Connected quantized Weyl algebras and quantum cluster algebras

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Abstract

We investigate a class of noncommutative algebras, which we call connected quantized Weyl algebras, with a simple description in terms of generators and relations. We already knew of two families, both of which arise from cluster mutation in mutation-periodic quivers, and we show that for generic values of a scalar parameter q these are the only examples.

We then investigate the ring-theoretic properties of these two families, determining their prime spectra, automorphism groups and some results on their Krull and global dimensions. The theory of ambiskew polynomial rings and generalised Weyl algebras is useful here and we obtain a description of the height 1 prime ideals in certain generalised Weyl algebras, along with some results on the dimension theory of these rings. We also investigate the semiclassical limit Poisson algebras of the connected quantized Weyl algebras, and compare the prime spectra and Poisson prime spectra of the corresponding rings.

We also show that the quantum cluster algebra without coefficients for an acyclic quiver is simple, and extend this result to find a simple localisation in the case where there are coefficients. Finally, we investigate quantum cluster algebra structures related to the connected quantized Weyl algebras discussed earlier, and use these to illustrate the previous result.

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Contents

1 Introduction

Informally, a **quantum algebra** is a family A_q of noncommutative k-algebras with a parameter $q \in k$ (where k is a field) such that A_1 is commutative. We say that A_q is a "noncommutative deformation" or "quantization" of the commutative ring A_1. Such a family A_q induces a Poisson algebra structure on A_1, known as the semiclassical limit Poisson bracket. This Poisson algebra structure can be thought of as a "first order approximation" to the noncommutative algebras.

The terminology (and only the terminology!) here comes from physics: in quantum mechanics, position (x) and momentum (p) satisfy $xp - px = i\hbar$, where $\hbar$ is the reduced Planck constant. If one "sends $\hbar$ to 0" - that is, increases the scale of x and p - one obtains a commutative relation, which is the relation that holds in classical physics.

One particular class of commutative rings and their quantizations that we will be interested in is that of cluster algebras and quantum cluster algebras, which were introduced, respectively, by Fomin and Zelevinsky in [11] and by Berenstein and Zelevinsky in [6]. These are constructed by starting with a quiver and a set of generators for a (quantum) algebraic torus, then using an iterative process known as seed mutation to produce more generators for the cluster algebra. This gives the (quantum) cluster algebra a combinatorial structure which can then be applied to algebraic questions (most famously the theory of total positivity and canonical bases in semisimple groups, which they were originally created for). From an algebraic point of view, this process produces ring presentations with large numbers of generators and relations - often infinitely many generators even when the resulting (quantum) cluster algebra is Noetherian - but very simple relations.

The definitions and already-known results described above, along with the other background material required, are found in Section 2.

The specific story of this thesis begins with the papers [14] and [13], the relevant aspects of which are reproduced in Sections 8.2.1 and 8.2.2. In these papers Fordy and Marsh describe a[1] Poisson algebra - a subalgebra of a cluster algebra, although they do not describe it as such - with an automorphism of finite order under which the Poisson bracket is invariant. When one passes to the analogous quantum algebra, one obtains an algebra with a generating set such that every pair of generators generates a subalgebra isomorphic either to a quantum

[1] I say "a", but actually this is a family of algebras, one for each odd positive integer, and the quantum algebra C_n^q is actually a family with two parameters, one running again over odd positive integers, the other over the base field of the algebra.

plane or a quantized Weyl algebra. This ring we denote by C_n^q, where n is an odd integer and q is a generic element of the base field k, and is the main object of study of the thesis.

The main focus of Section 4 is to classify the rings which share the property of C_n^q described above, that is, that they have a generating set such that every pair of generators generates a subalgebra isomorphic either to a quantum plane or a quantized Weyl algebra, and there are sufficiently many where it is a quantized Weyl algebra. It turns out that, provided q is sufficiently generic, there are very few such rings: there is another ring, which we denote L_m^q, where m is a positive integer and q is a generic element of k, such that L_m^q embeds naturally into C_n^q provided $m < n$, but these two are the only examples.

In the case when $q = 1$ we also obtain a classification result; here any ring satisfying this property is a polynomial ring over a Weyl algebra, making it a well understood ring.

In Section 5 we investigate various ring-theoretic properties of L_n^q and C_n^q, for suitably generic values of q. The main question is the prime spectrum of these rings, which for L_n^q turns out to be relatively straightforward: if n is odd, the prime spectrum of L_n^q is homeomorphic to that of $k[x]$, which is isomorphic to L_1^q; while if n is even, the prime spectrum of L_n^q is homeomorphic to that of the quantized Weyl algebra, which is isomorphic to L_2^q. For C_n^q, the prime spectrum is more complicated: there is a central element Ω, so $\Omega - \lambda$ generates a prime ideal for any $\lambda \in k$, but although this ideal is maximal for most values of λ, for countably many λ, there is a single maximal ideal strictly containing $(\Omega - \lambda)C_n^q$. Further, for each $a \geq 1$, two of these maximal ideals are such that the uniform rank of the corresponding simple factor of C_n^q equals a. We also determine the automorphism groups of L_n^q and C_n^q, and the Krull and global dimensions of these rings and some of their factor rings.

For many of these results about C_n^q, the technique is to pass to a localisation of C_n^q which has the structure of an ambiskew polynomial ring, prove a more general result, and then pass that result back to C_n^q. Ambiskew polynomial rings are a construction introduced by Jordan in the 1990s which include many classic examples of noncommutative algebras, most notably the enveloping algebra $U(sl_2)$ and its quantization $U_q(sl_2)$. As the name suggests, they are iterated skew polynomial extensions of a base ring in two variables, with a symmetry between the two indeterminates.

The ambiskew polynomial rings we consider will be conformal, meaning they have a normal Casimir element z. Factoring out a Casimir element from an ambiskew polynomial ring gives a construction called a generalised Weyl algebra. These had been introduced by Bavula prior to that of ambiskew polynomial rings; in fact, any ambiskew polynomial ring can be presented as a generalised Weyl algebra, though we will not make use of this.

In Section 3, we prove the results about ambiskew polynomial rings and generalised Weyl algebras which we need for Section 5, building on the work of Bavula, Jordan and Wells on the simplicity of these rings and their dimension theory. Specifically, we give conditions for a simple localisation of an ambiskew polynomial ring in the case of a central Casimir element, and describe the unique maximal ideal of a generalised Weyl algebra in the case where one condition of the simplicity criterion for generalised Weyl algebras developed by Bavula does not hold.

In Section 6, we investigate the Poisson prime spectra of the semiclassical limit Poisson algebras of the families $(L_n^q)_{q\neq 0}$ and $(C_n^q)_{q\neq 0}$. These correspond to the prime spectra of the corresponding noncommutative algebras, although in the case of C_n^P there are only two exceptional height 2 Poisson maximal ideals, as opposed to countably infinitely many in the noncommutative case. (However, we note that setting $q = 1$ in the formula that describes the λ such that $(\Omega - \lambda)C_n^q$ is not maximal gives only two distinct values for λ.)

In Sections 7 and 8 we return to the cluster algebras where the story began. Section 8 describes a quantum cluster algebra structure on L_n^q and determines a quantum cluster algebra that contains C_n^q. (It is not known whether there is a quantum cluster algebra structure on C_n^q itself, but it seems unlikely). Both these quantum cluster algebras arise from acyclic quivers, and in Section 7 we produce a simple localisation for a quantum cluster algebra arising from an acyclic quiver, extending a result of Zwicknagl.

The reader may notice that the presentation in this introduction is not the same order as the presentation in the thesis. The order in the thesis ensures that things build up mathematically, so no result requires a result from later in the thesis, whereas the order in this introduction is more-or-less chronological, showing the order in which the results were developed and how they led to each other.

2 Background

The aim of this section is to set up the definitions, notations and standard theorems about the various structures we will use.

2.1 Notation

Most notation is standard; the following are potential areas of confusion.

$\mathbb{N}$ denotes the natural numbers; we consider 0 to be a natural number. $\mathbb{N}^+$ denotes the strictly positive integers.

$A \subset B$ denotes "A is a subset of B"; it does not imply $A \neq B$.

2.2 Noncommutative Noetherian rings

2.2.1 Basics

The books [18] and [33] provide a fuller introduction to this topic.

For us, a **ring** is unital and associative but not necessarily commutative.

A **domain** is a ring in which every non-zero element is regular, i.e. not a zero-divisor. (It is not required to be commutative). A **field** is a commutative ring in which every non-zero element has a multiplicative inverse (a noncommutative ring with this property is a **division ring**). We will use the notation R^* for the set of non-zero elements of a domain R.

For our purposes, a **k-algebra** is a ring R with a homomorphism $k \to Z(R)$, where $Z(R)$ denotes the centre of R. Generally k will be a field, in which case the homomorphism is injective and we identify k with its image in R. If k is a field then a k-algebra is also a k-vector space. Most of our rings will be k-algebras for some (arbitrary) field k. When we talk of homomorphisms between k-algebras we will mean k-algebra homomorphisms unless otherwise stated.

Definition 2.1. A two-sided ideal P of a ring R is a **prime ideal** of R if, for any two-sided ideals A and B of R, $AB \subset P$ implies $A \subset P$ or $B \subset P$.

A two-sided ideal P of a ring R is a **completely prime ideal** of R if, for any elements a and b of R, $ab \in P$ implies $a \in P$ or $b \in P$ - that is, if the factor ring R/P is a domain.

From now on, we will use "ideal" to mean "two-sided ideal", and explicitly write "left ideal" or "right ideal" when discussing one-sided ideals.

We denote the set of prime ideals of a ring R, which is called the **spectrum** of R, by Spec R. For our purposes, we will think of this as a partially ordered set under inclusion, though one can put various topologies on it.

Proposition 2.2. *A completely prime ideal is a prime ideal.*

Proof. This follows from [33, 0.2.3(i)]. □

Remark. In commutative algebra the definition given above for "completely prime ideal" would normally be given as the definition of "prime ideal". However, for a commutative ring all prime ideals are completely prime, so there is no inconsistency.

Definition 2.3. A ring R is **left Noetherian** if it satisfies the ascending chain condition on left ideals, that is, if $I_1 \subset I_2 \subset I_3 \subset \cdots$ is a chain of left ideals of R then there exists $n \in \mathbb{N}$ such that $I_m = I_n$ for all $m \geq n$.

Similarly a ring is **right Noetherian** if it satisfies the ascending chain condition on right ideals.

A ring is **Noetherian** if it is both left and right Noetherian.

Proposition 2.4. *([18, Remark after Prop 1.2]). If R is a left (resp. right) Noetherian ring and I is a two-sided ideal of R then R/I is also left (resp. right) Noetherian.*

Definition 2.5. For $a \in \mathbb{N}^+$, q an element of some ring R, define $[a]_q := 1 + q + \cdots + q^{a-1}$. If R is a field, then $[a]_q = \frac{1-q^a}{1-q}$ and so if $q^a \neq 1$, $[a]_q \neq 0$

2.2.2 Localisation

In commutative algebra, given an integral domain R and a prime ideal P of R, one can **localise** R at P, constructing a ring R_P which contains R, but in which every non-zero element of $R \setminus P$ is a unit. There is then a one-to-one correspondence between ideals of R_P and ideals of R containing P, which is crucial to the study of prime ideals in R.

In noncommutative algebra, a similar technique works, but more care is needed. In the following definition, it is possible - with some extra work - to remove the requirement that $\mathcal{X}$ only contains regular elements of R. However, we will not need this, so state the definition in this form for simplicity.

Definition 2.6. ([33, 2.1.3]). Let R be a ring, and $\mathcal{X}$ a multiplicatively closed set of regular elements of R - that is, if $x, y \in \mathcal{X}$ then $xy \in \mathcal{X}$, and no element of $\mathcal{X}$ is a zero-divisor. Then a (right) **localisation** of R with respect to $\mathcal{X}$ is a ring extension Q of R such that:

(i) for all $x \in \mathcal{X}$, x is a unit in Q;

(ii) for all $q \in Q$, $q = rx^{-1}$ for some $r \in R$, $x \in \mathcal{X}$.

Remark. The term "quotient ring" is unhelpful, as it can refer both to the above construction and to the construction R/I where I is a two-sided ideal of R. We will use the terms "localisation" for the first and "factor ring" for the second throughout.

Definition 2.7. ([33, 2.1.6]). A set $\mathcal{X}$ of elements of a ring R is called a **right Ore set** if it is a multiplicatively closed set of regular elements of R and, for every $r \in R$ and $x \in \mathcal{X}$, there exist $r' \in R$ and $x' \in \mathcal{X}$ such that $rx' = xr'$.

Theorem 2.8. *([33, 2.1.12, 2.1.4]). Let R be a ring and $\mathcal{X}$ a multiplicatively closed set of regular elements of R. Then a right localisation of R with respect to $\mathcal{X}$ exists if and only if $\mathcal{X}$ is a right Ore set. If it does exist, it is unique up to isomorphism and we will denote it by $R_{\mathcal{X}}$.*

One can similarly define left localisations and left Ore sets, and the analogue of the above theorem holds; we denote the left localisation by ${}_{\mathcal{X}}R$.

Proposition 2.9. *([18, Proposition 6.5]). Let R be a ring and let $\mathcal{X}$ be a right and left Ore set in R. Then ${}_{\mathcal{X}}R = R_{\mathcal{X}}$.*

A key technique in investigating the prime spectrum of a ring is to localise, investigate the prime spectrum of the localisation and then pass back to the original ring. If the original ring is Noetherian then passing between the two is straightforward:

Theorem 2.10. *([33, 2.1.16(vii)]). Let R be a Noetherian ring, and $\mathcal{X}$ a right Ore set in R. Then there is a one-to-one inclusion-preserving correspondence between Spec $R_{\mathcal{X}}$ and $\{P \in \text{Spec } R : P \cap \mathcal{X} = \emptyset\}$ given by $P' \mapsto P' \cap R$, $P \mapsto PR_{\mathcal{X}}$.*

([18, Exercise 6C]). Furthermore, $R_{\mathcal{X}}$ is a Noetherian ring.

When R is not necessarily Noetherian we need to be more careful. We will need the following two results, both of which are immediate from the previous theorem if R is Noetherian.

Proposition 2.11. *Let R be a domain and $\mathcal{X}$ a right Ore set in R. If $I \cap \mathcal{X} \neq \emptyset$ for any non-zero ideal I of R, then $R_{\mathcal{X}}$ is simple. If in addition $R_{\mathcal{X}}$ is right Noetherian then the converse holds.*

Proof. Suppose first $J \cap \mathcal{X} \neq \emptyset$ for any ideal J of R. If $I \lhd R_{\mathcal{X}}$ then $I \cap R \lhd R$ and $I = (I \cap R)R_{\mathcal{X}}$ by [33, 2.1.16(iii)], so $I \neq 0 \implies I \cap R \neq 0 \implies I \cap R \cap \mathcal{X} \neq \emptyset \implies I \cap \mathcal{X} \neq \emptyset \implies I = R_{\mathcal{X}}$.

Suppose now that $R_{\mathcal{X}}$ is simple and right Noetherian, and let I be a non-zero ideal of R. Then $IR_{\mathcal{X}}$ is a non-zero two-sided ideal of $R_{\mathcal{X}}$ by [33, 2.1.16(vi)], and so $1 \in IR_{\mathcal{X}}$; but $IR_{\mathcal{X}} = \{is^{-1} : i \in I, s \in \mathcal{X}\}$, so $I \cap \mathcal{X} \neq 0$. □

The next result generalises [29, Lemma 3.1], where the Ore set $\mathcal{Y}$ is $\{y^i\}_{i \geq 1}$ for some regular $y \in R$.

Lemma 2.12. *Let R be a ring and $\mathcal{Y}$ a right and left Ore set in R such that any two elements of $\mathcal{Y}$ commute. If $R_{\mathcal{Y}}$ is simple and I is a non-zero ideal of R then $I \cap \mathcal{Y} \neq 0$.*

Proof. Let $J := \{y^{-1}iz^{-1} : y, z \in \mathcal{Y}, i \in I\} \subset R_{\mathcal{Y}}$. We claim that J is an ideal of $R_{\mathcal{Y}}$. If $j_1 = y_1^{-1}i_1z_1^{-1} \in J$ and $j_2 = y_2^{-1}i_2z_2^{-1} \in J$ then

$$\begin{aligned} j_1 + j_2 &= y_1^{-1}i_1z_1^{-1} + y_2^{-1}i_2z_2^{-1} \\ &= y_1^{-1}y_2^{-1}y_2i_1z_2z_2^{-1}z_1^{-1} + y_2^{-1}y_1^{-1}y_1i_2z_1z_1^{-1}z_2^{-1} \\ &= (y_1y_2)^{-1}(y_2i_1z_2 + y_1i_2z_1)(z_1z_2)^{-1} \qquad \text{since elements of } \mathcal{Y} \text{ commute.} \end{aligned}$$

This is an element of J since $\mathcal{Y}$ is multiplicatively closed and I is an ideal of R.

If $j = y^{-1}iz^{-1} \in J$ and $c = rx^{-1} \in R_{\mathcal{Y}}$ where $r \in R$, $x \in \mathcal{Y}$, then

$$\begin{aligned} cj &= rx^{-1}y^{-1}iz^{-1} \\ &= y'^{-1}r'iz^{-1} \in J \quad \text{for some } y' \in \mathcal{Y},\ r' \in R \text{ since } \mathcal{Y} \text{ is right Ore.} \end{aligned}$$

Similarly if $j = y^{-1}iz^{-1} \in J$ and $c = x^{-1}r \in {}_{\mathcal{Y}}R = R_{\mathcal{Y}}$ then $jc \in J$, so we've shown our claim.

So J is a non-zero ideal of the simple ring $R_{\mathcal{Y}}$, so $1 \in J$. Therefore $1 = y^{-1}iz^{-1}$ for some $y, z \in \mathcal{Y}$, $i \in I$ so $i = yz \in I \cap \mathcal{Y}$. □

2.2.3 Skew polynomial rings

One key construction of noncommutative rings for our purposes is that of **skew polynomial rings**. These are rings whose additive structure is the same as that of a polynomial ring, but where the multiplication has been "twisted" by an automorphism and a derivation of the base ring. More formally:

Definition 2.13. Let R be a ring and α an automorphism of R. Then a (left) α-**derivation** of R is a homomorphism of the additive abelian groups $\delta : R \mapsto R$ such that

$$\delta(rs) = \alpha(r)\delta(s) + \delta(r)s \text{ for all } r, s \in R$$

If R is a k-algebra then a k-algebra α-derivation additionally satisfies $\delta(\lambda) = 0$ for all $\lambda \in k$.

Remark. One can similarly define a right α-derivation, in which case various aspects of the following definition are also reversed. The book [18] uses the same convention as we do, while the book [33] uses the opposite convention. From now on, all α-derivations are left α-derivations, but we will happily cite the latter book and trust the reader - with this warning - to make the appropriate left/right changes.

Definition 2.14. ([18, p34]). Let R be a ring, α an automorphism of R, and δ an α-derivation of R.

We write $S = R[x; \alpha, \delta]$, and call S a **skew polynomial ring over** R if S is a ring extension of R with an $x \in S$ such that:

(i) S is a free left R-module with basis $\{1, x, x^2, \cdots\}$;

(ii) $xr = \alpha(r)x + \delta(r)$ for all $r \in R$.

If α is the identity automorphism or δ is the zero α-derivation, then we omit them and write $S = R[x; \delta]$ or $S = R[x; \alpha]$ - or, if both if these, occur, then S is just a polynomial ring over R and we write $S = R[x]$.

We write $S = R[x^{\pm 1}; \alpha]$, and call S a **skew Laurent polynomial ring over** R if S is a ring extension of R with an $x \in S$ such that:

(i) x is a unit in S;

(ii) S is a free left R-module with basis $\{\cdots, x^{-2}, x^{-1}, 1, x, x^2, \cdots\}$;

(iii) $xr = \alpha(r)x$ for all $r \in R$.

Again, if α is the identity automorphism then S is just the Laurent polynomial ring over R and we write $S = R[x^{\pm 1}]$.

Remark. It is not actually required for this construction that α be an automorphism, but we will not be considering more general skew polynomial rings, and the results below are easier to state if α is always assumed to be an automorphism.

Proposition 2.15. *Given a ring R, an automorphism α of R, and an α-derivation δ of R, the skew polynomial ring $R[x; \alpha, \delta]$ exists and is unique.*

Given a ring R and an automorphism α of R, the skew Laurent polynomial ring $R[x^{\pm 1}; \alpha]$ exists and is unique.

Proof. [18, Proposition 2.3] gives existence and [18, Corollary 2.5] gives uniqueness for skew polynomial rings, while [18, Exercises 1M and 1N, extending Lemma 1.11 and Corollary 1.12] give the results for skew Laurent polynomial rings. □

Proposition 2.16. *Let R be a ring, α an automorphism of R, and δ an α-derivation of R.*

(i) $R[x;\alpha,\delta]$ *can also be described as the ring generated by R and x subject to the relations* $ax = x\alpha(a) + \delta(a)$ *for all* $a \in R$.

(ii) If R is a domain then $R[x;\alpha,\delta]$ is a domain also.

(iii) If R is right (resp. left) Noetherian then $R[x;\alpha,\delta]$ is right (resp. left) Noetherian also.

Proof. (i) See [33, 1.2.4].

(ii) See [33, 1.2.9(i)].

(iii) See [33, 1.2.9(iv)].

□

Proposition 2.17. *Let R be a ring and α an automorphism of R.*

(i) $R[x^{\pm 1};\alpha]$ *can also be described as the ring generated by R and $x^{\pm 1}$ subject to the relations* $ax = x\alpha(a)$ *for all* $a \in R$, *plus the relation* $xx^{-1} = 1 = x^{-1}x$.

(ii) If R is a domain then $R[x^{\pm 1};\alpha]$ is a domain also.

(iii) If R is right (resp. left) Noetherian then $R[x^{\pm 1};\alpha]$ is right (resp. left) Noetherian also.

(iv) The set $\mathcal{X} = \{x^i : i \in \mathbb{N}\}$ is a right and left Ore set in $R[x;\alpha]$, and $R[x^{\pm 1};\alpha] = R[x;\alpha]_{\mathcal{X}}$.

Proof. (i) See [33, 1.4.3].

(ii) See [33, 1.4.5].

(iii) See [33, 1.4.5].

(iv) See [18, Exercise 10D].

□

Definition 2.18. An **iterated skew polynomial ring** over a ring R is a ring of the form

$$R[x_1;\alpha_1,\delta_1][x_2;\alpha_2,\delta_2]\cdots[x_n;\alpha_n,\delta_n]$$

where each α_i is an automorphism of $R[x_1;\alpha_1,\delta_1]\cdots[x_{i-1};\alpha_{i-1},\delta_{i-1}]$ and each δ_i is an α_i-derivation of the same ring.

An **iterated skew Laurent polynomial ring** over a ring R is a ring of the form

$$R[x_1^{\pm 1};\alpha_1][x_2^{\pm 1};\alpha_2]\cdots[x_n^{\pm 1};\alpha_n]$$

where each α_i is an automorphism of $R[x_1^{\pm 1};\alpha_1]\cdots[x_{i-1}^{\pm 1};\alpha_{i-1}]$.

Example 2.19. The **quantum plane**, sometimes denoted $k_q[x,y]$ or $\mathcal{O}_q(k^2)$, is the k-algebra generated by x and y subject to the relation $xy-qyx=0$, where $q\in k^\times$ is some scalar. This is an iterated skew polynomial ring $k[x][y;\alpha]$ where $\alpha(x)=qx$. The name and notation come from the fact that this is a quantization of the commutative ring $k[x,y]$ - in the sense that $\mathcal{O}_1(k^2)=k[x,y]$ - which in algebraic geometry is the coordinate ring of the plane k^2 and so is sometimes denoted $\mathcal{O}(k^2)$.

The (2-dimensional) **quantum torus**, sometimes denoted $k_q[x^{\pm 1},y^{\pm 1}]$ or $\mathcal{O}_q((k^\times)^2)$, is the k-algebra generated by $x^{\pm 1}$ and $y^{\pm 1}$ subject to the relation $xy-qyx=0$, where $q\in k^\times$ is some scalar. This is an iterated skew Laurent polynomial ring $k[x^{\pm 1}][y^{\pm 1};\alpha]$ where $\alpha(x)=qx$. In a similar fashion, this is a quantization of the commutative Laurent polynomial ring $k[x^{\pm 1},y^{\pm 1}]=O((k^\times)^2)$, the coordinate ring of the 2-dimensional torus.

Note that in the rest of this work, we will use a third different notation for the quantum plane, the quantum torus, and their higher-dimensional analogues: see Section 2.2.4.

The **first Weyl algebra**, usually denoted $A_1(k)$ or just A_1, is the k-algebra generated by x and y subject to the relation $xy-yx=1$. This is an iterated skew polynomial ring $k[x][y;\delta]$ where $\delta(x)=1$.

The **first quantized Weyl algebra**, usually denoted $A_1^q(k)$ or just A_1^q, is the k-algebra generated by x and y subject to the relation $xy-qyx=1-q$. This is an iterated skew polynomial ring $k[x][y;\alpha,\delta]$ where $\alpha(x)=qx$ and $\delta(x)=1-q$.

Definition 2.20. If R is a k-algebra, $x,y\in R$, $q\in k$, we will sometimes write $[x,y]_q:=xy-qyx$. This is called the **q-commutator** of x and y. If $[x,y]_q=0$ then we say x and y **q-commute**; we say x and y **skew-commute** if there exists some $q\in k$ such that x and y q-commute.

For instance, in the quantum plane $\mathcal{O}_q(k^2)$, $[x,y]_q=0$ and x^a skew-commutes with y^b for all $a,b\in\mathbb{N}$.

Remark. Often, the first quantized Weyl algebra is described as the k-algebra generated by x and y subject to the relation $xy - qyx = 1$. Provided $q \neq 1$, these two algebras are isomorphic. We use the above convention for two reasons: firstly, the relation can be rewritten $yx - q^{-1}xy = 1 - q^{-1}$, giving a symmetry between x and y up to replacing q by q^{-1}; secondly, since the case $q = 1$ is then commutative, this allows us to form a semiclassical limit Poisson algebra as in Section 2.4.

Proposition 2.21. *Let R be a k-algebra, α a k-algebra automorphism of R, and δ a k-algebra α-derivation of R. Suppose R is generated as a k-algebra by a set $X \subset R$. Then $R[x; \alpha, \delta]$ can also be described as the k-algebra generated by R and x subject to the relations $ax = x\alpha(a) + \delta(a)$ for all $a \in X$.*

Let R be a k-algebra and α a k-algebra automorphism of R. Suppose R is generated as a k-algebra by a set $X \subset R$. Then $R[x^{\pm 1}; \alpha]$ can also be described as the k-algebra generated by R and $x^{\pm 1}$ subject to the relations $ax = x\alpha(a)$ for all $a \in X$, plus the relation $xx^{-1} = 1 = x^{-1}x$.

Proof. Let S denote $R[x; \alpha, \delta]$, respectively $R[x^{\pm 1}; \alpha]$, and let T denote the k-algebra generated by R and x, respectively R and $x^{\pm 1}$, subject to the relations $ax = x\alpha(a) + \delta(a)$ for all $a \in X$ (with the notational convention that $\delta = 0$ in the skew Laurent case). So by Proposition 2.16 (i) or Proposition 2.17 (i) respectively, there is a surjection $\psi : T \to S$ which is the identity on R and sends x to x.

We note that if $a \in R$ is such that, in T, $ax = x\alpha(a) + \delta(a)$ and $b \in R$ is such that, in T, $bx = x\alpha(b) + \delta(b)$ then, again with calculations taking place in T:

$(a + b)x = ax + bx = x\alpha(a) + x\alpha(b) + \delta(a) + \delta(b) = x\alpha(a + b) + \delta(a + b)$;

$(ab)x = a(x\alpha(b) + \delta(b)) = x\alpha(a)\alpha(b) + \delta(a)\alpha(b) + a\delta(b) = x\alpha(ab) + \delta(ab)$; and

$(\lambda a)x = \lambda(x\alpha(a) + \delta(a)) = x\lambda\alpha(a) + \lambda\delta(a) = x\alpha(\lambda a) + \delta(\lambda a)$ for any $\lambda \in k$.

Therefore, the set $\{a \in R : ax = x\alpha(a) + \delta(a) \text{ in } T\}$ is a subalgebra of R. But this subalgebra contains X, and so equals R since X generates R.

So again by Proposition 2.16 (i) or Proposition 2.17 (i) respectively, there is a surjection $\phi : S \to T$ which is the identity on R and sends x to x. But since they are both surjections, ϕ must be the inverse of the map ψ, and so they must be isomorphisms. Since they are both the identity on R and send x to x, we can use these maps to identify S and T. □

Proposition 2.22. *Let k be a ring, X a set, F the free k-algebra generated by X, I an ideal of F generated by a set Y, and α a function $X \to F$. Using the universal property for F we can extend α to a k-algebra homomorphism $F \to F$, which we can then compose with the* *natural map $F \to F/I$ to get a k-algebra homomorphism $F \to F/I$. We claim this induces a k-algebra homomorphism $F/I \to F/I$ if and only if $\alpha(I) \subset I$. Further, if α defined an automorphism of F and $\alpha(I) = I$ then the induced homomorphism $F/I \to F/I$ is also an isomorphism.*

If in addition to the above we have a function $\delta : X \to F$, we can extend δ to a k-algebra α-derivation of F using the definition of an α-derivation. We claim this induces a k-algebra α-derivation of F/I if and only if $\delta(I) \subset I$.

Proof. Denote by π the natural map $F \to F/I$.

Define $\alpha^* : F/I \to F/I$ by $\alpha^*(x) = \pi(\alpha(x^*))$ where x^* is such that $\pi(x^*) = x$. This is well-defined since if $\pi(z^*) = \pi(x^*) = x$ then $z^* - x^* \in I$, so $\pi(\alpha(z^*)) = \pi(\alpha(x^*) + \alpha(z^* - x^*)) = \pi(\alpha(x^*)) = \alpha^*(x)$ since $\alpha(I) \subset I$. That α^* is a homomorphism follows from the fact that α and π are homomorphisms - for example, $\alpha^*(x + y) = \pi(\alpha(x^* + y^*)) = \pi\alpha(x^*) + \pi\alpha(y^*) = \alpha^*(x) + \alpha^*(y)$.

If $\alpha(I) = I$ and α is a automorphism then $\alpha^*(a) = 0$ iff $\alpha(a^*) \in I$ iff $a^* \in I$ iff $a = 0$, and so α^* is injective, while if $a \in F/I$, there exists $b^* \in F$ such that $\pi(\alpha(b^*)) = a$ since α is surjective, and so $\alpha^*(\pi(b^*)) = a$, so α^* is surjective, and so an automorphism.

The same argument as in the first part lets us define $\delta^* : F/I \to F/I$ by $\delta^*(x) = \pi(\delta(x^*))$ where $\pi(x^*) = x$. Then since $\delta^*(xy) = \pi(\delta(x^*y^*)) = \pi(\alpha(x^*)\delta(y^*) + \delta(x^*)y^*) = \alpha^*(x)\delta^*(y) + \delta^*(x)y$, δ^* is an α^*-derivation. □

When applying this proposition to a k-algebra defined by generators and relations, we will not formally pass up to the free algebra, but rather "check that all the relations are preserved by α", as in the following example. This is a shorthand, though – what we are actually doing is what is described above.

Example 2.23. The **quantum space** with parameters q_{ij} for $1 \leq i, j \leq n$, where $q_{ij} \in k$ and $q_{ij}q_{ji} = 1$, is the k-algebra generated by $x_1, \ldots, x_n$ subject to the relations $x_i x_j = q_{ij} x_j x_i$ for all i, j. This is an iterated skew polynomial ring $k[x_1][x_2; \alpha_2] \cdots [x_n; \alpha_n]$ where $\alpha_j(x_i) = q_{ij} x_i$ for $i < j$.

Proof. We prove this by induction on n. If $n = 1$ this is trivial. For $n > 1$, the subalgebra R_{n-1} generated by $x_1, \ldots, x_{n-1}$ is also a quantum space, and so by induction it is an iterated skew polynomial ring. We check that α_n defines an automorphism of R_{n-1}: if $i < j < n$ then $\alpha_n(x_i x_j - q x_j x_i) = q_{in} q_{jn} x_i x_j - q_{jn} q_{in} q x_j x_i = q_{in} q_{jn}(x_i x_j - q x_j x_i)$, so by Proposition 2.22 α_n does define an automorphism of R_{n-1}. Then by Proposition 2.21, $R_{n-1}[x_n; \alpha_n]$ is indeed the described quantum space. □

Example 2.24. The **second quantized Weyl algebra** with parameter q is the k-algebra generated by x_1, x_2, y_1, and y_2 subject to the relations $x_iy_i = qy_ix_i$ for $i = 1, 2$, $x_1x_2 = x_2x_1$, $y_1y_2 = y_2y_1$, and $x_iy_j = y_jx_i$ for $i \neq j$. This is an iterated skew polynomial ring $k[x_1][y_1; \alpha_1, \delta_1][x_2][y_2; \alpha_2, \delta_2]$ where $\alpha_i(x_i) = qx_i$ for $i = 1, 2$, $\alpha_2(x_1) = x_1$, $\alpha_2(x_2) = x_2$, $\delta_i(x_i) = 1 - q$ for $i = 1, 2$, $\delta_2(x_1) = \delta_2(x_2) = 0$.

Proof. Certainly the subalgebra S of this second quantized Weyl algebra generated by x_1, y_1 and x_2 equals $k[x_1][y_1; \alpha_1, \delta_1][x_2]$. We check that α_2 defines an automorphism of S: $\alpha_2(x_1y_1 - qy_1x_1 - (1-q)) = x_1y_1 - qy_1x_1 - (1-q)$, $\alpha_2(x_1x_2 - x_2x_1) = q(x_1x_2 - x_2x_1)$, and $\alpha_2(y_1x_2 - x_2y_1) = q(y_1x_2 - x_2y_1)$, so by Proposition 2.22 α_n does define an automorphism of S. Now we check that δ_2 defines an α_2-derivation of S: $\delta_2(x_1y_1 - qy_1x_1 - (1 - q)) = 0$, $\delta_2(x_1x_2 - x_2x_1) = (x_1(1-q)+0) - (0+(1-q)x_1) = 0$, and $\delta_2(y_1x_2 - x_2y_1) = (y_1(1-q)+0) - (0+(1-q)y_1) = 0$, so by Proposition 2.22, δ_2 does define an α_2-derivation of S. Then by Proposition 2.21, $S[y_2; \alpha_2, \delta_2]$ is indeed the second quantized Weyl algebra. □

Definition 2.25. Let R be a k-algebra, and let X be some finite set of elements of R, with some total ordering so that we can write $X = \{x_1, \dots, x_n\}$. Then the family $(x_1^{a_1} \cdots x_n^{a_n})_{a_i \in \mathbb{N}}$ is a **PBW basis** for R if it is a basis for R, that is, it spans R as a k-vector space and is linearly independent over k.

Remark. This is a family, as in [35, 1.1 and 6.5.1], rather than a set, so it can have repeated elements - but if it does, then it is certainly not linearly independent.

Proposition 2.26. *Let $R = k[x_1][x_2; \alpha_2, \delta_2] \cdots [x_n; \alpha_n, \delta_n]$ be an iterated skew polynomial ring. Then R has a PBW basis with respect to the set $\{x_1, \dots, x_n\}$.*

We conclude this section with some results about prime ideals in skew polynomial and skew Laurent rings. The first is a standard characterisation of simplicity in skew Laurent rings.

Definition 2.27. Let A be a ring, and α an automorphism of A. Then A is **α-simple** if it has no non-trivial α-stable (that is, $\alpha(I) \subset I$) ideals.

Definition 2.28. Let A be a ring, and $u \in A$ a unit. Then $a \mapsto u^{-1}au$ is an automorphism of A. Such an automorphism is called an **inner automorphism** of A.

Theorem 2.29. *([33, 1.8.5]). Let A be a ring, and α an automorphism of A. Then $A[x^{\pm 1}; \alpha]$ is simple if and only if A is α-simple and no power of α is an inner automorphism of A.*

The next result follows the proof of [33, 9.6.9(i)], and in fact follows from that result, but we prove it here to avoid unnecessarily setting up the machinery of tensor products.

Theorem 2.30. *Let k be a field, let A be a simple k-algebra such that $Z(A) = k$, and let I be an ideal of $A[t]$. Then I is generated by an element of $k[t]$.*

Proof. Consider the polynomials in I of minimal degree d. Their leading coefficients form an ideal of A, so there must be some monic polynomial p in I of degree d. Then for every $a \in A$, $ap - pa$ is an element of I of degree $< d$, so must be zero. Therefore each coefficient in p must be central in A, so in k, that is, $p \in k[t]$.

Now consider the polynomials in I which are not in $pA[t]$, and consider such a polynomial q of minimal degree among such polynomials. If q has leading coefficient a and degree e then $r = q - apt^{e-d}$ is an element of I of degree less than e, so by the minimality of e, $r \in pA[t]$, and so $q = apt^{e-d} + r \in pA[t]$, contradicting that $q \notin pA[t]$. Therefore no such q exists and $I = pA[t]$. □

Finally, we give a result that allows us to show that certain normal elements in a skew polynomial ring generate completely prime ideals.

Lemma 2.31. *([27, Proposition 1]). Let σ be an automorphism and δ a σ-derivation of a domain A. Let $R = A[x; \sigma, \delta]$. Let c be a normal element of R of the form $dx + e$, where $d, e \in A$ and $d \neq 0$. Let β be the automorphism of R such that $cr = \beta(r)c$ for all $r \in R$. Then $\beta(A) = A$, d is normal in A and $\beta(a)d = d\sigma(a)$ for all $a \in A$. Furthermore, if e is regular modulo Ad then R/Rc is a domain.*

2.2.4 Quantum spaces and quantum tori

Definition 2.32. Let k be a field, let $\mathbf{x} = (x_v)_{v \in Q}$ be a tuple indexed by some finite set Q, and let $\mathbf{L} = (L_{vw})_{v,w \in Q}$, where the L_{vw} are integers such that $L_{vw} = -L_{wv}$ for $v, w \in Q$, that is, $\mathbf{L}$ is a skew-symmetric integer matrix indexed by Q. Then the **quantum space** $S_q(k, \mathbf{x}, \mathbf{L})$ - which we will shorten to $S_q(\mathbf{x}, \mathbf{L})$ if the base field is unambiguous - is the k-algebra generated by the set $\mathbf{x}$ subject to the relations $x_v x_w = q^{L_{vw}} x_w x_v$ for all $v, w \in Q$.

If we denote by $\mathcal{W}$ the set of all monomials in $\mathbf{x}$ within $S_q(k, \mathbf{x}, \mathbf{L})$, we can define the **quantum torus** $T_q(k, \mathbf{x}, \mathbf{L})$ - or as before, $T_q(\mathbf{x}, \mathbf{L})$ if the base field is unambiguous - as $S_q(k, \mathbf{x}, \mathbf{L})_{\mathcal{W}}$. Alternatively, this is the k-algebra generated by $\mathbf{x}^{\pm 1} := \mathbf{x} \cup \{x_v^{-1} : v \in Q\}$, subject to the relations $x_v x_v^{-1} = x_v^{-1} x_v = 1$ as well as $x_v x_w = q^{L_{vw}} x_w x_v$ for all $v, w \in Q$.

If we have an ordering of the elements of Q then we can give $\mathbf{x}$ the same ordering, which allows us, by Propositions 2.21 and 2.22, to construct $S_q(k, \mathbf{x}, \mathbf{L})$ as an iterated skew polynomial ring over k, and $T_q(k, \mathbf{x}, \mathbf{L})$ as an iterated skew Laurent polynomial ring over k. This gives a PBW basis for $S_q(\mathbf{x}, \mathbf{L})$ or $T_q(\mathbf{x}, \mathbf{L})$ consisting of monomials in $\mathbf{x}$ or in $\mathbf{x}^{\pm 1}$ respectively.

In Section 2.2.3 we had the quantum plane, the quantum torus and the quantized Weyl algebra. The above are higher-dimensional versions of the first two, which we will use extensively. There are several higher-dimensional versions of the quantized Weyl algebra, with the one found in (e.g.) [2, 1.4] being perhaps the most common, but the following simpler version from [2, 1.5] is of more relevance to us:

Definition 2.33. Let k be a field, let $n \in \mathbb{N}^+$, let $\mathbf{\Lambda} = (\lambda_{ij})_{1 \leq i,j \leq n}$ be a multiplicatively skew-symmetric $n \times n$ matrix of non-zero elements of k, and let $\mathbf{q} = (q_i)_{1 \leq i \leq n}$ be an n-tuple of non-zero elements of k. Then the n^{th} quantized Weyl algebra $\mathcal{A}_n^{\mathbf{q},\lambda}$ is the k-algebra generated by a set of generators $\{x_i, y_i : 1 \leq i \leq n\}$ and relations, for $1 \leq i < j \leq n$,

$$x_i x_j = \lambda_{ij} x_j x_i \qquad\qquad y_j y_i = \lambda_{ji} y_i y_j$$

$$x_i y_j = \lambda_{ji} y_j x_i \qquad\qquad x_j y_i = \lambda_{ij} y_i x_j$$

$$x_j y_j = q_j y_j x_j + 1$$

In [2, 1.5] it is also shown that $\mathcal{A}_n^{\mathbf{q},\lambda}$ is an iterated skew polynomial ring over k.

2.2.5 Uniform rank

For the purposes of this section, all modules are right R-modules.

Definition 2.34. ([33, 2.2.1, 2.2.5]). Let M be an R-module. An **essential submodule** of M is a submodule N such that $N \cap X \neq 0$ for all nonzero submodules X of M.

A module M is a (right) **uniform module** if M is non-zero and every non-zero submodule of U is an essential submodule.

The next result gives a useful characterisation of uniform modules.

Lemma 2.35. *([33, 2.2.5]). A nonzero module U is uniform if and only if, given $u_1, u_2 \in U$ both nonzero, there exist $r_1, r_2 \in R$ such that $u_1 r_1 = u_2 r_2 \neq 0$.*

Definition 2.36. ([33, 2.2.10]). The **uniform rank** of a module M, (also variously called the Goldie rank; the rank; the uniform dimension; the Goldie dimension; or the dimension of M), is ∞ if M contains an infinite direct sum of non-zero submodules, or otherwise is the unique n such that M contains an essential submodule which is a finite direct sum of n uniform submodules.

The **right uniform rank of a ring** R is the uniform rank of R as a right R-module.

The next result confirms that the uniform rank is well-defined.

Theorem 2.37. *Let M be an R-module which does not contain an infinite direct sum of non-zero submodules. Then:*

(i) M contains an essential submodule which is a finite direct sum of uniform submodules, with say n summands ([33, 2.2.8]);

(ii) any direct sum of nonzero submodules has at most n summands ([33, 2.2.9(i)]); and

(iii) a direct sum of uniform submodules of M is essential in M if and only if it has n summands ([33, 2.2.9(ii)]).

Remark. Any Noetherian module - in particular, any finitely generated module over a Noetherian ring - must have finite uniform rank.

Example 2.38. ([33, 2.2.11]). Any right Ore domain has right uniform rank 1.

Let $R = k[x, y]/(x, y)^n$. Then R has right uniform rank n.

Let $R = M_n(S)$, where S has right uniform rank s. Then R has right uniform rank ns.

Remark. These examples illustrate that in some sense the uniform rank of a ring - at least for Noetherian rings - measures how far the ring is from being a domain.

Lemma 2.39. *([33, 2.2.12(v)]). Let $\mathcal{X}$ be a right Ore set of regular elements of a ring R. Then $R_{\mathcal{X}}$ and R have the same right uniform rank.*

2.2.6 Localisation under ring constructions

In the above, we have described a number of methods to construct new rings from an original ring R. We will often have an Ore set in R and wish to pass this Ore set to the new ring we have constructed. This section provides results that allow us to do this. First, we consider factor rings by some ideal I of R.

Lemma 2.40. *Let R be a ring, I a two-sided ideal in R, $\mathcal{X}$ a right Ore set in R such that $\{r \in R : rs \in I \text{ for some } s \in X\} = I$. Then the image $\bar{\mathcal{X}}$ of $\mathcal{X}$ in R/I is a right Ore set in R/I, and there is a surjective map*

$\theta : R_{\mathcal{X}} \to (R/I)_{\bar{\mathcal{X}}}$ *such that* $\theta(rx^{-1}) = \bar{r}\bar{x}^{-1}$ *and* $\ker\theta = I_X$.

Thus we can identify $(R/I)_{\bar{\mathcal{X}}}$ with $R_{\mathcal{X}}/I_{\mathcal{X}}$.

Proof. First we check that $\bar{\mathcal{X}}$ is a right Ore set in R/I. Let $\bar{a} \in R/I, \bar{s} \in \bar{\mathcal{X}}$, and pick lifts to a and s to R and $\mathcal{X}$ respectively. Then there exist $x \in X$ and $r \in R$ s.t. $ax = sr$. Then $\bar{a}\bar{x} = \bar{s}\bar{r}$ and $\bar{x} \in \bar{\mathcal{X}}$, so $\bar{\mathcal{X}}$ is a right Ore set in R/P.

To get existence of θ, we note that the composition of natural maps $R \to R/I \to (R/I)_{\bar{\mathcal{X}}}$ takes any element of $\mathcal{X}$ to an invertible element, and so the above map factors through $R_{\mathcal{X}}$, giving θ such that $\theta(rx^{-1}) = \bar{r}\bar{x}^{-1}$, which is clearly surjective.

We now need to consider $\ker\theta$. Suppose $\theta(rx^{-1}) = 0$, that is, $\bar{r}\bar{x}^{-1} = 0$ in $(R/I)_{\bar{\mathcal{X}}}$. By [18, Lemma 10.1(c)], this holds iff there exists $\bar{z} \in \bar{\mathcal{X}}$ such that $\bar{r}\bar{z} = 0$ in R/I, i.e. iff there exists $z \in \mathcal{X}$ s.t. $rz \in I$. But the condition $\{r \in R : rs \in$I for some $s \in X\} = I$ then tells us that $r \in I$. Thus $rx^{-1} \in \ker\theta$ iff $r \in I$, i.e. $\ker\theta = I_X$. □

Corollary 2.41. *Let R be a right Noetherian ring, P a prime ideal in R and $\mathcal{X}$ a right Ore set in R such that $\mathcal{X} \cap P = \emptyset$. Then the image $\bar{\mathcal{X}}$ of $\mathcal{X}$ in R/P is a right Ore set in R/P, and we can identify $(R/P)_{\bar{\mathcal{X}}}$ with $R_{\mathcal{X}}/P_{\mathcal{X}}$.*

Proof. By [18, Lemma 10.19], (R/P) is $\mathcal{X}$-torsionfree as an R-module, which is equivalent to $\text{ass}_{R/P}\mathcal{X} = \{r \in R : rs \in$P for some $s \in \mathcal{X}\} = P$, so we can apply Lemma 2.40. □

Next, we consider skew polynomial extensions.

Lemma 2.42. *Let R be a ring, α an automorphism of R, δ an α-derivation of R, and $\mathcal{X}$ a right Ore set in R such that $\alpha(\mathcal{X}) = \mathcal{X}$. Then $\mathcal{X}$ is a right Ore set in $R[x; \alpha, \delta]$.*

Proof. We note that if y is regular in R then y is regular in $R[x; \alpha, \delta]$, so $\mathcal{X}$ is a set of regular elements in $R[x; \alpha, \delta]$. Then the result follows from [15, Lemma 1.4]. □

Finally, we have a number of results describing the interactions between two Ore sets in R.

Lemma 2.43. *Let R be a ring, and $\mathcal{X}$ and $\mathcal{Y}$ two right Ore sets in R such that, for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, there exists $q_{xy} \in k^\times$ such that $xy = q_{xy}yx$. Then $\mathcal{X}$ is a right Ore set in $R_{\mathcal{Y}}$.*

By symmetry, $\mathcal{Y}$ is also a right Ore set in $R_{\mathcal{X}}$, and $(R_{\mathcal{X}})_{\mathcal{Y}} = (R_{\mathcal{Y}})_{\mathcal{X}}$.

Proof. Let $x \in \mathcal{X}$ and $r \in R_{\mathcal{Y}}$. We can write $r = sy^{-1}$ where $s \in R$ and $y \in \mathcal{Y}$. Since $\mathcal{X}$ is a right Ore set in R, there exist $s' \in R$ and $x' \in \mathcal{X}$ such that $sx' = xs'$. Then $sy^{-1}x' = q_{x'y}sx'y^{-1} = q_{x'y}xs'y^{-1}$, so defining $r' = q_{x'y}s'y^{-1}$ we have $rx' = xr'$. Therefore $\mathcal{X}$ is a right Ore set in $R_{\mathcal{Y}}$.

For the second part, $(R_{\mathcal{X}})_{\mathcal{Y}}$ is a right localisation of $R_{\mathcal{Y}}$ with respect to $\mathcal{X}$, since all $x \in \mathcal{X}$ are units in $(R_{\mathcal{X}})_{\mathcal{Y}}$, and $(rx^{-1})y^{-1} = (q_{yx}ry^{-1})x^{-1}$, and so we're done by the uniqueness of localisations. □

Lemma 2.44. *Let R be a ring, $\mathcal{X}$ a right Ore set in R, and $\mathcal{Y}$ a set of regular normal elements in R such that for all $y \in \mathcal{Y}$, if α_y is the automorphism defined by $ry = y\alpha_y(r)$, then $\alpha_y(\mathcal{X}) = \mathcal{X}$. Then the multiplicative closure $\mathcal{Z}$ of $\mathcal{X} \cup \mathcal{Y}$ is a right Ore set in R also. Further, $R_{\mathcal{Z}} = (R_{\mathcal{Y}})_{\mathcal{X}}$.*

Proof. Any element of $\mathcal{Z}$ is regular in R, since any element of $\mathcal{X}$ or $\mathcal{Y}$ is. Using the condition on the automorphisms α_y, any element of $\mathcal{Z}$ can be written xy where $x \in \mathcal{X}$ and y is in the multiplicative closure of $\mathcal{Y}$. Let $xy \in \mathcal{Z}$ and $r \in R$. Then since $\mathcal{X}$ is a right Ore set in R, there exist $x' \in \mathcal{X}$, $r' \in R$ such that $rx' = xr'$, so $rx'y = xr'y = xy\alpha_y(r')$. Thus $\mathcal{Z}$ satisfies the right Ore condition also.

Now $(R_{\mathcal{Y}})_{\mathcal{X}}$ is a ring extension of R such that any element of $\mathcal{Z}$ is a unit in $(R_{\mathcal{Y}})_{\mathcal{X}}$, and that any element of $(R_{\mathcal{Y}})_{\mathcal{X}}$ can be written as $ry^{-1}x^{-1} = r(xy)^{-1}$ for some $r \in R$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$. Therefore it is a right localisation of R with respect to $\mathcal{Z}$, and so by Theorem 2.8 we can identify $R_{\mathcal{Z}} = (R_{\mathcal{Y}})_{\mathcal{X}}$. □

Lemma 2.45. *Let R be a domain, and let $\mathcal{X}$ be a right Ore set in R. Suppose S is a ring such that $R \subset S \subset R_{\mathcal{X}}$. Then the right quotient ring $S_{\mathcal{X}}$ exists, and $S_{\mathcal{X}} = R_{\mathcal{X}}$.*

Proof. We claim that $R_{\mathcal{X}}$ is a right quotient ring of S with respect to $\mathcal{X}$ as in Definition 2.6. That is, since S is a domain and $R \subset S \subset R_{\mathcal{X}}$, we need to show that for all $x \in \mathcal{X}$, x is a unit in $R_{\mathcal{X}}$, and that every element of $R_{\mathcal{X}}$ can be written in the form sx^{-1} for some $s \in S$, $x \in \mathcal{X}$. Since $R_{\mathcal{X}}$ is a right quotient ring of R with respect to $\mathcal{X}$, x is a unit in $R_{\mathcal{X}}$ for all $x \in \mathcal{X}$, and every element of $R_{\mathcal{X}}$ can be written in the form rx^{-1} for some $r \in R$, $x \in \mathcal{X}$ - but since $R \subset S$, an element of this form is automatically of the form sx^{-1} for $s \in S$, $x \in \mathcal{X}$. □

Corollary 2.46. *Let R be a domain and let $\mathcal{X} \subset \mathcal{Y}$ be right Ore sets in R. Then $R_{\mathcal{X}} \subset R_{\mathcal{Y}}$ in a natural way, and so $(R_{\mathcal{X}})_{\mathcal{Y}} = R_{\mathcal{Y}}$.*

2.2.7 Dimensions

Definition 2.47. ([33, 7.1.2, 7.1.8]). Let R be a ring. A right R-module P is **projective** if there is a free module F and a module M such that $F \cong P \oplus M$.

A **finite projective resolution** of **length** n of a right R-module M is an exact sequence

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

where each P_i is projective.

The **projective dimension** of M, denoted pd M, is the minimal length of a finite projective resolution for M, or ∞ if no finite projective resolution for M exists.

The **right global dimension** of R is rgld $R := \sup\{\text{pd } M : M \text{ a right } R\text{-module}\}$.

The **left global dimension** of R is defined analogously.

Definition 2.48. ([33, 7.1.3]). A right R-module I is **injective** if, given a module M such that $I \subset M$, there exists $K \subset M$ such that $M = I \oplus K$.

A **finite injective resolution** of **length** n of a right R-module M is an exact sequence

$$0 \to M \to I_0 \to \cdots \to I_n \to 0$$

where each I_i is injective.

The **injective dimension** of M, denoted id M, is the minimal length of a finite injective resolution for M, or ∞ if no finite injective resolution for M exists.

Lemma 2.49. *([33, 7.1.8]). rgld* $R := \sup\{\textit{id } M : M \textit{ a right } R\textit{-module}\}$.

Thus the global dimension of a ring in some sense measures how "complex" its modules can get.

Theorem 2.50. *Let R be a ring.*

(i) If R is Noetherian then lgld $R =$ rgld R.

(ii) rgld $M_n(R) =$ rgld R.

(iii) If x is a regular normal nonunit in R then either rgld $R/xR = \infty$ or rgld $R/xR \leq$ rgld $R - 1$.

(iv) If σ is an automorphism of R and δ is a σ-derivation of R then

(a) rgld $R \leq$ rgld $R[x; \sigma, \delta] \leq$ rgld $R + 1$, provided rgld $R < \infty$;

(b) rgld $R[x; \sigma] =$ rgld $R + 1$;

(c) rgld $R \leq$ rgld $R[x^{\pm 1}; \sigma] \leq$ rgld $R + 1$, with equality in the second if $\sigma = 0$.

(v) If $\mathcal{X}$ is some right Ore set in R then rgld $R \geq$ rgld $R_{\mathcal{X}}$, with equality iff there is a simple right R-module M with pd $M = n$ and $M_{\mathcal{X}} \neq 0$.

Proof. (i) See [33, 7.1.11].

