages; money is explicitly introduced via a cash-in-advance constraint; the nominal shocks take the form of deviations by the monetary authority from a money growth rule (i.e., monetary surprises); and welfare implications of the inflation tax for the representative agent are derived. The main findings from this quantitative exercise were somewhat 'disappointing'. In fact, at the given calibration, the real effects of monetary shocks were found to be particularly weak, while the welfare costs of inflation turned out to be quite small. A few years later, in a chapter of Cooley's book "*Frontiers of Business Cycle Research*", Cooley and Hansen (1995) attempt to re-assess a similar experiment, adopting a *modified* cash-in-advance approach that allows endogenous movements in *consumption*-based velocity[3], and adding capital to their previous model. The results were "*decisively mixed*", and - at least from a quantitative point of view - money continued to be remain *nearly*-neutral in those types of models. In another section of the same chapter, Cooley and Hansen introduce nominal wage rigidities, wondering whether nominal rigidities could reproduce significant (short run) *non*-neutralities of monetary impulses. The result was that "*nominal contracts enable monetary shocks to have significant real effects*".

This last result stands as a prophecy calling for a new generation of DSGE models, where monetary policy has real effects because of the rigidities of prices and wages[4]. Early attempts to incorporate these types of nominal frictions into the RBC framework produced a distinct stream of literature, represented by the so-called New-Keynesian DSGE models. These new developments, which ultimately were addressed within formal imperfectly competitive environments are synthesised in the representative work by Woodford (2003). With the advent of a New-Keynesian paradigm an (involuntary?) shift in the way the problem of money is tackled occurred. In fact, in most of the cases cash balances enter the model in the form of money in the utility function (MIUF)[5], or an (external) money demand function is simply added to the model[6]; in some

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[3] The so-called cash/credit good model.

[4] See Fischer (1977), Taylor (1979) for early models of nominal rigidities in rational expectations setup, and Woodford (2003) for a treatment within the context of imperfectly competitive markets.

[5] For recent treatments of the MIUF approach to the business cycle see McCallum and Nelson (1999), Woodford (2003), Ireland (2004) and Gali (2008).

[6] This has been done by adding an LM curve to the model, as in King (1993), or simply added to the set of equations characterising the equilibrium, as in Gali (2008).

other cases money is considered nonessential for monetary policy purposes[7]. On the other hand, monetary policy is conducted through *interest rate rules* instead of involving the money supply.

One of the main purposes of this thesis is to show that this shift of focus, from the monetised RBC models explored by Cooley and Hansen (1989, 1995) to the New-Keyneisan monetary paradigm, left some corners unexplored. In particular, some important questions remain: 'What are the consequences of adopting different microfoundations for money demand?', 'How does the transmission mechanism of monetary shocks work?', 'What is the nature of monetary shocks?'. The present investigation keeps these questions up front, trying to persuade the reader that: a) one can start to address these questions even within a simple, perfectly competitive DSGE model; b) that trying to answer these questions might suggest an interesting research path for the assessment of the challenge raised almost 50 years ago by Friedman and Schwartz (1963).

1.3 Money and transactions

The program of microfoundations characterising the DSGE framework makes an extensive use of the so-called representative agent paradigm. The model economy is generally populated by a large number of identical and infinitely-lived households and firms, with the monetary and fiscal policies implemented by the government. The behaviour of individuals is characterized by rationality - i.e. the maximisation of objective functions taking into account preferences, technologies and all the available (relevant) information. Finally, the necessary clearing conditions for all markets close the models, determining the final allocation of resources.

In order to derive explicitly the optimal behaviour of the individuals in this economy, specific functional forms for the objective functions (e.g., utility, production and transaction technologies) must be specified. As noted by McCallum (1989), "[...] *There are very very few functional forms* [...] *that will permit derivation of explicit closed-form solutions* [for the endogenous vari-

[7] See Woodford (1998).

ables]". and then he concludes that "[...]*There is one reasonably attractive combination that will do so, however, which consequently has been featured in several papers* [...]." The 'reasonably attractive combination' McCallum refers to involves a log-linear specification for the utility function of the households and a Cobb-Douglas form for the production function. In their well known paper, King, Plosser and Rebelo (1988) stress the importance of adopting particular functional forms in order to obtain a steady state, which is compatible with balanced growth. The specifications adopted below take primarily into account these analytical concerns. In general, the adequacy of some functional forms has to be evaluated in relation to the specific object of investigation or its usefulness for comparison purposes.

When it comes to the relationship between money and economic activity, many different solutions have been proposed by the literature in this regard, and - especially if one wants to restrict himself to the representative agent framework - most of them involve simplifications and shortcuts of some kind. As noted by Svensson (1985), "*Ordinary assets have a value and are held because they give a return - dividends. In complete analogy, one can think of money as having a value and being held because it gives a return - liqudity services. Once these liquidity services have been specified, the price of money can be determined by an asset pricing equation as the price of other assets* [...]".

In this sense, a recurring theme in monetary economics has been to consider the association between money and transactions. When Brock (1990) wonders "*what is to be included in the category of transaction costs*", the list of modelling strategies he quotes is quite rich; in particular it includes "*(a) cash-in-advance (Clower) constraints; (b) general transactions cost functions derived from first principles; and (c) real balances placed into utility functions or production functions, and tradeoff functions between barter (underground) transactions and monetary transactions (Scheinkman (1980), Singleton (1986)).* [...] *Papers that derive real balances in an 'as if' utility function or production function are Feenstra (1986), Gray (1984), and Woodford (1986)*". This list could be enlarged to encompass other approaches as *shopping-time models* (see McCallum and Goodfriend 1987) and models of *search* (see Kiotaki and Wright (1989)).

Given the great variety of models offered by monetary theory, one needs some guidance in order to select those that can be 'easily' integrated within the RBC framework, or, more generally, that are suitable 'candidates' for the purposes of the present investigation. In particular, the motivation for 'inspecting the mechanism' (in the spirit of Campbell (1994)) suggests two criteria for the selection. First of all, the adopted modelling strategy has to facilitate the interpretation of the results (this implies that the role of money in providing *liquidity services* needs to be as clear as possible). Secondly, since the quantitative analysis is derived from *calibrated* models, a 'parsimonious' parametrisation would be ideal (in order to focus the sensitivity analysis on few parameters).

From this point of view, an ideal 'candidate' is represented by the *cash-in-advance* constraint. Actually, this approach not only satisfies the selection criteria stated above, but will be considered also a natural benchmark. Taking into account all these elements, all the other remaining models surveyed by Brock (1990) and Walsh (2003) can be classified as a 'second best', at most. In fact, all the most popular approaches to money used in the DSGE literature are either too 'implicit' (like the MIUF approach) - therefore not satisfying the 'interpretation' requirement -, or involving too many fuctional forms (like shopping-time models or search models) - hence involving too many parameters to calibrate. The restrictions imposed by all these considerations still leaves one with a number of potential 'candidates'. However, since one of the main questions driving this study is about the consequences of adopting different microfoundations for money demand, the criterion of *comparability* with the cash-in-advance constraint can be aduced as an additional requirement. From this point of view, and for the reasons explained in detail in the next section, a suitable model is represented by the so-called *real-resource-cost* approach.

1.4 Cash-in-advance and real-resource-cost: a suggested comparison

Although the cash-in-advance and the real-resource-cost approaches do share the view that money is a *catalyst* in the process of exchange, they differ in the way transaction technologies are specified.

"*Money buys goods, goods buy money, but goods do not buy goods*". This statement is at the origins of a seminal contribution in the microfoundations of money by Clower (1967). The practical translation of this general idea within DSGE models is obtained by modifying the traditional household's problem (which consists in maximising a utility function subject to a budget constraint) by introducing an additional constraint linking money holdings to expenditures.

Traditionally the general objection that has been raised to the use of cash-in-advance constraint is that it is too extreme, assuming that an entire category of goods (usually consumption goods) is bought using cash. This not only appears to be too restrictive, but has also immediate implications for the results of these models, where the velocity of money (or at least its *consumption*-based version) is constant and equal to unity by construction[8]. A second (related) objection raised to the specification of the cash-in-advance constraint regards the absence of any 'opporunity cost' variable - i.e., an interest rate - in the (implied) money demand function. In particular, the model has been criticised for not generating a microfounded LM curve, where the interest rate explicitly appears (see King (1993) and McCallum and Nelson (1999)). This criticism led some of its proponents to formulate alternative versions of the cash-in-advance device. An alternative specification can be found in Lucas and Stokey (1983), and adopted later by Cooley and Hansen (1995), where the cash-in-advance constraint is applied only to a sub-set of consumption goods (the complementary subset being purchase *on credit*). The main limitations of this modelling shortcut consists of including two types of consumption goods (i.e., *cash*- and *credit*-goods) as separate arguments within the utility function, whereas all consumption goods are provided by the same production technology. From this point of view, the homogeneity of consumption goods makes any theoretical justification for different preferences quite 'slippery'. The theoretical weakness of this attempt to reformulate the cash-in-advance constraint as a response to the critics, represented an important motivation for this thesis. In particular, instead of trying to reformulate the original intuition to overcome

[8]See the discussion in Svennson (1985) and Hodrick, Kocherlakota and Lucas (1991).

its (supposed?) empirical limitations, the type of cash-in-advance constraint on which the thesis is built on, consists in its 'extreme' version - namely, *all* consumption (or *all* investment) will be subject to the contraint. There are three main reasons behind this decision. Firstly, to inspect how the dynamics of ('extreme' versions of) CIA models are affected by particular market timing assumptions or under different specifications for the money supply process. Secondly, to assess the empirical performance of these 'extreme' CIA models, in order to check whether the criticisms are supported by the data. Finally, this 'extreme' feature makes the CIA model the 'benchmark model' for the whole investigation. In fact, the comparison of the CIA model with alternative micro-foundations - in principle, theoretically consistent with an endogenous velocity of money, or including an interest rate elasticity in the money demand - will be more informative.

The real-resource-cost approach assumes that real resources (i.e., a given amount of goods) must be used up in the process of exchange. The theoretical implementation of this approach in a general equilibrium setup is due to Brock (1974), while the analytical characteristics of the transaction costs function and its functional equivalence with the money-in-utility-function are extensively discussed by Feenstra (1986) and Wang and Yip (1992). More recent approaches are represented by an indeterminacy analysis by Carlstrom Fuerst (2001b) and a welfare analysis of transaction costs by Schmitt-Grohe and Uribe (2004), conducted in an imperfectly competitive environment. The general idea consists of modifying the traditional household's problem by explicitly introducing the costs of transactions into the household's budget constraint. Since the role of cash balances is to reduce transaction costs, a transaction technology that links money to the the level of expenditures must be specified. In particular this last feature makes the real-resource-cost approach very similar to the cash-in-advance idea[9]. From this point of view it passes the test of the 'interpretation' requirement described in the previous section. In particular, this characteristic is reinforced when it comes to the calibration issue. In order to calibrate the parameters related to the specific transaction

[9] Actually, one of the purposes of this thesis will be to show that the two models can be made comparable by an 'appropriate' calibration.

technology, all the other competing approaches derive the value of the parameters from the coefficients of the corresponding money demand function specification. This is certainly true for the calibration strategy of the preferences parameter in the cash/credit model of Cooley and Hansen (1995) and for the money-in-utility-fuction of Chari, Kehoe and McGrattan (2000). One might argue that (probably) there is nothing 'terribly wrong' from the point of view of this calibration procedure and that, after all, the purpose of calibration is to generate simulations replicating the relationship between the actual aggregate variables (and this includes the variables involved in the money demand function). However, the claim here is that the cash/credit and the MIUF approaches *must* be subject to this type of calibration strategy, given the absence, for these models, of an immediate economic meaning for the parameters appearing in the respective (microfounded) money demand functions. In the case of the real-resource-costs approach, instead, one can distinguish between unitary and total transaction costs, while the parameters characterising the transaction cost function exhibit an economic interpretation[10].

Finally, one last advantage of the real-resource-cost model concerns the treatment of money demand shocks. One limitation of the the cash/credit and the MIUF approaches consists in the fact that the researcher is forced (by construction) to introduce money demand shocks in the utility function; this strategy inevitably implies that money demand shocks must be interepreted as shocks to preferences[11]. Again, the same criticism that has been raised earlier about the theoretical weaknesses of a distinction between *cash-* and *credit-* goods, can be raised about the nature of money demand shocks: 'Why should individuals suddenly prefer to hold more cash in the MIUF model?', or 'Why should they prefer to buy *cash*-goods in the cash/credit model?'. On the contrary, a money demand shock in the real-resource-cost model has an economic explanation meaning: because the real cost of transactions increased.

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[10] One can adopt estimated parameters from other studies. This is the case for the transaction cost of Carlstrom and Fuerst (2001b), where the estimated parameters of the transaction cost function are adopted from the estimation in Marshall (1992).

[11] An alternative example is represented by Ireland (1996), where a shock to the velocity of money is simulated simply by multiplying the real balances appearing in the cash-in-advance constraint by an exogenous variable.

In fact, as will be shown along the analysis, in this model the money demand shock takes the form of a shock to unitary transaction costs.

These characteristics of the RRC model are extremely congenial for the content of this investigation. In fact, on the one hand, this approach displays features that, by construction, are not allowed in the 'extreme' CIA model - namely, endogenous velocity of money and an interest rate elasticity on the 'right hand side' of the money demand function. The comparison between the CIA model and the RRC model analysed in this thesis will show that these features are not essential for a monetary model to match the empirical evidence. On the other hand, differently from other approaches that allow an endogenous velocity and an interest rate elasticity, the real-resource-cost approach preserves the possibility of an economic interpretation. The whole thesis will show that this last characteristic is very important in order to interpret the simulation results.

According to the general criteria stated above, the cash-in-advance and the real-resource-cost models do emerge as comparable approaches for studying the transmission mechanism of shocks within an otherwise standard RBC model. However, once the models have been selected, the point becomes: how to undertake this comparison? On this subject this thesis will adopt two main criteria. The first criterion will be an *analytical* criterion, i.e., to study how both approaches fit into the DSGE framework. This involves deriving the conditions for the equilibrium, comparing the resulting optimality conditions and inspecting the elements determining the dynamics of the models[12]. A second criterion will consist of comparing the performance of these models with respect to the empirical evidence charaterisig the U.S. business cycle, using the same type of stylised facts traditionally adopted by the RBC school. The aim of the next section is to present the kind of evidence that will be used throughout the whole investigation. In particular, a close inspection of the characteristics of real and monetary factors of the U.S. business cycle,

[12]The analytical criterion adopted here is somewhat different than the more specific use Campbell (1994) makes of the word analytical. Given the fact that the monetary models presented here are relatively more complex than the baseline RBC model studied by Campbell, the main purpose is to let those analytical insights emerge by *comparing the dynamics* of these models, instead of deriving a complete *analytical solution.*

providing the other main motivation behind the final chapters of the thesis.

1.5 Money and the business cycle

This section presents some stylised facts about money and economic activity that will be used to assess the quantitative performance of the models considered in the investigation. The source used to analyse the key relationships is represented by the database on the U.S. Economy of the Federal Reserve Bank of St. Louis (*Federal Reserve Economic Data*, FRED2). The macroeconomic variables have been selected according to the theoretical counterparts appearing in the artificial economies. Moreover, in line with the RBC tradition, this study will consider *quarterly data*, in line with the proposed calibration strategy. Finally, the time period considered in the database spans over a period of 40 years (from the first quarter of 1964 to the last quarter of 2005). These extremes exclude the recent financial crisis and the period for which the data considered were not 'uniformly' available. Despite the intellectual challenge of recent events for the economic profession, the decision to exclude the relevant data period is due to the fact that the structure of the models analysed here is far too simple in order to capture the recent experience in a satisfactory way. All the time series of the variables appearing in this section are calculated as deviations from a long-run growth path. The trend has been removed using an H-P filter (with $\lambda = 1,600$) in order to 'isolate' quarterly fluctuations[13]. Table 1.1 reports the standard deviations of the detrended macroeconomic variables for the U.S. economy[14].

The data show that real variables are far more volatile than nominal variables. Consumption is slightly less volatile than output, while investment is much more volatile (more than 4 times). *Consumption*-based, *investment*-

[13]For quarterly data it is customary to use the value of $\lambda = 1,600$. For an illustration of the H-P filtering tecnique see Hoddrick and Prescott (1980) and Cooley and Prescott (1995). The H-P filter has been applied to natural logarithms of the original variables.

[14]The measures of real wages and real balances (M1) have been obtained by deflating the nominal counterparts by the GDP deflator and the CPI (measures derived using the latter are reported in brackets). The inflation rate has been derived using the quarterly variations in the GDP deflator and the CPI (with CPI-inflation reported in brackets). The nominal interest rates refer to the (quarterly equivalent) of the 3-months U.S. treasury bill.

VARIABLES	STD. DEV.
working hours	0.0043
real wage	0.0109 (0.0071)
consumption	0.0125
nominal interest rate	0.0030
inflation	0.0044 (0.0029)
real balances	0.0314 (0.0284)
output	0.0154
money growth	0.0089
investment	0.0699
consumption-velocity	0.0258
investment-velocity	0.0654
output-velocity	0.0277

Table 1.1: Standard deviations (U.S. economy, HP-filtered quarterly data. Source: Federal Reserve Bank of St. Louis).

VARIABLES	CORR. with OUTPUT
working hours	0.7077
real wage	0.5307 (0.5830)
consumption	0.8632
nominal interest rate	0.3522
inflation	0.3817 (0.1419)
real balances	0.3368 (0.3133)
output	1.0000
money growth	-0.1282
investment	0.9024
consumption- velocity	0.0713
investment- velocity	0.8056
output- velocity	0.2362

Table 1.2: Correlations with output (U.S. economy, HP-filtered quarterly data. Source: Federal Reserve Bank of St. Louis).

VARIABLES	CORR with MONEY GROWTH
working hours	-0.1957
real wage	0.2365 (0.1714)
consumption	0.0311
nominal interest rate	-0.4771
inflation	-0.3124 (-0.1940)
real balances	0.2264 (0.2021)
output	-0.1282
money growth	1.0000
investment	-0.1955
consumption- velocity	-0.2306
investment- velocity	-0.3250
output- velocity	-0.2781

Table 1.3: Correlations with money growth (U.S. economy, HP-filtered quarterly data. Source: Federal Reserve Bank of St. Louis).

based and *output*-based velocity have been obtained dividing the respective *nominal* aggregate variables by M1[15]. The data show that real balances and the alternative measures of velocity are quite volatile at quarterly frequencies (with a particularly high standard deviation of the *investment*-based velocity). Table 1.2 shows the correlation of these variables with respect to real output.

There is a positive correlation between inflation and output, and also between output and nominal interest rates, while all the measures of velocity are procyclical. There is a negative correlation between money growth and output (-0.1282). Table 1.3 reports a negative correlation between money growth and nominal interest rates (this is traditionally associated to the so-called *liquidity effect*) and a negative correlation between money growth and inflation. As will be shown later on in this study, (standard) models with flexible prices have a general weakness in capturing these two last features of the data. Consumption is positively (but weakly) related with money growth, while the link between money growth and investment is negative (and somewhat stronger).

[15] According to the definition adopted by the Federal Reserve Bank of St. Louis "*M1 includes funds that are readily accessible for spending [...]: (1) currency outside the U.S. Treasury, Federal Reserve Banks, and the vaults of depository institutions; (2) traveler's checks of nonbank issuers; (3) demand deposits; and (4) other checkable deposits (OCDs), which consist primarily of negotiable order of withdrawal (NOW) accounts at depository institutions and credit union share draft accounts.*" (source: FRED2).

Finally, money growth is negatively related with all the measures of velocity.

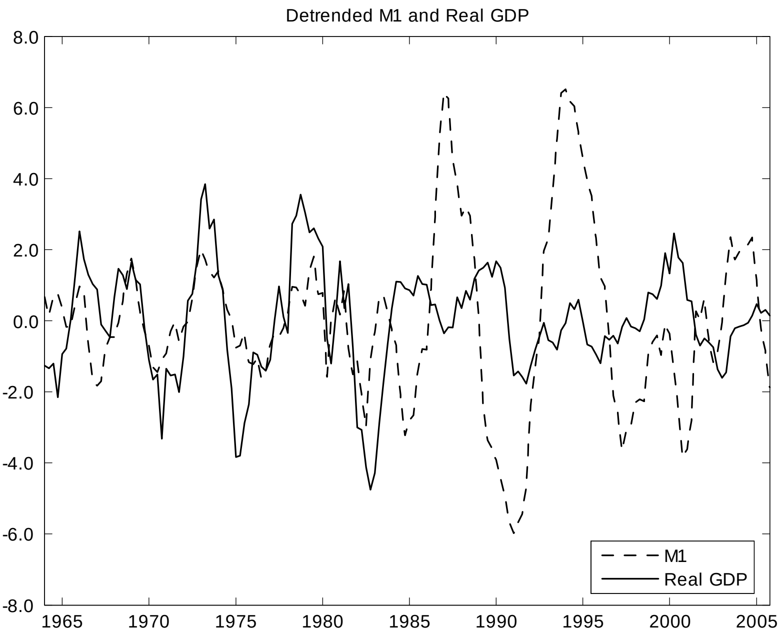

Figure 1.1: Detrended nominal money supply (M1) and Real GDP (2005 U.S.$ prices). Quarterly data (sample: 1964:1 - 2005:4).

Figure 1.1 plots the quarterly movements in the monetary aggregate M1 against real GDP for the U.S. Economy for the database used in this section. The relationship between these two variables looks quite strong[16]. Analysing this type of empirical evidence more in depth, Cooley and Hansen (1995) find that "*The cross correlation of output with monetary aggregates shows that output is more highly correlated with lagged values of the aggregates, implying that the monetary aggregates peak before output*".As emphasised by King (1991) the ability to capture this dynamic relationship is "*an important test for monetary models*".

[16] Cooley and Hansen (1995) obtain a similar graph, for an earlier period, using M1 and the broader monetary aggregate M2; their comment on that is quite interesting: "*No inference about the direction of causality can be made simply because these variables are highly correlated. Their causality could run in either direction. Nevertheless, models that seriously explore the role of monetary shocks are going to look for channels of causality from money to output*".

The traditional monetised versions of RBC models - in particular, those characterised by fully flexible prices - are based on the assumption that the monetary authority implements a monetary policy that takes the form of an *exogenous* money growth rule. In those same models, monetary injections by the government are transmitted *instantaneously* into the representative household's budget constraint by the monetary authority, via lump sum transfers. If one wants to interpret Cooley and Hansen's result that "*monetary aggregates peak before output*" as the evidence that the new money takes time to be put *effectively* in circulation, the instantaneous transfer assumption must be modified. This is the main motivation behind the introduction of the so-called 'monetary pipeline' in the last two chapters of this thesis. The main idea consists of altering the transmission mechanism of monetary shocks by assuming that the new money from the central bank will (start to) reach the private sector of the economy in the following period (i.e., in the following quarter). The speed at which this process occurrs will be 'regulated' through the calibration of a specific parameter. There are two main consequences of introducing this particular extension. On one hand it introduces a disturbance in the *inter*temporal allocation of resource by the representative agent. On the other hand, it 'splits' total money supply into two distinct components: a share will enter the households' accounts and the remaining share will be stored (i.e., trapped) within the 'pipeline' channel. The former will decrease the degree of the neutrality of money, while the latter will affect the measure of velocities.

Finally, one of the main features of the models that will be presented below is their relative simplicity. For this reason one would not expect a *perfect* match between these models and the real data. However, the type of empirical evidence discussed in this section represents an essential benchmark for a comparison *between* the different models, with respect to the data. On the other hand, the features of the data presented here generate some questions: 'how do monetary RBC models explain (at least some) of these stylised facts?', or 'can a closer inspection at the transmission mechanism of monetary shocks be of any help in trying to read through this aggregate evidence?'.

1.6 Money, transactions and the business cycle: variations on a theme

In order to address these questions, the thesis presents several variations on a theme involving 'small scale' Dynamic Stochastic General Equilibrium (DSGE) models, in which money is associated with transactions. In order to inspect the mechanisms implicit in the monetary models, this investigation makes use of a comparative approach along two dimensions. On one hand, an 'external' comparison between different microfounded monetary models - namely, the *cash-in-advance* (CIA) model and the *real-resource-cost* (RRC) model. On the other hand, an 'internal' comparison, where the performance of each of the two typologies is assessed with respect to its own extensions. Given the theoretical assumptions and the suggested calibration, the criteria that will be used to evaluate these models are both analytical and quantitative. One of the aspects under examination will be their relative ability to explain (some) features of the U.S. business cycle illustrated in the previous section. In particular, trying to address the following questions: does the way in which money is modelled matter quantitatively? Can monetary factors explain the high volatility of investment and velocities appearing in the data? How does the *inflation tax* work in these models?

The comparison between these two models is motivated by the fact that flexible price models adopting the CIA approach have been traditionally criticised for assuming a link between money and transactions which was too extreme. Moreover, the limitations of this model in terms of allowing endogenous velocity and an explicit interest rate elasticity of money demand led the profession to neglect its use, in favour of a type of microfoundations that could overcome the empirical limitations. However, in most of the cases, the switch towards alternative, more cryptic, approaches had the consequences of obscuring the role played by liquidity. Among these, and for the reasons stated above, the real-resource-cost approach to money and transactions exhibits characteristics that make it less cryptic. Hence, it is a natural candidate for a comparison with the CIA model.

In order to assess the results from the two models for the *same* types of extensions, the RRC model will be implemented in accordance with the CIA model (e.g., adopting the same category of transactions for which money is needed). The comparison will be made reliable by an 'appropriate' calibration and keeping the remaining features of the models (production, structure of shocks, etc.) unchanged.

In order to inspect the mechanism, the models need to fulfill some general requisites. First of all they have to be relatively simple in structure. The models of the economy presented are characterized by perfectly competitive markets and fully flexible prices.

The main purpose of Chapter 2, "*Money and transactions: two classical monetary models*", is to introduce and compare the CIA and RRC model in a context where labour is the only factor of production. The original contribution of the chapter is twofold: on the one hand it extends the analysis of flexible price models conducted by Woodford (2003) and Gali (2008) by comparing the impulse-reponse functions; on the other hand, it proposes an original calibration strategy for a quantitative assessment of the real resource cost model presented in Feenstra (1986) and Wang and Yip (1992).The decision to start from a simple framework is motivated by the need to emphasise more clearly the contribution of any extension to these baseline models, which is developed in further chapters. Despite the simplicity of the models, the results show that the modelling strategy for money demand is **relevant** for the response of the models to monetary shocks.

Chapter 3, "*Transaction technologies and the business cycle: a quantitative exploration*", keeps the comparison between these two monetary models at the centre of the analysis, introducing capital goods. This modification to households' and firms' setup allows a more accurate quantitative assesment of the CIA and RRC models and, at the same time, it conveys important information about the propagation mechanism of shocks. As it is well known, the high volatility of investment relatively to consumption's volatility at business cycle frequencies has been a major 'concern' for the RBC literature. Therefore, the possibility of analysing the dynamic contribution of capital goods for these monetary models seems a natural place to start for a quantitative assessment. The results will show that the presence of capital goods adds new insights in the effects of the inflation tax on *cash*- and *credit*-goods appearing in the two models. The comparison with the results reported in previous work by Cooley and Hansen (1989, 1995) reveals that the assumption of divisible labour adopted in this chapter performs better in terms of correlation of real variables with output and money growth, while the quantitative performance of the indivisible labour assumption (adopted by Cooley and Hansen) is superior in terms of standard deviations of consumption and working hours.

Chapter 4, "*Cash-in-advance and monetary injections. Some extensions*", focuses entirely on the CIA model, allowing investment to be subject to the cash-in-advance constraint. This extension is motivated by the fact that the definition of the monetary aggregate used in the analysis (M1) traditionally includes houesholds' deposits. Therefore, it seems sensible to assume that part of this liquidity goes to finance investment. On the other hand, the negative correlation between investment and money growth in the data (see Table 1.3) might reveal some effects of the inflation tax on investment. A second important contribution of this chapter goes in the direction of explaining the kind of empirical evidence reported above - namely, that movements in money supply *lead* movements in output. Here the conjecture looks as follows: Chapter 2 and 3 assume that new monetary injections by the monetary authority take the form of lump-sum transfers, which are *instantaneously* available to households. Consequently, the money supply channel is modified by introducing a (calibrated) lag in the transmission mechanism of monetary shocks: this is the essence of the 'pipeline' model. From a theoretical point of view, the original contribution of the extended CIA model consists of the integration of the seminal works of Stockman (1981) and Abel (1985) with a Lucas-type market timing assumption (after Lucas, 1982).

Finally, Chapter 5, "*Extending transaction technologies: a real resource cost approach to the business cycle*", incorporates the extensions of Chapter 4 into the RRC model. The original contribution in the case of the RRC model is built on the specifications of Feenstra (1986) and Wang and Yip (1992), extending transaction costs to investment.

Again, the comparison between the results in Chapters 5 and 6 with the results reported in previous work by Cooley and Hansen (1989, 1995) reveals that the assumption of divisible labour adopted in this chapter performs better in terms of correlation of real variables with output and money growth, while the quantitative performance of the indivisible labour assumption (adopted by

Cooley and Hansen) is superior in terms of standard deviations of consumption and working hours. Moreover, when the cash-in-advance constraint is extended to capital goods and the 'pipeline' mechanism is more rigid, the correlation between nominal interest rates and output exhibits the same 'sign' of the data, improving the results found in Cooley and Hansen (1989, 1995).

In all these chapters, the comparison between the CIA and the RRC models is conducted both on theoretical and quantitative grounds. Firstly, the assumptions implied by each approach are stated, then the resulting optimality conditions are derived. Then the models are calibrated (on quarterly basis) and impulse-response functions analysed. Finally, the *marginal* contribution of each implementation to the baseline models is evaluated against the empirical evidence presented above. A direct interpretation of the impulse-response functions will result much easier for the first two chapters, where the models are relatively simpler. Wherever models will become more complex, the comparison with the data will offer additional insights about their performance.

Chapter 6 concludes, summarising the results and suggests new directions for further research.

Chapter 2

Money and transactions: two classical monetary models

The main purpose of this chapter is to introduce and compare two micro-founded monetary models: the so-called *cash-in-advance* (CIA) model and the *real-resource-cost* (RRC) model. The models described in the chapter represent economies with endogenous supply of goods, where labour is the only factor of production[1]. This simple framework has been chosen as benchmark for recent discussions regarding Dynamic Stochastic General Equilibrium models involving money. In particular, an important contribution of this chapter consists of integrating the analysis of monetary (flexible price) models by Woodford (2003) and Gali (2008) through a detailed analysis of impulse-reponse functions. Secondly, the chapter also proposes an original calibration strategy for a quantitative assessment of the real resource cost model presented in Feenstra (1986) and Wang and Yip (1992).

The decision to start from a simple framework is motivated by the need to emphasise more clearly the contribution of any further extension to the baseline model, which will be developed in the next chapters. Furthermore, simplicity turns out to be useful at this stage, in order for the reader to become familiar with the overall structure of the analysis and the conventional notation.

In what follows, the assumptions implied by each approach are described

[1] This assumption will be relaxed later on, with the introduction of capital (see Chapters 3, 4 and 5).

and the resulting optimality conditions derived. The quantitative assessment is made by simulating the calibrated models and commenting on the relative inpulse-response functions, the standard deviations and the main correlations. Since the focus here is on money and transactions, particular attention is devoted to the monetary aspects of these economies. In particular, the performance of the different economies is assessed analysing the effects of sochastic shocks affecting either production (i.e., technology shocks), the money demand (i.e., shocks to transaction costs) or the money supply process (i.e., monetary policy innovations).

2.1 The CIA model

"*Money buys goods, goods buy money, but goods do not buy goods*". This statement is at the origins of a seminal contribution in the microfoundations of money by Clower (1967). The widespread use of the cash-in-advance assumption in dynamic sochastic general equilibrium models is mainly due to the theoretical contributions of Lucas (1982), Lucas and Stokey (1983, 1987), Svennson (1985). Empirical implementations have been proposed by Cooley and Hansen (1989, 1995) and Walsh (2003).

As will be shown below, the general idea consists of modifying the traditional household's problem (which is represented by a maximisation of a utility function subject to a budget constraint) by introducing an additional constraint linking money holdings to expenditures.

2.1.1 Households

The economy is populated by a large number of identical and infinitely-lived households. At time $t = 0$ the representative household seeks to maximize the following expected value of a discounted stream of period utility:

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u\left(c_t, l_t\right) \right\} \tag{2.1}$$

The expectational operator E_0 indicates that the expectations about present and future streams of utility are formed conditionally to the information available to the agent at time $t = 0$[2]. The objective function (2.1) assumes that utility at time t depends on real consumption c_t and leisure time l_t. Future utility is discounted by a constant discount factor β (with $0 < \beta < 1$).

The period utility function u is strictly concave and twice continuously differentiable. It is increasing in its arguments and decreasing in their marginal utility. Using u_j (u_{jj}) to denote the first (second) partial derivative of the function $u(j)$ with respect to its generic argument j, one can write: $u_c > 0$, $u_l > 0$, $u_{cc} < 0$, $u_{ll} < 0$. In addition to that, the Inada (1963) conditions are assumed to be holding: $\lim_{c\to 0} u_c = \infty$, $\lim_{l\to 0} u_l = \infty$, $\lim_{c\to\infty} u_c = 0$, $\lim_{l\to\infty} u_l = 0$.

Total time endowment is normalized to one, so that the following constraint applies to every period:

$$l_t + h_t^s = 1 \tag{2.2}$$

This means that at time t the agents will choose to split total time between leisure time l_t and working hours h_t^s (the superscript 's' is meant to indicate supply).

Using (2.2) one can reformulate (2.1) in terms of consumption and working hours:

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1 - h_t^s\right) \right\} \tag{2.3}$$

In this case the period utility u is decreasing in working time ($u_h < 0$) and increasing in the marginal *dis*utility of work ($u_{hh} > 0$). In deference to the real business cycle tradition, this last specification will be mantained throughout the analysis.

The explicit functional form chosen for period utility takes the form

$$u\left[c_t, 1 - h_t^s\right] \equiv \frac{c_t^{1-\Phi}}{1-\Phi} + \Psi \frac{\left(1 - h_t^s\right)^{1-\eta}}{1-\eta} \tag{2.4}$$

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[2] The dating convention implicitly adopted in (2.1) conforms to King and Rebelo (1999), i.e. there is no distinction between *planning time* for the individual and *calendar time* for the economy.

, where: $\Phi > 0$ is the coefficient of relative risk aversion (with $1/\Phi$ being the elasticity of intertemporal substitution); given $\eta > 0$, expression $1/\eta$ denotes the elasticity of intertemporal substitution for labour[3]; $\Psi > 0$ represents a preference parameter over leisure.

Constant relative risk aversion (CRRA) utility functions are quite common in the DSGE literature. Here, the adoption of a period utility function additively separable over consumption and leisure is mainly motivated by the purpose of separating the (direct) effect of money on consumption expenditures, from its (indirect) effect on working hours. Eventually, the use of a power utility for leisure allows one to calibrate η to investigate which values deliver a more realistic behaviour of labour supply[4].

Households do maximise the time separable utility function (2.3) subject to the following flow budget constraint (in nominal terms):

$$M_{t-1}^d + (1 + i_{t-1}) B_{t-1}^d + W_t h_t^s + T_t \geq P_t c_t + M_t^d + B_t^d \tag{2.5}$$

$\forall t \geq 0$; where: B_t^d denotes the nominal value of riskless bonds, which pay a one-period nominal (net) interest rate i_t; T_t represents nominal lump sum transfers from the government (taxes, if negative); M_t^d is individual money demand; W_t is hourly nominal wage and P_t represents the price of the homogeneous good produced in the economy. The superscripts 's' and 'd' characterise 'supply' and 'demand', respectively.

The right hand side of (2.5) represents individual's total nominal wealth. This encompasses financial wealth from the previous period plus labour income ($W_t h_t^s$) from current period and the exogenous lump sum transfers. Financial wealth is given by the nominal value of a portfolio of financial assets - namely, bonds and cash accumulated in period $t-1$ - inclusive of interest earnings

[3] In the literature this is also known as *labour supply elasticity*, after Frisch (1933).

[4] As stressed by Campbell (1994), the use of power utility is very convenient "*because it nests two popular special cases in the real business cycle literature: log-utility for leisure, in a model with divisible labour, and linear derived utility for leisure, in a model with indivisible labour in which workers choose lotteries over hours worked rather than choosing hours worked directly* [see Hansen (1985), Rogerson (1988)]". The former case corresponds to $\eta = 1$, the latter to $\eta = 0$. An explicit comparison of these two special cases can be found in Christiano and Eichenbaum (1992a) and King, Plosser and Rebelo (1988a).

(i_{t-1}) from bonds holdings[5]. As in the case of Woodford (2003), an implicit assumption here is that "*available financial assets completely span the relevant uncertainty faced by households about future income, prices, taste shocks, and so on, so that each household faces a single intertemporal budget constraint*". In other words, expression (2.5) implies that financial markets are complete.

Total wealth available in period t is allocated to the goods market, buying consumption goods at the prevailing price ($P_t c_t$), and to the financial markets, adjusting the portfolio of assets (given the prevailing interest rate, i_t).

In the context of a monetary economy, if the assumption of rational economic agents has to be maintained, what should matter to the individuals cannot just be their nominal wealth, expressed in currency units. What should matter is the command over the quantity of real resources nominal wealth can buy. Therefore, it would be useful to express the budget constraint in terms of units of output. Dividing both sides of (2.5) by the price level (P_t), the household budget constraint can be rewritten in real terms as

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t \geq c_t + m_t^d + b_t^d \tag{2.6}$$

$\forall t \geq 0$; where: $b_t^d \equiv B_t^d / P_t$ denotes the real value of riskless bonds; $I_t \equiv (1 + i_t)$ is the one-period nominal (gross) interest rate; $\tau_t \equiv T_t / P_t$ represents real lump sum transfers from the government; $m_t^d \equiv M_t^d / P_t$ is individual demand for real balances; $w_t \equiv W_t / P_t$ indicates real wage and $\Pi_t \equiv P_t / P_{t-1}$ is represents the (gross) inflation rate.

Resources not consumed in period t are saved in the form of bonds and cash balances, whose command over goods will become effective only at the beginning of the next period. Since this is true for every period (2.6) shows that the portfolio allocation decisions taken at time $t-1$ do in fact expose the real value of savings to changes in the price level from $t-1$ to t.

One might wonder why households should decide to hold cash balances: an

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[5] Traditionally, the idea of riskless bonds has been associated with *government* bonds. However, the bonds considered in these models are *private* bonds. In principle, nothing prevents the co-existence of private and government bonds in this context. However, under the assumption that both typologies of bonds would be risk free, by arbitrage they would have the same price in equilibrium. The assumption of riskless private bonds has been made by Woodford (2003) and Gali (2008). The reason to adopt it here is mere simplicity.

asset which - differently than bonds - does not pay any interest, and whose purchasing power (as any other nominal magnitude) is subject to the inflation dynamics. In actual economies individuals do hold money, despite the presence of other interest bearing assets and positive inflation rates. In order to translate this feature of the real world into the model, one needs to specify the reason why money is held.

The first option considered here is to integrate the household's problem by introducing a cash-in-advance constraint. Different modelling strategies for cash-in-advance constraints have been proposed in the literature, all of them implying precise assumptions about the market timing and the nature of the assets appearing in the constraint.

The particular type of cash-in-advance constraint adopted here is originally due to Lucas (1982). It states that purchases relative to a specific category of goods - in this case, consumption goods - can be made only in exchange of an equivalent amount of currency. In addition to that, it assumes that households are allowed to visit the financial markets *before* the goods markets, in order to gather the desired liquidity[6]. Finally, any lump sum (monetary) transfer to the household occurs at the beginning of the period, and is delivered *via* financial markets.

Considered all together, these assumptions correspond to a cash-in-advance constraint of the form:

$$M_{t-1}^d + (1 + i_{t-1})\, B_{t-1}^d - B_t^d + T_t \geq P_t c_t \tag{2.7}$$

$\forall t \geq 0$. The meaning of (2.7) is quite intuitive: the liquidity accumulated on the left hand side can purchase the nominal amount of goods and services appearing on the right hand side.

Using a terminology introduced by Kohn (1984), the market timing assumption implicit in (2.7) characterizes what can be defined a "liquid asset model": when the financial markets do open, bonds can be costlessly converted into cash at the convenience of the household. Given the fact that the model deals with risk-free assets, it seems reasonable to assume that bonds

[6] Here the term *financial markets* essentially refers to any operation by households involving financial assets (i.e., bonds and money).

share the same characteristics as money[7]. In case of a positive interest rate, this implies that households will decide to hold an amount of cash balances just sufficient to cover the desired consumption expenditures[8].

As emphasized by Salyer (1991) "*both Lucas (1982) and Svensson (1985)* [...] *assume that the transfer is received in the asset market*". This assumption appears in line with the way monetary expansions do occurr in the real world, i.e. affecting financial markets first. This last consideration justifies the inclusion of the monetary injection in (2.7).

One can derive the purchasing power of cash balances by dividing both sides of (2.7) by P_t. The result is a CIA constraint expressed in real terms:

$$\frac{m_{t-t}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t \geq c_t \tag{2.8}$$

$\forall t \geq 0$.

At time t the problem of the household is inherently dynamic: that one of choosing state-contingent claims for consumption (c_t), labour supply (h_t^s), real bonds holdings (b_t^d) and real balances (m_t^d), in order to maximize the expected utility (2.3), subject to the budget constraint (2.6) and to the cash-in-advance constraint (2.8)[9].

2.1.2 Firms

Given that there are many ways of modelling firms, and since the focus here is mainly on money, the idea is to keep the structure of the productive sector as simple as possible. After all, the first attempts to introduce money in an otherwise standard real economy are *monetised* versions of simple real business

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[7]In a context in which all bonds are private and all the individuals are alike, bonds are not exchanged in equilibrium. This does not prevent bonds being priced in equilibrium, according to to the prevailing market conditions when financial markets do open.

[8]This assumption might look quite innocuous. However, it can have crucial implications when monetary policy is conducted using an interest rate rule. For example, on this issue, Carlstrom and Fuerst (2001a) discuss the introduction of Lucas' market timing in the MIUF model.

[9]Moreover, *no-Ponzi game* conditions must hold to guarantee optimality.

cycle (RBC) models[10]. The model presented in this chapter makes things even easier: in fact production abstracts from capital. This simplification is not meaningful in itself, but only if considered in perspective: in order to understand how the present model is changed once capital is introduced (see Chapter 3), the results derived for this simple case will represent a useful benchmark.

The economy is populated by a large number of identical firms. They produce an homogeneous good using labour (i.e., working hours) supplied by the households. The real output produced in period t can be expressed by the following production function:

$$y_t = f\left(z_t, h_t^d\right) \tag{2.9}$$

$\forall t \geq 0$, where: y_t denotes *real* output; h_t^d are working hours demanded by the firm and z_t represents the level of technology.

In order to obtain a direct correspondence between the behaviour of individual firm and their aggregate counterpart, the production function is represented by a constant returns to scale technology. To satisfy this condition, the production technology is assumed to be *linear* in its only factor of production:

$$y_t = z_t h_t^d \tag{2.10}$$

$\forall t \geq 0$. In deference to the RBC literature, the variable z_t represents the total factor productivity, which evolves exogenously according to the law of motion

$$\ln z_t = (1 - \rho_z) \ln z + \rho_z \ln z_{t-1} + \epsilon_{z_t} \tag{2.11}$$

$\forall t \geq 0$; where: ρ_z is the autoregressive coefficient (with $0 \leq \rho_z \leq 1$), and ϵ_{z_t} is a random variable serially uncorrelated and normally distributed, with zero mean and constant variance ($\sigma^2_{\epsilon_z}$).

In period t firms sell their product in a perfectly competitive goods market, taking the price P_t of the homogenous good as given. Analogously, given the nominal wage W_t, they buy labour services from households in a perfectly

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[10] See King and Plosser (1984), Cooley-Hansen (1989, 1991) and Christiano (1991).

competitive labour market. In order to decide how much to produce - and, consequently, how much labour to buy - firms do maximise the following profit function:

$$\Gamma_t = P_t y_t - W_t h_t^d \tag{2.12}$$

$\forall t \geq 0$, where nominal profits (Γ_t) are defined as a difference between nominal revenues ($P_t y_t$) and nominal costs ($W_t h_t^d$). The assumptions of perfect competition and constant returns to scale do imply that the representative firm makes zero profits in equilibrium.

In every period t each firm solves a static problem: that of choosing working hours (h_t^d), which maximize profits (Γ_t) subject to the technology constraint (2.9).

2.1.3 Government

In this model the government operates as monetary and fiscal authority and its revenues and outlays in period t are combined in the following flow budget constraint (expressed in nominal terms):

$$M_t^s - M_{t-1}^s + B_t^g - (1 + i_{t-1}^g) B_{t-1}^g = P_t g_t + T_t \tag{2.13}$$

$\forall t \geq 0$, where: B_t^g denotes the face value of government debt outstanding, which pays a one-period nominal (net) interest rate i_t^g; T_t indicates governmental nominal lump sum transfers, net of taxes; M_t^s represents aggregate money supply[11]; and g_t denotes real government consumption. The superscripts s and g characterise 'supply' and 'government issues', respectively.

Since the focus here is in studying the impact of monetary shocks and not the impact of changes in government spending, g_t is set to zero (for all t

[11]Given that here the government acts as fiscal and monetary authority, the government budget constraint represents a *consolidated* budget constraint (i.e., inclusive of the central bank's balance sheet). For this reason, technically speaking, M^s in (2.13) should represent the monetary base. Given the absence of a proper transmission mechanism between monetary authority and private economy - and therefore the absence of an explicit money multiplier - here one should limit any consideration regarding monetary policy to changes in the monetary base. This point is not discussed in Cooley and Hansen (1995). In particular, in their quantitative exercise, they assume M^s to represent the aggregate M1.

periods). In this case (2.13) reduces to

$$M_t^s - M_{t-1}^s + B_t^g - (1 + i_{t-1}^g)B_{t-1}^g = T_t \tag{2.14}$$

$\forall t \geq 0$. As emphasized by Cooley and Hansen (1995), "*in this case, a money injection can be used to directly finance lump-sum transfers or to retire existing government debt. The first of these is analogous to the 'helicopter drop' described by Friedman (1968), and the second is a standard open market operation*".

If the government policy satisfies the present value budget constraint, markets are complete and households do *internalise* (2.14), then Ricardian equivalence holds in this model. A direct implication is that, given the initial stock of government debt (B_0^g) and a particular realization of the money supply process ($M_t^s - M_{t-1}^s$), the time path of B_t^g (for $t \geq 0$) and T_t (for $t \geq 0$) does not matter for the equilibrium allocations. As a consequence, 'helicopter drops' and open market operations are equivalent methods of injecting new money in this model. Therefore one can assume, with no loss of generality, that $B_0^g = 0$.

All together these assumptions imply that no government bonds are held in this economy and the government budget constraint then reduces to

$$M_t^s - M_{t-1}^s = T_t \tag{2.15}$$

$\forall t \geq 0$. Dividing both sides of (2.15) by the price level P_t, one obtains the equivalent expression in real terms :

$$m_t^s - \frac{m_{t-1}^s}{\Pi_t} = \tau_t \tag{2.16}$$

$\forall t \geq 0$; where: $\tau_t \equiv T_t/P_t$ represents real lump sum transfers; $m_t^s \equiv M_t^s/P_t$ is real money supply; and $\Pi_t \equiv P_t/P_{t-1}$ is the (gross) inflation rate.

The monetary authority is assumed to follow an *exogenous* money supply rule: namely, a constant money growth rule[12]. The reason for using such

[12] The following semantic convention is adopted: a monetary policy rule is defined as *exogenous* when it does not react to changes in other variables; viceversa, an *endogenous* rule indicates that the policy instrument reacts to deviations of other variables from the target. An example of exogenous rule is the so-called 'Friedman rule' (after Friedman (1968)); a common typology of endogenous rules is represented by 'Taylor rules' (after Taylor (1993)).

a rule in this context is twofold: it allows a direct comparison with similar flexible price monetary models that can be found in the literature; in the present context, it separates in a neat way the (exogenous) movements in money supply from the (endogenous) movements in money demand.

In each period, per capita nominal money supply is assumed to grow at the gross rate Θ_t. This implies:

$$\frac{M_t^s}{M_{t-1}^s} = \Theta_t \tag{2.17}$$

or, equivalently,

$$M_t^s = M_{t-1}^s + \theta_t M_{t-1}^s \tag{2.18}$$

where $\theta_t \equiv \Theta_t - 1$ defines the (net) money growth rate.

The money supply rule is implemented through monetary injections that take the form of lump sum transfers[13]. In order to satisfy the government budget constraint, a transfer occurring at time t must be proportional to the total money stock from the previous period according to:

$$T_t = \theta_t M_{t-1}^s \tag{2.19}$$

or, in real terms,

$$\tau_t = \frac{\theta_t m_{t-1}^s}{\Pi_t} \tag{2.20}$$

$\forall t \geq 0$.

To study the effects of a monetary surprise, the variable θ_t is assumed to evolve according to the law of motion

$$\ln \theta_t = (1 - \rho_\theta) \ln \theta + \rho_\theta \ln \theta_{t-1} + \epsilon_{\theta_t} \tag{2.21}$$

$\forall t \geq 0$; where: ρ_θ is the autoregressive coefficient (with $0 \leq \rho_\theta \leq 1$), and ϵ_{θ_t} is a random variable serially uncorrelated and normally distributed, with

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[13] In this model, lump sum injections represent additions (subtractions, if negative) to the outstanding monetary base by the monetary authority, regardless of the empirical counterpart used for M^s (monetary base or M1).

zero mean and constant variance ($\sigma^2_{\epsilon_\theta}$). With this specification, the average (net) growth rate of money supply chosen by the monetary authority is equal to θ.

2.2 The RRC model

The use of a cash-in-advance constraint is not the only way to model the link between money and transactions. The alternative approach that will be considered in this section consists of introducing transaction costs explicitly. This is done assuming that real resources (i.e., a given amount of goods) must be used up in the process of exchange. The theoretical implementation of the real resource cost (RRC) approach in a general equilibrium setup is due to Brock (1974), while the analytical characteristcs of the transaction costs function and its functional equivalence with the money-in-utility-function are extensively discussed by Feenstra (1986) and Wang and Yip (1992). The empirical performance of RRC models has been investigated by Sims (1989) and Marshall (1992) - using calibration and estimation methods, respectively.

The general idea consists of modifying the traditional household's problem (which consists in maximising a utility function subject to a budget constraint) by explicitly introducing the costs of transaction into the household's budget constraint. Since the role of cash balances is to reduce transaction costs, a transaction technology that links money to the the level of expenditures must be specified. Feenstra (1986) describes a generic transaction technology for the purchase of consumption goods as a liquidity cost function $\Upsilon[c, m]$, where c and m denote real consumption and real balances, respectively. The function Υ is assumed to be twice continuously differentiable and to satisfy the following properties:

i) $\Upsilon \geq 0, \qquad \Upsilon[0, m] = 0;$

ii) $\Upsilon_c \geq 0, \qquad \Upsilon_m \leq 0;$

iii) $\Upsilon_{cc} \geq 0, \qquad \Upsilon_{mm} \geq 0, \qquad \Upsilon_{cm} \leq 0;$

iv) $c + \Upsilon[c, m]$ *is quasi-convex with expansion paths having non-negative slope.*

Condition i) implies positive liqudity costs only for positive amounts of consumption. Properties ii) and iii) reflect the assumption that transaction costs rise at an increasing rate as consumption increases, and that money has a positive but diminishing marginal productivity in reducing transaction costs. The cross partial derivative $\Upsilon_{cm} \leq 0$ means that the marginal transaction costs of additional consumption do not increase with money holdings. Condition iv) defines a property of the iso-curves of the liquidity cost function, implying that $c + \Upsilon$ increases with income[14].

2.2.1 Households

Describing the analytical properties of Feenstra's transaction technology, the time subscripts have been deliberately omitted. In order to compare the RRC approach with the CIA model, it is essential to introduce a comparable set of timing assumptions. In other words, one needs to re-define appropriately the content of what Feenstra calls 'm'.

According to the timing assumptions for the cash-in-advance constraint described previously, households visit financial markets at the beginning of the period, in order to gather the liquidity needed for transaction purposes. In that case it is not just inherited cash balances that matter, but also the amount of bond holdings and interest payments. Analogously, one can assume that beginning-of-period liquidity can be used to reduce transaction costs in the RRC model. For this purpose an auxiliary variable (A_t) is introduced to denote total liquidity and to formulate the following liquidity constraint (in nominal terms):

$$M^d_{t-1} + (1 + i_{t-1})\, B^d_{t-1} - B^d_t + T_t \geq A_t \tag{2.22}$$

$\forall t \geq 0$. The liquidity constraint (2.22) looks essentially identical to the cash-in-advance constraint (2.7), except for the term A_t replacing $P_t c_t$ on the right hand side.

Dividing both sides of (2.22) by P_t one obtains the liquidity constraint in

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[14] Positive money holdings can be ensured by the additional assumption that $\lim_{m\to 0} \Upsilon_m = -\infty$.

real terms:

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t \geq a_t \tag{2.23}$$

$\forall t \geq 0$. In this way, Feenstra's real balances (m) are replaced by a variable expressing beginning-of-period liquidity in real terms (a_t).

Since the individuals are facing transaction costs in terms of real resources, households' budget constraint needs to be modified, including such costs on the left hand side:

$$\begin{aligned} &M_{t-1}^d + (1 + i_{t-1}) B_{t-1}^d + W_t h_t^s + T_t \\ &\qquad \geq P_t c_t + M_t^d + B_t^d + P_t \Upsilon(\omega_t, c_t, a_t) \end{aligned} \tag{2.24}$$

$\forall t \geq 0$; where the function $\Upsilon(\omega_t, c_t, a_t)$ represents total real resource costs of transactions, and ω_t is a unit transaction cost shock that will be described below.

One can express the budget constraint in real terms, dividing both sides of (2.24) by the price level P_t:

$$\frac{I_{t-1} b_{t-1}^d}{\Pi_t} + \frac{m_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t \geq c_t + m_t^d + b_t^d + \Upsilon(\omega_t, c_t, a_t) \tag{2.25}$$

$\forall t \geq 0$.

The transaction costs function appearing in the budget constraint is defined as

$$\Upsilon(\omega_t, c_t, a_t) \equiv \omega_t \Omega_1 \frac{(c_t)^{\Omega_2 + 1}}{(a_t)^{\Omega_2}} \tag{2.26}$$

$\forall t \geq 0$; where trasaction costs are positively related with real consumption (c_t) and decrease with liquidity in real terms (a_t); $\Omega_1 > 0$ is a scale parameter and $\Omega_2 > 0$ is an elasticity parameter[15]. The variable ω_t represents a stochastic transaction cost component, which follows the first-order autoregressive

[15] Note that expression (2.26) corresponds to a Cobb-Douglas transaction costs function, with constant returns to scale in its arguments.

process

$$\ln \omega_t = (1 - \rho_\omega) \ln \omega + \rho_\omega \ln \omega_{t-1} + \epsilon_{\omega_t} \tag{2.27}$$

$\forall t \geq 0$; where: ρ_ω is the autoregressive coefficient (with $0 \leq \rho_\omega \leq 1$), and ϵ_{ω_t} is a random variable serially uncorrelated and normally distributed, with zero mean and constant variance ($\sigma^2_{\epsilon_\omega}$).

To fully understand (2.26) it is useful to define the *unitary* real transaction cost (i.e., real cost associated with one unit of consumption, q_t) as follows:

$$q_t = \omega_t \Omega_1 \left(\frac{c_t}{a_t} \right)^{\Omega_2} \tag{2.28}$$

$\forall t \geq 0$. Using (2.26) and (2.28), one can re-write total transaction costs as:

$$\Upsilon_t = q_t c_t \tag{2.29}$$

$\forall t \geq 0$. Total transaction costs depend on the amount of consumption (c_t), while the unitary transaction costs (q_t) depend on the relative volume of consumption and liquidity as specified by (2.28).

In the RRC model the household's problem consists in maximising the utility function, subject to the budget constraint (2.25) and the liquidity constraint (2.23)[16].

Since the specification of households' utility function, the problem of the firm and the monetary policy are identical to those in the CIA model, these will be skipped here, to avoid unnecessary repetition.

2.3 The equilibrium

This section derives the equilibrium conditions for the CIA and the RRC models in terms of households' and firms' optimal choices. The necessary first order conditions are determined applying the Lagrangian method. The monetary policy rule and the market clearing conditions 'close' the general equilibrium

[16]Moreover, *no-Ponzi game* conditions must hold to guarantee optimality.

models.

2.3.1 The CIA model

Households

To state the households' problem in terms of the Lagrangian, it is useful to recall that the representative household in the CIA model seeks to maximise the utility stream

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\Phi}}{1-\Phi} + \Psi \frac{(1-h_t^s)^{1-\eta}}{1-\eta} \right) \tag{2.30}$$

, subject to the budget constraint

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t = c_t + m_t^d + b_t^d \tag{2.31}$$

$\forall t \geq 0$, and the cash-in-advance constraint

$$\frac{m_{t-t}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t = c_t \tag{2.32}$$

$\forall t \geq 0$[17].

Stating the problem in terms of the Lagrangian, the household chooses c_t, h_t^s, b_t^d and m_t^d in order to maximise

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[17] Given the analytical characteristics of utility function and budget constraint, the inequality in (2.6) has been replaced by an equality here (for a technical explanation see McCallum (1989)). Regarding (2.32), a positive nominal interest rate will 'force' rational agents to hold an amount of cash balances just sufficient to cover the desired consumption expenditures. Therefore the replacement of the inequality in (2.8) with an equality.

$$\mathcal{L}_t^{CIA} = E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \left(\frac{c_t^{1-\Phi}}{1-\Phi} + \Psi \frac{(1-h_t^s)^{1-\eta}}{1-\eta} \right) + \lambda_t^{CIA} \left[\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t - c_t - m_t^d - b_t^d \right] + \mu_t^{CIA} \left[\frac{m_{t-t}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t - c_t \right] \Bigg\}$$

,where λ_t^{CIA} and μ_t^{CIA} are the Lagrangian multipliers associated with the budget constraint and the cash-in-advance constraint, respectively[18].

The maximisation of the Lagrangian with respect to the choice variables (and after substituting for the Lagrangian multipliers) delivers the following optimality conditions[19]:

$$\frac{\Psi\left(1-h_t^s\right)^{-\eta}}{c_t^{-\Phi}} = \frac{w_t}{I_t} \tag{2.33}$$

$$1 = \beta E_t \left\{ \left(\frac{c_{t+1}}{c_t} \right)^{-\Phi} \frac{I_t}{\Pi_{t+1}} \right\} \tag{2.34}$$

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t = c_t + m_t^d + b_t^d \tag{2.35}$$

$$\frac{m_{t-t}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t = c_t \tag{2.36}$$

Expression (2.33) represents an *intra*temporal condition, which relates the marginal rate of substitution between leisure and consumption (on the left hand side), to the ratio of the respective marginal costs (on the right hand side). Because of the opportunity cost of saving, the gross nominal interest rate (I_t) acts like a 'tax' on consumption, affecting in turn the labour supply choice

[18] Here the purpose of the superscript 'CIA' on the multipliers λ and μ is to distinguish this Lagrangian multipliers from the ones that will be adopted forming the Lagrangian for the RRC model.

[19] Moreover, *transversality conditions* must hold to guarantee optimality.

(h_t^s) via the utility function. Expression (2.34) represents the *inter*temporal condition, which governs the degree of consumption smoothing through time[20]. Equations (2.35) and (2.36) are the constraints, obtained by derivation with respect to the Lagrangian multipliers.

Firms

In each period, the representative firm chooses the amount of working hours (h_t^d) that maximise profits

$$\Gamma_t = P_t e^{z_t} h_t^d - W_t h_t^d \tag{2.37}$$

given the production technology

$$y_t = z_t h_t^d \tag{2.38}$$

Given that firms solve a static problem (i.e., all decisions are taken with respect to one single period), the objective function can be expressed in real terms as

$$\gamma_t = z_t h_t^d - w_t h_t^d \tag{2.39}$$

$\forall t \geq 0$. Considering a generic time period t, the optimal choice of labour is obtained by imposing

$$\frac{\partial \gamma_t}{\partial h_t^d} = z_t - w_t = 0 \tag{2.40}$$

Making use of the production function (2.38), the first order condition (2.40) can be re-written as

$$\frac{y_t}{h_t^d} = w_t \tag{2.41}$$

The optimality condition (2.41) implies that firms demand working hours up to the point where marginal product of labour equals its real marginal cost

[20]Note that the real benefits from saving are expressed by $I_t/E_t\Pi_{t+1}$. This expression can be regarded as the real interest rate (after Fisher (1933)).

(i.e., the real wage, w_t).

Monetary policy

In period t real cash balances evolve to satisfy the government's budget constraint

$$\frac{m_{t-1}^s}{\Pi_t} + \tau_t = m_t^s$$

$\forall t \geq 0$. Given the money growth rate (θ_t) chosen by the monetary authority for period t, recall the characterization of the real lumpsum transfers as

$$\tau_t = \frac{\theta_t m_{t-1}^s}{\Pi_t}$$

$\forall t \geq 0$. Note that a monetary rule implying a constant (and positive) money growth rate, must be implemented through constant (and positive) nominal lump sum transfers T_t.

Market clearing conditions

For a general equilibrium characterisation of the model, the necessary market clearing conditions are required. In this model there are four markets: the goods market, the labour market, the money market and the bonds market. The market clearing condition for the goods market requires aggregate supply and aggregate demand of goods to be equal in every period:

$$y_t = c_t \tag{2.42}$$

$\forall t \geq 0$.The labour market clearing condition equates labour demand and labour supply for every period, according to

$$h_t^d = h_t^s \tag{2.43}$$

$\forall t \geq 0$. The money marlet clears in every period, when money demand by households is equal to the money supply:

$$m_t^d = m_t^s \tag{2.44}$$

$\forall t \geq 0$. Finally, since the bonds in this model are private bonds 'issued' by households, the assumption that all the individuals are alike implies that no bonds are actually exchanged in equilibrium. As a consequence, there will be no bonds outstanding (i.e., a zero net supply for this type of financial assets). Thus, the bonds market clearing condition corresponds to

$$b_t^d = b_t^s = 0 \tag{2.45}$$

$\forall t \geq 0$[21].

