Constructivity and Predicativity: Philosophical Foundations

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The candidate confirms that the work submitted is her own and that appropriate credit has been given where reference has been made to the work of others.

Abstract

The thesis examines two dimensions of constructivity that manifest themselves within foundational systems for Bishop constructive mathematics: *intuitionistic logic* and *predicativity*. The latter, in particular, is the main focus of the thesis. The use of intuitionistic logic affects the notion of *proof*: constructive proofs may be seen as very general *algorithms*. Predicativity relates instead to the notion of *set*: predicative sets are viewed as if they were constructed *from within and step by step*. The first part of the thesis clarifies the algorithmic nature of intuitionistic proofs, and explores the consequences of developing mathematics according to a constructive notion of proof. It also emphasizes intra-mathematical and pragmatic reasons for doing mathematics constructively. The second part of the thesis discusses predicativity. Predicativity expresses a kind of constructivity that has been appealed to both in the classical and in the constructive tradition. The thesis therefore addresses both classical and constructive variants of predicativity. It examines the origins of predicativity, its motives and some of the fundamental logical advances that were induced by the philosophical reflection on predicativity. It also investigates the relation between a number of distinct proposals for predicativity that appeared in the literature: strict predicativity, predicativity given the natural numbers and constructive predicativity. It advances a predicative concept of set as unifying theme that runs across both the classical and the constructive tradition, and identifies it as a forefather of a computational notion of set that is to be found in constructive type theories. Finally, it turns to the question of which portions of scientifically applicable mathematics can be carried out predicatively, invoking recent technical work in mathematical logic.

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Contents

Introduction

Recent times have witnessed transformations within mathematics so profound that it is natural to compare the present changes to the deep reformation of mathematics that started at the beginning of the 19th Century. A very prominent aspect is the significant level of technical ability and specialization that is required of a mathematician to produce new, interesting mathematical theorems, and the consequent fragmentation of mathematics itself into a plurality of distinct, highly specialised disciplines. A related important feature is the appearance of complex, very large proofs, that require whole teams of mathematicians, possibly aided by computers, and numerous years to complete. These changes are prompting reflection among mathematicians on practices that have traditionally fulfilled a minority role, in particular constructive mathematics, that uses intuitionistic rather than classical logic. This kind of mathematics is now pursued by an increasing number of mathematicians and computer scientists alike, due to its prominent algorithmic nature. In particular, it is at the heart of popular proof-assistants such as the system Coq (Coq n.d.). Large, highly specialized proofs make the verification process difficult and costly. The hope is that progress in the mechanization of mathematics could offer more economical routes to verification in the future, and, perhaps, even help with the discovery process. The perception of constructive mathematics' place within the overall mathematical enterprise is therefore changing; the constructive mathematician hopes that a more thorough realization of the importance of computational forms of mathematics will make his discipline a more dominant form of mathematics in years to come.

The principal aim of this thesis is to examine significant aspects of a prominent form of constructive mathematics: the mathematics that is developed in the style of Bishop (1967). This form of mathematics has motivated foundational systems like constructive type theory (Martin-Löf 1975) that are designed to make fully explicit the computational character of constructive mathematics. These systems feature two separate elements of constructivity: *the **logic**, that is **intuitionistic***, and *the concept of **set**, that is **predicative*** according to a distinctive notion of predicativity.

In an investigation of these two dimensions of constructivity that characterise foundational systems for constructive mathematics, the first and principal questions are: *What is constructive mathematics? What is predicativity?*

An answer to the first question requires a clarification of the principal consequences of relinquishing the principle of excluded middle. The most remarkable outcome of the use of intuitionistic logic is a modification of the notion of proof, that enables us to confer computational content on mathematical theorems. I shall therefore discuss an interpretation of intuitionistic logic that explicates the algorithmic nature of constructive proofs and also plays a role in relation to constructive type theory's predicativity. I shall examine prominent features of the techniques that are introduced to "constructivise" ordinary mathematical theorems; in addition, I shall expound significant aspects of the relation between constructive and classical mathematics.

The second question, related to what is predicativity, turns out to be more difficult to tackle. A number of characterisations of predicativity have been put forward within the literature, and a number of very different proposals have emerged over the years on what counts as predicative mathematics. In particular, the variant of predicativity that is embodied by constructive type theory is only one of three principal variants of predicativity. The most prominent variant of predicativity is often termed "predicativity given the natural numbers", and has been thoroughly analysed over the years by mathematical logicians. A characteristic of this form of predicativity is that it is framed within a classical context. The fact that some variants of predicativity make use of classical, while others of intuitionistic logic introduces further complexities to an analysis of predicativity. For these reasons a substantial part of this thesis will be devoted to a clarification of the very notion of predicativity and its different manifestations. The starting point of my analysis will be the historical development of this notion, from which I shall select some particularly significant aspects. In addition, I shall identify a predicative notion of set that acts as unifying theme, appearing in prominent contributions to both the classical and the constructive approaches to predicativity. Another form of predicativity that has been put forward in the relevant literature goes under the name of "strict predicativity", and introduces more stringent constraints on sets compared with predicativity given the natural numbers. It is in fact through an analysis of this more radical approach to predicativity that important aspects of constructive predicativity fully emerge.

My investigation focuses on particular aspects of predicativity and constructivity. Their choice is determined by my desire to bring some clarity on issues that I feel are especially pressing, although their thorough clarification lays beyond the remits of this thesis. The principal aim of this thesis is to explicate constructivity and predicativity, rather than assess the prospects of constructivism or predicativism. However, I shall consider the case of predicativism in the last chapter, by investigating the prospects of a form of indispensability argument for predicative mathematics. My interest in an analysis of this argument is determined by my interest in the following question: *how far can constructive and predicative mathematics reach?* A thorough investigation of this issue would require substantial technical and philosophical work, as explained in Chapter 7. I shall report on research in mathematical logic that suggests that the restriction to predicative and constructive mathematics may not induce a serious loss when we confine our attention to scientifically applicable mathematics.

A reason for my interest in those technical results is that they show that the exercise of reformulating ordinary mathematical theorems in weaker systems (constructive and/or predicative) is lucrative from both a mathematical and a philosophical perspective. Only by adopting a weaker perspective we seem to be able to single out a minimal conceptual apparatus that is required to perform large portions of ordinary mathematics. The gain is the possibility to draw a map of mathematics that differentiates portions of it that are more or less computational in nature, and distinguish the kinds of concepts, more elementary or more abstract, that they presuppose.

A motivation for embarking on this project of clarification of predicativity was the desire to determine which reasons may be adduced for predicativity. In the first part of the thesis I shall undertake to single out a number of reasons that constructive mathematicians have put forth for their adoption of intuitionistic logic. These are prominently reasons that are internal to the mathematics itself or motivated by the desire to develop an algorithmic form of mathematics. As a consequence they differ profoundly from traditional arguments for intuitionistic logic which move from philosophical reasons and deeply intersect with prominent philosophical debates, as the opposition between realism and anti-realism. The case of predicativity is, once more, more complex. I shall examine some of the motives that have been adduced for predicativity by a number of authors, starting from the fundamental contributions by Russell and Poincaré, Weyl, but also, more recently, Nelson and Parsons. A common interpretation of the debate on impredicativity frames it as a re-incarnation of the traditional opposition between realism and anti-realism. The writings by Poincaré and Nelson particularly encourage this reading. However, their writings simultaneously suggest also a different route to predicativity, which is intimately related to methodological considerations. In particular, Poincaré and Weyl express a preference for "older", more explicit methods in mathematics. One salient motivation for their preference for these more explicit methods is the desire to produce a safe, or error-free form of mathematics. Similar considerations appear in recent discussions by mathematicians on constructive predicativity. The thought is that methodological and intra-mathematics reasons offer a route to predicativity which is in agreement with the motives put forth by constructive mathematicians for the adoption of intuitionistic logic.

Summary of the thesis.

Part I: Constructive Mathematics

Chapter 1: The mathematical landscape.

In the first chapter I shall explain why a philosophical investigation of constructivity and predicativity is particularly relevant today. I shall propose a view of the contemporary mathematical landscape that suggests that computational forms of mathematics are gaining unprecedented attention within mathematics as a whole. As a consequence, there is urgent need for a philosophical clarification of these forms of mathematics. A central part of the chapter will be devoted to explaining in which sense the adoption of intuitionistic logic renders constructive mathematics algorithmic and how its algorithmic nature can be exploited to offer new applications for mathematics.

Chapter 2: Constructive Mathematics

The difference between classical and intuitionistic logic has been widely discussed in the philosophical as well as in the mathematical literature. However, less emphasis has been placed in clarifying the impact that the change of logic has for the mathematics itself. This is crucial if one wishes to ascertain that the adoption of a computational form of mathematics does not impair a sufficiently well developed form of mathematics. One of my purposes in this chapter is to clarify that constructive mathematics is a substantial field within mathematics as a whole, that is pursued by mathematicians for mathematical reasons, and that is fruitfully employed to produce a computational form of mathematics. The relevance of this field for mathematics and its applications is important as a motive for the present investigation. In this chapter I shall first of all address the question of what is constructive mathematics, and how it differs from other forms of mathematics that also use intuitionistic logic. I shall then give a brief indication of some of the most relevant strategies adopted by constructive mathematicians to develop this form of mathematics. Their interest for the present discussion lays in their offering instruments for the rephrasing of large portions of ordinary mathematics in a computationally informative way and for paving the way for similar adaptations of scientifically applicable mathematics that will be discussed in Chapter 7. I shall finally outline the motivation that prompts some constructive mathematicians to work constructively, emphasizing in particular intra-mathematical and pragmatic reasons.

Part II: Predicativity

The discussions by mathematicians of the Bishop school on constructive mathematics only focus on the role of intuitionistic logic: the claim is that the adoption of a more restricted concept of *proof* ensures that the resulting mathematics is algorithmic by default. However, a rich tradition within constructive mathematics introduces further differences with classical mathematics that arise by complying with a more stringent concept of *set*. In the case of constructive type theory, compliance with predicativity gives also rise to a distinctive form of computability. In this part of the thesis I investigate predicativity as it arises both within a classical and a constructive contexts. The strategy will be to start from the original texts on predicativity to begin unravelling the complexities involved with this notion.

Chapter 3: Origins of Predicativity.

In this chapter I highlight two characterisations of impredicativity. The first is better known, and amounts to identifying impredicativity with a form of vicious circularity in definitions. According to one rendering of this characterisation a definition is impredicative if it refers to a totality to which the definiendum belongs. Another characterisation of impredicativity has been particularly stressed by Poincaré and terms a set impredicative if its definition is not invariant. The notion of invariance is not clearly spelled out by Poincaré, but the intuition is that a set is invariant if, once it has been defined, it cannot be modified or "disturbed" by the definition of putative new elements of it. This second characterisation of impredicativity conceals a notion of set as "incompleted" that is typical of the constructive and predicative tradition, as further discussed in Chapter 5. I shall conclude this chapter by briefly outlining Russells "solution" to impredicativity with his ramified type theory and Weyl's development of predicative analysis in "Das Kontinuum".

Chapter 4: The logical analysis of Predicativity.

In this Chapter I shall briefly outline the principal outcomes of the logical analysis of predicativity (given the natural numbers) that preoccupied prominent logicians in the 1950s and the 1960s. A first observation is that under the logician's scrutiny, predicativity becomes first of all a property of a theory, and only subsequently, within the context of a given theory, a property of definitions or sets. It becomes then possible to assess if a certain portion of contemporary mathematics is predicative or not, by recasting it within the precise formulation of carefully fixed canonical systems. A fundamental contribution of the logical analysis of predicativity was the determination of the so-called limit of predicativity, by a complex proof-theoretic analysis. Another crucial outcome of that analysis, and also of the so-called programme of Reverse Mathematics, is the realization that large portions of contemporary mathematics can be carried out within predicative theories: the reach of predicative mathematics goes much beyond what might have been originally expected.

A significant aspect of the debate on predicativity is that a number of different proposals have been put forth under the name of predicativity. The most well-known is predicativity given the natural numbers, which is the predicativity that was carefully studied by logicians from the 1950s. It takes the natural number structure as "given" and introduces restrictions on sets beyond it, in particular on the set of natural numbers. Another variant of predicativity, constructive predicativity, is characterised by its use of intuitionistic logic, and is the variant that manifests itself in constructive type theory. Another form of predicativity, strict predicativity, has been discussed by Nelson and Parsons, and questions the predicativity of the natural number structure. The plurality of predicativity rises a number of questions, first of all on the relation between these forms of predicativity.

Chapter 5: On a predicative concept of set

An analysis of Poincaré and Weyl's discussions on predicativity highlights a crucial role, within this debate, of a particular concept of set, according to which sets are extensions of definitions or properties. This concept of set bears similarities to the concept of set that is to be found in the constructive tradition (e.g. constructive type theory). Here sets are obtained from a step by step construction, expressible through a finite rule, starting from some initial elements and some elementary operations over them. This predicative concept of set is remarkably different from the concept of arbitrary set which motivates Zermelo Fraenkel set theory. I shall argue that this notion of set is at the heart of predicativity and can be seen as offering a "common denominator" for each of the different kinds of predicativity that have emerged within the literature over the years. I shall single out the crucial contributions by Poincaré (1912) to setting out this conception of predicative set. I shall then detail the careful formulation by Weyl (1918) that fully explicates an arithmetical concept of set. A distinctive characteristic of Poincaré and Weyl's reflection on predicativity is the special status they ascribe to the natural number structure. Weyl's principal aim in developing a form of predicativism is, in fact, building a safe form of analysis on the basis of the unshakable natural number structure.

Chapter 6: Strict Predicativity

In this Chapter I shall analyse strict predicativity, which arises if one applies predicatively motivated restrictions to the natural number structure. Proponents of strict predicativity claim that the principle of mathematical induction hides a form of impredicativity. In this chapter I shall first of all explicate in detail Nelson and Parsons' motives for strict predicativity. Then I shall discuss Dummett's indefinite extensible concepts. I shall take inspiration from the latter to suggest that if impredicativity is found to affect the natural number structure, then one has two options: (i) maintain a classical view of universal quantification and abide to strict predicativism, or (ii) give an intuitionistic reading of universal quantification and proceed to a more encompassing form of mathematics. I shall propose to employ Dummett's discussion also to clarify a significant issue on the relation between predicativity given the natural numbers and constructive predicativity that was left open in previous chapters.

Chapter 7: Is impredicative mathematics Indispensable?

The main purpose of this chapter is to discuss recent claims by Solomon Feferman and Feng Ye according to which predicative mathematics is perhaps sufficient for all scientifically applicable mathematics. I shall see how the work by Feferman and Ye may be used to present a form of "indispensability" argument for predicative mathematics. Subsequently, I shall report on Ye's work that makes use of the techniques developed within Bishop constructive mathematics to reformulate within a strict predicativist context a substantial portion of scientifically applicable mathematics. The conclusions one may be able to draw from an indispensability argument for predicativism are less clear cut than desirable. The importance of this discussion for the philosophy of mathematics goes beyond the prospects of assessing the plausibility of predicativism. The possibility of reducing large portions of applicable mathematics to a form of mathematics that presupposes a small (strictly finitary) fragment of the natural number structure is remarkable. This ought to play a role in an understanding of the relations between different parts of mathematics and distinct mathematical structures, and in singling out the scope of different proof methods and assumptions. It may in fact help us clarify aspects of the complex relation between mathematics and science.

Part I

Part I: Constructive Mathematics

Chapter 1

The mathematical landscape

Mathematics is presently undergoing deep transformations comparable to the fundamental changes that took place at the turn of the 19th and 20th Centuries. In this chapter I shall discuss some of these alterations and the impact they are having on forms of mathematics that use intuitionistic rather than classical logic. Throughout this thesis I shall be concerned with the most successful form of mathematics based on intuitionistic logic, also known as Bishop's constructive mathematics, that will be the focus of the next chapter.[1] In the following I shall argue that the present surge of computer applications within mathematics has the potential to substantially modify the position constructive mathematics currently holds within mathematics as a whole, making it a more central enterprise.

The comparison between the present changes in mathematics and the transformations of the 19th and 20th Centuries is instructive, as it suggests a reading of today's alterations as the reappearance of themes from the past. One way of framing the changes that took place within mathematics starting from the 19th Century is as the slow emerging of a *new style* in mathematics, and the decay of an older one.[2]

[1] Unless otherwise stated I shall also simply write *constructive mathematics* to denote mathematics in the style of Bishop (1967). See Chapter 2 for a characterisation of this kind of mathematics.

[2] Edwards (see e.g. Edwards 1988, Edwards 2008) has particularly stressed the importance of Kronecker's work as repository of the older computational tradition. Kronecker's work is witness

A number of authors have emphasised that the new form of mathematics was prompted by a growing preference for conceptual reasoning and abstract characterizations of mathematical concepts, with a corresponding **de–emphasis on calculation**.[3] Another fundamental development was the increasing confidence in dealing with the **infinite**, which culminated with the emergence of Cantorian set theory (Cantor 1883, Cantor 1895, Cantor 1897). The axiomatic method, as emblematically exemplified by Hilbert's axiomatisation of geometry (Hilbert 1899), gained unprecedented primacy, due to the realisation of the wider applicability of general axiomatic characterisations of mathematical concepts. This in turn had an impact on the relation between mathematics and its **applications**, as mathematics became less dependent from the stimulus of specific applications. Overall, the changes were so dramatic that in a very influential article, Howard Stein (1988) wrote that the 19th Century witnessed *the second birth of mathematics*, the first having taken place at the time of the ancient Greeks.

It is tempting to claim that we are currently witnessing a *revival of the older style*, now enhanced and ameliorated by the influence of the substantial progress that has taken place within its opposite approach. Less abstract, more **computational** forms of mathematics flourish today due to stimuli originating directly from their **applications**. In particular, a computational form of mathematics is gaining prominence in view of the new emerging role of computers within mathematical practice, as further discussed below. In addition, the very concept of **set** is undergoing careful re-examination, with the aim of enabling a thoroughly computational form of

to a form of mathematics that predates the set-theoretic turn and was grounded solely on concepts that have clear algorithmic significance. From Edwards' constructive perspective, Kronecker's views, that have often been depicted as outmoded, appear therefore as anticipatory of today's new computational trend.

[3]See (Stein 1988). See also (Avigad & Reck 2001) for an analysis of the changes that took place between the 19th and early 20th Centuries and their impact on mathematical logic, especially proof theory.

mathematics. As argued in Chapter 5, the roots of the present constructive concept of set lay within the old form of mathematics, that was more directly bound to explicit description and symbolic representation. As witnessed by today's constructive and predicative set theories, like Martin-Löf type theory (Martin-Löf 1975, Martin-Löf 1982, Martin-Löf 1984), the tie with symbolic representation makes the ensuing notion of set ideal for computer applications.

The origins of the predicative notion of set are noteworthy: it emerged as a reaction primarily to the new abstract notion of set that emanated from Cantorian set theory, with its distinctive treatment of infinitary notions. The debates that gave birth to predicativity were witness to a strong opposition to the new methodology and to mathematical concepts that the developments of 19th Century mathematics had brought about. Mathematicians such as Poincaré and, especially, Weyl considered the paradoxes that emerged at the turn of the 20th Century as a signal that the new mathematics was unsafe, untrustworthy. They embarked on a task of clarifying a predicative concept of set that would offer a way out: a mean of perfecting and extending the older, algorithmic form of mathematics, while avoiding resort to unsafe new methodologies.[4]

A revival today of older forms of mathematics might be surprising, especially as it is prompted by some of the most advanced forms of technological development of this century. It is also remarkable that the new computational form of mathematics is proposed today for reasons that are very similar to the original ones: as a way of securing correct reasoning and flawless proofs, as further discussed in Section 1.1.2.

1.1 A changing mathematics

A prominent aspect of today's mathematics that particularly distinguishes it from the mathematics of a century ago, is the rapid growth of a *plurality* of practices.

[4]See in particular the discussion on Weyl's "Das Kontinuum" (Weyl 1918) in Chapter 5.

Mathematicians are exploring different ways of doing mathematics; therefore today, beside standard classical mathematics, we have fast expanding forms of constructive, computable, non–standard mathematics, to name a few.[5] In fact, a number of more prominent practices, as the constructive one, are rapidly flourishing and already comprise further sub–varieties within themselves, as discussed in Chapter 2, Section 2.4. This tendency to ***differentiation*** within mathematics is growing stronger, and follows years of proliferation of numerous varieties of logics. The latter have been prompted by a number of reasons, from philosophical motivations (as in the case of Brouwer and Heyting's intuitionism), to pragmatic reasons (as witnessed by the vast number of application–driven logics emerging within theoretical computer science today).

The incipient differentiation which characterises logic and mathematics today is witness to a more general phenomenon. At the beginning of the 20th Century the mathematical scene was dominated by figures like Poincaré, Hilbert and Weyl, whose fundamental contributions spanned across wide sections of the mathematical spectrum, deeply impacting other sciences, in particular physics. Today mathematics shares the fate of many other scientific enterprises, and, as it grows, it becomes more and more **specialised**. As vividly described by David & Hersh (2011), today's mathematics requires extremely well trained individuals who understand and practice technically demanding, difficult and often rather self–contained fields. The plurality of logical and mathematical practices is but one aspect of this new kind of mathematics: a complex, highly specialised and differentiated body of knowledge.

The differentiation and the technical complexity of today's mathematics poses unprecedented challenges for this growing discipline, with fundamental impact on both the ***discovery*** and ***verification processes***. The very nature of mathematics is dramatically changing, and this has repercussions on the way mathematicians find

[5]There are also developing forms of finitary, strictly finitary, paraconsistent mathematics, among others, that are more directly motivated by philosophical considerations.

and corroborate new proofs, consequently also affecting the fundamental concept of mathematical proof. Proofs are increasingly complex and technical, often the result of close cooperation among a number of authors. Sometimes they are very *large*, spanning across a number of journal articles and requiring years for their elaboration. A paradigmatic example is offered by the classification of finite simple groups. Its completion in 1981 was the conclusion of over 30 years of research, involving more than 100 mathematicians, with the published output of some 300 to 500 articles (Aschbacher 2005, Steingart 2012).

These changes open up the way for a new role for computers within mathematics.

1.1.1 The role of computers in mathematics

Not only the time span, number of authors, and size of the mathematical output are undergoing transformations, but we are witnessing also radical modifications in the process of proof itself. The ground–breaking case of the four colour theorem, which was proven in 1976 by Kenneth Appel and Wolfgang Haken with the help of a computer, has opened up the way for numerous examples of machine-aided proofs. An important case of computers' contribution to the discovery process in mathematics is Thomas Hales' computer–aided solution of Kepler's sphere packing problem. After having been stated by Johannes Kepler in 1611, the conjecture eluded mathematicians for almost 4 centuries. It was eventually solved by Thomas Hales and his team, following an approach suggested by Fejes Tóth in 1953. Hales' solution crucially made essential use of *computers*.

The potential for computers to advance today's mathematics has been advocated by a number of mathematicians. For example, Dana Scott has maintained that in today's mathematics there is an urgent need to solve complex and large proofs, "Big Proofs", and this requires computers and logic to work together to make progress.[6]

[6]See Scott's e–mail to the Foundations of Mathematics mailing list of 28-07-14, http://www.cs.nyu.edu/mailman/listinfo/fom, in which he reported on his opening talk at the

Scott has further expressed the hope that computers will be used to obtain more substantial and new, perhaps unexpected or surprising results, thus advancing and possibly redrawing completely the discovery process in mathematics. As also indicated by Scott, a more significant role for computers within mathematics also represents a major challenge, as computers have so far been mainly employed to verify known proofs or corroborate plausible conjectures.

The example of Hales' solution of Kepler's conjecture raises fundamental questions on the concept of proof and its relation to the mathematical community.[7] Davies (2011) reports that at a meeting of the Royal Society (convened in 2004 to discuss "The Nature of Mathematical Proof") Robert MacPherson, Editor of the Annals of Mathematics, described why the editors had felt compelled to accept Hales' proof, even though a team of experts had eventually abandoned the effort to check all its details after several years of intense work. This case is indicative that computer-assisted proofs are slowly gaining the status of mathematical proofs, notwithstanding concerns of full surveyability and human reproduction and understanding.[8]

"Vienna Summer of Logic" (9th–24th July 2014).

[7]The following questions come to mind: does a totally or partly computerised proof count as a mathematical proof? Do we require that a mechanised proof ought to be amenable to reduplication by a human being? Or, perhaps, should we impose the stronger constraint that it ought to be fully mastered by or, at least, understandable for a mathematician in order to be rightfully called a proof? See (Tymoczko 1998) for a discussion. See also (Corfield 2003) for an analysis of the impact of computers within today's mathematics and for a clear demand that contemporary philosophical reflection ought to be directly informed by contemporary mathematical practice.

[8]The following "Statement by the Editors on Computer-Assisted Proofs" can be read on the homepage of the Annals of Mathematics (http://annals.math.princeton.edu/board):

"Computer-assisted proofs of exceptionally important mathematical theorems will be considered by the Annals.

The human part of the proof, which reduces the original mathematical problem to one tractable by the computer, will be refereed for correctness in the traditional manner. The computer part may not be checked line-by-line, but will be examined for the methods by which the authors have

Examples as such highlight another aspect of the potential role of computers for today's mathematics: the size and complexity of many of today's proofs, not only cause hardship for the discovery process, but interfere in profound ways with the *verification process*, too. That is, a second difficulty faces contemporary mathematics, as it becomes more and more challenging, if not practically impossible, to accomplish the peer reviewing process that is necessary to ascertain the correctness of purported large proofs. Here again one might hope that computers could provide much needed assistance. The interaction between humans and computers might seem at first to introduce further intricacies: it could be argued that it becomes even more difficult to ascertain the correctness of a proof if it is the output of the interaction between a human and a machine. Conceivably, on the contrary, computers are likely to become a major game changer, by extending our capabilities in a number of ways, and offering new verification strategies. For example, the four colour theorem and the proof of Kepler's conjecture have already been formally verified by computer systems, and, it is argued, their verification has increased our confidence in the correctness of their proofs.[9]

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eliminated or minimized possible sources of error: (e.g., round-off error eliminated by interval arithmetic, programming error minimized by transparent surveyable code and consistency checks, computer error minimized by redundant calculations, etc. [Surveyable means that an interested person can readily check that the code is essentially operating as claimed]).

We will print the human part of the paper in an issue of the Annals. The authors will provide the computer code, documentation necessary to understand it, and the computer output, all of which will be maintained on the Annals of Mathematics website online."

[9]The four colour theorem has been formalised by Gonthier in 2005. Gonthier (n.d.) claims that the formalization of the four colour theorem is "an ultimate step" in the effort of clarifying the mathematical result, especially as the original proof contained a complex interaction between manual and mechanical components. See also (Gonthier 2008). On 10th of August 2014 Thomas Hales has announced the completion of the Flyspeck project, which has constructed a *formal* proof of the Kepler conjecture using a combination of the Isabelle and HOL Light proof assistants. The formal verification of the original proof has also required extensive and protracted team-work.

1.1.2 Formal verification

Formal verification is a substantial field in computer science, that customarily relates to the verification of a program by means of formalization, to eliminate errors.[10] In the case at hand, one considers instead the verification of *mathematical proofs*, carried out with the help of a "proof assistant". This is a software that typically implements (i.e. codifies) a version of type theory (e.g. a constructive type theory or a form of Higher Order Logic).[11]

The verification of a mathematical proof usually starts from an informal, known proof of a theorem that we wish to verify. The first task is to formulate the statement of the theorem which complies with the given formalism, as implemented in the proof assistant. The most common proof assistants today allow for an *interactive* execution of proofs: the user conducts the formalization of a proof, and is "assisted" by the software. The latter might suggest how to solve individual steps of the proof, perform mechanically simple routines, or retrieve the proof of a lemma from a rich library of already fully formalized and verified mathematical statements.[12] Most

[10]A program is a collection of instructions within a programming language that instruct the computer to perform a specific task.

[11]Most proof assistants today have a first central part "the core" that is designed on the basis of some type theory. They then introduce "higer levels" on top of the core, which implement a number of different formal systems, leaving to the user the choice of which one to employ in specific contexts. This is often expressed by stating that the proof assistants are "generic". Proof assistants are also designed to facilitate a number of different projects, like formalization of mathematics and verification of both mathematics and software, as well as applications to security. See, for example, (Coq n.d., HOL n.d., Isabelle n.d.).

[12]In order to prove interesting, non-trivial theorems, extensive work is required to produce all the necessary libraries of lemmata. Often the formalization process faces difficulties due to a number of defects of informal proofs. For example, informal proofs frequently omit trivial steps, which in many cases can be filled up by routine but tedious work. Sometimes, however, one needs substantial new work to clarify aspects of the informal proof. In addition, there are several cases in which the formalization brings to light mistakes in the informal argument, that also require

proof assistants utilize also automatic proof searches to speed up the formalization of elementary steps. The output of this long process is a mechanically verified proof of the original mathematical statement, in all details.[13] If mistakes were not found in the original argument, and if the implementation followed sufficiently closely the original proof, then the latter is also considered to be confirmed by the mechanization process.[14]

1.1.3 The status of constructive mathematics

The outline above draws a picture of mathematics as a multi–faceted, dynamic practice, which is fast changing. In particular, a prominent aspect of today's mathematics is the emergence of more complex and lengthy proofs. This indicates a new role for computers within mathematics, both within the discovery and the verification processes. In particular, proof assistants are already being employed to verify

correction.

[13]The formalisation by Hale and his group has spanned a number of years and has required the effort of numerous researchers. Another large enterprise is the formalisation of elementary group theory by Gonthier and collaborators (see e.g. Gonthier, Mahboubi, Rideau, Tassi & Thry 2007). In both cases the major outcome, beyond the particular verification exercise, is the substantial expansion of the library of verified mathematical theorems available within the chosen proof assistants. As stressed by (Gonthier et al. 2007), special care needs to be taken in making the results of the formalisation process as "portable" as possible. That is, one of the greatest challenges faced by developers of proof assistants today is to ensure that the libraries of theorems so produced can be shared among different platforms. A related challenge is to ensure that the formalized mathematics be readable by ordinary mathematicians, without requiring substantial familiarity with the ad hoc features of a specific proof assistant. See, for example, the Isar subsystem of the Isabelle proof assistant (http://isabelle.in.tum.de/doc/isar-ref.pdf, http://www.cl.cam.ac.uk/research/hvg/Isabelle/).

[14]Very challenging questions arise here regarding the relations between the original, informal proof and its formal counterpart: the fully formalized proof(s) will be syntactically very different from the original, informal proof, and the exact relation between them needs to be carefully discussed. Another crucial aspect that requires consideration is the relation between different formalizations of a single informal proof within different underlying formalisms and proof assistants.

complex and large proofs.

Constructive mathematics, that is, mathematics that uses intuitionistic logic, is one of the many emerging and fast expanding practices within mathematics. In this new mathematical landscape constructive mathematics sits in a very favourable position due to the distinctive *algorithmic nature* of its proofs (in a sense to be explained later). Constructive mathematics for this reason fits perfectly with computer systems and promises to be of relevance for the many applications of mathematics that require machine computation. In particular, constructive mathematics is at the heart of very popular and fast expanding proof assistants, like the system Coq (Coq n.d.).[15] The reasons for the increasing popularity of Coq are complex and plausibly include a number of sociological factors. However, one important aspect is the fact that constructive proofs carry *additional information* compared with classical proofs. The use of intuitionistic logic then allows us to obtain more informative proofs, proofs that are in fact programs.[16] In addition, pioneering work is looking into the possibility of exploiting the algorithmic nature of constructive proofs to help us develop programs that are free from mistakes (as discussed in Section 1.2.3). Finally, constructive proof assistants are more general, as they allow for the implementation of both classical and constructive proofs (in the latter case carrying over the additional computational information).[17]

The increasing popularity of constructive proof assistants and the desire to use them to support us in the development and the verification of complex and big proofs has drawn unprecedented attention to constructive mathematics. In particular, it has attracted prominent mathematicians who have manifested interest not only in the possibility of automatic verification of mathematical results, but in the com-

[15]The core of the proof assistant Coq codifies an intuitionistic type theory.

[16]This is the case, for example, of proofs carried out in Martin-Löf type theory. See Section 1.2.1 for a clarification of this point.

[17]The generality of a constructive approach will be discussed at some length in the next Chapter 2, Section 2.5.1.

putational nature of constructive mathematics.[18] This has important sociological consequences, as a change in how this practice is perceived within the mathematical community itself has the potential of further strengthening this particular form of mathematics.

The principal impetus for the prominence and the fast development of computer systems with a constructive core, like Coq, is the distinctive algorithmic nature of constructive mathematical proofs. In fact, as further clarified in Section 1.2.1, constructive proofs may be read as very general algorithms, which tell us how to carry out a construction step by step; for this reason, they are apt to computer applications, including computer assisted mathematics. In the following sections I shall further elucidate in which sense constructive proofs are algorithmic.

1.2 The computational content of mathematics based on intuitionistic logic

Constructive mathematicians claim that constructive proofs are more informative than their classical counterparts. They say they have “computational content” or “computational significance” not guaranteed by a classical proof of the same theorem. Douglas Bridges has dubbed constructive mathematics “algorithmic mathematics” (Bridges 2009). In this section, we shall look at the phenomenon that lies behind these labels.

Intuitionistic logic is the subsystem of the classical predicate calculus which is

[18]Formalisation of mathematics has recently been endorsed by Field Medalist Vladimir Voedovsky, who has started a successful project of formalization of advanced parts of mathematics in Coq. Voedovsky also proposes a variant of constructive (or Martin-Löf) type theory as a new foundation for mathematics (Univalent Foundations Program 2013). His programme has attracted considerable attention and has stimulated new significant research that is further unveiling the potential of constructive approaches.

obtained by omitting the principle of the Excluded Middle (**EM**), according to which for any formula A, $A \vee \neg A$ holds.[19] The absence of the principle of excluded middle has crucial consequences for the ensuing notion of **proof**. The difference between constructive and classical proofs is usually exemplified by considering the proofs of existential and disjunctive statements: statements of the form $\exists x A(x)$ and $A \vee B$, respectively. A constructive proof of an existential statement $\exists x A(x)$ ought to exhibit (at least in principle) a witness: an object, say t, for which $A(t)$ holds. Classically, one can also prove such a statement by contradiction: proving $\neg\forall x \neg A(x)$. However, the standard proof of the equivalence between $\exists x A(x)$ and $\neg\forall x \neg A(x)$ requires **EM**, and it is thus not generally acceptable constructively. As to disjunctive statements, to prove constructively a disjunction $A \vee B$, it does not suffice to prove that it is not the case that both $\neg A$ and $\neg B$ hold: we further need to say which of A or B holds. In fact, also the case of proofs of conditional and universally quantified statements is crucially different from the classical case, as further discussed below.[20]

These brief remarks already convey the thought that constructive proofs carry with them *more information* than classical proofs. As a result, constructive proofs are generally *more explicit* than their classical counterparts, in that they clearly show how to construct the objects they assert to exist, and "calculate" the relevant functions. This fact lays at the heart of the computational content of constructive mathematics that will be discussed in the next Section.

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[19] I shall discuss in more detail the differences between intuitionistic and classical logic and the impact they have on mathematics in Chapter 2.

[20] In Douglas Bridges' remembrance of Errett Bishop which opens (Crosilla & Schuster 2005), the author suggests that Bishop's "conversion" to constructive mathematics was probably prompted by discussions he had with students during an introductory course in logic. The students expressed uneasiness with classical (material) implication, and this drew Bishop's attention to intuitionistic logic.

1.2.1 The Brouwer–Heyting–Kolmogorov explanation of constructive proof

In order to clarify how constructive proofs work and give an idea of why a constructive proof is algorithmic, in the following I shall present the BHK (Brouwer–Heyting–Kolmogorov) interpretation of intuitionistic logic. In introductory texts on constructive mathematics, intuitionistic logic is usually presented together with the BHK interpretation of the connectives and quantifiers (see, for example, Dummett 1977, Troelstra & van Dalen 1988, van Atten 2014). This has the purpose of offering an informal clarification of the notion of *constructive proof*. The interpretation clarifies what a proof of a statement is in terms of the proofs of its logical constituents; for example, a proof of a conjunction, $A \wedge B$, is expressed in terms of the proofs of its conjuncts, A and B.[21]

The BHK interpretation is given by the following inductive clauses.

- Absurdity $\perp$ (contradiction) has no proof.

- A proof of a conjunction $A \wedge B$ is given by presenting a proof of A and a proof of B.

[21] To be more precise, the BHK interpretation is an explanation of what counts as a *canonical* proof of a mathematical statement. Canonical here means standard or prototypical. The thought is that the BHK interpretation clarifies the notion of canonical proof, and so lays the foundation for understanding proofs more generally, as, for example, proofs that contain "detours". The distinction between canonical and non canonical proofs is of no concern in this context, but becomes important especially when one considers formal counterparts of the BHK interpretation, as natural deduction systems, and, in particular, extensions of them by a notion of constructive set, as in constructive type theory (Martin-Löf 1975, Martin-Löf 1984). See also (Schroeder-Heister 2016, Dybjer & Palmgren 2016) for introductory texts. A related distinction between canonical and non-canonical elements of a set will be addressed in Chapter 6, for the particular case of the set of natural numbers.

- A proof of a disjunction $A \vee B$ is given by presenting either a proof of A or a proof of B.
- A proof of $A \to B$ is a construction (or method) which transforms any proof of A into a proof of B.
- A proof p of $(\forall x \in Z)A(x)$ is a construction (or method) transforming any (proof of) $d \in Z$ into a proof $p(d)$ of $A(d)$.
- A proof of $(\exists x \in Z)A(x)$ is a pair $\langle p, q\rangle$, with p an element of Z (the witness) and q a proof of $A(p)$.

Here the symbol $\perp$ represents a contradiction.[22] The notion of contradiction is required for *defining* negation, which is not primitive intuitionistically: $\neg A$ is defined as $A \to \perp$. According to the BHK interpretation, then, a proof of $\neg A$ is a construction which transforms any supposed proof of A into a proof of $\perp$ (a contradiction).

Note that the conditions for disjunction and existential quantification conform to the intuitions already reviewed above. For example, a proof of a disjunction requires us to present a proof of one of the disjuncts. This interpretation of disjunctive proof, together with the interpretation of the universal quantifier, are key to the constructive rejection of the principle of the excluded middle as a method of proof. The Principle of the Excluded Middle (**EM**), in fact, states that for any proposition A, $A \vee \neg A$ holds. According to the BHK interpretation, accepting **EM** as a general proof principle would require that we have a universal method for obtaining, for any proposition A, either a proof of A or a proof of $\neg A$ (that is, a method for obtaining a contradiction from a hypothetical proof of A). However, if this were the case we

[22] If working within a language which includes arithmetic, we can define $\perp$ as $0 = 1$. If we consider instead a language which does not include arithmetic, we shall take $\perp$ as a primitive symbol, denoting a contradiction.

should be able to decide so far unresolved conjectures, like, for example, Goldbach's conjecture, for which we have at present no solution.[23]

The BHK interpretation also satisfies the requirement discussed above that a proof of an existential statement ought to offer a *witness*: a proof of $(\exists x \in Z)A(x)$ is an ordered pair $\langle p, q \rangle$, with p a witness in Z (that is, an element of Z which satisfies the statement A) and q a proof of $A(p)$.

Note also the clauses for implication and universal quantification. These refer to a *construction* or *method*, transforming proofs into proofs, where the notion of construction (or method) is primitive. In fact, also the notion of proof in the BHK interpretation (as a whole) is primitive and not to be understood as proof in a formal system. It may be explicated informally in a number of ways; for example, as a method for solving a problem or doing a task, as in Kolmogorov (1932) (see also Martin-Löf 1984).

The BHK interpretation is an *informal* explanation of the notion of constructive proof, and, as such, it appeals to unspecified primitive notions of (informal) proof and construction. The latter in particular would seem to require further clarification if the interpretation is to offer an *explanation* of what counts as a constructive proof, and how it differs from a classical one.[24] The role of the BHK interpretation in introductory texts on constructive mathematics is therefore mainly pedagogical, as it offers an idea of the principal characters of the notion of constructive proof

[23] Golbach's conjecture states that every even integer greater than 2 can be expressed as the sum of two primes.

[24] This thought is further corroborated by the following observation. Suppose in setting out the conditions for the BHK interpretation one takes a classical perspective at the meta-level. That is, the right-hand side of the BHK interpretation is expressed by appeal to a classical interpretation of the logical constants, and the notion of construction is expressed in terms of a set-theoretic function. Then it turns out that one validates the principle of excluded middle, as clarified in (Troelstra & van Dalen 1988, Sundholm 2004), in particular (p. 9 and Exercise 1.3.4 in Troelstra & van Dalen 1988).

without requiring the appeal to more sophisticated logical tools. In this chapter the interpretation is used as an instrument for explaining why constructive proofs may be read as very general programs, without having to introduce more complex technical notions.

The notion of constructive proof that underlies the BHK interpretation can be precisely characterised by utilizing formal systems, like, for example, the natural deduction calculus (Gentzen 1935*a*, Gentzen 1935*b*).[25] In this chapter we are particularly interested in clarifying in which sense a constructive proof is algorithmic. For this purpose it is useful to consider a precisification of the BHK interpretation which expresses the algorithmic nature of constructive proofs more directly. This is know as the Curry-Howard correspondence, and exploits a structural similarity between constructive proofs on the one side and programs of a certain kind on the other. The latter are expressed within a formalism known as the typed lambda calculus (Church 1940, Barendregt 1981, Barendregt 1991), which is the core of prominent functional programming languages as well as of Martin-Löf type theory and the calculus of constructions.[26] Under the Curry-Howard correspondence the proofs and constructions that the BHK interpretation mentions become (very general) programs; as a consequence, the possible role of constructive proofs within concrete computer applications becomes evident. I shall give an illustration of this point in the next section, where I shall explicate the main features of the so-called Curry-Howard correspondence with an example.

[25] See also (Troelstra & van Dalen 1988, Schwichtenberg & Wainer 2012).

[26] The calculus of constructions is the formal system that underlies the core of the proof assistant Coq.

1.2.2 The BHK interpretation and the computational content of constructive mathematics

In this section I illustrate with an example one strategy for explicating the algorithmic nature of constructive proofs. I shall start with a statement (that can be proved constructively) and apply to it the BHK interpretation. This will supply us with a very general algorithm. Subsequently, I shall introduce the Curry-Howard correspondence that allows us to produce a "program" from the informal BHK algorithm.[27]

The following discussion is rather technical and is introduced here as a way of substantiating the claim that constructive proofs are of relevance for computer applications: they are very general programs. The technical details that I introduce here will not be required in subsequent chapters. However, in later chapters I shall refer back to the general idea of the Curry-Howard correspondence, that associates formulas in intuitionistic logic with "programs" within suitable type systems.

We are typically interested in the algorithmic content of statements of the form

$$(\star) \qquad (\forall x \in Z)(\exists y \in W)A(x, y),$$

that express the existence of a relation between elements of a set of inputs Z and a set of outputs W. For example, one might wish to consider the following statement:

$(\star\star)$ for every even natural number n, there is a natural number m such that $n = m + m$.

The aim is then to show that a constructive proof of $(\star\star)$ is in fact an algorithm that produces the half of any even natural number.

[27]The quotes in "program" refer to the fact that I shall produce in the first instance a very general program: an expression in the typed lambda calculus. This could also be called a "logician's program". A "real program", that may be run in computers can be obtained from that expression by translating it into a standard functional programming language as Haskell. This further step will be discusses in Section 1.2.3.

In the following I shall clarify how, given a statement of form ($\star$), the BHK interpretation produces an algorithm which, for any input a in Z, gives an output b in W such that $A(a, b)$ holds. In fact, under the BHK interpretation, a proof of a statement of the form $(\forall x \in Z)(\exists y \in W)A(x, y)$ is a construction p which takes an element a of Z to a proof p_a of the statement $(\exists y \in W)A(a, y)$. In turn, a proof p_a of the existential statement $(\exists y \in W)A(a, y)$ is an ordered pair $p_a = \langle b_a, c_a \rangle$, where b_a is an element of W (the witness of this existential statement) and c_a is a proof of $A(a, b_a)$. That is, according to the BHK interpretation, a proof of the statement above is a construction p that takes an element a of Z and produces a pair composed of a witness $b_a \in W$ and a proof c_a of the fact that $A(a, b_a)$ holds. If we give an algorithmic interpretation to the notion of construction that is appealed to within the BHK interpretation, then p can be read as an *algorithm* that for each element a of Z as input, produces a witness, b_a in W, of the formula A, and a proof c_a that A does in fact hold of a and b_a. In particular, in the case of the statement ($\star\star$), the algorithm produces the half of any even natural number and a proof that it is in fact the half of the given number.

The above considerations offer a general idea of why constructive mathematicians claim that constructive proofs can be read as algorithms. However, they do not clarify how to transform these very general algorithms into real programs that we can in fact apply in concrete circumstances.

This can be achieved by refining the BHK interpretation by means of the so-called Curry-Howard correspondence (Curry 1934, Curry & Feys 1958, Howard 1980). This is a correspondence which is peculiar to systems based on intuitionistic logic, and is at the heart of the development of constructive proof assistants such as Agda, Coq and Nuprl (Constable & et al. 1986, AGDA n.d., Coq n.d.).

The Curry-Howard correspondence

The Curry–Howard correspondence highlights and precisely specifies a structural similarity between pairs of syntactic objects in distinct theories. On the one side of the correspondence we have formal proof calculi for the intuitionistic logic, like, for example, the natural deduction calculus, that precisely codify the intuitionistic notion of proof. On the other side we have variants of constructive type theory, typically formulated as a suitable (extension of the) typed lambda calculus.

The idea is that these type theories express in a formal, accurate manner the algorithmic notions of proof and construction mentioned by the BHK interpretation. In particular, they give precise computational sense to the notion of "construction" that was mentioned at page 29, in terms of very general programs. We have the following pair of correspondences:

- on the one side **formulas** in the intuitionistic calculus and, on the other side, **types** in constructive type theories; and
- on the one side **constructive proofs** of such formulas and, on the other side, the **elements of the corresponding types**.

It is important to remark that the notion of type utilised in these type systems does not coincide with the familiar notion of set from classical ZF.[28] In addition, a function in these systems is not a set-theoretic function as a graph (i.e. an appropriate set of ordered pairs), but a primitive notion of function that better suits the computational needs: essentially a function is a "program". It is indeed the computational character of the type theories that clarifies why the correspondence explicates the computational content of constructive proofs.

To see how the correspondence works it is best to look at the particular case

[28] See (Barendregt 1981, Dybjer & Palmgren 2016, Troelstra & van Dalen 1988). See also Chapters 3 and 5 for more information on the ancestry of this notion of type.

of an implication.[29] The BHK condition for implication states that a proof of an implication $A \to B$ is a construction that transforms a proof of A into a proof of B. The thought is that *constructively* an implication $A \to B$ behaves as a *function* that transforms a proof of the formula A (i.e. a witness of the truth of A), into a proof of the formula B (i.e. a witness of the truth of B).[30] From a constructive perspective a formula corresponds to what makes it true: its proofs. That is, a formula may be seen as corresponding to a collection of proofs: the set or type of its proofs. The Curry-Howard correspondence makes this intuition precise by stating that a formula $A \to B$ corresponds to a type: the "function type" (usually also written $A \to B$). The latter is the collection of all (expressions denoting) functions with domain A and range B (or, equivalently, of all "programs" that take inputs of type A and produce outputs of type B).

This also suggests the motivation for the second pair of correspondences mentioned above: proofs as elements of a type. A proof of a formula C may be seen as a witness to the truth of C, and thus as belonging to the type that corresponds to C (the collection of all proofs of C). In the case of $A \to B$, a proof of this implication corresponds to (an expression that denotes) a function from the type of proofs of the formula A to the type of proofs of the formula B. The expression that denotes such a function is an element of the function type $A \to B$; typically this will be a term in the typed lambda calculus written as $\lambda x : A.t : B$.[31] That is, $\lambda x : A.t : B$ is an expression in the typed lambda calculus that represents a function from A to B, or, indeed a "program" that takes inputs in A and produces outputs in B.

[29] I shall here consider an implication instead of the formula ($\star$) above as it is considerably simpler.

[30] See also (Sundholm 1994), where proofs of a formula A are also seen as truth–makers for A.

[31] The notation $x : A$ indicates that the variable x is of type A, that is, x stands for (or can be substituted by) an element of the set A. In addition, λ is a binder that is used to express functions. See for example (Barendregt 1981, Seldin & Hindley 2008) for detailed treatments and (Troelstra 1999) for an introduction to the λ-calculus.

One can then extend considerations of this kind to all the other clauses of the BHK interpretation, to produce an interpretation of intuitionistic proofs as functions (or "programs") within a type lambda calculus. The surprising fact that the Curry-Howard correspondence clarifies is that there is a precise structural correspondence between the workings of intuitionistic logic and the workings of a computational calculus as the typed lambda calculus. Proving theorems in intuitionistic contexts is the same as programming within a very general functional programming language.

1.2.3 Program extraction from constructive proofs

The computational content of constructive proofs is at the heart of an ambitious project that is gaining currency nowadays: *program extraction from proofs*. The principal aim of this project is to extract "real" programs from mathematical proofs with the help of a constructive proof assistant. In Section 1.1.2 I have discussed the formalization within a proof assistant of mathematics, which uses logic and programming as instruments for the verification of mathematical theorems. Here, instead, one uses a proof assistant to transform *mathematical theorems into programming tools*. I shall now endeavour to give an explication of how the machinery introduced in the previous section can help us devise new ways of programming: instead of manually producing a program to carry out a given task, we first prove *constructively* a formula that describes that task, and then use it to obtain automatically a program from it.

Suppose we want a program that computes the half of every even number. Then the first task is to write down the statement:

($\star\star$) for every even natural number n, there is a natural number m such that $n = m + m$,

and prove it constructively. The aim is then to use our constructive proof assistant to transform its proof into a computer program.

Once ($\star\star$) has been appropriately formulated within the proof assistant, one begins to build an interactive constructive proof of it, as sketched in Section 1.1.2. The skeleton of the proof will essentially look like the proof p in the previous section. The crucial new step compared with Section 1.1.2 comes in now, as one endeavours to transform the formal proof into a *real program* in some functional programming language (for example Haskell or Scheme). The latter step can be obtained by exploiting the Curry-Howard correspondence presented in the previous section. If we apply the Curry-Howard correspondence to the case of the statement $(\star\star)$, we obtain a correspondence between a formal constructive proof of this statement, and a function or "program" within a relevant type system. By using this correspondence in the direction *from left to right*, that is from formulas to programs, the proof assistant can automatically transform the formalised proof into an expression in the relevant typed system. The output is a very general program, written in the notation of the typed lambda calculus; however, for concrete implementation into a computer it needs to be finally translated into a programming language like Haskell. This can also be done automatically by the proof assistant. Our constructive proof, say p, has therefore given rise to a program, π, that is ready to be run on appropriate inputs. In the particular case of the statement $(\star\star)$ above, the program will produce the half of any even number n.[32]

Researchers working on program extraction from proofs usually highlight a crucial advantage of this approach over traditional programming techniques. In customary programming, one devises a program so to meet its specification; for example, in the case of the statement $(\star\star)$ above, one writes a program that given an even number n produces its half. Once the program has been written and tested in a number of examples, it needs to be further verified, to ensure that it is correct (i.e.

