Skew Monoidal Categories and Grothendieck's Six Operations

by

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Abstract

In this thesis, we explore several topics in the theory of monoidal and skew monoidal categories.

In Chapter 3, we give definitions of dual pairs in monoidal categories, skew monoidal categories, closed skew monoidal categories and closed monoidal categories. In the case of monoidal and closed monoidal categories, there are multiple well-known definitions of a dual pair. We generalise these definitions to skew monoidal and closed skew monoidal categories.

In Chapter 4, we introduce semidirect products of skew monoidal categories. Semidirect products of groups are a well-known and well-studied algebraic construction. Semidirect products of monoids can be defined analogously. We provide a categorification of this construction, for semidirect products of skew monoidal categories. We then discuss semidirect products of monoidal, closed skew monoidal and closed monoidal categories, in each case providing sufficient conditions for the semidirect product of two skew monoidal categories with the given structure to inherit the structure itself.

In Chapter 5, we prove a coherence theorem for monoidal adjunctions between closed monoidal categories, a fragment of Grothendieck's 'six operations' formalism.

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Chapter 1

Introduction

Dual pairs (also known as exact pairings) are an important part of the theory of monoidal categories. See, for example, [9]; or [24], §4; or [10], §XIV.2. The notion of a dual pair in a closed monoidal category plays a role in Grothendieck's 'six operations' formalism. See, for example, [5]; or [1], where dualisable objects are referred to as rigid objects. In the theory of closed monoidal categories, the presence of a closed structure allows for various alternative, but equivalent, characterisations of a dualisable object. See, for example, [13], Chapter III, where dualisable objects are referred to as finite objects. In fact, there are characterisations of a dual pair which do not require the full structure of a closed monoidal category, or even just a monoidal category, to state. In Chapter 3, we give such characterisations of dual pairs in monoidal categories, skew monoidal categories, closed skew monoidal categories and closed monoidal categories.

Semidirect products of groups give a well-known method of constructing new examples of groups from smaller ones [23]. In Chapter 4, we introduce semidirect products of monoidal and skew monoidal categories as a categorification of semidirect products of groups, giving a method of constructing new examples of monoidal and skew monoidal categories from smaller ones. We also discuss how this construction interacts with monoidal, closed skew monoidal and closed monoidal categories, as well as dual pairs within these categories.

This chapter is based on an article due to be published in Cahiers de Topologie et Géométrie Différentielle Catégoriques [6]. The article itself is a note summarising several results and examples. The main paper, including all proofs and technical details, can be found on the arXiv (`arXiv:1510.08717 [math.CT]`).

Coherence theorems form an important part of our understanding of many categorical structures. In its most general form, the basic question which we seek to answer with a coherence theorem is as follows: Given a categorical structure of some sort (e.g. a monoidal category, a closed monoidal category, a monoidal adjunction between two closed monoidal categories), how can we tell whether a diagram constructed from the data of such a structure commutes? More specifically, we wish to describe the free such categorical structure generated by whichever data is appropriate and provide a method for determining whether two parallel morphisms in such a category are equal.

Mac Lane's original coherence theorem [18] provides, in the case of monoidal categories, the simplest possible answer to this question: that all such diagrams commute. In other words, in the free monoidal category generated by a set of objects, there is at most one morphism between any pair of objects.

Kelly and Mac Lane provided a coherence theorem for closed symmetric monoidal categories [12], based on earlier work by Lambek [16] [17]. However, in this case the result is more complicated; by introducing a closed structure, we make it possible to construct diagrams which do not commute. In order to answer the question of coherence in this setting, the notion of the 'graph' of a morphism is introduced, a concept closely tied to the extranaturality of the unit and counit for the closed monoidal structure. The coherence theorem in this case is that, for morphisms between a certain class of 'proper' objects, any pair of parallel morphisms with the same graph are equal.

Lewis provided a coherence theorem for a lax monoidal functor between two closed symmetric monoidal categories [21]. In this case the result is once again more complicated, and two notions of 'graph' are required: $\mathcal{G}$-graphs, which replace the graphs of Kelly and Mac Lane's coherence theorem and are related to the closed structure; and $\mathcal{D}$-graphs, which are new and are related to the lax monoidal functor. The coherence theorem in this case is that, for morphisms between a certain class of 'proper' objects, any pair of parallel morphisms with the same $\mathcal{G}$-graph and the same $\mathcal{D}$-graph are equal.

Došen and Petrić provided coherence theorems for lax monoidal endofunctors [2] and lax monoidal monads and comonads [3]. In the absence of any closed structure, the complications related to extranaturality which necessitated Kelly and Mac Lane's original notion of graph and Lewis's notion of $\mathcal{G}$-graph are avoided. Instead, sets and relations are used to describe the endofunctors, monads and comonads, in a role analogous to Lewis's $\mathcal{D}$-graphs. This is made more complicated by the possibility of iterating endofunctors, which is the main focus of these theorems.

In Chapter 5, we prove a coherence theorem for monoidal adjunctions between closed monoidal categories, a fragment of Grothendieck's 'six operations' formalism. A coherence theorem for monoidal adjunctions between closed monoidal categories combines both closed structures, with their attendant complications related to extranaturality, and induced lax monoidal monads and comonads, with their attendant complications related to iteration of endofunctors.

Chapter 2

Background

2.1 Monoidal and Skew Monoidal Categories

Monoidal categories, first introduced by Mac Lane [18], are ubiquitous in category theory. Examples include cartesian categories, such as the category of sets and functions or the category of topological spaces and continuous maps, where the tensor product is given by the cartesian product; categories of modules, such as the category of vector spaces and linear maps or the category of abelian groups and group homomorphisms, where the tensor product is given by the usual tensor product of modules; categories of endofunctors, where the tensor product is given by functor composition; the list goes on. For an overview of monoidal categories, see Leinster [20], §1.2.

Skew monoidal categories, first introduced by Szlachányi [25], are like ordinary monoidal categories, except that the associator and unitors are not required to be invertible. The main result of [25] is that, for a ring R, the closed skew monoidal structures on the category $\text{A}\textsc{b}_R$ of right R-modules, with the right-regular R-module as the unit $\mathcal{I}$, are precisely the right bialgebroids over R. Skew monoidal categories have subsequently been studied by others, such as Lack and Street, in other contexts (e.g. [14] [15]).

Definition 2.1.1 (skew monoidal category)**.** We will use the term 'skew monoidal category' to mean what is referred to by Szlachányi [25] as a 'right-monoidal category'. A skew monoidal category $\mathcal{C}$ consists of the following data.

- A category $\mathcal{C}$.
- An object $\mathcal{I} \in \operatorname{ob} \mathcal{C}$.
- A functor $(- \otimes -) \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.
- A natural transformation α, called the associator, with components

$$\alpha_{A,B,C} \colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C.$$

- A natural transformation λ, called the left unitor, with components

$$\lambda_A \colon A \to \mathcal{I} \otimes A.$$

- A natural transformation ρ, called the right unitor, with components

$$\rho_A \colon A \otimes \mathcal{I} \to A.$$

This is subject to the commutativity of the following diagrams.

$$
\begin{array}{ccccc}
 & & (A\otimes B)\otimes(C\otimes D) & & \\
 & \overset{\alpha_{A,B,C\otimes D}}{\nearrow} & & \overset{\alpha_{A\otimes B,C,D}}{\searrow} & \\
A\otimes(B\otimes(C\otimes D)) & & & & ((A\otimes B)\otimes C)\otimes D \\
{\scriptstyle A\otimes\alpha_{B,C,D}}\downarrow & & & & \uparrow{\scriptstyle \alpha_{A,B,C}\otimes D} \\
A\otimes((B\otimes C)\otimes D) & & \xrightarrow[\alpha_{A,B\otimes C,D}]{} & & (A\otimes(B\otimes C))\otimes D
\end{array}
\tag{2.1}
$$

$$
\begin{array}{ccc}
\mathcal{I}\otimes(B\otimes C) & \xrightarrow{\alpha_{\mathcal{I},B,C}} & (\mathcal{I}\otimes B)\otimes C \\
{\scriptstyle \lambda_{B\otimes C}}\nwarrow & & \nearrow{\scriptstyle \lambda_B\otimes C} \\
 & B\otimes C &
\end{array}
\tag{2.2}
$$

$$
\begin{array}{ccc}
A\otimes(\mathcal{I}\otimes C) & \xrightarrow{\alpha_{A,\mathcal{I},C}} & (A\otimes\mathcal{I})\otimes C \\
{\scriptstyle A\otimes\lambda_C}\nwarrow & & \swarrow{\scriptstyle \rho_A\otimes C} \\
 & A\otimes C &
\end{array}
\tag{2.3}
$$

$$
\begin{array}{ccc}
A\otimes(B\otimes\mathcal{I}) & \xrightarrow{\alpha_{A,B,\mathcal{I}}} & (A\otimes B)\otimes\mathcal{I} \\
{\scriptstyle A\otimes\rho_B}\searrow & & \swarrow{\scriptstyle \rho_{A\otimes B}} \\
 & A\otimes B &
\end{array}
\tag{2.4}
$$

$$
\begin{array}{ccc}
\mathcal{I} & = & \mathcal{I} \\
{\scriptstyle \lambda_{\mathcal{I}}}\searrow & & \nearrow{\scriptstyle \rho_{\mathcal{I}}} \\
 & \mathcal{I}\otimes\mathcal{I} &
\end{array}
\tag{2.5}
$$

Definition 2.1.2 (monoidal category)**.** A monoidal category is a skew monoidal category in which the associator α, the left unitor λ, and the right unitor ρ are all invertible.

Example 2.1.3. Introduced by Szlachányi [25], there is a skew monoidal category 'represented' by an oplax monoidal monad, defined as follows. Let T be an oplax monoidal monad on a monoidal category $\mathcal{C}$; that is, a monad on $\mathcal{C}$ whose underlying endofunctor has an oplax monoidal structure, such that the unit and multiplication are monoidal natural transformations. See definition 2.3.3 for the definition of oplax monoidal functor, and 2.3.6 for the definition of monoidal natural transformation. The objects of this skew monoidal category represented by T are the objects of $\mathcal{C}$. The monoidal unit is the monoidal unit of $\mathcal{C}$. The tensor product, denoted $(- \otimes_T -)$, is defined in terms of the tensor product in $\mathcal{C}$, denoted $(- \otimes -)$, as follows.

$$A \otimes_T B = A \otimes T(B)$$

This example is explored in more detail in Chapter 4, and appears again as Example 4.3.5.

Example 2.1.4. The category $\textsc{Set}$ of sets and functions is a monoidal category, with tensor product given by cartesian product and monoidal unit given by the terminal set $\{\star\}$.

Example 2.1.5. Choose a field $\mathbb{K}$. The category $\textsc{Vect}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces and $\mathbb{K}$-linear maps is a monoidal category, with tensor product given by ordinary tensor product of vector spaces and monoidal unit given by $\mathbb{K}$.

Example 2.1.6. The category $\textsc{Set}_\star$ of pointed sets and point-preserving functions is a monoidal category, with tensor product given by smash product and monoidal unit given by the two-element set.

$$(X, x_0) \otimes (Y, y_0) = (X \times Y/((x, y_0) \sim (x_0, y)), (x_0, y_0))$$

Example 2.1.7. The category $\textsc{Set}_\star$ of pointed sets and point-preserving functions is a skew monoidal category, with tensor product given by a 'biased disjoint union' and monoidal unit given by the one-element set.

$$(X, x_0) \otimes (Y, y_0) = (X + Y, y_0)$$

2.2 Additional Structures and Constructions

There are many kinds of additional structure that one may equip a monoidal category with, such as braidings, symmetries, closed structures, dual pairs, etc., or various combinations of these (e.g. closed symmetric monoidal structures). Some of these structures translate easily to skew monoidal catgories, while some do not. In particular, the definition of a closed skew monoidal category is straightforward, whereas the definition of a dual pair in a skew monoidal category is not. In chapter 3, we will give a definition of a dual pair in a skew monoidal category.

In order to state the definition of a closed skew monoidal category, we will need to be familiar with extranatural transformations. Extranatural transformations, first introduced by Eilenberg and Kelly [4], are a generalisation of ordinary natural transformations between functors which do not share the same source and target categories.

For comparison, recall the standard notion of naturality.

Definition 2.2.1 (naturality)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{D}$ be functors. A family of morphisms $\gamma_A\colon F(A) \to G(A)$ is said to be natural in A if, for each morphism $f\colon A \to B$ in $\mathcal{C}$, the following naturality square commutes.

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \gamma_A}\downarrow & & \downarrow{\scriptstyle \gamma_B} \\ G(A) & \xrightarrow[G(f)]{} & G(B) \end{array}$$

A family of morphisms which is natural in each variable separately, we call a natural transformation.

We can now define the notion of extranaturality.

Definition 2.2.2 (extranaturality)**.** Let $F\colon \mathcal{I} \to \mathcal{D}$ and $G\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be functors. A family of morphisms $\gamma_A\colon F(\star) \to G(A,A)$ is said to be extranatural in A if, for each morphism $f\colon A \to B$ in $\mathcal{C}$, the following extranaturality square commutes.

$$\begin{array}{ccccc} & & F(\star) & & \\ & {\scriptstyle \gamma_A}\swarrow & & \searrow{\scriptstyle \gamma_B} & \\ G(A,A) & & & & G(B,B) \\ & {\scriptstyle G(A,f)}\searrow & & \swarrow{\scriptstyle G(f,B)} & \\ & & G(A,B) & & \end{array}$$

Let $F\colon \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ and $G\colon \mathcal{I} \to \mathcal{D}$ be functors. A family of morphisms $\gamma_A\colon F(A,A) \to G(\star)$ is said to be extranatural in A if, for each morphism

$f\colon A \to B$ in $\mathcal{C}$, the following extranaturality square commutes.

$$
\begin{array}{ccccc}
 & & F(A,B) & & \\
 & {\scriptstyle F(A,f)} \swarrow & & \searrow {\scriptstyle F(f,B)} & \\
F(A,A) & & & & F(B,B) \\
 & {\scriptstyle \gamma_A} \searrow & & \swarrow {\scriptstyle \gamma_B} & \\
 & & G(\star) & &
\end{array}
$$

A family of morphisms which is natural or extranatural, as appropriate, in each variable separately, we call an extranatural transformation.

Example 2.2.3. Consider the category SET of sets and functions. There is a functor, which we will denote $[-,-]\colon \mathrm{SET}^{\mathrm{op}} \times \mathrm{SET} \to \mathrm{SET}$, defined so that $[X,Y]$ is the set of functions from X to Y. There is then a family of 'evaluation' morphisms, defined as follows.

$$\varepsilon_A^B\colon B \times [B,A] \to A \qquad \varepsilon_A^B(b,\phi) = \phi(b)$$

Since the following diagram commutes for all functions $f\colon A \to A'$, we say that ε is natural in the variable A.

$$
\begin{array}{ccc}
B \times [B,A] & \xrightarrow{B \times [B,f]} & B \times [B,A'] \\
{\scriptstyle \varepsilon_A^B} \downarrow & & \downarrow {\scriptstyle \varepsilon_{A'}^B} \\
A & \xrightarrow[f]{} & A'
\end{array}
$$

Since the following diagram commutes for all functions $f\colon B \to B'$, we say that ε is extranatural in the variable B.

$$
\begin{array}{ccccc}
 & & B \times [B',A] & & \\
 & {\scriptstyle B \times [f,A]} \swarrow & & \searrow {\scriptstyle f \times [B',A]} & \\
B \times [B,A] & & & & B' \times [B',A] \\
 & {\scriptstyle \varepsilon_A^B} \searrow & & \swarrow {\scriptstyle \varepsilon_A^{B'}} & \\
 & & A & &
\end{array}
$$

Since ε is natural in A and extranatural in B, we say that ε is an extranatural transformation.

The definition of an extranatural transformation is a key part of the definition of a closed skew monoidal category.

Definition 2.2.4 (left closed skew monoidal category)**.** We will use the term 'left closed skew monoidal category' to mean what is referred to by Szlachányi [25] as a 'left closed right-monoidal category'. A left closed skew monoidal category $\mathcal{C}$ consists of the following data.

- A skew monoidal category $\mathcal{C}$.
- A functor $(-\backslash-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$.
- An extranatural transformation η, called the coevaluation, with components
$$\eta_A^B\colon A \to B\backslash(B \otimes A).$$
- An extranatural transformation ε, called the evaluation, with components
$$\varepsilon_A^B\colon B \otimes (B\backslash A) \to A.$$

This is subject to the commutativity of the following diagrams.

$$\begin{array}{ccc} A \otimes B & = & A \otimes B \\ & {\scriptstyle A \otimes \eta_B^A} \searrow \quad \nearrow {\scriptstyle \varepsilon_{A\otimes B}^A} & \\ & A \otimes (A\backslash(A \otimes B)) & \end{array} \tag{2.6}$$

$$\begin{array}{ccc} A\backslash B & = & A\backslash B \\ & {\scriptstyle \eta_{A\backslash B}^A} \searrow \quad \nearrow {\scriptstyle A\backslash\varepsilon_B^A} & \\ & A\backslash(A \otimes (A\backslash B)) & \end{array} \tag{2.7}$$

This provides, for each object A of $\mathcal{C}$, an adjunction of the following form.

$$(A \otimes -) \dashv (A\backslash-)$$

The unit is given by η^A and the counit is given by ε^A.

Definition 2.2.5 (left closed monoidal category)**.** A left closed monoidal category is a left closed skew monoidal category in which the associator α, the left unitor λ, and the right unitor ρ are all invertible.

Remark. We use the $(-\backslash-)$ notation, used by Lambek [16] [17], instead of the standard $[-,-]$ notation for the internal hom of a closed monoidal category. We do this because having an infix symbol is more consistent with the $(- \otimes -)$ notation for the tensor product and makes complicated expressions simpler to read. We will also use the corresponding $(-/-)$ notation for right closed structures.

Example 2.2.6. Introduced by Szlachányi [25], there is a skew monoidal category 'corepresented' by a lax monoidal comonad, defined as follows. Let T be a lax monoidal comonad on a monoidal category $\mathcal{C}$; that is, a comonad on $\mathcal{C}$ whose underlying endofunctor has a lax monoidal structure, such that the counit and comultiplication are monoidal natural transformations. The objects of this skew monoidal category corepresented by T are the objects of $\mathcal{C}$. The monoidal unit is the monoidal unit of $\mathcal{C}$. The tensor product, denoted $(- \otimes_T -)$, is defined in terms of the tensor product in $\mathcal{C}$, denoted $(- \otimes -)$, as follows.

$$A \otimes_T B = T(A) \otimes B$$

If the skew monoidal category $\mathcal{C}$ has a left closed structure, then the skew monoidal category corepresented by T also has a left closed structure. The internal hom, denoted $(-\backslash_T-)$, is defined in terms of the internal hom in $\mathcal{C}$, denoted $(-\backslash-)$, as follows.

$$A\backslash_T B = T(A)\backslash B$$

Example 2.2.7. The category $\textsc{Set}$ of sets and functions is a left closed monoidal category, with the internal hom $X\backslash Y$ defined to be the set of functions $X \to Y$. The evaluation and coevaluation morphisms are defined as follows.

$$\varepsilon_Y^X \colon X \times (X\backslash Y) \to Y \qquad (x, f) \mapsto f(x)$$
$$\eta_Y^X \colon Y \to X\backslash(X \times Y) \qquad y \mapsto (x \mapsto (x, y))$$

Example 2.2.8. Choose a field $\mathbb{K}$. The category $\textsc{Vect}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces and $\mathbb{K}$-linear maps is a left closed (and also right closed) monoidal category, with the internal hom $U\backslash V$ defined to be the set of $\mathbb{K}$-linear maps $U \to V$, with pointwise addition and scalar multiplication. The evaluation and coevaluation morphisms are defined as follows.

$$\varepsilon_W^V \colon V \otimes (V\backslash W) \to W \qquad v \otimes f \mapsto f(v)$$
$$\eta_W^V \colon W \to V\backslash(V \otimes W) \qquad w \mapsto (v \mapsto v \otimes w)$$

As well as structures within monoidal and skew monoidal categories, there are constructions that can be made using monoidal and skew monoidal categories. In chapter 4, we introduce semidirect products of monoidal and skew monoidal categories as a categorification of semidirect products of monoids (or, perhaps more familiarly, of groups).

2.3 Functors and Natural Transformations

There are also various notions of functor between monoidal and skew monoidal categories.

Definition 2.3.1 (lax monoidal functor)**.** We will use the term 'lax monoidal functor' to mean what is referred to by Szlachányi as a 'right-monoidal functor' between skew monoidal categories. In the case that the skew monoidal categories are monoidal categories, this reduces to what is referred to by Leinster [20] as a 'lax monoidal functor'. A lax monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ between two skew monoidal categories consists of the following data.

- A functor $F\colon \mathcal{C} \to \mathcal{D}$.
- A natural transformation with components
$$\psi^F_{A,B}\colon F(A) \otimes F(B) \to F(A \otimes B).$$
- A morphism
$$\hat{\psi}^F : \mathcal{I} \to F(\mathcal{I}).$$

This is subject to the commutativity of the following diagrams.

$$\begin{array}{ccc}
F(A) \otimes (F(B) \otimes F(C)) & \xrightarrow{\alpha_{F(A),F(B),F(C)}} & (F(A) \otimes F(B)) \otimes F(C) \\
{\scriptstyle F(A) \otimes \psi^F_{B,C}} \downarrow & & \downarrow {\scriptstyle \psi^F_{A,B} \otimes F(C)} \\
F(A) \otimes F(B \otimes C) & & F(A \otimes B) \otimes F(C) \\
{\scriptstyle \psi^F_{A,B\otimes C}} \downarrow & & \downarrow {\scriptstyle \psi^F_{A\otimes B,C}} \\
F(A \otimes (B \otimes C)) & \xrightarrow[F(\alpha_{A,B,C})]{} & F((A \otimes B) \otimes C)
\end{array} \tag{2.8}$$

$$\begin{array}{ccccc}
 & & F(A) & & \\
 & {\scriptstyle \lambda_{F(A)}} \swarrow & & \searrow {\scriptstyle F(\lambda_A)} & \\
\mathcal{I} \otimes F(A) & & & & F(\mathcal{I} \otimes A) \\
 & {\scriptstyle \hat{\psi}^F \otimes F(A)} \searrow & & \nearrow {\scriptstyle \psi^F_{\mathcal{I},A}} & \\
 & & F(\mathcal{I}) \otimes F(A) & &
\end{array} \tag{2.9}$$

$$
\begin{array}{ccccc}
 & & F(A)\otimes F(\mathcal{I}) & & \\
 & {\scriptstyle F(A)\otimes\hat{\psi}^F}\nearrow & & \searrow{\scriptstyle \psi^F_{A,\mathcal{I}}} & \\
F(A)\otimes\mathcal{I} & & & & F(A\otimes\mathcal{I}) \\
 & {\scriptstyle \rho_{F(A)}}\searrow & & \swarrow{\scriptstyle F(\rho_A)} & \\
 & & F(A) & &
\end{array}
\tag{2.10}
$$

Example 2.3.2. The forgetful functor $U\colon \mathrm{VECT}_{\mathbb{K}} \to \mathrm{SET}$ which takes each $\mathbb{K}$-vector space to its underlying set is a lax monoidal functor. The structure maps are defined as follows.

$$\psi^U_{V,W}\colon U(V)\times U(W) \to U(V\otimes W) \qquad (v,w)\mapsto v\otimes w$$

$$\hat{\psi}^U\colon \{\star\} \to U(\mathbb{K}) \qquad \star\mapsto 1_{\mathbb{K}}$$

Definition 2.3.3 (oplax monoidal functor)**.** We will use the term 'oplax monoidal functor' to mean what is referred to by Szlachányi as a 'right-opmonoidal functor' between skew monoidal categories. In the case that the skew monoidal categories are monoidal categories, this reduces to what is referred to by Leinster [20] as a 'colax monoidal functor'. An oplax monoidal functor is a lax monoidal functor in which the structure maps ψ and $\hat{\psi}$ have their directions reversed. We will usually denote these structure maps as follows.

$$\varphi^F_{A,B}\colon F(A\otimes B) \to F(A)\otimes F(B)$$

$$\hat{\varphi}^F\colon F(\mathcal{I}) \to \mathcal{I}$$

Definition 2.3.4 (strong monoidal functor)**.** A strong monoidal functor is an oplax monoidal functor in which the structure maps φ and $\hat{\varphi}$ are both invertible.