2.3.2 The RRC model

Since the problem faced by the firms and the specification of the monetary policy rule are identical under the two monetary models, this section will focus only on those parts where the RRC model differs from the CIA model: namely, the first order conditions characterising the household's problem and (some of) the market clearing conditions.

Households

In the RRC model, households seek to maximise the usual intertemporal utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\Phi}}{1-\Phi} + \Psi \frac{(1-h_t^s)^{1-\eta}}{1-\eta} \right) \tag{2.46}$$

, subject to the budget constraint

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t = c_t + m_t^d + b_t^d + \Upsilon\left(\omega_t, c_t, a_t\right) \tag{2.47}$$

$\forall t \geq 0$, and the liquidity constraint

[21] The fact that private bonds are not exchanged does not prevent these assets to be priced in equilibrium. After all, the nominal interest rate appears in the households' first order conditions.

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1}b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t = a_t \tag{2.48}$$

$\forall t \geq 0$[22].

Stating the problem in terms of the Lagrangian method, the households choose c_t,h_t^s,b_t^d, m_t^d and a_t in order to maximise

$$\mathcal{L}_t^{RRC} = E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \left(\frac{c_t^{1-\Phi}}{1-\Phi} + \Psi \frac{(1-h_t^s)^{1-\eta}}{1-\eta} \right) + \lambda_t^{RRC} \left[\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1}b_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t - c_t - m_t^d - b_t^d - \Upsilon(\omega_t, c_t, a_t) \right] + \mu_t^{RRC} \left[\frac{m_{t-t}^d}{\Pi_t} + \frac{I_{t-1}b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t - a_t \right] \Bigg\}$$

, where λ_t^{RRC} and μ_t^{RRC} are the Lagrangian multipliers associated with the budget constraint (inclusive of real transaction costs) and the liquidity constraint, respectively.

The maximisation of the Lagrangian with respect to the choice variables delivers the following optimality conditions:

$$\frac{\Psi(1-h_t^s)^{-\eta}}{c_t^{-\Phi}} = \frac{w_t}{[1+\Upsilon_{c,t}]} \tag{2.49}$$

$$1 = \beta E_t \left\{ \left(\frac{c_{t+1}}{c_t} \right)^{-\Phi} \left(\frac{1+\Upsilon_{c,t}}{1+\Upsilon_{c,t+1}} \right) \frac{I_{t+1}}{\Pi_{t+1}} \right\} \tag{2.50}$$

$$\Upsilon_{a,t} = 1 - I_t \tag{2.51}$$

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1}b_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t = c_t + m_t^d + b_t^d + \Upsilon(\omega_t, c_t, a_t) \tag{2.52}$$

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[22] See footnote (17) for a discussion about replacing inequalities with equalities in the constraints.

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1}b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t = a_t \tag{2.53}$$

$\forall t \geq 0$[23].

The following definitions have been applied, in order to simplify the notation:

$$\Upsilon(\omega_t, c_t, a_t) \equiv c_t q_t \tag{2.54}$$

$$q_t \equiv \omega_t \Omega_1 (v_t)^{\Omega_2} \tag{2.55}$$

$$v_t \equiv \frac{c_t}{a_t} \tag{2.56}$$

$\forall t \geq 0$. Recalling the explicit functional form for $\Upsilon(\omega_t, c_t, a_t)$ in (2.26), the partial derivatives of the total transaction costs function with respect to real consumption and liquidity, are denoted respectively by

$$\Upsilon_{c,t} = (\Omega_2 + 1) q_t \tag{2.57}$$

$$\Upsilon_{a,t} = -\Omega_2 q_t v_t \tag{2.58}$$

$\forall t \geq 0$.

Expression (2.49) is the *intra*temporal condition for the RRC model, which relates the marginal rate of substitution between leisure and consumption (left hand side) to the ratio of the respective marginal costs (right hand side). Comparing the terms on the left hand side with those appearing in the equivalent CIA model expression (2.33), the marginal cost of consumption is represented now by the real cost of one unit of consumption plus marginal transaction costs ($\Upsilon_{c,t}$). Expression (2.50) refers to the *inter*temporal condition, which governs the degree of consumption smoothing thorugh time. This now depends on the (present and expected) marginal cost of consumption ($\Upsilon_{c,t}$ and $E_t\Upsilon_{c,t+1}$) and the opportunity cost of saving in the next period ($E_tI_{t+1}/E_t\Pi_{t+1}$). Expression

[23] Moreover, *transversality conditions* must hold to guarantee optimality.

(2.51) represents the arbitrage condition between liquidity and bonds, equating the reduction in marginal transaction costs by holding liquidity ($-\Upsilon_{a,t}$) to the net nominal interest rate on bonds (i_t). Equations (2.52) and (2.53) are simply the two constraints, obtained by derivation with respect to the Lagrangian multipliers.

Market clearing conditions

The market clearing conditions for labour services, money and bonds are the same as those derived in the CIA model. The only condition which is affected by the presence of transaction costs concerns the equilibrium in the goods market.

Given that real resources must be used up in transactions, total output (y_t) now must be equal to:

$$y_t = c_t + \Upsilon(\omega_t, c_t, a_t) \tag{2.59}$$

$\forall t \geq 0$. As stressed by Walsh (2003) "*the appropriate definition of the consumption variable* [in the RRC model] *needs to be considered if one attempts to use* [this] *framework to draw implications for actual macro time series.*". Given that deriving quantitative implications is one of the purposes of this work, using the definition of total transaction costs (2.54) one can rewrite the market clearing condition (2.59) as

$$y_t = c_t (1 + q_t) \tag{2.60}$$

$\forall t \geq 0$. The expression appearing on the right hand side of (2.60) represents total consumption. To distinguish total consumption in the RRC model from the analogous variable in the CIA model, a capital letter (C_t) will be used in the former:

$$C_t \equiv c_t (1 + q_t) \tag{2.61}$$

$\forall t \geq 0$.

Once the equilibrium conditions for the CIA model and RRC model have been derived, the analysis can now focus on the dynamics.

2.4 The dynamics

In order to explore and compare the dynamic performance of the two monetary models, subject to the random shocks described above, one needs to transform the non-linear system of equations characterising the general equilibrium into a linear system. This is done by taking a log-linear approximation around the deterministic steady state, applying the methodology described in Uhlig (1999). For each model, this section will take the following steps: firstly, presenting the equilibrium as obtained in the previous section; secondly, illustrating some steady state relationships; and finally by deriving the log-linear model.

2.4.1 The CIA model

In the case of the CIA model, the set of optimality conditions for households and firms, together with the specification of the monetary policy rule and the necessary market clearing conditions correspond to the following system of (non-linear) equations:

$$\frac{\Psi\left(1-h_t^s\right)^{-\eta}}{c_t^{-\Phi}}=\frac{w_t}{I_t} \tag{2.62}$$

$$1=\beta E_t\left\{\left(\frac{c_{t+1}}{c_t}\right)^{-\Phi}\frac{I_t}{\Pi_{t+1}}\right\} \tag{2.63}$$

$$\frac{m_{t-1}^d}{\Pi_t}+\tau_t=c_t \tag{2.64}$$

$$\frac{y_t}{h_t^d}=w_t \tag{2.65}$$

$$y_t=z_t h_t^d \tag{2.66}$$

$$\frac{m_{t-1}^s}{\Pi_t}+\tau=m_t^s \tag{2.67}$$

$$\tau_t\equiv\theta_t\frac{m_{t-1}^s}{\Pi_t} \tag{2.68}$$

$$y_t = c_t \tag{2.69}$$

$$h_t^d = h_t^s \tag{2.70}$$

$$m_t^d = m_t^s \tag{2.71}$$

$$\ln z_t = (1 - \rho_z) \ln z + \rho_z \ln z_{t-1} + \epsilon_{z_t} \tag{2.72}$$

$$\ln \theta_t = (1 - \rho_\theta) \ln \theta + \rho_\theta \ln \theta_{t-1} + \epsilon_{\theta_t} \tag{2.73}$$

$\forall t \geq 0$.[24]

Money demand and velocity of money

After all markets have cleared, the application of Walras' law implies that the evolution of real balances in the hands of the households follows

$$\frac{m_{t-1}^d}{\Pi_t} + \tau_t = m_t^d \tag{2.74}$$

$\forall t \geq 0$. Combining (2.74) with (2.64) one obtains the following expression for households' demand for real balances:

$$m_t^d = c_t \tag{2.75}$$

$\forall t \geq 0$. Expression (2.75) characterises money demand in the CIA model. In isolation, this money demand function does not appear to be related to any opportunity cost variable (i.e., the nominal interest rate). However, from a general equilibrium point of view this is incorrect: in fact, consumption (c_t) is linked to the nominal interest rate through the *intra*temporal and *in-*

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[24]The reason why the budget constraint does not appear in the non-linear system can be seen as a direct consequence of Walras' law.

*ter*temporal conditions ((2.62) and (2.63), respectively)[25]. A clearer implication of (2.75) is that the velocity of money is not responding to movements in the nominal interest rate. In fact, one can show that the *consumption-based* velocity is constantly equal to unity:

$$VEL\left(c_t\right) \equiv \frac{c_t}{m_t^s} = 1 \tag{2.76}$$

$\forall t \geq 0$. Using the goods market clearing condition, it is possible to check that the same applies to the *output*-based velocity:

$$VEL\left(y_t\right) \equiv \frac{y_t}{m_t^s} = 1 \tag{2.77}$$

$\forall t \geq 0$[26]. The use of (2.76) and (2.77) is mainly motivated by purposes of quantitative analysis. To find discussions about these velocity measures is not uncommon in the literature[27].

Steady state

Before turning to the log-linear system, it is useful to have a look at some long-run relationships implied by the model. When all the variables have reached their deterministic steady state, time subscripts can be 'removed' from the non-linear equations characterising the equilibrium. In this way it is possible to inspect how monetary factors impact the fundamental structure of the economy.

In steady state expressions (2.67) and (2.68) can be used to obtain

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[25] In support of this statement, Carlstrom and Fuerst (1995) discuss the *endogenous* movement in real balances for a CIA model where the monetary policy instrument is the nominal interest rate.

[26] The reason why m_t^d is replaced by m_t^s when deriving the expressions of the velocities from (2.75), emphasises the fact that, generally, different *empirical* measures of velocity are constructed using monetary aggregates at the denominator. Since in this chapter real balances used by households (m_t^d) coincide with the total money supply (m_t^s), the link between monetary aggregate and the relevant expenditure category looks trivial. However, this point will become extremely relevant in Chapters 4 and 5, where total money and households' money will not coincide.

[27] See Hodrick, Kocherlakota and Lucas (1991) and Cole and Ohanian (2002).

$$\Pi = \Theta \tag{2.78}$$

or, equivalently

$$\pi = \theta \tag{2.79}$$

The result indicates that the steady state inflation rate is determined by the money growth rate choosen by the monetary authority. A direct implication of this is that the (steady state) real quantity of money (m) is constant - i.e., *neutrality of money* holds in steady state. Given the household's subjective discount rate (β), the *inter*temporal condition (2.63) can be used to determine the long-run nominal interest rate:

$$I = \frac{\Theta}{\beta} \tag{2.80}$$

Since the consumption-labour choice in (2.62) depends on the level of nominal interest rate, it is possible to combine this with the steady state expression for (2.65) with (2.80) in order to obtain:

$$\frac{\Psi h}{(1-h)^{\eta}} = \frac{\beta}{\Theta} y c^{-\Phi} \tag{2.81}$$

Making use of (2.66) and (2.69) at steady state, expression (2.81) can be re-written as:

$$\frac{y^{\Phi}}{(1-y)^{\eta}} = \frac{\beta}{\Psi\Theta} \tag{2.82}$$

For positive values of the parameters Φ, η, β and Ψ the left hand side of expression (2.82) is positively related to real output, while the right hand side is negatively related to the money growth rate. As a consequence, a permanently higher money growth rate lowers output: therefore, superneutrality of money does not hold in this model.

The transmission mechanism from permanent changes in money growth to output can be analysed through the household's labour supply choice. In fact, by using the production function one obtains:

$$\frac{h^{\Phi}}{(1-h)^{\eta}} = \frac{\beta}{\Psi\Theta} \tag{2.83}$$

A permanent increase in the inflation tax on cash purchases causes a substitution from consumption to leisure, thereby reducing the supply of working hours (and therefore real output).

Log-linear approximation

Using the methodology decribed by Uhlig (1999) one can linearise the original model, taking a first order Taylor expansion around the steady state. In this and in the following chapters, the approximation is taken with respect to a steady state characterised by zero output growth and a positive inflation rate (an annual equivalent of 5%, corresponding to an equivalent constant positive money growth rate). The usefulness of the log-linearisation method is twofold: on the one hand, it allows one to solve the model applying standard solution methods for linear rational expectations models[28]; on the other hand, it redefines all the economic variables as percentage deviations from steady state, isolating their cyclical fluctuations. The result is a linear system of equations, where the variables with the 'hat' indicate percentage deviations of the original variables from their long-run values[29], while variables without time subscript indicate steady state values:

consumption/labour:

$$\left[\eta \frac{h}{(1-h)}\right] \hat{h}_t^s = \hat{w}_t - \Phi \hat{c}_t - \hat{\imath}_t \tag{2.84}$$

consumption/saving:

[28]For different approaches see Blanchard and Kahn (1980), Uhlig (1999), Sims (2002), Christiano (2002) and Klein (2000).

[29]The only exceptions are the inflation rate ($\hat{\pi}_t$), the nominal interest rate ($\hat{\imath}_t$) and the money growth rate ($\hat{\theta}_t$), where the 'hat' indicates deviations in levels. The fact that the original *net* rates are small numbers with respect to one, the correspondent *gross* rates (Π_t, I_t and Θ_t), can be log-linearised applying the following approximation: $\ln \Pi_t = \ln(1+\pi_t) \simeq \pi_t$.

$$\Phi E_t \hat{c}_{t+1} - \Phi \hat{c}_t = \hat{\imath}_t - E_t \hat{\pi}_{t+1} \tag{2.85}$$

money demand:

$$\hat{m}_t^d = \hat{c}_t \tag{2.86}$$

labour demand:

$$\hat{y}_t - \hat{h}_t^d = \hat{w}_t \tag{2.87}$$

output:

$$\hat{y}_t = \hat{z}_t + \hat{h}_t^d \tag{2.88}$$

money supply:

$$\frac{m^s}{\Pi} \hat{m}_{t-1}^s - \frac{m^s}{\Pi} \hat{\pi}_t + \tau \hat{\tau}_t = m^s \hat{m}_t^s \tag{2.89}$$

monetary injection:

$$\tau \hat{\tau}_t \equiv \Theta \frac{m^s}{\Pi} \hat{\theta}_t + \theta \frac{m^s}{\Pi} \hat{m}_{t-1}^s - \theta \frac{m^s}{\Pi} \hat{\pi}_t \tag{2.90}$$

goods market clearing condition:

$$\hat{y}_t = \hat{c}_t \tag{2.91}$$

labour market clearing condition:

$$\hat{h}_t^d = \hat{h}_t^s \tag{2.92}$$

money market clearing condition:

$$\hat{m}_t^d = \hat{m}_t^s \tag{2.93}$$

consumption-based velocity:

$$VE\hat{L}\left(c_t\right) \equiv \hat{c}_t - \hat{m}_t^s \tag{2.94}$$

output-based velocity:

$$V E\hat{L}\left(y_t\right) \equiv \hat{y}_t - \hat{m}_t^s \tag{2.95}$$

technology shock:

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \epsilon_{z_t} \tag{2.96}$$

monetary shock:

$$\hat{\theta}_t = \rho_\theta \hat{\theta}_{t-1} + \epsilon_{\theta_t} \tag{2.97}$$

$\forall t \geq 0$; where: $\epsilon_{z_t} \sim N\left(0, \sigma^2_{\epsilon_z}\right)$ and $\epsilon_{\theta_t} \sim N\left(0, \sigma^2_{\epsilon_\theta}\right)$. The definitions of *consumption*-based and *output*-based velocity have been also included, in order to obtain the related simulation results.

A quick look at the log-linearised equations reveals some important aspects of the model. In particular, expressions (2.89) and (2.90), together with the money market clearing condition (2.93), can be used to derive the inflation dynamics as:

$$\hat{\pi}_t = \hat{\theta}_t - \left(\hat{m}_t^d - \hat{m}_{t-1}^d\right) \tag{2.98}$$

$\forall t \geq 0$. This represents a key result in monetary models with flexible prices: fluctuations in the inflation rate around its steady state value are determined by the difference between money supply *growth* ($\hat{\theta}_t$) and money demand *growth* ($\hat{m}_t^d - \hat{m}_{t-1}^d$). In this model money is *neutral* even *outside* the steady state: a one-shot monetary shock increasing the quantity of money today will change the price level proportionally, leaving all real variables unaffected.

2.4.2 The RRC model

In the case of the RRC model, the set of optimality conditions for households and firms, together with the specification of the monetary policy rule and the necessary market clearing conditions correspond to the following system of (non-linear) equations:

$$\frac{\Psi\left(1 - h_t^s\right)^{-\eta}}{c_t^{-\Phi}} = \frac{w_t}{\left[1 + \Upsilon_{c,t}\right]} \tag{2.99}$$

$$1 = \beta E_t \left\{ \left(\frac{c_t}{c_{t+1}} \right)^{\Phi} \left(\frac{1 + \Upsilon_{c,t}}{1 + \Upsilon_{c,t+1}} \right) \frac{I_{t+1}}{\Pi_{t+1}} \right\} \tag{2.100}$$

$$1 - [1 + \Upsilon_{a,t}] = I_t - 1 \tag{2.101}$$

$$\frac{m_{t-1}^d}{\Pi_t} + \tau_t = a_t \tag{2.102}$$

$$\Upsilon_t \equiv c_t q_t \tag{2.103}$$

$$q_t \equiv \omega_t \Omega_1 (v_t)^{\Omega_2} \tag{2.104}$$

$$v_t \equiv \frac{c_t}{a_t} \tag{2.105}$$

$$[1 + \Upsilon_{c,t}] = 1 + (\Omega_2 + 1) q_t \tag{2.106}$$

$$[1 + \Upsilon_{a,t}] = 1 - \Omega_2 q_t v_t \tag{2.107}$$

$$\frac{y_t}{h_t^d} = w_t \tag{2.108}$$

$$y_t = z_t h_t^d \tag{2.109}$$

$$\frac{m_{t-1}^s}{\Pi_t} + \tau = m_t^s \tag{2.110}$$

$$\tau_t \equiv \theta_t \frac{m_{t-1}^s}{\Pi_t} \tag{2.111}$$

$$y_t = C_t \tag{2.112}$$

$$C_t \equiv c_t Q_t \tag{2.113}$$

$$h_t^d = h_t^s \tag{2.114}$$

$$m_t^d = m_t^s \tag{2.115}$$

$$\ln z_t = (1-\rho_z)\ln z + \rho_z \ln z_{t-1} + \epsilon_{z_t} \tag{2.116}$$

$$\ln \theta_t = (1-\rho_\theta)\ln \theta + \rho_\theta \ln \theta_{t-1} + \epsilon_{\theta_t} \tag{2.117}$$

$$\ln \omega_t = (1-\rho_\omega)\ln \omega + \rho_\omega \ln \omega_{t-1} + \epsilon_{\omega_t} \tag{2.118}$$

$\forall t \geq 0$. Where $Q_t \equiv 1 + q_t$ indicates gross transaction costs.

Money demand and velocity of money

After all markets have cleared, the application of Walras' law implies that the evolution of real balances in the hands of the households follows

$$\frac{m_{t-1}^d}{\Pi_t} + \tau_t = m_t^d \tag{2.119}$$

$\forall t \geq 0$. Combining (2.119) with (2.102) one obtains the following expression for households' demand for real balances:

$$m_t^d = a_t \tag{2.120}$$

$\forall t \geq 0$.In order to derive the expression of the money demand for the RRC model, one needs to combine (2.120), (2.101), (2.104), (2.105) and (2.107), to obtain

$$m_t^d = c_t \left(\frac{\Omega_1 \Omega_2 \omega_t}{I_t - 1} \right)^{\frac{1}{\Omega_2 + 1}} \tag{2.121}$$

$\forall t \geq 0$. Expression (2.121) shows that real balances respond positively to the expenditure variable (c_t) and a transaction costs shock (ω_t), and a negatively to the nominal interest rate ($i_t \equiv I_t - 1$). In addition to that, it implies a unitary elasticity with respect to consumption[30]. Differently than the CIA model, this money demand function allows endogenous variations in the velocity of money. However, one needs to be cautious in defining this magnitude in the RRC model. In order to compare the two models, one needs to define velocity in the same way. For this reason the *consumption*-based velocity in the RRC model takes the form:

$$VEL\left(C\right)_t \equiv \frac{C_t}{m_t^s} \tag{2.122}$$

$\forall t \geq 0$; where total consumption (C_t) includes real resources devoted to transactions. Using the goods market clearing condition, the *output*-based velocity corresponds to

$$VEL\left(y\right)_t \equiv \frac{y_t}{m_t^s} \tag{2.123}$$

$\forall t \geq 0$.

Steady state

As for the CIA model, expressions (2.110) and (2.111) can be used to obtain

$$\pi = \theta \tag{2.124}$$

, where the steady state inflation rate is determined by the money growth rate choosen by the monetary authority. As before, the (steady state) real quantity of money (m) is constant - i.e., *neutrality of money* holds in steady state of the RRC model as well. Given the household's subjective discount rate (β), the *inter*temporal condition (2.100) can be used to determine the long-run nominal interest rate:

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[30] For $\Omega_2 \longrightarrow \infty$ expression (2.121) collapses into $m_t^d = c_t$, which is the expression for the demand for money derived under the CIA model. It is worth noticing that, even in that case the RRC model does not become perfectly equivalent to the CIA model. In fact, in *this* RRC model $m_t^d = c_t$ would still imply positive total transaction costs $\Upsilon\left(c_t, a_t\right) = \omega_t \Omega_1 c_t$ on the right hand side of the households' budget constraint (which do not appear in the CIA model).

$$I = \frac{\Theta}{\beta} \tag{2.125}$$

, which is identical to the one found for the CIA model. In order to analyse the steady state properties of the RRC model one can start by re-arranging the steady state expression for the money demand (2.121) in order to obtain the following:

$$\mathbf{v} \equiv \frac{c}{m} = \left(\frac{\Theta/\beta - 1}{\Omega_1 \Omega_2} \right)^{\frac{1}{\Omega_2 + 1}} \tag{2.126}$$

Now, recalling the definition of the unitary transaction costs (2.104), it follows that a permanently higher money growth rate causes an increase in the unitary costs of transactions according to:

$$q \equiv \Omega_1 \left(\frac{\Theta/\beta - 1}{\Omega_1 \Omega_2} \right)^{\frac{\Omega_2}{\Omega_2 + 1}} \tag{2.127}$$

In order to inspect the effects of changes in the money growth rate on real output, one can combine expressions (2.99), (2.108) and the production function, at the steady state, in order to obtain:

$$\frac{1}{(1-h)^{\eta}} = \frac{1}{\Psi} \frac{c^{-\Phi}}{(1 + \Upsilon_c)} \tag{2.128}$$

Making use of the goods market clearing condition, $y \equiv cQ$, expression (2.128) can be re-written as

$$\frac{1}{(1-h)^{\eta}} = \frac{1}{\Psi} \frac{(y/Q)^{-\Phi}}{(1 + \Upsilon_c)} \tag{2.129}$$

Using the steady state value for the marginal transaction costs of consumption $\Upsilon_c = (1 + \Omega_2) q$ and recalling the definition of the gross unitary transaction costs (Q), one can re-write (2.129) as

$$\frac{y^{\Phi}}{(1-y)^{\eta}} = \frac{1}{\Psi} \frac{(1+q)^{\Phi}}{1 + q + \Omega_2 q} \tag{2.130}$$

, where - given positive values for the parameters Φ and η - the left hand side (LHS) is positively related with output. The value of the expression on the right hand side (RHS) of (2.129) varies with the level of unitary transaction costs (q). In particular:

$$\frac{d\,(RHS)}{dq} = \frac{1}{\Psi}\frac{\Phi\,(1+q)^{\Phi-1}\,(1+q+\Omega_2 q) - (1+\Omega_2)\,(1+q)^{\Phi}}{(1+q+\Omega_2 q)^2} \qquad (2.131)$$

In this case, the direction of change depends on the relative value of the parameters Φ and Ω_2. Because of the assumption of balanced growth this study adopts a log-utility function for consumption ($\Phi = 1$), and expression (2.131) reduces to:

$$\frac{d\,(RHS)}{dq} = -\frac{1}{\Psi}\frac{\Omega_2}{(1+q+\Omega_2 q)^2} < 0 \qquad (2.132)$$

Since $\Omega_2 > 0$, (2.132) implies that the RHS of (2.129) decreases with q.

Summing up: a permanently higher money growth rate increases the unitary transaction costs, which - according to (2.130) - reduce real output.

On the one hand, a permanent increase in the inflation tax causes a substitution from cash to credit purchases: whatever the means of payment used by households, the purchase of consumption goods becomes more expensive. On the other hand, the shift from cash to credit commands more real resources devoted to transactions. Whether a higher money growth will imply an increase or a decrease in output is going to depend on the relative weight of both effects on labour supply (i.e., on the value of the parameters defining preferences and transaction technologies).

Log-linear approximation

As for the CIA model, one can proceed log-linearising the RRC model around its steady state. As before, the variables with the 'hat' indicate percentage deviations of the original variables from their long-run values[31], while variables without time subscript indicate steady state values:

[31] The only exceptions are the inflation rate ($\hat{\pi}_t$), the nominal interest rate ($\hat{\imath}_t$) the money growth rate ($\hat{\theta}_t$), the unitary transaction costs ($\hat{q}_t$) and the marginal transaction costs ($\hat{\Upsilon}_{c,t}$ and $\hat{\Upsilon}_{a,t}$), where 'hat' indicates deviations in levels. See also footnote (30).

leisure/consumption:

$$\left[\eta\frac{h}{(1-h)}\right]\hat{h}_t^s = \hat{w}_t - \Phi\hat{c}_t - \hat{\Upsilon}_{c,t} \tag{2.133}$$

consumption/saving:

$$\Phi E_t\hat{c}_{t+1} - \Phi\hat{c}_t + E_t\hat{\Upsilon}_{c,t+1} - \hat{\Upsilon}_{c,t} = E_t\hat{\imath}_{t+1} - E_t\hat{\pi}_{t+1} \tag{2.134}$$

marginal costs of liquidity:

$$-\left[1+\Upsilon_{a,t}\right]\hat{\Upsilon}_{a,t} = I\hat{\imath}_t \tag{2.135}$$

money demand:

$$\hat{m}_t^d = \hat{a}_t \tag{2.136}$$

total transaction costs:

$$\Upsilon\hat{\Upsilon}_t = cq\hat{c}_t + cQ\hat{q}_t \tag{2.137}$$

unit transaction costs:

$$Q\hat{q}_t = q\hat{\omega}_t + q\Omega_2\hat{v}_t \tag{2.138}$$

liquidity ratio:

$$\hat{v}_t = \hat{c}_t - \hat{a}_t \tag{2.139}$$

marginal transaction cost of consumption:

$$\left[1+\Upsilon_c\right]\hat{\Upsilon}_{c,t} = \left[\Omega_2+1\right]Q\hat{q}_t \tag{2.140}$$

marginal transaction cost of liquidity:

$$\left[1+\Upsilon_a\right]\hat{\Upsilon}_{a,t} = -\Omega_2 Qv\hat{q}_t - \Omega_2 qv\hat{v}_t \tag{2.141}$$

labour demand:

$$\hat{y}_t - \hat{h}_t^d = \hat{w}_t \tag{2.142}$$

real output:

$$\hat{y}_t = \hat{z}_t + \hat{h}_t^d \tag{2.143}$$

money supply:s

$$\frac{m^s}{\Pi}\hat{m}_{t-1}^s - \frac{m^s}{\Pi}\hat{\pi}_t + \tau\hat{\tau}_t = m^s\hat{m}_t \tag{2.144}$$

monetary injection:

$$\tau\hat{\tau}_t \equiv \Theta\frac{m^s}{\Pi}\hat{\theta}_t + \theta\frac{m^s}{\Pi}\hat{m}_{t-1}^s - \theta\frac{m^s}{\Pi}\hat{\pi}_t \tag{2.145}$$

goods market clearing condition:

$$\hat{y}_t = \hat{C}_t \tag{2.146}$$

labour market clearing condition:

$$\hat{h}_t^d = \hat{h}_t^s \tag{2.147}$$

money maret clearing condition:

$$\hat{m}_t^d = \hat{m}_t^s \tag{2.148}$$

total consumption:

$$C\hat{C}_t \equiv c\hat{c}_t + \Upsilon\hat{\Upsilon}_t \tag{2.149}$$

consumption-based velocity:

$$VE\hat{L}\left(C\right)_t \equiv \hat{C}_t - \hat{m}_t^s \tag{2.150}$$

output-based velocity:

$$VE\hat{L}\left(y\right)_t \equiv \hat{y}_t - \hat{m}_t^s \tag{2.151}$$

technology shock:

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \epsilon_{z_t} \tag{2.152}$$

monetary shock:

$$\hat{\theta}_t = \rho_\theta \hat{\theta}_{t-1} + \epsilon_{\theta_t} \tag{2.153}$$

transaction cost shock:

$$\hat{\omega}_t = \rho_\omega \hat{\omega}_{t-1} + \epsilon_{\omega_t} \tag{2.154}$$

$\forall t \geq 0$; where: $\epsilon_{z_t} \sim N\left(0, \sigma^2_{\epsilon_z}\right)$, $\ln \epsilon_{\theta_t} \sim N\left(0, \sigma^2_{\epsilon_\theta}\right)$ and $\epsilon_{\omega_t} \sim N\left(0, \sigma^2_{\epsilon_\omega}\right)$ indicate the standard deviations of the shocks. The definitions of *consumption*-based and *output*-based velocity have been also included.

As for the CIA model, expressions (2.144) and (2.145), together with the money market clearing condition (2.148), lead to the same expression for the inflation dynamics:

$$\hat{\pi}_t = \hat{\theta}_t - \left(\hat{m}^d_t - \hat{m}^d_{t-1}\right) \tag{2.155}$$

$\forall t \geq 0$. Note that, as in the CIA model, money is *neutral* even *outside* the steady state: a one-shot monetary shock increasing the quantity of money today will change the price level proportionally, leaving all real variables unaffected.

2.5 Quantitative analysis

In the first part of this section numerical values are assigned to structural parameters and long-run relationships. The remaining coefficients in the linear approximations are derived using the the steady state relationships, implied by the original non-linear system. Given these calibration values, the last part of the section will compare the qualitative and quantitative impact of the stochastic shocks on the endogenous variables of the two monetary models. In particular, the analysis will focus on the *impulse-response* dynamics of the CIA and the RRC models.

2.5.1 Calibration

In order to derive the response of the baseline models to stochastic shocks, one needs to assign numerical values to the parameters appearing in the linear equations: this is the essence of calibration[32]. For the purpose of comparison, the following criteria have been taken into account : a) keeping (wherever possible) the values for the 'non-monetary' parameters in line with the RBC literature; b) 'choosing' the remaining parameters in order to make the CIA and the RRC models *comparable*.

Calibration of the CIA model

Table 2.1 reports the values for the parameters characterizing the utility function, some key long-run relationships and the stochastic shocks for the CIA model.

As shown by King, Plosser and Rebelo (1988), in a model characterised by utility being additively separable over consumption and leisure, log-utility for consumption is required to obtain constant steady state labour supply. The steady state labour supply has been set to one-third of the time endowment ($h = 0.33$), while the log-utility in consumption is guaranteed by setting the coefficient of relative risk aversion equal to unity ($\Phi = 1$). As explained by Campbell (1994), once the criteria for log-utility of consumption has been matched, the balanced growth result is not affected by the value of the labour supply elasticity parameter ($\eta > 0$). In order to get a logarithmic utility function in both arguments this parameter is also set to one. The value for the (quarterly) discount factor ($\beta = 0.989$), the autoregressive coefficient of the technology shock ($\rho_z = 0.95$) and its standard deviation ($\sigma_{\epsilon_z} = 0.007$) are set in line with the standard RBC literature[33] and are adopted from Cooley and Hansen (1995). The main reference for the monetary values is represented by Cooley and Hansen (1989), where the calibration is built around the monetary aggregate M1. The (exogenous) net nominal money growth rate is set in order

[32] See Prescott (1986).

[33] For a discussion about the value of the discount factor see Campbell (1994). The characteristics of the technology shock are derived from the statistical properties of the so-called 'Solow residual' (see Kydland and Prescott (1982)).

to deliver an annual net growth of 5% (this corresponds to $\theta = 0.0125$ quarterly); while the autoregressive parameter ($\rho_\theta = 0.5$) and standard deviation ($\sigma_{\epsilon_\theta} = 0.0089$) are derived from the estimation of an autoregressive process of M1[34].

The parameter values in Table 2.1 and the steady state relationships derived from the non-linear model are sufficient to derive all the remaining coefficients of the CIA linear system. Moreover, one can check whether the baseline calibration is able to generate steady state values (or ratios) compatible with the empirical evidence. All these results are reported in Table 2.2. Note that the long-run relationship between the money growth rate and inflation is one to one for this model. As stressed by Walsh (2003), this result confirms the long-run relationship between these two variables found by McCandless and Weber (1995). The baseline calibration implies a (quarterly) average net nominal interest rate (i) of 2.38% and a weight parameter for leisure of $\Psi = 1.9536$ in the utility function. The CIA model delivers an unrealistic result for the consumption share in aggregate output and the velocity of money. Regarding the former, the baseline calibration delivers a consumption share equal to 1, while in reality this is a lower number[35]. This is not surprising, since this model abstracts from capital. About the latter, the baseline calibration also obtains a value equal to 1, where Walsh (2003) reports a steady state value of 1.43 for the same ratio. In this case the discrepancy is due to the cash-in-advance specification, which implies $m = c$.

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[34] In order to check for robustness of the results, a sensitivity analysis exercise has been carried out, using different values for Θ and ρ_θ. However, these attempts did not change significantly the quantitative results reported in the next sections.

[35] Ahmed and Rogers (2000) report a value for the consumption-output ratio between 63.3 and 67.8.

parameter/variable	description	value
Φ	relative risk aversion	1
η	inverse of labour supply elasticity	1
β	discount factor	0.989
h	working hours	0.33
θ	net money growth rate	0.0125
ρ_z	autoregressive param. technology shock	0.95
σ_{ϵ_z}	s.d. technology shock	0.007
ρ_θ	autoregressive param. monetary shock	0.5
σ_{ϵ_θ}	s.d. monetary shock	0.0089

Table 2.1: Baseline calibration of CIA model (without capital).

parameter/variable	description	value
Θ	gross money growth rate	1.0125
Π	gross inflation rate	1.0125
π	net inflation rate	0.0125
I	gross nominal interest rate on bonds	1.0238
i	net nominal interest rate on bonds	0.0238
y	real output	0.3333
w	real wage	1
c	real consumption	0.3333
c/y	consumption share of output	1
Ψ	preference parameter for leisure	1.9536
m	real cash balances	0.3333
τ	real monetary injection	0.0041
c/m	consumption-based velocity	1
y/m	output-based velocity	1

Table 2.2: Steady state values of CIA model (without capital) at baseline calibration.

Calibration of the RRC model

As illustrated in the previous sections, the CIA model and the RRC model share the same basic (RBC) structure. The only dimension along which they differ concerns the way money is introduced. For reasons of comparison, the calibration of the RRC model adopts the same numerical values used for the CIA model, except for the transaction technology parameters. The baseline calibration of the RRC model is reported in Table 2.3 below.

Four additional parameters appear in Table 2.3, when compared with Table 2.1. The parameter Ω_2 reflects the curvature of the transaction cost function: setting $\Omega_2 = 1$ is equivalent to assume quadratic (total) transaction costs[36]. A key variable for a comparison with the CIA model is represented by the *liquidity ratio*, defined as $v \equiv c/m$. Because of real resources devoted to transaction, in the RRC model the *utility*-consumption (i.e., the consumption entering in the utility function of the representative household, denoted with c) does not coincide with *total* consumption (C)[37]. Setting $v = 1$ is equivalent to assuming that (in steady state) the representative household participates in the goods market, carrying an amount of real cash balances equivalent to the *utility*-consumption. In this context, this calibration strategy for v is at the core of the comparability between the CIA model and the RRC model[38].

The parameters characterising the transaction costs shock, are taken from an exercise proposed by Sims (1989), where the autoregressive parameter for the transaction cost shock ρ_ω and its standard deviation σ_{ϵ_ω} are set to 0.8 and

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[36] An analogy for the interpretation of the parameters governing the transaction cost function can be found in the literature about the adjustment costs of investment. See Casares and McCallum (2006) for a recent treatment.

[37] For this reason, the ratio v (despite the notation) does not represent, strictly speaking, the definition for the *consumption*-based velocity of money (defined, instead, by the ratio C/m). This is not the case in the CIA model, where consumption providing utility coincides with total consumption. This distinction in the RRC model is essential in order to assign empirical content to this particular velocity measure.

[38] In order to check for robustness of the results, a sensitivity analysis exercise has been carried out, using different values for Ω_2 and v. However, these attempts did not change significantly the quantitative results reported in the next sections.

0.01, respectively. Differently than the features of Cooley and Hansen (1989) monetary shock, the calibration of the transaction cost shock by Sims (1989) is not the result of estimation. These values are simply choosen by Sims (1989) in order to represent a 'persistent shock' after an initial 1% increase in the unit transaction costs[39].

The combination of the parameter values in Table 2.3 with the remaining steady state relationships characterising the RRC model, allow to compute the value for all the remaining coefficients. Given that most of the parameters values are identical to those used for the calibration of the CIA model, there are strong similarities between the results reported in Table 2.2 and Table 2.4 below. As for the CIA model, the long-run relationship between the money growth rate and inflation is one-to-one for the RRC model, and, becasue of the absence of capital, the consumption share of aggregate output (C/y) is also equal to 1. However, as anticipated before, an important difference concerns the value of *consumption*-based velocity, which results greater than 1 ($C/m = 1.0238$), and somewhat closer to the where steady state value of 1.43 reported by Walsh (2003). Here the discrepancy between the CIA model and the RRC model is due to the fact that part of consumption can be bought *on credit*, at the real (unitary) cost of $q = 2.38\%$. Given the particular calibration of the transaction cost function (where $\Omega_2 = 1$ and $v = 1$), in steady state the unit transaction costs of consumption are equal to the nominal interest rate ($i = 2.38\%$). As a consequence, in the RRC model, the part of output devoted to transaction purposes corresponds to $\Upsilon/y = 2.31\%$.

[39] In order to check for robustness of the results, a sensitivity analysis exercise has been carried out, using different values for ρ_ω. However, these attempts did not change significantly the quantitative results reported in the next sections.

parameter	**description**	**value**
Φ	relative risk aversion	1
η	inverse of labour supply elasticity	1
β	discount factor	0.989
h	working hours	0.3333
θ	net money growth rate	0.0125
Ω_2	elasticity param. of transaction cost function	1
υ	liquidity ratio	1
ρ_z	autoregressive param. technology shock	0.95
σ_{ϵ_z}	s.d. technology shock	0.007
ρ_θ	autoregressive param. monetary shock	0.5
σ_{ϵ_θ}	s.d. monetary shock	0.0089
ρ_ω	autoregressive param. transaction costs shock	0.8
σ_{ϵ_ω}	s.d. transaction costs shock	0.01

Table 2.3: Baseline calibration of RRC model (without capital).

parameter/variable	**description**	**value**
Θ	gross money growth rate	1.0125
Π	gross inflation rate	1.0125
π	net inflation rate	0.0125
I	gross nominal interest rate on bonds	1.0238
i	net nominal interest rate on bonds	0.0238
q	net unitary transaction costs	0.0238
Q	gross unitary transaction costs	1.0238
Ω_1	scale param. of transaction cost function	0.0238
Υ_c	marg. trans. costs of consumption	0.0475
Υ_a	marg. trans. costs of liquidity	-0.0238
y	real output	0.3333
w	real wage	1
c	real utility-consumption	0.3256
Ψ	preference parameter for leisure	1.9536
Υ	total transaction costs	0.0077
Υ/y	transaction costs share of output	0.0231
a	liquidity	0.3256
m	real cash balances	0.3256
τ	real monetary injection	0.004
C	total real consumption	0.3333
C/y	consumption share of output	1
C/m	consumption-based velocity	1.0238
y/m	output-based velocity	1.0238

Table 2.4: Steady state values of RRC model (without capital) at baseline calibration.

2.5.2 Impulse-response analysis

In what follows the dynamic response of the CIA and the RRC model are analysed and compared[40]. The figures below report the percentage deviation of the selected variables from their steady state value (which, for convenience, has been set to zero). The deviation from steady state of variables which do represent rates (e.g., inflation rate, interest rate, unit transaction costs), is measured in absolute terms. All the shocks take place at time zero and the time scale referrs to quarterly data.

Technology shock

Figure 2.1 shows the impact of the technology shock on real expenditure and money demand (real balances). Given that this real shock affects production in the same way in both models, the response of output (on impact and afterwards) coincides with the dynamics of the shock. Given that these models abstract from capital, total consumption evolves in the same way of the output response. Finally, comparing the expressions for the money demand derived in the previous sections, one realises that real balances respond one to one to the change in consumption[41]. As a consequence, the same changes in real expenditure (output and consumption) and real balances do explain the constant velocities in Figure 2.1.

---

[40] The simulations and the impulse-response function reported in this thesis have been obtain using the "Dynare" code (available at www.dynare.org)

[41] An important difference between money demand and consumption in the two models is given by the fact that in the RRC model real balances depend only on that part of consumption expenditures from which households derive their utility. However, the evolution of real balances in this model is equal to that one of the CIA model. This result is easily explained noting that an increase in total consumption in the RRC model is shared in fixed proportions between desired consumption and consumption used for transaction purposes. Therefore, the increase in the latter is just sufficient to provide the real resources needed to purchase the former. In other words, a positive technology shock provides more consumption goods along with a sufficient amount of real resources for covering the related transaction costs.

Figure 2.2 shows the impact of the technology shock on inflation and nominal interest rates. Remember that inflation is generated in both models as a difference between the growth in money supply and the growth in money demand. From Figure 2.1 it results a positive change in money demand. Therefore, if the government follows a constant money growth rule, a technology shock causes a temporary drop in inflation, *via* money demand dynamics. Given that the nominal interest rate is determined according to the Fisher relation, the change in consumption growth (matched by the money demand growth) cancels out with the drop in inflation. This explains the constant interest rate in Figure 2.2.

Figure 2.4 shows that the relative velocites of money do not react to the technology shock. Note that - becasue of a unitary elasticty of money demand with respect to desired consumption - real balances and real consumption increase by the same amount.

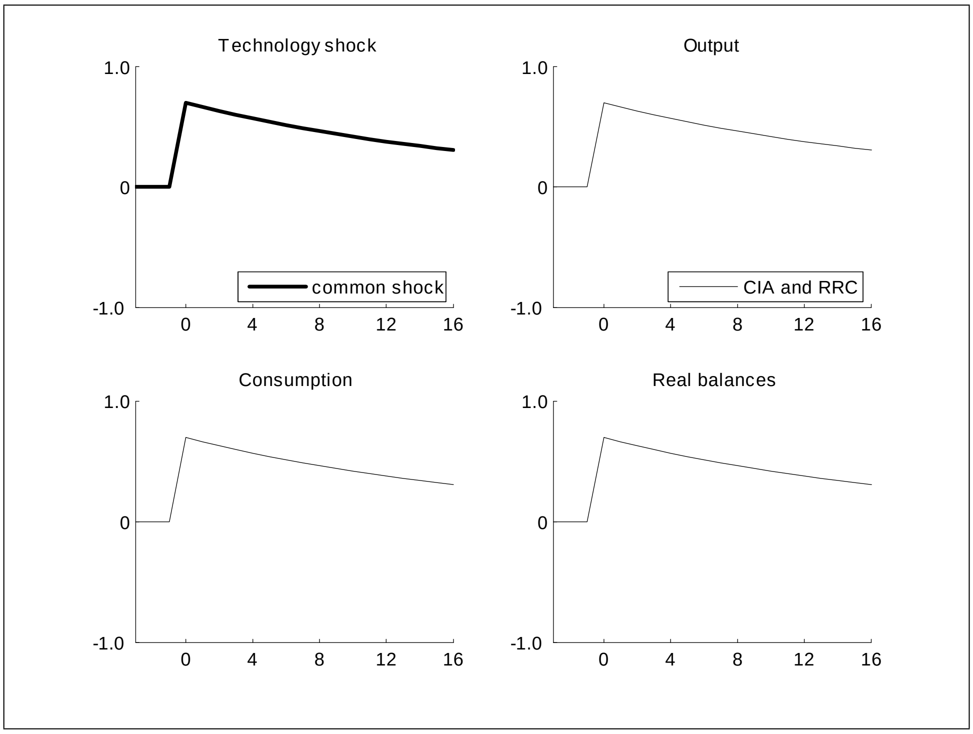

Figure 2.1: Impact of the technology shock on real expenditure (CIA vs. RRC, without capital)

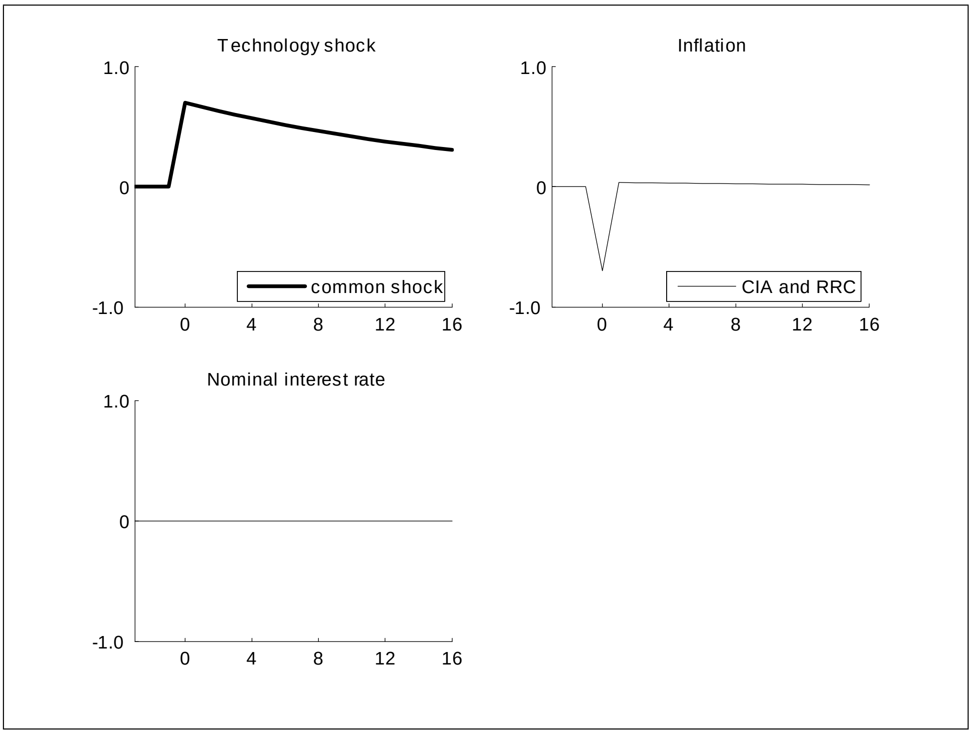

Figure 2.2: Impact of the technology shock on nominal variables (CIA vs. RRC, without capital)

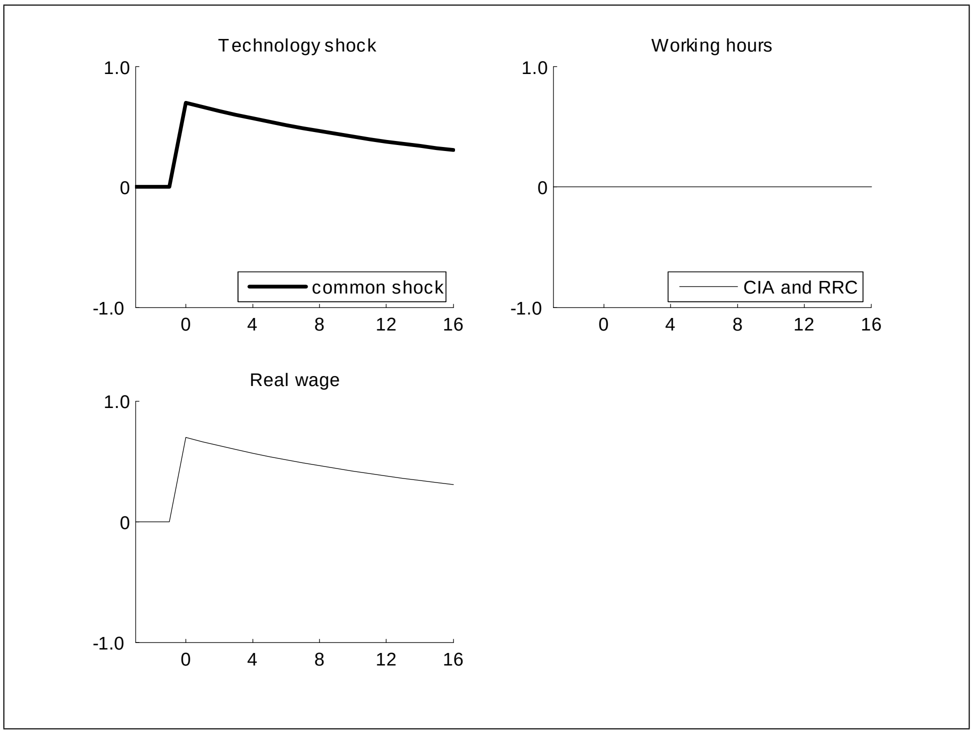

Figure 2.3: Impact of the technology shock on production factors (CIA vs. RRC, without capital)

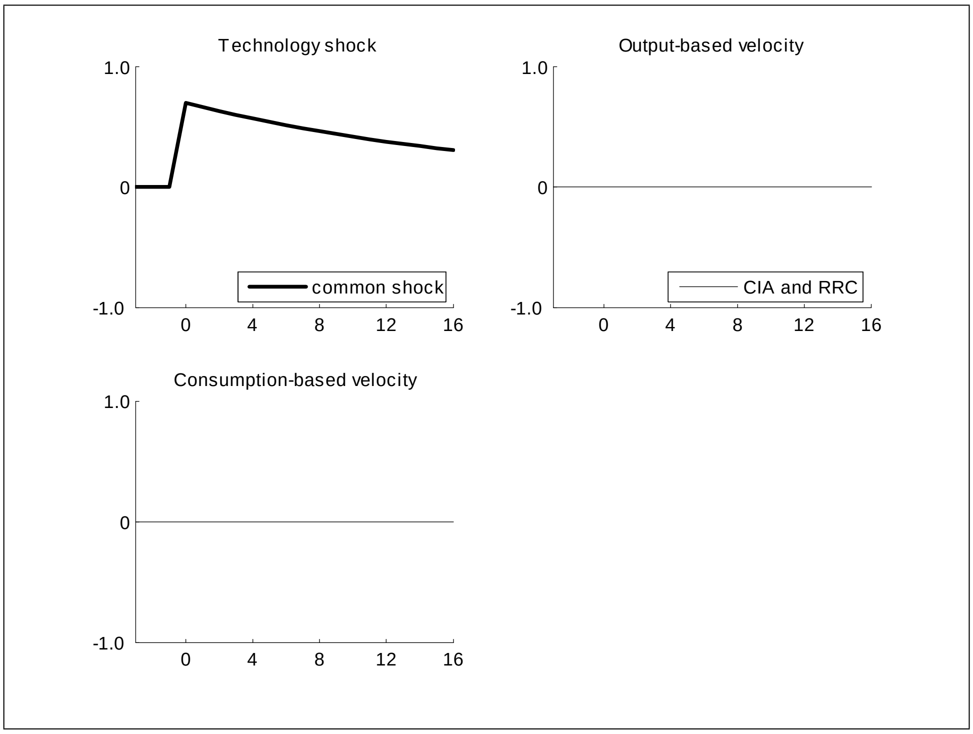

Figure 2.4: Impact of the technology shock on the velocity of money (CIA vs. RRC, without capital)

Monetary shock

Figure 2.5 shows the impact of the monetary shock on real expenditure and money demand (real balances). Because of the assumption of flexible prices, in both models a monetary shock produces real effects as long as it modifies expected inflation. In fact, in both models current and future consumption expenditure is connected with money holdings. This relationship generates a reciprocal link. On one hand, since every purchase where money is involved becomes more expensive in real terms, higher expected inflation induces a fall in real consumption tomorrow and today, through the mechanism of consumption smoothing. On the other hand, since in both models consumption yields utility, the tighter the link between desired consumption and cash, higher the value that households attach to money holdings. Since the two models differ explicitly along this last dimension, one should expect the models to react differently to the same monetary shock. One example is given by dynamics of real balances in Figure 2.5, where the fall in the RRC model is almost 3 times the one in the CIA model. Despite the smaller fall in the value of real balances, the consequent fall in real expenditure for the CIA model is about 25 times bigger. This reveals that real balances in the RRC model are *not so crucial* as in the CIA model for conducting transactions. In fact, what really matters for the level of households expenditures in the RRC model are the changes in real transaction costs, and not simply changes in real balances *per se*. This particularly shows up in the last two graphs in Figure 2.5: while in the CIA model velocity of money is unaffected by the shock to money growth, in the RRC model velocity reults highly countercyclical.

As anticipated before, the real effects of the monetary shock are guaranteed by the fact that it displays some degree of persistence. If the autoregressive parameter is set to zero ($\rho_\theta = 0$), only the *current* price level (and, therefore, only current inflation) will be affected, leaving the real magnitudes unchanged. In both models real effects of expected inflation propagate via intertemporal substitution.

Turning the attention to the nominal effects of a monetary shock (Figure 2.6), the response of inflation to an monetary shock is more than proportional. This is because in both models actual inflation is determined by the difference between the money supply growth and the change in real balances. When the shock occurs, the greater response of inflation in the RRC model, when compared with the CIA model, is given by the higher fall in real balances. However, inflation persistence is higher for the CIA model. Again, the behaviour of real balances is the key: one period after the shock the rate of change in real balances is reversed for both models. The correction in the second period is quicker in the case of the RRC model, while it dies out more slowly in the CIA model (see Figure 2.5).

As shown in Figure 2.6, the impact of the monetary shock on the nominal interest rate is less than propotional. Given the market timing assumptions adopted (i.e., the financial market opening first) the nominal interest rate represents the opportunity cost of holding money for both models. However, its response to the monetary shock depends on the transaction technologies. In the CIA model expected inflation drives the nominal interest up according to the Fisher equation, which links directly the marginal utility of consumption (and, therefore, real balances) today and tomorrow. In the RRC model, instead, the link between expected inflation and actual interest rate is explained by the marginal contribution of real balances to transaction costs. The reason why the movement of the nominal interest rate is so small (0.03 basis points) in the case of the RRC model, has to be found in the weak impact of any change in real balances (induced by a change in expected inflation) on the cost of transactions.

Figure 2.8 shows the impact of the monetary shock on the velocities of money. The increase in the velocity in the RRC model is essentially due to the big fall in real balances, while the CIA model velocities are constant (by construction). This endogenous movement in velocity is due to the fact that the agents in the RRC model can choose to adjust their portfolio, shifting from the cash-goods (subject to the inflation tax) to the *costly* credit-goods.

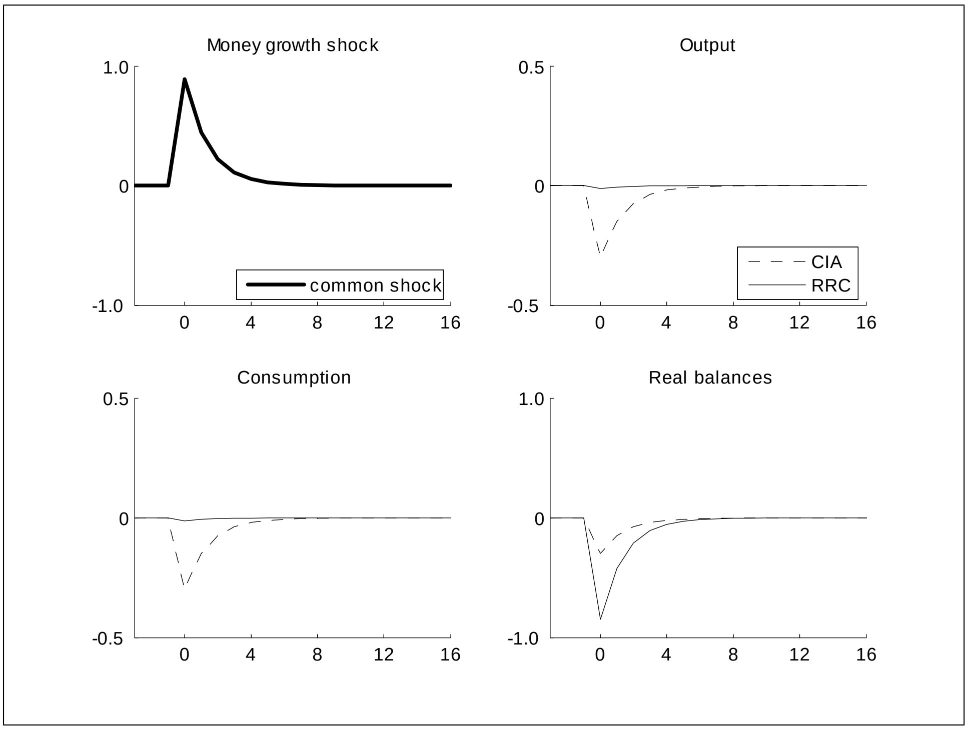

Figure 2.5: Impact of the monetary shock on real expenditure (CIA vs. RRC, without capital)

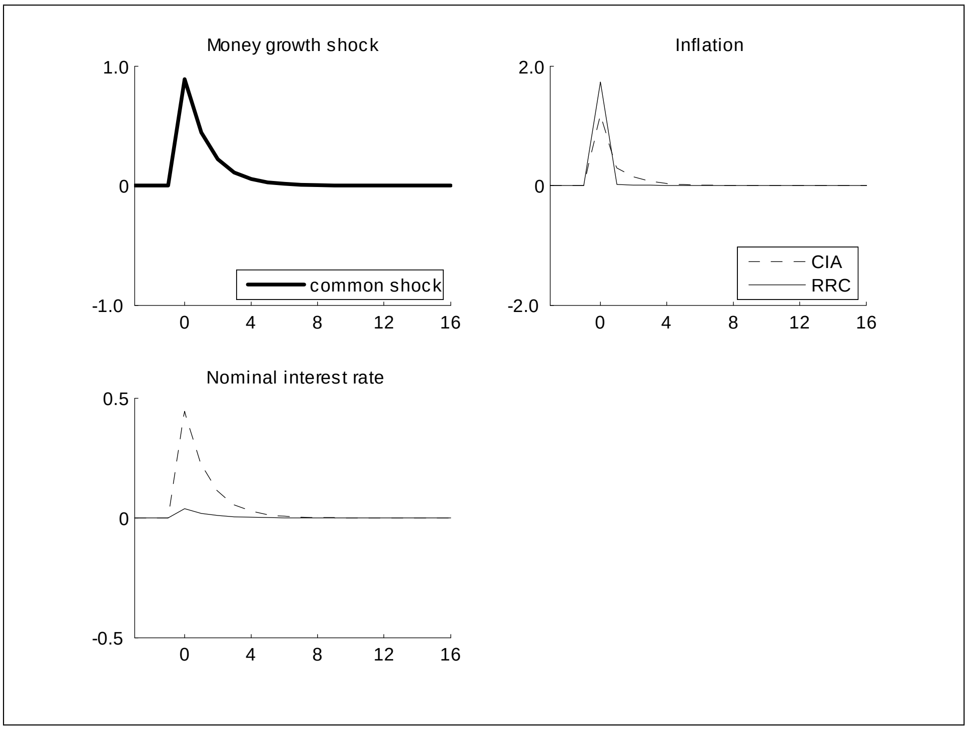

Figure 2.6: Impact of the monetary shock on nominal variables (CIA vs. RRC, without capital)

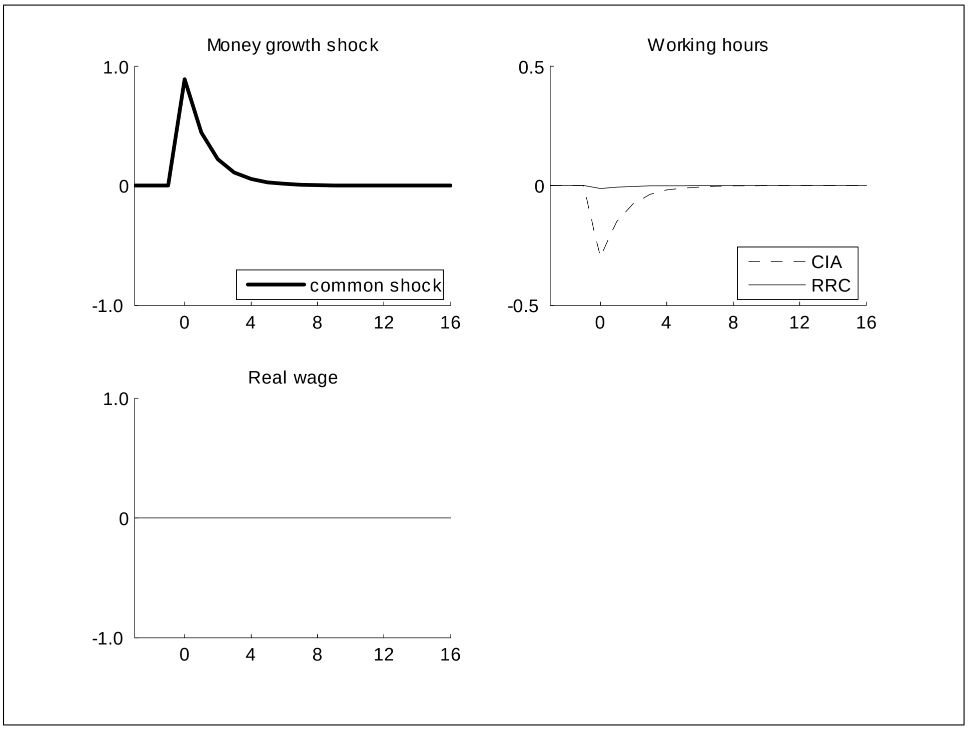

Figure 2.7: Impact of the monetary shock on production factors (CIA vs. RRC, without capital)

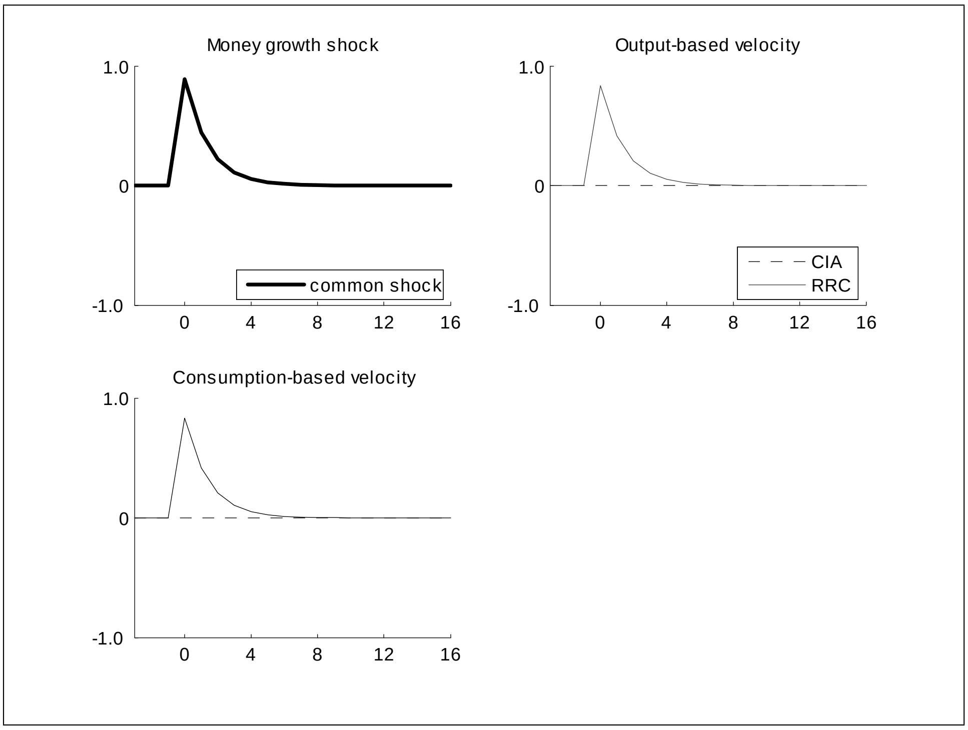

Figure 2.8: Impact of the monetary shock on the velocity of money (CIA vs. RRC, without capital)

Transaction costs shock

One additional source of shocks in the RRC model is represented by the transaction costs shock. There are two main consequences of this type of shock hitting the economy: it generates a negative effect on consumption (and output) and a higher demand for real balances. The former is due to the fact that now consumption purchases are more costly, the latter to the attempt of consumers to reduce transaction costs holding more cash.