[32]See (Crosilla, Seisenberger & Schwichtenberg 2011) for further simple examples of program extraction from proofs in the proof assistant Minlog (http://www.mathematik.uni-muenchen.de/ logik/minlog/). See (Schwichtenberg & Wainer 2012) for a detailed exposition of the theory underlying program extraction from proofs.

error-free). In the present setting, instead, one produces in the first instance a fully formalised mathematical proof of that statement, and subsequently automatically extracts a program from it. The advantage of this latter strategy is that the ensuing program is **automatically correct**, so that there is no need to carry out a separate verification, with consequent cost reduction. The reason for the correctness of the program can be gathered by recalling the example above of the BHK interpretation of the statement $(\forall x \in Z)(\exists y \in W)A(x,y)$. The BHK interpretation, in fact, produces not only an algorithm p that takes an element a of Z to a proof that $A(a, b_a)$ holds for a witness b_a, but also a proof, c_a, of this fact. Therefore c_a acts as our guarantee that the algorithm (and thus also the program that is extracted from it) is correct. This justifies the expression often found in the relevant literature that the extracted program is *correct by construction*.

The extensive work on program extraction from proofs of the last two decades shows that Bishop's insight was remarkable: the use of intuitionistic logic accords computational content to mathematical theorems.[33] Further, we can develop automatic tools which produce (correct) programs from constructive proofs, provided that we are willing to carry out the tedious task of formalising mathematical proofs.[34]

[33]Bishop's ideas were put forward well before the projects in computer assisted computation I discussed above were carried out. De Brujin's AUTHOMATH project and Martin–Löf's theory of types (Bruijn 1968, Martin-Löf 1975) represent other pioneering work which had lasting and more direct influence on such developments.

[34]It should be mentioned that in more recent years there has been intensive work also in the direction of understanding the computational content of classical mathematics. For example, the Curry–Howard correspondence has been extended to classical systems in (Parigot 1992, Curien & Herbelin 2000). To my knowledge, the techniques utilised in these cases typically make use of interpretations (or translations) of classical into constructive systems (see e.g. Berger & Schwichtenberg 1995), and then exploit the latter's algorithmic nature to extract computational content. For this reason in the following I shall refer to the computational content of constructive mathematics as "direct".

1.3 Conclusions

Mathematics today appears as a multi–faceted, dynamic practice, which is fast changing. In particular, mathematical proofs are becoming more complex and lengthy, therefore requiring new strategies for their discovery and verification. Computer systems are proving particularly valuable, as they have the potential to substantially improve both the discovery and the verification processes in mathematics. In addition, there is a surge of interest for *constructive* proof assistants, as they have additional features (as the ability to extract correct programs from proofs) that make them more versatile and appealing. As a consequence the mathematical community is expressing interest for computer systems and for forms of mathematics that are computational in nature, as constructive mathematics. The principal outcome of this new scenario is that among the plurality of mathematics that we witness today, the constructive one is gaining new terrain.

At the beginning of this chapter I highlighted some of the fundamental changes that took place in mathematics starting from the 19th Century. I claimed that the new computational form of mathematics that we witness today suggests that we are seeing a revival of an older mathematical style that had been supplanted by more abstract forms of mathematics that became dominant from the 20th Century. Quite surprisingly, the principal impetus for the new appreciation of an "older" mathematical style is cutting edge research whose primary aim is to verify software as well as mathematics, but also widen the applicability of mathematical methods to encompass computer programming. A likely outcome of this process is a change in status for constructive mathematics: from a deviant and perhaps outmoded form of mathematics it could soon gain a more central role within the mathematical landscape as a whole.

These changes within mathematics suggest a task for the philosopher of mathematics: to better understand the *de facto plurality of practices*, their motivations and the relations between them. There is a profound difference here between this way of addressing the philosophy of mathematics and more traditional approaches. In particular, there is a deep dissimilarity between the present way of casting constructive mathematic's new role within mathematics and philosophical arguments that aim at defending a constructive approach.[35]

In the next chapter I shall propose a view of constructive mathematics as directly motivated by a preference for a more algorithmic and explicit notion of proof. In the second part of this thesis I shall also suggest that predicativity may also be motivated by the desire to exploit a more algorithmic notion of set compared with the notion codified by Zermelo Frankel set theory.

[35]See also the introduction to (Mancosu 2008), for a discussion on a recent movement within the philosophy of mathematics that vindicates the importance of a philosophical discussion that is more directly influenced by the mathematical practice. The model there is the philosophy of science, that more directly draws from the practice and the history of a subject. See also (Corfield 2003). From the present perspective a clarification of the forms of mathematics, their historical origins, their character, are prior to an investigation of more traditional questions within the philosophy of mathematics.

Chapter 2

Constructive mathematics: an introduction

In the previous chapter I have expounded the consequences that the change of logic from classical to intuitionistic has for the ensuing notion of *proof*. I have argued that by eliminating the principle of the excluded middle we can confer computational content to the resulting mathematics. This is good news for constructive mathematics, as the changing nature of today's proofs demands the expansion of computer systems to assist us in the discovery and verification of mathematical theorems. This has the potential over time of modifying the perception of both mathematicians and philosophers on the position of constructive mathematics within the mathematical scene. I have also demonstrated the potential of constructive proofs as instruments for safe programming.

One might concede that a constructive form of mathematics may be attractive for those reasons, but worry that eliminating the principle of excluded middle with the purpose of enabling a computational interpretation of mathematics would simultaneously impair the development of a sufficiently broad form of mathematics. To counter this thought in the following I shall gather further evidence of the breadth of today's constructive mathematics, and argue that it is a fast growing, rich component of today's mathematics. The presence of constructive mathematics as a substantial field within mathematics determines its interest from a philosophical perspective, especially requiring a clarification of how is it that the elimination of the principle of excluded middle does not impair the constructive re-development of large portions of ordinary mathematics.

In the following I shall also address in more detail the question of the reasons that bring constructive mathematicians of the Bishop school to adopt this form of mathematics.[1] I shall argue that their principal reasons are intra-mathematical and pragmatic, rather than stemming from philosophical considerations. These very motivations highlight the potential of doing mathematics *also* from a constructive perspective, both for the development of mathematics as a whole and for the philosophy of mathematics. The constructive perspective, in fact, uncovers a whole new dimension of mathematical thought, as it allows us to develop new areas and study new more general structures than those that emerge when we routinely apply the principle of excluded middle. In addition, it offers the possibility of carrying out a fine analysis of ordinary mathematical concepts from a computational perspective.

In this chapter I shall be concerned with constructive mathematics in the style of Bishop (Bishop 1967), and in Section 2.2 I shall present a characterisation of this practice. This will be used in Section 2.4 to distinguish Bishop's constructive mathematics from both classical mathematics and other forms of mathematics that also use intuitionistic logic. In Section 2.3 I shall discuss common strategies that constructive mathematicians adopt to progress their field. Finally in Section 2.5 I shall address the motives that bring mathematicians and computer scientists today

[1] My principal focus in this chapter are the motivations adduced by mathematicians of the Bishop school, as, for example, Bishop himself, Bridges and Richman. A different approach to constructive mathematics has been proposed by Martin-Löf, whose writings display clear philosophical motivation. Although Martin-Löf's constructivism offers motivations for constructive mathematics that are more philosophical in nature compared with the considerations addressed in this chapter, I believe they present no conflict with the motives further discussed below.

to work constructively.

2.1 Bishop's constructive mathematics

The expression *constructive mathematics* is customarily used to refer to a form of mathematics that uses *intuitionistic* rather than classical logic. In fact, the terminology is not sufficiently uniform, and the adjectives "constructive" and "intuitionistic" are often interchanged and utilised to designate various forms of mathematics. In the following, I shall employ the expression "constructive mathematics" to denote a particular form of mathematics that uses intuitionistic logic: the mathematical practice "Bishop–style" originated in (Bishop 1967). This will be the main concern of this Chapter. The more elaborate "mathematics based on intuitionistic logic" will instead refer to *any* form of mathematics which uses intuitionistic rather than classical logic.[2]

Constructive mathematics was born when Errett Bishop's published "Foundations of constructive analysis" (Bishop 1967). Here a great deal of 20th Century analysis was developed on the basis of intuitionistic logic: from elementary analysis, to metric and normed spaces, abstract measure and integration, the spectral theory of self adjoint operators on a Hilbert space, Haar measure and duality on locally compact groups, and Banach algebras. This kind of mathematics has since witnessed great advances in analysis, topology and algebra, as well as the foundational systems designed to formalize it, as constructive type theory and constructive set theory.[3] A distinctive aspect of any form of mathematics based on the intuitionistic

[2] I prefer this complex expression to the simpler terminology "intuitionistic mathematics", as within the constructive mathematical literature the latter usually refers to mathematics based on intuitionistic logic *in the tradition of Brouwer* (Bridges & Richman 1987) (see also Section 2.4).

[3] See, for example, (Bishop 1967, Bishop & Cheng 1972, Friedman 1973, Martin-Löf 1975, Feferman 1975, Myhill 1975, Friedman 1977, Aczel 1978, Aczel 1982, Martin-Löf 1982, Martin-Löf 1984, Beeson 1985, Bishop & Bridges 1985, Aczel 1986, Bridges & Richman 1987, Sambin

logic is its computational character (see Chapter 1, Section 1.2.2); in particular, Bishop's constructive mathematics has been a fundamental source of inspiration for the theory and the applications to computer aided computation.[4] In fact, Bishop's principal motive for his constructive turn in the 1960's was the desire to develop a computational form of mathematics that would be suitable for implementation on a computer. The possibility to read constructive mathematics as a general and powerful form of programming is also at the heart of Martin-Löf type theory, often recognised as the most adequate foundational system for this kind of mathematics.[5]

2.2 A characterisation of Bishop's constructive mathematics: mathematics based *only* on intuitionistic logic

If prompted with the question: *"what is constructive mathematics?"* a mathematician would probably prove a number of prominent theorems that are recognised as constructive, and demonstrate by way of example the crucial differences between this practice and the classical one. Logicians starting from the 1970's have also

1987, Troelstra & van Dalen 1988, Mines, Richman & Ruitenburg 1988, Nordström, Petersson & Smith 1990, Sambin 2003, Crosilla & Schuster 2005, Calude & Ishihara 2005, Bridges & Vîţă 2006, Aczel & Rathjen 2008, Berger, Pattinson, Schuster & Zappe 2008, Bridges & Palmgren 2013, Univalent Foundations Program 2013, Crosilla 2015*b*, Lombardi & Quitté 2015).

[4]See, for example, (Martin-Löf 1975, Nordström et al. 1990, Coquand & Huet 1986, Constable & et al. 1986, AGDA n.d., Coq n.d.).

[5] In agreement with the logical tradition, the expression "foundational system" is used here to denote a formal system that suitably accommodates a form of mathematics, that is, a formal system within which we can naturally express or codify that form of mathematics. I shall however not suggest that such system also plays a"foundational" role for that form of mathematics in the philosophical sense, nor that there is a way of privileging one over another foundational system if both accommodate the relevant mathematics in suitable ways.

attempted to clarify the nature of constructive mathematics by introducing a number of foundational systems (i.e. set and type theories) that codify this practice.[6] They have further addressed the question of the relation between these different formalisations with each other, as well as with their respective classical counterparts.[7] Therefore to ascertain whether a given mathematical theorem or notion can be considered constructive we need either to rely on the constructive mathematician's insights on his field, or on the possibility of expressing it within a suitable foundational system. In fact, constructive mathematicians usually insist on the open-ended nature of constructive mathematics, as it is possible to foresee that as one develops this form of mathematics new mathematical entities will be recognised as satisfying the constructive perspective.[8]

It is however possible to offer a very general characterisation of constructive mathematics that, although not completely precise, suffices to single out this practice from other significant forms of mathematics. Constructive mathematicians Fred Richman and Douglas Bridges have stressed that *constructive mathematics* is simply *mathematics that uses only intuitionistic logic* (Richman 1990, Bridges 2009). Bridges (2009) terms constructive mathematics "algorithmic mathematics" and writes:

> Experience shows that the restriction to intuitionistic logic always forces mathematicians to work in a manner that, at least informally, can be de-

[6]See footnote 5 at page 46 for the notion of foundational system.

[7]See e.g. (Friedman 1973, Feferman 1975, Martin-Löf 1975, Myhill 1975, Friedman 1977, Aczel 1978, Aczel 1982, Martin-Löf 1982, Martin-Löf 1984, Beeson 1985, Aczel 1986, Griffor & Rathjen 1994, Rathjen & Tupailo 2006).

[8] The open-ended nature of mathematics is a typical presupposition of the intuitionistic tradition, as exemplified by Brouwer's notion of free choice sequences (Brouwer 1975, van Dalen 1999, van Atten 2007). An open-ended universe of sets is also clearly in agreement with Bishop's approach, as argued by (Simpson 2005). In addition, it is a crucial feature of Martin-Löf type theory (Martin-Löf 1975, Martin-Löf 1984), as witnessed in particular by the notion of reflecting universe.

> scribed as algorithmic; so *algorithmic mathematics appears to be equivalent to mathematics that uses only intuitionistic logic.*

This characterisation of constructive mathematics comprises two claims: first of all, constructive mathematics makes use of intuitionistic logic; secondly, it does not add any other assumption, axiom or principle that go beyond the sole change in logic.[9] The first part of this characterisation will be the focus of Section 2.2.1. The second component will be further clarified in Section 2.4, where I shall briefly outline other forms of mathematics that also use intuitionistic rather than classical logic, but, in addition, introduce assumptions that go beyond the pure change in logic.[10]

Bridges' quote is instructive in another respect, as the proposed characterisation of constructive mathematics is given on the basis of "experience", not by appeal to some a priori argument. As a matter of fact, it it turns out that in order to develop a form of mathematics that is algorithmic *it suffices* to modify the *methodology* of mathematics to comply with *intuitionistic logic*. It is therefore the aspiration to produce an algorithmic form of mathematics that enforces the shift to intuitionistic logic. Experience shows that this shift suffices to obtain the desired kind of mathematics: mathematics that uses intuitionistic logic is algorithmic "by default". I

[9] Constructive mathematicians such as Richman and Bridges often note that by using such a "formal" criterion to characterise constructive mathematics, one takes a considerably different perspective on mathematics compared with e.g. the views expressed by Brouwer, Bishop and others, who stressed the primacy of mathematics over logic. In fact, both the motives and the characterisation of constructive mathematics put forth by the Bishop school witness a substantial departure from more traditional approaches to intuitionistic logic, as further discussed in Section 2.5.

[10] In the second part of this thesis I shall further consider a second dimension on which constructive mathematics differs from classical mathematics: predicativity. This latter aspect of constructive mathematics is more prominent within the literature on foundational systems for constructive mathematics, and especially Martin-Löf type theory. The Bishop school typically focuses on the sole change of logic, and this will be the principal concern in this chapter, too.

shall return to the issue of the motives for constructive mathematics in more detail at the end of this chapter (Section 2.5).

2.2.1 The role of intuitionistic logic

Constructive mathematics shows that a change in logic, as unorthodox as the elimination of the principle of excluded middle, does not preclude access to a very rich form of mathematics. This realisation was first due to L. E. J. Brouwer, who introduced his intuitionistic mathematics at the beginning of the 20th Century as a way of better complying with his understanding of mathematics as a free activity of the human mind (Brouwer 1907, Brouwer 1908, Brouwer 1919, van Dalen 1999, van Atten 2004). Brouwer thus developed the first steps of a new kind of mathematics which used intuitionistic reasoning. This new way of reasoning was later formally codified first by Kolmogorov (1925) (in part) and, independently, by Heyting (1930), who produced the intuitionistic predicate calculus as we know it today.

As codified by Heyting (1930), intuitionistic logic is the subsystem of the classical predicate calculus which is obtained by omitting the principle of the Excluded Middle (**EM**), according to which for any formula A, $A \vee \neg A$ holds. In fact, even weakenings of the principle of excluded middle, like the Limited Principle of Omniscience (LPO), are considered unjustified from a constructive point of view, and for this reason omitted.[11] Note, however, that although intuitionistically one does not endorse the principle of excluded middle in its full generality, particular instances of it might

[11]The Limited Principle of Omniscience states that for any binary sequence $(\alpha_0, \alpha_1, ...)$, either $\alpha_i = 0$ for all i, or there is a k with $\alpha_k = 1$. A sequence $(\alpha_0, \alpha_1, ...)$ is binary if each α_i is either 0 or 1. Bishop (1967) noticed that many theorems in classical analysis can be obtained by adding LPO to a purely constructive argument, so that LPO may be seen as encapsulating the classical content of these theorems. LPO and other weakenings of **EM** have also been studied in recent times within the so called "constructive reverse mathematics project" (see Section 2.4.2). See e.g. (Ishihara 2006) for a survey of results. See also (van Atten 2014) for an introductory explication of the use of so-called "weak counterexamples" in mathematics based on intuitionistic logic.

turn out to be acceptable in specific cases, like, for example, in finitary cases.[12]

In spite of the fact that the excluded middle is not assumed as part of intuitionistic logic, its negation is also inadmissible from an intuitionistic perspective: we can not add the negation of the excluded middle to the intuitionistic predicate calculus without engendering a contradiction. This is due to the fact that intuitionistically we can derive the double negation of **EM**, that is, $\neg\neg(A \vee \neg A)$, for any formula A. Therefore a constructive mathematician does not deny the principle of excluded middle, although he will claim that **EM** lacks justification (so far), and that for this reason we should refrain from using it in general.

This fact, and the fact that constructive mathematics can be seen as arising by the simple elimination of the principle of excluded middle, has a crucial consequence that will be discussed in Section 2.4: every theorem in constructive mathematics is classically valid, too. In particular, constructive mathematics does not conflict with classical mathematics. In fact, it can be argued that this containment metaphor can also be reversed, and classical mathematics be viewed as a particular kind of constructive mathematics, as further discussed in Section 2.5.1.

2.3 Constructive strategies

Relinquishing the principle of excluded middle not only enables the development of a computational form of mathematics, but it has also given rise to entirely new areas of mathematical research.[13] The development of computational forms of mathematics is the principal concern in this work, therefore I shall prominently consider research in constructive mathematics that addresses the "constructivisation" of portions of

[12] For example, in the context of arithmetic many statements of a finite nature that are true in classical mathematics are true also constructively. In fact, constructive mathematics differs more substantially from classical mathematics especially when one considers infinitary objects (such as the real numbers).

[13] See also Section 2.5.1.

ordinary mathematics. In areas like, for example, analysis, the principal effort of the constructive mathematician is to express as much as possible of classical mathematics in a constructive form. Here the constructive mathematician will typically start from a given classical proof and will attempt to "constructivize it". In fact, constructive mathematicians often acknowledge the crucial role of classical mathematics as guide within the constructive practice. Given a classical theorem, the constructive mathematician will see it as presenting a task: to find a constructive proof of it or clarify why one is not available. Furthermore, often a careful analysis of the original classical proof helps in finding its constructive counterpart.

The ubiquity of the principle of excluded middle in classical mathematics might suggest that the "constructivization" of large portions of classical mathematics is bound to meet substantial obstacles: too many significant proofs of ordinary mathematics use the principle of excluded middle, or constructively unacceptable consequences of it.[14] Already Hilbert (see e.g., Hilbert 1926) objected to Brouwer's intuitionism that the absence of the principle of excluded middle would deprive the mathematician of a fundamental tool. For example, the trichotomy of the real numbers (i.e. the statement that every real number is either positive, negative or equal to zero), requires for its proof a constructively unjustified appeal to **EM**. Fifty years of work since Bishop's "Foundations of Constructive Analysis" have shown that we can develop a rich form of mathematics even without **EM**. How is this possible? What does it tell us about our ordinary mathematics and the concepts which we employ in its development?

In this section I shall expound some of the most common strategies that are employed by the constructive mathematician to work around the absence of the principle of excluded middle.

[14]See (Bauer 2016) for examples of constructivization and for a discussion of constructive mathematics from a constructive perspective.

2.3.1 Definitions

The shift to intuitionistic logic is noticeable both at the level of *logic* and at the level of *mathematical notions and proofs*. In both cases a striking difference between classical and constructive mathematics is that the availability of **EM** within the first allows for the identification of a number of statements and notions that are distinct according to the second. More precisely, given a classical notion this will have a number of equivalent formulations; however, many of these will turn out to be intuitionistically inequivalent. This is witness to a more general phenomenon. A characteristic of all forms of mathematics based on a logic weaker than the classical one is that they have a higher sensitivity to the formulation of their notions. Here slightly distinct definitions may turn out to have very different mathematical consequences. The constructive mathematician needs therefore to place particular care in choosing his definitions. In fact, it turns out that most classical definitions will carry over to constructive mathematics, although one needs to be particularly careful in selecting out of a number of equivalent ones, those that are appropriate for a given context.[15]

More radical changes are in place in some cases, like, for example, in intuitionistic and constructive approaches to topology (Johnstone 1983, Sambin 1987, Sambin 2003). The strategy here is to adopt a shift in perspective, and reformulate the most fundamental definitions; for example, the notion of *open set* is here taken as primitive and that of *point* is defined in terms of the former. As a result of this shift of perspective a new discipline, formal topology, has emerged in the last three decades that combines the use of intuitionistic logic with a form of predicativity.[16] A

[15]For example, in metric topology a closed set is often defined as the complement of an open set; an alternative definition is that for which a closed set, S, contains all limits of sequences in S. The latter definition turns out to be more useful in constructive analysis.

[16]See Chapter 4 for a clarification of a constructive notion of predicativity that is appealed to in formal topology.

compliance with predicativity requires the avoidance of any appeal to impredicative uses of the powerset operation, that is ubiquitous in standard treatments of topology. The outcome is a very concrete approach to one of the most abstract fields in mathematics.[17] Its concreteness, in particular, makes it ideal for applications to computer-aided proofs.[18]

2.3.2 Theorems

As clarified in the previous chapter, the change in logic has a profound impact on the notion of proof, and therefore deeply affects how one proves theorems. Sometimes given a classical theorem whose known proofs are non-constructive, one may find a slight modification of its classical proof that works constructively. In this case the original classical argument may remain substantially unchanged.

In other cases, nonetheless, more substantial changes are necessary. A typical approach is to modify the very statement of the theorem, therefore obtaining a ***constructive variant*** of the original classical statement. Sometimes the change

[17]The words "concrete" and "abstract" have different uses in mathematics and in philosophy, that are not completely unrelated. In mathematics a concrete structure (or a concrete concept) denotes a well-understood and sufficiently simple structure (or concept), like, for example, the natural or the real numbers. The term abstract instead refers to generalisations of these more concrete structures (or concepts), by appeal to set-theoretic or algebraic notions. An example is the notion of a complete totally ordered field. See also (Coquand & Lombardi 2006) for an elaboration of the notion of "concrete" in precise logical terms and its application to a study of constructive commutative algebra.

[18]Note that the change of perspective has proved fruitful. It has enabled a definition of real numbers as formal spaces that has been used to prove theorems in functional analysis and topology that evade constructive treatment along a more standard Bishop-style approach. See e.g. (Cederquist & Negri 1996, Cederquist, Coquand & Negri 1998) and (Ishihara 2006) for a comparison. See (Johnstone 1983) for a description of the related field of pointless topology and its history. In that expository article the author particularly emphasizes the importance of adopting a more general perspective for mathematical progress.

consists in using an alternative definition of some of the notions that occur in the statement of the theorem (as discussed above). Another option is to prove an *approximate* version of the classical theorem: for example, instead of proving the existence of a certain real number, one proves the existence of an approximation of it, within arbitrary precision. In cases in which the original statement is a conditional, one may suitably weaken the consequent or appropriately strengthen the antecedent of the implication, to obtain a theorem that is constructively provable and also sufficient for the applications one is interested to preserve.[19] Importantly, *classically* the new statement will be equivalent to the original one, so that the "variant" is such only from a constructive perspective. The variant, however, will be typically endowed with a direct computational content that the original proof did not have.

It is clear from these simple remarks that constructive mathematicians need to abandon familiar ways of reasoning. They need to acquire, with time, a distinctive ability to choose between alternative notions and find new proof strategies. Sometimes, like in the case of formal topology, they need to radically modify their perspective, to see new phenomena. However, in this respect constructive mathematics does not seem to differ remarkably from any other branch of advanced mathematics: in all cases a swift progress demands extensive familiarity with proof procedures and strategies that are specific to the individual field and therefore require adequate training.

The discussion above is also indicative that in order to *systematically* develop a computational form of mathematics we need to change perspective, and relinquish (for this purpose) the use of the excluded middle. Even if constructive proofs are after all particular classical proofs, more standard approaches to ordinary mathematics that routinely employ the principle of excluded middle do not enable the development of the necessary techniques that produce *by default* computational

[19]See, for example, constructive variants of the intermediate value theorem in analysis in (Bauer 2016). See also (Bridges & Richman 1987, Schuster & Schwichtenberg 2004).

mathematical theorems.

Attributes of proofs

One might worry that if not the nature of the research itself, perhaps the characteristics of the resulting mathematical proofs will be substantially different from the ordinary, and in negative respects: the new proofs might be longer, more complex, and less elegant than the original ones. This worry is prima facie justified, as it is well-known that a more frequent use of the principle of excluded middle at the turn of the 20th Century was in fact motivated by the desire to obtain shorter, more elegant proofs.

It is, nevertheless, difficult to reach general conclusions on this point: different individual proofs and, sometimes, different areas of research manifest different features. More importantly, judgements of complexity and elegance often hinge on personal taste, as well as mathematical background. And the anecdotal evidence is not all one way. Mathematicians working in constructive algebra have undertaken a number of case studies, which provide a useful contrast.[20] Here the authors have realised that a careful reformulation of a theorem's statement may improve on its classical proofs in many respects. Often it suffices to re-state the goal of the theorem in more elementary and algorithmic terms to easily obtain an adaptation of the original classical proof that altogether avoids non-constructive notions. According to the authors of this study, the resulting proof is typically *simpler*, *shorter* and fully *algorithmic*. Very often it is obtained by a careful analysis of the original classical proof, from which it preserves the general structure, eliminating unnecessary non-constructive components.[21]

[20] See (Coquand & Lombardi 2006, Lombardi & Quitté 2015). See also (Crosilla & Schuster 2014) for a discussion.

[21] Note that in the present context we have not only the elimination of the principle of excluded middle from classical theorems, but a careful replacement of abstract with more elementary and concrete statements. In fact, the very notion of concrete statement is amenable to formal (logical)

These examples from commutative algebra are instructive, as they show that the development of a constructive form of mathematics (alongside the classical one) has the potential to improve on classical mathematics in a number of respects. The constructive perspective allows for a more detailed analysis of mathematical concepts and proofs that populate ordinary mathematics; the outcome sometimes is a new algorithmic proof of a theorem that only appeals to very elementary concepts. It is the desire to obtain an algorithmic proof that suggests to re-state the goal of the theorem in more elementary terms, and it is the compliance with intuitionistic logic that guarantees that we do obtain an algorithmic proof.

This point is of fundamental importance to the present discussion of constructivity (and predicativity), as it suggests that with time we might be able to eliminate unnecessary abstract and non-algorithmic components from large parts of classical mathematics. There are two main reasons for being interested in this matter: a practical one and a philosophical one. On the one side, there is the potential to develop a form of mathematics that works better than the standard one in computerised applications, by combining the use of intuitionistic logic with the employment of more elementary notions. On the other side, there is the potential to reveal important insights into ordinary mathematics. In particular, ordinary mathematics makes frequent use of classical reasoning that eludes direct computational content, and frequently presupposes an abstract and non-computational concept of set. The work in constructive mathematics, as well as the one discussed in subsequent chapters on predicativity, is indicative that large portions of ordinary mathematics can be made algorithmic and their abstract assumptions explained away in terms of more elementary ones.[22]

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clarification within this context. There is therefore a deep analogy between these undertakings in constructive commutative algebra and the theme of predicativity that will be addressed in subsequent chapters.

[22]See Chapter 6 and Chapter 7 for further discussion on this point and for a clarification of the notion of ordinary mathematics.

These latter considerations suggest that constructive and predicative mathematics may play a role within the philosophy of mathematics as instruments for a conceptual analysis that applies across the whole spectrum of ordinary mathematics and delimits a portion of mathematics that is amenable to a more algorithmic and elementary treatment.[23] However, this role for constructive mathematics is only possible if the latter *does not conflict* with classical mathematics. The absence of conflict between constructive and classical mathematics is a characteristic of this form of mathematics based on intuitionstic logic that will be further addressed in the next section.

2.4 Varieties of constructive mathematics

In Section 2.2.1 I examined Bridges and Richman's characterisation of constructive mathematics as mathematics that uses only intuitionistic logic. Such a characterisation immediately allows us to discriminate the classical from the constructive practices: constructive mathematics is a kind of mathematics which originates by allowing only proofs which are carried out in intuitionistic logic, while classical mathematics also allows for proofs which make arbitrary uses of the principle of excluded middle. In the previous chapter I have clarified that the principal consequence of this difference is that constructive and classical mathematics employ different *methodologies*, as these two forms of mathematics allow for *different methods of proof*.

This clarifies one part of the characterisation above, the use of intuitionistic logic, but leaves unexplained the claim by Richman and Bridges that constructive mathematics uses *only* intuitionistic logic. To elucidate the latter point, a comparison with other kinds of mathematics, which differ in other respects from classical mathematics, is in order.

[23]This point will be further pressed in Section 2.4.2.

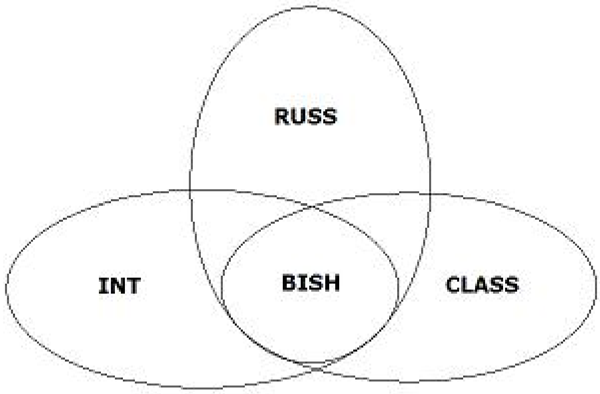
2.4.1 Exclusive use of intutionistic logic

The use of the word "only" in Bridges and Richman's characterisation of constructive mathematics is further justified by a comparison between this form of mathematics and other kinds of mathematics that also use intuitionistic logic. These are **Brouwer's intuitionistic** and **Russian constructive mathematics** (Brouwer 1975, Markov 1954, Shanin 1968), that are distinguished by their introducing concepts that produce a conflict with classical mathematics. In particular, Russian constructive mathematics may be framed as a particular form of computable mathematics, that also adheres to intuitionstic logic. Brouwers' intuitionistic mathematics instead differs from both classical and constructive mathematics especially for its treatment of the continuum, by the introduction of so-called free choice sequences (Brouwer 1975, van Dalen 1999, van Atten 2007).

The philosophical underpinnings of these forms of mathematics are extremely different; however, the mathematical practices themselves can be compared. Bishop's mathematics can be regarded as a minimum core of all of them (Feferman 1979, Bridges & Richman 1987, Ishihara 2006). Constructive mathematicians often explain the relation between these forms of mathematics as follows. Let **BISH** stand for an appropriate mathematical theory that codifies Bishop's constructive mathematics. One may characterize the other mathematical practices as the result of the addition of further specific principles to **BISH**. In particular, Russian constructive mathematics, **RUSS**, can be obtained by adding appropriate principles that represent that practice, in particular a principle known as a variant of Church's Thesis that ensures that all functions are computable. Brouwerian intuitionistic mathematics, **INT**, can be formally captured by adding to **BISH** principles that account for Brouwer's non-classical view of the real numbers as free choice sequences.[24]

[24]There are various ways of formally expressing the relationships between all of these kinds of mathematics. Here I shall follow Rathjen (2005). See also Bridges & Richman (1987). Let:

Russian constructive mathematics (**RUSS**) = **BISH** + **MP** + **CT**, In a picture, we have:



The main idea conveyed by this picture is that by looking at constructive mathematics as codified by some theory, **BISH** (however this is formalised), it turns out to be a sub–theory of classical mathematics, **CLASS**, as well as of each of the two other kinds of mathematics based on intuitionistic logic, **RUSS** and **INT**.[25] A for-

Brouwerian intuitionistic mathematics (**INT**) = $\mathbf{BISH} + \mathbf{BP_0} + \mathbf{BI}$,

Classical mathematics (**CLASS**) = $\mathbf{BISH} + \mathbf{EM}$.

Here **MP** is Markov's principle, stating that $[\forall n\, (\varphi(n) \vee \neg\varphi(n)) \wedge \neg\forall n \neg\varphi(n)] \to \exists n \varphi(n)$. Informally Markov's principle says that if φ is decidable (i.e. for each natural number n, $\varphi(n) \vee \neg\varphi(n)$) and φ is not false for every natural number, then it must be true for some number.

CT is Church's Thesis, according to which, if we are given a quantifier combination $\forall n\, \exists m\, \varphi(n, m)$, then we can find a *computable* function f such that: $\forall n\, \varphi(n, f(n))$. This can also be read as stating that all total functions are computable. The combination of **MP** and **CT** has the effect of producing a constructive variant of computable mathematics.

I shall not state $\mathbf{BP_0}$ in detail, but this will be a formula that states that every function from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$ is continuous. This principle is introduced to allow for a formal representation of Brouwer's theory of the creative subject, with the related notion of real numbers as free choice sequences (Brouwer 1975, van Atten 2007). Brouwer's mathematics also requires an appropriate form of induction (Dummett 1977), therefore the assumption of the principle of Bar Induction, **BI**, which is a form of transfinite induction on well-founded trees.

[25]In fact, it turns out that **BISH** is better seen as a proper subtheory of the intersection of all of these theories, as there are principles, like Ishihara's BD-N, which are compatible with each of Browerian intuitionism, Russian constructivism and Classical mathematics, but which are not

tiori, theorems obtained in **BISH** hold in each of the other varieties of mathematics, including the classical one. This observation is at the heart of the claim by constructive mathematicians that Bishop's constructive mathematics is a *generalisation* of classical mathematics, as well as of **RUSS** and **INT** (see Section 2.5.1).

A major difference between constructive mathematics and **INT** and **RUSS** is that *both the latter forms of mathematics prove statements that are classically false.* For example, in **INT** every function from the real numbers to the real numbers is continuous. This clearly differs from the case of constructive mathematics, as the latter does not prove anything which is classically false. There is no conflict between constructive mathematics and classical mathematics. I shall also write that constructive mathematics is ***compatible*** with classical mathematics. The picture above clarifies that the notion of compatibility has two components: the *absence of conflict* between constructive and classical mathematics and their *agreement*, in the sense that there is a *preservation of derivability* when moving from systems that codify constructive mathematics to those that systematize classical mathematics. In fact, constructive mathematics is compatible also with **INT** and **RUSS**, and its compatibility with all of these forms of mathematics has been exploited in the last 10 years or so within the so–called "constructive reverse mathematics programme", that I shall address next.

2.4.2 Constructive Reverse Mathematics

Independently initiated by Ishihara and Veldman (see, for example, Ishihara 2005, Ishihara 2006, Veldman 2005, Bridges & Palmgren 2013), the constructive reverse mathematics programme takes inspiration from Friedman and Simpson's (classical) reverse mathematics programme (Simpson 1999), but differs from it for its scope of action. The classical reverse mathematics programme aims at discovering which

accepted by constructive mathematicians of the Bishop school.

set existence axioms are needed in order to prove theorems of ordinary or core mathematics.[26] Often the theorems turn out to be equivalent to the axioms; hence the slogan "reverse mathematics". This programme uses the language of second order arithmetic $\mathbf{Z^2}$ and has isolated five main subsystems of it that frequently occur as the reversals of mathematical theorems. To classify a mathematical theorem one usually shows that it is equivalent, on the basis of the next weaker system, to the principal set existence axiom of one of these five systems. In this way one shows which set existence axioms are actually needed for a specific mathematical theorem. As a by–product of the analysis, one usually obtains more detailed and possibly involved, but also more informative proofs.

Constructive reverse mathematics originates from Bishop's observation, (see e.g., Bishop 1967, Bishop 1975) that (from a formal point of view) the classical *mathematical practice* can be recast in constructive terms by theorems of the form: $\mathbf{EM} \to \varphi$. In fact, Bishop singled out consequences of the principle of excluded middle, as the Limited Principle of Omniscience (**LPO**)[27] and the so–called Lesser Limited Principle of Omniscience (**LLPO**) (Bishop 1967, Bishop 1985), and showed that they are often sufficient to capture the classical element figuring in a non–constructive proof

[26] Set existence axioms are principles that allow us to define sets within a mathematical theory. Typical set-existence axioms are *comprehension axioms*, that allow for the formation of a set of all those objects that satisfy a given formula (of a certain complexity). The classical reverse mathematics programme is formulated within the context of a theory know as "second order arithmetic", of which it singles out some particularly interesting subsystems. Second order arithmetic, also termed $\mathbf{Z^2}$, is a theory that uses a two-sorted language, with variables for natural numbers and variables for sets, to formalise the theory of the natural numbers and their subsets. A particularly important comprehension principle is the principle of arithmetical comprehension. An arithmetical formula is a formula of $\mathbf{Z^2}$ with no occurrence of set quantifiers (but possibly with occurrences of number quantifiers). Arithmetical comprehension states that for each arithmetical formula, φ, we can form the set $\{x : \varphi(x)\}$. In Chapter 4, I shall further discuss the reverse mathematics programme in relation with predicativity.

[27] See footnote 11, page 49, for **LPO**.

in analysis. Pinning down their occurrences in classical proofs thus can be seen as a contribution to an analysis of the classical tradition, too, explaining where exactly the non–constructivity comes into play.

The thought underlying the constructive reverse mathematics programme is that the privileged standpoint of constructive mathematics, at the intersection of a number of different practices, allows it to act as a base from which to compare notions and results across *all* of these varieties of mathematics. Thus the constructive reverse mathematics programme aims at classifying not only theorems in classical mathematics but also theorems in recursive, Russian and Brouwerian (i.e. intuitionistic) mathematics. Constructive reverse mathematics is thus a very promising path in the direction of a fine-grained analysis of the mathematical practice as a whole. It also highlights a new trend within constructive mathematics, by introducing a very detailed attention to the ingredients of a proof.

As already pointed out at the end of Section 2.3.2, one of the reasons for being interested in these mathematical developments is because of the insights they offer on mathematics as a whole, including ordinary mathematics. As in the case of constructive commutative algebra, also the reverse mathematics projects (both classical and constructive) clarify why "weaker" standpoints in mathematics are valuable for mathematics and its philosophy. Such a weaker standpoint may be secured by working classically within a weak subsystem of $\mathbf{Z^2}$, like in classical reverse mathematics, or by working within constructive mathematics. For example, the classical reverse mathematics programme clarifies that many ordinary theorems that prima facie appeal to abstract (e.g. set-theoretic) notions, only require concepts that pertain to arithmetic.[28] The constructive reverse mathematics programme aims at pinning down the non-constructive component of ordinary mathematics. In so doing it helps clarify which theorems of ordinary mathematics are not amenable to direct computational interpretation and which may be offered computational interpretation after

[28]This point will be further discussed in Chapter 4.

suitable reformulation. In fact, given its position at the intersection of a number of mathematical practices, constructive mathematics offers the possibility of analysing all of them from a constructive point of view.

These considerations explicate my principal reason for being interested in forms of mathematics, like the constructive and predicative, that make fewer assumptions than classical mathematics. They help us clarify different components of contemporary mathematics, their relations with each other, and they enable us to carry out more elementary and algorithmic proofs. In the following section I shall discuss some of the motives that constructive mathematicians of the Bishop school have put forward for working constructively (Bishop 1967, Bishop 1975, Bishop 1985, Richman 1990, Richman 1994, Richman 1996, Bridges & Reeves 1999, Bridges & Palmgren 2013). These differ considerably from traditional arguments for intuitionistic logic the philosopher is more familiar with.

2.5 Reasons for constructive mathematics

Constructive mathematics is indebted to Brouwer's insights for his introduction of intuitionistic reasoning. However, already Bishop expressed severe criticism of Brouwers' philosophical views and, especially, his non-standard view of the continuum. Not only constructive mathematics differs considerably from Brouwerian mathematics, but their respective motivations are also extremely diverse. In the case of Brouwer the principal motive for the development of intuitionistic logic was a specific *philosophical perspective* that moved from a view of mathematics as free creation of the human mind. I shall argue that constructive mathematicians today are prompted instead by two main kinds of reasons, that I should like to term ***intra-mathematical*** and ***pragmatic reasons***. These motives move from within mathematics itself, or are determined by the desire to facilitate applications of mathematics in particularly relevant areas. In addition, typically constructive mathematicians express a preference for an algorithmic form of mathematics but do not offer arguments for the rejection of alternative forms of mathematics (e.g. the classical one).

Although the computational nature of constructive mathematics is arguably the most relevant reason for the development of this form of mathematics, I shall first of all discuss "intra-mathematical" reasons, which arise from within the mathematical practice itself. These have been advanced by Bishop, but have been particularly emphasized in a number of articles by Richman (Richman 1990, Richman 1994, Richman 1996). I shall single out two aspects of one notable intra-mathematical reason: **generalisation**.

2.5.1 Generalisation

Constructive mathematics is the result of the elimination of **EM** from classical mathematics and it does not add any further principles that conflict with classical mathematics. As a consequence, constructive mathematics may be seen as a *generalization* of classical mathematics. A first sense in which constructive mathematics may be viewed as a generalisation of classical mathematics is related to the fact that all constructive theorems are classically true, and, in addition, they can be given not only a classical but also a computational interpretation. This is the point that is more relevant for the present discussion and has been mentioned in previous sections. A second sense of generalisation is equally significant for the constructive mathematician, and relates to the fact that by relinquishing the principle of excluded middle we can discover new forms of mathematics. Here the motivation for the use of intuitionistic logic is clearly "internal" to mathematics: it is the sheer mathematical interest in very general mathematical "structures" which appear **only** if one uses intuitionistic logic. There are numerous cases of mathematical structures or notions that are profitably studied in an intuitionistic context because they are more general than if approached from a classical perspective. In these contexts the addition of the principle of excluded middle restricts the discussion to less general mathematical structures. To explain this point Richman (1990) uses the following simile: constructive mathematics is like group theory and classical mathematics is like Abelian group theory. The latter arises by the addition of the commutativity axiom and has the effect of focusing the mathematician's attention to particular mathematical structures. Similarly, in some cases the avoidance of the principle of excluded middle allows for the development of significant areas of mathematics that would be overly simplified by the imposition of the principle of excluded middle.

The best example is given by the notion of a *topos*. This is a fundamental categorical notion that has been thoroughly studied since its appearance in the 1960's within the developments of algebraic geometry. Its fruitfulness lies in its generality, that let us relate in new fundamental ways apparently unconnected areas of research. The important fact to observe is that toposes are very general categories, and, in fact, only particular instances of them (Boolean toposes) have a classical behaviour.[29]

Recent interest in homotopy type theory may be taken to offer further reasons for a constructive approach, again determined by the fruitfulness of a very general approach (Univalent Foundations Program 2013). Here once more we have that a more general, constructive approach allows for the revelation of unexpected relations

[29]The notion of elementary topos is particularly relevant in foundational contexts, as it plays a similar role (within category theory) as that played by standard set theory as foundation for ordinary mathematics. Indeed, the so called "internal language" of a topos is intuitionistic logic. It is because the notion of topos is interesting and fruitful that it has gained prominence within category theory. Indeed, its original motivation was geometric and topological, not logical. One of the principal reasons for the interest in the notion of topos is its generality, which enables us to relate in new fundamental ways apparently unconnected notions. In particular it allows us to relate mathematical phenomena that the more restricted context of a Boolean topos does not manifest. For references see, for example, the Programmatic Reading Guide supplement to the entry on Category Theory (Marquis 2015) of the Stanford Encyclopedia of Philosophy, available at the address: http://plato.stanford.edu/entries/category-theory/bib.html.

between apparently distant mathematical notions (or even areas of research) and for the development of a large new body of mathematics that is invisible from a classical perspective.

2.5.2 Computational content

The computational nature of constructive mathematics is the most prominent reason for working constructively, as already emphasized by Bishop. This reason has primarily a pragmatic component, as the desire to develop a kind of mathematics which is best suited to computational applications determines the choice to work with intuitionistic logic. There is, however, more to the preference for a computational form of mathematics than the desire to ease the implementation of mathematics on computers.[30] Arguably, the adherence to constructive, and, as further discussed later on in this thesis, predicative forms of mathematics is also motivated by deeper and more complex reasons. For example, constructive mathematicians often express a preference for a mathematical *style* that is more explicit and algorithmic. They typically show a strong uneasiness with proofs by contradiction of existential statements as they do not present us with a witness. This uneasiness is clearly not determined by the worry of being unable to produce a program that will run on a computer. It is instead expression of a desire to produce a form of mathematics that is considered more satisfactory because it is algorithmic and can be developed without recourse to highly abstract notions.[31] Therefore an algorithmic form of mathematics is pursued not only because of its applicability in computer systems but because of a preference for algorithmic methodology.

The attitude of the constructive mathematician may be compared with that of Poincaré and Weyl that I shall address in the second part of this thesis. In that case

[30] This suggests a more careful classification of the computational motive as not purely pragmatic.

[31] See Chapter 1, page 17 for a suggestion that there are a number of alternative styles in contemporary mathematics.

there is a preference for a mathematical style that is considered more satisfactory because it is more in agreement with the tradition, and also more secure. One of the reasons for Poincaré and Weyl's criticism of the "new" forms of mathematics was the worry of inconsistency. The thought was that a more careful formulation of a "constructive" (i.e. predicative) notion of set would enable the development of a safe form of mathematics. Some constructive mathematicians also claim that a constructive form of mathematics gives more confidence in the correctness of the resulting mathematics. Therefore, correctness is not only at the heart of the constructive enterprise through the development of constructive proof assistants for the verification of mathematics; a constructive way of reasoning is pursued also with the aim of gaining full confidence on its correctness.[32] The thought here is that an algorithmic way of reasoning is safer because it makes only steps that can be fully grasped and easily verified, it relies on simpler concepts, and therefore can be fully trusted.

The computational content of constructive mathematics and generalisation are among the reasons that the constructive mathematician adduces to explain why constructive mathematics is worth pursuing. These reasons are very different from well-known arguments for intutionistic logic that have been proposed within the philosophy of mathematics. The latter, in particular, characteristically also imply the rejection of classical mathematics. This is the case of Brouwer's arguments for intuitionism but also of an influential semantic argument for intuitionstic logic (Dummett 1975). According to the latter, intuitionistic logic is imposed by requirements that need to be satisfied by our language, in order to comply with its role as instrument of communication. As a result, since classical logic does not satisfy those requirements, one also obtains a stark rejection of classical logic.

The discussion above and in Chapter 1 proposes a very different perspective: we have *de facto* a plurality of mathematics, including the constructive one. The

[32]See (see, e.g., Martin-Löf 2008). See also (Crosilla 2015*a*).

latter is gaining more prominence within mathematics as a whole due to a number of factors, some of which purely sociological in nature. A fundamental task for the philosopher of mathematics is to understand the reasons for this plurality of practices and the relations between them, as a contribution to a general clarification of mathematics and mathematical thought. The motives for pursuing constructive mathematics that constructive mathematicians put forth can then be seen as a clarification of why this form of mathematics is worth pursuing. However, they do not seem to either suffice to propose the exclusive use of intuitionistic logic, nor, more crucially, to reject classical mathematics.

This leaves completely unresolved the issue of how to make sense of this plurality of forms of mathematics from a more traditional philosophical perspective, and account for it in a satisfactory manner. I contend that a clarification of the nature and scope of constructive (and classical) mathematics is prior to that work, and I hope that the discussion above has contributed to laying a foundation for subsequent investigations.

2.6 Conclusion

In Chapter 1 I have highlighted the differentiation of today's mathematics, and discussed a possible new position that constructive mathematics may acquire in the near future within the mathematical landscape as a result of deep changes that affect mathematics today.

In this Chapter I have first of all further specified what is Bishop's constructive mathematics, offering a characterisation of it that distinguishes it both from classical mathematics and from other forms of mathematics that use intuitionistic logic (INT and RUSS). I have also endavoured to clarify that constructive mathematics is a well developed, rich form of mathematics, that is witnessing fast progress. For this purpose I have sketched some common strategies that constructive mathematicians typically adopt to counter the absence of the excluded middle. These have given the way to a discussion of the motives that bring constructive mathematicians to work with intuitionistic logic. Here I have emphasised two kinds of reasons: pragmatic and intra-mathematical. The computational content of constructive mathematics offers an example of reason of the first kind. Among the latter is generalisation. All of these motives do not offer arguments for intutitionistic logic analogous to traditional philosophical arguments. The importance of generalisation for the philosophical discussion relates to the additional insights that a "weaker" perspective can offer on mathematics as a whole. The constructive perspective, in fact, allows us to see new mathematical structures as well as significant conceptual distinctions within known parts of mathematics. The adoption of weaker systems of reference allows for the realisation that large portions of today's mathematics are amenable to careful reformulation in terms of more algorithmic and elementary concepts, and therefore can be given full computational meaning. This observation is at the centre of the subsequent discussion on predicativity, that highlights the potential of combining the use of intuitionistic logic with suitable restrictions on sets, for both the mathematical and the philosophical investigation.

Part II

Part II: Predicativity

Chapter 3

Origins of Predicativity

3.1 Introduction

In Part I, I have examined the impact that constructive mathematics' compliance with intuitionistic logic has for its notion of proof: the exclusive use of this logic suffices to endow the theorems of constructive mathematics with a direct computational content. I have then endeavoured to clarify the impact that the adherence to intuitionistic logic has for mathematical practice. In particular, I have described some of the techniques that have been utilized to reproduce parts of ordinary mathematics in computational form. I have then examined Bridges and Richman's characterisation of constructive mathematics as mathematics that uses only intuitionistic logic. As constructive mathematics is fully compatible with the classical tradition it offers a further, refined insight into ordinary mathematics.

Bishop's mathematics has been the inspiration for the development of a number of foundational systems (i.e. set and type theories), introduced to clarify its underlying concept of set.[1] Here from an early stage the use of intuitionistic logic has

[1]See, for example, (Friedman 1973, Myhill 1973, Feferman 1975, Martin-Löf 1975, Myhill 1975, Friedman 1977, Aczel 1978, Aczel 1982, Martin-Löf 1982, Martin-Löf 1984, Beeson 1985, Aczel 1986, Aczel & Rathjen 2008).

been supplemented by a form of predicativity.[2] In particular, the most prominent foundational systems for constructive mathematics today, Martin-Löf type theory and Aczel and Myhill set theory (Martin-Löf 1984, Aczel & Rathjen 2008), not only use intuitionistic logic but also appropriately modify the notion of set compared with more standard classical foundational systems, like Zermelo Fraenkel (ZF) set theory.