(ii) See [33, 3.5.10 (vi)].

(iii) See [33, 7.3.5 (ii)].

(iv) (a) See [33, 7.5.3 (i)].

(b) See [33, 7.5.3 (iii)].

(c) See [33, 7.5.3 (ii, iv)].

(v) See [33, 7.4.3, 7.4.4].

□

Another method for measuring the "size" of a ring is its **Krull dimension**. For a commutative ring R, this is defined as maximal length of a chain of prime ideals in R, and is a key notion there. However, prime ideals are significantly rarer in noncommutative algebra, due to the existence of simple rings that are not division rings. For a commutative Noetherian ring, having Krull dimension 0 is equivalent to being Artinian, and so Krull dimension describes "how far the ring is from being Artinian", and it is this description that the noncommutative Krull dimension maintains. We will not give technical details of the construction, which can be found in [33, §6], but instead quote some results for calculating Krull dimension.

We will denote the (right) Krull dimension of a ring R by K. dim R. If R is Noetherian then this always exists by [33, 6.2.3].

Theorem 2.51. *Let R be a right Noetherian ring.*

(i) K. dim $R \geq$ the classical Krull dimension of R.

(ii) If R is commutative then K. dim $R =$ the classical Krull dimension of R.

(iii) K. dim $M_n(R) =$ K. dim R.

(iv) If σ is an automorphism of R and δ is a σ-derivation of R then

(a) K. dim $R \leq$ K. dim $R[x;\sigma,\delta] \leq$ K. dim $R+1$, with equality in the second if $\delta = 0$;

(b) *K. dim* $R \leq$ *K. dim* $R[x^{\pm 1};\sigma] \leq$ *K. dim* $R+1$, *with equality in the second if* $\sigma = 0$.

(v) *If* $\mathcal{X}$ *is some right Ore set in* R *then K. dim* $R \geq$ *K. dim* $R_{\mathcal{X}}$.

Proof. (i) See [33, 6.4.5].

(ii) See [33, 6.4.8].

(iii) See [33, 6.5.1 (iii)].

(iv) (a) See [33, 6.5.4 (i)].

(b) See [33, 6.5.4 (ii)].

(v) See [33, 6.5.3 (ii) (b)].

□

Lemma 2.52. *Let* $R \subset S$ *be rings, with* R *Noetherian, such that* S *is a faithfully flat right and left* R*-module. Then K. dim* $R \leq$ *K. dim* S *and rgld* $R \leq$ *rgld* S.

Proof. For Krull dimension, consider the map $\theta : I \mapsto IS = I \otimes S$ from right R-modules to right S-modules. Since ${}_RS$ is flat, this preserves inclusions and $J \otimes S/I \otimes S = (J/I) \otimes S$. Since S_R is faithfully flat, $J/I \otimes S = 0$ iff $J/I = 0$, and so θ preserves strict inclusions. Therefore by [33, 6.5.3(i)], K. dim $R \leq$ K. dim S.

For global dimension, this is [33, 7.2.6]. □

Lemma 2.53. *Let* R *and* S *be rings. Then K. dim* $(R \oplus S) = \sup\{$*K. dim* R, *K. dim* $S\}$ *(if K. dim* R *and K. dim* S *exist) and rgld* $(R \oplus S) = \sup\{$*rgld* R, *rgld* $S\}$.

Proof. For Krull dimension, we use [33, 6.1.14], noting that any left ideal of $R \oplus S$ must be of the form $I \oplus J$ where I is a left ideal of R and J is a left ideal of S.

For global dimension, this is [34, Exercise 157]. □

Theorem 2.54. *([8, Theorem A]). Let* $T = T_q(\mathbf{x}, \mathbf{L})$ *be a quantum torus. Given a set of the form* $\mathbf{y} = \{y_1, \cdots, y_m\}$, *where each* y_i *is a monomial in* $\mathbf{x}$, *let* $S(\mathbf{y})$ *denote the subalgebra of* T *generated by* $\mathbf{y}$. *Given a subalgebra* S *of* T, *let rk* $S = \inf\{|\mathbf{y}| : S = S(\mathbf{y})\}$, *where* $|\mathbf{y}|$ *is the cardinality of the set* $\mathbf{y}$. *Then*

$$\text{rgld } T = \text{K. dim } T = \sup\{\text{rk } S : S \text{ is a commutative subalgebra of } T\}$$

.

A third notion of "dimension" in noncommutative ring theory is that of Gelfand-Kirillov dimension, usually known as GK dimension. This is a measure of the rate of growth of the algebra with respect to (any) generating set. We will mostly use this as a tool, rather than as a goal in itself.

Lemma 2.55. *([33, 8.2.2]). Let R and S be k-algebras. If S is a subalgebra or a homomorphic image of R then GK dim $S \leq$ GK dim R.*

Lemma 2.56. *Let R and S be k-algebras which are Noetherian domains with GK dim $R =$ GK dim S, and let $\phi : R \to S$ be a surjective homomorphism. Then ϕ is an isomorphism.*

Proof. By [30, Corollary 3.16], the kernel of ϕ must have height 0, that is, $\ker \phi = 0$. □

Theorem 2.57. *([1, Theorem 1.3.1]). Let A be a finitely-generated k-algebra, let α be an automorphism of A and let δ be an α-derivation of A. Suppose there is a finite dimensional generating subspace B for A containing 1 such that $\delta(B) \subset B^2$ and $\alpha(B) \subset B$. Then*

$$\text{GK dim } A[x; \alpha, \delta] = \text{GK dim } A[x^{\pm 1}; \alpha] = \text{GK dim } A + 1.$$

2.3 Ambiskew polynomial rings and generalised Weyl algebras

One particular class of iterated skew polynomial rings we will be interested in is that of **ambiskew polynomial rings**. In the generality we will consider here, these are certain iterated skew polynomial rings of the form $R = A[y; \alpha][x; \alpha^{-1}, \delta]$.

Definition 2.58. Let A be a ring, α an automorphism of A, and v a central element of A. Then we can extend α^{-1} to $A[y; \alpha]$ by setting $\alpha^{-1}(y) = y$, and we can construct an α^{-1}–derivation δ of $A[y; \alpha]$ by setting $\delta(A) = 0$ and $\delta(y) = v$. (We use Proposition 2.22 to confirm that these constructions are valid: $\alpha^{-1}(ya - \alpha(a)y) = y\alpha^{-1}(a) - ay = 0$ and $\delta(ya - \alpha(a)y) = y\delta(a) + \delta(y)a - a\delta(y) - \delta(\alpha(a))y = va - av = 0$). Then the **ambiskew polynomial ring** $R(A, \alpha, v)$ is the iterated skew polynomial ring $A[y; \alpha][x; \alpha^{-1}, \delta]$. By Proposition 2.21 this is the ring generated by A, x and y subject to the relations

$$\begin{aligned} ya &= \alpha(a)y && \text{for all } a \in A; \\ xa &= \alpha^{-1}(a)x && \text{for all } a \in A; \\ xy - yx &= v. \end{aligned}$$

It is clear from these relations that we could alternatively write

$$R(A, \alpha, v) = A[x; \alpha^{-1}][y; \alpha, \gamma]$$

where $\alpha(x) = x$, $\gamma(A) = 0$, $\gamma(x) = -v$. The symmetry between the two iterated skew polynomial rings is the source of the term "ambiskew".

Remark. More general ambiskew polynomial rings - most notably, allowing v to be γ-normal in A for any automorphism γ of A which commutes with α - can be constructed: see e.g. [29].

On the other hand, when ambiskew polynomial rings were first introduced, in [25], the base ring was commutative and they were always what is now known as conformal.

Example 2.59. The first Weyl algebra is an ambiskew polynomial ring over k with α being the identity on k and $v = 1$. Similarly, the n^{th} Weyl algebra is an ambiskew polynomial ring over the $(n-1)^{\text{th}}$ Weyl algebra with α being the identity and $v = 1$.

Definition 2.60. Let A, α, v be as in Definition 2.58. Suppose there exists a central element $u \in A$ such that $v = u - \alpha(u)$. Then $z := xy - u = yx - \alpha(u)$, which we call a **Casimir element**, is central in $R(A, \alpha, v)$. In this case we call u a **splitting element** for R, and say R is **conformal** or a **conformal ambiskew polynomial ring**; if no such u exists then we say R is **singular**.

Proposition 2.61. *If u is a splitting element for R and $a \in A$ is such that $\alpha(a) = a$ then $u - a$ is also a splitting element for R; conversely if u and u' are splitting elements for R then $\alpha(u - u') = u - u'$.*

In particular, if $u \in A$ is a splitting element for R, R is a k-algebra and α is a k-automorphism then $u - \lambda$ is also a splitting element for any $\lambda \in k$.

Theorem 2.62. *([29, Theorem 3.10]). Let $R = R(A, \alpha, v)$ be an ambiskew polynomial ring, where A is a k-algebra for some field k of characteristic 0. Then R is simple if and only if:*

(i) A is α-simple, that is, A has no α-invariant non-zero proper ideals;

(ii) R is a singular ambiskew polynomial ring;

(iii) for all $m \geq 1$, $v^{(m)} := \sum_{i=0}^{m-1} \alpha^i(v)$ is a unit in A.

Definition 2.63. Suppose $R = R(A, \alpha, v)$ is a conformal ambiskew polynomial ring with splitting element u, and consider the ring R/zR where $z := xy - u$ as before. Then $A \cap zR = 0$ so we can identify elements of A with their images in R/zR, and denote by X and Y the images of x and y respectively in R/zR. So we find that R/zR is the ring generated by A, X and Y subject to the relations

$$\begin{aligned} Ya &= \alpha(a)Y && \text{for all } a \in A; \\ Xa &= \alpha^{-1}(a)X && \text{for all } a \in A; \\ XY &= u; \\ YX &= \alpha(u). \end{aligned}$$

Given a ring A, an automorphism α of A and a central element u of A, the ring $T = T(A, \alpha, u)$ generated by A, X and Y subject to the above relations is called a **generalized Weyl algebra** over A, following [3] and [4] (and subsequent papers). As for ambiskew polynomial rings, this definition can be extended to the case where u is normal.

Remark. Given a generalised Weyl algebra $T = T(A, \alpha, u)$ one can construct the ambiskew polynomial ring $R = R(A, \alpha, u - \alpha(u))$ which is conformal with splitting element u, so T is isomorphic to R/zR where z is the Casimir element.

Remark. Any ambiskew polynomial ring is isomorphic to a generalised Weyl algebra in the following way: let $R = R(A, \alpha, u)$ be an ambiskew polynomial ring, and then construct the generalised Weyl algebra $T = T(A[xy], \alpha, xy)$ where α is extended to $A[xy]$ by setting $\alpha(xy) = xy - v$. For proof, see [28, 2.6 Corollary].

Remark. If $T(A, \alpha, u)$ is a generalised Weyl algebra then there is a natural $\mathbb{Z}$-grading on $T(A, \alpha, u)$ with deg $Y = 1$, deg $X = -1$, deg $a = 0$ for $a \in A$.

Lemma 2.64. *([26, 5.2]). Let $T(A, \alpha, u)$ be a generalised Weyl algebra. Then* $X^mY^m = \Pi_{i=0}^{m-1}\alpha^{-i}(u)$ *and* $Y^mX^m = \Pi_{i=1}^{m}\alpha^{i}(u)$.

Lemma 2.65. *Let $T(A, \alpha, u)$ be a generalised Weyl algebra. If A is a domain then $T(A, \alpha, u)$ is a domain.*

Proof. $T(A, \alpha, u)$ embeds into $A[y^{\pm 1}; \alpha]$ via $a \mapsto a$ for $a \in A$, $x \mapsto uy^{-1}$ and $y \mapsto y$; if A is a domain then $A[y^{\pm 1}; \alpha]$ is also by Proposition 2.17 (ii). □

There is a simplicity criterion for generalised Weyl algebras similar to Theorem 2.62. This result generalises to the case where v is normal ([29, Theorem 5.4]).

Theorem 2.66. *([5, Theorem 4.2]). Let $T = T(A, \alpha, u)$ be a generalised Weyl algebra. Then T is simple if and only if* *(i) A is α-simple*

(ii) α has infinite order

(iii) u is regular

(iv) $uA + \alpha^m(u)A = A$ for all $m \geq 1$

2.4 Poisson algebras and the semiclassical limit

The quantum torus, quantum plane, and quantized Weyl algebra are all examples of **quantum algebras** - each is a noncommutative k-algebra with a scalar parameter q, such that when $q = 1$ the algebra is commutative. For example, if we consider the quantum plane, $\mathcal{O}_1(k^2) = k[x, y]$, the commutative polynomial ring in two variables. (This is the classical coordinate ring of the plane, hence the name "quantum plane").

The **semiclassical limit Poisson algebra** of such a family is a way to put additional structure on this commutative ring such that it "remembers" some aspects of the noncommutative family it came from. This has the advantage that the semiclassical limit, being commutative, is usually easier to work with, but preserves enough of the structure that investigating it still provides insight into the original quantum algebra. Sometimes, this link can be made more formal, though currently these tend to be in quite specific cases. The survey article [16] provides an excellent introduction and overview of this topic.

We now proceed with the formal definitions. We work over a fixed field k.

2.4.1 Poisson algebras

Definition 2.67. A **Poisson algebra** is a commutative k-algebra A together with a Lie bracket $\{-,-\}$ on A such that $\{a,-\}$ is a k-algebra derivation of A for all a. That is, $\{-,-\}$:

(i) is k-bilinear;

(ii) is antisymmetric;

(iii) satisfies the Jacobi identity, that is, $\{a, b\} + \{b, c\} + \{c, a\} = 0$ for all $a, b, c \in A$; and

(iv) satisfies the Leibniz rule, that is, $\{a, bc\} = b\{a, c\} + c\{a, b\}$ for all $a, b, c \in A$.

Many of the techniques we use in noncommutative algebra have analogues for Poisson algebras. Again, one of the most important is localisation.

Lemma 2.68. *Let A be a Poisson algebra, and let $\mathcal{X}$ be a multiplicatively closed set of regular elements in A. Then the Poisson bracket on A extends uniquely to $A_{\mathcal{X}}$, with*

$$\{ax^{-1}, by^{-1}\} = \{a, b\}x^{-1}y^{-1} - \{a, y\}bx^{-1}y^{-2} - \{x, b\}ax^{-2}y^{-1} + \{x, y\}abx^{-2}y^{-2}$$

where $a, b \in A$ and $x, y \in \mathcal{X}$.

Proof. That this defines a Poisson bracket is a straightforward check. The uniqueness follows from using the Leibniz rule to determine $\{ax^{-1}, by^{-1}\}$ from the Poisson bracket on A. $\square$

Definition 2.69. Let A be a Poisson algebra. Then an ideal P of A (as an associative algebra) is a **Poisson ideal** if it is also a Lie ideal, that is $\{a, p\} \in P$ for all $a \in A$, $p \in P$.

A Poisson ideal P of A is a **Poisson prime ideal** if $IJ \subset P$ implies either $I \subset P$ or $J \subset P$, where I and J are Poisson ideals of A. We denote by PSpec A the partially ordered (by inclusion) set of Poisson prime ideals of A.

We say a Poisson algebra is **Poisson simple** if it has no non-trivial Poisson ideals.

The **Poisson centre** of a Poisson algebra A is $Z_P(A) := \{a \in A : \{a, r\} = 0 \text{ for all } r \in A\}$. If $a \in Z_P(A)$ we say a is **Poisson central** in A.

A **Poisson homomorphism** between two Poisson algebras A and B is a ring homomorphism between A and B which is also a Lie algebra homomorphism. A **Poisson automorphism** of a Poisson algebra A is a Poisson homomorphism from A to A which is a bijection, and therefore is a ring isomorphism and a Lie isomorphism.

Proposition 2.70. *([10, Lemma 3.3.2]). Let A be a Noetherian Poisson algebra (A is Noetherian as an associative algebra) over a field of characteristic 0. Then an associative ideal of A is Poisson prime if and only if it is both a prime ideal and a Poisson ideal.*

Proposition 2.71. *Let A be a Noetherian Poisson algebra over a field of characteristic 0, and let $\mathcal{X}$ be a multiplicatively closed set of regular elements in A. Then there is a one-to-one inclusion-preserving correspondence between PSpec $A_{\mathcal{X}}$ and $\{P \in \text{PSpec } A : P \cap \mathcal{X} = \emptyset\}$ given by $P' \mapsto P' \cap A$, $P \mapsto PA_{\mathcal{X}}$.*

Proof. If P' is a Poisson ideal of $A_{\mathcal{X}}$ then $P' \cap A$ is a Poisson ideal of A: if $p \in P' \cap A$ and $a \in A$ then $\{p, a\} \in P'$ since P' is a Poisson ideal of $A_{\mathcal{X}}$ and $\{p, a\} \in A$ since $p, a \in A$; similarly, if P is a Poisson ideal of A then $PA_{\mathcal{X}}$ is a Poisson ideal of $A_{\mathcal{X}}$, since if $a \in P$, $x, y \in \mathcal{X}$ and $b \in A$ then all the terms in $\{ax^{-1}, by^{-1}\}$ are in $PA_{\mathcal{X}}$.

Therefore, the one-to-one inclusion-preserving correspondence from Theorem 2.10 between Spec $A_{\mathcal{X}}$ and $\{P \in \text{Spec } A : P \cap \mathcal{X} = \emptyset\}$ given by $P' \mapsto P' \cap A$, $P \mapsto PA_{\mathcal{X}}$ restricts to PSpec $A_{\mathcal{X}}$ and vice versa. $\square$

Definition 2.72. A **Poisson maximal ideal** in a Poisson algebra A is a Poisson ideal which is maximal among Poisson ideals, that is, there is no Poisson ideal strictly containing it other than A.

Proposition 2.73. *A Poisson maximal ideal is a Poisson prime ideal.*

Proof. This is immediate from [10, Lemma 3.3.2]. $\square$

2.4.2 The semiclassical limit

Definition 2.74. (See e.g. [9, III.5.4]). Let R be a commutative k-algebra and let $h \in R$. (Normally, we will take $R = k[t^{\pm 1}]$ and $h = t$ or $h = 1 - t$). Let A be an R-algebra with h regular in A such that A/hA is commutative. Then, for all $a, b \in A$, $ab - ba \in hA$, and since h is regular in A, $\frac{1}{h}(ab - ba)$ is well-defined. So define $\{a + hA, b + hA\} = \frac{1}{h}(ab - ba) + hA$. We call A/hA with this bracket the **semiclassical limit Poisson algebra** of the family $(A/(h-q)A)_{q \in k,\ h-q \text{ not a unit in } R}$.

In some sense this bracket gives a "first-order" impression of the noncommutative algebra: if $ab - ba = hx_1 + h^2 x_2 + \cdots$, then $\{a + hA, b + hA\} = x_1 + hA$, but all further terms are lost.

Proposition 2.75. *The bracket on A is a well-defined Poisson bracket on A/hA.*

Proof. If $a + hA = a' + hA$ then $a - a' = rh$ for some $r \in A$; then $\frac{1}{h}(a'b - ba') + hA = \frac{1}{h}(ab - ba) + \frac{1}{h}(rhb - brh) + hA = \frac{1}{h}(ab - ba) + (rb - br) + hA$; since A/hA is commutative, $rb - br \in hA$. Therefore $\{-, b + hA\}$ is well-defined for all b; similarly or by the antisymmetry, the bracket as a whole is well-defined.

It is a standard result that taking $\{a, b\} := ab - ba$ defines a Lie bracket on any algebra A, and this passes through to the bracket we have defined on A/hA.

Finally, $\{a + hA, bc + hA\} = \frac{1}{h}(abc - bca) + hA = \frac{1}{h}(bac - bca) + \frac{1}{h}(abc - bac) + hA = b\{a + hA, c + hA\} + c\{a + hA, b + hA\}$. $\square$

Example 2.76. Let A be the k-algebra generated by x, y, and $t^{\pm 1}$ such that t is central, so A is a $k[t^{\pm 1}]$-algebra, and $xy - tyx = 0$. Let $h = t - 1$; then A/hA is commutative, while for $q \neq 1$, $A/(h + 1 - q)A = A/(t - q)A \cong \mathcal{O}_q(k^\times)$. Then the semiclassical limit Poisson algebra of this family is $A/hA \cong k[x, y]$ with $\{x, y\} = yx$ (since $xy - (h+1)yx = 0$ so $xy - yx = hyx$).

Definition 2.77. Let $\mathbf{x} = (x_v)_{v \in Q}$ be a tuple indexed by some finite set Q, and let $\mathbf{L} = (L_{vw})_{v,w \in Q}$, where the L_{vw} are integers such that $L_{vw} = -L_{wv}$ for $v, w \in Q$, that is, $\mathbf{L}$

is a skew-symmetrix integer matrix indexed by Q. Then the **Poisson space**, which we denote $S_P(k, \mathbf{x}, \mathbf{L})$, or $S_P(,\mathbf{L})$ if the base field is unambiguous, is the Poisson algebra whose underlying associative algebra is the polynomial ring in the elements of $\mathbf{x}$, with Poisson bracket defined by $\{x_v, x_w\} = L_{vw}x_vx_w$.

Also the **Poisson torus**, which we denote $T_P(k, \mathbf{x}, \mathbf{L})$, or $T_P(\mathbf{x}, \mathbf{L})$ if the base field is unambiguous, is the Poisson algebra whose underlying associative algebra is the Laurent polynomial ring in the elements of $\mathbf{x}$, with Poisson bracket defined by $\{x_v, x_w\} = L_{vw}x_vx_w$. Alternatively, $T_P(k, \mathbf{x}, \mathbf{L}) = S_P(k, \mathbf{x}, \mathbf{L})_{\mathcal{W}}$, where $\mathcal{W}$ is the set of non-zero monomials in $\mathbf{x}$.

Alternatively, $S_P(k, \mathbf{x}, \mathbf{L})$ is the semiclassical limit of the family $(S_q(k, \mathbf{x}, \mathbf{L}))_{q\neq 0}$ while $T_P(k, \mathbf{x}, \mathbf{L})$ is the semiclassical limit of the family $(T_q(k, \mathbf{x}, \mathbf{L}))_{q\neq 0}$.

Theorem 2.78. *(This is analogous to [32, Proposition 1.3]). Let $T_P(k, \mathbf{x}, \mathbf{L})$ be a Poisson torus over a field of characteristic* 0*. Then TFAE:*

(i) If $(a_i)_{i\in Q}$ is a tuple such that $\sum_{i\in Q} L_{ij}a_i = 0$ then $a_i = 0$ for all $i \in Q$.

(ii) $T_P(k, \mathbf{x}, \mathbf{L})$ is Poisson simple.

(iii) $Z_P(T_P(k, \mathbf{x}, \mathbf{L})) = k$.

Proof. This is a direct consequence of [36, Lemma 1.2]; although the proof there is only stated in the case $k = \mathbb{C}$, it is still valid over arbitrary fields of characteristic 0. □

2.5 Quantum cluster algebras

2.5.1 Definition

This section follows the scope of the definition and much of the notation from [21, §2.3]. Results with proofs here were left as exercises in those notes. As discussed in the introduction, quantum cluster algebras are defined by an iterative process known as seed mutation, and create quantum algebras with large numbers of generators but relatively simple relations. They were first introduced in [6] (although their classical counterparts were a few years earlier in [11]).

We will use a less general setting than that which is set up in [6], in order to be able to describe the combinatorics in terms of quivers, since all the examples we are interested are covered by this setting; this also (hopefully!) simplifies the notation. Other than the Laurent phenomenon, all the results of this section that we quote from [6] are straightforward checks in the current setting.

Before we start, we fix a field k and a non-zero scalar $q \in k$.

Definition 2.79. A **quiver** Q consists of a finite set of vertices, which we also call Q, and, for each pair of vertices v, $w \in Q$, an integer B_{vw} representing the number of arrows from v to w. If B_{vw} is negative, then we interpret this as representing arrows from w to v, and so we insist that $B_{vw} = -B_{wv}$. We also require that $B_{vv} = 0$ for all $v \in Q$, that is, there are no loops in our quiver. (One could call this a directed multigraph without 1- or 2-cycles).

An **ice quiver**, or from now on just a **quiver** is a quiver (in the above sense) Q whose vertices have been partitioned into two subsets Q_{mut} and Q_{froz}, known as **mutable** and **frozen** vertices respectively. Pictorially, we represent this by putting a square around a frozen vertex. We also require that if v and w are both frozen vertices then $B_{vw} = 0$. If we restrict to looking at just Q_{mut} and arrows between vertices in Q_{mut} then we get a quiver with no frozen vertices which we call the **principal part** of Q.

We note the conditions above mean that all the information about Q is contained in the integers $(B_{vw})_{v\in Q, w\in Q_{\text{mut}}}$. We can treat this as an integer "matrix" $\mathbf{B}$ with rows parametrised by Q and columns by Q_{mut}. This matrix is skew-symmetric in the sense that the **principal part**, that is, the restriction of $\mathbf{B}$ to Q_{mut}, is skew-symmetric.

It will be helpful at times to give the vertices of Q an ordering, that is, we label the vertices of Q by $v_1, \ldots, v_n$, where $n = |Q|$. For convenience, we assume that vertices $v_1, \ldots, v_m \in Q_{\text{mut}}$ and $v_{m+1}, \ldots, v_n \in Q_{\text{froz}}$, where $m = |Q_{\text{mut}}|$. We will then use the shorthand $B_{ij} = B_{v_i v_j}$, for $v_i \in Q_{\text{mut}}$ and $v_j \in Q$.

Remark. In [6], quantum cluster algebras are defined with the matrix $\mathbf{B}$ only required to be skew-symmetrizable, that is, there exists some matrix $\mathbf{D} = (D_{uv})_{u,v\in Q}$ with $D_{uv} = 0$ if $u \neq v$ and $D_{uu} > 0$ for all $u \in Q$, such that $\mathbf{DB}$ is skew-symmetric.

Definition 2.80. A **(quantum) seed** is a triple $\mathbf{Q} = (Q, \mathbf{x}, \mathbf{L})$ where Q is an (ice) quiver, $\mathbf{x} = (x_v)_{v\in Q}$ is a tuple - called a **cluster** - indexed by Q, and $\mathbf{L} = (L_{vw})_{v,w\in Q}$ is a skew-symmetric integer matrix also indexed by Q, such that $\mathbf{B^T L} = d\mathbf{I}$ for some positive integer d. Here $\mathbf{I}$ is the matrix with rows parametrised by Q_{mut} and columns parametrised by Q such that for $v \in Q$ and $w \in Q_{\text{mut}}$, $I_{vw} = 1$ if $v = w$ while $I_{vw} = 0$ otherwise. This last requirement is known as the **compatibility condition**.

Remark. Given a quantum seed $\mathbf{Q} = (Q, \mathbf{x}, \mathbf{L})$ one can form the quantum space $S_q(\mathbf{x}, \mathbf{L})$ and its localisation the quantum torus $T_q(\mathbf{x}, \mathbf{L})$. In the absence of any other ambient ring (see Definition 2.91 later), we will work in $T_q(\mathbf{x}, \mathbf{L})$.

Remark. If Q is an ice quiver then the partition of Q into mutable and frozen vertices extends to $\mathbf{x}$ and then to $\mathbf{L}$, with the obvious notation $\mathbf{x}_{\text{mut}} := (x_v)_{v\in Q_{\text{mut}}}$, $\mathbf{x}_{\text{froz}} := (x_v)_{v\in Q_{\text{froz}}}$, $\mathbf{L}_{\text{mut}} := (L_{vw})_{v,w\in Q_{\text{mut}}}$ and $\mathbf{L}_{\text{froz}} := (L_{vw})_{v,w\in Q_{\text{froz}}}$.

Often the elements of $\mathbf{x}_{\text{froz}}$ are referred to as "coefficients".

Definition 2.81. (Quiver mutation) Let Q be a quiver and $v \in Q$ a vertex. Then we can construct a new quiver Q', which we call the **mutation of Q at v** with the same vertex set but different arrows. The arrows in Q' are given by

$$B'_{uw} = \begin{cases} -B_{uw} & u = v \text{ or } w = v \\ B_{uw} + \frac{|B_{uv}|B_{vw} + B_{uv}|B_{vw}|}{2} & \text{otherwise.} \end{cases}$$

Alternatively, and possibly more informatively, this process can be described in three stages:

- Reverse any arrows starting or ending at v.
- "Complete triangles", that is, for every pair of vertices u and w and pair of arrows u to v and v to w, add an arrow w to u.
- Remove 2-cycles, that is, if we have arrows u to w and w to u, remove one from each direction until they are all in the same direction.

Remark. Quiver mutation is a "local" process, i.e. when we mutate at a vertex v, the only vertices that change are those that have arrows to or from v: if $B_{uv} = 0$, then $B'_{uw} = B_{uw}$ for all $w \in Q$.

Definition 2.82. (Seed mutation) Let $\mathbf{Q} = (Q, \mathbf{x}, \mathbf{L})$ be a seed, and $v \in Q$ a vertex. Then we can construct a new seed $\mathbf{Q}'$, which we call the **mutation of $\mathbf{Q}$ at** v, by saying $\mathbf{Q}' = (Q', \mathbf{x}', \mathbf{L}')$, where Q' is as above and $\mathbf{x}'$ and $\mathbf{L}'$ are defined as follows:

Pick an ordering on the vertices of Q, with $v_k = v$.

Let $\mathbf{e}_i$ denote the ith basis vector in $\mathbb{Z}^n$, where $n = |Q|$. Let

$$\mathbf{b}_k^+ := -\mathbf{e}_k + \sum_{B_{ik}>0} B_{ik}\mathbf{e}_i$$

$$\mathbf{b}_k^- := -\mathbf{e}_k - \sum_{B_{ik}<0} B_{ik}\mathbf{e}_i$$

For $(a_1, \dots, a_n) \in \mathbb{Z}^n$, define

$$\lambda(a_1, \dots, a_n) = \frac{1}{2}\sum_{i<j} a_i a_j L_{ji}$$

$$M(a_1, \dots, a_n) = q^{\lambda(a_1, \dots a_n)} x_1^{a_1} \cdots x_n^{a_n}$$

Then define $\mathbf{x}'$ by $x'_v = M(\mathbf{b}_k^+) + M(\mathbf{b}_k^-)$, and $x'_u = x_u$ for $u \neq v$. Thus $x'_u \in T_q(\mathbf{x}, \mathbf{L})$ for all $u \in Q$.

Proposition 2.83. *x'_v is well-defined, that is, picking a different ordering on Q gives the same element of $T_q(\mathbf{x}, \mathbf{L})$.*

Proof. By [6, Lemma 4.4], $M(\mathbf{a})$ is well-defined for $\mathbf{a} \in \mathbf{Z}^n$. □

Remark. In the light of this, we can write $\mathbf{b}_v^{\pm}$ for $v \in Q$, and $M(\mathbf{b}_v^{\pm})$ is then well-defined. $\lambda(\mathbf{b}_v^{\pm})$ is not, but if we have picked an order on the vertices of Q then we will write $\lambda(\mathbf{b}_v^{\pm})$ to avoid having to identify k such that $v_k = v$.

Proposition 2.84. *([6, Proposition 4.7] and [6, Proposition 4.9]). There exists a matrix $\mathbf{L}' = (L'_{uv})_{u \in Q', v \in Q'}$ such that:*

(i) $x'_u x'_w = q^{L'_{uw}} x'_w x'_u$ for all $u, w \in Q$; and

(ii) $\mathbf{B}'^{\mathbf{T}} \mathbf{L}' = d\mathbf{I}$.

Therefore $\mathbf{Q}' = (Q', \mathbf{x}', \mathbf{L}')$ is a seed with $S_q(\mathbf{x}', \mathbf{L}') \subset T_q(\mathbf{x}, \mathbf{L})$.

Remark. It is a straightforward check that $\mathbf{L}'$ is given by $L'_{vv} = 0$, $L'_{vw} = \sum_{B_{uv} > 0} B_{uv} L_{uw} - L_{vw}$ for $w \neq v$, $L'_{wv} = -L'_{vw}$, and $L'_{uw} = L_{uw}$ when $u \neq v$ and $w \neq v$.

Proposition 2.85. *([6, Proposition 4.10]). Seed mutation is involutive, i.e. mutating at the same vertex v twice yields the original seed.*

Definition 2.86. Two seeds are **mutation equivalent** if there exists a sequence of seed mutations taking one to the other.

Remark. By Proposition 2.85, this is an equivalence relation.

Perhaps the most surprising property of cluster mutation is known as the **Laurent phenomenon**: however many mutations you perform, you never leave the quantum torus generated by the initial seed. It would not be an understatement to say that this is the "fundamental theorem of (quantum) cluster algebras".

Theorem 2.87 (**Laurent phenomenon**). *([6, Corollary 5.2]). Let $\mathbf{Q} = (Q, \mathbf{x}, \mathbf{L})$ be a seed, and let x_v° be a cluster variable from some seed which is mutation equivalent to $\mathbf{Q}$. Then $x_v^{\circ} \in T_q(\mathbf{x}, \mathbf{L})$.*

Definition 2.88. The **quantum cluster algebra** $A_q(\mathbf{Q})$ starting from a particular **initial seed** $\mathbf{Q} = (Q, \mathbf{x}, \mathbf{L})$ is the subalgebra of $T_q(\mathbf{x}, \mathbf{L})$ generated by the union of all the clusters belonging to seeds mutation equivalent to the initial seed.

2.5.2 Classical or commutative cluster algebras

Since the focus of this thesis is on noncommutative algebras, we have presented the quantum side of cluster algebras first, but we will have use for a commutative - or "classical" version as well. If one takes a quantum seed $\mathbf{Q}$ and sets the parameter q to be 1, then the torus $T_1(\mathbf{x}, \mathbf{L})$ becomes a commutative Laurent polynomial ring, and so the quantum cluster algebra $A_1(\mathbf{Q})$ is a commutative ring too. But then the matrix $\mathbf{L}$ becomes irrelevant, so we remove the requirement for it to exist. Formally:

Definition 2.89. A **classical seed** $\mathbf{Q} = (Q, \mathbf{x})$ consists of a quiver Q and a set of **cluster variables** $(x_v)_{v\in Q}$.

The **mutation** of a classical seed at $v \in Q$ is the seed $\mathbf{Q} = (Q', \mathbf{x}')$, where Q' is the quiver mutation of Q at v and $\mathbf{x}'$ is given by

$$x'_w = x_w \text{ if } v \neq w$$

$$x'_v = x_v^{-1}(\Pi_{b_{wv}>0} x_w^{b_{wv}} + \Pi_{b_{wv}<0} x_i^{-b_{wv}})$$

Two seeds are **mutation equivalent** if there exists a sequence of seed mutations taking one to the other.

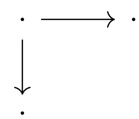
Given a classical seed $\mathbf{Q}$ the **classical cluster algebra** $A(\mathbf{Q})$ is the subalgebra of the Laurent polynomial $T_1(\mathbf{x})$ generated by the union of all the cluster variables belonging to seeds mutation equivalent to $\mathbf{Q}$.

This definition makes sense, since the Laurent phenomenon still holds:

Theorem 2.90. *([11, Theorem 3.1]). Let $\mathbf{Q} = (Q, \mathbf{x})$ be a seed, and let x_v° be a cluster variable from some seed which is mutation equivalent to $\mathbf{Q}$. Then $x_v^\circ \in T_1(\mathbf{x})$.*

Remark. As remarked earlier, from a historical point of view, this presentation is backwards - commutative cluster algebras were introduced first, in [11], a few years before quantum cluster algebras in [6].

Given a classical seed, one can create a quantum seed with the same quiver if and only if there exists a skew-symmetrix integer matrix $\mathbf{L}$ indexed by Q such that the compatibility condition $\mathbf{B^T L} = d\mathbf{I}$ holds. Since $d\mathbf{I}$ has rank equal to $|Q_{\text{mut}}|$, which is the maximum rank of $\mathbf{B}(Q)$, if the latter fails to have full rank then there is no quantum seed with the quiver Q. For example, in the following quiver $\mathbf{B}(Q)$ has rank 2 whereas $\mathbf{I}$ has rank 3 so no quantum seed with quiver Q exists:



If, however, such an $\mathbf{L}$ does exist, then it also induces a Poisson structure on the Laurent polynomial ring $T_1(\mathbf{x})$, given by $\{x_v, x_w\} = L_{vw}x_vx_w$. The matrix $\mathbf{L}$ mutates as in the quantum case, with $\{x'_v, x'_w\} = L'_{vw}x'_vx'_w$ by essentially the same proof, meaning this bracket extends to the cluster algebra $A(\mathbf{Q})$. Such a bracket is sometimes known as a **log-canonical Poisson bracket**. In this case, $A(\mathbf{Q})$ with the log-canonical Poisson bracket is the semiclassical limit of the family $(A_q(\mathbf{Q}))_{q\neq 0}$.

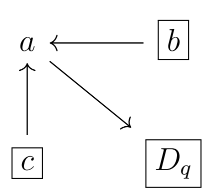
2.5.3 Some results on quantum cluster algebras

Definition 2.91. One frequent question regarding (quantum) cluster algebras is: what (quantum) algebras can be given a quantum cluster algebra structure? Given a k-algebra A which is a domain, a **seed within** A will be a seed $\mathbf{Q} = (Q, \mathbf{x}, \mathbf{L})$, such that $S_q(\mathbf{x}, \mathbf{L}) \subset A$ - that is, the elements of $\mathbf{x}$ are pairwise skew-commuting elements of A, and $\mathbf{L}$ describes those skew-commutators. Then we ask: if x_v° is a cluster variable from some seed which is mutation equivalent to $\mathbf{Q}$, does $x_v^\circ \in A$? If it does, then $A_q(\mathbf{Q}) \subset A$, so does the set of all such cluster variables generate A as a k-algebra? If it does, then $A_q(\mathbf{Q}) = A$, and we say the initial seed $\mathbf{Q}$ describes a cluster algebra structure on A.

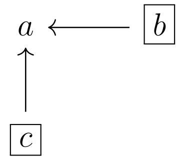
In this situation we will sometimes abuse notation to avoid naming the vertices of Q, and instead refer to them by the corresponding element of $\mathbf{x}$.

Example 2.92. [Quantized coordinate rings of $M_2(k)$ and $SL_2(k)$] The **quantized coordinate ring of** 2×2 **matrices** is the ring $\mathcal{O}_q(M_2(k)) := \langle a, b, c, d : ab = qba, ac = qca, bd = qdb, cd = qdc, bc = cb, ad - da = (q - q^{-1})bc\rangle$. This ring has a central element $D_q := ad - qbc$, called the *quantum determinant*; the factor ring $\mathcal{O}_q(SL_2(k)) := \mathcal{O}_q(M_2(k))/(D_q - 1)$ is called the **quantized coordinate ring of** $SL_2(k)$.

Both these rings have quantum cluster algebra structures. For $\mathcal{O}_q(M_2(k))$, one can the following as the initial seed:



For $\mathcal{O}_q(SL_2(k))$, one can take the following as the initial seed:



In both cases it is straightforward to check that the initial seed is indeed a seed (i.e. that the skew-commutation matrix, which is determined by the relations in the respective ring, is compatible with the quiver illustrated), and that $a' = d$, so the seeds above describe cluster algebra structures on the respective rings.

Other examples of rings with quantum cluster algebra structures include quantum Schubert cell algebras ([19]), quantum Grassmannians ([22], [23]), and quantum double Bruhat cells (conjectured in [6], proved in [20]).

Definition 2.93. The **neighbourhood** of a vertex $v \in Q$ is the subgraph of Q with vertex set $\{v\} \cup \{w \in Q : B_{wv} \neq 0\}$ and all edges from Q between those vertices.

We remarked earlier that quiver mutation was a local process in the sense that if one is mutating at v, and $B_{wv} = 0$, then $B'_{wu} = B_{wu}$ for all $u \in Q$. So the only vertices that are affected by the quiver mutation are the vertices in the neighbourhood of v. Therefore if we have a description for a class of seeds that describes the neighbours of each vertex, and wish to show that mutation remains in this class of seeds, one only needs to check that the mutated vertex and its neighbours still satisfy the description.

Definition 2.94. A quiver Q is **acyclic** if it contains no **cycles**, that is, vertices $v_0, \dots, v_n$ with $v_0 = v_n$ and arrows from v_i to v_{i+1} for $i = 0, \dots, n-1$.

Theorem 2.95. *([6, Theorem 7.5]). Let $\mathbf{Q} = (Q, \mathbf{x}, \mathbf{L})$ be a seed such that the principal part of Q - that is, Q with the frozen vertices removed - is acyclic, and assume that q is transcendental over $\mathbb{Z}$. The $A_q(\mathbf{Q})$ is generated as a k-algebra by $\mathbf{x} \cup \{x'_v : v \in Q_{mut}\}$.*

Definition 2.96. A (quantum) cluster algebra is said to be **of finite type** if the set of distinct seeds is finite.

Theorem 2.97. *([12, Theorem 1.4] in the commutative case; [6, Remark 6.3], combining [12, Theorem 1.4] with [6, Theorem 6.1], in the quantum case). A (quantum) cluster algebra starting from initial seed $\mathbf{Q}$ is of finite type if and only if there exists some seed $\mathbf{Q}'$ which is mutation equivalent to $\mathbf{Q}$, such that the principal part of Q' is a disjoint union of finitely many orientations of simply laced Dynkin diagrams (that is, Dynkin diagrams of type A, D or E).*

Remark. Let $\mathbf{Q} = (Q, \mathbf{x}, \mathbf{L})$ be a seed and let $v \in Q_{\text{mut}}$. Let $\bar{\mathbf{x}} = \mathbf{x} \backslash \{x_v\}$ and $\bar{\mathbf{L}} = \mathbf{L}$ restricted to $\bar{\mathbf{x}}$. Let $A = S_q(\bar{\mathbf{x}}, \bar{\mathbf{L}})$ or $T_q(\bar{\mathbf{x}}, \bar{\mathbf{L}})$, and let R be the subalgebra of $T_q(\mathbf{x}, \mathbf{L})$ generated by A, x_v and x'_v. Then R can be presented as a generalised Weyl algebra with u normal (that is, following the more general definition of [29] rather than the one given in Definition 2.63) over A with x_v and x'_v taking the roles of X and Y respectively.

3 Prime ideals in ambiskew polynomial rings and generalised Weyl algebras

3.1 Simple localisation for ambiskew polynomial rings with central Casimir elements

The aim of this section is to provide a simple localisation of the conformal ambiskew polynomial ring $R(A, \alpha, v)$. The main result, Theorem 3.3, is analogous to [29, Theorem 4.7], and follows a similar proof. That result uses a more general definition of an ambiskew polynomial ring which does not require v to be central, which in turn implies that a Casimir element will not be central either. If such a Casimir element exists then one can localise at the powers of the Casimir element, and [29, Theorem 4.7] gives conditions for this localisation to be simple. However, if v and the Casimir element z are central, then this localisation will not be simple, since $z - \lambda$ is another central Casimir element for any $\lambda \in k$ which will generate a non-zero ideal in the localisation. So in this situation the natural localisation to consider is the localisation at the set of non-zero elements of $k[z]$.

Definition 3.1. Let k be a field, A a k-algebra, α a k-automorphism of A and v a central element of A. Then define $v^{(0)} := 0$ and $v^{(m)} := \Sigma_{l=0}^{m-1} \alpha^l(v)$ for all $m \geq 1$.

Proposition 3.2. *([29, equation (2)]). Working in $R(A, \alpha, v)$, where A, α and v are as in Definition 3.1, for all $m \geq 0$, $xy^m - y^m x = v^{(m)} y^{m-1}$.*

Proof. When $m = 0$ this is trivial; otherwise,

$$\begin{aligned} xy^m - y^m x &= y(xy^{m-1} - y^{m-1}x) + vy^{m-1} \\ &= yv^{(m-1)}y^{m-2} + vy^{m-1} \qquad \text{by induction} \\ &= (\alpha(v^{(m-1)}) + v)y^{m-1} \\ &= v^{(m)}y^{m-1}. \end{aligned}$$

□

Theorem 3.3. *Let k be an algebraically closed field, A a k-algebra, α some k-automorphism of A, and v a central element of A. Suppose also that $\{c \in A : c \text{ central in } A \text{ and } \alpha(c) = c\} = k$.*

Let $R = R(A, \alpha, v)$ be the ambiskew polynomial ring defined by the above, and assume that R is conformal with splitting element u - so $z := xy - u$ is a Casimir element for R, $u + \lambda$ is also a splitting element for any $\lambda \in k$, and $z - \lambda$ is also a Casimir element for R for any $\lambda \in k$. Let $\mathcal{Z} = k[z]^* := k[z] \setminus \{0\}$.

Then the ring $S = R_{\mathcal{Z}}$ is simple if and only if the following hold:

(i) A is α-simple;

(ii) for all $m \geq 1$, α^m is not an inner automorphism of A;

(iii) for all $m \geq 1$, there exists a non-zero polynomial p (which may depend on m) over k such that $p(u) \in v^{(m)}A$.

To prove this, we use the following lemma which corresponds to, and uses the same proof as, [29, Lemma 4.1].

Lemma 3.4. *Let $\mathcal{Y} := \{y^i : i \in \mathbb{N}\}$. The ring S is simple if and only if $S_{\mathcal{Y}}$ is simple and, for all $m \geq 1$, there exists a non-zero polynomial p over k such that $p(u) \in v^{(m)}A$.*

Proof. For the statement of the lemma to make sense, we must show that $\mathcal{Y}$ is a right Ore set in S. Certainly $\mathcal{Y}$ is a right Ore set in $A[y;\alpha]$, since y is normal in that ring, then by Lemma 2.42, since $\alpha^{-1}(\mathcal{Y}) = \mathcal{Y}$, $\mathcal{Y}$ is a right Ore set in R, and hence by Lemma 2.43, $\mathcal{Y}$ is a right Ore set in S. Similarly, $\mathcal{Y}$ is a left Ore set in R and in S.

Suppose that S is simple. Then $S_{\mathcal{Y}}$ is simple by [18, Proposition 10.17(a)]. Let $m \geq 1$. Let J be the k-subspace of R spanned by the elements of the form $x^i a y^j$, where $i, j \in \mathbb{N}$ and $a \in A$, such that one of $i > 0$, $j \geq m$ or $a \in v^{(m)}A$. We claim J is a right ideal of R: certainly $Jy \subset J$, $JA \subset J$, and, using Proposition 3.2, $x^i a y^j x \in J$ if $i > 0$ or $j > m$. Then using Proposition 3.2 again, $ay^j x = x\alpha(a)y^j - v^{(j)}ay^{j-1}$, so $ay^j x \in J$ if $j = m$ or if $a \in v^{(m)}A$. So J is a right ideal of R. Let $I := \mathrm{ann}_R(R/J)$. This is a non-zero proper ideal of R contained in J. (It's non-zero because $y^m \in I$). Since $S = R_{\mathcal{Z}}$ is simple, there exists, by Lemma 2.12, some element of $\mathcal{Z} \cap I$. That is, there is some non-zero polynomial p such that $p(z) \in I \subset J$.

We note that $x \in J$, and $u^r xy = x\alpha(u^r)y \in J$ for $0 \leq r < n$. So since $p(xy - u) \in J, p(-u) \in J$. But $J \cap A = v^{(m)}A$, so $p(-u) \in v^{(m)}A$.

Conversely, suppose $S_{\mathcal{Y}}$ is simple and that, for all $m \geq 1$, there exists some non-zero polynomial p such that $p(u) \in v^{(m)}A$. Let I be some non-zero ideal of S; then by Lemma 2.12 $y^m \in I$ for some $m \geq 0$. Choose the least such m and suppose $m \neq 0$. Then there exists a non-zero polynomial p such that $p(u) \in v^{(m)}A$. By Proposition 3.2, $v^{(m)}y^{m-1} \in I$ and so $v^{(m)}Ay^{m-1} \subset I$, so $p(u)y^{m-1} \in I$. Assume that p has minimal degree such that $p(u)y^{m-1} \in I$, and assume that the degree of p is at least 1. So, since k is algebraically closed, factorise p as $p'(u)(u - \lambda)$ for some $\lambda \in k$, where $p'(u)$ necessarily has lower degree than p.

Consider $(z+\lambda)p'(u)y^{m-1} = p'(u)xy^m - p'(u)(u-\lambda)y^{m-1} = p'(u)xy^m - p(u)y^{m-1} \in I$. But $z+\lambda$ is invertible in S, so $p'(u)y^{m-1} \in I$, contradicting the minimality of the degree of p. Therefore p is a constant polynomial, and so $y^{m-1} \in I$; but this contradicts the minimality of m. Therefore $m = 0$, $1 \in I$ and so S is simple. $\square$

Proof of Theorem 3.3. As in [29], $R_{\mathcal{Y}}$ can be identified with $A[y^{\pm 1};\alpha][z]$; also, by Lemma 2.43, $(R_{\mathcal{Y}})_{\mathcal{Z}} = S_{\mathcal{Y}}$.

Suppose (i) - (iii) hold. Then by (i) and (ii) and [33, 1.8.5], $A[y^{\pm 1};\alpha]$ is simple. Further, the condition $\{c \in A : c \text{ central in } A \text{ and } \alpha(c) = c\} = k$ is equivalent to $Z(A[y^{\pm 1};\alpha]) = k$. So by Theorem 2.30, any prime ideal of $R_{\mathcal{Y}} = A[y^{\pm 1};\alpha][z]$ contains an element of $\mathcal{Z}$. Thus by Proposition 2.11, $(R_{\mathcal{Y}})_{\mathcal{Z}} = S_{\mathcal{Y}}$ is simple. Then by Lemma 3.4 and (iii), S is simple.

Conversely, suppose S is simple. Then, by Lemma 3.4, (iii) holds and $S_{\mathcal{Y}} = (R_{\mathcal{Y}})_{\mathcal{Z}}$ is simple. So by Lemma 2.12 any ideal of $R_{\mathcal{Y}} = A[y^{\pm 1};\alpha][z]$ must contain an element of $\mathcal{Z}$.

We claim this implies that $A[y^{\pm 1};\alpha]$ is simple: if I is a non-zero ideal of $A[y^{\pm 1};\alpha]$ then $Ik[z]$ is a non-zero ideal of $A[y^{\pm 1};\alpha][z]$, so by the above contains some non-zero element of $k[z]$; but this can only happen if $1 \in I$, so we've shown our claim.

Therefore, by [33, 1.8.5], (i) and (ii) hold. $\square$

Showing condition (ii) is often straightforward.

Lemma 3.5. *In the situation of Theorem 3.3, if $\alpha^m(v) \neq v$ for all $m \geq 1$, then condition (ii) holds. Further, if v is a regular non-unit and condition (i) holds then this is always true.*

Proof. v is central in A, so if α^m is inner then $\alpha^m(v) = v$. So if the latter is not true for any $m \geq 1$ then condition (ii) holds.

If v is a regular non-unit and $\alpha^m(v) = v$ then $v \cdots \alpha^{m-1}(v)$ is a central α-invariant non-zero non-unit in A, and so generates a nontrivial α-stable ideal of A. So if condition (i) holds and v is a regular non-unit then $\alpha^m(v) \neq v$ for all $m \geq 1$. $\square$

Example 3.6. Let k be an algebraically closed field of characteristic 0. The **universal enveloping algebra of $\mathfrak{sl}_2$**, denoted $U(\mathfrak{sl}_2)$, is the k-algebra generated by e, f, and h subject to the relations $he - eh = 2e$, $hf - fh = -2f$, and $ef - fe = h$. This algebra has been extensively studied so the results of this section do not discover anything that was not already well-known, but it is a useful example to illustrate the results of Sections 3.1-3.3. We will see the example for which these results were developed in Section 5.4.

If $A = k[h]$ and α is the k-automorphism of A defined by $\alpha(h) = h + 2$ then $U(\mathfrak{sl}_2) = R(A, \alpha, h)$ with $x = e$ and $y = f$. This is a conformal ambiskew polynomial ring with splitting element $u = \frac{-1}{4}(h-1)^2$ and associated central Casimir element $z = ef + \frac{1}{4}(h-1)^2$. (The usual Casimir element for $U(\mathfrak{sl}_2)$, which is usually denoted $\Omega = 2ef + 2fe + h^2 = 4ef - 2h + h^2$, satisfies $\Omega = 4z - 1$).

Since k is characteristic 0, the condition $\{c \in A : c \text{ central in } A \text{ and } \alpha(c) = c\} = k$ holds and A is α-simple. Condition (ii) is also clear (or one can use Lemma 3.5). Finally, $v^{(m)} = m(h + m - 1)$ so taking $p(t) = t + \frac{1}{4}m^2$, we have $p(u) = \frac{1}{4}(m + t - 1)(m - t + 1)$ and so condition (iii) is satisfied. Therefore we can apply Theorem 3.3 showing that $U(\mathfrak{sl}_2)_{k[z]\backslash 0}$ is simple.

3.2 Height two primes in generalised Weyl algebras

In Theorem 2.66 we gave a simplicity criterion for generalised Weyl algebras. In this section, we investigate what happens when one of the conditions from that theorem fails: specifically, if $uA + \alpha^m(u)A \neq A$ for some $m \geq 1$. The final result will assume that $uA + \alpha^m(u)A = A$ for all but one value of m, and in that situation $uA + \alpha^m(u)A$ will be a maximal ideal in A, but some of the intermediate results will have weaker hypotheses.

Lemma 3.7. *Suppose $Au + A\alpha^m(u) = A$. Let J be an ideal of T. If $Y^m \in J$ then $Y^{m-1} \in J$; if $X^m \in J$ then $X^{m-1} \in J$.*

Proof. If $Y^m \in J$ then $uY^{m-1} = XY^m \in J$ and $\alpha^m(u)Y^{m-1} = Y^mX \in J$. So since $uA + \alpha^m(u)A = A$, $Y^{m-1} \in J$.

Similarly, if $X^m \in J$ then $YX^m = \alpha(u)X^{m-1}$ and $X^mY = \alpha^{-(m-1)}X^{m-1} \in J$. Since $\alpha(u)A + \alpha^{-(m-1)}(u)A = \alpha^{-(m-1)}(uA + \alpha^m(u)A) = A$, $X^{m-1} \in J$. □

Corollary 3.8. *Suppose $Au + A\alpha^m(u) = A$ for $0 < m < n$. Let J be an ideal of T, such that either $Y^m \in J$ or $X^m \in J$, for some $0 \leq m \leq n-1$. Then $1 \in J$.*

Definition 3.9. Let $T = T(A, \alpha, u)$ be a generalised Weyl algebra such that, for some fixed $n \in \mathbb{N}$:

1. $Au + A\alpha^m(u) = A$ for $0 < m < n$;

2. $Au + A\alpha^n(u) \neq A$.

For any ideal I of A containing $Au + A\alpha^n(u)$, we define

$$I_m := \begin{cases} A & \text{if } m \geq n \text{ or } m \leq -n \\ \alpha^{-(n-1)}(I) \cap \cdots \cap \alpha^m(I) & \text{if } -(n-1) \leq m \leq 0 \\ \alpha^{-(n-1-m)}(I) \cap \cdots \cap I & \text{if } 0 \leq m \leq n-1 \end{cases}$$

and then $\mathcal{I} := \Sigma_{m \in \mathbb{N}}(I_mY^m + I_{-m}X^m)$.