The test of the existence of a relatively weak link between money and cosumption in the RRC model, is made by inspecting the effects of a 1% unit transaction costs shock (Figure 2.9). This leads to an increase in money demand which is less than 0.1%, together with a risible fall in consumption (less than - 0.02%). In the period after the shock, real balances start to converge back to equilibrium. With monetary authorities keeping the money supply growth constant, this causes a little inflation overshooting (see Figure 2.10). For the same reasons discussed above, the behaviour of expected inflation induces a little movement in the nominal interest rate. The movements in the velocity of money (Figure 2.12) is mainly due to the fall in expenditure (now more costly) and the increase in money demand by households', in the attempt to decrease the transaction costs.

Finally, whenever transaction costs increase in the RRC model - due to a fall in the value of real balances or to a transaction costs shock - a fall in desired consumption is always accompanied by an increase in the real resources produced by the economy for transaction purposes. One of the reasons why the latter does not dominate the former lies in its small dimension in terms of output share at the steady state (see Table 2.4).

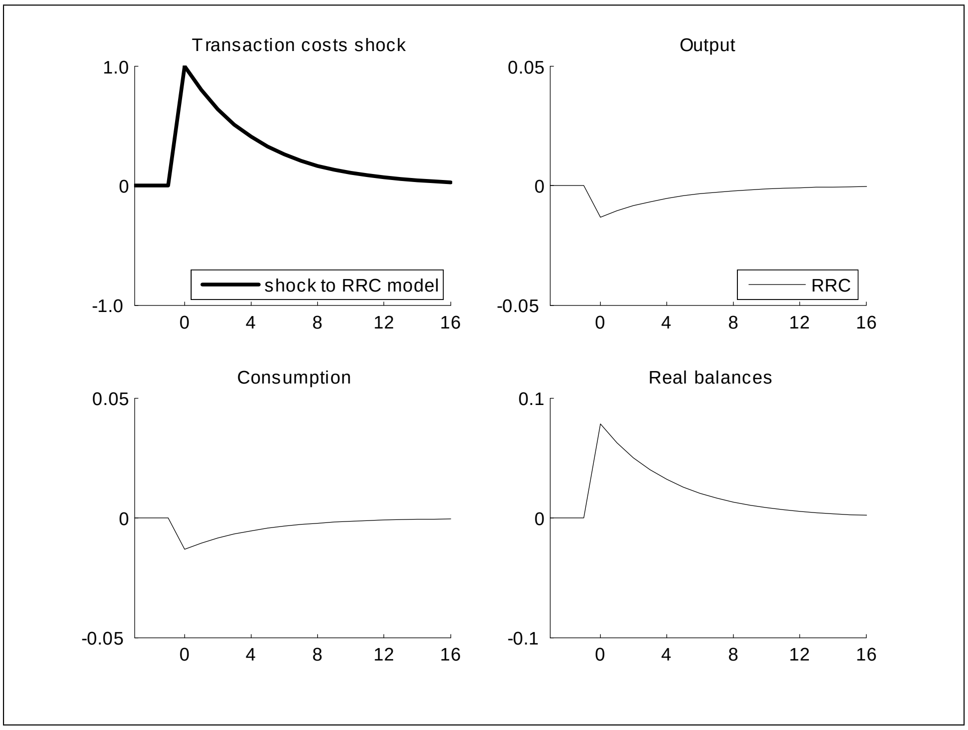

Figure 2.9: Impact of the transaction costs shock on real expenditure (RRC only, without capital)

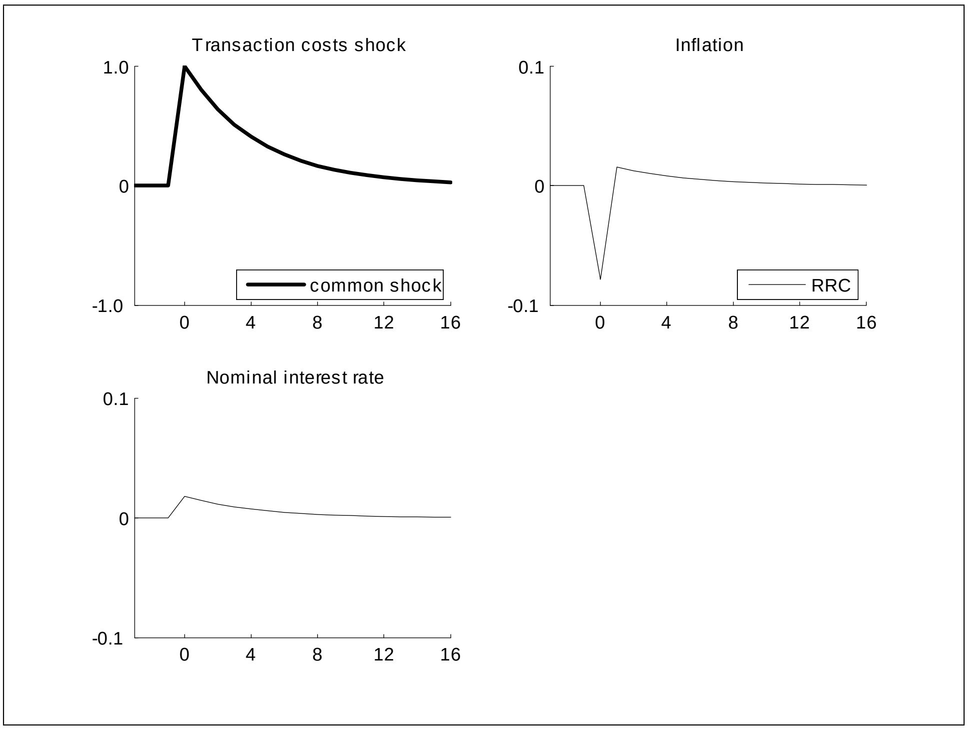

Figure 2.10: Impact of the transaction costs shock on nominal variables (RRC only, without capital)

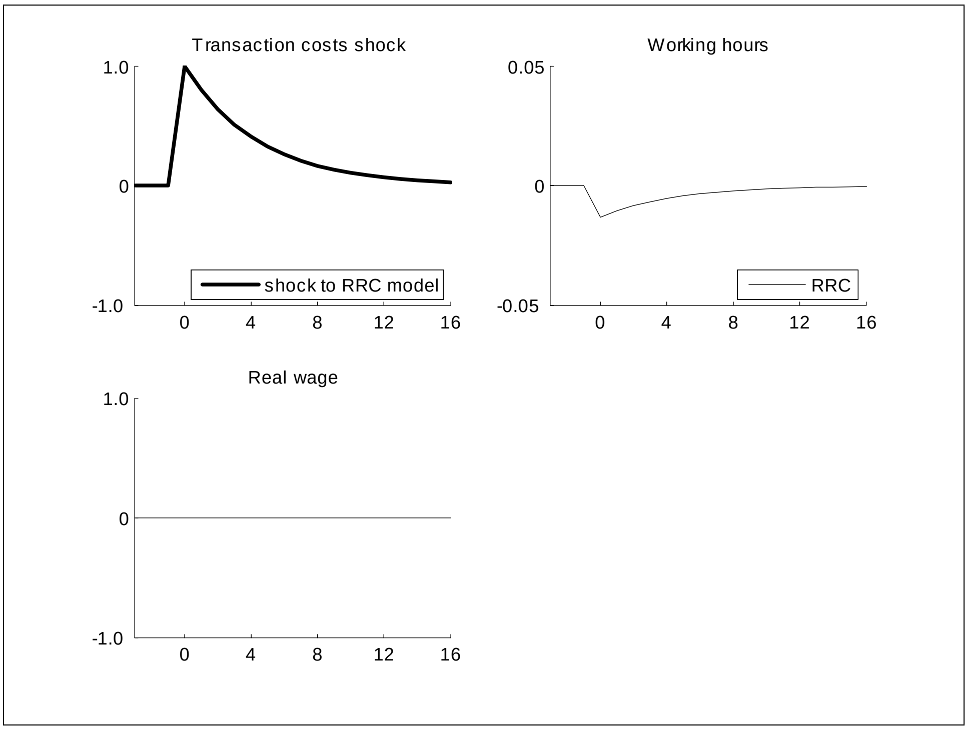

Figure 2.11: Impact of the transaction costs shock on production factors (RRC only, without capital)

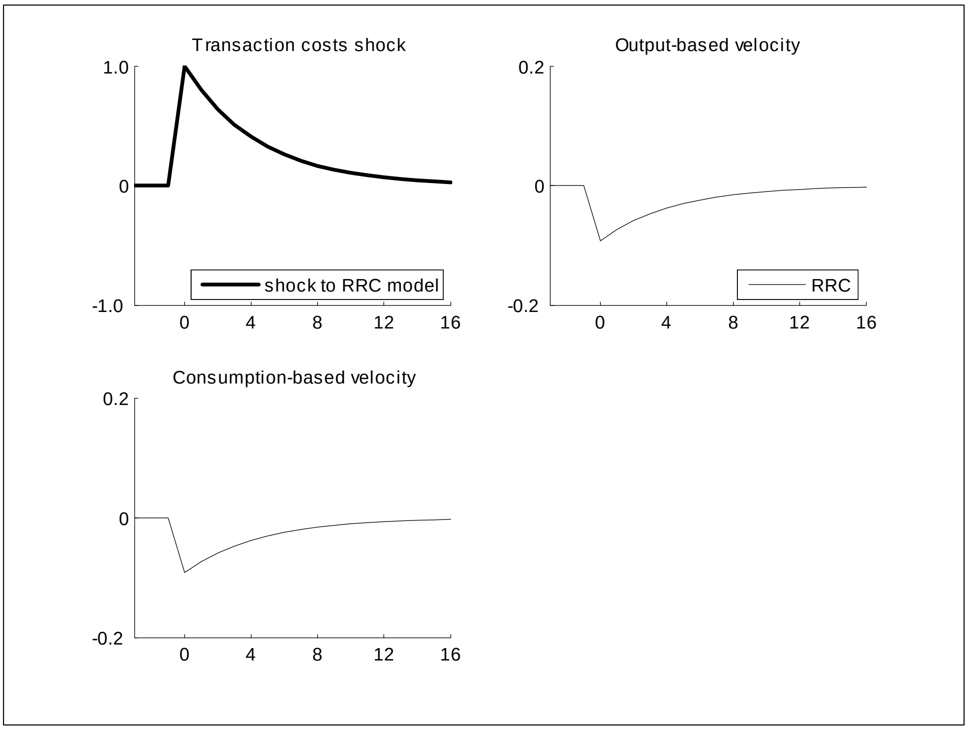

Figure 2.12: Impact of the transaction costs shock on the velocity of money (RRC only, without capital)

2.6 Conclusion

This chapter presented two classical monetary models where money is used to facilitate transactions: the cash-in-advance (CIA) model and the real-resource-costs (RRC) model. Both models were analysed and compared within a stochastic dynamic general equilibrium context characterised by perfect competition and flexible prices. In order to inspect the role of different transaction techologies when these economies were hit by stochastic shocks, the two monetary models were made comparable by adopting the same market timing assumptions (i.e., financial markets opening before the goods market) and an appropriate calibration.

The CIA and RRC model respond in the same way to a technology shock. What is crucial in the behaviour of money demand in both models is its unit elasticity with respect to the level of *desired* consumption.

When the model is hit by an autoregressive monetary shock, instead, the response differs along many dimensions. The main result is that the impact of actual and expected inflation on real variables depends on the specification of transaction technologies, which define the link between money and real expenditures. In particular, the stronger the link (as in the case of the CIA model), the greater the fall in real consumption due to the inflation tax. For this reason velocity is highly countercyclical in the RRC model, while it remains constantly at the steady state level in the CIA model. Another important consequence of the transaction technology specification concerns the volatility of the nominal interest rate: the weaker the link between money and consumption (as in the case of the RRC model) the lower the volatility.

The quantitative impact of a transaction costs shock in the RRC model appears to have a very small impact on real and nominal variables. This is mainly due to the fact that the calibration of the transaction cost function - from which the effects of this type of shock critically depend - has been targeted for a comparison with the CIA model (and mainly for studying the effects of the monetary shock).

The comparison of the CIA and the RRC models in this chapter reveals that these modelling strategies for money holdings matter for the dynamic response of the variables to the monetary shocks, but it does not for the technology shocks. In particular, the two models respond in the same way to a shock in total factor productivity because, despite the different microfoundations, the demand for real balances exhibits a unitary elasticity with respect to the transaction variable (consumption). On the other side, the same monetary shock has different effects on the two models. The impulse response functions show that the effects of the inflation tax generated by an autoregressive monetary shock depend on the possibility, for the agents in the model, to switch from money to other means of payment (e.g., costly credit). In fact, following the monetary shock, money demand falls more in RRC model than in the CIA model, given the possibility for the agents of the former to maximise their utility moving to (relatively) cheaper means of payment. These results explain the movements in velocity (in the RRC model) as a *portfolio reallocation* between cash and credit.

This chapter is useful in detecting the basic working principles of the CIA and RRC models. However, to have a real grasp of the relative quantitative performance one needs to check how far the simulation results are from the data. In order to do this, Chapter 3 will introduce capital goods into the analysis. On the one hand, this modification will inevitably introduce new dynamic features. On the other hand, it will bring the empirical assessment closer to the RBC approach, according to which the simulation results should be validated by matching the stylised facts characterising the *actual* business cycle.

Chapter 3

Transaction technologies and the business cycle: a quantitative exploration

The previous chapter focused on the characteristics of the *cash-in-advance* (CIA) and the *real-resource-cost* (RRC) models, investigating the respective dynamic properties, when labour is the only factor of production. This chapter keeps the comparison between these two monetary models at the center of the analysis and, at the same time it takes few steps further. The main novelty consists in the presence of capital goods. This implies a substantial modification of households' and firms' decisions. The motivation behind the introduction of capital is twofold: on the one hand, it allows a more accurate quantitative assesment of these models; on the other hand, it conveys important information about the propagation mechanism of shocks. The high volatility of investment with respect to consumption at business cycle frequencies (see Chapter 1, Table 1.1) lies behind the quantitative motivation[1], while the scope to analyse the dynamic contribution of capital goods, through a comparison with the results from the previous chapter, supports the theoretical motivation.

Examples of monetary models of the business cycle with capital, using a cash-in-advance or a real-resource-cost approach, are due to Cooley and

[1]In fact, this has been one of the main motivations for the the Real Business cycle (RBC) literature, since the seminal work of Kydland and Prescott (1982).

Hansen (1995), Carlstrom and Fuerst (2001b) and Walsh (2003). However, these examples present some limitations. The analysis of the results in Cooley and Hansen (1995) focuses almost entirely on the empirical performance of a cash-in-advance model, without discussing the transmission mechanism implied by their theoretical model. On the contrary, quantitative analysis is missing in the Carlstrom and Fuerst (2001) real-resource-cost model, where the main concern are indeterminacy issues. Finally, the theme developed by Walsh (2003) - although inspiring this investigation - is not discussing the dynamic implications of having capital goods and focuses mainly on the effects of monetary shocks[2].

The main contribution of this chapter is to overcome this limitations along two dimensions: a) understanding how the presence of capital goods affects the dynamics of the CIA and RRC models; b) exploring the quantitative properties of these two models against the real data.

The quantitative assessment conducted in the last part of this chapter extends the work of Cooley and Hansen (1989) by reporting the results for additional endogenous variables (e.g., the nominal interest rate, real balances and different measures of velocity of money), by inspecting the impulse-response functions of their CIA model and by reporting the correlation of endogenous variables with respect to money growth. Moreover, the set of simulation results for both models is richer than the one reported in Cooley and Hansen (1989, 1995).

The chapter is structured as follows: firstly, the assumptions implied by each approach are stated, then the resulting optimality conditions are derived. Finally, the models are calibrated (on quarterly basis) and outcomes from simulations are compared. As in the case of the previous chapter, the performance of the different economies is assessed analysing the effects of stochastic shocks affecting either production (i.e., technology shocks), the money demand (i.e., shocks to transaction costs) or the money supply process (i.e., monetary policy innovations).

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[2] For a brief review of these models, see Chapter 1.

3.1 The CIA model with capital

The main strength of the CIA model is that transactions that must be covered with cash are identified in a very precise way. This chapter assumes that consumption is a *cash good* (i.e., it is subject to the cash-in-advance constraint), while labour and capital are *credit goods*. As will be shown later on, the clarity of the CIA model in identifying the purchases for which money is needed makes the analysis of the propagation mechanism of shocks much 'easier'.

In what follows, the problem of the representative household, the problem of firms and some market clearing conditions are modified with respect to the previous chapter by the presence of investment in physical capital. At the same time, dating conventions, market timing assumptions and the way monetary policy is conducted are unaffected. Given the similarities, some details and explanations will not be repeated here. In any case, the reader is invited to refer back to the main features of the CIA model described in Chapter 2.

3.1.1 Households

The economy is populated by a large number of identical and infinitely-lived households. At time $t = 0$ the representative household seeks to maximize the following expected value of a discounted stream of period utility:

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u \left[c_t, l_t \right] \right\} \tag{3.1}$$

The expectational operator E_0 indicates that the expectations at time $t = 0$ about present and future streams of utility are formed conditionally to the information available to the agent. The objective function (3.1) assumes that utility at time t depends on real consumption c_t and leisure time l_t. Future utility is discounted by a constant discount factor β (with $0 < \beta < 1$).

The period utility function u is strictly concave and twice continuously differentiable. It is increasing in its arguments and decreasing in their marginal utility. Using u_j (u_{jj}) to denote the first (second) partial derivative of the function $u(j)$ with respect to its generic argument j, one can write: $u_c > 0$, $u_l > 0$, $u_{cc} < 0$, $u_{ll} < 0$. In addition to that, the Inada (1963) conditions are assumed to be holding: $\lim_{c\to 0} u_c = \infty$, $\lim_{l\to 0} u_l = \infty$, $\lim_{c\to\infty} u_c = 0$, $\lim_{l\to\infty} u_l = 0$.

Total time endowment is normalized to one, so that the following constraint applies to every period:

$$1 = l_t + h_t^s \tag{3.2}$$

This means that at time t the agents will choose to split total time between leisure time l_t and (supplied) working hours h_t^s.

Using (3.2) one can reformulate (3.1) in terms of consumption and working hours:

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u \left[c_t, 1 - h_t^s \right] \right\} \tag{3.3}$$

In this case u is decreasing in working time ($u_h < 0$) and increasing in the marginal *dis*utility of work ($u_{hh} > 0$). In deference to the real business cycle tradition, this last formulation will be mantained throughout the analysis.

The explicit functional form chosen for period utility takes the form of a constant relative risk aversion (CRRA) utility function:

$$u \left[c_t, 1 - h_t^s \right] \equiv \frac{c_t^{1-\Phi}}{1-\Phi} + \Psi \frac{\left(1 - h_t^s\right)^{1-\eta}}{1-\eta} \tag{3.4}$$

, where: $\Phi > 0$ is the coefficient of relative risk aversion (with $1/\Phi$ being the elasticity of intertemporal substitution); given $\eta > 0$, expression $1/\eta$ denotes the elasticity of intertemporal substitution for labour; $\Psi > 0$ represents a preference parameter over leisure.

The first modification to the CIA model presented in Chapter 2 regards the household budget constraint. At time t the economy now produces two types of goods: consumption goods (c_t) and capital goods (k_t). The capital stock is owned by households, who rent it to firms for production purposes. Therefore, a rental payment enters as an additional source of wealth on the left hand side of the budget constraint. Moreover, household expenditures in the goods market includes also investment goods ($P_t x_t$), which appear on the right hand side. For these reasons, the new budget constraint (in nominal terms) becomes:

$$M_{t-1}^{d} + (1 + i_{t-1}) B_{t-1}^{d} + W_t h_t^s + T_t + P_t r_t^k k_{t-1}^s \geq P_t c_t + M_t^d + B_t^d + P_t x_t \quad (3.5)$$

$\forall t \geq 0$; where: B_t^d denotes the nominal value of riskless bonds, which pay a one-period nominal (net) interest rate i_t; T_t represents nominal lump sum transfers from the government (taxes, if negative); M_t^d is individual money demand; W_t is hourly nominal wage and P_t represents the price of the homogeneous good produced in the economy[3]. At the end of each period households do receive a rental payment $(P_t r_t^k k_{t-1}^s)$ proportional to the capital stock they rented to the firms at the beginning of the period (k_{t-1}^s)[4].

Real investment is denoted by x_t and is defined as a change in the capital stock (net of capital depreciation):

$$x_t = k_t^s - (1 - \delta) k_{t-1}^s \quad (3.6)$$

$\forall t \geq 0$; where $0 < \delta < 1$ represents the constant real depreciation rate of capital[5].

In order to emphasise the fact that in each period households will be choosing the optimal level of capital stock to rent (k_t^s), one can use (3.6) into (3.5) to obtain:

$$M_{t-1}^{d} + (1 + i_{t-1}) B_{t-1}^{d} + W_t h_t^s + T_t + P_t r_t^k k_{t-1}^s + P_t (1 - \delta) k_{t-1}^s \geq P_t c_t + M_t^d + B_t^d + P_t k_t^s \quad (3.7)$$

[3]Note that the assumption of an homogeneous good, implies that consumption and investment goods are characterised by the same price P_t. In other words, firms are assumed to produce a single good that can be costlessly 'transformed' for private consumption use or re-invested in the productive process.

[4]The superscript 's' here is meant to indicate that capital is made available (i.e., supplied) by households to firms.

[5]The real depreciation rate respresents the part of capital stock disrupted or decayed in each period. This assumption implies that, in the steady state, a positive amount of investment will be required in order to maintain the capital stock.

$\forall t \geq 0$. The right hand side of (3.7) represents individual's total nominal wealth within the period t. This encompasses financial wealth accumulated in the previous period, the value of the capital stock net of real depreciation ($P_t(1-\delta)k_{t-1}^s$), labour income ($W_t h_t^s$), capital rental payments ($P_t r_t^k k_{t-1}^s$) and the exogenous lump sum transfers. Financial wealth is given by the nominal value of a portfolio of financial assets, namely bonds and cash balances from period $t-1$, inclusive of interest earnings (i_{t-1}) from bonds holdings.

Total wealth available in period t is allocated to the goods market (buying consumption goods and investment goods at the prevailing price P_t) and to the financial markets, adjusting the portfolio of assets (given the prevailing interest rate, i_t). As in Chapter 2, expression (3.7) implies that financial markets are complete.

Dividing both sides of (3.7) by the price level (P_t), the household budget constraint can be re-written in real terms as

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1}b_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t + r_t^k k_{t-1}^s + (1-\delta)k_{t-1}^s \geq c_t + m_t^d + b_t^d + k_t^s \quad (3.8)$$

$\forall t \geq 0$; where: $b_t^d \equiv B_t^d/P_t$ denotes the real value of riskless bonds; $I_t \equiv (1+i_t)$ is the one-period nominal (gross) interest rate; $\tau_t \equiv T_t/P_t$ represents real lump sum transfers from the government; $m_t^d \equiv M_t^d/P_t$ is individual demand for real balances; $w_t \equiv W_t/P_t$ indicates real wage and $\Pi_t \equiv P_t/P_{t-1}$ represents the (gross) inflation rate. The capital stock is represented by k_t^s, and its (real) return by r_t^k.

Resources not used in period t are saved in the form of bonds, cash balances and/or capital stock, whose command over goods will become effective only in the following period. Since this is true for every period (3.8) shows that the portfolio allocation decisions taken at time $t-1$ do in fact expose the real value of financial savings to changes in the price level from $t-1$ to t. Note that the same is not true for the real value of capital stock: infact the capital stock made available to firms *at the end* of previous period ($(1-\delta)k_{t-1}^s$), and the proportional real return from the *current* period ($r_t^k k_{t-1}$) are evaluated at the current price level (P_t)[6].

As for the CIA model of Chapter 2, the representative household is subject to a cash-in-advance constraint. The assumption that it applies only to the purchase of consumption goods continues to hold in this chapter[7], where households are allowed to visit the financial markets before the goods markets, in order to gather the desired liquidity. Any lump sum transfer by the monetary authority is also received *via* financial markets, at the beginning of each period.

Considered all together, these assumptions correspond to a cash-in-advance constraint of the form:

$$M_{t-t}^d + (1 + i_{t-1})\, B_{t-1}^d - B_t^d + T_t \geq P_t c_t \tag{3.9}$$

$\forall t \geq 0$.

The purchasing power of cash balances is obtained by dividing both sides of (3.9) by P_t. The result is a cash-in-advance constraint expressed in real terms:

$$\frac{m_{t-t}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t \geq c_t \tag{3.10}$$

$\forall t \geq 0$.

At time t the problem of the household is inherently dynamic: to choose state-contingent claims for consumption (c_t), labour supply (h_t^s), bonds holdings (b_t^d), money stock (m_t^d) and capital stock (k_t), which do maximize the expected utility (3.3), subject to the budget constraint (3.8) and to the finance constraint (3.10)[8].

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[6] The fact that the rental rate (r_t^k) paid by firms in the current period is proportional to k_{t-1} is due to the assumption that the capital stock used for production at time t is decided one period advance.

[7] For the analysis of the impact of a cash-in-advance constraint extended to encompass investment goods see Chapter 4.

[8] Moreover, *no-Ponzi game* conditions must hold to guarantee optimality.

3.1.2 Firms

The economy is populated by a large number of identical firms. The novelty with respect to Chapter 2 is represented here by the introduction of capital goods in the production function. From this point of view, the model is more in line with the RBC tradition than the model in Chapter 2. Now, in addition to labour (working hours) supplied by the households, firms produce the homogeneous good using rented capital too. The real output produced in period t can be expressed by the following production function:

$$y_t = f\left[z_t, h_t^d, k_{t-1}^d\right] \tag{3.11}$$

$\forall t \geq 0$, where: y_t denotes *real* output; h_t^d are working hours demanded by the firm; k_{t-1}^d is the capital stock rented from households[9] and z_t represents the "level" of technology.

In order to obtain a direct correspondance between the behaviour of individual firm and their aggregate counterpart, the production function is represented by a constant returns to scale technology. To satisfy this condition, the production technology is assumed to be a Cobb-Douglas type:

$$y_t = z_t\left(k_{t-1}^d\right)^{\alpha}\left(h_t^d\right)^{1-\alpha} \tag{3.12}$$

$\forall t \geq 0$, where $0 < a < 1$ represents the capital share. In deference to the RBC literature, the variable z_t represents the *total* factor productivity[10]. This variable evolves exogenously according to the law of motion

$$\log z_t = (1-\rho_z)\log z + \rho_z \log z_{t-1} + \epsilon_{z_t} \tag{3.13}$$

$\forall t \geq 0$; where: ρ_z is the autoregressive coefficient (with $0 \leq \rho_z \leq 1$), and ϵ_{z_t} is a random variable serially uncorrelated and normally distributed, with zero mean and constant variance $(\sigma^2_{\epsilon_z})$.

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[9]The superscript 'd' here is meant to indicate demand.

[10]Here the choice is to consider *total* factor productivity instead of a *labour augmenting* technology, also widely used by the RBC literature. The main reason is comparability with the models which are closer to the spirit of this investigation. See for example, Cooley and Hansen (1995).

In period t firms sell their product in a perfectly competitive goods market, taking the price P_t of the homogenous good as given. Analogously, given the nominal wage W_t, they buy labour services from households in a perfectly competitive labour market. Firms rent capital (k_{t-1}) from households, at the cost of a proportional rental rate ($P_t r_t^k k_{t-1}$). In order to decide how much to produce - and, consequently, how much labour to hire and capital to rent - firms do maximise a the following profit function:

$$\Gamma_t = P_t y_t - W_t h_t^d - P_t r_t^k k_{t-1}^d \tag{3.14}$$

$\forall t \geq 0$; where nominal profits (Γ_t) are defined as a difference between nominal revenues ($P_t y_t$) and nominal costs ($W_t h_t^d + P_t r_t^k k_{t-1}^d$). Note that the assumptions of perfect competition and constant returns to scale do imply that the representative firm makes zero profits in equilibrium.

In every period t each firms solves a static problem: that one of choosing working hours (h_t^d) and capital (k_{t-1}^d) which maximize profits (Γ_t) subject to the technology constraint (3.12).

3.1.3 Government

As in Chapter 2, the government operates as monetary and fiscal authority and its revenues and outlays in period t are combined in the following flow budget constraint (expressed in nominal terms):

$$M_t^s - M_{t-1}^s + B_t^g - (1 + i_{t-1}^g) B_{t-1}^g = P_t g_t + T_t \tag{3.15}$$

$\forall t \geq 0$, where: B_t^g denotes the face value of government debt outstanding, which pays a one-period nominal (net) interest rate i_t^g; T_t indicates governmental nominal lump sum transfers, net of taxes; M_t^s represents aggregate money supply; and g_t denotes real government consumption.

Since the focus here is on studying the impact of monetary shocks and not the impact of changes in government spending, g_t is set to zero (for all t periods). Moreover, Ricardian equivalence holds in this model. Therefore one can assume, with no loss of generality, that $B_0^g = 0$[11]. All together these assumptions imply that no government bonds are held in this economy and the government budget constraint then reduces to

$$M_t^s - M_{t-1}^s = T_t \tag{3.16}$$

$\forall t \geq 0$. Dividing both sides of (2.15) by the price level P_t, one obtains the equivalent expression in real terms :

$$m_t^s - \frac{m_{t-1}^s}{\Pi_t} = \tau_t \tag{3.17}$$

$\forall t \geq 0$; where: $\tau_t \equiv T_t/P_t$ represents real lump sum transfers; $m_t^s \equiv M_t^s/P_t$ is real money supply; and $\Pi_t \equiv P_t/P_{t-1}$ is the (gross) inflation rate.

The monetary authority is assumed to follow a constant money growth rule, according to which per capita nominal money supply is assumed to grow at the net rate θ_t in each period. This implies:

$$M_t^s = M_{t-1}^s + \theta_t M_{t-1}^s \tag{3.18}$$

$\forall t \geq 0$. The money supply rule is implemented through monetary injections that take the form of lump sum transfers accoring to:

$$T_t = \theta_t M_{t-1}^s \tag{3.19}$$

or, in real terms,

$$\tau_t = \frac{\theta_t m_{t-1}^s}{\Pi_t} \tag{3.20}$$

$\forall t \geq 0$.

To study the effects of a monetary surprise, the variable θ_t is assumed to evolve according to the law of motion

$$\ln \theta_t = (1 - \rho_\theta) \ln \theta + \rho_\theta \ln \theta_{t-1} + \epsilon_{\theta_t} \tag{3.21}$$

$\forall t \geq 0$; where: ρ_θ is the autoregressive coefficient (with $0 \leq \rho_\theta \leq 1$), and

[11] See Chapter 2, Section 2.1.3 for a more detailed explanation.

ϵ_{θ_t} is a random variable serially uncorrelated and normally distributed, with zero mean and constant variance ($\sigma^2_{\epsilon_\theta}$). With this specification, the average (net) growth rate of money supply chosen by the monetary authority is equal to θ.

3.2 The RRC model with capital

One point of strength of the RRC model is that it assumes that *not all* transactions (or cathegories of goods) must be subjected to cash holdings[12]. From this point of view, the RRC model adds realism in the way purchases of goods are made - i.e., a mix of credit and cash. However, as emphasised in the previous chapter, realism comes with a cost: the use of an *ad hoc* transaction cost function makes the interpretation of the results more 'difficult'. In order to shed some light on the properties of this monetary model, once capital goods are introduced, two sources of comparison will be available: the results for the RRC model (without capital) discussed in Chapter 2, and the comparison with a 'compatible' CIA model (with capital) developed in this chapter. For these reasons the assumption that transaction costs apply only to consumption goods will be mantained, together with the dating and market timing conventions. As for the CIA model described in the previous section, the presence of investment goods in the RRC model affects the problem of the representative household and firms, and some market clearing conditions. With the exception of the cash-in-advance constraint - which does not appear in the RRC model -, the specification of the utility function, the problem of the firm, the monetary policy are basically unchanged with respect to those described for the CIA model with capital. Therefore they will be skipped in what follows, in order to avoid unnecessary repetition. Regarding additional details and explanations strictly related to the RRC model, instead, the reader is invited to refer back to the main features of the RRC model described in Chapter 2.

[12]There are some exceptions though. In fact, Carlstrom and Fuerst (2001b) and Carlstrom (2004) make an attempt to identify *cash goods* and *credit goods* in a model characterised by real-resource-cost of transactions.

3.2.1 Households

As in the case of the CIA model, the presence of capital goods in the RRC model modifies the representative household budget constraint presented in Chapter 2. At time t the economy now produces two types of goods: consumption goods (c_t) and capital goods (k_t). The capital stock is owned by households, who rent it to firms for production purposes. In nominal terms:

$$M_{t-1}^{d}+(1+i_{t-1})\,B_{t-1}^{d}+W_t h_t^{s}+T_t+P_t r_t^{k} k_{t-1}^{s}+P_t\,(1-\delta)\,k_{t-1}^{s}$$
$$\geq P_t c_t+M_t^{d}+B_t^{d}+P_t\Upsilon\,(\omega_t,c_t,a_t)+P_t k_t^{s} \quad (3.22)$$

$\forall t\geq 0$. A rental payment enters as an additional source of wealth on the left hand side of the budget constraint. The value of the existing capital stock appears (as a real asset) on the left hand side, while investment expenditures consist in adding new capital to the existing stock (right hand side).

Dividing both sides of (3.22) by the price level (P_t), the household budget constraint can be re-written in real terms as

$$\frac{I_{t-1}b_{t-1}^{d}}{\Pi_t}+\frac{m_{t-1}^{d}}{\Pi_t}+w_t h_t^{s}+\tau_t+r_t^{k}k_{t-1}^{s}+(1-\delta)\,k_{t-1}^{s}$$
$$\geq c_t+m_t^{d}+b_t^{d}+\Upsilon\,(\omega_t,c_t,a_t)+k_t^{s} \quad (3.23)$$

As emphasised in Chapter 2, the main difference between the RRC model and the CIA model is represented by the presence of real resource costs of transactions $\Upsilon\,(\omega_t,c_t,a_t)$ on the right hand side. Moreover, a reliable comparison between these two models requires the same market timing assumption that financial markets open before goods markets. This is achieved through the auxiliary variable, A_t, which denotes total liquidity in the hands ot the household before visiting the goods market. In nominal terms:

$$M_{t-1}^{d}+(1+i_{t-1})\,B_{t-1}^{d}-B_t^{d}+T_t\geq A_t \quad (3.24)$$

$\forall t\geq 0$. Dividing both sides of (3.24) by P_t one obtains total liquidity in real terms:

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t \geq a_t \tag{3.25}$$

$\forall t \geq 0$.

A reliable comparison between the CIA and the RRC model also requires liquidity to facilitate transactions for the same typology of goods - i.e., consumption goods. Therefore, the transaction costs function appearing in the budget constraint is defined as

$$\Upsilon(c_t, a_t) \equiv \omega_t \Omega_1 \frac{(c_t)^{\Omega_2+1}}{(a_t)^{\Omega_2}} \tag{3.26}$$

$\forall t \geq 0$; where trasaction costs are positively related with real consumption (c_t) and decrease with liquidity in real terms (a_t); $\Omega_1 > 0$ is a scale parameter and $\Omega_2 > 0$ is an elasticity parameter. The variable ω_t represents a stochastic transaction cost component, which follows the first-order autoregressive process

$$\ln \omega_t = (1 - \rho_\omega) \ln \omega + \rho_\omega \ln \omega_{t-1} + \epsilon_{\theta_t} \tag{3.27}$$

$\forall t \geq 0$; where: ρ_ω is the autoregressive coefficient (with $0 \leq \rho_\omega \leq 1$), and ϵ_{ω_t} is a random variable serially uncorrelated and normally distributed, with zero mean and constant variance $(\sigma^2_{\epsilon_\omega})$.

To fully understand (3.26) it is useful to define the *unitary* real transaction cost (i.e., real cost associated with one unit of consumption, q_t) as follows:

$$q_t = \omega_t \Omega_1 \left(\frac{c_t}{a_t}\right)^{\Omega_2} \tag{3.28}$$

$\forall t \geq 0$. Using (3.26) and (3.28), one can re-write total transaction costs as:

$$\Upsilon_t = q_t c_t \tag{3.29}$$

$\forall t \geq 0$. Total transaction costs depend on the amount of consumption (c_t), while the unitary transaction costs (q_t) depend on the relative volume of consumption and liquidity as specified by (3.28).

At time t the problem of the household in the RRC model is inherently dynamic: that one to choose state-contingent claims for consumption (c_t), labour supply (h_t^s), bonds holdings (b_t^d), money stock (m_t^d) and capital stock (k_t), maximising the utility function, subject to the budget constraint (3.23) and the liquidity constraint (3.25)[13].

3.3 The equilibrium

This section derives the equilibrium conditions which characterize the two approaches: the CIA and the RRC model with capital. The household's and firm's optimal choices will be derived, while the monetary policy rule and the necessary market clearing conditions for the general equilibrium will close the models. The optimisation problems will be stated in terms of the Lagrangian method and then solved for the first order conditions.

3.3.1 The CIA model with capital

Households

To state the households' problem in terms of the Lagrangian, it is useful to recall that the representative household in the CIA model seeks to maximise the utility stream

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\Phi}}{1-\Phi} + \Psi \frac{(1-h_t^s)^{1-\eta}}{1-\eta} \right) \tag{3.30}$$

, subject to the budget constraint

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t + r_t^k k_{t-1}^s + (1-\delta) k_{t-1}^s = c_t + m_t^d + b_t^d + k_t^s \tag{3.31}$$

$\forall t \geq 0$, and the cash-in-advance constraint

[13] Moreover, *no-Ponzi game* conditions must hold to guarantee optimality.

$$\frac{m_{t-t}^{d}}{\Pi_t}+\frac{I_{t-1}b_{t-1}^{d}}{\Pi_t}-b_t^d+\tau_t=c_t \tag{3.32}$$

$\forall t\geq 0$[14].

Stating the problem in terms of the Lagrangian, the households choose c_t, h_t^s, b_t^d, m_t^d and k_t^s in order to maximise

$$\mathcal{L}_t^{CIA}=E_0\sum_{t=0}^{\infty}\beta^t\Bigg\{\left(\frac{c_t^{1-\Phi}}{1-\Phi}+\Psi\frac{(1-h_t^s)^{1-\eta}}{1-\eta}\right)$$
$$+\lambda_t^{CIA}\left[\frac{m_{t-1}^d}{\Pi_t}+\frac{I_{t-1}b_{t-1}^d}{\Pi_t}+w_th_t^s+\tau_t+r_t^kk_{t-1}^s+(1-\delta)\,k_{t-1}^s-c_t-m_t^d-b_t^d+k_t^s\right]$$
$$+\mu_t^{CIA}\left[\frac{m_{t-t}^d}{\Pi_t}+\frac{I_{t-1}b_{t-1}^d}{\Pi_t}-b_t^d+\tau_t-c_t\right]\Bigg\}$$

,where λ_t^{CIA} and μ_t^{CIA} are the Lagrangian multipliers associated with the budget constraint and the cash-in-advance constraint, respectively.

The maximisation of the Lagrangian with respect to the choice variables (after substituting for the Lagrangian multipliers) delivers the following optimality conditions:

$$\frac{\Psi\,(1-h_t^s)^{-\eta}}{c_t^{-\Phi}}=\frac{w_t}{I_t} \tag{3.33}$$

$$1=\beta E_t\left\{\left(\frac{c_{t+1}}{c_t}\right)^{-\Phi}\frac{I_t}{\Pi_{t+1}}\right\} \tag{3.34}$$

$$\mathbf{1}=\boldsymbol{\beta}\mathbf{E}_t\left\{\left(\frac{c_{t+1}}{c_t}\right)^{-\Phi}\frac{I_t}{I_{t+1}}R_{t+1}^k\right\} \tag{3.35}$$

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[14] See footnote (17) in Chapter 2 for a discussion about replacing inequalities with equalities in the constraints.

$$\frac{m_{t-1}^{d}}{\Pi_t}+\frac{I_{t-1}b_{t-1}^{d}}{\Pi_t}+w_t h_t^s+\tau_t+r_t^k k_{t-1}^s+(1-\delta)\,k_{t-1}^s$$
$$=c_t+m_t^d+b_t^d+k_t^s \quad (3.36)$$

$$\frac{m_{t-t}^{d}}{\Pi_t}+\frac{I_{t-1}b_{t-1}^{d}}{\Pi_t}-b_t^d+\tau_t=c_t \quad (3.37)$$

, where $R_t^k \equiv r_t^k+1-\delta$ represents the gross return on capital (net of depreciation)[15].

Expression (3.33) represents the *intra*-temporal condition, which relates the marginal rate of substitution between leisure and consumption (on the left hand side), to the ratio of the respective marginal costs (on the right hand side). Note that, because of the opportunity cost of saving, the gross nominal interest rate (I_t) acts like a 'tax' on consumption, affecting in turn the labour supply choice (h_t^s) via the utility function. Expression (3.34) refers to the *inter*-temporal condition, which governs the degree of consumption smoothing through time, taking into account the *real* opportunity cost of saving ($I_t/E_t\Pi_{t+1}$).

The 'new entry' in the set of first order condition is represented by equation **(3.35),** which relates the return on capital to the expected returns on bonds and the stochastic discount factor. The timing appearing in this expression reflects the particular market timing assumptions introduced with the CIA constraint, together with the fact that capital goods purchased at time t will not generate a return until period $t+1$. Finally, equations (3.36) and (3.37) are the constraints, obtained by derivation with respect to the Lagrangian multipliers.

Firms

In each period, the representative firm chooses the amount of working hours (h_t^d) and rent capital stock (k_{t-1}^d) that maximise the profit function

[15] Moreover, *transversality conditions* must hold to guarantee optimality.

$$\Gamma_t = P_t e^{z_t} h_t^d - W_t h_t^d - P_t r_t^k k_{t-1}^d \tag{3.38}$$

or, in real terms,

$$\gamma_t = e^{z_t} h_t^d - w_t h_t^d - r_t^k k_{t-1}^d \tag{3.39}$$

$\forall t \geq 0$. Given the production technology

$$y_t = z_t \left(k_{t-1}^d\right)^{\alpha} \left(h_t^d\right)^{1-\alpha} \tag{3.40}$$

and considering a generic time period t, one obtains the following first order conditions:

$$\frac{\partial \gamma_t}{\partial h_t^d} = e^{z_t} \left(k_{t-1}^d\right)^{\alpha} (1-\alpha) \left(h_t^d\right)^{-\alpha} - w_t = 0 \tag{3.41}$$

$$\frac{\partial \gamma_t}{\partial k_{t-1}^d} = e^{z_t} \alpha \left(k_{t-1}^d\right)^{\alpha-1} \left(h_t^d\right)^{1-\alpha} - r_t^k = 0 \tag{3.42}$$

for labour and capital, respectively[16].

Making use of the production function (3.40) and the definition of the gross return of capital $R_t^k \equiv r_t^k + 1 - \delta$, conditions (3.41) and (3.42) can be re-written as

$$(1-\alpha) \frac{y_t}{h_t^d} = w_t \tag{3.43}$$

$$\alpha \frac{y_t}{k_{t-1}^d} + 1 - \delta = R_t^k \tag{3.44}$$

The optimality condition (3.43) implies that firms demand working hours up to the point where marginal product of labour equals its marginal cost (i.e.,

[16] One might wonder why the maximisation with respect to capital involves a variable (k_{t-1}^d), which is actually pre-determined when used in the production process by firms. Another objection could be that there has been a change in the timing convention: in fact, households optimal allocations are all made with respect to variables chosen at time t. The suggestion here goes as follows. Given that in this model firms do not own the capital stock, their problem remains essentially a 'static' one. Therefore, they simply rent the capital stock made available from households at the beginning of the period (k_{t-1}^s), *producing up to the point* where its marginal product equals marginal cost.

the real wage w_t); while expression (3.44) implies that they rent capital up to the point where its marginal product equals the marginal cost (represented by the net rental rate $r_t^k = R_t^k - (1-\delta)$). Since the problem faced by the firms is the same under the different specifications for money demand, the results derived here for the CIA model with capital, will also apply to the firms in the 'analogous' RRC model.

Monetary policy

In period t real cash balances evolve to satisfy the government's budget constraint

$$\frac{m_{t-1}^s}{\Pi_t} + \tau_t = m_t^s$$

$\forall t \geq 0$. Given the money growth rate (θ_t) choosen by the monetary authority for period t, recall the characterization of the real lumpsum transfers as

$$\tau_t = \frac{\theta_t m_{t-1}^s}{\Pi_t}$$

$\forall t \geq 0$.

Market clearing conditions

For a general equilibrium characterisation of the model, the necessary market clearing conditions are required. In this model there are five markets: the goods market, the labour market, the money market, the bonds market and the capital market.

The capital market clears according to

$$k_{t-1}^d = k_{t-1}^s \tag{3.45}$$

$\forall t \geq 0$[17].

[17] The time subscript $t-1$ refers to the first order condition of the firm, with respect to cpaital goods. See also Uhlig (1999).

The labour market clearing condition equates labour demand and labour supply for every period, according to

$$h_t^d = h_t^s \tag{3.46}$$

$\forall t \geq 0$.

The money market clears in every period, when money demand by households is equal to the money supply:

$$m_t^d = m_t^s \tag{3.47}$$

$\forall t \geq 0$.

Since the bonds in this model are private bonds 'issued' by households, the assumption that all the individuals are alike implies that no bonds are actually exchanged in equilibrium. As a consequence, there will be no bonds outstanding (i.e., a zero net supply for this type of financial assets). Thus, the bonds market clearing condition corresponds to:

$$b_t^d = b_t^s = 0 \tag{3.48}$$

$\forall t \geq 0$.

Finally, The market clearing condition for the goods market requires aggregate supply and aggregate demand of goods to be equal in every period. Namely:

$$y_t = c_t + x_t \tag{3.49}$$

$\forall t \geq 0$. The difference between this market clearing condition and the correspondent expression in Chapter 2 consists in the presence of investment goods on the right hand side. At this stage it is probably useful to recall the definition of investment as net capital accumulation

$$x_t = k_t^s - (1-\delta)\, k_{t-1}^s$$

$\forall t \geq 0$.

3.3.2 The RRC model with capital

Since the problem faced by the firms and the specification of the monetary policy rule are identical under the two monetary models, this section will focus only on those parts where the RRC model differs from the CIA model: namely, the first order conditions characterising the household's problem and (some of) the market clearing conditions.

Households

In the RRC model, households seek to maximise the usual intertemporal utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\Phi}}{1-\Phi} + \Psi \frac{(1-h_t^s)^{1-\eta}}{1-\eta} \right)$$

, subject to the budget constraint

$$\frac{I_{t-1} b_{t-1}^d}{\Pi_t} + \frac{m_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t + r_t^k k_{t-1}^s + (1-\delta) k_{t-1}^s \geq c_t + m_t^d + b_t^d + \Upsilon(\omega_t, c_t, a_t) + k_t^s$$

$\forall t \geq 0$, and the liquidity constraint

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t = a_t \tag{3.50}$$

$\forall t \geq 0$.

Stating the problem in terms of Lagrangian method, the households choose c_t, h_t^s, b_t^d, m_t^d, a_t and k_t^s in order to maximise

$$\mathcal{L}_t^{RRC} = E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \left(\frac{c_t^{1-\Phi}}{1-\Phi} + \Psi \frac{(1-h_t^s)^{1-\eta}}{1-\eta} \right) + \lambda_t^{RRC} \Bigg[\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t + r_t^k k_{t-1}^s + (1-\delta) k_{t-1}^s - c_t - m_t^d - b_t^d - \Upsilon(\omega_t, c_t, a_t) - k_t^s \Bigg] + \mu_t^{RRC} \left[\frac{m_{t-t}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t - a_t \right] \Bigg\}\Bigg\}$$

, where λ_t^{RRC} and μ_t^{RRC} are the Lagrangian multipliers associated with the budget constraint (inclusive of real transaction costs) and the liqiduidity constraint, respectively.

The maximisation of the Lagrangian with respect to the choice variables delivers the following optimality conditions:

$$\frac{\Psi (1-h_t^s)^{-\eta}}{c_t^{-\Phi}} = \frac{w_t}{[1+\Upsilon_{c,t}]} \tag{3.51}$$

$$1 = \beta E_t \left\{ \left(\frac{c_{t+1}}{c_t} \right)^{-\Phi} \left(\frac{1+\Upsilon_{c,t}}{1+\Upsilon_{c,t+1}} \right) \frac{I_{t+1}}{\Pi_{t+1}} \right\} \tag{3.52}$$

$$\mathbf{1} = \boldsymbol{\beta} \mathbf{E}_t \left\{ \left(\frac{c_{t+1}}{c_t} \right)^{-\Phi} \left(\frac{1+\Upsilon_{c,t}}{1+\Upsilon_{c,t+1}} \right) R_{t+1}^k \right\} \tag{3.53}$$

$$\Upsilon_{a,t} = 1 - I_t \tag{3.54}$$

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t + r_t^k k_{t-1}^s + (1-\delta) k_{t-1}^s = c_t + m_t^d + b_t^d + \Upsilon(\omega_t, c_t, a_t) + k_t^s \tag{3.55}$$

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t = a_t \tag{3.56}$$

$\forall t \geq 0$[18].

The following definitions have been applied, in order to simplify the notation:

$$\Upsilon(\omega_t, c_t, a_t) \equiv c_t q_t \tag{3.57}$$

$$q_t \equiv e^{\omega_t} \Omega_1 (\upsilon_t)^{\Omega_2} \tag{3.58}$$

$$\upsilon_t \equiv \frac{c_t}{a_t} \tag{3.59}$$

$\forall t \geq 0$.

$\forall t \geq 0$. Recalling the explicit functional form for $\Upsilon(\omega_t, c_t, a_t)$ in (3.26), the partial derivatives of total transaction costs function with respect to real consumption and liquidity, are denoted respectively by

$$\Upsilon_{c,t} = (\Omega_2 + 1) q_t \tag{3.60}$$

$$\Upsilon_{a,t} = -\Omega_2 q_t \upsilon_t \tag{3.61}$$

respectively, $\forall t \geq 0$.

Expression (3.51) is the *intra*temporal condition for the RRC model, which relates the marginal rate of substitution between leisure and consumption (left hand side) to the ratio of the respective marginal costs (right hand side). Comparing the terms on the left hand side with those appearing in the equivalent CIA model expression (3.33), the marginal cost of consumption is represented now by the real cost of one unit of consumption plus marginal transaction costs ($\Upsilon_{c,t}$). Expression (3.52) refers to the *inter*temporal condition, which governs the degree of consumption smoothing thorugh time. This now depends on the (present and expected) marginal cost of consumption ($\Upsilon_{c,t}$ and $E_t \Upsilon_{c,t+1}$) and the opportunity cost of saving in the next period ($E_t I_{t+1} / E_t \Pi_{t+1}$). Equation (3.53) relates the return on capital to the stochastic discount factor, corrected by the marginal transaction costs. Expression (3.54) represents the arbitrage

[18] Moreover, *transversality conditions* must hold to guarantee optimality.

condition between liquidity and bonds, equating the *reduction* in marginal transaction costs by holding liquidity ($\Upsilon_{a,t}$) to the net nominal interest rate on bonds (i_t). Equations (3.55) and (3.56) are simply the two constraints, obtained by derivation with respect to the Lagrangian multipliers.

Market clearing conditions

The market clearing conditions for labour services, money and bonds are the same as those derived in the CIA model. The only condition which is affected by the presence of transaction costs concerns the equilibrium in the goods market.

Given that real resources must be used up in transactions, total output (y_t) now must be equal to:

$$y_t = c_t + \Upsilon\left(\omega_t, c_t, a_t\right) + x_t \tag{3.62}$$

$\forall t \geq 0$. As in Chapter 2, using the definition of total transaction costs (3.57) one can rewrite the market clearing condition (3.62) as

$$y_t = c_t\left(1 + q_t\right) + x_t \tag{3.63}$$

$\forall t \geq 0$. The expression appearing on the right hand side of (3.63) represents aggregated demand. To distinguish total consumption in the RRC model from the analogous variable in the CIA model, a capital letter (C_t) will be used in the former:

$$C_t \equiv c_t\left(1 + q_t\right) \tag{3.64}$$

$\forall t \geq 0$.

Once the equilibrium conditions for the CIA model and RRC model have been derived, the analysis can now focus on the dynamics.

3.4 The dynamics

In order to explore and compare the dynamic performance of the two monetary models, subject to the random shocks described above, one needs to transform the non-linear system of equations characterising the general equilibrium into a linear system. This is done by taking a log-linear approximation around the deterministic steady state, applying the methodology described in Uhlig (1999). For each model, this section will take the following steps: firstly, presenting the equilibrium as obtained in the previous section; secondly, illustrating some steady state relationships; and finally by deriving the log-linear model.

3.4.1 The CIA model with capital

Non-linear system

In the case of the CIA economy, the set of optimality conditions for households and firms, together with the specification of monetary policy and the necessary market clearing conditions characterise the dynamic general equilibrium model as a system of non linear equations:

$$\frac{\Psi\left(1-h_t^s\right)^{-\eta}}{c_t^{-\Phi}}=\frac{w_t}{\left[1+\Upsilon_{c,t}\right]} \tag{3.65}$$

$$1=\beta E_t\left\{\left(\frac{c_{t+1}}{c_t}\right)^{-\Phi}\frac{I_t}{\Pi_{t+1}}\right\} \tag{3.66}$$

$$1=\beta E_t\left\{\left(\frac{c_{t+1}}{c_t}\right)^{-\Phi}\frac{I_t}{I_{t+1}}R_{t+1}^k\right\} \tag{3.67}$$

$$\frac{m_{t-1}^d}{\Pi_t}+\tau_t=c_t \tag{3.68}$$

$$\frac{y_t}{h_t^d}=w_t \tag{3.69}$$

$$\alpha\frac{y_t}{k_{t-1}^d}+1-\delta=R_t^k \tag{3.70}$$

$$y_t=e^{z_t}\left(k_{t-1}^d\right)^{\alpha}\left(h_t^d\right)^{1-\alpha} \tag{3.71}$$

$$\frac{m_{t-1}^{s}}{\Pi_t} + \tau = m_t^s \tag{3.72}$$

$$\tau_t \equiv \theta_t \frac{m_{t-1}^{s}}{\Pi_t} \tag{3.73}$$

$$k_{t-1}^{d} = k_{t-1}^{s} \tag{3.74}$$

$$h_t^d = h_t^s \tag{3.75}$$

$$m_t^d = m_t^s \tag{3.76}$$

$$y_t = c_t + x_t \tag{3.77}$$

$$x_t = k_t^s - (1-\delta)\, k_{t-1}^{s} \tag{3.78}$$

$$\ln z_t = (1-\rho_z) \ln z + \rho_z \ln z_{t-1} + \epsilon_{z_t} \tag{3.79}$$

$$\ln \theta_t = (1-\rho_\theta) \ln \theta + \rho_\theta \ln \theta_{t-1} + \epsilon_{\theta_t} \tag{3.80}$$

$\forall t \geq 0$.[19]

Money demand and velocity of money

After all markets have cleared, the application of Walras' law implies that the evolution of real balances in the hands of the households follows

$$\frac{m_{t-1}^{d}}{\Pi_t} + \tau_t = m_t^d \tag{3.81}$$

$\forall t \geq 0$. Combining (3.81) with (3.68) one obtains the following expression

[19] The reason why the budget constraint does not appear in the non-linear system can be seen as a direct consequence of Walras' law.

for households' demand for real balances:

$$m_t^d = c_t \tag{3.82}$$

$\forall t \geq 0$. Expression (3.82) characterises money demand in the CIA model. Given the fact that consumption goods remain the only expenditure variable on the right hand side of the cash-in-advance constraint, the money demand function is identical to the expression (2.75) in Chapter 2. As a consequence, the *consumption-based* velocity is constantly equal to unity:

$$VEL\,(c_t) \equiv \frac{c_t}{m_t^s} = 1 \tag{3.83}$$

$\forall t \geq 0$[20]. However, using the goods market clearing condition in presence of investment goods, it is possible to check that the expression for the *output*-based velocity differs from the one derived in the previous chapter:

$$VEL\,(y_t) \equiv \frac{y_t}{m_t^s} = \frac{c_t + x_t}{m_t^s} \tag{3.84}$$

$\forall t \geq 0$. In fact, total output in this model does not coincide with the *cash good* only. As a result, this last measure of velocity will assume values generally different than 1 (in and outside of the steady state). For sake of completeness, a third measure of velocity is introduced, the so-called *investment*-based velocity:

$$VEL\,(x_t) \equiv \frac{x_t}{m_t^s} \tag{3.85}$$

$\forall t \geq 0$. The use of (3.85) is motivated by purposes of quantitative analysis. As seen in Chapter 2, to find measures of (3.83) and (3.84) is quite common in the literature. However, there are few studies that focus on ratios like (3.85)[21].

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[20] For a discussion about the use of total money supply (m_t^s) in the definitions of velocity see footnote (27) in Chapter 2.

[21] See Mankiw and Summers (1986)

Steady state

Before turning to the log-linear system, it is useful to have a look at some long-run relationships implied by the model. When all the variables have reached their deterministic steady state, time subscripts can be 'removed' from the non-linear equations characterising the equilibrium. In this way it is possible to inspect how monetary factors impact the fundamental structure of the economy.

In steady state expressions (3.72) and (3.73) can be used to obtain

$$\Pi = \Theta \tag{3.86}$$

or, equivalently

$$\pi = \theta \tag{3.87}$$

The result indicates that the steady state inflation rate is determined by the money growth rate choosen by the monetary authority. A direct implication of this is that the (steady state) real quantity of money (m) is constant - i.e., *neutrality of money* holds in steady state. Given the household's subjective discount rate (β), the *inter*temporal condition (3.66) can be used to determine the long-run nominal interest rate:

$$I = \frac{\Theta}{\beta} \tag{3.88}$$

One can combine (3.88) with (3.65), (3.69) and the production function (3.71), all evaluated at the steady state, in order to obtain the following expression:

$$\frac{\Psi h^{\Phi}}{(1-h)^{\eta}} = (1-\alpha)\frac{\beta}{\Theta}\left(\frac{y}{k}\right)^{\frac{\Phi-\alpha}{1-\alpha}}\left(\frac{c}{k}\right)^{-\Phi} \tag{3.89}$$

Using (3.70) and (3.77), it is possible to re-write (3.89) as

$$\frac{h^{\Phi}}{(1-h)^{\eta}} = \Delta\frac{\beta}{\Theta} \tag{3.90}$$

where:

$$\Delta \equiv \frac{(1-\alpha)}{\Psi}\left(\frac{1/\beta-1+\delta}{\alpha}\right)^{\frac{\Phi-\alpha}{1-\alpha}}\left(\frac{1/\beta-1+\delta}{\alpha}-\delta\right)^{-\Phi}$$

Given standard calibration (i.e., $0 < \beta < 1$ and δ small with respect to 1) one can check that $\Delta > 0$. As in the CIA model of the previous chapter, according to (3.90) a higher money growth rate implies a lower labour supply.

Using the production function, one can show that the steady state value of the capital stock corresponds to:

$$k = \left(\frac{y}{k}\right)^{\frac{1}{\alpha-1}} h \tag{3.91}$$

, which implies

$$y = \left(\frac{1/\beta-1+\delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} h \tag{3.92}$$

According to (3.92), and given standard calibration, one can show that output is positively related to working hours. Considering the negative impact of a higher money growth rate on labour supply shown in (3.90), one can conclude that superneutrality of money does not hold in this model.