Example 2.3.5. The free functor $F\colon \mathrm{SET} \to \mathrm{VECT}_{\mathbb{K}}$ which takes each set to the $\mathbb{K}$-vector space freely generated by its elements is a strong monoidal functor. The structure maps are defined as follows.

$$\varphi^F_{X,Y}\colon F(X\times Y) \cong F(X)\otimes F(Y) \qquad (x,y)\mapsto x\otimes y$$

$$\hat{\varphi}^F\colon F(\{\star\}) \cong \mathbb{K} \qquad \star\mapsto 1_{\mathbb{K}}$$

Of course, we also have a definition of the appropriate sort of natural transformation between these functors.

Definition 2.3.6 (monoidal natural transformation)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{D}$ be lax monoidal functors. A monoidal natural transformation is a natural transformation $\gamma\colon F \Rightarrow G$ such that the following diagrams commute.

$$\begin{array}{ccc} F(A) \otimes F(B) & \xrightarrow{\gamma_A \otimes \gamma_B} & G(A) \otimes G(B) \\ \Big\downarrow {\scriptstyle \psi^F_{A,B}} & & \Big\downarrow {\scriptstyle \psi^G_{A,B}} \\ F(A \otimes B) & \xrightarrow[\gamma_{B \otimes B}]{} & G(A \otimes B) \end{array} \tag{2.11}$$

$$\begin{array}{ccccc} & & \mathcal{I} & & \\ & \swarrow {\scriptstyle \hat{\psi}^F} & & \searrow {\scriptstyle \hat{\psi}^F} & \\ F(\mathcal{I}) & & \xrightarrow[\gamma_{\mathcal{I}}]{} & & G(\mathcal{I}) \end{array} \tag{2.12}$$

Monoidal natural transformations between oplax and strong monoidal functors are defined analogously.

There is one final definition involving monoidal categories which will be of interest to us: the monoidal adjunction. First, recall the notion of an ordinary adjunction.

Definition 2.3.7 (adjunction)**.** Let $L\colon \mathcal{D} \to \mathcal{C}$ and $R\colon \mathcal{C} \to \mathcal{D}$ be functors between categories. An adjunction $L \dashv R$ consists of the following data.

- A natural transformation θ, called the unit, with components

$$\theta_D\colon D \to RL(D).$$

- A natural transformation ζ, called the counit, with components

$$\zeta_C\colon LR(C) \to C.$$

This is subject to the commutativity of the following diagrams.

$$\begin{array}{ccccc} L(D) & & = & & L(D) \\ & \searrow {\scriptstyle L(\theta_D)} & & \nearrow {\scriptstyle \zeta_{L(D)}} & \\ & & LRL(D) & & \end{array} \tag{2.13}$$

$$\begin{array}{ccc} R(C) & = & R(C) \\ & {\scriptstyle \theta_{R(C)}} \searrow \quad \nearrow {\scriptstyle R(\zeta_C)} & \\ & RLR(C) & \end{array} \tag{2.14}$$

We can now define the notion of a monoidal adjunction.

Definition 2.3.8 (monoidal adjunction)**.** A monoidal adjunction this consists of a lax monoidal functor $L\colon \mathcal{D} \to \mathcal{C}$ with a lax monoidal right adjoint $R\colon \mathcal{C} \to \mathcal{D}$, such that the unit θ and counit ζ are monoidal natural transformations.

However, there is another, equivalent, definition of monoidal adjunction, which is the one we shall use. For a proof that these definitions are equivalent, see Kelly [11].

Definition 2.3.9 (monoidal adjunction)**.** A monoidal adjunction consists of a strong monoidal functor $L\colon \mathcal{D} \to \mathcal{C}$ with a right adjoint $R\colon \mathcal{C} \to \mathcal{D}$.

Example 2.3.10. The forgetful functor $U\colon \textsc{Vect}_{\mathbb{K}} \to \textsc{Set}$ which takes each $\mathbb{K}$-vector space to its underlying set is a lax monoidal functor. The free functor $F\colon \textsc{Set} \to \textsc{Vect}_{\mathbb{K}}$ which takes each set to the $\mathbb{K}$-vector space freely generated by its elements is a strong monoidal functor. These form a monoidal adjunction $F \dashv U$.

In chapter 5, we prove a coherence theorem for monoidal adjunctions between closed monoidal categories. This forms a fragment of Grothendieck's 'six operations' formalism. This is a formalisation which generalises the following scenario from algebraic geometry, for an introduction to which, see Hartshorne [8], Chapter II. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. This induces an inverse image–direct image adjunction between the categories of $\mathcal{O}_X$-modules and $\mathcal{O}_Y$-modules.

$$\mathcal{O}_Y\text{-}\textsc{mod} \underset{f_\star}{\overset{f^\star}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{O}_X\text{-}\textsc{mod}$$

The categories of $\mathcal{O}_X$-modules and $\mathcal{O}_Y$-modules are, among other things, left closed monoidal categories, and the adjunction $f^\star \dashv f_\star$ is a monoidal adjunction.

Chapter 3

Dual Pairs in Skew Monoidal Categories

3.1 Introduction

In §3.3, we recall the standard definition of a dual pair (L, R) in a monoidal category. This definition involves an evaluation morphism and a coevaluation morphism satisfying two relations known as the triangle identities.

$$\mathrm{ev}\colon L \otimes R \to \mathcal{I}$$

$$\mathrm{coev}\colon \mathcal{I} \to R \otimes L$$

It is known in the literature that, in a monoidal category, a dual pair (L, R) gives rise to an adjunction $(L\otimes -) \dashv (R\otimes -)$ (or, dually, an adjunction $(- \otimes R) \dashv (- \otimes L)$). See, for example, [9]. We denote the unit and counit of this adjunction as follows.

$$\theta_A\colon A \to R \otimes (L \otimes A)$$

$$\zeta_A\colon L \otimes (R \otimes A) \to A$$

In §3.4, we show that θ and ζ satisfy two conditions, each equivalent to one of the triangle identities. These conditions ensure that θ and ζ are defined entirely by their components at the monoidal unit. The morphism $\zeta_{\mathcal{I}}\colon L \otimes (R \otimes \mathcal{I}) \to \mathcal{I}$ is essentially the same as $\mathrm{ev}\colon L \otimes R \to \mathcal{I}$, and the morphism $\theta_{\mathcal{I}}\colon \mathcal{I} \to R \otimes (L \otimes \mathcal{I})$ is essentially the same as $\mathrm{coev}\colon \mathcal{I} \to R \otimes L$, and so we show that, in a monoidal category, any adjunction of this form which satisfies these two conditions must come from a dual pair, and we are therefore justified in taking this as the definition of a dual pair in a monoidal category. Since this definition does not involve inverting the associator, we may use it as the definition of a dual pair in a skew monoidal category.

It is known in the literature that, in a closed monoidal category, a dual pair (L, R) gives rise to a natural isomorphism of the following form.

$$\xi\colon (L\backslash -) \cong (R \otimes -)$$

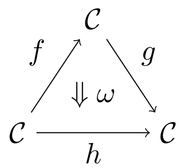
See, for example, [5]. In §3.5, we show that ξ satisfies a condition. By some standard facts about uniqueness of adjoints, we show that, in a closed skew monoidal category, any natural isomorphism of this form which satisfies this condition must come from a dual pair, and we are therefore justified in taking this as the definition of a dual pair in a closed skew monoidal category.

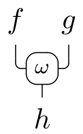
Just as the two conditions introduced in §3.4 ensure that, in the case of a monoidal category, θ and ζ are defined entirely by their components at the monoidal unit, so too does the condition introduced in §3.5 ensure that, in the case of a closed monoidal category, ξ is defined entirely by its component at the monoidal unit. And, just as there is a definition of a dual pair in a monoidal category involving morphisms ev and coev corresponding to $\zeta_{\mathcal{I}}$ and $\theta_{\mathcal{I}}$, so too is there a corresponding definition of a dual pair in a closed monoidal category involving an isomorphism $\hat{\xi}\colon L\backslash\mathcal{I} \cong R$ corresponding to $\xi_{\mathcal{I}}$. We give this definition in §3.6.

The end result, given in §3.7, is that, in a closed monoidal category, we have four different, but equivalent, definitions of a dual pair.

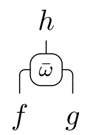
3.2 Notation

Before we begin, some notational conventions must be established. We will make use of string diagrams while working in various strict monoidal categories of endofunctors. The conventions used are that composition of functors (the monoidal product) is drawn from left to right, while composition of natural transformations (the usual composition of morphisms) is drawn from top to bottom. So, for example, the two diagrams below depict the same natural transformation; the first using a traditional globular diagram, the second using a string diagram with the conventions we will use.





We will denote the inverse of a morphism ω by $\bar{\omega}$, rather than ω^{-1}, when using string diagrams. For example, the inverse of the morphism ω, shown above, would be denoted as follows.

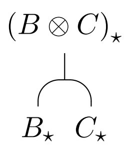


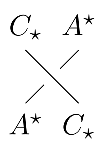
We use the following notation for the endofunctors induced by the tensor product.

$$A^{\star} = (A \otimes -)\colon \mathcal{C} \to \mathcal{C}$$

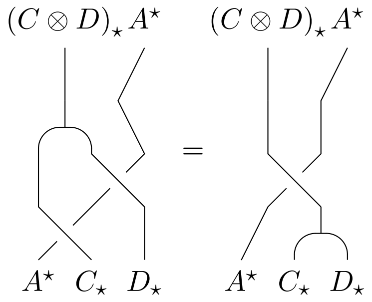
$$A_{\star} = (- \otimes A)\colon \mathcal{C} \to \mathcal{C}$$

We use the following notation for two natural transformations obtained by fixing all but one of the variables on which α depends. The third natural transformation in this sequence, which we would denote $\alpha_{A,B,-}$, will not be needed.

$\alpha_{-,B,C} =$ 

$\alpha_{A,-,C} =$ 

Viewing the objects of 2.1 as functors in the second variable gives the following equivalent condition.

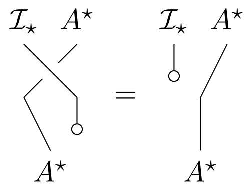


Remark. While this equation might suggest some topological intuition underlying this graphical calculus, we will neither prove nor make use of any such result. Instead, we will merely use these diagrams as convenient shorthand for traditional algebraic manipulations. In all proofs, we will take care to make explicit the algebraic justification behind each step.

We use the following notation for the right unitor; the left unitor will not be needed.

$\rho =$ 

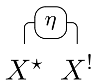
Viewing the objects of 2.4 as functors in the second variable gives the following equivalent condition.

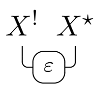


We use the following notation for the endofunctor induced by the internal hom.

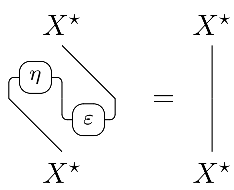
$$A^! = (A \backslash -) \colon \mathcal{C} \to \mathcal{C}$$

We use the following notation for the natural transformations obtained by fixing the variables on which η and ε depend extranaturally.

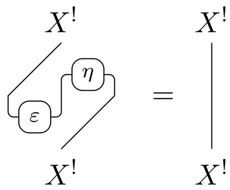
$\eta^X_- =$ 

$\varepsilon^X_- =$ 

Viewing the objects of 2.6 as functors in the natural variable gives the following equivalent condition.



Viewing the objects of 2.7 as functors in the natural variable gives the following equivalent condition.



3.3 Monoidal Categories

Recall the standard definition of a dual pair in a monoidal category. See, for example, [9]; or [24], §4; or [10], §XIV.2.

Definition 3.3.1 (dual pair in a monoidal category)**.** A dual pair in a monoidal category $\mathcal{C}$ consists of:

- a pair of objects $L, R \in \operatorname{ob}\mathcal{C}$,
- an evaluation morphism $\operatorname{ev}\colon L \otimes R \to \mathcal{I}$, and
- a coevaluation morphism $\operatorname{coev}\colon \mathcal{I} \to R \otimes L$;

such that the following two diagrams commute.

$$\begin{array}{ccc}
L & = & L \\
\downarrow{\scriptstyle \rho_L^{-1}} & & \uparrow{\scriptstyle \lambda_L^{-1}} \\
L \otimes \mathcal{I} & & \mathcal{I} \otimes L \\
\downarrow{\scriptstyle L \otimes \operatorname{coev}} & & \uparrow{\scriptstyle \operatorname{ev} \otimes L} \\
L \otimes (R \otimes L) & \xrightarrow[\alpha_{L,R,L}]{} & (L \otimes R) \otimes L
\end{array} \tag{3.1}$$

$$\begin{array}{ccc}
R & = & R \\
\downarrow{\scriptstyle \lambda_R} & & \uparrow{\scriptstyle \rho_R} \\
\mathcal{I} \otimes R & & R \otimes \mathcal{I} \\
\downarrow{\scriptstyle \operatorname{coev} \otimes R} & & \uparrow{\scriptstyle R \otimes \operatorname{ev}} \\
(R \otimes L) \otimes R & \xrightarrow[\alpha_{R,L,R}^{-1}]{} & R \otimes (L \otimes R)
\end{array} \tag{3.2}$$

We refer to L as the left dual and R as the right dual.

Finite-dimensional vector spaces are a prototypical example of objects with duals. Throughout this chapter, we will use this as a running example.

Example 3.3.2. Choose a field $\mathbb{K}$. The category $\textsc{Vect}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces and $\mathbb{K}$-linear maps is a closed symmetric monoidal category; in particular, it is a monoidal category, and the definition introduced in this chapter applies to it. Let U be a finite-dimensional $\mathbb{K}$-vector space with basis $\{u_i\}_{i\in I}$. Then U has a right dual V of the same dimension as U with basis $\{v^i\}_{i\in I}$. The evaluation and coevaluation morphisms are defined as follows.

$$\mathrm{ev}\colon U\otimes V\to\mathbb{K}\qquad u_i\otimes v^j\mapsto\begin{cases}1_{\mathbb{K}} & \text{if } i=j\\ 0_{\mathbb{K}} & \text{if } i\neq j\end{cases}$$

$$\mathrm{coev}\colon \mathbb{K}\to V\otimes U\qquad 1_{\mathbb{K}}\mapsto\sum_{i\in I}v^i\otimes u_i$$

This construction is independent of the choice of basis of U; two different choices of basis would result in two different constructions for the right dual, but they would be canonically isomorphic.

3.4 Skew Monoidal Categories

In §3.3, we gave a definition of a dual pair in a monoidal category. In this section, we will give a definition of a dual pair in a skew monoidal category. We will show that, in a monoidal category, where both definitions apply, these definitions agree.

3.4.1 Definition

In a monoidal category, there is a definition of a dual pair, equivalent to the definition introduced in §3.3, which doesn't involve the monoidal unit at all, or inverting the associator.

Proposition 3.4.1. *In a monoidal category, if we have a dual pair (L,R) then we have an adjunction $L^\star\dashv R^\star$. The unit and counit of this adjunction are given as follows.*

$$\theta_A\colon A\xrightarrow{\lambda_A}\mathcal{I}\otimes A\xrightarrow{\mathrm{coev}\otimes A}(R\otimes L)\otimes A\xrightarrow{\alpha^{-1}_{R,L,A}}R\otimes(L\otimes A)\tag{3.3}$$

$$\zeta_A\colon L\otimes(R\otimes A)\xrightarrow{\alpha_{L,R,A}}(L\otimes R)\otimes A\xrightarrow{\mathrm{ev}\otimes A}\mathcal{I}\otimes A\xrightarrow{\lambda^{-1}_A}A\tag{3.4}$$

Additionally, the following diagrams commute.

$$\begin{array}{ccccc} & & A \otimes B & & \\ & {\scriptstyle \theta_{A\otimes B}} \swarrow & & \searrow {\scriptstyle \theta_A \otimes B} & \\ R \otimes (L \otimes (A \otimes B)) & & & & (R \otimes (L \otimes A)) \otimes B \\ & {\scriptstyle R \otimes \alpha_{L,A,B}} \searrow & & \nearrow {\scriptstyle \alpha_{R,L\otimes A,B}} & \\ & & R \otimes ((L \otimes A) \otimes B) & & \end{array} \tag{3.5}$$

$$\begin{array}{ccccc} & & L \otimes ((R \otimes A) \otimes B) & & \\ & {\scriptstyle L \otimes \alpha_{R,A,B}} \nearrow & & \searrow {\scriptstyle \alpha_{L,R\otimes A,B}} & \\ L \otimes (R \otimes (A \otimes B)) & & & & (L \otimes (R \otimes A)) \otimes B \\ & {\scriptstyle \zeta_{A\otimes B}} \searrow & & \swarrow {\scriptstyle \zeta_A \otimes B} & \\ & & A \otimes B & & \end{array} \tag{3.6}$$

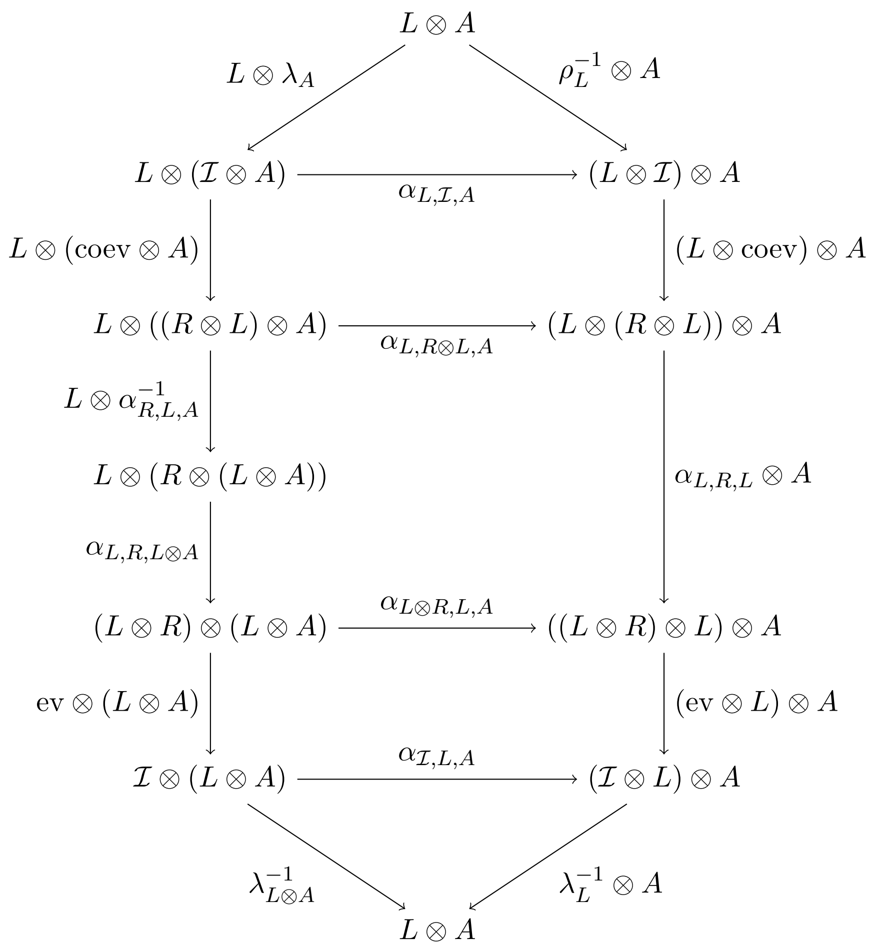
Proof. Explicitly, the triangle identities for the adjunction $L^\star \dashv R^\star$ are that the following morphisms are identities.

$$L \otimes A \xrightarrow{L\otimes\theta_A} L \otimes (R \otimes (L \otimes A)) \xrightarrow{\zeta_{L\otimes A}} L \otimes A \tag{3.7}$$

$$R \otimes A \xrightarrow{\theta_{R\otimes A}} R \otimes (L \otimes (R \otimes A)) \xrightarrow{R\otimes\zeta_A} R \otimes A \tag{3.8}$$

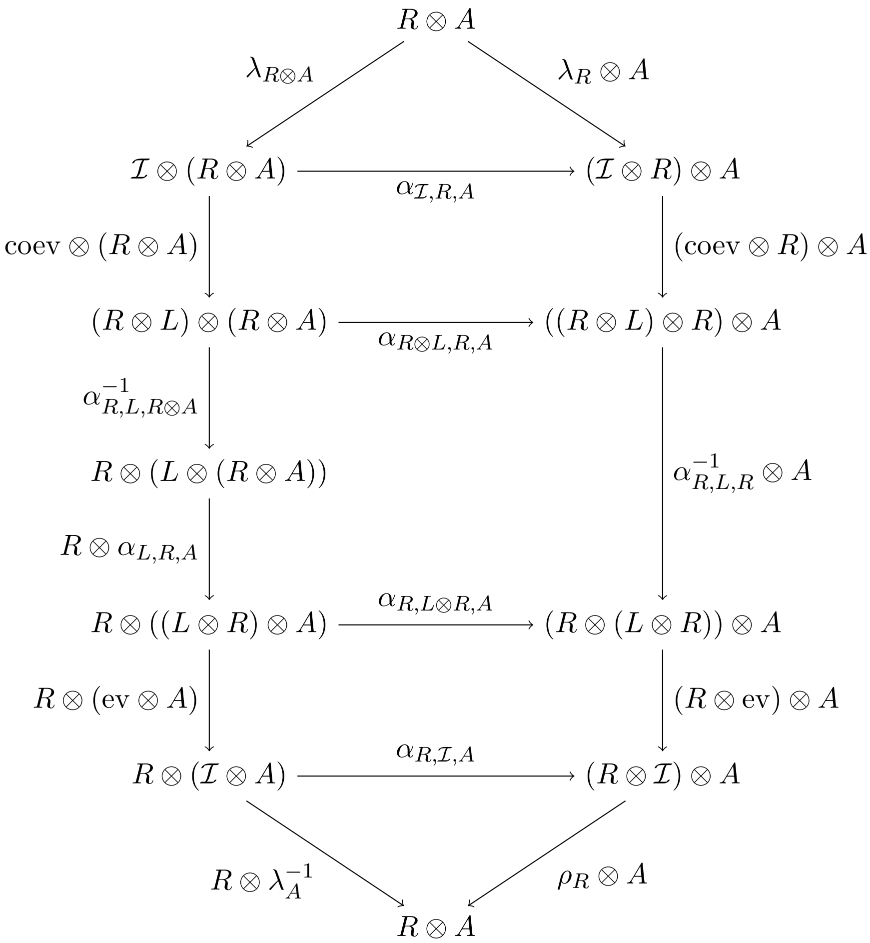
We must check the two triangle identities 3.7 and 3.8, and the two diagrams 3.5 and 3.6.

To see that the triangle identity 3.7 is satisfied, consider the following diagram.



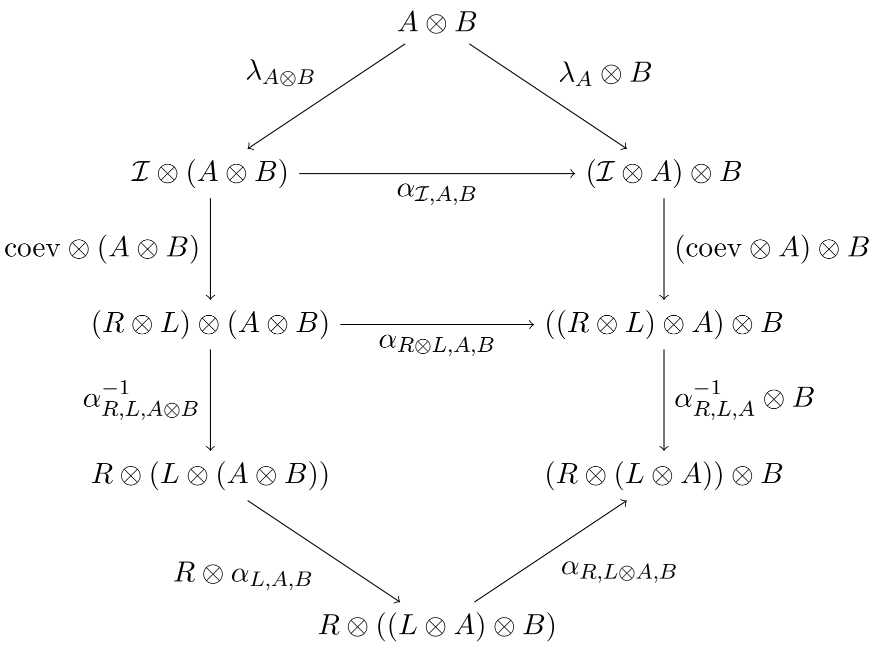
The anticlockwise path is $L \otimes \theta_A$ followed by $\zeta_{L\otimes A}$, which we are trying to show is an identity. The upper triangle commutes by condition 2.3. The upper square commutes by naturality of α. The pentagon commutes by condition 2.1. The lower square commutes by naturality of α. The lower triangle commutes by condition 2.2. The clockwise path is an identity by the triangle identity 3.1 for the dual pair (L, R).