Remark. This definition is X-Y symmetric in the following sense: we note that $T(A, \alpha, u) = T(A, \alpha^{-1}, \alpha(u))$ where the roles of X and Y have been switched. If I is an ideal of A containing $Au + A\alpha^n(u)$, then $J := \alpha^{-(n-1)}(I)$ is an ideal of A containing $A\alpha(u) + A\alpha^{-(n-1)}(u)$. So we can define J_m and $\mathcal{J}$ as in Definition 3.9. Then we have $J_m = I_{-m}$ for all m, and so $\mathcal{J} = \mathcal{I}$.

Proposition 3.10. *If $m \geq 0$, then $I_m \subset I_{m+1}$ and $\alpha(I_m) \subset I_{m+1}$; similarly, if $m \geq 0$ then $I_{-m} \subset I_{-m-1}$ and $\alpha^{-1}(I_{-m}) \subset I_{-m-1}$.*

Proof. This is immediate from the definition of I_m. □

Lemma 3.11. *Let T and I be as in Definition 3.9. Then $\mathcal{I}$ is a graded ideal of T.*

Proof. Since I_m is an ideal of A for each m, $\mathcal{I}A \subset \mathcal{I}$ and $A\mathcal{I} \subset \mathcal{I}$.

For all $m \geq 0$,

- $(I_m Y^m)Y = I_m Y^{m+1} \subset I_{m+1} Y^{m+1}$;
- $Y(I_m Y^m) = \alpha(I_m) Y^{m+1} \subset I_{m+1} Y^{m+1}$;
- $(I_{-m} X^m)X = I_{-m} X^{m+1} \subset I_{-(m+1)} X^{m+1}$;
- $X(I_{-m} X^m) = \alpha^{-1}(I_{-m}) X^{m+1} \subset I_{-(m+1)} X^{m+1}$.

And for all $m \geq 1$,

- $(I_m Y^m)X = I_m Y^{m-1}\alpha(u) = I_m \alpha^m(u) Y^{m-1} \subset (I_m \cap \alpha^{-(n-m)}(I)) Y^{m-1} = I_{m-1} Y^{m-1}$;
- $X(I_m Y^m) = (\alpha^{-1}(I_m)u) Y^{m-1} \subset (\alpha^{-1}(I_m) \cap I) Y^{m-1} = I_{m-1} Y^{m-1}$;
- $(I_{-m} X^m)Y = I_{-m} X^{m-1} u = (I_{-m} \alpha^{-(m-1)}(u)) X^{m-1} \subset (I_{-m} \cap \alpha^{-(m-1)}(I)) X^{m-1} = I_{-(m-1)} X^{m-1}$;
- $Y(I_{-m} X^m) = (\alpha(I_{-m})\alpha(u)) X^{m-1} \subset (\alpha(I_{-m}) \cap \alpha^{-(n-1)}(I)) X^{m-1} = I_{-(m-1)} X^{m-1}$.

Thus $\mathcal{I}$ is a graded ideal of T. □

Lemma 3.12. *Let T, I, and $\mathcal{I}$ be as in Definition 3.9. Then $Y^m X^m A + I = A$ for $0 \leq m \leq n-1$.*

Proof. We prove this by induction on m; when $m = 0$ the statement is trivial. So assume $m > 0$, in which case

$$\begin{aligned} Y^m X^m A + I &= \left(\Pi_{i=1}^{m} \alpha^i(u)\right) A + I \\ &= \left(\Pi_{i=1}^{m-1} \alpha^i(u)\right) (\alpha^m(u)A + uA) + I && \text{since } u \in I \\ &= Y^{m-1}X^{m-1}A + I && \text{since } \alpha^m(u)A + uA = A \\ &= A && \text{by induction.} \end{aligned}$$

□

Corollary 3.13. *$Y^{n-1}X^{n-1}A + I = A$, and so, since $X^{n-1}Y^{n-1} = \alpha^{-(n-1)}(Y^{n-1}X^{n-1})$, $X^{n-1}Y^{n-1}A + \alpha^{-(n-1)}(I) = A$.*

Definition 3.14. Let T be as in Definition 3.9, and let $x \in T$. For $m \in \mathbb{Z}$, define $x_m \in A$ to be the coefficient of Y^m (if $m \geq 0$) or X^{-m} (if $m \leq 0$) in x. That is, $x = x_{-r}X^r + \cdots + x_0 + \cdots + x_sY^s$, for some r and s (which will always exist) such that $x_t = 0$ if $t < r$ or $t > s$.

If $\mathcal{J}$ is a subset of T then for $m \in \mathbb{Z}$ define $J_m := \{x_m : x \in \mathcal{J}\}$.

Lemma 3.15. *Let T be as in Definition 3.9. If $\mathcal{J}$ is an ideal of T then J_m is an ideal of A.*

Proof. Given $a \in J_m$, pick some $y(a) \in \mathcal{J}$ such that $y(a)_m = a$, which must exist by the definition of J_m.

If $a \in J_m$ and $r \in A$ then $ry(a) \in \mathcal{J}$, and $(ry(a))_m = ra$, so $ra \in J_m$; similarly $y(a)\alpha^{-m}(r) \in \mathcal{J}$, and $(y(a)\alpha^{-m}(r))_m = ar$, so $ar \in J_m$. If $a, b \in J_m$ then $y(a) + y(b) \in \mathcal{J}$, so $a + b = (y(a) + y(b))_m \in J_m$. Therefore J_m is an ideal of A. □

Lemma 3.16. *Let T, I, and $\mathcal{I}$ be as in Definition 3.9. If I is a maximal ideal of A, then $\mathcal{I}$ is a maximal ideal of T.*

Proof. Let $\mathcal{J}$ be an ideal of T with $\mathcal{J} \supset \mathcal{I}$, and assume that $\mathcal{J} \neq T$.

Suppose there exists $x \in \mathcal{J}$ with $x_{n-1} \notin I_{n-1} = I$. Then, since I is maximal, $J_{n-1} = A$, so we may assume that in fact $x_{n-1} = 1$. Since $Y^m \in \mathcal{I} \subset \mathcal{J}$ for $m \geq n$, we may also assume that $x_m = 0$ for $m \geq n$. Now consider $y := X^{n-1}xX^{n-1} \in \mathcal{J}$. For $m \geq -(n-2)$, $y_m = 0$, so since $X^m \in \mathcal{J}$ for $m \geq n$, this tells us that $X^{n-1}Y^{n-1}X^{n-1} \in \mathcal{J}$.

However, $X^{n-1}Y^{n-1}X^{n-1}A + \alpha^{-(n-1)}(I)X^{n-1} = AX^{n-1}$ by Corollary 3.13, so since $\alpha^{-(n-1)}(I)X^{n-1} \subset \mathcal{I} \subset \mathcal{J}$, this means $X^{n-1} \in \mathcal{J}$, which is a contradiction since Corollary 3.8 then implies $1 \in \mathcal{J}$.

Therefore $J_{n-1} = I_{n-1} = I$.

Next, for any $x \in \mathcal{J}$, consider $y := Y^jxY^{i-j}$ for $0 \leq j \leq i \leq n-1$. This is an element of $\mathcal{J}$ with $y_{n-1} = \alpha^j(x_{n-1-i})$. So $\alpha^j(x_{n-1-i}) \in I$, that is, $x_{n-1-i} \in \alpha^{-j}(I)$. Combining this for

$0 \leq j \leq i$ and referring back to the definition of I_{n-1-i}, we get $x_{n-1-i} \in I_{n-1-i}$, and hence $J_{n-1-i} = I_{n-1-i}$ for $0 \leq i \leq n-1$.

By the same argument with X and Y reversed, $J_{-(n-1-i)} = I_{-(n-1-i)}$ for $0 \leq i \leq n-1$, and so $\mathcal{J} = \mathcal{I}$. Therefore $\mathcal{I}$ is maximal. □

Lemma 3.17. *Let T be as in Definition 3.9. Let $I = Au + A\alpha^n(u)$, and let $\mathcal{I}$ be as in Definition 3.9. Then any prime ideal $\mathcal{J}$ of T containing X^n and Y^n must contain $\mathcal{I}$.*

Proof. We claim first that if $X^n \in \mathcal{J}$ then $Y^{n-1}X^{n-1}\mathcal{I} \subset \mathcal{J}$. We note that $Y^{n-1}X^{n-1}$ commutes with elements of A, since by Lemma 2.64, $Y^{n-1}X^{n-1} = (\Pi_{j=1}^{n-1}\alpha^j(u))$.

Firstly, $Y^{n-1}X^n \in \mathcal{J}$, and so $Y^{n-1}X^{n-1}I_{-r}X^r \subset \mathcal{J}$ for all $r \geq 1$.

Secondly, $Y^{n-1}X^{n-1}u = Y^{n-1}X^nY \in \mathcal{J}$ and $Y^{n-1}X^{n-1}\alpha^n(u) = Y^{n-1}\alpha(u)X^{n-1} = Y^nX^n \in \mathcal{J}$; putting these two together, $Y^{n-1}X^{n-1}I \subset \mathcal{J}$, and since $I_r \subset I$ for all $r \geq 0$, $Y^{n-1}X^{n-1}I_rY^r \subset \mathcal{J}$ for all $r \geq 0$.

Combining these, we've shown our claim.

But now we know that $TY^{n-1}X^{n-1}\mathcal{I} \subset \mathcal{J}$, and $TY^{n-1}X^{n-1}$ and $\mathcal{I}$ are both left ideals of T. So, since $\mathcal{J}$ was assumed to be prime, either $TY^{n-1}X^{n-1} \subset \mathcal{J}$ or $\mathcal{I} \subset \mathcal{J}$.

Suppose $TY^{n-1}X^{n-1} \subset \mathcal{J}$. Then we note that $X^nY = \alpha^{-(n-1)}(u)X^{n-1}$ and $YX^n = \alpha(u)X^{n-1}$. So $\alpha^{-(n-1)}(I)X^{n-1} \subset \mathcal{J}$ and $X^{n-1}Y^{n-1}X^{n-1} \in \mathcal{J}$, so by Corollary 3.13, $AX^{n-1} \subset \mathcal{J}$. But this implies $1 \in \mathcal{J}$ by Lemma 3.8, which is a contradiction since $\mathcal{J}$ is prime.

Therefore if $\mathcal{J}$ is prime and contains both X^n and Y^n, we must have $\mathcal{I} \subset \mathcal{J}$. □

Theorem 3.18. *Let $T = T(A, \alpha, u)$ be a generalised Weyl algebra, with some $n \in \mathbb{N}^+$ such that:*

(i) $Au + A\alpha^m(u) = A$ *for* $m > 0$ *but* $m \neq n$*;*

(ii) $M := Au + A\alpha^n(u)$ *is a maximal ideal in* A*;*

(iii) A *is* α*-simple;*

(iv) α^m *is not an inner automorphism for any* $m \geq 1$*;*

(v) u *is regular.*

Then we can define ideals M_m of A for $m \in \mathbb{Z}$ as in Definition 3.9, and then the ideal $\mathcal{M}$ of T in Definition 3.9. Then $\mathcal{M}$ is the unique non-trivial prime ideal of T.

Proof. As remarked in [29, Notation 5.3], since u is regular, $\mathcal{X} = \{X^i : i \in \mathbb{N}\}$ and $\mathcal{Y} = \{Y^i : i \in \mathbb{N}\}$ are left and right Ore sets in T with $T_{\mathcal{X}} = A[X^{\pm 1}; \alpha^{-1}]$ and $T_{\mathcal{Y}} = A[Y^{\pm 1}; \alpha]$, and with the assumptions we have on A and α, $T_{\mathcal{X}}$ and $T_{\mathcal{Y}}$ are simple. So by Lemma 2.12 any non-zero prime ideal of T must contain some element of $\mathcal{X}$ and some element of $\mathcal{Y}$.

By Corollary 3.8 if an ideal of T contains X^r or Y^r for $0 < r < n$ it contains 1. By Lemma 3.7 if an ideal of T contains X^r (resp. Y^r) for $r > n$ it contains X^n (resp. Y^n). Therefore any non-zero prime ideal of T must contain X^n and Y^n. So by Lemma 3.17, any non-zero prime ideal of T must contain $\mathcal{M}$ - but $\mathcal{M}$ is maximal by Lemma 3.16, so we're done. □

Example 3.19. Let k be an algebraically closed field of characteristic 0, and let $A = k[h]$, $\alpha(h) = h + 2$, and $u = \frac{-1}{4}(h-1)^2$, as in Example 3.6, so $R = R(A, \alpha, h) = U(\mathfrak{sl}_2)$. There is a central Casimir element $z = xy - u$, so for each $\lambda \in k$ the factor rings $R/(z-\lambda)R$ are generalised Weyl algebras $T(A, \alpha, u + \lambda)$.

Conditions (iii) - (v) of Theorem 3.18 are clearly satisfied. For (i) and (ii), $A(u+\lambda) + A\alpha^m(u+\lambda) = A$ iff the polynomials $u + \lambda$ and $\alpha^m(u+\lambda)$ have no common roots. The roots of $u + \lambda$ are $h = \pm(2\lambda^{\frac{1}{2}} + 1)$; plugging these into $\alpha^m(u+\lambda) = \frac{-1}{4}(h + 2m - 1)^2 + \lambda$ gives $-m(\pm\lambda^{\frac{1}{2}} + m)$. Since we only care about $m > 0$, there is therefore one root in common if $\lambda = m^2$ and no roots otherwise; in the terms of the theorem, $A(u+\lambda) + A\alpha^m(u+\lambda)$ is maximal if $\lambda = m^2$ and equals A otherwise.

Therefore, if $\lambda = m^2$ then by Theorem 3.18 there is a unique maximal ideal of R strictly containing $(z-\lambda)R$, while otherwise, by Theorem 2.66, $(z-\lambda)R$ is maximal.

3.3 Uniform rank

We recall that the uniform rank of a ring in some sense measures how far a ring is from being a domain, and that for example the uniform rank of $k[x,y]/(x,y)^r = r$ for any $r \geq 1$. We observe that, in the setting of Theorem 3.18, that, modulo $\mathcal{M}$, $X^n = Y^n = 0$ but $X^{n-1} \neq 0 \neq Y^{n-1}$, and so one might expect the uniform rank of $T/\mathcal{M}$ to be n. Since if $n = 1$, $T/\mathcal{M} \cong A/M$, we clearly need A/M to be a ring of right uniform rank 1, and we will also assume that A/M is a domain; by the Remark after Theorem 2.37, this combination is equivalent to A/M being a right Ore domain. We will remain in this setting for the rest of this section.

Definition 3.20. Let $T = T(A, \alpha, u)$ satisfy the requirements of Theorem 3.18, with additionally the maximal ideal $M = Au + A\alpha^n(u)$ being such that A/M is a right Ore domain.

Write $M^{(-r)} := \alpha^{-r}(M)$, for $r \in \mathbb{N}$.

Let $I^{(r)} := \left(\bigcap_{\substack{i=0 \\ i \neq r}}^{n-1} M^{(-i)} \right) T + \mathcal{M}$. This is a graded right ideal in T which contains $\mathcal{M}$; so let $J^{(r)} := I^{(r)}/\mathcal{M}$, which is a graded right ideal in $T/\mathcal{M}$.

We aim to show that $J^{(0)} \oplus \cdots \oplus J^{(n-1)} = T/\mathcal{M}$, and that $J^{(r)}$ is a uniform $T/\mathcal{M}$-module, so therefore $T/\mathcal{M}$ has right uniform dimension n.

Lemma 3.21. *Let R be a ring, and let I and J be ideals generated by finitely many central elements $i_1, \ldots, i_r$ and $j_1, \ldots, j_s$ respectively. Then $IJ = JI$.*

Proof. Both IJ and JI are the ideal generated by the central elements $\{i_a j_b : 1 \leq a \leq r, 1 \leq b \leq s\}$ (noting $i_a j_b = j_b i_a$). □

Lemma 3.22. *Let R be a ring, and let I and J be right ideals in R such that $IJ = JI$ and $I + J = R$. Then $IJ = I \cap J$.*

Proof. For any right ideals, $(I+J)(I \cap J) \subset JI + IJ \subset I \cap J$. If $I + J = R$ then $I \cap J = JI + IJ$ and if $IJ = JI$ then $JI + IJ = IJ$, so if we have both then $I \cap J = IJ$. □

Lemma 3.23. *Let $T = T(A, \alpha, u)$, n and M satisfy the requirements of Definition 3.20.*

If $0 \leq i < j < n$ then $M^{(-i)} + M^{(-j)} = A$.

Proof.
$$\begin{aligned} M^{(-i)} + M^{(-j)} &= A\alpha^{-i}(u) + A\alpha^{n-i}(u) + A\alpha^{-j}(u) + A\alpha^{n-j}(u) \\ &= \alpha^{-j}(Au + A\alpha^{j-i}(u)) + A\alpha^{n-i}(u) + A\alpha^{n-j}(u) \\ &= A, \text{ since } Au + A\alpha^{j-i}(u) = A \text{ by assumption.} \end{aligned}$$
□

Lemma 3.24. *Let $T = T(A, \alpha, u)$, n and M satisfy the requirements of Definition 3.20.*

Let $i, j_1, \ldots, j_r$ be distinct integers between 0 and $n-1$ inclusive. Then

$$M^{(-i)} + \prod_{k=1}^{r} M^{(-j_k)} = A.$$

Proof. We prove this by induction on r. When $r = 1$, this is immediate from Lemma 3.23, so assume $r > 1$.

$$\begin{aligned}
M^{(-i)} + \prod_{k=1}^{r} M^{(-j_k)} &= M^{(-i)} + M^{(-i)}M^{(-j_r)} + \prod_{k=1}^{r} M^{(-j_k)} \\
&= M^{(-i)} + \left(M^{(-i)} + \prod_{k=1}^{r-1} M^{(-j_k)} \right) M^{(-j_r)} \\
&= M^{(-i)} + AM^{(-j_r)} && \text{by induction} \\
&= A && \text{by Lemma 3.23.}
\end{aligned}$$

□

Lemma 3.25. *Let $T = T(A, \alpha, u)$, n and M satisfy the requirements of Definition 3.20.*

Let $j_1, \ldots, j_r$ be distinct integers between 0 and $n-1$ inclusive. Then

$$\bigcap_{i=1}^{r} M^{(-j_i)} = \prod_{i=1}^{r} M^{(-j_i)}.$$

Proof. We prove this by induction on r. When $r = 1$ this is trivial, so assume $r > 1$.

Since $\prod_{i=1}^{r-1} M^{(-j_i)}$ and $M^{(-j_r)}$ commute by Lemma 3.21, and $\prod_{i=1}^{r-1} M^{(-j_i)} + M^{(-j_r)} = A$ by Lemma 3.24, the conditions of Lemma 3.22 are satisfied, and so,

$$\begin{aligned}
\bigcap_{i=1}^{r} M^{(-j_i)} &= \left(\bigcap_{i=1}^{r-1} M^{(-j_i)} \right) \cap M^{(-j_r)} \\
&= \left(\prod_{i=1}^{r-1} M^{(-j_i)} \right) \cap M^{(-j_r)} && \text{by induction} \\
&= \left(\prod_{i=1}^{r-1} M^{(-j_i)} \right) M^{(-j_r)} && \text{by Lemma 3.22} \\
&= \prod_{i=1}^{r} M^{(-j_i)}.
\end{aligned}$$

□

Lemma 3.26. *Let $T = T(A, \alpha, u)$, n and M satisfy the requirements of Definition 3.20.*

For $0 \leq r \leq n-1$, $J^{(0)} \oplus \cdots \oplus J^{(r)}$ is a direct sum of right $T/\mathcal{M}$-modules, and equals $\left(\left(\prod_{i=r+1}^{n-1} M^{(-i)} \right) T + \mathcal{M} \right) \Big/ \mathcal{M}$.

Proof. We prove these simultaneously by induction on r. The case $r = 1$ is just the definition of $J^{(0)}$.

We need to show two things: that $(J^{(0)} \oplus \cdots \oplus J^{(r-1)}) \cap J^{(r)} = 0$, and that $(J^{(0)} \oplus \cdots \oplus J^{(r-1)}) + J^{(r)} = \left(\left(\prod_{i=r+1}^{n-1} M^{(-i)} \right) T + \mathcal{M} \right) \Big/ \mathcal{M}$.

For the first,

$$\left(\left(\prod_{i=r}^{n-1} M^{(-i)}\right) T + \mathcal{M}\right) \cap I^{(r)}$$

$$= \left(\prod_{i=r}^{n-1} M^{(-i)}\right) T \cap \left(\prod_{\substack{i=0 \\ i\neq r}}^{n-1} M^{(-i)}\right) T + \mathcal{M} \quad \text{by definition of } I^{(r)}$$

$$= \left(\bigcap_{i=0}^{n-1} M^{(-i)}\right) T + \mathcal{M} \quad \text{by Lemma 3.25}$$

$$= \mathcal{M} \quad \text{by definition of } \mathcal{M}.$$

This is equivalent to $\left(\left(\left(\prod_{i=r}^{n-1} M^{(-i)}\right) T + \mathcal{M}\right) \Big/ \mathcal{M}\right) \cap J^{(r)} = 0$ as $T/\mathcal{M}$-modules, so by induction, $(J^{(0)} \oplus \cdots \oplus J^{(r-1)}) \cap J^{(r)} = 0$.

For the second,

$$\left(J^{(0)} \oplus \cdots \oplus J^{(r-1)}\right) + J^{(r)}$$

$$= \left(\left(\prod_{i=r}^{n-1} M^{(-i)}\right) T + \left(\prod_{\substack{i=0 \\ i\neq r}}^{n-1} M^{(-i)}\right) T + \mathcal{M}\right) \Big/ \mathcal{M} \quad \text{by induction}$$

$$= \left(\left(\prod_{i=r+1}^{n-1} M^{(-i)}\right) \left(M^{(-r)} + \prod_{i=0}^{r-1} M^{(-i)}\right) T + \mathcal{M}\right) \Big/ \mathcal{M}$$

$$= \left(\prod_{i=r+1}^{n-1} M^{(-i)} T + \mathcal{M}\right) \Big/ \mathcal{M} \quad \text{by Lemma 3.24.}$$

□

Corollary 3.27. *Taking $r = n-1$, we get $J^{(0)} \oplus \cdots \oplus J^{(n-1)} = T/\mathcal{M}$.*

Lemma 3.28. *Let $T = T(A, \alpha, u)$, n and M satisfy the requirements of Definition 3.20.*

Recall that $J^{(r)}$ is a graded right ideal of $T/\mathcal{M}$. Let $J_m^{(r)} = I_m^{(r)}/M_m$ for $m \in \mathbb{Z}$, so $J^{(r)} = \Sigma_{m\in\mathbb{N}}(J_{-m}^{(r)} X^m + J_m^{(r)} Y^m)$. Then

$$J_m^{(r)} := \begin{cases} 0 & \text{if } m \geq n-r \text{ or } m \leq -r-1 \\ \prod_{\substack{i=-m \\ i\neq r}}^{n-1} M^{(-i)} \Big/ \prod_{i=-m}^{n-1} M^{(-i)} & \text{if } -r \leq m \leq 0 \\ \prod_{\substack{i=0 \\ i\neq r}}^{n-1-m} M^{(-i)} \Big/ \prod_{i=0}^{n-1-m} M^{(-i)} & \text{if } 0 \leq m \leq n-r-1. \end{cases}$$

Proof.

$$\text{Recall that } M_m := \begin{cases} A & \text{if } m \geq n \text{ or } m \leq -n \\ \prod_{i=-m}^{n-1} M^{(-i)} & \text{if } -(n-1) \leq m \leq 0 \\ \prod_{i=0}^{n-1-m} M^{(-i)} & \text{if } 0 \leq m \leq n-1 \end{cases}$$

and that $I_m^{(r)} = \prod_{\substack{i=0 \\ i \neq r}}^{n-1} M^{(-i)} + M_m$. Then we apply Lemma 3.24, to get

$$I_m^{(r)} = \begin{cases} M_m & \text{if } m \geq n-r \text{ or } m \leq -r-1 \\ \prod_{\substack{i=-m \\ i \neq r}}^{n-1} M^{(-i)} & \text{if } -r \leq m \leq 0 \\ \prod_{\substack{i=0 \\ i \neq r}}^{n-1-m} M^{(-i)} & \text{if } 0 \leq m \leq n-r-1. \end{cases}$$

Then the result follows since $J_m^{(r)} = I_m^{(r)}/M_m$. □

Lemma 3.29. *Let $T = T(A, \alpha, u)$, n and M satisfy the requirements of Definition 3.20.*

Let $m \geq 0$. If $b+M_{-m}$ is a non-zero element of $J_{-m}^{(r)}$, then $bX^mY^m+\mathcal{M} = b\prod_{i=0}^{m-1} \alpha^{-i}(u)+\mathcal{M}$ is also a non-zero element of $J^{(r)}$.

Let $m \geq 0$. If $b+M_m$ is a non-zero element of $J_m^{(r)}$, then $bY^mX^m+\mathcal{M} = b\prod_{i=1}^{m} \alpha^{i}(u)+\mathcal{M}$ is also a non-zero element of $J^{(r)}$.

Therefore, if a is a non-zero element of $J^{(r)}$, then there exists $m \in \mathbb{N}$ such that one of aX^m or aY^m has non-zero coefficient in degree 0.

Proof. Suppose $b + M_{-m}$ is a non-zero element of $J_{-m}^{(r)}$, so $b \in I_{-m}^{(r)}$ with $b \notin M_{-m}$. Then $b \notin M^{(-r)}$. Also, by Lemma 3.28, we must have $0 \leq m \leq r$, so $\prod_{i=0}^{m-1} \alpha^{-i}(u) \notin M^{(-r)}$.

We note that, by assumption, M is a completely prime ideal of A, and so $M^{(-r)}$ is also a completely prime ideal of A for any $r \geq 0$.

Now $b\prod_{i=0}^{m-1} \alpha^{-i}(u) \notin M^{(-r)}$, since $M^{(-r)}$ is completely prime. Therefore $bY^mX^m = b\prod_{i=0}^{m-1} \alpha^{-i}(u) \notin M_0$ since $M_0 \subset M^{(-r)}$. Therefore $bY^mX^m + \mathcal{M}$ is a non-zero element of $J^{(r)}/\mathcal{M}$

Similarly, suppose $b + M_m \in J_m^{(r)}$ is a non-zero element of $J_m^{(r)}$, so $b \in I_m^{(r)}$ with $b \notin M_m$, and so $b \notin M^{(-r)}$. By Lemma 3.28, $0 \leq m \leq n-r-1$, so $\prod_{i=1}^{m} \alpha^{i}(u) \notin M^{(-r)}$. Therefore $b\prod_{i=1}^{m} \alpha^{i}(u) \notin M^{(-r)}$, and so is not in $\mathcal{M}_0$, and so $bY^mX^m + \mathcal{M} = b\prod_{i=1}^{m} \alpha^{i}(u) + \mathcal{M}$ is a non-zero element of $J^{(r)}$.

For the final part, pick m such that a has non-zero coefficient in degree m or $-m$, and apply the appropriate previous part. □

Lemma 3.30. *Let $T = T(A, \alpha, u)$, n and M satisfy the requirements of Definition 3.20.*

Let $a = a_{-r}X^r + \cdots + a_0 + \cdots + a_{n-r}Y^{n-r} \in I^{(r)}$, so $a + \mathcal{M} \in J^{(r)}$. Then $a\prod_{\substack{i=0\\i\neq r}}^{n-1}\alpha^{-i}(u) + \mathcal{M} = a_0\prod_{\substack{i=0\\i\neq r}}^{n-1}\alpha^{-i}(u) + \mathcal{M}$, which is non-zero in $J^{(r)}$ iff $a_0 \notin M_0$.

Proof. For $j > 0$, the coefficient of X^j in $a\prod_{\substack{i=0\\i\neq r}}^{n-1}\alpha^{-i}(u)$ is $a_{-j}\prod_{\substack{i=0\\i\neq r}}^{n-1}\alpha^{-i-j}(u)$. Since $0 < j \leq r$, $\alpha^{-(r-j)-j}(u) = \alpha^{-r}(u)$ is one of the terms in that product. And since $a_{-j} \in \prod_{\substack{i=m\\i\neq r}}^{n-1}M^{(-i)}$, $a_{-j}\prod_{\substack{i=0\\i\neq r}}^{n-1}\alpha^{-i-j}(u) \in \prod_{i=m}^{n-1} M^{(-i)} = \mathcal{M}_{-j}$.

Similarly, for $j > 0$ the coefficient of Y^j in $a\prod_{\substack{i=0\\i\neq r}}^{n-1}\alpha^{-i}(u)$ is $a_j\prod_{\substack{i=0\\i\neq r}}^{n-1}\alpha^{j-i}(u)$. Since $0 < j \leq n - r - 1$, $\alpha^{j-(j+r))}(u) = \alpha^{-r}(u)$ is one of the terms in that product. And since $a_j \in \prod_{\substack{i=m\\i\neq r}}^{n-1}M^{(-i)}$, $a_j\prod_{\substack{i=0\\i\neq r}}^{n-1}\alpha^{-i-j}(u) \in \prod_{i=m}^{n-1} M^{(-i)} = \mathcal{M}_j$.

Therefore $a\prod_{\substack{i=0\\i\neq r}}^{n-1}\alpha^{-i}(u) + \mathcal{M} = a_0\prod_{\substack{i=0\\i\neq r}}^{n-1}\alpha^{-i}(u) + \mathcal{M}$. To complete the proof, we note that if $a_0 \notin M_0$ then $a_0 \notin M^{(-r)}$, and that $\prod_{\substack{i=0\\i\neq r}}^{n-1}\alpha^{-i}(u) \notin M^{(-r)}$ also. Since $A/M^{(-r)}$ is a domain, $a_0\prod_{\substack{i=0\\i\neq r}}^{n-1}\alpha^{-i}(u) \notin M^{(-r)} \supset M_0$, and therefore is non-zero in $J^{(r)}$. □

Lemma 3.31. *Let $T = T(A, \alpha, u)$, n and M satisfy the requirements of Definition 3.20. Then $J^{(r)}$ is a uniform $T/\mathcal{M}$-module for each $0 \leq r \leq n - 1$.*

Proof. We use Lemma 2.35, that is, $J^{(r)}$ is uniform iff for all non-zero $a_1, a_2 \in J^{(r)}/\mathcal{M}$, there exist $r_1, r_2 \in T/\mathcal{M}$ such that $a_1r_1 = a_2r_2 \neq 0$.

So let a_1, a_2 be non-zero elements of $J^{(r)}$. By Lemmas 3.29 and 3.30, there exist r_1', r_2' such that a_1r_1' and a_2r_2' are homogeneous of degree 0. Next:

$$J_0^{(r)} = \frac{\prod_{\substack{i=0\\i\neq r}}^{n-1}M^{(-i)}}{\prod_{i=0}^{n-1} M^{(-i)}} \cong \frac{A}{M^{(-r)}}$$

Since $A/M^{(-r)} \cong A/M$, and the latter is a right uniform ring by assumption, $A/M^{(-r)}$ is a uniform right $A/M^{(-r)}$-module. So $J_0^{(r)}$ is a uniform right $A/M^{(-r)}$-module, and so $J_0^{(r)}$ is a uniform right $A/\mathcal{M}_0$-module. Therefore there exist $r_1'', r_2'' \in A/\mathcal{M}_0$ such that $a_1r_1'r_1'' = a_2r_2'r_2'' \neq 0$, and thus $J^{(r)}$ is a uniform $T/\mathcal{M}$-module. □

Theorem 3.32. *Let $T(A, \alpha, u)$ be a generalised Weyl algebra such that, for some fixed $n \in \mathbb{N}$:*

1. *$Au + A\alpha^m(u) = A$ for $0 < m < n$ and for $n < m$;*
2. *$M := Au + A\alpha^n(u)$ is a maximal ideal in A;*

3. A/M is a uniform ring and a domain.

Then the factor ring $T/\mathcal{M}$ has uniform rank n.

Proof. Each $J^{(r)}$ is a uniform submodule of $T/\mathcal{M}$ by Lemma 3.31, and $J^{(0)} \oplus \cdots \oplus J^{(n-1)} = T/\mathcal{M}$ by Corollary 3.27. Therefore $T/\mathcal{M}$ has uniform rank n. □

Example 3.33. The exceptional height 2 maximal ideals of $U(\mathfrak{sl}_2)$ we found in Example 3.19, which strictly contain $(z - m^2)U(\mathfrak{sl}_2)$, have uniform rank m.

3.4 Global dimensions

The aim of this section is to generalise the work of [25], predominantly §3, and [24], predominately §6 and 7, to the case of a noncommutative base ring sufficiently to be able to determine the global dimensions of the rings we consider in Section 5. We could not obtain a complete generalisation of those results in the general setting, but the results that are available we present here.

The setup will be as follows: let k be an algebraically closed field, let K be a simple ring with centre k, and let Z be a commutative finitely generated k-algebra which is an integral domain. Let $A = K \otimes_k Z$, let α be a k-automorphism of A such that $\alpha(K) = K$ (we automatically have $\alpha(Z) = Z$ since $Z(A) = Z$), and let $u \in Z$ be such that $\alpha(u) \neq u$. So we can construct the ambiskew polynomial ring $R = R(A, \alpha, u - \alpha(u))$.

By [33, 9.6.9] the map $I \mapsto K \otimes I$ is a one-to-one correspondence between the ideals of Z and the two-sided ideals of A, and this correspondence sends prime ideals to prime ideals.

We will at various times also assume that A is α-simple; we will state this when needed.

Write R_x for the localisation of R at the set $\{x^i : i \geq 1\}$, and similiarly R_y. Note that $R_x = A[x^{\pm 1}; \alpha^{-1}][z]$ and $R_y = A[y^{\pm 1}; \alpha][z]$, where $z = xy - u$ as in Definition 2.60.

Definition 3.34. An R-module X is x-torsion if, for all $m \in X$, there exists $r \geq 0$ such that $x^r m = 0$.

An R-module X is said to be xy-torsion if it is both x-torsion and y-torsion.

Definition 3.35. We will say $A = K \otimes_k Z$ is nice if, in addition to the requirements above:

1. All maximal ideals of Z are isomorphic in the sense that, for any two maximal ideals M, M' of Z, there exists a k-automorphism γ of Z such that $\gamma(M) = M'$. Examples of such Z include polynomial rings in finitely many variables over a field, and Laurent polynomial rings in finitely many variables over a field.

2. K is a constructible k-algebra as in [33, 9.4.12], that is, it can be obtained from k by a finite number of ring extensions, each being either an almost normalising extension ([33, 1.6.10]), which includes skew polynomial or skew Laurent extensions), or a finite module extension. In particular, this holds if K can be constructed as an iterated skew polynomial or skew Laurent ring over k.

Lemma 3.36. *Suppose that Z is finitely generated as a commutative k-algebra and K is a constructible k-algebra. Then the only primitive ideals of $A = K \otimes_k Z$ are the ideals $K \otimes M$ where M is a maximal ideal of Z.*

Proof. In the first case, by [33, 9.4.21], A satisfies the Nullstellensatz over Z. Then by the primitive property ([33, 9.2.3]) and the correspondence above, the result follows. □

Definition 3.37. (cf. [25, 3.1]).

Let M be a maximal right ideal of A such that there exists $\lambda \in k$ such that $u - \lambda \in M$. (We note that this λ must be unique). Let $V(M)$ denote the following right R-module:

$V(M) = \bigoplus_{i\geq 0} \frac{A}{\alpha^{-i}(M)}$ as an A-module;

$(a + \alpha^{-i}(M))y = \alpha^{-1}(a) + \alpha^{-(i+1)}(M)$ for $i \geq 0$, $a \in A$;

$(a + \alpha^{-i}(M))x = \alpha(a)(\alpha(u) - \lambda) + \alpha^{-(i-1)}(M)$ for $i \geq 1$, $a \in A$; and

$(a + M)x = 0$ for $a \in A$.

(We note that $\alpha(a)\alpha(u - \lambda) \in \alpha(M)$, and so this last is consistent with the pattern from the previous line; however, we must define it separately since $0 + \alpha(M) \notin V(M)$).

It is easy to check that this is a well-defined R-module structure.

Lemma 3.38. *Let M be a maximal right ideal of A such that there exists $\lambda \in k$ such that $u - \lambda \in M$. Suppose M is such that $\alpha^i(Z \cap M) \not\subset \alpha^j(Z \cap M)$ for $i, j \geq 0$ with $i \neq j$. Then any non-zero submodule of $V(M)$ must have the form*

$W_j = \bigoplus_{i\geq j} \frac{A}{\alpha^{-i}(M)}$ *where $j \geq 0$ is such that $u - \lambda \in \alpha^{-j}(M)$.*

Proof. Firstly, we check that any such W is in fact a submodule. The only non-trivial check is that $(a + \alpha^{-j}(M))x \in W_j$ for $a \in A$.

$$\begin{aligned}(a + \alpha^{-j}(M))x &= \alpha(a)(\alpha(u) - \lambda) + \alpha^{-(j-1)}(M) \\ &= (\alpha(u) - \lambda)\alpha(a) + \alpha^{-(j-1)}(M) \\ &= 0 + \alpha^{-(j-1)}(M) \qquad \text{as } \alpha(u) - \lambda \in \alpha^{-(j-1)}(M), \\ &\qquad\qquad\qquad\qquad \text{since } u - \lambda \in \alpha^{-j}(M).\end{aligned}$$

So any such W_j is a submodule of $V(M)$. We note also that $1 + \alpha^{-j}(M)$ generates W_j as an R-module.

Now let V' be any non-zero submodule of $V(M)$.

Any element of $V(M)$ can be written in the form $\Sigma_{i=0}^{n}(a_i + \alpha^{-i}(M))$, for some $a_i \in A$ and $n \in \mathbb{N}$. Define the **lower degree** of an element of V written in this form to be $\inf\{i : a_i \notin \alpha^{-i}(M)\}$, and define the **length** of an element of V written in this form to be $|\{i : a_i \notin \alpha^{-i}(M)\}|$.

Let p be an element of V' which has minimal lower degree amongst elements of V', and call this degree j.

Now since M is a maximal right ideal of A, $A/\alpha^{-j}(M)$ is a simple right A-module, so there exists $a \in A$ such that $pa = (1 + \alpha^{-j}(M)) + \Sigma_{i=j+1}^{n}(a_i + \alpha^{-i}(M))$, for some $a_i \in A$ and $n \in \mathbb{N}$.

Considering $pax = (\alpha(u) - \lambda + \alpha^{-(j-1)}(M)) + \cdots$, since p has minimal lower degree among elements of V', we see that $\alpha(u) - \lambda \in \alpha^{-(j-1)}(M)$, and so $u - \lambda \in \alpha^{-j}(M)$.

We now aim to show that $1 + \alpha^{-j}(M) \in V'$. Let $p' := \Sigma_{i=j}^{n}(a_i + \alpha^{-i}(M)) \in V'$ be an element of V' with lower degree j, and suppose p' has minimal length among elements of V' with lower degree j. Suppose p' has length > 1, that is, there exists $k > j$ with $a_k \notin \alpha^{-k}(M)$. Then since $\alpha^{-j}(M) \cap Z \neq \alpha^{-k}(M) \cap Z$, there exists $z \in \alpha^{-k}(M) \cap Z$ with $z \notin \alpha^{-j}(M)$. Then since $\alpha^{-j}(M)$ is a maximal right ideal of A, there exists $b \in A$ with $zb + \alpha^{-j}(M) = 1 + \alpha^{-j}(M)$.

Now consider $p'zb = (a_j + \alpha^{-j}(M)) + \Sigma_{i=j+1}^{n}(za_ib + \alpha^{-i}(M))$. This has smaller length than p', since $za_kb \in \alpha^{-k}(M)$, which is a contradiction to the minimality of the length of p'. Therefore p' has length 1, and so $1 + \alpha^{-j}(M) \in V'$, and we obtain $V' = W_j$. □

Corollary 3.39. *Let M be a maximal right ideal of A such that there exists $\lambda \in k$ such that $u - \lambda \in M$. If M is such that $\alpha^i(Z \cap M) \not\subset \alpha^j(Z \cap M)$ for $i, j \geq 0$ with $i \neq j$, then $V(M)$ has a unique simple quotient, determined by the least $j \geq 0$ is such that $u - \lambda \in \alpha^{-j}(M)$. We denote this simple module by $L(M)$.*

We note here that if $z = xy - (u - \lambda)$ with $u - \lambda \in M$, z annihilates $V(M)$ and so also $L(M)$.

Lemma 3.40. *(cf. [25, 3.10]). Suppose every maximal ideal of Z has infinite order under α, i.e. if N is a maximal ideal of Z then $\alpha^i(N) \neq N$ for all $i \geq 1$. Let X be a simple right R-module which is xy-torsion. Then X is isomorphic to $L(M)$ for some M such that $L(M) \neq V(M)$.*

Proof. This is proved as in [25, 3.10]. □

Lemma 3.41. *Suppose $A = K \otimes_k Z$ is nice in the sense of Definition 3.35, and rgld $A = d$.*

Then for any maximal ideal $M \subset Z$, there exists a maximal right ideal J of A such that pd $A/J = d$ and $M \subset J$.

Proof. By [33, 1.6.14] and [33, 1.1.3], K is (left and right) Noetherian, so by the Hilbert basis theorem, $A = K \otimes_k Z$ is Noetherian. Then by [7, Prop 1.1], there exists a maximal right ideal J of A such that pd $A/J = d$.

By Lemma 3.36, $\mathrm{ann}_A(A/J) = K \otimes M_0$ for M_0 some maximal ideal of Z, and so $M_0 \subset J$. Then by assumption, there exists a k-automorphism γ of Z such that $\gamma(M_0) = M$. Then $M \subset (\gamma \otimes \mathrm{id}_K)(J)$ and pd $A/(\gamma \otimes \mathrm{id}_K)(J) = d$. So $(\gamma \otimes \mathrm{id}_K)(J)$ is the ideal we require. □

Theorem 3.42. *Suppose $A = K \otimes_k Z$ is nice in the sense of Definition 3.35, Z is α-simple, and rgld $A = d$. Then if $\lambda \in k$ and $j \in \mathbb{N}$ are such that:*

1. *$A(u - \lambda) + A\alpha^m(u - \lambda) = A$ for $0 < m \neq j$;*

2. *$A(u - \lambda) + A\alpha^j(u - \lambda) = M$ is a maximal ideal in A.*

Then there exists a maximal right ideal J of A containing M such that $L(J)$ has projective dimension $d + 2$ (as an R-module). Therefore, rgld $R = d + 2$.

Proof. By Lemma 3.41, there exists a maximal right ideal J of A containing M such that pd $(A/J)_A = d$. Since Z is α-simple, $\alpha^i(Z \cap M) \neq Z \cap M$ for all M and all $i \neq 0$, and we know $u - \lambda \in \alpha^{-j}(M)$, so by Corollary 3.39, $L(J)$ is finitely generated over A. Furthermore, as an A-module, $L(J) = \bigoplus_{i=0}^{j-1} A/\alpha^{-i}(M)$, and so p.d. $L(J)_A = d$. Therefore by [33, 7.9.16] (twice), pd $L(J)_R = d + 2$, and so rgld $R \geq d + 2$.

On the other hand, by [33, 7.5.3(i)], since rgld $A = d$, rgld $R \leq d + 2$, and so rgld $R = d + 2$. □

We now turn our attention to the factor rings $T(u) := R/zR$, where $z = xy - u$.

Lemma 3.43. *Suppose we know that rgld R_x = rgld $R_y = d + 1$. Let X be a simple right R-module which is not xy-torsion. Then pd $X_R \leq d + 1$.*

Proof. Either $X_x \neq 0$ or $X_y \neq 0$, so wlog $X_y \neq 0$. If pd $X_R = d + 2$ then rgld $R = d + 2 >$ rgld R_y contradicting Theorem 2.50 (v). □

Theorem 3.44. *Suppose A is nice and Z is α-simple, and also that rgld $T(u) \neq \infty$. Then rgld $T(u) = d + 1$ iff there exists a maximal ideal N of Z containing both u and $\alpha^j(u)$ for some $j \geq 1$.*

Proof. Firstly, we note that by Theorem 2.50 (iii), rgld $T(u) \leq d+1$, since rgld $R \leq d+2$.

If there exists such an N, then by Theorem 3.42, there exists a maximal right ideal M of A containing N such that pd $L(J)_R = d+2$ and z annihilates $L(J)$. Then by [33, 7.3.5(i)] pd $L(J)_{T(u)} = d+1$, since we cannot have pd $L(J)_{T(u)} = \infty$. So rgld $T(u) = d+1$.

Conversely, if there doesn't exist such an N, then for any maximal ideal M of A containing u, $L(M) = V(M)$. So by Lemma 3.40, any simple R-module X annihilated by z - that is, any simple $T(u)$-module - is not xy-torsion. So by Lemma 3.43, pd $X_R \leq d+1$, and by [33, 7.3.5(i)] pd $X_{T(u)} \leq d$. Finally, by [7, Prop 1.1], it suffices to check the global dimension of $T(u)$ on simple $T(u)$-modules. □

Lemma 3.45. *Let J be a right ideal of A such that $\alpha(J) = J$ and pd $(A/J)_A = r$. Then rgld $T(u) \geq r+1$ for all u.*

Proof. Define a right R-module W_u in a similar fashion to before: as an A-module, $W_u = A/J$, and

$(a+J)y = \alpha^{-1}(a) + J$ for $a \in A$;

$(a+J)x = \alpha(a)u + J$ for $a \in A$.

Again, it is easy to check this is a well-defined R-module structure. And $a(xy-u) = 0$ for all $a \in A$, so W_u is a $T(u)$-module. Applying [33, 7.9.16] twice, pd $(W_u)_R = r+2$, then by [33, 7.3.9], either pd $(W_u)_{T(u)} = \infty$ or pd $(W_u)_{T(u)} = r+1$; either way, rgld $T(u) \geq r+1$. □

Note that this implies that we must always have rgld $T(u) \geq 1$.

4 Connected quantized Weyl algebras - definition and classification

4.1 Definition and first properties

Recall the higher quantized Weyl algebras $\mathcal{A}_n^{\mathbf{q},\mathbf{\Lambda}}$ of Definition 2.33. These have the following property: they are generated by a finite set of elements X, together with a relation of the form $xy - q_{xy}yx = r_{xy}$ for each pair x, y of elements in X. (For $A_n^{\mathbf{q},\mathbf{\Lambda}}$, $X = \{x_i, y_i : 1 \leq i \leq n\}$, and $r_{xy} = 0$ unless $\{x, y\} = \{x_i, y_i\}$ for some i).

We ask the question: what other rings have this property? A first observation is $A_n^{\mathbf{q},\mathbf{\Lambda}}$ is built up of several copies of A_1^q (for different values of q), with any two generators from different copies of A_1^q skew-commuting, so we should first investigate those rings which cannot be split up in this fashion. More formally:

Definition 4.1. Let R be a k-algebra, where k is a field, and $X \subset R$ be a finite generating set for R. We say X presents R as a **quantum space with Weyl relations** if:

1. for every two elements x and y of X, they satisfy a relation of the form $xy - q_{xy}yx = r_{xy}$, where $q_{xy} \in k^\times$ and $r_{xy} \in k$;
2. R has a PBW basis with respect to X, that is, we can write $X = \{x_1, \ldots, x_n\}$ (putting an ordering on X) such that the set of monomials of the form $x_1^{a_1} \cdots x_n^{a_n}$ form a basis for R.

Definition 4.2. If X presents R as a quantum space with Weyl relations then we can form the **associated graph** $G(R, X)$ of this presentation as follows: let the vertex-set of $G(R, X)$ be X, and say that $x, y \in X$ have an edge between them iff $r_{xy} \neq 0$.

A **connected quantized Weyl algebra** is a quantum space with Weyl relations whose associated graph is connected.

Example 4.3. The first quantized Weyl algebra is a connected quantized Weyl algebra. Higher quantized Weyl algebras are quantum spaces with Weyl relations, but not - at least with respect to the normal presentation - connected quantized Weyl algebras.

Proposition 4.4. *Let R be a k-algebra, and suppose $X \subset R$ presents R as a quantum space with Weyl relations. Then*

(i) $q_{yx} = q_{xy}^{-1}$ and $r_{yx} = -r_{xy}q_{xy}^{-1}$ for all $x, y \in X$, that is, the relations $xy - q_{xy}yx = r_{xy}$ and $yx - q_{yx}xy = r_{yx}$ are the same relation;

(ii) If $x, y, z \in X$ with $r_{xy} \neq 0$ then $q_{zx}q_{zy} = 1$;

(iii) If $x, y, z \in X$ with $r_{xy} \neq 0$ and $r_{yz} \neq 0$ then $q_{xy} = q_{yz} = q_{zx}$;

(iv) R is generated by X with respect to the relations $\{xy - q_{xy}yx = r_{xy} : x, y \in X\}$;

(v) If $Y \subset X$ and S is the subalgebra of R generated by Y, then Y presents S as a quantum space with Weyl relations.

Proof. (i) We have $xy - q_{xy}yx = r_{xy}$, so $yx - q_{xy}^{-1}xy = -q_{xy}^{-1}r_{xy}$.

But we also have $yx - q_{yx}xy = r_{yx}$.

Subtracting these two equations, we get $xy(q_{yx} - q_{xy}^{-1}) = r_{yx} + q_{xy}^{-1}r_{xy}$. If $q_{yx} \neq q_{xy}^{-1}$ then this violates the PBW basis condition, so we must have $q_{yx} = q_{xy}^{-1}$, in which case we also get $r_{yx} + q_{xy}^{-1}r_{xy} = 0$.

(ii) Pick an ordering of X, and let $a < b < c$ be elements of X. Consider $cba \in R$. We can write this in terms of this basis in two ways, by writing it first as $(cb)a$ or as $c(ba)$. These give us, respectively:

$cba = q_{cb}q_{ca}q_{ba}abc + r_{cb}a + r_{ca}q_{cb}b + r_{ba}q_{cb}q_{ca}c$;

$cba = q_{ba}q_{ca}q_{cb}abc + r_{ba}c + r_{ca}q_{ba}b + r_{cb}q_{ba}q_{ca}a$.

Comparing coefficients of a gives us $r_{cb}(1 - q_{ba}q_{ca}) = 0$ (1), comparing coefficients of b gives us $r_{ca}(q_{cb} - q_{ba}) = 0$ (2), and comparing coefficients of c gives us $r_{ba}(q_{cb}q_{ca} - 1) = 0$ (3). Relabelling appropriately, these three give us the desired result in all cases (when combined with (i)). For example, if $x < y < z$ then we take $a = x$, $b = y$, $c = z$; by (i), $r_{xy} \neq 0$ implies $r_{yx} \neq 0$, so (3) implies $q_{zy}q_{zx} = 1$ as desired.

(iii) From parts (i) and (ii), $r_{xy} \neq 0$ implies $q_{yz} = q_{zx}$ and $r_{yz} \neq 0$ implies $q_{xy} = q_{zx}$.

(iv) Pick an ordering on X such that R has a PBW basis with respect to that ordering. Let F be the free k-algebra on X, and let I be the ideal of F generated by $\{xy - q_{xy}yx = r_{xy} : x, y \in X\}$. These relations show that the monomials in X form a spanning set for F/I, so the natural surjection $F/I \to R$ must be an isomorphism, since it maps a spanning set for F/I to a basis for R.

(v) Order Y by restricting our order on X to Y. Relations between elements of Y of the appropriate form hold in S because they hold in R. As in (iv), this shows that the monomials in Y span S, while the monomials in Y are linearly independent since the monomials in Y are a subset of the monomials in X which form a basis for R, so are linearly independent.

□

Proposition 4.5. *Let R be a k-algebra, $X \subset R$ a generating set, and let $q_{xy} \in k^{\times}$, $r_{xy} \in k$, each for $x, y \in X$, be such that $q_{yx} = q_{xy}^{-1}$ and $r_{yx} = r_{xy}q_{xy}^{-1}$. Then the following are equivalent:*

1. *X presents R as a quantum space with Weyl relations, with $xy - q_{xy}yx = r_{xy}$ for $x, y \in X$.*
2. *We can write $X = \{x_1, \dots, x_n\}$ such that $R = R_n := k[x_1][x_2; \alpha_2, \delta_2] \cdots [x_n; \alpha_n, \delta_n]$, where α_i is the k-automorphism of $R_{i-1} := k[x_1] \cdots [x_{i-1}; \alpha_{i-1}, \delta_{i-1}]$ given by $\alpha_i(x_j) = q_{x_i x_j}$, $1 \le j \le i-1$, and δ_i is the α_i-derivation of R_{i-1} given by $\delta_i(x_j) = r_{x_i x_j}$, $1 \le j \le i - i$.*

Proof. 2 $\implies$ 1: By Proposition 2.21, the definitions of the α_i and δ_i show that R_n has the appropriate presentation, and R_n has a PBW basis with respect to the same ordering of X by 2.26.

1 $\implies$ 2: We must first show that the given α_i and δ_i do extend to valid automorphisms and α_i-derivations respectively on R_{i-1}. By Proposition 4.4(iv) and Proposition 2.22 it suffices to check that $\alpha_i(x)\alpha_i(y) - q_{xy}\alpha_i(y)\alpha_i(x) = r_{xy}$ and $\delta_i(x)y + \alpha_i(x)\delta_i(y) - q_{xy}\delta_i(y)x - q_{xy}\alpha_i(y)\delta_i(x) = 0$ for all $x, y \in X$. The first of these becomes $q_{x_i x}q_{x_i y}(xy - q_{xy}yx) = r_{xy}$, equivalently $q_{x_i x}q_{x_i y} = 1$, which holds by Proposition 4.4(ii); while the second becomes $r_{x_i x}y + q_{x_i x}r_{x_i y}x - q_{xy}r_{x_i y}x - q_{xy}q_{x_i y}r_{x_i x}y = 0$, which again follows from Proposition 4.4(ii).

So we have shown that R_i exists for all i. Now we claim by induction that R_i equals the subalgebra of R generated by $x_1, \dots, x_i$. For $i = 1$ this is trivial, and for higher i it follows direct from the definition of $R_{i-1}[x_i; \alpha_i, \delta_i]$ given the choices of α_i and δ_i and the PBW basis condition, which guarantees that the subalgebra of R generated by $x_1, \dots, x_i$ is free as a left module over R_{i-1}. □

Corollary 4.6. *Any quantum space with Weyl relations is a Noetherian domain.*

Corollary 4.7. *Let X present R as a quantum space with Weyl relations. Then*

$$\textit{GK dim.}R = |X|$$

Proof. This follows from Proposition 4.5 by repeated application of Theorem 2.57. □

Proposition 4.8. *Let X present R as a quantum space with Weyl relations. Then R has a PBW basis with respect to any ordering of X.*

Proof. Given an ordering on X, the relations that hold in R show that any element of R can be written as a sum of monomials in X with respect to that ordering, so those monomials span X.