Predicativity is the focus of this second part of the thesis. The principal aim of the remaining chapters is to clarify what is predicativity and thus lay down a foundation for a further investigation of the relation between predicativity and constructive mathematics.

The reasons for the introduction of predicative constraints in foundational systems for Bishop's constructive mathematics will be further discussed in Chapter 4, Section 4.4.2. Here it suffices to mention that predicativity embodies a notion of constructivity that is deeply-rooted in the mathematical tradition, and is perhaps more fundamental than the notion of constructivity that originates by the adoption of the intuitionistic logic, as it affects the notion of set. It is a form of constructivity that has found expression both in classical and in intuitionistic settings. In the latter, the desire to obtain a fully algorithmic form of mathematics seems to naturally promote also adherence to predicativity, as witnessed, for example, by Bishop's constructive analysis in (Bishop 1967). A careful inspection of the proofs in (Bishop 1967) reveals that virtually all of the constructive analysis developed in that book is in fact carried out without any need for impredicative notions (Myhill 1975).[3]

[2]As further expounded in Chapter 4, a number of variants of predicativity have been proposed in the mathematical literature. The notion of predicativity that is appealed to in the case of constructive mathematics is usually termed constructive or generalised predicativity (see Chapter 4, Section 4.4.2).

[3]A significant portion of the recent literature in constructive mathematics has aimed at obtaining constructive and predicative renderings of both classical and intuitionistic impredicative proofs. See, for example (Aczel 2006, Coquand 1992, Curi 2001, Curi 2003, Curi 2006, Curi 2007, Ishihara

If we take the perspective put forward by constructive foundational systems as Martin-Löf type theory, then constructive mathematics is the result of combining two components: (i) the use of only intuitionistic logic, and (ii) a form of predicativity. From a *classical, impredicative* perspective, one could also say that constructive mathematics is the result of a *double restriction* compared with classical mathematics:

- **Intuitionistic Logic:** the logic is the intuitionistic logic;
- **Predicativity:** sets conforms to appropriate predicative constraints.

A clarification of the notion of predicative set will be one of the central aims of this part of the thesis, however, an intuition can be given as follows. Predicativity can be roughly characterised as a constraint on the way mathematical objects are *defined*, that enables us to conceive of the predicatively definable mathematical objects as "**built up from within**" and in stages. As we typically codify mathematical objects in terms of sets, predicativity can be seen as a restriction acting on sets: they also are "built up from within" and in stages, from a limited initial stock of "simple objects".[4]

In very schematic form, again from a classical perspective, the restriction to intuitionistic logic effected by the Bishop school substantially modifies the notion of **proof**. The adherence to predicative constraints that is witnessed in the foundational systems for constructive mathematics induces, in addition, a modification of the notion of **set**. A crucial observation is that both constructive and predicative mathematics are fully compatible with, and in fact refinements of, classical impred-

& Palmgren 2006, Maietti 2005, Palmgren 2005, Palmgren 2006*b*, Palmgren 2006*a*, Sambin 1987, Sambin 2003, Vickers 2005, Vickers 2006, Vickers 2007).

[4]For example, according to one of the notions of predicativity that we shall discuss below, sets are constructed by means of repeated application of simple logical operations acting on the natural numbers. See Chapter 5, Section 5.4.

icative mathematics.[5] As a consequence they can also be seen as offering a refined view of ordinary mathematics.

3.2 The origins of predicativity

The starting point of my analysis of predicativity will be the historical development of this notion, from which I shall select some particularly significant aspects. By examining the origins of predicativity, I shall focus on developments that are largely independent from the use of intuitionistic logic that characterises constructive mathematics. In fact, the discussion will also centre on proposals, like Weyl's, that explicitly endorsed classical logic. The principal reason for this is that I should like to demonstrate a direct continuity between some of the ideas expressed within classical predicativity and the constructive one.

The relation between predicativity and logic is a point that requires further clarification. As predicativity introduces constraints on how to define sets, the most significant aspects of predicativity may be seen as independent from the use of a specific logic.[6] In fact, predicative approaches have originally been developed within the *classical* tradition; as a consequence, the cohabitation of predicativity with

[5]For the notion of compatibility see Chapter 2, page 60. In fact, all the predicative formal systems contemplated in this thesis, even the classical ones to be discussed in later chapters, are compatible with classical impredicative mathematics. This might at first seem surprising since, as further argued in Chapter 5, one of their principal motivation was the development of a concept of set that is radically different from the concept of set that motivates ZF set theory.

[6]The case of constructive predicativity is, in fact, more involved, as in the case of Martin-Löf type theory there is a deep relation between predicativity and intuitionistic logic, due to the Curry-Howard isomorphism (Martin-Löf 2008). See also Chapter 4, Section 4.4.2. However, some prominent features of constructive predicativity can be discussed independently from the use of intuitionistic logic. In fact, in Chapter 5 I shall argue that even in the case of constructive predicativity there is a continuity of ideas that run through some of the initial discussions on predicativity (that were cast from a classical perspective) up to forms of constructive predicativity.

intuitionistic logic that we witness in the case of constructive mathematics is not a necessary feature of either mathematics based on intuitionistic logic or predicativity. The relation between predicativity and constructivity is, at least in most cases, orthogonal to the question of the role of the intuitionistic logic for constructivity. To witness this independence, there are examples of mathematical theories which take one revision and not the other on board, and others which take both, or none. More importantly, the sole use of intuitionistic logic does not guarantee adherence to predicativity: for example, there are intuitionistic theories of sets in the style of Zermelo Fraenkel (Friedman 1973, Beeson 1985) which are fully impredicative.[7] This makes an analysis of constructive predicativity particularly complex, but in no way can one omit due consideration of the classical tradition in an elucidation of the constructive one.

Another clarification is necessary before examining the origins of predicativity. I wish to draw a distinction between predicativity and predicativism. The term predicative denotes a characteristic of (mathematical) definitions and of the objects that are so defined.[8] The term "predicativism" denotes a philosophical position within the philosophy of mathematics which rejects as unjustified those mathematical notions that can not be defined predicatively. That is, predicativism might be characterised as maintaining that mathematics fully coincides with the mathematics that can be developed on the basis of predicatively definable notions. Hermann Weyl may be seen as proposing a form of predicativism in "Das Kontinuum" (Weyl 1918), and so may Edward Nelson in "Predicative Arithmetic" (Nelson 1986).[9] The term

[7]The full impredicativity of intuitionistic Zermelo Fraenkel set theory (e.g. Friedman's system IZF) is evinced from the fact that its proof-theoretic strength equals that of ZF (Friedman 1973).

[8]The notion of predicativity is typically first discussed in relation to definitions, and then also extended to those objects that can be predicatively defined. In Chapter 4 we shall see that also a theory may be termed predicative.

[9]As further clarified below, these authors can be seen as proposing very different forms of predicativism.

"predicativity", by contrast, is often also used to refer to a (possibly philosophically neutral) adoption of predicative constraints. That is, predicativity often refers to a form of mathematics that implements predicative constraints, but is not bound to predicativism. One might, for example, consider as legitimate those forms of mathematics which use impredicative concepts, but express a preference for predicative mathematics on specific grounds, for example their clarity or lack of ambiguity. In fact, Solomon Feferman has often proffered his interest in predicativity, but has denied advocating predicativism.[10] There is here a clear similarity of attitude with that of many constructive mathematicians today, who declare a preference for constructive mathematics, without advocating constructivism (see Chapter 2).

3.3 Poincaré and Russell

The notion of predicativity has its origins in the writings of Poincaré and Russell (see, for example, Poincaré 1906, Russell 1906*a*, Russell 1908, Poincaré 1909, Poincaré 1912), and was instigated by the discovery of the paradoxes in Cantor's and Frege's set theory. Adherence to predicativity resulted in the creation by Russell of ramified type theory, which has profoundly influenced the development of logic and computer science. A rigorous study of predicativity in the particular case of analysis was initiated by Weyl (1918), who showed in some detail how to develop a predicative account of the continuum.

These first elaborations of a notion of predicativity were spurred by the paradoxes, but were part of a more general debate that was stirred by the deep methodological changes in mathematics starting from the 19th Century that were mentioned in Chapter 1. Arguably, the discovery of the paradoxes made more urgent the ongoing clarification of the new methodology. Predicativism can be seen as sharing important characteristics with logicism, Hilbert's programme and intuition-

[10] See e.g. (Feferman 2004).

ism. These influential philosophical programmes emerged between the end of the 19th and the beginning of the past century, in an attempt to bring clarity to a fast changing mathematics.[11] The foundational programmes attempted to either clarify the nature of mathematics and give its contemporary practice full justification, like Frege's logicism and Hilbert's programme, or, in the case of intuitionism and predicativism, reform mathematics, as its current practice was seen as lacking justification. Predicativism therefore represents a perhaps less known chapter in the fundamental contributions to the philosophy of mathematics that distinguish the turn of the 20th Century.

The notion of predicativity emerged in an animated discussion between Poincaré and Russell which spanned from 1905 to 1912. Notwithstanding the remarkably different views of these two authors, for instance, on the role of logic and formalization within mathematics, they both converged on holding **impredicativity** responsible for the onset of the **paradoxes**, and attempted to clarify a notion of predicativity, adherence to which would avoid inconsistencies.

Through Russell and Poincaré's confrontation a number of ways of capturing impredicativity and explaining its perceived problematic character emerged. Russell (1906*b*) introduced the term **predicative** to denote a *propositional function* that defines a class.[12] Today it is common to liken the notion of propositional function

[11]The foundational programmes were anticipated by discussions on the methodology of mathematics which did not take the form of clear philosophical programmes, and instead addressed individual features of the mathematical methodology, often stimulated by specific technical issues. Criticism of the new methods of proof was put forward, for example, by Kronecker, and, from a different point of view, also by the French analysts Baire, Lesbegue and Borel, as well as Poincaré. These discussions were prompted by the new kind of mathematics and not only preceded but deeply influenced the foundational programmes (see e.g. Michel 2008, van Dalen 1999). A thorough analysis of these pre-foundational debates and their relation with predicativity would substantially advance the present analysis, but will have to be postponed to another occasion.

[12] Note that the term "class" in both Russell and Poincaré is used to refer to a generic collection of elements, and hence should be carefully distinguished from a (proper) class in contemporary set

to that of open formula (i.e. a formula with a free variable).[13] From a contemporary perspective Russell's introduction of the term predicative may be seen as an attempt to mark a distinction between those formulas that do give rise to a class, the class of those x which satisfy φ (also written as $\{x : \varphi(x)\}$), and those which do not. The introduction of the term predicative is at the heart of Russell's influential analysis of the paradoxes, which imputes them to the illegitimate assumption that *any propositional function gives rise to a class*, the class of all the objects satisfying it. For Russell the paradoxes showed that some propositional functions, the impredicative ones, do not give rise to a class. He therefore set up to determine ways of clarifying the distinction between predicative and impredicative classes.

Russell and Poincaré's attempts to clarify predicativity produced two principal ways of explaining this notion. The first one is expressed in terms of circularity, while the second one features a lack of "invariance". I shall outline each of them in turn in the next Sections.

3.3.1 Circularity

According to one characterisation of impredicativity, a definition is impredicative if *it involves a vicious circularity, or self-reference.* More precisely:

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theory. In informal discussions within the latter, the term class is usually employed to refer to a "proper class", that is, a collection that is not a set.

[13]See e.g. (Feferman 2005). The interpretation of the notions of proposition and propositional function in Russell is a matter of heated debate among Russell scholars. A common interpretation is the one suggested by (Gödel 1944) (p. 452, footnote 9), who interprets a propositional function as a proposition in which one or several constituents are designated as arguments. For example, if we write the formula $P(a)$ to denote a proposition that holds of an object denoted by a, then $P(x)$ will denote a propositional function, that takes arguments denoted by $a, b, ...$ and produces the propositions denoted by $P(a), P(b), ...$, respectively. Gödel also assimilates a Russellian propositional function to a concept; in fact, propositional functions share the unsaturated character of Fregean concepts.

A definition is impredicative if *it defines an entity by reference to a totality to which the entity itself belongs.*[14]

This characterisation of impredicativity originates in Jules Richard's analysis of the paradox that bear his name (Richard 1905), and was subsequently advanced by Poincaré (1906), and then endorsed by Russell in a number of writings.[15] The characterisation of impredicativity in terms of vicious circularity is often further elaborated in a logically more precise form as follows: **A definition is impredicative if it defines an entity by *quantifying over a totality which includes the entity to be defined*.** Given this notion of impredicative definition, one can express a notion of impredicative entity: this is an entity that can *only* be defined through an impredicative definition. In fact, in Russell and Poincaré's discussion the notion of impredicativity is not restricted to the case of sets, as it applies to a number of different kinds of entities, like propositions, properties, etc. **Predicativity** is then defined as **the negation of impredicativity**.

Russell famously introduced his **"Vicious–Circle Principle" (VCP)** to ban impredicative definitions. This had a number of formulations, like, for example:

"no totality can contain members defined in terms of itself" (Russell 1908, p. 237).

Another is to be found in (Russell 1973, p. 198):

> ... whatever in any way concerns *all* or *any* or *some* of a class must not be itself one of the members of a class.

[14]See (Gödel 1944, p. 455). Note that in the original literature one finds frequently the word "totality" instead of "set". The underlying thought is that a totality is a collection whose extent we can determine in a precise and unambiguous way (see also Section 3.3.2). This has important similarities with today's notion of set as codified in Zermelo Fraenkel (ZF) set theory. However, I shall make use of Russell's terminology rather than utilize the word "set", because the latter has also additional connotations that should not be presupposed in this discussion. See also footnote 12, page 79, for a clarification of the use of the term "class".

[15]See page 89 for a description of Richard's paradox.

The latter formulation clearly highlights the fundamental link between impredicativity and quantification.[16]

In subsequent chapters I shall examine in some detail the reasons that induce the predicativist to perceive this form of circularity as problematic. In the next Section I clarify the present characterisation of impredicativity by considering some examples. I also examine early analysis of these examples and draw principally from (Russell 1908, Whitehead & Russell 1910, 1912, 1913). For the first example, however, I rely on an early analysis of impredicativity by Carnap (1931).

Examples: circularity

1. *The logicist definition of natural number:*

$$N(n) := \forall F[F(0) \wedge \forall x(F(x) \rightarrow F(Suc(x))) \rightarrow F(n)].$$

According to this definition, the concept of natural number is defined by reference to all properties, F, of the natural numbers. A circularity arises here as the property N itself is within the range of the first quantifier. As a consequence, N is defined by reference to itself. The difficulty with this definition is often explained as follows[17]: suppose we wish to check a particular case, that is, whether $N(x)$ holds for a specific natural number x, say for $x = 5$. In order to do so, it would seem that we need to check if for *each* property of the natural numbers, F, F holds of 5, that is:

$\forall F[F(0) \wedge \forall x(F(x) \rightarrow F(Suc(x))) \rightarrow F(5)]$.

[16]This is probably the best know formulation of the VCP; however, Russell gave other formulations, some of which, like the first one above, do not directly involve quantification, but, rather, self-reference or simply the reference to, rather than quantification over, a totality that includes the definiendum. This plurality of formulation of the VCP induces difficulties for an exegesis of Russell's thought, as noted already by Gödel (1944).

[17]See for example Carnap (1931).

However, the property "to be a natural number", which is expressed by the predicate N, is one of the properties of the natural numbers. That is, to find out whether $N(5)$ holds, we need to be able to clarify whether the following holds:

$$N(0) \wedge \forall x(N(x) \rightarrow N(Suc(x))) \rightarrow N(5)?$$

Hence it would seem that we need first to ascertain whether the property of being a natural number holds of 5 in order to assess whether it holds of 5. This is an unacceptable form of circularity.

2. *The Liar.* Russell (1908) first of all observes that the sentence "I'm lying" is the same as: "There is a proposition which I am affirming and which is false." This in turn can be rephrased as a universally quantified statement as: "It is not true for all propositions p that if I affirm p, p is true." A paradox then arises if we take this statement as affirming a proposition, which must then come under the scope of the universal quantifier. In fact, Russell further claims that "whatever we suppose to be the totality of propositions, statements about this totality generate new propositions which, on pain of contradiction, must lie outside the totality." The thought here is that an impredicative definition would seem to *generate* a new element of the very class that was used to define it. Russell's conclusion is that the totality of all proposition is *illegitimate* and statements such as "all propositions" are *meaningless.*

   Russell's analysis of this paradox is particularly interesting, as it bears significant similarities to the thought underlying an alternative characterisation of predicativity as invariance that will be discussed in the next Section. It also clearly sets out which conclusions Russell draws from paradoxes as this. As further mentioned in Section 3.3.2 Russell's conclusions differ from Poincaré's in important respects.

3. *Napoleon's qualities.* Another example (Whitehead & Russell 1910, 1912, 1913,

p. 59) is given by the sentence: ***Napoleon had all the qualities of a great general.*** This example and example 1 are particularly noteworthy because they do not involve paradoxes. Here the property "having all the qualities of a great general" refers to all such properties, and thus, when applied to a great general, also to itself. We might wish to compare the sentence above with the sentences: *Napoleon was Corsican*, or *Napoleon was brave.* These are utterly unproblematic, as the properties "being Corsican" and "being brave" do not refer to other properties (including themselves). We can see the difficulty arising with this example if we observe that in "having all the qualities of a great general", the locution "all qualities" refers to the totality of all qualities, which therefore would seem to include the one referred to by this very expression. Russell's conclusion here is that if we take "all qualities" to refer to a presumed totality of all qualities, then with the expression "having all the qualities of a great general" we "generate" a new element of that totality. But this violates the VCP, as the new quality is defined by reference to itself.[18]

4. *Russell's paradox.* So far we have considered cases of impredicativity which involve properties rather than classes. An example which involves classes is the famous **Russell's "set"**, that can be so defined in modern terminology:

$$R = \{x \mid x \notin x\}.$$

Here R can be seen as arising from an application of the **Unrestricted Comprehension schema**: given any formula φ in the language of set theory, we can form the set of all the x's that satisfy φ, that is, $\{x \mid \varphi(x)\}$. In R's definition, in particular, one takes φ to be $x \notin x$.

In his analysis of this paradox Russell (1908, p. 225) observes that R is defined impredicatively as it *refers* to the class of all classes. If we wanted to block the

[18]This example is often used by Russell to explain how ramification works in ramified type theory. See also Section 3.4.

paradox by deciding that no class is a member of itself, then R would become the class of all classes. But then the question arises whether R is an element of itself, and we have to decide that R is not a member of itself, that is, that R is not a class. Russell draws the conclusion that there is no class of all classes, since if we supposed there is, this would give rise to new classes lying outside the supposed totality of all classes.

This analysis by Russell confirms the thought already suggested in (Russell 1906*b*) (see also page 79 above) that a solution to the paradoxes can be reached by countering the assumption that any propositional function gives rise to a set. As further discussed in Section 3.4, Russell's remedy to the paradoxes, his type theory, introduces restrictions to the ranges of significances of the propositional functions utilized to define a set.

5. *Least Upper Bound principle.* Finally an often mentioned example from analysis: the **Least Upper Bound principle** (LUB).

> Every bounded, non–empty subset M of the real numbers has a least upper bound.

This is impredicative since we define a subset of the real numbers by quantifying on all subsets of the real numbers. As noted by Weyl (1918), if impredicativity is seen as problematic, this particular example is critical, as it goes at the very heart of analysis. Not only does it show that impredicativity is crucially used in the ordinary theory of the continuum, but it also would seem to impose some constraints on any predicative attempt to develop mathematics. In fact, it would seem that if by banning impredicativity one also impairs the availability of the (LUB), then any reasonable development of analysis would be blocked from the start. As we shall see below, Weyl's "Das Kontinuum" (Weyl 1918) is particularly relevant also for Weyl's recognition that we can develop a substantial portion of analysis also without the (LUB).[19]

3.3.2 Invariance

While Russell's discussion of the paradoxes and the VCP has attracted vast attention within the philosophical literature, another characterisation of predicativity proposed in later writings by Poincaré is less known today (Poincaré 1909, Poincaré 1912). This is of particular relevance for an account of constructive predicativity, as it is deeply interrelated with a predicative concept of set which is more apt to a constructive setting (as discussed in Chapter 5).[20]

The context of Poincaré's discussion of impredicativity in (Poincaré 1909, Poincaré 1912) is a reflection on the role of infinity in mathematics.[21] In (Poincaré 1909, Poincaré 1912), the French mathematician observes that the antinomies are particularly poignant in the case of infinite sets and sees their arising as consequence of an unjustified assumption of actual infinity.[22] Infinity for Poincaré is *unboundedness*, or *potential infinity*:

> There is no actual infinity, and when we speak of an infinite collection, we mean a collection to which we can add continuously new elements.[23]

Poincaré saw the assumption of actual infinity as particularly problematic because, for him, we can only reason about objects that can be defined by a finite number

[19] See (Feferman 1988*a*) for details.

[20] Poincaré's thought also strongly influenced the logical analysis of this notion that begun in the 1950's. See Chapter 4. See also (Cantini 1981) for an insightful analysis of Poincaré's notion of invariance and a formal systematization of it.

[21] The centrality of the discussion on infinity and its role within the reflections on the methodological changes that took place in the 19th Century is well-known and has been briefly hinted at in Chapter 1.

[22] This marks a difference with Russell, that with examples as the Liar emphasized that paradoxes arise also in finitary contexts.

[23] (Poincaré 1909, p. 463), my translation.

of words (Poincaré 1909, p. 464). As clarified in (Poincaré 1912), this view is in turn motivated by the conviction that the existence of mathematical objects is not independent from a thinking subject. The thinking subject, observes Poincaré, is a human being, or "something similar to it", and thus its finitude imposes a requirement of definability within a finite number of words.

I shall mention further Poincaré's definabilism in Chapter 5.[24] Here Poincaré's discussion on actual and potential infinity is relevant because it is deeply interconnected with his new characterisation of predicativity. Already in (Poincaré 1909), the French mathematician highlights what I should like to call a sort of "instability" of impredicative collections. Poincaré's thought bears similarities to an observation by Russell that was mentioned in the previous section: impredicative definitions seem to "generate" new elements of a totality over which they generalise.[25] For example, the quality "having all the qualities of a great general" generalises over all such qualities, and in so doing seems to produce a new element of the (presumed) totality of all qualities. Russell and Poincaré draw apparently different conclusions from this observation. For Russell, impredicatively defined totalities are *illegitimate*, and quantification over them is *meaningless*. The adoption of the VCP in (Russell 1908) has as effect that classes defined by impredicative definitions are *empty*. Poincaré instead sees the instability of impredicatively defined collections as indicating that their *definitions* are *illegitimate*, as they treat as immutable, or *invariant*, classes that are instead open-ended and *unfinished or incomplete*.[26]

The French mathematician therefore proposes a new characterisation of predicativity that is first of all phrased in terms of *predicative classifications* (or classes), as follows.

[24]See Chapter 5 for a description of definabilism.

[25]See the discussion of the examples of the Liar, "Napoleon's qualities" and Russell's paradox in Section 3.3.1.

[26]There is a strong similarity between Poincaré's ideas and Dummett's discussion of indefinitely extensible concepts. The latter will be examined in Chapter 6.

A predicative classification is one that can not be "disordered" (or disrupted) by the introduction of new elements.

Poincaré (1909, p. 463) writes:

> Hence a distinction between two species of classifications, which are applicable to the elements of infinite collections: the *predicative* classifications, that can not be disordered by the introduction of new elements; the *non predicative* classifications which the introduction of new elements forces to remain without end.[27]

A characterisation of predicative *definitions* is then given by claiming that a definition is after all a classification: it separates the objects that satisfy, from those that do not satisfy that definition, and it arranges them in two distinct classes. Consequently a ***definition is impredicative if it defines a classification that is not predicative***.

The very idea of classes without end and disordered by the introduction of new elements is very hard to grasp from a contemporary perspective, that is so much influenced by the very static concept of set that is codified by ZF set theory.[28] I shall endeavour to clarify a predicative concept of set inspired by Poincaré's remarks in Chapter 5. For now the crucial point to focus on is that for Poincaré the paradoxes highlighted the illegitimacy of impredicative definitions, because such definitions

[27]My translation; italics by Poincaré. The word "disordered" translates the French "bouleverseé".

[28]The claim is that it is difficult to capture Poincaré's notion of set within a context as ZFC, not that it is difficult to express invariance. The latter has strong affinities with the concept of absoluteness in classical set theory. Poincaré's reflections on predicativity have also strongly influenced the first identifications of predicativity with the hyperarithmetical hierarchy within the logical analysis of predicativity that will be discussed in Chapter 4, Section 4.2. See, for example, (Kreisel 1960) for discussion. Linnebo's (2013) modal set theory offers a way of capturing a constructivistic (or potentialist) concept of set from a modal perspective.

treat as completed (French "arrêté") infinite classes that are instead "in fieri", open ended or incomplete by their very nature.

In the following I shall refer to this characterisation of predicativity as "**invariance**": *a collection is predicative if it is "invariant under extension".*

I wish to further elucidate Poincaré's characterization of predicativity in terms of invariance by considering some examples.

Examples: invariance

1. Poincaré (1909) gives the following example of a *predicative* collection: *the collection of natural numbers that are less than 10.* Poincaré clarifies that we can recognise whether a natural number is less than 10 or not independently of the relation between this number and other numbers. He states that when we have defined the first 100 numbers, we know which ones are smaller and which ones are greater or equal to 10. If we then *introduce* the number 101, the numbers less then 100 that were smaller than 10 are still so, and similarly for those that were greater or equal to 10.

   It is instructive to see that Poincaré elucidates the predicativity of this collection by highlighting the importance of stable relations between a newly defined element of a class and other elements of it. In addition, Poincaré's explanation of this example offers further indication of his "constructivistic" conception of classes, as the French mathematician refers to our "introducing" the number 100.

2. An example of impredicative collection is obtained by an application of Richard's paradox. This paradox allows for the definition by diagonalization of the least non–definable real number, r, by reference to the totality of all definable real numbers. More precisely, let us consider all the real numbers that are definable in English by a finite number of words and let D be their collection. This is

countable as each definition can be given by a finite number of symbols. We can then list all the elements of D, and mimic Cantor's diagonal proof of the non-denumerability of the real numbers to produce a new real number, r, that is different from each element of D. But one can easily express in English a rendering of the "algorithm" that allows for the definition of r, so that r turns out to be a definable real number after all, and a contradiction arises.

The usual diagnosis of this paradox is in terms of circularity: the difficulty lies in the fact that we define r by reference to the whole D, thus also to r itself. For Russell, then, the paradox shows that D is ill-formed, and quantification over it is meaningless. The new diagnosis suggested by Poincaré underlines the fact that the newly defined real number r would then appear to *extend* the totality of definable real numbers that was used in its definition. It is the instability of the collection of definable real numbers that becomes for Poincaré symptom of impredicativity and, as in Russell's analysis, imparts the illegitimacy of quantification over such a class.

It is worth noting that it would seem that all the cases of circular definitions discussed in Section 3.3.1 also give rise to a lack of invariance, as they undermine the possibility of determining in a definitive way the boundary of a collection of entities in terms of which the definiendum is specified.[29] An important aspect that is in common to both characterisations is that they are negative, in that they tells us what predicativity is not (variance, circularity), but they do not specify what counts as a predicative definition. Substantial work has been carried out since Russell and Poincaré's early writings to further clarify in more positive terms what is predicativity.[30] The first major contribution to an elucidation of this notion has

[29]See (Cantini 1981) for a comparison between the strength of a formalism there introduced to capture Poincaré's notion of invariance and systems introduced within the "Kreisel-Feferman-Schütte" analysis of predicativity that will be outlined in Chapter 4.

[30]See in particular Chapter 4.

been offered by Russell himself, with his ramified type theory.

3.4 The developments of predicativity I: Russell's ramified type theory

The analysis of the paradoxes turned out to be extremely fruitful for the development of mathematical logic, starting from Russell's own implementation of the vicious circle principle through his type theory.[31] Russell's type theory is one of a number of fundamental technical achievements that were instigated by the philosophical reflection on predicativity. The latter, in fact, is an emblematic example that stands out in the history of the philosophy of mathematics and of logic for the fruitfulness of interactions between philosophy and logic.

In Russell's ramified type theory the vicious circle principle is applied to prescribe a hierarchical regimentation of sets, starting from a domain of individuals. The fundamental idea underlying Russell's type theory was already adumbrated in (Russell 1906*b*), where the author introduced the term predicative to distinguish propositional functions that denote from those that do not denote a set (see page 79). In (Russell 1908), to prevent any occurrence of impredicativity, Russell made two moves *simultaneously*.[32] He introduced:

- **a type restriction** for "ranges of significance of propositional functions";
- and an **order regimentation** for propositional functions.

The first amounts to associating a ***range of significance*** to each propositional function, that is, a set of all arguments to which the propositional function can be meaningfully applied. In Russell's terms: "within this range of arguments, the

[31] See (Cantini 2009) for a rich discussion of the impact of the paradoxes on mathematical logic.

[32] The distinction between these two separate moves is retrospective, and due to Chwistek (1922) and Ramsey (1926), as further discussed below.

function is true or false; outside this range, it is nonsense." (Russell 1908, p. 247) The ranges of significance then form types, and these are arranged in **levels**: first we have a type of individuals, and then types which are ranges of significance of propositional functions defined on the individuals, and so on. For example, one can take as individuals the natural numbers, and build predicatively sets of natural numbers of increasing type level. In fact, Russell deliberately left open what exactly constitutes the first level of the type–theoretic hierarchy.[33] The crucial point is that as a consequence of this regimentation of the notion of set, in type theory quantification is only allowed "locally", i.e. over a given type, not over "all sets". In addition, expressions such as $x \in x$ and $x \notin x$ are simply ill–formed, since in $z \in w$, z must be of the next-lower level than w. Accordingly, Russell's paradox does not carry through.

This has important consequences, as it seems to suffice to block not only Russell's paradox, but more generally all set–theoretic paradoxes.[34] Chwistek and Ramsey observed that if one implements only this restriction, then one obtains a formalism

[33]Russell (1908, p. 237) wrote: "It is unnecessary, in practice, to know what objects belong to the lowest type, or even whether the lowest type of variable occurring in a given context is that of individuals or some other. For in practice only the relative types of variables are relevant; thus the lowest type occurring in a given context may be called that of individuals, so far as that context is concerned." This quotation suggests a relativity of predicative constraints introduced by the type theoretic structure that bears similarities with a form of relativity of predicativity further discussed in Chapter 4.5, Section 4.5.2.

[34]It might be worth recalling the distinction between set–theoretic and semantic paradoxes. Set-theoretic paradoxes are usually described as those that concern the notion of set, while semantic (or epistemological) paradoxes involve "semantic" notions such as truth, definability, etc. Typical examples of the first kind are Russell's and Burali-Forti paradoxes. The second type include the Liar, as well as Richard's paradox of the least non-definable real number that was mentioned in Section 3.3.2. Peano (1902-1906) and Ramsey (1926) observed that these paradoxes can be seen as pertaining to two different kinds; only those of the first kind seem to affect our conception of set and ought concern us when discussing the foundations of set theory (see also (Carnap 1931)).

that is interesting in its own right. Today this goes under the name of *simple type theory* and its formulation was subsequently simplified by Church (Church 1940) (see also (Coquand 2015)). Simple type theory seems sufficient to block all set theoretic paradoxes; however, it does not eliminate all impredicativity.[35] For example, it does not eliminate the impredicativity to be found in the Liar paradox (page 83) and in the example above on "Napoleon's qualities" (page 83).

The second move, ramification, can be seen as arising from the desire to eliminate all impredicativity. For Russell, one of the lessons of the paradoxes was that impredicative totalities, as, for example, the totality of all propositions, are illegitimate, and hence quantifiers as $\forall F$ make no sense when F ranges over such totalities (see Examples 1, 2 and 3 in Section 3.3.1). One of Russell's aims with his type theory was to represent ordinary mathematical discourse without appealing to these problematic uses of quantification. His solution was to introduce, alongside a notion of ***level*** for **ranges of significance**, a notion of ***order*** for **propositional functions** and require that *a propositional function can only quantify over propositional functions of lower order than its own.* Thus in ramified type theory, one has **first order** propositional functions, **second order** ones, etc.; in addition, the second order propositional functions can quantify on the first order ones, but not the other way around, and similarly for higher orders. In this way one obtains a stratification of propositions and properties, as first order, second order and so on. To clarify how ramification works, let us consider Example 3 from page 83 again. Recall the properties "being Corsican" and "being brave". These are unproblematic, as they

[35] It is worth noting that Zermelo Fraenkel set theory includes the separation schema, that is a restriction of the unrestricted comprehension schema that was mentioned at page 84. The separation schema allows for the formation of *subsets* of a given set by comprehension. That is, given a set A, and φ a formula in the language of ZF, $\{x \in A : \varphi(x)\}$ is a set. This restriction seems sufficient to block the set–theoretic paradoxes, but it does not eliminate all impredicativity; this is because there is no constraint on the formula φ, which might contain unrestricted quantifiers and thus contravene to the VCP.

do not refer to other properties. They can then be taken to be first order properties. However, "having all the qualities of a great general" is a property of order two, as it refers to first order properties. But as "having all the qualities of a great general" can only refer to properties of a lower order, we can not reproduce the undesirable circularity of the original example.

To see how ramification eliminates the threat of the semantic paradoxes, let us consider Example 2, the Liar. This is analysed as follows by Russell (1908, p. 238):

> if Epimenides asserts "all first-order propositions affirmed by me are false", then he asserts a second order proposition; he may assert this truly, without asserting truly any first order proposition, and thus no contradiction arises.

In conclusions, by introducing a type restriction and an order for propositional functions, ramified type theory eliminates all impredicativity and therefore seems to block all vicious circularity, and thus paradoxes of both set–theoretic and semantic nature.

3.4.1 Reducibility and the natural numbers

Russell's type theory was a first fundamental contribution to the clarification of the complex question of what is predicativity in precise, logico–mathematical terms. Its impact on mathematical logic and, more recently, computer science, has been immense. However, as a way of developing a predicative form of mathematics Russell's type theory encountered substantial difficulties, and eventually surrendered to the assumption, in (Russell 1908), of the axiom of reducibility.[36] This axiom was

[36]The introduction of the axiom of reducibility has the effect that the hierarchy of propositional functions (first-order, second-order, etc.) collapses at level 1. Reducibility in fact states that for any propositional function of any order, there is a propositional function of the first-order level which is equivalent to it.

strenuously criticised for being introduced for purely pragmatic reasons. The main problem in the present context is that it may be seen as restoring impredicativity within ramified type theory (Feferman 2005).

Before looking at another attempt, by Weyl (1918), to develop analysis predicatively, I wish to briefly describe how Russell justified the need to introduce his infamous axiom of reducibility in (Russell 1908). This is of relevance to the discussion in Chapter 6 of a variant of predicativity discussed in recent times by Nelson and Parson (Nelson 1986, Parsons 1992, Parsons 2008), that questions the predicativity of the natural number structure. Russell (1908, p. 241) recalls that in his type theory propositional functions are assigned an order. He then notes that propositional functions are used to state properties of mathematical objects, therefore we can not quantify on "all properties of x" in ramified type theory. Russell then writes (Russell 1908, p. 241-242):

> it is absolutely necessary, if mathematics is to be possible, that we should have some method of making statements which will usually be equivalent to what we have in mind when we (inaccurately) speak of "all properties of x." This appears in many cases, but especially in connection with mathematical induction.[37]

Russell then discusses the definition of natural number in Example 1, page 82. This is clearly impredicative, as already noted, as it features a quantifier on all

[37] The principle of mathematical induction, or, simply, induction, is required to reason inductively on the natural numbers. It states that if we can show that a property holds of 0, and that whenever it holds of a natural number, n, it also holds of its successor, then we can conclude that that property holds of *every* natural number. In modern logical terminology, within the familiar Peano Arithmetic (**PA**), induction reads as follows:

$$[\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(Suc(x)))] \to \forall x \varphi(x),$$

where φ is an *arbitrary* formula in the language of **PA**, and $Suc(x)$ is the successor of x.

properties of the natural numbers (including itself). In ramified type theory, that definition therefore gives way to a reformulation in which the quantifier $\forall F$ ranges over propositional functions belonging to some fixed order. This, however, allows us to obtain only partial renderings of the notion of natural number, one for each order of propositional functions. We can not obtain therefore a general **definition** of the concept of natural number, since that would require a universal quantifier unrestricted by any order. As a consequence, many proofs by induction do not carry through in their usual form, as they require the full generality of a universal quantifier ranging on all propositions; for example, we can not prove in full generality in ramified type theory that if m and n are finite numbers, then so is $m + n$.[38]

In addition, a development of analysis in ramified type theory is also compromised by the fact that, as in the case of the natural numbers, also for real numbers one obtains real numbers for each order, and thus can not prove statements on *all* real numbers.

To conclude this section, ramified type theory had two principal deficiencies: (1) it turned out to make the derivation of ordinary mathematics awkward, if not impossible, (2) as already the fundamental definition of the natural numbers cannot be recast in ramified type theory, it also failed to fulfil Russell's aim of vindicating the logicist program.[39] This prompted Russell and Whitehead to introduce the axiom of reducibility, whose addition, however, did not retain the predicative nature of the original ramified type theory.

[38] See e.g. (Feferman 2000*b*) for a simple exposition of the ideas underlying ramified type theory and the difficulties it encounters.

[39] See also (Gödel 1944) for similar complaints.

3.5 The developments of predicativity II: Weyl's "Das Kontinuum"

A fresh attempt at developing analysis from a predicative point of view was proposed by Weyl in his book "Das Kontinuum" (Weyl 1918). Weyl developed (a portion of) analysis from a predicative point of view without resorting to ramification; in so doing he clarified that we can work predicatively without engendering unnecessary complications, and that we can achieve more than previously thought.

Weyl's principal motivation for his investigation in "Das Kontinuum" was a deep dissatisfaction with the current status of the foundations of mathematics. The new methodology that had recently been introduced in mathematics seemed to Weyl to lack justification. In particular, the efforts of philosophical programmes as formalism and logicism had left unanswered the difficulties that plagued contemporary mathematics, and in particular the new mathematics' reliance on impredicativity. Weyl found that the most worrying instances of impredicativity were those that arose at the very heart of mathematics, in analysis, as witnessed by the impredicativity of the Least Upper Bound principle.[40]

Weyl was clearly influenced by Russell's analysis of impredicativity; however, there are significant dissimilarities with Russell's approach. First of all Weyl's investigations into predicativity focus directly on mathematical practice. Weyl hoped to achieve a reformulation of the concept of set that would warrant a non-circular, safe foundation for a mathematical analysis of the continuum. This was meant to offer a partial solution to what Weyl in the opening of his booklet calls *Pythagoras' problem*: the difficulty of clarifying in absolutely rigorous terms the concept of the continuum.

Secondly, a fundamental difference with Russell is that Weyl took as given, as requiring no definition or explanation, the natural number structure with full math-

[40]See page 85 for the Least Upper Bound principle.

ematical induction. Weyl (1918, p. 48) wrote: " *the idea of iteration, i.e., of the sequence of the natural numbers, is an ultimate foundation of mathematical thought, which can not be further reduced*". This aspect of Weyl's thought bears similarities with Poincaré's views, who, criticising Russell's introduction in (Russell 1908) of the reducibility axiom, stressed that the natural number structure with induction is beyond doubt, it is *synthetic a priori*; thus requiring no reduction or foundation (Poincaré 1909).[41]

After assuming the natural number structure as "given", Weyl imposed restrictions motivated by predicativity concerns at the next level of idealization beyond the natural numbers: the continuum. As the real numbers can be represented by sets or sequences of rational numbers, Weyl's fundamental question was: which sets or sequences of rational numbers can be justified predicatively? As the rational numbers, in turn, can be represented as ordered pairs of natural numbers, this questioned the justification of the *powerset of the natural numbers*. Weyl therefore explained how to "produce" predicative subsets of the natural numbers: he started from the natural numbers with full mathematical induction as a base, and used the ordinary logical operations applied to them to ascend step by step from the natural numbers to predicative sets of natural numbers. Weyl called this the "mathematical process".[42]

Weyl was fully aware of the difficulties introduced by ramification for the development of mathematics, and he severely criticized the axiom of reducibility; he thus refrained from both.[43] In modern terminology Weyl introduced restrictions on how

[41]The role of the natural numbers for Poincaré and Weyl's predicativism will be further discussed in Chapter 5 and in Chapter 6.

[42]See Chapter 5, Section5.4, for a detailed discussion of Weyl's "mathematical process".

[43]In Section 6 of (Weyl 1918) the author also considers an iteration of the mathematical process that resembles Russell's ramified hierarchy: one produces via the mathematical process all the sets of the first level, and then uses them in a new application of the mathematical process to form new sets of the second level, etc. Weyl, however, concludes that: "A 'hierarchical' version of analysis

we form subsets of the natural numbers, that, in practice, justify only applications of the comprehension schema to *arithmetical formulas*, that is, those formulas that do not quantify over sets (but may quantify over natural numbers). In this way one justifies sets of the form $\{x : \varphi(x)\}$ only if φ does not contain set quantifiers. This restriction prevents vicious–circular definitions of subsets of the natural numbers: the restriction to number quantifiers in the comprehension principle does not allow for the definition of a new set by quantifying over a totality of sets to which the definiendum belongs.[44]

I wish to highlight two aspects of Weyl's contribution. First, Weyl succeeded in reducing to predicative methodology a significant segment of analysis, including portions which prima facie required impredicativity.[45] This opened up the way for a

is artificial and useless. It loses sight of its proper object, i.e. number [...]. Clearly, we must take the other path [...] *to abide the narrower iteration procedure.*" (Weyl 1918, p. 32)

[44] See Feferman (1988*a*) for a precise account of Weyl's achievements. As remarked by Feferman (1988*a*) (see also (Feferman 2000*b*)) there are some ambiguities in Weyl's formulations, so that it is not completely settled which system he had in mind. A frequent claim is that Weyl's system essentially corresponds to a subsystem of second order arithmetic known as ACA_0 (Simpson 2009). Here "second order arithmetic" refers to a theory that uses a two-sorted language (with variables for natural numbers and variables for sets) to formalise the natural numbers and their subsets. The claim that Weyl's system essentially corresponds to ACA_0 can be justified by Feferman's contention that an inspection of the proofs in "Das Kontinuum" shows that ACA_0 suffices to carry out all of Weyl's constructions. Therefore, even if Weyl had a different system in mind, in particular a stronger one, still, his mathematics can be carried out within ACA_0. The system ACA_0 has first and second order variables, the first ranging on the natural numbers and the latter on subsets of the natural numbers. It has the usual Peano's axioms for 0 and successor, and defining equations for all primitive recursive functions. Furthermore, ACA_0 includes the principle of arithmetical comprehension and the induction axiom, stating that we can reason by induction on sets of natural numbers. Remarkably, although this system is expressed in the second order language, it is very weak (proof theoretically), as it can be shown to have the same proof theoretic strength as Peano Arithmetic.

[45] In particular, a substantial obstacle was the impredicativity of the Least Upper Bound principle

rich literature in mathematical logic that aimed at clarifying the extent of predicative mathematics, as further discussed in Chapter 4, Section 4.3.

Second, as witnessed by the following quotation, Weyl saw only predicative mathematics as fully justified; as he quickly became aware that not all of ordinary mathematics could be so recovered, he was ready to give up the rest, as (so far) not fully justified. Weyl (1918, p. 1) wrote:

> The house of analysis [...] is to a large degree built on sand. I believe that I can replace this shifting foundation with pillars of enduring strength. They will not, however, support everything which today is generally considered to be securely grounded. I give up the rest, since I see no other possibility.

This very bold attitude towards the foundational problem bears similarities with Brouwer's intuitionist attempt, which later on Weyl temporarily joined, but eventually abandoned as he felt it was too "awkward" and inadequate to deal with mathematics' applications.

3.5.1 After "Das Kontinuum"

The interest in predicativity sharply decreased soon after the publication of Weyl's book for a number of reasons. The most widespread assessment of predicativism was that it did not allow for a sufficiently thorough development of ordinary mathematics: a too high price to pay.[46] In addition, Chwistek and Ramsey noted that simple type theory seemed to suffice to overcome the set-theoretic paradoxes (Chwistek 1922, Ramsey 1926); therefore predicativity's principal justification, to

(see Example 5). The LUB is used in a crucial way to prove the completeness of the real number system. Weyl overcame this difficulty by using a form of sequential completeness instead of order completeness, as the first does not require an appeal to the LUB. See (Feferman 1988*a*) for details.

[46] Prominent witnesses to this attitude are for example (Zermelo 1908) and (Gödel 1944).

avoid inconsistencies, seemed lost.[47] Peharps the most significant circumstance that determined predicativity's loss of appeal was the rapid accreditation of impredicative set theory as the standard foundation, especially in the form of the Zermelo-Fraenkel system with choice, ZFC.

3.6 Conclusion

In this chapter I have outlined the origins of predicativity in the reflections that followed the substantial extension of the mathematical methodology in the 19th and early 20th Centuries. Predicativity was directly instigated by the discovery of the set-theoretic paradoxes, and an analysis of the paradoxes helped Poincaré and Russell devise strategies that would make mathematics safe. The first discussions on predicativity culminated in a ban on circularity, with the introduction of Russell's VCP. In subsequent years Poincaré further stressed a form of instability, or lack of invariance, of sets that are defined impredicatively. These discussions alone did not, however, fully specify a notion of predicativity, and rather imposed generic bans on circular or non-invariant definitions.

The first fundamental step towards a thorough clarification of predicativity was the creation by Russell of ramified type theory. Here one introduces appropriate syntactic restrictions to the way sets (there called types) are defined. The types are stratified according to levels (restricting the range of significance of propositional functions) and orders (restricting the quantifiers occurring in propositional functions to lower orders). This ensures that in defining a new type we only appeal to already defined types. In particular, one does not refer to the type that is being defined, nor to totalities of which it is a member. Russell's ramified type theory therefore gives justice to the "building up from within and in stages" metaphor suggested at the beginning of this chapter.

[47]See Section 3.4, and footnote 34, page 92.

Ramification, however, made the mathematics awkward; therefore a more local approach, driven more directly by the mathematical practice, was proposed by Weyl, who focused on the fundamental case of analysis. Weyl's predicativism can be seen as proposing a safe concept of set, grounded on the natural numbers and the iteration of simple operations over them. Weyl showed how to develop a substantial part of 19th Century analysis from this constrained perspective. His efforts in clarifying the prospects of a predicative form of mathematics have been substantially furthered by the work of prominent mathematical logicians starting from the 1950's. Weyl also clearly outlined a form of predicativism, renouncing to those portions of analysis that could not be predicatively recovered. This deeply contrasts with the attitude of the logicians that in the 1950's revived the discussion on predicativity, as further analysed in the next chapter.

Chapter 4

The logical analysis of predicativity

Poincaré's polemic attacks addressed at the "logisticians" may make the historical development of predicativity rather surprising, as the interest in predicativity was revived by work in mathematical logic starting from the 1950's. This is after all not that surprising, as both Russell's ramified type theory and Weyl's analysis of the continuum had already clarified the potential of employing the most recent advances in mathematical logic to sharpen the philosophical discussion.[1]

The history of predicativity is witness to a remarkable example of cross-fertilisation between philosophy of mathematics and mathematical logic. A critical reflection on the new abstract methods that were introduced in mathematics in the 19th Century gave rise to a proposal for a renewal of older methodologies; it also encouraged a novel philosophical programme, predicativism, in the philosophy of mathematics. The latter stimulated further technical advances, in an attempt to develop a form of mathematics that would comply with that philosophical stance. Later on, the

[1] It is also worth noting that substantial progress in the mathematical understanding of predicativity came from within the Hilbertian proof-theoretic tradition, that was harshly criticised not only by Poincaré but also by Weyl (1918).

desire to gain a fine understanding of the mathematical and philosophical implications of the proposed more restricted methodologies, determined the development of new mathematical notions and techniques, as further outlined in this chapter. The mathematical output of those technical investigations subsequently gained a life of its own, promoting further substantial developments in mathematical logic and computer science.

The new interactions between mathematics and philosophy that emerged from the 1950's, witness, however, profound differences with the first analysis of predicativity at the beginning of the 20th Century. The most remarkable aspect is a modification of the stated aims. In this respect, already Gödel manifested a shift of attitude in his influential appraisal of Russell's contribution to mathematical logic in (Gödel 1944). There Gödel clearly expressed the view that predicativity is a fruitful concept within mathematics, but that it should be pursued "independently of the question whether impredicative definitions are admissible."[2] His remarks indicate the beginning of a new phase for predicativity starting from the 1950's; this is characterised by a **logical analysis** of this notion that, although of relevance for the philosophical debate on the foundations of mathematics, is carried out independently of predicativism.

A possible explanation for this change of attitude is that the worries for the, by then, less "new" methodology in mathematics had subsided, and impredicative methods and set theory had become standard. It is therefore not surprising that the new approaches to predicativity were more technically driven and less foundationally committed.[3] Perhaps the most relevant aspect of this new phase is the fact that

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[2]As example of the fruitfulness of predicativity, Gödel mentions his constructible hierarchy, that was inspired by Russell's ramified type theory and allowed for fundamental proofs of the consistency of the axiom of choice and of the continuum hypothesis with ZF set theory. This will be discussed in the next Section 4.1.

[3]This is not to say that foundational issues were not at the centre of a significant portion of mathematical logic. In fact, even a cursory inspection of the logical literature from the 1950's

different questions were now driving the discussion compared with the early days of predicativity. The main question was not any more one of ultimate justification of (a portion of) mathematics, but one of clarification of the boundaries and power of predicativity. More precisely, the fundamental question became whether predicative mathematics was "already (formally) sufficient to obtain the full range of arithmetical consequences realized by impredicative mathematics" (Feferman 1964, p. 4).

The most notable outcome of the logical analysis of predicativity was the realisation that *large portions of ordinary mathematics are within the realms of predicativity.*[4] This realisation has important consequences for the philosophical debate on predicativity, as further discussed below.

In this chapter I shall first of all review Gödel's constructible hierarchy, which witnesses the fruitfulness of predicativity for mathematical logic. I shall then set out the principal milestones of the logical analysis of predicatvity. I shall conclude the chapter with a discussion of a by-product of that analysis: a clarification that there are distinct forms of predicativity that relate to substantially different forms of mathematics. This plurality of predicativity suggests a relativity of predicativity with respect to initial assumptions, as clarified in Section 4.5.2.

4.1 Gödel's constructible hierarchy

A crucial technical contribution that influenced subsequent discussions on predicativity came with Gödel's constructible hierarchy (Gödel 1938, Gödel 1940). The

and 1960's witnesses that foundational questions were very prominent, so that complex and subtle new technical results were in many cases cast through a foundational light. In this respect the logical landscape was very different from today, as now technical outcomes are very often aimed at thoroughly independently of any wider context.