To see that the triangle identity 3.8 is satisfied, consider the following diagram.



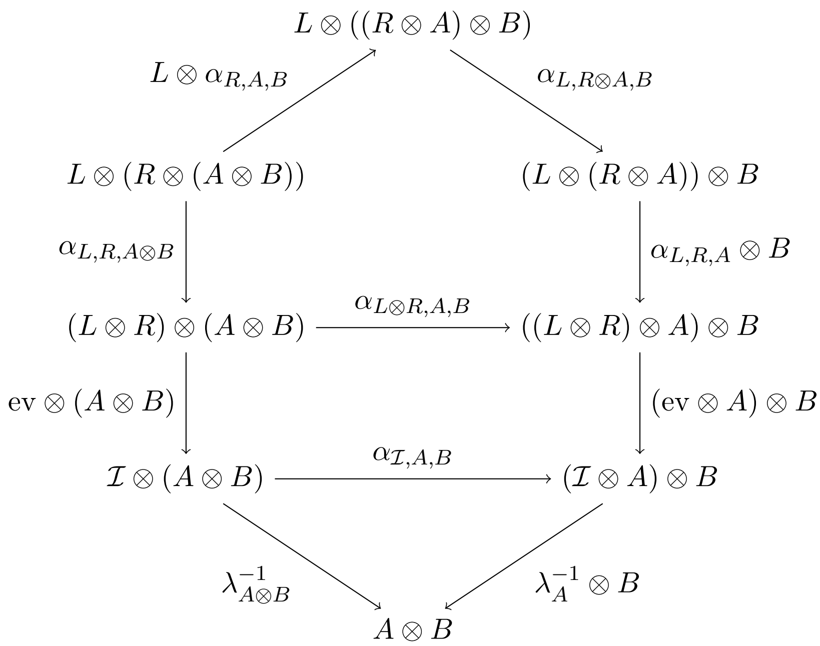
The anticlockwise path is $\theta_{A\otimes R}$ followed by $R \otimes \zeta_A$, which we are trying to show is an identity. The upper triangle commutes by condition 2.2. The upper square commutes by naturality of α. The pentagon commutes by condition 2.1. The lower square commutes by naturality of α. The lower triangle commutes by condition 2.3. The clockwise path is an identity by the triangle identity 3.2 for the dual pair (L, R).

To see that diagram 3.5 commutes, consider the following diagram.



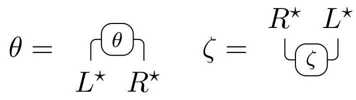
The anticlockwise path to $R \otimes (L \otimes (A \otimes B))$ is $\theta_{A\otimes B}$. The clockwise path to $(R \otimes (L \otimes A)) \otimes B$ is $\theta_A \otimes B$. The triangle commutes by condition 2.2. The square commutes by naturality of α. The pentagon commutes by condition 2.1.

To see that diagram 3.6 commutes, consider the following diagram.

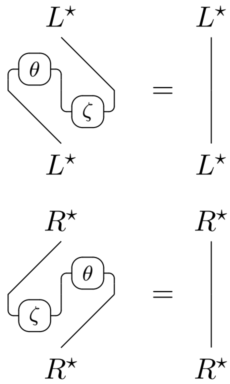


The anticlockwise path from $L \otimes (R \otimes (A \otimes B))$ is $\zeta_{A \otimes B}$. The clockwise path from $(L \otimes (R \otimes A)) \otimes B$ is $\zeta_A \otimes B$. The pentagon commutes by condition 2.1. The square commutes by naturality of α. The triangle commutes by condition 2.2. $\square$

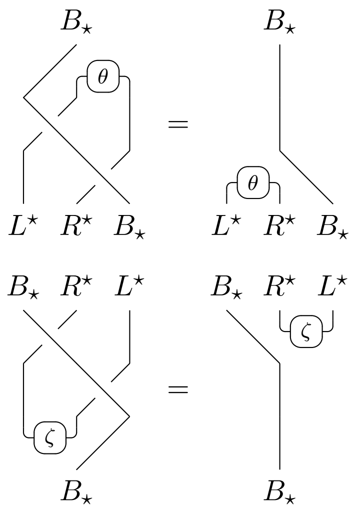
We use the following notation for for the unit and counit of the adjunction $L^\star \dashv R^\star$.



Viewing the objects of the two triangle identities 3.7 and 3.8 as functors in the variable A gives the following equivalent conditions.



Viewing the vertices of diagrams 3.5 and 3.6 as functors in the variable A gives the following equivalent conditions.



We have shown that an adjunction of this form exists whenever we have a dual pair in the usual sense. Now we must show the converse: that every such adjunction is of this form. Then we can conclude that the two notions of dual pair are equivalent.

Proposition 3.4.2. *In a monoidal category, given two objects L and R, if we have an adjunction $L^\star \dashv R^\star$ satisfying conditions 3.5 and 3.6, then we have a dual pair (L, R). The evaluation and coevaluation of this dual pair are given as follows.*

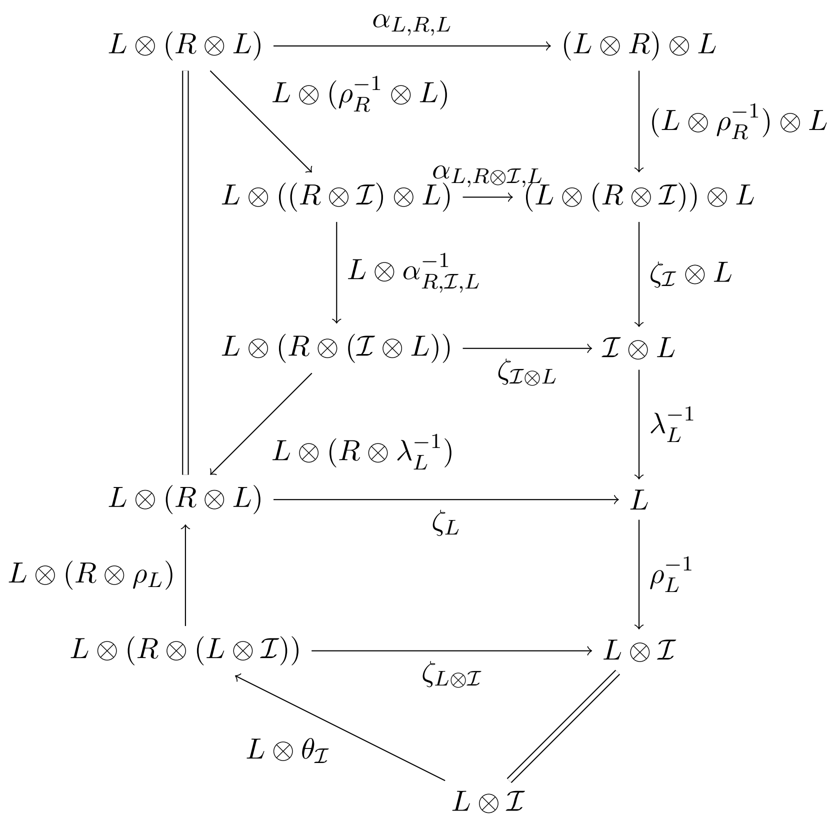
$$\mathrm{coev}\colon \mathcal{I} \xrightarrow{\theta_{\mathcal{I}}} R \otimes (L \otimes \mathcal{I}) \xrightarrow{R \otimes \rho_L} R \otimes L \tag{3.9}$$

$$\mathrm{ev}\colon L \otimes R \xrightarrow{L \otimes \rho_R^{-1}} L \otimes (R \otimes \mathcal{I}) \xrightarrow{\zeta_{\mathcal{I}}} \mathcal{I} \tag{3.10}$$

Additionally, θ and ζ are determined by coev *and* ev *as in equations 3.3 and 3.4.*

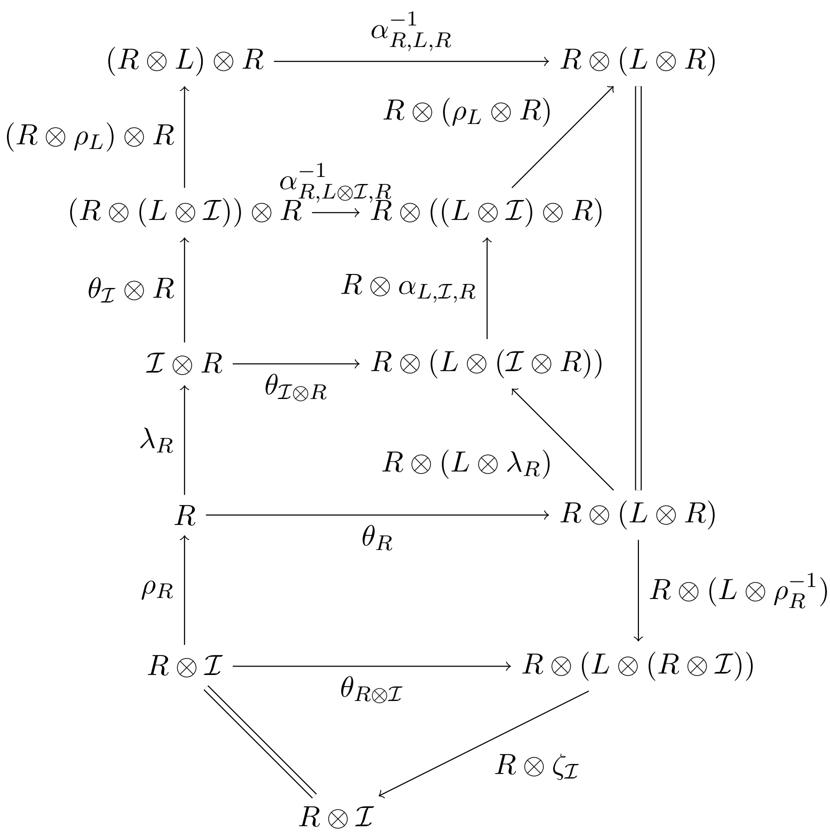
Proof. We must check the two triangle identities 3.1 and 3.2, and the two equations 3.3 and 3.4.

To see that the triangle identity 3.1 is satisfied, consider the following diagram.



The path from L to L clockwise around the outside is the morphism 3.1 we wish to show is an identity. The four squares on the right commute by, from top to bottom: naturality of α, condition 3.6, naturality of ζ, and naturality of ζ. The square on the left commutes by condition 2.3. The triangle commutes by the triangle identity 3.7 for the adjunction $L^\star \dashv R^\star$.

To see that the triangle identity 3.2 is satisfied, consider the following diagram.



The path from R to R clockwise around the outside is the morphism 3.2 we wish to show is an identity. The four squares on the left commute by, from top to bottom: naturality of α, condition 3.5, naturality of θ, and naturality of θ. The square on the right commutes by condition 2.3. The triangle commutes by the triangle identity 3.8 for the adjunction $L^\star \dashv R^\star$.

To see that equation 3.3 holds, consider the following diagram.

$$
\begin{array}{ccc}
A & \xrightarrow{\theta_A} & R \otimes (L \otimes A) \\
{\scriptstyle \lambda_A}\downarrow & & \downarrow{\scriptstyle R \otimes (L \otimes \lambda_A)} \\
\mathcal{I} \otimes A & \xrightarrow{\theta_{\mathcal{I} \otimes A}} & R \otimes (L \otimes (\mathcal{I} \otimes A)) \\
{\scriptstyle \theta_{\mathcal{I}} \otimes A}\downarrow & & \downarrow{\scriptstyle R \otimes \alpha_{L,\mathcal{I},A}} \\
(R \otimes (L \otimes \mathcal{I})) \otimes A & \xrightarrow[\alpha^{-1}_{R,L\otimes\mathcal{I},A}]{} & R \otimes ((L \otimes \mathcal{I}) \otimes A) \\
{\scriptstyle (R \otimes \rho_L) \otimes A}\downarrow & & \downarrow{\scriptstyle R \otimes (\rho_L \otimes A)} \\
(R \otimes L) \otimes A & \xrightarrow[\alpha^{-1}_{R,L,A}]{} & R \otimes (L \otimes A)
\end{array}
$$

The anticlockwise path around the outside is the morphism 3.3 we wish to show is an equal to θ_A. The upper square commutes by naturality of θ. The middle square commutes by condition 3.5. The lower square commutes by naturality of α. The right path is an identity by condition 2.3.

To see that equation 3.4 holds, consider the following diagram.

$$
\begin{array}{ccc}
L \otimes (R \otimes A) & \xrightarrow{\alpha_{L,R,A}} & (L \otimes R) \otimes A \\
{\scriptstyle L \otimes (\rho_R^{-1} \otimes A)}\downarrow & & \downarrow{\scriptstyle (L \otimes \rho_R^{-1}) \otimes A} \\
L \otimes ((R \otimes \mathcal{I}) \otimes A) & \xrightarrow{\alpha_{L,R\otimes\mathcal{I},A}} & (L \otimes (R \otimes \mathcal{I})) \otimes A \\
{\scriptstyle L \otimes \alpha^{-1}_{R,\mathcal{I},A}}\downarrow & & \downarrow{\scriptstyle \zeta_{\mathcal{I}} \otimes A} \\
L \otimes (R \otimes (\mathcal{I} \otimes A)) & \xrightarrow[\zeta_{\mathcal{I}\otimes A}]{} & \mathcal{I} \otimes A \\
{\scriptstyle L \otimes (R \otimes \lambda_A^{-1})}\downarrow & & \downarrow{\scriptstyle \lambda_A^{-1}} \\
L \otimes (R \otimes A) & \xrightarrow[\zeta_A]{} & A
\end{array}
$$

The clockwise path around the outside is the morphism 3.4 we wish to show is an equal to ζ_A. The upper square commutes by naturality of α. The middle square commutes by condition 3.6. The lower square commutes by naturality of ζ. The left path is an identity by condition 2.3. □

We have shown that, in a monoidal category, an adjunction of the form $L^\star \dashv R^\star$ satisfying conditions 3.5 and 3.6 is the same as a dual pair. This justifies the following definition.

Definition 3.4.3 (dual pair in a skew monoidal category)**.** A dual pair in a skew monoidal category $\mathcal{C}$ consists of:

- a pair of objects $L, R \in \mathrm{ob}\,\mathcal{C}$, and
- an adjunction $L^\star \dashv R^\star$;

such that the two diagrams 3.5 and 3.6 commute. We refer to L as the left dual and R as the right dual.

Recall our running example of finite-dimensional vector spaces.

Example 3.4.4. Choose a field $\mathbb{K}$. The category $\textsc{Vect}_\mathbb{K}$ of $\mathbb{K}$-vector spaces and $\mathbb{K}$-linear maps is a closed symmetric monoidal category; in particular, it is a skew monoidal category, and the definition introduced in this chapter applies to it. Let U be a finite-dimensional $\mathbb{K}$-vector space with basis $\{u_i\}_{i\in I}$. Then U has a right dual V of the same dimension as U with basis $\{v^i\}_{i\in I}$. The unit and counit for the adjunction $U^\star \dashv V^\star$ are defined as follows.

$$\theta_W \colon W \to V \otimes (U \otimes W) \qquad w \mapsto \sum_{i\in I} v^i \otimes u_i \otimes w$$

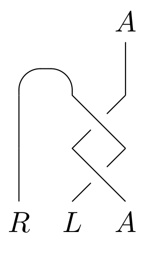
$$\zeta_W \colon U \otimes (V \otimes W) \to W \qquad u_i \otimes v^j \otimes w \mapsto \begin{cases} w & \text{if } i = j \\ 0_W & \text{if } i \neq j \end{cases}$$

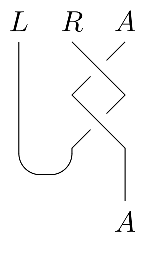
This construction is independent of the choice of basis of U; two different choices of basis would result in two different constructions for the right dual, but they would be canonically isomorphic.

3.4.2 A Non-Example

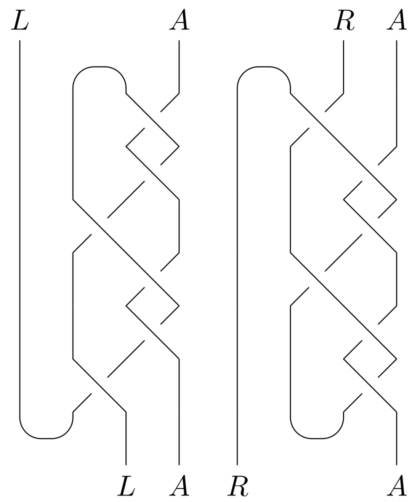
In this section, we will give an example showing that the two conditions 3.5 and 3.6 are a necessary part of the definition of a dual pair. We will be in the setting of the (strict) monoidal category of tangles. In this category, we will give an example of an adjunction of the form $L^\star \dashv R^\star$ satisfying neither condition 3.5 nor condition 3.6. Thus, this will not correspond to a dual pair (L, R).

Choose L and R to be a dual pair in the usual sense. Then define the unit and counit of the adjunction $L^\star \dashv R^\star$ as follows.

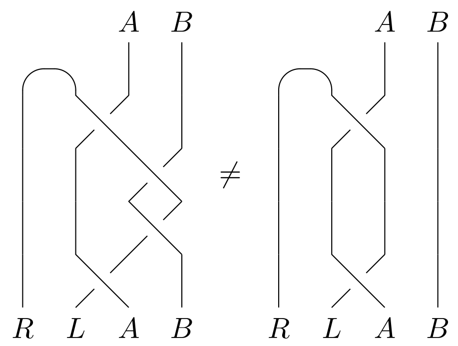
$\theta_A =$ 

$\zeta_A =$ 

This does give an adjunction $L^\star \dashv R^\star$. The triangle identities 3.7 and 3.8 correspond to the following tangles being identities, which they are.

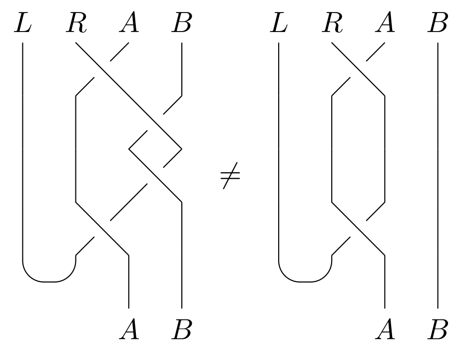


To see that condition 3.5 is not satisfied, note the following.



$$\theta_{A\otimes B} = \quad \neq \quad = \theta_A \otimes B$$

To see that condition 3.6 is not satisfied, note the following.



$$\zeta_{A\otimes B} = \quad \neq \quad = \zeta_A \otimes B$$

3.4.3 Invertibility of α

In this section, we will prove a result concerning the associator α and its interaction with dual pairs in a skew monoidal category.

Proposition 3.4.5 (invertibility of α)**.** *In a skew monoidal category, if we have a dual pair $L^\star \dashv R^\star$, then the natural transformation*

$$\alpha_{R,-,-} \colon (R \otimes (- \otimes -)) \Rightarrow ((R \otimes -) \otimes -)$$

is invertible.

Corollary 3.4.6. *Let $\mathcal{C}$ be a skew monoidal category. If each object of $\mathcal{C}$ has a left dual, then the associator α is invertible.*

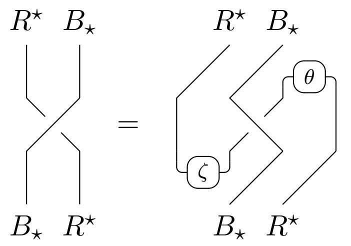
Proof. The inverse of $\alpha_{R,A,B}$ is the adjunct under the adjunction $L^\star \dashv R^\star$ of the following morphism.

$$L \otimes ((R \otimes A) \otimes B) \xrightarrow{\alpha_{L,R\otimes A,B}} (L \otimes (R \otimes A)) \otimes B \xrightarrow{\zeta_A \otimes B} A \otimes B$$

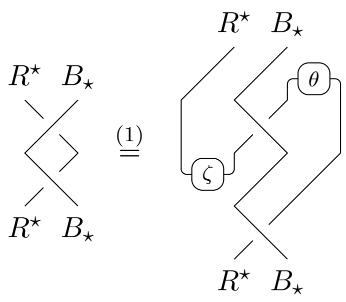
Explicitly, the inverse of $\alpha_{R,A,B}$ is defined as follows.

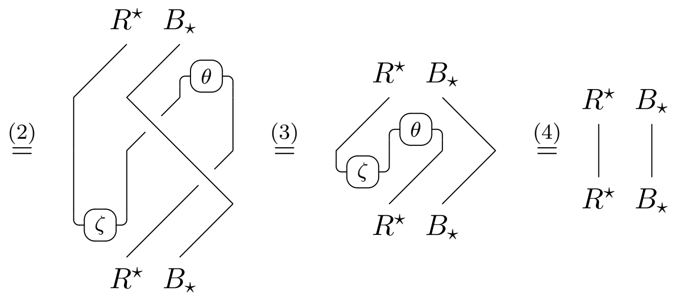
$$\begin{aligned}\alpha^{-1}_{R,A,B} \colon (R \otimes A) \otimes B &\xrightarrow{\theta_{(R\otimes A)\otimes B}} R \otimes (L \otimes ((R \otimes A) \otimes B)) \\ &\xrightarrow{R\otimes\alpha_{L,R\otimes A,B}} R \otimes ((L \otimes (R \otimes A)) \otimes B) \\ &\xrightarrow{R\otimes(\zeta_A \otimes B)} R \otimes (A \otimes B)\end{aligned}$$

We use the following notation for a natural transformation obtained by fixing all but one of the variables on which α^{-1} depends.

$\alpha^{-1}_{R,-,B} =$ 

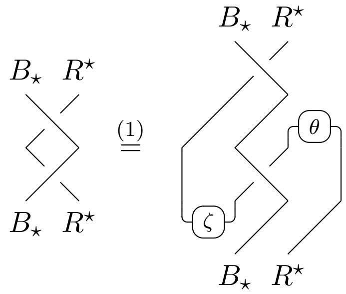
To see that $\alpha^{-1}_{R,A,B}$ is a right inverse of $\alpha_{R,A,B}$, note the following.

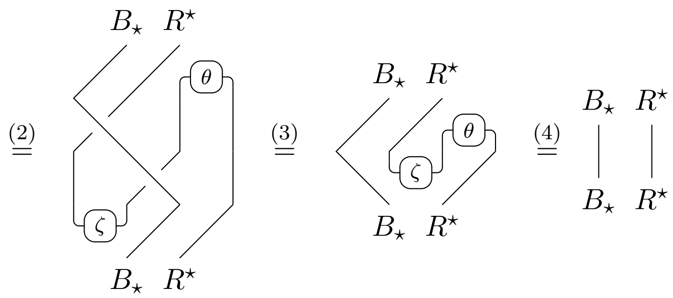




Step (1) is the definition of α^{-1}. Step (2) is naturality. Step (3) is condition 3.5. Step (4) is the triangle identity 3.8 for the adjunction $L^\star \dashv R^\star$.

To see that $\alpha^{-1}_{R,A,B}$ is a left inverse of $\alpha_{R,A,B}$, note the following.





Step (1) is the definition of α^{-1}. Step (2) is naturality. Step (3) is condition 3.6. Step (4) is the triangle identity 3.8 for the adjunction $L^\star \dashv R^\star$. □

3.5 Closed Skew Monoidal Categories

In §3.4, we gave a definition of a dual pair in a skew monoidal category. In this section, we will give a definition of a dual pair in a closed skew monoidal category. We will show that, in a closed skew monoidal category, where both definitions apply, these definitions agree.