Now suppose that the monomials in X with respect to our ordering are not linearly independent, and let p be some non-trivial sum of monomials in X which equals 0 in R. Pick x and $y \in X$ which are adjacent in our ordering, and consider a new ordering on X which has x and y switched. Then the relation $yx = q_{yx}xy + r_{yx}$ allows us to rewrite p with respect to the new ordering, and there is a one-to-one correspondence between the monomials of highest total degree in p and the monomials of highest total degree in our rewriting of p, so p is still a non-trivial sum of monomials. So the monomials in X with respect to our new ordering are not linearly independent. But since the symmetric group on n elements is generated by the transpositions of adjacent numbers, this means that the monomials in X with respect to any ordering are not linearly independent, a contradiction to our assumptions. □

Corollary 4.9. *Let X present R as a quantum space with Weyl relations, and write $X = \{x_1, \dots, x_n\}$ for any ordering of the elements of X. Then R can be written as an iterated skew polynomial ring $R[x_1][x_2; \alpha_2, \delta_2] \cdots [x_n; \alpha_n, \delta_n]$, where the α_i and δ_i are automorphisms and α_i-derivations respectively of the appropriate rings, as in Proposition 4.5.*

4.2 Classification

Definition 4.10. The ring L_n^q, where $n \in \mathbb{N}$ and $q \in k^\times$, is generated by $X_n = \{x_1, \dots, x_n\}$ with relations

$x_i x_{i+1} - q x_{i+1} x_i = 1 - q$, for $1 \le i \le n-1$;
$x_i x_j - q x_j x_i = 0$, for $i < j$, $j \ne i+1$, $j - i \equiv 1 \mod 2$;
$x_i x_j - q^{-1} x_j x_i = 0$, for $i < j$, $j \ne i+1$, $j - i \equiv 0 \mod 2$.

Definition 4.11. The ring C_n^q, where n is an odd positive integer and $q \in k^\times$, is generated by $Y_n = \{x_1, \dots, x_n\}$ with relations

$x_i x_{i+1} - q x_{i+1} x_i = 1 - q$, for $1 \le i \le n-1$;
$x_n x_1 - q x_1 x_n = 1 - q$;
$x_i x_j - q x_j x_i = 0$, for $i < j$, $j \ne i+1$, $j - i \equiv 1 \mod 2$;
$x_i x_j - q^{-1} x_j x_i = 0$, for $i < j$, $j \ne i+1$, $j - i \equiv 0 \mod 2$, $(i, j) \ne (1, n)$.

Proposition 4.12. *Provided $q \ne 1$, L_n^q and C_n^q are both connected quantized Weyl algebras with respect to the generating sets X_n and Y_n respectively.*

Proof. The main part is to show that they are quantum spaces with Weyl relations with respect to these generating sets; if they are, then the associated graphs are an n-vertex path for L_n^q and an n-vertex cycle for C_n^q, which are both connected. (It is here that the condition $q \neq 1$ is required, since L_n^1 and C_n^1 are commutative polynomial rings. It is possible to use relations of the form $x_i x_j - q x_j x_i = 1$ rather than $x_i x_j - q x_j x_i = 1 - q$, in which case L_n^1 and C_n^1 are indeed connected quantized Weyl algebras; however, many of the calculations are easier in the form we use, and as we shall see later, the case $q = 1$ is not our primary concern anyway.)

By induction and Proposition 4.5, it suffices to show that the maps α_n and δ_n of Proposition 4.5 define valid automorphisms and α_n-derivations on L_{n-1}^q, and so by Proposition 2.22 it suffices to show that they preserve the defining relations of L_{n-1}^q.

We first consider the automorphism and derivation of L_{n-1}^q required for L_n^q. Define $\alpha_n(x_i) = q^{(-1)^{n-i}} x_i$.

For $1 \leq i \leq n-2$,

$$\begin{aligned} \alpha_n(x_i x_{i+1} - q x_{i+1} x_i - (1-q)) &= q^{(-1)^{n-i}(-1)^{n-i-1}}(x_i x_{i+1} - q x_{i+1} x_i) - (1-q) \\ &= (1-q) - (1-q) \\ &= 0 \text{ as required.} \end{aligned}$$

For $i < j-1 < n-1$,

$$\begin{aligned} \alpha_n(x_i x_j - q^{(-1)^{j-i+1}} x_j x_i) &= q^{(-1)^{n-i}(-1)^{n-j}}(x_i x_j - q^{(-1)^{j-i+1}} x_j x_i) \\ &= 0 \text{ as required.} \end{aligned}$$

These combine to show that α_n is a k-automorphism of L_{n-1}^q.

Now define $\delta_n(x_{n-1}) = 1 - q^{-1}$, and $\delta_n(x_i) = 0$ otherwise.

For $i < n-2$,

$$\delta_n(x_i x_{i+1} - q x_{i+1} x_i - (1-q)) = 0 \text{ as required.}$$

Considering $i = n-2$,

$$\begin{aligned} \delta_n(x_{n-2} x_{n-1} - q x_{n-1} x_{n-2} - (1-q)) &= \alpha_n(x_{n-2})(1-q^{-1}) - q(1-q^{-1})x_{n-2} \\ &= 0 \text{ as required.} \end{aligned}$$

For $i < j < n-1$,

$$\delta_n(x_i x_j - q^{(-1)^{j-i+1}} x_j x_i) = 0 \text{ as required.}$$

For $i < n-1$,

$$\begin{aligned} \delta_n(x_i x_{n-1} - q^{(-1)^{n-1-i+1}} x_j x_i) &= \alpha_n(x_i)(1-q^{-1})x_i - q^{(-1)^{n-i}}(1-q^{-1})x_i \\ &= 0 \text{ as required.} \end{aligned}$$

These combine to show that δ_n is an α_n-derivation of L^q_{n-1}. So L^q_n is indeed a connected quantized Weyl algebra.

Now let n be odd. We note that the automorphism of L^q_{n-1} required for C^q_n is the same α_n defined above. Define $\gamma_n(x_1) = 1-q$, $\gamma_n(x_{n-1}) = 1-q^{-1}$, and $\gamma_n(x_i) = 0$ otherwise.

$$\begin{aligned}
\gamma_n(x_1x_2 - qx_2x_1 - (1-q)) &= (1-q)x_2 - q\alpha_n(x_2)(1-q) \\
&= (1-q)x_2 - qq^{-1}x_2(1-q) \\
&= 0 \text{ as required.}
\end{aligned}$$

For $2 < j < n-1$,

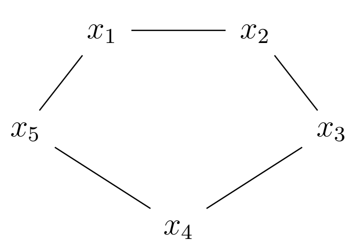
$$\begin{aligned}
\gamma_n(x_1x_j - q^{(-1)^{j-1+1}}x_jx_1) &= (1-q)x_j - (1-q)q^{(-1)^j}\alpha_n(x_j) \\
&= (1-q)x_j - (1-q)qq^{(-1)^j(-1)^{n-j}}x_j \\
&= (1-q)x_j - (1-q)qq^{(-1)^n} \\
&= 0 \text{ as required.} \\
\delta_n(x_1x_{n-1} - qx_{n-1}x_1) &= (1-q)x_{n-1} + \alpha_n(x_1)(1-q^{-1}) \\
&\quad - q(1-q^{-1})x_1 - q\alpha_n(x_{n-1})x_1 \\
&= (1-q)x_{n-1} + qx_1(1-q^{-1}) - q(1-q^{-1})x_1 - qq^{-1}x_{n-1} \\
&= 0 \text{ as required.}
\end{aligned}$$

These, together with the fact that $\delta_n(x_i) = \gamma_n(x_i)$ for $i \neq 1$, show that γ_n is an α_n-derivation of L^q_{n-1}. So C^q_n is indeed a connected quantized Weyl algebra. □

Remark. The associated graph of (L^q_n, X_n) is an n-vertex path:

$$x_1 \;\text{———}\; x_2 \;\text{———}\; \cdots \;\text{———}\; x_{n-1} \;\text{———}\; x_n$$

The associated graph of (C^q_n, Y_n) is an n-vertex cycle (depicted for $n = 5$):



Proposition 4.13. *Let X present R as a connected quantized Weyl algebra. Then there exists $q \in k^\times$ such that $q_{xy} \in \{q, q^{-1}\}$ for all $x \neq y \in X$.*

Proof. We prove this by induction on $n = |X|$. When $n = 1$ or $n = 2$ this is trivial. When $n > 2$, pick $x \in X$ such that removing x from $G(R, X)$ does not disconnect the graph, and let S be the subalgebra of R generated by $X \setminus \{x\}$. Then by Proposition 4.4 (v), $X \setminus \{x\}$ presents S as a connected quantized Weyl algebra, so by induction, there exists $q \in k^{\times}$ such that $q_{yz} \in \{q, q^{-1}\}$ for all $y \neq z \in X \setminus \{x\}$.

Now since $G(R, X)$ is connected, there exists $y \in X$ such that $r_{xy} \neq 0$. So by Proposition 4.4 (ii), for all $z \in X \setminus \{x, y\}$ we have $q_{zx} = q_{yz} \in \{q, q^{-1}\}$. Finally since $n > 2$ and $G(S, X \setminus \{x\})$ is connected, there must be $t \in X \setminus \{x, y\}$ such that $r_{ty} \neq 0$, and so by Proposition 4.4 (ii) we have $q_{yx} = q_{xt} \in \{q, q^{-1}\}$. □

Definition 4.14. Let X present R as a quantum space with Weyl relations, and let $q \in k^{\times}$. We say (R, X) is a **q-quantum space with Weyl relations**, or **q-qswr** for short, if $q_{xy} \in \{q, q^{-1}\}$ for all $x \neq y \in X$.

If X in fact presents R as a connected quantized Weyl algebra then we say (R, X) is a **connected q-quantized Weyl algebra**, or **q-cqwa** for short.

Remark. Not every quantum space with Weyl relations is a q-qswr, but by Proposition 4.13, any connected quantized Weyl algebra is a q-qcwa for some $q \in k$.

Remark. Any q-qswr (resp. q-qcwa) is a q^{-1}-qswr (resp. q^{-1}-qcwa).

The next result, which is a basic result from graph theory, will allow us to prove results on connected quantized Weyl algebras by induction: using this and Proposition 4.4 (iv), if X presents R as a connected quantized Weyl algebra, then there exists $v \in X$ such that $X \setminus \{v\}$ presents the subalgebra S of R generated by $X \setminus \{v\}$ as a connected quantized Weyl algebra, so we can apply induction to $(S, X \setminus \{v\})$.

Proposition 4.15. *Let G be a connected graph. Then there exists $v \in G$ such that $G \setminus \{v\}$ is still connected.*

Proof. Any connected graph has a spanning tree (i.e. a subgraph with the same vertices but possibly fewer edges which is connected and acyclic); any tree has a leaf (a vertex of degree 1). Removing this leaf leaves the spanning tree connected, so must leave the original graph connected. □

We will be aiming to show that a q-cqwa is always isomorphic to one of our known examples (when $q^2 \neq 1$, these are L_n^q and C_n^q). To perform the induction described above, we will want to replace $(S, X \setminus \{v\})$ by our known example in such a way that adding v back in gives us a q-qswr again. The following definition turns out to be what we need, as Proposition 4.20 shows.

Definition 4.16. Let R be a k-algebra, $q \in k^{\times}$, and X and Y be generating sets for R such that (R, X) and (R, Y) are both q-qswrs. (We note that we must have $|X| = |Y|$ by Proposition 4.7). Then we say that Y is **q-compatible** with X if, given any ring extension $T \supset R$ and $z \in T$ such that, for all $x \in X$, there exists $q_{xz} \in \{q, q^{-1}\}$, $r_{xz} \in k$ such that $xz - q_{xz}zx = r_{xz}$, then, for all $y \in Y$, there exists $q_{yz} \in \{q, q^{-1}\}$ and $r_{yz} \in k$ such that $yz - q_{yz}zy = r_{yz}$.

If X is q-compatible with Y and Y is q-compatible with X then we say X and Y are **q-compatible**.

Suppose further we have a bijection $\phi : X \to Y$. We say Y is **graph-q-compatible** with X if: for $x \in X$ and $x' \in X$, $r_{\phi(x)\phi(x')}$ is nonzero if and only if $r_{xx'}$ is nonzero, and $r_{\phi(x)z}$ is nonzero if and only if r_{xz} is nonzero. If X is also **graph-q-compatible** with Y then we say X and Y are **graph-q-compatible**. If this is the case, $G(R, X)$ and $G(R, Y)$ are isomorphic graphs (but this is a stronger condition than that).

Proposition 4.17. *Let R be a k-algebra, $q \in k^{\times}$, and X, Y and Z be generating sets for R such that (R, X), (R, Y) and (R, Z) are all q-qswrs. If X and Y are q-compatible and Y and Z are q-compatible then X and Z are q-compatible; the same is true if "q-compatible" is replaced by "graph-q-compatible".*

Proof. This is immediate from the definitions. □

Definition 4.18. Let (R, X) and (S, Y) be q-qswrs. We say a ring isomorphism $\phi : R \to S$ is a **q-qswr isomorphism** if $\phi(X)$ and Y are q-compatible. If $\phi(X)$ and Y are graph-q-compatible then we say ϕ is a **graph-q-qswr isomorphism** while if $\phi(X) = Y$ then we say ϕ is a **strong q-qswr isomorphism**.

Remark. If (R, X) and (S, Y) are strongly q-qswr isomorphic q-qswrs, and Y' is another generating set for S such that (S, Y') is a q-qswr, then (R, X) and (S, Y') are (graph-)q-isomorphic if and only if Y and Y' are (graph-)q-compatible.

Proposition 4.19. *Let (R, X) and (S, Y) be q-qswrs. If $\phi : R \to S$ is a (graph) q-qswr isomorphism then ϕ^{-1} is also a (graph) q-qswr isomorphism.*

Let (R, X), (S, Y), and (T, Z) be q-qswrs. If $\phi : R \to S$ and $\psi : S \to T$ are (graph) q-qswr isomorphisms then $\psi \circ \phi : R \to T$ is a (graph) q-qswr isomorphism.

Proof. Since $\phi(X)$ and Y are (graph) q-compatible, their images X and $\phi^{-1}(Y)$ under ϕ must be (graph) q-compatible also.

Since $\phi(X)$ and Y are (graph) q-compatible, their images $\psi \circ \phi(X)$ and $\psi(Y)$ must be (graph) q-compatible; therfore since $\psi(Y)$ and Z are (graph) q-compatible, by Proposition 4.17, $\psi \circ \phi(X)$ and Z are (graph) q-compatible. □

Proposition 4.20. *Let (R, X) be a q-qswr. Let $Y \subset X$ and let S be the subalgebra of R generated by Y, so by Proposition 4.4 (v), (S, Y) is a q-qswr. Let Y' be a generating set for S such that (S, Y') is a q-qswr and Y and Y' are q-compatible. Finally let $X' := Y' \cup (X \setminus Y)$. Then (R, X') is a q-qswr and X and X' are q-compatible.*

The same is true if "q-compatible" is replaced by "graph-q-compatible".

Proof. Certainly, X' generates R, since Y' generates S and $S \cup (X \setminus Y)$ generates R. Also, relations of the form $xy - q_{xy}yx = r_{xy}$, with $q_{xy} = q^{\pm 1}$ if $x \neq y$, hold for all $x, y \in X'$: if $x, y \in Y'$ or $x, y \in (X \setminus Y)$ then this follows from (S, Y') or (R, X) (respectively) being q-qswrs; while if $x \in (X \setminus Y)$ and $y \in Y$ or vice versa, this follows from the q-compatibility of Y and Y', the fact that R is a ring extension of S, and the fact that (R, X) is a q-qswr.

To show that R has a PBW basis with respect to X', let $m = |S|$ and $n = |R|$, and order X such that $Y = \{x_1, \ldots, x_m\}$; then by Corollary 4.9, R can be written as $S[x_{m+1}; \alpha_{m+1}, \delta_{m+1}] \cdots [x_n; \alpha_n, \delta_n]$; meanwhile, if we write $Y' = \{y_1, \ldots, y_m\}$ for any ordering then $S = k[y_1] \cdots [y_m; \alpha_m, \delta_m]$; putting these together, and applying Proposition 4.5, we're done.

To show that X' is q-compatible with X, let $T \supset R$ be a ring extension of the appropriate form, and let $x \in X'$. If $x \in Y'$ then q-compatibility of Y and Y' gives a relation of the required form, while if $x \notin Y'$ then $x \in X$, so a relation of the required form already holds. The same proof shows that X is q-compatible with X', so X and X' are q-compatible.

To show that X and X' are graph-q-compatible if Y and Y' are graph-q-compatible, let $\phi : Y \to Y'$ be the designated bijection, and extend ϕ to a bijection $X \to X'$ by setting $\phi(x) = x$. Then for $x, x' \in X$, $r_{\phi(x)\phi(x')}$ is nonzero if and only if $r_{xx'}$ by the graph-q-compatibility of Y and Y' if at least one of x and x' is in Y, or because $\phi(x) = x$ and $\phi(x') = x'$ otherwise; while for $x \in X$, $z \in T$ where T is a ring extension of R of the appropriate form, $r_{\phi(x)z}$ is nonzero if and only if r_{xz} is nonzero by the graph-q-compatibility of Y and Y' if $x \in Y$, or by $x = \phi(x)$ if not. □

A lot of the above is unnecessary if we are only considering q-qswrs with $q^2 \neq 1$: here we only need to rescale elements of our generating set by non-zero scalars, in which case we get graph-q-compatibility. However, when we consider the case $q^2 = 1$ later we will get q-compatibility without graph-q-compatibility.

Lemma 4.21. *Let (S, Y) be a q-qswr, and let $y' = \lambda_y y$ for all $y \in Y$, where each $\lambda_y \in k^\times$. Then $Y' := \{y' : y \in Y\}$ is graph-q-compatible with Y.*

Proof. First we must show that (R, Y') is a q-qswr. Certainly Y' generates R, and any monomial in Y' is a scalar multiple of one in Y, so if the monomials in Y form a basis for R then the monomials in Y' do so too. Next, if $x, y \in Y$ with $xy - q_{xy}yx = r_{xy}$ then $x'y' - q_{xy}y'x' = \lambda_x\lambda_y r_{xy}$, so setting $q_{x'y'} = q_{xy}$ and $r_{x'y'} = \lambda_x\lambda_y r_{xy}$, we have $x'y' - q_{x'y'}y'x' = r_{x'y'}$ as required, and further, $r_{x'y'}$ is nonzero if and only if r_{xy} is.

Let T be a ring extension of S and let $z \in T$ be such that, for all $y \in Y$, there exists $q_{yz} \in \{q, q^{-1}\}$, $r_{yz} \in k$ such that $yz - q_{yz}zy = r_{yz}$. Then $y'z - q_{yz}zy' = \lambda_y r_{yz}$, and so Y' is q-compatible with Y; the reverse holds by symmetry. Further, $r_{y'z} = \lambda_y r_{yz}$, so $r_{y'z}$ is nonzero if and only if r_{yz} is; □

Our aim is to show that, for $q^2 \neq 1$, any q-cqwa is either L_n^q or, if n is odd, C_n^q. There will be two parts: first, show that, up to graph-q-qswr isomorphism, these are the only possibilities whose associated graphs are the associated graphs of L_n^q or C_n^q (which are an n-vertex path and an n-vertex cycle respectively); second, show that those associated graphs are the only possibilities for a q-cqwa.

Lemma 4.22. *Let $q \in k^\times$ be such that $q^2 \neq 1$.*

(i) Let (R, X) be a q-qswr such that $G(R, X)$ is an n-vertex path. Then R is graph-q-qswr isomorphic to L_n^q.

(ii) Suppose our base field k has square roots, and let (R, X) be a q-qswr such that $G(R, X)$ is an n-vertex cycle, where n is odd. Then R is graph-q-qswr isomorphic to C_n^q.

Proof. (i) We prove this by induction on n. We will find a set X' which is q-compatible with X such that (R, X') is strongly q-qswr isomorphic to L_n^q.

When $n = 1$, (R, X) is strongly q-qswr isomorphic to L_1^q since they are both polynomial rings in one variable over k.

When $n = 2$, we can write $X = \{x, y\}$ such that R is generated by x and y subject to the relation $xy - qyx = r$ for some $r \neq 0$. Letting $y' = (1 - q)r^{-1}y$, $X' := \{x, y'\}$ is graph-q-compatible with X by Lemma 4.21, and $xy' - qy'x = 1 - q$ so (R, X') is strongly q-qswr isomorphic to L_2^q.

When $n > 2$, let z be one of the end vertices of the path, and let S be the subalgebra of R generated by $Y := X \setminus \{z\}$, so by Proposition 4.4 (v), S is a q-qswr. By induction,

S is graph-q-qswr isomorphic to L^q_{n-1}, so let Y' be a generating set for S such that (S, Y') is strongly q-qswr isomorphic to L^q_{n-1} and Y and Y' are graph-q-compatible. Then let $X' := Y' \cup \{z\}$, so (R, X') is a q-qswr and X and X' are graph-q-compatible by Proposition 4.20. In particular $G(R, X')$ is an n-vertex path with z being one of the end vertices.

Let $x \in X'$ and $y \in X'$ be the two vertices nearest to z, that is, $r_{xy} \neq 0$ and $r_{yz} \neq 0$. Either $q_{xy} = q$ or $q_{xy} = q^{-1}$; since $L^q_{n-1} \cong L^{q^{-1}}_{n-1}$ and all q-qswr properties are invariant under switching q for q^{-1}, by switching q and q^{-1} if necessary we can assume $q_{xy} = q$.

Set $z' := (1-q)r_{yz}^{-1}z$ and set $X'' := Y' \cup \{z'\}$, so (R, X'') is a q-qswr and X'' and X' are graph-q-compatible by Proposition 4.20.

By Proposition 4.4 (iii), $q_{yz} = q_{zx} = q$, and then by Proposition 4.4 (ii), $q_{zt} = q_{yt}^{-1}$ for $t \notin \{y, z\}$; therefore $q_{yz'} = q_{z'x} = q$ and $q_{z't} = q_{yt}^{-1}$ for $t \notin \{y, z\}$. Also $r_{yz'} = 1 - q$ and $r_{tz'} = 0$ for $t \in X''$ with $t \neq y$. Checking this with the definition of L^q_n (Definition 4.10), (R, X'') is strongly q-qswr isomorphic to L^q_n, and so since X'' is graph q-compatible with X, we're done.

(ii) When $n = 1$, R is strongly q-qswr isomorphic to C^q_1 since they are both polynomial rings in one variable over k.

Otherwise, let $z \in X$, and let S be the subalgebra of R generated by $X \setminus \{z\}$, so by Proposition 4.4 (v), S is a q-qswr. By the previous part, S is graph-q-qswr isomorphic to L^q_{n-1}, so let Y' be a generating set for S such that (S, Y') is strongly q-qswr isomorphic to L^q_{n-1} and Y and Y' are graph-q-compatible. Then let $X' := Y' \cup \{z\}$, so (R, X') is a q-qswr and X and X' are graph-q-compatible by Proposition 4.20. In particular $G(R, X')$ is an n-vertex cycle.

There are two pairs (x, y) with $x, y \in X'$ such that $r_{xy} \neq 0$ and $r_{yz} \neq 0$; consider the pair such that $q_{xy} = q$. By Proposition 4.4 (iii), $q_{yz} = q_{zx} = q$, and then by Proposition 4.4 (ii), $q_{zt} = q_{yt}^{-1}$ for $t \notin \{y, z\}$.

Let $w \in X'$ be such that $w \neq y$ but $r_{zw} \neq 0$. Relabel the elements of Y' with $y := y_1$, $x := y_2, \ldots, w := y_{n-1}$, so y_i and y_{i+1} are always adjacent in $G(S, Y')$. $r_{y_i y_{i+1}} = 1 - q$ but $r_{y_i y_j} = 0$ if $|i - j| > 1$, while $r := r_{zy_1} \neq 0$ and $s := r_{y_{n-1}z} \neq 0$ but $r_{zy_i} = 0$ otherwise.

Let $y'_i := ry_i$ if i is even, $y'_i := sy_i$ if i is odd, and $z' := (1-q)z$. So $X'' := \{y'_i : 1 \leq i \leq n-1\} \cup \{z'\}$ is graph-q-compatible with X', $q_{t'u'} = q_{tu}$ for $t, u \in X'$. If $t, u \in X'$

then $r_{t'u'} = 0$ if $r_{tu} = 0$, and otherwise we have $r_{zy_1} = r_{y_iy_{i+1}} = r_{y_{n-1}z} = (1-q)rs$.

Finally let $t' := (rs)^{-\frac{1}{2}}t$ for $t \in X''$, so $X''' := \{t' : t \in X''\}$ is graph-q-compatible with X'' and so with X, and (R, X''') is strongly q-qswr isomorphic to C_n^q.

□

Lemma 4.23. *Let X present R as a connected quantized Weyl algebra with $q_{ab}^2 \neq 1$ for some $a, b \in X$, and let $x \in X$. Then there are at most two elements $y \in X$ such that $r_{xy} \neq 0$, that is, no vertex of $G(R, X)$ has degree greater than 2.*

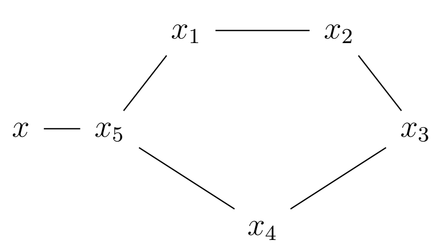
Proof. Suppose there are three such elements, y, z, and w. Then by Proposition 4.4 (iii) applied three times, $q_{yx} = q_{xz} = q_{wx} = q_{xy}$ and so $q_{xy}^2 = 1$, contradicting Proposition 4.13 and the assumption. □

Theorem 4.24. *Let k be a field with square roots, and let (R, X) be a q-cqwa with $q^2 \neq 1$. Then if n is even, (R, X) is graph-q-qswr isomorphic to L_n^q, while if n is odd, (R, X) is graph-q-qswr isomorphic to either L_n^q or C_n^q.*

Proof. We prove this by induction on $n := |X|$. When $n = 1$ or $n = 2$ then $G(R, X)$ is an n-vertex path, so we're done by by Lemma 4.22.

So suppose $n > 2$. Let $x \in X$ be such that x does not disconnect $G(R, X)$, and let S be the subalgebra of R generated by $Y := X \setminus \{x\}$. Then by induction, there is Y' such that (S, Y') is a q-qswr, Y and Y' are graph-q-compatible, and (S, Y') is strongly q-qswr isomorphic to either L_{n-1}^q or C_{n-1}^q, with the latter only possible if n is even. Let $X' := Y' \cup \{x\}$, so by Proposition 4.20, (R, X') is a q-qswr and X and X' are graph-q-compatible. In particular, $G(R, X')$ is isomorphic to $G(R, X)$ and so is connected.

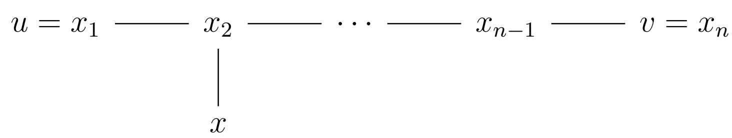
Suppose first that n is even and (S, Y') is strongly q-qswr isomorphic to C_{n-1}^q. Then there must exist $y \in Y'$ such that $r_{xy} \neq 0$, but such a y would then have degree 3 in $G(R, X')$, contradicting Lemma 4.23.



Here x_5 contradicts Lemma 4.23.

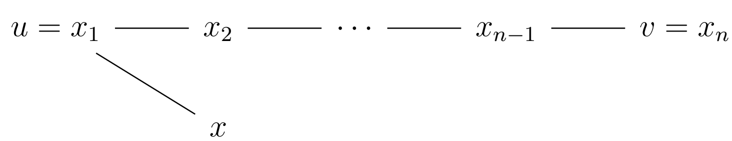
So (S, Y') must be strongly q-qswr isomorphic to L_{n-1}^q. Let u and v denote the two elements of Y' of degree 1 in $G(S, Y')$, that is, the end vertices of the path. If $y \in Y$,

$y \notin \{u, v\}$ then $r_{yx} = 0$, since if not then this y has degree 3 in $G(R, X')$, contradicting Lemma 4.23.



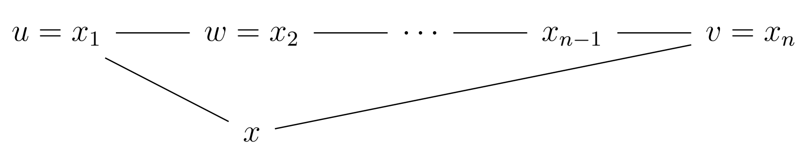
Here x_2 contradicts Lemma 4.23.

So since $G(R, X')$ is connected, at least one of r_{ux} and r_{vx} must be nonzero. If only one is, then $G(R, X')$ is an n-vertex path, and so by Lemma 4.22 (i), (R, X') is graph-q-isomorphic to L_n^q, while if both are non-zero and n is odd then $G(R, X')$ is an n-vertex cycle and so by 4.22 (ii) (R, X') is graph-q-isomorphic to C_n^q.



Here (R, X') is graph-q-isomorphic to L_n^q by Lemma 4.22 (i).

There remains one case: if $n > 2$ is even and both $r_{ux} \neq 0$ and $r_{vx} \neq 0$. In this case, let $w \in Y'$ be such that $r_{wu} \neq 0$, as the diagram below illustrates. Then Proposition 4.4 (iii) gives $q_{xu} = q_{uw}$ and Proposition 4.4 (ii) using $r_{xv} \neq 0$ gives $q_{xu} = q_{uv}$. But checking the definition of L_{n-1}^q, $q_{uv} = q_{uw}^{-1}$, since u, v, and w correspond to x_1, x_{n-1}, and x_2 respectively. This contradicts the assumption that $q^2 \neq 1$, so this case cannot arise.



When n is even, this fails (as described above) since $q^2 \neq 1$.

□

So we have completed the classification of q-cqwas in the case $q^2 \neq 1$. The classification when $q^2 = 1$ is very different. On the one hand, one can construct a q-qswr generated by a set X given any choice of values of r_{xy} for $x, y \in X$ (providing $r_{yx} = r_{xy} q_{xy}^{-1}$); on the other, when $q = 1$, it turns out that any 1-qswr is in fact isomorphic to a polynomial ring over a Weyl algebra.

Theorem 4.25. *Let X be a finite set, let $q = \pm 1$, and let $r_{xy} \in k$ for $x, y \in X$ be such that $r_{yx} = qr_{xy}$. Let R be the ring generated by X subject to the relations $xy - qyx = r_{xy}$ for each $x, y \in X$. Then X presents R as a quantum space with Weyl relations.*

Proof. We show this by induction on $|X|$. Let $\{x_1, \ldots, x_n\}$ be an ordering of X with respect to which R has a PBW basis. Let S be the subalgebra of R generated by $Y = \{x_1, \ldots, x_{n-1}\}$. By induction and Proposition 4.5, it suffices to check that the automorphism of S given by $\alpha(x_i) = qx_i$ and the α-derivation of S given by $\delta(x_i) = r_{x_n x_i}$ are well-defined, that is, by Proposition 2.22, that they preserve the defining relations of S.

Any such defining relation is of the form $x_i x_j - q x_j x_i = r_{ij}$. Noting that $q^2 = 1$, $\alpha(x_i x_j - q x_j x_i) = q^2 x_i x_j - q^3 x_j x_i = x_i x_j - q x_j x_i = r_{x_i x_j}$ as required, while $\delta(x_i x_j - q x_j x_i) = q x_i r_{x_n x_j} + r_{x_n x_i} x_j - q^2 x_j r_{x_n x_i} - q r_{x_n x_j} x_i = 0$ as required, so we're done. □

Lemma 4.26. *Let (R, X) be a 1-qswr. Let $x, y \in X$, $\mu \in k^\times$, and $\lambda \in k$. Define $x' := \mu x - \lambda y$, and define $X' := \{x'\} \cup X \setminus \{x\}$. Then X and X' are 1-compatible.*

Proof. First, using Proposition 4.21 to replace x with μx, we may assume $\mu = 1$.

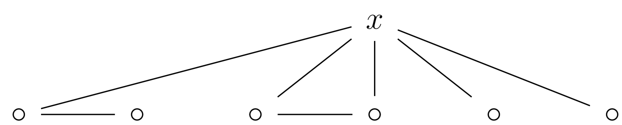
First we need to check that (R, X') is a 1-qswr. Certainly X' generates R. Given $z \in X \setminus \{x\}$, $xz - zx = r_{xz}$ and $yz - zy = r_{yz}$, so $x'z - zx' = r_{xz} - \lambda r_{yz}$. So the elements of X' satisfy relations of the appropriate form.

Order X with x and then y first, and order X' the same way but with x replaced by x'. Then we know the monomials in X' span R; we need to check they are linearly independent. Suppose they are not, so p is some non-trivial sum of monomials in X' that equals 0 in R. Then we can use $x' = x - \lambda y$ together with $xy - yx = r_{xy}$ to write p as a sum of monomials in X, and by considering the terms of highest degree in x' in p, which correspond exactly with the terms of highest degree in x in our rewriting of p, this is still a non-trivial sum. But then p cannot be 0, since the monomials in X are linearly independent, so we have a contradiction.

Now let T be a ring extension of R and let $z \in T$ be such that, for all $w \in X$, $wz - zw = r_{wz}$ for some $r_{wz} \in k$. Then $x'z - zx' = r_{xz} - \lambda r_{yz}$. So X' is 1-compatible with X; the reverse is also true by symmetry. □

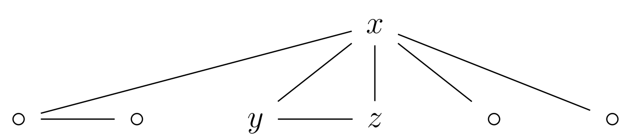
Theorem 4.27. *Let (R, X) be a 1-qswr. Then there exists X' such that (R, X') is a q-qswr, X and X' are 1-compatible, and $G(R, X')$ is isomorphic to a disjoint union of 2-vertex paths and single vertices, and so (R, X) is 1-qswr isomorphic to a polynomial ring over a Weyl algebra.*

Proof. Pick any $x \in X$, let $Y = X \setminus \{x\}$, and let S be the subalgebra of R generated by Y. Then by induction, there exists Y' such that (S, Y') is a 1-qswr, Y and Y' are 1-compatible and $G(S, Y')$ is isomorphic to a disjoint union of 2-vertex paths and single vertices. Let $X' = Y' \cup \{x\}$, so by Proposition 4.20, (R, X') is a 1-qswr and X and X' are 1-compatible.

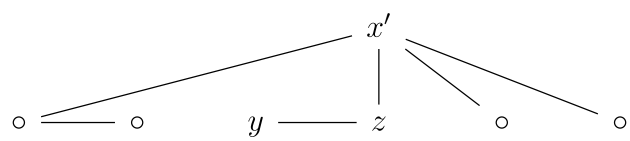


A typical example of the setting.

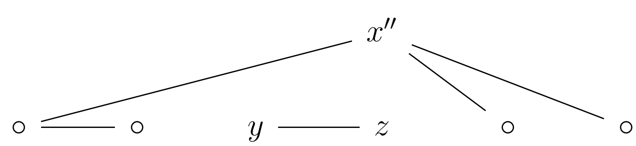
Suppose $y, z \in Y'$ are such that r_{xy} and r_{zy} are nonzero - so by the properties of $G(S, Y')$, $r_{yt} = 0 = r_{zt}$ for all $t \in X' \setminus \{x, y, z\}$. (But r_{xz} may be zero or not.) Then let $x' = r_{zy}x - r_{xy}z$, and let $X'' = Y' \cup \{x'\}$, so by Lemma 4.26 X' and X'' are 1-compatible. We note that $r_{x'y} = r_{zy}r_{xy} - r_{xy}r_{zy} = 0$ and $r_{x'z} = r_{zy}r_{xy}$, while for $t \in X' \setminus \{x, y, z\}$, $r_{x't} = r_{xt}$. So replacing x by x' reduces its degree in the associated graph by 1.



Here we can replace x by $x' = r_{zy}x - r_{xy}z$; the new graph is then:

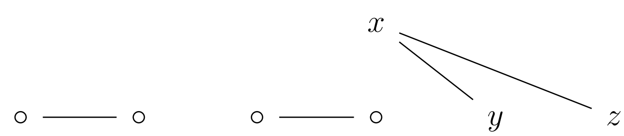


Now we can replace x' by $x'' = r_{yz}x - r_{xz}y$; the new graph is then:

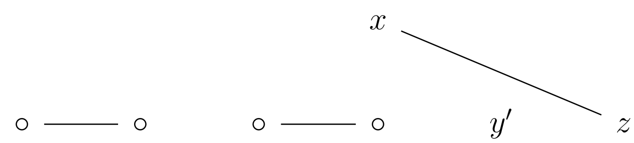


We can similarly remove the leftmost edge.

Therefore, we may assume that all of the neighbours of x in $G(R, X')$ are isolated vertices in $G(S, Y')$. Suppose there are two such neighbours y and z, that is, r_{xy} and r_{xz} are nonzero but $r_{yt} = r_{zt} = 0$ for all other t. Let $y' = r_{xz}y - r_{xy}z$, and let $X'' = \{y'\} \cup X' \setminus \{y\}$, so by Lemma 4.26 X' and X'' are 1-compatible. We note that $r_{xy'} = r_{xz}r_{xy} - r_{xy}r_{xz} = 0$, $r_{y'z} = 0$, and $r_{y't} = 0 = r_{yt}$ otherwise. So replacing y by y' reduces the degree of x in the associated graph by 1.



Here we can replace y by $y' = r_{xz}y - r_{xy}z$; the new graph is then:



So we may assume that x has at most one neighbour in $G(R, X')$. But then $G(R, X')$ is of the required form.

Finally we note that if x and y are such that $r_{xy} \neq 0$, by Proposition 4.21 we may replace x by $x' := r_{xy}^{-1}x$, so $r_{x'y} = 1$. Therefore if $G(R, X')$ is of the described form then (R, X) is 1-qswr isomorphic to a polynomial ring over a Weyl algebra. □

Theorem 4.28. *Let $r \geq 1$, $s \geq 0$ be integers. Then $A_r[z_1, \ldots, z_s]$ is 1-qswr isomorphic to a 1-cqwa.*

Proof. Let G be the associated graph of $A_{r-1}[z_1, \ldots, z_{s+1}]$, and let $G' = G \cup \{x\}$, where x is adjacent in G' to every vertex of G. Then by Theorem 4.25, there exists a 1-qswr (R, X) with $G(R, X) \cong G'$, which is therefore a 1-cqwa since G' is connected. Then following through the proof of Theorem 4.27, one sees that $G(R, X)$ is 1-qswr isomorphic to $A_r[z_1, \ldots, z_s]$. □

Remark. This does not hold for $r = 0$ and $s > 1$, since any 1-cqwa with more than 2 generators is noncommutative, but $A_0[z_1, \ldots, z_s]$ is a commutative polynomial ring.

Remark. The classification of -1-cqwas is harder, since neither Lemma 4.23, restricting the possible associated graphs, nor Lemma 4.26, allowing us to change generators to modify the associated graph, applies. One might conjecture that -1-cqwas - indeed -1-qswrs - are classified by their associated graphs, but even this seems unlikely: consider a -1-cqwa whose associated graph is a 4-cycle, so r_{12}, r_{23}, r_{34} and r_{41} are non-zero. If one attempts to rescale the generators using Lemma 4.21, the quantity $r_{12}r_{23}^{-1}r_{34}r_{41}^{-1}$ is invariant.

5 Connected quantized Weyl algebras - ring-theoretic properties

5.1 Normal elements in L_n^q

We note that there is a natural embedding $L_m^q \hookrightarrow L_n^q$ for $m \leq n$ by extending the natural embedding $X_m \hookrightarrow X_n$, and similarly a natural embedding $L_m^q \hookrightarrow C_n^q$ for $m \leq n$ by extending the natural embedding $X_m \hookrightarrow Y_n$. We will use these embeddings implicitly from now on.

We define an automorphism $\theta : C_n^q \to C_n^q$ by extending the map $x_n \mapsto x_1$, $x_i \mapsto x_{i+1}$ otherwise. If $m < n$ then the restriction of θ to L_m^q gives a partial function, which we also call θ, defined on L_{m-1}^q, whose range $\theta(L_{m-1}^q)$ is the subalgebra of L_m^q generated by $x_2, \ldots, x_m$. Furthermore, this definition of θ (as a partial function $L_{m-1}^q \to L_{m-1}^q$) is independent of the choice of n, as is the definition of $\theta(x_i)$

To reduce the number of calculations we need to make, recall the **opposite ring** of a ring R, usually denoted R^{op}, which has the same elements and additive structure as R, but the product of two elements in R^{op} is their product in the opposite order in R. When discussing this we will continue to suppress multiplication in R, using $\cdot^{\text{op}}$ to denote multiplication in the opposite ring, so $a \cdot^{\text{op}} b = ba$.

This is useful because there is an isomorphism $*_m : L_m^q \to (L_m^q)^{\text{op}}$ given by extending the map $x_i \mapsto x_{m+1-i}$. We note that whenever they are both defined, $*_n = \theta^{n-m} \circ *_m$.

Definition 5.1. Define elements $z_n \in L_n^q$ recursively by setting $z_{-1} = 0$, $z_0 = 1$, and for $n > 0$, $z_n = z_{n-1}x_n - z_{n-2}$.

Proposition 5.2. *We describe some basic properties of these elements.*

(i) $z_n = x_1\theta(z_{n-1}) - \theta^2(z_{n-2})$ *for all* $n > 0$.

(ii) $*_n(z_n) = z_n$ *for all* $n \geq 0$.

(iii) $x_i z_n = z_n x_i$ *(for* $1 \leq i \leq n$, $n > 0$ *odd);*

$x_i z_n = q^{(-1)^{i-1}} z_n x_i$ *(for* $1 \leq i \leq n$, $n > 0$ *even).*

Thus, z_n *is normal in* L_n^q*, and central if* n *is odd.*

(iv) $z_i z_j = q_{ij} z_j z_i$*, where* $q_{ij} = \begin{cases} 1, & \text{if } \max\{i,j\} \text{ is odd or both } i,j \text{ even;} \\ q, & \text{if } i \text{ odd, } j \text{ even, } i < j; \\ q^{-1}, & \text{if } i \text{ even, } j \text{ odd, } i > j. \end{cases}$

(v) If $1 \leq i + r \leq j$*, with* $i \geq 1$ *and* $r \geq 0$*,* $\theta^r(z_i)z_j = p_{ijr}z_j\theta^r(z_i)$*, where*

$$p_{ijr} = \begin{cases} 1 & \text{if } j \text{ is odd or } i \text{ is even;} \\ q & \text{if } j \text{ is even, } i \text{ is odd and } r \text{ is even;} \\ q^{-1} & \text{if } j \text{ is even, } i \text{ is odd and } r \text{ is odd.} \end{cases}$$

(vi) $z_i x_j = x_j z_i$ *(for* i *even,* $i + 1 < j$*);*

$z_i x_j = q^{(-1)^j} x_j z_i$ *(for* i *odd,* $i + 1 < j$*).*

Proof. (i) We prove this by induction on n.

When $n = 1$, $x_1\theta(z_{n-1}) - \theta^2(z_{n-2}) = x_1 = z_1$ as required.

When $n = 2$, $x_1\theta(z_{n-1}) - \theta^2(z_{n-2}) = x_1x_2 - 1 = z_2$ as required.

When $n > 2$,

$$\begin{aligned} z_n &= z_{n-1}x_n - z_{n-2} \\ &= x_1\theta(z_{n-2})x_n - \theta^2(z_{n-3})x_n - x_1\theta(z_{n-3}) + \theta^2(z_{n-4}) \quad \text{by induction} \\ &= x_1\theta(z_{n-1}) - \theta^2(z_{n-2}). \end{aligned}$$

(ii) We prove this by induction on n. When $n = 0$ or $n = 1$ this is trivial, so assume $n > 1$.

$$\begin{aligned} *_n(z_n) &= *_n(x_1\theta(z_{n-1}) - \theta^2(z_{n-2})) \\ &= *_n(x_1) \cdot^{\mathrm{op}} *_n\theta(z_{n-1}) - *_n\theta^2(z_{n-2}) \\ &= x_n \cdot^{\mathrm{op}} *_{n-1}(z_{n-1}) - *_{n-2}(z_{n-2}) && \text{since } *_n = \theta^{n-m} \circ *_m \\ &= x_n \cdot^{\mathrm{op}} z_{n-1} - z_{n-2} && \text{by induction} \\ &= z_{n-1}x_n - z_{n-2} \\ &= z_n && \text{as required.} \end{aligned}$$

(iii) We prove this by induction on n. When $n = 1$ this is trivial, since $z_1 = x_1$.

For the case $n = 2$,

$$\begin{aligned} x_1z_2 &= x_1x_1x_2 - x_1 \\ &= x_1(qx_2x_1 + 1 - q) - x_1 \\ &= qx_1x_2x_1 - qx_1 \\ &= qz_2x_1. \end{aligned}$$

$$\begin{aligned} *_2(x_2z_2 - q^{-1}z_2x_2) &= z_2x_1 - q^{-1}x_1z_2 && \text{by (ii)} \\ &= 0 && \text{by the previous calculation.} \end{aligned}$$

Therefore since $*_2$ is an isomorphism, $x_2z_2 - q^{-1}z_2x_2 = 0$, and we've proved the result when $n = 2$.

If n is even and $i \leq n-2$ then

$$\begin{aligned} x_iz_n &= x_i(z_{n-1}x_n - z_{n-2}) \\ &= z_{n-1}x_ix_n - z_{n-2}q^{(-1)^{i-1}}x_i \qquad \text{by induction} \\ &= q^{(-1)^{i-1}}z_{n-1}x_nx_i - q^{(-1)^{i-1}}z_{n-2}x_i \\ &= q^{(-1)^{i-1}}z_nx_i. \end{aligned}$$

If $i > 2$ and n is even then

$$\begin{aligned} *_n(x_iz_n - q^{(-1)^{i-1}}z_nx_i) &= z_nx_{n-i+1} - q^{(-1)^{i-1}}x_{n-i+1}z_n \quad \text{by (ii)} \\ &= 0 \qquad \text{by the previous calculation.} \end{aligned}$$

Meanwhile if n is odd and $i \leq n-2$ then

$$\begin{aligned} x_iz_n &= x_i(z_{n-1}x_n - z_{n-2}) \\ &= q^{(-1)^{i-1}}z_{n-1}x_ix_n - z_{n-2}x_i \quad \text{by induction} \\ &= z_{n-1}x_nx_i - z_{n-2}x_i \\ &= z_nx_i. \end{aligned}$$

If $i > 2$ and n is odd then

$$\begin{aligned} *_n(x_iz_n - z_nx_i) &= z_nx_{n-i+1} - x_{n-i+1}z_n \quad \text{by (ii)} \\ &= 0 \qquad \text{by the previous calculation.} \end{aligned}$$

We are almost done, but there remains the case when $n = 3$ and $i = 2$, which is not covered by any of the above:

$$\begin{aligned} x_2z_3 &= x_2x_1x_2x_3 - x_2x_1 - x_2x_3 \\ &= q^{-1}x_1x_2x_2x_3 + (1-q^{-1})x_2x_3 - q^{-1}x_1x_2 - (1-q^{-1}) - x_2x_3 \\ &= x_1x_2x_3x_2 + q^{-1}(1-q)x_1x_2 - q^{-1}x_2x_3 - q^{-1}x_1x_2 - (1-q^{-1}) \\ &= x_1x_2x_3x_2 - x_1x_2 - x_3x_2 - q^{-1}(1-q) - (1-q^{-1}) \\ &= z_3x_2. \end{aligned}$$

(iv, v) (iv) is a special case of (v). For j odd this is immediate from part (iii).

For j even, we prove this by induction on i for fixed j. When $i = 0$ this is trivial, while when $i = 1$ this is immediate from part (iii).

For all odd $i > 1$, $\theta^r(z_i)z_j = \theta^r(z_{i-1})x_{i+r}z_j - \theta^r(z_{i-2})z_j$

$$= q^{(-1)^r}\theta^r(z_{i-1})z_jx_{i+r} - q^{(-1)^r}z_j\theta^r(z_{i-2})$$
$$= q^{(-1)^r}z_j\theta^r(z_{i-1})x_{i+r} - q^{(-1)^r}z_j\theta^r(z_{i-2})$$
$$= q^{(-1)^r}z_jz_i.$$

Meanwhile, for all even $i > 1$, $\theta^r(z_i)z_j = \theta^r(z_{i-1})x_{i+r}z_j - \theta^r(z_{i-2})z_j$

$$= q^{(-1)^(r+1)}\theta^r(z_{i-1})z_jx_{i+r} - z_j\theta^r(z_{i-2})$$
$$= z_j\theta^r(z_{i-1})x_i - z_j\theta^r(z_{i-2})$$
$$= z_j\theta^r(z_i).$$

(vi) We prove this by induction on i. When $i = 0$ this is trivial and when $i = 1$ this follows from $z_1 = x_1$ and the definition of L_n^q.

When $i > 1$ is even and $j > i + 1$,

$$\begin{aligned} z_ix_j &= z_{i-1}x_ix_j - z_{i-2}x_j \\ &= q^{(-1)^{j-i}}z_{i-1}x_jx_i - x_jz_{i-2} && \text{by induction and the definition of } L_n^q \\ &= q^{(-1)^{j-i}}q^{(-1)^j}x_jz_{i-1}x_i - x_jz_{i-2} && \text{by induction} \\ &= x_jz_i && \text{since } i \text{ is even.} \end{aligned}$$

When $i > 1$ is odd and $j > i + 1$,

$$\begin{aligned} z_ix_j &= z_{i-1}x_ix_j - z_{i-2}x_j \\ &= q^{(-1)^{j-i-1}}z_{i-1}x_jx_i - q^{(-1)^j}x_jz_{i-2} && \text{by induction and the definition of } L_n^q \\ &= q^{(-1)^{j-i-1}}x_jz_{i-1}x_i - q^{(-1)^j}x_jz_{i-2} && \text{by induction} \\ &= q^{(-1)^j}x_jz_i && \text{since } i \text{ is odd.} \end{aligned}$$

□

Proposition 5.3. *For all $i \geq 1$,*

(i) $z_ix_{i+1} = x_{i+1}z_i + (1-q)z_{i-1}$ *(if i is even);*

(ii) $z_ix_{i+1} = qx_{i+1}z_i + (1-q)z_{i-1}$ *(if i is odd);*

(iii) $x_1\theta(z_i) = \theta(z_i)x_1 + (1-q)\theta^2(z_{i-1})$ *(if i is even);*

(iv) $x_1\theta(z_i) = q\theta(z_i)x_1 + (1-q)\theta^2(z_{i-1})$ *(if i is odd).*

Proof. (i) For all even $i \geq 1$,

$$
\begin{aligned}
z_i x_{i+1} &= z_{i-1} x_i x_{i+1} - z_{i-2} x_{i+1} && \text{by definition of } z_i \\
&= z_{i-1}(q x_{i+1} x_i + 1 - q) - z_{i-2} x_{i+1} && \text{by the definition of } L_n^q \\
&= x_{i+1} z_{i-1} x_i - x_{i+1} z_{i-2} + (1-q) z_{i-1} && \text{by Proposition 5.2 (vi)} \\
&= x_{i+1} z_i + (1-q) z_{i-1}.
\end{aligned}
$$

(ii) For all odd $i \geq 1$,

$$
\begin{aligned}
z_i x_{i+1} &= z_{i-1} x_i x_{i+1} - z_{i-2} x_{i+1} && \text{by definition of } z_i \\
&= z_{i-1}(q x_{i+1} x_i + 1 - q) - z_{i-2} x_{i+1} && \text{by the definition of } L_n^q \\
&= q x_{i+1} z_{i-1} x_i - q x_{i+1} z_{i-2} + (1-q) z_{i-1} && \text{by Proposition 5.2 (vi)} \\
&= q x_{i+1} z_i + (1-q) z_{i-1}.
\end{aligned}
$$

(iii) For all even $i \geq 1$,

$$
\begin{aligned}
*_{i+1}(x_1 \theta(z_i) - \theta(z_i) x_1) &= z_i x_{i+1} - x_{i+1} z_i && \text{by Proposition 5.2 (ii)} \\
&= (1-q) z_{i-1} && \text{by (i)} \\
&= *_{i+1}((1-q)\theta^2(z_{i-1})).
\end{aligned}
$$

Since $*_{i+1}$ is an isomorphism, $x_1 \theta(z_i) = \theta(z_i) x_1 + (1-q)\theta^2(z_{i-1})$.

(iv) For all odd $i \geq 1$,

$$
\begin{aligned}
*_{i+1}(x_1 \theta(z_i) - q\theta(z_i) x_1) &= z_i x_{i+1} - q x_{i+1} z_i && \text{by Proposition 5.2 (ii)} \\
&= (1-q) z_{i-1} && \text{by (i)} \\
&= *_{i+1}((1-q)\theta^2(z_{i-1}))
\end{aligned}
$$

Since $*_{i+1}$ is an isomorphism, $x_1 \theta(z_i) = q\theta(z_i) x_1 + (1-q)\theta^2(z_{i-1})$.

□

Corollary 5.4. *For all* $i \geq 1$*,*

(i) $x_{i+1} z_i = q z_i x_{i+1} + (1-q) z_{i+1}$ *(if i is even);*

(ii) $q x_{i+1} z_i = q z_i x_{i+1} + (1-q) z_{i+1}$ *(if i is odd);*

(iii) $\theta(z_i) x_1 = q x_1 \theta(z_i) + (1-q) z_{i+1}$ *(if i is even);*

(iv) $q\theta(z_i) x_1 = q x_1 \theta(z_i) + (1-q) z_{i+1}$ *(if i is odd).*

Proof. (i), (ii) Subtract $(1-q)(z_{i+1} + z_{i-1}) = (1-q) z_i x_{i+1}$ from both sides in Proposition 5.3 (i), (ii).

(iii), (iv) Subtract $(1-q)(\theta^2(z_{i+1})+z_{i-1}) = (1-q)x_1\theta(z_i)$ from both sides in Proposition 5.3 (iii), (iv).