[4]Here the expression "ordinary mathematics" refers to mainstream mathematics. See Section 4.3, page 112, for further clarification of this notion.

latter can be seen as taking forward Russell's idea of ramified type theory to the transfinite (Gödel 1944, p. 464).[5]

The *cumulative hierarchy* is well-known, as it is often considered the "standard interpretation" of set theory. It is defined by a transfinite recursion on the ordinals as follows: we start from the empty set, we apply the powerset operation to go from any level of the hierarchy to its successor and then we collect all previously defined sets at limit ordinals.[6] The ***constructible* hierarchy** is defined by recursion on the ordinals so to mimic the the cumulative hierarchy, but it differs from that at successor steps: it replaces the full powerset operation with an operation of "definable powerset". That is, at each successor step one takes only those subsets of previously constructed sets that are definable purely in terms of previously constructed sets. Therefore at each step, the new sets are obtained *predicatively* from those introduced at *previous stages* of the hierarchy.[7] As remarked by Gödel (1944, p. 464), the constructible hierarchy can be seen as reducing all kinds of impredicativity to

[5]Gödel's constructible hierarchy is also indebted to (Hilbert 1926), where an inconclusive attempt at settling the Continuum Hypothesis is sketched.

[6]The cumulative hierarchy, V, is defined as follows:

$$V_0 := \emptyset,$$

$$V_{\alpha+1} := Pow(V_\alpha),$$

$$V_\lambda := \bigcup_{\xi<\lambda} V_\xi \quad (\lambda \quad limit),$$

$$V := \bigcup_{\alpha \in On} V_\alpha,$$

where $Pow(X)$ is the powerset of X, i.e. the set of all subsets of X, and On is the class of all ordinals. Note that the adjective "cumulative" refers to the fact that for each $\beta < \alpha$, $V_\beta \subseteq V_\alpha$.

[7] More precisely, given a set A, its definable powerset, $Def(A)$, is the collection of *all* subsets of A which are *definable over* $\langle A, \in \rangle$. In particular, all elements of $Def(A)$ are subsets of A which are definable by formulas in the language of set theory whose quantifiers range over A, and whose parameters are elements of A. The underlying intuition is that at successor steps in the hierarchy, one only appeals to sets that "have already been constructed", that is, to previous levels of the

one special kind: "the existence of certain large ordinal numbers (or well-ordered sets) and the validity of recursive reasoning for them". The thought is that although at a *local* level, that is, at each step of the hierarchy, all new sets are defined predicatively, the hierarchy as a whole manifests an element of impredicativity, given by the assumption of the class of all ordinals. The latter is required to iterate the definition sufficiently far, but it is problematic because the ordinals are order types of well ordered sets, and therefore require impredicativity for their very definition. This feature of the constructible hierarchy suggests a relativity of predicativity that will be further discussed in Section 4.5.2.

4.2 Γ_0 and the limit of predicativity

Renewed interest on predicativity emerged from the 1950's, when fresh attempts were made to obtain a clearer demarcation of the boundary between predicative and impredicative mathematics. The literature from the 1950's and 1960's shows the complexity of the task, and witnesses the involvement of a number of prominent logicians (such as Feferman, Kleene, Kreisel, Gandy, Schütte, Spector, Wang), who utilised new technical tools in a number of attempts at separating the predicative from the impredicative. The new approach to predicativity aimed at two main

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hierarchy. The constructible hierarchy, L, is defined as follows:

$$L_0 := \emptyset,$$

$$L_{\alpha+1} := Def(L_\alpha),$$

$$L_\lambda := \bigcup_{\xi<\lambda} L_\xi \quad (\lambda \quad limit),$$

$$L := \bigcup_{\alpha \in On} L_\alpha.$$

(Gödel 1938) showed that the constructible hierarchy is a model of ZF; in addition he proved that it is a model also of the axiom of choice and of the continuum hypothesis, thus settling the question of their consistency with ZF.

targets:

- **Limit:** a determination of the limit of predicativity; and
- **Extent:** a clarification of which parts of contemporary mathematics can be re-cast within a predicative setting.

I shall examine the first point in this section, and postpone a discussion of the second to the next section.

The first attempts at a logical analysis of predicativity highlighted a connection between predicativity and the recently developed concept of the hyperarithmetic hierarchy.[8] The hyperarithmetical hierarchy has a fundamental place in the development of mathematical logic because of its prominence within a number of fundamental areas in mathematical logic: definability theory, recursion theory and (admissible) set theory. This witnesses the centrality within logic of themes that pertain to the predicativity debate. The proposed identification of the realm of predicativity with the hyperarithmetic hierarchy, however, turned out to rely on the assumption of the countable ordinals up to the first non recursive ordinal, ω_1^{CK}, along which to iterate the construction of the hierarchy.[9] This is a less problem-

[8]This consists of a hierarchy of sets of natural numbers which can be equivalently characterised in a number of ways. The simplest characterisation is in terms of definability, and sees the hyperarithmetical sets as those sets of natural numbers that can be defined both by a Σ^1_1 and by a Π^1_1 formulas. A Σ^1_1 formula is one of the form: $\exists X\, \varphi(X)$, with φ an arithmetical formula, that is, a formula that does not quantify over sets (but may quantify over natural numbers). A Π^1_1 formula is one of the form $\forall X\, \varphi(X)$, with φ an arithmetical formula. The formulas that are equivalently definable by a Σ^1_1 and by a Π^1_1 formulas are called Δ^1_1 formulas. Note that above the upper case letter X denotes a second order variable, it thus stands for a set of natural numbers. There are deep connections between the hyperarithmetic hierarchy and an initial fragment of Gödel's constructible hierarchy. As already discussed, the latter can be seen as motivated by Russell's concept of ramification. See (Kreisel 1960) for discussion of the way the hyperarithmetical hierarchy also captures and makes more precise the intuitions underlying Poincaré's notion of invariance.

[9]See (Feferman 2005).

atic assumption than that of the whole class of ordinals that we saw in the case of Gödel's constructible hierarchy, but still relies on the impredicative notion of being a well-ordering relation.

For this reasons Kreisel (1958) suggested a different analysis of predicativity which required the definition of an appropriate hierarchy of formal systems that would canonically represent predicative reasoning. One could then introduce a notion of *predicatively provable ordinal*, that is, an ordinal that can be proved to be the order type of a wellordering relation *within* the given hierarchy of systems. The introduction of the notion of predicatively provable ordinal has the purpose of guaranteeing that one progresses up along the hierarchy to a stage α only if α has been recognised as provable at a *previous stage* of the hierarchy. A remarkable consequence of this new course of inquiry is that it shifted the centre of research from definability issues to provability issues. The celebrated upshot of that research is the logical analysis of predicativity by Feferman and Schütte (independently) following lines indicated by Kreisel (Kreisel 1958, Feferman 1964, Schütte 1965*b*, Schütte 1965*a*). Here Russell's original idea of ramification had a crucial role, as a transfinite progression of systems of ramified second order arithmetic indexed by ordinals was used to determine a precise limit for predicativity.[10] This turned out to be expressed in terms of an ordinal, called Γ_0, which was the least non-predicatively provable ordinal.[11]

[10]Second order arithmetic is a system that uses a two-sorted language, with variables for natural numbers and variables for sets. It formalises the theory of the natural numbers and their subsets (see also footnote 44, page 99). In the present context, the subsystems of second order arithmetic that make up the levels of the hierarchy are characterised by a principle of ramified comprehension, inspired by Russell's idea of ramification. Each level of the hierarchy therefore is predicatively justified. In addition, the introduction of the notion of predicatively provable ordinal has the purpose of guaranteeing that one progresses up along the hierarchy to a stage α only if α has been recognised as provable at a previous stage of the hierarchy.

[11]This ordinal is also the proof-theoretic ordinal assigned to the progression of ramified systems mentioned above. In the branch of proof theory known as ordinal analysis, suitable (countable) ordinals, termed "proof-theoretic ordinals", are assigned to theories as a way of measuring their

Therefore, in proof theory Γ_0 is often referred to as **the limit of predicativity**.[12]

Once the strength of the canonical systems of ramified second order arithmetic (also known as ramified analysis) was determined, the aim was to use this to assess the predicativity of other formal systems. In particular, as ramified systems are cumbersome to work in, one needed a way of assessing the predicativity of systems that better suited the practical needs of a codification of ordinary mathematics. The notion of *proof–theoretic reducibility* was introduced for this purpose. The intuition underlying this notion is that in order to assess the predicativity of a new formal system it suffices to "translate" (that is, proof-theoretically reduce) it into one of the ramified systems. The latter, thus, acted as canonical systems of reference in terms of which the predicativity of other systems could be assessed. The outcome was a notion of *predicative justification*: a formal system is considered predicatively justifiable if it is proof–theoretically reducible to a system of ramified second order arithmetic indexed by an ordinal less then Γ_0.[13]

The above analysis of predicativity is part of a more general program suggested

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consistency strength and computational power. The "proof-theoretic strength" of a theory is then expressed in terms of such ordinals. For example, the proof-theoretic ordinal assigned to Peano Arithmetic is ϵ_0.

[12]The countable ordinal Γ_0 is relatively small in proof-theoretic terms. As a way of comparison, it is well above the ordinal ϵ_0 which encapsulates the proof-theoretic strength of Peano Arithmetic, but it is much smaller than the ordinal assigned to a well–known theory, called ID_1, of one inductive definition. The latter ordinal is known in the literature as the Bachmann–Howard ordinal. The theory ID_1 is the first (and weakest) of a hierarchy of theories of (iterated) inductive definitions, whose strength has been investigated in (Buchholz, Feferman, Pohlers & Sieg 1981). Their strength is well below that of second order arithmetic, which is in turn much weaker than full set theory. For surveys on proof theory and ordinal analysis see for example (Rathjen 1998, Rathjen 1999, Rathjen 2006).

[13]See (Feferman 2005) for a more accurate informal account of this notion of predicativity and for further references. It should be noted that the notion of predicative reducibility is not without difficulties. Due to the technicalities involved, I shall have to postpone a discussion of this point to a different context.

by Kreisel and further advanced by Feferman, of understanding "what rests on what" in mathematics (Feferman 2000*a*). This makes essential use of the notion of proof–theoretic reduction of a theory to another; its purpose is to clarify in exact terms how some forms of abstract mathematics can be reduced to more elementary ones, by looking at formal systems which codify those forms of mathematics and studying their relationship with more elementary ones (Feferman 1988*a*). For example, Kreisel and Feferman considered the reduction of the *infinitary* to the *finitary*, of the *nonconstructive* to the *constructive* and of the *impredicative* to the *predicative*.[14]

An important remark regarding the logical analysis of predicativity is that the limit it imposes is an **"external limit"**. As clearly acknowledged by Feferman (see, e.g., Feferman 1964), one takes an impredicative stance and attempts to clarify the extent of predicativity from above, so to speak, or, using a metaphor due to Rathjen in a different context, from an eagle-eye perspective (Rathjen 2005). The convinced predicativist will not recognize the limit Γ_0 as it lies beyond his reach, its very definition being impredicative. This further clarifies the deep change in attitude between the early discussions on predicativity and the logical analysis, as the latter is an attempt at understanding predicativity rather than arguing for it.

4.3 Predicativity and ordinary mathematics: the extent of predicativity

The determination of the limit of predicativity allows us to assess which theories can be considered predicative, on the basis of a precise comparison with canonical predicative theories. The underlying contention is that the mathematics that can be carried out within a predicative theory is to be considered predicative, and that those portions of mathematics that elude treatment within any of those theories are

[14]See (Feferman 2005) for further references.

impredicative. This is a great achievement especially as it allows for an assessment of the predicativity of theories that are not ramified, and therefore easier to work with. However, by itself the determination of the limit of predicativity leaves unanswered the question of the exact extent of predicative mathematics within ordinary mathematics. That is, to shed light on which theorems of ordinary mathematics can be proved predicatively, we need to complement the proof-theoretic analysis with a different kind of enquiry.

A second crucial contribution to the clarification of predicativity has been a logical analysis of ordinary mathematics, to elucidate which parts of ordinary mathematics can be expressed in predicative terms. Here Weyl's pioneering work in "Das Kontinuum" constituted fundamental reference, especially for Feferman's investigations (Feferman 1988*b*, Feferman 2005). More precisely, Feferman (1988*b*) has carefully analysed Weyl's text and has proposed a system, W, which codifies in modern terms Weyl's system in "Das Kontinuum". System W is particularly weak proof-theoretically as it is no stronger than Peano Arithmetic, however Feferman has verified that large portions of contemporary analysis can be carried out on its basis. Another source of insight are results obtained within Friedman and Simpson's programme of Reverse Mathematics (Simpson 2009) that was mentioned in Chapter 2, Section 2.4.2.

In attempting to clarify the extent of predicativity it is first of all important to clarify what is intended with the expression "**ordinary mathematics**". This is mainstream mathematics, that is so characterised, for example, by (Simpson 1999, p. 1): "that body of mathematics which is prior to or independent of the introduction of abstract set-theoretic concepts". That is: "geometry, number theory, calculus, differential equations, real and complex analysis, countable algebra, the topology of complete separable metric spaces, mathematical logic and computability theory".[15] The principal outcome of the studies outlined above is that large parts

[15] According to Simpson, excluded from ordinary mathematics are instead "those branches of

of ordinary mathematics can be framed within predicative systems.[16] Perhaps more surprisingly, it also turns out that if a theorem can be established predicatively, it can already be carried out within a system as weak as Peano Arithmetic.[17] Similarly as in the case of constructive mathematics that was discussed in Chapter 2 (especially Section 2.3.2), also here the unexpected result is that, once analysed in more detail, the apparent necessity of certain features of ordinary mathematics, like impredicativity or classical reasoning, turn out to be a by-product of the context in which mathematical theorems are proved, and might also depend on the specific formulation of their statements. In the case at hand, many instances of prima facie impredicativity become amenable to predicative treatment once we work within a sufficiently weak system.[18] In addition, like in constructive mathematics, we need to rely on *individual case studies* for our findings, so that any general conclusion can only be achieved on the basis of a thorough investigation of the mathematical practice.

4.3.1 Fruitfulness of predicativity

The logical analysis of predicativity is a further remarkable example of the *fruitfulness* of predicativity. This fruitfulness relates first of all to mathematics, as, for

mathematics that were created by the set-theoretic revolution which took place approximately a century ago. We have in mind such branches as general topology, abstract functional analysis, the study of uncountable discrete algebraic structures, and of course abstract set theory itself." (Simpson 1999, p. 1)

[16]See (Simpson 1999) for details.

[17]See (Feferman 1988*b*) and (Feferman 2005) for an informal discussion and further references.

[18]There has been in fact extensive cross-fertilisation between reverse and constructive mathematics. Simpson (1999), however, also emphasizes a difference with constructive mathematics, in that the aim in reverse mathematics is "to draw out the set existence assumptions which are implicit in the ordinary mathematical theorems *as they stand*". Bishop's goal, according to Simpson, is instead "to replace ordinary mathematical theorems by their "constructive" counterparts." (Simpson 1999, p. 137)

example, a prominent chapter in proof theory was instigated by attempts to clarify the limit of predicativity. Secondly, the fruitfulness of predicativity extends to the philosophy of mathematics. As indicated by Kreisel and Feferman, the clarification of the limit of predicativity helps us better understand how far the predicative perspective can go. As further discussed in Chapter 5, one way of framing predicativity is in terms of definitions that "generate" from below mathematical entities, starting from a limited stock of initial objects (e.g. the natural numbers) and by some simple operations over them (e.g. the ordinary logical operations). The determination of the limit of predicativity by mathematical methods can then be seen as an instrument for assessing how far this limited apparatus can reach.

This has two main philosophical benefits: it can be used to determine the plausibility of (a form of) predicativism, or it can be employed to single out certain portions of mathematics that rely on a selected limited stock of assumptions. As to the first point, as indicated by Feferman (2005, p. 29), "[t]he logical problem in each case is to characterize exactly the limits of that particular stance. The potential value for philosophy then is to be able to say in sharper terms what arguments may be mounted for or against taking such a stance."

As to the second point, there is a similarity, as already indicated, with the case of constructive mathematics that was discussed in Chapter 2. In that chapter I claimed that the generality of constructive mathematics allows for a fine analysis of mathematics.[19] Similarly here, the adoption of weaker underlying theories helps us determine natural boundaries between parts of ordinary mathematics that are distinguishable in terms of the assumptions they make and the methodology they employ.[20] The suggestion is that there are different parts of today's mathematics that rely on qualitatively different kinds of assumptions. Some assumptions are

[19] It also enables us to see more: mathematical structures and distinctions that are "invisible" to a classical eye.

[20] See also the discussion in Chapter 2, Section 2.3.2.

more elementary, as, for example, the natural number structure, and others are more involved and abstract, as the sets of ordinary set theory. It is standard practice to think of ZFC set theory as a foundational system for virtually all of mathematics. However, the "power" of ZFC, that is useful in many contexts, does not enable us to clarify how much mathematics can be obtained from the simple assumption of the natural number structure. It also does not clarify which of the frequent appeals to more abstract structures that figure within ordinary mathematics may in fact be eliminated.[21]

One might further wonder whether these considerations can also offer an insight on whether different portions of mathematics, e.g. more abstract or more elementary ones, have distinct roles within the application of mathematics to science. In particular, a natural question that arises is whether the above mentioned results also suggest that we can dispense from employing more abstract concepts in developing the mathematics that is needed for science. Recourse to the more abstract parts of mathematics would then perhaps play an auxiliary role, for example, simplifying proofs (see Chapter 7 for discussion). In this regard, Feferman has put forth the working hypothesis that all of *scientifically applicable analysis can be developed predicatively*. More precisely, Feferman has proposed the hypothesis that all scientifically applicable analysis can be developed in the system W of (Feferman 1988*b*), which codifies in modern terms Weyl's system in "Das Kontinuum". As System

[21] There is a potential benefit for the epistemology of mathematics, as knowledge of more elementary concepts would seem to afford easier explanation than that of more abstract notions. Consider, for example, (Shapiro 1997). Here within an ante-rem structuralism, the epistemology of less abstract parts of mathematics is accounted for in terms of pattern recognition. For more abstract mathematical structures, however, Shapiro, introduces a more problematic notion of implicit definition. The latter, in turn, requires the assumption of a primitive notion of coherence that is difficult to spell out. Clarifying that pattern recognition accounts for a vast part of ordinary mathematics, and possibly for all of scientific applications, would seem to alleviate some of the difficulties of the ante-rem structuralist.

W is as weak as Peano Arithmetic, and the latter is well within the boundary of predicative mathematics, one clearly obtains the more general formulation above of Feferman's hypothesis.[22] I shall further discuss the issue of the applicability of predicative mathematics to science in Chapter 7, in which I shall also report on work by Ye (2011) which suggests that even a small fragment of Peano Arithmetic may suffice for much of contemporary applicable mathematics. I shall also claim that more work is needed to clarify this issue, but that its consideration would greatly enrich the present debate on the indispensability of mathematics to science.

Independence results

Given these reassuring results on the extent of predicative mathematics, one might wonder if its reaches can account for all of ordinary mathematics. It turns out that it does not: some exceptional ordinary theorems, like, for example, the Cantor-Benedixson theorem (Simpson 1999) escape predicativity. In addition, a number of combinatorial statements can not be carried out predicatively, like, for example, variants of the Ramsey colouring theorem, and some simple consequences of Kruskal's theorem about embeddings of finite trees.[23] This is an important insight if predicativity is pursued with the intent of shedding light on our mathematical practice. It appears, however, to be bad news for predicativism. In fact, if this philosophical position bans impredicative mathematics, then it has to offer very good arguments to outweigh the loss. This seems particularly compelling if the portions of mathematics that are not amenable to predicative treatment may be considered "ordinary" according to a sufficiently uncontroversial notion of ordinary mathematics. How bad is this news for predicativism will depend on which arguments are put forth for this philosophical position, and which role within them the

[22] See Footnote 12, page 110, for a comparison between the strength of Peano Arithmetic and predicative analysis.

[23] See (Simpson 1999, Simpson 2002) for more details and bibliographic references.

essentially impredicative portion of mathematics may play. For example, one may wish to appeal to a form of indispensability argument that aims at showing that predicative mathematics is sufficient for current scientific applications. In this case, one would have to clearly assess any possible role of the impredicative portion of mathematics for applications. I shall further discuss these issues in Chapter 7.

4.4 Plurality of predicativity

The logical analysis of predicativity aimed at determining the limits and the extent of a notion of predicativity ***given the natural numbers***. Here one takes an approach to predicativity like Weyl's by assuming the structure of the natural numbers with full (i.e. unrestricted) mathematical induction, and imposing predicativity constraints on the formation of subsets of the natural numbers.[24] With Kreisel and Feferman the study of predicativity became an endeavour to clarify what is implicit in the acceptance of the natural number structure with full induction. In fact, the approach proposed by the logicians in the 1960's goes well beyond Weyl's. According to a common interpretation of Weyl's "Das Kontinuum", Weyl only needed arithmetical comprehension, thus going no further than Peano Arithmetic in proof-theoretic strength.[25] The systems envisaged by Kreisel, Feferman and Schütte, instead, allow for ramified comprehension iterated along predicative ordinals, going therefore much further than Weyl's system. However, what is in common to these two renderings of predicativity is that they start from the natural number structure as given, and introduce predicatively motivated restrictions above it, therefore imposing limits already at the level of the subsets of the natural numbers. The principal difference between them is in that the second form of predicativity engenders from an iteration

[24]In the following I shall omit the adjective "mathematical" when referring to mathematical induction if no ambiguity arises. See footnote 37, page 95 for a definition of mathematical induction.

[25]See footnote 44, page 99 for a discussion of Weyl's system, and footnote 11, page 109 for the notion of proof-theoretic strength.

along predicative ordinals of a ramified mechanism that allows for the formation of predicative subsets of the natural numbers. The justification "from below" of the iteration through these ordinals is in fact one of the most important achievements of the proof-theoretic analysis of predicativity. As already indicated, Weyl did not resort to ramification, as he thought it would make the mathematics too artificial. The more generous Γ_0 approach, instead, was made possible by the logical advances that had in the meantime taken place. In the 1960's one could use canonical ramified theories for the meta-mathematical analysis of predicativity and, via proof-theoretic reducibility, develop the mathematics within more flexible type-free theories.

In the following I shall primarily be concerned with Weyl's approach, and in fact refer to both forms of predicativity as ***"predicativity given the natural numbers"***. The principal reason for this choice is that Weyl's approach is not only simpler to discuss within an informal account, but also particularly revealing for the philosophical analysis.[26] If a distinction between Weyl's original approach and the one originating within the proof-theoretic analysis of the 1960's is required, I shall refer to the first one as **arithmetical predicativity** and to latter as Γ_0 **predicativity**.

The assumption of the natural number structure as starting point for predicativity is not uncontroversial. In fact, as reviewed in Chapter 3, Section 3.4.1, Russell held a very different view, as, due to his logicist ambitions, he saw the very concept of natural number as requiring justification. He eventually introduced both an axiom of infinity and the axiom of reducibility in his type theory to ensure smooth applications of mathematical induction. Different incarnations of predicativity have appeared in the literature in more recent times, some of which have questioned the assumption of the natural number system, in particular, the appeal, within it, to full

[26]In Section 4.3 I have also mentioned Feferman's claim that in fact, when it comes to ordinary mathematics, Weyl's approach to predicativity seems not to be inferior than the more generous Γ_0 one.

induction. Others, originating in the constructive tradition, have instead extended well beyond the acceptance of the natural numbers, admitting more complex inductive structures; however, they have combined this with a rejection of the principle of excluded middle.

An example of the first kind of predicativity is given by Edward Nelson's "Predicative Arithmetic" (Nelson 1986) and Charles Parsons' criticism of the impredicativity of standard explanations of the notion of natural number (Parsons 1992). This will be discussed in the next section, while Section 4.4.2 will describe constructive predicativity.

4.4.1 Strict predicativity

In the book "Predicative Arithmetic" Nelson (1986) proposes a form of predicativity that is more restrictive than predicativity given the natural numbers. Compliance with this notion of predicativity gives rise to a subsystem of Peano Arithmetic that introduces severe restrictions to the induction principle. Nelson's principal motivation for his predicative arithmetic is a complaint that already the natural number structure hides a form of impredicativity. In the following I shall present a brief summary of Nelson's views. A more detailed analysis of Nelson's position is deferred to Chapter 6.

According to Nelson, already the whole natural number structure equipped with full mathematical induction is predicatively problematic on grounds of circularity, and the principal culprit is the induction principle. Nelson's concern are instances of the induction principle that use unrestricted number quantifiers.[27] These are needed to prove even very elementary facts about numbers. However, the presence

[27]The context of discussion here is Peano Arithmetic, in which one has only number quantifiers, i.e. quantifier that range over the natural numbers. In particular, there are no quantifiers ranging over sets. An unrestricted quantifier in this context is therefore a quantifier that ranges on all natural numbers.

of unrestricted number quantifiers in these instances of induction is suggestive, according to Nelson, that those applications of induction require the prior *assumption as given of the natural number structure.* As for Russell also for Nelson the natural number structure can not be taken for granted, although for reasons that are deeply different from Russell's.[28] Consequently, as the natural numbers are not assumed as given, their definition ought not to refer to the natural number structure itself, on pain of circularity. For Nelson, however, some uses of induction with unrestricted quantifiers are necessary for the definition of natural number, and this gives rise to a vicious circle.

I shall further examine Nelson's complaint on induction in Chapter 6. Here it is important to remark that Nelson's considerations bring him to develop a form of arithmetic that introduces strict constraints on the induction principle. The result is a form of mathematics that resides within the context of what is known as bounded arithmetic; the latter is usually seen as capturing the concept of feasible mathematics, that is, mathematics that can be carried out in practice (Buss 1986). Given their weakness, systems of bounded arithmetic allow for a fine study of question of computational complexity, and are therefore at the heart of a lively field at the intersection between logic and computer science (Dean 2016).

Another form of strict predicativity has been discussed by Parsons e.g. in (Parsons 1992, Parsons 2008).[29] The similarity with Nelson's predicativity lies in the fact that Parsons also claims that impredicativity already manifests itself in the induction principle in arithmetic. Therefore the assumption of the natural number structure within predicativity given the natural numbers does not comply with a thorough predicativist perspective. The thought is then that if one wished to fully comply with predicativity, one ought to impose restrictions on the principle of

[28]Nelson puts forward a formalist stance, as further expounded in Chapter 6, Section 6.5. Note that in Nelson's discussion there is a complex interplay between concepts and sets that are their extension, as further discussed in Chapter 6.

[29]Parson's views will be further examined in Chapter 6.

induction.

Notwithstanding the similarity of objections to the natural number structure, there are also differences in Nelson's and Parsons' notions of predicativity. For example, a difference between Nelson's and Parsons' positions arises as the first author rejects as impredicative the totality of the exponentiation function, while the latter admits it within his strict predicativity.[30] I shall further discuss this point in Chapter 6. Here it suffices to observe that this difference of perspectives is of particular relevance for strict predicativism. If, as it seems, these forms of predicativism differ in crucial respects, then they will also sanction different fragments of arithmetic.

The significant aspect that Nelson and Parsons' discussion bring about is that the assumption as given of the natural number structure that is at the heart of predicativity given the natural numbers has not gone unchallenged. The worry is that if one were to completely conform to predicativity, then one would have to impose restrictions to the principle of induction that lays at the heart of this structure. Exactly which restriction is perhaps controversial, but full induction, so is contended, is impredicative.

4.4.2 Constructive predicativity

Themes stemming from the original predicativity debates play a prominent role within foundational systems for constructive mathematics, and in particular in Martin-Löf type theory (Martin-Löf 1975).[31] This (unramified) type theory differs in two fundamental respects compared with classical set theory: it uses intuitionistic logic and also conforms to a notion of predicativity.

[30]A function is total if it is everywhere defined, or, equivalently, $f : X \to Y$ is total if for every element x of X there is an element y of Y such that $y = f(x)$.

[31]In the following I shall call "Martin-Löf type theory" also "constructive type theory". Another common terminology is "intuitionistic type theory" (Martin-Löf 1975). For an introduction to Martin-Löf type theory see (Dybjer & Palmgren 2016, Crosilla 2006).

One might wonder whether combining a form of predicativity with the restriction to intuitionistic logic might make constructive mathematics far too weak. We have seen in Chapter 2 that the constructive reformulation strategies of classical statements introduced, for example, in (Bishop 1967), naturally induce also the elimination of impredicativity.[32] The principal reason for this fact is that Bishop was interested in a computational form of mathematics, so that he carefully replaced abstract set theoretic notions with more concrete ones. In particular, the natural numbers held for Bishop a fundamental position.[33] It is widely held that the restriction to predicativity does not impair the development of constructive mathematics, and ongoing work is further enlarging the extent of predicative constructive mathematics.

From a proof-theoretic perspective, it turns out that constructive predicativity is the most generous of all the forms of predicativity considered so far, as it allows for systems whose proof-theoretic strength well exceeds Γ_0 (Palmgren 1998, Rathjen, Griffor & Palmgren 1998, Rathjen 2005). The reason is that while one introduces a constraint on the logic, one also allows for more generous set-construction principles, as further clarified below.

Traditionally, constructive foundational theories have manifested a more 'liberal' approach to predicativity compared with Γ_0 predicativity. Here the driving idea is that so-called *generalised inductive definitions* ought to be allowed in the realm

[32]See Chapter 2, Section 2.3. Other areas of mathematics, like, for example, topology, have proved more difficult to reproduce within predicative reasoning, and have therefore required a more substantial reformulation of their primitive concepts. The result, formal topology, was briefly discussed in Chapter 2, page 52.

[33]Bishop (1967) insisted on an intensional notion of set that was grounded on the integers and some simple constructions on them (as product, function space). This witnesses his desire to develop a mathematics that would unveil the numerical content of ordinary proofs. The informal notion of set utilised by Bishop has been source of inspiration for Martin-Löf's type theory, and in particular its intensional equality. The latter has recently been further analysed, clarified and generalised by work in homotopy type theory (Univalent Foundations Program 2013).

of constructive mathematics.[34] An inductive definition of a set can be informally presented by giving some rules for *generating* the elements of the set; in addition, one has a condition stating that an element is in the set only if it has been generated according to the given rules. For example, one may give an inductive definition of the set of natural numbers, by stating that:

1. 0 is a natural number,
2. if n is a natural number, then so is its successor, $suc(n)$,
3. nothing else is a natural number.[35]

The intuition underlying the predicative justification of inductive definitions is related to the fact that they can be expressed by means of *finite rules*, and would seem to allow for a specification of a set which proceeds from the 'bottom up': we start from some initial stock of elements, and by applying the rules we produce new elements of the set, that in turn are used to produce new ones, again by application of the rules, and so on... The idea underlying generalised inductive definition is that once we have inductively defined a set, say the natural numbers, then we can further use another inductive definition to extend this, and so on. We thus build a first subset of the set of natural numbers according to the given rules, then use this to build a new one, and so on. The idea that seems to underlie the acceptance of inductive definitions from a constructive perspective is that they seem to ensure that at no stage in the building up of the new set, we need to presuppose sets "outside" the set under construction. The thought is that we rely exclusively on increasingly more complex fragments of the very set under definition: we proceed from within.

An important observation is that the proof-theoretic strength of so-called theories of inductive definitions goes well beyond Feferman and Schütte's bound (and

[34]The work of Lorenzen, Myhill and Wang is particularly relevant in this respect. See for example (Lorenzen & Myhill 1959).

[35]I shall further examine the inductive definition of the natural number set in Chapter 6.

thus also very much beyond Peano Arithmetic), as shown in (Buchholz et al. 1981).[36] Following this line of reasoning, relatively strong constructive theories are considered predicative in today's foundations of constructive mathematics (Palmgren 1998, Rathjen 2005).[37]

This disparity of outcomes is rather striking and suggests that the use of intuitionistic logic might have a crucial role here in enabling a constructive and predicative justification of notions that from a classical perspective are impredicative. In fact theories of inductive definitions are discussed in (Feferman 1964), where they are considered unacceptable from a predicative point of view on grounds of circularity. The underlying idea is that in the build up of an inductive set, we need to refer to the very set we are defining, thus contravening the VCP. The essence of this objection to inductive definitions is related to the objection to induction by the strict predicativist, which will be investigated in Chapter 6. In that Chapter I shall propose an explanation of why an appeal to *intuitionistic logic* may be seen as justifying inductive definitions.[38]

[36] Theories of inductive definitions formalize inductive definitions over the natural numbers, therefore replacing the powerset of the natural numbers with a more "constructive" notion. See Feferman's introduction to (Buchholz et al. 1981) for a clarification of their role within the discussions on predicativity and the developments of proof theory in the 1970's and 80's.

[37] In particular, one has constructive theories whose proof-theoretic strength exceeds that of all the subsystems of second order arithmetic considered in the Reverse Mathematics programme. This has brought Rathjen (2005) to suggest that these "strong" constructively predicative theories can be seen as offering justification for all of ordinary mathematics.

[38] I have not found a detailed discussion of the role of intuitionistic logic for a justification of inductive definitions in the relevant literature, either mathematical or philosophical. In fact, my impression is that there is widespread uncertainty on this point among a number of mathematical logicians. The only article I am aware of that mentions the question of why inductive definitions may be justified from a constructivistic perspective is (Parsons 1992). This considers the constructivistic position put forth by Lorenzen (1955), that does not involve the use of intuitionistic logic. Parson's discussion therefore seems to point towards a different perspective from the one I shall put forth in Chapter 6, as I shall propose that intuitionistic logic may play a crucial role, given

A crucial role for intuitionistic logic within constructive predicativity is particularly visible in the case of Martin-Löf type theory. Here compliance with predicativity is manifested in two ways: (1) the availability of inductive definitions (but not stronger constructions as, e.g. powerset); (2) the Curry-Howard correspondence.[39] In fact, the circumstances of the appearance of predicativity within Martin-Löf type theory are noteworthy, as they bear surprising similarities to how predicativity entered the mathematical scene at the beginning of the 20th Century: a paradox due to Girard (1972). A first formulation of type theory countenanced an impredicative type of all types, that gave rise to a paradox analogous to the Burali-Forti paradox. Constructive type theory was promptly corrected by Martin-Löf (1975), and the paradox clarified an unexpected connection between the Curry-Howard correspondence and predicativity.

Martin-Löf type theory (Martin-Löf 1975) incorporates a form of Curry-Howard correspondence.[40] The Curry-Howard correspondence in the case of constructive type theory is framed *within the theory itself*, as it spells out a structural similarity between two components of this theory: the propositions on the one side (that play the role of the formulas in our discussion in Chapter 1), and the sets on the other side (that play the role of the types).[41] The other remarkable aspect is that the

the predicativist typical understanding of classical domains of quantification.

[39]See Chapter 1, Section 1.2.1. To be more precise, predicativity in constructive type theory has one additional manifestation: reflection. The latter is witnessed by the role of universes in this type theory. A discussion of this point is, however, beyond the aims of this thesis.

[40]A detailed clarification of Martin-Löf type theory would require extensive discussion, therefore in the following I shall only briefly mention the most relevant aspects of this theory, to give an idea of the relation between the Curry-Howard correspondence and predicativity. See (Dybjer & Palmgren 2016, Nordström et al. 1990) for an introduction and an exposition of type theory, respectively.

[41]The terms “proposition” and “set” refer here to “first class objects” of the theory, that is, to the objects that are defined by the rules that make up the formal system. As propositions are objects of the theory, in type theory one does not have the standard set up of ZFC, in which we

correspondence that was discussed in Chapter 1 becomes now an isomophism, and therefore *identifies* propositions with sets. An analysis of Girard's paradox shows that it is induced by a combination of the Curry-Howard *isomorphism* with impredicativity, where impredicativity here takes the form of arbitrary quantification over propositions. Girard's paradox can be seen as offering two ways out: either to renounce to the identification of propositions and sets, or to relinquish impredicativity. In the case of type theory the choice was to keep the Curry-Howard correspondence. The reasons are complex, and relate to Martin-Löf's desire to offer a clear inductive justification of type theory, as well as to preserve the direct computational content of mathematical statements (see e.g. Dybjer 2012, Martin-Löf 2008).[42] The alternative has also been successfully explored, as the calculus of constructions, the type theory that underlines the Coq system, features instead a form of impredicativity. The Curry-Howard isomophism that characterises Martin-Löf type theory is replaced in the calculus of construction by a one direction correspondence: to each proposition corresponds a set, but not, in general, vice versa. This seems sufficient to prevent the derivation of Girard's paradox, and also is sufficient to endow the theorems of this calculus with computational content.

It might be interesting to observe that the rationale in devising the calculus of constructions has been to try and obtain a theory as powerful as possible, without engendering inconsistency. In fact, attempts to expand as much as possible Martin-Löf type theory while remaining faithful to the Curry-Howard isomorphism have also been proposed. Here the crucial ingredients have been a combination of inductive definitions with reflecting universes, and universe operators.[43]

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first specify a logical calculus (e.g. classical logic) and then add the set-theoretic axioms. In type theory the intuitionistic logic arises from the rules that explain the behaviour of the sets, which are also the propositions.

[42]See also the discussion in Chapter 6.

[43]See (Rathjen 2005) for discussion and references.

4.5 Analysis of Predicativity

The apparent plurality of notions of predicativity discussed in the previous sections seems first of all to require a more sophisticated analysis of predicativity. If, as it would seem, these are indeed different incarnations of one notion of predicativity, then we need to better spell out the different assumptions that are implicit in each of these approaches and give rise to different analysis of the same notion. A clarification of these points is not only needed to obtain an understanding of predicativity but it is also crucial for an assessment of predicativism, as further indicated below.

The logical analysis of predicativity allows for a comparison between these different forms of predicativity. Let us first of all suppose that we can fix appropriate canonical theories of reference for each form of predicativity.[44] Then if we grant proof theory (ordinal analysis) as a tool for measuring from the outside, so to speak, the strength of these canonical theories, we can compare their corresponding proof-theoretic strength. We have the following, in order of increasing strength: at the bottom strict predicativity, then predicativity given the natural numbers and then constructive predicativity, with substantial gaps between their respective proof-theoretic strength. These gaps in proof-theoretic strength suggest that these notions of predicativity may be related to different forms of mathematics. Thus strict predicativity only justifies weak fragments of theories that codify the theory of the natural numbers, like Peano Arithmetic. Predicativity given the natural num-

[44]This is not as easy as it might seem. In particular, the case of strict predicativity requires some care. As further discussed in Chapter 6, the principal difficulty with strict predicativity is that it lacks the clear formulation that we have for predicativity given the natural numbers. In fact, as noted above, there are two distinct proposals for strict predicativity, one put forth by Nelson, and one by Parsons. For the present considerations, however, this is not a serious issue, as it suffices to place a generous upper bound that encompasses both proposals, as long as we remain below Peano arithmetic. As to constructive predicativity, I shall take here the view that the canonical theories of reference are the strongest systems to date in Martin-Löf type theory (Rathjen 2005).

bers accords all of arithmetic; in fact, following the proof-theoretic analysis of the 1960's, much more than that. Constructive type theory allows for the justification of even stronger theories.

This is particularly significant for predicativism. If each of these variants of predicativity is taken as imposing unnegotiable constraints on which portions of contemporary mathematics are justified, it might draw the divide between acceptable and unacceptable mathematics in remarkably different places. In other terms, as predicativism imposes that only predicative notions are justified, a form of predicativism would seem to be bound to espouse one and only one of these forms of predicativity, and consequently validate one and only one form of mathematics. However, if the reasons adduced for the elected form of predicativity are very similar to the reasons given for (at least one of) the other forms of predicativity, perhaps the corresponding forms of predicativism would be difficult to separate. In other terms, the question then arises whether a form of instability could plague predicativism, provided that sufficiently similar arguments support the various forms of predicativism. In fact, one might further fear that if there is an instability that forces one form of predicativism (e.g. predicativism given the natural numbers) to collapse into a weaker one (strict predicativism), then any serious difficulty faced by the latter would turn into a serious objection to the first, too.

A predicativist, perhaps, would object to the very idea of a plurality of predicativities and argue that, for example, only predicativity given the natural numbers ought to be given the name "Predicativity", the strict and the constructive approaches in effect manifesting other kinds of restrictions, motivated by different concerns and representing altogether different foundational stances in the philosophy of mathematics. He could, for example, appeal to specific syntactic expressions of predicativity, like the VCP, or to the special role of the natural numbers through the historical development of predicativity, going back to Poincaré and Weyl's positions. This could be taken to justify the exclusive use of "Predicativism" to designate predicativism given the natural numbers.

This is a difficult issue, made particularly involved by the complexity of the early discussions on predicativity, the plurality of perspectives that they expressed and the converging of a number of distinct issues within the remit of predicativity. We have seen, for example, the plurality of formulations by Russell of the VCP, and the close connection that Poincaré, but not Russell, perceived between predicativity and (a proper treatment of) infinity. In the next chapter I shall draw a line from Poincaré's notion of set to constructive predicativity. A notion of set that shares important characteristicts with Poincaré's is also prominent in Weyl's "Das Kontinuum", as further discussed in Chapter 5. In addition, a very similar notion of set is at the heart of Parson's complaint on the impredicativity of induction, giving rise to a form of strict predicativity. There seems to be therefore at least one way of arguing that all of these forms of predicativity belong to one general notion of predicativity. In the subsequent discussion I shall also argue that despite this continuity there are also significant differences between these forms of predicativity and I shall endeavour to elucidate some of their dissimilarities.

4.5.1 Base and constraints

In attempting to clarify the similarities as well as the differences between the forms of predicativity presented above, it might be useful to introduce a distinction that is modelled after the case of predicativity given the natural numbers. I shall distinguish between a **base** and **predicativity constraints** that are imposed beyond the base. One way of explicating this distinction is by looking at the **base** as an underlying conceptual apparatus that is taken for granted.[45] For example, in predicativity given the natural numbers one takes the full natural number structure as base. The **predicative constraints**, instead, are conditions imposed on those sets

[45] I wish to present the notion of base in very general terms, leaving it open for the possibility of further clarification of this notion in philosophically less neutral terms.

that lie beyond the base (e.g. sets of elements of the base). For example, if one defines sets of elements of the base by appeal to a comprehension principle, the constraints could impose compliance with arithmetical comprehension, or a form of ramified comprehension. From an impredicative perspective the constraints can be seen as restrictions acting on the "formation" of sets that populate higher levels of abstraction, compared with the base. In particular, from a standard set-theoretic perspective, we would say that the constraints act at levels of the set theoretic hierarchy that are higher than that of the base, to enable the "formation" only of predicative sets. In the case of predicativity given the natural numbers one applies predicatively motivated constraints starting from the notion of powerset of the natural numbers.

The principal reason for introducing this distinction is that it helps us clarify the difference between the variants of predicativity that were mentioned in the previous section. Strict predicativity may then be framed as arising if one takes as base a strictly finitist fragment of the natural number structure, and introduces appropriate restrictions to whatever lies beyond it.[46] Now the comparison between predicativity given the natural numbers and strict predicativity becomes more perspicuous, as their difference can be framed principally as a difference in the choice of the base: the whole natural number structure for the first, a suitable substructure of it for the second. The distinction between base and constraints is, however, less appropriate for the more complex case of constructive predicativity, due to the "interference" there with the logic. This suggests to take this distinction as simply highlighting *some* significant aspects of predicativity; it is a useful instrument assisting us in the

[46] This is admittedly very vague. As already remarked in footnote 44 at page 127, the very notion of strict predicativity needs to be further sharpened. A possible way of phrasing the base is in terms of a substructure of the natural number structure that has a restricted form of induction. In the case of the strict predicativity suggested by Parsons, one may also introduce ramification at the level of constraints.

philosophical analysis of predicativity.

Let us consider again the issue of predicativism and let us examine the cases of predicativism given the natural numbers and strict predicativism. I suggested above that there is a worry that the first form of predicativism might collapse into the second, if sufficiently uniform reasons can be adduced for the constraints introduced in each case (e.g. non vicious circularity, invariance). If now we avail ourselves of the distinction between the base and the constraints, we may suggest that the differences between these philosophical positions lay especially within their distinct choices of bases: it is here that a predicativist given the natural numbers will have to concentrate his efforts to try and stabilise his position. Here, in addition, more work will need to be carried out to clarify in which sense the base may be taken for granted. For example, one may argue along epistemological or ontological lines for a defence of the choice of the base.[47]

The distinction between base and constraints may also help us express a claim often found in the relevant literature on predicativity, according to which the latter is a relative rather than an absolute concept, as expanded in the next section.[48]

4.5.2 Relativity of predicativity

The logical analysis of predicativity suggests that the predicativity of a definition depends on the context in which it is embedded. In fact, the proof-theoretic analysis of predicativity utilises some canonical formal systems to help determine if a definition, and the entity that is thereby defined, are predicative. To assess if a given entity, e.g. a set or a function, is predicative, one needs to check whether it can be defined within a predicative system. In practice this means that we need

[47]See Chapter 7 for additional discussion.

[48]Brief hints at a form of relativity of the notion of predicativity are to be found, for example, in (Gödel 1944, Kreisel 1960). The thought that predicativity is relative has been also expressed by Parsons (1992) and Feferman (2005).

to show that a predicative system can prove the existence of that entity (e.g. by way of appropriate comprehension principles). The logical analysis of predicativity therefore clarifies that a definition should not be considered in isolation, but within a wider context. In case of formal theories, one would have to pay particular attention to the set-existence principles available. In fact, a remarkable effect of the logical analysis of predicativity was to shift considerations of predicativity from individual definitions to whole theories. It is a theory, now, that is predicative or impredicative, and individual definitions are assessed for their predicativity relative to the theories they are embedded into, not in isolation. In particular, the discussion in Section 4.3 suggests the importance of adopting a sufficiently weak theory as basis for our analysis.[49]

A different, more striking form of relativity appears when we consider the possi-

[49] An example of relativity of predicativity with respect to a context arises by considering the operation of powerset, that given a set, A, gives the set of all of A's subsets. It is often claimed that this is impredicative. However, on the basis of a *finitary* theory of sets, e.g. the theory ZF with the axiom of infinity replaced by its negation, powerset does not introduce impredicativity (according to the notion of predicativity given the natural numbers). This theory, in fact, has an axiom of powerset, but is of the same strength as Peano Arithmetic (and thus predicative according to this notion of predicativity).

A similar situation arises in the case of a variant of Zermelo Fraenkel set theory based on intuitionistic logic that is known as constructive set theory (Myhill 1975, Aczel 1978, Aczel & Rathjen 2008). Here one has that the core system, CZF, is predicative according to the constructive understanding of predicativity, as it has the same proof-theoretic strength as a system of one inductive definition. But the addition of the principle of excluded middle to CZF gives rise to the full system ZF, that is, a highly impredicative theory. Thus it would seem that the principle of excluded middle in this case acts as a strong impredicative principle. The analysis of the proof that CZF + **EM** = ZF suggests that the presence of the excluded middle in fact modifies the significance of other axioms of CZF and in particular of a principle, known as subset collection, that replaces, in that context, the powerset axiom. This is weaker than powerset on the basis of intuitionistic logic, but it becomes equivalent to powerset if **EM** is added. In this context, where one has, among other principles, an axiom of infinity, powerset in turn gives rise to full impredicativity.

bility of a plurality of predicativities. Here one could claim that the *assessment* of what counts as predicative is *relative to the chosen base*. Thus in predicativity given the natural numbers a given (definition of) an entity is predicative only relative to the prior assumption of the natural numbers as base. This suggests a more general pattern. We may now classify as a form of predicativity also Gödel constructible hierarchy (see Section 4.1). As already noticed by Gödel, the hierarchy is predicative at each step, but impredicative on the whole, because of the assumption of the class of ordinals. One way of framing the local predicativity of the constructible hierarchy is by claiming that the hierarchy is predicative relative to the class of ordinal numbers.

An advantage of framing predicativity in these terms, is that one can then separate the tasks that face a predicativist as: (1) arguing for the constraints, and (2) arguing for a specific base. More importantly, it allows us to distinguish between different parts of mathematics, that require different assumptions and forms of reasoning. Here in particular, we can single out some assumptions (e.g. the natural number structure, the class of ordinals) and see how far we can go from them by allowing ourselves a limited number of arguably uncontroversial moves (as expressed by the constraints).

4.6 Conclusion

In this chapter I have aimed at clarifying the fruitful interaction between philosophy of mathematics and mathematical logic that was induced by reflections on predicativity. I have also aimed at conveying the thought that predicative mathematics is a central part of today's mathematics as it accounts for large portions of ordinary mathematics. In addition, predicatively motivated notions are at the core of the logical enquiry. I have in particular outlined two main components of the analysis of predicativity by logical means: the determination of its proof-theoretic limit, and the establishment of its extent within ordinary mathematics. By clarifying how far predicativity goes we also distinguish areas of today's mathematics that are erected on the basis of very different kinds of assumptions, and that make use of different reasoning tools. For example in the case of the analysis of predicativity given the natural numbers, this allows us to see how far we can extend beyond arithmetic without requiring new conceptual leaps to different, more abstract notions. This has strong affinities with the perspective I have offered on constructive mathematics in Chapter 2, where I have argued that a serious consideration of the constructive stance can help in the philosophical analysis of mathematics in general.

Perhaps the most surprising outcome of the above investigation into predicativity is that not one but a plurality of variants of predicativity have appeared within the mathematical literature, so that it turns out that different predicatively inclined mathematicians will answer differently to the question whether a given definition, or a portion of mathematics is predicative or not. In fact, this may be seen as rising a challenge to forms of predicativism, to ensure that sufficient argumentation is offered to support not only predicative constraints but also the choice of base.

In the next Chapter 5 I shall endeavour to uncover a common theme that may be seen as running through all of these forms of predicativity, relating to a predicative concept of set.

Chapter 5

On a predicative concept of set

In the previous two chapters I have begun an investigation into a second fundamental dimension of constructivity that characterises constructive mathematics: predicativity. From a classical, impredicative perspective predicativity introduces crucial restrictions to what counts as a set. These are witnessed by the introduction of a combination of type restrictions and ramification in (Russell 1908) and, differently, by Weyl's development of an arithmetical form of analysis in (Weyl 1918). I have then explored the fruitful interaction between mathematical logic and philosophy that originated from a reflection on predicativity. From an external, impredicative perspective, one can determine limits for the strength of predicative theories; in addition, one can assess predicative systems' ability to re-capture theorems of ordinary mathematics. I have indicated the outcomes of a logical analysis of predicativity that shows that a substantial portion of ordinary mathematics is already predicatively reducible.