As in Chapter 2, this result sheds some light on the monetary transmission mechanism of the CIA model: a permanent increase in the inflation tax on cash purchases causes a substitution from consumption to leisure, thereby reducing the supply of working hours (and therefore real output).

Log-linear approximation

Using the methodology decribed by Uhlig (1999) one can *linearise* the original model, taking a first order Taylor expansion around the steady state. The usefulness of the log-linearisation method is twofold: on the one hand, it allows one to solve the model applying standard solution methods for linear rational expectations models; on the other hand, it re-defines all the economic variables as percentage deviations from steady state, isolating their cyclical fluctuations. The result is a linear system of equations, where the variables with the 'hat' indicate percentage deviations of the original variables from their long-run values[22], while variables without time subscripts indicate steady state values:

consumption/labour:

$$\left[\eta\frac{h}{(1-h)}\right]\hat{h}_t^s = \hat{w}_t - \Phi\hat{c}_t - \hat{\imath}_t \tag{3.93}$$

consumption/saving:

$$\Phi E_t\hat{c}_{t+1} - \Phi\hat{c}_t = \hat{\imath}_t - E_t\hat{\pi}_{t+1} \tag{3.94}$$

capital/bonds

$$\Phi E_t\hat{c}_{t+1} - \Phi\hat{c}_t = \hat{\imath}_t - E_t\hat{\imath}_{t+1} + E_t r_{t+1}^k \tag{3.95}$$

money demand:

$$\hat{m}_t^d = \hat{c}_t \tag{3.96}$$

labour demand:

$$\hat{y}_t - \hat{h}_t^d = \hat{w}_t \tag{3.97}$$

capital demand:

$$\alpha\frac{y}{k}\hat{y}_t - \alpha\frac{y}{k}k_{t-1}^d = R^k r_t^k \tag{3.98}$$

output:

$$\hat{y}_t = \hat{z}_t + \alpha\hat{k}_{t-1} + (1-\alpha)\,\hat{h}_t^d \tag{3.99}$$

money supply:

$$\frac{m}{\Pi}\hat{m}_{t-1}^s - \frac{m^s}{\Pi}\hat{\pi}_t + \tau\hat{\tau}_t = m^s\hat{m}_t^s \tag{3.100}$$

[22] As in the Chapter 2, the only exceptions are the inflation rate ($\hat{\pi}_t$), the nominal interest rate ($\hat{\imath}_t$) and the money growth rate ($\hat{\theta}_t$), where the 'hat' indicates deviations in levels. The fact that the original *net* rates are small numbers with respect to one, the correspondent *gross* rates (Π_t, I_t and Θ_t), can be log-linearised applying the following approximation: $\ln\Pi_t = \ln(1+\pi_t) \simeq \pi_t$.

monetary injection:

$$\tau\hat{\tau}_t \equiv \Theta\frac{m^s}{\Pi}\hat{\theta}_t + \theta\frac{m^s}{\Pi}\hat{m}^s_{t-1} - \theta\frac{m^s}{\Pi}\hat{\pi}_t \tag{3.101}$$

capital market clearing condition:

$$\hat{k}^d_{t-1} = \hat{k}^s_{t-1} \tag{3.102}$$

labour market clearing condition:

$$\hat{h}^d_t = \hat{h}^s_t \tag{3.103}$$

money market clearing condition:

$$\hat{m}^d_t = \hat{m}^s_t \tag{3.104}$$

goods market clearing condition:

$$y\hat{y}_t = c\hat{c}_t + x\hat{x}_t \tag{3.105}$$

investment

$$\hat{x}_t = \frac{1}{\delta}\hat{k}_t - \frac{(1-\delta)}{\delta}\hat{k}_{t-1} \tag{3.106}$$

consumption-based velocity:

$$VE\hat{L}\left(c\right)_t \equiv \hat{c}_t - \hat{m}^s_t \tag{3.107}$$

investment-based velocity:

$$VE\hat{L}\left(x\right)_t \equiv \hat{x}_t - \hat{m}^s_t \tag{3.108}$$

output-based velocity:

$$VE\hat{L}\left(y\right)_t \equiv \hat{y}_t - \hat{m}^s_t \tag{3.109}$$

technology shock:

$$\hat{z}_t = \rho_z\hat{z}_{t-1} + \epsilon_{z_t} \tag{3.110}$$

monetary shock:

$$\hat{\theta}_t = \rho_\theta \hat{\theta}_{t-1} + \epsilon_{\theta_t} \tag{3.111}$$

, where: $\epsilon_{z_t} \sim N\left(0, \sigma^2_{\epsilon_z}\right)$ and $\ln \epsilon_{\theta_t} \sim N\left(0, \sigma^2_{\epsilon_\theta}\right)$. The velocity definitions have been also included, in order to obtain the related simulation results.

Note that expressions (3.100) and (3.101), together with the money market clearing condition (3.104), deliver the same result of Chapter 2 in terms of inflation dynamics:

$$\hat{\pi}_t = \hat{\theta}_t - \left(\hat{m}^d_t - \hat{m}^d_{t-1}\right) \tag{3.112}$$

$\forall t \geq 0$; where fluctuations in the inflation rate around its steady state value are determined by the difference between money supply *growth* ($\hat{\theta}_t$) and money demand *growth* ($\hat{m}^d_t - \hat{m}^d_{t-1}$). As in Chapter 2, money is *neutral* even *outside* the steady state: a one-shot monetary shock increasing the quantity of money today will change the price level proportionally, leaving all real variables unaffected.

3.4.2 The RRC model with capital

In the case of the RRC model, the set of optimality conditions for households and firms, together with the specification of the monetary policy rule and the necessary market clearing conditions correspond to the following system of (non-linear) equations:

$$\frac{\Psi\left(1 - h^s_t\right)^{-\eta}}{c_t^{-\Phi}} = \frac{w_t}{\left[1 + \Upsilon_{c,t}\right]} \tag{3.113}$$

$$1 = \beta E_t \left\{ \left(\frac{c_{t+1}}{c_t}\right)^{-\Phi} \left(\frac{1 + \Upsilon_{c,t}}{1 + \Upsilon_{c,t+1}}\right) \frac{I_{t+1}}{\Pi_{t+1}} \right\} \tag{3.114}$$

$$1 = \beta E_t \left\{ \left(\frac{c_{t+1}}{c_t}\right)^{-\Phi} \left(\frac{1 + \Upsilon_{c,t}}{1 + \Upsilon_{c,t+1}}\right) R^k_{t+1} \right\} \tag{3.115}$$

$$1 - \left[1 + \Upsilon_{a,t}\right] = I_t - 1 \tag{3.116}$$

$$\frac{m_{t-1}^{d}}{\Pi_t} + \tau_t = a_t \tag{3.117}$$

$$\Upsilon_t \equiv c_t q_t \tag{3.118}$$

$$q_t \equiv e^{\omega_t}\Omega_1 \left(\upsilon_t\right)^{\Omega_2} \tag{3.119}$$

$$\upsilon_t \equiv \frac{c_t}{a_t} \tag{3.120}$$

$$[1 + \Upsilon_{c,t}] = 1 + \left(\Omega_2 + 1\right) q_t \tag{3.121}$$

$$[1 + \Upsilon_{a,t}] = 1 - \Omega_2 q_t \upsilon_t \tag{3.122}$$

$$\frac{y_t}{h_t^d} = w_t \tag{3.123}$$

$$\alpha \frac{y_t}{k_{t-1}^d} + 1 - \delta = R_t^k \tag{3.124}$$

$$y_t = e^{z_t}\left(k_{t-1}^d\right)^{\alpha}\left(h_t^d\right)^{1-\alpha} \tag{3.125}$$

$$\frac{m_{t-1}^{s}}{\Pi_t} + \tau = m_t^s \tag{3.126}$$

$$\tau_t \equiv \theta_t \frac{m_{t-1}^{s}}{\Pi_t} \tag{3.127}$$

$$k_{t-1}^d = k_{t-1}^s \tag{3.128}$$

$$h_t^d = h_t^s \tag{3.129}$$

$$m_t^s = m_t^d \tag{3.130}$$

$$y_t = C_t + x_t \tag{3.131}$$

$$C_t \equiv c_t Q_t \tag{3.132}$$

$$x_t = k_t - (1-\delta)\, k_{t-1} \tag{3.133}$$

$$\ln z_t = (1-\rho_z)\ln z + \rho_z \ln z_{t-1} + \epsilon_{z_t} \tag{3.134}$$

$$\ln \theta_t = (1-\rho_\theta)\ln \theta + \rho_\theta \ln \theta_{t-1} + \epsilon_{\theta_t} \tag{3.135}$$

$$\ln \omega_t = (1-\rho_\omega)\ln \omega + \rho_\omega \ln \omega_{t-1} + \epsilon_{\omega_t} \tag{3.136}$$

$\forall t \geq 0$. Where $Q_t \equiv 1 + q_t$ indicates gross transaction costs.

Money demand and veocity of money

After all markets have cleared, the application of Walras' law implies that the evolution of real balances in the hands of the households follows

$$\frac{m_{t-1}^d}{\Pi_t} + \tau_t = m_t^d \tag{3.137}$$

$\forall t \geq 0$. Combining (3.137) with (3.117) one obtains the following expression for households' demand for real balances:

$$m_t^d = a_t \tag{3.138}$$

$\forall t \geq 0$.

As in Chapter 2, in order to derive the expression of the money demand for the RRC model, one needs to combine (3.116), (3.117), (3.119), (3.120) and (3.122), to obtain

$$m_t^d = c_t \left(\frac{\Omega_1 \Omega_2 \omega_t}{I_t - 1}\right)^{\frac{1}{\Omega_2+1}} \tag{3.139}$$

$\forall t \geq 0$. Expression (3.139) shows that real balances respond positively to the expenditure variable (c_t) and a transaction costs shock (ω_t), and a negatively to the nominal interest rate ($i_t \equiv I_t - 1$). In addition to that, it implies a unitary elasticity with respect to consumption. Differently than the CIA model, this money demand function allows endogenous variations in the ratio c_t/m_t^d. However, as for the RRC model in Chapter 2, the *consumption*-based velocity of money is defined as

$$VEL\left(C\right)_t \equiv \frac{C_t}{m_t^s} \tag{3.140}$$

$\forall t \geq 0$; where total consumption (C_t) includes real resources devoted to transactions. For sake of comparison with the CIA model with capital, *output*-based and *investment*-based velocity are also reported:

$$VEL\left(x\right)_t \equiv \frac{x_t}{m_t^s} \tag{3.141}$$

$$VEL\left(y\right)_t \equiv \frac{y_t}{m_t^s} \tag{3.142}$$

$\forall t \geq 0$.

Steady state

As for the CIA model, expressions (3.126) and (3.127) can be used to obtain

$$\pi = \theta \tag{3.143}$$

, where the steady state inflation rate is determined by the money growth rate chosen by the monetary authority. As before, the (steady state) real quantity of money (m) is constant - i.e., *neutrality of money* holds in steady state of the RRC model as well. Given the household's subjective discount rate (β), the *inter*temporal condition (3.114) can be used to determine the long-run nominal interest rate:

$$I = \frac{\Theta}{\beta} \tag{3.144}$$

, which is identical to the one found for the CIA model.

In order to analyse the steady state properties of the RRC model one can start by rearranging the expression for the money demand (3.139) in order to obtain the following:

$$v \equiv \frac{c}{m} = \left(\frac{\Theta/\beta - 1}{\Omega_1\Omega_2}\right)^{\frac{1}{\Omega_2+1}} \tag{3.145}$$

Now, recalling the definition of the unitary transaction costs (3.119), it follows that a permanently higher money growth rate causes an increase in the cost of transactions according to:

$$q = \Omega_1 \left(\frac{\Theta/\beta - 1}{\Omega_1\Omega_2}\right)^{\frac{\Omega_2}{\Omega_2+1}} \tag{3.146}$$

Combining (3.113) with (3.123) and the production function (3.125), all evaluated at the steady state, one obtains the following expression:

$$\frac{\Psi h^{\Phi}}{(1-h)^{\eta}} = (1-\alpha)\frac{1}{1+\Upsilon_c}\left(\frac{y}{k}\right)^{\frac{\Phi-\alpha}{1-\alpha}}\left(\frac{c}{k}\right)^{-\Phi} \tag{3.147}$$

Using (3.70) and the market clearing condition (3.131) one obtains that the consumption/capital ratio is equivalent to:

$$\frac{c}{k} = \frac{\left(\frac{y}{k} - \delta\right)}{Q} \tag{3.148}$$

Deriving the expression for the output/capital ratio from (3.124) and using (3.148) one can re-write (3.147) as:

$$\frac{\Psi h^{\Phi}}{(1-h)^{\eta}} = (1-\alpha)\frac{Q^{\Phi}}{1+\Upsilon_c}\left(\frac{1/\beta - 1 + \delta}{\alpha}\right)^{\frac{\Phi-\alpha}{1-\alpha}}\left(\frac{1/\beta - 1 + \delta}{\alpha} - \delta\right)^{-\Phi} \tag{3.149}$$

The left-hand side (LHS) of (3.149) is positively related with working hours. The value of the expression on the right hand side (RHS), instead, varies with the level of gross unitary transaction costs (Q) and marginal transaction costs of consumption (Υ_c). Making use of the definition of Q and (3.121) it is possible to re-write (3.149) as

$$\frac{h^{\Phi}}{(1-h)^{\eta}} = \Delta \frac{(1+q)^{\Phi}}{1+q+\Omega_2 q} \tag{3.150}$$

where, as in the CIA model,

$$\Delta \equiv \frac{(1-\alpha)}{\Psi} \left(\frac{1/\beta - 1 + \delta}{\alpha}\right)^{\frac{\Phi-\alpha}{1-\alpha}} \left(\frac{1/\beta - 1 + \delta}{\alpha} - \delta\right)^{-\Phi}$$

To determine the effect of an increase in q (due to a higher money growth rate) on working hours one can differentiate the right-hand side (RHS) of (3.150) with respect to q as follows:

$$\frac{d\,(RHS)}{dq} = \Delta \frac{\Phi\,(1+q)^{\Phi-1}\,(1+q+\Omega_2 q) - (1+\Omega_2)\,(1+q)^{\Phi}}{(1+q+\Omega_2 q)^2} \tag{3.151}$$

Given that $\Delta > 0$ (for standard calibration), the sign of (3.151) depends on the value of the parameters $\Phi > 0$ and $\Omega_2 > 0$. Since the assumption of balanced growth implies a log-utility function for consumption ($\Phi = 1$), in this chapter the right-hand side (RHS) of (3.150) is lowered when unitary transaction costs increase, according to:

$$\frac{d\,(RHS)}{dq} = -\Delta \frac{\Omega_2}{(1+q+\Omega_2 q)^2} < 0 \tag{3.152}$$

Finally, using the production function, the value of capital at steady state and the exression for the output/capital ratio it is possible to show that real output is positively related to working hours according to:

$$y = \left(\frac{1/\beta - 1 + \delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} h \tag{3.153}$$

Considering the positive impact of a higher money growth rate on unitary transaction costs, the negative relationship between these and the labour supply, and expression (3.153) all together, one can conclude that superneutrality of money does not hold in this model.

Summing up: a permanent increase in the inflation tax causes a substitution from cash to credit purchases: whatever the means of payment used by households, the purchase of consumption goods is now more expensive. As in

Chapter 2, the shift from cash to credit commands more real resources devoted to transactions. Whether a higher money growth will imply an increase or a decrease in output is going to depend on the relative weight of both effects on labour supply (i.e., on the value of the parameters defining preferences and transaction technologies).

Log-linear approximation

As for the CIA model, one can proceed log-linearising the RRC model around its steady state. As before, the variables with the 'hat' indicate percentage deviations of the original variables from their long-run values[23], while variables without time subscripts indicate steady state values:

leisure/consumption:

$$\left[\eta\frac{h}{(1-h)}\right]\hat{h}_t^s = \hat{w}_t - \Phi\hat{c}_t - \hat{\Upsilon}_{c,t} \tag{3.154}$$

consumption/saving:

$$\Phi E_t\hat{c}_{t+1} - \Phi\hat{c}_t + E_t\hat{\Upsilon}_{c,t+1} - \hat{\Upsilon}_{c,t} = E_t\hat{\imath}_{t+1} - E_t\hat{\pi}_{t+1} \tag{3.155}$$

capital/bonds:

$$\mathbf{\Phi E}_t\hat{\mathbf{c}}_{t+1} - \mathbf{\Phi}\hat{\mathbf{c}}_t + \mathbf{E}_t\hat{\mathbf{\Upsilon}}_{c,t+1} - \hat{\mathbf{\Upsilon}}_{c,t} = \mathbf{E}_t\mathbf{r}_{t+1}^k \tag{3.156}$$

marginal costs of liquidity:

$$-\left[1+\Upsilon_{a,t}\right]\hat{\Upsilon}_{a,t} = I\hat{\imath}_t \tag{3.157}$$

money demand:

$$\hat{m}_t^d = \hat{a}_t \tag{3.158}$$

total transaction costs:

[23] The only exceptions are the inflation rate ($\hat{\pi}_t$), the nominal interest rate ($\hat{\imath}_t$) the money growth rate ($\hat{\theta}_t$), the unitary transaction costs ($\hat{q}_t$) and the marginal transaction costs ($\hat{\Upsilon}_{c,t}$ and $\hat{\Upsilon}_{a,t}$), where 'hat' indicates deviations in levels. See also footnote (23).

$$\Upsilon\hat{\Upsilon}_t = cq\hat{c}_t + cQ\hat{q}_t \tag{3.159}$$

unit transaction costs:

$$Q\hat{q}_t = q\hat{\omega}_t + q\Omega_2\hat{v}_t \tag{3.160}$$

liquidity ratio:

$$\hat{v}_t = \hat{c}_t - \hat{a}_t \tag{3.161}$$

marginal transaction cost of consumption:

$$\left[1 + \Upsilon_c\right]\hat{\Upsilon}_{c,t} = \left[\Omega_2 + 1\right] Q\hat{q}_t \tag{3.162}$$

marginal transaction cost of liquidity:

$$\left[1 + \Upsilon_a\right]\hat{\Upsilon}_{a,t} = -\Omega_2 Qv\hat{q}_t - \Omega_2 qv\hat{v}_t \tag{3.163}$$

labour demand:

$$\hat{y}_t - \hat{h}_t^d = \hat{w}_t \tag{3.164}$$

marginal product of capital

$$\alpha\frac{y}{k}\hat{y}_t - \alpha\frac{y}{k}k_{t-1}^d = R^k r_t^k \tag{3.165}$$

real output:

$$\hat{y}_t = \hat{z}_t + \alpha\hat{k}_{t-1}^d + (1-\alpha)\,\hat{h}_t^d \tag{3.166}$$

money supply:

$$\frac{m^s}{\Pi}\hat{m}_{t-1}^s - \frac{m^s}{\Pi}\hat{\pi}_t + \tau\hat{\tau}_t = m^s\hat{m}_t^s \tag{3.167}$$

monetary injection:

$$\tau\hat{\tau}_t \equiv \Theta\frac{m^s}{\Pi}\hat{\theta}_t + \theta\frac{m^s}{\Pi}\hat{m}_{t-1}^s - \theta\frac{m^s}{\Pi}\hat{\pi}_t \tag{3.168}$$

capital market clearing condition:

$$\hat{k}^d_{t-1} = \hat{k}^s_{t-1} \tag{3.169}$$

labour market clearing condition:

$$\hat{h}^d_t = \hat{h}^s_t \tag{3.170}$$

money market clearing condition:

$$\hat{m}^s_t = \hat{m}^d_t \tag{3.171}$$

goods market clearing condition:

$$y\hat{y}_t = C\hat{C}_t + x\hat{x}_t \tag{3.172}$$

investment:

$$\hat{x}_t = \frac{1}{\delta}\hat{k}_t - \frac{(1-\delta)}{\delta}\hat{k}_{t-1} \tag{3.173}$$

total consumption:

$$C\hat{C}_t \equiv c\hat{c}_t + \Upsilon\hat{\Upsilon}_t \tag{3.174}$$

consumption-based velocity:

$$VE\hat{L}\left(C\right)_t \equiv \hat{C}_t - \hat{m}^s_t \tag{3.175}$$

investment-based velocity:

$$VE\hat{L}\left(x\right)_t \equiv \hat{x}_t - \hat{m}^s_t \tag{3.176}$$

output-based velocity:

$$VE\hat{L}\left(y\right)_t \equiv \hat{y}_t - \hat{m}^s_t \tag{3.177}$$

technology shock:

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \epsilon_{z_t} \tag{3.178}$$

monetary shock:

$$\hat{\theta}_t = \rho_\theta \hat{\theta}_{t-1} + \epsilon_{\theta_t} \tag{3.179}$$

transaction cost shock:

$$\hat{\omega}_t = \rho_\omega \hat{\omega}_{t-1} + \epsilon_{\omega_t} \tag{3.180}$$

$\forall t \geq 0$; where: $Q \equiv 1 + q$ indicates (steady state) gross transaction costs, and $\epsilon_{z_t} \sim N\left(0, \sigma^2_{\epsilon_z}\right)$, $\ln \epsilon_{\theta_t} \sim N\left(0, \sigma^2_{\epsilon_\theta}\right)$ and $\epsilon_{\omega_t} \sim N\left(0, \sigma^2_{\epsilon_\omega}\right)$ indicate the standard deviations of the shocks. The definitions of consumption-based and output-based velocity have also been included.

Expressions (3.167) and (3.168), together with the money market clearing condition (3.171), lead to the same expression for the inflation dynamics:

$$\hat{\pi}_t = \hat{\theta}_t - \left(\hat{m}^d_t - \hat{m}^d_{t-1}\right) \tag{3.181}$$

$\forall t \geq 0$, which is identical to the one derived under the CIA model. As in the case of the CIA model, money is *neutral* even *outside* the steady state: a one-shot monetary shock increasing the quantity of money today will change the price level proportionally, leaving all real variables unaffected.

3.5 Quantitative analysis

In the first part of this section numerical values are assigned to structural parameters and long-run relationships. The remaining coefficients in the linear approximations are derived using the the steady state relationships, implied by the original non-linear system. Given these calibration values, the last part of this section will compare the qualitative and quantitative impact of the stochastic shocks on the endogenous variables of the two monetary models. In particular, the analysis will focus on the *impulse-response* dynamics and the relative match of the CIA and the RRC model with respect to the empirical evidence presented in Chapter 1.

3.5.1 Calibration

In order to derive the response of the baseline models to stochastic shocks, one needs to assign numerical values to the parameters appearing in the linear equations. For the purpose of comparison, the following criteria have been taken into account : a) keeping (wherever possible) the values for the 'non-monetary' parameters in line with the RBC literature; b) 'choosing' the remaining parameters in order to make the CIA and the RRC models *comparable*. Given that the link between cash balances and consumption has not changed from the previous chapter, here the novelty is represented by the parameters regarding capital and the implied changes for the aggregate demand shares of output.

Calibration of the CIA model

Table 3.1 reports the values for the parameters characterising the utility function, some key long-run relationships and the stochastic shocks for the CIA model.

Regarding the utility function, the parameterisation adopted in Chapter 2 is unchanged: the steady state labour supply is set to one-third of the time endowment, while the log-utility in consumption and labour is satisfied by setting $\Phi = 1$ and $\eta = 1$, respectively. The parameter relative to the capital depreciation has been chosen in order to get a quarterly depreciation of 1.9% ($\delta = 0.019$), which is the same value used by Cooley and Hansen (1995). The labour share of output has been set in line with the long-run empirical evidence about this value[24]. In particular, the value reported in 3.1 ($\alpha = 0.36$) is adopted from Walsh (2003), and corresponds (roughly) to setting the capital share of output to the remaining one-third. The value for the (quarterly) discount factor ($\beta = 0.989$), the autoregressive coefficient of the technology shock ($\rho_z = 0.95$) and its standard deviation ($\sigma_{\epsilon_z} = 0.007$) are in line with the standard RBC literature[25] and are adopted from Cooley and Prescott

[24] In particular, Cooley and Prescott (1995, p.22) report that "*about two-thirds of the fluctuations in aggregate output are attributable to fluctuations in the labor input.*".

[25] For a discussion about the value of the discount factor see Campbell (1994). The characteristics of the technology shock are derived from the statistical properties of the

(1995). As in Chapter 2, the main reference for the monetary values is the CIA model by Cooley and Hansen (1989). This implies a quarterly net nominal money growth rate for M1 corresponding to $\theta = 0.0125$ and a monetary shock characterised by an autoregressive parameter $\rho_\theta = 0.5$, with standard deviation $\sigma_{\epsilon_\theta} = 0.0089$.[26]

One can check whether the baseline calibration of the CIA model with capital is able to generate steady state values compatible with the empirical evidence. These results are reported in Figure 3.2. Note that the long-run relationship between the money growth rate and inflation, the nominal interest rate and the parameter for leisure in the utility function have not changed from Chapter 2. However, with the introduction of capital goods, the real expenditure shares of output begin to converge towards more 'realistic' results than Chapter 2. In particular, the consumption share of output is around 77.3%, leaving the remaining 23.7% to investment.

Regarding the velocity of money the limitations of a cash-in-advance constraint applied exclusively to consumption goods continue to appear in a unitary *consumption*-based velocity. If one considers an output-based measure of velocity, instead, the baseline calibration delivers a value greater than one ($y/m = 1.2938$). This difference is due to the fact that investment purchases are not covered with cash (i.e., investment is a *credit good*). Finally, note that because the steady state value of consumption is now higher than it was in Chapter 2 (where labour was the only factor of production), the real value of the liquidity injections by the monetary authority ($\tau = 0.0128$) is also higher[27].

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so-called Solow residual (see Kydland and Prescott (1982)).

[26] In order to check for robustness of the results, a sensitivity analysis exercise has been carried out, using different values for Θ and ρ_θ. However, these attempts did not change significantly the quantitative results reported in the next sections.

[27] See Chapter 2, Figure 2.2 for a comparison of this and the other calibration values.

parameter/variable	**description**	**value**
Φ	relative risk aversion	1
η	inverse of labour supply elasticity	1
β	discount factor	0.989
δ	capital depreciation rate	0.019
α	capital income share of output	0.36
h	working hours	0.3333
θ	net money growth rate	0.0125
ρ_z	autoregressive param. technology shock	0.5
σ_{ϵ_z}	s.d. technology shock	0.007
ρ_θ	autoregressive param. monetary shock	0.5
σ_{ϵ_θ}	s.d. monetary shock	0.0089

Table 3.1: Baseline calibration of CIA model (with capital).

parameter/variable	**description**	**value**
Θ	gross money growth rate	1.0125
Π	gross inflation rate	1.0125
π	net inflation rate	0.0125
I	gross nominal interest rate on bonds	1.0238
i	net nominal interest rate on bonds	0.0238
R^k	gross (net) return on capital	1.0111
$r^k - \delta$	net return on capital	0.0111
r^k	marginal product of capital	0.0301
y/k	output/capital ratio	0.0836
k	capital stock	16.082
x	investment	0.3055
y	real output	1.3456
w	real wage	2.5836
c	real consumption	1.0401
c/y	consumption share of output	0.773
x/y	investmet share of output	0.227
Ψ	preference parameter for leisure	1.656
m	real cash balances	1.0401
τ	real monetary injection	0.0128
c/m	*consumption*-based velocity	1
x/m	*investment*-based velocity	0.2938
y/m	*output*-based velocity	1.2938

Table 3.2: Steady state values of CIA model (with capital) at baseline calibration.

Calibration of the RRC model

In this chapter (as in Chapter 2), the CIA model and the RRC model share the same basic structure. Again, for the seek of comparison, the calibration of the RRC model adopts the same numerical values used for the CIA model, except for the transaction technology parameters. The baseline calibration of the RRC model is reported in Table 3.3 below.

Four additional parameters appear in 3.3, when it is compared with 3.1. As in Chapter 2, Ω_2 is set in order to obain quadratic (total) transaction costs ($\Omega_2 = 1$), while the *liquidity ratio* (v) is set in order to make the comparison with the CIA model feasible[28]. This implies setting $v = 1$. The parameters characterising the transaction costs shock, are taken from an exercise proposed by Sims (1989), where the autoregressive parameter for the transaction cost shock ρ_ω and its standard deviation σ_{ϵ_ω} are set to 0.8 and 0.01, respectively[29]. As in Chapter 2, Table 3.4 shows that the baseline calibration results in a value for the unitary transaction costs in line with the nominal interest rate ($q = i = 2.38\%$), corresponding to a scale parameter $\Omega_1 = 0.0238$[30].

Looking at Table 3.5, the long-run relationship between the money growth rate and inflation and the consumption and investment shares of output are the same as in the CIA model. However, the consumption-output ratio includes a proportion of consumption goods devoted to transactions equivalent to the 1.8% of total output. The values for the velocity measures differ from the CIA model. In particular, the RRC model presents higher values along all the relative dimensions. Here the discrepancy between the CIA model and the RRC model is due to the fact that in the latter *only utility*-consumption is fully 'covered' by cash ($v = 1$), therefore the level of real balances agents require in steady state is lower than the analogous proportions in the CIA approach.

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[28] See Chapter 2, section 2.51 for a discussion on this issue.

[29] See Chapter 2, Section 2.51 for a discussion about the calibration of this type of shock.

[30] In order to check for robustness of the results, a sensitivity analysis exercise has been carried out, using different values for Ω_2 and v. However, these attempts did not change significantly the quantitative results reported in the next sections.

parameter/variable	description	value
Φ	relative risk aversion	1
η	inverse of labour supply elasticity	1
β	discount factor	0.989
δ	capital depreciation rate	0.019
α	capital share of output	0.36
h	working hours	0.3333
θ	net money growth rate	0.0125
ρ_z	autoregressive param. technology shock	0.95
σ_{ϵ_z}	s.d. technology shock	0.007
ρ_θ	autoregressive param. monetary shock	0.5
σ_{ϵ_θ}	s.d. monetary shock	0.0089
Ω_2	elasticity param. of transaction cost function	1
v	liquidity ratio	1
ρ_{ϵ_ω}	autoregressive param. transaction costs shock	0.8
σ_{ϵ_ω}	s.d. transaction costs shock	0.01

Table 3.3: Baseline calibration of RRC model (with capital).

parameter/variable	description	value
q	net unitary transaction costs	0.0238
Q	gross unitary transaction costs	1.0238
Υ_c	marg. trans. costs of consumption	0.0475
Υ_a	marg. trans. costs of liquidity	−0.0238
Ω_1	scale param. of transaction cost function	0.0238

Table 3.4: Steady state values of the RRC model (with capital), at baseline calibration: transaction costs function.

parameter/variable	**description**	**value**
Θ	gross money growth rate	1.0125
Π	gross inflation rate	1.0125
π	net inflation rate	0.0125
I	gross nominal interest rate on bonds	1.0238
i	net nominal interest rate on bonds	0.0238
R^k	gross return on capital (net of deprec.)	1.0111
r^k	marginal product of capital	0.0301
y/k	output/capital ratio	0.0836
k	capital	16.082
x	investment	0.3055
y	real output	1.3456
w	real wage	2.5836
c	real utility-consumption	1.0159
x/y	investment share of output	0.227
Ψ	preference parameter for leisure	1.6954
Υ	total transaction costs	0.0241
C	total real consumption	1.0401
C/y	consumption share of output	0.773
Υ/y	transaction costs share of output	0.018
a	liquidity	1.0159
m	real cash balances	1.0159
τ	real monetary injection	0.0125
C/m	consumption-based velocity	1.0238
x/m	investment-based velocity	0.2938
y/m	output-based velocity	1.3245

Table 3.5: Steady state values of RRC model (with capital) at baseline calibration.

3.5.2 Impulse-response analysis

In what follows the dynamic response of the CIA and the RRC model are analysed and compared. The figures below report the percentage deviation of the selected variables from their steady state value (which, for convenience, has been set to zero). The deviation from steady state of variables which do represent rates (e.g., inflation rate, interest rate, unit transaction costs), is measured in absolute terms. All the shocks take place at time zero and the time scale refers to quarterly data.

Technolocy shock

Figure 3.1 shows the impact of the technology shock on real expenditure and money demand (real balances). Given that this real shock affects production in the same way in both models the response of output (on impact and afterwards) coincides with the dynamics of the shock. However, given the presence of capital goods the effects of the technology shock on consumption are different from those derived in Chapter 2: in particular, while the dynamics of investment follows the dynamics of the shock quite closely, consumption remains very persistent after 16 quarters. Also on impact these two components of aggregate demand react very differently: in fact, invesment responds about 10 times more than consumption. Finally, since the money demand functions link real balances to consumption in both models these respond one to one to the change in consumption.

These elements can be used to explain the constant *consumption*-based velocity and the pro-cyclical *investment*- and *output*-based velocities in Figure 3.4. The technology shock impacts on the production function causing a strong response in output and investment and a 'softer' (but more persistent) increase in consumption. Given that money is intrinsically linked to consumption, the change in the *investment*- and *output*-based velocity is due to the *net* difference between the increase in investment and output, with respect to the change in consumption.

Figure 3.2 shows the impact of the technology shock on inflation and nominal interest rates. As in Chapter 2, inflation is generated in both models as a difference between the growth in money supply and the growth in money demand. If the government follows a constant money growth rule, a technology shock causes a temporary drop in inflation, *via* money demand dynamics. However, differently from Chapter 2, the drop in inflation is less severe and the (little) overshooting is now absent. This result is due to the different behaviour of real balances (i.e., of consumption) in the presence of capital goods. Given that the nominal interest rate is determined according to the Fisher relation, the change in consumption growth (matched by the money demand growth) cancels out with the drop in inflation. This explains the constant interest rate in Figure 3.2.

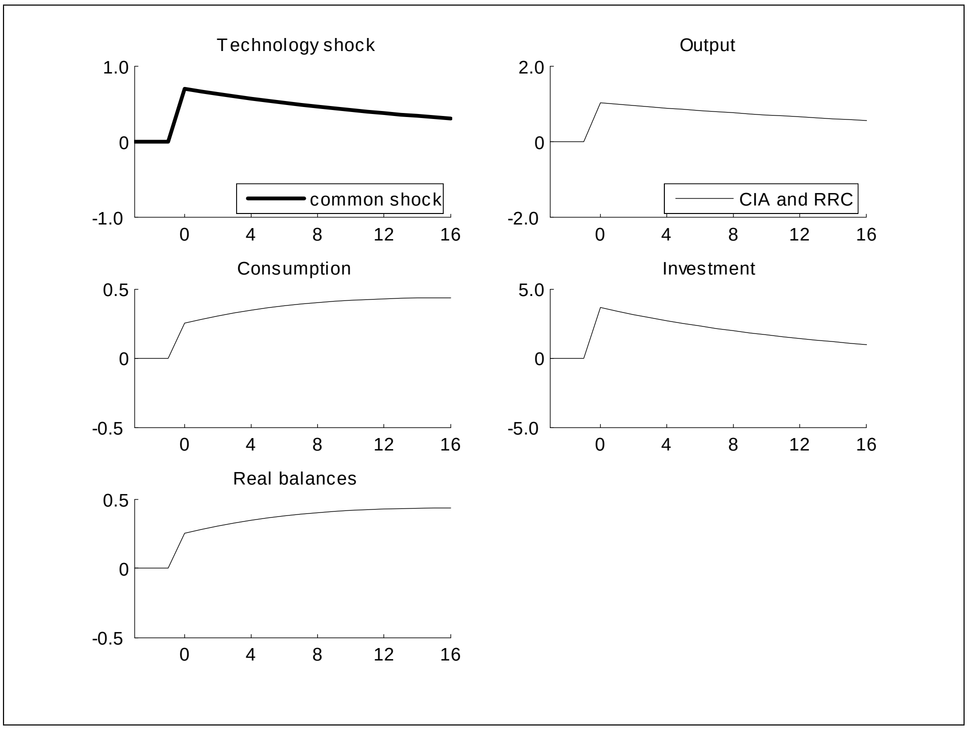

Figure 3.1: Impact of the technology shock on real expenditure (CIA vs. RRC, with capital)

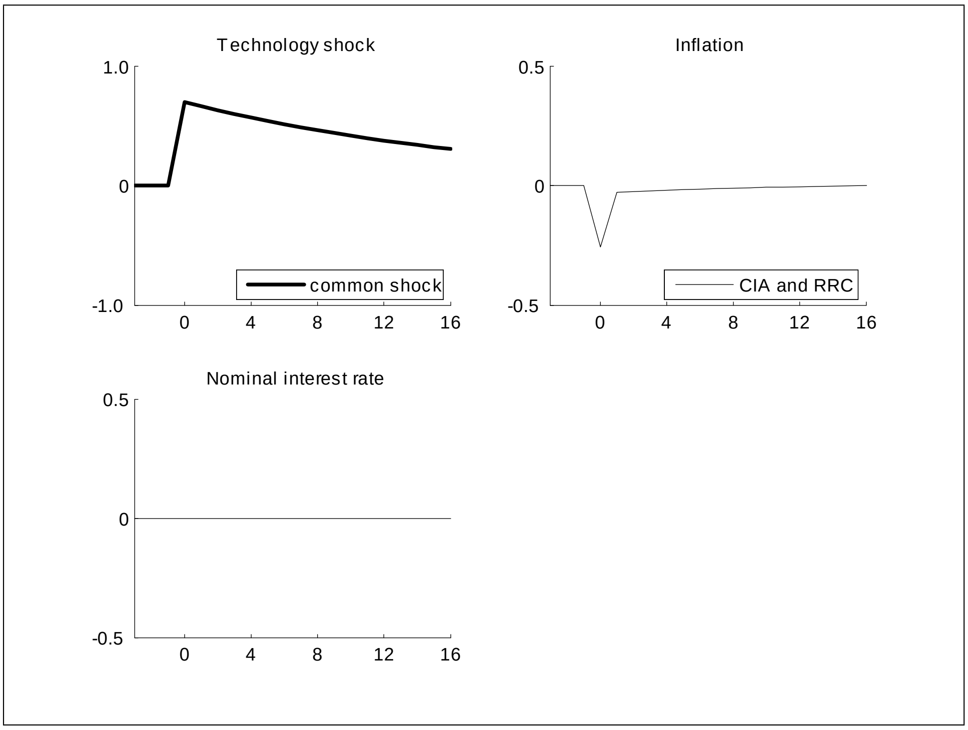

Figure 3.2: Impact of the technology shock on nominal variables (CIA vs. RRC, with capital)

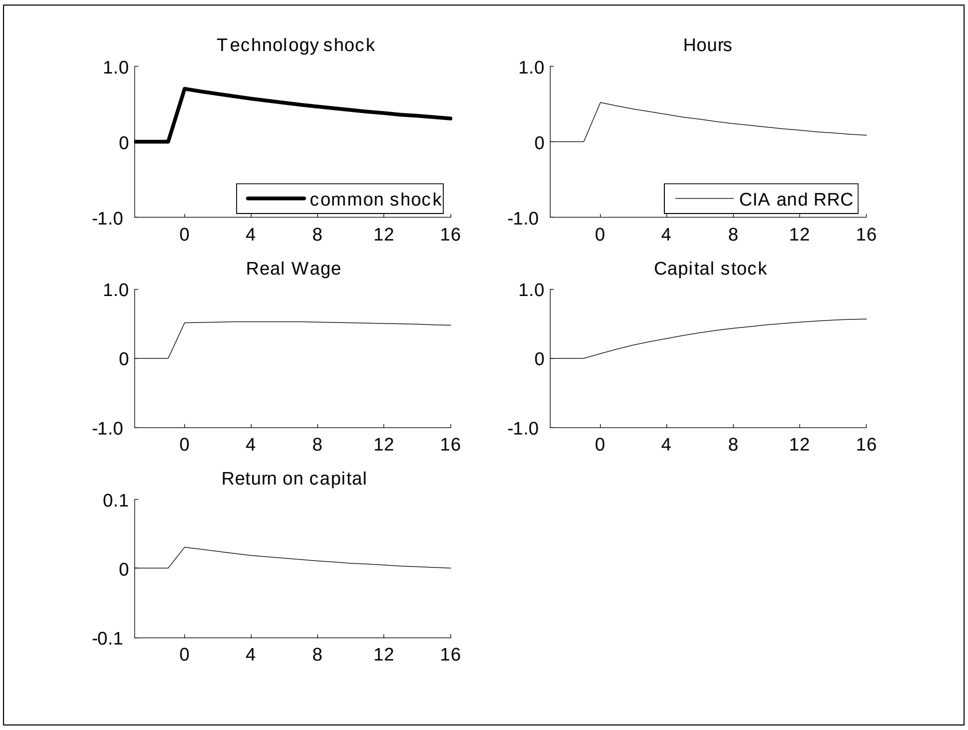

Figure 3.3: Impact of the technology shock on production factors (CIA vs. RRC, with capital)

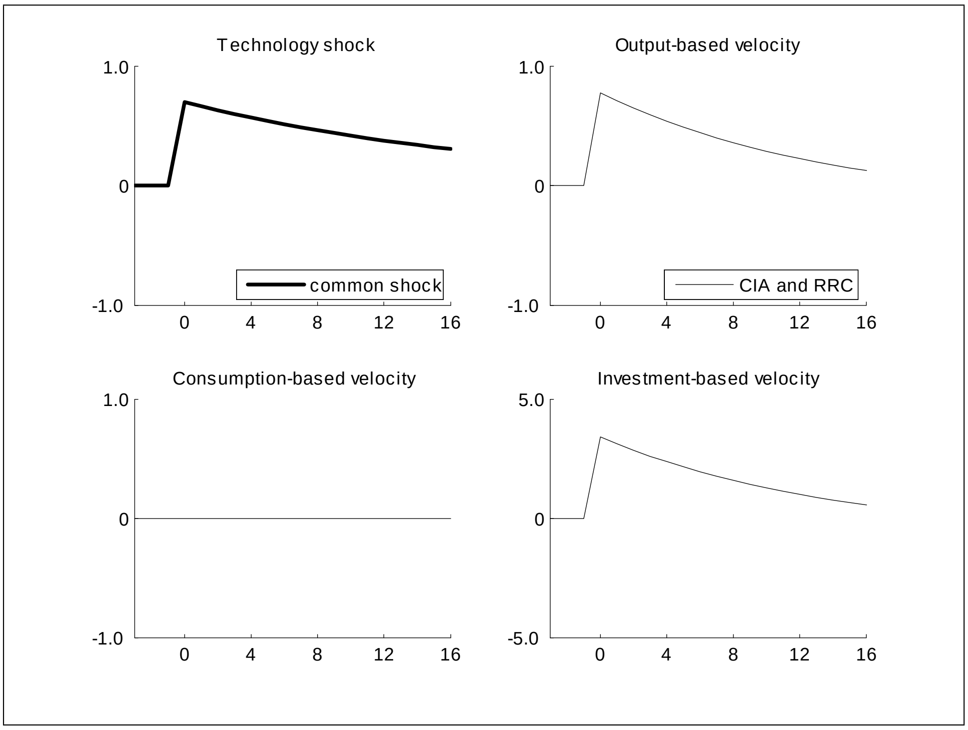

Figure 3.4: Impact of the technology shock on the velocity of money (CIA vs. RRC, with capital)

Monetary shock

Figure 3.5 shows the impact of the monetary shock on real expenditure and money demand (real balances). Qualitatively, the results exhibit similarities with those derived in a model without capital. As in Chapter 2, because of the assumption of flexible prices, in both models a monetary shock produces real effects as long as it modifies expected inflation. On one hand, since every purchase where money is involved becomes more expensive in real terms, higher expected inflation induces a fall in real consumption tomorrow and today, through the mechanism of consumption smoothing. On the other hand, since in both models consumption yields utility, the tighter the link between desired consumption and cash, the higher the value that households attach to money holdings. Since the two models differ explicitly along this last dimension, one should expect these to react differently to the same monetary shock. Again, the main difference emerges by examining closely the dynamics of real balances in Figure 3.5. Here the fall is identical to that in a model without capital, with real balances in the RRC model falling almost 3 times than in the CIA model. Despite the smaller fall in the value of real balances, the fall in real expenditure for the CIA model is about 25 times bigger than the RRC model. This is not surprising, since the way of modelling money demand is unchanged with respect to the models without capital: real balances remain more crucial in the CIA model than in the RRC model. However, as for the case of the technology shock, the presence of investment goods modifies the quantitative impact of the monetary shock. In fact, the deviations of consumption from steady state are bigger than those in Chapter 2 for both models. The reason is due to the fact that now households have the possibility to switch expenditures towards another good - i.e., investment - which is not subject to the inflation tax. Given the different impact of the inflation tax for the two models, the increase in investment is 15 times bigger in the CIA model. For all these reasons, with the only exception of *consumption*-based velocity the CIA model, the monetary shock causes all velocities to increase in Figure 3.5.

As anticipated before, the real effects of the monetary shock are guaranteed by the fact that it displays some degree of persistence. If the autoregressive parameter is set to zero ($\rho_\theta = 0$), only the *current* price level (and, therefore, only current inflation) will be affected, leaving the real magnitudes unchanged.

In both models real effects of expected inflation propagate via intertemporal substitution. As shown in Figure 3.6, the response of inflation to an monetary shock is more than proportional. This is because in both models actual inflation is determined by the difference between the money supply growth and the change in real balances. When the shock occurs, the greater response of inflation in the RRC model, when compared with the CIA model, is given by the higher fall in real balances. However, inflation persistence is higher for the CIA model. Again, the behaviour of real balances is the key: one period after the shock the rate of change in real balances is reversed for both models. The correction i the second period is quicker in the case of the RRC model, while it dies out more slowly in the CIA model (see Figure 3.5).

As shown in Figure 3.6, the impact of the monetary shock on the nominal interest rate is less than propotional. Given the market timing assumptions adopted (i.e., the financial market opening first) the nominal interest rate represents the opportunity cost of holding money for both models. However, its response to the monetary shock depends on the transaction technologies. In the CIA model expected inflation drives the nominal interest up according to the Fisher equation, which links directly to the marginal utility of consumption (and, therefore, real balances) today and tomorrow. Given the higher persistence of the inflation process in the CIA model, when the shock occurs, expected inflation in the CIA model is higher than expected inflation in the RRC model. This explains one channel through which monetary shocks trasmit to nominal rates. An additional channel is due to the fact that the link between expected inflation and actual interest rate in the RRC model is explained by the marginal contribution of real balances to transaction costs. The reason why the movement of the nominal interest rate is so small (0.03 basis points) in the case of the RRC model, has to be found in the weak impact of any change in real balances (induced by a change in expected inflation) on the cost of transactions.

Note that the change in velocities in this chapter involves also the CIA model. This result is due to the presence of an additional *credit-* good in this model (in contrast to chapter 2). Figure 3.6 shows that the movements in velocity are affected by the possibility for the agents to switch from *cash-* to *credit*-goods, in both models.

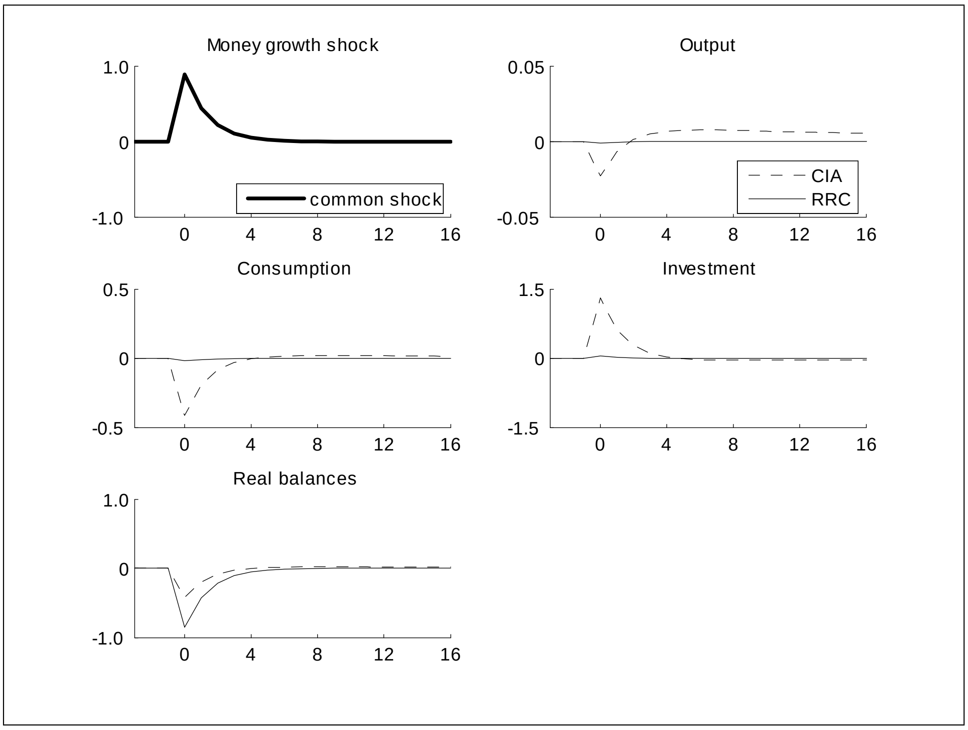

Figure 3.5: Impact of the monetary shock on real expenditure (CIA vs. RRC, with capital)

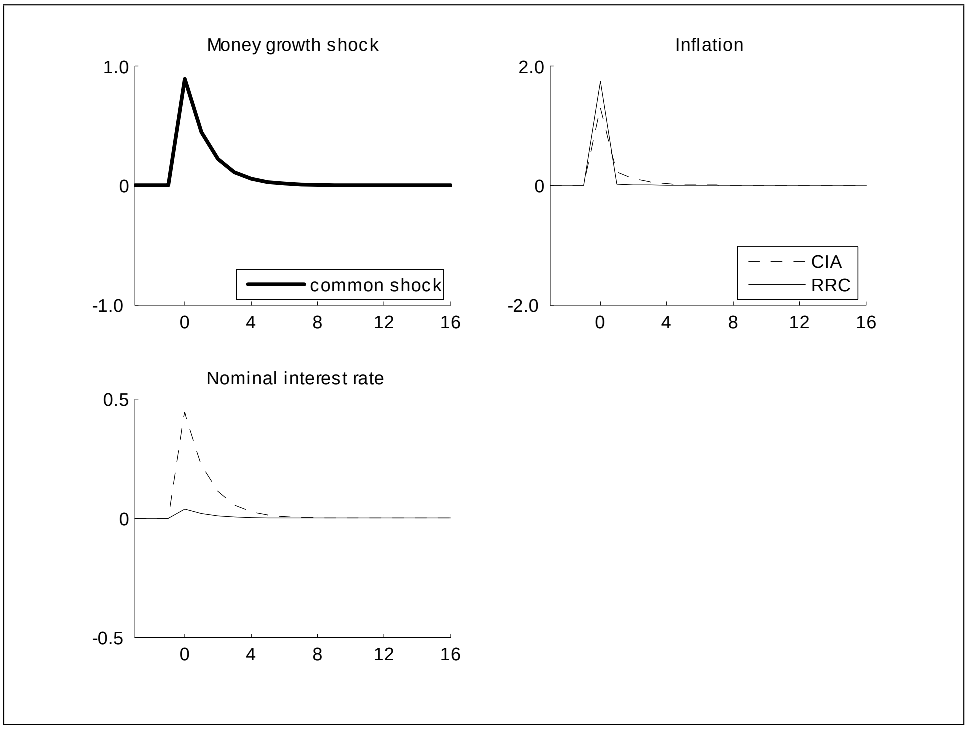

Figure 3.6: Impact of the monetary shock on nominal variables (CIA vs. RRC, with capital)

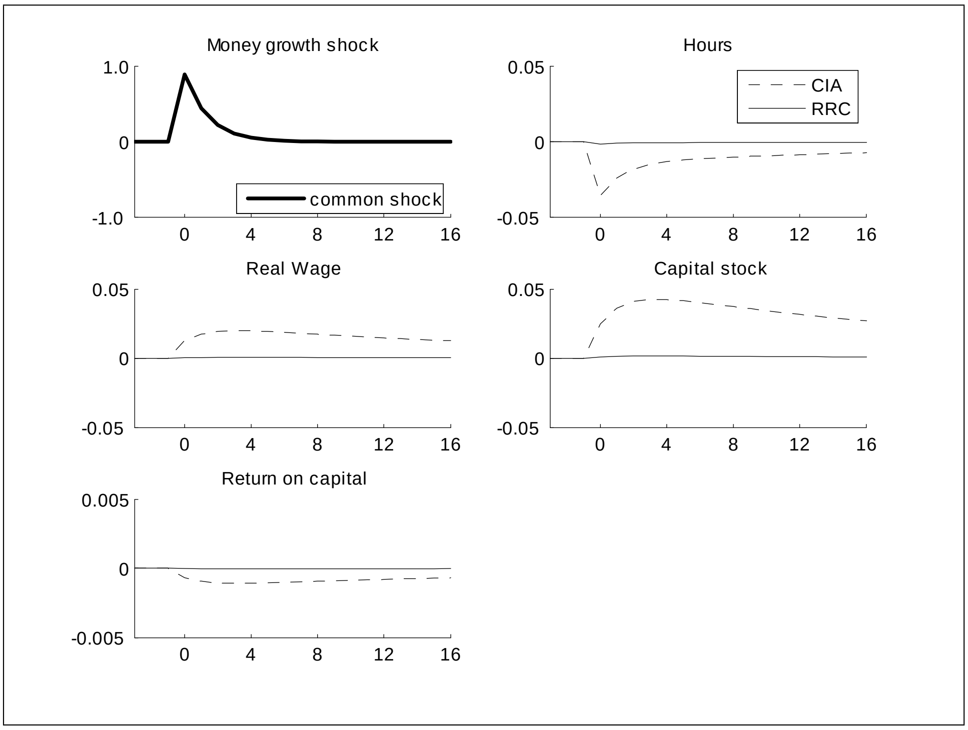

Figure 3.7: Impact of the monetary shock on production factors (CIA vs. RRC, with capital)

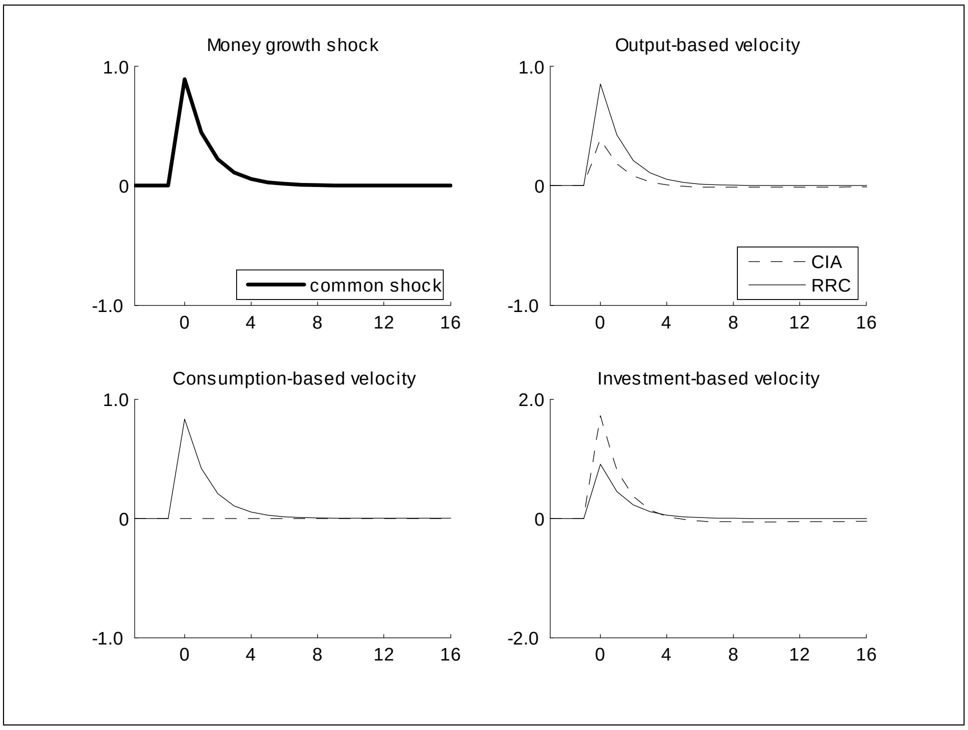

Figure 3.8: Impact of the monetary shock on the velocity of money (CIA vs. RRC, with capital)

Transaction costs shock

One additional source of shocks in the RRC model is represented by the transaction costs shock.There are two main consequences of this type of shock hitting the economy: it generates a negative effect on consumption (and output) and a higher demand for real balances. The former is due to the fact that now consumption purchases are more costly, the latter to the attempt of consumers to reduce transaction costs holding more cash. Given that the money demand function is unchanged relative to the one appearing in Chapter 2, the link between money and consumption in the RRC model is unchanged. In particular, a 1% unit transaction costs shock (Figure 3.9) continues to geerate an increase in money demand which is less than 0.1%, together with a risible fall in consumption (less than - 0.02%). However, the fact that purchases in capital goods are not subject to transaction costs, induces households to address part of their resources towards investment. In other words, transaction costs shocks in the RRC model operate in a way analogous to monetary shocks in the CIA model. In the period after the shock, real balances start to converge back to equilibrium. With monetary authorities keeping the money supply growth constant, this causes a little inflation overshooting (see Figure 3.10). For the same reasons discussed above, the behaviour of expected inflation induces a little movement in the nominal interest rate.

Finally, whenever transaction costs increase in the RRC model - due to a fall in the value of real balances or to a transaction costs shock - a fall in desired consumption is always accompanied by an increase in the real resources produced by the economy for transaction purposes. One of the reasons why the latter does not dominate the former lies in its small dimension in terms of output share at the steady state (see Table 3.5).