$\square$

Corollary 5.5. *For all $i \geq 1$,*

(i) $z_i^a x_{i+1} - x_{i+1} z_i^a = (1-q)[a]_{q^{-1}} z_{i-1} z_i^{a-1}$ *(if i is even);*

(ii) $z_i^a x_{i+1} - q^a x_{i+1} z_i^a = (1-q)[a]_q z_{i-1} z_i^{a-1}$ *(if i is odd);*

(iii) $z_i x_{i+1}^a - x_{i+1}^a z_i = (1-q)[a]_q z_{i-1} x_{i+1}^{a-1}$ *(if i is even);*

(iv) $z_i x_{i+1}^a - q^a x_{i+1}^a z_i = (1-q)[a]_q z_{i-1} x_{i+1}^{a-1}$ *(if i is odd);*

(v) $x_1^a \theta(z_i) - \theta(z_i) x_1^a = (1-q)[a]_q x_1^{a-1} \theta^2(z_{i-1})$ *(if i is even);*

(vi) $x_1^a \theta(z_i) - q^a \theta(z_i) x_1^a = (1-q)[a]_q x_1^{a-1} \theta^2(z_{i-1})$ *(if i is odd).*

Proof. In all cases, this is by induction on a, and $a=1$ comes from the appropriate section of Lemma 5.3. We note that $[a-1]_q + q^{a-1} = [a]_q$ and $q[a-1]_q + 1 = [a]_q$.

(i) For all even $i \geq 1$,

$$\begin{aligned} z_i^a x_{i+1} &= z_i^{a-1} x_{i+1} z_i + (1-q) z_i^{a-1} z_{i-1} && \text{by Proposition 5.3 (i)} \\ &= (x_{i+1} z_i^{a-1} + (1-q)[a-1]_{q^{-1}} z_{i-1} z_i^{a-2}) z_i && \text{by induction} \\ &\qquad + (1-q) q^{-(a-1)} z_{i-1} z_i^{a-1} \\ &= x_{i+1} z_i^a + (1-q)[a]_{q^{-1}} z_{i-1} z_i^{a-1}. \end{aligned}$$

(ii) For all odd $i \geq 1$,

$$\begin{aligned} z_i^a x_{i+1} &= q z_i^{a-1} x_{i+1} z_i + (1-q) z_i^{a-1} z_{i-1} && \text{by Proposition 5.3 (ii)} \\ &= q(q^{a-1} x_{i+1} z_i^{a-1} + (1-q)[a-1]_q z_{i-1} z_i^{a-2}) z_i && \text{by induction} \\ &\qquad + (1-q) z_{i-1} z_i^{a-1} \\ &= q^a x_{i+1} z_i^a + (1-q)[a]_q z_{i-1} z_i^{a-1}. \end{aligned}$$

(iii) For all even $i \geq 1$,

$$\begin{aligned} z_i x_{i+1}^a &= x_{i+1} z_i x_{i+1}^{a-1} + (1-q) z_{i-1} x_{i+1}^{a-1} && \text{by Proposition 5.3 (iii)} \\ &= x_{i+1}(x_{i+1}^{a-1} z_i + (1-q)[a-1]_q z_{i-1} x_{i+1}^{a-2}) && \text{by induction} \\ &\qquad + (1-q) z_{i-1} x_{i+1}^{a-1} \\ &= x_{i+1}^a z_i + (1-q)(q[a-1]_q + 1) z_{i-1} x_{i+1}^{a-1} \\ &= x_{i+1}^a z_i + (1-q)[a]_q z_{i-1} x_{i+1}^{a-1}. \end{aligned}$$

(iv) For all odd $i \geq 1$,

$$\begin{aligned} z_i x_{i+1}^a &= q x_{i+1} z_i x_{i+1}^{a-1} + (1-q) z_{i-1} x_{i+1}^{a-1} && \text{by Proposition 5.3 (iv)} \\ &= q x_{i+1}(q^{a-1} x_{i+1}^{a-1} z_i + (1-q)[a-1]_q z_{i-1} x_{i+1}^{a-2}) && \text{by induction} \\ &\qquad + (1-q) z_{i-1} x_{i+1}^{a-1} \\ &= q^a x_{i+1}^a z_i + (1-q)(q[a-1]_q + 1) z_{i-1} x_{i+1}^{a-1} \\ &= q^a x_{i+1}^a z_i + (1-q)[a]_q z_{i-1} x_{i+1}^{a-1}. \end{aligned}$$

(v) For all even $i \geq 1$,

$$\begin{aligned} *_{i+1}(x_1^a \theta(z_i) - \theta(z_i) x_1^a) &= z_i x_{i+1}^a - x_{i+1}^a z_i && \text{by Proposition 5.2 (ii)} \\ &= (1-q)[a]_q z_{i-1} x_{i+1}^{a-1} && \text{by (iii)} \\ &= *_{i+1}((1-q)[a]_q x_1^{a-1} \theta^2(z_{i-1})) \end{aligned}$$

Therefore, since $*_{i+1}$ is an isomorphism, $x_1^a \theta(z_i) - \theta(z_i) x_1^a = (1-q)[a]_q x_1^{a-1} \theta^2(z_{i-1})$.

(vi) For all odd $i \geq 1$,

$$\begin{aligned} *_{i+1}(x_1^a \theta(z_i) - q^a \theta(z_i) x_1^a) &= z_i x_{i+1}^a - q^a x_{i+1}^a z_i && \text{by Proposition 5.2 (ii)} \\ &= (1-q)[a]_q z_{i-1} x_{i+1}^{a-1} && \text{by (iii)} \\ &= *_{i+1}((1-q)[a]_q x_1^{a-1} \theta^2(z_{i-1})) \end{aligned}$$

Therefore, since $*_{i+1}$ is an isomorphism, $x_1^a \theta(z_i) - q^a \theta(z_i) x_1^a = (1-q)[a]_q x_1^{a-1} \theta^2(z_{i-1})$.

□

5.2 The central element Ω in C_n^q

Recall that C_n^q is a skew polynomial ring over L_{n-1}^q. So the elements $z_1, \ldots, z_{n-1}$ are all elements of C_n^q. However, the element $z_{n-1}x_n - z_{n-2}$ of C_n^q is not normal in C_n^q, so we need a replacement.

Lemma 5.6. *In C_n^q, $z_{n-1}x_n = x_n z_{n-1} + (1-q)(z_{n-2} - \theta(z_{n-2}))$.*

Proof.

$$\begin{aligned} z_{n-1}x_n &= x_1\theta(z_{n-2})x_n - \theta^2(z_{n-3})x_n \\ &= x_1\theta(z_{n-2}x_{n-1}) - \theta^2(z_{n-3}x_{n-2}) \\ &= x_1\theta(qx_{n-1}z_{n-2} + (1-q)z_{n-3}) - \theta^2(x_{n-2}z_{n-3} + (1-q)z_{n-4}) \\ &= qx_1x_n\theta(z_{n-2}) + (1-q)x_1\theta(z_{n-3}) - x_n\theta^2(z_{n-3}) + (1-q)\theta^2(z_{n-4}) \\ &= (x_nx_1 - (1-q))\theta(z_{n-2}) - x_n\theta^2(z_{n-3}) + (1-q)z_{n-2} \\ &= x_nz_{n-1} + (1-q)z_{n-2} - (1-q)\theta(z_{n-2}). \end{aligned}$$

□

Definition 5.7. In C_n^q, define $\Omega_n = z_{n-1}x_n - z_{n-2} - q\theta(z_{n-2})$. In general we will shorten this to Ω unless the context makes the ambient ring unclear.

Lemma 5.8. *$\theta(\Omega) = \Omega$.*

Proof.
$$\begin{aligned}
\theta(\Omega) &= \theta(z_{n-1}x_n) - \theta(z_{n-2}) - q\theta^2(z_{n-2}) \\
&= \theta(x_n z_{n-1} + (1-q)(z_{n-2} - \theta(z_{n-2}))) && \text{by Lemma 5.6} \\
&\qquad - \theta(z_{n-2}) - q\theta^2(z_{n-2}) \\
&= x_1\theta(z_{n-1}) - q\theta(z_{n-2}) - \theta^2(z_{n-2}) \\
&= x_1\theta(z_{n-2})x_n - x_1\theta(z_{n-3}) - \theta^2(z_{n-3})x_n && \text{by definition of } z_i \\
&\qquad + \theta^2(z_{n-4}) - q\theta(z_{n-2}) \\
&= z_{n-1}x_n - z_{n-2} - q\theta(z_{n-2}) && \text{by Proposition 5.2 (i)} \\
&= \Omega.
\end{aligned}$$
□

Theorem 5.9. *$x_i\Omega = \Omega x_i$ for $1 \le i \le n$.*

Proof. It suffices to check that $x_i\Omega = \Omega x_i$ for one valid i, using the automorphism θ and Lemma 5.8.

When $n > 3$,
$$\begin{aligned}
x_2\Omega &= x_2 z_{n-1}x_n - x_2 z_{n-2} - qx_2\theta(z_{n-2}) \\
&= q^{-1}z_{n-1}x_2x_n - z_{n-2}x_2 - q\theta(z_{n-2})x_2 \\
&= z_{n-1}x_nx_2 - z_{n-2}x_2 - q\theta(z_{n-2})x_2 \\
&= \Omega x_2.
\end{aligned}$$

When $n = 3$,
$$\begin{aligned}
x_1\Omega &= x_1x_1x_2x_3 - x_1x_1 - qx_1x_2 - x_1x_3 \\
&= x_1(qx_2x_1 + 1 - q)x_3 - x_1x_1 - qx_1x_2 - x_1x_3 \\
&= qx_1x_2x_1x_3 - x_1x_1 - qx_1x_2 - qx_1x_3 \\
&= x_1x_2(x_3x_1 - (1-q)) - x_1x_1 - qx_1x_2 - qx_1x_3 \\
&= x_1x_2x_3x_1 - x_1x_1 - x_1x_2 - qx_1x_3 \\
&= x_1x_2x_3x_1 - x_1x_1 - qx_2x_1 - (1-q) - x_3x_1 + (1-q) \\
&= \Omega x_1.
\end{aligned}$$
□

5.3 Prime ideals in L_n^q

Definition 5.10. Denote by $\mathbf{z}_n$ the set $\{z_1, \ldots, z_n\}$

$$\text{Let } A_{ij} = \begin{cases} 0, & \text{if } max\{i,j\} \text{ is odd or both } i,j \text{ even} \\ 1, & \text{if } i \text{ odd, } j \text{ even, } i<j \\ -1, & \text{if } i \text{ even, } j \text{ odd, } i<j \end{cases}.$$

Let $\mathbf{A}_n$ be the matrix $(A_{ij})_{i,j=1}^n$. This corresponds with Proposition 5.2 (iv), so that the quantum space $S_q(\mathbf{z}_n, \mathbf{A}_n)$ is naturally contained within L_n^q.

Lemma 5.11. *(i) Let $\mathcal{X}_n$ denote the set of non-zero monomials in $\mathbf{z}_n$ and their non-zero scalar multiples. This is a right Ore set in L_n^q and $(L_n^q)_{\mathcal{X}_n} = T_q(k, \mathbf{z}_n, \mathbf{A}_n)$.*

(ii) $(L_n^q)_{\mathcal{X}_{n-1}} = T_q(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[z_n; \alpha_n]$ where $\alpha_n(z_i) = q^{A_{ni}} z_i$. (If n is odd then α_n is the identity automorphism).

Proof. (i) We show that $\mathcal{X}_n$ is a right Ore set in L_n^q by induction on n. When $n = 1$, $\mathcal{X}_1 = k[x]^*$, which is an Ore set in L_1^q since the latter is commutative. So by induction, suppose we know $\mathcal{X}_{n-1}$ is a right Ore set in L_{n-1}^q. We recall that $L_n^q = L_{n-1}^q[x_n; \sigma_n, \delta_n]$ where $\sigma_n(x_i) = q^{\pm 1} x_i$ for each i. Since $\sigma_n(\mathcal{X}_{n-1}) = \mathcal{X}_{n-1}$, by Lemma 2.42, $\mathcal{X}_{n-1}$ is a right Ore set in L_n^q. Now $\mathcal{X}_n$ is the multiplicative closure of $\mathcal{X}_{n-1} \cup \{z_n\}$, so by Lemma 2.44, $\mathcal{X}_n$ is a right Ore set in L_n^q also, since z_n is normal in L_n^q.

Since $S_q(k, \mathbf{z}_n, \mathbf{A}_n) \subset L_n^q$ and $S_q(k, \mathbf{z}_n, \mathbf{A}_n)_{\mathcal{X}_n} = T_q(k, \mathbf{z}_n, \mathbf{A}_n)$, we have $T_q(k, \mathbf{z}_n, \mathbf{A}_n) \subset (L_n^q)_{\mathcal{X}_n}$. Then, since $x_i = z_{i-1}^{-1}(z_i + z_{i-2})$, we have $x_i \in T_q(k, \mathbf{z}_n, \mathbf{A}_n)$ for $1 \leq i \leq n$, and so $T_q(k, \mathbf{z}_n, \mathbf{A}_n) = (L_n^q)_{\mathcal{X}_n}$.

(ii) We have already shown that $\mathcal{X}_{n-1}$ is a right Ore set in L_n^q. By the same argument as in (i), $T_q(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[z_n; \alpha_n] \subset (L_n^q)_{\mathcal{X}_{n-1}}$, and then the calculation $x_i = z_{i-1}^{-1}(z_i + z_{i-2})$ shows $x_i \in T_q(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[z_n; \alpha_n]$ for $1 \leq i \leq n$, and so $T_q(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[z_n; \alpha_n] = (L_n^q)_{\mathcal{X}_n}$.

□

Lemma 5.12. *Let K be a field, and $q \in K$ be such that q is not a root of unity. Then $T_q(K, \mathbf{z}_n, \mathbf{A}_n)$ is simple if and only if n is even.*

Proof. Given integers $m_1, \ldots, m_n$, we say property (*) holds if $A_{1j}m_1 + \cdots + A_{nj}m_n = 0$ for $j = 1, \ldots, n$. By [32, Proposition 1.3], $T_q(K, \mathbf{z}_n, \mathbf{A}_n)$ is simple if and only if the only n-tuple $m_1, \ldots, m_n$ which satisfies property (*) is $m_1 = \cdots = m_n = 0$.

If n is odd then $A_{nj} = 0$ for all j, so taking $m_1 = \cdots = m_{n-1} = 0$ and m_n non-zero means (*) holds, so $T_q(K, \mathbf{z}_n, \mathbf{A}_n)$ cannot be simple. (Alternatively, we can observe directly that z_n is then a central non-unit, so generates a non-trivial two-sided ideal).

If n is even, suppose $m_1, \ldots, m_n$ are such that property (*) holds. Considering $j = 2$ tells us that $m_1 = 0$, since $A_{i2} = 0$ unless $i = 1$ and $A_{12} = 1$. Then considering $j = 4$ tells us that $m_3 = 0$, since $A_{i4} = 0$ unless $i = 1$ or $i = 3$, and both A_{14} and $A_{34} = 1$; since $m_1 = 0$ we must also have $m_3 = 0$. Continuing similarly, $m_k = 0$ for all odd k. Similarly, considering $j = n-1$ tells us that $m_n = 0$, since $A_{i,n-1} = 0$ unless $i = n$, and $A_{n,n-1} = -1$, and then considering $j = n-3$ tells us $m_{n-2} = 0$, and continuing similarly $m_k = 0$ for all even k. Thus $m_k = 0$ for all k, and so by the cited proposition, $T_q(K, \mathbf{z}_n, \mathbf{A}_n)$ is simple. □

Lemma 5.13. *Let I be a (two-sided) ideal in some ring extension R of L_n^q (e.g. $R = L_m^q$, $m \geq n$, or $R = C_m^q$, odd $m > n$), and assume that q is not a root of unity.*

Let $w \in R$ be such that $wx_i = q^{r_i} x_i w$ for $1 \leq i \leq n$, where $r_i \in \mathbb{Z}$ for each i, and let $t_i \in \mathbb{N}$ for each i.

Suppose I contains $t = z_1^{t_1} \cdots z_{n-1}^{t_{n-1}} w$. Then I contains w.

Proof. For any element u of the form $u = z_1^{u_1} \cdots z_{n-1}^{u_{n-1}} w$, define $c(u) := \Sigma_{i=1}^{n-1} i u_i$. Let $t \in I$ be such that $c(t)$ is minimal among elements of I of this form. If $c(t) = 0$ then $t = w$ and so $w \in I$ as required, so suppose $c(t) > 0$. Then there exists some $i < n$ such that $t_i \neq 0$. Take the biggest such i and consider tx_{i+1}. By Corollary 5.5 (i), (ii), together with Proposition 5.2 (vi), there exist $m, m' \in \mathbb{Z}$ such that

$$\begin{aligned} tx_{i+1} &= z_1^{t_1} \cdots z_i^{t_i} w x_{i+1} \\ &= q^m x_{i-1} z_1^{t_1} \cdots z_i^{t_i} w + q^{m'}(1-q)[t_i]_{q^{-1}} z_1^{t_1} \cdots z_{i-1}^{t_{i-1}+1} z_i^{t_i - 1} w, \text{ if } i \text{ is even, or} \\ tx_{i+1} &= z_1^{t_1} \cdots z_i^{t_i} w x_{i+1} \\ &= q^m x_{i-1} z_1^{t_1} \cdots z_i^{t_i} w + q^{m'}(1-q)[t_i]_q z_1^{t_1} \cdots z_{i-1}^{t_{i-1}+1} z_i^{t_i - 1} w, \text{ if } i \text{ is odd.} \end{aligned}$$

Since q is not a root of unity, $[t_i]_{q^{-1}}$ and $[t_i]_q$ are both non-zero, so in either case, we can define $t' := z_1^{t_1} \cdots z_{i-1}^{t_{i-1}+1} z_i^{t_i+1} w \in I$, and then $c(t') < c(t)$, contradicting the minimality of t. □

Corollary 5.14. *Recall the set $\mathcal{X}_{n-1}$ from Lemma 5.11, of non-zero scalar multiples of monomials in $\mathbf{z}_{n-1}$, and let I be a proper ideal of L_n^q. Then $I \cap \mathcal{X}_{n-1} = \emptyset$.*

Proof. A generic element of $\mathcal{X}_{n-1}$ is of the form $\lambda z_1^{t_1} \cdots z_{n-1}^{t_{n-1}}$, where $\lambda \in k^\times$ and $t_i \in \mathbb{N}$; if such an element is in I, then taking $w = \lambda$, we can apply Lemma 5.13 to get $\lambda \in I$, a contradiction since I is proper. □

Proposition 5.15. *Any prime ideal of L_n^q is completely prime.*

Proof. This follows from [17, Theorem 2.3]. Conditions (a) and (b) are clear. For condition (c), note that $\delta_i \alpha_i(z_j) = \alpha_i \delta_i(z_j) = 0$ if $j < i - 1$. If i is odd then, by Proposition 5.3,

$\delta_i\alpha_i(z_{i-1}) = (1-q)z_{i-2} = q^{-1}\alpha_i\delta_i(z_{i-1})$, while if i is even, $\delta_i\alpha_i(z_{i-1}) = q(1-q)z_{i-2} = q\alpha_i\delta_i(z_{i-1})$. Then condition (d) and the condition on the subgroup of $k^\times$ generated by the λ_{ij} - which is just the subgroup of $k^\times$ generated by q - follow from the fact that q is not a root of unity. □

Lemma 5.16. *If n is odd, $(z_n - \lambda)L_n^q$ is a completely prime ideal of L_n^q for each $\lambda \in k$.*

If n is even, $z_n L_n^q$ is a completely prime ideal of L_n^q.

Proof. We show this by induction using Lemma 2.31; the case $n = 0$ is trivial. If $n > 0$ and then in the setting of that Lemma we have $c = z_n - \lambda$; $d = z_{n-1}$; and $e = -z_{n-2} - \lambda$, where $\lambda = 0$ if n is even and $\lambda \in k$ if n is odd. By induction, $z_{n-1}L_{n-1}^q$ is a completely prime ideal in L_{n-1}^q. Since z_{n-1} has degree 1 in x_{n-1} but e has degree 0 in x_{n-1}, e is non-zero, and so regular, modulo $z_{n-1}L_{n-1}^q$. Therefore we can apply Lemma 2.31, and cL_n^q is a completely prime ideal in L_n^q. □

Theorem 5.17. *Let n be odd, assume that k is algebraically closed, and assume q is not a root of unity. Then the prime ideals of L_n^q are 0 and the ideals $(z_n - \lambda)L_n^q$ for each $\lambda \in k$.*

Proof. We know that L_n^q is a domain, so together with Lemma 5.16, the given ideals are all prime ideals of L_n^q.

By Lemma 5.11, $(L_n^q)_{\mathcal{X}_{n-1}} = T_q(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[z_n]$. Let I be a non-zero ideal of $(L_n^q)_{\mathcal{X}_{n-1}}$. By Lemma 5.12, $T_q(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})$ is a simple ring, so we can apply Theorem 2.30, so $I = p(L_n^q)_{\mathcal{X}_{n-1}}$, where $p \in k[z_n]$. Since z_n is central and k is algebraically closed, if I is prime then $p = z_n - \lambda$ for some $\lambda \in k$.

We recall Theorem 2.10: there is a one-to-one correspondence between $\{P \in \text{Spec } L_n^q : P \cap \mathcal{X}_{n-1} = \emptyset\}$ and Spec $(L_n^q)_{\mathcal{X}_{n-1}}$ given by $P \mapsto P(L_n^q)_{\mathcal{X}_{n-1}}$. Since $(z_n - \lambda)L_n^q$ maps to $(z_n - \lambda)(L_n^q)_{\mathcal{X}_{n-1}}$ under this correspondence, using the fact that this correspondence is bijective we therefore have $\{P \in \text{Spec } L_n^q : P \cap \mathcal{X}_{n-1} = \emptyset\} = \{(z_n - \lambda)L_n^q : \lambda \in k\} \cup \{0\}$.

But by Corollary 5.14, $\{P \in \text{Spec } L_n^q : P \cap \mathcal{X}_{n-1} = \emptyset\} = \text{Spec } L_n^q$, and so we're done. □

Lemma 5.18. *Let n be even, $\lambda \in k^\times$, and define $P_\lambda := z_n L_n^q + (z_{n-1} - \lambda)L_n^q$. Then P_λ is an ideal of L_n^q with L_n^q/P_λ isomorphic to $L_{n-1}^q/(z_{n-1} - \lambda)L_{n-1}^q$; in particular P_λ is a completely prime ideal of L_{n-1}^q.*

Proof. Firstly, z_{n-1} is central modulo $z_n L_n^q$, since it commutes with x_i for $1 \leq i \leq n-1$, and by Corollary 5.4 (ii) $x_n z_{n-1} - z_{n-1}x_n \in z_n L_n^q$. Therefore P_λ is an ideal of L_n^q.

Secondly, since $x_n = \lambda^{-1}z_{n-2}$ modulo P_λ, L_n^q/P_λ is generated by $\overline{x_1}, \ldots, \overline{x_{n-1}}$. So there are homomorphisms $L_n^q/P_\lambda \to L_{n-1}^q/(z_{n-1} - \lambda)L_{n-1}^q$ and vice versa given by $\bar{x}_i \mapsto \bar{x}_i$ for

$1 \leq i \leq n-1$ (it is easy to check that these are well-defined), and these are inverses to each other, and so isomorphisms. □

Theorem 5.19. *Let n be even, and assume k is algebraically closed and q is not a root of unity. Then the prime ideals in L_n^q are 0, $z_n L_n^q$ and $(z_{n-1}-\lambda)L_n^q + z_n L_n^q$ for each $\lambda \in k^\times$.*

Proof. We know that L_n^q is a domain, so together with Lemmas 5.16 and 5.18, the given ideals are all (completely) prime ideals of L_n^q.

By Lemma 5.11 $(L_n^q)_{\mathcal{X}_{n-1}} = T_q(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[z_n; \alpha]$ for an appropriate automorphism α. Let $T = T_q(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[z_n; \alpha]$ and consider the prime spectrum of T. The set $\mathcal{U} = \{z_n^i : i \in \mathbb{N}\}$ is an Ore set in T, and $T_{\mathcal{U}} = T_q(k, \mathbf{z}_n, \mathbf{A}_n)$, which is simple by Lemma 5.12. Therefore any ideal of T must contain some power of z_n; since z_n is normal in T, any prime ideal of T must contain z_n.

The ideal $z_n T$ is a completely prime ideal of T, since $T/z_n T \cong T_q(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})$ which is a domain. Also, $T_q(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1}) = T_q(k, \mathbf{z}_{n-2}, \mathbf{A}_{n-2})[z_{n-1}^{\pm 1}]$, which is a Laurent polynomial ring over a simple ring by Lemma 5.12, so we can apply Theorem 2.30: any ideal I of $T_q(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})$ is of the form $pT_q(k, \mathbf{z}_{n-2}, \mathbf{A}_{n-2})$ with $p \in k[z_{n-1}^{\pm 1}]$. Since k is algbraically closed, if I is prime then $p = z_{n-1} - \lambda$ for some $\lambda \in k^\times$, and in this case $T_q(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})/I \cong T_q(k, \mathbf{z}_{n-2}, \mathbf{A}_{n-2})$, which is simple, so such an I is maximal.

Therefore the prime spectrum of T consists of: 0; $z_n T$; and $z_n T + (z_{n-1} - \lambda)T$, for each $\lambda \in k^\times$.

Now we apply Theorem 2.10, recalling that $T = (L_n^q)_{\mathcal{X}_{n-1}}$: there is a one-to-one correspondence between $\{P \in \text{Spec } L_n^q : P \cap \mathcal{X}_{n-1} = \emptyset\}$ and $\text{Spec } T$ given by $P \mapsto PT$. Since by Corollary 5.14, $\{P \in \text{Spec } L_n^q : P \cap \mathcal{X}_{n-1} = \emptyset\} = \text{Spec } L_n^q$, this correspondence is between $\text{Spec } L_n^q$ and $\text{Spec } T$. But this correspondence sends the known prime ideals of L_n^q, as listed above, to the prime ideals of T, and therefore those ideals are the only prime ideals of L_n^q. □

5.4 Prime ideals in C_n^q

In this section, we will always assume n is odd, so C_n^q exists.

We begin by summarising for easy reference the results of Proposition 5.3, Corollary 5.4 and Lemma 5.6.

Proposition 5.20. *For $i < n-1$,*

$z_i x_{i+1} = x_{i+1} z_i + (1-q) z_{i-1}$ *(if i is even) (1);*

$z_i x_{i+1} = q x_{i+1} z_i + (1-q) z_{i-1}$ *(if i is odd) (2);*

$x_{i+1} z_i = q z_i x_{i+1} + (1-q) z_{i+1}$ *(if i is even) (3);*

$qx_{i+1}z_i = qz_ix_{i+1} + (1-q)z_{i+1}$ (if i is odd) (4);
$x_1\theta(z_i) = \theta(z_i)x_1 + (1-q)\theta^2(z_{i-1})$ (if i is even) (5);
$x_1\theta(z_i) = q\theta(z_i)x_1 + (1-q)\theta^2(z_{i-1})$ (if i is odd) (6);
$\theta(z_i)x_1 = qx_1\theta(z_i) + (1-q)z_{i+1}$ (if i is even) (7);
$q\theta(z_i)x_1 = qx_1\theta(z_i) + (1-q)z_{i+1}$ (if i is odd) (8);
$z_{n-1}x_n = x_nz_{n-1} + (1-q)(z_{n-2} - \theta(z_{n-2}))$ (9).

Lemma 5.21. *For $i < n-1$,*
$q\theta(z_i)z_i - \theta(z_{i-1})z_{i+1} = q\theta(z_{i-1})z_{i-1} - q\theta(z_{i-2})z_i$ (if i is odd); or
$\theta(z_i)z_i - \theta(z_{i-1})z_{i+1} = q\theta(z_{i-1})z_{i-1} - \theta(z_{i-2})z_i$ (if i is even).

Proof. When i is odd:

$$\begin{aligned}
q\theta(z_i)z_i - \theta(z_{i-1})z_{i+1} = {} & q\theta(z_{i-1})x_{i+1}z_i - q\theta(z_{i-2})z_i - \theta(z_{i-1})z_ix_{i+1} \\
& + \theta(z_{i-1})z_{i-1} \\
= {} & q\theta(z_{i-1})x_{i+1}z_i - q\theta(z_{i-2})z_i - q\theta(z_{i-1})x_{i+1}z_i \\
& - (1-q)\theta(z_{i-1})z_{i-1} + \theta(z_{i-1})z_{i-1} \qquad \text{by 5.20 (2)} \\
= {} & q\theta(z_{i-1})z_{i-1} - q\theta(z_{i-2})z_i.
\end{aligned}$$

When i is even:

$$\begin{aligned}
\theta(z_i)z_i - \theta(z_{i-1})z_{i+1} = {} & \theta(z_{i-1})x_{i+1}z_i - \theta(z_{i-2})z_i - \theta(z_{i-1})z_ix_{i+1} \\
& + \theta(z_{i-1})z_{i-1} \\
= {} & \theta(z_{i-1})x_{i+1}z_i - \theta(z_{i-2})z_i - \theta(z_{i-1})x_{i+1}z_i \\
& - (1-q)\theta(z_{i-1})z_{i-1} + \theta(z_{i-1})z_{i-1} \qquad \text{by 5.20 (1)} \\
= {} & q\theta(z_{i-1})z_{i-1} - \theta(z_{i-2})z_i.
\end{aligned}$$

□

Corollary 5.22. *In L_m^q, for $0 \le i \le m-1$, i odd, $q\theta(z_i)z_i - \theta(z_{i-1})z_{i+1} = q^{\frac{i+1}{2}}$.*
In L_m^q, for $0 \le i \le m-1$, i even, $\theta(z_i)z_i - \theta(z_{i-1})z_{i+1} = q^{\frac{i}{2}}$.

Proof. When $i = 0$, this is trivial, and otherwise it follows by induction using Lemma 5.21. □

Lemma 5.23. $\theta^{-1}(z_{n-1})z_{n-1} = qz_{n-1}\theta^{-1}(z_{n-1}) + (1-q)(q^{\frac{n-1}{2}} - qz_{n-2}^2)$.

Proof. $\theta^{-1}(z_{n-1})z_{n-1} = \theta^{-1}(z_{n-1})z_{n-2}x_{n-1} - \theta^{-1}(z_{n-1})z_{n-3}$

$$
\begin{aligned}
&= qz_{n-2}\theta^{-1}(z_{n-1})x_{n-1} - z_{n-3}\theta^{-1}(z_{n-1}) \\
&= qz_{n-2}x_{n-1}\theta^{-1}(z_{n-1}) + qz_{n-2}(1-q)(\theta^{-1}(z_{n-2}) - z_{n-2}) \\
&\qquad - qz_{n-3}\theta^{-1}(z_{n-1}) - (1-q)z_{n-3}\theta^{-1}(z_{n-1}) \\
&= qz_{n-1}\theta^{-1}(z_{n-1}) + (1-q)(qz_{n-2}\theta^{-1}(z_{n-2}) - z_{n-3}\theta^{-1}(z_{n-1}) - qz_{n-2}^2) \\
&= qz_{n-1}\theta^{-1}(z_{n-1}) + (1-q)(q^{\frac{n-1}{2}} - qz_{n-2}^2).
\end{aligned}
$$

(The last equality is by Corollary 5.22, which gives $q\theta(z_{n-2})z_{n-2} - \theta(z_{n-3})z_{n-1} = q^{\frac{n-1}{2}}$). □

To describe the prime ideals in C_n^q we will use the more general results we proved in Section 3. We cannot apply these results directly to C_n^q, but we can apply them to a localisation of C_n^q and then pass back to C_n^q.

Lemma 5.24. *Let* $A := (L_{n-2}^q)_{\mathcal{X}_{n-2}}$, $v := (1-q)(q^{\frac{n-3}{2}}z_{n-2}^{-1} - z_{n-2})$, *and let* α *be the automorphism of* A *given by* $\alpha(x_i) = q^{(-1)^i}x_i$.

Define $\phi : R(A, \alpha, v) \to (C_n^q)_{\mathcal{X}_{n-2}}$ *by* $\phi(a) = a$ *for* $a \in A$; $\phi(x) = \theta^{-1}(z_{n-1})$; *and* $\phi(y) = z_{n-2}^{-1}z_{n-1}$. *Then* ϕ *is an isomorphism.*

Proof. By Lemma 5.11, $\mathcal{X}_{n-2}$ is a right Ore set in L_{n-2}^q, then by Lemma 2.42 (applied twice) it is a right Ore set in C_n^q, so $(C_n^q)_{\mathcal{X}_{n-2}}$ exists.

We check ϕ is well-defined, i.e. it preserves the defining relations of $R(A, \alpha, v)$:

$\phi(ya) = z_{n-2}^{-1}z_{n-1}a = \alpha(a)z_{n-2}^{-1}z_{n-1} = \phi(\alpha(a)y)$;

$\phi(xa) = \theta^{-1}(z_{n-1})a = \alpha^{-1}(a)\theta^{-1}(z_{n-1}) = \phi(\alpha^{-1}(a)x)$;

$$
\begin{aligned}
\phi(xy) &= \theta^{-1}(z_{n-1})z_{n-2}^{-1}z_{n-1} \\
&= q^{-1}z_{n-2}^{-1}\theta^{-1}(z_{n-1})z_{n-1} \\
&= z_{n-2}^{-1}z_{n-1}\theta^{-1}(z_{n-1}) + q^{-1}z_{n-2}^{-1}(1-q)(q^{\frac{n-1}{2}} - qz_{n-2}^2) \\
&= z_{n-2}^{-1}z_{n-1}\theta^{-1}(z_{n-1}) - (1-q)(z_{n-2} - q^{\frac{n-3}{2}}z_{n-2}^{-1}) \\
&= \phi(yx + v).
\end{aligned}
$$

Thus ϕ is well-defined.

Since $x_{n-1} = z_{n-2}^{-1}(z_{n-1} - z_{n-3})$ and $x_n = (\theta^{-1}(z_{n-1}) - \theta(z_{n-3}))z_{n-2}^{-1}$, the k-algebra $(C_n^q)_{\mathcal{X}_{n-2}}$ is generated by A, z_{n-1}, and $\theta^{-1}(z_{n-1})$, and therefore ϕ is surjective.

We use a GK dimension argument to complete the proof. By repeated application of Theorem 2.57, GK dim $R(A, \alpha, v) = n$, while by Corollary 4.7, GK dim $C_n^q = n$. Therefore by two applications of Lemma 2.55, GK dim $R(A,\alpha,v) \geq$ GK dim $(C_n^q)_{\mathcal{X}_{n-2}} \geq$ GK dim C_n^q, and so GK dim $(C_n^q)_{\mathcal{X}_{n-2}} = n$ also. Therefore, by Lemma 2.56, ϕ is an isomorphism. □

From now on we will identify $R(A,\alpha,v)$ and $(C_n^q)_{\mathcal{X}_{n-2}}$ via this isomorphism.

Lemma 5.25. *$R(A,\alpha,v)$ is a conformal ambiskew polynomial ring with splitting element $u = q^{\frac{n-3}{2}} z_{n-2}^{-1} + \lambda + qz_{n-2}$ for any $\lambda \in k$. The corresponding Casimir element is equal to $\Omega - \lambda$, and thus $R(A,\alpha,v)/(\Omega-\lambda)R(A,\alpha,v)$ is a generalised Weyl algebra.*

Proof. By definition, u is a splitting element for $R(A,\alpha,v)$ if $u - \alpha(u) = v$. For $u = q^{\frac{n-3}{2}} z_{n-2}^{-1} + \lambda + qz_{n-2}$, we have

$$\begin{aligned} u - \alpha(u) &= q^{\frac{n-3}{2}} z_{n-2}^{-1} + \lambda + qz_{n-2} - q \cdot q^{\frac{n-3}{2}} z_{n-2}^{-1} - \lambda - z_{n-2} \\ &= (1-q)(q^{\frac{n-3}{2}} z_{n-2}^{-1} - z_{n-2}) \\ &= v \text{ as required.} \end{aligned}$$

The Casimir element is given by $xy - u$:

$$\begin{aligned} xy - u &= \theta^{-1}(z_{n-1})z_{n-2}^{-1}z_{n-1} - (q^{\frac{n-3}{2}} z_{n-2}^{-1} + \lambda + qz_{n-2}) \\ &= \theta^{-1}(z_{n-1})z_{n-2}^{-1}z_{n-2}x_{n-1} - \theta^{-1}(z_{n-1})z_{n-2}^{-1}z_{n-3} - (q^{\frac{n-3}{2}} z_{n-2}^{-1} + \lambda + qz_{n-2}) \\ &= \theta^{-1}(z_{n-1})x_{n-1} - q^{-1}z_{n-2}^{-1}z_{n-3}\theta^{-1}(z_{n-1}) - (q^{\frac{n-3}{2}} z_{n-2}^{-1} + \lambda + qz_{n-2}) \\ &= \Omega + \theta^{-1}(z_{n-2}) + qz_{n-2} - q^{-1}z_{n-2}^{-1}z_{n-3}\theta^{-1}(z_{n-1}) - (q^{\frac{n-3}{2}} z_{n-2}^{-1} + \lambda + qz_{n-2}) \\ &= \Omega - \lambda + (q-q)z_{n-2} + q^{-1}z_{n-2}^{-1}(qz_{n-2}\theta^{-1}(z_{n-2}) - z_{n-3}\theta^{-1}(z_{n-1}) - q^{\frac{n-1}{2}}) \\ &= \Omega - \lambda. \end{aligned}$$

Recall that by Corollary 5.22, $q\theta(z_{n-2})z_{n-2} - \theta(z_{n-3})z_{n-1} = q^{\frac{n-1}{2}}$, which gives the last equality after applying θ^{-1}. □

Lemma 5.26. *Let $\lambda \in k$, let $u = q^{\frac{n-3}{2}} z_{n-2}^{-1} + \lambda + qz_{n-2}$, let A and α be as in Lemma 5.24, let q be not a root of unity, and let $m \geq 1$. Then $Au + A\alpha^m(u) = A$ unless $\lambda = \pm q^{\frac{n-3}{4}} q^{\frac{1-m}{2}}(q^m+1)$, in which case $Au + A\alpha^m(u)$ is a maximal ideal of A.*

Proof. First, we note that $A = T_q(\mathbf{z}_{n-3}, \mathbf{A}_{n-3})[z_{n-2}^{\pm 1}]$, and by Lemma 5.12, $T_q(\mathbf{z}_{n-3}, \mathbf{A}_{n-3})$ is simple. Therefore the maximal ideals of A are of the form $(z_{n-2} - \lambda)A$, for $\lambda \in k^\times$, and these ideals are completely prime.

We also recall that $\alpha(z_{n-2}) = q^{-1}\alpha(z_{n-2})$.

Now suppose $Au + A\alpha^m(u)$ is proper. Then there exists some maximal ideal M of A such that $Au \subset M$ and $A\alpha^m(u) \subset M$. By the above, there exists $\mu \in k^\times$ such that $M = (z_{n-2} - \mu)A$. Then, since $u = 0 = \alpha^m(u)$ modulo M, we obtain

$q^{\frac{n-3}{2}} \mu^{-1} + \lambda + q\mu = 0$; and

$q^{\frac{n-3}{2}}q^m\mu^{-1} + \lambda + q^{1-m}\mu = 0.$

Eliminating μ^{-1}, we get $(q^m - 1)\lambda + (q^{m+1} - q^{1-m})\mu = 0$, that is, $\lambda = -q^{1-m}(1 + q^m)\mu$, since $q^m - 1 \neq 0$ by assumption; in particular, since $\mu \neq 0$, $\lambda \neq 0$ also.

Substituting back in, we get

$$\begin{aligned} q^{\frac{n-3}{2}}q^{1-m}(1+q^m)^2\lambda^{-1} - (1+q^m)\lambda + q^m\lambda &= 0 \\ q^{\frac{n-3}{2}}q^{1-m}(1+q^m)^2 - \lambda^2 &= 0 \end{aligned}$$

Therefore $\lambda = \pm q^{\frac{n-3}{4}}q^{\frac{1-m}{2}}(1+q^m)$.

Conversely, if $\lambda = \epsilon q^{\frac{n-3}{4}}q^{\frac{1-m}{2}}(1+q^m)$ with $\epsilon = \pm 1$, then

$u = (1 + \epsilon q^{\frac{n-3}{4}}q^{\frac{m-1}{2}}z_{n-2}^{-1})(qz_{n-2} + \epsilon q^{\frac{n-3}{4}}q^{\frac{1-m}{2}})$; and

$\alpha^m(u) = (1 + \epsilon q^m q^{\frac{n-3}{4}}q^{\frac{m-1}{2}}z_{n-2}^{-1})(q^{1-m}z_{n-2} + \epsilon q^{\frac{n-3}{4}}q^{\frac{1-m}{2}})$.

Therefore $Au + A\alpha^m(u) \subset (z_{n-2} + \epsilon q^{\frac{n-3}{4}}q^{\frac{m-1}{2}})A$, and in fact, since $(qz_{n-2} + \epsilon q^{\frac{n-3}{4}}q^{\frac{1-m}{2}})$ and $(1 + \epsilon q^m q^{\frac{n-3}{4}}q^{\frac{n-1}{2}}z_{n-2}^{-1})$ generate distinct prime ideals of A, $Au + A\alpha^m(u) = (z_{n-2} + \epsilon q^{\frac{n-3}{4}}q^{\frac{m-1}{2}})$, which is maximal. □

Lemma 5.27. *Let R be a simple ring with centre k, let $A = R[t^{\pm 1}]$, and let α be an automorphism of A such that $\alpha(t) = \lambda t$, where $\lambda \in k^\times$ is not a root of unity. Then A is α-simple.*

Proof. First, by Theorem 2.30, any non-zero ideal I of A is generated by a non-zero element of $k[t^{\pm 1}]$.

Let I be an α-stable ideal of A, and let $p(t)$ be an element of $I \cap k[t^{\pm 1}]$ that, among non-zero elements of $I \cap k[t^{\pm 1}]$, has the minimal number of non-zero terms. If $p(t)$ has only one non-zero term then $p(t) = a_r t^r$ for some $r \in \mathbb{Z}$ and $a_r \in k^\times$, which is a unit in A and so $I = A$. Otherwise, $p(t) = \sum_{i=1}^{s} a_i t^{r_i}$, where $s > 1$, the integers r_i for $1 \leq i \leq s$ are all distinct, and $a_i \in k^\times$ for $1 \leq i \leq s$. Then $p(t) - \lambda^{-r_s}\alpha(p(t)) = \sum_{i=1}^{s}(1 - \lambda^{r_i - r_s})a_i t^{r_i}$. If $i \neq s$, $1 - \lambda^{r_i - r_s} \neq 0$ since $r_i \neq r_s$ and λ is not a root of unity. So $p(t) - \lambda^{-r_s}\alpha(p(t))$ is an element of $I \cap k[t^{\pm 1}]$ with $s - 1$ non-zero terms, a contradiction to the minimality of $p(t)$. □

Lemma 5.28. *Work in the Laurent polynomial ring $k[t^{\pm 1}]$. Suppose λ_i, $\mu_i \in k$ for $i = 1, 2$, with each $\mu_i \neq 0$. Let $u = \lambda_1 t^{-1} + \lambda_2 t$ and $v = \mu_1 t^{-1} + \mu_2 t$. Then there exists a polynomial $p(x) \in k[x]$ such that $p(u) \in vk[t^{\pm 1}]$.*

Proof. Let
$$\begin{aligned} a &= u^2 - \left(\frac{\lambda_1^2}{\mu_1}t^{-1} + \frac{\lambda_2^2}{\mu_2}t\right) v \\ &= \lambda_1^2 t^{-2} + 2\lambda_1\lambda_2 + \lambda_2^2 t^2 - \left(\frac{\lambda_1^2}{\mu_1}t^{-1} + \frac{\lambda_2^2}{\mu_2}t\right)\left(\mu_1 t^{-1} + \mu_2 t\right) \\ &= \lambda_1^2 t^{-2} + 2\lambda_1\lambda_2 + \lambda_2^2 t^2 - \lambda_1^2 t^{-2} - \lambda_2^2 t^2 - \frac{\lambda_1^2\mu_2}{\mu_1} - \frac{\lambda_2^2\mu_1}{\mu_2} \\ &= 2\lambda_1\lambda_2 - \frac{\lambda_1^2\mu_2}{\mu_1} - \frac{\lambda_2^2\mu_1}{\mu_2} \in k. \end{aligned}$$

Therefore $p(x) = x^2 - a$ is a polynomial such that $p(u) = \left(\frac{\lambda_1^2}{\mu_1}t^{-1} + \frac{\lambda_2^2}{\mu_2}t\right) v \in vk[t^{\pm 1}]$. □

Lemma 5.29. *For any $\lambda \in k$, $(\Omega - \lambda)C_n^q$ is a completely prime ideal in C_n^q.*

Proof. We show this using Lemma 2.31. In the setting of that Lemma, we have $R = C_n^q$, $A = L_{n-1}^q$, $c = \Omega - \lambda$, $d = z_{n-1}$ and $e = -z_{n-2} - q\theta(z_{n-2}) - \lambda$. By considering total degree, we cannot have $e \in dR$, and so, since by Lemma 5.16, R/dR is a domain, e is regular modulo Ad. Therefore we can apply Lemma 2.31 to show that R/cR is a domain, that is, $(\Omega - \lambda)C_n^q$ is a completely prime ideal in C_n^q. □

Theorem 5.30. *The prime ideals of C_n^q (n odd) are: 0; $(\Omega - \lambda)C_n^q$, for all $\lambda \in k$; and for $\lambda = \pm q^{\frac{n-3}{4}} q^{-\frac{a-1}{2}}(q^a + 1)$ ($1 \le a \in \mathbb{N}$), maximal ideals P_λ strictly containing $(\Omega - \lambda)C_n^q$ corresponding to the ideals found in Theorem 3.18. Further, C_n^q/P_λ has uniform dimension a.*

Proof. First, we describe the prime ideal structure of the ambiskew polynomial ring $R = R(A, \alpha, v) = (C_n^q)_{\mathcal{X}_{n-2}}$, where A, α and v are as in Lemma 5.24.

We note the following properties of A, α and v:

(i) A is α-simple;

(ii) for $m \ge 1$, α^m is not an inner automorphism of A;

(iii) in particular, for $m \ge 1$, $\alpha^m \ne 1$;

(iv) u is a regular element of A.

Of these, (i) is given by Lemma 5.27; (ii) holds since z_{n-2} is central in A but not fixed by any power of α; and (iv) holds since A is a domain.

We claim that $R(A, \alpha, v)$ is of the form required for Theorem 3.3 to apply. Conditions (i) and (ii) of that Theorem are points (i) and (ii) above. Condition (iii) follows from Lemma 5.28, recalling $u = q^{\frac{n-3}{2}} z_{n-2}^{-1} + qz_{n-2}$ and $v = (1-q)(q^{\frac{n-3}{2}} z_{n-2}^{-1} - z_{n-2})$, so for all

m, u and $v^{(m)} = \sum_{i=0}^{m-1} \alpha^i(v)$ are of the correct form to apply that lemma with $v^{(m)}$ in the role of v. Finally, the condition $\{c \in A : c \text{ central in } A \text{ and } \alpha(c) = c\} = k$ is clear since $Z(A) = k[z_{n-2}^{\pm 1}]$, $\alpha(z_{n-2}) = q^{-1}z_{n-2}$, and q is not a root of unity.

By Lemma 5.25, Ω is a central Casimir element for $R(A, \alpha, v)$. So applying Theorem 3.3, $R_{k[\Omega]^*}$ is simple, and so any non-zero prime ideal of R must contain some element of $k[\Omega]^*$.

Since k is algebraically closed any such element can be written as a product of linear terms, and since Ω is central, any prime ideal must therefore in fact contain a linear term, i.e. $\Omega - \lambda$ for some λ. Further, by Lemma 5.29, $(\Omega - \lambda)R$ is a completely prime ideal of R.

Lemma 5.26 gives the conditions on $Au + A\alpha^m(u)$ required for Theorem 2.66 and Theorem 3.18 to apply; the other conditions for those theorems are given by points (i) to (iv) at the start of the proof. So if $\lambda = \pm q^{\frac{n-3}{4}} q^{-\frac{a-1}{2}}(q^a + 1)$ for some integer $a \geq 1$, then by Theorem 3.18 there is a unique non-zero prime ideal S_λ of R strictly containing $(\Omega - \lambda)R$; while if $\lambda \neq \pm q^{\frac{n-3}{4}} q^{-\frac{a-1}{2}}(q^a + 1)$ for any integer $a \geq 1$ then by Theorem 2.66, $R/(\Omega - \lambda)R$ is simple.

Therefore, the prime ideals of R are: 0; $(\Omega - \lambda)R$, for each $\lambda \in k$; and for each λ of the form $\lambda = \pm q^{\frac{n-3}{4}} q^{-\frac{a-1}{2}}(q^a + 1)$, where $a \geq 1$ is an integer, a maximal ideal S_λ containing $(\Omega - \lambda)R$.

Next, we note that by Lemma 5.13, $\{P \in \operatorname{Spec} C_n^q : P \cap \mathcal{X}_{n-2} = \emptyset\} = \operatorname{Spec} C_n^q$, and so by Theorem 2.10 there is a 1-1 correspondence between $\operatorname{Spec} C_n^q$ and $\operatorname{Spec} R$ given by $P \mapsto PR$, $P' \mapsto P' \cap C_n^q$.

By Lemma 5.29, for any $\lambda \in k$, $(\Omega - \lambda)C_n^q$ is a prime ideal of C_n^q, and this is sent to $(\Omega - \lambda)R$ under the correspondence above. So the prime ideals of C_n^q are: 0; $(\Omega - \lambda)C_n^q$ for any $\lambda \in k$; and for each λ of the form $\lambda = \pm q^{\frac{n-3}{4}} q^{-\frac{a-1}{2}}(q^a + 1)$ for some integer $a \geq 1$, maximal ideals P_λ corresponding to the maximal ideals S_λ of R, with $S_\lambda \supset (\Omega - \lambda)R$ and so $P_\lambda \supset (\Omega - \lambda)C_n^q$.

By Theorem 3.32, R/S_λ has right uniform rank a when $\lambda = \pm q^{\frac{n-3}{4}} q^{-\frac{a-1}{2}}(q^a + 1)$. So, since by Corollary 2.41, $(C_n^q/P_\lambda)_{\mathcal{X}_{n-2}^-} = (C_n^q)_{\mathcal{X}_{n-2}}/(P_\lambda)_{\mathcal{X}_{n-2}} = R/S_\lambda$, and by Lemma 2.39, the right uniform ranks of $(C_n^q/P_\lambda)_{\mathcal{X}_{n-2}^-}$ and C_n^q/P_λ are equal, C_n^q/P_λ has right uniform rank a when if $\lambda = \pm q^{\frac{n-3}{4}} q^{-\frac{a-1}{2}}(q^a + 1)$. $\square$

5.5 Automorphism groups of L_n^q and C_n^q

We have already met the k-automorphism θ of C_n^q defined by $\theta(x_n) = x_1$, $\theta(x_i) = x_{i+1}$ otherwise. It is clear from the defining relations of C_n^q that there is a k-automorphism ι of C_n^q defined by $\iota(x_i) = -x_i$ for each i.

Similarly, it is clear from the defining relations of L_n^q that for each $\nu \in k^\times$ there is a

k-automorphism ι_ν of L_n^q defined by $\iota(x_i) = \nu^{(-1)^i} x_i$ for each i.

To show that these are the only automorphisms of C_n^q, respectively L_n^q, we will need the following observation about the elements z_n and Ω.

Lemma 5.31. *For each $n \geq 0$, the element z_n of L_n^q can be written as $x_1 \cdots x_n +$ smaller terms, where the smaller terms have degree at most 1 in each x_i, $1 \leq i \leq n$.*

For each odd $n \geq 1$, the element Ω of C_n^q can be written as $x_1 \cdots x_n +$ smaller terms, where the smaller terms have degree at most 1 in each x_i, $1 \leq i \leq n$.

Proof. Let us say that a **balanced term of degree** $< n$ (with respect to a PBW basis of monomials in elements $x_1, \ldots, x_n$) is a (scalar multiple of a) monomial of total degree $< n$ with degree at most 1 in each x_1, for $1 \leq i \leq n$.

We show the first part by induction; certainly this is true for $n = 0$ (where the statement is vacuous) and $n = 1$ (since $z_1 = x_1$). For $n > 1$, by induction, $z_{n-1} = x_1 \cdots x_{n-1} +$ balanced terms of degree $< n-1$, so $z_{n-1}x_n = x_1 \cdots x_n$ + balanced terms of degree $< n$. Also, by induction, all the terms of z_{n-2} are balanced terms of degree $< n$, so, recalling that $z_n = z_{n-1}x_n - z_{n-2}$, we can write $z_n = x_1 \cdots x_n$ + balanced terms of degree $< n$, as required.

Similarly, in C_n^q, by the previous part $z_{n-1}x_n = x_1 \cdots x_n +$ balanced terms of degree $< n$, and all the terms of z_{n-2} and $q\theta(z_{n-2})$ are balanced terms of degree $< n$, and so recalling that $\Omega = z_{n-1}x_n - z_{n-2} - q\theta(z_{n-2})$, we can write $\Omega = x_1 \cdots x_n$ + balanced terms of degree $< n$, as required. ☐

Theorem 5.32. *(i) Let C_r denote the cyclic group with r elements. The k-automorphism group of C_n^q is isomorphic to $C_n \times C_2$, with generators by θ and ι. Since n is odd we therefore have $Aut_k C_n^q \cong C_{2n}$.*

(ii) The k-automorphism group of L_n^q is isomorphic to $k^\times$ via $k \ni \nu \mapsto \iota_\nu \in Aut_k(L_n^q)$.

Proof. (i) For this part, to simplify notation let $x_{n+i} = x_i$, or alternatively consider the indices on the x_i modulo n.

Firstly, we note that $\theta\iota = \iota\theta$ and that $\theta^n = \iota^2 = 1$, so the subgroup of $\mathrm{Aut}_k(C_n^q)$ generated by θ and ι is isomorphic to $C_n \times C_2$.

Let $\psi : C_n^q \to C_n^q$ be a k-automorphism. We note that ψ must send height 1 primes to height 1 primes, and so since the only elements that generate $(\Omega - \lambda)C_n^q$ are non-zero constant multiples of $\Omega - \lambda$, we must have $\psi(\Omega) = \mu\Omega - \lambda$ for some $\mu \in k^\times$, $\lambda \in k$.

By Lemma 5.31, $\Omega = x_1 \cdots x_n +$terms of degree $< n$, where the terms of smaller degree have degree at most 1 in each x_i. If p is such a term of degree $< n$, then $\psi(p)$ must have total degree less than the total degree of $\psi(x_1 \cdots x_n)$. Therefore, the total degree of $\psi(\Omega)$ equals the total degree of $\psi(x_1 \cdots x_n)$, which equals the sum of the total degrees of $\psi(x_1), \dots, \psi(x_n)$. Since we've already observed that the total degree of $\psi(\Omega)$ is n, each of $\psi(x_1), \dots, \psi(x_n)$ must therefore have total degree 1 (noting that they cannot have total degree 0).