3.5.1 Lemmas

Before mentioning dual pairs in closed skew monoidal categories, we will prove some lemmas which we will need.

In any closed skew monoidal category, we can define a morphism

$$\nu_{A,B,C} \colon (A\backslash B) \otimes C \to A\backslash(B \otimes C)$$

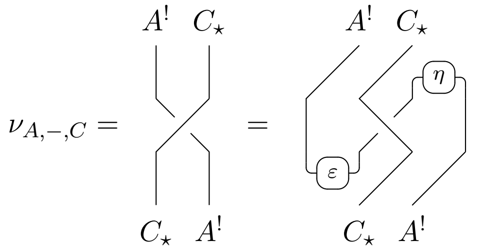
as the adjunct under the adjunction $A^\star \dashv A^!$ of the following morphism.

$$A \otimes ((A\backslash B) \otimes C) \xrightarrow{\alpha_{A,A\backslash B,C}} (A \otimes (A\backslash B)) \otimes C \xrightarrow{\varepsilon^A_B \otimes C} B \otimes C$$

Explicitly, the morphism $\nu_{A,B,C}$ is defined as follows.

$$\begin{aligned}\nu_{A,B,C} \colon (A\backslash B) \otimes C &\xrightarrow{\eta^A_{(A\backslash B)\otimes C}} A\backslash(A \otimes ((A\backslash B) \otimes C)) \\ &\xrightarrow{A\backslash\alpha_{A,A\backslash B,C}} A\backslash((A \otimes (A\backslash B)) \otimes C) \\ &\xrightarrow{A\backslash(\varepsilon^A_B \otimes C)} A\backslash(B \otimes C)\end{aligned}$$

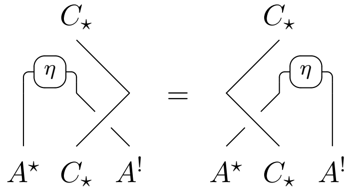
We use the following notation for a natural transformation obtained by fixing all but one of the variables on which ν depends.



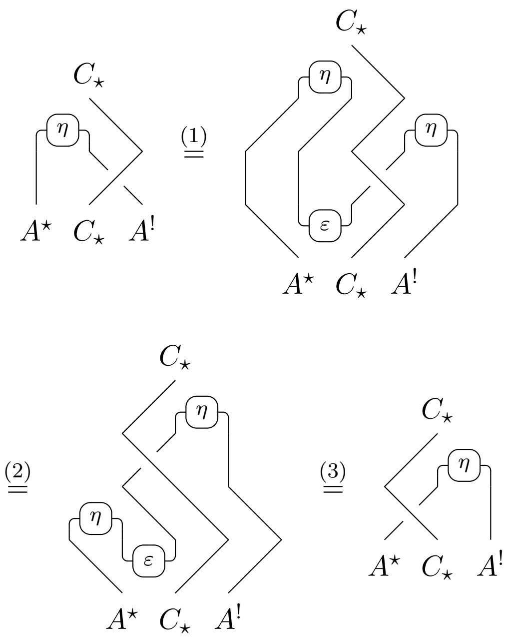
Lemma 3.5.1 (ν-η Lemma)**.** *The following diagram commutes.*

$$\begin{array}{ccc} & B \otimes C & \\ \swarrow \eta^A_{B\otimes C} & & \eta^A_B \otimes C \searrow \\ A\backslash(A \otimes (B \otimes C)) & & (A\backslash(A \otimes B)) \otimes C \\ A\backslash\alpha_{A,B,C} \searrow & & \swarrow \nu_{A,A\otimes B,C} \\ & A\backslash((A \otimes B) \otimes C) & \end{array}$$

Proof. Viewing the vertices of this diagram as functors in the variable B gives the following equivalent condition.

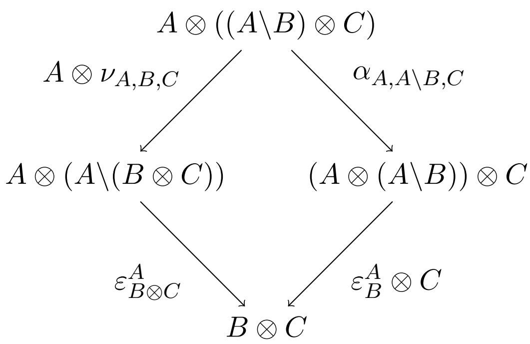


We can prove this as follows.

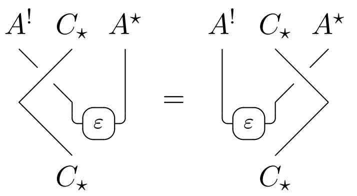


Step (1) is the definition of ν. Step (2) is naturality. Step (3) is the triangle identity 2.6 for the adjunction $A^\star \dashv A^!$. □

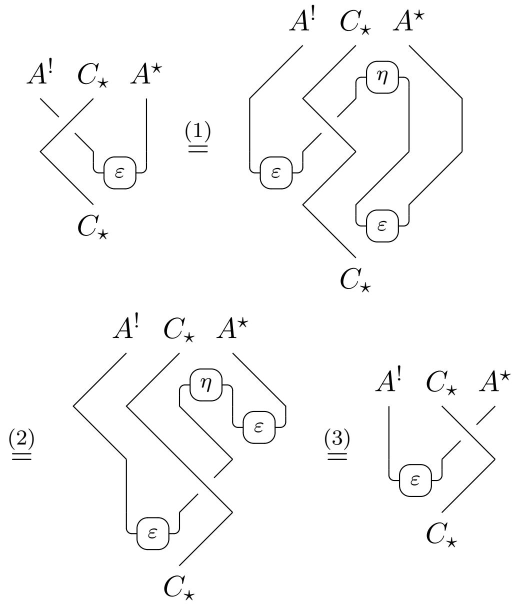
Lemma 3.5.2 (ν-ε Lemma)**.** *The following diagram commutes.*



Proof. Viewing the vertices of this diagram as functors in the variable B gives the following equivalent condition.

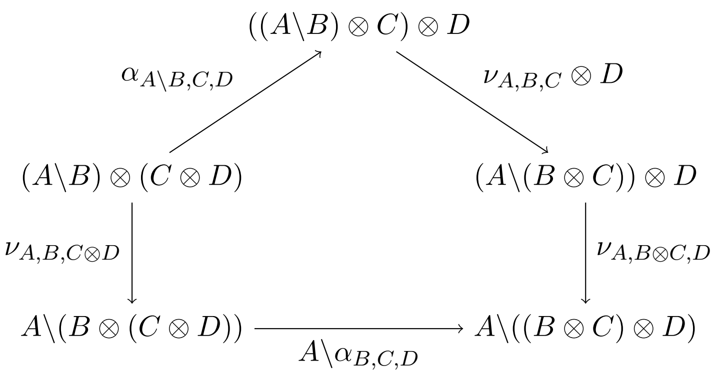


We can prove this as follows.

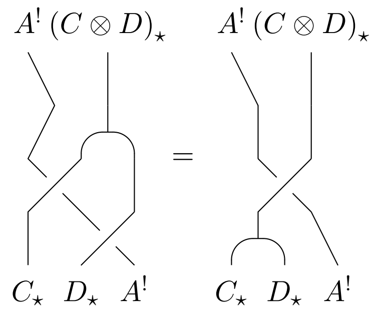


Step (1) is the definition of ν. Step (2) is naturality. Step (3) is the triangle identity 2.6 for the adjunction $A^\star \dashv A^!$. ☐

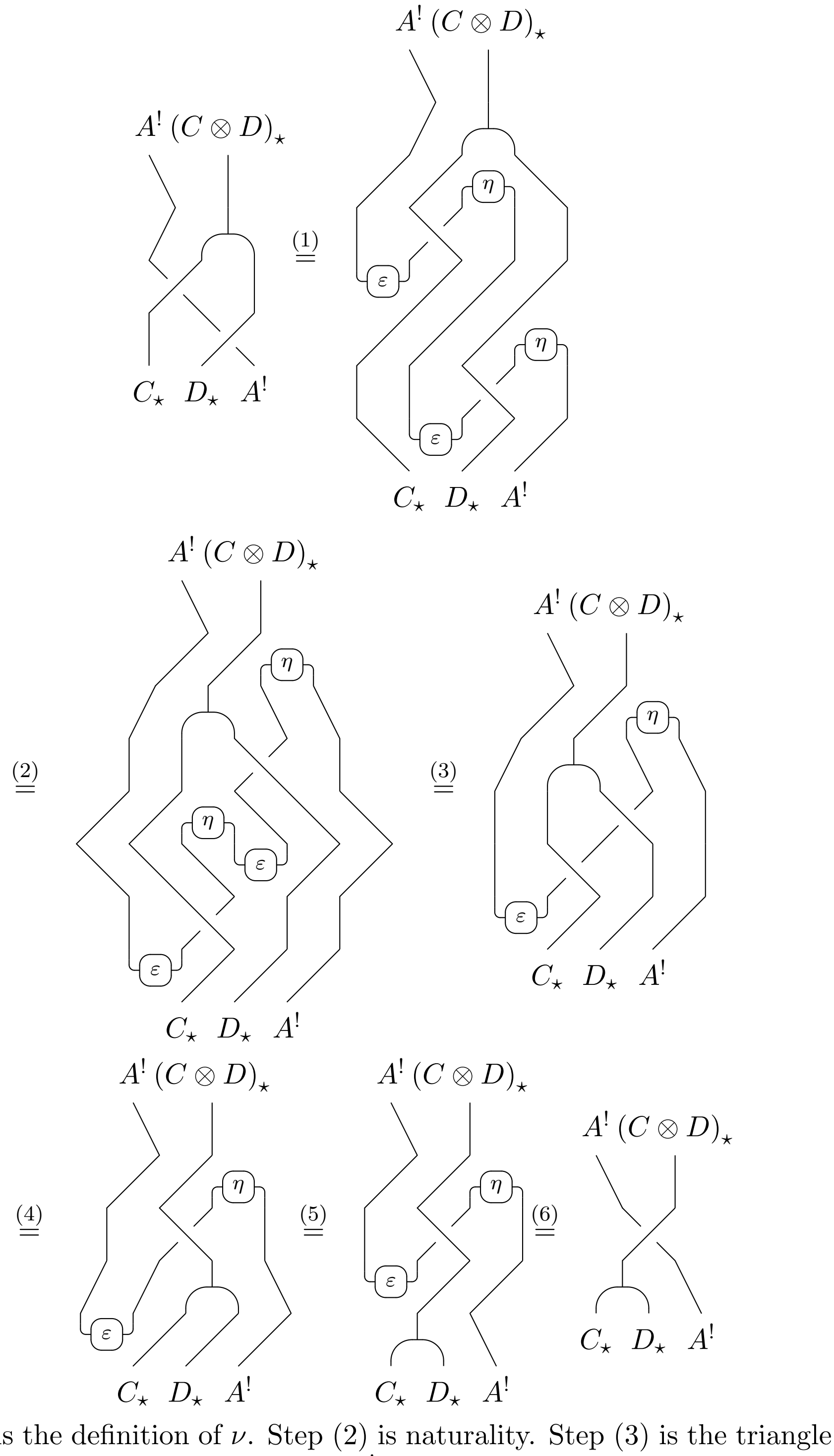
Lemma 3.5.3 (α-ν pentagon identity). *The following diagram commutes.*



Proof. Viewing the vertices of this diagram as functors in the variable B gives the following equivalent condition.

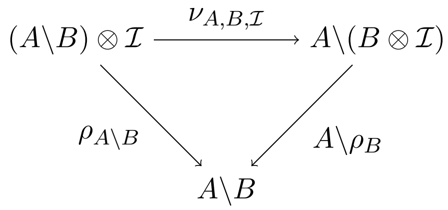


We can prove this as follows.

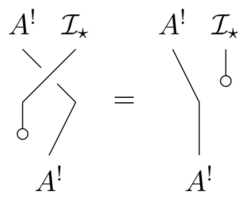


Step (1) is the definition of ν. Step (2) is naturality. Step (3) is the triangle identity 2.6 for the adjunction $A^\star \dashv A^!$. Step (4) is condition 2.1. Step (5) is naturality. Step (6) is the definition of ν. $\square$

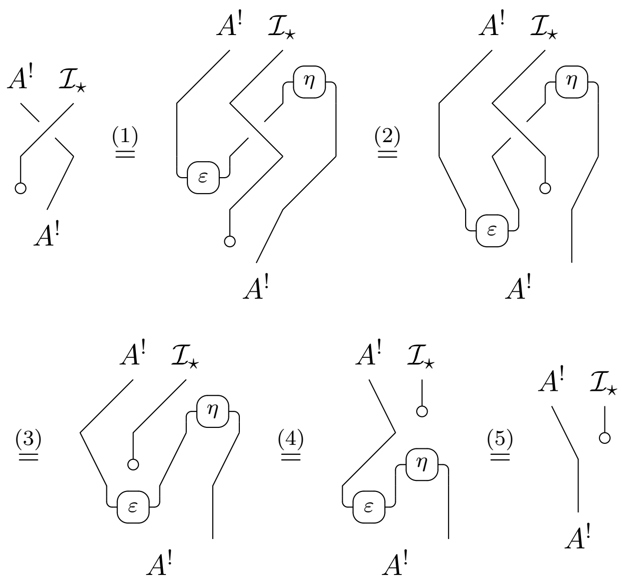
Lemma 3.5.4 (ρ-ν triangle identity)**.** *The following diagram commutes.*



Proof. Viewing the vertices of this diagram as functors in the variable B gives the following equivalent condition.



We can prove this as follows.



Step (1) is the definition of ν. Step (2) is naturality. Step (3) is condition 2.4. Step (4) is naturality. Step (5) is the triangle identity 2.6 for the adjunction $A^\star \dashv A^!$. □

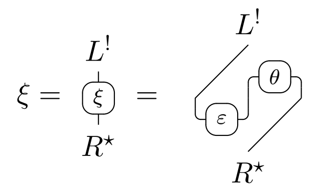
3.5.2 Uniqueness of Adjoints

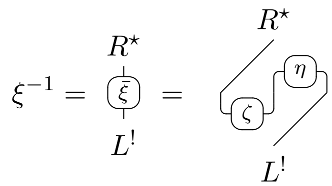
Recall some standard facts about uniqueness of adjoints, which we will make use of later. Let L and R be two objects in a closed skew monoidal category.

If we have an adjunction $L^\star \dashv R^\star$, then there is a natural isomorphism $\xi\colon L^! \cong R^\star$ between the two right adjoints of $L^\star$. The components of this isomorphism, and of its inverse, are given as follows.

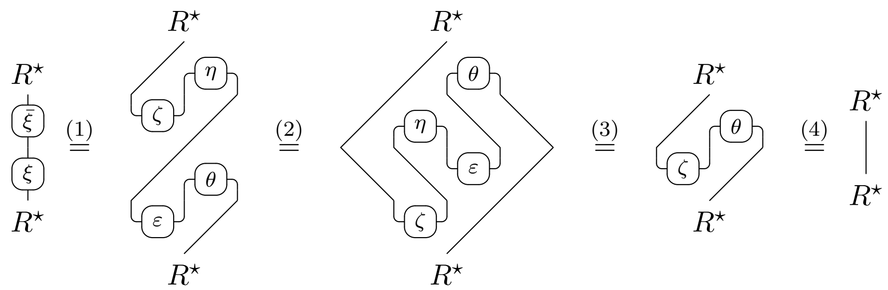
$$\xi_A\colon L\backslash A \xrightarrow{\theta_{L\backslash A}} R \otimes (L \otimes (L\backslash A)) \xrightarrow{R\otimes\varepsilon^L_A} R \otimes A \tag{3.11}$$

$$\xi_A^{-1}\colon R \otimes A \xrightarrow{\eta^L_{R\otimes A}} L\backslash(L \otimes (R \otimes A)) \xrightarrow{L\backslash\zeta_A} L\backslash A \tag{3.12}$$



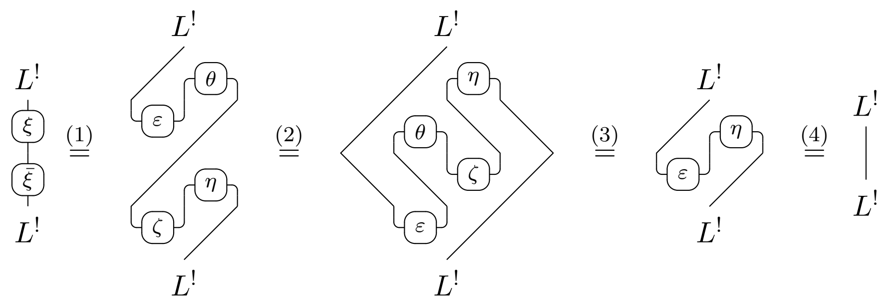


To see that ξ^{-1} is a right inverse of ξ, note the following.



Step (1) is the definitions of ξ^{-1} and ξ. Step (2) is naturality. Step (3) is the triangle identity 2.6 for the adjunction $L^\star \dashv L^!$. Step (4) is the triangle identity 3.8 for the adjunction $L^\star \dashv R^\star$.

To see that ξ^{-1} is a left inverse of ξ, note the following.

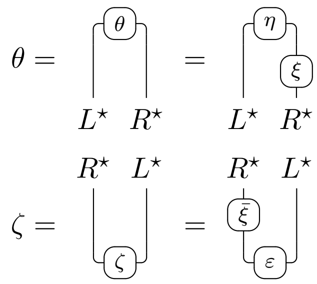


Step (1) is the definitions of ξ and ξ^{-1}. Step (2) is naturality. Step (3) is the triangle identity 3.7 for the adjunction $L^\star \dashv R^\star$. Step (4) is the triangle identity 2.7 for the adjunction $L^\star \dashv L^!$.

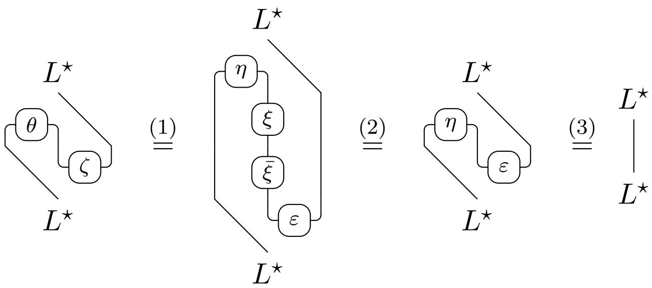
Similarly, if we have a natural isomorphism $\xi\colon L^! \cong R^\star$, then there is an adjunction $L^\star \dashv R^\star$. The unit and counit for this adjunction are given as follows.

$$\theta_A\colon A \xrightarrow{\eta^L_A} L\backslash(L \otimes A) \xrightarrow{\xi_{L\otimes A}} R \otimes (L \otimes A) \tag{3.13}$$

$$\zeta_A\colon L \otimes (R \otimes A) \xrightarrow{L\otimes\xi^{-1}_A} L \otimes (L\backslash A) \xrightarrow{\varepsilon^L_A} A \tag{3.14}$$

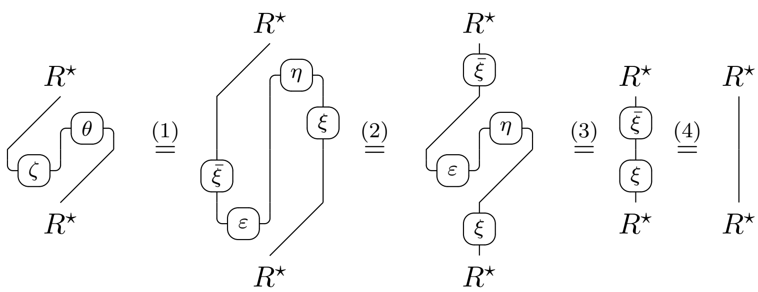


To see that the triangle identity 3.7 holds, note the following.



Step (1) is the definitions of θ and ζ. Step (2) is ξ^{-1} being a left inverse of ξ. Step (3) is the triangle identity 2.6 for the adjunction $L^\star \dashv L^!$.

To see that the triangle identity 3.8 holds, note the following.



Step (1) is the definitions of θ and ζ. Step (2) is naturality. Step (3) is the triangle identity 2.7 for the adjunction $L^\star \dashv L^!$. Step (4) is ξ^{-1} being a right inverse of ξ.

It can easily be checked that this gives a one-to-one correspondence between such adjunctions and such isomorphisms.

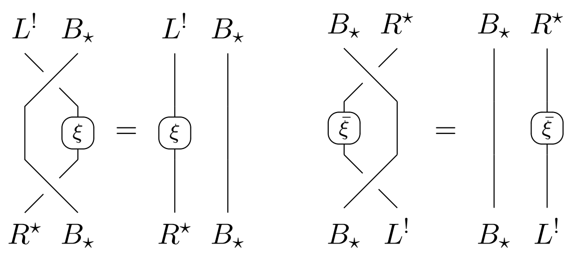
3.5.3 Definition

In a closed skew monoidal category, there is a definition of a dual pair, equivalent to the definition introduced in §3.4, involving the closed structure.

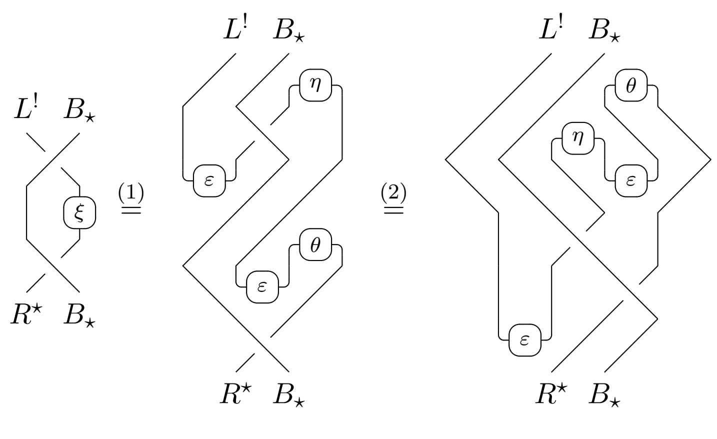
Proposition 3.5.5. *In a closed skew monoidal category, if we have a dual pair $L^\star \dashv R^\star$ and a corresponding natural isomorphism $\xi \colon L^! \cong R^\star$, then the following diagram commutes.*

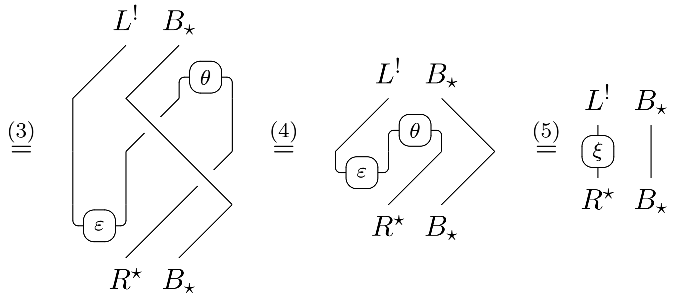
$$\begin{array}{ccc} (L\backslash A)\otimes B & \xrightarrow{\nu_{L,A,B}} & L\backslash(A\otimes B) \\ {\scriptstyle \xi_A\otimes B}\big\downarrow & & \big\downarrow{\scriptstyle \xi_{A\otimes B}} \\ (R\otimes A)\otimes B & \xleftarrow[\alpha_{R,A,B}]{} & R\otimes(A\otimes B) \end{array} \tag{3.15}$$

Proof. Viewing the vertices of diagram 3.15 as functors in the variable A, and considering Lemma (3.4.5), gives either of the following two equivalent conditions.



We can prove the first of these as follows.





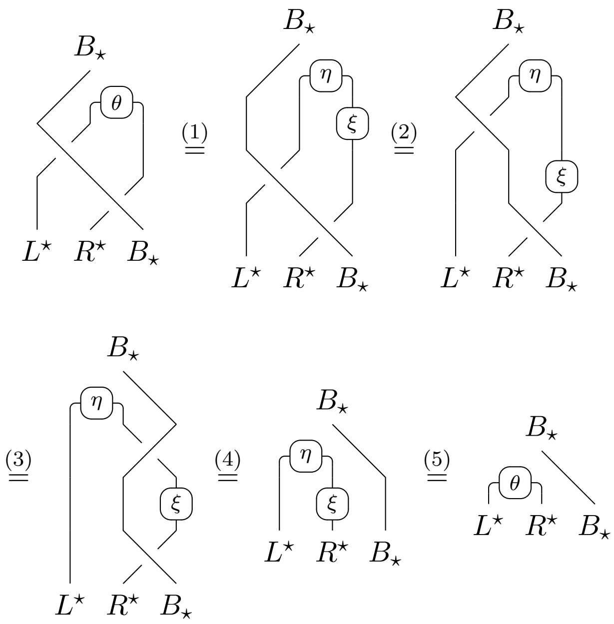
Step (1) is the definition of ν and ξ. Step (2) is naturality. Step (3) is the triangle identity 2.6 for the adjunction $L^\star \dashv L^!$. Step (4) is condition 3.5. Step (5) is the definition ξ. □

We have shown that a natural isomorphism of this sort exists whenever we have a dual pair. Now we must show the converse: that every such natural isomorphism is of this form. Then we can conclude that the two notions of dual pair are equivalent.