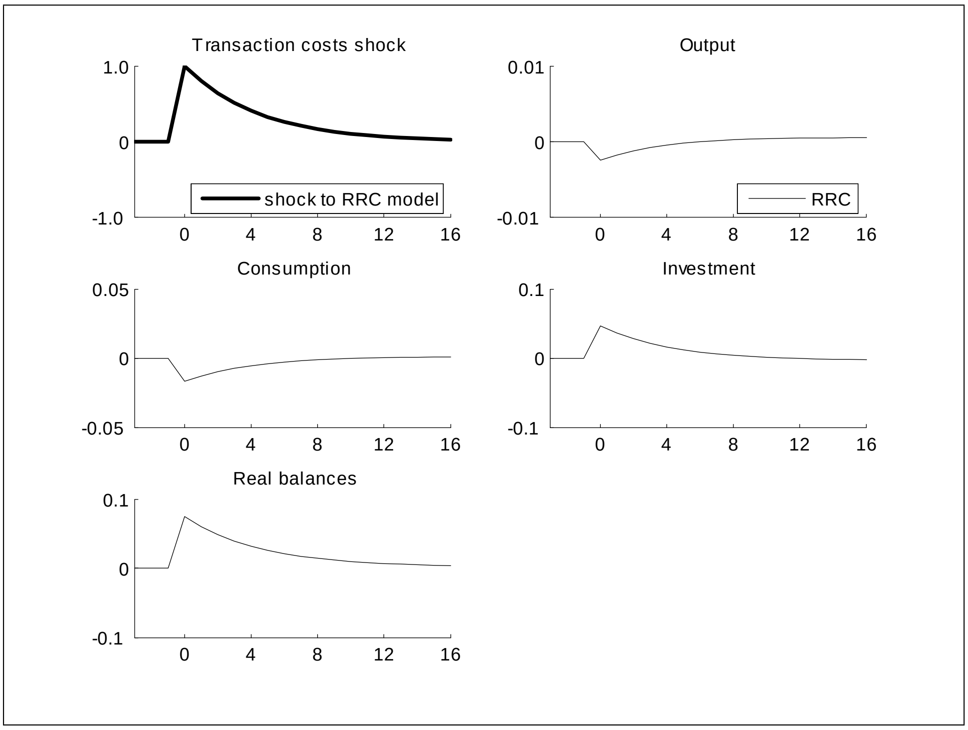

Figure 3.9: Impact of the transaction costs shock on real expenditure (RRC only, with capital)

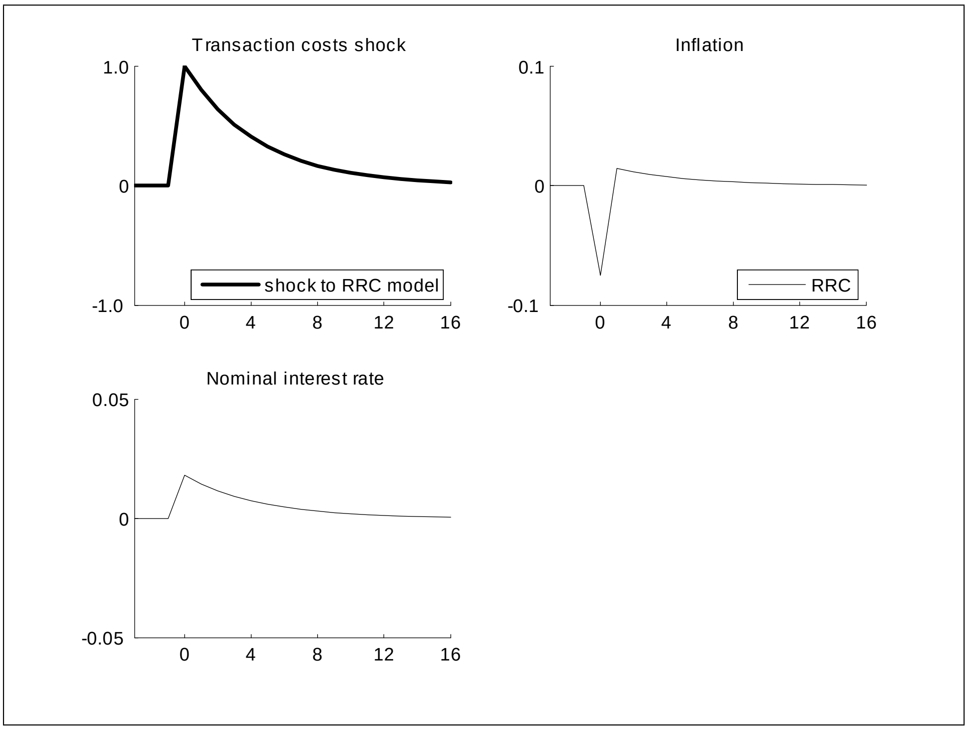

Figure 3.10: Impact of the transaction costs shock on nominal variables (RRC only, with capital)

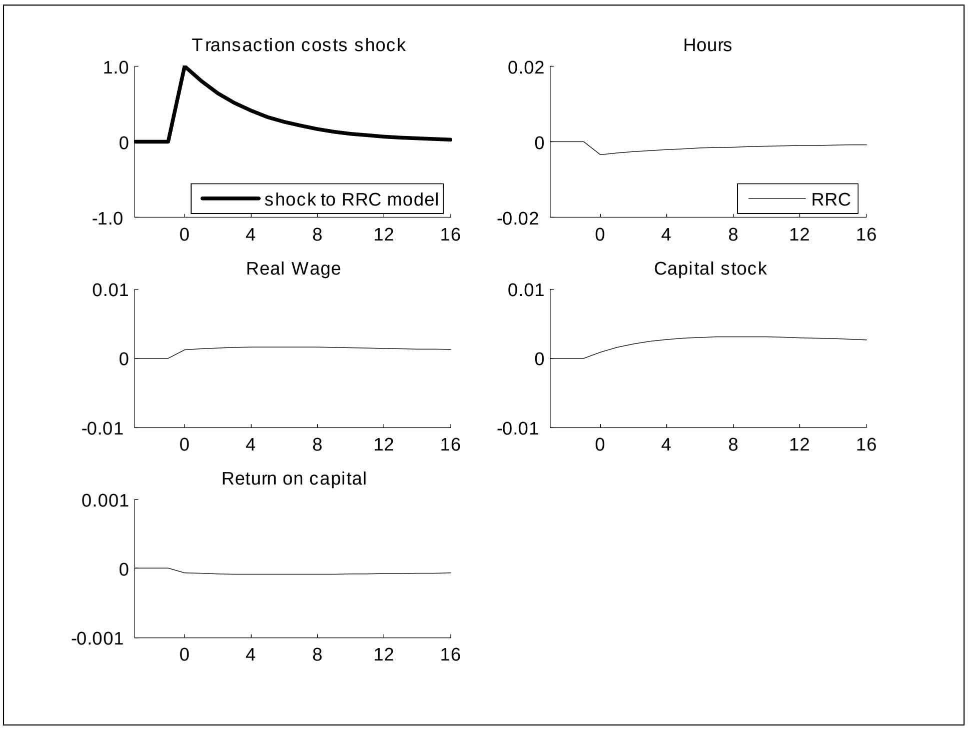

Figure 3.11: Impact of the transaction costs shock on production factors (RRC only, with capital)

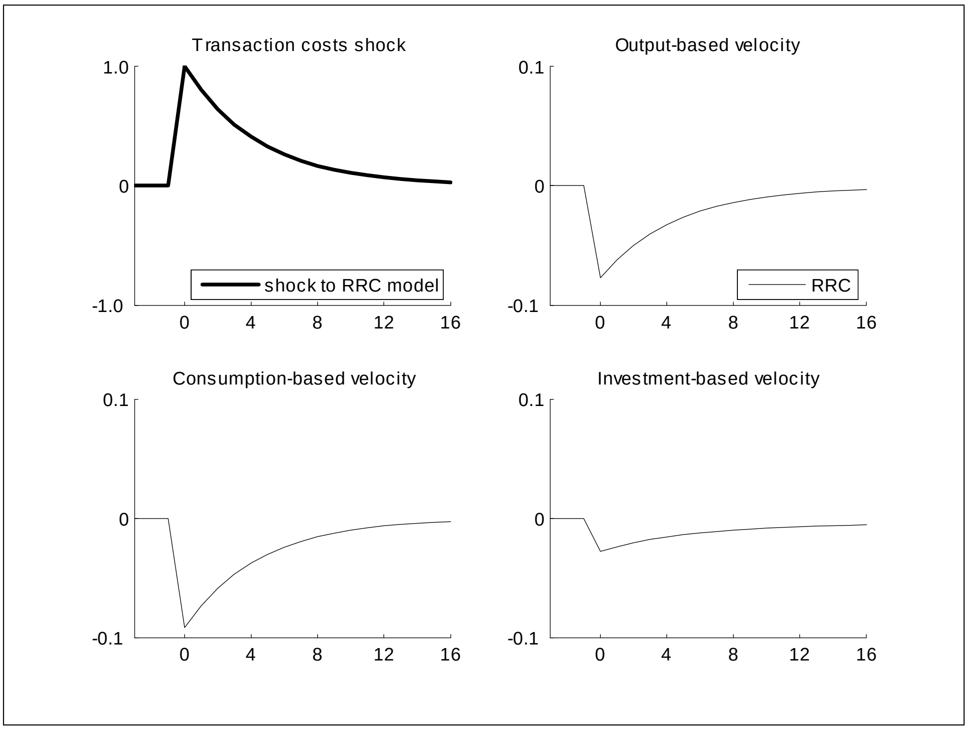

Figure 3.12: Impact of the transaction costs shock on the velocity of money (RRC only, with capital)

3.5.3 Additional simulation results

The impulse-reponse analysis conducted above helped to analyse the effects of the horthogonal shocks on the variables of interest. In this part the quantitative exploration will focus on the comparison of the simulation results with the characteristics of the actual U.S. time series for the variables appearing in the model. Following the spirit of Cooley and Hansen (1995), the performance of the CIA and RRC model will be assessed along three dimensions: the standard deviation of the variables in the simulated models and their correlation with output and money growth. Since both monetary models abstract from many real world features and rigidities, one should not expect a perfect match. In fact the aim of this comparison should be helpful in suggesting whether the models go in the direction the data suggest and, eventually, which of the two models is closer to the empirical evidence. In practice, this corresponds to a quantitative analysis *at the margin*.

Table 3.6 reports the standard deviations of the variables of the U.S. economy (second column), together with the standard deviations of the artificial variables resulting from the simulations of the CIA and the RRC model (third and fifth column, respectively)[31]. The results available from the analysis by Cooley and Hansen (1989) and Cooley and Hansen (1995) are also reported (under CH1989 and CH1995, respectively). Along this dimension, the two models generally exhibit similar results, with some exceptions though. Starting from the similarities, the two models deliver the same numerical results for working hours (0.0127), real wage (0.0302), output (0.0382) and the money growth rate (0.0103). With the exception of the money growth rate, all these are generally three times more volatile than the respective real counterparts. A second group of variables, for which the two models deliver similar results, encompasses consumption, investment, real balances, and *investment-* and *output*-based velocity. For all these last variables the RRC model performs slightly better than the CIA model: in some cases the values are quite close to the data (as in the case of real balances and *output*-based velocity), in some others the match is poorer (as for consumption and investment). Despite the RRC model

[31] For the description of the data reported in the following Tables, refer to Chapter 1, Section 1.5.

allows *consumption*-based velocity to vary, the standard deviation remains 2.5 times lower than the data. When it comes to inflation and nominal interest rates, the two models start to exhibit some differences, with the CIA model performing slightly better for inflation (0.0136 against 0.0044) and decisively better regarding interest rates (0.0051 against 0.0030).

Table 3.7 reports the correlations with output for the variables of the U.S. economy (second column), together with the correlations with output of the artificial variables resulting from the simulations of the CIA and the RRC model (third and fifth column, respectively). The results available from the analysis by Cooley and Hansen (1989) and Cooley and Hansen (1995) are also reported (under CH1989 and CH1995, respectively). In general, also here the two models behave quite similarly, even if the relative performance is decisively mixed. The two models deliver similar numerical results for the variables of the labour market and the goods market, inflation, real balances and velocity. In some of these cases the values are very close to the data for both models (like in the case of working hours, consumption, investment and *investment*-based velocity), while for some others the correlation sign is the opposite of the data (inflation). In the case of nominal interest rates, both models do have the opposite correlation with respect to the data, with the RRC model being the less worse (-0.0007).

Table 3.8 reports the correlations with money growth for the variables of the U.S. economy (second column), together with the correlations with money growth of the artificial variables resulting from the simulations of the CIA and the RRC model (third and fifth column, respectively). The results available from the analysis by Cooley and Hansen (1989) and Cooley and Hansen (1995) are also reported (under CH1989 and CH1995, respectively). Here, with the only exception of inflation, the two models behave quite differently. In general, both models seem to perform quite poorly. In fact, 8 variables out of 12 exhibit a correlation sign which is the opposite with respect to the one suggested by the data. Finally, the relative performance of the two models is decisively mixed.

variables	STD. DEV.				
	data	CIA	CH1989	RRC	CH1995
working hours	0.0043	0.0127	0.0133	0.0127	0.0135
real wage	0.0109 (0.0071)	0.0302	-	0.0302	-
consumption[32]	0.0125	0.0280	0.0062	0.0276	0.0053
nominal interest rate	0.0030	0.0051	-	0.0005	0.0058
inflation	0.0044 (0.0029)	0.0136	-	0.0176	0.0123
real balances	0.0314 (0.0284)	0.0280	-	0.0293	-
output	0.0154	0.0382	0.0173	0.0382	0.0169
money growth	0.0089	0.0103	-	0.0103	0.0087
investment	0.0699	0.0975	0.0569	0.0964	0.0590
consumption-based vel	0.0258	0.0000	-	0.0098	-
investment-based vel	0.0654	0.0861	-	0.0845	-
output-based vel	0.0277	0.0196	-	-	-

Table 3.6: Standard deviations (CIA vs. RRC, with capital).

variables	CORR. with OUTPUT				
	data	CIA	CH1989	RRC	CH1995
working hours	0.7077	0.7299	0.9800	0.7309	0.9870
real wage	0.5307 (0.5830)	0.9578	-	0.9581	-
consumption	0.8632	0.8705	0.7200	0.8820	0.6760
nominal interest rate	0.3522	-0.0054	-	-0.0007	-0.014
inflation	0.3817 (0.1419)	-0.0726	-	-0.0516	-0.1380
real balances	0.3368 (0.3133)	0.8705	-	0.8302	-
output	1.0000	1.0000	1.0000	1.0000	1.0000
money growth	-0.1282	-0.0054	-	-0.0002	-
investment	0.9024	0.8769	0.9700	0.8884	0.9750
consumption-based vel.	0.0713	0.0000	-	-0.0001	-
investment-based vel.	0.8056	0.7101	-	0.7253	-
output-based vel.	0.2362	0.7101	-	0.6485	-

Table 3.7: Correlations with output (CIA vs. RRC, with capital).

variables	CORR with MONEY GROWTH				
	data	CIA	CH1989	RRC	CH1995
working hours	-0.1957	-0.0378	-	-0.0015	-0.0300
real wage	0.2365 (0.1714)	0.0090	-	0.0004	-
consumption	0.0311	-0.1653	-	-0.0068	-0.6000
nominal interest rate	-0.4771	1.0000	-	0.8244	0.7200
inflation	-0.3124 (-0.1940)	0.9284	-	0.8614	0.9200
real balances	0.2264 (0.2021)	-0.1653	-	-0.3351	-
output	-0.1282	-0.0054	-	-0.0002	-0.0100
money growth	1.0000	1.0000	-	1.0000	1.0000
investment	-0.1955	0.1521	-	0.0062	0.1600
consumption-based vel.	-0.2306	0.0000	-	0.9877	-
investment-based vel.	-0.3250	0.2259	-	0.1233	-
output-based vel.	-0.2781	0.2259	-	0.4572	-

Table 3.8: Correlations with money growth (CIA vs. RRC, with capital).

The quantitative exercises by Cooley and Hansen (1989, 1995) deliver a lower variability of consumption and output, generally more in line with the data. This is mainly due to the assumption of indivisible labour, adopted in their work. However, the CIA and RRC models - with divisible labour - presented in this chapter exhibit a correlation of consumption and working hours more in line with the data. The same is true for the inflation tax: in fact the correlation between money growth and consumption in the CIA and the RRC model is still negative, but lower (and therefore 'closer' to the data) than the one obtained by Cooley and Hansen (1995).

3.5.4 CIA vs. RRC: a discussion

As anticipated before, given the relative simplicity of the models at hand, one should not expect the simulated data to match up *perfectly* with the real data. However, some general comments can be added at this stage. The volatility of the variables for which the RRC model and the CIA model display similar results is generally higher than the one shown by the real data: the bigger discrepancies concern inflation (with the RRC model displaying a volatility four times higher than the data), the labour market (where working hours and real wage in the models result three times more volatile), followed by the goods market (with consumption and output twice as volatile). The exceptions are represented by the volatility of real balances and the *output*-based velocity, for which both models match pretty well the real counterparts. The variables for which the two models differ the most from each other are represented by inflation and nominal interest rates. In these cases the results show a better performance CIA model, even if the match is not particularly good. Regarding the correlation with output, where both models deliver an equally satisfactory match with the data in the case of the labour market and the goods market. The reason for such a good match in those fields lies in the characteristics of the technology shock: in fact, the high value of the autoregressive parameter of this shock (0.95) and its immediate effects on output and factors' demand drives these results. On the contrary, with the only exception of the *investment*-based velocity, both models perform very poorly on the monetary side. In particular, nominal interest rate inflation in the models exhibit a negative correlation with output, whereas the data show a positive relationship instead. The limitations of these models become quite evident when the correlation of the variables with money growth in the simulated economies is compared with the real data. In this case all the correlations involving monetary variables exhibit an opposite sign with respect to the data, while the match is quite poor for the variables that display a correlation sign in line with the data. The reasons for such an unsatisfactory result seem to suggest that the improvement should focus on the monetary transmission mechanism (or in the money market in general). On this issue two main implementations can be proposed. On one hand, one could think about modifying the money demand side, extending the transactive role of money to investment goods. On the other hand, the hypothesis of monetary injections implemented through direct lump sum transfers to households do not seem to be a very realistic representation of monetary expansions. The next chapters will try to deal with both issues, extending both monetary models at hand along these directions. For the moment it is usefull to bare in mind the quantitative results derived in this chapter, in order to assess the relative improvement of the extended models developed in the following chapters.

3.6 Conclusion

This chapter presented two monetary models with capital where money is used to facilitate consumption transactions: the cash-in-advance (CIA) model and the real-resource-costs (RRC) model. Both models were analysed and compared within a stochastic dynamic general equilibrium context characterised by perfect competition and flexible prices. In order to inspect the role of different transaction techologies when these economies were hit by the same stochastic shocks, the two monetary models were made comparable adopting the same market timing assumptions (i.e., financial markets opening before the goods market) and an appropriate calibration.

The CIA and RRC model respond in the same way to a technology shock. What is crucial in the behaviour of money demand in both models is its unit elasticity with respect to the level of *desired* consumption.

When the model is hit by an autoregressive monetary shock, instead, the response differs along many dimensions. The main result is that the impact of actual and expected inflation on real variables depends on the specification of transaction technologies, which define the link between money and real expenditures. In particular, the stronger the link (as in the case of the CIA model), the greater the fall in real consumption due to the inflation tax and the stronger the portfolio re-allocation of resources towards real investment. As in Chapter 2 agents in the RRC model can use real resources to conduct transactions, therefore they will try to substitute cash resources (subject to inflation tax) for credit payments. Another important consequence of the transaction technology specification concerns the volatility of the nominal interest rate: the weaker the link between money and consumption (as in the case of the RRC model) the lower the volatility of nominal interest rates.

The quantitative impact of a transaction costs shock in the RRC model with capital on real and nominal variables remains generally weak. This is mainly due to the fact that the calibration of the transaction cost function - from whom the effects of this type of shock critically depend - has been targeted for a comparison with the CIA model (in order to assess the impact of the other type of shocks).

Regarding the correlation with output, where both models deliver an equally satisfactory match with the data in the case of the labour market and the goods market. The reason for such a good match in those fields lies in the characteristics of the technology shock: in fact, the high value of the autoregressive parameter of this shock (0.95) and its immediate effects on output and factors' demand drives these results. On the contrary, with the only exception of the *investment*-based velocity, both models perform very poorly on the monetary side. The limitations of these models become quite evident when the correlation of the variables with money growth in the simulated economies is compared with the real data. In this case all the correlations involving monetary variables exhibit an opposite sign with respect to the data, while the match is quite poor for the variables that display a correlation sign in line with the data.

Here the impulse response analysis permits an answer to the following question: 'How does the presence of capital goods affect the transmission mechanism of shocks'? From this point of view, the results found in Chapter 2 are confirmed and, in some ways, reinforced. In fact, since in Chapter 3 consumption continues to be the only good linked to money, the presence of capital goods introduces a *credit*-good in these economies. This implementation allows the agents in *both* models to allocate their wealth between *cash*- and *credit*-goods, according to the opportunity costs generated by the inflation tax. This is particularly evident in the results characterising the response of the CIA model to the monetary shock: in particular, consumption falls more than in Chapter 2 because total expenditure moves in favour of investment. Ultimately, this reallocation of real resources 'softens' the fall in output with respect to the model without capital by 10 times. The switch from cash- to credit-goods, however, does not happen in the same way for the two models. Again, the key element to read these results is the utility of consumption: the households in the RRC model can continue to enjoy consumption simply switching to an alternative way of payment (costly credit), while the households in the CIA model need to evaluate how much utility to 'give up' shifting to the good that is not subject to the inflation tax. In this context the movements in the different measures of velocity for both models correspond again to a *portfolio reallocation* between cash and credit.

When the simulation results of the models in Chapter 3 are examined against the data, the performance is quite mixed. In general, the technology shock (calibrated in line with the RBC literature) 'dominates' the standard deviations and the correlations of output with the variables in the goods and labour market. In particular, the main findings are that the standard deviation of output, consumption and investment are higher than the data and are not influenced by the different microfoundations of money. This is mainly due to the fact that both models exhibit a unitary elasticity of the demand for real balances with respect to consumption. Therefore, both models respond in the same way to technological shocks. As a consequence, the correlation of the artificial variables with respect to output replicates the charateristic RBC results: technolgical shocks are important and reverse-causation hypothesis for money and output. When it comes to the correlations of variables with money growth both monetary models fail in several dimensions. In fact, in the two-thirds of the cases, the correlation of the artificial variables with respect to the money growth exhibits the 'wrong' sign with respect to the correlations shown by the actual U.S. data. Part of the quantitative failure is certainly due to the fact that these models abstract from important features of the real world: for example, the fact that prices are not fully flexible as assumed in the artificial models. The assumption of full price flexibility certainly explains the absence of a liquidity effect or the strength of the inflation tax. On the other hand, some other results are quite puzzling. In particular, the 'wrong' sign of the correlation of *all* the velocity measures with money growth and of the correlation between the latter and investment raised a fundamental question: 'Do the movements in *velocities* and *investment* depend on other factors than those related to the inflation tax?'.

Finally, the comparison with the results reported in previous work by Cooley and Hansen (1989, 1995) reveals that the assumption of divisible labour adopted in this chapter performs better in terms of correlation of real variables with output and money growth, while the quantitative performance of the indivisible labour assumption (adopted by Cooley and Hansen) is superior in terms of standard deviations of consumption and working hours.

On this issue two main implementations can be proposed. On the one hand, one could think about modifying the money demand side, extending the transactive role of money to investment goods. The fact that the monetary aggregate M1 in the real world includes *savings* it is reasonable to assume that part of this liquidity is destined to cover investment expenditures. On the other hand, the hypothesis of monetary injections implemented through direct lump sum transfers to households does not seem to be a very realistic representation of how monetary expansions take place in reality. These extensions will be integrated with the CIA model and the RRC model in Chapters 4 and 5, respectively.

Chapter 4

Cash in advance and monetary injections: some extensions

This chapter focuses exclusively on the *cash-in-advance* (CIA) model, suggesting two extensions. As emphasised in the previous chapters, there is no unique way in which the link between money and transactions can be specified. On the *money demad* side Chapters 2 and 3 focused on the characteristics of a particular type of cash-in-advance constraint: the one where the *cash good* was represented by consumption, with labour and capital being regarded as *credit goods*. This chapter investigates the consequences of modifying this assumption, allowing investment to be subject to the cash-in-advance constraint, along with consumption. This extension is motivated by noticing that the monetary aggregate M1 used in the empirical analysis of this thesis includes "[...] *funds that are readily accessible for spending [...]: (1) currency outside the U.S. Treasury, Federal Reserve Banks, and the vaults of depository institutions; (2) traveler's checks of nonbank issuers; (3) demand deposits; and (4) other checkable deposits (OCDs), which consist primarily of negotiable order of withdrawal (NOW) accounts at depository institutions and credit union share draft accounts* [...]". Given this 'broad' characterisation of liquidity, it does not appear unrealistic to assume that part of it is used to cover a larger set of goods, than merely consumption[1]. On the *money supply* side, the previ-

[1] Clearly, the inclusion of investment in the set of cash goods does not represent an exhaustive solution regarding the association between monetary aggregates and the related

ous chapters assumed that monetary injections took the form of lump sum transfers: once implemented, these transfers were *instantaneously* available to households. This is clearly unrealistic, especially given the nature of monetary shocks in type of models considered here. For this reason, the injection mechanism is modified in the following way: lump sum transfers take some time to enter households' accounts. Put in terms of Friedman's language: 'helicopter drops' take time to reach the 'ground'. As will be shown below, the degree of 'rigidity' this device imposes on the propagation mechanism of the monetary shock, can be modelled in order to allow a sensitivity analysis check.

Examples of models with a cash-in-advance constraint including investment are due to Stockman (1981) and Abel (1985). The former analyses this implementation in a deterministic setup, while the latter studies its implications for the dynamics in a stochastic setup. The contribution of this chapter to the literature consists in inspecting how modifications to the way money is injected impacts on the dynamics and the quantitative properties of this type of model. The convenience of using the cash-in-advance constraint for modelling money demand (i.e., monetary transactions are idetified in a very precise way) is still valid here: on a certain extent, the clarity of the CIA model in identifying the purchases for which money is needed continues to make the analysis of the propagation mechanism of shocks much 'easier'[2].

As in the previous chapter, the quantitative assessment conducted in the last part extends the work of Cooley and Hansen (1989) by reporting the results for additional endogenous variables (e.g., the nominal interest rate, real balances and different measures of velocity of money), by inspecting the impulse-response functions of their CIA model and by reporting the correlation of endogenous variables with respect to money growth. Moreover, the set of simulation results for both models is richer than the one reported in Cooley and Hansen (1989, 1995).

This chapter is structured as follows: firstly, the assumptions implied by

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expenditure categories. However, this extension can be considered as a first, approximative attempt of bringing the structure of the theoretical model in line with the definition of money used in the quantitative assessment.

[2] At least when compared with the (extended) real-resource-cost model, developed in Chapter 5.

each approach are stated, then the resulting optimality conditions are derived. Finally, the models are calibrated (on quarterly basis) and outcomes from simulations are compared. The model performance is assessed analysing the effects of stochastic shocks affecting production (i.e., technology shocks) and the money supply process (i.e., monetary policy innovations). The problem of the representative firm and the monetary policy rule are the same as those that characterised the CIA model in the previous chapters. At the same time, dating conventions and market timing assumptions are unaffected by the extensions considered in this chapter. Given the similarities some details and explanations will not be repeated here. In any case, the reader is invited to refer back to the main features of the baseline CIA model described in Chapter 2.

4.1 Households

The economy is populated by a large number of identical and infinitely-lived households. At time $t = 0$ the representative household seeks to maximize the following expected value of a discounted stream of period utility:

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u \left[c_t, l_t \right] \right\} \tag{4.1}$$

The expectational operator E_0 indicates that the expectations at time $t = 0$ about present and future streams of utility are formed conditionally to the information available to the agent. The objective function (4.1) assumes that utility at time t depends on real consumption c_t and leisure time l_t. Future utility is discounted by a (constant) discount factor β (with $0 < \beta < 1$).

The utility function u is strictly concave and twice continuously differentiable. It is increasing in its arguments and decreasing in their marginal utility. Using u_x to denote the partial derivative of the function u with respect to its generic argument x, one can write: $u_c > 0$, $u_l > 0$, $u_{cc} < 0$, $u_{ll} < 0$. In addition to that, also the Inada (1963) conditions are assumed to be holding: $\lim_{c \to \infty} u_c = \infty$, $\lim_{l \to 0} u_l = \infty$, $\lim_{c \to \infty} u_c = 0$, $\lim_{l \to \infty} u_l = 0$.

Total time endowment is normalized to 1, so that the following constraint applies to every period:

$$1 = l_t + h_t^s \tag{4.2}$$

This means that at time t the agents will choose to split total time between leisure time l_t and (supplied) working hours h_t^s.

Using (4.2) one can reformulate (4.1) in terms of consumption and working hours:

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u \left[c_t, 1 - h_t^s \right] \right\} \tag{4.3}$$

As in the previous chapters, u is decreasing in working time ($u_h < 0$) and increasing in the marginal *dis*utility of work ($u_{hh} > 0$). In deference to the real business cycle tradition, this last formulation will be mantained throughout the analysis.

The explicit functional form chosen for period utility takes the form of a constant relative risk aversion (CRRA) utility function:

$$u \left[c_t, 1 - h_t^s \right] \equiv \frac{c_t^{1-\Phi}}{1-\Phi} + \Psi \frac{\left(1 - h_t^s\right)^{1-\eta}}{1-\eta} \tag{4.4}$$

, where: $\Phi > 0$ is the coefficient of relative risk aversion (with $1/\Phi$ being the elasticity of intertemporal substitution); given $\eta > 0$, expression $1/\eta$ denotes the elasticity of intertemporal substitution for labour; $\Psi > 0$ represents a preference parameter over leisure.

The representative household is subject to a budget constraint. An important modification to the CIA model presented in Chapters 2 and 3 consists in the origin of the monetary lump sum transfers appearing on the left hand side. To emphasise this difference, the monetary injection will be denoted by T_t^H, where the superscipt 'H' is meant to denote 'households', to characterise any monetary sum entering the households budget constraint. The rest of the budget constraint is identical to the one appearing in Chapter 3 and corresponds to (in nominal terms):

$$M_{t-1}^{d} + (1 + i_{t-1}) B_{t-1}^{d} + W_t h_t^s + T_t^H + P_t r_t^k k_{t-1}^s \geq P_t c_t + M_t^d + B_t^d + P_t x_t \quad (4.5)$$

$\forall t \geq 0$; where: B_t^d denotes the nominal value of (risk free) bonds holdings, which pay a one-period nominal (net) interest rate i_t; T_t^H represents nominal lump sum transfers and M_t^d is individual money demand. The capital stock (k_{t-1}^s) is owned by households, who rent it to firms at the beginning of the period for production purposes. Therefore, the rental payment ($P_t r_t^k k_{t-1}^s$) enters as an additional source of wealth on the left hand side of the budget constraint. Household's expenditures in the goods market includes consumption ($P_t c_t$) and investment goods ($P_t x_t$), which appear on the right hand side. W_t is hourly nominal wage and P_t represents the price of the homogeneous good produced in the economy.

Real investment is denoted by x_t and is defined as a change in the capital stock (net of capital depreciation):

$$x_t = k_t^s - (1 - \delta) k_{t-1}^s \quad (4.6)$$

$\forall t \geq 0$, where $0 < \delta < 1$ represents the (constant) real depreciation of capital and the superscript 's' here is meant to indicate that capital is made available (i.e., supplied) by households to firms.

In order to emphasise the fact that in each period households will be choosing the optimal level of capital stock to rent (k_t^s), one can use (4.6) into (4.5) to obtain:

$$M_{t-1}^{d} + (1 + i_{t-1}) B_{t-1}^{d} + W_t h_t^s + T_t^H + P_t r_t^k k_{t-1}^s + P_t (1 - \delta) k_{t-1}^s \geq P_t c_t + M_t^d + B_t^d + P_t k_t^s \quad (4.7)$$

The right hand side of (4.7) represents the individual's total nominal wealth within the period t. This encompasses financial wealth accumulated in the previous period, the value of the capital stock net of real depreciation $(P_t (1-\delta) k^s_{t-1})$, labour income $(W_t h^s_t)$, capital rental payments $(P_t r^k_t k^s_{t-1})$ and the exogenous lump sum transfers. Financial wealth is given by the nominal value of a portfolio of financial assets, namely bonds and cash balances from period $t-1$, inclusive of interest earnings (i_{t-1}) from bonds holdings.

Total wealth available in period t is allocated to the goods market (buying consumption goods and investment goods at the prevailing price P_t) and to the financial markets, adjusting the portfolio of assets (given the prevailing interest rate, i_t). As in the previous chapters, expression (4.7) implies that financial markets are complete.

Dividing both sides of (4.7) by the price level (P_t), the household budget constraint can be re-written in real terms as

$$\frac{m^d_{t-1}}{\Pi_t} + \frac{I_{t-1} b^d_{t-1}}{\Pi_t} + w_t h^s_t + \tau^H_t + r^k_t k^s_{t-1} + (1-\delta) k^s_{t-1} \geq c_t + m^d_t + b^d_t + k^s_t \quad (4.8)$$

$\forall t \geq 0$; where: $b^d_t \equiv B^d_t / P_t$ denotes the real value of riskless bonds; $I_t \equiv (1+i_t)$ is the one-period nominal (gross) interest rate; $\tau^H_t \equiv T^H_t / P_t$ represents real lump sum transfers; $m^d_t \equiv M^d_t / P_t$ is individual demand for real balances; $w_t \equiv W_t / P_t$ indicates real wage and $\Pi_t \equiv P_t / P_{t-1}$ represents the (gross) inflation rate. The capital stock is represented by k^s_t, and its (real) return by r^k_t.

Resources not used in period t are saved in the form of bonds, cash balances and/or capital stock, whose command over goods will become effective only in the following period. Since this is true for every period (4.8) shows that the portfolio allocation decisions taken at time $t-1$ do in fact expose the real value of financial savings to changes in the price level from $t-1$ to t. Note that the same is not true for the real value of capital stock: in fact the capital stock made available to firms *at the end* of previous period $((1-\delta) k^s_{t-1})$, and the proportional real return from the *current* period $(r^k_t k^s_{t-1})$ are evaluated at the current price level (P_t).

The representative household is subject to a cash-in-advance constraint. As anticipated before, on the one hand the constraint appearing in Chapters

2 and 3 is modified, to include the purchase of investment goods, in the spirit of Stockman (1981). On the other hand, the lump sum transfer appearing in this constraint is denoted by T_t^H, coherently with the notation adopted in the budget constraint (see (4.5)). The market timing assumption (according to which households are allowed to visit the financial markets before the goods markets) continues to hold, together with the fact that households receive the lump sum monetary injection *via* financial markets, at the beginning of each period.

Considered all together, these assumptions correspond to a cash-in-advance constraint of the form:

$$M_{t-t}^d + (1 + i_{t-1}) B_{t-1}^d - B_t^d + T_t^H \geq P_t c_t + P_t x_t \tag{4.9}$$

$\forall t \geq 0$. The purchasing power of cash balances is obtained by dividing both sides of (4.9) by P_t. The result is a cash-in-advance constraint expressed in real terms:

$$\frac{m_{t-t}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t^H \geq c_t + x_t \tag{4.10}$$

$\forall t \geq 0$.

At time t the problem of the household is inherently dynamic: to choose state-contingent claims for consumption (c_t), labour supply (h_t^s), bonds holdings (b_t^d), money stock (m_t^d) and capital stock (k_t), which do maximize the expected utility (4.3), subject to the budget constraint (4.8) and to the finance constraint (4.10)[3].

4.2 Firms

The economy is populated by a large number of identical firms. As in Chapter 3, firms produce an homogeneous good buying labour (working hours) and renting capital from households. The real output produced in period t can be expressed by the following production function:

[3]Moreover, *no-Ponzi game* conditions must hold to guarantee optimality.

$$y_t = f\left[z_t, h_t^d, k_{t-1}^d\right] \tag{4.11}$$

$\forall t \geq 0$, where: y_t denotes *real* output; h_t^d are working hours and k_{t-1}^d is the capital stock (where the superscript d is meant to indicate 'demand'); z_t represents the 'level' of technology.

In order to obtain a direct correspondance between the behaviour of individual firms and their aggregate counterpart, the production function is represented by a constant returns to scale technology. To satisfy this condition, the production technology is assumed to be a *Cobb-Douglas* type:

$$y_t = z_t \left(k_{t-1}^d\right)^\alpha \left(h_t^d\right)^{1-\alpha} \tag{4.12}$$

$\forall t \geq 0$, where $0 < a < 1$ represents the capital share. In deference to the RBC literature, the variable z_t represents *total* factor productivity. This variable evolves exogenously according to the law of motion

$$\log z_t = (1 - \rho_z) \log z + \rho_z \log z_{t-1} + \epsilon_{z_t} \tag{4.13}$$

$\forall t \geq 0$; where: ρ_z is the autoregressive coefficient (with $0 \leq \rho_z \leq 1$), and ϵ_{z_t} is a random variable serially uncorrelated and normally distributed, with zero mean and constant variance $(\sigma_{\epsilon_z}^2)$.

In period t firms sell their product in a perfectly competitive goods market, taking the price P_t of the homogenous good as given. Analogously, given the nominal wage W_t, they buy labour services from households in a perfectly competitive labour market. Firms rent capital (k_{t-1}) from households, at the cost of a proportional rental rate $(P_t r_t^k k_{t-1})$. In order to decide how much to produce - and, consequently, how much labour to hire and capital to rent - firms maximise the following profit function:

$$\Gamma_t = P_t y_t - W_t h_t^d - P_t r_t^k k_{t-1}^d \tag{4.14}$$

$\forall t \geq 0$; where nominal profits (Γ_t) are defined as the difference between nominal revenues $(P_t y_t)$ and nominal costs $(W_t h_t^d + P_t r_t^k k_{t-1}^d)$. Note that the assumptions of perfect competition and constant returns to scale do imply that the representative firm makes zero profits in equilibrium.

In every period t each firm solves a static problem: that one of choosing working hours (h_t^d) and capital (k_{t-1}^d) which maximize profits (Γ_t) subject to the technology constraint (4.12).

4.3 Government

As in the previous chapters , the government operates as monetary and fiscal authority and its revenues and outlays in period t are combined in the following flow budget constraint (expressed in nominal terms):

$$M_t^s - M_{t-1}^s + B_t^g - (1 + i_{t-1}^g)B_{t-1}^g = P_t g_t + T_t \tag{4.15}$$

$\forall t \geq 0$, where: B_t^g denotes the face value of government debt outstanding, which pays a one-period nominal (net) interest rate i_t^g; T_t indicates governmental nominal lump sum transfers; M_t^s represents aggregate money supply; and g_t denotes real government consumption.

Since the focus here is on studying the impact of monetary shocks and not the impact of changes in government spending, g_t is set to zero (for all t periods). Moreover, Ricardian equivalence holds in this model. Therefore one can assume, with no loss of generality, that $B_0^g = 0$[4]. All together these assumptions imply that no government bonds are held in this economy and the government budget constraint then reduces to

$$M_t^s - M_{t-1}^s = T_t \tag{4.16}$$

in each period t.

$\forall t \geq 0$. Dividing both sides of (4.16) by the price level P_t, one obtains the equivalent expression in real terms :

$$m_t^s - \frac{m_{t-1}^s}{\Pi_t} = \tau_t \tag{4.17}$$

$\forall t \geq 0$; where: $\tau_t \equiv T_t/P_t$ represents real lump sum transfers; $m_t^s \equiv M_t^s/P_t$ is real money supply; and $\Pi_t \equiv P_t/P_{t-1}$ is the (gross) inflation rate.

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[4] See Chapter 2 for a more detailed explanation.

The monetary authority is assumed to follow a constant money growth rule, according to which per capita nominal money supply is assumed to grow at the net rate θ_t in each period. This implies:

$$M_t^s = M_{t-1}^s + \theta_t M_{t-1}^s \tag{4.18}$$

$\forall t \geq 0$. The money supply rule is implemented through monetary injections that take the form of lump sum transfers according to:

$$T_t = \theta_t M_{t-1}^s \tag{4.19}$$

or, in real terms,

$$\tau_t = \frac{\theta_t m_{t-1}^s}{\Pi_t} \tag{4.20}$$

$\forall t \geq 0$.

To study the effects of a monetary surprise, the variable θ_t is assumed to evolve according to the law of motion

$$\log \theta_t = (1 - \rho_\theta) \log \theta + \rho_\theta \log \theta_{t-1} + \epsilon_{\theta_t} \tag{4.21}$$

$\forall t \geq 0$; where: ρ_θ is the autoregressive coefficient (with $0 \leq \rho_\theta \leq 1$), and ϵ_{θ_t} is a random variable serially uncorrelated and normally distributed, with zero mean and constant variance ($\sigma_{\epsilon_\theta}^2$). With this specification, the average (net) growth rate of money supply chosen by the monetary authority is equal to θ.

4.4 Monetary injections: the 'pipeline' model

An important difference with the previous chapters consists in modifying the way in which monetary impulses are propagated to the private economy. There are certainly many 'realistic' ways of modelling the *trasmission mechanism* of monetary policy. However, given that one of the main purposes of this work is to investigate this mechanism keeping the representative agent model as simple as possible, the modification to the monetary channel represents a small (and, at the same time, meaningful) step.

As the reader might have noticed, in this chapter nominal monetary injections (T_t) by the government do not correspond to the monetary transfers received by households (denoted by T_t^H). The justification for this 'separation' is that in the real world monetary expansions implemented by central banks do not coincide *instantaneously* with accredited lump sum transfers in households' accounts. In fact, as anticipated in the introduction, this process actually might take some time. In order to model this feature, the simplest assumption one can think of is in terms of *lags* in the process of monetary injections. In the context of the present model, this can be achieved by introducing the following (in nominal terms):

$$N_{t-1} + T_t = T_t^H + N_t \tag{4.22}$$

$\forall t \geq 0$. Expression (4.22) can be considered as a prototypical transmission mechanism, whose essence derives from the working principle of a 'pipeline'. At the beginning of period t the 'pipeline' contains a given nominal amount (N_{t-1}) of money in circulation, augmented by the new money injected by the monetary authority (T_t). Total money accumulated on the left hand side of (4.22) is *partly* passed on to households' in the form of a monetary transfer (T_t^H), while the remaining part (N_t) is stored for next period.

For (4.22) to play any meaningful role in the transmission mechanism, additional assumptions about outflows T_t^H from the 'pipeline' is required. First of all, one must require that $T_t^H > 0$. This restriction has two important implications. On the one hand, it guarantees in every period a positive monetary injection in the representative household's account; on the other hand, it excludes the possibility that money accumulates *indefinitely* within the 'pipeline'[5].

The second assumption is relative to the *timing*. According to (4.22) inflows and outflows occur at the same time. In theory, this timing assumption does *not* exclude the possibility $T_t = T_t^H$, which actually represents another 'disruptive' condition for the transmission mechanism at hand. In fact, the eventuality of inflows and outflows being equal would correspond to the case

[5] This last result is obtained by excluding the case $T_t^H = 0$. In fact, with a monetary authority pursuing a constant money growth rule, positive injections T_t in every period would lead to an infinite accumulation.

of 'no money in the pipeline'[6]. To ensure a positive amount of money in the 'pipeline', the model assumes that the new cash entering the representative household's budget constraint is a function of the money stored in the 'pipeline' at the end of the previous period. therefore, households lump sum transfers can be expressed (in nominal terms) by the following:

$$T_t^H = \xi N_{t-1} \tag{4.23}$$

$\forall t \geq 0$, where the parameter $0 < \xi \leq 1$ represents the *share* of stored 'pipeline' money (N_{t-1}) passed on to households'. Note that when $\xi = 1$ *all* the stored 'pipeline' money during the previous period is passed on to consumers[7].

Dividing both sides of (4.22) by the price level, one obtains the equivalent expression in real terms

$$\frac{n_{t-1}}{\Pi_t} + \tau_t - \tau_t^H = n_t \tag{4.24}$$

$\forall t \geq 0$, where: n_{t-1} and n_t indicate the real value of 'pipeline' money at the beginning and at the end of period, respectively, with Π_t being the (gross) inflation rate; while τ_t and τ_t^H represent real *inflows* and *outflows*, respectively. Analogously, one can obtain (4.23) in real terms:

$$\tau_t^H = \xi \frac{n_{t-1}}{\Pi_t} \tag{4.25}$$

$\forall t \geq 0$.

4.5 The equilibrium

This section derives the equilibrium conditions which characterise the *extended* CIA model. The household's and firm's optimal choices will be derived, while

[6] In practice this would bring the analysis back to the case of government monetary injections entering households budget constraint straight away, which characterised Chapters 2 and 3.

[7] According to (4.22) this in turn implies $T_t = N_t$, i.e. *stored* 'pipeline' money in period t is equal to the monetary injections by the government occurring in the same period.

the monetary policy rule, the 'pipeline' model and the necessary market clearing conditions for the general equilibrium will close the models. The optimisation problems will be stated in terms of the Lagrangian method and then solved for the first order conditions.

4.5.1 Households

To state households' problem in terms of the Lagrangian, it is useful to recall that the representative household in the extended CIA model seeks to maximise the utility stream

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\Phi}}{1-\Phi} + \Psi \frac{(1-h_t^s)^{1-\eta}}{1-\eta} \right) \tag{4.26}$$

, subject to the budget constraint

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t^H + r_t^k k_{t-1}^s + (1-\delta) k_{t-1}^s = c_t + m_t^d + b_t^d + k_t^s \tag{4.27}$$

$\forall t \geq 0$, and the (extended) cash-in-advance constraint

$$\frac{m_{t-t}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t^H = c_t + k_t^s - (1-\delta) k_{t-1}^s \tag{4.28}$$

$\forall t \geq 0$[8].

Stating the problem in terms of the Lagrangian, the households choose c_t, h_t^s, b_t^d, m_t^d and k_t^s in order to maximise

[8] See footnote (17) in Chapter 2 for a discussion about replacing inequalities with equalities in the constraints.

$$\mathcal{L}_t = E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \left(\frac{c_t^{1-\Phi}}{1-\Phi} + \Psi \frac{(1-h_t^s)^{1-\eta}}{1-\eta} \right)$$
$$+ \lambda_t \Bigg[\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t^H + r_t^k k_{t-1}^s + (1-\delta) k_{t-1}^s$$
$$- c_t - m_t^d - b_t^d + k_t^s \Bigg]$$
$$+ \mu_t \left[\frac{m_{t-t}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t^H - c_t - k_t^s + (1-\delta) k_{t-1}^s \right] \Bigg\}$$

,where λ_t and μ_t are the Lagrangian multipliers associated with the budget constraint and the cash-in-advance constraint, respectively.

The maximisation of the Lagrangian with respect to the choice variables (after substituting for the Lagrangian multipliers) delivers the following optimality conditions:

$$\frac{\Psi (1-h_t^s)^{-\eta}}{c_t^{-\Phi}} = \frac{w_t}{I_t} \tag{4.29}$$

$$1 = \beta E_t \left\{ \left(\frac{c_{t+1}}{c_t} \right)^{-\Phi} \frac{I_t}{\Pi_{t+1}} \right\} \tag{4.30}$$

$$\mathbf{1} = \boldsymbol{\beta} \mathbf{E}_t \left\{ \left(\frac{c_{t+1}}{c_t} \right)^{-\Phi} \frac{I_t}{I_{t+1}} \left[R_{t+1}^k + (I_{t+1} - 1)(1-\delta) \right] \right\} \tag{4.31}$$

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t^H + r_t^k k_{t-1}^s + (1-\delta) k_{t-1}^s$$
$$= c_t + m_t^d + b_t^d + k_t^s \tag{4.32}$$

$$\frac{m_{t-t}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t^H = c_t + k_t^s - (1-\delta) k_{t-1}^s \tag{4.33}$$

, where $R_t^k \equiv r_t^k + 1 - \delta$ represents the gross return on capital (net of depreciation)[9].

Expression (4.29) represents the *intra*-temporal condition, which relates the marginal rate of substitution between leisure and consumption (on the left hand side), to the ratio of the respective marginal costs (on the right hand side). Note that, because of the opportunity cost of saving, the gross nominal interest rate (I_t) acts like a "tax" on consumption, affecting in turn the labour supply choice (h_t^s) via the utility function. Expression (4.30) refers to the *inter*-temporal condition, which governs the degree of consumption smoothing through time, taking into account the *real* opportunity cost of saving ($I_t/E_t\Pi_{t+1}$). Equation (4.31) relates the arbitrage condition between capital goods and bonds to the stochastic discount factor. When investment is subject to the cash-in-advance constraint, its arbitrage conditions with respect to bonds become similar to consumption: the nominal interest rate becomes an opportunity cost, therefore the condition that links the rental rate of capital to bonds returns must reflect this condition. Finally, equations (3.36) and (3.37) are the constraints, obtained by derivation with respect to the Lagrangian multipliers. Note that the latter differs from the analogous expressions in Chapters 2 and 3, given that now investment goods are subject to the cash-in-advance constraint.

4.5.2 Firms

In each period, the representative firm chooses the amount of working hours (h_t^d) and rent capital stock (k_{t-1}^d) that maximise the profit function

$$\Gamma_t = P_t e^{z_t} h_t^d - W_t h_t^d - P_t r_t^k k_{t-1}^d \tag{4.34}$$

or, in real terms,

$$\gamma_t = e^{z_t} h_t^d - w_t h_t^d - r_t^k k_{t-1}^d \tag{4.35}$$

$\forall t \geq 0$. Given the production technology

[9] Moreover, *transversality conditions* must hold to guarantee optimality.

$$y_t = z_t \left(k_{t-1}^d\right)^\alpha \left(h_t^d\right)^{1-\alpha} \tag{4.36}$$

and considering a generic time period t, one obtains the following first order conditions:

$$\frac{\partial \gamma_t}{\partial h_t^d} = e^{z_t} \left(k_{t-1}^d\right)^\alpha (1-\alpha) \left(h_t^d\right)^{-\alpha} - w_t = 0 \tag{4.37}$$

$$\frac{\partial \gamma_t}{\partial k_{t-1}^d} = e^{z_t} \alpha \left(k_{t-1}^d\right)^{\alpha-1} \left(h_t^d\right)^{1-\alpha} - r_t^k = 0 \tag{4.38}$$

for labour and capital, respectively.

Making use of the production function (4.36) and the definition of the gross return of capital $R_t^k \equiv r_t^k + 1 - \delta$, conditions (4.37) and (4.38) can be re-written as

$$(1-\alpha) \frac{y_t}{h_t^d} = w_t \tag{4.39}$$

$$\alpha \frac{y_t}{k_{t-1}} + 1 - \delta = R_t^k \tag{4.40}$$

As in the previous chapters, the optimality condition (4.39) implies that firms demand working hours up to the point where the marginal product of labour equals its marginal cost (i.e., the real wage w_t); while expression (4.40) implies that they rent capital up to the point where its marginal product equals the marginal cost (represented by the net rental rate $r_t^k = R_t^k - (1-\delta)$).

4.5.3 Monetary policy and monetary injections

In period t real cash balances evolve to satisfy the government's budget constraint

$$\frac{m_{t-1}^s}{\Pi_t} + \tau_t = m_t^s$$

$\forall t \geq 0$. Given the money growth rate (θ_t) chosen by the monetary authority for period t, recall the characterization of the real lumpsum transfers as

$$\tau_t = \frac{\theta_t m^s_{t-1}}{\Pi_t}$$

$\forall t \geq 0$.

As described in the previous section, the monetary policy 'channel' is modified in order to incroporate a transmission mechanism. This is done through the 'pipeline' device (in real terms):

$$\frac{n_{t-1}}{\Pi_t} + \tau_t - \tau^H_t = n_t$$

$\forall t \geq 0$. Real outflows (τ^H_t) are represented by the following expression

$$\tau^H_t = \xi \frac{n_{t-1}}{\Pi_t}$$

$\forall t \geq 0$, where $0 < \xi \leq 1$ represents a calibrated 'pipeline valve'.

4.5.4 Market clearing conditions

For a general equilibrium characterisation of the model, the necessary market clearing conditions are required. In this model there are five markets: the goods market, the labour market, the money market, the bonds market and the capital market.

The capital market clears according to

$$k^d_{t-1} = k^s_{t-1} \tag{4.41}$$

$\forall t \geq 0$[10].

The labour market clearing condition equates labour demand and labour supply for every period, according to

$$h^d_t = h^s_t \tag{4.42}$$

$\forall t \geq 0$.

[10] The time subscript $t-1$ refers to the first order condition of the firm, with respect to cpaital goods. See also Uhlig (1999).

In every period *total* money supply (M_t^s) has to be equal to *total* money demand. But total money demand now corresponds to the cash held by households (M_t^d) *plus* the liquidity stored in the 'pipeline' (N_t). Therefore, the resulting market clearing condition will differ with respect to Chapters 2 and 3. In real terms, the equilibrium in the money market is achieved (in real terms) when

$$m_t^d + n_t = m_t^s \tag{4.43}$$

$\forall t \geq 0$.

Since the bonds in this model are private bonds 'issued' by households, the assumption that all the individuals are alike implies that no bonds are actually exchanged in equilibrium. As a consequence, there will be no bonds outstanding (i.e., a zero net supply for this type of financial assets). Thus, the bonds market clearing condition corresponds to:

$$b_t^d = b_t^s = 0 \tag{4.44}$$

$\forall t \geq 0$. Finally, the market clearing condition for the goods market requires aggregate supply and aggregate demand of goods to be equal in every period. Namely:

$$y_t = c_t + x_t \tag{4.45}$$

$\forall t \geq 0$, where $x_t = k_t^s - (1 - \delta)\, k_{t-1}^s$ represents net investment in period t.

4.6 The dynamics

In order to explore and compare the dynamic performance of the extended CIA model, subject to the random shocks described above, one needs to transform the non-linear system of equations characterising the general equilibrium into a linear system. This is done by taking a log-linear approximation around the deterministic steady state, applying the methodology described in Uhlig (1999). For each model, this section will take the following steps: firstly, presenting the equilibrium as obtained in the previous section; secondly, illustrating some steady state relationships; and finally by deriving the log-linear model.

The set of optimality conditions for households and firms, together with the specification of monetary policy and the necessary market clearing conditions characterise the dynamic general equilibrium model as a system of non-linear equations:

$$\frac{\Psi\left(1-h_t^s\right)^{-\eta}}{c_t^{-\Phi}}=\frac{w_t}{I_t} \tag{4.46}$$

$$1=\beta E_t\left\{\left(\frac{c_{t+1}}{c_t}\right)^{-\Phi}\frac{I_t}{\Pi_{t+1}}\right\} \tag{4.47}$$

$$1=\beta E_t\left\{\left(\frac{c_{t+1}}{c_t}\right)^{-\Phi}\frac{I_t}{I_{t+1}}\left[R_{t+1}^k+\left(I_{t+1}-1\right)\left(1-\delta\right)\right]\right\} \tag{4.48}$$

$$\frac{m_{t-1}^d}{\Pi_t}+\tau_t^H=c_t+x_t \tag{4.49}$$

$$\frac{y_t}{h_t^d}=w_t \tag{4.50}$$

$$\alpha\frac{y_t}{k_{t-1}^d}+1-\delta=R_t^k \tag{4.51}$$

$$y_t=e^{z_t}\left(k_{t-1}^d\right)^{\alpha}\left(h_t^d\right)^{1-\alpha} \tag{4.52}$$

$$\frac{m_{t-1}^s}{\Pi_t}+\tau=m_t^s \tag{4.53}$$

$$\tau_t\equiv\theta_t\frac{m_{t-1}^s}{\Pi_t} \tag{4.54}$$

$$\frac{n_{t-1}}{\Pi_t}+\tau_t-\tau_t^H=n_t \tag{4.55}$$

$$\tau_t^H=\xi\frac{n_{t-1}}{\Pi_t} \tag{4.56}$$

$$k_{t-1}^d=k_{t-1}^s \tag{4.57}$$

$$h_t^d = h_t^s \tag{4.58}$$

$$m_t^d + n_t = m_t^s \tag{4.59}$$

$$y_t = c_t + x_t \tag{4.60}$$

$$x_t = k_t^s - (1-\delta)\, k_{t-1}^s \tag{4.61}$$

$$\ln z_t = (1-\rho_z) \ln z + \rho_z \ln z_{t-1} + \epsilon_{z_t} \tag{4.62}$$

$$\ln \theta_t = (1-\rho_\theta) \ln \theta + \rho_\theta \ln \theta_{t-1} + \epsilon_{\theta_t} \tag{4.63}$$

$\forall t \geq 0$.

4.6.1 Money demand and velocity of money

At time t there are *two* sources of money demand in this model: the demand for cash by households (m_t^d) and the quantity of money demanded (i.e., stored) by the 'pipeline' (n_t). Given that the latter is represented by a rigid transmission mechanism, the attention of this section will be focused mainly on the former.

After all markets have cleared, the application of Walras' law implies that the evolution of real balances in the hands of the households follows

$$\frac{m_{t-1}^d}{\Pi_t} + \tau_t^H = m_t^d \tag{4.64}$$

$\forall t \geq 0$. Combining (4.64) with (4.49) one obtains the following expression for households' demand for real balances:

$$m_t^d = c_t + x_t \tag{4.65}$$

$\forall t \geq 0$. This money demand function differs from the one appearing in the CIA models of Chapters 2 and 3, because of the appearance of investment on its right hand side.

A direct implication of (4.59) is that money demanded by households does not coincide with total money supply. This suggests that none of the different measures of velocity, derived by linking households expenditure to total money in circulation (m_t^s)[11], will ever be equal to unity in the extended CIA model considered in this chapter[12]. The *consumption-*, *investment-* and *output*-based velocities are defined as

$$VEL\left(c_t\right) \equiv \frac{c_t}{m_t^s} \tag{4.66}$$

$$VEL\left(x_t\right) \equiv \frac{x_t}{m_t^s} \tag{4.67}$$

$$VEL\left(y\right)_t \equiv \frac{y_t}{m_t^s} \tag{4.68}$$

$\forall t \geq 0$, respectively.

4.6.2 Steady state

Before turning to the log-linear system, it is useful to have a look at some long-run relationships implied by the model. When all the variables have reached their deterministic steady state, time subscripts can be 'removed' from the non-linear equations characterising the equilibrium. In this way it is possible to inspect how monetary factors impact on the fundamental structure of the economy.

In steady state, expressions (4.53) and (4.54) can be used to obtain

$$\Pi = \Theta \tag{4.69}$$

or, equivalently

[11] For a discussion about the use of total money supply (m_t^s) in the definitions of velocity see footnote (27) in Chapter 2.

[12] Note that: given the extended cash-in-advance constraint used in this chapter, the absence of the pipeline' model for the transmission of monetary shocks would imply an *output-based* velocity of money constantly equal to unity.

$$\pi = \theta \tag{4.70}$$

The result indicates that the steady state inflation rate is determined by the money growth rate choosen by the monetary authority. As for the previous chapters, a direct implication of this is that the (steady state) real quantity of money (m) is constant - i.e., *neutrality of money* holds in steady state. Given the household's subjective discount rate (β), the *inter*temporal condition (4.47) can be used to determine the long-run nominal interest rate:

$$I = \frac{\Theta}{\beta} \tag{4.71}$$

As in Chapters 2 and 3 the consumption-labour choice in (4.46) depends on the level of nominal interest rate. Using (4.46), (4.71) and the production function, one obtains:

$$\frac{h^{\Phi}}{(1-h)^{\eta}} = \frac{(1-\alpha)}{\Psi}\frac{\beta}{\Theta}\left(\frac{y}{k}\right)^{\frac{\Phi-\alpha}{1-\alpha}}\left(\frac{c}{k}\right)^{-\Phi} \tag{4.72}$$

Making use of (4.51) and the goods market clearing condition (4.60) one can re-write (4.72) as follows:

$$\frac{h^{\Phi}}{(1-h)^{\eta}} = \frac{(1-\alpha)}{\Psi}\frac{\beta}{\Theta}\left(\frac{R^k-1+\delta}{\alpha}\right)^{\frac{\Phi-\alpha}{1-\alpha}}\left(\frac{R^k-1+\delta}{\alpha}-\delta\right)^{-\Phi} \tag{4.73}$$

Combining (4.47) and (4.48), and using the nominal interest rate derived in (4.71), one can express the gross real return on capital as:

$$R^k = \frac{\beta}{\Theta}\left(\frac{1}{\beta}-(1-\delta)\right)+1-\delta \tag{4.74}$$

Plugging (4.74) into (4.73) and re-arranging terms, one obtains the following expression:

$$\frac{h^{\Phi}}{(1-h)^{\eta}} = \frac{(1-\alpha)}{\Psi}\frac{\beta}{\Theta}\left(\frac{\Theta}{\beta}\left(\frac{1/\beta-1+\delta}{\alpha}\right)\right)^{\frac{\Phi-\alpha}{1-\alpha}}\left(\frac{\Theta}{\beta}\left(\frac{1/\beta-1+\delta}{\alpha}\right)-\delta\right)^{-\Phi} \tag{4.75}$$

Again, as in chapters 2 and 3, the assumption of balanced growth implies the choice $\Phi = 1$, obtaining:

$$\frac{h}{(1-h)^{\eta}} = \frac{(1-\alpha)}{\Psi}\left(\frac{1/\beta - 1 + \delta}{\alpha}\right)\left(\frac{\beta\alpha}{\Theta\,(1/\beta - 1 + \delta) - \beta\alpha\delta}\right) \tag{4.76}$$

Given standard calibration (i.e., $0 < \beta < 1$ and δ small with respect to 1) the left-hand side of (4.76) is positively related with working hours, while the right hand side is negatively related with the money growth rate. As in the previous chapters there is a negative relation between labour supply and money growth.

In order to inspect the steady state relationship between money growth ad output one can use the production function, the output/capital ratio (4.51) and the expression for the gross real return on capital (4.74) to derive the following expression:

$$y = \left(\frac{\Theta}{\beta}\left(\frac{1/\beta - 1 + \delta}{\alpha}\right)\right)^{\frac{\alpha}{\alpha-1}} h \tag{4.77}$$

Given the assumption of $0 < \alpha < 1$ (and given the relationship between money growth and labour supply derived above, under the assumption of balanced growth) one can conclude that a higher money growth rate decreases real output.

As in the previous chapters, these results shed some light on the monetary transmission mechanism: on the one hand, a higher money growth rate discourages investment because now the inflation-tax extends also to capital goods; on the other hand it reduces labour supply, since households substitute consumption goods (subject to the inflation-tax) with leisure. Both effects have a negative impact on real output.

4.6.3 Log-linear approximation

Using the methodology decribed by Uhlig (1999) one can *linearise* the original model, taking a first order Taylor expansion around the steady state. The usefulness of the log-linearisation method is twofold: on the one hand, it allows one to solve the model applying standard solution methods for linear rational expectations models; on the other hand, it re-defines all the economic variables as percentage deviations from steady state, isolating their cyclical fluctuations. The result is a linear system of equations, where the variables with the 'hat' indicate percentage deviations of the original variables from their long-run values[13], while variables without time subscript indicate steady state values:

consumption/labour:

$$\left[\eta\frac{h}{(1-h)}\right]\hat{h}_t^s = \hat{w}_t - \Phi\hat{c}_t - \hat{\imath}_t \tag{4.78}$$

consumption/saving:

$$\Phi E_t\hat{c}_{t+1} - \Phi\hat{c}_t = \hat{\imath}_t - E_t\hat{\pi}_{t+1} \tag{4.79}$$

capital/bonds

$$I\Phi E_t\hat{c}_{t+1} - I\Phi\hat{c}_t = \beta R^k E_t\hat{r}_{t+1}^k + \left[\beta I\left(1-\delta\right)-1\right]E_t\hat{\imath}_{t+1} \tag{4.80}$$

money demand:

$$m^d\hat{m}_t^d = c\hat{c}_t + c\hat{x}_t \tag{4.81}$$

labour demand:

$$\hat{y}_t - \hat{h}_t^d = \hat{w}_t \tag{4.82}$$

capital demand:

$$\alpha\frac{y}{k}\hat{y}_t - \alpha\frac{y}{k}\hat{k}_{t-1}^d = R^k\hat{r}_t^k \tag{4.83}$$

real output:

[13] As in previous chapters, the only exceptions are the inflation rate ($\hat{\pi}_t$), the nominal interest rate ($\hat{\imath}_t$) and the money growth rate ($\hat{\theta}_t$), where the 'hat' indicates deviations in levels. The fact that the original *net* rates are small numbers with respect to one, the correspondent *gross* rates (Π_t, I_t and Θ_t), can be log-linearised applying the following approximation: $\ln\Pi_t = \ln\left(1+\pi_t\right) \simeq \pi_t$.

$$\hat{y}_t = \hat{z}_t + \alpha \hat{k}^d_{t-1} + (1-\alpha)\,\hat{h}^d_t \tag{4.84}$$

money supply:

$$\frac{m^s}{\Pi}\hat{m}^s_{t-1} - \frac{m^s}{\Pi}\hat{\pi}_t + \tau\hat{\tau}_t = m^s\hat{m}^s_t \tag{4.85}$$

monetary injection:

$$\tau\hat{\tau}_t \equiv \Theta\frac{m^s}{\Pi}\hat{\theta}_t + \theta\frac{m^s}{\Pi}\hat{m}^s_{t-1} - \theta\frac{m^s}{\Pi}\hat{\pi}_t \tag{4.86}$$

'pipeline':

$$\frac{n}{\Pi}\hat{n}_{t-1} - \frac{n}{\Pi}\hat{\pi}_t + \tau\hat{\tau}_t - \tau^H\hat{\tau}^H_t = n\hat{n}_t \tag{4.87}$$

'pipeline' outflow:

$$\hat{\tau}^H_t \equiv \hat{n}_{t-1} - \hat{\pi}_t \tag{4.88}$$

capital market clearing condition:

$$\hat{k}^d_{t-1} = \hat{k}^s_{t-1} \tag{4.89}$$

labour market clearing condition:

$$\hat{h}^s_t = \hat{h}^d_t \tag{4.90}$$

money market clearing condition:

$$m^s\hat{m}^s_t = n\hat{n}_t + m^d\hat{m}^d_t \tag{4.91}$$

goods market clearing condition:

$$y\hat{y}_t = c\hat{c}_t + x\hat{x}_t \tag{4.92}$$

investment

$$\hat{x}_t = \frac{1}{\delta}\hat{k}^s_t - \frac{(1-\delta)}{\delta}\hat{k}^s_{t-1} \tag{4.93}$$

consumption-based velocity:

$$V E \hat{L} (c)_t \equiv \hat{c}_t - \hat{m}^s_t \tag{4.94}$$

investment-based velocity:

$$V E \hat{L} (x)_t \equiv \hat{x}_t - \hat{m}^s_t \tag{4.95}$$

output-based velocity:

$$V E \hat{L} (y)_t \equiv \hat{y}_t - \hat{m}^s_t \tag{4.96}$$

technology shock:

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \epsilon_{z_t} \tag{4.97}$$

monetary shock:

$$\hat{\theta}_t = \rho_\theta \hat{\theta}_{t-1} + \epsilon_{\theta_t} \tag{4.98}$$

, where: $\epsilon_{z_t} \sim N\left(0, \sigma^2_{\epsilon_z}\right)$ and $\ln \epsilon_{\theta_t} \sim N\left(0, \sigma^2_{\epsilon_\theta}\right)$.