Suppose $i \neq j$, and that $\psi(x_i)$ and $\psi(x_j)$ both have non-zero degree in x_k for some k. Then $\psi(x_i)\psi(x_j) - q^{\pm 1}\psi(x_j)\psi(x_i) = c(1-q^{\pm 1})x_k^2 +$ other terms, where $c \in k^\times$; however, we note that for any $i \neq j$, one of $x_i x_j - q x_j x_i$ and $x_j x_i - q x_i x_j$ has total degree ≤ 0, so applying ψ, one of $\psi(x_i)\psi(x_j) - q\psi(x_j)\psi(x_i)$ and $\psi(x_j)\psi(x_i) - \psi(x_i)\psi(x_j)$ must have total degree ≤ 0. This is a contradiction, and therefore $\psi(x_i)$ and $\psi(x_j)$ cannot both have non-zero degree in x_k for any k.

Therefore the sets $\{r : \psi(x_i) \text{ has non-zero degree in } x_r\}$ for $1 \leq i \leq n$ form a partition of $\{1, \dots, n\}$; since each of them has at least one element and there are n of them, they each have precisely one element.

That is, there exists a permutation π of $\{1, \cdots, n\}$ such that $\psi(x_i) = \mu_{\pi(i)} x_{\pi(i)} + \lambda_{\pi(i)}$ for some $\mu_i \in k^\times$, $\lambda_i \in k$. But we can apply [31, Proposition 2.1] to show that $\lambda_i = 0$ for each i.

Let $1 \leq i \leq n$, and let $a = \pi^{-1}(i)$, $b = \pi^{-1}(i+1)$. So one of $x_a x_b - q x_b x_a$ and $x_a x_b - q^{-1} x_b x_a$ must be an element of k.

$$\begin{aligned}\psi(x_a x_b - q^{-1} x_b x_a) &= \mu_i \mu_{i+1}(x_i x_{i+1} - q^{-1} x_{i+1} x_i) \\ &= \mu_i \mu_{i+1}(x_i x_{i+1} - q^{-2} x_i x_{i+1} + q^{-2}(1-q))\end{aligned}.$$

Therefore $x_a x_b - q^{-1} x_b x_a$ cannot have degree ≤ 0, and so $x_a x_b - q x_b x_a$ equals either 0 or $1 - q$.

$$\begin{aligned}\psi(x_a x_b - q x_b x_a) &= \mu_i \mu_{i+1}(x_i x_{i+1} - q x_{i+1} x_i) \\ &= (1-q)\mu_i \mu_{i+1}\end{aligned}$$

We cannot have $\mu_i \mu_{i+1} = 0$, so we must have $\mu_i \mu_{i+1} = 1$, and $\pi^{-1}(i) + 1 = \pi^{-1}(i+1)$ modulo n.

Since n is odd, $\mu_i \mu_{i+1} = 1$ for each i implies either $\mu_i = 1$ for all i or $\mu_i = -1$ for all i. Therefore there exists $\mu \in \{\pm 1\}$ and $r \in \mathbb{Z}$ such that, for $1 \leq i \leq n$, $\psi(x_i) = \mu x_{i+r}$. If $\mu = 1$ then $\psi = \theta^r$ while if $\mu = -1$ then $\psi = \iota\theta^r$. Therefore $\mathrm{Aut}_k(C_n^q)$ is generated by θ and ι, as required.

(ii) Let $\psi : L_n^q \to L_n^q$ be a k-automorphism. As in part (i), since ψ sends prime ideals of height 1 to prime ideals of height 1, and the same is true of their generators, $\psi(z_n) = \mu z_n - \lambda$ for some $\mu \in k^\times$, $\lambda \in k$. Then by the same argument using Lemma 5.31 and [31, Proposition 2.1] as was applied to Ω in part (i), applied here to z_n, there exists $\mu_i \in k^\times$, for $1 \leq i \leq n$, and some permutation $\pi \in S_n$, such that $\psi(x_i) = \mu_{\pi(i)} x_{\pi(i)}$ for $1 \leq i \leq n$.

Again in a similar fashion to part (i), by considering the images under ψ of the skew commutators between $x_{\pi^{-1}(i)}$ and $x_{\pi^{-1}(i+1)}$, we must have $\mu_i \mu_{i+1} = 1$ for $1 \leq i \leq n-1$ and $\pi(i+1) = \pi(i) + 1$ for $1 \leq i \leq n-1$. Therefore $\pi(i) = i$.

Thus $\psi(x_i) = \mu_i x_i$ for some $\mu_i \in k^\times$, fixed $r \in \mathbb{Z}$, such that $\mu_i \mu_{i+1} = 1$ for $1 \leq i \leq n-1$. But then we have $\psi = \iota_{\mu_2}$.

Finally, $k^\times \ni \nu \mapsto \iota_\nu \in \mathrm{Aut}_k(R)$ is clearly an injective homomorphism $k^\times \to \mathrm{Aut}_k(L_n^q)$, and is surjective by the above, and so an isomorphism.

□

5.6 Krull and global dimensions in L_n^q

Many of these results hold for both Krull and global dimension; in this case dim means either. For paired references the first reference is for Krull dimension and the second is for global dimension.

Definition 5.33. Let $\mathbf{\Gamma} \in \{-,+\}^{n-1}$. Then $\mathbf{z}_{n-1,\mathbf{\Gamma}}$ is the $(n-1)$-tuple of elements of L_n^q defined by:

(i) $(\mathbf{z}_{n-1,\mathbf{\Gamma}})_{n-1} = \begin{cases} z_{n-1} & \text{if } \mathbf{\Gamma}_{n-1} = - \\ \theta(z_{n-1}) & \text{if } \mathbf{\Gamma}_{n-1} = + \end{cases}$;

(ii) for $1 \leq i \leq n-2$, if $(\mathbf{z}_{n,\mathbf{\Gamma}})_{i+1} = \theta^r(z_{i+1})$ then $(\mathbf{z}_{n,\mathbf{\Gamma}})_i = \begin{cases} \theta^r(z_i) & \text{if } \mathbf{\Gamma}_i = - \\ \theta^{r+1}(z_i) & \text{if } \mathbf{\Gamma}_i = + \end{cases}$.

If $1 \leq m < n-1$, then $\mathbf{z}_{m,\mathbf{\Gamma}}$ is the m-tuple of elements of L_n^q with $(\mathbf{z}_{m,\mathbf{\Gamma}})_i = (\mathbf{z}_{n-1,\mathbf{\Gamma}})_i$ for $1 \leq i \leq m$.

For $1 \leq m \leq n-1$, let $\mathbf{A}_{m,\mathbf{\Gamma}}$ be the $m \times m$ matrix such that, for all $1 \leq i, j \leq m$,

$$(\mathbf{z}_{m,\mathbf{\Gamma}})_i (\mathbf{z}_{m,\mathbf{\Gamma}})_j = q^{(\mathbf{A}_{m,\mathbf{\Gamma}})_{ij}} (\mathbf{z}_{m,\mathbf{\Gamma}})_j (\mathbf{z}_{m,\mathbf{\Gamma}})_i$$

(This exists by, and can be found from, Proposition 5.2 (v)).

With this definition, for any $1 \leq m \leq n-1$, $S_q(\mathbf{z}_{m,\mathbf{\Gamma}}, \mathbf{A}_{m,\mathbf{\Gamma}}) \subset L_n^q$.

Finally, for $1 \leq m \leq n-1$ let $\mathcal{X}_{m,\mathbf{\Gamma}}$ denote the set of non-zero monomials in $\mathbf{z}_{m,\mathbf{\Gamma}}$ and their non-zero scalar multiples.

Example 5.34. If $\mathbf{\Gamma} = (-, -, \cdots, -)$ then $\mathbf{z}_{n-1,\mathbf{\Gamma}} = (z_1, z_2, \ldots, z_{n-1}) = \mathbf{z}_{n-1}$.

We will need the following generalisation of Lemma 5.11.

Lemma 5.35. *Let $n \geq 1$. Let $\mathbf{\Gamma} \in \{-,+\}^{n-1}$. Then:*

(i) $\mathcal{X}_{n-1,\mathbf{\Gamma}}$ is a right Ore set in L_n^q and $(L_n^q)_{\mathcal{X}_{n-1,\mathbf{\Gamma}}} = T_q(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[z_n; \alpha_n]$ where α_n is the automorphism of $T_q(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})$ given by $\alpha_n(\theta^r(z_i)) = q^{(-1)^r A_{ni}} z_i$. (If n is odd then α_n is the identity automorphism).

(ii) If $\mathbf{\Gamma}_{n-1} = -$, then $\mathcal{X}_{n-1,\mathbf{\Gamma}}$ is a right Ore set in L_{n-1}^q, while if $\mathbf{\Gamma}_{n-1} = +$, then $\mathcal{X}_{n-1,\mathbf{\Gamma}}$ is a right Ore set in $\theta(L_{n-1}^q)$; either way, the localisation equals $T_q(\mathbf{z}_{n-1,\mathbf{\Gamma}}, \mathbf{A}_{n-1,\mathbf{\Gamma}})$.

(iii) If n is even then $\mathcal{X}_{n-1,\mathbf{\Gamma}}$ is a right Ore set in C_{n+1}^q and $(C_{n+1}^q)_{\mathcal{X}_{n-1,\mathbf{\Gamma}}}$ is an ambiskew polynomial ring $R(A, \alpha, u)$. If $(\mathbf{\Gamma}_{n-1})_{n-1} = -$ then $A = (L_{n-1}^q)_{\mathcal{X}_{n-1,\mathbf{\Gamma}}}$, $u = q^{\frac{n-3}{2}} z_{n-2}^{-1} + \lambda + q z_{n-2}$, $x = \theta^{-1}(z_{n-1})$, and $y = z_{n-2}^{-1} z_{n-1}$, while if $(\mathbf{\Gamma}_{n-1})_{n-1} = +$ then $A = (\theta(L_{n-1}^q))_{\mathcal{X}_{n-1,\mathbf{\Gamma}}}$, $u = q^{\frac{n-3}{2}} \theta(z_{n-2}^{-1}) + \lambda + q\theta(z_{n-2})$, $x = z_{n-1}$, and $y = \theta(z_{n-2})^{-1}\theta(z_{n-1})$.

Proof. (i) We show that $\mathcal{X}_{n-1,\mathbf{\Gamma}}$ is a right Ore set in L_n^q by induction on n. When $n = 1$, $\mathcal{X}_1 = k^\times$, which is an Ore set in L_1^q.

So by induction, suppose we know $\mathcal{X}_{n-2,\mathbf{\Delta}}$ is a right Ore set in L_{n-1}^q for any $\mathbf{\Delta} \in \{-,+\}^{n-2}$. This tells us that $\mathcal{X}_{n-2,\mathbf{\Gamma}}$ is a right Ore set in L_{n-1}^q for any $\mathbf{\Gamma} \in \{-,+\}^{n-1}$ such that $\mathbf{\Gamma}_{n-1} = -$, while $\mathcal{X}_{n-2,\mathbf{\Gamma}}$ is a right Ore set in $\theta(L_{n-1}^q)$ (that is, the subring of L_n^q generated by $x_2, \ldots, x_n$) for any $\mathbf{\Gamma} \in \{-,+\}^{n-1}$ such that $\mathbf{\Gamma}_{n-1} = +$.

There are two cases depending on the value of $\mathbf{\Gamma}_{n-1}$. If $\mathbf{\Gamma}_{n-1} = -$, then $\mathcal{X}_{n-1,\mathbf{\Gamma}}$ is the multiplicative closure of $\mathcal{X}_{n-2,\mathbf{\Gamma}} \cup \{z_{n-1}\}$, so by Lemma 2.44, $\mathcal{X}_{n-1,\mathbf{\Gamma}}$ is a right Ore set in L_{n-1}^q, since z_n is normal in L_{n-1}^q. Then we recall that $L_n^q = L_{n-1}^q[x_n; \sigma_n, \delta_n]$ where $\sigma_n(x_i) = q^{\pm 1} x_i$ for each i, so since $\sigma_n(\mathcal{X}_{n-1,\mathbf{\Gamma}}) = \mathcal{X}_{n-1,\mathbf{\Gamma}}$, by Lemma 2.42, $\mathcal{X}_{n-1,\mathbf{\Gamma}}$ is a right Ore set in L_n^q.

If $\mathbf{\Gamma}_{n-1} = +$, the proof is similar, using Corollary 4.9 to show that L_n^q is a skew polynomial extension $\theta(L_{n-1}^q)[x_1; \sigma, \delta]$ for appropriate σ, δ.

Since $S_q(k, \mathbf{z}_{n-1,\mathbf{\Gamma}}, \mathbf{A}_{n-1,\mathbf{\Gamma}})[z_n; \alpha_n] \subset L_n^q$ and

$$S_q(k, \mathbf{z}_{n-1,\mathbf{\Gamma}}, \mathbf{A}_{n-1,\mathbf{\Gamma}})_{\mathcal{X}_{n-1,\mathbf{\Gamma}}}[z_n; \alpha_n] = T_q(k, \mathbf{z}_{n-1,\mathbf{\Gamma}}, \mathbf{A}_{n-1,\mathbf{\Gamma}})[z_n; \alpha_n]$$

we have $T_q(k, \mathbf{z}_{n-1,\mathbf{\Gamma}}, \mathbf{A}_{n-1,\mathbf{\Gamma}})[z_{n-1}, \alpha_n] \subset (L_n^q)_{\mathcal{X}_{n-1,\mathbf{\Gamma}}}$. Then, using powers of θ applied to the relations $x_i = z_{i-1}^{-1}(z_i + z_{i-2})$ and $x_1 = \theta(z_{i-1})^{-1}(z_i + \theta^2(z_{i-2})$, we have $x_i \in T_q(k, \mathbf{z}_{n-1,\mathbf{\Gamma}}, \mathbf{A}_{n-1,\mathbf{\Gamma}})[z_n; \alpha_n]$ for $1 \leq i \leq n$, and so $T_q(k, \mathbf{z}_{n-1,\mathbf{\Gamma}}, \mathbf{A}_{n-1,\mathbf{\Gamma}})[z_n; \alpha_n] = (L_n^q)_{\mathcal{X}_{n-1,\mathbf{\Gamma}}}$.

(ii) We showed in (i) that $\mathcal{X}_{n-1,\mathbf{\Gamma}}$ is a right Ore set in the appropriate ring, and the second part is proved as in (i).

(iii) If $\mathbf{\Gamma}_{n-1} = -$, this follows from Lemma 5.24 and localising both sides at $\mathcal{X}_{n-3,\mathbf{\Gamma}}$. If $\mathbf{\Gamma}_{n-1} = +$, then the same is true but we must apply θ before localising.

□

Lemma 5.36. *Let* $\mathbf{\Gamma} \in \{-,+\}^{n-1}$. *Then* $\dim T_q(\mathbf{z}_{n,\mathbf{\Gamma}}, \mathbf{A}_{n,\mathbf{\Gamma}}) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd} \end{cases}$.

Proof. Assume first that n is even. We will aim to apply Theorem 2.54.

If $\mathbf{b} \in \mathbb{Z}^n$ is a row vector, write $\mathbf{z}_{n,\mathbf{\Gamma}}^{\mathbf{b}} := \prod_{i=1}^n (\mathbf{z}_{n,\mathbf{\Gamma}})_i^{b_i}$.

If $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$, then $\mathbf{z}_{n,\mathbf{\Gamma}}^{\mathbf{a}}$ and $\mathbf{z}_{n,\mathbf{\Gamma}}^{\mathbf{b}}$ commute if and only if $\mathbf{a}\mathbf{A}_{n,\mathbf{\Gamma}}\mathbf{b}^T = 0$.

Suppose r is a positive integer such that there exists a set $B = \{\mathbf{b_1}, \cdots, \mathbf{b}_r\}$ of r linearly independent integer vectors such that the subalgebra $S(B)$ of $T_q(\mathbf{z}_{n,\mathbf{\Gamma}}, \mathbf{A}_{n,\mathbf{\Gamma}})$ generated by $\{\mathbf{z}_{n,\mathbf{\Gamma}}^{\mathbf{b}_i} : 1 \leq i \leq r\}$ is commutative. Then let $\mathbf{B}$ be the matrix with rows $\mathbf{b_1}, \mathbf{b_2}, \ldots \mathbf{b_r}$. Since the $\mathbf{b}_i$ are linearly independent, $\text{rank}(\mathbf{B}) = r$, and since $S(B)$ is commutative, $\mathbf{B}\mathbf{A}_{n,\mathbf{\Gamma}}\mathbf{B}^T = \mathbf{0}$.

We recall the following standard properties of rank (where the matrices are such that the products are defined):

(i) if Y is an $n \times n$ matrix of rank n then rank (XY) = rank (X) and rank (YZ) = rank (Z);

(ii) $\text{rank}(XY) + \text{rank}(YZ) \leq \text{rank}(Y) + \text{rank}(XYZ)$.

Taking $X = \mathbf{B}$, $Y = \mathbf{A}_{n,\mathbf{\Gamma}}$, $Z = \mathbf{B}^T$, and noting that $\mathbf{A}_{n,\mathbf{\Gamma}}$ has rank n, we get $2r \leq n$.

Finally we note that since all the elements of $\{(\mathbf{z}_{n,\mathbf{\Gamma}})_i : i \text{ odd}\}$ commute, we can achieve equality, and thus dim $T_q(\mathbf{z}_{n,\mathbf{\Gamma}}, \mathbf{A}_{n,\mathbf{\Gamma}}) = n/2$.

Now, if n is odd, let $\mathbf{\Gamma}' \in \{+,-\}^{n-2}$ such that $\mathbf{\Gamma}'_i = \mathbf{\Gamma}_i$ for $1 \leq i \leq n-2$. Then

$$T_q(\mathbf{z}_{n,\mathbf{\Gamma}}, \mathbf{A}_{n,\mathbf{\Gamma}}) = \begin{cases} T_q(\mathbf{z}_{n-1,\mathbf{\Gamma}'}, \mathbf{A}_{n-1,\mathbf{\Gamma}'})[z_n] & \text{if } \mathbf{\Gamma}_n = - \\ T_q(\theta(\mathbf{z}_{n-1,\mathbf{\Gamma}'}), \mathbf{A}_{n-1,\mathbf{\Gamma}'})[z_n] & \text{if } \mathbf{\Gamma}_n = + \end{cases}$$

Therefore, by Theorem 2.51 (iv) (b) or Theorem 2.50 (iv) (c), $\dim T_q(\mathbf{z}_{n,\mathbf{\Gamma}}, \mathbf{A}_{n,\mathbf{\Gamma}}) = (n-1)/2 + 1 = (n+1)/2$. □

Corollary 5.37. *Let* $\mathbf{\Gamma} \in \{-, +\}^{n-1}$*. Then:*

(i) $\dim(L^q_{n-1,\mathbf{\Gamma}})_{\mathcal{X}_{n-1,\mathbf{\Gamma}}} = (n-2)/2$ *if* n *is even.*

(ii) $\dim(L^q_{n-1})_{\mathcal{X}_{n-1,\mathbf{\Gamma}}} = (n-1)/2$ *if* n *is odd.*

(iii) $\dim(L^q_{n})_{\mathcal{X}_{n-1,\mathbf{\Gamma}}} = (n+2)/2$ *if* n *is even.*

(iv) $\dim(L^q_{n})_{\mathcal{X}_{n-1,\mathbf{\Gamma}}} = (n+1)/2$ *if* n *is odd.*

Proof. (i), (ii) By Lemma 5.35 (ii), $(L^q_{n-1})_{\mathcal{X}_{n-1,\mathbf{\Gamma}}} = T_q(\mathbf{z}_{n-1,\mathbf{\Gamma}}, \mathbf{A}_{n-1,\mathbf{\Gamma}})$, so apply Lemma 5.36.

(iii), (iv) By Lemma 5.35 (i), $(L^q_{n})_{\mathcal{X}_{n-1,\mathbf{\Gamma}}} = T_q(\mathbf{z}_{n-1,\mathbf{\Gamma}}, \mathbf{A}_{n-1,\mathbf{\Gamma}})[z_n;\alpha]$ for some (possibly identity) automorphism α of $T_q(\mathbf{z}_{n-1}, \mathbf{A}_{n-1})$, so apply Lemma 5.36 together with Theorems 2.51 (iv) (a) and 2.50 (iv) (b).

□

Lemma 5.38. *For* $i \geq 1$*,*

$$z_i\theta(z_i) = q\theta(z_i)z_i + (1-q)q^{\frac{i-1}{2}} \quad \textit{if } i \textit{ is odd;}$$

$$z_i\theta(z_i) = q\theta(z_i)z_i + (1-q)q^{\frac{i}{2}} \quad \textit{if } i \textit{ is even.}$$

Proof. If i is odd,

$$\begin{aligned} z_i\theta(z_i) &= z_i\theta(z_{i-1})x_{i+1} - z_i\theta(z_{i-2}) \\ &= \theta(z_{i-1})z_ix_{i+1} - \theta(z_{i-2})z_i \\ &= q\theta(z_{i-1})x_{i-1}z_i + (1-q)\theta(z_{i-1})z_{i-1} && \text{by Proposition 5.3 (ii)} \\ &\qquad - q\theta(z_{i-2})z_i - (1-q)\theta(z_{i-2})z_i \\ &= q\theta(z_{i-1})z_i + (1-q)q^{\frac{i-1}{2}} && \text{by Corollary 5.22.} \end{aligned}$$

If i is even,

$$\begin{aligned} z_i\theta(z_i) &= z_i\theta(z_{i-1})x_{i+1} - z_i\theta(z_{i-2}) \\ &= q\theta(z_{i-1})z_ix_{i+1} - \theta(z_{i-2})z_i \\ &= q\theta(z_{i-1})x_{i+1}z_i + (1-q)q\theta(z_{i-1})z_{i-1} && \text{by Proposition 5.3 (i)} \\ &\qquad - q\theta(z_{i-2})z_i - (1-q)\theta(z_{i-2})z_i \\ &= q\theta(z_i)z_i + (1-q)q^{\frac{i}{2}} && \text{by Corollary 5.22.} \end{aligned}$$

□

Corollary 5.39. *For* $a \geq 1$ *and* $i \geq 1$*,*

$$z_i^a\theta(z_i) = q^a\theta(z_i)z_i^a + (1-q)q^{\frac{i-1}{2}}[a]_q z_i^{a-1} \quad \textit{if } i \textit{ is odd;}$$

$$z_i^a\theta(z_i) = q^a\theta(z_i)z_i^a + (1-q)q^{\frac{i}{2}}[a]_q z_i^{a-1} \quad \textit{if } i \textit{ is even.}$$

Proof. Let $s(i) = \begin{cases} (1-q)q^{\frac{i-1}{2}} & \text{if } i \text{ is odd} \\ (1-q)q^{\frac{i}{2}} & \text{if } i \text{ is even} \end{cases}$, so $z_i\theta(z_i) = q\theta(z_i)z_i + s(i)$. Then we proceed by induction, with the case $a = 1$ being the just-stated equality. For $a > 1$,

$$\begin{aligned} z_i^a\theta(z_i) &= qz_i^{a-1}\theta(z_i)z_i + s(i)z_i^{a-1} \\ &= q^a\theta(z_i)z_i^a + qs(i)[a-1]_q z_i^{a-1} + s(i)z_i^{a-1} \quad \text{by induction} \\ &= q^a\theta(z_i)z_i^a + s(i)[a]_q z_i^{a-1}. \end{aligned}$$

□

Lemma 5.40. *Assume $q \in k$ is not a root of unity.*

Let R be a prime factor ring of L_n^q (including $R = L_n^q$), or let R be a factor ring of C_{n+1}^q (including $R = C_{n+1}^q$). Then

$$\dim R = \sup\{\dim(R_{\mathcal{X}_{n-1,\mathbf{\Gamma}}}) : \mathbf{\Gamma} \in \{-,+\}^{n-1}\}.$$

Proof. The proof is the same in both cases.

For $j = 0, \dots, n-1$, define $\mathcal{V}_{j,\mathbf{\Gamma}}$ to be the set of non-zero monomials in $\{(z_{n-1,\mathbf{\Gamma}})_i : n-j \leq i \leq n-1\}$. As in the proof of Lemma 5.35, repeated application of Lemma 2.44 and Lemma 2.42, using Corollary 4.9, together with Lemma 5.13 and Corollary 2.41 if R is neither L_n^q nor C_{n+1}^q, shows that $\mathcal{V}_{j,\mathbf{\Gamma}}$ is a right Ore set in R.

We say a j**-fold localisation** of R is a localisation of the form $R_{\mathcal{V}_{j,\mathbf{\Gamma}}}$ for $\mathbf{\Gamma} \in \{-,+\}^{n-1}$. The only 0-fold localisation of R is R itself while $\mathcal{V}_{j,\mathbf{\Gamma}} = \mathcal{X}_{j,\mathbf{\Gamma}}$ so our aim is to show that $\dim R = \sup\{\dim S\colon S \text{ a } (n-1)\text{-fold localisation of } R\}$.

Given a j-fold localisation S of R, there exist two $(j+1)$-fold localisations of R which contain S, which we denote S_- and S_+. (One of these is $R_{\mathcal{V}_{j+1,\mathbf{\Gamma}}}$ and the other is $R_{\mathcal{V}_{j+1,\mathbf{\Gamma}'}}$ where $\mathbf{\Gamma}' = \mathbf{\Gamma}$ except that $\mathbf{\Gamma}'_{n-(j+1)} = -\mathbf{\Gamma}_{n-(j+1)}$). If $(\mathbf{z}_{n-1,\mathbf{\Gamma}})_{n-j} = \theta^r(z_{n-j})$, then $S_- = S_{\theta^r(z_{n-(j+1)})}$ and $S_+ = S_{\theta^{r+1}(z_{n-(j+1)})}$, where for $y \in S$, S_y denotes the localisation of S at the set $\{y^i : i \in \mathbb{N}\}$.

We claim that $S_i \oplus S_+$ is a faithfully flat extension of S. Let M be an S-module which is $\theta^r(z_{n-j})$-torsion and $\theta^{r+1}(z_{n-j})$-torsion. If $M \neq 0$, let $a \in \mathbb{N}$ be minimal such that there exists $m \in M$ nonzero with $\theta^r(z_{n-j}^a)m = 0$ and $\theta^{r+1}(z_{n-j})m = 0$, and suppose $a \geq 1$. Then by Corollary 5.39, $s(i)[a]_q\theta^r(z_i^{a-1})m = 0$, where $s(i)$ is as Corollary 5.39. Both $s(i)$ and $[a]_q$ are non-zero scalars, so $\theta^r(z_i^{a-1})m = 0$, contradicting the minimality of a. Therefore $a = 0$ and $m = 0$, so $M = 0$. Therefore $S_i \oplus S_+$ is a faithfully flat extension of S.

Therefore, by Lemmas 2.52 and 2.53, $\dim S \leq \dim(S_i \oplus S_+) \leq \sup\{\dim S_-, \dim S_+\}$, while by Theorems 2.51 (v) and 2.50 (v), $\dim S_- \leq \dim S$ and $\dim S_+ \leq \dim S$. Therefore $\dim S = \sup\{\dim S_-, \dim S_+\}$. Combining this for all S, we're done. □

Theorem 5.41. *(a) If n is even,*

(i) $\dim L_n^q = (n+2)/2$*;*

(ii) $\dim L_n^q / z_n L_n^q = n/2$*;*

(iii) $\dim L_n^q / ((z_{n-1} - \lambda) L_n^q + z_n L_n^q) = (n-2)/2$ *for all* $\lambda \in k^\times$*.*

(b) If n is odd,

(i) $\dim L_n^q = (n+1)/2$*;*

(ii) $\dim L_n^q / (z_n - \lambda) L_n^q = (n-1)/2$*, for all* $\lambda \in k$*.*

Proof. We combine Lemma 5.35 and Corollary 2.41 to localise at $\mathcal{X}_{n-1,\mathbf{\Gamma}}$ for any $\mathbf{\Gamma}$, Lemma 5.37 to determine the dimension of the localisation and Lemma 5.40 to pass back to the original ring. □

5.7 Global dimensions in C_n^q

We aim to apply the results of Section 3.4 to determine the global dimension of C_n^q and some of its factor rings. Fix n odd and fix $q \in k^\times$ such that q is not a root of unity.

If $\mathbf{\Gamma} \in \{-, +\}^{n-2}$, assume without loss of generality that $\mathbf{\Gamma}_{n-2} = -$. (If $\mathbf{\Gamma}_{n-2} = +$ we apply θ^{-1} to all that follows). Then define $S = (C_n^q)_{\mathcal{X}_{n-2,\mathbf{\Gamma}}}$. Then we recall that by Lemma 5.35 (iii), S is an ambiskew polynomial ring with $A = (L_{n-2}^q)_{\mathcal{X}_{n-2,\mathbf{\Gamma}}}$, $u = q^{\frac{n-3}{2}} z_{n-2}^{-1} + \lambda + q z_{n-2}$, $x = \theta^{-1}(z_{n-1})$, and $y = z_{n-2}^{-1} z_{n-1}$. Letting $K = (L_{n-3}^q)_{\mathcal{X}_{n-3,\mathbf{\Gamma}}}$ or $K = (\theta(L_{n-3}^q))_{\mathcal{X}_{n-3,\mathbf{\Gamma}}}$ as appropriate, and $Z = k[z_{n-2}^{\pm 1}]$, we have $A = K \otimes_k Z$, and this K and Z satisfy the conditions of Lemma 3.41, with rgld $A = (n-1)/2$; also $\alpha(z_{n-2}) = q^{-1} z_{n-2}$, so Z is α-simple.

As in Section 3.4, write $T(u) := S/(xy - u)$, so $T(u)$ is a generalised Weyl algebra over A, and write $d =$ rgld A.

Corollary 5.42. *rgld* $S = d + 2 = (n+3)/2$*.*

Proof. We apply Theorem 3.42. The existence of appropriate λ and j is shown in Lemma 5.26, while by Lemma 5.37 rgld $A = (n-1)/2$, so rgld $S = (n-1)/2 + 2 = (n+3)/2$. □

Corollary 5.43. *rgld* $T(u) = \infty$ *iff u has a repeated irreducible factor.*

Proof. Consider the ambiskew polynomial ring $S' = R(Z, \alpha, u - \alpha(u))$, and write $T'(u) = S'/(xy - u)R$. We note that $z_1, \ldots, z_{n-3}$ commute or skew-commute with x, y and z_{n-2} and each other, so $T(u)$ can be written as a iterated skew Laurent extension of $T'(u)$. So by Theorem 2.50 (iv) (c), rgld $T(u) = \infty$ iff rgld $T'(u) = \infty$. And by [24, 7.8] rgld $T'(u) = \infty$ iff u has a repeated irreducible factor. □

Corollary 5.44. *rgld $T(u) \geq d = (n-1)/2$ for any u.*

Proof. Let $\mathbf{v} = \mathbf{z}_{n-2,\mathbf{\Gamma}}$ (so if $\mathbf{\Gamma}_i = -$ for all i, $\mathbf{v} = \mathbf{z}_{n-2}$), and let J be the right ideal of A generated by $\{(\mathbf{v}_2 - 1,\ \mathbf{v}_4 - 1,\ \ldots,\ \mathbf{v}_{n-3} - 1\}$. Clearly, $\alpha(J) = J$. Let C denote the (commutative) subalgebra of A generated by $\{\mathbf{v}_1, \mathbf{v}_3, \ldots, \mathbf{v}_{n-2}\}$. We note that, as a (right) C-module, $A/J \cong C$, and so is finitely generated and free. (To see this, it's clear that as an C-module, A/J is a quotient of C, but if x is a non-zero element of J it must have non-zero degree in some $\mathbf{v}_{2i}$, and so $J \cap C = \{0\}$).

Then by $(n-3)/2$ applications of [33, 7.9.16], we have pd $(A/J)_A = (n-3)/2$, and so by Lemma 3.45, rgld $T(u) \geq (n-1)/2$ for any u. □

Corollary 5.45. *Let $u = q^{\frac{n-3}{2}} z_{n-2}^{-1} + \lambda + q z_{n-2}$. Then there are three cases:*

(i) $\lambda = \pm q^{\frac{n-3}{4}} q^{-\frac{a-1}{2}} (q^a + 1)$ *for some integer $a \geq 1$, in which case $T(u)$ is not simple and rgld $T(u) = d + 1 = (n+1)/2$;*

(ii) $\lambda = \pm 2 q^{\frac{n-1}{4}}$*, in which case $T(u)$ is simple and rgld $T(u) = \infty$. (We note that this is the case $a = 0$ from (i));*

(iii) otherwise, $T(u)$ is simple and rgld $T(u) = d = (n-1)/2$.

Proof. (ii) $u = q^{\frac{n-3}{2}} z_{n-2}^{-1} + \lambda + q z_{n-2}$. By the quadratic formula, u has a repeated irreducible factor iff $\lambda^2 - 4q^{\frac{n-1}{2}} = 0$, i.e. $\lambda = \pm 2q^{\frac{n-1}{4}}$. So the result follows by Corollary 5.43.

(i), (iii) There exists a maximal ideal N of Z containing both u and $\alpha^j(u)$ iff $Au + A\alpha^j(u) \neq A$. By 5.26, this occurs iff $\lambda = \pm q^{\frac{n-3}{4}} q^{-\frac{a-1}{2}} (q^a + 1)$ for some integer $a \geq 1$ (which then equals j), so the result follows by Corollary 3.44. This also shows that rgld $T(u) \leq d$ otherwise.

□

Corollary 5.46. *(a) rgld $C_n^q = (n+3)/2$.*

(b) Let $R = C_n^q/(\Omega - \lambda) C_n^q$. Then there are three cases:

(i) $\lambda = \pm q^{\frac{n-3}{4}} q^{-\frac{a-1}{2}} (q^a + 1)$ *for some integer $a \geq 1$ in which case R is not simple and rgld $R = (n+1)/2$;*

(ii) $\lambda = \pm 2q^{\frac{n-1}{4}}$ *in which case R is simple and rgld $R = \infty$;*

(iii) otherwise, R is simple and rgld $R = (n-1)/2$.

Proof. This is immediate by applying Lemma 5.40 to the previous results from this section.

□

5.8 Krull and global dimensions in C_3^q

We apply the results of [24] to $(C_3^q)_{z_1}$ by considering it as an ambiskew polynomial ring as in Lemma 5.24. By Lemma 5.40, $\dim((C_3^q)/I)_{z_1} = \dim(C_3^q)/I$, for any prime ideal I of C_3^q, where dimdenotes either Krull dimension or right global dimension.

Write $R = C_3^q$ and $S = (C_3^q)_{z_1}$.

Lemma 5.47. *K. dim $S/I = 2 - \text{ht } I$.*

Proof. If ht $I = 2$, S/I is finite dimensional over k, and thus Artinian, so K. dim $S/I = 0$.

By [24, 5.4], if ht $I = 1$ then K. dim $S/I = 1$.

When $I = 0$ we use [25, 5.6] to show that all finite dimensional right S-modules are semisimple; then by [24, 3.7], we have K.dim $S = 2$.

The condition in [25, 5.6] is that for every maximal ideal M of A there is at most one positive integer d such that $u - \alpha^d(u) \in M$, and if such an integer exists, $M^2 + (u - \alpha^d(u))A = M$.

For the first part, $u - \alpha^d(u) = qz_1(1 - q^{-d}) + z^{-1}(1 - q^d)$. By the quadratic formula (assuming char $k \neq 2$) this has roots

$$\pm\frac{\sqrt{-4q(1-q^{-d})(1-q^d)}}{2q(1-q^{-d})}$$

If two roots of this form are equal then we must have $(1-q^d)(1-q^{-s}) = (1-q^{-d})(1-q^s) = q^{d-s}(q^d-1)(q^{-s}-1)$ for some $d, s \in \mathbb{N}$. Thus $q^{d-s} = 1$ and so since q is not a root of unity, $d = s$. The requirement $M^2 + (u - \alpha^d(u))A = M$ simply requires that $u - \alpha^d(u)$ doesn't have a repeated root, which again since q is not a root of unity is the case. □

Corollary 5.48. *K.dim $R/I = 2 - \text{ht } I$.*

Corollary 5.49. *(a) rgld $R = 3$.*

(b) Let $I = (\Omega - \lambda)R$. Then there are three cases:

(i) $\lambda = \pm q^{-\frac{a-1}{2}}(q^a + 1)$ for some integer $a \geq 1$ in which case R/I is not simple and rgld $S/I = 2$;

(ii) $\lambda = \pm 2q^{\frac{1}{2}}$ in which case S/I is simple and rgld $R/I = \infty$ (we note that this is the case $a = 0$ from (i));

(iii) otherwise, R/I is simple and rgld $R/I = 1$.

(c) If I is a height 2 prime in R, rgld $R/I = 0$.

Proof. (a), (b) This is immediate from Corollary 5.46, or by applying the results of [24].

(c) If I is a height 2 prime ideal in S, S/I is simple Artinian, so by the Artin-Wedderburn theorem it is a matrix ring over a division ring, and so by Theorem 2.50 (ii), rgld $S/I = 0$; then we apply Lemma 5.40 to get rgld $R/I = 0$.

□

6 Poisson algebras associated to connected quantized Weyl algebras

For each positive integer n, $(L_n^q)_{q\neq 0}$ is a family of noncommutative algebras in the sense of Definition 2.74, with L_n^1 commutative, so we can construct the semiclassical limit Poisson algebra, which we denote L_n^P. As a commutative ring, $L_n^P = k[x_1, \cdots, x_n]$, with Poisson bracket given by

$$\{x_i, x_{i+1}\} = 1 + x_{i+1}x_i \quad \text{for } 1 \leq i < n;$$
$$\{x_i, x_j\} = x_j x_i \quad \text{if } 1 \leq i < i+1 < j \leq n \text{ and } j-i \text{ is odd};$$
$$\{x_i, x_j\} = -x_j x_i \quad \text{if } 1 \leq i < i+1 < j \leq n \text{ and } j-i \text{ is even}.$$

Similarly, for each odd positive integer n, $(C_n^q)_{q\neq 0}$ is a family of algebras in the sense of Definition 2.74, with C_n^1 commutative, so we can construct the semiclassical limit Poisson algebra, which we denote C_n^P. As a commutative ring, $C_n^P = k[x_1, \cdots, x_n]$, with Poisson bracket given by

$$\{x_i, x_{i+1}\} = 1 + x_{i+1}x_i \quad \text{for } 1 \leq i < n;$$
$$\{x_n, x_1\} = 1 + x_1 x_n$$
$$\{x_i, x_j\} = x_j x_i \quad \text{if } 1 \leq i < i+1 < j \leq n \text{ and } j-i \text{ is odd};$$
$$\{x_i, x_j\} = -x_j x_i \quad \text{if } 1 \leq i < i+1 < j \leq n,\ j-i \text{ is even, and } (i,j) \neq (1,n).$$

The aim of this section is to determine the Poisson prime ideals of L_n^P and C_n^P. For the former, they precisely correspond with the prime ideals in the corresponding noncommutative algebra (at least for generic q), but for the latter they do not. (Since C_n^P has prime ideals which are not completely prime, this is unsurprising).

6.1 Analogues of the normal elements in L_n^P and C_n^P

Definition 6.1. As in the noncommutative case, L_m^P embeds naturally into L_n^P and C_n^P if $m < n$. Then again as in the noncommutative case, we define $z_0 = 1$, $z_1 = x_1$, and for $n \geq 3$, define $z_n \in L_n^P$ by $z_n = z_{n-1}x_n - z_{n-2}$.

Similarly, we define $\theta : C_n^P \to C_n^P$ to be the Poisson automorphism given by $x_i \mapsto x_{i+1}$ for $1 \leq i \leq n-1$ and $x_n \mapsto x_1$, and then we define $C_n^P \ni \Omega_n := z_{n-1}x_n - z_{n-2} - \theta(z_{n-2})$. (As before, if the context is clear then we will write just Ω).

Remark. If we treat L_n^P and L_n^q for $q \neq 0$ as factor rings of a larger algebra A as in Definition 2.74, then the $z_i \in L_n^P$ are images of the same elements of A as the $z_i \in L_n^q$ are, and so we can read off brackets involving the z_i from calculations of commutators in L_n^q. The same is true for $\Omega_n \in C_n^P$.

Proposition 6.2. *(i) For* $1 \le i \le j$, $\{x_i, z_j\} = \begin{cases} 0 & \text{if } j \text{ is odd;} \\ x_i z_j & \text{if } i \text{ is odd and } j \text{ is even;} \\ -x_i z_j & \text{if } i \text{ is even and } j \text{ is even.} \end{cases}$

(ii) For all $1 \le i$, $\{z_i, x_{i+1}\} = \begin{cases} -z_{i-1} & \text{if } i \text{ is even;} \\ z_i x_{i+1} - z_{i-1} & \text{if } i \text{ is odd.} \end{cases}$

(iii) For all $1 \le i$, $\{x_1, \theta(z_i)\} = \begin{cases} -\theta^2(z_{i-1}) & \text{if } i \text{ is even;} \\ x_1\theta(z_i) - \theta^2(z_{i-1}) & \text{if } i \text{ is odd.} \end{cases}$

(iv) Denote by $\mathbf{z}_n$ *the set* $\{z_1, \dots, z_n\}$

Let $A_{ij} = \begin{cases} 0, & \text{if } max\{i,j\} \text{ is odd or both } i,j \text{ are even} \\ 1, & \text{if } i \text{ odd, } j \text{ even, } i < j \\ -1, & \text{if } i \text{ even, } j \text{ odd, } j < i \end{cases}$,

and then let $\mathbf{A}_n = (A_{ij})_{i,j=1}^n$. *(This is the same as in Definition 5.10).*

Then for $1 \le i \le j$, $\{z_i, z_j\} = A_{ij} z_i z_j$, *so* $S_P(\mathbf{z}_n, \mathbf{A}_n) \subset L_n^P$.

(v) For $1 \le i \le n$, $\{x_i, \Omega_n\} = 0$.

(vi) In C_n^P, $\{z_{n-1}, x_n\} = z_{n-2} - \theta(z_{n-2})$.

(vii) In C_n^P, $\{\theta^{-1}(z_{n-1}), z_{n-1}\} = \theta^{-1}(z_{n-1})z_{n-1} - 1 + z_{n-2}^2$.

Proof. (i) This comes from Proposition 5.2 (iii).

(ii) This comes from Proposition 5.2 (iv).

(iii) This comes from Proposition 5.3 (i), (ii).

(iv) This comes from Proposition 5.3 (iii), (iv).

(v) This comes from Theorem 5.9.

(vi) This comes from Lemma 5.6.

(vii) This comes from Lemma 5.23.

□

6.2 Poisson prime ideals in L_n^P

Lemma 6.3. *(This is analogous to Lemma 5.11). Let $\mathcal{X}_n$ denote the set of non-zero scalar multiples of non-zero monomials in $\mathbf{z}_n$, and for n odd, let $\mathcal{Y}_n$ denote the multiplicative closure of $\mathcal{X}_{n-1} \cup k[z_n]^*$.*

(i) $(L_n^P)_{\mathcal{X}_{n-1}} = T_P(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[z_n]$ *where* $\{z_i, z_n\} = A_{in} z_i z_n$. *(We note that if n is odd then z_n is Poisson central in this ring).*

(ii) For n odd, $(L_n^P)_{\mathcal{Y}_n} = T_P(k(z_n), \mathbf{z}_{n-1}, \mathbf{A}_{n-1})$.

Proof. (i) Let B denote $T_P(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[z_n]$ with the given Poisson bracket. Then we have $B = S_P(k, \mathbf{z}_n, \mathbf{A}_n)_{\mathcal{X}_{n-1}}$, and since $S_P(k, \mathbf{z}_n, \mathbf{A}_n) \subset L_n^P$, $B \subset (L_n^P)_{\mathcal{X}_{n-1}}$, and the calculation $x_i = z_{i-1}^{-1}(z_i - z_{i-2})$ shows $x_i \in B$ for $1 \leq i \leq n$, so $B = (L_n^P)_{\mathcal{X}_{n-1}}$.

(ii) $S_P(k, \mathbf{z}_n, \mathbf{A}_n)_{\mathcal{Y}_n} = T_P(k(z_n), \mathbf{z}_{n-1}, \mathbf{A}_{n-1})$, and so as before $T_P(k(z_n), \mathbf{z}_{n-1}, \mathbf{A}_{n-1}) \subset (L_n^P)_{\mathcal{Y}_n}$, and $x_i \in T_P(k(z_n), \mathbf{z}_{n-1}, \mathbf{A}_{n-1})$ for $1 \leq i \leq n$, so $T_P(k(z_n), \mathbf{z}_{n-1}, \mathbf{A}_{n-1}) = (L_n^P)_{\mathcal{Y}_n}$.

□

Lemma 6.4. *(This is analogous to Lemma 5.12). Let K be any field of characteristic* 0. *Then $T_P(K, \mathbf{z}_n, \mathbf{A}_n)$ is Poisson simple if and only if n is even.*

Proof. The matrix $\mathbf{A}_n$ and the condition from Theorem 2.78 are the same as in Lemma 5.12, where we already showed that $\mathbf{A}_n$ satisfies that condition. □

Lemma 6.5. *(This is analogous to Lemma 5.13). Let I be a Poisson ideal in some Poisson algebra $R \supset L_n^P$ (e.g. $R = L_m^P$, $m \geq n$, or $R = C_m^P$, $m > n$ and m odd). If I contains z_i for some $1 \leq i \leq n-1$, then I contains* 1.

Proof. By Proposition 6.2 (ii), if $z_i \in I$ then $z_{i-1} \in I$, so by induction, since $z_0 = 1$, $1 \in I$. □

Corollary 6.6. *(This is analogous to Corollary 5.14). Let I be a prime ideal of L_n^P. Then $I \cap \mathcal{X}_{n-1} = \emptyset$.*

Proof. Suppose I contains an element of $\mathcal{X}_{n-1}$. Then since I is prime, I contains z_i for some $1 \leq i \leq n-1$, and so by Lemma 6.5, I contains 1, a contradiction. □

Lemma 6.7. *(This is analogous to Lemma 5.16). If n is odd, $(z_n - \lambda)L_n^P$ is a Poisson prime ideal of L_n^P for each $\lambda \in k$.*

If n is even, $z_n L_n^P$ is a Poisson prime ideal of L_n^P.

Proof. First, by Proposition 6.2 (i), for n odd $(z_n - \lambda)$ is Poisson central in L_n^P, so $(z_n - \lambda)L_n^P$ is a Poisson ideal of L_n^P, while for n even $\{z_n, L_n^P\} \subset z_n L_n^P$ so $z_n L_n^P$ is a Poisson ideal of L_n^P. Therefore, if we can show these ideals are prime ideals then we're done by Proposition 2.70.

We show this by induction using Lemma 2.31; the case $n = 0$ is trivial. If $n > 0$ and then in the setting of that Lemma we have $c = z_n - \lambda$; $d = z_{n-1}$; and $e = -z_{n-2} - \lambda$, where $\lambda = 0$ if n is even and $\lambda \in k$ if n is odd. By induction, $z_{n-1}L_{n-1}^P$ is a completely prime ideal in L_{n-1}^P. Since z_{n-1} has degree 1 in x_{n-1} but e has degree 0 in x_{n-1}, e is non-zero, and so regular, modulo $z_{n-1}L_{n-1}^P$. Therefore we can apply Lemma 2.31, and cL_n^P is a completely prime ideal in L_n^P. □

Theorem 6.8. *(This is analogous to Theorem 5.17). Let n be odd, and assume k is algebraically closed and characteristic* 0. *Then the non-trivial Poisson prime ideals of L_n^P are the ideals $(z_n - \lambda)L_n^P$ for $\lambda \in k$.*

Proof. We know that L_n^P is a domain, so together with Lemma 6.7, the given ideals are all Poisson prime ideals of L_n^P.

By Lemma 6.3 (i), $(L_n^P)_{\mathcal{X}_{n-1}} = T_P(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[z_n]$. By Lemma 6.4, $T_P(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})$ is a simple Poisson algebra, so $(z_n - \lambda)(L_n^P)_{\mathcal{X}_{n-1}}$ is a maximal Poisson ideal of $(L_n^P)_{\mathcal{X}_{n-1}}$ for each $\lambda \in k$.

Also, by Lemma 6.3 (ii), $((L_n^P)_{\mathcal{X}_{n-1}})_{k[z_n]^*} = (L_n^P)_{\mathcal{Y}_n} = T_P(k(z_n), \mathbf{z}_{n-1}, \mathbf{A}_{n-1})$, which is simple by Lemma 6.4, so any Poisson ideal of $(L_n^P)_{\mathcal{X}_{n-1}}$ must contain an element of $k[z_n]^*$. Therefore PSpec $(L_n^P)_{\mathcal{X}_{n-1}}$ consists of 0 together with $(z_n - \lambda)(L_n^P)_{\mathcal{X}_{n-1}}$ for each $\lambda \in k$.

By Proposition 2.71, there is a one-to-one correspondence between $\{P \in \text{PSpec } L_n^P : P \cap \mathcal{X}_{n-1} = \emptyset\}$ and PSpec $(L_n^P)_{\mathcal{X}_{n-1}}$ given by $P \mapsto P(L_n^P)_{\mathcal{X}_{n-1}}$. Since $(z_n - \lambda)L_n^P$ maps to $(z_n - \lambda)(L_n^P)_{\mathcal{X}_{n-1}}$ under this correspondence, using the fact that this correspondence is bijective we therefore have $\{P \in \text{PSpec } L_n^P : P \cap \mathcal{X}_{n-1} = \emptyset\} = \{(z_n - \lambda)L_n^P : \lambda \in k\} \cup \{0\}$.

But by Corollary 6.6, $\{P \in \text{PSpec } L_n^P : P \cap \mathcal{X}_{n-1} = \emptyset\} = \text{PSpec } L_n^P$, and so we're done. □

Lemma 6.9. *(This is analogous to Lemma 5.18). Let n be even, $\lambda \in k^\times$, and define $P_\lambda := z_n L_n^P + (z_{n-1} - \lambda)L_n^P$. Then P_λ is a Poisson ideal of L_n^P with L_n^P/P_λ isomorphic to $L_{n-1}^P/(z_{n-1} - \lambda)L_{n-1}^P$; in particular P_λ is a Poisson prime ideal of L_{n-1}^P.*

Proof. Firstly, z_{n-1} is Poisson central modulo $z_n L_n^P$, since it Poisson commutes with x_i for $1 \leq i \leq n-1$, and by Proposition 6.2 (ii) $\{z_{n-1}, x_n\} = z_{n-1}x_n - z_{n-2} = z_n \in z_n L_n^P$. Therefore P_λ is a Poisson ideal of L_n^q.

Secondly, since $x_n = \lambda^{-1}z_{n-2}$ modulo P_λ, L_n^P/P_λ is generated by $\bar{x_1}, \dots, \bar{x_n}$. So there are homomorphisms $L_n^P/P_\lambda \to L_{n-1}^P/(z_{n-1}-\lambda)L_{n-1}^q$ and vice versa given by $\bar{x_i} \mapsto \bar{x_i}$ for $1 \le i \le n-1$ (it is easy to check that these are well-defined), and these are inverses to each other, and so isomorphisms. □

Theorem 6.10. *(This is analogous to Theorem 5.19). Let n be even, and assume k is algebraically closed and characteristic 0. Then the prime ideals in L_n^P are 0, $z_nL_n^P$ and $(z_{n-1}-\lambda)L_n^P + z_nL_n^P$ for each $\lambda \in k^\times$.*

Proof. We know that L_n^P is a domain, so together with Lemmas 6.7 and 5.18, the given ideals are all (completely) prime ideals of L_n^q.

By Lemma 6.3 (i) $(L_n^P)_{\mathcal{X}_{n-1}} = T_P(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[z_n]$. Let $T = T_P(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[z_n]$ and consider PSpec T. The set $\mathcal{U} = \{z_n^i : i \in \mathbb{N}\}$ is an Ore set in T, and $T_{\mathcal{U}} = T_P(k, \mathbf{z}_n, \mathbf{A}_n)$, which is Poisson simple by Lemma 6.4. Therefore any Poisson ideal of T must contain some power of z_n, so any Poisson prime ideal of T must contain z_n.

Further, the ideal z_nT is a Poisson prime ideal of T, since $T/z_nT \cong T_P(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})$ which is a domain. Also, $T_P(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1}) = T_P(k, \mathbf{z}_{n-2}, \mathbf{A}_{n-2})[z_{n-1}^{\pm 1}]$, which is a Laurent polynomial ring over a Poisson simple ring by Lemma 6.4, so as in the proof of Theorem 6.8, the Poisson prime ideals of $T_P(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})$ are 0 and the ideals generated by $(z_n - \lambda)$ for $\lambda \in k^\times$.

Therefore the prime spectrum of T consists of: 0; z_nT; and $z_nT + (z_{n-1}-\lambda)T$, for each $\lambda \in k^\times$.

Now we apply Proposition 2.71, recalling that $T = (L_n^P)_{\mathcal{X}_{n-1}}$: there is a one-to-one correspondence between $\{P \in \text{PSpec } L_n^P : P \cap \mathcal{X}_{n-1} = \emptyset\}$ and PSpec T given by $P \mapsto PT$. Since by Corollary 6.6, $\{P \in \text{PSpec } L_n^P : P \cap \mathcal{X}_{n-1} = \emptyset\} = \text{PSpec } L_n^q$, this correspondence is between PSpec L_n^P and PSpec T. But this correspondence sends the known prime ideals of L_n^P, as listed above, to the prime ideals of T, and therefore those ideals are the only prime ideals of L_n^P. □

6.3 Poisson prime ideals in C_n^P

Lemma 6.11. *Let I be a Poisson ideal of C_n^P containing $\mu_1\theta^{i+1}(z_{n-i-2}) + \mu_2 z_i$, where $0 \le i \le n-2$ and $\mu_i \in k$ for $i = 1, 2$. Then I also contains $\mu_1\theta^{i+2}(z_{n-i-3}) + \mu_2 z_{i+1}$.*

Therefore, if I contains $\mu_1\theta(z_{n-2}) + \mu_2$, I also contains $\mu_1 + \mu_2 z_{n-2}$.

Proof. When i is odd, using Proposition 6.2 (ii) and (iii),

$$\begin{aligned}\{\mu_1\theta^{i+1}(z_{n-i-2})+\mu_2 z_i, x_{i+1}\} &= -\mu_1\theta^{i+1}(z_{n-i-2})x_{i+1}+\mu_1\theta^{i+1}(z_{n-i-3})-\mu_2 z_{i-1}\\ &= -(\mu_1\theta^{i+1}(z_{n-i-2})+\mu_2 z_i)x_{i+1}\\ &\quad\; +\mu_1\theta^{i+1}(z_{n-i-3})+\mu_2 z_i x_{i+1}-\mu_2 z_{i-1}\\ &= -(\mu_1\theta^{i+1}(z_{n-i-2})+\mu_2 z_i)x_{i+1}+\mu_1\theta^{i+1}(z_{n-i-3})+\mu_2 z_{i+1}.\end{aligned}$$

Therefore $\mu_1\theta^{i+1}(z_{n-i-3})+\mu_2 z_{i+1}\in I$ as desired.