In this chapter I shall argue that, from a different perspective, predicativity may also be seen as arising not from constraints imposed on our ordinary sets, but from a radically different conception of set. To elucidate this thought, I shall analyse in more detail Poincaré's late writings on predicativity and Weyl's "Das Kontinuum". My purpose is to single out a predicative conception of set that is deeply rooted in an older form of mathematics that pre-dates the deep methodological changes that revolutionised mathematics from the 19th Century. Simultaneously, this concept of set also aims at introducing improvements to that tradition. A fundamental aspect of this concept of set is its tie to a **definition** or *uniform description*, that allows for a step by step "generation" of the set. One of the outcomes of a slow process of clarification of this conception of set, has given rise to a notion of predicative set that is "algorithmic", in that it offers a procedure for the step by step "construction" of its elements. In fact, surprising similarities arise especially between Weyl's notion of set and the notion that we find in prominent foundational systems for constructive mathematics today. The thought, once more, is that the older, more explicit mathematical style that was supplanted by the advent of abstract forms of mathematics had the potential to offer a notion of set that turns out to be ideal for a computational form of mathematics.

In this chapter I shall further explicate this predicative conception of set, by analysing in some detail how it evolved through Poincaré and Weyl's reflections on predicativity. In particular, a clarification of Poincaré's concept of set is crucial for an elucidation of Poincaré's characterisation of predicativity in terms of invariance that was discussed in Chapter 3, Section 3.3.2. However, before addressing Poincaré and Weyl's conception of set, it is important to clarify its relation with an alternative notion of set to which both Poincaré and Weyl reacted: the notion of arbitrary set that is often seen as underpinning Zermelo Fraenkel set theory with choice (Section 5.1.1).

The opposition between predicative and arbitrary sets witnesses an opposition between alternative views of mathematical entities, requiring or not the availability in principle of a (uniform) description or a definition. The requirement that all mathematical entities be definable in a uniform way is often termed definabilism.[1] In Section 5.3 I shall propose that Poincaré's discussion hints at a particular form of

[1]See e.g. (Maddy 1997).

definabilism, by imposing first of all the requirement that all mathematical objects be definable through a *finite* number of words. This requirement places the natural number structure at the heart of the predicative mathematical universe, as a finite definition can be coded in terms of the natural numbers. Secondly, closer examination of the later writings by Poincaré points towards a more sophisticated form of definabilism, according to which sets are *genetically constructed* from their elements by means of finitary definitions. Poincaré's discussion is informal and his proposal is insufficiently clear from a contemporary perspective. In Section 5.4 I shall therefore focus on Weyl's specification of a concept of set as extension of a property that is set up in full detail in (Weyl 1918). I shall highlight two salient components of Weyl's discussion: the central role assigned also by the German mathematician to the natural number structure with full induction, and the "mathematical process" that starting from the natural numbers gives rise to sets through a step by step specification of arithmetical properties. As mentioned above, from a contemporary perspective, surprising similarities emerge between the predicative concept of set proposed by Poincaré and Weyl and the concept of set that is codified in Martin-Löf type theory. I shall briefly discuss this in Section 5.5.

A reason recommends a closer investigation of the original texts on predicativity. The previous chapter has left us with a difficulty. We have seen that a number of alternative proposals go under the name of predicativity; however, they reach very different conclusions on which mathematical theories are predicative. For example, constructive predicativity legitimizes the use of generalised inductive definitions in conjunction with intuitionistic logic, while the same are considered unjustified from the perspective of predicativity given the natural numbers. This may cause difficulties for predicativism, as discussed in the previous chapter. But, more importantly, it may be seen as challenging the very idea that we can gather all of these varieties of predicativity under one denomination. The present chapter aims at singling out one theme in the predicativity debate that runs through both the classical and the constructive tradition. The concept of set characterised below appears in the writings by Poincaré, Weyl, Parsons, and seems particularly close to the one that figures in Martin-Löf type theory. It would seem that an appeal to a predicative conception of set justifies one sense in which all of these different approaches to predicativity may be brought under a unitary framework.

5.1 Sets as extensions of predicates

Reading Poincaré and Weyl one can not but notice that they both assumed a conception of set that differs considerably from the one we are most familiar with, given our acquaintance with ZFC.[2] The concept of set underlying Poincaré and Weyl's discussion is rooted in the **logical conception of set**, for which **a set is the extension of a concept**, that is, the collection of all the objects that satisfy (or belong to) a given concept. As further evinced below, this conception of predicative set also features typical aspects of the "older" form of mathematics that pre-dated the deep methodological changes that took place at the turn of the 19th and 20th Centuries, as reviewed in Chapter 1, Section 1.1. In particular, it retains a requirement of *finitary definability* that in some respects resembles the requirement of explicit definability that was typically imposed on mathematical entities, and in particular on functions, *prior* to the introduction of abstract set theory.

Poincaré typically expressed his views in linguistic terms, and saw a set as given by a **definition** composed of a finite number of words. Weyl made use of Husserlian terminology and referred to judgements affirming **properties** and relations between mathematical objects.[3] The distinction between each of these ways of framing the

[2]The significant role of a different conception of set (compared with ZFC's) within the early debates on predicativity has been underlined in (Parsons 1992, Parsons 2002).

[3]Also Russell's recourse to the notion of propositional function that we encountered in Chapter 3 suggests a similar concept of set, although the case of Russell is more complex; as indicated by Parsons (1992, p. 153) : "Russell is perhaps not so clear because of his unclear conception of

concept of set, as extension of a definition, or a property, is of historical and conceptual relevance; however, within the limits of the present context it seems reasonable to follow Parsons (1992) and assimilate the thought of Weyl to that of Poincaré. In the following I shall endeavour to clarify a common theme that appears across these authors' work, giving rise to a conception of set that Parsons (1992) describes in contemporary terms as that for which **sets are extensions of predicates**.[4]

A crucial feature of this conception of set is that the predicates (definitions, properties) are prior to the corresponding sets. As clarified by (Parsons 1992, p. 154), the priority of the predicates over the sets is often framed in epistemological terms, in that the understanding of the predicate is before the apprehension of its extension as an object. Another fundamental characteristic, expressed very neatly by Weyl, is that the "properties" can be thought of as if they were built up in stages, starting from the natural numbers.

In subsequent sections I shall analyse Poincaré and Weyl's conceptions of set in detail. In the following I shall first address the emergence of a prominent alternative to it, given by the concept of arbitrary set. This seems necessary to appropriately elucidate Poincaré and Weyl's proposals as they arise as a reaction to the concept of arbitrary set.

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the relation of propositional functions to language". Russell's concept of set will not be discussed below.

[4]To the contemporary eye the word "predicate" suggests that more should be said regarding the background language and formal system(s) in which the predicate is expressed. The plan in the following is to clarify first Poincaré's views, that are expressed in very general and informal terms. I shall then address Weyl's discussion. This offers the possibility to clearly express the notion of set as extension of a predicate in a way that also satisfies a contemporary perspective. One can, for example, recast Weyl's notion of set in terms of the extension of a predicate that is expressible in the language of second order arithmetic, and within a subsystem of the latter that is characterised by the arithmetical comprehension principle (ACA_0). See also footnote 44, Chapter 3 for details.

5.1.1 Sets in transition

Traditional approaches to logic (e.g. syllogistic logic) typically grant centrality to the notion of concept. It is therefore unsurprising that the first formulations of a modern notion of set were typically framed in terms of *extensions of concepts*. Accordingly, throughout the 19th Century a set was usually conceived of as the extensional counterpart of a concept, that is, as the collection of all and only the objects that satisfy a given concept (Ferreirós 1996). In addition, the possibility of faithful linguistic description of the concepts was typically taken for granted. In fact, some of the distinctions we make today were not available at the turn of the 20th Century, when it was natural to hope that a number of different renderings of the notion of set (e.g. in terms of concepts or linguistict descriptions) would eventually coincide.[5]

A crucial consequence of the set-theoretic antinomies that emerged at the end of the 19th Century was the growing awareness that more care was required in formulating the relation between concepts, sets, and their linguistic representations.[6] The antinomies then can be seen as placing strain on the naive thought that there is truthful mirroring between the three realms: concepts, mathematical objects (e.g. sets), and their linguistic descriptions. A well-known reaction to the set-theoretic

[5]See (Ferreirós 2011) (especially Section 1.4) for a brief discussion of the relation between the logical notion of set as extension of a concept and one in terms of linguistic definitions. "Concepts may be available in at least two ways, as abstractly given, or by means of a linguistic specification; since by 1890 and even as late as 1910 notions were still unclear and in flux, Frege and others (including the French analysts Borel, Baire, Lebesgue) could hope that both avenues for availability might coincide." (Ferreirós 2011, p. 367) Ferreirós further elucidates that it was then also expected that it would be possible to account for *all* sets of natural numbers in either of these ways.

[6]As observed by (Ferreirós 1996), the antinomies contributed to a reconsideration of the then common view according to which the notion of set is ultimately a *logical* notion. A distinction between logical and mathematical notions started to appear only at that time. Ferreirós (1996, p. 63) sees the antinomies as producing the "divorce between logic and set theory".

paradoxes is exemplified by Zermelo (1908), who observed that one can not take arbitrarily concocted concepts to give rise to sets.[7] As described in Chapters 3 and 4, the attempts made to clarify this point turned out to shape much of contemporary set theory and logic.

I propose that we distinguish between two opposite attitudes to the challenge of the paradoxes: for some the antinomies had the effect of drastically severing the traditional tie between sets and concepts and their linguistic representations, while for others they imposed a reinforced link between sets and concepts (and for some also their linguistic representations).[8]

One reply to the antinomies was to develop formal tools that would characterise a very liberal conception of set, one that had arisen especially, but not exclusively, through the work of Cantor.[9] The resulting sets were eventually fully emancipated

[7]Zermelo (1908, p. 200) writes: "In particular, in view of the "Russell antinomy" of the set of all sets that do not contain themselves as elements, it no longer seems admissible today to assign to an arbitrary logically definable notion [Begriff] a set, or class, as its extension."

[8]In the following I shall predominantly focus on the predicativist position, and especially on Poincaré, so that the linguistic dimension will typically be a component of the view. However, I shall try as much as possible to ensure that my discussion does not depend on particular features of specific languages, as a thorough discussion of this point from a contemporary perspective would introduce substantial complexities. As my analysis of this predicative concept of set has also the ambition of clarifying aspects of a contemporary constructive concept of set, it is interesting to note the attitude of contemporary constructivists on the relation between mathematical conceptualization and its linguistic representation. As it is well known, the intuitionist tradition started by Brouwer emphasizes the importance of the conceptual component of mathematics, eschew as much as possible from the linguistic component. Constructivists today usually do not share the anti-linguistic attitude of the early intuitionists, although they often assign a certain prominence to the conceptual dimension, stressing the role of intuition and evidence within the mathematical activity. Linguistic representation has gained, however, more prominence in recent years due to the relation between constructive mathematics and computer computation, as the latter requires very detailed attention to the linguistic and formal dimensions.

[9]See e.g. (Ferreirós 1999, Ferreirós 2011) for an assessment of the fundamental contributions

from concepts and definitions. The outcome was a notion of "arbitrary" set, and one of the aims, after the antinomies were discovered and their full impact realised, was to ensure that arbitrary sets were safe. This prompted a careful formulation of an axiomatic system that would avoid the set-theoretic antinomies. As a consequence, Zermelo set theory was fabricated so to capture all the set-theoretic operations that had been proposed out of the needs of the expanding contemporary mathematics, without at the same time giving rise to any known paradox. The outcome, Zermelo set theory and its subsequent extension, ZFC, aim at capturing a conception of set that does not require the availability of prior concepts nor linguistic descriptions for each set. One possible way of justifying this notion of arbitrary set is through an appeal to what has been called quasi-combinatorialism (Bernays 1935), as further discussed in Section 5.2.1.

Another, opposite reaction to the antinomies was not to divorce sets from concepts, but to appropriately strengthen their link. For some, in fact, the antinomies acted as a warning against the very detachment of sets from their conceptual counterparts, as the paradoxes were deemed the result of too loose a connection between sets and our conceptualisation of them. The ensuing concept of predicative set reinforced, rather than severed, the traditional tie of sets with concepts and their linguistic presentations: strictures were imposed on how the concepts, and the corresponding definitions or predicates, ought to be formed so to obtain a consistent notion of set.

These two alternative attitudes to the discovery of the set-theoretic antinomies underlie the opposition between the concepts of arbitrary set and set as extension of a predicate that I shall now endeavour to further clarify. In the following I shall indicate the main characteristics of a concept of arbitrary set, outline the concerns that arise from a predicative perspective, and briefly sketch a prominent justification of this notion of set in terms of Bernay's "quasi-combinatorialism". The main focus

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by e.g. Riemann and Dedekind.

will be, however, the predicative concept of set, which will be thoroughly discussed in subsequent sections.[10]

5.2 Absolutely arbitrary sets

The concept of set as extension of a predicate profoundly differs from a concept of arbitrary set that slowly made its way within the fast changing mathematics of the 19th Century, reaching completion with Cantorian set theory. Cantor's "definition" of set in (Cantor 1895, §1) contains paradigmatic aspects of this conception: *"By a set (German: Menge) we understand any collection (German: Zuzammenfassung) into a whole of definite and well-distinguished objects of our intuition or thought"*. The most significant aspect of this new conception of set which is relevant in the present context is *the lack of uniform definability of the set in terms of its elements*, which is instead characteristic of the predicative notion of set that we shall examine in subsequent sections.[11]

A note is in order. Given a common association between a set-theoretic perspective with mathematical realism, one might be puzzled by my reference to Cantor's "definition", as it introduces a mention of "our intuition or thought". This indeed helps me clarify that the aspect I wish to focus on here is merely the availability even in principle of a *uniform description* of the set, or the lack of it. I maintain that this is a significant aspect that can be considered independently of the separate issue of the nature of the mathematical entities and their relation with us, and in particular, whether the uniform description is imposed by us or not.

[10]The concept of arbitrary set and "quasi-combinatorialism" have been extensively discussed in the literature. See e.g. (Maddy 1997). See also (Ferreirós 2011).

[11]The aspect I wish to highlight is that this definition by Cantor manifests the detachment of the sets from a uniform law; the latter was instead a fundamental component of Cantor's first "definition" of set in (Cantor 1883, p. 204), according to which sets are seen as elements that can be linked into a whole *by some law* (Ferreirós 1996).

The origins of the concept of arbitrary set are usually traced back to the methodological alterations that took place within mathematics and, especially, but not exclusively, analysis, in the 19th Century.[12] It is hard to overemphasise the radical nature of the new conception of set that emerged within set theory. The new set-theoretic perspective, in fact, countenances the mathematical treatment of sets that are no more required to be definable. This, however, makes it difficult to elucidate what arbitrary sets are, as also emphasized by Ferreirós (2011). Arbitrary sets encompass "non–definable" sets: sets that are taken to exist but for which we do not have *even in principle* a linguistic definition. Ferreirós observes that we have numerous examples of definable sets, but a specific (or "concrete", to use the mathematician's terminology) "non definable" set is simply not amenable to be presented as an example.[13] A typical arbitrary set is the powerset of the natural numbers, that includes ***absolutely all*** (definable and "non–definable") subsets of the natural numbers.[14]

The powerset of the natural numbers is a clear witness to a new form of arbitrariness that appears in mathematics at the end of the 19th Century. Wang (1954, p. 243) clarifies the importance of the conceptual leap that this form of arbitrariness manifests. He stresses the difference between a "moderate arbitrariness" that

[12] (See e.g. Ferreirós 2011, Stein 1988, Wang 1954).

[13] "The more one reflects on this matter, the more obvious it becomes; eventually one may come to think that the idea of a "concrete" example of an "arbitrary" anything is an oxymoron." (Ferreirós 2011, p. 364) See also footnote 17, page 53 for the mathematical use of the word "concrete".

[14] It is remarkable that from a classical, impredicative perspective the opposition between the two notions of predicative and arbitrary set manifests itself in emblematic form very low in the set-theoretic hierarchy, already at the level of the powerset of the natural numbers. It is this simple case that will be main focus of my discussion in this chapter. Note that Ferreirós also discusses the role of the Axiom of Choice for the notion of arbitrary set. His main argument being that ZFC's formalization is rather poor in capturing the concept of arbitrary set that motivates that set theory.

mathematicians often appeal to, and the "really arbitrary" that arises in set theory:

> mathematicians often speak of arbitrary functions and arbitrary curves when they have no precise definition of these notions and actually have in mind only certain special functions and special curves.

Set theory, instead, aims at capturing a notion that I should like to call "absolutely arbitrary set", a paradigmatic example of which is the powerset of an infinite set. Wang (1954) discusses the case of the powerset of the natural numbers, and briefly summarises Cantor's proof of the indenumerability of the powerset of the natural numbers, noting its impredicativity. He also observes that Cantor's proof only shows that for any enumeration f of the powerset of the natural numbers, we can find a new set x that is different from all the sets enumerated by f. This means that given any enumeration f of the powerset of the natural number, we can extend f to a new more encompassing enumeration of it. From this to conclude that the powerset of the natural numbers is indenumerable we need, however, to further *assume* that the powerset of the natural numbers exists.[15]

> From the fact that no enumeration can exhaust all sets of positive integers, Cantor infers that the set of all sets of positive integers is absolutely indenumerable. In order to justify this inference, we have to assume that there is a set which includes all sets of positive integers [...].

Wang's discussion may be seen as indicating that Cantor's proof of the indenumerability of the powerset of the natural numbers comes at the cost of assuming

[15]Wang (1954, p. 244) expresses scepticism regarding the notion of arbitrary set encoded by ZFC, on the grounds that it is unclear whether such arbitrariness is needed within the mathematical practice. He writes that the use of uncountable (indenumerable) and impredicative sets "remains a mystery which has shed little light on any problems of ordinary mathematics. There is no clear reason why mathematics could not dispense with impredicative or absolutely indenumerable sets." See also Chapter 7, Section 7 for a discussion of this point.

a set whose elements are both the definable and the non-definable subsets of the natural numbers. Provided that this set exists, then Cantor's proof shows that it is indenumerable.[16] The important aspect to remark is that this requires the assumption of the existence of a totality of all definable and non-definable sets of natural numbers, although we might be unable to specify individual elements of this totality. This clearly differs from assuming that the natural numbers form a set, as in this case we can, in principle, exhibit individual elements.

The impredicativity and the arbitrariness of the powerset of an infinite set is also addressed in a fundamental article by Myhill (1975), in which the author sets out the details of a constructive set theory that notwithstanding its use of intuitionistic logic bears strong formal affinities with ZF set theory. Myhill replaces the powerset axiom of ZF with a constructively weaker axiom of exponentiation, as the first is seen as lacking constructive justification.[17] Myhill's criticism of the powerset axiom of ZF is particularly clear, and deserves quoting:

> Power set seems expecially nonconstructive and impredicative compared with the other axioms [of set theory]: it does not involve, as the others do, putting together or taking apart sets that one has already constructed but rather selecting out of the totality of all sets, all those that stand in the relation of inclusion with a given set. (Myhill 1975, p. 351)

In this Section I have offered an intuition of the concept of arbitrary set, and

[16]It is interesting to note that also Poincaré (1912) discussed Cantor's related proof of the indenumerability of the real numbers within his criticism of the arbitrary notion of set adopted in set theory. Poincaré remarked that Cantor's proof shows that there is no way of defining a bijection between the natural numbers and the real numbers without producing a new real number that would require an extension of the bijection. His conclusion is that this makes the possibility of comparing the cardinality of any two sets, and, in particular, the existence of $\aleph_1$ doubtful.

[17]The axiom of exponentiation allows us to collect in a set all the functions from a set A to a set B. This is constructively weaker than the full powerset, and allows for the formulation of set theories that are constructive predicative (Myhill 1975, Aczel 1978) (see also (Crosilla 2015*b*)).

indicated some of the difficulties that arise from a predicatively inclined perspective. In the next Section 5.2.1 I shall briefly consider a prominent view that may be taken to justify the notion of arbitrary set.

5.2.1 Quasi-combinatorialism

A very influential exposition of a view that justifies the powerset of an infinite set is given in (Bernays 1935), where it is termed **quasi-combinatorialism**: this arises by ***taking as a set any possible combination of given objects***. We can see how quasi-combinatorialism works by considering first a finite set, B. We observe that we can form the set of all possible combinations of its elements, $\mathbf{P}(B)$, which contains $2^{|B|}$ elements, each a subset of B.[18] As $P(B)$ gathers together all possible combinations of all the elements of B, this process can be seen as an example of "combinatorial reasoning" (Bernays 1935). The distinctive characteristic of this kind of reasoning is that it allows for the formation of the powerset of $B, \mathbf{P}(B)$, without requiring the uniform definition of each of the elements of $\mathbf{P}(B)$: *each element of the powerset can be imagined as the result of an "independent determination".*

Quasi-combinatorialism amounts to doing *exactly the same thing for infinite sets*, too: we take all possible combinations of all the elements of an infinite set A and collect them into a whole, the powerset of A, without requiring the uniform definability of each of its elements.[19]

A fundamental motive that underlies quasi-combinatorialism is the thought that *there is no substantial difference between finite and infinite sets.* As further clarified below, both Poincaré and Weyl instead insisted on the deep asymmetry between the finitary and the infinitary cases. It is in particular the belief that infinitary mathematical notions only afford safe treatment through a finitary (and genetic) description that determines the development of their predicative notion of set.

[18] Here $|B|$ denotes the cardinality of B, that is, the number of its elements.

[19] See also Maddy (1997) for discussion of quasi-combinatorialism.

It might be useful to expound in full generality a predicativist's remonstrations against quasi-combinatorialism. In the case of a set B of finite cardinality it is typically expected that we could give, at least in principle, an explicit definition of each of the subsets of B, if we were sufficiently pressed.[20] In this case the combinatorial formation of $P(B)$ works as a more economical procedure, that could then be replaced by the explicit definition of each of the elements of $P(B)$, the latter offering full justification for the first. On the contrary, in the case of an infinite set A, like, for example, the natural numbers, it is no more to be expected that we could offer a way of describing or singling out each of the elements of $\mathbf{P}(A)$, were we requested to do so. The principal complaint is that we are not offered here a precise description of each subset of A. Therefore, if we were asked to pin down exactly the extension of the predicate "subset of A" we would seem to be unable to do so, as we do not possess a uniform rule that explicates how the elements of $P(A)$ relate to $P(A)$ and to each other. One option for justifying $P(A)$ is to assume at the start the set-theoretic universe, out of which we select the subsets of A.[21]

5.3 Poincaré on sets and definitions

A thorough clarification of the differences between the notion of set appealed to by the early predicativists and the arbitrary notion of set can be gathered by expanding further on the views of Poincaré (Poincaré 1909, Poincaré 1912). Poincaré is particularly explicit in framing the opposition between rival conceptions of sets as related to the specific difficulties that arise when we wish to work with infinite sets in mathematics. Poincaré (1912), in particular, portrays the conflict on predicativity as deeply interconnected with contrary views on the nature of infinity, which manifest the traditional opposition between potential and actual infinity. The notion of

[20] There is clearly an element of idealization in this contention, that is questioned by forms of predicativity like Nelson's strict predicativism.

[21] I shall further discuss this point in the next section.

arbitrary set for the French mathematician witnesses full acceptance of actual infinity, while a requirement of definability of the set through a finite definition reflects the belief that only potential infinity is justified.

In the following, I wish, however, to refrain from using the opposition between actual and potential infinity. An appeal to this traditional opposition, in fact, seems unhelpful in this discussion, in as much as one does not also offer a clear account of this distinction that would make it a profitable instrument in the understanding of the opposition between arbitrary and definable sets.[22] I shall rather try to clarify the opposition between arbitrary sets and (a particular kind of) finitarily definable sets. First, however, I shall briefly outline Poincaré's reasons for the finitary requirement.

For Poincaré, as human finite beings we can only safely work with mathematical entities that are definable by a *finite number of words* (see also Chapter 3, Section 3.3.2, page 87).[23] In other terms, it is our finite nature that imposes that a mathematical treatment of infinite classes is only possible through a finite definition. Sets

[22]The opposition between actual and potential infinity is identified by some authors with the availability or not of the principle of excluded middle when reasoning about infinite collections. Some of Feferman's writings on predicativity, for example, may suggest such an understanding of this opposition (see e.g. (Feferman 2005)). This identification has the remarkable advantage of clarifying in precise mathematical terms the extent of the distinction between potential and actual infinity; however, it also seems to elude some of the most significant aspects of this distinction from a philosophical perspective. In particular, taken by itself, it does not clarify the distinction between views of the set-theoretic universe which manifest a potential or an actual understanding of infinity. See e.g. (Linnebo 2013) for a modal account that affords an explication of a potentialist perspective.

[23]Poincaré writes that as the thinking subject is a human being, "or something that resembles a human being", it is a finite being. As a consequence, infinity is the possibility of creating as many finite objects as one wishes. That is, infinity is *potential infinity.* Poincaré also clarifies that there is an element of idealization in this position, as he assumes a form of definability *in principle*, not actual definability, as it would be required by a finite being. This is clearly a very delicate issue that would need to be fully assessed if one were to offer a philosophical defence of definabilist positions on the basis of considerations of finitude of the human mind or the human language.

then are *the extension of finite definitions* for Poincaré. In fact, as further discussed below, this view by Poincaré is deeply interrelated with the further contention that a class *is given by a definition* and does not exist independently from it.

The requirement of finite definability of sets places Poincaré's discussion on predicativity within the larger context of a position on the methodology of mathematics that is often referred to as definabilism (or definitionism). Maddy (1997, p. 116) so characterises definabilism:

> the requirement that all mathematical things be definable in a certain uniform way. I leave open the exact specification of this 'uniform way' to allow for different versions of the same general maxim.

This was a widespread position at the time of Poincaré and is particularly well represented by the French analysts Borel, Lesbegue and Baire, whose criticism of the axiom of choice moved directly from the contention that "it is impossible to demonstrate the existence of an object without defining it" (Baire, Borel, Hadamard & Lebesgue 1905, p. 314).[24]

Poincaré's discussion on predicativity further specifies the finitary requirement by imposing a genetic definition of a set in terms of (the definition of) its elements. This relates to his requirement of invariance of a definition, and may be seen as further specifying the "uniformity" of definitions that Maddy alludes to in the quotation above. I shall further clarify this point in Section 5.3.2 below. First I wish to consider Poincaré's (1912) discussion of definitions and give an example of impredicative

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[24] Ferreirós (2011, p. 375) writes: "It was because many mathematicians understood sets as concept-extensions that they found it unacceptable to postulate the existence of sets such as those guaranteed by Choice. (Similarly, many thought that functions ought to be given by explicit formulas–a view that was strongly promoted from Berlin, as early as 1870.) To be more precise, many mathematicians of the period were inclined towards a constructivist notion of mathematical existence (for the real numbers) and showed definabilist preferences concerning sets of reals, all of which caused them to object to the Axiom of Choice."

definition.

5.3.1 Impredicative definitions

Poincaré's discussions on predicativity are part of a more general discussion on legitimate methods in mathematics. Poincaré therefore addresses the issue of predicativity by analysing different kinds of definitions within the mathematical practice, and assessing the threat that impredicativity poses for correct reasoning in mathematics. For example, in Poincaré (1912) distinguishes between two principal kinds of definitions: ***direct definitions***, and ***definitions by postulates***. The first kind of definitions, the *direct definitions*, are characterised by the fact that one could (in principle) replace each term in the definition by its own definition, and, by going all the way down, reach identities that hold purely in virtue of tautologies.[25] These are ideal definitions, but are not always available. The second kind of definitions is given by means of *postulates*. Poincaré describes this case by explaining that in definitions by postulate we know that the definiendum belongs to a class, however, we need to further specify the definiendum by appeal to an additional condition, a postulate, that it needs to satisfy. Here the principal difficulty is in ensuring the consistency of the postulate, without which the definition would be illegitimate.

Within our context, the most significant part of this discussion by Poincaré on definitions by postulates arises when he claims that some of these definitions are source of disagreement between mathematicians. The disputed definitions by postulate express a relation between the definiendum and *all* the objects of a class to which the definiendum is supposed to belong. In this case some mathematicians, but not others, claim that there is a vicious circle. Poincaré gives the following very general example: suppose we wish to define an object X by postulating a relation R between this object and all the elements of a "class", G, to which it is supposed

[25]The notion of direct definition coincides with what is often termed explicit definition in the philosophical literature.

to belong.[26]

In the following I shall first of all offer a very general description of this example by calling G a "class", and suggesting one way of clarifying the difficulty that is involved with its definition. In the next section I shall further analyse this example by more carefully distinguishing between a definition of a class and its extension.[27] This second reading better explicates the difficulties that are involved with impredicative definitions, given the notion of set as extension of a predicate here under examination.

The definition of X in terms of G gives rise to a vicious circle, since it presupposes all the elements of G, and therefore also X itself. It is interesting to see how Poincaré (1912) frames the difficulty with impredicative definitions of this kind. Poincaré compares here the perspectives of "Cantorians" and "Pragmatists": the first are depicted as defending a form of mathematical platonism and the latter as holding predicativist positions, thus representing largely Poincaré's standpoint. In particular, the latter are also represented as holding a traditional form of idealism, according to which mathematical objects do not exist independently from a (human) thinking subject.[28]

The French mathematician so expresses the difficulty that arises in this case: we can not define an object X in this way without *knowing* all the individuals of G, and thus without knowing X itself. The Cantorians, observes Poincaré, do not see a difficulty here, as from their perspective the class G is *given*, and the only purpose of the definition is to discern, by reference to all the elements of G, the particular individual that is in the relation R with all the $G's$.[29]

[26]The case of Richard's paradox that was discussed in Chapter 3, Section 2 offers a more concrete example that falls under this scheme.

[27]Poincaré took sets to be definitions, therefore his discussion may be difficult to frame within a contemporary perspective that separates the two aspects.

[28]Poincaré's comparison between these alternative positions concludes with the disenchanted observation that the two points of view are incompatible, and the disagreement irreconcilable.

[29]Note that X's presupposing G is here expressed in epistemological terms. In this text, Poincaré

This point is important and deserves further clarification. According to Poincaré, for the Cantorians the elements of G are already given, consequently the definition suffices to single out X from the other elements of G. However, for the predicativists the elements are "constructed" through their definitions, they do not exist independently from these definitions, and therefore a circularity is problematic, as it requires the assumption of what we are about to construct.

There are a number of points that Poincaré's discussion raises here. In particular, there is the contention that impredicative definitions may be seen as legitimate from a realist perspective, but illegitimate from an anti-realist perspective. This is a frequent reading of the debate on impredicativity, made particularly prominent by two early influential articles that address predicativity (Carnap 1931, Gödel 1944). A frequent interpretation of the predicativity versus impredicativity debate is in terms of an opposition between realist and constructivistic attitudes. A common claim is than that realism offers a way of justifying impredicativity, by granting the mathematical entities on which impredicative definitions generalise. A definition in this context has only a descriptive role: to select out of the totality of mathematical entities, those that satisfy the definition. If, however, one does not presume that mathematical entities are "given" in the first instance, for example, if one takes an idealist perspective similarly to Poincaré's Pragmatist, then an impredicative definition is highly problematic. A definition here is a construction of a *new* mathematical entity, and thus it can not appeal to its definiendum.

I shall further discuss this example in the next two sections, in which I shall try and generalise from this particular way of framing the opposition between predicativity and impredicativity. For this purpose I shall adopt a philosophically less

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sees the existence of the mathematical objects as dependent on their definability and (hence) also on their knowability. Poincaré also writes that the Cantorian thus also *knows all of the elements of G*. An epistemic reading of realism is not unusual also within more recent philosophical reflections by constructive mathematicians, as, for example, in the writings of Bishop. See (Billinge 2003) for critical discussion.

committed prespective, while availing myself of some of Poincaré's ideas. I shall therefore detach from Poincaré's discussion his particular view of the nature of the mathematical entities, and rather examine the general features of a conception of set as extension of a definition that is suggested by Poincaré's text. From this perspective I wish to clarify the significance of Poincaré's notion of invariance.

5.3.2 Poincaré's "genre" and incomplete definitions

For Poincaré definitions are **classifications**: they separate the objects that satisfy, from those that do not satisfy that definition, and they arrange them in two distinct classes. Poincaré (1912) also assimilates the definition of a class to traditional classifications by ***"genus proximum et differentiam specificam"***: when defining a class, one should specify (1) a *"genre"* that is in common to all the elements of the class and, (2) some individuating characteristics that are specific of each element of the class. It is the second point that is worth our attention, as it may be seen as offering a refinement of the definabilist requirement that a definition of a set be finitary.

Let us clarify this point by means of the example above. We aim at defining an object X by postulating a relation R between this object and all the elements of a class to which it is supposed to belong. Poincaré here refers to the relevant class as given by a "genre" G: this is a general description of a property that all the elements of the class satisfy.[30] From a contemporary perspective, the "genre" G acts as a definition of a class, and the latter is G's extension. The crucial point is that G does not specify how to define or "construct" each individual element of its extension. One could also say that the "genre" G gives only an **incomplete definition** or a **mere specification** of its extension; that is, a description of some

[30] As already mentioned, there is a conflation in Poincaré's discussion between G as referring to the definition of a class and G as referring to the extension of this definition. In the following I shall distinguish the two aspects.

general characteristics shared by all its elements, without an explicit definition of the individual elements.

The incomplete nature of the "genre" is the reason why we run into difficulties in defining X. The thought is that prior to defining X we need to fix the extension of G, as X is defined by postulating a relation, R, with all of the G's.[31] However, as we define X, this turns out to be a new element of the extension of G. Therefore it seems that the definition of X **enlarges** G's extension and **disorders** G itself.

It is worth pointing out the implicit assumptions that seem to be required for finding fault with the impredicative definition of X. First of all, we assume that X is specifiable only in terms of its genre G, so that we do not have a definition of it that is independent from G. Secondly, there is the contention that the extension of G ought to be fixed *before* we generalise over it to define X. In contemporary terms this may be rephrased by stating that the extension of a definition needs to be fully determined prior to acting as domain of quantification. Finally, the dependence of the set on its definition also plays a role, as this is framed in such a way that we can envisage situations in which the extension of a definition might not be fixed once for all. The thought seems to be that if we give only an incomplete definition of a set, if we specify no more than the "genre", then the extension of such a definition is not fixed once for all, and might be enlarged by subsequent definitions.

Poincaré's own assessment appears to be that a realist attitude to mathematical entities grants the individual elements of a set independently from the definition of the set itself, so that a specification of the "genre" suffices to fix the relevant domain of quantification. However, from a Pragmatist or "constructivistic" perspective of the mathematical entities, we cannot concede the prior availability of X, and thus we exclusively rely on the genre G for its specification. In this case, the difficulty with impredicative definitions clearly arises.

On reflection, it seems that the move to realism Poincaré alludes to is but *one*

[31]See the next section for a clarification of the idea of "fixing" the extension of a genre.

possible way of according impredicativity. Any way of "detaching" the elements of a set A from A's definition would also seem to work. This is the case, for example, of quasi-combinatorialism, where the elements of a set are assumed as given independently from the specification of the set itself. The discussion above also suggests that an appeal to realism by itself does not suffice to eliminate the difficulties with impredicativity, unless that form of realism also ensures an appropriate "detachment" of sets from definitions, as indicated above.[32]

"Genre" and invariance

Poincaré's discussion on the "genre" also helps clarify the notion of invariance that was introduced in Chapter 3, Section 3.3.2. Given the view that a definition is a classification, a distinction between predicative and impredicative definitions was framed as follows by Poincaré: a definition is **predicative** if it defines a class that is not "disordered" by the introduction of new elements, and impredicative otherwise.

If we take a view according to which a definition is prior to the class it defines, then we can envisage situations in which the class of all the objects that satisfy a certain definition might not be fixed in advance. Therefore, if we hold this conception of set, then it becomes important to guarantee that the tie that links the set to its definition does not cause an instability of the resulting set itself. This is particularly important in view of the possibility that the set so defined could in turn act as domain of quantification in subsequent definitions of other sets: if so its definition needs to fix in uncontroversial terms what belongs to its extension and what does not. Poincaré's notion of invariance here plays a crucial role as it is introduced with the aim of ensuring that an invariant definition fixes in uncontroversial terms what belongs to the extension of the definition. Its purpose is to fix once and for all the

[32]See e.g. (Parsons 2002) for a discussion of alternative routes to impredicativity already proposed within the early debates on this issue. I shall further examine another possible way out in Chapter 6, where I shall examine Dummett's notion of indefinite extensible concepts.

relations between all the elements of a set with each other as well as with the rest of the universe of sets; as a consequence, subsequent definitions of new sets will not modify these relations nor the set itself.[33]

As we have seen before, the genre does not suffice to fix in a definitive way its extension, in all those cases in which the elements of the set do not possess independent specifications. Therefore the genre needs to be complemented by an invariant specification of the elements of the set. In fact, Poincaré (1912) suggests that a proper definition of a set should start with a description of some initial elements, use them to construct new ones, and then use the latter to produce new ones, and so on. It is in this sense that Poincaré refines definabilism by requiring not just a finitary definition of mathematical entities, but a "genetic" construction of the set in terms of the definitions of its elements. This also determines a *uniform* tie between a set and its elements: the set is formed from the definition of its elements according to fixed rules, not arbitrary individual choices.

Poincaré's discussion clarifies the difficulties that impredicative definitions cause from a predicativist perspective, and gestures towards possible solutions. From a contemporary perspective, this requires a more detailed analysis. We shall see in the next section that Weyl (1918) offers a clarification of this process of "construction" of a set in precise, mathematical terms.

[33]A note is in order. In the discussion above I have predominantly drawn from (Poincaré 1912), although there are similarities with the views to be found in (Poincaré 1909). There are, however, also noticeable differences in emphasis between these two texts. In (Poincaré 1909) the discussion on predicativity highlights more prominently the epistemological dimension. A possible reading of (Poincaré 1909) is in semantic terms. Impredicativity here arises because of a difficulty in fixing the meaning of mathematical expressions. Specifically, the difficulty with impredicative definitions has to do with their *lack of meaning*, as their extension can not be fixed once and for all, being modified by the very definition. If we were to rephrase the example above according to this understanding of (Poincaré 1909), then the definition of X would be *meaningless*, as there is no way of fixing in a definitive way the referent of the genre G, and thus that of X. As a consequence the definition is also illegitimate.

5.4 Weyl's Mathematical Process

Weyl's overall philosophical stance in (Weyl 1918) is remarkably different from Poincaré's, influenced as it is by Husserl's phenomenology. The terminology is also importantly different, as is the emphasis on the linguistic component. In fact, many important distinctions that the contemporary eye does not find clearly expressed in Poincaré are clearly formulated by Weyl.[34] However, notwithstanding the differences, there are important similarities between the ensuing notions of set these two mathematicians propose, mainly in two respects. Firstly, both mathematicians view sets as extensional counterparts of something that is prior to sets: a definition or a property. In addition, they express a genetic build up of sets from their elements. Secondly, both authors highlight the importance of the natural number structure, and the profound difference between this structure and other infinitary structures (see Section 5.4.2).

The book (Weyl 1918) begins with the observation that although the critique of the 19th Century of the foundations of classical analysis was right, and contributed "immense advance in the rigor of thought", what had replaced the old edifice, if one focuses on the "ultimate principles", is even more unclear than what was there before. As recalled at page 100, the opening of (Weyl 1918, p. 1) is very explicit, setting up as a goal for the book to correct these mistakes, and replace analysis' present "shifting foundation with pillars of enduring strength".

A crucial instrument in the resulting clarification of the foundations of analysis is the notion of set, and Weyl (1918) carefully explicates a notion of set as extension of a **property** that is at the centre of his reconstruction of analysis. Weyl is more detailed than Poincaré in describing his notion of set, and his discussion makes use of a precise logical analysis of the mathematical language. This marks a significant

[34] For example, Weyl clearly distinguishes between the realm of the mathematical objects, the conceptual sphere and the linguistic one.

difference with Poincaré, whose polemics against the "logisiticians" is well-known. Weyl criticises the notion of set that underlies the new kind of mathematics, but at the same time is willing to make use of the recent advances in logic to rectify its foundations.[35]

The first chapter of "Das Kontinuum" is devoted to clarifying Weyl's notion of set[36], while the second chapter of the book is dedicated to an analysis of the concepts of natural number and, especially, the continuum. The view that sets are extensions of properties is clearly stated at page 20, where Weyl writes:

> Finite sets can be described in two ways: either in *individual* terms, by exhibiting each of their elements, or in *general* terms, on the basis of a rule, i.e., by indicating properties which apply to the elements of the set and to no other objects. In the case of infinite sets, the first way is impossible (and this is the very essence of the infinite). [...]
>
> *To every primitive or derived property P there corresponds a* ***set*** (P)*. The expressions "An object a has the property P" [...] and "a is an element of the set (P)" have the same significance.*

Weyl expounds the formation of derived properties from primitive ones in the first chapter of the book, and also states an extensional criterion of identity between sets,

[35]Weyl (1918, p. 23) writes: "But what was positively erected in place of the old is, *if one's glance is directed to the ultimate principles*, even more unclear and assailable than what it replaced– although it is certain that most of the achievements of modern critical research can be used anew as building material for a definitive founding of analysis."

[36]Weyl also discusses the notion of function, that is taken as primitive. This point would deserve further analysis, especially in view of the fact that the concept of function is usually primitive also in the constructive tradition. Due to space constraints, I shall postpone an analysis of the role of a primitive notion of function within predicativity and constructivity to subsequent work. For technical work that relates to this issue see also (Cantini & Crosilla 2008, Cantini & Crosilla 2010, Cantini & Crosilla 2012).

identifying sets (P) and (P') for properties P and P', if and only if the same objects satisfy the properties P and P'.[37]

The "production" in stages of new sets from properties is called the *"mathematical process"* by Weyl (1918, p. 22). This can be briefly described as follows: we start from some given category of primitive objects and proceed by forming judgements affirming the fact that certain properties and relations hold of objects of this category.[38] The next step is given by taking combinations of these judgements by means of the logical operations, with the crucial constraint that quantifiers are only allowed to range over the objects of the primitive category. This process is then further iterated to obtain more complex judgements. A set then ensues by taking the extension of a property affirmed by such a judgement.

In the following I shall expound in more detail Weyl's mathematical process to gain further insight into his notion of set.

5.4.1 Ascending from an initial category to sets

The starting point of the mathematical process is given by selecting a suitable "category of primitive objects", a particular example of which is the natural number

[37] Weyl's introduction of an extensionality criterion is important, as properties of sets (similarly to definitions for Poincaré) are intensional entities, and we may have a many-one relation between a number of properties and a set. Postulating an extensional identity criterion for sets is therefore necessary to ensure a development of standard mathematical notions within Weyl's system. One may in fact worry that the intensionality of properties might be an unwelcome feature of Weyl's system. In fact, the availability of intensionality at the level of properties is a very intriguing aspect of Weyl's treatment that would require further investigation. It is important to note that a form of intensionality is also at the heart of a contemporary constructive notion of set, and has there proved extremely useful. See Section 5.5 below.

[38] See the next section for the notion of category. Weyl clarifies the notion of judgement in the first line of (Weyl 1918, p. 1): "A *judgement* affirms a state of affairs. If this state of affairs obtains, then the judgement is true; otherwise it is untrue. [...] A judgement involving properties asserts that a certain object possesses a certain property [...]."

structure. Categories in general play a similar role as Russell's "ranges of significance" (see 91, Section 3.4), eliminating the possibility of meaningless applications of properties.[39] Weyl (1918, p. 5) so further clarifies the concept of category: "each property is always affiliated with a definite category of object in such a way that the *proposition* "a has that property" is *meaningful*, i.e., expresses a judgement and thereby affirms a state of affairs, only if a is an object of that category."[40] Weyl (1918, p. 8) requires that the *initial categories of objects* are "*immediately given* (i.e. exhibited in intuition)."[41] From a purely formal point of view, Weyl's initial categories play a similar role as Russell's type of individuals: they account for the initial objects that are used to start the mathematical process (or the type theoretic hierarchy, respectively). Significantly, Weyl requires that the initial category determines "a complete system of definite self-existent objects", that is, the extension of the category is fully determined and its elements can be considered as fully determinate (Weyl 1918, p. 8). This ensures that the initial category is a suitable domain of quantification, and that the principle of excluded middle can be applied to it. A paradigmatic example of initial category is the natural number structure.[42]

Weyl's fundamental contention is that when we move from an initial category (e.g. the natural numbers) to sets of its elements, we can no longer treat the latter

[39]See also Husserl's concept of categories of meaning (Husserl 1900/1, Husserl 1913).

[40]As an example of meaningless propositions, Weyl considers the "heterological" paradox, arising by considering words that denote properties that they do not possess. For example, the English word "short" is not heterological, it is "autological", as it is short, while the word "long" is heterological, as it is short. Weyl so concludes the discussion on this paradox: "Formalism regards this as an insoluble contradiction; but in reality this is a matter of scholasticism of the worst sort: for the slightest consideration shows that absolutely no sense can be attached to the question of whether the word "heterological" is itself auto- or heterological" (Weyl 1918, p. 6-7).

[41]There is a clear similarity here with Poincaré's insistence that the natural numbers (and in particular the principle of mathematical induction) are "synthetic a priori". See also the discussion in Chapter 6.

[42]The case of the natural numbers will be discussed in more detail in Section 5.4.2 below.

as definite. The reasons for this will be clearer when we discuss the case of the natural numbers in the next section. The important point to observe at this stage is that while the elements of the initial category are given in intuition, sets of them are presented as extensions of properties of the elements of the initial category. These considerations also determine the constraint on quantification, that is only allowed on elements of the initial category.

To ascend from the initial category to sets of its elements one considers primitive properties of and relations between the objects of the initial category. Corresponding to these one has judgements that assert that the properties (and relations) hold of the relevant objects. Weyl here considers also judgement schemes, that include variables that can be substituted by objects of the appropriate category to give rise to true or false judgements. Primitive properties and relations give rise to primitive judgements (or judgement schemes).[43] In Section 2, Chapter 1 of (Weyl 1918) the German mathematician delineates six "principles of the combination of judgements". These correspond to ways of combining judgements, starting from primitive ones, by simple operations that "define the logical functions of the concepts "not", "and", "or", and "there is" in an exact way" (Weyl 1918, p. 12). They also include a substitution rule for variables. The application of the six principles gives rise to new judgements that can then be used once more together with principles 1-6 to determine new ones, and so on. Essentially Weyl shows how to compose new judgements from given ones by means of iterated applications of the usual first order operations.[44] Crucially *quantification is restricted to the original initial category.*

[43]To these Weyl also adds identity judgements that state identities between objects of the category.

[44]As clarified at page 48 of (Weyl 1918), and further stressed by (Feferman 2000*b*), the roots of this analysis by Weyl are to be found in (Weyl 1910). There Weyl was prompted by the desire to clarify the concept of "definite property" used by Zermelo in his separation schema. As it is well known, the separation schema was introduced to constrain the comprehension principle that, unrestricted, had given rise to Russell's paradox. Zermelo formulated the separation schema so

The parallelism between properties on the one side and sets (their extensions) on the other allows for the formation of sets as extension of properties. Therefore Weyl offers a mathematically precise account of a predicative notion of set as extension of a property, one that differs radically from the arbitrary notion of set that was discussed in Section 5.1.1.

Indeed, the opposition to the notion of arbitrary set is a the heart of Weyl's criticism of the new form of mathematics. Weyl (1918, p. 23) writes:

> The notion that an infinite set is a "gathering" brought together by infinitely many individual arbitrary acts of selection, assembled and then surveyed as a whole by consciousness, is nonsensical; "inexhaustibility" is essential to the infinite. [...] Therefore I contrast the concept of set and function formulated here in an exact way with the *completely vague* concept of function which has become canonical in analysis since Dirichelet and, together with it, the prevailing concept of set.

The aim of (Weyl 1918) therefore is to give a secure, indubitable foundation for mathematics, one that proceeds genetically from the safety of an initial category to the notion of set of elements of that initial category, and so on. The mathematical process guarantees that the ensuing notion of set is free from vicious circles and therefore correct. Weyl is also very clear in stating that what can not be so accounted for in this way, needs to be relinquished.

In the next Section I wish to further explicate the mathematical process in the case of analysis that lays at the centre of Weyl's efforts in "Das Kontinuum", and

to allow for the formation of subsets of a given set which are definable by a "definite property". However, he did not clearly specify what counts as a definite property. As indicated by Feferman, Weyl (1910) clear analysis of the concept of "definite property" expresses it in terms of the first order set theoretic language of set theory. Feferman also observes that Weyl's analysis antedates the analogous one by Skolem, although the latter is usually credited for the fundamental clarification of this aspect of ZF set theory.

clarify the role within it of the natural number system as initial category.

5.4.2 The natural numbers structure

Notwithstanding the full generality of the first pages, Weyl at page 15 focuses on the case in which the initial category is that of the natural numbers: "the arithmetic of natural numbers supplies us with an example of a domain of individuals". The formation of judgements can then again be seen as governing the production of sets from the natural numbers by means of some "characteristic properties" that are obtained from primitive properties and relations by applying the first order logical operations (p. 20). As the quantifiers are now crucially restricted to the natural numbers, from a contemporary perspective Weyl's account can be seen as allowing for the definition of sets of natural numbers by means of *arithmetical comprehension* only (see page 99).

The importance of the natural number structure as an initial category in (Weyl 1918) should not be underestimated. As we saw at page 92, Russell left it deliberately open what constitutes the first level of his type theory, the individuals. At first also Weyl aims at a very general account of his mathematical process, and does not fully specify what constitutes an initial category of objects. Nevertheless, he soon clarifies that the natural number structure has a unique role within mathematics, as it is *presupposed in every mathematical discipline*, and *it is in any case an initial category.*

The comparison with Russell is again instructive, as Russell aimed at a *definition* of the natural number structure, to witness its logical nature; Weyl, however, in a similar vein as Poincaré before him, takes the natural numbers as starting point, as intuitively given (i.e. synthetic a priori). In fact, at page 48 Weyl stresses that any attempt at founding the concept of natural number on logic or on the concept of set is deceptive, given the fundamental role in mathematics of the concept of natural number.[45]

The natural number structure is characterised by Weyl in terms of the operation of successor and mathematical induction, the latter expressed in terms of a principle of iteration that allows us to progress from a property P holding of 0, to P holding of its successor, then P holding of the successor of the successor of 0, and so on, for all natural numbers.[46] Weyl (p. 37) clarifies that the principle of iteration "exploits the characteristic feature of the natural numbers, whose sequence is the general scheme of a procedure consisting in the iteration (endlessly repeated performance) of an elementary process." That is, the natural numbers are here seen as the fundamental example of an iterative process given by the repeated application of the successor function. As a consequence, we can reason inductively on the natural numbers.

To further clarify this point, Weyl explains that the natural numbers are "without exception, individuals" (Weyl 1918, p. 27). The concept of individual is elucidated by Weyl at page 15 in terms of the availability of a characteristic property that singles out the individual as unique. In modern terminology, one could express this in terms of the availability of a *canonical representation* that only that individual satisfies. In the case of the arithmetic of natural numbers, the fundamental relation that underlies this discipline is that of successor: it is its iteration that allows for the unique characterisations of each natural number as individual. For example, there is one and only one number that is the successor of no natural number, the number 0. Then 1 can be characterized as the unique successor of 0, and so on. Weyl expresses this with a "structuralist slogan":

> [...] it is impossible for a number to be given otherwise than through its position in the number sequence, i.e. by indicating its characteristic

[45] I shall further address this point in Chapter 6.