Proposition 3.5.6. *In a closed skew monoidal category, if we have a natural isomorphism $\xi\colon L^! \cong R^\star$ satisfying condition 3.15, then the corresponding adjunction $L^\star \dashv R^\star$ forms a dual pair.*

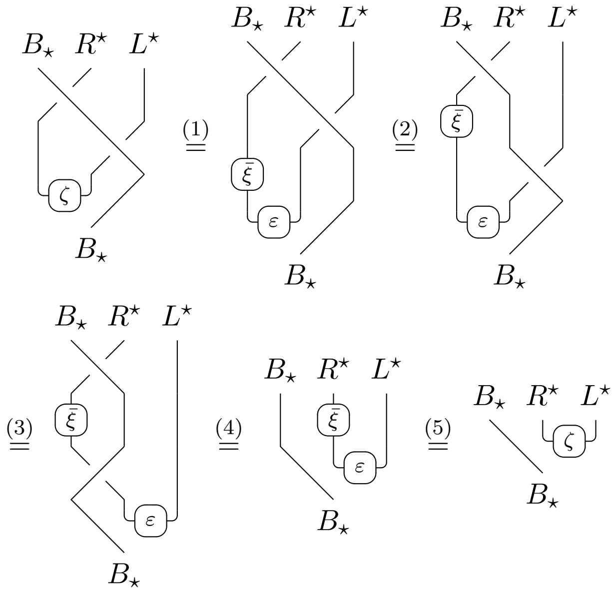
Proof. We must check conditions 3.5 and 3.6.

To see that condition 3.5 holds if condition 3.15 holds, note the following.



Step (1) is the definition of θ. Step (2) is naturality. Step (3) is Lemma 3.5.1. Step (4) is condition 3.15. Step (5) is the definition of θ.

To see that condition 3.6 holds if condition 3.15 holds, note the following.



Step (1) is the definition of ζ. Step (2) is naturality. Step (3) is Lemma 3.5.2. Step (4) is condition 3.15. Step (5) is the definition of ζ. ☐

We have shown that, in a closed skew monoidal category, an isomorphism satisfying condition 3.15 is the same as a dual pair. This justifies the following definition.

Definition 3.5.7 (dual pair in a closed skew monoidal category)**.** A dual pair in a closed skew monoidal category consists of:

- a pair of objects $L, R \in \operatorname{ob}\mathcal{C}$, and
- a natural isomorphism $\xi\colon L^! \cong R^\star$;

such that the diagram 3.15 commutes. We refer to L as the left dual and R as the right dual.

Recall our running example of finite-dimensional vector spaces.

Example 3.5.8. Choose a field $\mathbb{K}$. The category $\mathrm{V{\scriptsize ECT}}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces and $\mathbb{K}$-linear maps is a closed symmetric monoidal category; in particular, it is a closed skew monoidal category, and the definition introduced in this chapter applies to it. Let U be a finite-dimensional $\mathbb{K}$-vector space with basis $\{u_i\}_{i\in I}$. Then U has a right dual V of the same dimension as U with basis $\{v^i\}_{i\in I}$. The natural isomorphism $\xi\colon U^! \cong V^\star$ is defined as follows.

$$\xi_W\colon U\backslash W \cong V \otimes W \qquad f \mapsto \sum_{i\in I} v^i \otimes f(u_i)$$

This construction is independent of the choice of basis of U; two different choices of basis would result in two different constructions for the right dual, but they would be canonically isomorphic.

3.5.4 Invertibility of ν

In this section, we will prove a result concerning the natural transformation ν and its interaction with dual pairs in a closed skew monoidal category.

Proposition 3.5.9 (invertibility of ν)**.** *In a closed skew monoidal category, if we have a dual pair $\xi\colon L^! \cong R^\star$, then the natural transformation*

$$\nu_{L,-,-}\colon ((L\backslash -) \otimes -) \Rightarrow (L\backslash(- \otimes -))$$

is invertible.

Corollary 3.5.10. *Let $\mathcal{C}$ be a closed skew monoidal category. If each object of $\mathcal{C}$ has a right dual, then the natural transformation ν is invertible.*

Proof. By condition 3.15, the inverse is defined as follows.

$$\begin{aligned}\nu^{-1}_{L,A,B}\colon L\backslash(A \otimes B) &\xrightarrow{\xi_{A\otimes B}} R \otimes (A \otimes B)\\ &\xrightarrow{\alpha_{R,A,B}} (R \otimes A) \otimes B\\ &\xrightarrow{\xi_A^{-1}\otimes B} (L\backslash A) \otimes B\end{aligned}$$

□

3.6 Closed Monoidal Categories

In §3.5, we gave a definition of a dual pair in a closed skew monoidal category. In this section, we will give a definition of a dual pair in a closed monoidal category. We will show that, in a closed monoidal category, where both definitions apply, these definitions agree.

3.6.1 Definition

In a closed monoidal category, there is a definition of a dual pair, equivalent to the definition introduced in §3.5, involving the closed structure and the monoidal unit.

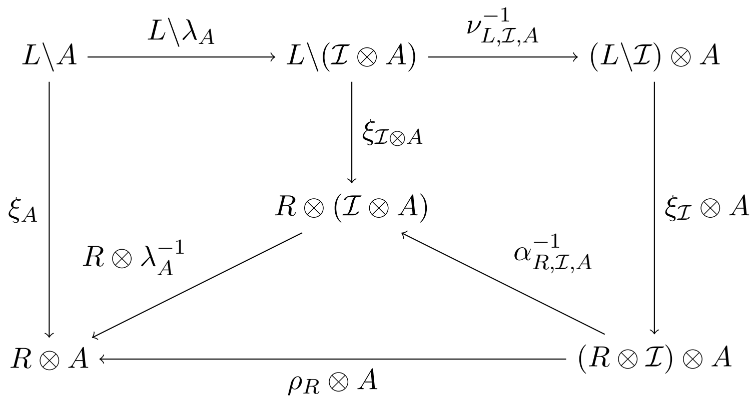
Proposition 3.6.1. *In a closed monoidal category, if we have a dual pair $\xi\colon L^! \cong R^\star$, then we have an isomorphism*

$$\hat{\xi}\colon L\backslash\mathcal{I} \xrightarrow{\xi_{\mathcal{I}}} R \otimes \mathcal{I} \xrightarrow{\rho_R} R,$$

and ξ is determined by $\hat{\xi}$, as follows.

$$\xi_A\colon L\backslash A \xrightarrow{L\backslash\lambda_A} L\backslash(\mathcal{I} \otimes A) \xrightarrow{\nu^{-1}_{L,\mathcal{I},A}} (L\backslash\mathcal{I}) \otimes A \xrightarrow{\hat{\xi}\otimes A} R \otimes A \tag{3.16}$$

Proof. Consider the following diagram.



The leftmost square commutes by naturality of ξ. The rightmost square commutes by condition 3.15. The triangle commutes by condition 2.3. □

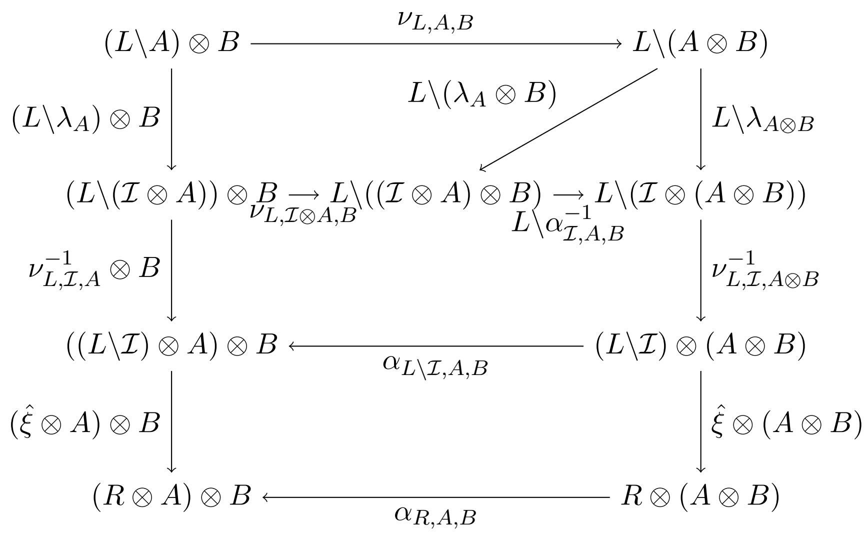
We have shown that an isomorphism of this sort exists whenever we have a dual pair. Now we must show the converse: that every such isomorphism can be extended to a natural isomorphism corresponding to a dual pair. Then we can conclude that the two notions of dual pair are equivalent. In order to ensure that ξ_A, as defined in equation 3.16, is an isomorphism, we must also require that the natural transformation $\nu_{L,\mathcal{I},-}$ be invertible. In fact, in §3.6.2, we will show that it suffices to only assume that the morphism $\nu_{L,\mathcal{I},L}$ is invertible.

Proposition 3.6.2. *In a closed monoidal category, if we have an isomorphism $\hat{\xi}\colon L\backslash\mathcal{I} \cong R$ and if the natural transformation*

$$\nu_{L,\mathcal{I}-}\colon ((L\backslash\mathcal{I}) \otimes -) \to (L\backslash(\mathcal{I} \otimes -))$$

is invertible, then we have a dual pair $\xi\colon L^! \cong R^\star$, with ξ defined by equation 3.16.

Proof. To see that condition 3.15 holds, consider the following diagram.



The upper square commutes by naturality of ν. The triangle commutes by condition 2.2. The pentagon commutes by Lemma 3.5.3. The lower square commutes by naturality of α. □

We have shown that, in a closed monoidal category, an isomorphism $\hat{\xi} \colon L\backslash\mathcal{I} \cong R$ for an object L for which the natural transformation $\nu_{L,\mathcal{I},-}$ is invertible is the same as a dual pair. This justifies the following definition.

Definition 3.6.3 (dual pair in a closed monoidal category)**.** A dual pair in a closed monoidal category consists of:

- a pair of objects $L, R \in \operatorname{ob}\mathcal{C}$, and
- a isomorphism $\hat{\xi} \colon L\backslash\mathcal{I} \cong R$;

such that the natural transformation $\nu_{L,\mathcal{I},-}$ is invertible. We refer to L as the left dual and R as the right dual.

Recall our running example of finite-dimensional vector spaces.

Example 3.6.4. Choose a field $\mathbb{K}$. The category $\textsc{Vect}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces and $\mathbb{K}$-linear maps is a closed symmetric monoidal category; in particular, it is a closed monoidal category, and the definition introduced in this chapter applies to it. Let U be a finite-dimensional $\mathbb{K}$-vector space with basis $\{u_i\}_{i\in I}$. Then U has a right dual V of the same dimension as U with basis $\{v^i\}_{i\in I}$. The natural isomorphism $\hat{\xi} \colon U\backslash\mathbb{K} \cong V$ is defined as follows.

$$\hat{\xi} \colon U\backslash\mathbb{K} \cong V \qquad f \mapsto \sum_{i\in I} f(u_i) \cdot v^i$$

This construction is independent of the choice of basis of U; two different choices of basis would result in two different constructions for the right dual, but they would be canonically isomorphic.

3.6.2 Invertibility of ν

We have shown that, in a closed monoidal category, if an object L has a right dual, then the right dual is always isomorphic to $L\backslash\mathcal{I}$. Thus, we may, without loss of generality, assume that the right dual is $L\backslash\mathcal{I}$.

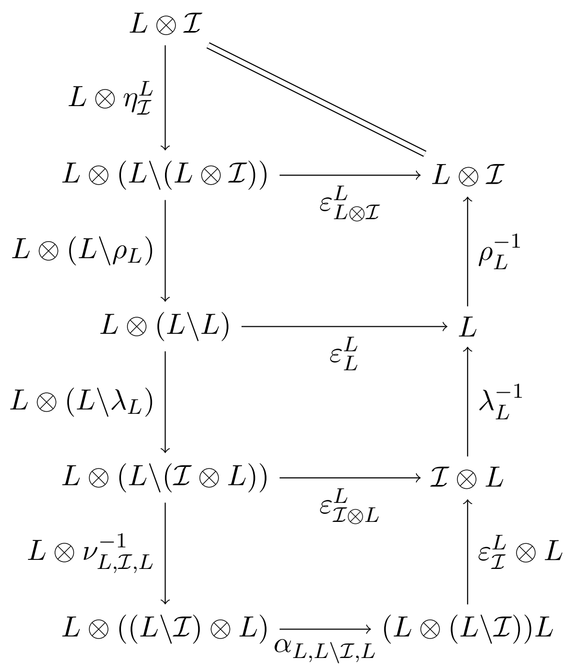
Furthermore, we will show that we can weaken the condition that the natural transformation $\nu_{L,\mathcal{I},-}$ is invertible to the condition that the morphism $\nu_{L,\mathcal{I},L}$ is invertible.

Lemma 3.6.5. *In a closed monoidal category, if we have an object L for which the morphism $\nu_{L,\mathcal{I},L}$ is invertible, then we have a dual pair (in the sense introduced in §3.3) $(L, L\backslash\mathcal{I})$, with evaluation and coevaluation morphisms defined as follows.*

$$\mathrm{ev}\colon L\otimes(L\backslash\mathcal{I}) \xrightarrow{\varepsilon^L_{\mathcal{I}}} \mathcal{I}$$

$$\mathrm{coev}\colon \mathcal{I} \xrightarrow{\eta^L_{\mathcal{I}}} L\backslash(L\otimes\mathcal{I}) \xrightarrow{L\backslash\rho_L} L\backslash L \xrightarrow{L\backslash\lambda_L} L\backslash(\mathcal{I}\otimes L) \xrightarrow{\nu^{-1}_{L,\mathcal{I},L}} (L\backslash\mathcal{I})\otimes L$$

Proof. To see that condition 3.1 holds, consider the following diagram.



The anticlockwise path around the outside from L to L is the morphism 3.1 we wish to show is an identity. The triangle commutes by the triangle identity 2.7 for the adjunction $L^\star \dashv L^!$. The middle two squares commute by naturality of ε. The lower square commutes by Lemma 3.5.2.

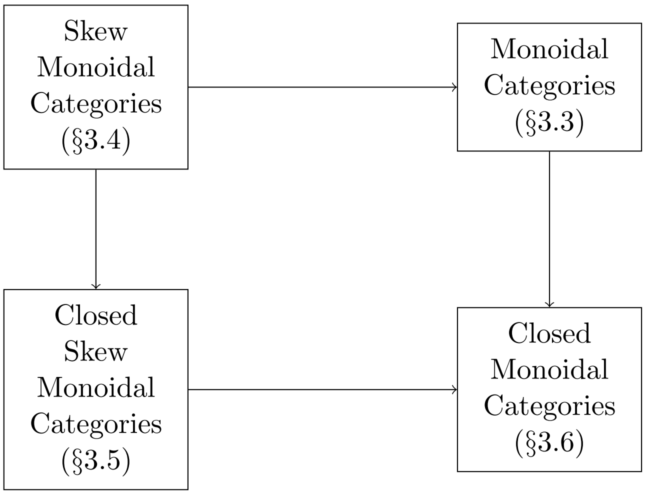
To see that condition 3.2 holds, consider the following diagram.

$$
\begin{tikzcd}
L\backslash\mathcal{I} \arrow[rr, "\eta^L_{L\backslash\mathcal{I}}"] \arrow[d, "\lambda_{L\backslash\mathcal{I}}"'] & & L\backslash(L\otimes(L\backslash\mathcal{I})) \arrow[d, "L\backslash(L\otimes\lambda_{L\backslash\mathcal{I}})"] \\
\mathcal{I}\otimes(L\backslash\mathcal{I}) \arrow[rr, "\eta^L_{\mathcal{I}\otimes(L\backslash\mathcal{I})}"] \arrow[d, "\eta^L_{\mathcal{I}}\otimes(L\backslash\mathcal{I})"'] & & L\backslash(L\otimes(\mathcal{I}\otimes(L\backslash\mathcal{I}))) \arrow[d, "L\backslash\alpha_{L,\mathcal{I},L\backslash\mathcal{I}}"] \\
(L\backslash(L\otimes\mathcal{I}))\otimes(L\backslash\mathcal{I}) \arrow[rr, "\nu_{L,L\otimes\mathcal{I},L\backslash\mathcal{I}}"] \arrow[d, "(L\backslash\rho_L)\otimes(L\backslash\mathcal{I})"'] & & L\backslash((L\otimes\mathcal{I})\otimes(L\backslash\mathcal{I})) \arrow[d, "L\backslash(\rho_L\otimes(L\backslash\mathcal{I}))"] \\
(L\backslash L)\otimes(L\backslash\mathcal{I}) \arrow[rr, "\nu_{L,L,L\backslash\mathcal{I}}"] \arrow[d, "(L\backslash\lambda_L)\otimes(L\backslash\mathcal{I})"'] & & L\backslash(L\otimes(L\backslash\mathcal{I})) \arrow[d, "L\backslash(\lambda_L\otimes(L\backslash\mathcal{I}))"] \\
(L\backslash(\mathcal{I}\otimes L))\otimes(L\backslash\mathcal{I}) \arrow[rr, "\nu_{L,\mathcal{I}\otimes L,L\backslash\mathcal{I}}"] \arrow[d, "\nu^{-1}_{L,\mathcal{I},L}\otimes(L\backslash\mathcal{I})"'] & & L\backslash((\mathcal{I}\otimes L)\otimes(L\backslash\mathcal{I})) \arrow[dd, "L\backslash\alpha^{-1}_{\mathcal{I},L,L\backslash\mathcal{I}}"] \\
((L\backslash\mathcal{I})\otimes L)\otimes(L\backslash\mathcal{I}) \arrow[d, "\alpha^{-1}_{L\backslash\mathcal{I},L,L\backslash\mathcal{I}}"'] & & \\
(L\backslash\mathcal{I})\otimes(L\otimes(L\backslash\mathcal{I})) \arrow[rr, "\nu_{L,\mathcal{I},L\otimes(L\backslash\mathcal{I})}"] \arrow[d, "(L\backslash\mathcal{I})\otimes\varepsilon^L_{\mathcal{I}}"'] & & L\backslash(\mathcal{I}\otimes(L\otimes(L\backslash\mathcal{I}))) \arrow[dl, "L\backslash(\mathcal{I}\otimes\varepsilon^L_{\mathcal{I}})"'] \arrow[dd, "L\backslash\lambda^{-1}_{L\otimes(L\backslash\mathcal{I})}"] \\
(L\backslash\mathcal{I})\otimes\mathcal{I} \arrow[r, "\nu_{L,\mathcal{I},\mathcal{I}}"] \arrow[d, "\rho_{L\backslash\mathcal{I}}"'] & L\backslash(\mathcal{I}\otimes\mathcal{I}) \arrow[dl, "L\backslash\lambda^{-1}_{\mathcal{I}}"'] & \\
L\backslash\mathcal{I} & & L\backslash(L\otimes(L\backslash\mathcal{I})) \arrow[ll, "L\backslash\varepsilon^L_{\mathcal{I}}"]
\end{tikzcd}
$$

The left path is the morphism 3.2 we wish to show is an identity. The right path is an identity by conditions 2.3 and 2.2. The remaining clockwise path around the outside is an identity by the triangle identity 2.7 for the adjunction $L^\star \dashv L^!$. The six squares commute by, from top to bottom: naturality of η, Lemma 3.5.1, naturality of ν, naturality of ν, naturality of ν, and naturality of λ. The pentagon commutes by Lemma 3.5.3. The triangle commutes by Lemma 3.5.4 and condition 2.5. $\square$

3.7 Conclusion

We have considered definitions of dual pairs in monoidal categories, skew monoidal categories, closed skew monoidal categories and closed monoidal categories. Consider the following diagram, where arrows indicate specialisation; that is, any definition which makes sense in one structure can be transported along an arrow to a more specific structure, in which it will still make sense.



In §3.3, we recalled the definition of a dual pair in a monoidal category. In §3.4, we gave a definition of a dual pair in a skew monoidal category and showed that, in any monoidal category, it agrees with the definition given in §3.3. In §3.5, we gave a definition of a dual pair in a closed skew monoidal category and showed that, in any closed skew monoidal category, it agrees with the definition given in §3.4. In §3.6, we gave a definition of a dual pair in a closed monoidal category and showed that, in any closed monoidal category, it agrees with the definition given in §3.5. It is known in the literature that, in a closed monoidal category, the two definitions of dual pair introduced in §3.3 and §3.6 are equivalent, at least in the symmetric case. See, for example, [13].

Thus, in a closed monoidal category, the most specific of the four structures we have considered, in which all four definitions apply, we have shown that all four definitions agree. This can be summarised with the following theorem.

Theorem 3.7.1. *In a closed monoidal category, given a pair of objects L and R, the following are equivalent.*

- *A pair of morphisms*

$$\mathrm{ev}\colon L \otimes R \to \mathcal{I} \qquad \mathrm{coev}\colon \mathcal{I} \to R \otimes L$$

satisyfing conditions 3.1 and 3.2.

- *An adjunction*
$$L^\star \dashv R^\star$$
satisfying conditions 3.5 and 3.6.

- *A natural isomorphism*
$$\xi\colon L^! \cong R^\star$$
satisfying condition 3.15.

- *An isomorphism*
$$\hat{\xi}\colon L\backslash\mathcal{I} \cong R$$
and an inverse to the morphism
$$\nu_{L,\mathcal{I},L}\colon (L\backslash\mathcal{I})\otimes L \to L\backslash(\mathcal{I}\otimes L).$$

Furthermore, if the above conditions hold, then the natural transformation

$$\nu_{L,-,-}\colon ((L\backslash -)\otimes -) \Rightarrow (L\backslash(-\otimes -))$$

is invertible.

Chapter 4

Semidirect Products of Skew Monoidal Categories

4.1 Introduction

The main ingredient of a group semidirect product is an action of a group G on another group K; this can be concisely defined as a group homomorphism from G to the group of group automorphisms on K. It is straightforward to generalise this to semidirect products of monoids. In this chapter, we will categorify semidirect products of monoids to semidirect products of skew monoidal categories.

In §4.2, we will explain how to categorify the notion of actions of monoids to actions of skew monoidal categories. Instead of a monoid homomorphism from a monoid to the endomorphism monoid of another monoid, we use a monoidal functor from a monoidal category to the endofunctor category of another monoidal category. Some care must be taken in choosing which sorts of functors between monoidal categories we wish to consider. The majority of this section is devoted to examining the coherence data and conditions involved in this definition.

In §4.3, we will explain how to categorify the notion of semidirect products of monoids to semidirect products of skew monoidal categories. The majority of this section is devoted to proving the coherence conditions involved in this definition.

In §4.4, §4.5, §4.6 and §4.7, we will give sufficient conditions for a semidirect product skew monoidal category to be a monoidal category, be a left closed skew monoidal category, contain a dual pair, and be a right closed skew monoidal category, respectively.

Several examples in this chapter will involve generalised metric spaces, as described by Lawvere [19]. Because of this, we will now give a brief description of the closed symmetric monoidal category of generalised metric spaces, which we will denote by Met.

Let $[0, \infty]$ denote the set of non-negative real numbers and positive infinity.

Definition 4.1.1 (generalised metric space)**.** A generalised metric space M consists of the following.

- A set $\mathrm{ob}(M)$.
- A metric function $M\colon \mathrm{ob}(M) \times \mathrm{ob}(M) \to [0, \infty]$.

In addition to this, the metric must satisfy the following conditions.