When i is even, again using Proposition 6.2 (ii) and (iii),

$$\begin{aligned}\{\mu_1\theta^{i+1}(z_{n-i-2})+\mu_2 z_i, x_{i+1}\} &= \mu_1\theta^{i+1}(z_{n-i-3})+\mu_2 z_i x_{i+1}-\mu_2 z_{i-1}\\ &= \mu_1\theta^{i+1}(z_{n-i-3})-\mu_2 z_{i+1}.\end{aligned}$$

Therefore $\mu_1\theta^{i+1}(z_{n-i-3})+\mu_2 z_{i+1}\in I$ as desired.

The second part follows by repeated application of the first. □

Lemma 6.12. *Assume k is algebraically closed. Let I be a Poisson prime ideal of C_n^P which contains z_{n-1}. Then I contains $\Omega-\lambda$ for some $\lambda\in k$.*

Proof. If $I=C_n^P$ this is trivial. Otherwise, by Proposition 6.2 (vi), $z_{n-2}-\theta(z_{n-2})\in I$. But then $I\cap L_{n-1}^P$ is a nontrivial Poisson ideal of L_{n-1}^P strictly containing $z_{n-1}L_{n-1}^P$, and so by Theorem 6.10, $I\cap L_{n-1}^P$ must contain some non-unit element of $k[z_{n-2}^{\pm1}]$. Then since I is a prime ideal, it must contain $z_{n-2}-\mu$ for some $\mu\in k$.

Putting these together gives $z_{n-1}x_n+z_{n-2}-\theta(z_{n-2})-2(z_{n-2}-\mu)\in I$, and therefore $\Omega+2\mu\in I$. □

Lemma 6.13. *Let I be a nontrivial Poisson ideal of C_n^P which contains z_{n-1} and $\Omega-\lambda$ for some $\lambda\in k$. Then $\lambda=\pm2$.*

Proof. By Proposition 6.2 (vi), $z_{n-2}-\theta(z_{n-2})\in I$, while by the definition of Ω, $-z_{n-2}-\theta(z_{n-2})-\lambda\in I$. Putting these together, $-2z_{n-2}-\lambda\in I$ and $-2\theta(z_{n-2})-\lambda\in I$; therefore, by Lemma 6.11, $-2-\lambda z_{n-2}\in I$. But then if $\lambda\neq\pm2$, $I=C_n^P$, contradicting our assumption. □

Lemma 6.14. *(i) Let $\mathcal{Z}_n$ denote the multiplicative closure of $\mathcal{X}_{n-1}\cup k[\Omega]^*$. Then $(C_n^P)_{\mathcal{Z}_n}=T_P(k(\Omega),\mathbf{z}_{n-1},\mathbf{A}_{n-1})$.*

(ii) $(C_n^P)_{\mathcal{X}_{n-1}}=T_P(k,\mathbf{z}_{n-1},\mathbf{A}_{n-1})[\Omega]$, where Ω is Poisson central.

(iii) $(C_n^P)_{\mathcal{X}_{n-2}}=T_P(k,\mathbf{z}_{n-2},\mathbf{A}_{n-2})[z_{n-1}][\theta^{-1}(z_{n-1})]$, with appropriate Poisson bracket.

Proof. For the first two, as in the proof of Lemma 6.3, $S_P(k,\mathbf{z}_{n-1},\mathbf{A}_{n-1})[\Omega]\subset C_n^P$, giving $T_P(k(\Omega),\mathbf{z}_{n-1},\mathbf{A}_{n-1})\subset(C_n^P)_{\mathcal{Z}_n}$ and $T_P(k,\mathbf{z}_{n-1},\mathbf{A}_{n-1})[\Omega]\subset(C_n^P)_{\mathcal{X}_{n-1}}$

Then $x_i = z_i^{-1}(z_{i+1} + z_{i-1})$ gives $x_i \in T_P(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[\Omega] \subset T_P(k(\Omega), \mathbf{z}_{n-1}, \mathbf{A}_{n-1})$ for $1 \leq i \leq n-1$; this then gives $\theta(z_{n-2}) \in T_P(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[\Omega]$, and so $x_n = z_{n-2}^{-1}(\Omega + z_{n-2} + \theta(z_{n-2})) \in T_P(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[\Omega]$, giving equality in the two equations in the previous paragraph.

For the third, $S_P(k, \mathbf{z}_{n-2}, \mathbf{A}_{n-2})[z_{n-1}][\theta^{-1})(z_{n-1})] \subset C_n^P$, so $B := T_P(k, \mathbf{z}_{n-2}, \mathbf{A}_{n-2})[z_{n-1}][\theta^{-1})(z_{n-1})] \subset (C_n^P)_{\mathcal{X}_{n-2}}$; the relation $x_i = z_{i-1}^{i-1}(z_i + z_{i-2})$ gives $x_i \in B$ for $1 \leq i \leq n-1$, while the relation $x_n = z_{n-2}^{-1}(\theta^{-1}(z_{n-1}) + \theta(z_{n-2}))$ gives $x_n \in B$. □

Lemma 6.15. *(This is analogous to Lemma e̊fomegaminuslambdaiscp).*

For any $\lambda \in k$, $(\Omega - \lambda)C_n^P$ is a Poisson prime ideal in C_n^P.

Proof. Since $\Omega - \lambda$ is Poisson central in C_n^P, $(\Omega - \lambda)C_n^P$ is a Poisson ideal of C_n^P, so, if we can show that it's a prime ideal then we're done by Proposition 2.70.

We show this using Lemma 2.31. In the setting of that Lemma, we have $R = C_n^P$, $A = L_{n-1}^q$, $c = \Omega - \lambda$, $d = z_{n-1}$ and $e = -z_{n-2} - \theta(z_{n-2}) - \lambda$. By considering total degree, we cannot have $e \in dR$, and so, since by Lemma 5.16, R/dR is a domain, e is regular modulo Ad. Therefore we can apply Lemma 2.31 to show that R/cR is a domain, that is, $(\Omega - \lambda)C_n^P$ is a prime ideal in C_n^P. □

Theorem 6.16. *(This is analogous to Theorem 5.30). The Poisson prime ideals in C_n^P (n odd) are: 0, $(\Omega - \lambda)C_n^P$, and two exceptional Poisson maximal ideals containing $\Omega \pm 2$.*

Proof. By Lemma 6.14, $(C_n^P)_{\mathcal{Z}_n} = T_P(k(\Omega), \mathbf{z}_{n-1}, \mathbf{A}_{n-1})$, which is simple by Lemma 6.4. Therefore any Poisson prime ideal P of C_n^P must contain some element of $\mathcal{Z}_n$, and so either $\Omega - \lambda$ for some $\lambda \in k$, or some element of $\mathbf{z}_{n-1}$. If $z_{n-1} \in P$ then there exists $\lambda \in k$ such that $\Omega - \lambda \in P$ by Lemma 6.12, while if $z_i \in P$ for $1 \leq i < n-1$, then $1 \in P$ by Lemma 6.5. Therefore any Poisson prime ideal of C_n^P must contain $\Omega - \lambda$, and since, by Lemma 6.14, $(C_n^P)_{\mathcal{X}_{n-1}} = T_P(k, \mathbf{z}_{n-1}, \mathbf{A}_{n-1})[\Omega]$, $(\Omega - \lambda)(C_n^P)_{\mathcal{X}_{n-1}}$ is a Poisson maximal ideal in $(C_n^P)_{\mathcal{X}_{n-1}}$.

Therefore by Proposition 2.71, noting that by Lemma 6.15, $(\Omega - \lambda)C_n^P$ is a Poisson prime ideal in C_n^P, we obtain $\{P \in \text{PSpec } C_n^P : P \cap \mathcal{X}_{n-1}\} = \{0\} \cup \{(\Omega - \lambda)C_n^P : \lambda \in k\}$.

By Lemmas 6.5, 6.12, and 6.13, the only situation in which a Poisson prime ideal of C_n^P can have non-empty intersection with $\mathcal{X}_{n-1}$ is if it contains $\Omega \pm 2$ and z_{n-1}.

Let $\lambda = \pm 2$, and I be a Poisson prime ideal of C_n^P containing $\Omega - \lambda$ and z_{n-2}. Then applying the logic of the previous paragraph to $\theta^{-1}(C_n^P)$, I must also contain $\theta^{-1}(z_{n-2})$. And by the proof of Lemma 6.13, I must also contain $2z_{n-2} + \lambda$.

Therefore any Poisson prime ideal of C_n^P strictly containing $(\Omega - \lambda)C_n^P$ must contain z_{n-1}, $\theta^{-1}(z_{n-1})$ and $2z_{n-2} + \lambda$. Therefore the same must be true of any Poisson prime ideal of $(C_n^P)_{\mathcal{X}_{n-2}}$ strictly containing $(\Omega - \lambda)(C_n^P)_{\mathcal{X}_{n-2}}$.

By Lemma 6.14, $(C_n^P)_{\mathcal{X}_{n-2}} = T_P(k, \mathbf{z}_{n-2}, \mathbf{A}_{n-2})[z_{n-1}][\theta^{-1}(z_{n-1})]$, with appropriate Poisson bracket. Let I be the ideal of $(C_n^P)_{\mathcal{X}_{n-2}}$ generated by z_{n-1}, $\theta^{-1}(z_{n-1})$, and $2z_{n-2}+\lambda$. This is a Poisson ideal: the only non-trivial check is that $\{\theta^{-1}(z_{n-1}), z_{n-1}\} = \theta^{-1}(z_{n-1})z_{n-1}-1+z_{n-2}^2 \in I$, but this holds for either value of λ since $z_{n-2}^2 - 1 = (z_{n-2}-1)(z_{n-2}+1)$. Therefore $(C_n^P)_{\mathcal{X}_{n-2}}/I = T_P(k, \mathbf{z}_{n-2}, \mathbf{A}_{n-2})$, which is Poisson simple, and so I is a Poisson maximal ideal in $(C_n^P)_{\mathcal{X}_{n-2}}$. Further, I contains $\Omega - \lambda$, and we've already shown that is the only Poisson prime ideal of $(C_n^P)_{\mathcal{X}_{n-2}}$ strictly containing $(\Omega-\lambda)(C_n^P)_{\mathcal{X}_{n-2}}$.

Therefore, by Proposition 2.71, $\{P \in \text{PSpec } C_n^P : P \cap \mathcal{X}_{n-2} = \emptyset\}$ consists of the ideals described in the statement of the Theorem. But by Lemma 6.5, $\{P \in \text{PSpec } C_n^P : P \cap \mathcal{X}_{n-2} = \emptyset\} = \text{PSpec } C_n^P$, and so we're done. □

Remark. There is a general conjecture that, given a family of quantum algebras A_q parametrised by a scalar q, and the semiclassical limit Poisson algebra A of that family, then there should be an inclusion-preserving bijection between Spec A_q, for suitably generic q, and PSpec A. (The bijection is inclusion-preserving if and only if it is a homeomorphism with respect to the Zariski topologies on Spec A_q and PSpec A).

When n is odd, provided q is not a root of unity and the base field k is algebraically closed and characteristic 0, by Theorems 5.17 and 6.8, there is a bijection between Spec L_n^q and PSpec L_n^P sending 0 to 0 and $(z_n-\lambda)L_n^q$ to $(z_n-\lambda)L_n^P$, and this bijection is inclusion-preserving. Similarly, when n is even and under the same conditions, by Theorems 5.19 and 6.10 there is an inclusion-preserving bijection between Spec L_n^q and PSpec L_n^P.

There is not, however, an inclusion-preserving bijection between Spec C_n^q and PSpec C_n^P, since the former has infinitely many height 2 prime ideals whereas the latter only has 2 such. The natural map is a surjection from Spec C_n^q to PSpec C_n^P: the values of λ for which exceptional height 2 primes exist in C_n^q are all such that, when one sets q to 1, they equal ± 2, so the countably many exceptional height 2 prime ideals in C_n^q each map under this surjection to one of the two exceptional height 2 Poisson prime ideals in C_n^P.

There is, however, an inclusion-preserving bijection between PSpec C_n^P and the set of completely prime ideals in Spec C_n^q; since for a Noetherian ring no Poisson prime ideal can fail to be completely prime, this is perhaps the best we can hope for.

7 Acyclic quantum cluster algebras and prime ideals

Throughout this section we only consider quantum cluster algebras starting from an acyclic initial seed $\mathbf{Q}$.

The aim of this section is to find a minimal multiplicatively closed set $\mathcal{S}$ such that $A_q(\mathbf{Q})_{\mathcal{S}}$ is simple. In the case where there are no coefficients, it turns out that $A_q(\mathbf{Q})$ is itself simple; this result had already appeared in [37, Theorem 5.1], with the peculiar-looking additional condition

$$\sum_{j=1}^{n}(B^{-1})_{ij}(\max(B_{ij},0),\min(B_{ij},0)) \neq 0 \text{ for } 1 \leq i \leq n.$$

First, we find a minimal multiplicatively closed set $\mathcal{S}$ such that $S_q(\mathbf{x},\mathbf{L})_{\mathcal{S}}$ is simple.

7.1 Simple localisations of $S_q(\mathbf{x},\mathbf{L})$

Definition 7.1. Let $\mathbf{Q} = (Q, \mathbf{x}, \mathbf{L})$ be a (quantum) seed.

Define $\mathcal{W}$ to be the set of monomials in $\mathbf{x}$ with nonnegative degree in each variable, so $T_q(\mathbf{x},\mathbf{L}) = S_q(\mathbf{x},\mathbf{L})_{\mathcal{W}}$.

Define $\mathcal{X} := \{zw \in S_q(\mathbf{x},\mathbf{L}) : z \in Z(T_q(\mathbf{x},\mathbf{L}))\backslash\{0\}, w \in \mathcal{W}\}$, and $\mathcal{Y} := \mathcal{X} \cap S_q(\mathbf{x}_{\text{froz}},\mathbf{L}_{\text{froz}})$.

Proposition 7.2. *([6, 11.2]).*

Write $T = T_q(\mathbf{x},\mathbf{L})$, and assume q is not a root of unity.

(i) The monomials in $\mathbf{x}$ that are central in T form a k-basis for $Z(T)$.

(ii) $I \mapsto IT$ and $J \mapsto J \cap Z(T)$ define a one-to-one correspondence between the two-sided ideals of $Z(T)$ and the two-sided ideals of T.

Proposition 7.3. *(i) The localisation $S_q(\mathbf{x},\mathbf{L})_{\mathcal{X}}$ is simple Noetherian.*

(ii) $Z(T_q(\mathbf{x},\mathbf{L})) \subset T_q(\mathbf{x}_{froz},\mathbf{L}_{froz})$.

Proof. (i) Since $\mathcal{W} \subset \mathcal{X}$, $S_q(\mathbf{x},\mathbf{L})_{\mathcal{X}} = T_q(\mathbf{x},\mathbf{L})_{\mathcal{X}}$.

Let I be a non-zero two-sided ideal of $T_q(\mathbf{x},\mathbf{L})$. By Propostion 7.2 (ii), I contains an element $z \in Z(T_q(\mathbf{x},\mathbf{L}))$. Let $w \in \mathcal{W}$ be such that $zw \in S_q(\mathbf{x},\mathbf{L})$; such a w exists by taking w to be the product of the denominators of terms in z. Then $zw \in I$, so $I \cap \mathcal{X} \neq \emptyset$. Since this holds for any I, by Proposition 2.11, $T_q(\mathbf{x},\mathbf{L})_{\mathcal{X}}$ is simple, and so $S_q(\mathbf{x},\mathbf{L})_{\mathcal{X}}$ is simple.

Since $S_q(\mathbf{x},\mathbf{L})$ is Noetherian $S_q(\mathbf{x},\mathbf{L})_{\mathcal{X}}$ is also Noetherian.

(ii) Let $\mathbf{x}^{\mathbf{a}}$ be a monomial in $T_q(\mathbf{x},\mathbf{L})$. Then $\mathbf{x}^{\mathbf{a}}$ is central if and only if $\mathbf{La}=0$, so if $\mathbf{x}^{\mathbf{a}}$ is central then $\mathbf{B^TLa}=0$, so by the compaitibility condition from Definition 2.80, there exists an integer $d>0$ such that $d\mathbf{Ia}=0$. That is, for any mutable $u\in Q$, $da_u=0$, and so $a_u=0$. Therefore $\mathbf{x}^{\mathbf{a}}\in T_q(\mathbf{x}_{\text{froz}},\mathbf{L}_{\text{froz}})$; so by Proposition 7.2 (i), $Z(T_q(\mathbf{x},\mathbf{L}))\subset T_q(\mathbf{x}_{\text{froz}},\mathbf{L}_{\text{froz}})$.

□

Lemma 7.4. *Let $\mathbf{Q}=(Q,\mathbf{x},\mathbf{L})$ be a seed, and $\mathcal{X}$ be as above. Then $A_q(\mathbf{Q})_{\mathcal{X}}$ exists and equals $S_q(\mathbf{x},\mathbf{L})_{\mathcal{X}}$. Therefore any prime ideal of $A_q(\mathbf{Q})$ contains an element of $\mathcal{X}$.*

Proof. We recall the Laurent phenomenon (Theorem 2.87), which tells us that $S_q(\mathbf{x},\mathbf{L})\subset A_q(\mathbf{Q})\subset T_q(\mathbf{x},\mathbf{L})$.

With $\mathcal{W}$ as above also, we have $\mathcal{W}\subset\mathcal{X}$, so by Corollary 2.46, $T_q(\mathbf{x},\mathbf{L})\subset S_q(\mathbf{x},\mathbf{L})_{\mathcal{X}}$. Thus $S_q(\mathbf{x},\mathbf{L})\subset A_q(\mathbf{Q})\subset S_q(\mathbf{x},\mathbf{L})_{\mathcal{X}}$, so we can apply Lemma 2.45 to get that $A_q(\mathbf{Q})_{\mathcal{X}}$ exists and equals $S_q(\mathbf{x},\mathbf{L})_{\mathcal{X}}$.

Since $S_q(\mathbf{x},\mathbf{L})_{\mathcal{X}}$ is simple Noetherian, we can apply Lemma 2.11 to show that any prime ideal of $A_q(\mathbf{Q})$ contains an element of $\mathcal{X}$. □

Lemma 7.5. *Let Q be an acyclic quiver. Then Q contains a sink.*

Proof. Suppose not. Pick a vertex $v_1\in Q$, then define recursively a sequence of vertices $v_1,v_2,\ldots$ such that there is an arrow $v_i\to v_{i+1}$ for all $i\geq 1$. This is possible since each v_i is not a sink. Since Q is finite, there must exist i and j such that $v_i=v_j$. But then $v_i\to v_{i+1}\to\cdots\to v_j=v_i$ is a cycle in Q, a contradiction. □

Lemma 7.6. *Let Q be an acyclic quiver. Then there exists an ordering of the vertices of Q, $v_1,\ldots,v_n$, such that there are no arrows from v_j to v_i for $j>i$.*

Proof. By the previous lemma, Q has a sink. Call this vertex v_n. Then by induction (the case where Q has one vertex is trivial, and a subgraph of an acyclic graph is still acyclic), there exists an ordering of $Q\setminus v_n$ with no arrows from v_j to v_i for $n>j>i$. And since v_n is a sink, there are no arrows from v_n to v_j for $j<n$. □

Lemma 7.7. *Let $\mathbf{Q}=(Q,\mathbf{x},\mathbf{L})$ be a seed and let $v\in Q$. Then there exists $s\in\mathbb{Z}$ and $\lambda\in k^\times$ such that*

$$x_vx'_v-q^sx'_vx_v=\lambda\prod_{B_{vw}<0}x_w^{-B_{vw}}$$

Remark. The precise value of scalar λ does depend on the ordering on Q, but its existence does not.

Proof. By definition of x'_v, there exist non-zero scalars μ_+ and μ_- such that

$$x'_v = x_v^{-1}\mu_+ \prod_{B_{vw}>0} x_w^{B_{vw}} + x_v^{-1}\mu_- \prod_{B_{vw}<0} x_w^{-B_{vw}}$$

Using the skew-commutation relations between x_v and the other x_w, we get

$$x'_v = \mu_+ q^{\sum\limits_{B_{vw}>0} -L_{vw}B_{vw}} \prod_{B_{vw}>0} x_w^{B_{vw}} x_v^{-1} + \mu_- q^{\sum\limits_{B_{vw}<0} L_{vw}B_{vw}} \prod_{B_{vw}<0} x_w^{-B_{vw}} x_v^{-1}.$$

Thus, setting $s = \sum\limits_{B_{vw}>0} L_{vw}B_{vw}$, we get

$$\begin{aligned} x_v x'_v - q^s x'_v x_v &= \mu_- \left(1 - q^{\sum\limits_{B_{vw}<0} L_{vw}B_{vw}} q^{\sum\limits_{B_{vw}>0} L_{vw}B_{vw}}\right) \prod_{B_{vw}<0} x_w^{-B_{vw}} \\ &= \mu_- \left(1 - q^{\sum\limits_{w\in Q} L_{vw}B_{vw}}\right) \prod_{B_{vw}<0} x_w^{-B_{vw}} \\ &= \mu_- \left(1 - q^d\right) \prod_{B_{vw}<0} x_w^{-B_{vw}}. \end{aligned}$$

In this d is the integer such that $\mathbf{B^T L} = d\mathbf{I}$, which is strictly positive and so $\lambda := \mu_-(1-q^d)$ is a non-zero scalar. □

Corollary 7.8. *With s and λ as above, and $a \geq 1$,*

$$x_v^a x'_v - q^{as} x'_v x_v^a = \lambda q^{-(a-1)s} [a]_{q^{t-s}} \prod_{B_{vw}<0} x_w^{-B_{vw}} x_v^{a-1}$$

In particular, if q is not a root of unity, there exists a non-zero scalar μ such that

$$x_v^a x'_v - q^{as} x'_v x_v^a = \mu \prod_{B_{vw}<0} x_w^{-B_{vw}} x_v^{a-1}$$

Proof. We prove this by induction on a; if $a = 1$ then this is immediate from the previous lemma. Write $\mathbf{x}_- := \lambda \prod\limits_{B_{vw}<0} x_w^{-B_{vw}}$, and note that $x_v \mathbf{x}_- = q^{\sum\limits_{B_{vw}<0} -L_{vw}B_{vw}} \mathbf{x}_- x_v$. So write $t := \sum\limits_{B_{vw}<0} - L_{vw}B_{vw}$. With this notation,

$$\begin{aligned} x_v^a x'_v &= q^{(a-1)s} x_v x'_v x_v^{a-1} + q^{(a-1)s} [a-1]_{q^{t-s}} x_v \mathbf{x}_- x_v^{a-2} \\ &= q^{as} x'_v x_v^a + q^{(a-1)s} \mathbf{x}_- x_v^{a-1} + q^{(a-2)s} [a-1]_{q^{t-s}} q^t \mathbf{x}_- x_v^{a-1} \\ &= q^{as} x'_v x_v^a + q^{(a-1)s} \mathbf{x}_- x_v^{a-1} + q^{(a-1)s} [a-1]_{q^{t-s}} q^{t-s} \mathbf{x}_- x_v^{a-1} \\ &= q^{as} x'_v x_v^a + q^{(a-1)s} [a]_{q^{t-s}} q^{t-s} \mathbf{x}_- x_v^{a-1}. \end{aligned}$$

□

Theorem 7.9. *Let $\mathbf{Q} = (Q, \mathbf{x}, \mathbf{L})$ be a seed such that the principal part of Q - that is, Q with the frozen vertices removed - is acyclic. Assume that q is not a root of unity. Let $\mathcal{Y}$ be as in Definition 7.1. Then any non-zero ideal of $A_q(\mathbf{Q})$ contains an element of $\mathcal{Y}$.*

Conversely, any element of $\mathcal{Y}$ is normal in $A_q(\mathbf{Q})$, so generates a non-trivial ideal of $A_q(\mathbf{Q})$. If $y \in \mathcal{Y}$ is irreducible, that is, there do not exist $y', y'' \in \mathcal{Y} \setminus k^\times$ such that $y = y'y''$, then y generates a prime ideal of $A_q(\mathbf{Q})$.

Proof. Let I be a non-zero ideal of $A_q(\mathbf{Q})$, so, by Lemma 7.4, I must contain some element of $\mathcal{X}$.

Order the mutable vertices of Q using Lemma 7.6. Let $Q^+ = \{$mutable vertices of $Q\} \cup \{0\}$. Order Q^+ by extending the order on the mutable vertices of Q by saying $0 < v$ for all v. ('0' will represent being an element of $\mathcal{Y}$).

For $y \in \mathcal{X}$ define $m(y) \in Q^+$ to be 0 if $y \in \mathcal{Y}$, and otherwise $m(y)$ is the maximal vertex v such that y has non-zero degree in x_v.

Let $v \in Q^+$ be minimal such that $m(y) = v$ for some $y \in I \cap \mathcal{X}$. If $v = 0$ then we're done. Otherwise, let $z \in I \cap \mathcal{X}$ have minimal degree in x_v among elements of $\{y \in I \cap \mathcal{X} : m(y) = v\}$. Write $z = y \prod_w x_w^{a_w} x_v^{a_v}$, where $y \in \mathcal{Y}$.

Now since x'_v skew-commutes with x_w for all $w \neq v$, and commutes with y for all $y \in \mathcal{Y}$,, by Corollary 7.8 there exists an integer t and a non-zero scalar ν such that

$$zx'_v - q^t x'_v z = \nu y \left(\prod_{w<v} x_w^{a_w} \right) \left(\prod_{B_{vw}<0} x_w^{-B_{vw}} \right) x_v^{a_v - 1}$$

But $B_{vw} < 0 \implies w < v$, so writing $a'_w = \begin{cases} a_w & \text{if } B_{vw} \geq 0 \\ a_w - B_{vw} & \text{if } B_{vw} < 0 \end{cases}$, we get

$$zx'_v - q^t x'_v z = \nu y \left(\prod_{w<v} x_w^{a'_w} \right) x_v^{a_v - 1}.$$

Therefore, $zx'_v - q^t x'_v z \in I \cap \mathcal{X}$, and if $a_v > 1$ then $m(zx'_v - q^t x'_v z) = v$ and $zx'_v - q^t x'_v z$ contradicts the minimality of the degree of x_v in z. If $a_v = 1$ then $m(zx'_v - q^t x'_v z) < v$ contradicting the minimality of v. Either way we have a contradiction, so we must have had $v = 0$ and I contains an element of $\mathcal{Y}$. □

Corollary 7.10. *In the setting of Theorem 7.9, $A_q(\mathbf{Q})_{\mathcal{Y}}$ is simple.*

Proof. Every element of $\mathcal{Y}$ is normal in $A_q(\mathbf{Q})$, so $\mathcal{Y}$ is a right Ore set in $A_q(\mathbf{Q})$ and the localisation exists. Now apply Lemma 2.11. □

Corollary 7.11. *Let $\mathbf{Q} = (Q, \mathbf{x}, \mathbf{L})$ be a seed with no frozen variables such that Q is acyclic. Then $A_q(\mathbf{Q})$ is simple.*

Having described the height 1 primes, a natural question is whether there exist quantum cluster algebra structures on the factor rings. In the case of a central coefficient, there exists a natural candidate:

Conjecture 7.12. *([21, Proposition 3.3]). Let $\mathbf{Q} = (Q, \mathbf{x}, \mathbf{L})$ be a seed with $v \in Q_{froz}$ such that $L_{vw} = 0$ for all w, so x_v is central in $A_q(\mathbf{Q})$. Then removing v from $\mathbf{Q}$ gives valid initial data for a quantum cluster algebra; it is conjectured that this is a quantum cluster algebra on $A_q(\mathbf{Q})/(x_v - 1)$.*

7.2 Examples

We use the standard notation that frozen vertices are represented by boxes.

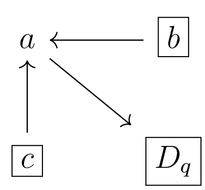
Example 7.13 (The first quantized Weyl algebra). The first quantized Weyl algebra $A_1^{q^2} := k\langle x, y : xy - q^2yz = 1 - q^2\rangle$ has a quantum cluster algebra structure given by

$$x \longrightarrow \boxed{q^{-1}z}$$

where $z := xy - 1$ is the normal element in $A_1^{q^2}$, and mutation at x turns out to give y.

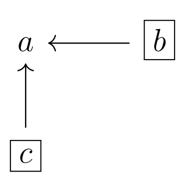
So Theorem 7.9 recovers the fact that the prime ideal generated by z is in fact the only height one prime in $A_1^{q^2}$.

Example 7.14. We recall that $\mathcal{O}_q(M_2(k))$ has a quantum cluster algebra structure with the following as the initial seed:



Here the centre of the quantum torus is the Laurent polynomial ring in D_q and bc^{-1}, so the set $\mathcal{Y}$ consists of non-zero polynomials in b, c, and D_q which are homogeneous in b and c.

Similarly, we recall that $\mathcal{O}_q(SL_2(k))$ has a quantum cluster algebra structure with the following as the initial seed:



Here the centre of the quantum torus is the Laurent polynomial ring in bc^{-1}, so the set $\mathcal{Y}$ consists of non-zero homogeneous polynomials in b and c.

We note also that since $\mathcal{O}_q(SL_2(k)) = \mathcal{O}_q(M_2(k))/(D_q - 1)$, the quantum cluster structure on $\mathcal{O}_q(SL_2(k))$ is an example of Theorem 7.12.

We will see further examples in Section 8.

8 Quantum cluster algebra structures on connected quantized Weyl algebras

8.1 A quantum cluster algebra structure on $L_n^{q^2}$

We aim to give a quantum cluster algebra presentation of the ring $L_n^{q^2}$. The initial seed will be the following, where $\hat{z_i}$ is a non-zero scalar multiple (the precise value to be defined later) of z_i for all i.

$$\hat{z_1} \longrightarrow \hat{z_2} \longrightarrow \cdots \longrightarrow \hat{z_{n-1}} \longrightarrow \boxed{\hat{z_n}}$$

Proposition 8.1. *Let $\mathbf{Q} = (Q, \mathbf{x}, \mathbf{L})$ be a seed, and $v \in Q$ a vertex.*

Let $\tilde{\mathbf{Q}}$ be the seed formed by taking $\mathbf{Q}$ and adding an extra vertex w, any arrows from w to other vertices of Q that we choose, and a central cluster variable x_w. We can mutate this seed at the vertex v, giving a seed $\tilde{\mathbf{Q}}'$. Now construct a seed $\hat{\mathbf{Q}}'$ from $\tilde{\mathbf{Q}}'$ by replacing all instances of x_w in x_v' by 1, removing w and any arrows to or from it from $\tilde{Q}'$, and removing x_w from $\tilde{\mathbf{x}}'$. Then the seeds $\hat{\mathbf{Q}}'$ and $\mathbf{Q}'$ are identical.

Informally, "adding vertices labelled by 1 doesn't change anything".

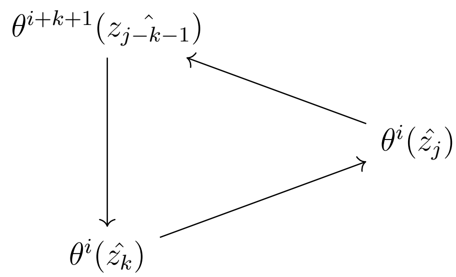
Proof. If u and t are vertices in Q then $\tilde{B}'_{ut}$ depends only on $\tilde{B}_{ut}$, $\tilde{B}_{uv}$ and $\tilde{B}_{tv}$, which are all equal to their counterparts in $\mathbf{B}$, and so $\tilde{B}'_{ut} = B'_{ut}$. Thus $\hat{B}'_{ut} = B'_{ut}$ for any u, $t \in \hat{Q}'$, and since $\hat{Q}'$ and Q' have the same vertex set, these quivers are identical.

If we pick an ordering on the vertices of Q then we note that $\lambda(\mathbf{b}_v^{\pm}) = \lambda(\tilde{\mathbf{b}}_v^{\pm})$, since $\tilde{L}_{uw} = 0$ for all u, and so $M(\tilde{\mathbf{b}}_v^{\pm}) = x_w^{(\tilde{b}_v^{\pm})_w} M(\mathbf{b}_v^{\pm})$. Thus $\hat{x_v'} = x_v'$ as elements of $T_q(\mathbf{x}, \mathbf{L})$, and so $\hat{\mathbf{x}'} = \mathbf{x}'$ as subsets of $T_q(\mathbf{x}, \mathbf{L})$, and so also $\hat{\mathbf{L}}' = \mathbf{L}'$. □

Definition 8.2. For integers $j \geq 0$, define $t(j) := \begin{cases} j/2 & \text{if } j \text{ is even} \\ (j-1)/2 & \text{if } j \text{ is odd} \end{cases}$.

Define $\hat{z_j} := q^{-t(j)} z_j$, that is, a rescaled version of z_j, where $z_j \in L_j^q$ is as in Definition 5.1.

Consider the following picture, where $0 \leq k \leq j-1$ and $i + j \leq n$. If any of the variables associated to vertices are $\theta^r(z_0)$ for some r - that is, if $k = 0$ or $k = j-1$ - then we remove that vertex and any arrows to or from it. When $j = 1$, and so $k = 0$ *and* $k = j-1$, then we remove both those vertices, leaving just the vertex labelled by $\theta^i(\hat{z_j}) = x_{i+1}$.



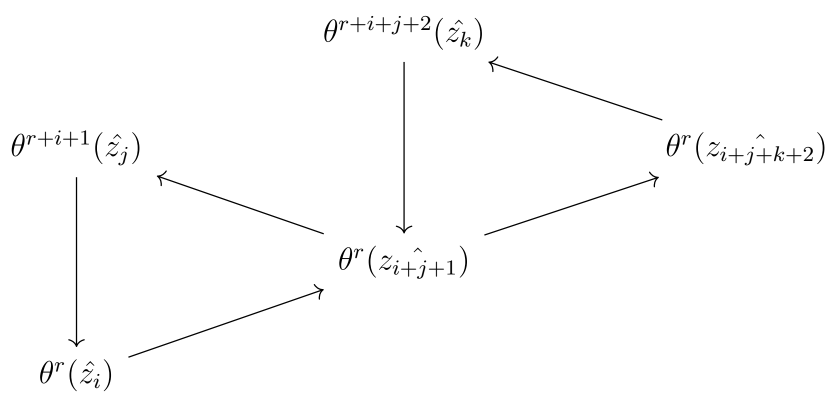
We say that the vertex labelled $\theta^i(\hat{z}_j)$ has a **smaller neighbourhood**, while the other two vertices each have a **larger neighbourhood**.

A $L_n^{q^2}$**-seed** is a seed contained within $L_n^{q^2}$ with n vertices in the quiver and

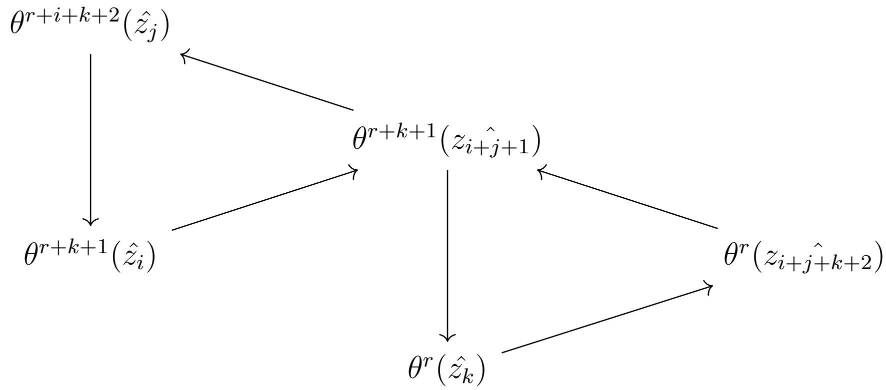
1. the variable associated to each vertex is $\theta^i(\hat{z}_j)$ for some $0 \leq i, j$ with $i + j \leq n$;
2. it has precisely one frozen vertex, with associated variable $\hat{z_n}$;
3. each unfrozen vertex has a smaller neighbourhood and a larger neighbourhood, and no arrows to or from it otherwise;
4. the frozen vertex has a smaller neighbourhood and no arrows to or from it otherwise.

Lemma 8.3. *The neighbourhood of any mutable vertex in a $L_n^{q^2}$-seed is one of the following:*

(i)



(ii)

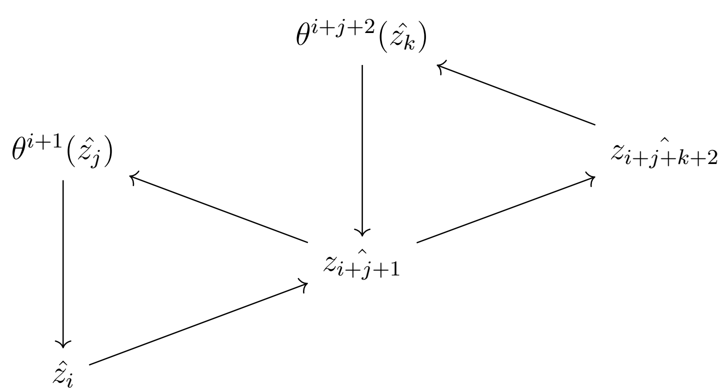


where $i,\ j,\ k \geq 0$ are such that $i+j+k+2 \leq n$, and the vertex we are considering has $\theta^r(z_{\widehat{i+j+1}})$ associated to it.

If any of i, j, or k equal 0 then some of the vertices in this diagram will not be present, but we can use Proposition 8.1 to add vertices "labelled by 1" in their place to make the neighbourhood of this form.

Proof. The vertex must have both a smaller neighbourhood and a larger neighbourhood; the two cases arise from there being two vertices in Definition 8.2 which have a larger neighbourhood. $\square$

Lemma 8.4. *Suppose $i,\ j,\ k \geq 0$ are such that $i+j+k+2 \leq n$, and we have a vertex with $z_{\widehat{i+j+1}}$ associated to it, such that its neighbourhood is the following:*



Then ${z_{\widehat{i+j+1}}}'$, i.e. the new variable obtained by mutating at $z_{\widehat{i+j+1}}$, equals $\theta^{i+1}(z_{\widehat{j+k+1}})$.

As before, if any of i, j, k are 0 then some of the vertices in this diagram will not be present, but we can use Proposition 8.1 to add vertices "labelled by 1" in their place without changing anything.

To show this we will need the following calculation:

Lemma 8.5. *For all non-negative integers i, j, k, and also for $k = -1$, the following holds:*

$z_{i+j+1}\theta^{i+1}(z_{j+k+1}) = \theta^{i+1}(z_j)z_{i+j+k+2} + q^j z_i \theta^{i+j+2}(z_k)$ *(if j is even);*

$z_{i+j+1}\theta^{i+1}(z_{j+k+1}) = \theta^{i+1}(z_j)z_{i+j+k+2} + q^{j+1} z_i \theta^{i+j+2}(z_k)$ *(if i and j are both odd);*

$z_{i+j+1}\theta^{i+1}(z_{j+k+1}) = q^2\theta^{i+1}(z_j)z_{i+j+k+2} + q^{j+1} z_i \theta^{i+j+2}(z_k)$ *(if i is even and j is odd).*

Proof. We consider first the case when $k = 0$. In this case we wish to show

$$z_{i+j+1}\theta^{i+1}(z_{j+1}) = \theta^{i+1}(z_j)z_{i+j+2} + q^j z_i \quad \text{if } j \text{ is even;}$$

$$z_{i+j+1}\theta^{i+1}(z_{j+1}) = \theta^{i+1}(z_j)z_{i+j+2} + q^{j+1} z_i \quad \text{if } j \text{ odd and } i \text{ odd;}$$

$$z_{i+j+1}\theta^{i+1}(z_{j+1}) = q^2\theta^{i+1}(z_j)z_{i+j+2} + q^{j+1} z_i \quad \text{if } j \text{ odd and } i \text{ even.}$$

We note that, by Proposition 5.2 (v), $\theta^{i+1}(z_j)z_{i+j+2} = q^{s_{ij}} z_{i+j+2}\theta^{i+1}(z_j)$, with

$$s_{ij} = \begin{cases} 0 & \text{if } j \text{ is even;} \\ 0 & \text{if } i+j \text{ is odd;} \\ 1 & \text{if } i \text{ and } j \text{ are both odd.} \end{cases}$$

These skew commutators allow us to rewrite the relations slightly, so that we wish to show

$$z_{i+j+1}\theta^{i+1}(z_{j+1}) = z_{i+j+2}\theta^{i+1}(z_j) + q^j z_i \quad \text{if } j \text{ is even;} \quad (1)$$

$$z_{i+j+1}\theta^{i+1}(z_{j+1}) = q^2 z_{i+j+2}\theta^{i+1}(z_j) + q^{j+1} z_i \quad \text{if } j \text{ is odd.} \quad (2)$$

We show these by induction on j. If $j = 0$ then (1) becomes just $z_{i+1}x_{i+2} = z_{i+2} + z_i$, i.e. the definition of z_{i+2}.

If $j > 0$ is even then

$$\begin{aligned}
&z_{i+j+1}\theta^{i+1}(z_{j+1}) - z_{i+j+2}\theta^{i+1}(z_j) \\
&\quad = z_{i+j+1}\theta^{i+1}(z_j)x_{i+j+2} - z_{i+j+1}x_{i+j+2}\theta^{i+1}(z_j) - z_{i+j+1}\theta^{i+1}(z_{j-1}) + z_{i+j}\theta^{i+1}(z_j) \\
&\quad = z_{i+j+1}\theta^{i+1}(z_j)x_{i+j+2} - z_{i+j+1}\theta^{i+1}(z_j)x_{i+j+2} \\
&\qquad + (1-q^2)z_{i+j+1}\theta^{i+1}(z_{j-1}) - z_{i+j+1}\theta^{i+1}(z_{j-1}) + z_{i+j}\theta^{i+1}(z_j)
\end{aligned}$$

(since $x_{i+j+2}\theta^{i+1}(z_j) = \theta^{i+1}(x_{j+1}z_j) = \theta^{i+1}(z_j x_{j+1} - (1-q^2)z_{j-1}))$ by Proposition 5.3

$$\begin{aligned}
&\quad = z_{i+j}\theta^{i+1}(z_j) - q^2 z_{i+j+1}\theta^{i+1}(z_{j-1}) \\
&\quad = q^j z_i \text{ by induction.}
\end{aligned}$$

If $j > 0$ is odd then

$$
\begin{aligned}
&z_{i+j+1}\theta^{i+1}(z_{j+1}) - q^2 z_{i+j+2}\theta^{i+1}(z_j) \\
&\quad = z_{i+j+1}\theta^{i+1}(z_j)x_{i+j+2} - q^2 z_{i+j+1}x_{i+j+2}\theta^{i+1}(z_j) - z_{i+j+1}\theta^{i+1}(z_{j-1}) + q^2 z_{i+j}\theta^{i+1}(z_j) \\
&\quad = z_{i+j+1}\theta^{i+1}(z_j)x_{i+j+2} - z_{i+j+1}\theta^{i+1}(z_j)x_{i+j+2} \\
&\qquad + (1-q^2)z_{i+j+1}\theta^{i+1}(z_{j-1}) - z_{i+j+1}\theta^{i+1}(z_{j-1}) + q^2 z_{i+j}\theta^{i+1}(z_j) \\
&\quad \text{(since } q^2 x_{i+j+2}\theta^{i+1}(z_j) = \theta^{i+1}(z_j x_{j+1} - (1-q^2)z_{j-1})) \text{ by Proposition 5.3} \\
&\quad = q^2(z_{i+j}\theta^{i+1}(z_j) - z_{i+j+1}\theta^{i+1}(z_{j-1})) \\
&\quad = q^2 q^{j-1} z_i \text{ by induction} \\
&\quad = q^{j+1} z_i.
\end{aligned}
$$

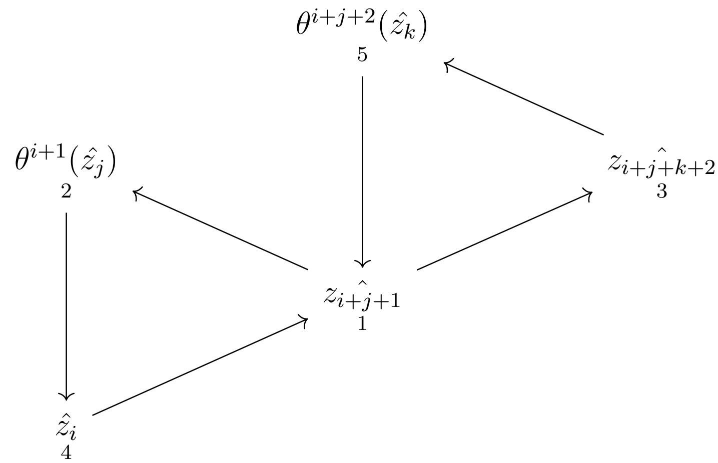
So we have shown the result in the case $k = 0$. We note that with the convention $z_{-1} = 0$, the result makes sense for $k = -1$ as well. and in that case is simply the skew-commutation relation between z_{i+j+1} and $\theta^{i+1}(z_j)$ from Proposition 5.2 (v).

Now we show the general case by induction on k: if $\lambda = \begin{cases} q^2 & \text{if } j \text{ is odd and } i \text{ is even} \\ 1 & \text{otherwise} \end{cases}$ and $\mu = \begin{cases} q^j & \text{if } j \text{ is even} \\ q^{j+1} & \text{if } j \text{ is odd} \end{cases}$ (note that these are independent of the value of k) then, just using the definition of z_r for various r and reordering,

$$
\begin{aligned}
&z_{i+j+1}\theta^{i+1}(z_{j+k+1}) - \lambda\theta^{i+1}(z_j)z_{i+j+k+2} - \mu z_i\theta^{i+j+2}(z_k) \\
&\quad = z_{i+j+1}\theta^{i+1}(z_{j+k})x_{i+j+k+2} - z_{i+j+1}\theta^{i+1}(z_{j+k-1}) \\
&\qquad - \lambda(\theta^{i+1}(z_j)z_{i+j+k+1}x_{i+j+k+2} - \theta^{i+1}(z_j)z_{i+j+k}) \\
&\qquad - \mu(z_i\theta^{i+j+2}(z_{k-1})x_{i+j+k+2} - z_i\theta^{i+j+2}(z_{k-2})) \\
&\quad = (z_{i+j+1}\theta^{i+1}(z_{j+k}) - \lambda\theta^{i+1}(z_j)z_{i+j+k+1} - \mu z_i\theta^{i+j+2}(z_{k-1}))x_{i+j+k+2} \\
&\qquad - (z_{i+j+1}\theta^{i+1}(z_{j+k-1}) - \lambda\theta^{i+1}(z_j)z_{i+j+k} - \mu z_i\theta^{i+j+2}(z_{k-2})) \\
&\quad = 0 \text{ by induction.}
\end{aligned}
$$

□

Proof of Lemma 8.4. We choose the following ordering on the vertices of the neighbourhood of our vertex:



Therefore $\mathbf{b}_1^+ = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \end{pmatrix}^T$ and $\mathbf{b}_1^- = \begin{pmatrix} -1 & 0 & 0 & 1 & 1 \end{pmatrix}^T$.

The table below calculates $\lambda(\mathbf{b}_1^+)$ and $\lambda(\mathbf{b}_1^-)$, depending on the parity of i, j and k.

i	j	k	L_{12}	L_{13}	L_{23}	L_{14}	L_{15}	L_{45}	$\lambda(\mathbf{b}_1^+)$	$\lambda(\mathbf{b}_1^-)$
even	even	even	0	2	0	0	0	0	1	0
even	even	odd	0	0	0	0	-2	0	0	-1
even	odd	even	2	0	0	0	0	0	1	0
even	odd	odd	2	0	-2	0	0	0	2	0
odd	even	even	0	0	0	-2	0	0	0	-1
odd	even	odd	0	0	0	-2	0	2	0	-2
odd	odd	even	0	2	2	0	0	0	0	0
odd	odd	odd	0	0	0	0	-2	-2	0	0

We now calculate $q^{t(j+k+1)}M(\mathbf{b}_1^+)$ and $q^{t(j+k+1)}M(\mathbf{b}_1^-)$, since we want to show $M(\mathbf{b}_1^+) + M(\mathbf{b}_1^-) = q^{-t(j+k+1)}\theta^{i+1}(z_{j+k+1})$

We know $q^{t(j+k+1)}M(\mathbf{b}_1^+) = q^a z_{i+j+1}^{-1}\theta^{i+1}(z_j)z_{i+j+k+2}$, where $a = \lambda(\mathbf{b}_1^+) + t(i+j+1) - t(j) - t(i+j+k+2) + t(j+k+1)$. The following table calculates a, again depending on the parity of i, j and k:

i	j	k	$\lambda(\mathbf{b}_1^+)$	$t(i+j+1)$	$-t(j)$	$-t(i+j+k+2)$	$t(j+k+1)$	a
even	even	even	1	$i+j/2$	$-j/2$	$-(i+j+k+2)/2$	$j+k/2$	0
even	even	odd	0	$i+j/2$	$-j/2$	$-(i+j+k+1)/2$	$j+k+1/2$	0
even	odd	even	1	$i+j+1/2$	$-(j-1)/2$	$-(i+j+k+1)/2$	$j+k+1/2$	2
even	odd	odd	2	$i+j+1/2$	$-(j-1)/2$	$-(i+j+k+2)/2$	$j+k/2$	2
odd	even	even	0	$i+j+1/2$	$-j/2$	$-(i+j+k+1)/2$	$j+k/2$	0
odd	even	odd	0	$i+j+1/2$	$-j/2$	$-(i+j+k+2)/2$	$j+k+1/2$	0
odd	odd	even	0	$i+j/2$	$-(j-1)/2$	$-(i+j+k+2)/2$	$j+k+1/2$	0
odd	odd	odd	0	$i+j/2$	$-(j-1)/2$	$-(i+j+k+1)/2$	$j+k/2$	0

Similarly, we know $q^{t(j+k+1)}M(\mathbf{b}_1^-) = q^b z_{i+j+1}^{-1} z_i \theta^{i+j+2}(z_k)$, where $b = \lambda(\mathbf{b}_1^-) + t(i+j+1) - t(j) - t(i+j+k+2) + t(j+k+1)$. The following table calculates b depending on the parity of i, j and k:

i	j	k	$\lambda(\mathbf{b}_1^-)$	$t(i+j+1)$	$-t(i)$	$-t(k)$	$t(j+k+1)$	b
even	even	even	0	$i+j/2$	$-i/2$	$-k/2$	$j+k/2$	j
even	even	odd	-1	$i+j/2$	$-i/2$	$-(k-1)/2$	$j+k+1/2$	j
even	odd	even	0	$i+j+1/2$	$-i/2$	$-k/2$	$j+k+1/2$	$j+1$
even	odd	odd	0	$i+j+1/2$	$-i/2$	$-(k-1)/2$	$j+k/2$	$j+1$
odd	even	even	-1	$i+j+1/2$	$-(i-1)/2$	$-k/2$	$j+k/2$	j
odd	even	odd	-2	$i+j+1/2$	$-(i-1)/2$	$-(k-1)/2$	$j+k+1/2$	j
odd	odd	even	0	$i+j/2$	$-(i-1)/2$	$-k/2$	$j+k+1/2$	$j+1$
odd	odd	odd	0	$i+j/2$	$-(i-1)/2$	$-(k-1)/2$	$j+k/2$	$j+1$

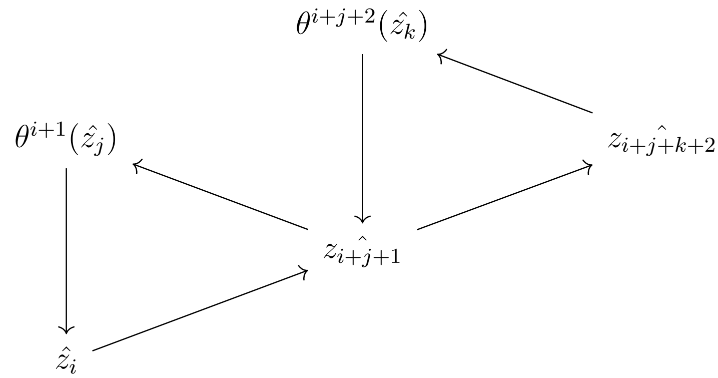
Combining these together, we get

$$q^{t(j+k+1)} z_{i\hat{+}j+1}{}' = \begin{cases} z_{i+j+1}^{-1}(\theta^{i+1}(z_j) z_{i+j+k+2} + q^j z_i \theta^{i+j+2}(z_k)) & (j \text{ even}) \\ z_{i+j+1}^{-1}(\theta^{i+1}(z_j) z_{i+j+k+2} + q^{j+1} z_i \theta^{i+j+2}(z_k)) & (j \text{ odd, } i \text{ odd}) \\ z_{i+j+1}^{-1}(q^2\theta^{i+1}(z_j) z_{i+j+k+2} + q^{j+1} z_i \theta^{i+j+2}(z_k)) & (j \text{ odd, } i \text{ even}) \end{cases}$$

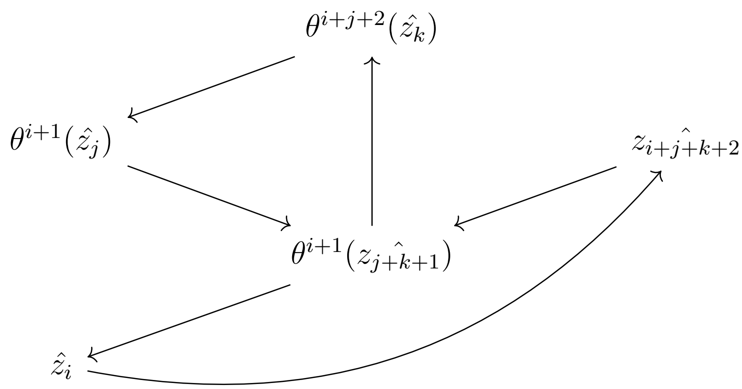
So by Lemma 8.5 we get $z_{i\hat{+}j+1}{}' = \theta^{i+1}(z_{j\hat{+}k+1})$, as required. □

Lemma 8.6. *Any mutation of a $L_n^{q^2}$-seed yields another $L_n^{q^2}$-seed.*

Proof. If we have an $L_n^{q^2}$-seed and a mutable vertex in it, then the neighbourhood of our vertex will be of one of the two forms from Lemma 8.3. Suppose first that the neighbhourhood is of the first form. By applying an appropriate power of θ to all the calculations involved, we may assume $r = 0$, so the neighbourhood is:



Lemma 8.4, together with an easy check of the quiver mutation, shows that mutating at $z_{\widehat{i+j+1}}$ gives the following local picture:



Furthermore, if all the vertices in the picture satisfied the requirements of Definition 8.2 before the mutation, they continue to do so after the mutation: the four vertices (including the one we mutate at) that started with larger neighbourhoods in the picture still have larger neighbourhoods in the picture, while the two vertices that started with smaller neighbourhoods in the picture still have smaller neighbhourhoods in the picture, and there are no other arrows that have not been accounted for in the above. So if the previous seed was an $L_n^{q^2}$-seed, then the mutated seed is too.

Further, the neighbourhood is now in the second form from Lemma 8.3; since seed mutation is involutive, starting from a vertex whos neighbourhood is of the second form within a $L_n^{q^2}$-seed gives a $L_n^{q^2}$-seed with the neighbourhood of the mutated vertex being of the first form.