[46] See (Adams & Luo 2010) for a precise mathematical characterisation of the principle of iteration. See also (Cantini & Crosilla 2008) for a constructive set theory with operations that includes a primitive operation of iteration inspired by Weyl.

property" (Weyl 1918, p. 27).

Weyl's discussion suggests that the natural numbers are paradigmatic in virtue of the fact that starting from 0 the iteration of the successor operation allows us to *characterise uniquely each natural number in elementary terms, and by exclusive appeal to its predecessors.* Weyl in particular very clearly emphasises the role of induction in the characterization of the natural numbers: not only we need an initial element, 0, and a successor operation but also mathematical induction, to ensure that we iterate the successor operation sufficiently far to capture *all* the natural numbers.[47]

This is not the case, according to Weyl, for the real numbers, that require the notion of subset of the natural numbers for their mathematical treatment, and are therefore not individuals. Similarly as for Poincaré's notion of invariance, the mathematical process is a way of extending beyond the initial structure of the natural numbers by predicatively justified means (i.e., by arithmetical comprehension). In this respect Weyl's "Das Kontinuum" can be seen as contributing also to a precisification of Poincaré's requirement of predicative definability, by giving a detailed analysis of this notion in terms of arithmetical definability. According to the distinction between base and predicativity constraints that was introduced in Chapter 4, Section 4.5.1, Weyl clearly singles out as the base the natural number structure. The mathematical process, then, details the predicativity constraints, and counts as the fundamental tool that allows for an expansion from the base by the iteration of appropriately constrained operations, enabling us to characterise a predicative notion of set. On reflection, the principal ingredients of the notion of predicativity with which Weyl refines Poincaré's predicativity are: (1) the natural number structure, (2) simple arithmetical operations, corresponding to the application of arithmetical comprehension, and (3) the iteration of this process.[48]

[47]This point will be further discussed in some detail in Chapter 6.

[48]The iteration was then further extended within the logical analysis of predicativity to allow

As already hinted at, Weyl's mathematical process and its ensuing notion of set is the prerequisite for his reconstruction on secure grounds of fundamental portions of analysis. Predicativity then appears as a way of casting an *arithmetical eye* on the continuum.[49] The justification of Weyl's mathematical analysis of the continuum ultimately relies on the justification of the natural number structure (with induction), and the logical operations described by the mathematical process. These are the building blocks that are used to produce genetically a notion of set as extension of an arithmetical property.

Weyl's notion of set and its application in "Das Kontinuum" to analysis are a fundamental step in understanding how far this very limited conceptual apparatus can extend and which parts of ordinary mathematics their assumption can justify. The results obtained since in mathematical logic have further sharpened Weyl's analysis, as expounded in Chapter 4. There is another aspect that makes both Poincaré and Weyl's accounts particularly fascinating, the unexpected similarity with aspects of the notion of set that underlies Martin-Löf type theory. A thorough discussion of the latter is well beyond the aims of the present work. However, in the next section I should like to highlight some aspects that are particularly significant.[50]

5.5 Preludes to a constructive notion of set

A recurrent theme in the constructive mathematical literature is that for which a set is defined in terms of a *law* or a *rule* that specifies the elements of the set and

for iterations along the predicatively provable ordinals (see Section 4.2).

[49] Weyl expresses the conviction that we can not hope to gain exact mathematical understanding of the "intuitive continuum". The latter is the continuum as "given in the intuition of space", the arithmetical continuum is instead "constructed in a logical conceptual way" (p. 49). The exact mathematical (i.e. arithmetical) continuum represents as far as we can go in our attempt to give precise mathematical treatment of the intuitive continuum, in a way that is useful for science.

[50] See e.g. (Martin-Löf 1984, Nordström et al. 1990) for detailed treatments of type theory, and (Dybjer & Palmgren 2016, Crosilla 2006) for more elementary introductions.

their identity conditions. This view, for example, is presented in Bishop's (1967) monograph. According to Bishop a set can be specified by explaining:

- Existence: how to construct the elements of the set;
- Identity: when two elements of the set are equal.

In this thesis I have largely neglected the important issue of identity, due to space constraints. I wish, however, to briefly observe that the notion of set as extension of a definition or a property that emerges from Poincaré and Weyl's writings is very modern in this respect. This notion of set manifests, in fact, a form of intensionality at the level of definitions (or properties), as more than one definition (or property) may correspond to the same set. In particular, Weyl's mathematical process shows how to obtain sets as extensions of properties, and a set may be the extension of a number of properties. Weyl therefore introduces an extensionality criterion for sets: given properties P and P', we identify the extension of P and the extension of P', if and only if the same objects satisfy the properties P and P'.[51] This has surprising similarities to Martin-Löf type theory, that was introduced to formally capture Bishop's notion of set in (Bishop 1967). There are, in fact, intensional and extensional variants of constructive type theory (see, for example, Nordström et al. 1990). The first offers advantages for computer computation (like, for example, decidability of type checking), while the second facilitates the treatment of mathematical notions.[52] It is the intensional variant of type theory that has attracted most attention in very recent times, as the full potential of intensional equality has been realised and is currently being exploited within the Homotopy Type Theory project (Univalent Foundations Program 2013).

[51] See also footnote 37 at page 160.

[52] In fact, variants of intuitionistic type theory have been proposed that exploit both intensional and extensional components within a single formalism (Maietti & Sambin 2005, Maietti 2007, Maietti 2009).

As to the condition of existence, there is a similarity between Poincaré's notion of "genre" and the notion of category in (Martin-Löf 1984). At the core of constructive type theory (Martin-Löf 1984) there is a distinction between *categories* and *sets*. The fundamental difference between sets and categories lies in that the definition of a set is more exacting than that of a category. In particular, the definition of a set requires rules that specify step by step (i.e. inductively) the construction of the set in terms of its elements.[53] There is no such requirement for the definition of a category. Martin-Löf writes (Martin-Löf 1984, p. 21):

> A category is defined by explaining what an object of the category is and when two such objects are equal. A category need not be a set, since we can grasp what it means to be an object of a given category even without exhaustive rules for forming its objects.

In addition, (Martin-Löf 1984, p. 22):

> To define a category it is not necessary to prescribe how its objects are formed, but just to grasp what an (arbitrary) object of the category is. Each set determines a category, namely the category of elements of the set, but not conversely: for instance, the category of sets and the category of propositions are not sets, since we cannot describe how all their elements are formed.

It is tempting to see a similarity between Martin-Löf's categories and Poincaré's "genre", the general specification of a set that is short of a complete definition (see Section 5.3). As already clarified above, according to the predicativist, it does not suffice to give a general condition that is common to all the elements of the set, in particular in the case of infinite sets. We also need to offer a definition that allows for the "generation" of the elements of the set step by step. In the case of

[53] See Chapter 6 for an example of inductive definition in the particular case of the set of natural numbers.

type theory, the thought is that there is a crucial distinction between a set and a category. The first has a structure, can be pictured as constructed step by step from the elements according to a uniform rule. A category, instead, gathers together a collection of unstructured objects. Specifying a category suffices to recognise an object as belonging to the category, when prompted with one, but it does not offer a description (of a process of construction) of all of its elements. Martin-Löf also insists that a set and not a category is to be thought as a domain of quantification.

Another observation relates to Weyl's mathematical process. This may be seen as making precise sense of Poincaré's discussion on invariant or complete definitions. In fact, Weyl's process bears surprising similarities with the inductive specification of the sets in Martin-Löf type theory. An important difference is that in type theory one has more than the natural numbers as starting point, as one allows for generalised inductive definitions (see Chapter 4, Section 4.4.2.) In addition, the process of construction itself is more general, as it employs so-called dependent types.[54] However, the overall set-up has remarkable similarities with Weyl's strategy. In type theory we start from some initial (inductively defined) set, among which are the natural numbers. We then apply the logical operations to obtain complex propositions (that are essentially a generalization of Weyl's properties). As mentioned in Chapter 4, Section 4.4.2, the Curry-Howard correspondence is in fact an isomorphism in type theory, which enables an identification of propositions with sets.[55] The correspondence that Weyl saw between properties and sets, that enables us to see sets as extensions of properties, is therefore here strengthened to an identifi-

[54]Another important difference lies in the fact that Weyl considers the natural numbers as individuals, fully determinate, and thus admits classical logic, while Martin-Löf type theory is intuitionistic.

[55]See Chapter 1, Section 1.2.2 for the Curry-Howard correspondence. Note that in Chapter 1 I have formulated the Curry-Howard correspondence in terms of formulas and types. In the case of Martin-Löf type theory we need to speak of propositions and sets. The underlying idea of the correspondence is unchanged.

cation of these two levels. The significant fact is that this identification is at the heart of the computational reading of the notion of set in type theory. The availability of an inductive description makes a set in type theory a datatype in a very general programming language. We can program with sets. These observations are indicative that the notion of set as extension of a predicate better complies with the computational vocation of type theory than the abstract notion of arbitrary set that underpins ZFC set theory. The availability of a step by step description of the "construction" of a set makes it computational and ideal for programming.

5.6 Conclusion

In this chapter I have analysed the opposition between two alternative conceptions of set: set as arbitrary and set as extension of a predicate. An arbitrary set may be seen as a collection of elements brought together independently of a law or a rule. A set as extension of a predicate is characterised by a tie with a predicate (or a definition or a property) which determines its extension step by step. It is the latter notion of set that progresses through the predicative tradition and intersects with the attempts by Poincaré and Weyl to clarify the notion of predicativity and the nature of predicative analysis, respectively.

In this chapter, I have recalled Poincaré's notion of set as extension of a definition, and its role in clarifying the characterisation of predicativity in terms of invariance. The latter is crucially appealed to in order to ensure the stability of the extension of definitions, that is particularly needed whenever a set is used as domain of quantification in defining new mathematical entities. Poincaré gestures towards a genetic construction of sets that anticipates in a number of respects today's constructive notion of set as exemplified in Martin-Löf type theory.

Weyl's mathematical process is an exemplary attempt at clarifying in full detail a notion of arithmetical set, and exploring how far the assumption of the natural number structure and simple logical operations can bring us. A fundamental contribution by Weyl is the explicit recognition of the paradigmatic role of the natural number structure within mathematics.[56] Therefore with Weyl we have the official birth of predicativity given the natural numbers. Weyl's mathematical process establishes a parallelism between properties and sets, and engenders an inductive structure at the level of properties that is then mirrored by their extensions, the sets. This has remarkable similarities with the Curry-Howard correspondence that is at the heart of Martin-Löf type theory.

[56]Weyl's intuitions have been largely confirmed by the results in mathematical logic discussed in Chapter 4, Section 4.3, as substantial portions of ordinary mathematics can be reduced to arithmetic.

Chapter 6

The natural number structure and strict predicativity

In Chapter 5 I have highlighted the fundamental role of the natural number structure within the predicativism of Poincaré and Weyl. As further discussed in Section 6.1, for these authors the natural number set equipped with the principle of induction is at the centre of all of mathematics. In fact, as detailed in Chapter 5 an appeal to this structure together with simple logical operations enables for the articulation of a predicative notion of set whose principal aim is to offer a secure and uncontroversial foundation for mathematics (Weyl 1918). This comes at a price, as restrictions need to be introduced at the level of those sets that are beyond the natural number structure itself, therefore affecting the notion of power set of the natural numbers. The latter is at the heart of analysis and therefore a new, more careful development of analysis had to be carried out. In fact, Weyl (1918) clearly states that whatever can not be accounted for in this way ought to be renounced.

In the present Chapter I shall consider proposals that appeal to predicative themes to suggest an even more radical perspective, as they question the very natural number structure. One way of motivating these positions is by requiring that predicativity constraints ought to be applied all the way through. The contention is that if they are so applied, then the principle of induction, that is essentially required for the development of arithmetic, comes into question. The conclusion is that we can offer no non–circular justification of the whole natural number structure. Therefore, if we wish to fully comply with predicativity we ought to introduce restrictions already at the level of the natural number structure itself.

We can then schematically distinguish two possible predicativist strategies with respect to the natural number structure. We might accept the natural number structure in its entirety and adopt predicativity constraints starting from the subset of the set of natural numbers. Or we may object to the very natural number structure and rely on an appropriate substructure of it to deliver a predicative form of mathematics. According to the terminology introduced in Chapter 4 these two strategies may be distinguished by the different bases their proponents are ready to accept.[1]

The first strategy may be motivated by a number of reasons, like, for example, Poincaré and Weyl's contention that the natural number structure is synthetic a priori. Appropriate argumentation will need to be offered in this case to clarify why exactly the natural number structure, and no less and no more, should be accepted as our base. In Chapter 7 I shall explore another possible way of arguing for the assumption of the natural number structure as base. This arises if we claim that this structure is necessary in order to develop any reasonable portion of mathematics. An argument along these lines seems particularly persuasive if it relies on considerations of which form of mathematics is necessary for science. A justification of the choice of the whole natural number structure as base for a form of predicativity would require therefore sufficient evidence that exactly the natural number structure is necessary for the development of our best scientific theories. In Chapter 7 I shall

[1]In this chapter I shall be primarily concerned with strict predicativism, that arises by adopting the second strategy. Predicativism given the natural numbers and constructive predicativism may be seen as implementing strategies of the second kind. In the following, for simplicity, I shall refer to the first of these two options.

argue that more work is required to fully assess the strength of “minimal” theories for the development of all the mathematics that is required for science, so that it is unclear, so far, what the outcome of this strategy would be.

In the rest of this Chapter I shall consider positions that arise if we object to the natural number structure. I shall consider two (non-exclusive) approaches. The first is related to the desire to offer an absolutely correct and safe form of mathematics. The thought here is that all but the most trivial assumptions ought to be considered with suspicion in building up a correct or safe form of mathematics. The claim is that the principle of induction is also to be placed under scrutiny, as it does not seem to be among those most trivial assumptions. Therefore, we ought to offer a clarification or a justification of this principle. The contention is that this can not be done without giving rise to a form of vicious circularity: we seem to require the assumption of induction in any attempt to justify it. The consequence of this strategy is that it imposes the application of predicativity constraints all the way through, thus without exempting the natural number structure itself. This gives rise to forms of mathematics that introduce severe restriction to mathematical induction, as discussed in Section 6.2 below.

Another route also brings an objection to the assumption of the natural numbers as base. Here the thought is that the natural number concept requires an explanation, and that the latter has to be non-circular (Section 6.6). One could in fact claim that the natural numbers play a fundamental role within mathematics as a basic domain of quantification. As a consequence, the concept of natural number requires a genuine clarification that avoids any reference to this very domain; however, it is argued, any attempt at explaining it results in circularity. The perhaps surprising conclusion that is drawn here is not that we ought to impose predicativity constraints all the way through, but that we ought to change our conception of quantification domain, shifting to intuitionistic logic. This strategy will be the focus of Section 6.6.

In the following I shall first of all examine Poincaré and Weyl's objections to any attempts to justify induction (Section 6.1). I shall then examine Nelson's criticism of the induction principle in (Nelson 1986) (Section 6.3). This will be the starting point for an analysis of a similar complaint by Parsons (Section 6.4). I shall also consider the role of induction in the inductive definition of natural numbers (Section 6.4.2), and then examine Nelson's thought, once more, to clarify with an example the role of induction in the definition of natural numbers (Section 6.5). In the final Section 6.6, I shall explore ideas suggested by Dummett's discussion on indefinitely extensible concepts (Dummett 1991, Dummett 1993), and see which consequences one might wish to draw from them from a predicativist perspective.

To conclude, I shall suggest the following thought: strict predicativism can be seen as lying at the "intersection" between predicativity given the natural numbers and constructive predicativity. I shall suggest that *if*, as argued by Nelson and Parsons, impredicativity is found to affect the natural number structure and *if* this is seen as problematic, then one has two options: (i) maintain a classical view of universal quantification and abide to strict predicativism, or (ii) give an intuitionistic reading of universal quantification and proceed to a more encompassing form of mathematics.

6.1 Circularity and Mathematical induction

The natural number structure plays a fundamental role within the predicativism of Poincaré and Weyl. A crucial component of the natural number structure is the principle of mathematical induction, or, simply, induction. This allows one to reason inductively on the natural numbers: if we can show that a property holds of 0, and that whenever it holds of a natural number, n, it also holds of its successor, then we can conclude that that property holds of *every* natural number. In modern logical terminology, within the familiar Peano Arithmetic (**PA**), induction reads as follows:

$$[\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(Suc(x)))] \rightarrow \forall x \varphi(x),$$

where φ is an *arbitrary* formula in the language of **PA**, and $Suc(x)$ is the successor of x.

Weyl is very clear in stating that the principle of induction is a fundamental component of the natural number structure. From a contemporary perspective Weyl (1918, p. 37) appears to suggest that the validity of induction is due to the fact that the natural numbers are a prototypical inductively defined structure: they can be pictured as a sequence produced through an iterated application of the successor operation.[2] That is, Weyl states very perspicuously that it is in virtue of the fact that the natural numbers can be seen as if they were produced by repeated applications of the successor operation that we can reason by induction on them. In (Weyl 1918) induction is expressed in terms of a principle of iteration that allows us to argue inductively on the natural numbers.[3] Echoing Poincaré, who claimed that the mathematical induction principle is synthetic a priori, Weyl (1918, p. 48) highlights that iteration is a form of *pure intuition*; in fact, this explains its fundamental role within mathematics: "*the idea of iteration, i.e., of the sequence of the natural numbers, is an ultimate foundation of mathematical thought*". When setting out his mathematical process Weyl claims that the natural numbers are not only the primitive category that is required in order to justify analysis, but they are "presupposed in every mathematical discipline" (Weyl 1918, p. 25). Weyl then observes that even in disciplines such as geometry, analysis, group theory, and so on, "the natural numbers are, from the start, related to the objects under consideration." (Weyl 1918, p. 25) In the particular case in which the natural numbers are the only initial category, "then we arrive at *pure number theory*, which forms the centerpiece of mathematics; its concepts and results are clearly of significance

[2] In Section 6.4.2 I shall expound in detail the inductive definition of the natural numbers.

[3] See also Chapter 5, Section 5.4.2.

for *every* mathematical discipline." (Weyl 1918, p. 25)

As discussed in Chapter 5, Section 5.4, in "Das Kontinuum" Weyl shows how to build a *safe* and unshakable notion of set from the natural numbers by means of "characteristic properties" that are obtained from primitive properties and relations of the natural numbers, that is, by applying arithmetical comprehension. Weyl therefore condemns Dedekind's attempts to reduce the concept of natural number to that of set as *viciously circular*, since, from Weyl's perspective, the intuition of iteration is required in order to grasp the very concept of set. In this respect Weyl claims to be in full agreement with Poincaré, and writes (Weyl 1918, p. 48):

> "*the idea of iteration, i.e. of the sequence of the natural numbers, is an ultimate foundation of mathematical thought.* For if it is true that the basic concepts of set theory can be grasped only through this "pure" intuition, it is unnecessary and deceptive to turn around then and offer a set-theoretic foundation for the concept "natural numbers."

Before Weyl, in a number of writings Poincaré objected to various attempts to reduce the natural number structure to more fundamental concepts (see, for example, (Poincaré 1906)).[4] Poincaré observed that in order to either prove the principle of induction, or show that the definition of the natural number structure is non–contradictory, one needs to *assume* the very principle of induction, thus resulting in a vicious circularity. One of Poincaré's main targets is Russell. In Chapter 3, Section 3.4.1, I have mentioned the treatment of the natural numbers in Russell (1908); there, in an attempt to establish the logicality of the concept of natural number, Russell adopted Frege's definition (see Chapter 3, Example 1). To impose predictivity constraints and block a variety of paradoxes, Russell (1908) introduced

[4] A discussion of the role of intuition in Poincaré is beyond the remit of this thesis. However, it is important to note that intuition (as, for example, manifested in the principle of induction) plays a crucial role in Poincaré's philosophy of mathematics, as it ensures creativity in mathematics, the worry being that application of logic alone would reduce mathematics to mere calculation.

ramified type theory (see Chapter 3, Section 3.4). This, however, makes the mathematics awkward, in particular giving rise to natural numbers of different orders and preventing proofs by induction of simple arithmetical facts. When developing his ramified type system, Russell felt compelled to introduce the axiom of reducibility to allow for proofs by induction of the necessary generality. Poincaré was very dismissive of this solution, a crucial objection to Russell's strategy being that it introduces a new postulate, reducibility, that is less evident than the principle that it is supposed to clarify.

For Poincaré and Weyl, the above considerations enforced the *assumption of mathematical induction*, since any attempt to prove or reduce it ought to presuppose it already, or, indeed, require to resort to dubious postulates. In addition, as promptly recognized by all parties, including Russell, mathematical induction is essential for the development of even elementary parts of arithmetic (see Chapter 3, Section 3.4.1, and Section 6.5 below). Therefore in (Weyl 1918) the natural number structure in its entirety, with full mathematical induction, was accepted as a starting point, and predicative restrictions were introduced at the next level of mathematical idealization, affecting the concept of property of the natural numbers (and hence that of set of natural numbers).

Poincaré and Weyl's charge of vicious circularity in any attempt to justify induction has seen new life in recent times, as a number of authors have denounced the *impredicativity of induction*. If induction is impredicative, this will certainly substantiate the difficulties noticed by Poincaré and Weyl in attempts to explain induction away.

6.2 Worries about mathematical induction: strict predicativism

As noted in Chapter 4, Section 4.4.1, Nelson (1986) has objected to the assumption of the natural number structure on the grounds that it is already impredicative. Nelson (1986) consequently set up a form of "predicative arithmetic" that drastically restricts the principle of induction on the natural numbers, and, consequently, the realm of justified mathematics.[5]

Parsons has also maintained the impredicative character of the natural number structure with full induction (Parsons 1992, Parsons 2008).[6]

[5]Interpretability in Robinson's system Q seems to be the criteria for predicativity that Nelson applies in his Predicative Arithmetic (Nelson 1986). System Q is a weak subsystem of **PA** with axioms for successor, addition and multiplication, but without induction. The views expressed in (Nelson 1986) then justify up to weak subsystems of **PA** that allow for forms of *bounded* induction (see e.g. (Buss 1986)). It has been argued that these systems capture the concept of feasibility (Buss 1986). As further discussed below, a distinctive characteristic of Nelson's predicativity is that it does not countenance the totality of the exponentiation function.

[6]Parsons' notion of strict predicativity is less precisely formulated than Nelson's. Notwithstanding the similarity of arguments, there is a substantial difference of outcomes between Parsons and Nelson's approaches to strict predicativity. Parsons, in fact, seems persuaded that the totality of exponentiation can be justified from a strictly predicative perspective. To corroborate this conviction, Parsons refers to recent work that addresses the strength of predicative versions of Frege's arithmetic. The literature on this topic is very rich. See, for example, (Heck 1996, Burgess & Hazen 1998, Ferreira & Wehmeier 2002, Linnebo 2004, Burgess 2005, Visser 2009). That research points towards systems that are substantially weaker than primitive recursive arithmetic, but stronger than Nelson's, in particular they do account for the totality of exponentiation (Visser 2009). My understanding is that Parsons believes that more work needs to be carried out to fully clarify the extent of strict predicativity. He seems however inclined to accept at least elementary arithmetic as justified from a strict predicative perspective. Elementary arithmetic is a subsystem of Peano Arithmetic that is weaker than primitive recursive arithmetic, but stronger than Nelson's predicative arithmetic. It is a very robust and a well understood system (see also

Nelson and Parsons' forms of predicativity give rise to different forms of mathematics that lie below primitive recursive arithmetic. According to an influential analysis by Tait (1981), the latter allows for the formalization of finitary reasoning. Therefore the above mentioned forms of predicativity have also been called strict predicativity, as they give rise to a more constrained perspective than finitism. In fact, Nelson's predicativism strongly resembles a form of strict finitism. Both Nelson and Parsons contend that a thorough avoidance of impredicativity forces us to accept as justified only a fragment of finitist arithmetic.

In the following I shall examine Nelson and Parsons' objections to the induction principle. However, before doing so, it is necessary to consider a natural objection. If the adoption of predicativity constraints all the way through brings us to a form of strict finitism, then one might worry that the resulting mathematics would be far too weak to be of any interest at all. This would then make the unrestricted adoption of predicativity constraints simply implausible and force us to argue for a more reasonable (and generous) starting base (e.g. the whole natural number structure). In reply one might observe that a thriving field at the intersection between mathematical logic and computer science has emerged in the last few decades, that requires very weak systems to investigate phenomena related to computational complexity (see e.g. Dean 2016). As a consequence strict predicative mathematics is highly interesting. This observation, however, is unlikely to dispel the worry, if the resulting mathematics does not have sufficient breadth to account for the applicability of mathematics to science. As further examined in Chapter 7, substantial work is needed to clarify which form of predicativism, if any, is sufficient for our best scientific theories. In particular, Ye (2011) has argued that perhaps already systems that can be considered within the realm of strict predicativism (according

Chapter 7). If Parsons' strict predicativity allows for elementary arithmetic, then it is sufficient to justify the form of arithmetic that has been used by Ye (2011) to develop substantial parts of contemporary analysis (see Chapter 7).

to Parsons) might suffice for expressing all scientifically applicable mathematics. In other terms, we can not a priori rule out the adequacy for science of some version of strict predicativism, and this makes this kind of objection, so far, difficult to press.

6.3 Nelson's criticism of mathematical induction

Nelson's motivation for his predicative arithmetic is briefly announced in less than two pages at the opening of (Nelson 1986), and then expanded on by further comments scattered through out the book. While the technical work which Nelson has contributed to promoting is currently enjoying attention within mathematics and computer science, his motivation has often caused puzzlement and perplexity.

Nelson (1986) presents his main objections to the "impredicativity of induction" in the first two pages of his book. In a rather dense paragraph he writes:

> The reason for mistrusting the induction principle is that it involves an impredicative concept of number. It is not correct to argue that induction only involves the numbers from 0 to n; the property of n being established may be a formula with bound variables that are thought of as ranging over all numbers. That is, the induction principle assumes that the natural number system is given. A number is conceived to be an object satisfying every inductive formula; for a particular inductive formula, therefore, the bound variables are conceived to range over objects satisfying every inductive formula, including the one in question.

To ease an understanding of Nelson's thought it might be useful to separate the following claims.

1. There is a difficulty with the principle of induction due to its involving an impredicative concept of number.

2. The natural numbers are conceived of as those objects that satisfy every inductive formula.[7]

3. In some cases of application (use) of the induction principle the formula on which we carry out the induction contains unbounded quantifiers, that is, quantifiers that range over *all* the natural numbers.

4. Suppose that one has a proof by induction on a formula A that contains unbounded quantifiers, as in 3. By 2, the bound variables in A are thought of as ranging over objects that satisfy every inductive formula. A is one of them.

5. Therefore the bound variables in A are thought of as ranging over objects that satisfy A.

Nelson stops here, but I imagine the conclusion he draws is that this is an unbearable form of circularity.

Even in this more systematic form, Nelson's criticism of induction is far from clear. There is evidently a form of circularity involved in uses of induction, but what is precisely the problem with it?

Reading Nelson we might be tempted to find in his discussion the old objection to the *Fregean definition* of natural numbers that was discussed in Chapter 3, Section 1. Statements 1 and 2 in my reconstruction of his thought may suggest this interpretation. In Chapter 3, Section 1, within a second order context, the Fregean definition of natural number was so formulated:

$$N(n) := \forall F[F(0) \land \forall x(F(x) \to F(Suc(x))) \to F(n)],$$

where F is a predicate variable.[8]

[7] Note that this is a rendering of the familiar Fregean definition of natural numbers, that was discussed in Chapter 3, Example 1.

[8] Nelson works within the *first order* context of **PA**, and thus presents the definition of natural number as follows: n is a natural number if it satisfies every inductive formula. We say that φ

From a logical perspective, the impredicativity of this definition is due to the fact that we have an unrestricted quantification at the **second order**. This is given by the quantifier $\forall F$ at the front of the expression, that ranges over predicates (standing for properties of natural numbers). A worry here is that for this quantifier to make sense we need to already have available as a set (a totality) the collection of all properties of the natural numbers, including N. This seems a justified worry and points 1 and 2 in my reconstruction of Nelson's thought (page 182) suggest that this might be a worry for Nelson, too. However, the second order form of the definition of natural number does not seem to be the principal worry in Nelson's text, as it would not directly explain his insistence on the impredicativity of *induction* in case of unrestricted quantifiers.[9]

More precisely, Nelson's text seems to convey a worry related to the occurrence of unrestricted quantifiers in *specific uses* of induction (see points 3–5 in my reconstruction at page 182). Recall **PA**'s formulation of induction at page 177. Since the induction principle introduces no constraint on the complexity of the formula φ,

is inductive if the following holds: φ is satisfied by 0 and it is satisfied by the successor of x, whenever it is satisfied by x. A dissimilarity between **PA**'s formulation of induction and Frege's definition of natural number is that the first is schematic, in that, unlike the Fregean definition above, it does not explicitly quantify on formulas at the second order. One then might worry that the conclusions I draw in the following on Frege's definition will not necessarily apply to Nelson's context. The issue of the relation between **PA**'s formulation of induction and the second order version of it (where we have a second order quantifier over formulas) is of relevance to this debate, but will be presently omitted (see e.g. Isaacson 1987). The principal reason for this omission is that in subsequent parts of this Chapter I shall consider the claim that already a not explicitly second order form of induction involves impredicativity (Section 6.4.2), so that my analysis applies generally.

[9] As further discussed below, it is also better if the main complaint does not depend on particular syntactic features of the definition of natural numbers. In fact, if Nelson's worry were only the second order form of the Fregean definition of natural number, then if we could offer another, not explicitly second order definition of the natural numbers, this might suffice to eliminate all concerns (see Section 6.4).

one might apply induction to a formula φ that contains unbounded (or unrestricted) *number* quantifiers; for example φ could be: $\forall n \chi(n)$ or $\exists n \psi(n)$ (where χ and ψ may contain further number quantifiers). Note that as Nelson's context is the first order theory **PA**, the concern in this case is the range of quantifiers that are **first order**: quantifiers in **PA** range only over natural numbers, not over sets or properties of them. Here the issue seems to be that in *some particular uses* of the induction principle in which the formula we induct on has *unrestricted quantifiers*, the natural number structure needs to be *presupposed* in order to act as domain of quantification. That is, the natural numbers need to be assumed as *given* in their totality prior to those applications of the induction principle. This seems to be the content of point 3 in my reconstruction at page 182. However, *if* we furthermore suppose that induction plays a role in the *definition* of the natural numbers (points 1–2), we have an apparent circularity: we need induction to define the natural numbers, but we need the set of natural numbers to act as domain of quantification in those instances of induction that are needed for the definition of the concept of natural number.

If my interpretation of Nelson's discussion is correct, the charge of vicious circularity relates to *uses* of induction. One might find this problematic, since the induction principle is typically appealed to as a principle of *proof*: to prove that a property holds of *all* the natural numbers. It is therefore unclear what relation this bears with impredicativity as a characteristic of *definitions*.

In the following Sections 6.4–6.5, I shall address the question of whether induction plays a *"definitional role"*, that is, whether it contributes in an essential way to the *definition* of the concept of natural number. To anticipate the conclusion of my analysis (Section 6.5), I shall observe that *if* the whole natural number structure is already available, then the principle of induction is a principle of proof. However, from a perspective that does not assume the natural numbers as given, the uses of induction alluded to by Nelson can be seen as performing also a definitional role.

Induction, in fact, allows us to prove that certain numbers can be computed (or that certain expressions can be regarded as natural numbers), therefore also determining the extension of the natural number concept.

6.4 Parsons and roles of induction

To clarify the "definitional" role of induction I shall first of all discuss ideas brought forward by Parsons in (Parsons 1992, Parsons 2008) that closely resemble Nelson's charge of impredicativity of induction.

It is important to note that Parsons' discussion presupposes a form of the notion of set as extension of a predicate that was discussed in Chapter 5. This point will be further discussed below.[10]

One possible concern with Nelson's objection to the impredicativity of induction is that it presupposes a crucial role for induction within the definition of the natural numbers. Such a role is clear in the case of the Fregean definition of the concept of

[10] At a later stage in this chapter I shall also make use of the notion of set as extension of a concept, as this is the notion that is used by Dummett (1993) (and also alluded to by Nelson (1986)). In the previous Chapter 5 I have argued that the notions of set appealed to by Poincaré and Weyl can be seen as particular precisifications of Parsons' notion of set as extension of a predicate, and I have endeavoured to clarify Weyl's own way of specifying this notion. Prima facie it would seem that depending on how we frame concepts, the notions of set as extension of a concept and set as extension of a predicate may diverge in considerable respects. In particular, it seems possible to frame concepts so to give rise to a notion of set as extension of a concept that does not forbid impredicativity. This is for example hinted at by Gödel (1944) and also by Carnap (1931) (where it is attributed to Ramsey). These considerations suggest to carefully distinguish these two notions. However, for the present analysis the most important aspect that I wish to highlight is that both notions introduce a tie between a concept or a predicate on the one side, and its corresponding set on the other. In other terms, for the present discussion the crucial aspect I am interested in, is the assumption that the predicate or the concept are prior to their extension. The different requirements that are imposed by Parsons on the one side, and Dummett on the other on the respective notion of set will become clear as we proceed in the discussion.

natural number, since, as noted by Parsons (1992, p. 139), that definition in fact reduces mathematical induction to a definition of the predicate 'natural number'. But *can we assume the same role for induction in other definitions as well?* Parsons (1992) addresses this question. According to Parsons (1992, p. 139), who cites Dummett (1963) for rising a similar point, the impredicativity of induction that we observe in the Fregean definition "remains if induction is treated not as a definition but as integral to an informal explanation of the predicate "natural number", even if second order logic may not be used, for example if we think of the introduction of N as an inductive definition." I shall discuss in detail the inductive definition of the natural numbers in Section 6.4.2 below. For now it suffices to mention that an inductive definition of a set may be seen as describing the construction of a set *step by step*, by appeal to a finite number of rules. The thought is that an explanation of the natural number predicate that is effected by means of an inductive definition also requires induction to fully determine the extension of that predicate.

Parsons' aim is to dispel a worry: if we could offer a definition of natural number that, contrary to the Fregean one, is not expressed by an appeal to second order logic, then perhaps also the difficulties with the alleged impredicativity of induction might subside. The strategy in (Parsons 1992) is to claim that the impredicativity of the Fregean definition is due to ***uses** of induction*: impredicative applications of induction are essential to develop even elementary arithmetic. As a consequence, the Fregean definition is not only prima facie impredicative, but we need to make use of that impredicativity. Once we have clarified this point, then the argument for the impredicativity of induction *will also apply to any other definition of the natural numbers that involves a similar role for induction.* Parsons (1992) therefore extends his complaint to other *explications* of the natural number predicate as, for example, its inductive definition, as, he argues, it will have to make a similar appeal to induction and therefore will be plagued with the same difficulties.[11]

[11]Parsons distinguishes between an explicit definition and an explication of the predicate "natural

In order to assess Parsons' argument we need to clarify if:

1. there are uses of induction that make the concept of natural number impredicative, as also claimed by Nelson;

2. *other* definitions, beyond the Fregean definition of natural number, require an appeal to uses of induction of the kind mentioned at point 1.

Before addressing these points, it is important to recall which conclusions Parsons wishes to draw from the alleged impredicativity of induction. Parsons (1992, p. 139) writes: "The observation that the notion of natural number is already impredicative [...] seriously weakens the case for the claim, deriving from Poincaré, that impredicativity is a sign of a vicious circle and altogether to be avoided." Therefore in that article Parsons does not argue for strict predicativism.

6.4.1 Defining the natural numbers

I shall consider examples of problematic uses of induction in the next Section 6.5. In this Section I shall be concerned with the second point above, Parsons' claim that

number". Frege's is an explicit definition of that predicate, as it is of the form: $N(n) = \varphi(n)$, with $\varphi(n)$ a second order formula that defines $N(n)$. However, Parsons suggests that e.g. the inductive "definition" of natural number (to be discussed shortly) may not be regarded as a definition of the natural number predicate, but, rather, an explication of it. For this reason he phrases his argument as applying to any *explication* of the natural number predicate. The issue of the status of the inductive "definition" of natural number is very interesting, and would deserve further thought. As discussed in Chapter 5 the role of definitions, and in particular of explicit definitions, in mathematics is a topic that is intimately related with the debate on impredicativity, as already intuited by Poincaré. However, I shall avoid discussing this issue in this thesis. In the following I shall also term the inductive definition of the natural numbers a definition, in agreement with current mathematical terminology. Mathematical logicians see it as a particular case of what is known as an"inductive definition". The practice of calling this a definition is due to the fact that the rules uniquely determine the natural number *set* (up to a relevant equality relation, e.g. extensionality) and this is usually considered sufficient for the mathematical practice.

other explications of the natural number predicate involve a problematic circularity. In particular, I shall expound in some detail the inductive definition of the natural numbers to which Parsons refers in (Parsons 1992) and discuss within this context the charge of impredicativity of induction.

It is important to note that this is short of what would be required to establish the impredicativity of the natural number concept: that would necessitate an argument to the effect that we can not offer a strictly predicative definition of natural number. Parsons (1992) claims that both the Fregean and the inductive definitions of the natural numbers are impredicative due to the role of induction in those definitions. In addition, he addresses an alternative "set-theoretic" definition of natural number proposed by George (1987) and dismisses it as it shifts to the notion of finite set essentially the same difficulty with induction. This clearly leaves open the possibility that other definitions of natural number may be offered that may be shown not to incur in the same difficulties. In (Parsons 2008, p. 295) the author claims that "certain explanations of the predicate "natural number" [...] are impredicative." He also adds that "it might be claimed in addition that the explanations in question are the philosophically most attractive ones, or even that no explanation is in sight that is not impredicative." He observes that the latter claim represents his position in (Parsons 1992). In Section 50 of (Parsons 2008) the author expands on (Parsons 1992) and discusses other proposals that might be seen as offering a predicative justification of the natural numbers like, for example, further proposals that take as primitive a concept of finite set, instead of one of natural number.[12] This issue is complex and, as further indicated in Section 6.5.2, I believe that its assessment requires a careful analysis of strict predicativism, to be carried out not

[12]In particular Parsons discusses a proposal by Feferman and Hellman (Feferman & Hellman 1995, Feferman & Hellman 2000). An assessment of this proposal goes beyond the aims of this Chapter, and is particularly complex, as Feferman and Hellman do not specify with sufficient clarity the notion of predicativity they adopt. It is therefore difficult to fully assess their criticism of Parsons' claims.

only from a conceptual but also from a technical perspective. This ought first of all set out a notion of strict predicativity and determine its limits, in a similar way as in the case of the logical analysis of predicativity given the natural numbers (see Chapter 4). Without a precise characterisation of strict predicativity and without such an analysis it is difficult to make any significant progress on the question whether alternative explanations of the natural numbers could be considered strictly predicative.

For these reasons, in the following I shall not attempt to either argue for or even assess the stronger thesis according to which any explication of the concept of natural number does involve a form of circularity of the kind discussed in this Chapter. I shall be content by scrutinizing the weaker thesis that some explanations of the concept of natural number manifest a form of circularity. In addition, I shall consider the claim that the inductive definition of natural number is one such. I shall also suggest that this definition is particularly apt to a predicativist and constructivist perspective. Therefore, unless another, strictly predicative and equally appealing definition of natural number is provided, a predicativist that does not offer independent arguments for the exemption of the natural number structure from predicativity constraints, would have to either adopt strict predicativity or, as further discussed below, perhaps, move to intuitionistic logic (Section 6.6).

6.4.2 The inductive definition of the natural numbers

To clarify what Parsons (1992) terms the inductive definition of natural number, it might be useful to first present a different, more typical definition.[13] In a set-theoretic context, the set of natural numbers is introduced by postulating an axiom of infinity. A common way of presenting the axiom of infinity is by stating that *a least inductive set exists.* A set A is inductive if it contains 0 and it is closed with

[13] For an introduction to inductive definitions see e.g. (Aczel 1977).

respect to the successor operation: if $n \in A$ then also $suc(n) \in A$, where $suc(n)$ is $n \cup \{n\}$.[14] Induction clearly follows from this definition, as if a predicate holds of 0 and it holds of $suc(n)$ whenever it holds of n, then its extension is inductive. As the set of natural numbers is the least inductive set, it is a subset of the extension of such a predicate, that therefore applies to every natural number.

Note that the definition above has two components: first one considers all sets that include 0 and are closed with respect to the successor operation, secondly one adds that the natural number set is the smallest such set. The second condition is required to weed out all those "things" that might also belong to inductive sets, but that are not natural numbers.[15] This definition seems to presuppose the pre-existence of the set-theoretic universe from which we select out of all sets those that satisfy certain conditions.

In the following I shall consider a different perspective, one that does not appeal to a previously given domain of natural numbers, or, more generally, of sets. That is, I shall consider a perspective that arises if we suppose that we might not yet have the natural numbers available to us. The natural numbers can instead be viewed as if they were the result of a process of construction, starting from 0 and taking its successor, repeatedly. Similarly as in Weyl's mathematical process, it is as if the mathematical entities were produced starting from some initial object(s) and by iteration of some elementary operations.[16] The advantage of taking this perspective is that we now can clarify what is required for obtaining the natural

[14]This is a natural set-theoretic rendering of the Fregean definition of natural numbers that was discussed above. Such a definition of the natural numbers is termed "Pair Down" by George & Velleman (1998), to contrast it with the more constructive, inductive or "Build Up" definition to be considered below. George & Velleman (1998) offer a very clear description of the difference between the standard set-theoretic definition of natural number and the inductive one.

[15]A celebrated example of non-natural number that we might wish to weed out from inductive sets is Caesar, should he happen to belong to an inductive set.

[16]See Chapter 5, Section 5.4. The crucial difference with Weyl is that he also assumed the natural numbers as given.

number structure and, especially, the role within it of the principle of induction.[17] If we assume this perspective, then a natural ally is the notion of set as extension of a predicate that was discussed in the previous chapter, as the predicate can be taken as describing a procedure that step by step "constructs" the natural numbers.[18]

From this perspective definitions as the one at page 190 are problematic. In fact, the very metaphor of weeding out those sets that are "too large" becomes empty: we do not have available at the start sets from which to eliminate any non-natural numbers. If we wish to give a definition of natural number that is in keeping with

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[17] The question of what is the related metaphysical picture underlying this perspective and whether talk of construction processes ought to be understood literally or only metaphorically is very interesting; however it can not be addressed here. My aim is to clarify which conceptual presuppositions underlie the natural number structure and, in particular, what is the role of the principle of mathematical induction within it. If we start from a definition of the natural numbers as the one at page 190, than the role of the principle of induction is obscured by the fact that it already holds as a consequence of that very definition. We therefore need to take a different perspective, and define the numbers as if they were not already available. One might subsequently wish to explore a number of philosophical options that might further substantiate this picture.

[18] Another reason for assuming a perspective that does not presupposes the natural numbers as given is offered by Parsons. Parsons in (Parsons 1992, Parsons 2008) argues for a structuralist position that does not "give individual identities to the objects playing the role of $0, 1, 2$, etc; therefore there should be no unique answer to the question from what domain the natural numbers are picked out or even whether there is one." (Parsons 1992, p. 144). The idea is not to "exclude the case where there is a previously given domain". However, Parsons wishes to consider a view that does not presuppose that "some infinite structure is given to us independently of our knowledge of the kind of structure the natural numbers instantiate." (Parsons 2008, p. 268) Parsons then adds:

> If there is no such previously given infinite structure, then it is as if we had arrived at the concept of natural number by pulling ourselves up by our conceptual bootstraps, so as to understand the notion of some such structure and convince ourselves of its possibility without having in advance the conception of a domain of objects from which the objects of the structure are picked out. (Parsons 2008, p. 268–9)

this very general perspective, the first condition, expressing the closure of a set under 0 and the successor operation, will be restated as describing a generation process for the natural numbers, starting from 0; while the second condition will be replaced by an appropriate *extremal clause*, that states that *nothing else is a natural number*.

The inductive definition of the natural number is best exemplified by considering Martin-Löf type theory (Martin-Löf 1975, Martin-Löf 1984). See Appendix 6.7 for a precise account of the rules for the natural number set in (Martin-Löf 1984). In this type theory one has three principal kinds of rules: ***introduction***, ***elimination*** and equality rules. The introduction rules introduce the *canonical elements* of the set one wishes to define, in this case the natural number set. A canonical element of a set is a prototypical or standard element of it. In the case of the natural number set, a natural number is canonical if it is of the form 0 or $suc(n)$, for n canonical. It is non-canonical otherwise. For example, $2 + 2$ is a non-canonical element of the natural numbers, as we need to perform a computation to reduce it to canonical form. There are therefore two introduction rules for the natural number set: one tells us that 0 is a natural number, and the other that the successor of a natural number is a natural number. The formal presentation of the elimination rule for the natural numbers is more complex (see Appendix 6.7), but its essence is to express the induction principle and grant recursive reasoning on the natural numbers. The equality rules introduce suitable identity conditions. [19]

The contention in the following is that through proof by induction, one role of the elimination rules is to tell us which natural numbers enter into the natural number structure. Therefore, the role of induction is not to weed out unwanted elements but, rather, to "allow in" desirable ones. A specific case of application of induction, will be discussed in Section 6.5. Here I shall give an intuition of this idea in full generality and from a more standard perspective.

[19]The equality rules are very important, as indicated in Chapter 5, Section 5.5, but do not concern us here.

We may suppose that the inductive definition of the natural numbers defines the extension of a concept, e.g. expressed by a *new* predicate, NAT. The introduction rules tell us that 0 is a NAT, and that whenever n satisfies NAT, so does its successor, $suc(n)$. In other words, this establishes what counts as a canonical element of the extension of NAT: a canonical element is of the form fixed by the introduction rule, either 0 or $suc(n)$ for n canonical. Therefore, if we are prompted with a canonical expression we have a fixed procedure to ascertain that it is indeed a canonical element of the extension of NAT. The introduction rules, however, do not tell us whether expressions that denote non-canonical elements can be allowed in, nor, more generally, what exactly does belong to the extension of NAT. We might be tempted to think that we need to take "all" $NATs$; however, talk of "all" $NATs$ presupposes that we have somehow closed off the domain of this universal quantifier, that we already have an idea of what "all $NATs$" means.

Seen from a more "standard" perspective, one which takes the natural number structure as given, we can present the process of definition of NAT as a way of gaining right to larger and larger fragments of the classical, externally visible structure of the natural numbers by way of computations. That is, by successive applications of induction we enlarge the extension of the predicate NAT (see Section 6.5) to eventually cover all of the natural numbers.

There is here an important correlation with the discussion in Chapter 5, Section 5.3.2, where I stressed a distinction that emerges in Poincaré's writings between a complete definition of a set and a mere specification, or the "genre" (page 154). Poincaré claims that from a platonistic perspective the mere specification will suffice to select out of the mathematical objects the elements of the set so specified. From a predicativist perspective, instead, that does not assume the mathematical objects as given, one needs to offer also a description of (or at least a rule for constructing) the elements of the set. In the case at hand, it would be tempting to suggest that the introduction rules for the natural numbers offer only the "genre", they specify the "canonical form" of the natural numbers, so that when prompted with a canonical natural number n we can recognise it as such. However, they do not fix in a definitive way the extension of the natural number concept, they do not prescribe where to close off, or which "objects" the natural number structure definitely comprises. For example, they do not tell us whether also $2 + 2$ is a natural number or not. We need to appeal to the recursion principle (i.e. the elimination rule) to perform a computation and show that $2 + 2$ equals $Suc(Suc(Suc(Suc(0))))$. The elimination rules (and the equality rules) are therefore necessary to reach a complete definition of the natural number structure (see Sections 6.5 and 6.6 for further clarifications on this point).

I shall consider a concrete example that corroborates the definitional role of induction in the next Section 6.5. Here I conclude by observing that the inductive definition of the natural numbers appears as particularly apt to a predicativist (and constructivistic) stance. In fact, it seems to capture the idea that we do not wish to presuppose "external" assumptions (e.g. the existence of some other sets, or the universe of sets, or, indeed, the definiendum) while defining the natural numbers. We start from the weakest possible assumptions, 0, and an operation of successor, and repeatedly apply it to gain larger and larger fragments of the natural number structure, and finally close off with induction. The definition prima facie appears as entirely "from within", and therefore it would be tempting to conclude that it also complies with the VCP as well as Poincaré's invariance: we expand step by step from what we have constructed so far without at any time disrupting what has already been defined.

Nelson and Parsons' complaints on induction suggest that this impression is misguided because of impredicative uses of induction (Parsons 1992, Parsons 2008). If this is the case, then it is plausible that any other explanation that is acceptable from a predicativist or constructivistic perspective ought to be subject to similar objections.

Parsons claims that we need induction to determine the extension of the natural number predicate. However, the case of $2+2$ that was mentioned above does not suffice to confirm this claim. In this particular case we also have a much simpler way of showing that $2+2$ may be brought to canonical form: we simply apply the successor to 0 twice and then twice again. We therefore need to gain some confidence that Parsons' claim is justified: appeals to instances of induction with unrestricted quantifiers are necessary to determine the extension of the natural number predicate. In the next Section 6.5, I shall go back to Nelson's "Predicative Arithmetic" and utilize his discussion on exponentiation to clarify this point.

6.5 Nelson on exponentiation

Nelson opens his book by formulating the induction principle and giving two examples of statements that can be proved by induction. These are two existential statements and are chosen to illustrate the difficulties he perceives with induction.

The first is given by the formula:

$$\theta(n) \ = \exists m(2 \times m = n \times (n+1));$$

the second is:

$$\pi(n) \ = \exists m(m \neq 0 \wedge m \text{ is divisible by all numbers from 1 to n}).$$

There is a crucial difference between an application of induction to the first and one to the second of these formulas. In the first case, for each n, $\theta(n)$ asserts the existence of a number m that is bounded in terms of n, that is, it is not bigger than $n \times (n+1)$. In the second case, however, we can recover no bound for m in terms of n, that is, there is no way, given n to predict how big m is going to be. This observation is crucial for an understanding of Nelson's approach. His predicative arithmetic introduces appropriate restrictions that have the effect of allowing only for suitably bounded forms of induction; as a consequence, in predicative arithmetic we can establish $\forall n\theta(n)$, but not $\forall n\pi(n)$.

This point is closely related to Nelson's criticism of exponentiation.[20] It is well known that exponentiation quickly gives rise to unfeasible computations, as we can easily realise by considering short expressions such as $5^{5^{5^{5^{5}}}}$. The question then is: how do we persuade ourselves that the expression $5^{5^{5^{5^{5}}}}$ denotes a natural number? If we think of the natural numbers as inductively defined as in the previous Section 6.4.2, then it is clear that $5^{5^{5^{5^{5}}}}$ is not in canonical form, as it is not a successor of a canonical natural number. The question above may now be rephrased as follows: how can we persuade ourselves that $5^{5^{5^{5^{5}}}}$ can be brought to canonical form? The thought is that we can not hope to reach a conclusion on this expression by repeated applications of the successor, due to feasibility constraints. However, a proof by induction will do. I shall now attempt to further clarify this point.