- For all $m \in \mathrm{ob}(M)$,
$$M(m, m) = 0.$$
- For all $m_1, m_2, m_3 \in \mathrm{ob}(M)$,
$$M(m_1, m_2) + M(m_2, m_3) \geq M(m_1, m_3).$$

Definition 4.1.2 (morphism of generalised metric spaces)**.** Let M and N be generalised metric spaces. A morphism $f\colon M \to N$ is a distance-non-increasing map. Explicitly, this is a function

$$f\colon \mathrm{ob}(M) \to \mathrm{ob}(N)$$

satisfying the condition that, for all $m_1, m_2 \in \mathrm{ob}(M)$,

$$M(m_1, m_2) \geq N(f(m_1), f(m_2)).$$

Definition 4.1.3 (tensor product of generalised metric spaces)**.** Let M and N be generalised metric spaces. The tensor product $M \otimes N$ is defined as follows.

- An object of $M \otimes N$ consists of an object of M and an object of N.

$$\mathrm{ob}(M \otimes N) = \mathrm{ob}(M) \times \mathrm{ob}(N)$$

- The metric is defined as follows.

$$(M \otimes N)((m_1, n_1), (m_2, n_2)) = M(m_1, m_2) + N(n_1, n_2)$$

Definition 4.1.4 (internal hom of generalised metric spaces)**.** Let M and N be generalised metric spaces. The internal hom $M\backslash N$ is defined as follows.

- An object of $M\backslash N$ is a morphism $M \to N$.

$$\mathrm{ob}(M\backslash N) = \textsc{Met}(M, N)$$

- The metric is defined as follows.

$$(M\backslash N)(f, g) = \sup_{m \in M} N(f(m), g(m))$$

4.2 Actions

In this section, we will explain how to categorify the notion of actions of monoids to actions of skew monoidal categories. We will then spend some time going through, in detail, all of the data making up such an action. We will assume familiarity with the concepts of strong, lax and oplax monoidal functors, and monoidal natural transformations.

As is often the case when categorifying, there are some choices to be made as to the direction certain morphisms should take and whether or not they should be invertible. We will focus on one such choice, which happens to be convenient for our definition of semidirect products of skew monoidal categories, and call it simply a weak action.

Definition 4.2.1 (weak action)**.** Let $\mathcal{X}$ and $\mathcal{C}$ be skew monoidal categories. A 'weak action' of $\mathcal{X}$ on $\mathcal{C}$ is a lax monoidal functor Γ from $\mathcal{X}$ to $[\mathcal{C}, \mathcal{C}]_{\text{oplax}}$, the strict monoidal category of oplax monoidal endofunctors on $\mathcal{C}$ and monoidal natural transformations between them with tensor product given by functor composition; i.e. $(F \otimes G)(C) = F(G(C))$.

There is quite a bit of data involved in this definition, so we will spend some time going through the structure maps involved.

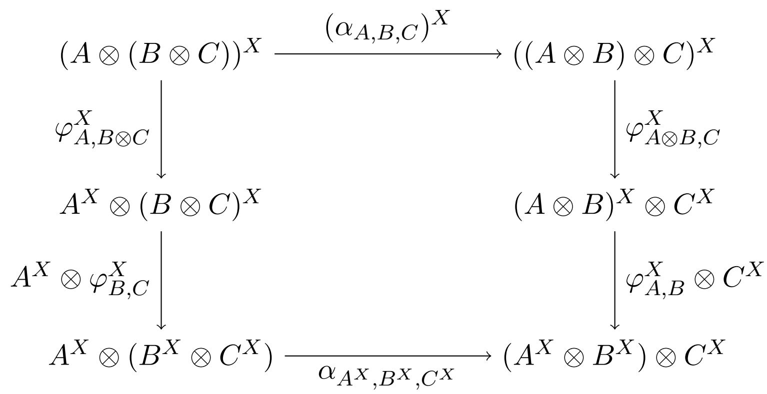
Firstly, we have the action of Γ on objects. For every object $X \in \mathcal{X}$, we have an oplax monoidal endofunctor on $\mathcal{C}$, denoted as follows.

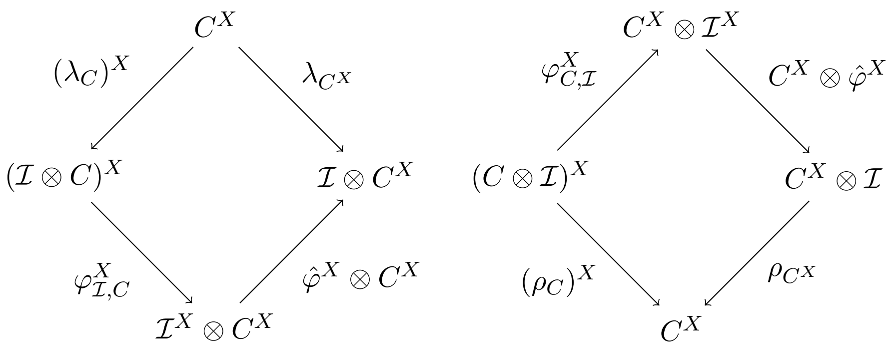
$$(-)^X : \mathcal{C} \to \mathcal{C}$$

We will denote the structure maps for this oplax monoidal endofunctor as follows.

$$\varphi^X_{B,C} : (B \otimes C)^X \to B^X \otimes C^X \qquad \hat{\varphi}^X : \mathcal{I}^X \to \mathcal{I}$$

The conditions that these must satisfy are that the following three diagrams must commute.





Secondly, we have the action of Γ on morphisms. For every morphism $f : X \to Y$ in $\mathcal{X}$, we have a monoidal natural transformation $(-)^f : (-)^X \to (-)^Y$. The conditions that this natural transformation must satisfy in order to be a monoidal natural transformation are that the following two diagrams must commute.

$$\begin{array}{ccc} (B\otimes C)^X & \xrightarrow{(B\otimes C)^f} & (B\otimes C)^Y \\ {\scriptstyle \varphi^X_{B,C}}\downarrow & & \downarrow{\scriptstyle \varphi^Y_{B,C}} \\ B^X\otimes C^X & \xrightarrow[B^f\otimes C^f]{} & B^Y\otimes C^Y \end{array} \qquad \begin{array}{ccc} \mathcal{I}^X & \xrightarrow{\mathcal{I}^f} & \mathcal{I}^Y \\ {\scriptstyle \hat\varphi^X}\searrow & & \swarrow{\scriptstyle \hat\varphi^Y} \\ & \mathcal{I} & \end{array}$$

In addition to this, the functor Γ itself is lax monoidal. We will denote the structure maps for Γ as follows.

$$\psi^{X,Y}_C : (C^Y)^X \to C^{X\otimes Y} \qquad \hat\psi_C \colon C \to C^{\mathcal{I}}$$

The conditions that these must satisfy are that the following three diagrams must commute. Note that the monoidal category $[\mathcal{C},\mathcal{C}]_{\mathrm{oplax}}$, which is the target of Γ, is strict, so some of the edges in these diagrams are identities.

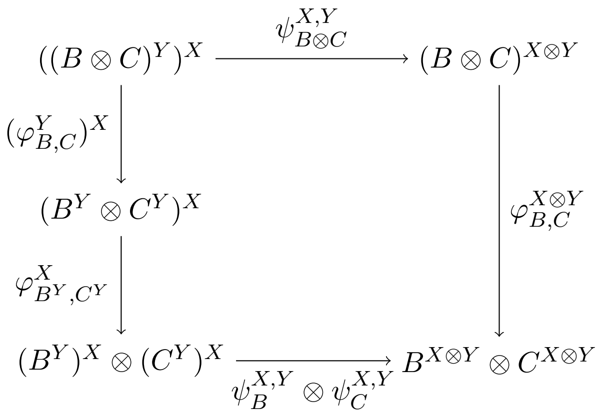
$$\begin{array}{ccc} ((C^Z)^Y)^X & = & ((C^Z)^Y)^X \\ {\scriptstyle (\psi^{Y,Z}_C)^X}\downarrow & & \downarrow{\scriptstyle \psi^{X,Y}_{C^Z}} \\ (C^{Y\otimes Z})^X & & (C^Z)^{X\otimes Y} \\ {\scriptstyle \psi^{X,Y\otimes Z}_C}\downarrow & & \downarrow{\scriptstyle \psi^{X\otimes Y,Z}_C} \\ C^{X\otimes(Y\otimes Z)} & \xrightarrow[C^{\alpha_{X,Y,Z}}]{} & C^{(X\otimes Y)\otimes Z} \end{array}$$

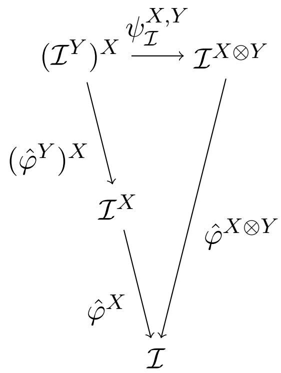
$$\begin{array}{ccccc} & & C^X & & \\ & /\!/ & & \searrow{\scriptstyle C^{\lambda_X}} & \\ C^X & & & & C^{\mathcal{I}\otimes X} \\ & {\scriptstyle \hat\psi_{C^X}}\searrow & & \nearrow{\scriptstyle \psi^{\mathcal{I},X}_C} & \\ & & (C^X)^{\mathcal{I}} & & \end{array} \qquad \begin{array}{ccccc} & & (C^{\mathcal{I}})^X & & \\ & {\scriptstyle (\hat\psi_C)^X}\nearrow & & \searrow{\scriptstyle \psi^{X,\mathcal{I}}_C} & \\ C^X & & & & C^{X\otimes\mathcal{I}} \\ & \backslash\!\backslash & & \swarrow{\scriptstyle C^{\rho_X}} & \\ & & C^X & & \end{array}$$

Finally, the components of the structure maps for Γ are morphisms in $[\mathcal{C},\mathcal{C}]_{\mathrm{oplax}}$, which means that they are monoidal natural transformations. The structure map

$$\psi^{X,Y}_C : (C^Y)^X \to C^{X\otimes Y}$$

being a monoidal natural transformation corresponds to the following two diagrams commuting.

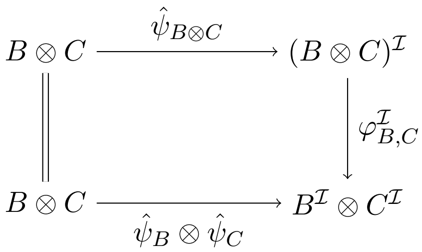


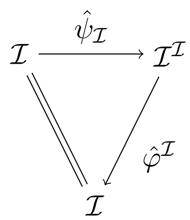


The structure map

$$\hat{\psi}_C \colon C \to C^{\mathcal{I}}$$

being a monoidal natural transformation corresponds to the following two diagrams commuting.





4.3 Skew Monoidal Categories

In this section, we will explain how to categorify the notion of semidirect products of monoids to semidirect products of skew monoidal categories.

Definition 4.3.1 (semidirect product)**.** Given a weak action of a skew monoidal category $\mathcal{X}$ on a skew monoidal category $\mathcal{C}$, we can define a semidirect product skew monoidal category, $\mathcal{C} \rtimes \mathcal{X}$. The underlying category of $\mathcal{C} \rtimes \mathcal{X}$ is $\mathcal{C} \times \mathcal{X}$, and we will denote an object $(C, X) \in \mathcal{C} \rtimes \mathcal{X}$ by $\langle C, X \rangle$. The tensor product is defined as follows.

$$\langle B, X \rangle \otimes \langle C, Y \rangle = \langle B \otimes C^X, X \otimes Y \rangle$$

The monoidal unit is defined as follows.

$$\mathcal{I} = \langle \mathcal{I}, \mathcal{I} \rangle$$

In order to define the associator and unitors, it suffices to define their images under the projection functors $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{C}}$.

$$\mathcal{C} \xleftarrow{\pi_{\mathcal{C}}} \mathcal{C} \rtimes \mathcal{X} \xrightarrow{\pi_{\mathcal{X}}} \mathcal{X}$$

The associator α is defined as follows. The component

$$\alpha_{\langle A,X\rangle,\langle B,Y\rangle,\langle C,Z\rangle} \colon \langle A, X\rangle \otimes \langle\langle B, Y\rangle \otimes \langle C, Z\rangle\rangle \to \langle\langle A, X\rangle \otimes \langle B, Y\rangle\rangle \otimes \langle C, Z\rangle$$

is the morphism whose images under $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{C}}$ are the following pair of morphisms, respectively.

$$X \otimes (Y \otimes Z) \xrightarrow{\alpha_{X,Y,Z}} (X \otimes Y) \otimes Z$$

$$\begin{aligned} A \otimes (B \otimes C^Y)^X &\xrightarrow{A\otimes\varphi^X_{B,C^Y}} A \otimes (B^X \otimes (C^Y)^X) \\ &\xrightarrow{\alpha_{A,B^X,(C^Y)^X}} (A \otimes B^X) \otimes (C^Y)^X \\ &\xrightarrow{(A\otimes B^X)\otimes\psi^{X,Y}_C} (A \otimes B^X) \otimes C^{X\otimes Y} \end{aligned}$$

The left unitor λ is defined as follows. The component

$$\lambda_{\langle C,X\rangle} \colon \langle C, X\rangle \to \mathcal{I} \otimes \langle C, X\rangle$$

is the morphism whose images under $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{C}}$ are the following pair of morphisms, respectively.

$$X \xrightarrow{\lambda_X} \mathcal{I} \otimes X$$

$$C \xrightarrow{\lambda_C} \mathcal{I} \otimes C \xrightarrow{\mathcal{I}\otimes\hat{\psi}_C} \mathcal{I} \otimes C^{\mathcal{I}}$$

The right unitor ρ is defined as follows. The component

$$\rho_{\langle C,X\rangle} \colon \langle C, X\rangle \otimes \mathcal{I} \to \langle C, X\rangle$$

is the morphism whose images under $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{C}}$ are the following pair of morphisms, respectively.

$$X \otimes \mathcal{I} \xrightarrow{\rho_X} X$$

$$C \otimes \mathcal{I}^X \xrightarrow{C\otimes\hat{\varphi}^X} C \otimes \mathcal{I} \xrightarrow{\rho_C} C$$

In order to show that $\mathcal{C} \rtimes \mathcal{X}$ is a skew monoidal category, we must show that the pentagon identity 2.1, the three triangle identities 2.2, 2.3 and 2.4, and the unitor identity 2.5 hold. However, in order to show that a diagram commutes in $\mathcal{C} \rtimes \mathcal{X}$, it suffices to show that its images under the projection functors $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{C}}$ commute. And, since the images of the associator and unitors under the projection functor $\pi_{\mathcal{X}}$ are just the associator and unitors in $\mathcal{X}$, which is a skew monoidal category, the images of the pentagon diagram, the three triangle diagrams and the unitor diagram under the projection functor $\pi_{\mathcal{X}}$ do commute. Hence, we only need to show that the images of the pentagon diagram, the three triangle diagrams and the unitor diagram under the projection functor $\pi_{\mathcal{C}}$ commute.

Lemma 4.3.2. *The pentagon identity 2.1 holds.*

Proof. We will consider the pentagon identity as it applies to the four objects $\langle A, W\rangle$, $\langle B, X\rangle$, $\langle C, Y\rangle$ and $\langle D, Z\rangle$. Throughout this proof, we will denote tensor products by juxtaposition, for notational convenience. In this case, the five different bracketings which form the vertices of the pentagon are as follows.

$$\langle A,W\rangle(\langle B,X\rangle(\langle C,Y\rangle\langle D,Z\rangle)) = \langle A(B(CD^Y)^X)^W, W(X(YZ))\rangle$$
$$\langle A,W\rangle((\langle B,X\rangle\langle C,Y\rangle)\langle D,Z\rangle) = \langle A((BC^X)D^{XY})^W, W((XY)Z)\rangle$$
$$(\langle A,W\rangle(\langle B,X\rangle\langle C,Y\rangle))\langle D,Z\rangle = \langle (A(BC^X)^W)D^{W(XY)}, (W(XY))Z\rangle$$
$$((\langle A,W\rangle\langle B,X\rangle)\langle C,Y\rangle)\langle D,Z\rangle = \langle ((AB^W)C^{WX})D^{(WX)Y}, ((WX)Y)Z\rangle$$
$$(\langle A,W\rangle\langle B,X\rangle)(\langle C,Y\rangle\langle D,Z\rangle) = \langle (AB^W)(CD^Y)^{WX}, (WX)(YZ)\rangle$$

The image under $\pi_{\mathcal{C}}$ of the pentagon diagram is shown in Figure 4.1. □

Lemma 4.3.3. *The triangle identities 2.2, 2.3 and 2.4 hold.*

Proof. In the case of the first triangle identity 2.2, the three different bracketings which form the vertices of the triangle are as follows.

$$(\mathcal{I} \otimes \langle B, Y\rangle) \otimes \langle C, Z\rangle = \langle (\mathcal{I} \otimes B^{\mathcal{I}}) \otimes C^{\mathcal{I}\otimes Y}, (\mathcal{I} \otimes Y) \otimes Z\rangle$$
$$\mathcal{I} \otimes (\langle B, Y\rangle \otimes \langle C, Z\rangle) = \langle \mathcal{I} \otimes (B \otimes C^Y)^{\mathcal{I}}, \mathcal{I} \otimes (Y \otimes Z)\rangle$$
$$\langle B, Y\rangle \otimes \langle C, Z\rangle = \langle B \otimes C^Y, Y \otimes Z\rangle$$

The image under $\pi_{\mathcal{C}}$ of the first triangle diagram is shown in Figure 4.2.

In the case of the second triangle identity 2.3, the three different bracketings which form the vertices of the triangle are as follows.

$$(\langle A, X\rangle \otimes \mathcal{I}) \otimes \langle C, Z\rangle = \langle (A \otimes \mathcal{I}^X) \otimes C^{X\otimes\mathcal{I}}, (X \otimes \mathcal{I}) \otimes Z\rangle$$
$$\langle A, X\rangle \otimes (\mathcal{I} \otimes \langle C, Z\rangle) = \langle A \otimes (\mathcal{I} \otimes C^{\mathcal{I}})^X, X \otimes (\mathcal{I} \otimes Z)\rangle$$
$$\langle A, X\rangle \otimes \langle C, Z\rangle = \langle A \otimes C^X, X \otimes Z\rangle$$

The image under $\pi_{\mathcal{C}}$ of the second triangle diagram is shown in Figure 4.3.

In the case of the third triangle identity 2.4, the three different bracketings which form the vertices of the triangle are as follows.

$$(\langle A, X\rangle \otimes \langle B, Y\rangle) \otimes \mathcal{I} = \langle (A \otimes B^X) \otimes \mathcal{I}^{X\otimes Y}, (X \otimes Y) \otimes \mathcal{I}\rangle$$

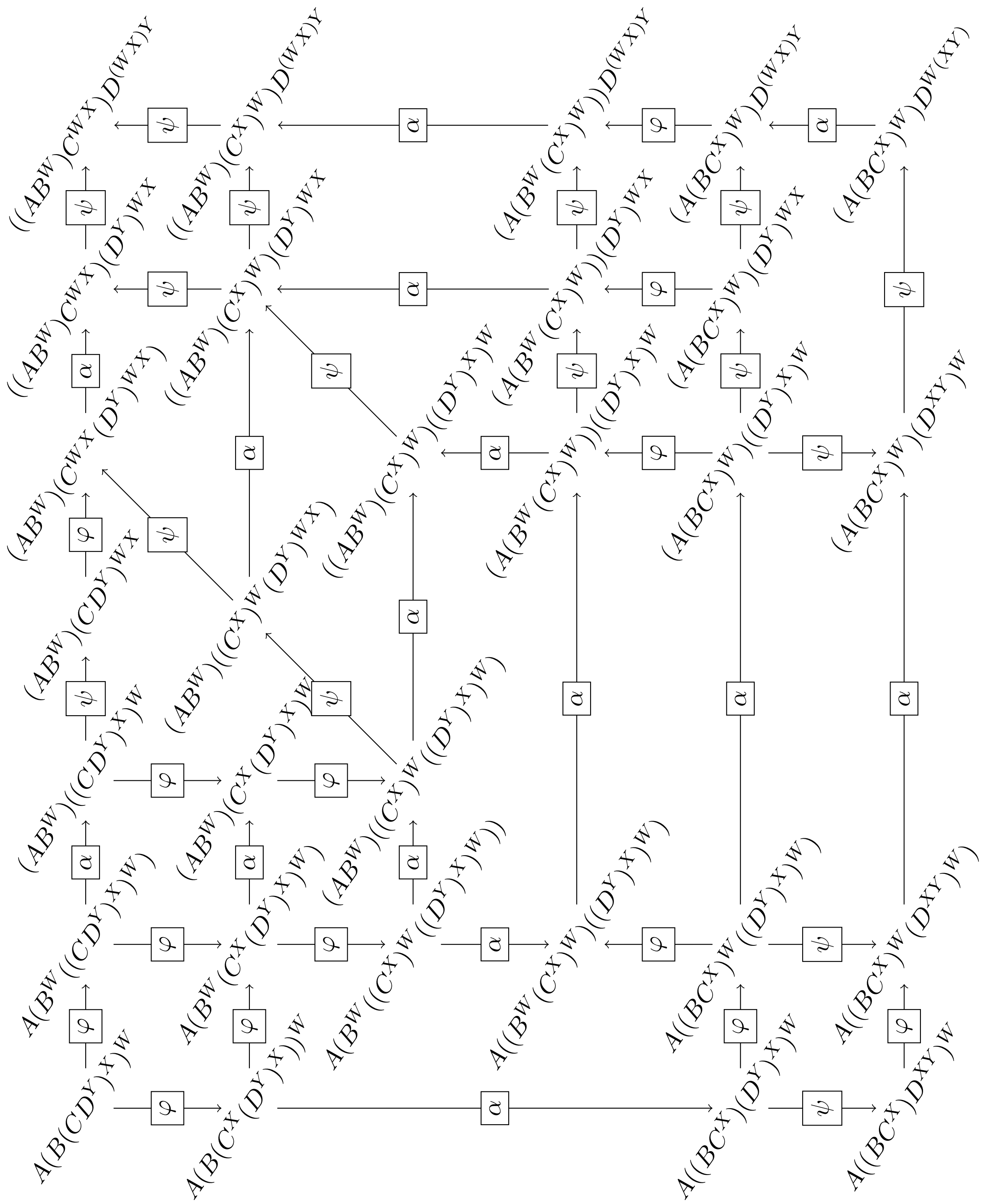


Figure 4.1: Proof of the pentagon identity in a semidirect product skew monoidal category.

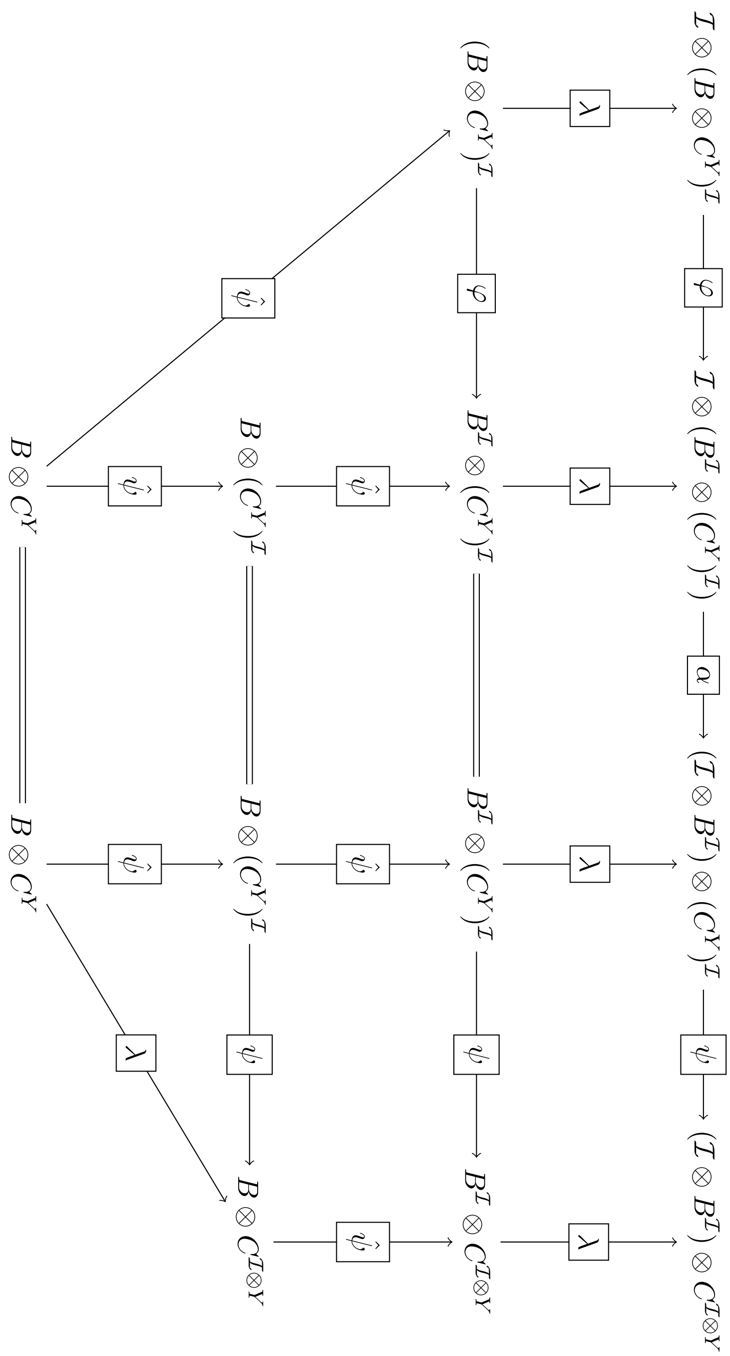


Figure 4.2: Proof of the first triangle identity in a semidirect product skew monoidal category.