Either way, if we start with an $L_n^{q^2}$-seed and mutate it we get a $L_n^{q^2}$-seed. □

Theorem 8.7. *Let the following diagram describe a seed* $\mathbf{Q}$ *within* $L_n^{q^2}$*:*

$$\hat{z}_1 \longrightarrow \hat{z}_2 \longrightarrow \cdots \longrightarrow z_{\widehat{n-1}} \longrightarrow \boxed{\hat{z}_n}$$

Then $A_q(\mathbf{Q}) = L_n^{q^2}$.

Proof. The initial seed is clearly an $L_n^{q^2}$-seed, so all seeds mutation equivalent to it are $L_n^{q^2}$-seeds also. So all the cluster variables are all elements of $L_n^{q^2}$, so the quantum cluster algebra is a subalgebra of $L_n^{q^2}$. And the new cluster variable obtained by mutating at $\hat{z}_i$ equals x_{i+1} for $1 \leq i \leq n-1$, so together with $\hat{z}_1 = x_1$ we have all the generators of $L_n^{q^2}$. So the quantum cluster algebra does in fact equal $L_n^{q^2}$. □

Corollary 8.8. *The commutative cluster algebra with initial seed*

$$z_1 \longrightarrow z_2 \longrightarrow \cdots \longrightarrow z_{n-1} \longrightarrow \boxed{z_n}$$

is isomorphic to a polynomial ring in n *variables.*

Proof. Take $q = 1$. □

Remark. I have no idea whether this is already known.

We can use this, together with Theorem 7.9, to recover some of the results of Section 5.3.

Corollary 8.9. *If* n *is even,* k *is algebraically closed, and* q *is not a root of unity, then any prime ideal of* L_n^q *contains* z_n*, while if* n *is odd, any prime ideal of* L_n^q *contains* $z_n - \lambda$ *for some* λ*. Further, if* n *is odd then* $L_n^q/(z_n - 1)$ *is simple.*

Proof. Let $\mathbf{Q} = (Q, \mathbf{x}, \mathbf{L})$ denote the initial seed described above. If n is even, $Z(T_q(\mathbf{x}, \mathbf{L})) = k$, while if n is odd, $Z(T_q(\mathbf{x}, \mathbf{L})) = k[z_n^{\pm 1}]$. So the set $\mathcal{Y}$ of Definition 7.1 is $\{z_n^i : i \geq 0\}$ if n is even and $k[z_n]^*$ if n is odd. The result then follows by Theorem 7.9, together with the algebraic closure of k allowing complete factorisation of elements of $k[z_n]$ into linear terms.

Theorem 7.12 tells us that $L_n^q/(z_n - 1)$ has a quantum cluster algebra structure with initial seed

$$\bar{\hat{z}}_1 \longrightarrow \bar{\hat{z}}_2 \longrightarrow \cdots \longrightarrow \bar{\hat{z}}_{n-1}$$

By Corollary 7.11, this quantum cluster algebra is simple, so $L_n^q/(z_n - 1)$ is simple. □

8.2 A quantum cluster algebra containing $C_n^{q^2}$

In Section 4 we defined connected quantized Weyl algebras in terms of their generators and relations. However, the ring C_n^q (and so by extension its subring L_n^q) first arose in the work of Fordy and Marsh in [14] and [13], as a subalgebra of a cluster algebra arising from certain mutation-periodic quivers. The next two subsections provide an exposition of the parts of this work relevant to our topic, first in the context of commutative Poisson algebras, following the example of the papers just mentioned, and then the analogous quantum algebras.

8.2.1 Mutation-periodic quivers

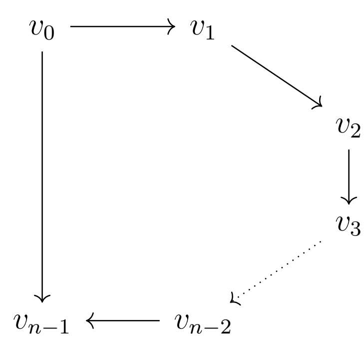
Definition 8.10. Let Q be a quiver with vertices $v_0, \dots, v_{n-1}$.

Let $\phi(Q)$ be the quiver with the same vertex set as Q but with $\phi(B)_{v_i v_j} = B_{v_{i-1} v_{j-1}}$, where the latter indices are taken modulo n - that is, if we draw the vertices in a circle labelled clockwise, then $\phi(Q)$ is Q rotated one place clockwise.

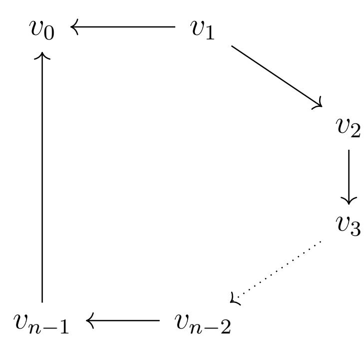
Let $Q^{(s)}$ denote the quiver obtained from Q by mutating at v_0, then v_1, and so on up to v_{s-1} - so $Q = Q^{(0)}$ and $Q' = Q^{(1)}$.

We say - following [14] - that Q is **mutation-periodic with period** s if $\phi^s(Q) = Q^{(s)}$.

Example 8.11. The quiver $P_n^{(1)}$, depicted below, is mutation-periodic with period 1 with the ordering given:



since the quiver obtained by mutating at v_0 is



Definition 8.12. Given two quivers Q_1, Q_2 with the same vertex set, with arrows given by $\mathbf{B}_1$ and $\mathbf{B}_2$ respectively, we can add these quivers by defining $Q_1 + Q_2$ to be the quiver with again the same vertex set and arrows given by $(B_+)_{uv} := (B_1)_{uv} + (B_2)_{uv}$.

Definition 8.13. Let $R_n^{(k)}$, $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$ denote the quiver with n vertices, ordered v_0, …, v_{n-1}, and a single arrow from v_0 to v_k.

If Q is a quiver with vertices v_0, …, v_{n-1}, let $\psi(Q)$ denote the quiver with the same vertex set as Q but any arrows at vertex v_0 reversed.

Let $P_n^{(k)} = \sum_{i=0}^{n-k-1} \phi^i(R_n^{(k)}) - \sum_{i=n-k}^{n-1} \phi^i(R_n^{(k)})$, for $1 \leq k < \frac{n}{2}$, and $P_n^{(k)} = \sum_{i=0}^{n-k-1} \phi^i(R_n^{(k)})$ if $k = \frac{n}{2}$.

Proposition 8.14. *$P_n^{(k)}$ is a mutation-periodic quiver of period 1.*

Proof. The terms which give arrows in $P_n^{(k)}$ to or from vertex 0 are $R_n^{(k)}$ and $-\phi^{n-k}(R_n^{(k)})$. Both of these are arrows away from $P_n^{(k)}$, so mutating at vertex 0 simply reverses these arrows. Thus:

$$\begin{aligned}
P_n^{(k)\prime} &= P_n^{(k)} - 2R_n^{(k)} + 2\phi^{n-k}(R_n^{(k)}) \\
&= \left(\sum_{i=0}^{n-k-1} \phi^i(R_n^{(k)})\right) - R_n^{(k)} + \phi^{n-k}(R_n^{(k)}) - \left(\sum_{i=n-k}^{n-1} \phi^i(R_n^{(k)})\right) + \phi^{n-k}(R_n^{(k)}) - \phi^n(R_n^{(k)}) \\
&= \sum_{i=1}^{n-k} \phi^i(R_n^{(k)}) - \sum_{i=n-k+1}^{n} \phi^i(R_n^{(k)}) \\
&= \phi(P_n^{(k)})
\end{aligned}$$

□

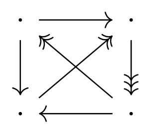
It turns out that every mutation-periodic quiver of period 1 is made up of quivers of this form:

Theorem 8.15. *[14, Thm 6.7] Let $b_i \in \mathbb{Z}$ for $1 \leq i \leq r$. Then $\sum_{i=0}^{r} b_i P_n^{(i)}$ is mutation-periodic of period 1 iff the b_i all have the same sign; however if it is not mutation-periodic then one can add "correction terms" to make it mutation-periodic: that is, there exist $e_{k,i} \in \mathbb{Z}$ such that $Q = \sum_{i=0}^{r} b_i P_n^{(i)} + \sum_{k=1}^{r-1} \sum_{i=0}^{r-2k} e_{k,i}\phi^k(P_{n-2k}^{(i)})$ is mutation-periodic. Furthermore, every period 1 mutation-periodic quiver is of this form.*

Given a mutation-periodic quiver of period 1, one can form a (commutative) seed simply by associating to v_i the variable w_i in the function field $F = k(w_0, \dots, w_{n-1})$. Then seed mutation at v_0 gives a new seed, with quiver isomorphic to the initial quiver, but "rotated" one place and with a new element $w_0' \in F$ at $v_0 = \phi(v_{n-1})$. This seed mutation extends ϕ to F, with $\phi(w_i) = w_{i+1}$ for $0 \leq i < n-1$ and $\phi(w_{n-1}) = w_0'$. With this in mind, we define $w_n := w_0'$. Repeating this process - this time, of course, mutating at v_1 instead - we get $w_{n-1} := w_1'$, and so on, giving a sequence $(w_i)_{i \in \mathbb{N}}$ of rational functions from F - actually, by the Laurent phenomenon, Laurent polynomials in the w_i. Since seed mutation is involutive, we can also go in the other direction - so starting by mutating the initial seed at v_{n-1}, extending the sequence w_i to $i \in \mathbb{Z}$. And with this choice of nomenclature, for each i the map $\phi : F \to F$ maps $(w_i, \dots, w_{n+i-1})$ to $(w_{i+1}, \dots, w_{n+i})$.

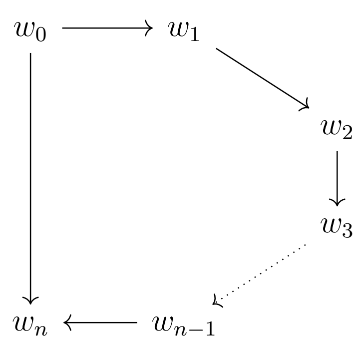
Remark. If we pick initial values for $w_0, \ldots, w_{n-1}$ then this sequence of functions becomes a sequence of rational numbers defined by a recurrence relation of the form $w_{n+i}w_i = \Pi_1 + \Pi_2$, where the Π_j are monomials in $w_{i+1}, \ldots, w_{n+i-1}$. If we take $w_0 = \cdots = w_{n-1} = 1$ then by the Laurent property, we get the surprising result that $w_i \in \mathbb{N}$ for all i.

For example, the Somos 4 sequence defined by $w_i w_{i+4} = w_{i+1}w_{i+3} + w_{i+2}^2$, $w_1 = w_2 = w_3 = w_4 = 1$, always takes integer values, since it arises from the following mutation-periodic quiver:



8.2.2 The quiver $P_{n+1}^{(1)}$

If n is an odd integer, then we form a commutative seed with quiver $P_{n+1}^{(1)}$, which we denote $(P_{n+1}^{(1)}, \mathbf{w})$. in the following natural fashion:



Proposition 8.16.

Let $L_{v_i v_j} = \begin{cases} 0 & \text{if } i+j \text{ is even} \\ 1 & \text{if } i+j \text{ is odd, } 0 \le i < j \le n \\ -1 & \text{if } i+j \text{ is odd, } 0 \le j < i \le n \end{cases}$

Then $\mathbf{L}$ *is compatible with* $P_{n+1}^{(1)}$*, and so* $\mathbf{P}_{n+1} := (P_{n+1}^{(1)}, \mathbf{w}, \mathbf{L})$ *is a quantum seed. In addition, any other integer matrix compatible with* $P_{n+1}^{(1)}$ *is an integer multiple of* $\mathbf{L}$.

If we mutate $\mathbf{P}_{n+1}$ *at vertex* w_0*, so* $P_{n+1}^{(1)\prime} = \phi(P_{n+1})$*, then* $L'_{\phi(v)\phi(w)} = L_{vw}$*: therefore, there is a well-defined map* $\phi : T_q(\mathbf{w}, \mathbf{L}) \to T_q(\mathbf{w}', \mathbf{L}')$ *given by* $w_i \mapsto w_{i+1}$ *for* $0 \le i \le n-1$*,* $w_n \mapsto w_0'$.

Proof. $(\mathbf{B^T L})_{jk} = \sum_i B_{v_i v_j} L_{v_i v_k} = B_{v_{j-1} v_j} L_{v_{j-1} v_k} + B_{v_{j+1} v_j} L_{v_{j+1} v_k}$, where addition of subsubscripts is taken modulo $n+1$.

If $0 < j < n$ then $(\mathbf{B^T L})_{jk} = L_{v_{j-1}v_k} - L_{v_{j+1}v_k}$. Checking this with the definition of $\mathbf{L}$ gives $(\mathbf{B^T L})_{jk} = 0$ when $j \neq k$ and $(\mathbf{B^T L})_{jj} = 2$.

If $j = 0$ then $(\mathbf{B^T L})_{jk} = -L_{v_n v_k} - L_{v_1 v_k}$; this equals 0 when $k \neq 0$ and 2 when $k = 0$.

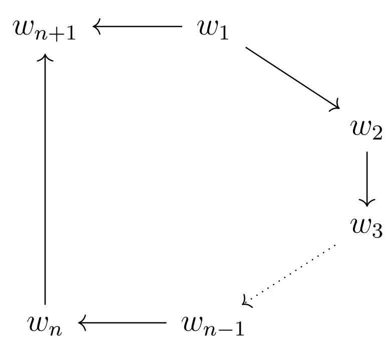
If $j = n$ then $(\mathbf{B^T L})_{jk} = L_{v_{n-1}v_k} + L_{v_0 v_k}$; this equals 0 when $k \neq n$ and 2 when $k = n$.

Thus $\mathbf{B^T L} = \mathbf{2I}_{n+1}$ and $\mathbf{L}$ is compatible with the above quiver.

Let $\mathbf{A}$ be such that $\mathbf{B^T A} = \lambda \mathbf{I}_{n+1}$, for some non-zero scalar λ. Then by uniqueness of inverses for square matrices, $\mathbf{A} = \frac{\lambda}{2}\mathbf{L}$; $\mathbf{A}$ is then an integer matrix only if $\lambda \in 2\mathbf{Z}$, and the uniqueness of $\mathbf{L}$ is shown.

If neither v nor w equals v_n - so neither $\phi(v)$ nor $\phi(w)$ equals v_0 then $L'_{\phi(v)\phi(w)} = L_{\phi(v)\phi(w)} = L_{vw}$ directly from the definition of $\mathbf{L}$. Otherwise, the definition of seed mutation give us $w'_0 = w_0^{-1} + q w_0^{-1} w_1 w_n$. From this we get that $w_i w'_0 = w'_0 w_i$ if i is even and $w_i w'_0 = q w'_0 w_i$ if i is odd, and so $L'_{v_i v_0} = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd} \end{cases}$, so $L'_{v_i v_0} = L_{v_{i-1} v_n}$ as desired. □

With this in mind, we will write $w_{n+1} := w'_0$, so the mutated seed $\mathbf{P}'_{n+1}$ is given by:



Furthermore, the rule for defining the skew-commutators holds with this notation, that is, $\mathbf{L}'$ is given by $L'_{v_i v_j} = \begin{cases} 0 & \text{if } i + j \text{ is even} \\ 1 & \text{if } i + j \text{ is odd, } 1 \leq i < j \leq n+1 \\ -1 & \text{if } i + j \text{ is odd, } 1 \leq j < i \leq n+1 \end{cases}$

Repeating this process, as in the commutative case, by mutating $\mathbf{P}'_{n+1}$ at v_1 and so on, we get a sequence of cluster variables w_i for all $i \in \mathbb{N}$, defined recursively by $w_i := w_{i-n-1}^{-1} + q w_{i-n-1}^{-1} w_{i-n} w_{i-1}$.

Similarly (or using the involutive property of seed mutation), mutating $\mathbf{P}_{n+1}$ at v_n gives $\phi^{-1}(\mathbf{P}_{n+1})$, with $w_{-1} := w'_n = w_n^{-1} + q^{-1} w_n^{-1} w_0 w_{n-1}$, and repeating this lets us define w_{-i} for all $i \in \mathbb{N}$.

As in the commutative case, we extend the map ϕ to the quantum seeds, so $\phi(w_i) = w_{i+1}$.

Definition 8.17. Next we consider what happens when we mutate $\mathbf{P}_{n+1}$ at some other vertex. If we mutate at v_1 then

$x_1 := w_1' = q^{\frac{1}{2}}(w_0 w_1^{-1} + w_1^{-1} w_2)$.

Similarly if we mutate $\mathbf{P}_{n+1}$ at v_i for $1 \leq i \leq n-1$ we get a mutated variable

$x_i := w_i' = q^{\frac{1}{2}}(w_{i-1} w_i^{-1} + w_i^{-1} w_{i+1})$.

If we mutate $\phi(\mathbf{P}_{n+1})$ at v_n - so mutating first at v_0 and then at v_n - then we get

$x_n := w_n' = q^{\frac{1}{2}}(w_{n-1} w_n^{-1} + w_n^{-1} w_{n+1})$.

(The following proposition explains the notation x_i).

Proposition 8.18. *(i)* $x_{i+1} = \phi(x_i)$*, for* $1 \leq i \leq n-1$.

(ii) $\phi(x_n) = x_1$.

(iii) *(a)* $x_1 x_2 = q^2 x_2 x_1 + (1 - q^2)$.

(b) $x_1 x_i = q^2 x_i x_1$ *for* $1 < i < n$, i *even.*

(c) $x_1 x_i = q^{-2} x_i x_1$ *for* $1 < i < n$, i *odd.*

(iv) *The elements* $x_1, \ldots, x_n$ *of* $T_q(\mathbf{w}, \mathbf{L})$ *satisfy the defining relations for* $C_n^{q^2}$.

Proof. (i) $\phi(x_i)$ is obtained by mutating first at v_0 and then at v_{i+1}, so when $i = n-1$, $x_{i+1} = \phi(x_i)$ follows directly from the definition. If $1 < i+1 < n$, mutating at v_0 does not change v_{i+1} or any of its neighbours, so cannot change w_{i+1}', so $\phi(x_i) = x_{i+1}$.

(ii)
$$\begin{aligned}
\phi(x_n) &= q^{\frac{1}{2}}(w_n w_{n+1}^{-1} + w_{n+1}^{-1} w_{n+2}) \\
&= q^{\frac{1}{2}}(w_n w_{n+1}^{-1} + q^{-1} w_{n+2} w_{n+1}^{-1}) \\
&= q^{\frac{1}{2}}(w_n(w_0^{-1} + q w_0^{-1} w_1 w_n)^{-1} + (q^{-1} w_1^{-1} + w_1^{-1} w_2 w_{n+1}) w_{n+1}^{-1}) \\
&= q^{\frac{1}{2}}(w_n(1 + q w_1 w_n)^{-1} w_0 + q^{-1} w_1^{-1}(1 + q w_1 w_n)^{-1} w_0 + w_1^{-1} w_2) \\
&= q^{\frac{1}{2}}((1 + q w_1 w_n)^{-1} w_n w_0 + q^{-1}(1 + q w_1 w_n)^{-1} w_1^{-1} w_0 + w_1^{-1} w_2) \\
&= q^{\frac{1}{2}}((1 + q w_1 w_n)^{-1}(w_n w_1 + q^{-1}) w_1^{-1} w_0 + w_1^{-1} w_2) \\
&= q^{\frac{1}{2}}((1 + q w_1 w_n)^{-1}(q w_1 w_n + 1) w_0 w_1^{-1} + w_1^{-1} w_2) \\
&= w_0 w_1^{-1} + w_1^{-1} w_2 \\
&= x_1
\end{aligned}$$

(iii) (a) $x_1x_2 = q(w_0w_1^{-1} + w_1^{-1}w_2)(w_1w_2^{-1} + w_2^{-1}w_3)$

$$= q(w_0w_1^{-1}w_1w_2^{-1} + w_0w_1^{-1}w_2^{-1}w_3 + w_1^{-1}w_2w_1w_2^{-1} + w_1^{-1}w_2w_2^{-1}w_3)$$

$$= q(q^2w_1w_2^{-1}w_0w_1^{-1} + q^2w_2^{-1}w_3w_0w_1^{-1} + w_1w_2^{-1}w_1^{-1}w_2 + q^2w_2^{-1}w_3w_1^{-1}w_2)$$

$$= q^2x_2x_1 + q(1-q^2)w_1w_2^{-1}w_1^{-1}w_2$$

$$= q^2x_2x_1 + (1-q^2)w_1w_1^{-1}w_2^{-1}w_2$$

$$= q^2x_2x_1 + (1-q^2)$$

(b) For $1 < i < n$, i even,

$$x_1x_i = q(w_0w_1^{-1} + w_1^{-1}w_2)(w_{i-1}w_i^{-1} + w_i^{-1}w_{i+1})$$

$$= q(w_0w_1^{-1}w_{i-1}w_i^{-1} + w_0w_1^{-1}w_i^{-1}w_{i+1} + w_1^{-1}w_2w_{i-1}w_i^{-1} + w_1^{-1}w_2w_i^{-1}w_{i+1})$$

$$= \begin{array}{l} q(q^2w_{i-1}w_i^{-1}w_0w_1^{-1} + q^2w_i^{-1}w_{i+1}w_0w_1^{-1} \\ \quad + q^2w_{i-1}w_i^{-1}w_1^{-1}w_2 + q^2w_i^{-1}w_{i+1}w_1^{-1}w_2) \end{array}$$

$$= q^2x_ix_1.$$

(c) For $1 < i < n$, i odd,

$$x_1x_i = q(w_0w_1^{-1} + w_1^{-1}w_2)(w_{i-1}w_i^{-1} + w_i^{-1}w_{i+1})$$

$$= q(w_0w_1^{-1}w_{i-1}w_i^{-1} + w_0w_1^{-1}w_i^{-1}w_{i+1} + w_1^{-1}w_2w_{i-1}w_i^{-1} + w_1^{-1}w_2w_i^{-1}w_{i+1})$$

$$= \begin{array}{l} q(q^{-2}w_{i-1}w_i^{-1}w_0w_1^{-1} + q^{-2}w_i^{-1}w_{i+1}w_0w_1^{-1} \\ \quad + q^{-2}w_{i-1}w_i^{-1}w_1^{-1}w_2 + q^{-2}w_i^{-1}w_{i+1}w_1^{-1}w_2) \end{array}$$

$$= q^{-2}x_ix_1.$$

(iv) This follows directly from (iii) together with the properties $\phi(x_i) = x_{i+1}$ and $\phi^n(x_i) = x_i$.

□

Remark. The subalgebra of $A_q(\mathbf{P}_{n+1})$ generated by the x_i is therefore a factor ring of $C_n^{q^2}$ - it might not be the full ring.

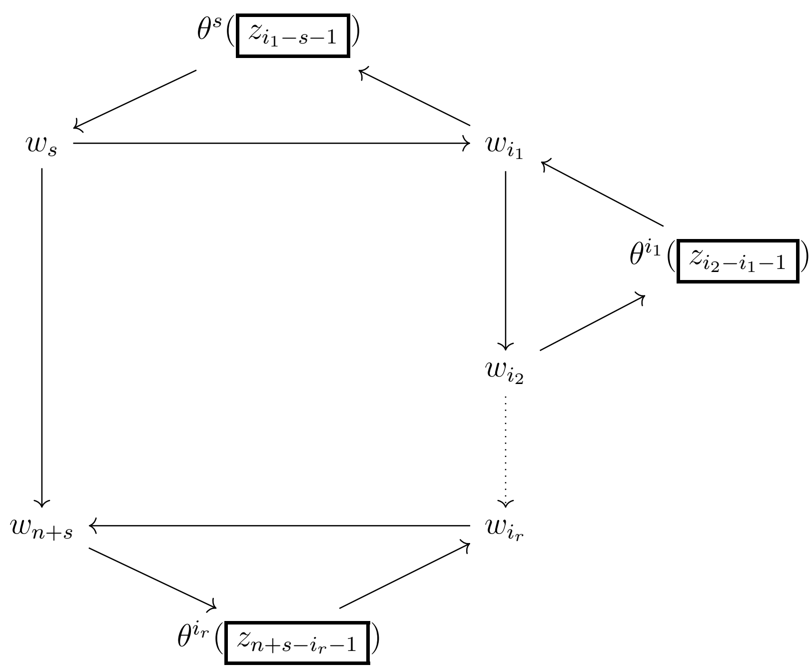
Remark. This raises the question of whether there might be a quantum cluster algebra structure on C_n^q itself. Although proving non-existence of such structures is in general hard, it seems unlikely here given what we know about the prime ideal structure of C_n^q: firstly, quantum cluster algebras tend to be more well-behaved than C_n^q, with no prime ideals that are not completely prime; second, we know by Corollary 7.10 that if the quiver is acyclic - and most quivers are mutation-equivalent to an acyclic one - $\Omega - \lambda$ must be a frozen cluster variable, and the only one, which limits the possibilities for an initial seed quite considerably.

Extending the results of Section 7 might give an approach to settling this question.

8.2.3 The set of cluster variables

The aim of this section is to show that any cluster variable of the cluster algebra $A_q(\mathbf{P}_{n+1})$ is one of the w_i, $i \in \mathbb{Z}$, or is of the form $\theta^i(\hat{z}_j)$, where $0 \leq i \leq n-1$, $1 \leq j \leq n-1$, and $\hat{z}_j$ is as in Definition 8.2. As just remarked, we might not actually have $\theta^i(\hat{z}_j)$, we may have its image in some factor ring of $C_n^{q^2}$ instead, but we will abuse notation slightly and write $\theta^i(\hat{z}_j)$ anyway.

Definition 8.19. To do this we claim that any seed obtained by mutating the initial seed is of the following form, which we call a $C_n^{q^2}$**-seed**:



In this diagram:

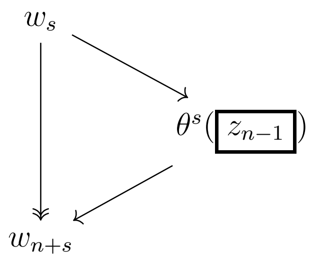
$\boxed{z_b}$ represents a $L_b^{q^2}$-seed, as in Definition 8.2, except that we do not freeze $\hat{z}_b$, and instead the arrows to or from the $\boxed{z_b}$ are drawn to or from $\hat{z}_b$.

$\theta^a(\boxed{z_b})$ is similar but with θ^a applied to every variable of the $L_b^{q^2}$-seed.

r is an integer between 0 and $n-1$, and $s = i_0 < i_1 < \cdots < i_r < i_{r+1} = n+s$ are integers.

If $i_r = i_{r-1} + 1$ then we would get a $\boxed{1}$ which we remove.

If $r = 0$ then we interpret this as follows:



where the double-headed arrow represents an arrow of weight 2.

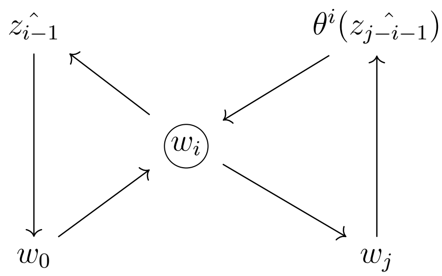
We call w_s a **quasi-source**, w_{n+s} a **quasi-sink**, and the other w_i **link nodes**.

We note that since there are b vertices in the quiver for a $\theta^a(L_b^{q^2})$-seed, there are $n+1$ vertices in a quiver of this form, as there should be.

We also note that applying ϕ to every vertex of a $C_n^{q^2}$-seed yields another $C_n^{q^2}$-seed.

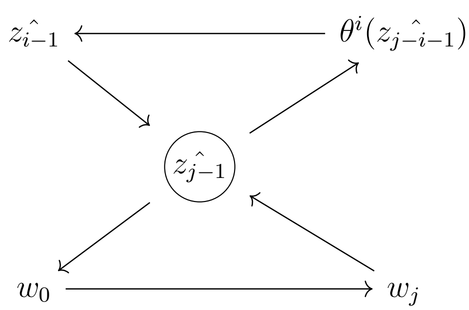
Proposition 8.20. *The neighbourhoods of a vertex (the vertex we are considering is circled in the diagrams below) in a $C_n^{q^2}$-seed take the following form, up to translation by ϕ, and removing vertices labelled by $\theta^l(\hat{z}_0)$ for some l:*

(i)



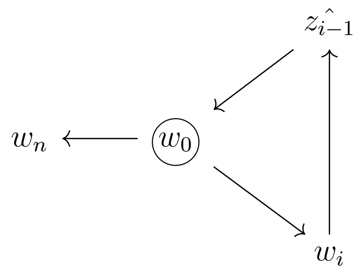
where $0 < i < j < n$, w_0 can be a quasi-source or a link node, w_i is a link node, and w_j can be a quasi-sink or a link node (but since $j < n$ at least one of w_0 and w_j must be a link node).

(ii)



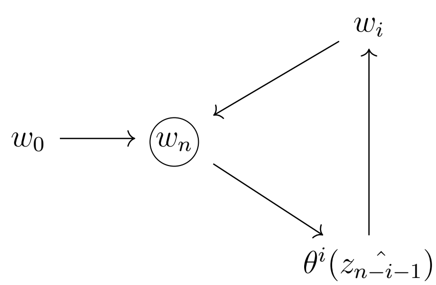
where $0 < j < n$, $0 \leq i \leq j$, w_0 can be a quasi-source or a link node, and w_j can be a quasi-sink or a link node (but since $j < n$ at least one of w_0 and w_j must be a link node).

(iii)



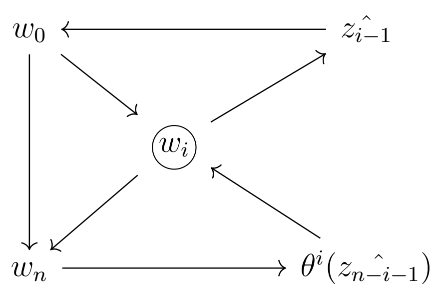
where $0 < i < n$, w_0 *is a quasi-source,* w_n *is a quasi-sink, and* w_i *is a link node.*

(iv)



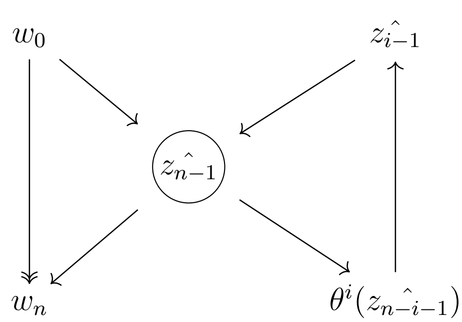
where again $0 < i < n$, w_0 *is a quasi-source,* w_n *is a quasi-sink, and* w_i *is a link node.*

(v)



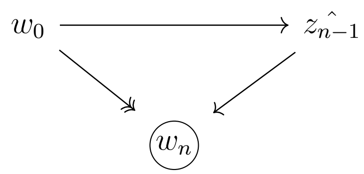
where again $0 < i < n$, w_0 *is a quasi-source,* w_n *is a quasi-sink, and* w_i *is a link node.*

(vi)

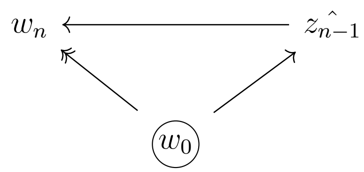


where $0 \leq i \leq n - 1$.

(vii)

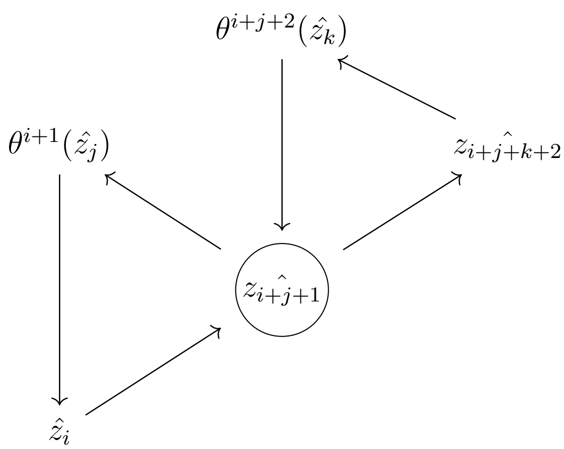


(viii)



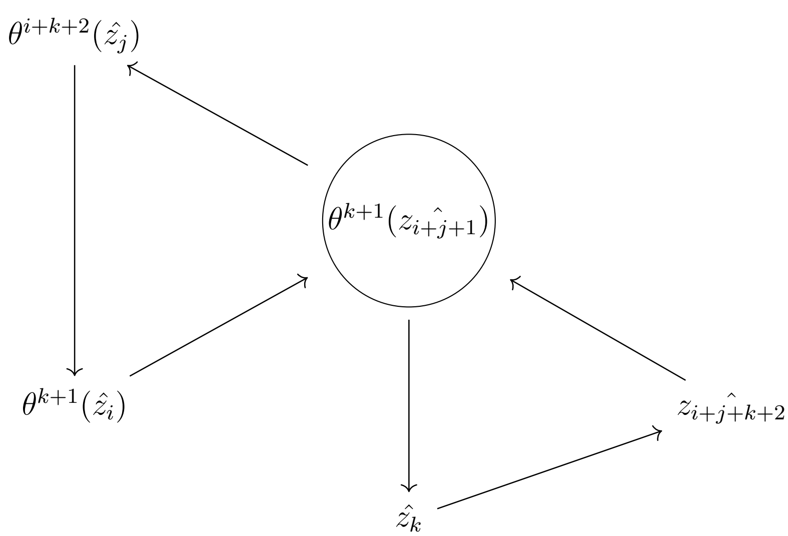
(In the last three it is clear that w_0 *is a quasi-source and* w_n *is a quasi-sink.)*

(ix)



where $0 \leq i, j, k$ *and* $i + j + k + 2 \leq n - 1$.

(x)



where $0 \leq i, j, k$ *and* $i + j + k + 2 \leq n - 1$.

Proof. Every vertex in the pictures in Definition 8.19 has a neighbourhood that is in one of these forms, whatever the value of r in that definition. $\square$

Proposition 8.21. *The skew commutation relations involving the x_i and the w_j are:*

$w_i w_j = w_j w_i$ *if $i + j$ is even, $i < j$;*

$w_i w_j = q w_j w_i$ *if $i + j$ is odd, $i < j$;*

$w_i x_j = q x_j w_i$ *if $i + j$ is even, $i < j$;*

$w_i x_j = q^{-1} x_j w_i$ *if $i + j$ is odd, $i < j$;*

$x_i w_j = q w_j x_i$ *if $i + j$ is even, $i < j$;*

$x_i w_j = q^{-1} w_j x_i$ *if $i + j$ is odd, $i < j$.*

So, for $i \leq j$, we have:

$w_i \theta^j(z_k) = \theta^j(z_k) w_i$ *if k is even;*

$w_i \theta^j(z_k) = q\theta^j(z_k) w_i$ *if k is odd and $i + j$ is odd;*

$w_i \theta^j(z_k) = q^{-1}\theta^j(z_k) w_i$ *if k is odd and $i + j$ is even.*

While for $k < i$, we have:

$z_k w_i = w_i z_k$ *if k is even;*

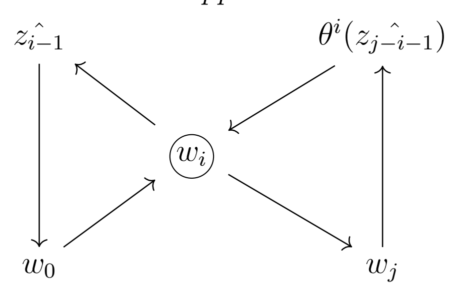
$z_k w_i = q^{-1} w_i z_k$ *if k is odd and i is even;*

$z_k w_i = q w_i z_k$ *if k is odd and i is odd.*

Proof. For the relations involving w_i and x_j, we use $x_i = q^{\frac{1}{2}}(w_{i-1}w_i^{-1} + w_i^{-1}w_{i+1})$, and then check that the relations hold in $T_q(\mathbf{w}, \mathbf{L})$. For example, if $i < j$ and $i+j$ is odd, w_i commutes with w_{j-1} and w_{j+1}, and $w_i w_j^{-1} = q^{-1} w_j^{-1} w_i$, so since $x_j = q^{\frac{1}{2}}(w_{j-1}w_j^{-1} + w_j^{-1}w_{j+1})$, $w_i x_j = q^{-1} x_j w_i$. The remaining cases are similar.

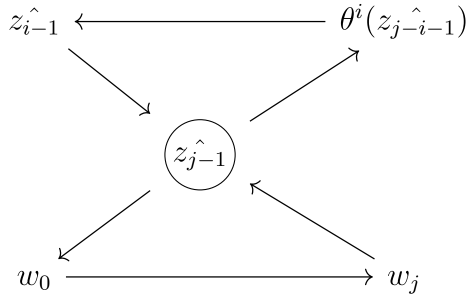
The remaining relations follow, either by induction on k or by considering the numbers of x_r of each parity in each term of z_k. $\square$

Lemma 8.22. *Suppose the circled vertex has the following neighbourhood:*



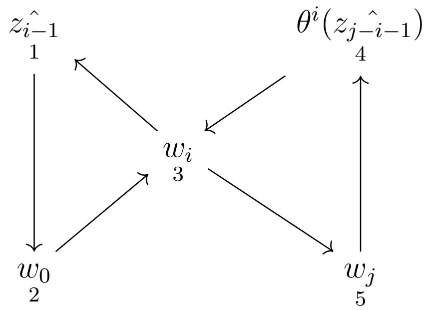
where $0 < i < j < n$, w_0 can be a quasi-source or a link node, w_i is a link node, and w_j can be a quasi-sink or a link node (but since $j < n$ at least one of w_0 and w_j must be a link node).

Mutating at the circled vertex gives



where w_0 and w_j are of the same type (quasi-source, quasi-sink or link node) they started as.

Proof. The quiver mutation is easy to check. For the variable mutation, label the vertices as follows:



Therefore $\mathbf{b}_3^+ = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \end{pmatrix}^T$ and $\mathbf{b}_3^- = \begin{pmatrix} 0 & 1 & -1 & 1 & 0 \end{pmatrix}^T$.

We know $w_i' = q^{\lambda(\mathbf{b}_3^+)} z_{\widehat{i-1}} w_i^{-1} w_j + q^{\lambda(\mathbf{b}_3^-)} w_0 w_i^{-1} \theta^i(z_{\widehat{j-i-1}})$, and we wish to show $w_i' = z_{\widehat{j-1}}$, so we wish to show $z_{j-1} = q^a z_{i-1} w_i^{-1} w_j + q^b w_0 w_i^{-1} \theta^i(z_{j-i-1})$, where $a = t(j-1) + \lambda(\mathbf{b}_3^+) - t(i-1)$ and $b = t(j-1) + \lambda(\mathbf{b}_3^+) - t(j-i-1)$.

The following table calculates $\lambda(\mathbf{b}_3^\pm)$, depending on the parity of i and j:

i	j	L_{15}	L_{13}	L_{35}	L_{24}	L_{23}	L_{34}	$\lambda(\mathbf{b}_3^+)$	$\lambda(\mathbf{b}_3^-)$
even	even	1	1	0	-1	0	-1	0	0
odd	even	0	0	1	0	1	0	1	1
even	odd	1	-1	1	0	0	0	-1	0
odd	odd	0	0	0	1	1	-1	0	-1

The following table calculates a and b, depending on the parity of i and j:

i	j	t(j-1)	-t(i-1)	$\lambda(\mathbf{b}_3^+)$	-t(j-i-1)	$\lambda(\mathbf{b}_3^-)$	a	b
even	even	(j-2)/2	-(i-2)/2	0	-(j-i-2)/2	0	(j-i)/2	i/2
odd	even	(j-2)/2	-(i-1)/2	1/2	-(j-i-1)/2	1/2	(j-i)/2	i/2
even	odd	(j-1)/2	-(i-2)/2	-1/2	-(j-i-1)/2	0	(j-i)/2	i/2
odd	odd	(j-1)/2	-(i-1)/2	0	-(j-i-2)/2	-1/2	(j-i)/2	i/2

So we wish to show:

$z_{j-1} = q^{(j-i)/2} z_{i-1} w_i^{-1} w_j + q^{i/2} w_0 w_i^{-1} \theta^i(z_{j-i-1})$

We prove this by induction on j; the base cases needed are $j = i$, which is clear, and $j = i+1$, which we prove by induction on i. Here the base case is $i = 1$, where the RHS is $q^{1/2} w_1^{-1} w_2 + q^{1/2} w_0 w_1^{-1} = x_1 = z_1$ which equals the LHS.

For the induction step,

$$q^{1/2} z_{i-2} w_{i-1}^{-1} w_i + q^{(i-1)/2} w_0 w_{i-1}^{-1} = z_{i-1} \qquad \text{by the induction hypothesis}$$

$$q^{-1/2} z_{i-2} w_i + q^{(i-1)/2} w_0 = z_{i-1} w_{i-1}$$

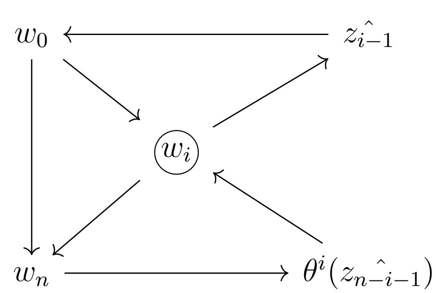
$$\begin{aligned} q^{1/2} z_{i-1} w_i^{-1} w_{i+1} + q^{i/2} w_0 w_i^{-1} &= z_{i-1} x_i - q^{1/2} z_{i-1} w_{i-1} w_i^{-1} + q^{i/2} w_0 w_i^{-1} \\ &= z_{i-1} x_i - q^{1/2}(q^{-1/2} z_{i-2} w_i + q^{(i-1)/2} w_0) w_i^{-1} + q^{i/2} w_0 w_i^{-1} \\ &= z_{i-1} x_i - z_{i-2} \\ &= z_i. \end{aligned}$$

Now the induction step for the induction on j:

$$\begin{aligned} & q^{(j-i)/2} z_{i-1} w_i^{-1} w_j + q^{i/2} w_0 w_i^{-1} \theta^i(z_{j-i-1}) \\ &= q^{(j-i)/2} z_{i-1} w_i^{-1} (q^{-1/2} w_{j-1} x_{j-1} - q^{-1} w_{j-2}) + q^{i/2} w_0 w_i^{-1} (\theta^i(z_{j-i-2}) x_{j-1} - \theta^i(z_{j-i-3})) \\ &= (q^{(j-i-1)/2} z_{i-1} w_i^{-1} w_{j-1} + q^{i/2} w_0 w_i^{-1} \theta^i(z_{j-i-2})) x_{j-1} \\ &\qquad - (q^{(j-i-2)/2} z_{i-1} w_i^{-1} w_{j-2} + q^{i/2} w_0 w_i^{-1} \theta^i(z_{j-i-3})) \\ &= z_{j-2} x_{j-1} - z_{j-3} \text{ by the induction hypothesis} \\ &= z_{j-1} \end{aligned}$$

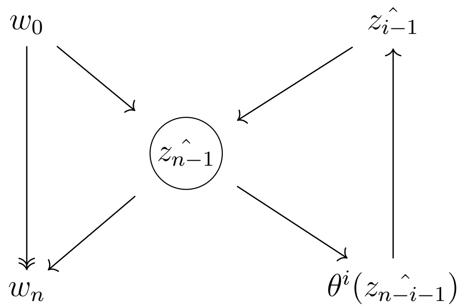
□

Corollary 8.23. *Suppose the circled vertex has the following neighbourhood:*



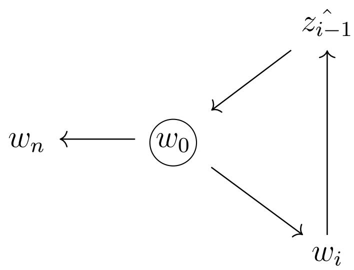
where $0 < i < n$.

Mutating at the circled vertex gives

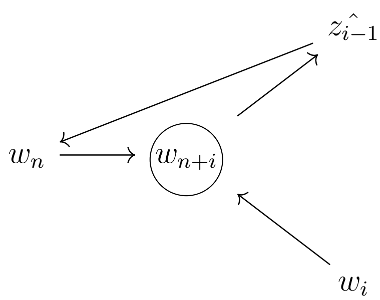


Proof. The quiver mutation is as usual easy to check. To check the variable mutation we note that the calculations in Lemma 8.22 hold for $j = n$, and are precisely the calculations needed here (since the "extra" arrow from w_0 to w_n does not affect the variable mutation). $\square$

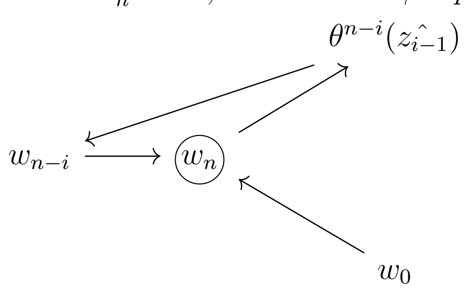
Lemma 8.24. *Suppose the circled vertex has the following neighbourhood:*



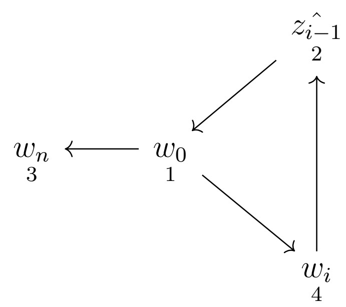
where $0 < i < n$, so w_n is a quasi-sink, w_0 is a quasi-source and w_i is a link node. Mutating at the circled vertex gives



where w_n is a link node, w_{n+I} is a quasi-sink and w_i is a quasi-source, and so the seed remains a $C_n^{q^2}$-seed, since this is ϕ^i applied to



Proof. The quiver mutation is easy to check. For the variable mutation, label the vertices as follows:



Therefore $\mathbf{b}_1^+ = \begin{pmatrix} -1 & 0 & 1 & 1 \end{pmatrix}^T$ and $\mathbf{b}_1^- = \begin{pmatrix} -1 & 1 & 0 & 0 \end{pmatrix}^T$.

We know $w_n = q^{\lambda(\mathbf{b}_3^+)} w_0^{-1} z_{\hat{i-1}} + q^{\lambda(\mathbf{b}_3^-)} w_0^{-1} w_n w_i$, and wish to show $w'_n = w_{n+i}$, so we wish to show $w_{n+i} = q^a w_0^{-1} z_{i-1} + q^b w_0^{-1} w_n w_i$, where $a = \lambda(\mathbf{b}_3^+) - t(i-1)$ and $b = \lambda(\mathbf{b}_3^-)$. The following table calculates a and b:

i	L_{13}	L_{14}	L_{34}	L_{12}	$\lambda(\mathbf{b}_3^+)$	$\lambda(\mathbf{b}_3^-)$	-t(i-1)	a	b
even	0	1	-1	-1	0	0	-(i-2)/2	1	-(i-1)/2
odd	1	1	0	0	2/2	0	-(i-1)/2	1	-(i-1)/2

So we wish to show:

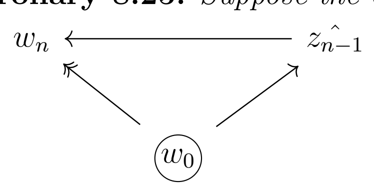
$w_{n+i} = q^{-(i-1)/2} w_0^{-1} z_{i-1} + q w_0^{-1} w_n w_i$

We prove this by induction on i. The base cases are $i = 0$, which is clear, and $i = 1$, which becomes $w_{n+1} = w_0^{-1} + q w_0^{-1} w_n w_1$ which is the definition of w_{n+1}. For the induction step:

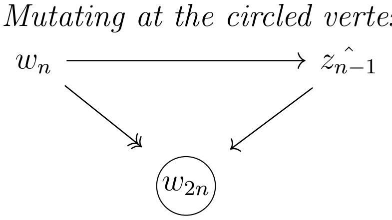
$$\begin{aligned} & q^{-(i-1)/2} w_0^{-1} z_{i-1} + q w_0^{-1} w_n w_i \\ &= q^{-(i-1)/2} w_0^{-1} (z_{i-2} x_{i-1} - z_{i-3}) + q w_0^{-1} w_n (q^{-1/2} w_{i-1} x_{i-1} - q^{-1} w_{i-2}) \\ &= q^{-1/2} (q^{-(i-2)/2} w_0^{-1} z_{i-2} + q w_0^{-1} w_n w_{i-1}) x_{i-1} - q^{-1} (q^{-(i-3)/2} w_0^{-1} z_{i-3} + w_0^{-1} w_n w_{i-2}) \\ &= q^{-1/2} w_{n+i-1} x_{n+i-1} - q^{-1} w_{n+i-2} \text{ by induction and } x_j = x_{n+j} \\ &= w_{n+i} \end{aligned}$$

□

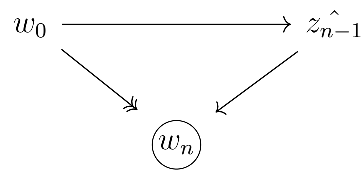
Corollary 8.25. *Suppose the circled vertex has the following neighbourhood:*



Mutating at the circled vertex gives:



which is ϕ^n applied to



Proof. The quiver mutation is as usual easy to check. To check the variable mutation we note that the calculations in Lemma 8.24 hold for $i = n$, and are precisely the calculations needed here. □

Theorem 8.26. *Mutating a $C_n^{q^2}$-seed gives a $C_n^{q^2}$ seed.*

Proof. Proposition 8.20 gave 10 cases for the neighbourhood of a vertex in a $C_n^{q^2}$-seed. Lemma 8.22 shows that (i) mutates to (ii), and vice versa since seed mutation is involutive; Corollary 8.23 shows that (iii) mutates to (iv) and vice versa; Lemma 8.24 shows that (v) mutates to (vi) and vice versa; Corollary 8.25 shows that (vii) mutates to (viii) and vice versa; and Lemma 8.6 shows that (ix) mutates to (x) and vice versa. So all the cases are covered, so any mutation of a $C_n^{q^2}$-seed gives another $C_n^{q^2}$-seed. □

Remark. The above all still holds in the commutative case. Further, in the commutative case one can take n to be even, since there is no requirement for there to exist a compatible matrix $\mathbf{L}$, and the above all still holds.

8.2.4 Some ring-theoretic properties

For ease of notation, fix n odd and q a non-zero scalar in k which is not a root of unity, and then write $Q := A_q(\mathbf{P}_{n+1})$.

Proposition 8.27. *Q is generated by $\{w_0; x_1, x_2, \ldots, x_n\}$.*

Proof 1. By Theorem 8.26 and the definition of a quantum cluster algebra, Q is generated by $\{w_i : i \in \mathbb{Z}\} \cup \{\theta^i(z_j) : i \in \mathbb{Z}, 1 \leq j \leq n-1\}$. The definitions of the z_j and the fact that $\theta^n = 1$ allow us to reduce this to $\{w_i : i \in \mathbb{Z}\} \cup \{x_1, \ldots, x_n\}$. The relations of the form $x_i w_i - q w_i x_i = q^{\frac{1}{2}}(q^{-1} - q) w_{i+1}$ and $x_i w_i - q^{-1} w_i x_i = q^{\frac{1}{2}}(1 - q^{-2}) w_{i-1}$ then allow us to remove all but one of the w_i from the generating set, wlog w_0. □

Proof 2. By Theorem 2.95, Q is generated by $\{w_{-1}, \ldots, w_{n+1}; x_1, \ldots, x_{n-1}\}$; if we add x_n this is still true, then we apply the same argument as in Proof 1 to reduce this generating set to $\{w_0; x_1, x_2, \ldots, x_n\}$. □

Theorem 8.28. *Q is simple.*

Proof. The quiver $P^{(1)}_{n-1}$ is acyclic, so this follows directly from Corollary 7.11. □

Theorem 8.29. *Q is Noetherian.*

Proof. We construct an iterated skew polynomial ring in $n+2$ generators that surjects onto Q.

We start with $R_0 := k[v_1][v_2; \alpha_1]$, where $\alpha_1(v_1) = qv_1$.

Define $R_1 := R_0[y_1; \beta_1, \delta_1]$ where $\beta_1(v_1) = q^{-1}v_1$, $\beta_1(v_2) = qv_2$, $\delta_1(v_1) = q^{-\frac{1}{2}}(q - q^{-1})v_2$, and $\delta_2(v_2) = 0$.

Define $R_2 := R_1[y_2; \beta_2, \delta_2]$ where $\beta_2(v_1) = q^{-1}v_1$, $\beta_2(v_2) = qv_2$, $\beta_2(y_1) = q^2 y_1$, $\delta_2(v_1) = 0$, $\delta_2(v_2) = q^{-\frac{1}{2}}(1 - q^2)v_1$ and $\delta_2(y_1) = 1 - q^2$.

For $3 \le i \le n-1$, define $R_i := R_{i-1}[y_i; \beta_i, \delta_i]$, where $\beta_i(v_j) = \begin{cases} qv_j & \text{if } i+j \text{ is even} \\ q^{-1}v_j & \text{if } i+j \text{ is odd} \end{cases}$ for $j = 1, 2$; $\beta_i(y_j) = \begin{cases} q^2 y_j & \text{if } i+j \text{ is odd} \\ q^{-2} y_j & \text{if } i+j \text{ is even} \end{cases}$ for $1 \le j < i$; $\delta_i(v_j) = 0$ for $j = 1, 2$ and $1 \le j \le i-1$; $\delta_i(y_j) = 0$ for $1 \le j < i-1$; and $\delta_i(y_{i-1}) = 1 - q^2$.

Finally define $R_n := R_{n-1}[y_n; \beta_n, \delta_n]$, where $\beta_n(v_j) = \begin{cases} qv_j & \text{if } n+j \text{ is even} \\ q^{-1}v_j & \text{if } n+j \text{ is odd} \end{cases}$ for $j = 1, 2$; $\beta_n(y_j) = \begin{cases} q^2 y_j & \text{if } n+j \text{ is odd} \\ q^{-2} y_j & \text{if } n+j \text{ is even} \end{cases}$ for $1 \le j < i$; $\delta_n(v_j) = 0$ for $j = 1, 2$; $\delta_n(y_j) = 0$ for $1 < j < n-1$; $\delta_n(y_{n-1}) = 1 - q^2$; and $\delta_n(y_1) = 1 - q^{-2}$.

At each stage it is straightforward to check that β_i is an automorphism of R_{i-1} and δ_i is a β_i-derivation of R_{i-1}. (It suffices to check that β_i and δ_i preserve the defining relations of R_{i-1}).

Then we define $\gamma : R_n \to Q$ by $\gamma(v_i) = w_i$ for $i = 1, 2$, and $\gamma(y_i) = x_i$ for $1 \le i \le n$. It is straightforward to check that γ is well defined, i.e. that the defining relations of R_n are all satisfied by their images in Q. And γ is surjective, by Proposition 8.27. So Q is a homomorphic image of R_n, which is a Noetherian ring, and so is Noetherian itself. □

9 References

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