If we allow for the full principle of induction then we can prove that crucial operations on the natural numbers are well-defined or total.[21] For example, we can show that if n and m are natural numbers, then $n + m$ is also a natural number.[22] Furthermore, induction allow us to prove that if n and m are natural numbers, so is n^m. Therefore, *if* we have full induction, we can ensure that, indeed, the expression $5^{5^{5^{5^{5}}}}$ denotes a natural number.

However, from a "constructivistic" position, the question is whether we are enti-

[20]Nelson's reasons for a ban on exponentiation have often been source of puzzlement. In the following I shall offer my understanding of the difficulty with exponentiation, on the basis of informal considerations. My interpretation of Nelson's thought has been enhanced by remarks to be found in an influential article by Leivant (1991), that proposes a new characterisation of polynomial time computability. See, in particular, the section entitled: "A foundational delineation of the main theorem".

[21]A function is total if it is everywhere defined, or, equivalently, $f : X \to Y$ is total if for every element x of X there is an element y of Y such that $y = f(x)$.

[22]In Chapter 3, Section Reducibility we saw that these forms of induction were also a worry for Russell, as they are not available in a suitable form within ramified type theory.

tled to *assume* these instances of induction. Nelson believes that some applications of induction are in fact justified, but others are not. Nelson's reasons for this claim are deeply interrelated to his formalist philosophy of mathematics, that I shall endeavour to clarify at the end of this section. I wish however to first attempt a clarification of this point that does not require adherence to that particular perspective.

The crucial thought is that only those instances of induction that can be explained away in terms of the initial conceptual apparatus given by 0 and successor ought to be justified. We can, for example, make sense of the totality of addition and multiplication simply in terms of applying the successor operation sufficiently far. The reason for this is analogous to the one that makes $\theta(n)$ above acceptable for Nelson: we can bound the result of these operations in terms of their inputs. As a consequence, a proof by induction of the totality of addition and multiplication on the natural numbers can be seen as a useful device to make our computations and proofs more economical, faster. But the ultimate justification of the totality of these functions lays on the possibility of considering simply the initial conceptual "kit" given by 0 and successor.

The case of exponentiation, so it is argued, is different, as we can not produce similar bounds in terms of the inputs. The contention is that for exponentiation we have no other option but to resort to a proof by induction of its totality. In fact, Nelson's claim is that an inductive proof of totality of exponentiation requires *a universal quantification on the natural numbers.* As Leivant (1991, p. 405) clarifies:

> [...] the proof by induction that exponentiation is well-defined over $\mathbb{N}$ presupposes that addition is well–defined for *arbitrary* elements of $\mathbb{N}$, i.e., that $\mathbb{N}$ is already obtained as a completed totality.

The problem here seems to be the following: suppose we take a genetic constructivistic view of the natural numbers. Then we are not entitled to forms of induction in which the inductive formula contains universal quantifiers on the set of natural numbers, as this very set is, so to speak, in the process of construction. From this perspective, when we *prove* the totality of the exponentiation function, we are in fact showing that expressions that are obtained by using this function can be reduced to the canonical ones. This way of looking at the natural numbers suggests that through an inductive proof of the totality of exponentiation we are in fact *defining new natural numbers*: we allow ourselves to add to the natural number structure we have built up so far all the new numbers that can be computed by arbitrary applications of that function. Consequently, if the inductive proof of the totality of the exponentiation function essentially requires a universal quantification on the natural numbers, such a proof is problematic, as it quantifies on the set of natural numbers and, by doing so, extends it to comprise new natural numbers.[23]

As discussed below, the issue whether a proof of totality of exponentiation does contravene to predicativity requires further investigations. At present, the above discussion establishes at least that the *usual* inductive definitions of the totality of exponentiation are radically different from the usual proofs of totality of addition and multiplication.[24] The relevant aspect is that contrary to the case of addition and multiplication, the usual inductive proofs of totality of exponentiation seem to depend not on the inputs but on the outputs of the exponentiation function at previous (smaller) arguments. The thought is then that within a proof of the totality of exponentiation we have to presuppose that expressions with exponentiation are already reducible to canonical form.[25]

[23]This point will be further discussed in Section 6.6.

[24]See the thoughtful discussion in Cardone (1995).

[25]A detailed analysis of these aspects may be obtained by introducing forms of tiering or ramification within arithmetic, or a natural device that separates variables that act as inputs from those that act as outputs of computations. See for example (Bellantoni & Cook 1992, Leivant 1991, Ostrin & Wainer 2005, Wainer & Williams 2005). In the treatment by Ostrin and Wainer and Wainer and Williams, one can formally represent the idea that induction is only allowed for those numbers

6.5.1 Philosophical perspectives

I conclude this section with a brief account of the philosophical view of mathematics Nelson gestures at in (Nelson 1986). This offers a particular way of substantiating the genetic constructivistic picture of mathematical entities that I have instead adopted only for the purposes of my philosophical analysis. Nelson claims that numbers originated "in sequences of tally marks that were used to count things". Subsequently, positional notation was invented, and "[i]t has been universally assumed, on the basis of scant evidence, that decimals are the same kind of thing as sequences of tally marks, only expressed in a more practical and efficient notation." (Nelson 1986, p. 172) The reason for this assumption, according to Nelson, is the overwhelming belief that mathematical expressions (e.g. sequences of tally marks or decimals) denote abstract objects. Nelson calls this belief the "semantic view", and repeatedly claims throughout the book that it is this view that induces our uncritical belief in the natural number concept. Nelson, however, does not wish to make any such "semantic" assumption, and proffers instead a formalist philosophy of mathematics. Mathematics' subject matter "is the expressions themselves with the rules for manipulating them – nothing more. From this point of view, the invention of positional notation was *the creation of a new kind of number*".[26] (Nelson 1986, p. 172)

We find here, once more, a familiar theme that we have also encountered in Poincaré and Weyl: the opposition to any form of "platonistic" assumption of mathematical entities. Nelsons' concerns are more radical than those already discussed in previous chapters, as they do not spare the natural numbers, but their motivation

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that have previously been defined or proved to exist. It is interesting to note that in this context a proof of totality of exponentiation requires an alternation of universal and existential quantifier (i.e. it is Π_2).

[26]Italics mine. See (Marion 2008) for a discussion of the striking similarity between these remarks by Nelson and views expressed by Wittgenstein (1978, Sections 12, 51).

appears to be similar in many respects.

The principal reason for Nelson's objection to these "semantic" assumptions is that they seem to be appealed to as a way of justifying our belief in the correctness of mathematics. Nelson believes that the consistency of even elementary parts of mathematics, as arithmetic, should not be taken for granted. According to Nelson "semantic" assumptions are, however, at the heart of proofs of consistency of systems as weak as arithmetic. The thought seems to be that in view of Gödel's incompleteness phenomenon, the proof of consistency of any formalization of mathematics that is sufficiently strong to define the concept of natural numbers requires already a system that exceeds in some sense the resources of the original one. Nelson's aim is therefore to develop a very weak form of arithmetic that could in principle be proved consistent within a strictly predicative theory. Nelson frames these thoughts in terms of a "modified Hilbert's programme" whose purpose is to justify as much mathematics as possible taking as starting point a demonstrably consistent elementary mathematics. There is only a brief hint at this thought in (Nelson 1986), however, it is important to observe that the worry of inconsistency, that motivated the first discussions on predicativity is still very much alive in Nelson.

6.5.2 The limit of strict predicativity

The previous sections aimed at clarifying the strict predicativist criticism of induction. I have particularly discussed the views of Nelson, and taken into consideration the example of exponentiation. Suppose that we agree with Nelson and we think that, indeed, there is an issue with induction as *some* of its applications are impredicative for the reasons examined above. And suppose that we also accept that those instances of induction have a definitional role: we need them to "fill up" the natural number structure. Therefore as ways of clarifying the concept of natural number they are problematic.

Can we conclude from all this that exponentiation is indeed impredicative? It seems clear that by way of the previous discussion we have in no way determined which functions on the natural numbers are total according to a strictly predicative perspective and which ones are not. Or, equivalently, we have not offered yet any means of determining from the outside, so to speak, the limit of strict predicativity. One might suggest that, given Nelson's discussion of strict predicativity, its external limit could be placed somewhere around a system that allows for the totality of addition and multiplication, but not exponentiation. For example, the system of bounded arithmetic $I\Delta_0$ would represent a strictly predicative system. In addition, any system whose strength is greater or equal to $I\Delta_0$ extended by the totality of exponentiation is clearly impredicative.

Nelson's criticism, in fact, suggests that there is a crucial difference between exponentiation on the one side and addition and multiplication on the other. However, only a very careful technically informed project can answer the question of whether indeed the totality of exponentiation offers a case of impredicativity.

A reason for doubting that we can easily reach any clear conclusion on this point is that the logical analysis of predicativity has uncovered innumerable cases in which a prima facie impredicativity could be eventually eliminated by a more careful analysis. As already mentioned in Chapter 4, Parsons has recently suggested that we may be able to offer a strict predicative justification for exponentiation. Parsons' conviction seems to rely on the fact that exponentiation can be granted on the basis of predicative ramified versions of Frege's arithmetic.[27] The thought is that such systems might play a similar role, within the strict predicativist context, of the subsystems of ramified second order arithmetic that were used for the logical analysis of predicativity (see Chapter 4, Section 4.2); they could be regarded as canonical systems, on the basis of which to assess the (im)predicativity of mathematical notions. If one could argue for the adequacy of those systems as canonical systems for

[27] See e.g. (Heck 1996, Burgess & Hazen 1998, Ferreira & Wehmeier 2002, Linnebo 2004, Burgess 2005, Visser 2009).

strict predicativity, then Nelson's complaint about exponentiation would appear to be motivated by distinct, though related concerns, having to do with feasibility.[28]

In conclusion, it would seem that a careful analysis of the limit of strict predicativity needs to address the fundamental issue of the choice of what in Chapter 4 I called the *base*: which initial "objects" and which initial operations are we granted from a purely predicativist perspective. In addition, we need to clarify how far we can iterate permissible operations.

6.6 On Dummett's indefinite extensible concepts and the impredicativity of the natural numbers

There are important similarities between Nelson's and Parson's complaints on induction, and themes arising within Dummett's discussion on *indefinitely extensible concepts* (Dummett 1991, Dummett 1993).[29] There are differences, too, as further evinced below. Dummett (1991) introduces the notion of indefinitely extensible concept within an assessment of Frege's logicist programme. From Frege's logicism Dummett borrows the notion of set, that is the logical notion of set as extension of a concept mentioned in Chapter 5. Dummett (1991) offers an argument for intuitionistic logic based on the notion of indefinitely extensible concept. He claims that there are indefinitely extensible concepts; in particular, all *infinite sets* are extensions of

[28] Some care in assessing this issue is also intimated by a comparison between finitism and strict finitism. In an influential logical analysis of finitism, (Tait 1981) has argued that Primitive Recursive Arithmetic (PRA) signs the limits of finitism. Kreisel (1958) has instead claimed that one can reach full Peano Arithmetic from a finitary perspective. The disparity of outcomes in this case suggests particular care in addressing also the more complex case of strict finitism.

[29] See Section 6.6.1 for a clarification of this notion. As further evinced below, there are deep similarities also with Poincaré's thought.

such concepts. The fundamental consequence that Dummett (1991) draws from this claim is that the extensions of indefinitely extensible concepts are unsuitable as domains of *classical* quantification. Dummett's perhaps surprising conclusion is not that we need to appropriately restrict the notion of legitimate domain of quantification, as the strict predicativist claims; it is rather that in mathematics we ought to reason *intuitionistically*.

In the following, I shall not attempt to offer an exegesis of Dummett's texts, nor an assessment of his position. The purpose of this Section is instead to merge aspects of Nelson and Parson's discussion on strict predicativity with themes that are suggested by Dummett's argument for intuitionistic logic based on indefinite extensibility. This will present us with the following picture. Suppose that with Nelson and Parsons (and (Dummett 1963)) we find an impredicativity already in the concept of natural number, and, furthermore, that we perceive this as a problem. Suppose further that we think that this impredicativity requires that unrestricted classical quantification on the natural numbers ought to be avoided. Then we have two options: either to persist with classical logic, and restrict legitimate domains of quantification to strictly predicative ones (according to a suitable notion of strict predicativity); or, following Dummett's (1991) suggestion, abandon classical logic in favour of the intuitionistic one. The latter strategy is suggestive that a change in logic makes the impredicativity of the natural numbers tolerable in some sense. It therefore rises the question of why constructive predicativism does introduce some form of predicativity beyond the natural numbers, in addition to abiding to intuitionistic logic.[30]

[30]This point will be briefly discussed in Section 6.6.3.

6.6.1 Existence of indefinitely extensible concepts

Dummett's starting point in (Dummett 1991) is Frege's notion of set as extension of a concept and the central idea is that some concepts, like, for example, the concept "class that is not a member of itself" are **indefinitely extensible concepts**.

> An indefinitely extensible concept is one such that, if we can form a definite conception of a totality of all whose members fall under that concept, we can, by reference to that totality, characterize a larger totality of all whose members fall under it." (Dummett 1993, p. 26)

Let R denote the extension of the concept "class that is not a member of itself". If we ask whether R is an element of R we engender Russell's paradox. We have seen Russell's (1908) analysis of the paradox that bears his name in Chapter 3, Section 3.3.1. Russell there suggests that the paradox would induce an enlargement of R, were R considered as a set. Russell therefore concludes that R ought not to be a set.

Dummett draws a different conclusion from Russell's paradox. Given R as above, by Russell's paradox we are prompted with a new class, $R \cup \{R\}$, that extends R and also satisfies the given concept. Dummett's claim is that Russell's paradox implies not that some concepts do not have an extension, rather that *some concepts are indefinitely extensible.*

This conclusion has strong affinities with Poincaré's analysis of Richard's paradox that motivates his notion of invariance (see Chapter 3, Section 3.3.2). In fact, the example of the "genre", G, that we encountered in Chapter 5 Section 5.3.2 readily provides another instance of an indefinitely extensible concept. Let us frame that example within the Fregean terminology that distinguishes between a concept and its extension. We are given a concept G, and we wish to define a new element, X, of the extension of G (say $ext(G)$), by reference to all of $ext(G)$. The definition of X for Poincaré has the effect of enlarging the extension of G itself. Indefinitely extensible concepts like G have a strong affinity with Poincaré's *incomplete definitions* (see Chapter 5).

In Chapter 5 I particularly emphasized the constructivistic outlook of Poincaré's conception of set. Poincaré asserts that from a realist perspective impredicativity is unproblematic. More generally, it would seem that any conception of set that does not tie a set to its definition and to the definition of its elements seems to be in principle immune to the difficulties Poincaré perceives.

Dummett's discussion is more carefully framed then Poincaré's, and avoids any direct reference to a constructivistic outlook; however it also assumes that the defining condition for a set (or its concept) is prior to the set itself. As a consequence, truths about sets should be established on the basis of reasoning on definitions or concepts, by means of valid proofs, not as a result of facts that are independently established.[31]

6.6.2 Classical and intuitionistic quantification

In addition to the notion of indefinite extensible concept, Dummett (1991, p. 314) introduces the notion of **definite concept**: "[a] concept is definite provided that it has a definite criterion of application – it is determinate what has to hold good of an object for it to fall under the concept – and a definite criterion of identity – it is determinate what is to count as one and the same such object."[32] The distinction

[31]Dummett (1991, p. 303) argues for this particular, logical notion of set, since he believes it is philosophically unrivaled: it explains the applicability of mathematics, its apparent necessity and how we could know mathematical truths. I shall not address these claims here. In Chapter 5 I have suggested that in some contexts, when computability issues are considered particularly relevant, such a notion of set turns out to be more adequate than more abstract notions of set. In addition, in Chapter 7 I shall emphasize the importance of clarifying whether more abstract notions of set can be dispensed of when developing scientifically applicable mathematics.

[32]See e.g. (Linnebo 2013) for a proposal that makes the above informal characterisation of definite concept more precise and (Feferman 2012, Linnebo 2013, Rathjen 2016) for the evaluation

between definite and indefinitely extensible concepts is of primary importance in relation to the notion of domain of quantification.

First, Dummett considers the case of empirical concepts, like the concept of a *star*, that apply to concrete objects. He observes that a realist view of the external world makes quantification over their extension unproblematic. That is, a sharp criterion for whether the concept *applies* to a given object and a sharp *identity criteria* suffice to make available standard **classical** quantification on the extension of that concept. According to Dummett (1991, p. 314), in the empirical case provided that the concept is definite, reality will of itself determine the truth or falsity of statements that quantify over the extensions of such concepts. "On this view, *reality* dispels all haziness: we need do nothing further to eliminate it."

Secondly, Dummett considers the case of mathematical statements: his main contention is that in this case reality does not determine their truth-values "without any need for us to circumscribe the domain of quantification or to specify what objects belong to it." (Dummett 1991, p. 314) Dummett (1993, p. 25) insists that in the case of mathematics, we need to "specify the domain outright, or form some conception of it before interpreting the primitive predicates of a theory as applying to elements of that domain." He also writes that we need to "contrive adequate means of laying down just what elements the domain is to comprise" (Dummett 1991, p. 315). Dummett's text is suggestive of a distinction between what is required for us to *recognize* that a certain mathematical entity is e.g. a real number and offering means of *circumscribing* the domain of real numbers. The latter amounts to saying which real numbers belong to the domain.

A crucial component of Dummett's discussion is the thought that it is *only classical quantification* that requires a circumscribed domain. In the classical case, we

of the definiteness of mathematical statements within the context of appropriate formal systems. In particular, (Feferman 2012, Rathjen 2016) address the question of the lack of definiteness of the Continuum Hypothesis.

need to offer a complete specification of the domain of quantification, one that determines in a definitive way which elements are in the domain. Classical quantification requires that the conception of the domain be completely definite. In fact, any haziness about what elements it does or does not contain must vitiate the assumption that quantification can be interpreted classically. (Dummett 1991, p. 313) However, if we admit that there are indefinitely extensible concepts in mathematics, these are such that their extension is indeterminate, it is hazy and thus not suitable as domain of *classical* quantification.

Dummett's solution to this difficulty is to renounce to classical logic for such domains. One could justify this move by claiming that if we consider intuitionistic rather than classical quantification, the availability of only applicability and identity criteria is sufficient to determine a quantification domain. There is no further need to circumscribe the domain. The reason for this is not spelled out by Dummett, but it seems plausible to assume that it relies on the intuitionistic semantics of the universal and existential quantifiers. According to the BHK interpretation (see Chapter 3, Section 1.2.1), a universally quantified statement $\forall x \in ZA(x)$ is true provided that we have a method or a construction transforming any $d \in Z$ into a proof $p(d)$ of $A(d)$. The suggestion seems to be that an intuitionistic semantics for the universal quantifier may be read as requiring only the ability to recognize if any d satisfies the conditions that make it an element of Z (criteria of application and identity) and then check if it satisfies A. The contention is that this does not require the actual determination of each element of the domain (and bivalence for statements about the domain) nor a full determination of the extension of its corresponding concept.

More on quantification

It is worth pondering on this opposition between classical and intuitionistic quantification. Dummett understands classical quantification as requiring a definite domain of quantification. This is in fact the understanding of quantification that underlies the historical debates on impredicativity. A full acknowledgement of the role of this understanding of quantification for the perceived difficulties with impredicativity is to be found in (Carnap 1931).[33] Gödel (1944) also referred to Carnap's analysis within his assessment of Russell's mathematical logic. He claimed that problems with impredicativity arise if a universal quantifier of the form $\forall x \in D\varphi(x)$ is read as (a possibly infinite) conjunction, the conjunction of all the statements $\varphi(a), \varphi(b), \ldots,$ for each element $a, b, \ldots$ of the set D.[34] Gödel's assessment then is that such a view is problematic from a constructivistic perspective, but unproblematic from a realist point of view.[35] However, Gödel (mentioning Carnap) also alludes to the availability of alternatives. He writes:

> [...] one may, on good grounds, deny that reference to a totality necessarily implies reference to all single elements of it, or, in other words,

[33]Carnap (1931) suggested that there are alternative ways of understanding quantification that do not cause the difficulties the predicativist perceives with impredicativity. He hinted at logical validity as granting legitimacy to quantified statements, including impredicative ones. His gesturing in (Carnap 1931) is insufficiently clear from today's perspective. For a contemporary approach to impredicative theories that is inspired by Carnap's thought see e.g. (Fruchart & Longo 1997). Note also that the general problem Carnap raised has received renewed attention in the philosophical debate in recent years, intersecting with a very rich discussion that was prompted by Dummett's indefinitely extensibility argument. See, for example, (Rayo & Uzquiano 2006).

[34]The informal interpretation of the *classical* universal quantifier *on a given domain* as a possibly infinite conjunction is common in mathematical logic. See also e.g. (Priest 2008, p. 458). Note that this seems in agreement with the standard Tarskian conditions for the universal quantifier, by further application of rules for conjunction introduction.

[35]The relevant aspect here is that this understanding of universal quantification is problematic from a constructive point of view, because it is read as requiring the availability of each individual element of the domain (and the decidability of statements about it). In fact, (classical) quantification is often explained by metaphors as "running through", or "surveying" the entire domain, and this seems to require the prior existence of each element of the domain (and the decidability of relative statements). But these are problematic in general from a constructivist perspective.

that "all" means the same as an infinite logical conjunction.

One might hope that intuitionistic logic might come to the rescue here. A full development of Dummett's suggestion would need extensive work. Here I shall put forward some preliminary thoughts on how one might wish to proceed.[36] The suggestion is to look at the BHK interpretation of the universal quantifier clause. Here a proof of $\forall d \in ZA$, is a *uniform* method that transforms a proof that d is in Z in a proof that it satisfies the property A. Therefore in order to prove that the quantified statement holds it might suffice to show that A holds of a *generic* element d of Z. This is very much in agreement with standard mathematical practice, as we typically prove a universally quantified statement of the form "for all real numbers .... " by showing its validity for a generic real number. In other terms, this understanding of the universal quantifier opens up the possibility in principle of establishing universal statements without the requirement that the domain be definite. It would seem sufficient to be able to single out *one* generic element (with respect to the particular problem under consideration) to carry out the proof of the universal statement. The difficult issue is how do we ascertain that we can in fact fix such a generic element in all cases. Equivalently, the question is how do we ensure that the proofs we give are in fact independent from our choice of generic elements. Here predicativity constraints may be brought back into the picture to guarantee sufficient uniformity of the domain (see discussion in Section 6.6.3.) Another option is to argue for a justification of impredicativity together with intuitionistic logic. This is done in (Fruchart & Longo 1997), by appeal to technical results (among which a genericity theorem) within a polymorphic second order lambda calculus.

Is the concept of natural number indefinitely extensible?

If we accept Dummett's reasoning so far, the crucial question is which mathematical concepts are indefinitely extensible, beyond the cases of R and G above. Already

[36] I shall here make use of the discussion in (Fruchart & Longo 1997).

my inclusion of G within the cohort of indefinitely extensible concepts suggest that the distinction between definite and indefinitely extensible concepts is not meant to capture the separation between paradoxical and non-paradoxical concepts. As in the case of impredicativity the paradoxes are the contingent motivation that brought to our attention a deeper distinction, that between definite and indefintely extensible concepts. Dummett's surprising claim is that not only R but fundamental mathematical domains such as the real and the natural numbers are the extensions of indefinitely extensible concepts. As a consequence, intuitionistic logic ought to be used in reasoning about them, too.

In the following I shall focus on the case of the natural numbers, that is certainly bound to be the less persuasive of the two cases. Dummett's reason for considering the natural number concept as an example of indefinitely extensible concept is that the natural numbers are a fundamental mathematical domain which is required in order to make sense of further quantification within the mathematical discourse. The contrast with the concept "prime number" makes this point clearer, as in this latter case we are presupposing already a domain of natural numbers and we separate from it the sub–domain of prime numbers. However, if we are to specify what the concept natural number means, we ought not to appeal to the natural numbers themselves. Consequently, this is one of those cases in which if we wish to use classical quantification we need more than just a criteria of application and one of identity: we need to circumscribe the domain, or to know which natural numbers there are.

Dummett's contention is that in this case, like in the case of Russell's class R, there is a form of circularity in our attempts to circumscribe the natural number set. I presume that Dummett's point is essentially the one raised by Nelson, Parsons and (Dummett 1963): to fully circumscribe the natural number set, we need to appeal to induction and therefore incur in a form of vicious circularity that is similar to the one that manifests itself in the case of Russell's paradox. In fact, the discussion on predicativity suggests a similarity between the predicativist's requirement of complete definition and Dummett's requirement of circumscription of a domain of classical quantification (see Chapter 5.3.2 page 154). Sharp application (and identity) criteria suffice to fully determine the extension of a concept in those cases in which the elements of that extension are given in some sense; however, for infinite sets we need a more informative process.

It might be useful to go back once more to the inductive definition of the natural numbers discussed in Section 6.4.2. Here it would seem that the introduction rules offer criteria of application and the equality rules offer suitable identity criteria. They suffice to recognize a specified mathematical entity as an element of this set, as a natural number. To "circumscribe" the set of natural numbers, however, we need to appeal to appropriate elimination rules, that clarify exactly which elements belong to it. That is, we need a principle of induction. We have, however, seen that according to at least some predicativists this appeal to induction on the natural numbers is problematic, it is circular, in that it seems to presuppose the natural numbers as given. The natural numbers from this perspective can not be circumscribed by appeal to an elimination rule (i.e. induction) without giving rise to circularity. According to the predicativist, we need to introduce restrictions on the principle of induction and revert to weaker forms of arithmetic.

However, Dummett's discussion seems to open up another way out. Instead of restricting the domain of quantification, we may shift to intuitionistic logic.

6.6.3 Ways out

Suppose that one is persuaded by Dummett's argumentation and sees the natural number concept as indefinitely extensible. Suppose that, in addition, one believes that an indefinitely extensible concept lacks the necessary definiteness that is required for its extension to act as domain of classical quantification (according to the predicativist's understanding of classical quantification). Assume, finally, that one does not offer any alternative independent justification for the assumption of the natural number set as a suitable domain of quantification. Then it would seem that we are faced with two main options. We may substantially restrict the principle of induction, so to obtain a definite concept of natural number without giving rise to vicious circularity. This is the route pursued by the strict predicativist. Alternatively, we may adopt Dummett's strategy and claim that although classical quantification on the extension of indefinitely extensible concepts is problematic, intuitionistic quantification is not, and shift to an intuitionistic theory.

There are three points that arise from this observation.

In Chapter 4, Section 4.4.2, I introduced constructive predicativity and reported that constructive predicativity endorses generalised inductive definitions, even if these are considered impredicative from a classical perspective. Dummett's "way out" could offer one strategy for explaining why from an intuitionistic perspective inductive definitions may be considered acceptable.

For simplicity, let us consider the case of the inductive definition of the natural numbers. It is useful to recall the discussion at page 195, in which I claimed the apparent naturalness of the inductive definition of natural numbers from a predicativist perspective. We start from the weakest possible assumptions, 0, and an operation of successor, and repeatedly apply the latter to gain larger and larger fragments of the natural number structure, to finally close off with induction. I have then examined the strict predicativist's objection to the predicativity of this definition: it makes uses of induction with unrestricted universal quantifiers and therefore presupposes as fixed and "completed" the extension of the very natural number concept that it is in the process of defining. Dummett's suggestion is that the completeness of a domain of quantification is only required if we use classical logic. If we work within an intuitionistic setting, the appeal to the set of natural numbers within those contested uses of induction becomes unproblematic; this is because intuitionistic quantification does not need in general to presuppose as completed the "process" of definition of the natural numbers. We can be seen as referring only to that portion of the natural number set that has being "constructed" *so far*.

The important remark is that the constructivist will claim that we can offer a similar justification also for more complex, possibly iterated inductive definitions. Given the step by step character of inductive definitions, any appeal to the definiendum within those definitions can be considered harmless as we only refer to the part of the extension of the definition that we have constructed so far. This explains the sense in which the circularity that arises in the case of these definitions can be considered tolerable from a constructive perspective.[37] This seems also to indicate that the characterisation of predicativity in terms of invariance is the most significant within a constructive context. The desire to avoid vicious circularity seems instead a more suitable characterisation in a classical context.

There is another issue that arises from the discussion above. If Dummett's reasoning is correct, it would seem that the charge of vicious circularity of impredicativity does not retain the same force in intuitionistic contexts. As a consequence one might suggest that we are granted not only intuitionistic reasoning, but perhaps also some form of impredicativity.[38] In fact, Fruchart & Longo (1997) offer a persuasive argument towards a "Carnapian" route that justifies some forms of impredicativity within an intuitionistic formal system.[39] One might then wonder what could be the reason for the compliance with predicativity in a constructive, intuitionistic context like Martin-Löf's type theory.

I can see a number of ways of arguing for predicativity in this setting. One option is to argue that we need to introduce predicativity constraints to guarantee the uniformity of the domains of quantification that is required for the BHK reading

[37]See also (Parsons 1992) for a different analysis of this issue.

[38]Here impredicativity is to be intended with respect to constructive predicativity.

[39] The context of that discussion is a second order lambda calculus known as system F, and the principal feature that is analysed is a form of polymorphism.

of the universal quantifier. As discussed in Section 6.6.2, an intuitionistic reading of the universal quantifier may legitimize proofs of universal statements that are carried out for a generic element. This clearly has the advantage of requiring us to exhibit only one particular, though prototypical (with respect to the given problem), element of the domain. The question this rises is: how do we ensure that the chosen element is in fact prototypical? In constructive type theory, the restriction to inductively defined sets seems to offer a solution. For simplicity, let us consider again the case of the set of natural numbers (as defined in constructive type theory). Here every element of the natural number set may be brought into canonical form, and the canonical form of the natural numbers may be seen as ensuring the required uniformity. In other terms we have a general description for all the natural numbers that ensures the safety of this process. By way of comparison, in more complex cases, like, for example, the powerset of the natural numbers, or the real numbers, we do not seem to have in general similar resources.[40]

Another (related) option is to stress the importance of granting safe mathematics "from within". One might claim that the adoption of predicativity, and in particular of the Curry-Howard correspondence, is determined by the desire to gain confidence in the correctness of the resulting mathematics without any appeal to "external" assumptions: i.e. from within. The comparison with impredicative calculi like, for example, a second order lambda calculus known as system F, is instructive. To gain confidence in the consistency of system F one relies on normalisation proofs, which show that every computation terminates. However, all known proofs of normalisation (so far) appeal to impredicative reasoning in the meta-theory.[41] Therefore one would seem to have to presume the correctness of impredicative reasoning in order to justify it. The case of constructive type theory differs because of the availability

[40] An alternative to adopting predicativity to grant uniformity is discussed in (Fruchart & Longo 1997). This, however, seems to presuppose the consistency of impredicative system, that will be discussed next.

[41] See e.g. (Martin-Löf 2008, Dybjer 2012).

of an inductive (or "bottom up") specification of all the sets that are formed in that theory. The contention then is that this inductive structure of the sets in type theory grants its correctness without an appeal to an "external" proof of normalization in the meta-theory (Martin-Löf 2008, Dybjer 2012).

Finally, the last issue relates to predicativism. Suppose that a predicativist agrees with the analysis above, and is faced with two options: either adopting strict predicativity or change the logic and allow for a more generous notion of predicativity. The question is which arguments could be offered to eliminate the impasse. I believe this requires more thought, but the following considerations might turn in favour of the intuitionistic shift. As constructive predicativity is the most generous form of mathematics, it would seem reasonable to opt for it, as it allows for a more encompassing form of mathematics. In addition, an approach that only changes the logic rather than restricting the mathematical objects, has arguably advantages to offer in terms of its naturalness: the mathematics is much more similar to classical mathematics. In reply a strict predicativist could attempt to show that strictly predicative mathematics is in fact sufficient for the most salient part of mathematics: the one that is necessary to science (see Chapter 7). The indispensability of predicative mathematics to science will be the topic of the next chapter.

6.7 Appendix: The natural numbers in Martin-Löf type theory

In type theory one has not only introduction and elimination rules for the elements of a set, but also corresponding equality rules, whose main purpose is to define the equality relation between elements of the set to be defined. I shall here present the natural number rules in Martin-Löf type theory (Martin-Löf 1984). My presentation here borrows from (Crosilla 2006).

1. $\mathbb{N}$-Formation

$$\mathbb{N}\ set \qquad\qquad \mathbb{N} = \mathbb{N}$$

2. $\mathbb{N}$-Introduction

$$0 \in \mathbb{N} \qquad\qquad 0 = 0 \in \mathbb{N}$$

$$\frac{a \in \mathbb{N}}{suc(a) \in \mathbb{N}} \qquad\qquad \frac{a = b \in \mathbb{N}}{suc(a) = suc(b) \in \mathbb{N}}$$

3. $\mathbb{N}$-Elimination

$$\frac{c \in \mathbb{N} \qquad d \in C(0) \qquad \begin{matrix} (x \in \mathbb{N},\ y \in C(x)) \\ e(x,y) \in C(suc(x)) \end{matrix}}{\mathbf{R}(c,d,e) \in C(c)}$$

$$\frac{c = f \in \mathbb{N} \qquad d = g \in C(0) \qquad \begin{matrix} (x \in \mathbb{N},\ y \in C(x)) \\ e(x,y) = h(x,y) \in C(suc(x)) \end{matrix}}{\mathbf{R}(c,d,e) = \mathbf{R}(f,g,h) \in C(c)}$$

4. $\mathbb{N}$-Equality

$$\frac{d \in C(0) \qquad \begin{matrix} (x \in \mathbb{N},\ y \in C(x)) \\ e(x,y) \in C(suc(x)) \end{matrix}}{\mathbf{R}(0,d,e) = d \in C(0)}$$

$$\frac{a \in \mathbb{N} \qquad d \in C(0) \qquad \begin{matrix} (x \in \mathbb{N},\ y \in C(x)) \\ e(x,y) \in C(suc(x)) \end{matrix}}{\mathbf{R}(suc(a),d,e) = e(a,\mathbf{R}(a,d,e)) \in C(suc(a))}$$

The elimination rule may be clarified as follows.

In the last two rules $C(z)$ is a family of sets depending on $\mathbb{N}$ that is, if z is a natural number, then $C(z)$ is a set according to Martin-Löf type theory (see (Martin-Löf 1984, Crosilla 2006) for an explanation of the notion of dependent family of sets). In addition, we say that a natural number is canonical if it is of the form 0 or $suc(n)$ for n canonical. It is non-canonical otherwise. Therefore, for example, $2+2$ is a non-canonical element of $\mathbb{N}$ (see (Martin-Löf 1984, Crosilla 2006)).

Given an arbitrary element, c, of $\mathbb{N}$ (that is an element which is possibly non–canonical), we can read $C(c)$ as a proposition for which we require a proof (see (Martin-Löf 1984, Crosilla 2006) for an explanation of the notion of proposition in Martin-Löf type theory). The rule then enables us to prove $C(c)$ by induction. We form first of all a proof d of $C(0)$. Then provided that for $x \in \mathbb{N}$, y is a proof of $C(x)$, we give a proof $e(x, y)$ of $C(suc(x))$. The rule gives us a proof $\mathbf{R}(c, d, e)$ (depending from c, d and e), of the proposition $C(c)$.

More precisely, $\mathbf{R}(c, d, e)$ is computed as follows.

1. Take an arbitrary element, c, of $\mathbb{N}$ and compute its canonical value;

2. if the result of the computation is $c = 0 \in \mathbb{N}$, then compute $d \in C(0)$, hence obtaining a new canonical element f of $C(0)$. Note that $c = 0 \in \mathbb{N}$, so that $C(c) = C(0)$ and so f will be a canonical element also of $C(c)$;

3. if instead the computation produces an element of $\mathbb{N}$ of the form $suc(a)$ for $a \in \mathbb{N}$, then proceed as follows: substitute a for x and $\mathbf{R}(a, d, e)$ for y in e, hence obtaining $e(a, \mathbf{R}(a, d, e)) \in C(suc(a))$. Note that $C(suc(a)) = C(c)$, so that $e(a, \mathbf{R}(a, d, e)) \in C(c)$. Compute the latter, thus obtaining a canonical element g, of $C(c)$;

4. if a has value 0, then $\mathbf{R}(a, d, e) \in C(c)$ by (2); otherwise proceed again as in (3).

Chapter 7

Is predicative mathematics indispensable?

In Chapter 5 I have sketched Poincaré's (1912) views on the debate on impredicativity. For Poincaré this is a manifestation of the classical opposition between realism and idealism. The predicativist perceives difficulties with impredicative definitions due to his "constructivistic" attitude towards them. From this perspective, the purpose of a definition is to introduce a new mathematical entity, and therefore it ought not to quantify on a collection of entities that includes the definiendum. A realist philosophy of mathematics, instead may be seen as making available the domain of quantification on which the definition quantifies, thus eliminating the perceived difficulty. From this perspective, the purpose of a definition is to select a particular element out of the domain, and, it is contended, there is no reason to object to an impredicative way of accomplishing this aim. In Chapter 6 I have also explained the role that Nelson's formalist attitude to the philosophy of mathematics plays in his rejection of any form of impredicativity. It is then natural to wonder whether the debate on impredicativity is after all a new manifestation of traditional debates in the philosophy of mathematics and whether, in particular, taking a stance on impredicativity requires taking a stance on these more complex issues.

Undoubtedly, this may be one way of framing the debate on impredicativity, and the previous chapters suggest it is also faithful to how prominent mathematicians that have addressed this issue have thought about it. However, reading the original texts on predicativity one also gains a different impression. For example, Poincaré's discussion on kinds of definitions suggests a concern with methodological considerations. It is therefore natural to propose that compliance with predicativity may be induced also from a perspective that is in line with the reflections on constructive mathematics that were put forth in Chapter 2. In that chapter I have presented motives for doing mathematics constructively, and indicated that constructive mathematics may be motivated by intra-mathematical reasons, like its greater generality compared with classical mathematics, and by the preference for a more explicit and algorithmic form of mathematics. These motives may also be applied to the case of predicative mathematics, as all the varieties discussed in this thesis are fully compatible with classical mathematics, of which they are, in fact, a part.[1] In Chapter 6 I have indicated that since its inception predicativity has been further motivated by the desire to secure a safe form of mathematics "from within". By adopting a more elementary conceptual apparatus (and, in the case of constructive predicativity, a more algorithmic notion of proof) we seem to gain more confidence in the correctness of the resulting mathematics. In addition, we may gain the benefit of a computational form of mathematics that is suitable for machine implementation (especially in the constructive and strict predicative cases). In particular, the predicative notion of set discusses in the previous chapter, with its inductive specification, seems particularly apt to computational interpretations. It therefore would seem that we can offer purely intra-mathematical and pragmatic reasons also for the development of a constructive and predicative form of mathematics, too. In particular, it may be argued that a constructive mathematician who is predicatively inclined does not

[1] In particular, all the formal systems mentioned in this thesis can be interpreted in subsystems of ZFC. See Chapter 2, Section 2.4.1, page 60 for the notion of compatibility.

need to take a side on complex issues as the nature of mathematical entities, or the question of their existence. He might be only motivated by reasons that directly lie within his own practice.

In Chapter 2, I have suggested that the motives adduced by constructive mathematicians for their practice, if taken by themselves, are unlikely to offer an argument for the exclusive adoption of intuitionistic logic in mathematics. It is natural to wonder if predicative mathematics (of any variant) holds a stronger position in this respect. In particular, whether the reasons mentioned above may be sufficient to support a form of predicativism.

Let us consider the motive for predicativity of producing a safer, more convincing form of mathematics. If a predicativist were to suggest that we ought to abandon impredicativity on this ground, one could retort that as long as an inconsistency has not been found within impredicative systems as ZF, we ought to allow for them. Unless one offers good reasons for suspecting that classical mathematics may be inconsistent, there is not sufficient ground for abandoning it on this basis. Those considerations adduced by the predicativist might indicate reasons for a *preference* for predicative mathematics. However, by themselves, they do not seem to offer any means for objecting to impredicative methods. Therefore the predicativist seems to require different arguments for the exclusive use of predicative mathematics.

The main focus of this chapter is to examine the prospects of a defense of predicativism on the basis of a variant of indispensability argument which claims that predicative mathematics is indispensable to science. I shall primarily attempt to clarify what such an argument may gain for the predicativist and I shall report on recent technical research that may help assess the relation between predicative mathematics and scientifically applicable mathematics. I shall also argue that a substantial amount of work is still required before reaching any definitive conclusion on this matter. Given my tentative conclusions, the discussion in this chapter is more speculative and should be seen primarily as a contribution to setting up the task for a strategy along these lines. I shall argue that even if this strategy might not, by itself, lead to a full defense of predicativism, it is a worthwhile enquiry bound to enrich the philosophy of mathematics, and in particular the discussion on indispensability arguments. Before examining a variant of indispensability arguments for predicativism, I shall outline other ways in which a predicativist might proceed and the difficulties he might encounter.

7.1 Predicativism

The distinction introduced in Chapter 4 between base and constraints suggests two tasks for a defense of predicativism. First, one needs to argue for predicativity constraints. This line of argument aims at objecting to impredicativity. Second, one needs to argue for the base. The latter task is required in view of the plurality of notions of predicativity, and the principal aim is to argue for one particular variant of predicativity against the others. The worry is that in view of the similarity of arguments that support the various forms of predicativism, a form of instability might arise that forces the stronger forms (e.g. predicativism given the natural numbers) to collapse into the weakest one (strict predicativism).

I shall not examine the first point in detail in this thesis as I shall instead consider the prospects of a different route that attempts to by-pass the complexity of that task. I wish however to briefly suggest where one might wish to focus on. If we take the route to predicativity that I have suggested in this thesis, it would seem that a crucial component of the predicativist thesis is the deep tie that links a set to the genetic construction of its elements. This was incorporated into the notion of set as extension of a predicate, where the predicate acts as a list of instructions for specifying the genetic "construction" of the set from its elements. In addition to arguing for this notion of set, the predicativist would have to counter alternatives.

The most standard strategy to obviate to the difficulties that the predicativist perceives with impredicativity is to "detach" the notion of set from prior definitions, predicates, properties or concepts. One possibility is given by appealing to Bernays' quasi-combinatorialism that was briefly discussed in Chapter 5.[2] This might be combined with a form of set-theoretic platonism.

Another, less standard, alternative is not to free sets from any uniform tie with their elements, but to make use of that uniformity. The thought is that if we shift from classical to intuitionistic logic, then quantification may not require the prior availability of each individual element of the domain. What might suffice is the possibility of recognising that an object does belong to the domain, and here, it may be argued, all that is needed is a kind of uniformity.[3]

There are other options. One could perhaps consider holding to the logical notion of set as extension of a concept, but reinforce, not constrain, the concepts. Hints at a platonist reading of concepts along lines may be found in (Carnap 1931) (where they are attributed to Ramsey) and in (Gödel 1944). Alternatively, one may argue for the adoption of second order logic to grant a form of impredicativity.

The predicativist will have to offer ways of objecting to each of these strategies if he wishes to offer a defense of predicativism that rejects impredicativity.

7.1.1 Securing the base

In addition to arguing for predicative constraints and against impredicative alternatives, the predicativists needs to address the second point mentioned at the beginning of the previous section: the request for an argument for the base. In the following I shall consider the case of predicativism given the natural numbers, as this variant seems to offer particular difficulties. This form of predicativism, in fact, combines the acceptance of the natural number structure with predicatively motivated restrictions to sets that lay beyond that structure.

[2]See also (Maddy 1997).

[3]See Chapter 6, section 6.6.3 and (Fruchart & Longo 1997).

In Chapter 6 I examined Parson's claim that if applied all the way through, that is, also to the natural number structure, predicative restrictions give rise to some form of strictly predicative mathematics. The predicativist given the natural numbers then needs to offer independent arguments to support his belief in the natural number structure as base, exempt from predicativity constraints, and needs to ensure that in so doing he does not justify impredicativity at higher levels than that structure.

One could perhaps appeal to the canonical representation of the natural numbers as a way of justifying the assumption of their structure as base. As clarified in Chapter 5, page 164, Weyl claims that the natural numbers supply us with a "domain of individuals". In that chapter I suggested to express this in terms of the canonicity of the natural numbers, for which we can fix a unique representation in terms of an initial element, 0, and the iteration of the successor operation (see page 165).[4] As discussed in Chapter 5 the crucial difference that the predicativist wishes to capture between the natural numbers on the one side, and the powerset of the natural numbers and other uncountable sets on the other, is the lack of uniform describability of the latter. Therefore we had better embed the canonical describability of the natural numbers within an account of the natural number structure as base. As a starting point towards a justification of the base, we may wish to claim that we are entitled to this structure as base in virtue of the canonical representability of its elements: there is no arbitrariness in the natural numbers, while there is in the powerset of the natural numbers. The discussion in Chapter 6 clarifies, however, that there is a leap from the consideration of individual natural numbers and their structure: we also need to account for the principle of induction, which

[4]One might worry that the availability of a plurality of alternative representations for the natural numbers might put strain on the very idea of a canonical representation of these numbers. One could fix a canonical representation or, as in Weyl, take the individuality of the natural numbers as conferred by their unique position within the natural number structure.

plays a crucial role in enabling us to close off, and obtain the whole natural number structure, not just a fragment of it. The worry is that any justification we can offer for our assumption of induction would seem to be different and perhaps wholly unrelated to the canonicity of the natural numbers. Therefore we have to ensure that a justification of induction neither clashes with the reasons for the introduction of predicativity constraints, nor gives the way for a justification of impredicativity.[5] This is indicative that a defense of predicativism is very complex.

To witness the difficulties involved in securing the base, I shall sketch a possible strategy in relation to a particular way of understanding predicativism given the natural numbers. One way of framing predicativism given the natural numbers is in terms of what I should like to call a *"moderate platonism"*. This is an intermediate position in the philosophy of mathematics, laying in between the more radical constructivism and platonism.[6] The predicativist holds a platonist position on the natural numbers and a constructivistic (or definabilist) position on the remaining sets, starting from the powerset of the natural numbers. The natural numbers are seen as existent mind-independently, but the other sets are seen as dependent on our constructions or definitions, and because of this need to undergo restrictions compared with the classical conception.

The cohabitation of features belonging to traditionally opposing philosophical positions within one induces difficulties for predicativism given the natural numbers.

Maddy (1997) explores three routes to mathematical realism, that she terms Gödelian, Quinean and set theoretic realisms, and concludes that they are all wanting. The first two forms of realism could prima facie inspire a predicativist's attempts to stabilize his position as a moderate platonism. The predicativist would need to offer appropriate elaborations of these forms of realism that support *all and*

[5] See (Clark 1993, Parsons 2008) for discussion.

[6] See, e.g.,(Feferman 1964) for such a characterisation of predicativism given the natural numbers.

no more than platonism for the natural numbers. Provided that this is possible, the predicativist will then need to fence his position from a number of objections that have been risen against each of the original positions (including Maddy's), if they also apply to the moderate case.[7] My aim in the following is to suggest that a preliminary analysis indicates that such strategies face notable difficulties.

The first strategy is complex, and aims at adapting to the present context a form of Gödelian realism. According to (Maddy 1997) Gödel's realism is rooted in a strong analogy between mathematics and physical science. In (Gödel 1964) Gödel argued for an extrinsic justification of the more theoretical parts of mathematics because, like similar components of physical science, they are justified by their consequences. The analogy between mathematics and science brings Gödel to appeal to "another, more basic form of mathematical insight that plays a role parallel to that of sense perception" (Maddy 1997, p. 90). Gödel (1951, p. 320) writes: "The truth, I believe, is that [mathematical] concepts form an objective reality of their own, which we cannot create or change, but only perceive and describe".[8] He also writes: "The similarity between mathematical intuition and a physical sense is very striking. It is arbitrary to consider 'This is red' as immediate datum, but not so to consider the proposition expressing modus ponens or complete induction (or perhaps some simpler propositions from which the latter follows)." (Gödel 1953/9, 359)

The predicativist could then hope that and appeal to a suitable form of mathematical intuition could allow for a platonist assumption of the natural number structure with induction. This strategy faces a number of difficulties, first of all the challenge of clarifying in a sufficiently perspicuous way the underlying notion of intuition. The principal worry in this specific context is that unless such a notion

[7]There are other possible defences of platonism that take different routes all together, including Maddy's naturalism in (Maddy 1997), or forms of structuralism (Shapiro 1997). However, they prima facie do not seem suitable candidates for the purposes at hand, as they seem "devised" to justify much more than moderate platonism.

[8]Quoted in (Maddy 1997, p. 90).

of intuition is sufficiently different from a Gödelian one, it seems arduous to utilize it to support the mind-independent existence of *all and only* the natural numbers. In fact, (Gödel 1964, p. 268) writes:

> despite their remoteness from sense experience we do have something like a perception of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., mathematical intuition, than in sense perception, which induces us to build up physical theories [...].

One concern is that unless suitable constraints are introduced to prevent the intuition of sets beyond the set of natural numbers, one would be pushed all the way up with Gödel to account for "all" sets. In an attempt to secure no more than the natural number set, the predicativist may wish to appeal to the reasons he has adduced for predicative constraints. However, if Nelson and Parsons' criticism of induction is correct, an appeal to arguments against impredicative definitions, if applied all the way through, would drive us to more stringent forms of predicativism that do not spare the natural number structure.

One could perhaps appeal here, too, to the canonical representation of the natural numbers, and suggest that we can profitably characterise a different, more fundamental form of intuition that is grounded on the availability of a representation for each of the natural numbers. Provided that we can find a way of spelling this out, the principal problem with a strategy along these lines is, again, the leap from the intuition (however specified) of any individual natural number and the intuition of the totality of them (see Chapter 6). The latter requires intuition of a closure principle, mathematical induction, that differs from the intuition of individual numbers. In other terms, the step from individual numbers to their totality involves a form of idealization that goes well beyond the actual intuitability of the canonical representation of any individual natural number: if there is a way of justifying induction in terms of a form of intuition, this seems to be of a different kind as the one appealed to for individual natural numbers. If one claims that because of this difference there is no way of accounting for the principle of induction in a satisfactory way, then we are brought to a weaker form of predicativism. In the quote above Gödel appealed to the non-arbitrariness of the proposition expressing complete induction. An elaboration of this thought could perhaps offer a strategy that complements the intuition of the natural numbers with (a different, "'Gödelian") intuition of induction; however, the worry is that this might concede too much, as it would seem that it could equally well justify the assumption of much stronger set theoretic constructs. For example, it could justify generalised inductive definitions or a form of iteration on the ordinals. As already discussed in Chapter 4, Section 4.1, the combination of such iteration with predicative constraints gives rise to Gödel's constructible universe, that goes well beyond predicativity given the natural numbers.

To summarise, the principal concern for a strategy along these lines is the risk of instability: unless we find a perfectly calibrated account of the base in its entirety, including mathematical induction, and nothing more, then a form of instability could plague predicativism given the natural numbers, with the risk of being pushed all the way up with Gödel or all the way down with Nelson. As argued in Chapter 6 given the overall motivation underlying predicativity, the most serious worry is that of predicativism sliding down with Nelson.