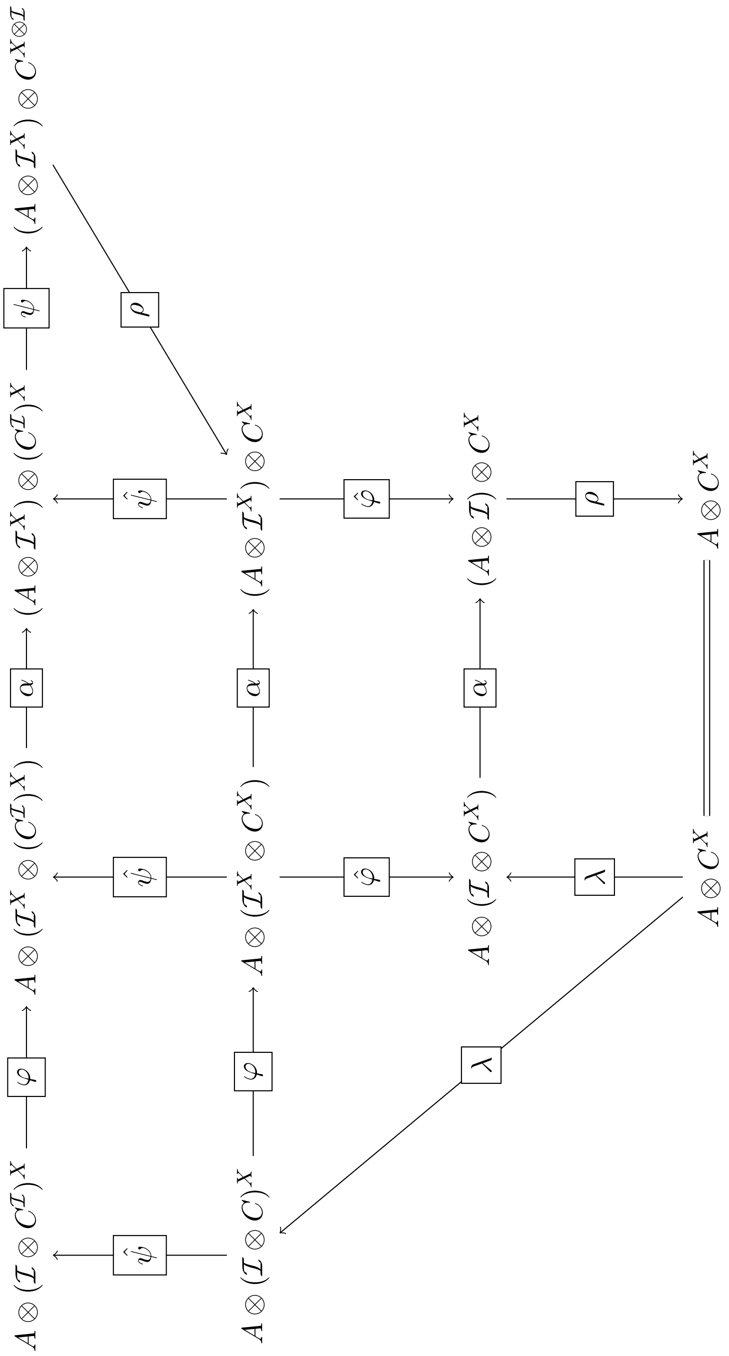


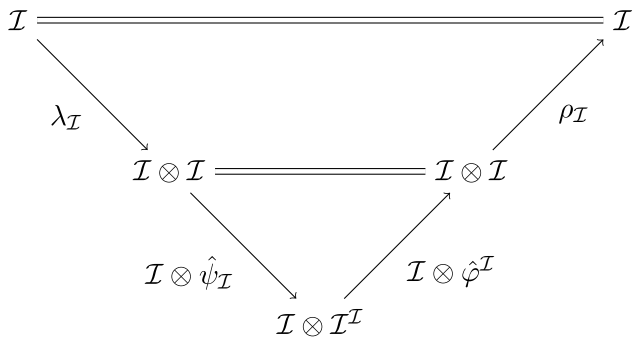
Figure 4.3: Proof of the second triangle identity in a semidirect product skew monoidal category.

$$\langle A, X\rangle \otimes (\langle B, Y\rangle \otimes \mathcal{I}) = \langle A \otimes (B \otimes \mathcal{I}^Y)^X, X \otimes (Y \otimes \mathcal{I})\rangle$$
$$\langle A, X\rangle \otimes \langle B, Y\rangle = \langle A \otimes B^X, X \otimes Y\rangle$$

The image under $\pi_{\mathcal{C}}$ of the third triangle diagram is shown in Figure 4.4.
□

Lemma 4.3.4. *The unitor identity 2.5 holds.*

Proof. The image under $\pi_{\mathcal{C}}$ of the unitor diagram is the following.



□

We will now give some examples of semidirect product skew monoidal categories.

Example 4.3.5. Let $\mathcal{X} = \{\star\}$, the monoidal category with one object and one morphism. Let $\mathcal{C}$ be a monoidal category. Then a weak action of $\mathcal{X}$ on $\mathcal{C}$ endows the endofunctor $(-)^\star$ with the structure of an oplax monoidal monad on $\mathcal{C}$; in fact, all oplax monoidal monads are of this form. Given such a weak action, the resulting semidirect product $\mathcal{C} \rtimes \mathcal{X}$ is a skew monoidal structure on $\mathcal{C} \times \{\star\} \cong \mathcal{C}$, with tensor product defined as follows.

$$\langle B, \star\rangle \otimes \langle C, \star\rangle = \langle B \otimes C^\star, \star\rangle$$

This is what is referred to by Szlachányi [25] as the skew monoidal category 'represented' by the oplax monoidal monad $(-)^\star$.

Example 4.3.6. This example involves generalised metric spaces, as described by Lawvere [19]. Let $\mathcal{X}$ be $[0, \infty]$, the category whose objects are the non-negative real numbers and positive infinity, with a unique morphism $x \to y$ if and only if $x \leq y$, with min as the tensor product and ∞ as the monoidal unit. Let $\mathcal{C}$ be Met, the closed symmetric monoidal category of generalised metric spaces. Then there is a weak action of $\mathcal{X}$ on $\mathcal{C}$, given by truncation, in which the underlying set of M^x is the same as the underlying set of M, but with a new truncated metric, defined as follows.

$$M^x(m, m') = \min(x, M(m, m'))$$

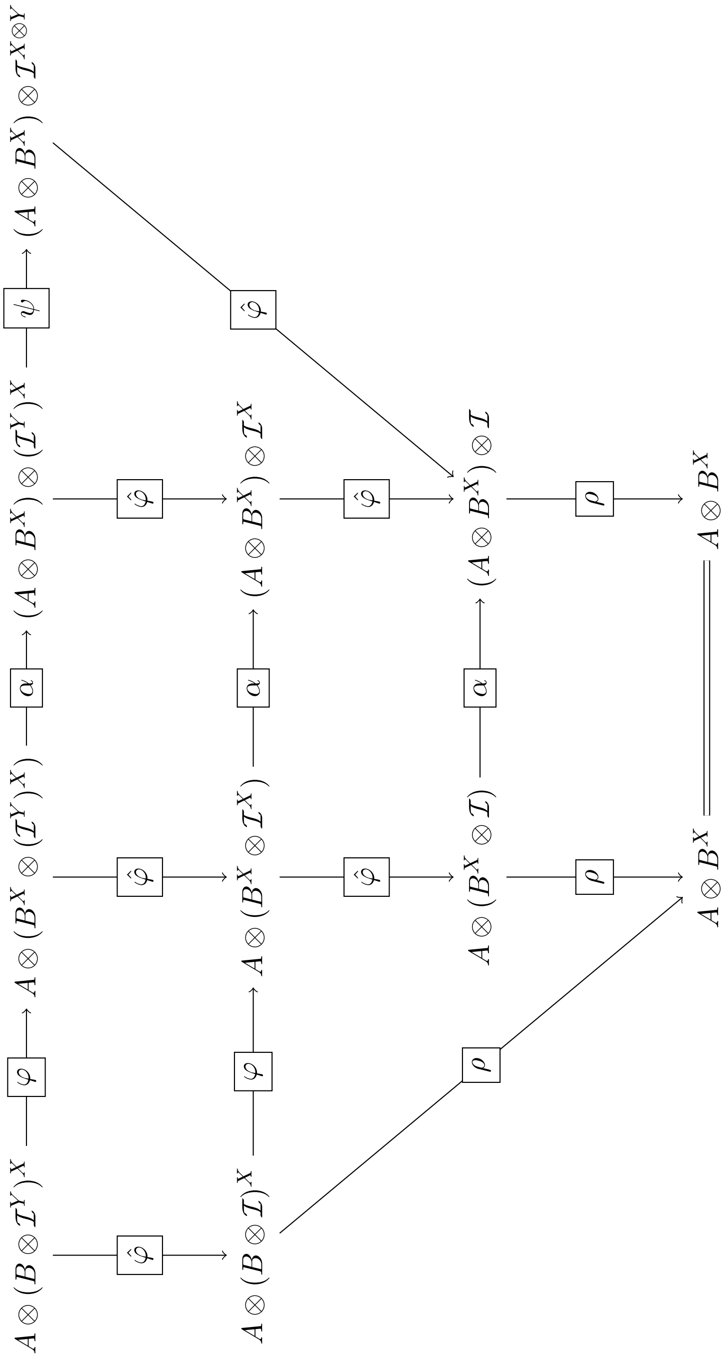


Figure 4.4: Proof of the third triangle identity in a semidirect product skew monoidal category.

The structure map φ for this action is not invertible; this is a consequence of the following inequality not being an equality.

$$\min(x, M(m, m') + N(n, n')) \le \min(x, M(m, m')) + \min(x, N(n, n'))$$

The resulting semidirect product $\mathcal{C} \rtimes \mathcal{X}$ is a skew monoidal structure on Met $\times [0, \infty]$, with tensor product defined as follows.

$$\langle M, x\rangle \otimes \langle N, y\rangle = \langle M \otimes N^x, \min(x, y)\rangle$$

The metric on the generalised metric space $M \otimes N^x$ is given as follows.

$$(M \otimes N^x)((m, n), (m', n')) = M(m, m') + \min(x, N(n, n'))$$

4.4 Monoidal Categories

In this section, we will give a sufficient condition for a semidirect product skew monoidal category to be a monoidal category.

Definition 4.4.1 (strong action)**.** Let $\mathcal{X}$ and $\mathcal{C}$ be monoidal categories. A 'strong action' of $\mathcal{X}$ on $\mathcal{C}$ is a weak action of $\mathcal{X}$ on $\mathcal{C}$ in which all of the structure maps below are invertible.

$$\varphi^X_{B,C} \colon (B \otimes C)^X \to B^X \otimes C^X \qquad \hat{\varphi}^X : \mathcal{I}^X \to \mathcal{I}$$

$$\psi^{X,Y}_C : (C^Y)^X \to C^{X \otimes Y} \qquad \hat{\psi}_C \colon C \to C^{\mathcal{I}}$$

Equivalently, this is the same as a strong monoidal functor Γ from $\mathcal{X}$ to $[\mathcal{C}, \mathcal{C}]_{\text{strong}}$, the strict monoidal category of strong monoidal endofunctors on $\mathcal{C}$ and monoidal natural transformations between them with tensor product given by functor composition; i.e. $(F \otimes G)(C) = F(G(C))$.

Theorem 4.4.2. *Let $\mathcal{X}$ and $\mathcal{C}$ be monoidal categories. Let there be a strong action of $\mathcal{X}$ on $\mathcal{C}$. Then the semidirect product $\mathcal{C} \rtimes \mathcal{X}$ is a monoidal category.*

Proof. In order to show that a skew monoidal category is a monoidal category, it suffices to show that the coherence data is invertible. Each coherence morphism in $\mathcal{C} \rtimes \mathcal{X}$ is a pair of morphisms, whose first component is a composite of coherence data in $\mathcal{C}$ and the structure maps of the action, and whose second component is a single coherence morphism in $\mathcal{X}$. Since all of these are invertible, it follows that the coherence data in $\mathcal{C} \rtimes \mathcal{X}$ is invertible, and that $\mathcal{C} \rtimes \mathcal{X}$ is a monoidal category. ☐

The semidirect products of monoidal categories introduced here are a special case of the distributive laws for pseudomonads introduced by Marmolejo [22]. A monoidal category can be regarded as a pseudomonad in the

1-object 3-category obtained as the delooping of the cartesian monoidal 2-category Cat. A strong action of one monoidal category on another is then a distributive law between these pseudomonads which is partially trivial in a particular sense.

In a distributive law of this kind, there is not only an action of $\mathcal{X}$ on $\mathcal{C}$, but also an action of $\mathcal{C}$ on $\mathcal{X}$. These combine into a single functor of the form

$$\mathcal{X} \times \mathcal{C} \to \mathcal{C} \times \mathcal{X}$$

which must satisfy some coherence conditions generalising those we have already defined. The semidirect products of monoidal categories introduced here can be viewed as distributive laws in which the action of $\mathcal{C}$ on $\mathcal{X}$ is trivial.

We will now give some examples of semidirect product monoidal categories.

Example 4.4.3. Let $\mathcal{C}$ be a closed cartesian category. Let $\mathcal{X} = \mathcal{C}^{\mathrm{op}}$. Then there is a strong action of $\mathcal{X}$ on $\mathcal{C}$ using the internal hom, defined as follows.

$$C^X = X \backslash C$$

The resulting semidirect product $\mathcal{C} \rtimes \mathcal{X}$ is a monoidal structure on $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$, with tensor product defined as follows.

$$\langle B, X \rangle \otimes \langle C, Y \rangle = \langle B \times (X \backslash C), X \times Y \rangle$$

Example 4.4.4. Let $\mathcal{C}$ be a closed cartesian category with finite coproducts. Let $\mathcal{X}$ be $\mathcal{C}$, with binary coproducts as the tensor product and the initial object as the monoidal unit. Choose an object J in $\mathcal{C}$. Then there is a strong action of $\mathcal{X}$ on $\mathcal{C}$ using the internal hom, defined as follows.

$$C^X = (X \backslash J) \backslash C$$

The resulting semidirect product $\mathcal{C} \rtimes \mathcal{X}$ is a monoidal structure on $\mathcal{C} \times \mathcal{C}$, with tensor product defined as follows.

$$\langle B, X \rangle \otimes \langle C, Y \rangle = \langle B \times ((X \backslash J) \backslash C), X + Y \rangle$$

Example 4.4.5. Choose a category $\mathcal{J}$, and a monoidal category $\mathcal{M}$. Let $\mathcal{X}$ be $[\mathcal{J}, \mathcal{J}]$, the strict monoidal category of endofunctors of $\mathcal{J}$ and natural transformations between them with tensor product given by functor composition; i.e. $(F \otimes G)(J) = G(F(J))$. Let $\mathcal{C}$ be $[\mathcal{J}, \mathcal{M}]$; this category inherits a monoidal structure from that of $\mathcal{M}$. Then there is a strong action of $\mathcal{X}$ on $\mathcal{C}$ given by composition, defined as follows.

$$C^X = C \circ X$$

The resulting semidirect product $\mathcal{C} \rtimes \mathcal{X}$ is a monoidal structure on $[\mathcal{J}, \mathcal{M}] \times [\mathcal{J}, \mathcal{J}]$, with tensor product defined as follows.

$$\langle C, G \rangle \otimes \langle B, F \rangle = \langle C \otimes (B \circ G), F \circ G \rangle$$

Example 4.4.6. As a specific case of the previous example, let $\mathcal{J}$ be a group G, considered as a 1-object groupoid, and let $\mathcal{M}$ be $\textsc{Vect}$, the category of vector spaces and linear maps. Then $\mathcal{X}$ is a category whose objects are endomorphisms of G and $\mathcal{C}$ is $\textsc{Rep}(G)$, the category of linear representations of G. The resulting semidirect product $\mathcal{C} \rtimes \mathcal{X}$ is a monoidal structure on $\textsc{Rep}(G) \times [G, G]$, with tensor product defined as follows.

$$\langle U, f \rangle \otimes \langle V, g \rangle = \langle U \otimes f^\star(V), g \circ f \rangle$$

Example 4.4.7. This example involves generalised metric spaces, as described by Lawvere [19]. Let $\mathcal{X}$ be the strict closed symmetric monoidal category $\{F \to T\}$ of truth values, with tensor product given by logical conjuction and internal hom given by logical implication. Let $\mathcal{C}$ be $\textsc{Met}$, the symmetric monoidal category of generalised metric spaces. Then there is a strong action of $\mathcal{X}$ on $\mathcal{C}$, in which the underlying set of M^x is the same as the underlying set of M, but with a new metric, defined as follows.

$$\begin{aligned} M^T(m, m') &= M(m, m') \\ M^F(m, m') &= \begin{cases} 0 & \text{if } M(m, m') = 0 \\ \infty & \text{if } M(m, m') > 0 \end{cases} \end{aligned}$$

The resulting semidirect product $\mathcal{C} \rtimes \mathcal{X}$ is a monoidal structure on $\textsc{Met} \times \{F \to T\}$, with tensor product defined as follows.

$$\langle M, x \rangle \otimes \langle N, y \rangle = \langle M \otimes N^x, x \wedge y \rangle$$

The metric on the generalised metric space $M \otimes N^x$ with underlying set $M \times N$ is given as follows.

$$\begin{aligned} (M \otimes N^T)((m, n), (m', n')) &= M(m, m') + N(n, n') \\ (M \otimes N^F)((m, n), (m', n')) &= \begin{cases} M(m, m') & \text{if } N(n, n') = 0 \\ \infty & \text{if } N(n, n') > 0 \end{cases} \end{aligned}$$

4.5 Left Closed Structures

In this section, we will give a sufficient condition for a semidirect product skew monoidal category to be left closed.

We might hope for the following to hold. Compare this to Conjecture 4.7.1.

Conjecture 4.5.1. *Let $\mathcal{X}$ and $\mathcal{C}$ be left closed skew monoidal categories. Let there be a weak action of $\mathcal{X}$ on $\mathcal{C}$. Then the semidirect product $\mathcal{C} \rtimes \mathcal{X}$ is a left closed skew monoidal category.*

However, this conjecture is false, as we will now show. Consider Example 4.4.7; we will show that tensoring on the left by $\langle 1, F\rangle$ does not preserve colimits. Denote by D_t the generalised metric space consisting of two points separated by a distance t in each direction. There is an obvious morphism $D_s \to D_t$ whenever $s \geq t$. Consider the following diagram.

$$\langle D_{\frac{1}{1}}, T\rangle \to \langle D_{\frac{1}{2}}, T\rangle \to \langle D_{\frac{1}{3}}, T\rangle \to \langle D_{\frac{1}{4}}, T\rangle \to \cdots$$

Since colimits are calculated pointwise, the colimit of this diagram is the object $\langle D_0, T\rangle$. The image of this diagram under the functor $(\langle 1, F\rangle \otimes -)$ is the following diagram.

$$\langle D_\infty, F\rangle \to \langle D_\infty, F\rangle \to \langle D_\infty, F\rangle \to \langle D_\infty, F\rangle \to \cdots$$

The colimit of this diagram is the object $\langle D_\infty, F\rangle$. Thus, if tensoring on the left by $\langle 1, F\rangle$ were to preserve colimits, we would expect an isomorphism of the following form.

$$\langle 1, F\rangle \otimes \langle D_0, T\rangle \cong \langle D_\infty, F\rangle$$

However, the left hand side of this equation evaluates as follows.

$$\langle 1, F\rangle \otimes \langle D_0, T\rangle = \langle D_0, F\rangle$$

Thus, no such isomorphism exists, so tensoring on the left by $\langle 1, F\rangle$ cannot preserve colimits, and so $\mathcal{C} \rtimes \mathcal{X}$ cannot be left closed. This provides a counterexample to Conjecture 4.5.1.

However, the following weaker result does hold.

Theorem 4.5.2. *Let $\mathcal{X}$ and $\mathcal{C}$ be left closed skew monoidal categories. Let there be a weak action of $\mathcal{X}$ on $\mathcal{C}$ such that each oplax monoidal endofunctor $(-)^X$ has a right adjoint, denoted $(-)_X$. Then the semidirect product $\mathcal{C} \rtimes \mathcal{X}$ is a left closed skew monoidal category, with internal hom defined as follows.*

$$\langle A, X\rangle \backslash \langle C, Z\rangle = \langle (A\backslash C)_X, X\backslash Z\rangle$$

Proof. We have the following natural isomorphism of hom sets.

$$\begin{aligned}
(\mathcal{C} \rtimes \mathcal{X})(\langle A, X\rangle \otimes \langle B, Y\rangle, \langle C, Z\rangle) &= (\mathcal{C} \rtimes \mathcal{X})(\langle A \otimes B^X, X \otimes Y\rangle, \langle C, Z\rangle) \\
&= \mathcal{C}(A \otimes B^X, C) \times \mathcal{X}(X \otimes Y, Z) \\
&\cong \mathcal{C}(B^X, A\backslash C) \times \mathcal{X}(Y, X\backslash Z) \\
&\cong \mathcal{C}(B, (A\backslash C)_X) \times \mathcal{X}(Y, X\backslash Z) \\
&= (\mathcal{C} \rtimes \mathcal{X})(\langle B, Y\rangle, \langle (A\backslash C)_X, X\backslash Z\rangle) \\
&= (\mathcal{C} \rtimes \mathcal{X})(\langle B, Y\rangle, \langle A, X\rangle \backslash \langle C, Z\rangle)
\end{aligned}$$

□

We will now give an example of a left closed semidirect product monoidal category.

Example 4.5.3. This example involves generalised metric spaces, as described by Lawvere [19]. Let $\mathcal{X}$ be $[0, \infty)$ the category whose objects are the non-negative real numbers, with a unique morphism $x \to y$ if and only if $x \geq y$. This category has a closed symmetric monoidal structure, with tensor product given by addition and internal hom given by truncated subtraction, defined as follows.

$$x \otimes y = x + y \qquad x \backslash y = \max(0, y - x)$$

Let $\mathcal{C}$ be Met, the closed symmetric monoidal category of generalised metric spaces. Then there is a strong action of $\mathcal{X}$ on $\mathcal{C}$, given by scaling, in which the underlying set of M^x is the same as the underlying set of M, but with a new scaled metric, defined as follows.

$$M^x(m, m') = e^x \cdot M(m, m')$$

Each functor $(-)^x$ has a right adjoint (in fact, an inverse), $(-)_x$, in which the underlying set of M_x is the same as the underlying set of M, but with a new scaled metric, defined as follows.

$$M_x(m, m') = e^{-x} \cdot M(m, m')$$

The resulting semidirect product $\mathcal{C} \rtimes \mathcal{X}$ is a left closed monoidal structure on Met $\times [0, \infty)$, with tensor product and internal hom defined as follows.

$$\langle M, x \rangle \otimes \langle N, y \rangle = \langle M \otimes N^x, x + y \rangle$$

$$\langle M, x \rangle \backslash \langle P, z \rangle = \langle (M \backslash P)_x, \max(0, z - x) \rangle$$

The metric on the generalised metric space $M \otimes N^x$ with underlying set $M \times N$ is given as follows.

$$(M \otimes N^x)((m, n), (m', n')) = M(m, m') + e^x \cdot N(n, n')$$

The metric on the generalised metric space $(M \backslash P)_x$ with underlying set $\mathcal{C}(M, P)$ is given as follows.

$$(M \backslash P)_x(f, g) = e^{-x} \cdot \sup_{m \in M} P(f(m), g(m))$$

4.6 Dual Pairs

In this section, we will give a sufficient condition for the existence of a dual pair in a semidirect product skew monoidal category, in the sense introduced in Chapter 3, §3.4.