Instead of log-linearising the equilibrium expression for the cash-in-advance constraint (4.49), the log-linear version of the money demand function (4.65) appears in the log-linear system (see (4.81)). As in the previous chapters, velocity definitions have also been included, in order to obtain the related simulation results.

Expressions (4.85) and (4.86), together with the money market clearing condition (4.91), can be used to deliver the expression for the inflation dynamics. However in this context, different form previous chapters, the presence of the 'pipeline' re-defines money demand according to its components ($\hat{m}^d_t$ and $\hat{n}_t$):

$$\hat{\pi}_t = \hat{\theta}_t - \left[\left(\frac{m^d}{m^s} \hat{m}^d_t - \frac{m^d}{m^s} \hat{m}^d_{t-1} \right) + \left(\frac{n}{m^s} \hat{n}_t - \frac{n}{m^s} \hat{n}_{t-1} \right) \right] \tag{4.99}$$

$\forall t \geq 0$; where fluctuations in the inflation rate around its steady state value are determined by the difference between money supply *growth* ($\hat{\theta}_t$) and the *growth* in the money demand components. Again, different from previous chapters, money is *non*-neutral: if the monetary shock would not be persistent (i.e., $\rho_\theta = 0$) a change in the quantity of money *today* would still have real effects, because of the *delay* introduced by the 'pipeline' mechanism.

4.7 Quantitative analysis

In the first part of this section numerical values are assigned to structural parameters and long-run relationships. The remaining coefficients in the linear approximations are derived using the the steady state relationships, implied by the original non-linear system. Given these calibration values, the last part of this section will compare the qualitative and quantitative impact of the stochastic shocks on the endogenous variables of the two monetary models. In particular, the analysis will focus on the *impulse-response* dynamics and the relative match of the extended CIA model with respect to the empirical evidence presented in Chapter 1.

4.7.1 Calibration

In order to derive the response of the baseline models to stochastic shocks, one needs to assign numerical values to the parameters appearing in the linearised model. For the purpose of comparison with the quantitative results obtained previously, all parameters values have been set in line with the calibration of the CIA model in Chapter 3. The only modifications relate to the extended link between cash balances and aggregate demand and the existence of the 'pipeline' mechanism.

Table 4.1 reports the values for the parameters characterizing the utility function, some long-run relationships and the stochastic shocks for the extended CIA model.

Log-utility for consumption and leisure imply setting the coefficients of relative risk aversion and the labour supply elasticity equal to unity ($\Phi = 1$, $\eta = 1$). The steady state labour supply has been set to one-third of the time endowment ($h = 0.33$). The parameter relative to the capital depreciation (δ) has been set in order to get a quarterly depreciation of 1.9%. The share of capital to total income ($\alpha = 0.36$) has been chosen in order to deliver a labour share of 2/3 and a capital share of 1/3. The value for the (quarterly) discount factor ($\beta = 0.989$), the autoregressive coefficient of the technology shock ($\rho_z = 0.95$) and its standard deviation ($\sigma_{\epsilon_z} = 0.007$) are in line with the standard RBC literature reported in Chapter 2. The (exogenous) net nominal money growth rate is set to $\theta = 0.0125$; while the autoregressive parameter ($\rho_\theta = 0.5$) and standard deviation ($\sigma_{\epsilon_\theta} = 0.0089$) are derived from the estimation of an autoregressive process of M1 by Cooley and Hansen (1989)[14]. Finally, the parameter regulating the 'pipeline' (ξ) is initially set to 1.00. For a second run of simulations, this number will be replaced by 0.05 (reported in brackets in Table 4.1) with the purpose of conducting a sensitivity analysis exercise. This will help to inspect the quantitative implications of variations in the degree of rigidity in the money supply process.

One can use the parameter values assigned in Table 4.1 and the steady state relationships derived from the equilibrium to derive all the remaining coefficients of the linear system. Moreover, one can check whether the baseline calibration is able to generate steady state values (or ratios) compatible with the empirical evidence. All these results are reported in Table 4.2 and Table 4.3.

As in Chapters 2 and 3, the long-run relationship between the money growth rate and inflation is one to one for this model. The baseline calibration implies a (quarterly) average net nominal interest rate (i) of 2.38% and a weight parameter for leisure $\Psi = 1.6067$ in the utility function. With the cash-in-advance constraint extended to capital goods, the real expenditure shares of output do not differ from the results reported in Chapter 3 (see 3.2). Table 4.3 reports the steady state value for all monetary variables influenced by the values taken by ξ in the sensitivity analysis[15]. Because of the presence of the 'pipeline' technology for the transmission mechanism, all measures relative to the velocity of money are smaller than 1, when *total* money supply

[14] As in the original paper by Cooley and Hansen 1989, in order to check for robustness of the results, a sensitivity analysis exercise has been carried out, using different values for Θ and ρ_θ. However, these attempts did not change significantly the quantitative results reported in the next sections.

[15] Reflecting the convention adopted in Table 4.1, the numbers in brackets correspond to the values derived when $\xi = 0.05$.

is used[16]. As reported in Chapter 2, when M1 is used Walsh (2003) reports a value bigger than 1 for the steady state of *consumption*-based velocity. Finally, note that, because of the 'pipeline' device, the ratio between households real balances and total real balances (Ξ) varies between 0.9877 and 0.80, when ξ takes values 1 and 0.05, respectively.

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[16]In models in which *all* money is in the hands of the households, like in Chapters 2 and 3 (or as in the original models by Stockman (1981) and Abel (1986)) at least one of the measures for the velocity of money considered here assumes value 1.

parameter/variable	**description**	**value**
Φ	relative risk aversion	1
η	inverse of labour supply elasticity	1
β	discount factor	0.989
δ	capital depreciation rate	0.019
α	capital income share of output	0.36
h	working hours	0.3333
θ	net money growth rate	0.0125
ρ_z	autoregressive param. technology shock	0.95
σ_{ϵ_z}	s.d. technology shock	0.007
ρ_θ	autoregressive param. monetary shock	0.5
σ_{ϵ_θ}	s.d. monetary shock	0.0089
ξ	'pipeline' parameter	1.00 (0.05)

Table 4.1: Baseline calibration of CIA model (with capital).

parameter/variable	**description**	**value**
Θ	gross money growth rate	1.0125
Π	gross inflation rate	1.0125
π	net inflation rate	0.0125
I	gross nominal interest rate on bonds	1.0238
i	net nominal interest rate on bonds	0.0238
R^k	gross (net) return on capital	1.0118
$r^k - \delta$	net return on capital	0.0118
r^k	marginal product of capital	0.0308
y/k	output/capital ratio	0.0857
k	capital stock	15.5025
x	investment	0.2945
y	real output	1.3280
w	real wage	2.5497
c	real consumption	1.0334
c/y	consumption share of output	0.7782
x/y	investmet share of output	0.2218
Ψ	preference parameter for leisure	1.6067

Table 4.2: Steady state values of CIA model (with capital) at baseline calibration.

m^d	households' (real) balances	1.3280
τ^H	households (real) monetary injection	0.0164
n	'pipeline' (real) balances	0.0166 (0.3320)
m^s	total (real) cash balances	1.0401 (1.6600)
τ	(real) monetary injection	0.0128 (0.0205)
Ξ	households' real balances share	0.9877 (0.8000)
c/m^s	*consumption*-based velocity	0.7686 (0.6226)
x/m^s	*investment*-based velocity	0.2191 (0.1774)
y/m^s	*output*-based velocity	0.9877 (0.8000)

Table 4.3: Steady state values of CIA model (with capital) at baseline calibration: monetary variables.

4.7.2 Impulse-response analysis

In what follows the dynamic responses of the extended CIA model are analysed. The figures below report the percentage deviation of the selected variables from their steady state value (which, for convenience, has been set to zero). The deviation from steady state of variables which do represent rates (e.g., inflation rate, interest rate, unit transaction costs), is measured in absolute terms. All the shocks take place at time zero and the time scale refers to quarterly data.

Technology shock

Figure 4.1 shows the impact of the technology shock on real expenditure and money demand (households' real balances). Since the production function and the utility function have not been changed, the response of consumption and investment are the same as those derived for Chapter 3. The only exception is represented by the dynamics of real balances. This is due to the fact that the demand for real balances now encompasses also investment goods. On impact the response of households' real balances is bigger when investment goods are subject to the cash-in-advance constraint, because now they 'cover' an amount of transactions that corresponds to real output.

Figure 4.2 shows the impact of the technology shock on inflation and nominal interest rates. Inflation is generated as a difference between the growth in money supply and the growth in money demand. Therefore, if the government follows a constant money growth rule, a technology shock causes a temporary drop in inflation, *via* money demand dynamics.

Given the drop in inflation, the other real balances - namely, 'total' and 'pipeline' balances - follow the dynamics of Households' real balances. These elements, together with the results described in Figure 4.1, can be used to explain the constant *output*-based velocity, the pro-cyclical *investment*-based velocity and the anti-cyclical *output*-based velocity in Figure 4.4.

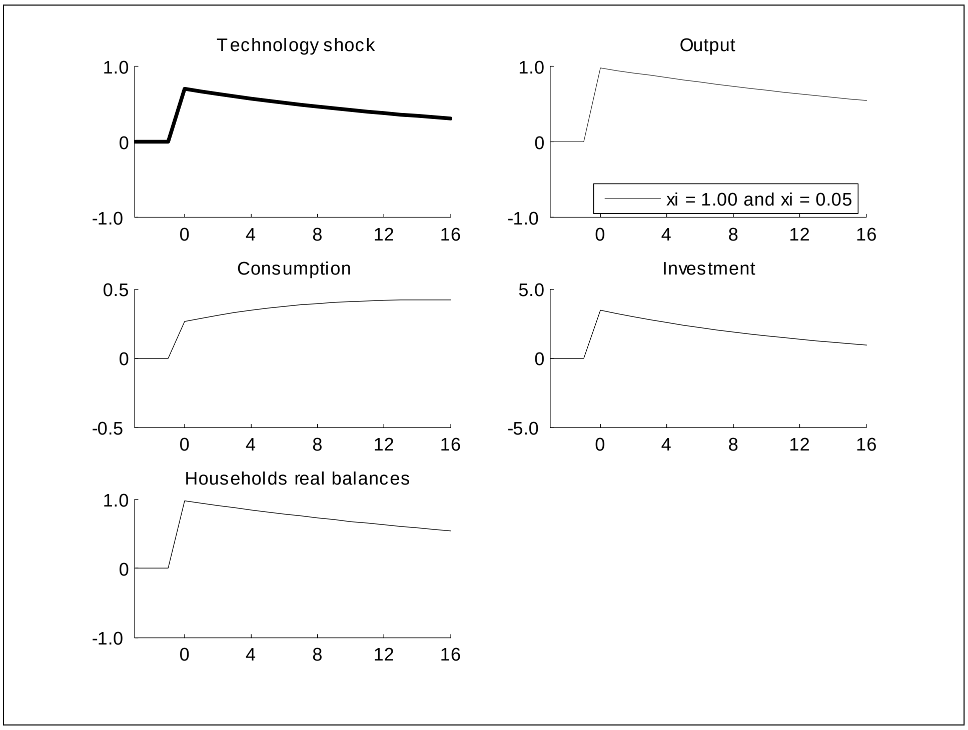

Figure 4.1: Impact of the technology shock on real expenditure (extended CIA)

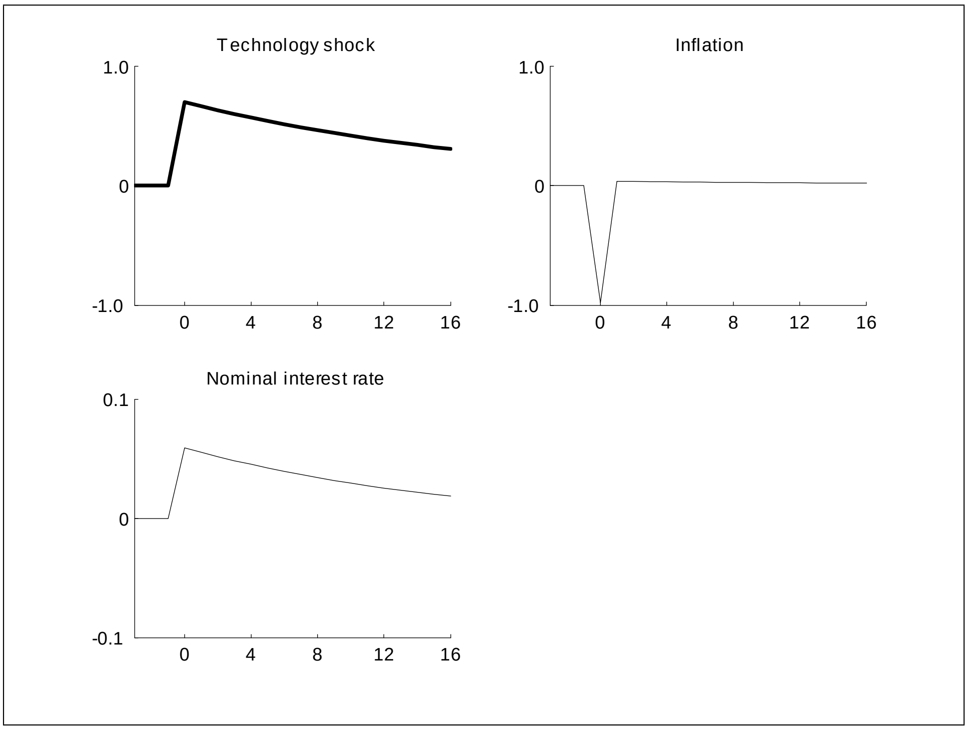

Figure 4.2: Impact of the technology shock on nominal variables (extended CIA)

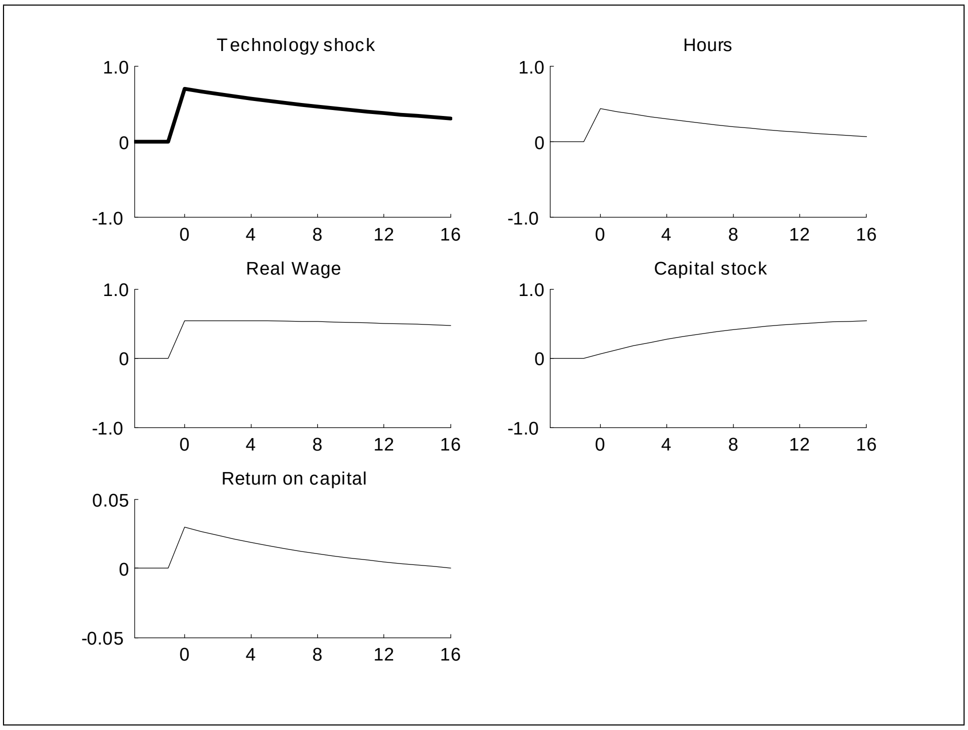

Figure 4.3: Impact of the technology shock on production factors (extended CIA)

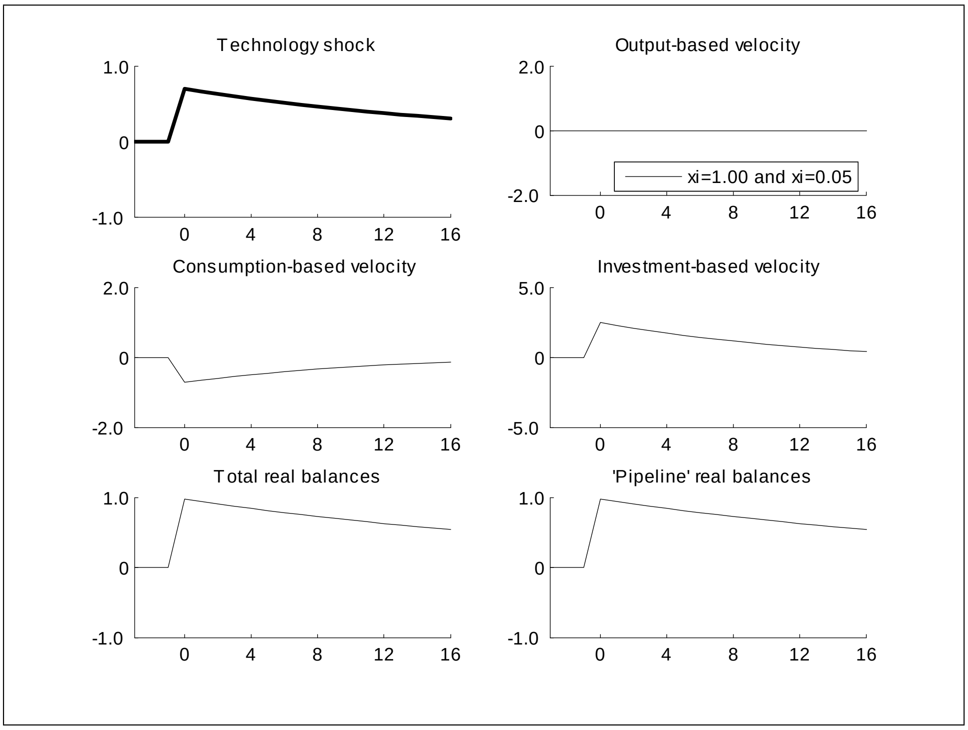

Figure 4.4: Impact of the technology shock on the velocity of money (extended CIA)

Monetary shock

Figure 4.5 shows the impact of the monetary shock on real expenditure and households' real balances, for different values of the parameter regulating the 'pipeline valve'. Because of the assumption of flexible prices, in both models a monetary shock produces real effects as long as it modifies expected inflation. In this model, not only current and future consumption expenditure is connected with money holdings, but money is used also to buy investment. On the one hand, higher expected inflation makes every (future) purchase where money is involved more expensive in real terms, therefore induces a fall in real consumption tomorrow (and today), through the mechanism of consumption smoothing. On the other hand, the (rational) households anticipate the fall in real investment expenditures - and, therefore, a future fall in output - trying to smooth the loss in consumption between periods. Since the 'pipeline' mechanism regulates the flow of new monetary injections, introducing a delay between the shock (occurring at time 0) and the time these enter in households' accounts, it is crucial for the dynamics of private expenditure. In particular, when households' receive *all* the new money with one period of delay ($\xi = 1.00$), there is a negative impact on both consumption and investment. In this case, the fall in investment is 200 times bigger than the fall in consumption. This is due to the fact that households derive direct utility from the latter, preferring to sacrifice consumption. In practice a result opposite to that of Chapter 3: now that consumption and investment are both treated in the same way (i.e., both are *cash goods*) because of the inflation tax, consumers switch from investment to consumption. In Chapter 3 instead, the inflation tax applied only to consumption, and there households were switching towards the *credit good*. Figure 4.5 shows that this result depends on the 'pipeline' parameter. In particular, once this parameter is set to 0.05 (which corresponds, in steady state, to 20% of total money in the 'pipeline') the result is reversed: there is a positive (but very small) reaction of consumption and a (small) fall in investment. The fall in output is largely determined by the dynamics of investment. Since the cash-in-advance applies to output, households' real balances dynamics coincides with the latter.

Another difference with the previous chapters regards the behaviour of total real balances. After a monetary shock the new money does not reach the households' budget constraint immediately, but accumulates in the 'pipeline'. this is represented by the last graph in Figure 4.8. When the 'pipeline' parameter is set to 1, 'pipeline' real balances increase by almost 80%. This is due to the fact that, at the steady state, xi = 1.00 corresponds to a smaller proportion of total money in the 'pipeline' (therefore a higher proportion in the households' accounts). In fact, if the money stored from previous period in the pipeline is *completely* passed on to households in the current period, in the same period only the new money injected by the Government is stored in the 'pipeline'. The increase of the latter (in real terms) is responsible for a separation of the total money supply and households' money demand dynamics. Therefore, a positive monetary shock has the effect of increasing the value of total real balances in this model. These results are confirmed also by the (negative) response of velocities in Figure 4.8.

As anticipated before, the real effects of the monetary shock are guaranteed by the fact that it displays some degree of persistence. However, a key difference with the previous chapters refers to the non-neutrality of contemporary monetary shocks. In fact, if the autoregressive parameter is set to zero ($\rho_\theta = 0$), because of the structure of the 'pipeline' mechanism, the *current* price level (and, therefore, current inflation) is not affected. This is due to the fact that any new monetary injection will produce its effects in the next period, regardless of the autoregressive nature of the shock.

Turning the attention to the nominal effects of a monetary shock (Figure 4.6), the response of inflation to a monetary shock is more than proportional. This is because in both models actual inflation is determined by the difference between the money supply growth and the change in money demand. As stressed above, the latter depends on the value of the 'pipeline parameter'. When the shock occurs, the greater response of inflation for $\xi = 1.00$ is given by the higher fall in real balances. However, because of the lag introduced by the 'pipeline', the peak in the inflation process is reached one period after the shock. This can be explain by the fact that in the period when government injects new money in the economy, the movement in inflation is exclusively due to the effect of anticipated inflation on money demand (see Figure 4.5).

As shown in Figure 4.6, the impact of the monetary shock on the nominal interest rate is less than propotional. However, the behaviour of the interest rates follows closely inflation. Given the market timing assumptions adopted (i.e., the financial market opening first) the nominal interest rate represents the opportuniy cost of holding money. In the CIA model expected inflation drives the nominal interest up according to the Fisher equation, which directly links the marginal utility of consumption (and, therefore, real balances) today and tomorrow.

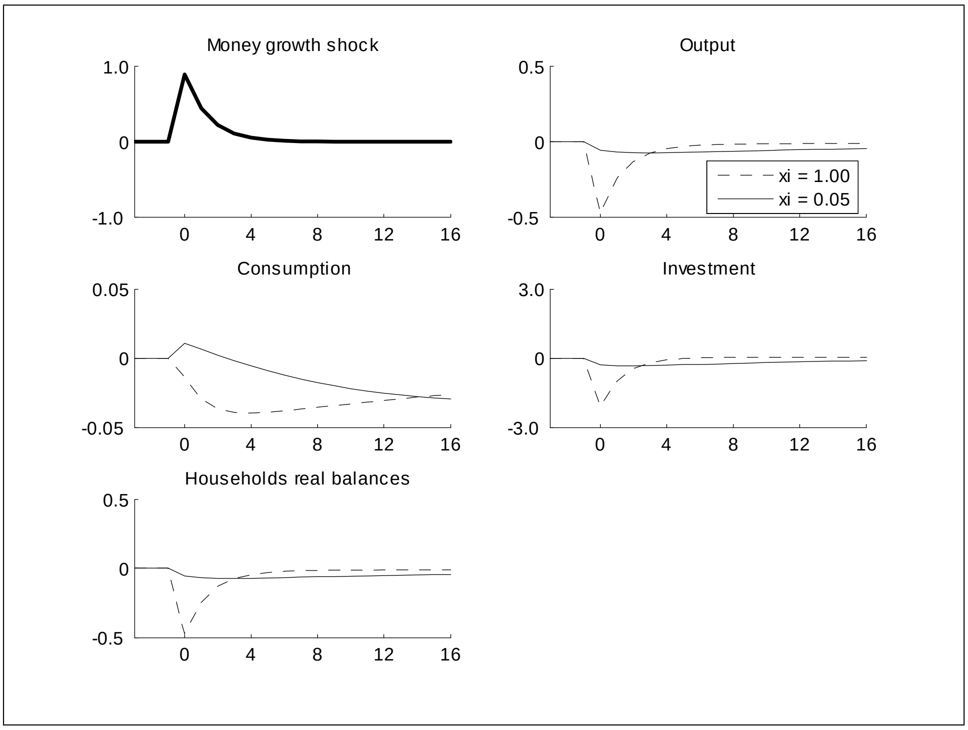

Figure 4.5: Impact of the monetary shock on real expenditure (extended CIA)

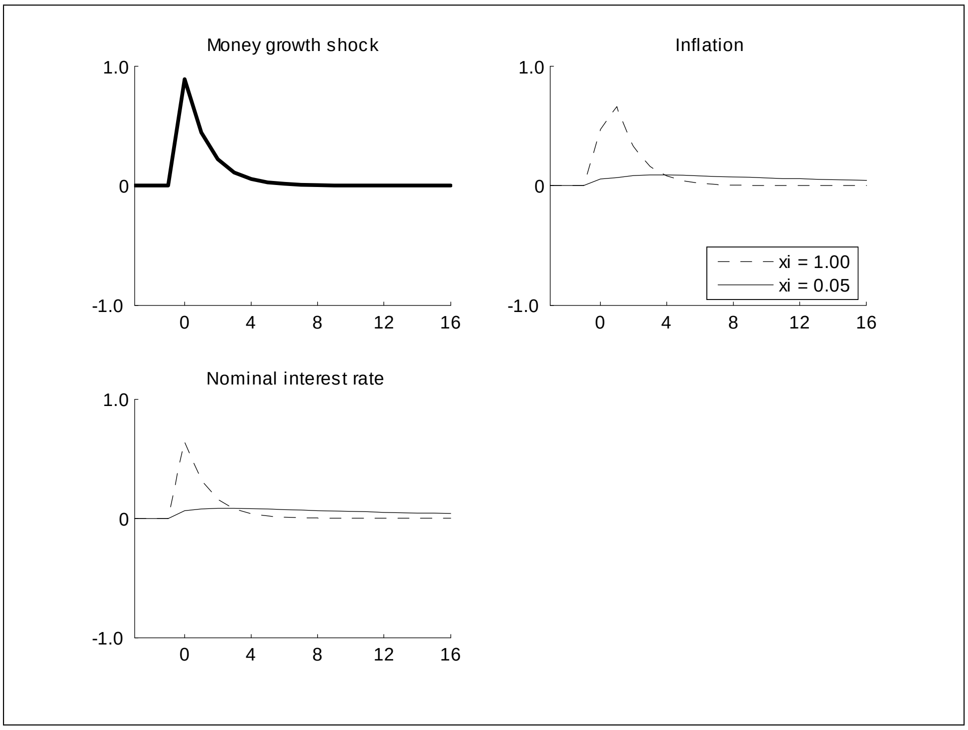

Figure 4.6: Impact of the monetary shock on nominal variables (extended CIA)

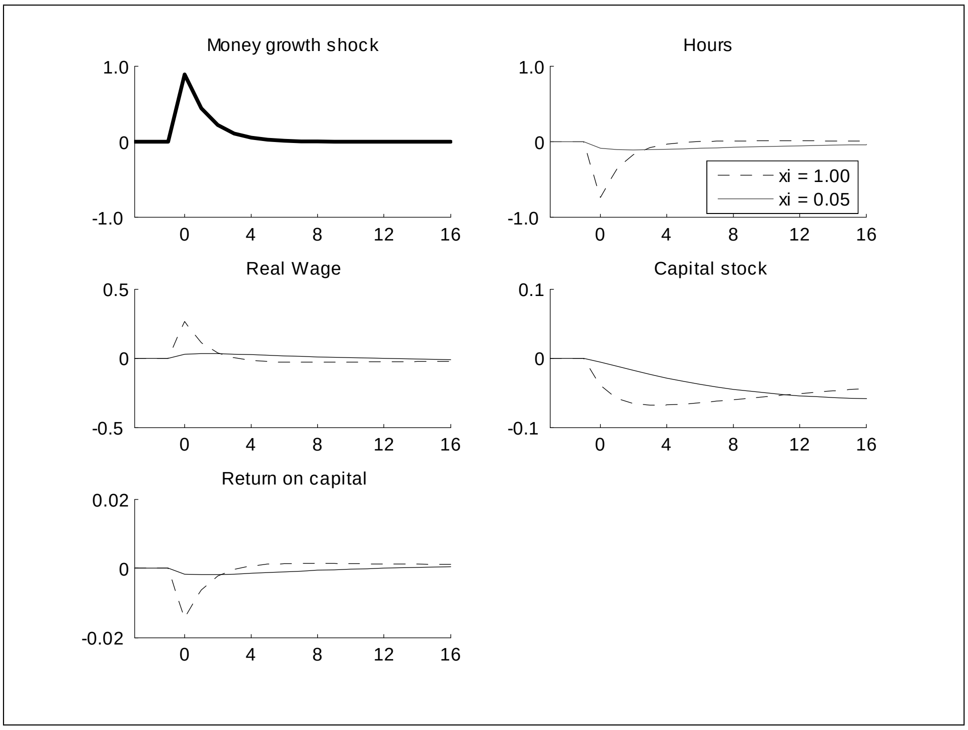

Figure 4.7: Impact of the monetary shock on production factors (extended CIA)

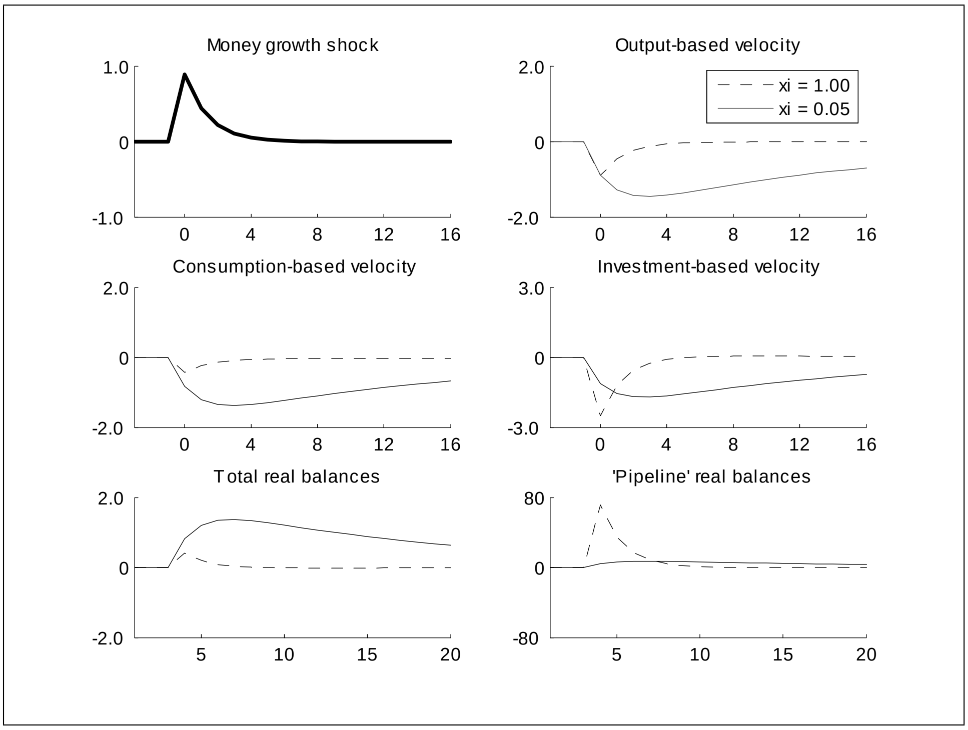

Figure 4.8: Impact of the monetary shock on the velocity of money (extended CIA)

4.7.3 Additional simulation results

The impulse-response analysis conducted above helped to analyse the effects of the orthogonal shocks on the variables of interest. In this part the quantitative exploration will focus on the comparison of the simulation results with the characteristics of the actual U.S. time series for the variables appearing in the model. Following the spirit of Cooley and Hansen (1995), the performance of the CIA and RRC model will be assessed along three dimensions: the standard deviation of the variables in the simulated models and their correlation with output and money growth. Since both monetary models abstract from many real world features and rigidities, one should not expect a perfect match. In fact the aim of this comparison should be helpful in suggesting whether the models go in the direction the data suggest and, eventually, which of the two model is closer to the empirical evidence. In practice, this corresponds to a quantitative analysis *at the margin*.

Table 4.4 reports the standard deviations of the variables of the U.S. economy (second column) together with the standard deviations of the artificial variables resulting from the simulations of the extended CIA model (the third and fourth column report the sensitivity results for the 'pipeline' parameter)[17]. The results available from the analysis by Cooley and Hansen (1989) and Cooley and Hansen (1995) are also reported (under CH1989 and CH1995, respectively). Along this dimension, the model is quite sensitive to changes in the 'pipeline' parameter. In fact, only for 5 variables, the two different calibrations of ξ deliver similar results: real wage, consumption, output, investment and money growth rate. With the exception of the money growth rate, all these are generally more volatile than the respective real counterparts. Regarding working hours the model's performance improves when $\xi = 0.05$. In terms of standard deviations, the model performs pretty well for the monetary variables - i.e., nominal interest rates, real balances and velocities. In these last cases the match is either really close to the data (e.g., nominal interest rates in the case $\xi = 0.05$) or not too far (real balances and *investment*-based velocity when $\xi = 1.00$). Finally, in the case of *consumption*- and *output*-based velocity, the

[17]For the description of the data reported in the following Tables, refer to Chapter 1, Section 1.5.

real data standard deviation (0.0258 and 0.0277, respectively) falls within the range of the sensitivity analysis. At a first glance the standard deviations of the model's variables are closer to the data when the rigiditiy in the money supply process is higher (i.e., $\xi = 0.05$).

Table 4.5 reports the correlations with output of the variables of the U.S. economy (second column) together with the correlations with output of the artificial variables resulting from the simulations of the extended CIA model (the third and fourth column report the sensitivity results for the 'pipeline' parameter)[18]. The results available from the analysis by Cooley and Hansen (1989) and Cooley and Hansen (1995) are also reported (under CH1989 and CH1995, respectively). The two different parametrisations of the 'pipeline' deliver similar numerical results for the variables of the labour market, the goods market and inflation. In some of these cases the values are very close to the data for both models (like in the case of working hours, consumption and investment), while for some others the correlation sign is the opposite of the data (inflation and *consumption*-based velocity). In the case of nominal interest rates, the model is very close to the data when the rigidity in the money supply process is higher (i.e., $\xi = 0.05$), while the correlation of output with money growth is superior in the case of a one-period-lag in the monetary injections (i.e., $\xi = 1.00$). Summing up: in terms of correlation with output (and with the exception of money growth and interest rates) the two parametrisations work quite similarly.

Table 4.6 reports the correlations with money growth for the variables of the U.S. economy (second column), together with the correlations with money growth of the artificial variables resulting from the simulations of the extended CIA model (the third and fourth column report the sensitivity results for the 'pipeline' parameter). The results available from the analysis by Cooley and Hansen (1989) and Cooley and Hansen (1995) are also reported (under CH1989 and CH1995, respectively). Here the results are very sensitive to the 'pipeline' parameters. In general, the model performs quite well in terms of working hours, output and velocities. The model performs very poorly instead

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[18] For the description of the data reported in the following tables, refer to Chapter 1, Section 1.5.

regarding inflation and interest rates, where the correlation sign is the opposite of the data. Moreover, the simulations results for consumption and real wage show a lower correlation with money growth than the data. Finally, in the case of real balances, the real data standard deviation (0.2264) falls within the range of the sensitivity analysis.

variables	STD. DEV.				
	data	xi = 1.00	xi = 0.05	CH1989	CH1995
working hours	0.0043	0.0136	0.0112	0.0133	0.0135
real wage	0.0109 (0.0071)	0.0304	0.0303	-	-
consumption	0.0125	0.0269	0.0269	0.0062	0.0053
nominal interest rate	0.0030	0.0077	0.0033	-	0.0058
inflation	0.0044 (0.0029)	0.0134	0.0103	-	0.0123
real balances	0.0314 (0.0284)	0.0371	0.0594	-	-
output	0.0154	0.0372	0.0369	0.0173	0.0169
money growth	0.0089	0.0103	0.0103	-	0.0087
investment	0.0699	0.0951	0.0926	0.0569	0.0590
consumption-based vel	0.0258	0.0184	0.0508	-	-
investment-based vel	0.0654	0.0680	0.0833	-	-
output-based vel	0.0277	0.0103	0.0494	-	-

Table 4.4: Standard deviations (extended CIA model).

variables	CORR. with OUTPUT				
	data	xi = 1.00	xi = 0.05	CH1989	CH1995
working hours	0.7077	0.6370	0.6911	0.9800	0.987
real wage	0.5307 (0.5830)	0.9382	0.9635	-	-
consumption	0.8632	0.8845	0.8918	0.7200	0.676
nominal interest rate	0.3522	0.0480	0.3807	-	-0.014
inflation	0.3817 (0.1419)	-0.1877	-0.1507	-	-0.138
real balances	0.3368 (0.3133)	0.9617	0.5575	-	-
output	1.0000	1.0000	1.0000	1.0000	1.0000
money growth	-0.1282	-0.1488	-0.0292	-	-
investment	0.9024	0.8861	0.8870	0.9700	0.975
consumption-based vel.	0.0713	-0.6464	-0.1786	-	-
investment-based vel.	0.8056	0.7138	0.5883	-	-
output-based vel.	0.2362	0.1488	0.0772	-	-

Table 4.5: Correlations with output (extended CIA model).

variables	CORR. with MONEY GROWTH				
	data	xi = 1.00	xi = 0.05	CH1989	CH1995
working hours	-0.1957	-0.6160	-0.1453	-	-0.0300
real wage	0.2365 (0.1714)	0.0943	0.0192	-	-
consumption	0.0311	-0.0151	0.0044	-	-0.6000
nominal interest rate	-0.4771	0.9772	0.3744	-	0.7200
inflation	-0.3124 (-0.1940)	0.5920	0.1099	-	0.9200
real balances	0.2264 (0.2021)	0.1278	0.3080	-	-
output	-0.1282	-0.1488	-0.0292	-	-0.0100
money growth	1.0000	1.0000	1.0000	-	1.0000
investment	-0.1955	-0.2476	-0.0570	-	0.1600
consumption-based vel.	-0.2306	-0.2802	-0.3577	-	-
investment-based vel.	-0.3250	-0.4157	-0.3917	-	-
output-based vel.	-0.2781	-1.0000	-0.3917	-	-

Table 4.6: Correlations with money growth (extended CIA model).

The quantitative exercises by Cooley and Hansen (1989, 1995) deliver a lower variability of consumption and output, generally more in line with the data. As noted previously, this is mainly due to the assumption of indivisible labour, adopted in their work. The standard deviation of the nominal interest rate in Cooley and Hansen (1995) remains higher than the one reported in the data (from this point of view, the extended CIA model with a more rigid pipeline performs better). However, the extended CIA model - with divisible labour - presented in this chapter exhibit a correlation of consumption and working hours more in line with the data. The same is true for the inflation tax: in fact the correlation between money growth and consumption - in the extended CIA model with a more rigid 'pipeline' - exhibits the same 'sign' as the the data, subverting the result obtained by Cooley and Hansen (1995).

4.7.4 The extended CIA model: some comments

As anticipated before, given the relative simplicity of the models at hand, one should not expect the simulated data to match up *perfectly* with the real data. However, some general comments can be added at this stage. The volatility of the variables for the extended CIA model is generally higher than the one shown by the real data: the bigger discrepancies concern inflation (with a volatility three times higher than the data, when the friction created by 'pipeline' is greater), the labour market (where working hours and real wage in the models result three times as volatile than the data), followed by the goods market (with all the components of aggregate demand being, on average, twice as volatile). However, in all the other cases, the performance is mixed and seems to depend on the degree of rigidity in the 'pipeline' mechanism. For most of these remaining variables, with the exception of the nominal interest rates, a pretty good match is achieved when the 'pipeline' is less rigid (i.e., when *all* the new money is passed on households in the next period). Finally, the volatility of the different measures of velocity depends crucially on the value of the parameter that regulates the 'pipeline'. Regarding the correlation with output, the range considered by the sensitivity analysis delivers a satisfactory match with the data, especially in the case of the labour market and the goods market. The reason for such a good match in those fields lies in the characteristics of the technology shock: in fact, the high value of the autoregressive parameter of this shock (0.95) and its immediate effects on output and factors' demand drives these results. On the other hand, the model performs quite decently on the monetary side. For some of these variables - such as the nominal interest rates - a higher rigidity in the money supply process seems to work better, while in others (like the correlation between mney growth and output) less rigidity fits better the empirical evidence. The (negative) exceptions are represented by inflation and the *consumption*-based velocity, where the data suggest pro-cyclicality, while the model displays a negative sign. When the correlation of the variables with money growth in the simulated economies is compared with the real data, the results are encouraging. In particular many variables display the right correlation within the sensitivity range, while some others (such as real wages and consumption) display a lower correlation than the data. The match with the data is pretty good in the case of output, investment and velocities. The bad news is limited to inflation and nominal interest rates, where the correlation with money growth exhibits the 'wrong' sign with respect to the data. This is mainly due to the assumption of flexible prices in the model. On one hand, a monetary injection transmits immediately to higher inflation, and this implies a strong correlation of inflation with money growth. On the other hand, the nominal interest rates respond strongly to the evolution of expected inflation, showing the absence of any liquidity shock.

4.8 Conclusion

This chapter presented a flexible-price monetary model where both consumption and investment goods are subject to a *cash-in-advance* constraint and the money supply process is 'disturbed' by the presence of a rigid transmission mechanism. The model can be considered, at the same time, as an implementation and a quantitative assessment of the seminal papers by Stockman (1981) and Abel (1985). The implementation is represented by the adoption of two structural assumptions: the association of the extension in the *cash-in-advance* constraint with a Lucas-type market timing assumption (i.e., financial markets opening before the goods markets); and the presence of a lag in the transmission of monetary shocks to the private economy. The quantitative assessment is made relative to some key descriptive statistics concerning selected characteristics of the U.S. business cycle.

The main findings are that the standard deviation of the artifical variables related to the goods market (output, consumption and investment) are higher than the data and are not influenced by the rigidity in the money supply process. The reason is that they mainly respond to the (highly persistent) technology shock. On the contrary, the volatility in the monetary variables is strongly affected by the sensitivity analysis conducted on the 'pipeline' mechanism and in most of the cases the match with the data lies within the sensitivity range. The fact that the technology shock maintains a relative 'dominance' on the behaviour of the variables related with the goods market is confirmed by a (strong) correlation with output, which seems to match the data quite closely.

On the other hand, the correlation of the artificial variables with the money growth rate shows the strong explanatory power of the 'pipeline' mechanism for the behaviour of the nominal variables. However, the model fails along two dimensions: it is not able to explain a negative correlation between inflation and money growth and it is not able to produce a liquidity effect.

Finally, the comparison with the results reported in previous work by Cooley and Hansen (1989, 1995) reveals that the assumption of divisible labour adopted in this chapter performs better in terms of correlation of real variables with output and money growth, while the quantitative performance of the indivisible labour assumption (adopted by Cooley and Hansen) is superior in terms of standard deviations of consumption and working hours Moreover, differently than Cooley and Hansen (1989, 1995), when the cash-in-advance constraint is extended to capital goods and the pipeline mechanism is more rigid, the correlation between nominal interest rates and output exhibits the same 'sign' of the data.

The relative success of the combination of the 'pipeline' model with the extension of the *cash-in-advance* specification suggests that the model can be improved in two directions. The first represents the possibility to allow monetary transfers to firms, as well as to households. The possibility for the productive sector to cover production costs with the new money might generate, under specific conditions, the same effects of a positive techology shock. In principle, given an appropriate calibration, one could obtain an initial fall (and/or a persistent dynamics) for the inflation rate and, at the same time, a liquidity effect for the nominal interest rates.

Chapter 5

Extending transaction technologies: a real resource cost approach to the business cycle

The purpose of this chapter is to extend the real-resource-cost (RRC) model in the same directions to which the previous chapter extended the baseline cash-in-advance (CIA) model. If the extension of the cash-in-advance constraint to investment goods had already been considered as a theoretical option by the literature (see the seminal contributions of Stockman (1981) and Abel (1985)), there is no trace of attempts to state the problem in terms of the real-resource-cost model. Therefore, from this point of view, this implementation represents a novelty in the field. Contrary to the modification of the cash-in-advance constraint, there is no *unique* way of extending transaction costs to investment. As will be shown below, the modelling strategy adopted here takes into account a twin goal. Firstly, agents' choice about resource allocations in the goods market must allow *separability* between consumption and investment purchases. Secondly, comparability of results with the extended CIA model presented in Chapter 4 must be preserved.

The other important extension that will be maintained from the previous chapter concerns the money supply process. The extended CIA model presented in Chapter 4 revealed the quantitative importance for a flexible price model of the presence of rigidities in the money supply channel. Consequently, it appears reasonable to test the performance of the extended RRC model including the same modification to the transmission mechanism of monetary shocks. This will be achieved by introducing the monetary 'pipeline' technology.

The quantitative assessment conducted in the last part extends the work of Cooley and Hansen (1989) by reporting the results for additional endogenous variables (e.g., the nominal interest rate, real balances and different measures of velocity of money), by inspecting the impulse-response functions of their CIA model and by reporting the correlation of endogenous variables with respect to money growth. Moreover, the set of simulation results for both models is richer than the one reported in Cooley and Hansen (1989, 1995).

The chapter is structured as follows: firstly, the assumptions implied by each approach are stated, then the resulting optimality conditions are derived. Finally, the models are calibrated (on quarterly basis) and outcomes from simulations are compared. The model performance is assessed analysing the effects of stochastic shocks affecting production (i.e., technology shocks), the money demand (i.e., shocks to transaction costs) and the money supply process (i.e., monetary policy innovations). The problem of the representative firm and the monetary policy rule are the same that characterised the RRC model in the previous chapters. At the same time, dating conventions and market timing assumptions are unaffected by the extensions considered in this chapter. Given the similarities some details and explanations will not be repeated here. In any case, the reader is invited to refer back to the main features of the baseline RRC model described in Chapters 2.

5.1 Households

At time $t = 0$ the representative household seeks to maximize the following expected value of a discounted stream of period utility:

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u \left[c_t, l_t \right] \right\} \tag{5.1}$$

The expectational operator E_0 indicates that the expectations at time $t = 0$ about present and future streams of utility are formed conditionally to the information available to the agent. The objective function (5.1) assumes that utility at time t depends on real consumption c_t and leisure time l_t. Future utility is discounted by a (constant) discount factor β (with $0 < \beta < 1$).

The utility function u is strictly concave and twice continuously differentiable. It is increasing in its arguments and decreasing in their marginal utility. Using u_x to denote the partial derivative of the function u with respect to its generic argument x, one can write: $u_c > 0$, $u_l > 0$, $u_{cc} < 0$, $u_{ll} < 0$. In addition to that, also the Inada (1963) conditions are assumed to be holding: $\lim_{c\to\infty} u_c = \infty$, $\lim_{l\to 0} u_l = \infty$, $\lim_{c\to\infty} u_c = 0$, $\lim_{l\to\infty} u_l = 0$.

Total time endowment is normalized to 1, so that the following constraint applies to every period:

$$1 = l_t + h_t^s \tag{5.2}$$

This means that at time t the agents will choose to split total time between leisure time l_t and (supplied) working hours h_t^s.

Using (5.2) one can reformulate (5.1) in terms of consumption and working hours:

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u \left[c_t, 1 - h_t^s \right] \right\} \tag{5.3}$$

As in the previous chapters, u is decreasing in working time ($u_h < 0$) and increasing in the marginal *dis*utility of work ($u_{hh} > 0$). In deference to the real business cycle tradition, this last formulation will be mantained throughout the analysis.

The explicit functional form chosen for period utility takes the form of a constant relative risk aversion (CRRA) utility function:

$$u \left[c_t, 1 - h_t^s \right] \equiv \frac{c_t^{1-\Phi}}{1-\Phi} + \Psi \frac{\left(1 - h_t^s\right)^{1-\eta}}{1-\eta} \tag{5.4}$$

, where: $\Phi > 0$ is the coefficient of relative risk aversion (with $1/\Phi$ being the elasticity of intertemporal substitution); given $\eta > 0$, expression $1/\eta$ denotes the elasticity of intertemporal substitution for labour; $\Psi > 0$ represents a preference parameter over leisure.

The representative household is subject to a budget constraint. With respect to the RRC model presented in Chapters 2 and 3, two important extensions characterise the RRC model in this chapter. The first consists in introducing transactions costs for investment goods, represented by the transaction cost function $\Upsilon^x(\omega_t^x, x_t, a_t^x)$. The second consists in the origin of the monetary lump sum transfers entering the household budget constraint, and denoted by T_t^H (where, as in Chapter 4, the superscipt 'H' is meant to denote 'households'). with these two modifications, the budget constraint can be written (in nominal terms) as follows:

$$\begin{aligned} M_{t-1}^d + (1+i_{t-1}) B_{t-1}^d + W_t h_t^s + T_t^H + P_t r_t^k k_{t-1}^s \\ \geq P_t c_t + P_t x_t + M_t^d + B_t^d + P_t \Upsilon^c(\omega_t^c, c_t, a_t^c) + P_t \Upsilon^x(\omega_t^x, x_t, a_t^x) \end{aligned} \quad (5.5)$$

$\forall t \geq 0$, where: B_t^d denotes the nominal value of (risk free) bonds holdings, which pay a one-period nominal (net) interest rate i_t; T_t^H represents nominal lump sum transfers and M_t^d is individual money demand. The other nominal variables are represented by the hourly nominal wage (W_t) and the price of the homogeneous good produced in the economy (P_t). The capital stock (k_{t-1}^s) is owned by households, who rent it to firms at the beginning of the period for production purposes. Therefore, the rental payment ($P_t r_t^k k_{t-1}^s$) enters as an additional source of wealth on the left hand side of the budget constraint. Two transaction costs functions appear on the left hand side of the budget constraint: these are denoted by $\Upsilon^c(\omega_t^x, c_t, a_t^c)$ and $\Upsilon^x(\omega_t^x, x_t, a_t^x)$, with superscripts 'c' and 'x' characterising elements of total transaction cost functions applying to *consumption* and *investment*, respectively. Finally, household's expenditures in the goods market includes consumption ($P_t c_t$) and investment goods ($P_t x_t$), which appear on the right hand side[1].

As explained in Chapter 2, the real resource cost (RRC) approach assumes that transactions are costly. These transaction costs take the form of real resources (i.e., goods), which must be used up in the process of exchange,

[1]For explanatory purposes, expression (5.5) distinguishes between the cost of consumption and investment goods from related transaction costs. Rigorously, one should define consumption and investment expenditure as $P_t(1+\Upsilon^c) c_t$ and $P_t(1+\Upsilon^x) x_t$.

while the role of liquidity is to reduce these costs. As in Chapter 2 one can proceed re-defining total real costs as follows:

$$\Upsilon^c\left(\omega_t^c, c_t, a_t^c\right) \equiv q_t^c c_t \tag{5.6}$$

$$\Upsilon^x\left(\omega_t^x, x_t, a_t^x\right) \equiv q_t^x x_t \tag{5.7}$$

$\forall t \geq 0$; where: q_t^c and q_t^x represent the *unitary* real transaction costs (i.e., real cost associated with *one unit* of good) for consumption and investment, respectively. Compatibly with Feenstra (1986) specification (see Chapter 2), the following functional form is adopted for unitary costs:

$$q_t^c = \omega_t^c \Omega_1^c \left(\frac{c_t}{a_t^c}\right)^{\Omega_2^c} \tag{5.8}$$

$$q_t^x = \omega_t^x \Omega_1^x \left(\frac{x_t}{a_t^x}\right)^{\Omega_2^x} \tag{5.9}$$

$\forall t \geq 0$; where, as before, superscripts 'c' and 'x' characterising elements of total transaction cost functions applying to *consumption* and *investment*. As in Chapter 2, ω_t is a unit transaction cost shock, while a_t denotes liquidity in real terms (described below); $\Omega_1 > 0$ is a scale parameter and $\Omega_2 > 0$ is an elasticity parameter. As anticipated in the introduction, the presence of transaction costs for investment represents the major contribution of this chapter. However, the modelling strategy adopted here - namely, to cosider two *separate* transaction costs functions - is not the *only* possibility for extending transaction costs to investment. An alternative way could have been to consider a unique transaction cost function, where transaction costs were proportional to *total* expenditures in the goods market[2]. The specification

[2]In that case one possible specification for the transaction cost function, in real terms, would have been:

$$\Upsilon\left(\omega_t, c_t, x_t, a_t\right) = q_t\left(c_t + x_t\right)$$

$\forall t \geq 0$. With unitary transaction costs defined as:

$$q_t = \omega_t \Omega_1 \left(\frac{c_t + x_t}{a_t}\right)^{\Omega_2}$$

adopted here, instead, treats consumption and investment purchases - inclusive of transaction costs - 'separately'. Moreover, the distinction between a_t^c and a_t^x allows households to share optimally the allocation of total liquidity between consumption and investment expeditures, according to their relative opportunity (and transaction) costs. All these elements suggest that a *separable* specification looks a reasonable starting point. If one wants to preserve the comparability with the extended CIA model in Chapter 4, specifications represent by (5.8) and (5.9) then represent an (almost) necessary pre-requisite.

Given that the model deals with an homogeneous good, there is no reason why consumption and investment purchases should entail different transaction cost functions. As a consequence, the parameters appearing in (5.8) and (5.9) will satisfy the following conditions: $\Omega_1^c = \Omega_1^x = \Omega_1$ and $\Omega_2^c = \Omega_2^x = \Omega_2$. For the same reason, a shock occurring at the transaction costs level is going to affect consumption and investment purchases symmetrically: this corresponds to set $\omega_t^c = \omega_t^x = \omega_t$. As in the previous chapters, the transaction cost shock is assumed to follow the first-order autoregressive process

$$\log \omega_t = (1 - \rho_\omega) \log \omega + \rho_\omega \log \omega_{t-1} + \epsilon_{\theta_t} \tag{5.10}$$

$\forall t \geq 0$ and $0 \leq \rho_\omega \leq 1$, with ϵ_{ω_t} being a random variable serially uncorrelated and normally distributed, with zero mean and constant variance $(\sigma^2_{\epsilon_\omega})$.

The parameter restrictions, together with the definitions (5.6) and (5.7), and the functional forms (5.8) and (5.9), give rise to the following *total* transaction costs (in real terms):

$$\Upsilon^c(\omega_t, c_t, a_t^c) = \omega_t \Omega_1 \frac{(c_t)^{\Omega_2+1}}{(a_t^c)^{\Omega_2}} \tag{5.11}$$

$$\Upsilon^x(\omega_t, x_t, a_t^x) = \omega_t \Omega_1 \frac{(x_t)^{\Omega_2+1}}{(a_t^c)^{\Omega_2}} \tag{5.12}$$

$\forall t \geq 0$.

In the case of the cash-in-advance constraint, it was not simply money

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$\forall t \geq 0$.

held from the previous period that mattered in financing consumption expenditures, but all the financial resources available at the beginning of the period. Analogously, in the real-resource-cost context, one can assume that total liquidity reduces transaction costs. For this purpose the auxiliary variables, A_t^c and A_t^x, are introduced to denote total liquidity in the hands ot the household at the beginning of period t - i.e., before the opening of the goods market. As a result, the liquidity constraint for the extended RRC model can be defined (in nominal terms) as:

$$M_{t-1}^d + (1 + i_{t-1}) B_{t-1}^d - B_t^d + T_t^H \geq A_t^c + A_t^x \tag{5.13}$$

$\forall t \geq 0$. In addition to the cash balances that the individuals decided to hold from the previous period (M_{t-1}^d), a variety of liquidity sources do appear on the right hand side of (5.13). The portfolio adjustment involving bonds is given by the market timing assumption of the financial markets opening before the goods markets. In case of a positive interest rate, this implies that households will decide to hold an amount of cash balances just sufficient to reduce the transaction cost of the desired transactions.

The novelty with respect to the RRC model presented in Chapters 2 and 3 (where transaction costs were associated with consumption goods only) consists in the presence of two 'types' of liquidity on the right hand side of (5.13). This is due to the fact that households decide to allocate part of total liquidity (A_t^c) to reduce transaction costs of consumption, and the remaining part (A_t^x) to reduce transaction costs associated with the purchase of investment goods. One can derive the liquidity constraint in real terms by dividing both sides of (5.13) by P_t:

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t^H \geq a_t^c + a_t^x \tag{5.14}$$

$\forall t \geq 0$; where: $b_t^d \equiv B_t^d / P_t$ denotes the real value of (risk free) bonds holdings; $I_t \equiv (1 + i_t)$ is the one-period nominal (gross) interest rate; $\tau_t^H \equiv T_t^H / P_t$ represents the household's monetary transfer; $m_t^d \equiv M_t^d / P_t$ is individual demand for real balances; $\Pi_t \equiv P_t / P_{t-1}$ is defined as the (gross) inflation rate.

As in previous chapters, real investment is denoted by x_t and is defined as a change in the capital stock, net of capital depreciation:

$$x_t = k_t^s - (1-\delta)\,k_{t-1}^s \tag{5.15}$$

$\forall t \geq 0$, where $0 < \delta < 1$ represents the (constant) real depreciation rate of capital. In order to emphasise the fact that in each period households will be choosing the optimal level of capital stock to rent (k_t^s), one can use (5.15) into (5.5) to obtain:

$$\begin{aligned} &M_{t-1}^d + (1+i_{t-1})\,B_{t-1}^d + W_t h_t^s + T_t^H + P_t r_t^k k_{t-1}^s + P_t\,(1-\delta)\,k_{t-1}^s \\ &\geq P_t c_t + P_t k_t^s + M_t^d + B_t^d + P_t \Upsilon^c\,(\omega_t, c_t, a_t^c) + P_t \Upsilon^x\left(\omega_t, k_t^s, k_{t-1}^s, a_t^x\right) \end{aligned} \tag{5.16}$$

The right hand side of (5.16) represents the individual's total nominal wealth within the period t. This encompasses financial wealth accumulated in the previous period, the capital stock (net of real depreciation) owned during the period ($P_t\,(1-\delta)\,k_{t-1}^s$), labour income ($W_t h_t^s$), capital rental payments ($P_t r_t^k k_{t-1}^s$) and the exogenous lump sum transfers (T_t^H). Financial wealth is given by the nominal value of a portfolio of financial assets, namely bonds and cash balances from period $t-1$, inclusive of interest earnings (i_{t-1}) from bonds holdings.

Dividing both sides of (5.16) by the price level (P_t), the household budget constraint can be re-written in real terms as

$$\begin{aligned} &\frac{I_{t-1} b_{t-1}^d}{\Pi_t} + \frac{m_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t^H + r_t^k k_{t-1}^s + (1-\delta)\,k_{t-1}^s \\ &\qquad \geq c_t + k_t^s + m_t^d + b_t^d + \Upsilon^c\,(\omega_t, c_t, a_t^c) + \Upsilon^x\left(\omega_t, k_t^s, k_{t-1}^s, a_t^x\right) \end{aligned} \tag{5.17}$$

$\forall t \geq 0$; where: $b_t^d \equiv B_t^d/P_t$ denotes the real value of (risk free) bonds holdings; $I_t \equiv (1+i_t)$ is the one-period nominal (gross) interest rate; $\tau_t^H \equiv T_t^H/P_t$ represents the houshold's monetary transfer; $m_t^d \equiv M_t^d/P_t$ is individual demand for real balances; $w_t \equiv W_t/P_t$ indicates *real* wage and $\Pi_t \equiv P_t/P_{t-1}$ is defined as the (gross) inflation rate.

Resources not used in period t are saved in the form of bonds, cash balances and capital stock, whose command over goods will become effective only in the following period. Since this is true for every period (5.17) shows that the portfolio allocation decisions taken at time $t-1$ do in fact expose the real value of financial savings to changes in the price level from $t-1$ to t. As emphasised in Chapter 3, the same is not true for the real value of capital stock: infact the capital stock inherited from the previous period (k_{t-1}) and its proportional real return ($r_t^k k_{t-1}$) are evaluated at the current price level (P_t).

At time t the problem of the household is inherently dynamic: to choose optimally state-contingent claims for consumption (c_t), labour supply (h_t^s), bonds holdings (b_t^d), money stock (m_t^d), capital stock (k_t^s), *consumption*-liquidity (a_t^c) and *investment*-liquidity. The optimisation will be carried out in order to maximise the expected utility (5.3), subject to the budget constraint (5.17) and the liquidity constraint (5.14)[3].

5.2 Firms

The economy is populated by a large number of identical firms. As in Chapter 3, firms produce an homogeneous good buying labour (working hours) and renting capital from households. The real output produced in period t can be expressed by the following production function:

$$y_t = f\left[z_t, h_t^d, k_{t-1}^d\right] \tag{5.18}$$

$\forall t \geq 0$, where: y_t denotes *real* output; h_t^d are working hours and k_{t-1}^d is the capital stock (where the superscript d is meant to indicate 'demand'); z_t represents the 'level' of technology.

In order to obtain a direct correspondance between the behaviour of individual firms and their aggregate counterpart, the production function is represented by a constant returns to scale technology. To satisfy this condition, the production technology is assumed to be a *Cobb-Douglas* type:

$$y_t = z_t \left(k_{t-1}^d\right)^{\alpha} \left(h_t^d\right)^{1-\alpha} \tag{5.19}$$

$\forall t \geq 0$, where $0 < a < 1$ represents the capital share. In deference to

[3] Moreover, *no-Ponzi game* conditions must hold to guarantee optimality.

the RBC literature, the variable z_t represents *total* factor productivity. This variable evolves exogenously according to the law of motion

$$\log z_t = (1-\rho_z)\log z + \rho_z \log z_{t-1} + \epsilon_{z_t} \tag{5.20}$$

$\forall t \geq 0$; where: ρ_z is the autoregressive coefficient (with $0 \leq \rho_z \leq 1$), and ϵ_{z_t} is a random variable serially uncorrelated and normally distributed, with zero mean and constant variance ($\sigma^2_{\epsilon_z}$).