The predicativist has, however, another option altogether: he might attempt to claim that predicative mathematics is justified because it is required for the applicability of mathematics to science, and that impredicative mathematics lacks so far justification, as there is no actual *need* to appeal to it, that is, *impredicativity is dispensable*. The possibility that impredicativity turns out to be dispensable has been briefly discussed by Feferman in (Feferman 1993*b*, Feferman 2005) and, more extensively, by Ye e.g. in (Ye 2008, Ye 2011) and will be the focus of the reminder of this chapter.

7.2 Indispensability

Weyl's aim in "Das Kontinuum" was to clarify how far can we proceed in developing analysis from the bare assumption of the natural number structure and some elementary operations over them. Later on the mathematical logicians of the 1950's and 60's also addressed the question of how much mathematics is predicatively expressible (according to predicativity given the natural numbers); this eventually came under more precise scrutiny within Feferman's work and the Reverse Mathematics Programme, as discussed in Chapter 4, Section 4.3. Already (Weyl 1918) offered an indication that not all of contemporary mathematics is expressible according to predicative standards. This first assessment has been further confirmed by the logical analysis of predicativity: each of the forms of predicativity that have been presented in Chapter 4 Section 4.4 is short of accounting for all of contemporary mathematics, in particular a substantial part of set theory evades predicative mathematics according to any of these varieties of predicativity.

In this section I shall again focus prominently on predicativity given the natural numbers and strict predicativity, but it should be clear how to adapt the discussion to constructive predicativity.

Granted that predicativity can not capture the whole range of contemporary mathematics, one may wonder whether the restriction to "core" or "ordinary" mathematics would accord better prospects for predicative mathematics. As further discussed below, one difficulty arises here in clarifying which criteria one ought to apply in selecting a portion of mathematics as the "core". A number of mathematicians have appealed to a notion of *ordinary mathematics*, others have invoked a vague notion of *interesting mathematics*.

As mentioned in Chapter 5, Wang (1954, p. 244) clearly expressed a common sentiment among a number of mathematical logicians in the 1950's, according to which the debate on impredicativity had left unresolved the question of which role impredicativity plays within "core" mathematics. Wang (1954, p. 244) observed that the use of uncountable (or indenumerable) and impredicative sets "remains a mystery which has shed little light on any problems of ordinary mathematics. There is no clear reason why mathematics could not dispense with impredicative or absolutely indenumerable sets."

The concept of "ordinary mathematics" can be elucidated as in (Simpson 1999, p. 1): "that body of mathematics which is prior to or independent of the introduction of abstract set-theoretic concepts". That is: "geometry, number theory, calculus, differential equations, real and complex analysis, countable algebra, the topology of complete separable metric spaces, mathematical logic and computability theory".[9] We have seen in Chapter 4, Section 4.3 that a large portion of ordinary mathematics can be carried out within predicative systems, in fact, systems that are no stronger than Peano Arithmetic suffice.

In his fundamental article, Feferman (1964, p. 3-4) writes:

> It is well known that a number of algebraic and analytic arguments can be systematically recast into a form which can be subsumed under elementary (first order) number theory. [...] It is thus not at first sight inconceivable that predicative mathematics is already (formally) sufficient to obtain the full range of arithmetical consequences realized by impredicative mathematics.

As Feferman quickly clarifies, not every elementary statement can be so obtained. The logical analysis of predicativity in (Feferman 1964) readily provides us with

[9] See also Chapter 4, Section 4.3, page 112. Rathjen (1999) writes: "Roughly speaking, by ordinary mathematics we mean main-stream, non-set-theoretic mathematics, i.e. the core areas of mathematics which make no essential use of the concepts and methods of set theory and do not essentially depend on the theory of uncountable cardinal numbers."

a counterexample: the very arithmetical statement expressing the consistency of predicative analysis.[10] However, Feferman suggests that one could argue that "*all mathematically interesting statements about the natural numbers, as well as many analytic statements, which have so far been obtained by impredicative methods can already be obtained by predicative ones*".

These quotations suggest a strategy for supporting predicativism: by showing that predicative mathematics is already sufficient to develop "core" mathematics. Provided that some agreement can be reached on what counts as "core" mathematics, the hope is that a logical analysis of *that* portion of mathematics could show that essentially impredicative mathematics is not needed.

In recent years Feferman and Ye (Feferman 1993*b*, Feferman 2005, Ye 2008, Ye 2011) have offered evidence that might support the claim that impredicative mathematics is formally dispensable with respect to "core" mathematics, if "core" mathematics is framed in terms of the mathematics that is *applicable to science*. Here I write "formally dispensable" to indicate the availability of a reformulation of impredicative mathematics within some predicative system.[11]

We are not yet in a position to decide in a definitive way whether scientifically applicable mathematics is predicatively expressible, as more work is required to clarify whether predicative mathematics suffices for today's science's needs. However, it is interesting to explore which conditional claims can a predicativist make to support his position.

If a thorough and detailed logical investigation of the mathematics that is necessary for science were carried out, I can envisage the following main possible outcomes.

1. It could turn out that predicative mathematics is already sufficient to develop all of scientifically applicable mathematics. This would count as supporting predicativism, by showing that its revisionary component is, after all, inno-

[10] See Chapter 4.

[11] This point will be further discussed below.

cent. In addition, if one could clearly show that the part of mathematics that is needed for science is exactly captured by predicativity given the natural numbers, then one would have offered also a way of stabilising this position against strict predicativism. In addition, one would have shown that from a purely formal point of view, the impredicative component of ordinary mathematics is dispensable.

2. It could become apparent that scientifically applicable mathematics requires impredicativity. Now predicativism would count as a drastic restriction of a fundamental part of mathematics, and could be opposed on these grounds.

Before examining in more detail an argument for the indispensability of predicative mathematics to science, a general consideration is in order: it is important to observe that this strategy, if successful, would offer *no explanation* of why a form of predicativity is indispensable to scientifically applicable mathematics. In particular, such an argument would bear no relation with the motives for predicativity that were discussed so far in this thesis. An indispensability argument would allow us to conclude that, as it happens, this portion of mathematics, rather than another, is the one that is applicable to contemporary science. Any other philosophical position that would also single out that portion of mathematics as indispensable to science would be equally supported by the argument.

7.3 Is predicative mathematics indispensable?

The central aim of this section is to formulate an argument for the indispensability of predicative mathematics to our best scientific theories and clarify which conclusions may be drawn from it ***if*** one were to prove the formal indispensability of predicative mathematics. The argument is modelled after arguments invoked in a different debate, to assess the ontological commitment of mathematics by evaluating its indispensability to our best scientific theories.[12] In this thesis I shall not enter the discussion on the merits of indispensability arguments within that debate. In fact, I shall largely avoid addressing the complex question of whether, and in which sense, considerations on indispensability to our best scientific theories can bestow justification to a mathematical practice. Instead, I shall grant to the predicativist that a strategy along these lines is available, that is, that an appeal to indispensability to science is legitimate in the present context. I shall then conclude that we have so far no determinate evidence that predicative mathematics is indispensable to science. If, however, we were to accept the suggestions by Feferman and Ye that progress made so far makes it reasonable to expect that scientifically applicable mathematics can be carried within predicative systems, it seems possible that, as suggested by Ye, very weak systems will suffice. Then indispensability arguments would have the positive effect of strengthening predicativism against perhaps the most serious criticism: its inadequacy to express the mathematics required by science. In addition, the logical analysis would clarify whether strict predicativism is already formally sufficient for all scientific applications. This will have an impact on the question of which form of predicativism, if any, is to be endorsed.

In the following I shall begin my analysis by setting out a general form of indispensability argument; then I shall consider adaptations of it which produce an argument that may help the predicativist.

7.3.1 Indispensability arguments

The controversies that have surrounded indispensability arguments have not spared their very formulation. Here I shall adopt a formulation that has the advantage of being particularly succinct and easy to adapt to our case.

[12] The entry on "Indispensability arguments in the philosophy of mathematics" of the Stanford Encyclopedia of Philosophy contains an up to date bibliography on this subject (Colyvan 2015).

Standard Indispensability (SI):

(P1) We ought to believe in a mathematical theory M that is indispensable to our best scientific theories.

(P2) We ought to have ontological commitment to all and only the entities that are postulated by a mathematical theory M that we believe in.

(P3) Mathematical entities are postulated by the mathematical theory that is indispensable to our best scientific theories.

(C) We ought to have ontological commitment to mathematical entities.

As a first approximation, the predicativist might reformulate the argument for his purpose as follows:

Indispensability of Predicativity (IP):

(P1) We ought to believe in a form of mathematics that is indispensable to our best scientific theories.

(P2) Predicative mathematics is indispensable to our best scientific theories.

(C) We ought to believe in predicative mathematics.

A first task for the predicativist wishing to use an argument of this kind is to clarify what a form of mathematics is. In full generality this is hard to achieve. However it suffices to observe that in the present context we are only interested in the particular cases of variants of predicative and impredicative mathematics.[13] The next task then is to clarify what each of these particular forms of mathematics is. One temptation would be to avail oneself of the logical analysis of predicativity and, for example, define predicative mathematics given the natural numbers as a collection of definitions and theorems that are expressible within predicatively reducible theories (see Chapter 4, Section 4.2). However, the logical analysis of

[13]Therefore a clarification of the expression "form of mathematics" may be effected by a disjunction between strict predicative, predicative given the natural numbers, constructive predicative and three notions of impredicative mathematics, each capturing the "complement" of a form of predicative mathematics with respect to contemporary mathematics.

predicativity is carried out from the outside, so to speak, by appeal to impredicative notions. In particular, the very ordinal Γ_0 lays beyond the predicativist's reach. As a consequence, the predicativist that wishes to employ an indispensability argument needs to use a different strategy. A natural approach is to fix a canonical theory which is sufficiently strong to carry out the mathematics one wishes to express, while being predicative according to the relevant standards. For example, for predicativity given the natural numbers, one could choose the system ACA_0, or Feferman's W (Feferman 1988*b*).[14] These are clearly within the predicativist-given-the-natural-number's remit, and they are also appropriate for the development of scientifically applicable mathematics.

The next question is how to express the indispensability of a form of mathematics, M, to a scientific theory, say T. This would have first of all to take into account the adequacy of mathematics to T.

Adequacy (of M to T): within the form of mathematics M we can define all the mathematical notions and prove all the theorems that are employed by the scientific theory T.

Feferman (1993*a*) makes use of a notion of "applicable mathematics", and writes:

> Of course, there are results of theoretical analysis which cannot be carried out predicatively, either because they are essentially impredicative in their very formulation, or because they are independent of predicative systems [...] However, none of those affects the working hypothesis because they do not figure in the applicable mathematics.

Applicable mathematics could be seen as gathering together the notions and theorem that all our best scientific theories make use of. In fact, a first task for the predicativist wishing to appeal to an indispensability argument for predicativity would be to clarify in full detail what is applicable mathematics, which theorems

[14]See note 44, page 99 for details on these systems.

and notions belong to it.

Adequacy does not suffice to express indispensability, as if predicative mathematics were adequate to scientifically applicable mathematics, so would be also impredicative mathematics. What we need here is a notion of indispensability of a form of mathematics that also introduces some minimality requirement: a form of mathematics M is indispensable to scientifically applicable mathematics if it suffices to carry out this body of mathematics, but it is also "minimal" in some appropriate sense. Minimality ought to encode the idea that no less comprehensive form of mathematics can already be used to carry out scientifically applicable mathematics. Roughly minimality could be so expressed:

Minimal adequacy: a form of mathematics M is minimally adequate with respect to scientifically applicable mathematics if no other weaker form of mathematics M' can express all the notions and derive all the theorems that are appealed to by scientifically applicable mathematics.

There are difficulties in formulating this requirement in sufficiently general and precise terms. One, for example, would have to spell out the notion of "weaker form of mathematics" in a way that is sufficiently precise and flexible to allow for different choices of canonical systems. Once more, given our limited aims, we may suppose that this can be done by suitable choices of canonical systems for predicativity and comparison of which assumptions they make. In particular, the aspect we wish to capture is the difference in strength between strict predicativism and the stronger forms of predicativism (as well as impredicativity). Then one could in principle fix suitable canonical systems that (possibly modulo suitable proof-theoretic interpretations) would enable us to pin down the difference between strict predicative and stronger systems in terms of the amount of induction on the natural numbers one allows for.

A reformulation of the argument (IP) that takes into account both adequacy and a suitable notion of minimality could then be proposed as follows.

Dispensability of Impredicativity (DI):

(P1) We ought to believe in a form of mathematics that is indispensable to our best scientific theories.

(P2') A form of mathematics is indispensable to our best scientific theories if it is adequate and minimal for scientifically applicable mathematics.

(P3') Predicative mathematics is adequate and minimal for scientifically applicable mathematics.

(P4) We ought to reject a form of mathematics that is adequate but not minimal for scientifically applicable mathematics.

(P5) Impredicative mathematics is adequate but not minimal for scientifically applicable mathematics.

(C) We ought to believe in predicative mathematics and reject impredicative mathematics.

Given this argument, what conclusions can we draw from it? By granting the predicativist an appeal to this kind of arguments I have, for the sake of argument, already conceded the substantial claim that (P1) is true. I have not clearly spelled out the notion of scientifically applicable mathematics, nor given sufficient evidence for premise (P2'). However, I shall presuppose that we can in principle determine which mathematics is scientifically applicable by a careful and extensive investigation of case studies of application of mathematics to physics. If one could offer sufficient evidence for (P3'), then predicativism would be substantially strengthened, as the common criticism that it imposes unjustified and unbearable restrictions to the mathematical practice would not hold. At least for the crucial case of scientifically applicable mathematics predicativism would not cause any loss. This would be in fact a remarkable gain for predicativism. However, if premise (P3') would turn out to be false, we would have a serious objection to predicativism. In Section 7.3.2 I shall further report on technical work that might offer supporting evidence for premise (P3').

One might wonder whether the conclusion of (DI) gives us not only a very much needed support for predicativism but also the resources to (1) stabilize predicativism given the natural numbers and (2) *reject* impredicativity. Here we enter more treacherous terrain. Suppose that we could establish premises (P3') and (P5). Then we would have the remarkable outcome that scientifically applicable mathematics, that prima facie uses impredicativity, could already be reduced to predicative form. From a purely mathematical point of view, we would have an important reducibility result that would also seem to imply that impredicative mathematics can be dispensed of. A subtler question is whether we could also claim that this result implies that we could do without impredicative mathematics within our best scientific theories. That is to say, the difficulty is to clarify what could justify premise (P4).

Here I suggest to distinguish between *formal dispensability* and *dispensability proper*. The question than can be rephrased as whether the formal dispensability of a form of mathematics, that is, the possibility of reducing scientifically applicable mathematics to predicative systems, also implies the dispensability proper, that is, the possibility of doing without that form of mathematics within our best scientific theories. This is a considerably more complex issue that can not be fully addressed in this thesis.

As preliminary thought, in order to address this question one might wish to combine a careful analysis of the role of applicable mathematics within our best scientific theories, with considerations of theoretical virtues of scientific theories. In particular, one may wish to borrow ideas that have emerged within the literature on standard indispensability arguments. A first thought is that an application of Ockham's razor could be employed in this case to cast the non-minimal mathematical practices as "recreational mathematics", and thus dispensable. However, a worry quickly emerges: an appeal to a form of Ockham's razor ignores other characters of a scientific theory as a whole that one might wish to consider when choosing between different form of mathematics. These relate to aspects that contribute to the overall attractiveness of a scientific theory, like, for example, simplicity, economy, explanatory power etc.[15] This suggests that the notion of indispensability captured by adequacy and minimality is not sophisticated enough: we also need to carefully assess whether there is a gap between a formal adequacy of a form of mathematics (in the sense of the possibility of expressing scientifically applicable mathematics within a certain system) and its adequacy to science, that is, its performing the role that our best scientific theories require for that form of mathematics. Consequently, in an overall assessment of an indispensability argument of this kind, we would need to embark on the complex task of clarifying the role of a given form of mathematics within scientific theories. This goes beyond assessing the availability of a formal reducibility of a form of mathematics to a "weaker" one. A difficulty in our case is that if we wish to gain any useful insight on which portion of ordinary mathematics is in fact indispensable to our best science, we can not allow for too generous a rounding up, one that includes substantial portions of set theory, for example.[16] This makes this task particularly difficult. As a consequence, it would seem that one will need to either carefully argue for premise (P4), or, more plausibly, appropriately weaken it to account for a suitable "rounding up", if this can in fact be determined.

There is, however, a very general conclusion that one may be able to draw, if we could show that some predicative form of mathematics is formally indispensable (i.e. adequate and minimal) for scientifically applicable mathematics. The predicativist might wish to distinguish between a part of *impredicative* mathematics that may turn out to be profitably employed within scientific applications and a part that is bound to play no role within it. Let us call the second *strongly impredicative* mathematics. Such a divide might be not as sharp as desirable until more work has been carried out, both technical and philosophical. However, if premise (P3')

[15]See, for example, (Quine 1986, Field 1980). An explanatory role of mathematics within science has been at the centre of recent debates on standard indispensaiblity arguments.

[16]See (Quine 1986).

were found to be true, we can expect that one could give some general rounding up of predicative mathematics that is useful for science without reaching the strongly impredicative. One way to re-phrase this is by claiming that if we could enforce a demarcation between scientifically applicable and non-applicable mathematics, then we could also hope to single out a portion of impredicative mathematics that plays no significant role at all within applicable mathematics; no theoretical virtue can rescue this part of mathematics, and if (P3') were true, strongly impredicative mathematics may be dispensed of.

The discussion so far clarifies that the crucial task for the predicativist is to offer support to premise (P3'), that will be the focus of the next section.

7.3.2 Supporting the indispensability of predicative mathematics

Having addressed the issue of which kind of conclusion a predicativist may draw from a "dispensability" argument, it is now time to look at the prospects of deciding whether (IP) and (DI)'s crucial premises (P2) and (P3'), respectively, are true. There are two points the predicativist needs to clarify:

1. Is predicative mathematics (formally) adequate to science?
2. Which is the minimal form of predicative mathematics that is (formally) adequate to science?

If the first question could be answered positively, then an answer to the latter question would represent a first step in stabilizing the predicativist's position, by clarifying the formal relation between possibly different predicative theories that are adequate to science.

As to the first question, as already mentioned at in Chapter 4 Section 4.3, Feferman has argued that the case can be made that all scientifically applicable mathematics can be codified by predicative theories. Over the years he has extensively tested the following **working hypothesis**:

All of scientifically applicable analysis can be developed predicatively.

More precisely, Feferman has argued that all scientifically applicable analysis can be developed in the system W of (Feferman 1988*b*), which codifies in modern terms Weyl's system in "Das Kontinuum". As system W is of the same proof theoretic strength as Peano Arithmetic, this gives us an upper bound for scientifically applicable mathematics that would enable the justification of no more than arithmetic as a base.[17]

Feferman further points to the work carried out in Reverse Mathematics and in Bishop style constructive mathematics as additional evidence in support of his thesis, as the results obtained there also confirm that large portions of contemporary analysis can be carried out on the basis of theories whose strength does not exceed that of Peano Arithmetic.[18] In fact, Feferman observes that the Reverse Mathematics project has shown that "an exceptional amount of analysis is already accounted for

[17]Feferman (1993*b*, p. 443) writes: "In considering what mathematics is actually used in science it suffices to restrict attention to physics since, among all the sciences, that subject makes the heaviest use of mathematics and there is hardly any branch of mathematics, that has some scientific application, which is not applied there. It would be foolish to claim detailed knowledge of the vast body of mathematics that has been employed in mathematical physics. However, in general terms one can say that it makes primary use of mathematical analysis on Euclidean, complex, and Riemannian spaces, and of functional analysis on various Hilbert and Banach spaces. Any logical foundation for scientifically applicable mathematics should, at a minimum, cover all of 19th century mathematical analysis of (piece-wise) continuous functions on the former kind of spaces and should then go on to cover the theory of (Lebesgue) measurable functions and basic parts of 20th century functional analysis on the latter spaces."

[18]Of fundamental importance in this context is Friedman's article (Friedman 1977), that pioneers the idea of restricting set theoretic induction to give rise to proof theoretically weak constructive set theories that are simultaneously mathematically very expressive. The strategy of weakening set induction to account for no more than mathematical induction has been extensively exploited in proof theory in the last few decades. See also (Feferman 1979).

on finitistically justifiable grounds" (Feferman 1993*b*) (see also (Simpson 1988)).[19]

The question then naturally arises of how much scientifically applicable mathematics can be carried out in systems that are weaker than arithmetic. It is in the work by Ye that this question has been directly addressed (Ye 1999, Ye 2000, Ye 2008, Ye 2011).

A conjecture of finitism

Ye (2011) aims at showing that substantial portions of analysis only require a fragment of Primitive Recursive Arithmetic, that essentially corresponds to Elementary Arithmetic (EA). This is a fragment of quantifier-free primitive recursive arithmetic

[19] Feferman (1993*b*) discusses the impact of the mathematical results mentioned above for *standard* indispensability arguments in the philosophy of mathematics. Feferman's principal claim is that standard indispensability arguments are "vitiated". Feferman (1998, p. 297) writes: "...if one accepts the indispensability arguments, practically nothing philosophically definitive can be said of the entities which are then supposed to have the same status–ontologically and epistemologically–as the entities of natural science. That being the case, what do the indispensability arguments amount to? As far as I'm concerned, they are completely vitiated." The main reason for Feferman's conclusion is the following: Feferman's working hypothesis is that scientifically applicable mathematics may be developed in the system W of (Feferman 1988*b*). That system is proof-theoretically reducible to Peano Arithmetic, which, in turn, is reducible to intuitionistic arithmetic, i.e. Heyting Arithmetic (see Chapter 4, page 112 for system W). According to Feferman, the latter allow for an understanding of infinity in terms of potential rather than actual infinity. Therefore, according to Feferman, we gain no clear insight on ontological commitment. The reason for mentioning Feferman's discussion is that one might worry that the indispensability arguments to be discussed below will also be facing Feferman's own criticism. The reasons why this is not the case is that the kind of indispensability arguments discussed in this chapter substantially differ from the original ones: their aim is not an assessment of the ontological commitment of scientifically applicable mathematics, but a clarification of whether predicative mathematics includes all scientifically applicable mathematics. Feferman's main contention that predicative mathematics seems sufficient for scientifically applicable mathematics supports rather than undermine the present arguments.

(PRA), that proves the totality of only *elementary* recursive functions.[20]

Primitive recursive arithmetic (PRA) is often seen as codifying the limit of finitary reasoning, following Tait's analysis (Tait 1981). Ye's choice of EA instead of PRA brings us to a more constrained context than a form of finitism. This is due to his desire to single out the *minimum* necessary for applications: "The reason for restricting to elementary recursive functions here is to recognize the fact that, in scientific applications, perhaps elementary recursive functions are all the functions we actually need." (Ye 2011, p. 40)[21] The choice of Elementary Arithmetic is not arbitrary. EA has often been considered a remarkable theory as it is simultaneously

[20]In the following for simplicity (contravening to standard conventions) I shall also use the abbreviation EA to refer to Elementary Arithmetic without presupposing a specific formalisation of the underlying theory. EA proves the totality of addition, multiplication and exponentiation. In particular, closed terms in EA are constructed from numerals and elementary recursive functions by composition, bounded primitive recursion, finite sum, and finite product. There are a number of formal systems that have been proposed to formalize elementary arithmetic. See (Ye 2011), Section 2.1.1 for details of Ye's system **SF** of strict finitism. See also (Avigad 2003) for an introductory discussion of elementary arithmetic, its formulations and relevant conservative extensions. Note also that in the following I shall conform to Ye's use of "strict finitism" to denote a position that countenances no more than elementary arithmetic. This use of the term "strict finitism" somehow diverges from other uses of it, that refer to an even more stringent philosophy of mathematics, that does not countenances the totality of exponentiation. See also Chapter 3, Section 6.

[21] Ye's choice of system has further philosophical aims, as the availability of a very weak system for formalising applicable mathematics is crucial for his nominalistic strategy. Ye proceeds in two steps. First he argues for the (probable) eliminability of infinitary notions in favour of strictly finitary ones. This is supported by his extensive technical work that shows that a conspicuous part of analysis is expressible within a strictly finitary system. Second, he claims that from this, one can achieve the elimination of abstract mathematical notions in favour of concrete ones. The latter reduction is to be achieved by an interpretation of strict finitism along psychologistic lines. Ye (2008, p. 32) writes: "According to this philosophy, human mathematical practices are human brains' cognitive activities, and what really exist in human mathematical practices are human brains and mathematical concepts and thoughts inside brains realized as neural circuitries (and there are no alleged abstract mathematical entities 'outside the brains')."

proof theoretically extremely weak, but mathematically very robust, in the sense that it is difficult to come up with natural mathematical statements that are not expressible within EA.[22]

Ye proposes the following **Conjecture of Finitism (CF):** *"Strict finitism is in principle sufficient for formulating current scientific theories about natural phenomena above the Planck scale and for conducting proofs and calculations in those theories"* (Ye 2011, p. 38). In fact, the claim Ye defends throughout the book is that his system SF that codifies Elementary Arithmetic is sufficient for the development of scientifically applicable mathematics.

Ye offers two main considerations to support his conjecture. First of all, he offers some "intuitive reasons", which relate to the finitary nature of applications to science: when we engage in science we only need finitary and discrete magnitudes, and approximations, the appeal to infinitary notions and continuity only having an instrumental role. Secondly, he shows that one can develop large portions of analysis within his system SF. He then concludes that the applicability of infinitary mathematics to science is rooted in the applicability of the strictly finitary one. As to the first point, Ye writes:

[22] As reported in (Avigad 2003, p. 258), on April 16, 1999, Harvey Friedman posted the following conjectured to the Foundations of Mathematics mailing list: "Every theorem published in the Annals of Mathematics whose statement involves only finitary mathematical objects (i.e., what logicians call an arithmetical statement) can be proved in elementary arithmetic." For example, in particular, the conjecture implies that Fermat's last theorem is derivable in EA. Avigad (2003) reports a number of case studies that indicate some progress towards assessing the prospects of this conjecture. However, caution is required in discussing this point. Avigad (2003, p. 259) also writes: "We are a long way from settling even the more restricted conjecture [regarding Fermat's last theorem]; making real progress towards that end will require combining a deep understanding of some of the most advanced methods of modern number theory with the proof theorist's penchant for developing mathematics in restricted theories. But the conjectures are interesting because many proof theorists consider them plausible, whereas, I suspect, most mathematicians would lay long odds against them."

> In almost all mathematical applications, the physical entities we deal with are finite and discrete. Macroscopically, the universe is believed to be finite; microscopically, current well-established physics theories describe only things above the Planck scale (about $10^{-35}m$, $10^{-45}s$ etc.). Except for the theories about the microscopic structure of space-time, such as the theories of quantum gravity, all scientific theories in a broad sense, from physics to cognitive psychology and population studies, describe only finite things within the finite range from the Planck scale to the cosmological scale. In these theories, infinity and continuity in mathematics are idealizations to gloss over microscopic details or generalize beyond an unknown finite limit, in order to get simplified mathematical models of finite and discrete natural phenomena. (Ye 2011, p. 1)

This can be summarised by claiming that we have reasons to believe that physical quantities and states in the actual applications can be represented by the functions available to strict finitism because we will only need finite precision in representing physical quantities above the Planck scale.

As to the second point, Ye needs to show how to actually eliminate *all* appeals to infinitary notions within the mathematics that is applied to science. (Ye 2011) comes to a partial achievement of this goal. Ye (2011) develops large portions of contemporary analysis on the basis of SF. I shall outline below some of the strategies adopted by the author to recast in strictly finitary terms large portions of analysis. First, however, it is necessary to block a possible objection to the overall project that Ye himself considers and addresses. One might observe that we might be able to translate ordinary mathematical constructions into strictly finitary ones, and then derive within this more constrained context the corresponding parts of analysis; however, the resulting mathematics might turn out to be substantially different from the original. If this were the case, one might worry that we would be unable to retain the confidence that the applicability to science that we have so often tested for the original theory will also translate into applicability of the strictly finitary version of mathematics. To counter this objection Ye clarifies that:

> the theorems in strict finitism have very similar syntactical formats as the corresponding theorems in classical mathematics. [...] a physics theory formulated with strict finitism states the same physical facts and regularities as the original one formulated with classical mathematics. They are actually the same physics theory with different mathematical formalisms. Therefore, the development of an applied classical mathematical theory within strict finitism implies that the applications of that theory can be automatically translated into the applications of strict finitism.

These considerations are similar in spirit to those that are put forth by the constructive mathematician to clarify that the change of logic from classical to intuitionistic does not induce a distortion of ordinary mathematics (see Chapter 2). In fact, Ye's work takes as starting point the development of Bishop style mathematics, of which he further refines the techniques to allow for a finitary treatment. In particular, as in the case of constructive mathematics, also here there is no conflict with the results obtained within standard classical mathematics. This is of fundamental importance not only for Ye's claim relating the applicability of the mathematics developed so far in SF; but it is also required to make more plausible the claim that progress can be made to produce strictly finitary counterparts of other classical results that have so far not been tackled. In addition, the "miniaturization" achieved so far of classical results to the context of SF does not make use of any "ad hoc" devices, but of a combination of a number of strategies that can be further applied.

Like Feferman before him, Ye is also optimistically cautious and claims that "[m]ore work has to be done in developing applied mathematics within strict finitism, as well as in analysing what could be a counterexample to the conjecture, in order to support the conjecture better. However, based on the reasons we already have, a positive answer to the conjecture seems plausible." (Ye 2011, p. 28) The basis of this belief is given by the fact that "the general techniques used here seem to show that applied mathematics within strict finitism can advance much further." (Ye 2011, p. 27)

Finitary strategies

It is important to clarify why Ye is cautiously confident that more progress can be attained towards confirming (CF). Ye's work takes largely inspiration from Bishop's style mathematics, by using extensively the techniques developed in that context to "constructivise" classical results. Therefore, large parts of his treatment follow rather closely (Bishop & Bridges 1985). However, some ingenuity is required, as sometimes the definitions need to be stated more carefully and the recursive constructions in the proofs need to be explicitly carried out within the more limited apparatus available.[23] In addition, the last part of the book (Ye 2011) extends the results in (Bishop & Bridges 1985) by addressing directly the mathematical needs of quantum mechanics (see also (Ye 2000)).

One might still be wary and wonder how is it possible that a formalism as limited as Ye's allows for a suitable treatment of analysis, given the substantial use of sets and functions within contemporary analysis. The strategies adopted here is analogous to those used by e.g. Feferman in his predicative development of analysis

[23] In Elementary Arithmetic one can only apply bounded primitive recursion on numerical terms and induction on quantifier-free formulas. See Section 2.2.2 and 2.2.3 (Ye 2011) for an indication of general strategies the author adopts to ease the treatment of ordinary notions within a strict finitist context and to address difficulties related to restrictions on the induction. Interestingly Ye also uses Bishop's "numerical quantifiers" to ease the treatment of ordinary notions in his strictly finitary system. These were developed by Bishop in (Bishop 1970) in an attempt to apply the techniques of Gödel's Dialectica interpretation (Gödel 1958) to offer a foundation for constructive mathematics, but, to my knowledge, have since been largely forgotten.

as well as in constructive mathematics (Feferman 1988*b*, Bishop & Bridges 1985). One crucial idea is to utilise sets and functions in the "metalanguage" whenever possible. This strategy is so explained by Feferman (1993*b*, p. 446):

> "In all of the indicated formal systems one can speak within the language of these systems about arbitrary real numbers, functions of real numbers, sets of real numbers, etc. Only the existence principles (closure conditions) concerning these objects are much more restricted than in the case of systems of set theory like Zermelo's.

The idea is to utilize expressions referring to sets in a fashion that is similar to how expressions involving classes are usually introduced in ZF. Classes in ZF are abbreviations for formulas or conditions that some sets may satisfy, they are not first class objects of the theory, but 'syntactic sugar' that we employ to simplify the development of the mathematics. Similarly, here sets are "conditions for classifying terms of various types" and functions are "terms that apply to terms satisfying some conditions and produce other terms satisfying some other conditions" (Ye 2011, p. 80). Sets and functions can be appealed to in order to express *conditional statements* of the form: "provided that we can construct a term so and so, then ....". This in particular clarifies that no direct appeal to sets and functions is made, although they are used in the actual development of the mathematics to gain generality and economy. It is also clear that given this use of sets and functions, quantification over them requires particular care. For example a quantification: "for all sets A" corresponds to a statement: "for all formulas so and so", and "for all $a \in A$" is an abbreviation for a statement of the form: "whenever we can construct a term, a, that satisfies the formula so and so, then... ".[24] In cases of quantification that is not nested, one can understand quantification as a form of schematic claim defined by

[24]Note that typically one needs to specify conditions for being a member of a set, and identity conditions for sets.

formulas of some form. However, in the case of nested quantifiers and combinations with other logical constants, particular care is required.

Consequences for predicativism

The above is only a glimpse of some of the strategies utilised in (Ye 2011) to develop a strictly finitary form of analysis that is workable and sufficiently similar to the classical one. The principal reason for reviewing here very briefly how the author proceeds in constructing a form of strict finitist analysis is to offer some support to the author's claim that although more work is required to fully test the conjecture (CF), still the fundamental mathematical mechanism that is required to further extend the work accomplished so far is in place.

One can conclude that Ye's case for (CF) is based then on three components: (1) some general considerations related to the finitary nature of applications to science, (2) the success so far encountered in miniaturizing to the strictly finitary case large portions of ordinary analysis and (3) the conviction that the procedures utilized so far can be further extended.

It is then natural to ask: how can a predicativist make use of this work to support predicativism by an appeal to indispensability and dispensability arguments as (IP) and (DI)?

It is difficult to draw general conclusions on this issue, as each of the three points above seem to require further investigation before one can reach any definitive conclusion. In particular, we would need to embark on a detailed analysis of which portions of mathematics are in fact used in our best scientific theories, to see how far Ye's technical work has achieved and what is still left to do. As to the chances of further extending the present work, one would need to ensure that the restricted form of induction available in strict finitism suffices for developing any further parts of mathematics that might be required for science. However, for what concerns the purely mathematical component, a preliminary analysis suggests that it is at least possible that the mechanism that is in place will suffice to grant a formal reduction of scientifically applicable mathematics to system SF. Interestingly, system SF codifies Elementary Arithmetic, which seems to be a system Parsons is ready to accept as strictly predicative. As a consequence, strict predicativism (à la Parsons) could turn out to be formally adequate for scientifically applicable mathematics. This is clearly a remarkable prospect, and a very valuable insight on ordinary mathematics. If this were the case, we would also have that both the other forms of predicativism would be formally adequate for science, although not minimal. In particular, this would suffice to block a possible objection that may be risen to constructive mathematics: the possibility that constructive mathematics is not adequate to science. The work of Feferman, Ye and the Bishop school makes it plausible that constructive mathematics has all the necessary tools to be formally adequate for science.[25]

More crucially, one might object to the "intuitive reasons" that Ye appeals to, which relate to the finitary nature of applications to science.[26] An assessment of the latter is required to press any argument for the dispensability of impredicative mathematics, as are the issues of theoretical virtues of scientific theories that were discussed at page 238.

This preliminary analysis of "dispensability" arguments highlights that the task for the predicativist is extremely complex and the chances of success are so far unclear.[27] However, once more I wish to highlight the benefit that an analysis as

[25]See also the discussion in (Billinge 1998).

[26]For example, see (Ye n.d.).

[27] The (in)dispensability arguments discussed above differ considerably from another kind of indispensability argument that may be mounted against predicativity and take inspiration from contemporary research by Harvey Friedman into the incompleteness phenomenon. A discussion of this point is particularly complex and would require substantial space. Here I simply wish to emphasise that a different choice of the concept of "interesting mathematics" might have a radically different impact on the overall outcome of discussions of indispensability. This second kind of argument moves from within mathematics itself and exploits recent debates on whether the assumption of large cardinal axioms in set theory can be justified in view of their elementary consequences.

the one undertaken by Feferman and Ye holds for a clarification of the concepts that are involved in mathematics and of their role, within mathematics and science. I wish to conclude with an observation.

7.3.3 Indispensability and platonism?

Indispensability arguments have traditionally been presented in very general terms, stating that mathematical objects exist, without clarifying in detail which mathematical objects are deemed to exist, according to those arguments. There is a reason for this, as for nominalism the existence of only one mathematical entity is bad news. It would seem therefore that the platonist can be content with arguing that at least one mathematical object exists.

A very general estimate of the ontological commitment implied by indispensability arguments was once offered by Quine (1986) in terms of Zermelo set theory. According to the investigations reported on in the previous sections, however, this

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If this were the case, then the argument would aim at establishing that impredicative mathematics is indispensable for "core" mathematics, as strong forms of impredicativity are required to explain elementary, lower level mathematics. Friedman's recent examples of "concrete" forms of incompleteness have sparked a debate that has questioned not only the choice of the portion of mathematics that is deemed "interesting" but also the naturalness of the examples themselves and whether the impredicative notions involved ought to be taken at face value. Friedman is presently working to produce further examples to show that strongly impredicative notions are needed in the proof of natural elementary statements. The principal aim of these examples is to corroborate the thought that the *ordinary* mathematics of the future will require the assumption of large cardinal notions. If this were the case then a fundamental consequence would be that the often presupposed demarcation between elementary and strongly impredicative mathematics would be fully unjustified: even elementary mathematics would involve impredicativity in an essential way. The outcomes of this work and the relevant discussion are therefore of fundamental importance for an overall assessment of the prospects of predicativism. However, I shall have to omit a discussion of this point as the debate is still insufficiently developed to allow for a self-contained treatment, and the possible outcomes seem at present widely open.

seems far too generous. Zermelo set theory would seem to already embody elements of the strongly impredicative component of mathematics.

Feferman (1993*b*) writes:

> Answers given in the past to these [indispensability] questions have been extremely broad, on the order of: mathematical analysis is indispensable to science, the real numbers and functions and sets of reals are the basic objects of analysis, set theory provides our best account of the real number continuum and of functions and sets in general, so the entities and principles of set theory are justified by science. This sweeping passage leaves undetermined just which of those entities and principles are thereby justified, except perhaps to say that the farther reaches of set theory are evidently unnecessary for science and so may be disregarded.

A research project aiming at clarifying exactly which portion of contemporary mathematics is needed for science is a valuable contribution to the philosophy of mathematics, though difficult to pursue. It would be especially useful to fully clarify if arithmetic already formally suffices for all scientifically applicable mathematics.

The discussion in the previous sections suggests that the platonist ought to be careful in his appeal to indispensability arguments. The strategy of supporting platonism by a recourse to indispensability arguments may leave the platonist with a very meagre ontolgy. This is particularly significant as indispensability arguments have often been deemed the most compelling arguments for platonism. However, if it turns out that these arguments can, at best, support a form of moderate platonism, or, perhaps, a restricted form of finitism, this would modify our perception of what platonism in fact is. Platonism is often contrasted with revisionist approaches to mathematics, and is also often seen as a natural ally of standard set theory and impredicative methods. However, if platonism were supported by indispensability arguments, and if arithmetic were shown to be formally sufficient to develop all of scientifically applicable mathematics, then this would place strain on this common association between platonism and impredicativity. A platonist who wanted to grant ZFC set theory would have to turn to different arguments to support his views.[28]

The discussion in this chapter also clarifies once more that the relation between realism and impredicativity needs to be spelled out carefully. It is usually common to assume that a realistic attitude to the mathematical entities resolves the difficulties with impredicativity. However, we seem to require a more detailed account of this realistic attitude, what it comprises, as well as how it is spelled out. We need to ensure that that form of realism, like Poincaré's Cantorian's realism, does in fact grant sufficient room for impredicativity. There is here a strong affinity with the conclusions the logician draws from an analysis of impredicative systems like, for example, Girard's system F or the calculus of constructions. Here to prove the consistency of these calculi one normally carries out a normalization proof, showing that every computation terminates. All known proofs so far require impredicative reasoning in the meta-theory. For example, the normalization proof of Girard's system relies on set-theoretic reasoning at the meta-level. This suggests that in order to justify impredicative systems we need to resort to impredicativity at the meta-level.

7.4 Conclusion

In this chapter I have first of all suggested that a predicatively inclined mathematician may pursue predicative mathematics on the basis of the sort of methodological preferences and intra-mathematical reasons that constructive mathematicians typically adduce for working with intuitionistic logic. This further supports the thought that predicativity is a natural component of constructive mathematics in the Bishop tradition.

[28]See e.g. (Maddy 1997).

I have advanced some preliminary thoughts on how a predicativist could proceed in supporting his philosophical stance. I have suggested that an argument for predicativism would have to carefully account not only for the introduction of the constraints but also for the choice of its base. In this respect, I have considered the particular case of predicativism given the natural numbers and suggested that the discussion on strict predicativism from the previous chapter suggests that an argument for the natural number structure as base is particularly complex to achieve. It requires finding the right balance between complying with predicativity constraints for sets of natural numbers, while accepting the natural number structure as base, therefore also accounting for the principle of induction.

The predicativist may suggest that suitable forms of indispensability arguments might come to help in delivering a defense of predicativism and also in stabilising the base of stronger forms of predicativism, in particular predicativism given the natural numbers. I have proposed possible formulations of indispensability arguments that aim at assessing the role of predicative mathematics with respect to *our best scientific theories*. I have also reported on work by, among others, Feferman and Ye that gives support to the claim that very weak theories may already be formally adequate for expressing scientifically applicable mathematics. This work, if fully carried out, would give a much needed support to predicativism, in that it would block (at lest within the context of scientifically applicable mathematics) the frequent objection that it cripples the mathematics too far. Therefore from a *purely formal point of view*, an appeal to indispensability to science seems to offer the potential to clarify whether predicative mathematics of some sort is sufficient for science. It would also help us determine which base is indispensable, and therefore offer a way of justifying the choice of the base. In the particular case of predicativism given the natural numbers, if full arithmetic were found to be indispensable for science, then this could stabilise its position; if, however, a fragment of arithmetic would turn out to be sufficient, the predicativist given the natural numbers will need to offer a different kind of argument for his choice of base.

The crucial question is which conclusions may be obtained from the formal indispensability of a form of mathematics. Suppose that in the near future we were able to show that Ye's system SF is already formally sufficient to capture scientifically applicable mathematics. This does not imply in any way that impredicative mathematics is dispensable from science.[29] As mentioned above, more work (not only technical but also philosophical) needs to be carried out for a full evaluation of the relation between formal indispensability of a form of mathematics to science and its actual capability to fulfil an appropriate role within science.

My overall impression is that it is doubtful that indispensability arguments may be successfully used to reject impredicative mathematics in general. To go from the formal adequacy and minimality of a portion of ordinary mathematics with respect to scientifically applicable mathematics to the dispensability of what lays beyond it, is a huge step. If fully carried out, a research project of this kind could clarify the centrality of predicative mathematics within applicable mathematics. But the rejection of impredicativity does not follow from this. In particular, beyond the difficult task of assessing the role of mathematics within our best scientific theories (as discussed above), a full defense of predicativism, and thus the rejection of impredicative mathematics, will also require a defence of premise (P1) that I have accorded to the predicativist for the sake of argument.

Beyond the purpose of addressing the prospects of predicativism itself, the discussion in this chapter had the aim of clarifying once more the potential of a careful analysis of mathematics on the basis of weak systems. The complex task of producing new proofs of ordinary theorems within constructive and predicative systems has required the development of sophisticated tools that would have not been normally required (and perhaps found) if we were working within a more standard context.[30]

[29]Incidentally, neither Feferman nor Ye make a claim of this kind.

[30]See Chapter 2, Sections 2.3 and 2.4.2 and Chapter 4 4.3.

This has both a mathematical and a philosophical benefit. We are uncovering the computational content of large portions of mathematics, thus paving the way for computer applications. In addition, we are explicating the relation between different parts of mathematics, and their underlying assumptions. For example, we have realised the surprising fact that large portions of infinitary and impredicative mathematics can be carried out by appealing only to a restricted finitary apparatus. This offers the potential for a clarification of which concepts or entities are required for which portions of mathematics, and lays the foundation for an analysis of how to justify them. In addition, by investigating the more specific question of which portions of mathematics are indispensable to our best scientific theories, we may contribute to an understanding of the complex relation between mathematics and science.

Conclusion

In this thesis I have laid down the foundations for a philosophical study of two dimensions of constructivity that are to be found in foundational systems for constructive mathematics: intuitionistic logic and predicativity. In the following I shall review the principal outcomes of the thesis and also lay out suggestions for future work.

In the first part of the thesis, I have clarified in which sense constructive mathematics is algorithmic, by demonstrating the workings of the BHK interpretation and the Curry-Howard correspondence. I have then explored the impact that the compliance with a constructive notion of proof has for mathematics. I have especially focused my attention on techniques that have been developed to progress this form of mathematics. These techniques are among a number of tools that have been extensively used to reduce substantial portions of ordinary mathematics not only to constructive, but also to predicative formulations. They enable a fine analysis of ordinary mathematics in terms of a constructive core. Finally, I have examined intra-mathematical and pragmatic reasons that motivate constructive mathematicians to adopt intuitionistic logic in their research.

Further work needs to be carried out to elucidate the relation between classical and intuitionistic forms of mathematics; this would benefit from careful investigation of specific case studies. A thorough philosophical evaluation of the motives induced by constructive mathematicians for the adoption of constructive mathematics is also important, and could offer a contribution to contemporary debates on logical pluralism.

In the second part of the thesis I have begun an investigation of predicativity. The starting point was a distinction between two characterisations of predicativity: one in terms of circularity and one in terms of invariance. The characterisation in terms of circularity is at the heart of Russell's ramified type theory, whose introduction had enduring impact on mathematical logic and computer science. It also figured prominently within the logical analysis of predicativity that aimed at clarifying, from an impredicative perspective, the limits of predicativity given the natural numbers.

Numerous issues would deserve further investigation in this context, like, for example, the process of reflection that enables the ascent along the progression of systems of ramified second order arithmetic. Here a comparison with Feferman's notion of "unfolding" (that was introduced to avoid any reference to ordinals) would be particularly illuminating.[31] In addition, the notion of proof-theoretic reducibility requires further consideration.

The characterisation in terms of invariance gestured at by Poincaré is bound up with a constructive concept of set that deeply differs from ZFC's. This predicative concept of set may be seen as offering a unifying theme that brings together the three variants of predicativity that were discussed in this thesis: strict predicativity, predicativity given the natural numbers and constructive predicativity. It is recognisable in the writings by Poincaré, Weyl, Parsons and also figures in constructive type theory. The central character of this conception of set is the dependence of a set on a (finitary) definition which explicates a process of "generation" of the elements of the set. The resulting predicative sets are clearly badly affected by impredicative methods of definition. Poincaré's discussion on invariance may be seen as pointing towards a possible strategy for "stabilizing" sets as extensions of definitions.

The characterisation of predicativity in terms of invariance was here considered especially in relation to the underlying concept of set. Future work will have to fur-

[31] See (Feferman 2005) for references.

ther analyse this notion, its relation with the alternative characterisation in terms of circularity, and its role within the logical analysis of predicativity. Possible relations with formal aspects of constructive type theory should also be explored.

Another fundamental contribution to the study of predicativity and to an explication of the predicative concept of set was Weyl's "Das Kontinuum". Weyl's mathematical process may be seen as fully explicating the construction of predicative sets from an initial base of "uncontroversial" elements and by application of simple logical operations. The contemporary eye can not but notice with surprise the similarity of Weyl's mathematical process with the inductive construction of propositions and sets in constructive type theory.

In my discussion I have introduced the notion of predicativity "base" as an instrument for clarifying the differences between predicativity given the natural numbers and strict predicativity. I have deliberately left open how to frame the base from a philosophical perspective. It would be desirable to offer a more precise delineation of the base, including an analysis of possible philosophical interpretations of it. A philosophical challenge is then to clarify what is the philosophical difference between the entities appealed to in the base, and the ones that are "generated" under predicative constraints.

While Weyl's mathematical process assumes as base the natural number structure with full mathematical induction, Nelson and Parsons object to it on grounds of circularity. I have analysed in some detail Nelson's complaint on induction, and drawn similarities with Parson's discussion on the circularity of the inductive definition of the natural number structure.

Substantial work is still required to fully clarify the very notion of strict predicativity. In particular, an assessment of exactly how far strict predicative restrictions ought to go requires a philosophically driven logical analysis, similar to the logical analysis of Γ_0 predicativity. As suggested by Parsons, results obtained within the investigations on predicative versions of Frege's arithmetic are likely to offer very useful insights.

In my attempts to further elucidate strict predicativity I was drawn to an analysis of Dummett's indefinite extensible concepts, which has brought to the unexpected insight that a different way out, compared with strict predictivity, is available to those who share Nelson and Parsons' worry on induction: instead of restricting induction one could opt for intuitionistic logic. This would have the apparent benefit of granting a more generous portion of mathematics and a *relatively* more standard approach. The discussion of Dummett's "way out" had the advantage of fully explicating the role within the debate on impredicativity of a specific understanding of classical domains of quantification. The intuitionistic "way out", that is proposed to avoid the impasse given by the combination of this view of quantification domains with the predicative concept of set, seems also to help resolve a difficulty that was caused by the plurality of notions of predicativity: the fact that inductive definitions are considered predicative from a constructive but impredicative from a classical perspective. The thought is that given a different understanding of the workings of quantification, from a constructive perspective some circularity may be tolerable.

We are therefore left with a task: explicate the reasons for the introduction of predicativity constraints in constructive systems. More work is required to fully assess the latter point. The writings by Martin-Löf are particularly relevant in this case. In addition, one ought to compare these motivations with Fruchart and Longo's "Carnapian" route to intuitionistic impredicative type theories that was only quickly outlined in Chapter 6.

A legacy of Weyl's "Das Kontinuum" is his investigation of the mathematical extent of predicative mathematics (given the natural numbers). Weyl's work was subsequently extended by Feferman and the reverse mathematics project and the outcome was the surprising realisation that predicative mathematics (in the sense of Weyl) suffices to carry out large portions of ordinary mathematics.

I have repeatedly emphasised the possible benefits of these insights for the philosophy of mathematics; this deserves, in fact, to be worked out in detail. One option is to consider the impact of these technical results on the epistemology of mathematics. Another possible direction of investigation aims at clarifying which are the minimal mathematical assumptions that are required to carry out scientifically applicable mathematics. I have suggested a kind of indispensability argument that ought to help clarify what would be required to show that a form of predicative mathematics is indispensable to science. I have reviewed technical results that might bring some support for the formal indispensability of weak predicative forms of mathematics. I have then emphasized the difficulties that lay ahead in drawing precise conclusions from a formal indispensability result, should that be confirmed. My analysis in Chapter 7 points towards a vast possible research project that ought to employ both mathematical and philosophical instruments to further clarify the relation between predicative mathematics and the needs of our best scientific theories.

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