In period t firms sell their product in a perfectly competitive goods market, taking the price P_t of the homogenous good as given. Analogously, given the nominal wage W_t, they buy labour services from households in a perfectly competitive labour market. Firms rent capital (k_{t-1}) from households, at the cost of a proportional rental rate ($P_t r^k_t k_{t-1}$). In order to decide how much to produce - and, consequently, how much labour to hire and capital to rent - firms maximise the following profit function:

$$\Gamma_t = P_t y_t - W_t h^d_t - P_t r^k_t k^d_{t-1} \tag{5.21}$$

$\forall t \geq 0$; where nominal profits (Γ_t) are defined as the difference between nominal revenues ($P_t y_t$) and nominal costs ($W_t h^d_t + P_t r^k_t k^d_{t-1}$). Note that the assumptions of perfect competition and constant returns to scale do imply that the representative firm makes zero profits in equilibrium.

In every period t each firm solves a static problem: that of choosing working hours (h^d_t) and capital (k^d_{t-1}) which maximize profits (Γ_t) subject to the technology constraint (5.19).

5.3 Government

As in the previous chapters, the government operates as monetary and fiscal authority and its revenues and outlays in period t are combined in the following flow budget constraint (expressed in nominal terms):

$$M^s_t - M^s_{t-1} + B^g_t - (1+i^g_{t-1})B^g_{t-1} = P_t g_t + T_t \tag{5.22}$$

$\forall t \geq 0$, where: B^g_t denotes the face value of government debt outstanding, which pays a one-period nominal (net) interest rate i^g_t; T_t indicates governmental nominal lump sum transfers; M_t^s represents aggregate money supply; and g_t denotes real government consumption.

Since the focus here is on studying the impact of monetary shocks and not the impact of changes in government spending, g_t is set to zero (for all t periods). Moreover, Ricardian equivalence holds in this model. Therefore one can assume, with no loss of generality, that $B_0^g = 0$[4]. All together these assumptions imply that no government bonds are held in this economy and the government budget constraint then reduces to

$$M_t^s - M_{t-1}^s = T_t \tag{5.23}$$

in each period t.

$\forall t \geq 0$. Dividing both sides of (4.16) by the price level P_t, one obtains the equivalent expression in real terms :

$$m_t^s - \frac{m_{t-1}^s}{\Pi_t} = \tau_t \tag{5.24}$$

$\forall t \geq 0$; where: $\tau_t \equiv T_t/P_t$ represents real lump sum transfers; $m_t^s \equiv M_t^s/P_t$ is real money supply; and $\Pi_t \equiv P_t/P_{t-1}$ is the (gross) inflation rate.

The monetary authority is assumed to follow a constant money growth rule, according to which per capita nominal money supply is assumed to grow at the net rate θ_t in each period. This implies:

$$M_t^s = M_{t-1}^s + \theta_t M_{t-1}^s \tag{5.25}$$

$\forall t \geq 0$. The money supply rule is implemented through monetary injections that take the form of lump sum transfers accoring to:

$$T_t = \theta_t M_{t-1}^s \tag{5.26}$$

or, in real terms,

$$\tau_t = \frac{\theta_t m_{t-1}^s}{\Pi_t} \tag{5.27}$$

[4]See Chapter 2 for a more detailed explanation.

$\forall t \geq 0$.

To study the effects of a monetary surprise, the variable θ_t is assumed to evolve according to the law of motion

$$\log \theta_t = (1 - \rho_\theta) \log \theta + \rho_\theta \log \theta_{t-1} + \epsilon_{\theta_t} \tag{5.28}$$

$\forall t \geq 0$; where: ρ_θ is the autoregressive coefficient (with $0 \leq \rho_\theta \leq 1$), and ϵ_{θ_t} is a random variable serially uncorrelated and normally distributed, with zero mean and constant variance ($\sigma^2_{\epsilon_\theta}$). With this specification, the average (net) growth rate of money supply chosen by the monetary authority is equal to θ.

5.4 The 'pipeline' model

The way in which monetary impulses are propagated to the private economy is modified using the 'pipeline' technology introduced in the previous chapter. The reader is invited to refer back to Chapter 4 (Section 4.4) for a detailed description of the implied transmission mechanism. Briefly: nominal monetary injections (T_t) by the government do not correspond to the monetary transfers received by households (denoted by T_t^H). This represents the fact that the process through which the 'new liquidity' enters circulation might take some time. This can be achieve by introducing a lag in the mechanism of monetary injections, according to the following (in nominal terms):

$$N_{t-1} + T_t = T_t^H + N_t \tag{5.29}$$

$\forall t \geq 0$. At the beginning of period t the 'pipeline' contains a given nominal amount (N_{t-1}) of money in circulation, augmented by the new money injected by the monetary authority (T_t). Total money accumulated on the left hand side of (5.29) is partially passed on to households in form of a monetary transfer (T_t^H), while the remaining part (N_t) is stored for next period. As in the previous chapter, the restriction $T_t^H > 0$ must be imposed to guarantee the existence of this particular type of transmission device. Moreover, to ensure a positive amount of money in the 'pipeline', the model assumes that the new cash entering the representative household's budget constraint is a function of the money stored in the 'pipeline' at the end of the previous period[5]. Therefore, households lump sum transfers can be expressed (in nominal terms) by the following:

$$T_t^H = \xi N_{t-1} \tag{5.30}$$

$\forall t \geq 0$, where the parameter $0 < \xi \leq 1$ represents the *share* of stored 'pipeline' money (N_{t-1}) passed on to households.

Dividing both sides of (5.29) by the price level, one obtains the equivalent expression in real terms

$$\frac{n_{t-1}}{\Pi_t} + \tau_t - \tau_t^H = n_t \tag{5.31}$$

$\forall t \geq 0$, where: n_{t-1} and n_t indicate the real value of 'pipeline' money at the beginning and at the end of period, respectively, with Π_t being the (gross) inflation rate; while τ_t and τ_t^H represent real *inflows* and *outflows*, respectively. Analogously, one can obtain (5.30) in real terms:

$$\tau_t^H = \xi \frac{n_{t-1}}{\Pi_t} \tag{5.32}$$

$\forall t \geq 0$.

5.5 The equilibrium

This section derives the equilibrium conditions which characterise the *extended* RRC model. The household's and firm's optimal choices will be derived, while the monetary policy rule, the 'pipeline' model and the necessary market clearing conditions for the general equilibrium will close the models. The optimisation problems will be stated in terms of the Lagrangian method and then solved for the first order conditions.

[5]For a detailed analysis of the consequences of violating these restrictions, see Chapter 4 (Section 4.4).

5.5.1 Households

To state households' problem in terms of the Lagrangian, it is useful to recall that the representative household in the extended CIA model seeks to maximise the utility stream

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\Phi}}{1-\Phi} + \Psi \frac{(1-h_t^s)^{1-\eta}}{1-\eta} \right) \tag{5.33}$$

, subject to the budget constraint

$$\begin{aligned} \frac{I_{t-1} b_{t-1}^d}{\Pi_t} + \frac{m_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t^H + r_t^k k_{t-1}^s + (1-\delta) k_{t-1}^s \\ \geq c_t + k_t^s + m_t^d + b_t^d + \Upsilon^c (\omega_t, c_t, a_t^c) + \Upsilon^x \left(\omega_t, k_t^s, k_{t-1}^s, a_t^x \right) \end{aligned} \tag{5.34}$$

$\forall t \geq 0$, and the liquidity constraint

$$\frac{m_{t-1}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t^H \geq a_t^c + a_t^x \tag{5.35}$$

$\forall t \geq 0$[6].

Stating the problem in terms of Lagrangian method, the households choose c_t, h_t^s, b_t^d, m_t^d, a_t^c, a_t^x and k_t in order to maximise

$$\begin{aligned} \mathcal{L}_t = E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \left(\frac{c_t^{1-\Phi}}{1-\Phi} + \Psi \frac{(1-h_t^s)^{1-\eta}}{1-\eta} \right) \\ & + \lambda_t \Bigg[\frac{I_{t-1} b_{t-1}^d}{\Pi_t} + \frac{m_{t-1}^d}{\Pi_t} + w_t h_t^s + \tau_t^H + r_t^k k_{t-1}^s + (1-\delta) k_{t-1}^s \\ & - c_t - k_t^s - m_t^d - b_t^d - \Upsilon^c (\omega_t, c_t, a_t^c) - \Upsilon^x \left(\omega_t, k_t^s, k_{t-1}^s, a_t^x \right) \Bigg] \\ & + \mu_t \left[\frac{m_{t-t}^d}{\Pi_t} + \frac{I_{t-1} b_{t-1}^d}{\Pi_t} - b_t^d + \tau_t^H - a_t^c - a_t^x \right] \Bigg\} \Bigg\} \end{aligned}$$

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[6]See footnote (17) in Chapter 2 for a discussion about replacing inequalities with equalities in the constraints.

, where λ_t and μ_t are the Lagrangian multipliers associated with the budget constraint (inclusive of real transaction costs) and the liqiduidity constraint, respectively.

The maximisation of the Lagrangian with respect to the choice variables delivers the following optimality conditions:

$$\frac{\Psi\left(1-h_t^s\right)^{-\eta}}{c_t^{-\Phi}}=\frac{w_t}{\left[1+\Upsilon_{c,t}^c\right]} \tag{5.36}$$

$$1=\beta E_t\left\{\left(\frac{c_{t+1}}{c_t}\right)^{-\Phi}\left(\frac{1+\Upsilon_{c,t}^c}{1+\Upsilon_{c,t+1}^c}\right)\frac{I_{t+1}}{\Pi_{t+1}}\right\} \tag{5.37}$$

$$1=\beta E_t\left\{\left(\frac{c_{t+1}}{c_t}\right)^{-\Phi}\left(\frac{1+\Upsilon_{c,t}^c}{1+\Upsilon_{c,t+1}^c}\right)\left(\frac{E_tR_{t+1}^k-E_t\Upsilon_{k2,t+1}^x}{1+\Upsilon_{k1,t}^x}\right)\right\} \tag{5.38}$$

$$\Upsilon_{a,t}^c=1-I_t \tag{5.39}$$

$$\Upsilon_{a,t}^x=1-I_t \tag{5.40}$$

$$\begin{aligned}\frac{I_{t-1}b_{t-1}^d}{\Pi_t}+\frac{m_{t-1}^d}{\Pi_t}+w_th_t^s+\tau_t^H+r_t^kk_{t-1}^s+(1-\delta)\,k_{t-1}^s\\ =c_t+k_t^s+m_t^d+b_t^d+\Upsilon^c\left(\omega_t,c_t,a_t^c\right)+\Upsilon^x\left(\omega_t,k_t^s,k_{t-1}^s,a_t^x\right)\end{aligned} \tag{5.41}$$

$$\frac{m_{t-1}^d}{\Pi_t}+\frac{I_{t-1}b_{t-1}^d}{\Pi_t}-b_t^d+\tau_t^H=a_t^c+a_t^x \tag{5.42}$$

$\forall t\geq 0$, where:

$$\Upsilon^c\left(\omega_t,c_t,a_t^c\right)\equiv q_t^cc_t \tag{5.43}$$

$$\Upsilon^x\left(\omega_t,k_t^s,k_{t-1}^s,a_t^x\right)\equiv q_t^xx_t \tag{5.44}$$

$$q_t^c = \omega_t \Omega_1 \left(v_t^c\right)^{\Omega_2} \tag{5.45}$$

$$q_t^x = \omega_t \Omega_1 \left(v_t^x\right)^{\Omega_2} \tag{5.46}$$

$$v_t^c \equiv \frac{c_t}{a_t^c} \tag{5.47}$$

$$v_t^x \equiv \frac{x_t}{a_t^x} \tag{5.48}$$

$\forall t \geq 0$. The partial derivatives of total transaction costs functions $\Upsilon^c\left(\omega_t, c_t, a_t^c\right)$ and $\Upsilon^x\left(\omega_t, k_t^s, k_{t-1}^s, a_t^x\right)$ with respect to the arguments c_t, k_t, a_t^c, and a_t^x are denoted by

$$\Upsilon_{c,t}^c = \left(\Omega_2 + 1\right) q_t^c \tag{5.49}$$

$$\Upsilon_{k1,t}^x = \left(\Omega_2 + 1\right) q_t^x \tag{5.50}$$

$$E_t \Upsilon_{k2,t+1}^x = \left(\Omega_2 + 1\right)\left(1 - \delta\right) E_t q_{t+1}^x \tag{5.51}$$

$$\Upsilon_{a,t}^c = -\Omega_2 q_t^c v_t^c \tag{5.52}$$

$$\Upsilon_{a,t}^x = -\Omega_2 q_t^x v_t^x \tag{5.53}$$

$\forall t \geq 0$. The indexes $k1$ and $k2$ indicate the marginal transacion costs of adjusting capital in the current period and in the next period, respectively[7].

Expression (5.36) is the *intra*temporal condition for the RRC model, which relates the marginal rate of substitution between leisure and consumption (left hand side) to the ratio of the respective marginal costs (right hand side). As in Chapters 2 and 3, the marginal cost of consumption is represented by the real cost of one unit of consumption plus marginal transaction costs ($\Upsilon_{c,t}$). Expres-

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[7] Moreover, *transversality conditions* must hold to guarantee optimality.

sion (5.37) refers to the *inter*temporal condition, which governs the degree of consumption smoothing through time. This depends on the (present and expected) marginal cost of consumption ($\Upsilon_{c,t}$ and $E_t\Upsilon_{c,t+1}$) and the opportunity cost of saving in the next period ($E_tI_{t+1}/E_t\Pi_{t+1}$). Equation (3.53) represents the relation between the return on capital, its marginal transaction costs and the stochastic discount factor. This expression shares the basic idea of the extended cash-in-advance constraint developed in Chapter 4, where the marginal cost of capital was represented by the nominal interest rate instead. Expressions (5.39) and (5.40) represent the arbitrage conditions between bonds with liquidity devoted to reduce consumption and investment transaction costs, respectively. The reduction in the marginal transaction costs by holding liquidity for these two purposes is equated *in both cases* to the (unique) nominal interest rate on bonds (i_t), on the respective right hand sides. This implies that, in equilibrium, the marginal benefits of money in reducing the two types of transaction costs must be equal: $\Upsilon^c_{a,t} = \Upsilon^x_{a,t}$. Equations (5.41) and (5.42) are simply the two constraints, obtained by derivation with respect to the Lagrangian multipliers.

5.5.2 Firms

In each period, the representative firm chooses the amount of working hours (h^d_t) and rent capital stock (k^d_{t-1}) that maximise the profit function

$$\Gamma_t = P_t e^{z_t} h^d_t - W_t h^d_t - P_t r^k_t k^d_{t-1} \tag{5.54}$$

or, in real terms,

$$\gamma_t = e^{z_t} h^d_t - w_t h^d_t - r^k_t k^d_{t-1} \tag{5.55}$$

$\forall t \geq 0$. Given the production technology

$$y_t = z_t \left(k^d_{t-1}\right)^\alpha \left(h^d_t\right)^{1-\alpha} \tag{5.56}$$

and considering a generic time period t, one obtains the following first order conditions:

$$\frac{\partial \gamma_t}{\partial h_t^d} = e^{z_t} \left(k_{t-1}^d\right)^{\alpha} (1-\alpha) \left(h_t^d\right)^{-\alpha} - w_t = 0 \tag{5.57}$$

$$\frac{\partial \gamma_t}{\partial k_{t-1}^d} = e^{z_t} \alpha \left(k_{t-1}^d\right)^{\alpha-1} \left(h_t^d\right)^{1-\alpha} - r_t^k = 0 \tag{5.58}$$

for labour and capital, respectively.

Making use of the production function (5.56) and the definition of the gross return of capital $R_t^k \equiv r_t^k + 1 - \delta$, conditions (5.57) and (5.58) can be re-written as

$$(1-\alpha) \frac{y_t}{h_t^d} = w_t \tag{5.59}$$

$$\alpha \frac{y_t}{k_{t-1}} + 1 - \delta = R_t^k \tag{5.60}$$

As in the previous chapters, the optimality condition (5.59) implies that firms demand working hours up to the point where marginal product of labour equals its marginal cost (i.e., the real wage w_t); while expression (5.60) implies that they rent capital up to the point where its marginal product equals the marginal cost (represented by the net rental rate $r_t^k = R_t^k - (1-\delta)$).

5.5.3 Monetary policy and monetary injections

In period t real cash balances evolve to satisfy the government's budget constraint

$$\frac{m_{t-1}^s}{\Pi_t} + \tau_t = m_t^s$$

$\forall t \geq 0$. Given the money growth rate (θ_t) choosen by the monetary authority for period t, recall the characterization of the real lumpsum transfers as

$$\tau_t = \frac{\theta_t m_{t-1}^s}{\Pi_t}$$

$\forall t \geq 0$.

As described in the previous section, the monetary policy 'channel' is modified in order to incroporate a transmission mechanism. This is done through the 'pipeline' device (in real terms):

$$\frac{n_{t-1}}{\Pi_t} + \tau_t - \tau_t^H = n_t$$

$\forall t \geq 0$. Real outflows (τ_t^H) are represented by the following expression

$$\tau_t^H = \xi \frac{n_{t-1}}{\Pi_t}$$

$\forall t \geq 0$, where $0 < \xi \leq 1$ represents a calibrated 'pipeline valve'.

5.5.4 Market clearing conditions

For a general equilibrium characterisation of the model, the necessary market clearing conditions are required. In this model there are five markets: the goods market, the labour market, the money market, the bonds market and the capital market.

The capital market clears according to

$$k_{t-1}^d = k_{t-1}^s \tag{5.61}$$

$\forall t \geq 0$.

The labour market clearing condition equates labour demand and labour supply for every period, according to

$$h_t^d = h_t^s \tag{5.62}$$

$\forall t \geq 0$.

In every period *total* money supply (M_t^s) has to be equal to *total* money demand. But total money demand now corresponds to the cash held by households (M_t^d) *plus* the liquidity stored in the 'pipeline' (N_t). Therefore, the resulting market clearing condition will differ with respect to Chapters 2 and 3. In real terms, the equilibrium in the money market is achieved (in real terms) when

$$m_t^d + n_t = m_t^s \tag{5.63}$$

$\forall t \geq 0$.

Since the bonds in this model are private bonds 'issued' by households, the assumption that all the individuals are alike implies that no bonds are actually exchanged in equilibrium. As a consequence, there will be no bonds outstanding (i.e., a zero net supply for this type of financial assets). Thus, the bond market clearing condition corresponds to:

$$b_t^d = b_t^s = 0 \tag{5.64}$$

$\forall t \geq 0$.Given that real resources must be used up in transactions, total output (y_t) now must be equal to:

$$y_t = c_t + \Upsilon^c\left(\omega_t, c_t, a_t^c\right) + x_t + \Upsilon^x\left(\omega_t, x_t, a_t^x\right) \tag{5.65}$$

$\forall t \geq 0$. Recalling the definition of unit transaction costs, one can rewrite (5.65) as

$$y_t = c_t\left(1 + q_t^c\right) + x_t\left(1 + q_t^x\right) \tag{5.66}$$

$\forall t \geq 0$, where $x_t = k_t^s - (1-\delta)\, k_{t-1}^s$ represents net investment in period t. Expression (5.66) reveals another important assumption of this model: real resources devoted to the exchange have the same 'nature' of the good involved in the transaction itself. In order to stress the fact that, in this way, the national income identity preserves its original meaning - i.e., aggregate demand corresponds to total aggregate consumption and investment -, one can re-write the last condition as:

$$y_t = C_t + X_t \tag{5.67}$$

$\forall t \geq 0$; where $C_t \equiv c_t\left(1 + q_t^c\right)$ and $X_t \equiv x_t\left(1 + q_t^x\right)$ indicate *total* consumption and investment, respectively.

5.6 The dynamics

In order to explore and compare the dynamic performance of the extended RRC model, subject to the random shocks described above, one needs to transform the non-linear system of equations characterising the general equilibrium into a linear system. This is done by taking a log-linear approximation around the deterministic steady state, applying the methodology described in Uhlig (1999). For each model, this section will take the following steps: firstly, presenting the equilibrium as obtained in the previous section; secondly, illustrating some steady state relationships; and finally by deriving the log-linear model.

The set of optimality conditions for households and firms, together with the specification of monetary policy and the necessary market clearing conditions characterise the dynamic general equilibrium model as a system of non-linear equations:

$$\frac{\Psi\left(1-h_t^s\right)^{-\eta}}{c_t^{-\Phi}} = \frac{w_t}{\left[1+\Upsilon_{c,t}^c\right]} \tag{5.68}$$

$$1 = \beta E_t \left\{ \left(\frac{c_{t+1}}{c_t}\right)^{-\Phi} \left(\frac{1+\Upsilon_{c,t}^c}{1+\Upsilon_{c,t+1}^c}\right) \frac{I_{t+1}}{\Pi_{t+1}} \right\} \tag{5.69}$$

$$1 = \beta E_t \left\{ \left(\frac{c_{t+1}}{c_t}\right)^{-\Phi} \left(\frac{1+\Upsilon_{c,t}^c}{1+\Upsilon_{c,t+1}^c}\right) \left(\frac{E_t R_{t+1}^k - E_t \Upsilon_{k2,t+1}^x}{1+\Upsilon_{k1,t}^x}\right) \right\} \tag{5.70}$$

$$1 - \left(1+\Upsilon_{a,t}^c\right) = I_t - 1 \tag{5.71}$$

$$1 - \left(1+\Upsilon_{a,t}^x\right) = I_t - 1 \tag{5.72}$$

$$\frac{m_{t-1}^d}{\Pi_t} + \tau_t^H = a_t^c + a_t^x \tag{5.73}$$

$$\Upsilon_t^c \equiv q_t^c c_t \tag{5.74}$$

$$\Upsilon_t^x \equiv q_t^x x_t \tag{5.75}$$

$$q_t^c = \omega_t \Omega_1 \left(v_t^c\right)^{\Omega_2} \tag{5.76}$$

$$q_t^x = \omega_t \Omega_1 \left(v_t^x\right)^{\Omega_2} \tag{5.77}$$

$$v_t^c \equiv \frac{c_t}{a_t^c} \tag{5.78}$$

$$v_t^x \equiv \frac{x_t}{a_t^x} \tag{5.79}$$

$$\Upsilon_{c,t}^c = \left(\Omega_2 + 1\right) q_t^c \tag{5.80}$$

$$\Upsilon_{k1,t}^x = \left(\Omega_2 + 1\right) q_t^x \tag{5.81}$$

$$E_t \Upsilon_{k2,t+1}^x = \left(\Omega_2 + 1\right)\left(1 - \delta\right) E_t q_{t+1}^x \tag{5.82}$$

$$\Upsilon_{a,t}^c = -\Omega_2 q_t^c v_t^c \tag{5.83}$$

$$\Upsilon_{a,t}^x = -\Omega_2 q_t^x v_t^x \tag{5.84}$$

$$\frac{y_t}{h_t^d} = w_t \tag{5.85}$$

$$\alpha \frac{y_t}{k_{t-1}^d} + 1 - \delta = R_t^k \tag{5.86}$$

$$y_t = e^{z_t} \left(k_{t-1}^d\right)^{\alpha} \left(h_t^d\right)^{1-\alpha} \tag{5.87}$$

$$\frac{m_{t-1}^s}{\Pi_t} + \tau = m_t^s \tag{5.88}$$

$$\tau_t \equiv \theta_t \frac{m_{t-1}^s}{\Pi_t} \tag{5.89}$$

$$\frac{n_{t-1}}{\Pi_t} + \tau_t - \tau_t^H = n_t \tag{5.90}$$

$$\tau_t^H = \xi \frac{n_{t-1}}{\Pi_t} \tag{5.91}$$

$$k_{t-1}^d = k_{t-1}^s \tag{5.92}$$

$$h_t^d = h_t^s \tag{5.93}$$

$$m_t^d + n_t = m_t^s \tag{5.94}$$

$$y_t = C_t + X_t \tag{5.95}$$

$$C_t \equiv c_t Q_t^c \tag{5.96}$$

$$X_t \equiv x_t Q_t^x \tag{5.97}$$

$$x_t = k_t - (1-\delta)\, k_{t-1} \tag{5.98}$$

$$\ln z_t = (1-\rho_z) \ln z + \rho_z \ln z_{t-1} + \epsilon_{z_t} \tag{5.99}$$

$$\ln \theta_t = (1-\rho_\theta) \ln \theta + \rho_\theta \ln \theta_{t-1} + \epsilon_{\theta_t} \tag{5.100}$$

$$\log \omega_t = (1-\rho_\omega) \log \omega + \rho_\omega \log \omega_{t-1} + \epsilon_{\theta_t} \tag{5.101}$$

$\forall t \geq 0$; where $\Upsilon_t^c \equiv \Upsilon^c(\omega_t, c_t, a_t^c)$ and $\Upsilon_t^x \equiv \Upsilon^x(\omega_t, x_t, a_t^x)$, indicate *total* transaction costs of consumption and investment, respectively; and $Q_t^c \equiv 1 + q_t^c$ and $Q_t^x \equiv 1 + q_t^x$ indicate gross transaction costs of consumption and investment, respectively.

5.6.1 Money demand and velocity of money

At time t there are *two* sources of money demand in this model: the demand for cash by households (m_t^d) and the quantity of money demanded (i.e., stored) by the 'pipeline' (n_t). Given that the latter is represented by a rigid transmission mechanism, the attention of this section will be focused mainly on the former.

After all markets have cleared, the application of Walras' law implies that the evolution of real balances in the hands of the households follows

$$\frac{m_{t-1}^d}{\Pi_t} + \tau_t^H = m_t^d \tag{5.102}$$

$\forall t \geq 0$. Combining (5.102) with (5.73) one obtains the following expression for households' demand for real balances:

$$m_t^d = a_t^c + a_t^x \tag{5.103}$$

$\forall t \geq 0$. This money demand function differs from the one appearing in the RRC models of Chapters 2 and 3, because of the distinction between liquidity used for consumption and investment purposes.

In order to derive the expression for the money demand for the extended RRC model, one needs to combine (5.71) and (5.72), (5.76) and (5.77), (5.78) and (5.79), (5.83) and (5.84), to obtain

$$a_t^c = c_t \left(\frac{\Omega_1 \Omega_2 \omega_t}{I_t - 1} \right)^{\frac{1}{\Omega_2 + 1}} \tag{5.104}$$

$$a_t^x = x_t \left(\frac{\Omega_1 \Omega_2 \omega_t}{I_t - 1} \right)^{\frac{1}{\Omega_2 + 1}} \tag{5.105}$$

$\forall t \geq 0$. Expression (5.104) and (5.105) represent the money demand for consumption and investment purposes, respectively.

Using (5.103) one finally obtains the following *total* money demand function:

$$m_t^d = (c_t + x_t) \left(\frac{\Omega_1 \Omega_2 \omega_t}{I_t - 1} \right)^{\frac{1}{\Omega_2 + 1}} \tag{5.106}$$

$\forall t \geq 0$. Expression (3.139) shows that real balances respond positively to the expenditure variables (c_t and x_t) and to the transaction costs shock (ω_t). As in Chapters 2 and 3, there is a negative relationship between real balances and the nominal interest rate ($i_t \equiv I_t - 1$). In addition to that, (5.106) implies a unitary elasticity with respect to consumption and investment[8].

As for the RRC model in Chapter 2 and 3, the *consumption*-based velocity of money is defined as

$$VEL\left(C\right)_t \equiv \frac{C_t}{m_t^s} \tag{5.107}$$

$\forall t \geq 0$[9]. where total consumption (C_t) includes real resources devoted to consumption transactions, while *output*-based velocity of money is defined as usual:

$$VEL\left(y\right)_t \equiv \frac{y_t}{m_t^s} \tag{5.108}$$

$\forall t \geq 0$. The novelty in this chapter is represented by the expression for *investment*-based velocity of money:

$$VEL\left(x\right)_t \equiv \frac{X_t}{m_t^s} \tag{5.109}$$

$\forall t \geq 0$, where total consumption (X_t) includes real resources devoted to investment transactions.

5.6.2 Steady state

Before turing to the (log-)linear system, it is useful to have a look at some long-run relationships implied by the model. When all the variables have reached their deterministic steady state, time subscripts can be 'removed' from the non-linear equations characterising the equilibrium. In this way it is possible to inspect how monetary factors impact on the fundamental structure of the economy.

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[8] Note that these do not correspond to *total* aggregate consumption and *total* aggregate investment, but just *desired* levels.

[9] For a discussion about the use of total money supply (m_t^s) in the definitions of velocity see footnote (27) in Chapter 2.

In steady state expressions (5.88) and (5.89) can be used to obtain

$$\Pi = \Theta \tag{5.110}$$

or, equivalently

$$\pi = \theta \tag{5.111}$$

The result indicates that the steady state inflation rate is determined by the money growth rate chosen by the monetary authority. As for the previous chapters, a direct implication of this is that the (steady state) real quantity of money (m) is constant - i.e., *neutrality of money* holds in steady state. Given the household's subjective discount rate (β), the *inter*temporal condition (5.69) can be used to determine the long-run nominal interest rate:

$$I = \frac{\Theta}{\beta} \tag{5.112}$$

In order to analyse the steady state properties of the RRC model one can start by finding the steady state value of v^c and v^x using (5.104) and (5.105), respectively:

$$\mathbf{v}^c = \left(\frac{I_t - 1}{\Omega_1\Omega_2}\right)^{\frac{1}{\Omega_2+1}} \tag{5.113}$$

$$\mathbf{v}^x = \left(\frac{I_t - 1}{\Omega_1\Omega_2}\right)^{\frac{1}{\Omega_2+1}} \tag{5.114}$$

It follows that $v^c = v^x$. it follows that the unitary transaction costs are also the same:

$$q^c = \Omega_1 \left(\frac{\Theta/\beta - 1}{\Omega_1\Omega_2}\right)^{\frac{\Omega_2}{\Omega_2+1}} \tag{5.115}$$

$$\mathbf{q}^x = \boldsymbol{\Omega}_1 \left(\frac{\Theta/\beta - 1}{\Omega_1\Omega_2}\right)^{\frac{\Omega_2}{\Omega_2+1}} \tag{5.116}$$

As in Chapters 2 and 3, it follows that a permanently higher money growth rate causes an increase in all the cost of transactions[10].

Combining (5.68) with (5.85) and the production function (5.87), all evaluated at the steady state, one obtains the following expression:

$$\frac{\Psi h^{\Phi}}{(1-h)^{\eta}}=(1-\alpha)\frac{1}{1+\Upsilon_c}\left(\frac{y}{k}\right)^{\frac{\Phi-\alpha}{1-\alpha}}\left(\frac{c}{k}\right)^{-\Phi} \tag{5.117}$$

Using the market clearing condition (5.95) one obtains that the consumption/capital ratio is equivalent to:

$$\frac{c}{k}=\frac{\left(\frac{y}{k}\right)}{Q}-\boldsymbol{\delta} \tag{5.118}$$

Deriving the expression for the output/capital ratio from (5.86) and using the expressions (5.118), (5.80), (5.81), (5.82), one can re-write (5.117) as:

$$\frac{h^{\Phi}}{(1-h)^{\eta}}=\boldsymbol{\Delta}\left(1+(\Omega_2+1)q\right)^{\frac{\Phi-1}{1-\alpha}}\left(\frac{(1/\beta-1+\delta)\left(1+(\Omega_2+1)q\right)}{\alpha\left(1+q\right)}-\delta\right)^{-\Phi} \tag{5.119}$$

where

$$\Delta\equiv\frac{(1-\alpha)}{\Psi}\left(\frac{1/\beta-1+\delta}{\alpha}\right)^{\frac{\Phi-\alpha}{1-\alpha}}$$

The left-hand side (LHS) of (5.119) is positively related with working hours. The value of the expression on the right hand side (RHS), instead, varies with the level of unitary transaction costs (q).

Since the assumption of balanced growth implies a log-utility function for consumption ($\Phi=1$), expression (5.119) reduces to the following:

$$\frac{h}{(1-h)^{\eta}}=\Delta\left(\frac{\alpha\left(1+q\right)}{(1/\beta-1+\delta)\left(1+(\Omega_2+1)q\right)-\delta\alpha\left(1+q\right)}\right)<0 \tag{5.120}$$

To determine the effect of an increase in q (due to a higher money growth rate) on working hours, one can differentiate the right-hand side (RHS) of

[10] Given the results in the equations (5.113) - (5.116), from now onwards the following notation will be used: $v^c=v^x=v$, $q^c=q^x=q$ and $Q^c=Q^x=Q$.

(5.120) with respect to q, in order to get the following:

$$\frac{d\left(RHS\right)}{dq}=-\Delta\frac{\alpha\Omega_2\left(1/\beta-1+\delta\right)}{\left[\left(1/\beta-1+\delta\right)\left(1+(\Omega_2+1)q\right)-\delta\alpha\left(1+q\right)\right]^2} \tag{5.121}$$

Given standard calibration ($\Delta>0$), and given log-utility, the sign of (5.121) is negative. This implies that the right-hand side (RHS) of (5.120) is lowered when unitary transaction costs increase. Therefore, according to (5.120), a higher money growth rate (Θ) increases q, decreasing labour supply (h).

Finally, using the production function, the value of capital at steady state and the exression for the output/capital ratio it is possible to show that real output is positively related to working hours according to:

$$\mathbf{y}=\left(\frac{\left(1/\beta-1+\delta\right)\left(1+(\Omega_2+1)q\right)}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}\mathbf{h} \tag{5.122}$$

Given that $0<\alpha<1$, considering the positive impact of a higher money growth rate on unitary transaction costs and the negative relationship between these and the labour supply, expression (5.122) implies that superneutrality of money does not hold in this model.

Summing up: a permanent increase in the inflation tax causes a substitution from cash to credit purchases: whatever the means of payment used by households, the purchase of consumption and investment goods is now more expensive. As in Chapter 2 and 3, the shift from cash to credit commands more real resources devoted to transactions; whether a higher money growth will imply an increase or a decrease in output is going to depend on the relative weight of both effects on labour supply (i.e., on the value of the parameters defining preferences and transaction technologies).

As in the previous chapters, these results shed some light on the monetary transmission mechanism: on the one hand, a higher money growth rate increases q^x discouraging investment; on the other hand it reduces labour supply, since households substitute consumption goods (made more expensive by an increase in q^c) with leisure. Clearly, both effects have a negative impact on real output.

5.6.3 Log-linear approximation

To inspect the dynamics of the CIA model, when the economy is subject to stochastic disturbances, the nonlinear system is transformed into a linear one, following the methodology decribed in Uhlig (1999). The result is a linear system of equations, where the variables denoted with *hats* indicate deviations from the point around which the first order Taylor expansion is evaluated, i.e., the steady state.

leisure/consumption:

$$\left[\eta\frac{h}{(1-h)}\right]\hat{h}_t^s = \hat{w}_t - \Phi\hat{c}_t - \hat{\Upsilon}_{c,t}^c \tag{5.123}$$

consumption/saving:

$$\Phi E_t\hat{c}_{t+1} - \Phi\hat{c}_t + E_t\hat{\Upsilon}_{c,t+1}^c - \hat{\Upsilon}_{c,t}^c = E_t\hat{\imath}_{t+1} - E_t\hat{\pi}_{t+1} \tag{5.124}$$

capital/bonds:

$$\begin{aligned} & (1+\Upsilon_{k1}^x)\,\Phi E_t\hat{c}_{t+1} - (1+\Upsilon_{k1}^x)\,\Phi\hat{c}_t + (1+\Upsilon_{k1}^x)\,E_t\hat{\Upsilon}_{c,t+1}^c && (5.125)\\ & -(1+\Upsilon_{k1}^x)\,\hat{\Upsilon}_{c,t}^c + (1+\Upsilon_{k1}^x)\,E_t\hat{\Upsilon}_{k1,t+1}^x && (5.126)\\ = \; & \beta R^k E_t\hat{r}_{t+1}^k - \beta\,(1+\Upsilon_{k2}^x)\,E_t\hat{\Upsilon}_{k2,t+1}^x && (5.127)\end{aligned}$$

marginal costs of liquidity for consumption:

$$-\left[1+\Upsilon_{a,t}^c\right]\hat{\Upsilon}_{a,t}^c = I\hat{\imath}_t \tag{5.128}$$

marginal costs of liquidity for investment:

$$-\left[1+\Upsilon_{a,t}^x\right]\hat{\Upsilon}_{a,t}^x = I\hat{\imath}_t \tag{5.129}$$

money demand:

$$m^d\hat{m}_t^d = a^c\hat{a}_t^c + a^x\hat{a}_t^x \tag{5.130}$$

total transaction costs of consumption:

$$\Upsilon^c \hat{\Upsilon}^c_t = cq^c \hat{c}_t + cQ^c \hat{q}^c_t \tag{5.131}$$

total transaction costs of investment:

$$\Upsilon^x \hat{\Upsilon}^x_t = cq^x \hat{x}_t + cQ^x \hat{q}^x_t \tag{5.132}$$

unit transaction costs of consumption:

$$Q\hat{q}^c_t = q^c \hat{\omega}_t + q^c \Omega^c_2 \hat{v}^c_t \tag{5.133}$$

unit transaction costs of investment:

$$Q\hat{q}^x_t = q^x \hat{\omega}_t + q^x \Omega^x_2 \hat{v}^x_t \tag{5.134}$$

consumption/liquidity ratio:

$$\hat{v}^c_t = \hat{c}_t - \hat{a}^c_t \tag{5.135}$$

investment/liquidity ratio:

$$\hat{v}^x_t = \hat{c}_t - \hat{a}^x_t \tag{5.136}$$

marginal transaction cost of consumption:

$$\left[1 + \Upsilon^c_c\right] \hat{\Upsilon}^c_{c,t} = \left[\Omega^c_2 + 1\right] Q^c \hat{q}^c_t \tag{5.137}$$

marginal transaction cost of capital today:

$$\left[1 + \Upsilon^x_{k1}\right] \hat{\Upsilon}^x_{k1,t} = \left[\Omega^x_2 + 1\right] Q^x \hat{q}^x_t \tag{5.138}$$

marginal transaction cost of capital tomorrow:

$$\left[1 + \Upsilon^x_{k2}\right] E_t \hat{\Upsilon}^x_{k2,t+1} = \left[\Omega^x_2 + 1\right] \left[1 - \delta\right] Q^x E_t \hat{q}^x_{t+1} \tag{5.139}$$

marginal transaction cost of liquidity for consumption:

$$\left[1 + \Upsilon^c_a\right] \hat{\Upsilon}^c_{a,t} = -\Omega^c_2 Q^c v^c \hat{q}^c_t - \Omega^c_2 q^c v^c \hat{v}^c_t \tag{5.140}$$

marginal transaction cost of liquidity for investment:

$$[1+\Upsilon_a^x]\,\hat{\Upsilon}_{a,t}^x = -\Omega_2^x Q^x v^x \hat{q}_t^x - \Omega_2^x q^x v^x \hat{v}_t^x \tag{5.141}$$

labour demand:

$$\hat{y}_t - \hat{h}_t^d = \hat{w}_t \tag{5.142}$$

marginal product of capital

$$\alpha\frac{y}{k}\hat{y}_t - \alpha\frac{y}{k}k_{t-1} = R^k r_t^k \tag{5.143}$$

real output:

$$\hat{y}_t = \hat{z}_t + \alpha\hat{k}_{t-1} + (1-\alpha)\,\hat{h}_t^d \tag{5.144}$$

money supply:

$$\frac{m^s}{\Pi}\hat{m}_{t-1}^s - \frac{m^s}{\Pi}\hat{\pi}_t + \tau\hat{\tau}_t = m^s\hat{m}_t \tag{5.145}$$

monetary injection:

$$\tau\hat{\tau}_t \equiv \Theta\frac{m^s}{\Pi}\hat{\theta}_t + \theta\frac{m^s}{\Pi}\hat{m}_{t-1}^s - \theta\frac{m^s}{\Pi}\hat{\pi}_t \tag{5.146}$$

labour market clearing condition:

$$k_{t-1}^d = k_{t-1}^s \tag{5.147}$$

labour market clearing condition:

$$\hat{h}_t^d = \hat{h}_t^s \tag{5.148}$$

money market clearing condition:

$$m^d\hat{m}_t^d + n\hat{n}_t = m^s\hat{m}_t^s \tag{5.149}$$

goods market clearing condition:

$$y\hat{y}_t = C\hat{C}_t + X\hat{X}_t \tag{5.150}$$

total consumption:

$$C\hat{C}_t \equiv c\hat{c}_t + \Upsilon^c \hat{\Upsilon}_t^c \tag{5.151}$$

total investment:

$$X\hat{X}_t \equiv x\hat{x}_t + \Upsilon^x \hat{\Upsilon}_t^x \tag{5.152}$$

investment

$$\hat{x}_t = \frac{1}{\delta}\hat{k}_t - \frac{(1-\delta)}{\delta}\hat{k}_{t-1} \tag{5.153}$$

consumption-based *velocity:*

$$VE\hat{L}\left(C\right)_t \equiv \hat{C}_t - \hat{m}_t^s \tag{5.154}$$

investment-based velocity:

$$VE\hat{L}\left(X\right)_t \equiv \hat{X}_t - \hat{m}_t^s \tag{5.155}$$

output-based velocity:

$$VE\hat{L}\left(y\right)_t \equiv \hat{y}_t - \hat{m}_t^s \tag{5.156}$$

technology shock:

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \epsilon_{z_t} \tag{5.157}$$

monetary shock:

$$\hat{\theta}_t = \rho_\theta \hat{\theta}_{t-1} + \epsilon_{\theta_t} \tag{5.158}$$

transaction cost shock:

$$\hat{\omega}_t = \rho_\omega \hat{\omega}_{t-1} + \epsilon_{\omega_t} \tag{5.159}$$

, where: , where $\epsilon_{z_t} \sim N\left(0, \sigma_{\epsilon_z}^2\right)$, $\ln \epsilon_{\theta_t} \sim N\left(\left(1-\rho_\theta\right)\theta, \sigma_{\epsilon_\theta}^2\right)$ and $\epsilon_{\omega_t} \sim N\left(0, \sigma_{\epsilon_\omega}^2\right)$.

Expressions (5.145) and (5.146), together with the money market clearing condition (5.149), can be used to deliver the expression for the inflation dynamics. However in this context, as in Chapter 4, the presence of the 'pipeline' re-defines money demand according to its components ($\hat{m}_t^d$ and $\hat{n}_t$):

$$\hat{\pi}_t = \hat{\theta}_t - \left[\left(\frac{m^d}{m^s} \hat{m}_t^d - \frac{m^d}{m^s} \hat{m}_{t-1}^d \right) + \left(\frac{n}{m^s} \hat{n}_t - \frac{n}{m^s} \hat{n}_{t-1} \right) \right] \quad (5.160)$$

$\forall t \geq 0$; where fluctuations in the inflation rate around its steady state value are determined by the difference between money supply *growth* ($\hat{\theta}_t$) and the *growth* in the money demand components. As in Chapter 4, money is *non*-neutral: if the monetary shock would not be persistent (i.e., $\rho_\theta = 0$) a change in the quantity of money *today* would still have real effects, because of the *delay* introduced by the 'pipeline' mechanism.

5.7 Quantitative analysis

In the first part of this section numerical values are assigned to structural parameters and long-run relationships. The remaining coefficients in the linear approximations are derived using the the steady state relationships, implied by the original non-linear system. Given these calibration values, the last part of this section will compare the qualitative and quantitative impact of the stochastic shocks on the endogenous variables of the two monetary models. In particular, the analysis will focus on the *impulse-response* dynamics and the relative match of the extended RRC model with respect to the empirical evidence presented in Chapter 1.

5.7.1 Calibration

In order to derive the response of the baseline models to stochastic shocks, one needs to assign numerical values to the parameters appearing in the linear equations. This chapter gives the opportunity of comparing the extended RRC model developed here with the extended CIA model of Chapter 4. As done for the comparisons in Chapters 2 and 3, the values for the parameters have been chosen in order to undertake a reliable comparison. In fact, the CIA model and the RRC model - even when considered in their extended versions - do share the same basic structure. The only dimension along which they differ concerns the way money is introduced. In particular, the calibration of the RRC model adopts the same numerical values used for the CIA model in Chapter 4, except for the transaction technology parameters.

The baseline calibration of the RRC model is reported in Table 5.1 below.

Five additional parameters appear in Table 5.1, when it is compared with Table 4.1 . The parameter Ω_2 is set in order to obain quadratic (total) transaction costs ($\Omega_2 = 1$). The key value for a comparison with the performance of the extended CIA model is represented by the *liquidity ratios* of the extended RRC model (v^c and v^x). Along this dimension the normalisation is done by imposing: $c + y = m^d$. In the extended CIA model households covered all consumption and investment expenditures with their money (m^d). The condition $(c + y)/m^d = 1$ here, is equivalent to assume that (in steady state) the representative household participates to the goods market, carrying an amount of real cash balances equal to the *desired* consumption and investment[11]. Given the assumption that transaction costs functions for consumption and investment do share the same parameters, the arbitrage conditions between liquidity and bonds imply: $v^c = v^x = 1$. This makes sense, given that in the liquidity constraint the two costs of *credit* (i.e., the real-resources devoted to consumption and investment transactions) 'compete' with the same nominal interest rate (i.e., the opportunity cost of liquidity). The parameters characterising the transaction costs shock, are taken from an exercise proposed by Sims (1989), where the autoregressive parameter for the transaction cost shock (ρ_ω) and its standard deviation (σ_{ϵ_ω}) are set to 0.8 and 0.01, respectively[12].

Log-utility for consumption and leisure imply setting the coefficients of relative risk aversion and the labour supply elasticity equal to unity ($\Phi = 1$, $\eta = 1$). The steady state labour supply has been set to one-third of the time endowment ($h = 0.33$). The parameter relative to the capital depreciation (δ) has been set in order to get a quarterly depreciation of 1.9%. The share of capital to total income ($\alpha = 0.36$) has been chosen in order to deliver a labour share of 2/3 and a capital share of 1/3. The value for the (quarterly)

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[11] In order to check for robustness of the results, a sensitivity analysis exercise has been carried out, using different values for Ω_2 and $(c + y)/m^d$. However, these attempts did not change significantly the quantitative results reported in the next sections.

[12] See Chapter 2, Section 2.51 for a discussion about the calibration of this type of shock.

discount factor ($\beta = 0.989$), the autoregressive coefficient of the technology shock ($\rho_z = 0.95$) and its standard deviation ($\sigma_{\epsilon_z} = 0.007$) are in line with the standard RBC literature reported in Chapter 2. The (exogenous) net nominal money growth rate is set to $\theta = 0.0125$; while the autoregressive parameter ($\rho_\theta = 0.5$) and standard deviation ($\sigma_{\epsilon_\theta} = 0.0089$) are derived from the estimation of an autoregressive process of M1 by Cooley and Hansen (1989). Finally, the parameter regulating the 'pipeline' (ξ) is initially set to 1.00. For a second run of simulations, this number will be replaced by 0.05 (reported in brackets in Table 5.1) with the purpose of conducting a sensitivity analysis exercise. This will help to inspect the quantitative implications of variations in the degree of rigidity in the money supply process.

One can use the parameter values assigned in Table 5.1 and the steady state relationships derived from the equilibrium to derive all the remaining coefficients of the linear system. Moreover, one can check whether the baseline calibration is able to generate steady state values (or ratios) compatible with the empirical evidence.

Table 5.2 shows that the baseline calibration results in a value for the unitary transaction costs in line with the nominal interest rate ($q = i = 2.38\%$), corresponding to a scale parameter $\Omega_1 = 0.0238$. Table 5.3 reports the (usual) result for long-run relationship between the money growth rate and inflation, while consumption and investment shares of output ($C/y = 0.7781$ and $X/y = 0.2219$, respectively) are also almost identical to those derived in Chapter 3 and 4. Because all output is now subject to transaction costs, the output share devoted to transaction activities is back to the levels found in the model without capital ($\Upsilon/y = 0.023$).

Table 5.4 reporst the steady state value for all monetary variables influenced by the values taken by ξ in the sensitivity analysis[13]. Because of the presence of the 'pipeline' technology for the transmission mechanism, *consumption*- and *investment*-based velocity are smaller than 1, like in the extended CIA model of Chapter 4. However, because of the specific calibration adopted for the *liquidity ratio* ($v^c = v^x = 1$) the *output*-based measure is bigger than 1. Finally,

[13]Reflecting the convention adopted in Chaper 4, the numbers in brackets correspond to the values derived when $\xi = 0.05$.

note that, because of the 'pipeline' device, the ratio between households real balances and total real balances (Ξ) varies between 0.9877 and 0.80, when ξ takes values 1 and 0.05, respectively.

parameter/variable	**description**	**value**
Φ	relative risk aversion	1
η	inverse of labour supply elasticity	1
β	discount factor	0.989
δ	capital depreciation rate	0.019
α	capital share of output	0.36
h	working hours	0.33
θ	net money growth rate	0.0125
ρ_z	autoregressive param. technology shock	0.95
σ_{ϵ_z}	s.d. technology shock	0.007
ρ_θ	autoregressive param. monetary shock	0.5
σ_{ϵ_θ}	s.d. monetary shock	0.0089
Ω_2	elasticity param. of transaction costs	1
υ^c	consumption/liquidity ratio	1
υ^x	investment/liquidity ratio	1
ρ_{ϵ_ω}	autoregressive param. transaction costs shock	0.8
σ_{ϵ_ω}	s.d. transaction costs shock	0.01
ξ	'pipeline' parameter	1.00 (0.05)

Table 5.1: Baseline calibration of (extended) RRC model.

parameter/variable	description	value
q^c	net unitary transaction costs of consumption	0.0238
q^x	net unitary transaction costs of investment	0.0238
Q^c	gross unitary transaction costs of consumption	1.0238
Q^x	gross unitary transaction costs of investment	1.0238
Υ_c^c	marg. trans. costs of consumption	0.0475
Υ_{k1}^x	marg. trans. costs of capital	0.0475
Υ_{k2}^x	marg. trans. costs of (net) capital	-0.0466
Υ_a^c	marg. trans. costs of liquidity for consumption	-0.0238
Υ_a^x	marg. trans. costs of liquidity for investment	-0.0238
Ω_1	scale param. of transaction costs	0.0238

Table 5.2: Steady state values of (extended) RRC model, at baseline calibration: transaction costs function(s).

parameter/variable	**description**	**value**
Θ	gross money growth rate	1.0125
Π	gross inflation rate	1.0125
π	net inflation rate	0.0125
I	gross nominal interest rate on bonds	1.0238
i	net nominal interest rate on bonds	0.0238
R^k	gross return on capital (net of deprec.)	1.0686
r^k	marginal product of capital	0.0686
y/k	output/capital ratio	0.0876
k	capital	14.9566
x	investment	0.2842
y	real output	1.3109
w	real wage	2.5170
c	real utility-consumption	0.9963
Υ^c	total transaction costs of consumption	0.0237
Υ^x	total transaction costs of investment	0.0068
C	total real consumption	1.0200
X	total real investment	0.2909
C/y	consumption share of output	0.7781
X/y	investment share of output	0.2219
Ψ	preference parameter for leisure	1.6078
Υ/y	transaction costs share of output	0.0232

Table 5.3: Steady state values of (extended) RRC model, at baseline calibration.

a^c	consumption-liquidity	0.9963
a^x	investment-liquidity	0.2842
m^d	households' (real) balances	1.2805
τ^H	households (real) monetary injection	0.0158
n	'pipeline' (real) balances	0.0160 (0.3320)
m^s	total (real) cash balances	1.2965 (1.6600)
τ	(real) monetary injection	0.0160 (0.0205)
Ξ	households' real balances share	0.9877 (0.8000)
C/m^s	*consumption*-based velocity	0.7867 (0.6226)
X/m^s	*investment*-based velocity	0.2244 (0.1774)
y/m^s	*output*-based velocity	1.0111 (0.8000)

Table 5.4: Steady state values of (extended) RRC model at baseline calibration: monetary variables.

5.7.2 Impulse-response analysis

In what follows the dynamic responses of the RRC model are analysed. The figures below report the percentage deviation of the selected variables from their steady state value (which, for convenience, has been set to zero). The deviation from steady state of variables which do represent rates (e.g., inflation rate, interest rate, unit transaction costs), is measured in absolute terms. All the shocks take place at time zero and the time scale refers to quarterly data.

Technology shock

Figure 5.1 shows the impact of the technology shock on real expenditure and money demand (households' real balances). Since the production function and the utility function have not been changed, the response of consumption and investment are the same as those derived for Chapters 3 and 4. The only difference with the previous models is represented by the dynamics of real balances. This is due to the fact that the money in this model is used to reduce transaction costs on consumption and investment. If this represents a similarity between the model developed here and the extended CIA model in Chapter 4, the analogy must be handled 'with care'. The difference lies in the money demand equation, where the 'weights' in front of consumption and investment differ with respect to those appearing in the extended cash-in-advance constraint. For this reason, the impact of the technology shock on the demand for real balances is *twice as* strong as Chapter 3, but it is *half* with respect to Chapter 4. For velocity considerations, it is worth noticing that consumption increases less than household real balances.

Figure 5.2 shows the impact of the technology shock on inflation and nominal interest rates. Inflation is generated as a difference between the growth in money supply and the growth in money demand. Therefore, if the government follows a constant money growth rule, a technology shock causes a temporary drop in inflation, *via* money demand dynamics.

Given the drop in inflation, the other real balances - namely, 'total' and 'pipeline' balances - follow the dynamics of households' real balances. These elements, together with the results described in Figure 5.1, can be used to explain the pro-cyclicality of *investment*- and *output*-based velocity and the anti-cyclical *consumption-based* velocity in Figure 5.4.

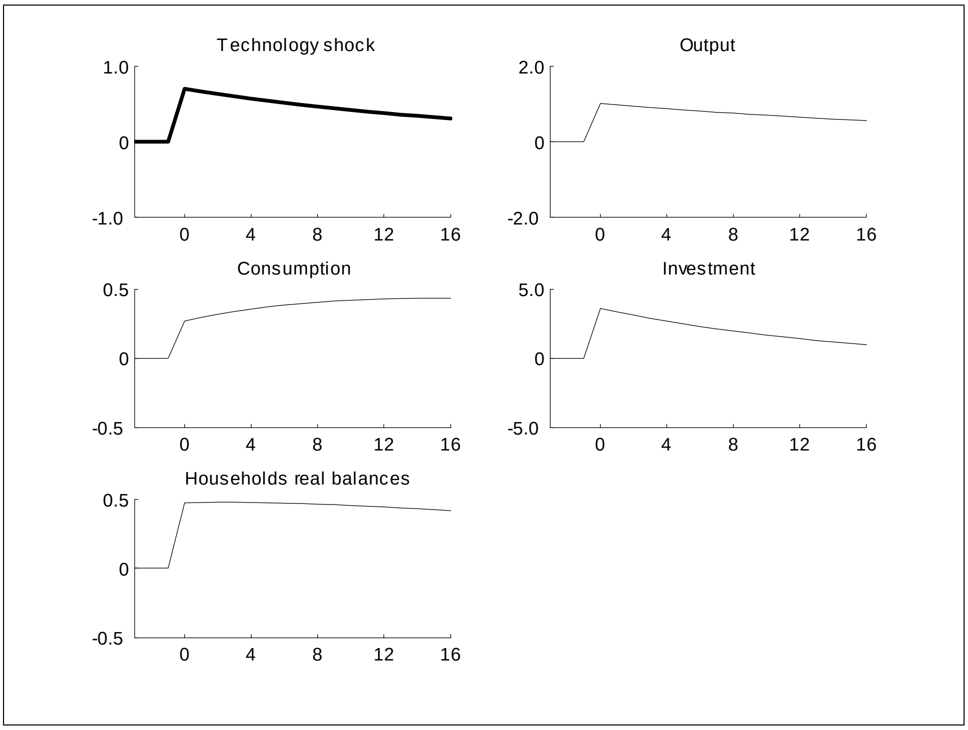

Figure 5.1: Impact of the technology shock on real expenditure (extended RRC)

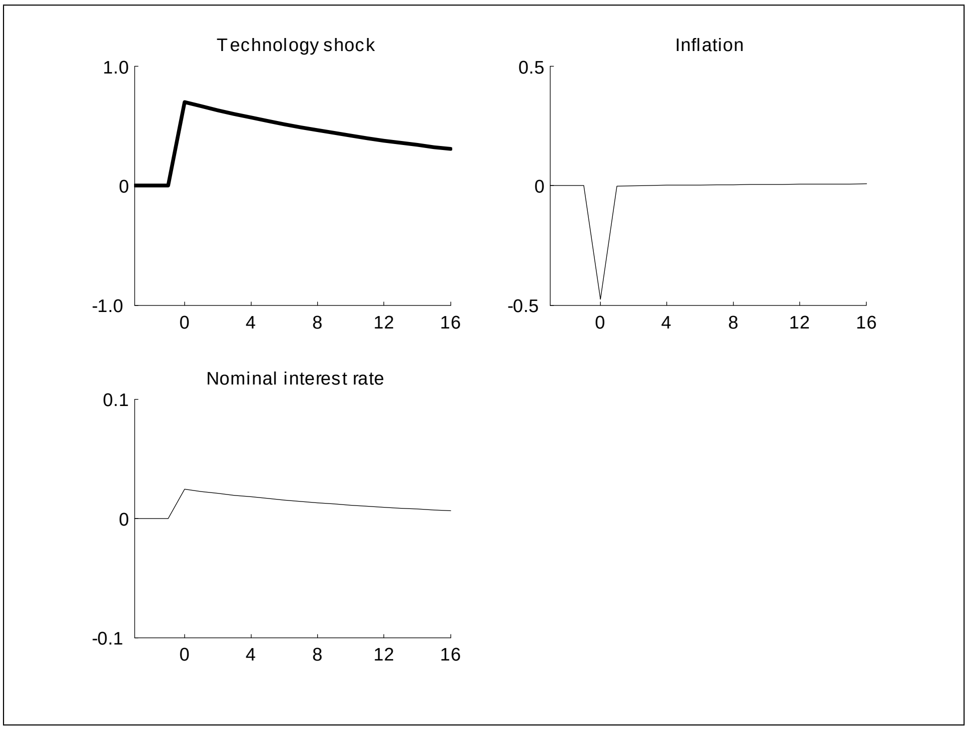

Figure 5.2: Impact of the technology shock on nominal variables (extended RRC)

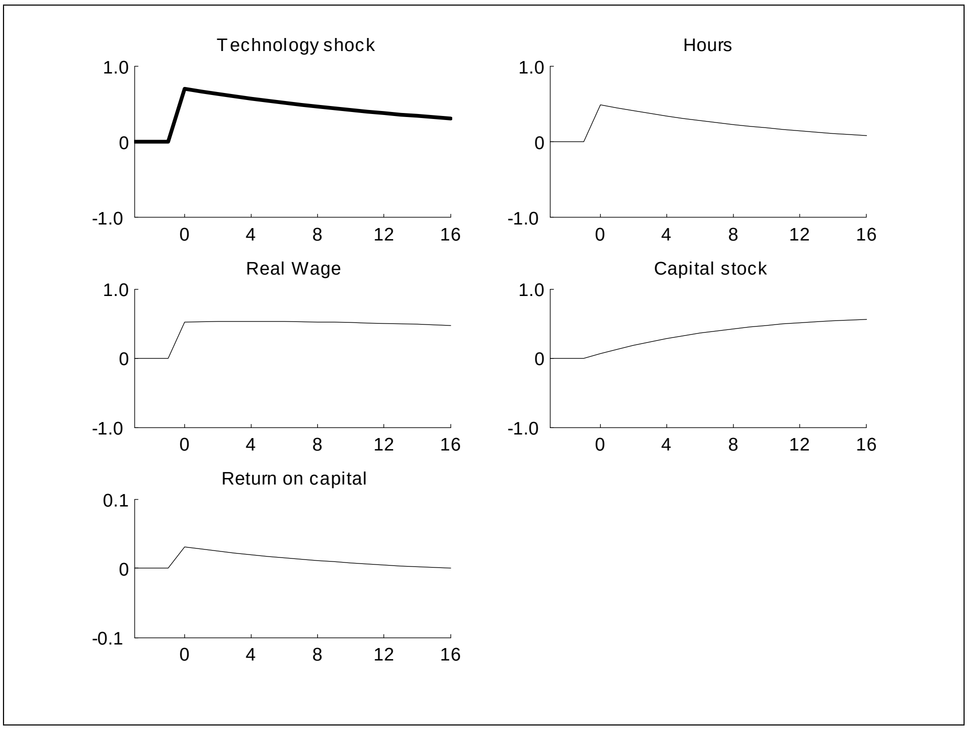

Figure 5.3: Impact of the technology shock on production factors (extended RRC)

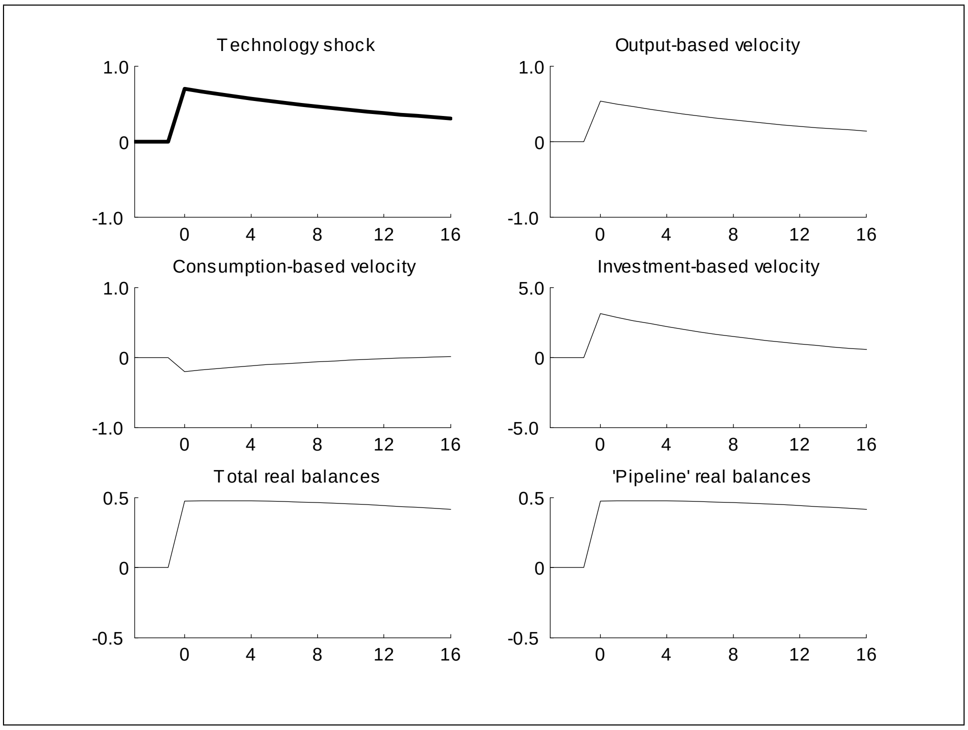

Figure 5.4: Impact of the technology shock on the velocity of money (extended RRC)

Monetary shock

Figure 5.5 shows the impact of the monetary shock on real expenditure and money demand (real balances). Because of the assumption of flexible prices, in both models a monetary shock produces real effects as long as it modifies expected inflation. In fact, in both models current and future consumption expenditure is connected with money holdings. This relationship generates a reciprocal link. On the one hand, since every purchase where money is involved becomes more expensive in real terms, higher expected inflation induces a fall in real consumption tomorrow and - through the mechanism of consumption smoothing - today. On the other hand, since in both models consumption yields utility, the tighter the link between desired consumption and cash, the higher the value that households attach to money holdings.

In the case of the extended RRC model, a monetary shock causes an increase in consumption and a decrease in investment. The amplitude of these movements is determined by the parameter that regulates the mechanism of the 'pipeline'. As in the case of the extended CIA model, a restriction in the monetary channel (i.e., when $\xi = 0.05$) reduces these fluctutations around the steady state. The fact that money does not cover expenditures fully (like in the case of the extended CIA), is responsible for the opposite direction of consumption ad investment: in other words, the inflation tax is not fully operative in the RRC model, given that households can buy goods on credit (paying a real price augmented by transaction costs). This is shown clearly by the last graph in Figure 5.5, where households' demand for real balances falls considerably for both values of the 'pipeline' parameter. The inflation tax reduces the real value of money and increases the unitary transaction costs. Given the model calibration, at this new costs of credit, households do prefer to use real resources to finance a small increase in consumption (which gives direct utility), at the expenses of capital goods.

With households' real balances falling between 2% and 1% (in the case when $\xi = 1.00$ and $\xi = 0.05$, respectively), and a falling output, one should expect velocity of money to result high counte-cyclical in case of a monetary shock. This is certainly true for the case in which $\xi = 1.00$, for which all the velocities in Figure 5.8 increase. However, this is not so straighforward when the 'pipeline' parameter is changed to $\xi = 0.05$. In this case, on impact, *con-*

sumption- and *output*-based velocity increase, but fall immediately after under the long run equilibrium; while *investment*-based velocity remains always pro-cyclical (within the time range considered). The difference is explained by the fact that total real balances (which are used to derive the velocity measures) are affected by the 'pipeline' real balances. Even if the scale of the last graph in Figure 5.8 does not help to quantify the movement in the real 'pipeline' money, the numerical results report, on impact, an increase of 3.46%. When $\xi = 0.05$, households' real balances fall by nearly 1%. Thus, the explanation of an increase in total real balances (considering the appropriate 'weights' from the money market clearing condition).

As anticipated before, the real effects of the monetary shock are guaranteed by the fact that it displays some degree of persistence. However, a key difference with the previous chapters cosists of the non-neutrality of contemporary monetary shocks. In fact, if the autoregressive parameter is set to zero ($\rho_\theta = 0$), because of the structure of the 'pipeline' mechanism, the *current* price level (and, therefore, current inflation) is not affected. This is due to the fact that any new monetary injection will produce its effects in the next period, regardless of the autoregressive nature of the shock.

Turning attention to the nominal effects of a monetary shock (Figure 5.6), the response of inflation to a monetary shock is more than proportional when $\xi = 1.00$, while is nearly 1% when the 'pipeline' rigidity degree is increased ($\xi = 0.05$). As usual, this comes from the difference between the money supply growth and the change in households' real balances. As in Chapter 3, when the shock occurs, the extended RRC model experiences a considerable fall in households' real balances. The inflation process resulting from a monetary shock does not look particularly persistent. But again, the scale of the relative graph in Figure 5.6 is not helpful. A closer look at the simulation values show that the process where $\xi = 0.05$ dies out very slowly, after the impact in the first period.

As shown in Figure 5.6, the impact of the monetary shock on the nominal interest rate is less than proportional. Given the market timing assumptions adopted (i.e., the financial market opening first) the nominal interest rate represents the opportunity cost of holding money. However, its response to the monetary shock depends on the transaction technologies. In particular, as shown in Chapter 3, in the RRC model, the link between expected inflation and actual interest rate is explained by the marginal contribution of real balances to transaction costs. The reason why the movement of the nominal interest rate is so small (between 0.05 and 0.07 basis points) has to be found in the weak impact of any change in real balances on the costs of transactions.

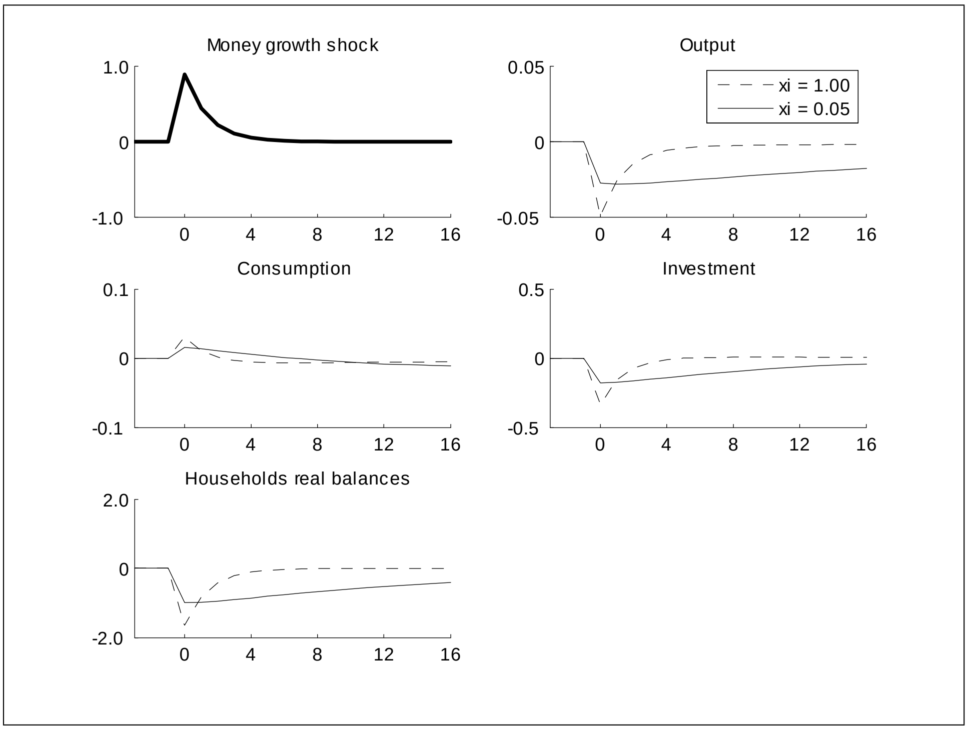

Figure 5.5: Impact of the monetary shock on real expenditure (extended RRC)

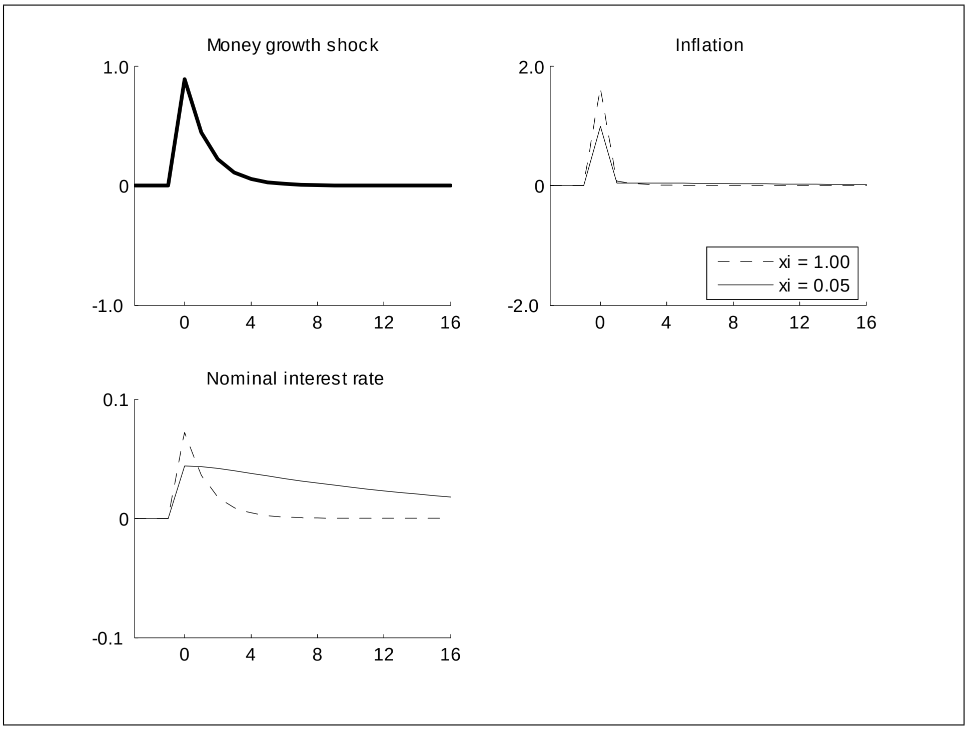

Figure 5.6: Impact of the monetary shock on nominal variables (extended RRC)

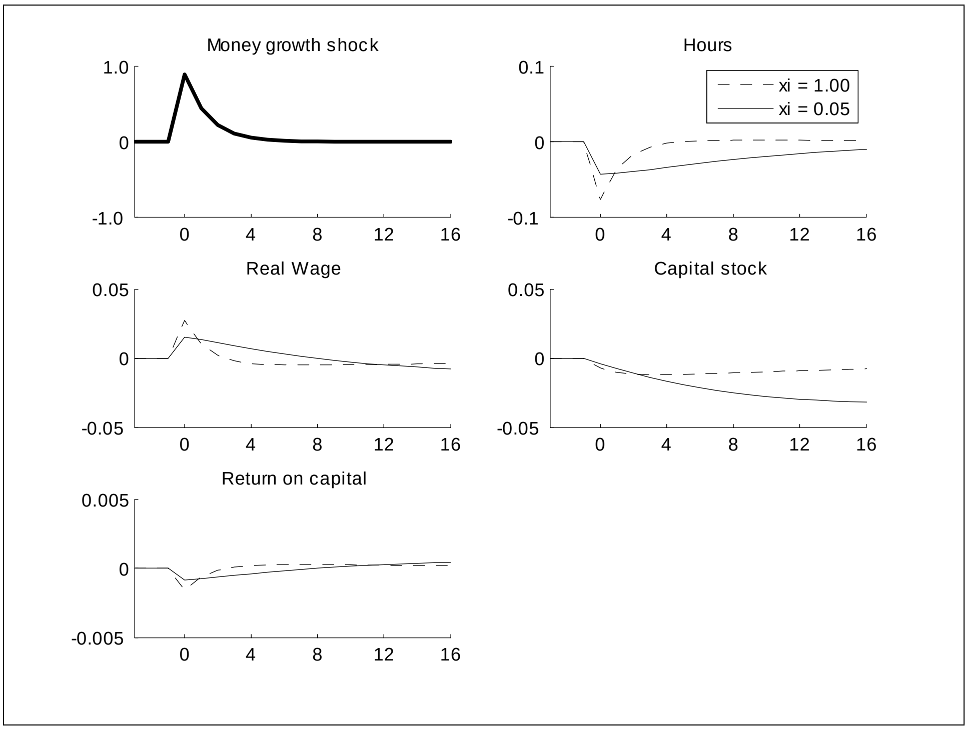

Figure 5.7: Impact of the monetary shock on production factors (extended RRC)

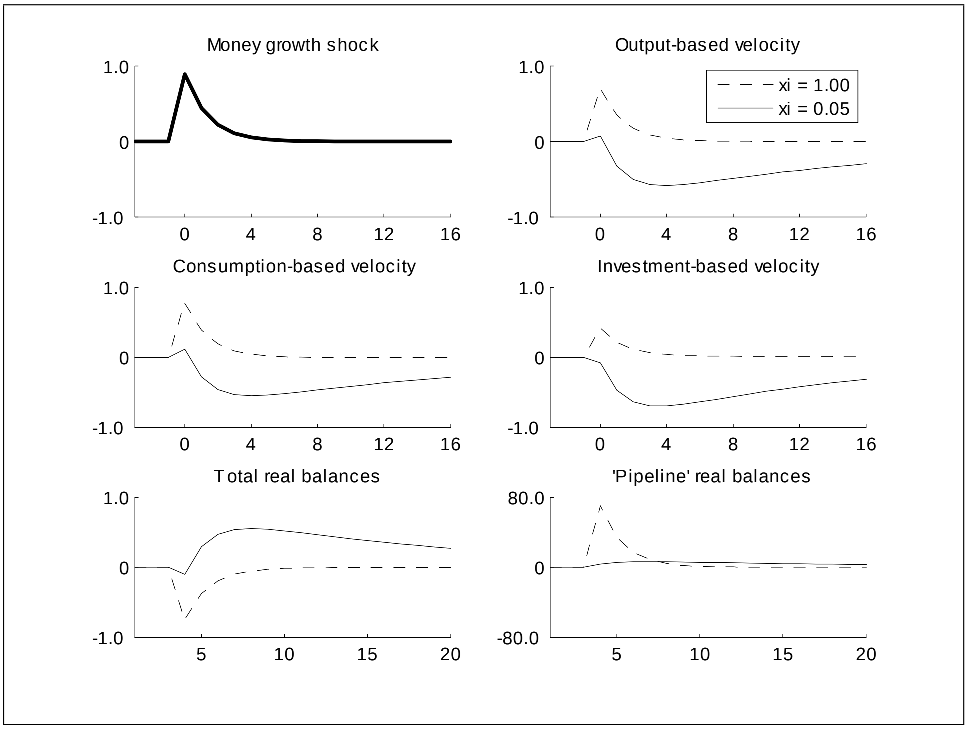

Figure 5.8: Impact of the monetary shock on the velocity of money (extended RRC)

Transaction costs shock

One additional source of shocks in the RRC model is represented by the transaction costs shock. There are two main consequences of this type of shock hitting the economy: it generates a negative effect on output and a higher demand for households' real balances. The former is due to the fact that in this chapter investment and consumption goods are more costly, the latter due to the attempt of consumers to reduce transaction costs by holding more cash. However, the extension of the transaction technologies implies the same increase in the real cost of consumption and investment. The qualitative results are similar to those generated by a monetary shock: i.e., given the model calibration, at this new costs of credit, households do prefer to use real resources to finance a small increase in consumption (which gives direct utility), at the expense of capital goods.

As in Chapter 3, the link between money and cosumption remains weak also in the extended RRC model. A 1% unit transaction costs shock (Figure 5.9) leads to an increase in money demand which is less than 0.1%. In the period after the shock, real balances start to converge back to equilibrium. With monetary authorities keeping the money supply growth constant, this causes a little inflation overshooting (see Figure 5.10). For the same reasons discussed in Chapter 3, the behaviour of expected inflation induces a little movement in the nominal interest rate.

Finally, whenever transaction costs increase in the RRC model - due to a fall in the value of real balances or to a transaction costs shock - a fall in desired consumption is always accompanied by an increase in the real resources produced by the economy for transaction purposes. One of the reasons why the latter does not dominate the former lies in its small dimension in terms of output share at the steady state (see Table 5.3).

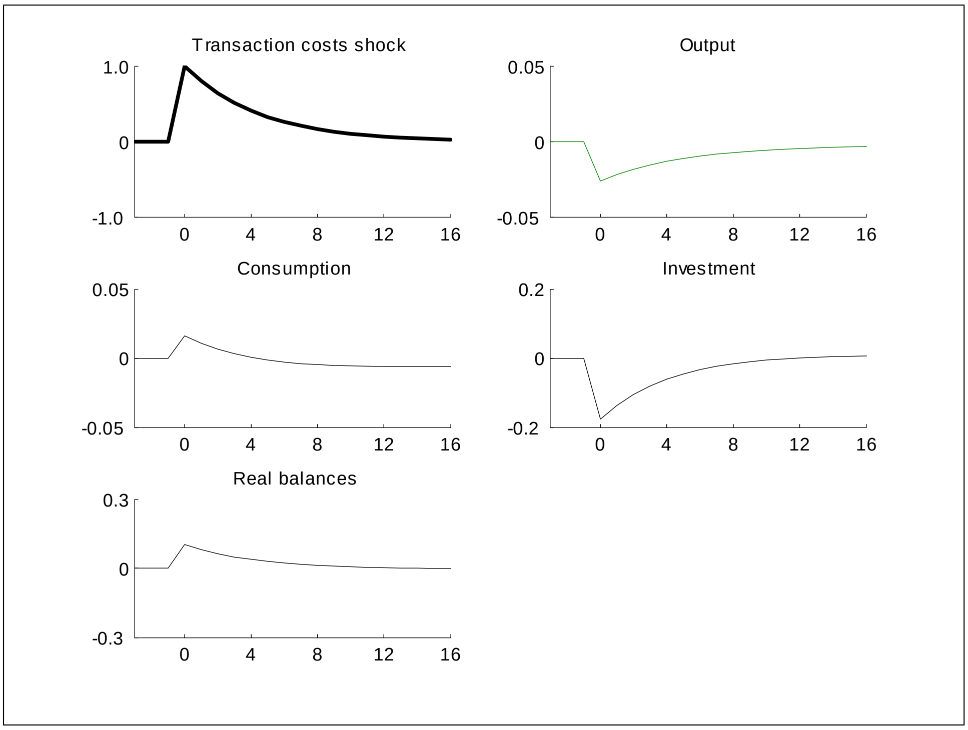

Figure 5.9: Impact of the transaction costs shock on real expenditure (extended RRC)

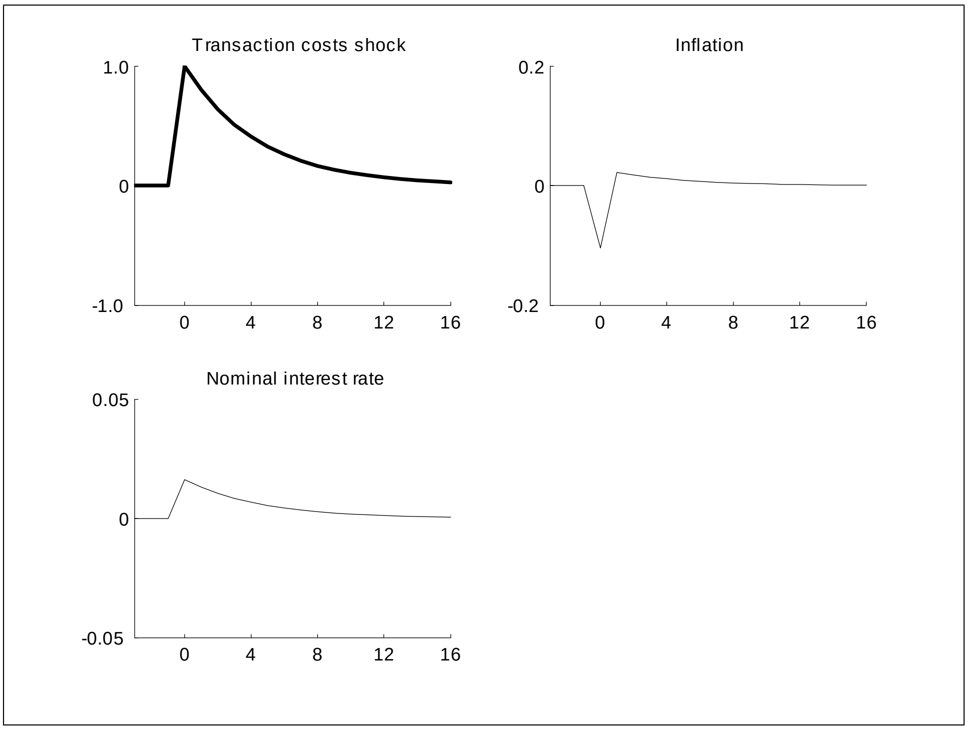

Figure 5.10: Impact of the transaction costs shock on nominal variables (extended RRC)

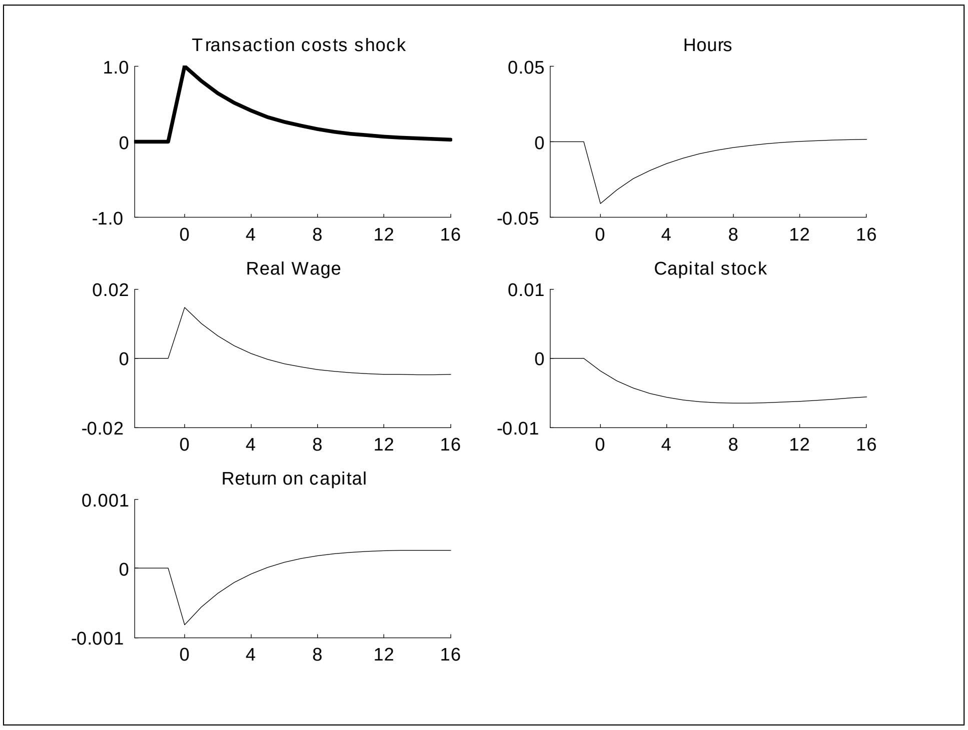

Figure 5.11: Impact of the transaction costs shock on production factors (extended RRC)

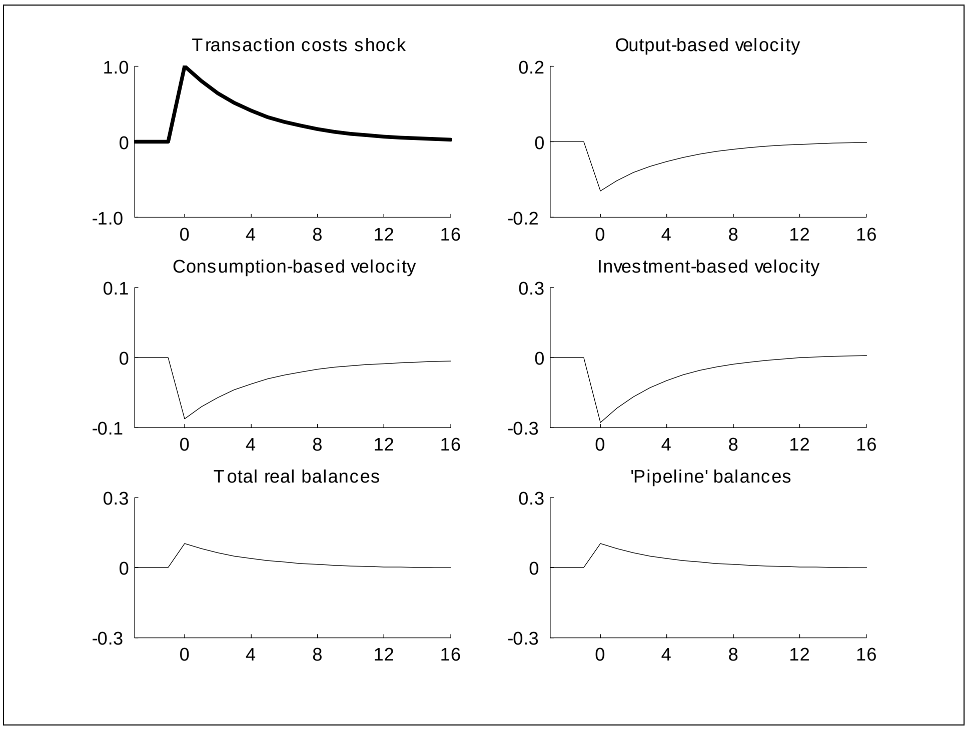

Figure 5.12: Impact of the transaction costs shock on the velocity of money (extended RRC)

5.7.3 Additional simulation results

The impulse-reponse analysis conducted above helped to analyse the effects of the orthogonal shocks on the variables of interest. In this part the quantitative exploration will focus on the comparison of the simulation results with the characteristics of the actual U.S. time series for the variables appearing in the model. Following the spirit of Cooley and Hansen (1995), the performance of the CIA and RRC model will be assessed along three dimensions: the standard deviation of the variables in the simulated models and their correlation with output and money growth. Since both monetary models abstract from many real world features and rigidities, one should not expect a perfect match. In fact the aim of this comparison should be helpful in suggesting whether the models go in the direction the data suggest and, eventually, which of the two model is closer to the empirical evidence. In practice, this corresponds to a quantitative analysis *at the margin*.

Table 5.5 reports the standard deviations of the variables of the U.S. economy (second column) together with the standard deviations of the artificial variables resulting from the simulations of the extended CIA model (the third and fourth column report the sensitivity results for the 'pipeline' parameter)[14]. The results available from the analysis by Cooley and Hansen (1989) and Cooley and Hansen (1995) are also reported (under CH1989 and CH1995, respectively). Along this dimension, the two different parameterisations deliver similar results, and in most of the cases these results are not too close to the real data. In fact variables relative to the labour market, the goods market and inflation are generally *more* volatile than the data, while some other variables (e.g., interest rates) are *less* volatile. Marginally, the model with more rigidity in the money supply process delivers a superior performance.

Table 5.6 reports the correlations with output of the variables of the U.S. economy (second column) together with the correlations with output of the artificial variables resulting from the simulations of the extended RRC model (the third and fourth column report the sensitivity results for the 'pipeline' parameter). The results available from the analysis by Cooley and Hansen

[14] For the description of the data reported in the following Tables, refer to Chapter 1, Section 1.5.

(1989) and Cooley and Hansen (1995) are also reported (under CH1989 and CH1995, respectively). In general, the two different parameterisations of the 'pipeline' deliver different numerical results. The exceptions are represented by the variables of the labour market and the goods market, where the extended RRC model performs generally well. The model delivers the wrong correlation sign in the case of inflation and *consumption*-based velocity. A good performance characterises the nominal interest rate though, the real data (0.3522) fall within the range of the sensitivity analysis. Despite most of the correlations in the model are far from those appearing in the real sample, the model characterised by a more rigid money supply process (i.e., $\xi = 0.05$) is characterised by a better marginal fit.

Table 5.7 reports the correlations with money growth for the variables of the U.S. economy (second column), together with the correlations with money growth of the artificial variables resulting from the simulations of the extended RRC model (the third and fourth column report the sensitivity results for the 'pipeline' parameter). The results available from the analysis by Cooley and Hansen (1989) and Cooley and Hansen (1995) are also reported (under CH1989 and CH1995, respectively). Here the results are very sensitive to the 'pipeline' parameters. In general, the model performs quite poorly. In the cases where the correlation sign is not in opposition to the data (which seem to be happening when $\xi = 1.00$), almost all the variables in the extended RRC model deliver results that are not close to the empirical evidence. Even when there is a high rigidity in the money supply process, in the majority of the cases the RRC model exhibits a lower correlation with respect the data.

variables	STD. DEV.				
	data	xi = 1.00	xi = 0.05	CH1989	CH1995
working hours	0.0043	0.0120	0.0120	0.0133	0.0135
real wage	0.0109 (0.0071)	0.0302	0.0302	-	-
consumption[15]	0.0125	0.0275	0.0275	0.0062	0.0053
nominal interest rate	0.0030	0.0011	0.0016	-	0.0058
inflation	0.0044 (0.0029)	0.0171	0.0112	-	0.0123
real balances	0.0314 (0.0284)	0.0279	0.0324	-	-
output	0.0154	0.0378	0.0378	0.0173	0.0169
money growth	0.0089	0.0103	0.0103	-	0.0087
investment[16]	0.0699	0.0952	0.0953	0.0569	0.0590
consumption-based vel	0.0258	0.0108	0.0199	-	-
investment-based vel	0.0654	0.0779	0.0810	-	-
output-based vel	0.0277	0.0103	0.0243	-	-

Table 5.5: Standard deviations (extended RRC model).

variables	CORR. with OUTPUT				
	data	xi = 1.00	xi = 0.05	CH1989	CH1995
working hours	0.7077	0.7254	0.7251	0.9800	0.987
real wage	0.5307 (0.5830)	0.9618	0.9616	-	-
consumption	0.8632	0.8902	0.8901	0.7200	0.676
nominal interest rate	0.3522	0.4972	0.3286	-	-0.014
inflation	0.3817 (0.1419)	-0.0672	-0.0936	-	-0.138
real balances	0.3368 (0.3133)	0.9204	0.7703	-	-
output	1.0000	1.0000	1.0000	1.0000	1.0000
money growth	-0.1282	-0.0154	-0.0126	-	-
investment	0.9024	0.8876	0.8875	0.9700	0.975
consumption-based vel.	0.0713	-0.1116	-0.0264	-	-
investment-based vel.	0.8056	0.2457	0.7356	-	-
output-based vel.	0.2362	0.7439	0.5267	-	-

Table 5.6: Correlations with output (extended RRC model).

variables	CORR. with MONEY GROWTH				
	data	xi = 1.00	xi = 0.05	CH1989	CH1995
working hours	-0.1957	-0.0719	-0.0590	-	-0.0300
real wage	0.2365 (0.1714)	0.0094	0.0077	-	-
consumption	0.0311	0.0116	0.0086	-	-0.6000
nominal interest rate	-0.4771	0.7702	0.4762	-	0.7200
inflation	-0.3124 (-0.1940)	0.8533	0.8047	-	0.9200
real balances	0.2264 (0.2021)	-0.3115	0.0802	-	-
output	-0.1282	0.0154	-0.0126	-	-0.0100
money growth	1.0000	1.0000	1.0000	-	1.0000
investment	-0.1955	-0.0392	-0.0313	-	0.1600
consumption-based vel.	-0.2306	0.8325	-0.1185	-	-
investment-based vel.	-0.3250	0.0635	-0.0689	-	-
output-based vel.	-0.2781	0.4969	-0.1265	-	-

Table 5.7: Correlations with money growth (extended RRC model).

The quantitative exercises by Cooley and Hansen (1989, 1995) deliver a lower variability of consumption and output, generally more in line with the data. As noted previously, this is mainly due to the assumption of indivisible labour, adopted in their work. The standard deviation of the nominal interest rate in Cooley and Hansen (1995) remains higher than the one reported in the data (from this point of view, the extended CIA model with a more rigid pipeline performs better). However, the extended CIA model - with divisible labour - presented in this chapter exhibit a correlation of consumption and working hours more in line with the data. The same is true for the inflation tax: in fact the correlation between money growth and consumption - in the extended RRC model with the 'pipeline' - exhibits the same 'sign' as the the data, subverting the result obtained by Cooley and Hansen (1995).

5.7.4 The extended RRC model: some comments

As anticipated before, given the relative simplicity of the models at hand, one should not expect the simulated data to match up *perfectly* with the real data. However, some general comments can be added at this stage. The volatility of the variables for the extended RRC model is generally higher than the one shown by the real data: the bigger discrepancies concern inflation (with a volatility four times higher than the data, when the friction created by 'pipeline' is lower), the labour market (where working hours and real wage in the models result is three times as volatile as the data), followed by the goods market (with all the components of aggregate demand being, on average, twice as volatile). In all the other cases, where the performance with respect to the data seems to depend on the degree of rigidity in the 'pipeline' mechanism, in general better results are achieved when this rigidity is higher (i.e., $\xi = 0.05$). However, for most of these variables the match is poor. The only (positive) exception is due to the behaviour of real balances, where the volatility of the extended RRC model is in line with the data. Regarding the correlation with output, in general the match with the data is not too bad, especially in the case of the labour market (working hours) and the goods market (consumption and investment). The reason for such a good match in those fields lies in the characteristics of the technology shock: in fact, the high value of the autoregressive parameter of this shock (0.95) and its immediate effects on output and factors demand drives these results. On the other hand, the model performance on the monetary side remains mixed. For some of these variables - such as the nominal interest rates - a higher rigidity in the money supply process generates a close match with the data, while in other cases (like the correlation between money growth and output) less rigidity goes in the 'right' direction with respect to the data, even if the results are too far from a satisfactory fit. The worst result here concerns the *consumption*-based velocity, where the data suggest pro-cyclicality, while the model displays a negative sign. When the correlation of the variables with money growth in the simulated economies is compared with the real data, the results are *not* so ecouraging. In particular the model fails to match the data in correspondence of the same variables it failed for in Chapter 3: in this case all the correlations involving monetary variables exhibit either an opposite sign with respect to the data, or a quite poor match.

5.8 Conclusion

This chapter presented a flexible-price monetary model where both consumption and investment goods are subject to a *real-resource costs* of transactions. In addition to that, the money supply process was 'disturbed' by the presence of a rigid transmission mechanism. The model can be considered as an original translation of the cash-in-advance specifications by Stockman (1981) and Abel (1985) into the context of the real-resource-cost approach. The treatment is characterised by the adoption of two structural assumptions: the use of a Lucas-type market timing assumption (i.e., financial markets opening before the goods markets); and the presence of a lag in the transmission of monetary shocks to the private economy. The quantitative assessment is made relative to some key descriptive statistics concerning selected characteristics of the U.S. business cycle as reported in Chapter 1.

The main findings are that the standard deviation of the artifical variables related to the goods market (output, consumption and investment) are higher than the data and are not influenced by the rigidity in the money supply process. The reason is that they mainly respond to the (highly persistent) technology shock. On the contrary, the volatility in the monetary variables is strongly affected by the sensitivity analysis conducted on the 'pipeline' mechanism, but here the result is the opposite: the standard deviations appearing in the data are higher than those generated by the model's simulations. The fact that the technology shock maintains a relative 'dominance' on the behaviour of the variables related with the goods market is confirmed by a (strong) correlation of these variables with output, matching the data quite closely.

On the other hand, the correlation of the artificial variables with the money growth rate is strongly affected by the calibration of the 'pipeline' mechanism. This influence extends beyond the behaviour of nominal variables, involving also consumption and investment. However, the model fails along many dimensions. In fact, an opposite signed correlation with the money growth rate compared to the data characterises not only inflation and interest rates (a result very common for flexible price models), but also measures of velocity, real balances and output. Moreover, when the correlations exhibit the 'right' sign, their scale from the simulated economy are generally far from the empirical evidence shown in Chapter 1.

On the *money demand* side, the set of transactions for which money is needed has been extended to incorporate investment goods. This implementation was essentially motivated by the intuition that a negative correlation between investment and money growth in the data (see Chapter 1, Table 1.3) might reveal some effects of the inflation tax on investment. The CIA model incorporated this extension by implementing the seminal contributions of Stockman (1981) and Abel (1985) by applying a Lucas-type market timing assumption (after Lucas (1982)). On the other hand, the RRC model derives an (original) *comparable* re-formulation of the real-resource-cost approach.

At the same time, the *money supply* process of these models has been modified by introducing a specific rigidity in the transmission mechanism of monetary shocks. This rigidity assumed the form of a liquidity 'pipeline' tranferring the new monetary injections into the households' budget constraint, with a *delay.* On the one hand, the main motivation for the adoption of this device was realism: in reality, monetary injections by the central banks do not reach households straight away. On the other hand, this type of rigidity could be used to investigate whether the empirical results by Cooley and Hansen (1995), about the fact that "*money peaks before output*", were actually due to rigidities in the monetary transmission mechanism (for example, at the level of the financial intermediaries, receiving the new money first).

The analysis of the impulse-response functions showed that both models behave in the same way under a technology shock. In fact, both the CIA and the RRC models continue to exhibit a unitary velocity with respect to *utility*-consumption and *desired* investment. However, as in Chapters 2 and 3, they respond differently to the monetary shock.

In the extended CIA model (Chapter 4) the inflation tax causes a fall in consumption and investment. However, while investment falls considerably (almost by 3%), the reduction in consumption is negligible (less than 0.05%). This revealed the following result: when consumption and investment are both subject to the inflation tax, households prefer to maintain their original level of utility, sacrificing capital goods. Since the inflation tax impacts (negatively) on the two components of aggregate demand, output falls. Despite the fall in all of the expenditure components, one should expect an increase in the corresponding measures of velocity. However, the result is quite the opposite. This is explained by the fact that total money supply (used to construct the empirical counterparts of the velocities) is made of two *distinct* components: the share in the hands of the households and the share within the 'pipeline'. When monetary injections do not reach the households' accounts instantaneously, the new money accumulates in the 'pipeline', causing (under appropriate calibration of a 'pipeline' parameter) an increase in the value of total real balances. This increase dominates the fall in the expenditure components, and in the velocity ratios. This last tendency is confirmed also in the case of the extended RRC model (Chapter 5). However, in this model the inflation tax causes a fall in the value of 'households real balances' three times bigger than the extended CIA model. This is due to the fact that consumption and investment goods can be bought *on credit* in the real-resource-cost framework. Contrary to the case of the extended CIA model, however, the inflation tax causes consumption and investment to move in opposite directions (with an increase in the latter).

When compared with the empirical evidence, the extended CIA and RRC models still exhibit the RBC 'dichotomy' which emerged in the early chapters: namely, the volatility and the correlation with output of real expenditure and working hours are essentially driven by the technology shock, while nominal variables are mainly affected by the monetary shock. However, it is in the correlation between the artificial variables and money growth that the extended CIA model shows a superiority with respect to the extended RRC model. In fact the simulation results of the former are much closer to the data the correlation of output, investment and all the velocity measures, when these are compared with those in Chapter 3. On the contrary, the marginal improvements for the RRC model are not so evident. In particular, this last model continues to exhibit too many correlation signs in opposition to the real data.

The restrictions imposed on calibration, in order to make the RRC model comparable with the other models derived in the previous chapters, might have played a role in driving the results. A further investigation, for example conducting a sensitivity analysis with respect to the parameters characterising the transaction costs function, could help in clarifying whether the RRC model has still some potential explanatory power, in the context of flexible price models.

Finally, the comparison with the results reported in previous work by Cooley and Hansen (1989, 1995) reveals that the assumption of divisible labour adopted in this chapter performs better in terms of correlation of real variables with output and money growth, while the quantitative performance of the indivisible labour assumption (adopted by Cooley and Hansen) is superior in terms of standard deviations of consumption and working hours Moreover, when transaction costs are extended to capital goods and the pipeline mechanism introduced, the correlation between nominal interest rates and output exhibits the same 'sign' of the data - improving the results of Cooley and Hansen (1989, 1995)

Chapter 6

Conclusion

This thesis analyses and compares two monetary models - the *cash-in-advance* (CIA) model and the *real-resource-cost* (RRC) model - using a Dynamic Stochastic General Equilibrium (DSGE) framework, with perfectly competitive markets and flexible prices. The purpose of the investigation is twofold: first of all, to find out whether different microfoundations of money do actually matter for understanding and explaining business cycle phenomena; and, secondly, to check whether the suggested methodology for inspecting the mechanism - from the selection criteria of the monetary models to the quantitative assessment of the simulations - could be helpful in providing an answer. Certainly, these last methodological concerns provide effective guidance in organising ideas and in structuring the whole thesis from the very begining. In addition to the detailed discussion of the results from each chapter presented above, this final chapter summarises the main findings and provides some suggestions for future research.

6.1 Main findings

What are the implications of adopting the cash-in-advance constraint or the real-resource-cost approach for the transmission mechanism of shocks? In monetised versions of the RBC model characterised by fully flexible prices, the answer depends on the mix between *cash-* and *credit-*goods present in these economies.

In all the chapters the CIA and the RRC models respond in the same way to a shock in total factor productivity. This result is due to the fact that, despite the different specification for the microfoundations of money, these two models exhibit a unitary elasticity of money demand (i.e., real balances) with respect to the transaction variable.

However, in the case of a monetary shock, it is necessary to distinguish between the different contexts. The impulse response fuctions show that, in general, the effects of the inflation tax generated by an (expansionary) autoregressive monetary shock depend on the possibility, for the agents in the model, to switch from money to other means of payment - namely, *costless* credit in the case of the CIA model and *costly* credit in the case of the RRC approach. When consumption goods are linked with money in models without capital (Chapter 2), the agents in the CIA model are forced to buy *all* goods with cash. For this reason, following the monetary shock, money demand falls more in RRC model than in the CIA model, given the possibility for the agents of the former to maximise their utility moving to (relatively) cheaper means of payment. When capital goods are introduced in this framework (Chapter 3), these results are confirmed and, in some way, reinforced. Again, the key element to read in these results is the utility of consumption: the households in the RRC model can continue to enjoy consumption simply switching to an alternative way of payment (costly credit), while the households in the CIA model need to evaluate how much *direct* utility of consumption to 'give up' when shifting to the good which is not subject to the inflation tax (investment). The extensions introduced in Chapters 4 and 5 affect the relationship between *cash*- and *credit*-goods in two ways: modifying the dynamic evolution of the inflation tax (and, therefore, the intertemporal allocation of resources) and, at the same time, extending the influence of the inflation tax on investment. As a result, investment falls in both models. Consumption increases in the case of the RRC model, while it can increase or decrease in the CIA model (depending on the degree of rigidity in the money supply channel).

In the context of Chapters 2 and 3 the response of the different measures of velocity to an expansionary monetary shock reflect the *portfolio reallocation* between *cash*- and *credit*-goods in both models. In Chapters 4 and 5, instead, the corresponding measures of velocity *increase,* despite the negative impact of the inflation tax on consumption and investment. The reason why velocities do not reflect portfolio allocation anymore is explained by the fact that monetary injections do not reach the households' accounts instantaneously, and part of the new money accumulates in the 'monetary pipeline'. This causes an increase in the value of *total* real balances (used to construct the empirical counterparts of the velocities) that dominates the fall in the expenditure components.

Finally, the impact of a transaction cost shock in the RRC model is extremely weak in all chapters. This result might be due to the particular calibration adopted in order to make this model comparable with the CIA approach. In any case, to calibrate the type of transaction cost function presented in this thesis is not an easy task. Given the general difficulty in finding reliable proxies for transaction costs in the real world, the suggested approach remains a sensitivity analysis exercise[1].

When compared with the empirical evidence, in general *all* the CIA and RRC approaches considered in this thesis exhibit the 'dichotomy' typical of the RBC literature: namely, the volatility of real expenditure and working hours (and the respective correlation with output) are essentially driven by the technology shock, while nominal variables are mainly affected by the monetary shock. In the context of these simple models, these results tend to reinforce the 'standard' RBC conclusions: technological shocks are important and *reverse-causation* characterises the relationship between money and output.

When it comes to the correlations of the endogenous variables with money growth, again it is necessary to distinguish between the different contexts. In the case in which only consumption is linked with money (Chapter 3) both monetary models fail along many dimensions. In fact, for two-thirds of the endogenous variables the correlation with respect to money growth exhibits the 'wrong' correlation sign with respect to the data. When transaction technologies are extended to investment and rigidites are introduced in the money supply process (Chapters 4 and 5), the extended CIA model reveals its *superiority* with respect to the extended RRC model.

When compared with the results reported in previous work by Cooley and Hansen (1989, 1995) the assumption of divisible labour adopted in this thesis performs better in terms of correlation of real variables with output and money

[1] See Aiyagari and Gertler (1991).

growth, while the quantitative performance of the indivisible labour assumption (adopted by Cooley and Hansen) remains superior in terms of standard deviations of consumption and working hours. However, when money is extended to the purchase of capital goods and the 'pipeline' mechanism is introduced (Chapter 4 and 5), the correlation between nominal interest rates and output exhibits the same 'sign' of the data, improving the empirical results found in Cooley and Hansen (1989, 1995).

6.2 Further research

The models developed in this thesis are based on many simplistic assumptions. On the other hand, the results show that a further improvement in the quantitative performance of this type of monetary model is possible, once the transmission mechanism of moentary shocks is taken more seriously. This section suggests some possible directions this investigation can take.

One major limitation of the models presented in this thesis concerns the use of the monetary aggregate M1. In reality fluctuations in M1 can be due either to movements in the monetary base (M0) or in the *money multiplier*. On the one hand, in the type of models considered in this thesis the modifications to the basic transmission mechanism of monetary shocks introduced in the last two chapters are too simple to distinguish between these two elements with precision. On the other hand, one could interpret the 'pipeline mechanism' as a first attempt to separate the monetary aggregate M1 in two components: a component available to the households and another component 'stored' within the 'pipeline'. This observation suggests inevitably an analogous feature of the real world: the distinction in monetary base (M0) between *currency*, held by public, and *reserves,* held by commercial banks. However, the absence of a proper money multiplier in Chapters 4 and 5 does not allow such a detailed 'disaggregation' in the total money supply. If one wants to allow a more precise distinction between exogenous movements in M1, due to the monetary authority intervention, and endogenous movements in the monetary aggregate, due to the optimal allocation of monetary resources among the agents of the model, a more sophisticated transmission mechanism is needed. This can be done by replacing the rigid 'monetary pipeline' used in this thesis with an optimising representative *bank*, which collects deposits from households, makes loans to firms, and receives the new monetary base directly from the central bank. In this way the main components of M1 would be defined explicitly within the model. The basic flexible-price framework that can be used for this purpose is represented by the so-called 'limited-participation' models[2]. These models overcome the difficulty of modelling different types of agents by modelling the representative households as a representative *family*, who preserves the analytical advantage of the representative agent framework. The idea is to extend the models of this thesis by assuming a cash-in-advance contraint for households and firms, and modelling the demand for reserves by the representative bank via a liquidity cost function. One could start by assuming that the lucrative activity for the bank (i.e., providing loans to the firms) involves some extra *liquidity costs* for the intermediary (e.g., balance sheet adjustments, costly portfolio re-allocations, etc.). On the one hand, the adoption of the liquidity cost function (with bank reserves reducing financial costs) would force banks to hold money in equilibrium[3]. On the other hand, the adoption of a functional form of the type adopted in this thesis, allows one to study whether the effect of *liqudity cost shocks* occurring at the level of the banking sector are quantitatively important. The conjecture here is that this modelling strategy could in principle overcome two main limitations of flexible-price monetary models. The first regards an (independent) access of firms to liquidity through loans: the literature on 'limited participation' models showed that this feature can generate a liquidity effect. Perhaps an optimising banking sector could allocate the new money to firms and households in such a way that the inflation tax might be 'overturned'. Secondly, allowing liquidity shocks within the transmission mechanism could help to distinguish between fluctuations in M1 due to policy interventions and movements in the monetary

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[2] The first-limited participations models are due to the seminal contributions of Grossman and Weiss (1983) and Rotemberg (1984). The application of the original intuition within the DSGE framework is due to the theoretical contributions of Lucas (1990) and Fuerst (1992). Calibrated versions of these models have been assessed by Christiano and Eichanbaum (1995) and Dotsey and Ireland (1995), while a recent treatment is due to Gertler and Kiyotaki (2010).

[3] For the same reason, Gillman and Kejak (2004) develops a model of banks involving a cash-in-advance constraint for reserves.

aggregate due to liquidity factors. At least two extensions are connected with these issues. One is to study how a model with an optimising transmission mechanism for monetary policy would modify the performance of traditional monetary policy rules, in the spirit of Christiano and Gust (1999). The second consists in checking whether the presence of an optimising banking sector is likely to introduce nominal and/or real indeterminacies in the model, given 'conventional' stabilisation monetary policies (see Taylor 2009).

References

Abel, A. B. 1985. "Dynamic Behavior of Capital Accumulation in a Cash-in-Advance Model." *Journal of Monetary Economics* 16(1): 55-71.

Ahmed, S. and J. H. Rogers. 2000. "Inflation and the great ratios: Long term evidence from the U.S." *Journal of Monetary Economics* 45(1): 3-35.

Aiyagari, S. R., M. Gertler. 1991. "Asset returns with transactions costs and uninsured individual risk." *Journal of Monetary Economics*. 27(3): 311-331.

Blanchard O. and C. Kahn. 1980. "The solution of linear difference models under rational expectations." *Econometrica* 48(5): 1305-1311.

Brock, W. A. 1974. "Money and Growth: The Case of Long-Run Perfect Foresight." *International Economic Review* 15(3): 750-777.

Brock, W. A. 1990. "Overlapping Generations Models with Money and Transaction Costs." In *Handbook of Monetary Economics*, ed. B. Friedman and F. Hahn. Vol. 1, 263-295. Amsterdam: North-Holland.

Campbell, J. Y. 1994. "Inspecting the Mechanism: An Analytical Approach to the Stochastic Growth Model." *Journal of Monetary Economics* 33(3): 463-506.

Carlstrom, C. T. 2004. "Comments on 'Optimal fiscal and monetary policy under imperfect competition'." *Journal of Macroecoomics* 26(2): 219-222.

Carlstrom, C. T., and T. S. Fuerst. 1995. "Interest Rate Rules vs. Money Growth Rules: A Welfare Comparison in a Cash-in-Advance Economy." *Journal of Monetary Economics* 36(2): 247-267.

Carlstrom, C. T., and T. S. Fuerst. 2001a. "Timing and real indeterminacy in monetary models." *Journal of Monetary Economics*, Elsevier 47(2): 285-298.

Carlstrom, C. T., and T. S. Fuerst. 2001b. "Real Indeterminacy in Monetary Models with Nominal Interest Rate Distortions," *Review of Economic Dynamics* 4(4): 767-789.

Casares, M., and B. T. McCallum. 2006. "An optimizing IS-LM framework with endogenous investment." *Journal of Macroeconomics* 28(4): 621-644.

Chari, V. V., P.J. Kehoe and E. R. McGrattan. 2000. "Sticky Price Models of the Business Cycle: can the Contract Multiplier Solve the Persistence Problem?" *Econometrica* 68(5): 1151-1179.

Chari, V. V., P.J. Kehoe and E. R. McGrattan. 2009. "New Keynesian Models: Not Yet Useful for Policy Analysis." *American Economic Journal: Macroeconomics* 1(1): 242-266.

Christiano, L. J. 1991. "Modelling The Liquidity Effect of a Money Shock." Federal Reserve Bank of Minneapolis *Quarterly Review* 15(1): 3-34.

Christiano, L. J. 2002. "Solving Dynamic Equilibrium Models by a Method of Undetermined Coefficients." *Computational Economics* 20(1-2): 21-55.

Christiano, L. J., and M. Eichenbaum. 1992. "Current Real Business Cycle Theories and Aggregate Labor-Market Fluctuations." *American Economic Review* 82(3): 430-450.

Christiano, L. J., and M. Eichenbaum. 1995. "Liquidity Effects, Monetary Policy, and the Business Cycle." *Journal of Money, Credit, and Banking* 27(4): 1113-1136.

Christiano, L. J., and C. J. Gust. 1999. "Taylor Rules in a Limited Participation Model." NBER Working Papers 7017, National Bureau of Economic Research, Inc.

Clower, R. W. 1967. "A Reconsideration of the Microfoundations of Monetary Theory." *Western Economic Journal* 6(1): 1-9.

Cole, H. L., and L.E. Ohanian. 2002. "Shrinking money: the demand for money and the nonneutrality of money." *Journal of Monetary Economics* 49(4): 653-686.

Cooley, T. F., and G. D. Hansen. 1989. "The Inflation Tax in a Real Business Cycle Model." *American Economic Review* 79(4): 733-748.

Cooley, T. F., and G. D. Hansen. 1991. "The Welfare Costs of Moderate Inflations." *Journal of Money, Credit, and Banking* 23(3): 483-503.

Cooley, T. F., and G. D. Hansen. 1995. "Money and the Business Cycle." In *Frontiers of Business Cycle Research*, ed. T. F. Cooley, 175-216. Princeton, N.J.: Princeton University Press.

Cooley, T. F., and E. C. Prescott. 1995. "Economic Growth and Business Cycles." In *Frontiers of Business Cycle Research*, ed. T. F. Cooley, 175-216. Princeton, N.J.: Princeton University Press.

Dotsey, M., and P. Ireland. 1995. "Liquidity Effects and Transaction Technologies." *Journal of Money, Credit, and Banking* 27(4): 1441-1457.

Feenstra, R. C. 1986. "Functional Equivalence between Liquidity Costs and the Utility of Money." *Journal of Monetary Economics* 17(2): 271-291.

Fischer, S. 1977. "Long-Term contracts, Rational Expectations, and the Optimal Money Supply Rule." *Journal of Political Economy* 85(1): 191-206.

Fisher, I. 1933. "The Debt-Deflation Theory of Great Depressions." *Econometrica* 1(4): 337-357.

Friedman, M. 1968. "The Role of Monetary Policy." *American Economic Review* 58(1): 1-17.

Friedman, M., and A. J. Schwartz. 1963. A Monetary History of the United States, 1867-1960. Princeton: Princeton University Press.

Frisch, R. 1933 "Propagation Problems and Impulse Response Problems in Dynamic Economics." In *Economic Essays in Honour of Gustav Cassel: October 20, 1933*. London. Geroge Allen & Unwin.

Fuerst, T. S. 1992. "Liquidity, Loanable Funds, and Real Activity." *Journal of Monetary Economics* 29(1): 3-24.

Gali, J. 2008. Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework. Princeton, N.J.: Princeton University Press.

Gertler, M., and N. Kiyotaki. 2010. "Financial Intermediation and Credit Policy in Business Cycle Analysis." In *Handbook of Monetary Economics*, eds. B. Friedman and M. Woodford. Vol. 3A. Amsterdam: Elsevier.

Gillman, M., and M. Kejak. 2004. "The Demand for Bank Reserves and Other Monetary Aggregates." *Economic Inquiry* 42(3): 518-533.

Gray, J. 1984. "Dynamic instability in rational expectations models: An attempt to clarify." *International Economic Review* 25: 93-122.

Grossman, S., and L. Weiss. 1983. "A Transaction-Based Model of the Monetary Transmission mechanism." *American Economic Review* 73(5): 871-880.

Hansen, G. D. 1985 "Indivisible Labor and the Business Cycle." *Journal of Monetary Economics* 16: 309-327.

Hodrick, R. J., N. R. Kocherlakota, and D. Lucas. 1991. "The Variability of Velocity in Cash-in-Advance Models." *Journal of Political Economy* 99(2): 358-84.

Hodrick, R. J., and C. E. Prescott 1980. "Post-War U.S. Business Cycles." Carnegie Mellon University Working Paper. Reproduced.

Hoover, K. D. 1995. Macroeconometrics: Developments, Tensions, and Prospects. Dordrecht: Kluwer.

Hume, D. 1752. "Of Money" In D. Hume, *Whitings on Economics*, ed. E. Rotwein, Madison: University of Wisconsin Press, 1970.

Inada, K. 1963. "On a Two-Sector Model of Economic Growth: Comments and a Generalization." *Review of Economic Studies* 30(2): 119-127.

Ireland, P.N. 1996. "The Role of countercyclical Monetary Policy." *Journal of Political Economy* 104(4): 704-723.

Ireland, P.N. 2004. "Money's Role in the Monetary Business Cycle." *Journal of Money, Credit, and Banking* 36(6): 969-983.

King, R. G. 1991. "Money and Business Cycles." Working Paper, University of rochester. Reproduced.

King, R. 1993. "Will the New Keynesian Macroeconomics Resurrect the IS-LM Model?." *Journal of Economic Perspectives* 7: 67-82.

King, R. G., and C. I. Plosser. 1984. "Money, Credit, and Prices in a Real Business Cycle." *American Economic Review* 74(3): 363-380.

King, R. G., C. I. Plosser, and S. Rebelo. 1988. "Production, Growth and Business Cycles. I. The Basic Neoclassical Model." *Journal of Monetary Economics* 21(2/3): 195-232.

King, R. G., and S. T. Rebelo. 1999. "Resuscitating real business cycles." In *Handbook of Macroeconomics*, ed. J. B. Taylor and M. Woodford. Vol. 1B, 927-1007. Amsterdam: North-Holland.

Kiyotaki, N., and R. Wright. 1989. "On Money as a Medium of Exchange." *Journal of Political Economy* 97(4): 927-954.

Klein, P. 2000. "Using the generalized Schur form to solve a multivariate linear rational expectations model." *Journal of Economic Dynamics and Control* 24(10): 1405-1423.

Kohn, M. 1984 "The Finance (Cash-in-Advance) Constraint Comes of Age: A Survey of Some Recent Developments in the Theory of Money." Working Paper No. 84-1, Dartmouth College.

Kydland, F. E., and E. C. Prescott. 1982. "Time To Build and Aggregate Fluctuations." *Econometrica* 50(6): 1345-1370.

Kydland, F. E., and E. C. Prescott. 1990. "Business Cycles: Real Facts and a Monetary Myth." Federal Reserve Bank of Minneapolis *Quarterly Review* 14 (Spring): 3-18.

Lucas, R. E. Jr. "Liquidity and interest rates." *Journal of Economic Theory* 50(2): 237-264.

Lucas, R. E. Jr. 1972. "Expectations and the Neutrality of Money." *Journal of Economic Theory* 4(2): 103-124.

Lucas, R. E. Jr. 1982. "Interest Rates and Currency Prices in a Two-Country World." *Journal of Monetary Economics* 10(3): 335-359.

Lucas, R. E. Jr. 1987. Models of Business Cycles. Oxford: Blackwell.

Lucas, R. E. Jr., and N. Stokey. 1983. "Optimal fiscal and Monetary Policy in an Economy without Capital." *Journal of Monetary Economics* 12(1): 55-93.

Lucas, R. E. Jr., and N. Stokey. 1987. "Money and Interest in a Cash-in-Advance Economy." *Econometrica* 55(3): 491-514.

Mankiw, N. G., and L. H. Summers. 1986. "Money Demand and the Effect of Fiscal Policies." *Journal of Money, Credit, and Banking* 18(4): 415-429.

Marshall, D. A. 1992. "Inflation and Asset Returns in a Monetary Economy." *Journal of Finance* 48(4): 1315-1342.

McCallum, B.T. 1989. "Real Business Cycles Models." in *Modern Business Cycle Theory*, ed. R. J. Barro, 16-50. Cambridge: Harvard University Press.

McCallum, B. T., and M. S. Goodfriend. 1987. "Demand for Money: Theoretical Studies." In *The New Palgrave Dictionary of Economics*, ed. P. Newmann, M. Milgate, and J. Eatwell, 775-781. Houndmills, U.K.: Palgrave Macmillan.

McCallum, B. T., and E. Nelson. 1999. "An Optimizing IS-LM Specification for Monetary Policy and Business Cycle Analysis." *Journal of Money, Credit, and Banking* 31(3): 296-316.

McCandless, G. T. Jr., and W. E. Weber. 1995. "Some Monetary Facts." Federal Reserve Bank of Minneapolis *Quarterly Review* 19(3): 2-11.

Prescott, E. C. 1986. "Theory ahead of business cycle measurement." Federal Reserve Bank of Minneapolis *Quarterly Review* 1042, Fall 1986: 9-22.

Rogerson, R. 1988 "Indivisible Labor, Lotteries and Equilibrium." *Journal of Monetary Economics* 21: 3-16.

Rotemberg, J. J. 1984. "A Monetary Equilibrium Model with Transaction Costs." *Journal of Political Economy* 92(1): 40-58.

Salyer, K.D. 1991. "The Timing of Markets and Monetary Transfers in Cash-in-Advance Economies." *Economic Enquiry* 29: 762-773.

Scheinkman, J. 1980 "Discussion." In *Models of Monetary Economies*, eds. J. Kareken and N. Wallace. Minneapolis Federal Reserve Bank.

Schmitt-Grohe, S. Uribe, M. 2004. "Optimal fiscal and monetary policy under imperfect competition." *Journal of Macroeconomics* 26(2):183-209.

Sims, C. A. 1989. "Models and their uses." *American Journal of Agricultural Economics* 71: 489-494.

Sims, C. A. 2002. "Solving Linear Rational Expectations Models." *Computational Economics* 20(1-2): 1-20.

Singleton, K. 1986. "Speculation and the volatility of foreign currency exchange rates." Graduate School of Industrial Administration, Carnegie-Mellon University.

Stockman, A. 1981. "Anticipated Inflation and the Capital Stock in a Cash-in-Advance Economy." *Journal of Monetary Economics* 8(3): 387-393.

Svensson, L.E.O. 1985. "Money and Asset Prices in a Cash-in-Advance Economy." *Journal of Political Economy* 93(5): 919-944.

Taylor, J. B. 1979. "Staggered Wage Setting in a Macro Model." *American Economic Review* 69(2): 108-113.

Taylor, J. B. 1993. "Discretion versus Policy Rules in Practice." Carnegie-Rochester Conferences Series on Public Policy 39 (Dec.): 195-214.

Taylor, J. B. 2009. "Getting Off Track: How Government Actions and Interventions Caused, Prolonged, and Worsened the Financial Crisis." Hoover Institution Press.

Uhlig, H. 1999. "A Toolkit for Analysing Nonlinear Dynamic Stochastic Models Easily." In *Computational Methods for the Study of Dynamic Economies.* eds. R. Marimon and A. Scott, 30-61. Oxford: Oxford University Press.

Walsh, C. E. 2003. Monetary Theory and Policy. Second Edition. MIT Press Books. The MIT Press.

Wang, P., and C. K. Yip. 1992. "Alternative Approaches to Money and Growth." *Journal of Money, Credit, and Banking* 24(4): 553-562.

Woodford, M. 1986. "Asset bubbles and fiat money." Graduate School of Business, The University of Chicago.

Woodford, M. 1998. "Doing Without Money: Controlling Inflation in a Post-Monetary World." *Review of Economic Dynamics* 1(1):173-219.

Woodford, M. 2003. Interest and Prices: Foundations of a Theory of Monetary Policy. Princeton, N.J.: Princeton University Press.