Theorem 4.6.1. *Let $\mathcal{X}$ and $\mathcal{C}$ be skew monoidal categories. Let there be a weak action of $\mathcal{X}$ on $\mathcal{C}$. Assume there are dual pairs*

$$(W \otimes -) \dashv (X \otimes -) \qquad (A \otimes -) \dashv (B \otimes -)$$

in $\mathcal{X}$ and $\mathcal{C}$. Furthermore, assume that there is an adjunction

$$(-)^W \dashv (-)^X$$

whose unit and counit are monoidal natural transformations

$$\bar{\theta}_C \colon C \to (C^W)^X \qquad \bar{\zeta}_C \colon (C^X)^W \to C$$

such that the following two diagrams commute.

$$\begin{array}{ccccc} & & C^Y & & \\ & \bar{\theta}_{C^Y} \swarrow & & \searrow C^{\theta_Y} & \\ ((C^Y)^W)^X & & & & C^{X\otimes(W\otimes Y)} \\ & (\psi_C^{W,Y})^X \searrow & & \nearrow \psi_C^{X,W\otimes Y} & \\ & & (C^{W\otimes Y})^X & & \end{array} \tag{4.1}$$

$$\begin{array}{ccccc} & & (C^{X\otimes Y})^W & & \\ & (\psi_C^{X,Y})^W \nearrow & & \searrow \psi_C^{W,X\otimes Y} & \\ ((C^Y)^X)^W & & & & C^{W\otimes(X\otimes Y)} \\ & \bar{\zeta}_{C^Y} \searrow & & \swarrow C^{\zeta_Y} & \\ & & C^Y & & \end{array} \tag{4.2}$$

Then there is a dual pair

$$(\langle A, W\rangle \otimes -) \dashv (\langle B^X, X\rangle \otimes -)$$

in $\mathcal{C} \rtimes \mathcal{X}$.

The unit θ is defined as follows. The component

$$\theta_{\langle C,Y\rangle} \colon \langle C, Y\rangle \to \langle B^X, X\rangle \otimes (\langle A, W\rangle \otimes \langle C, Y\rangle)$$

is the morphism whose images under $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{C}}$ are the following pair of morphisms, respectively.

$$\theta_Y \colon Y \to X \otimes (W \otimes Y)$$

$$\begin{aligned} C &\xrightarrow{\bar{\theta}_C} (C^W)^X \\ &\xrightarrow{(\theta_{C^W})^X} (B \otimes (A \otimes C^W))^X \\ &\xrightarrow{\varphi^X_{B,A\otimes C^W}} B^X \otimes (A \otimes C^W)^X \end{aligned}$$

The counit ζ is defined as follows. The component

$$\zeta_{\langle C,Y\rangle} \colon \langle A, W\rangle \otimes (\langle B^X, X\rangle \otimes \langle C, Y\rangle) \to \langle C, Y\rangle$$

is the morphism whose images under $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{C}}$ are the following pair of morphisms, respectively.

$$\zeta_Y \colon W \otimes (X \otimes Y) \xrightarrow{\zeta} Y$$

$$\begin{aligned} A \otimes (B^X \otimes C^X)^W &\xrightarrow{A\otimes\varphi^W_{B^X,C^X}} A \otimes ((B^X)^W \otimes (C^X)^W) \\ &\xrightarrow{A\otimes(\bar{\zeta}_B\otimes\bar{\zeta}_C)} A \otimes (B \otimes C) \\ &\xrightarrow{\zeta_C} C \end{aligned}$$

Proof. In order to show that this data constitutes a dual pair, we must show that diagrams 3.5 and 3.6, from Chapter 3, §3.4, commute. However, in order to show that a diagram commutes in $\mathcal{C} \rtimes \mathcal{X}$, it suffices to show that its images under the projection functors $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{C}}$ commute. And, since the images of the components of the unit and counit in for the dual pair

$$(\langle A, W\rangle \otimes -) \dashv (\langle B^X, X\rangle \otimes -)$$

in $\mathcal{C} \rtimes \mathcal{X}$ under the projection functor $\pi_{\mathcal{X}}$ are just the components of the unit and counit for the dual pair

$$(W \otimes -) \dashv (X \otimes -)$$

in $\mathcal{X}$, the images of diagrams 3.5 and 3.6 under the projection functor $\pi_{\mathcal{X}}$ do commute. Hence, we only need to show that the images of the diagrams 3.5 and 3.6 under the projection functor $\pi_{\mathcal{C}}$ commute.

In the following two diagrams, we will denote tensor products by juxtaposition, for notational convenience. The image under $\pi_{\mathcal{C}}$ of diagram 3.5 is shown in Figure 4.5, and image under $\pi_{\mathcal{C}}$ of diagram 3.6 is shown in Figure 4.6.

□

In the case where we have a semidirect product of two monoidal categories, this results in the following simple corollary.

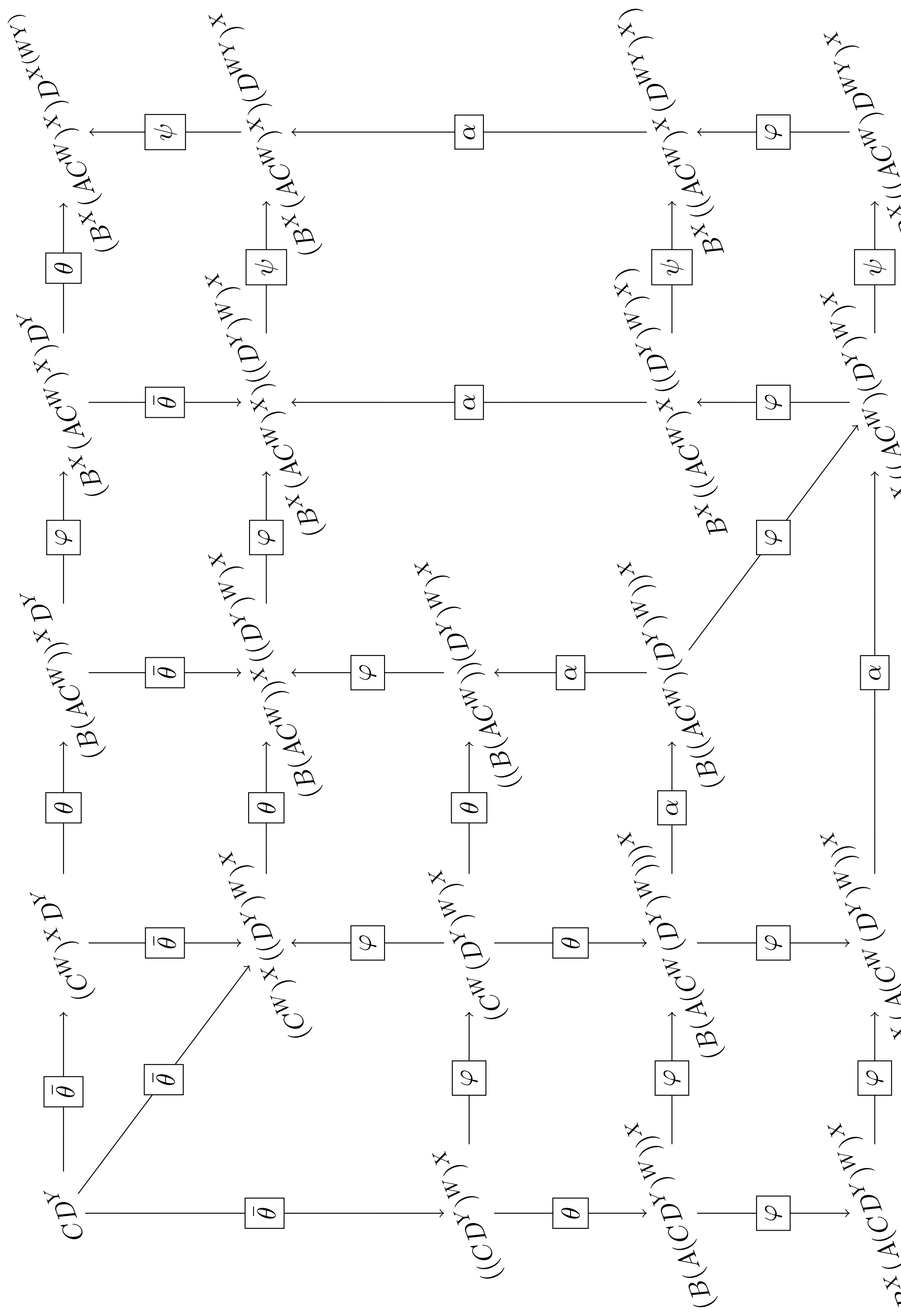


Figure 4.5: Proof that diagram 3.5 commutes in a semidirect product skew monoidal category.

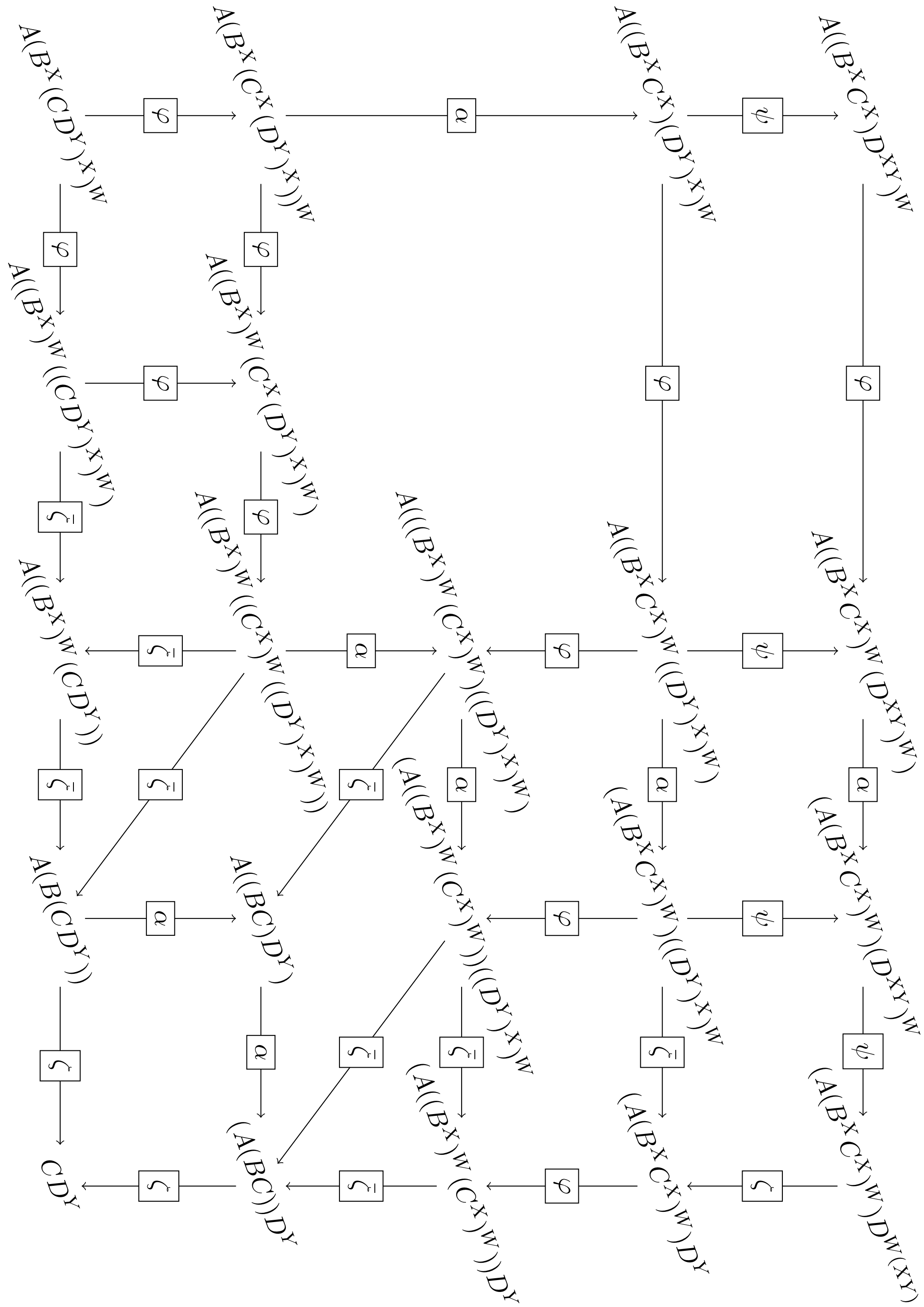


Figure 4.6: Proof that diagram 3.6 commutes in a semidirect product skew monoidal category.

Corollary 4.6.2. *Let $\mathcal{X}$ and $\mathcal{C}$ be monoidal categories. Let there be a strong action of $\mathcal{X}$ on $\mathcal{C}$. Assume there are dual pairs (W, X) in $\mathcal{X}$ and (A, B) in $\mathcal{C}$. Then there is a dual pair $(\langle A, W\rangle, \langle B^X, X\rangle)$ in $\mathcal{C} \rtimes \mathcal{X}$.*

Proof. Strong monoidal functors preserve duals; see [9] for details. In particular, the strong monoidal functor $\Gamma\colon \mathcal{X} \to [\mathcal{C}, \mathcal{C}]_{\text{strong}}$ preserves duals. This means that Γ sends the dual pair (W, X) in $\mathcal{X}$ to an adjunction $(-)^W \dashv (-)^X$ in $[\mathcal{C}, \mathcal{C}]_{\text{strong}}$. The unit and counit for this adjunction are defined as follows.

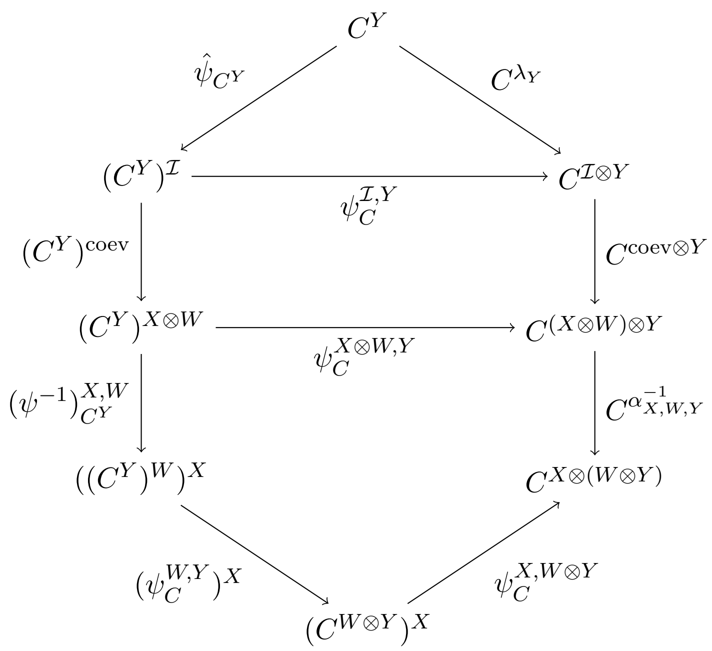
$$\bar{\theta}_C\colon C \xrightarrow{\hat{\psi}_C} C^{\mathcal{I}} \xrightarrow{C^{\text{coev}}} C^{X\otimes W} \xrightarrow{(\psi^{-1})_C^{X,W}} (C^X)^W$$

$$\bar{\zeta}_C\colon (C^W)^X \xrightarrow{\psi_C^{W,X}} C^{W\otimes X} \xrightarrow{C^{\text{ev}}} C^{\mathcal{I}} \xrightarrow{\hat{\psi}_C^{-1}} C$$

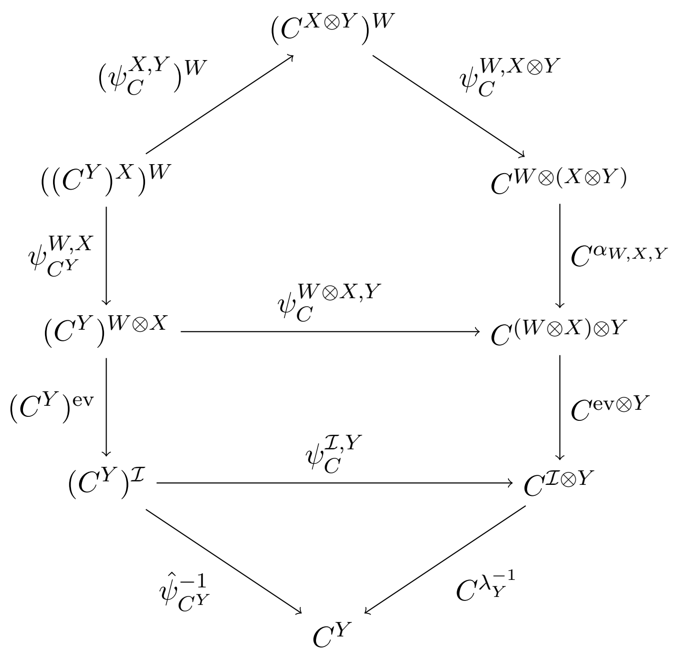
Since these are morphisms in $[\mathcal{C}, \mathcal{C}]_{\text{strong}}$, they are monoidal natural transformations.

We must check that the two diagrams 4.1 and 4.2 commute.

To see that diagram 4.1 commutes, consider the following diagram.



To see that diagram 4.2 commutes, consider the following diagram.



□

We will now give an example of a dual pair in a semidirect product monoidal category.

Example 4.6.3. Choose a monoidal category $\mathcal{M}$. Let $\mathcal{C}$ be the category $\mathcal{M}^{\mathbb{Z}}$ of integer-indexed collections of objects in $\mathcal{M}$; this category inherits a monoidal structure from that of $\mathcal{M}$. Given an object $C \in \operatorname{ob}\mathcal{C}$, denote its constituent objects by $C_i \in \operatorname{ob}\mathcal{M}$ for $i \in \mathbb{Z}$. Let $\mathcal{X}$ be $\mathbb{Z}$, considered as a category whose objects are the integers, with no non-trivial morphisms, with addition as the tensor product and 0 as the monoidal unit. Then there is a strong action of $\mathcal{X}$ on $\mathcal{C}$ given by shifting, defined as follows.

$$(C^x)_i = C_{i+x}$$

The resulting semidirect product $\mathcal{C} \rtimes \mathcal{X}$ is a monoidal structure on $\mathcal{M}^{\mathbb{Z}} \times \mathbb{Z}$, with tensor product defined as follows.

$$\langle B, x\rangle \otimes \langle C, y\rangle = \langle B \otimes C^x, x + y\rangle$$

The object $B \otimes C^x$ is defined as follows.

$$(B \otimes C^x)_i = B_i \otimes C_{i+x}$$

Choose an object $w \in \operatorname{ob}\mathcal{X}$. Then w has a right dual $-w$. Choose an object $A \in \operatorname{ob}\mathcal{C}$ such that each object $A_i \in \operatorname{ob}\mathcal{M}$ has a right dual $B_i \in \operatorname{ob}\mathcal{M}$.

Then these objects together form an object $B \in \mathrm{ob}\,\mathcal{C}$, which is right dual to A. Then the object $\langle A, w\rangle$ has a right dual $\langle B^{-w}, -w\rangle$. The object B^{-w} is defined as follows.

$$(B^{-w})_i = B_{i-w}$$

4.7 Right Closed Structures

In this section, we will produce some examples of semidirect product monoidal categories which are right closed but not left closed.

Throughout this section, we will use the term 'right closed skew monoidal category' to mean a skew monoidal category $\mathcal{C}$ in which tensoring on the right has a right adjoint. We will denote the internal hom as follows.

$$(-/-)\colon \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$$

Note that this is distinct from the internal hom of a left closed skew monoidal category $(-\backslash-)$; in particular, here the contravariant variable appears on the right. Using this notation, the hom-tensor adjunction is the following natural isomorphism of hom sets.

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(A, C/B)$$

Because of the non-invertibility of the associator α, right closed structures in skew monoidal categories behave very differently to left closed structures. For this reason, it should not be surprising that this section proceeds differently to §4.5, in which we consider left closed structures.

We might hope for the following to hold. Compare this to Conjecture 4.5.1.

Conjecture 4.7.1. *Let $\mathcal{X}$ and $\mathcal{C}$ be right closed skew monoidal categories. Let there be a weak action of $\mathcal{X}$ on $\mathcal{C}$. Then the semidirect product $\mathcal{C} \rtimes \mathcal{X}$ is a right closed skew monoidal category.*

However, this conjecture is false, as we will now show. Consider Example 4.4.7; we will show that tensoring on the right by an arbitrary object $\langle M, x\rangle$ does not preserve coproducts. Consider the following coproduct diagram.

$$\langle 1, F\rangle \qquad \langle 0, T\rangle$$

Since colimits are calculated pointwise, the colimit of this diagram is the object $\langle 1, T\rangle$. The image of this diagram under the functor $(- \otimes \langle M, x\rangle)$ is the following diagram.

$$\langle M^F, F\rangle \qquad \langle 0, x\rangle$$

The colimit of this diagram is the object $\langle M^F, x\rangle$. Thus, if tensoring on the right by $\langle M, x\rangle$ were to preserve colimits, we would expect an isomorphism of the following form.

$$\langle 1, T\rangle \otimes \langle M, x\rangle \cong \langle M^F, x\rangle$$

However, the left hand side of this equation evaluates as follows.

$$\langle 1, T\rangle \otimes \langle M, x\rangle = \langle M^T, x\rangle$$

Thus, no such isomorphism exists, so tensoring on the right by $\langle M, x\rangle$ cannot preserve colimits, and so $\mathcal{C} \rtimes \mathcal{X}$ cannot be right closed. This provides a counterexample to Conjecture 4.7.1.

However, we can describe a class of semidirect product skew monoidal categories which are right closed.

Theorem 4.7.2. *Let $\mathcal{X}$ be a right closed skew monoidal category. Let $\mathcal{C}$ be a monoidal category with binary coproducts as the tensor product and the initial object as the monoidal unit. Let there be a weak action of $\mathcal{X}$ on $\mathcal{C}$. Furthermore, assume that $\mathcal{X}$ has finite products and that there is another functor*

$$(- \lhd -)\colon \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{X}$$

and a natural isomorphism of hom sets of the following form.

$$\mathcal{C}(B^X, C) \cong \mathcal{X}(X, C \lhd B)$$

Then the skew monoidal category $\mathcal{C} \rtimes \mathcal{X}$ is right closed, with internal hom defined as follows.

$$\langle C, Z\rangle / \langle B, Y\rangle = \langle C, (C \lhd B) \times (Z/Y)\rangle$$

Proof. We have the following natural isomorphism of hom sets.

$$\begin{aligned}
(\mathcal{C} \rtimes \mathcal{X})(\langle A, X\rangle \otimes \langle B, Y\rangle, \langle C, Z\rangle) &= (\mathcal{C} \rtimes \mathcal{X})(\langle A + B^X, X \otimes Y\rangle, \langle C, Z\rangle) \\
&= \mathcal{C}(A + B^X, C) \times \mathcal{X}(X \otimes Y, Z) \\
&\cong \mathcal{C}(A, C) \times \mathcal{C}(B^X, C) \times \mathcal{X}(X \otimes Y, Z) \\
&\cong \mathcal{C}(A, C) \times \mathcal{X}(X, C \lhd B) \times \mathcal{X}(X, Z/Y) \\
&\cong \mathcal{C}(A, C) \times \mathcal{X}(X, (C \lhd B) \times (Z/Y)) \\
&= (\mathcal{C} \rtimes \mathcal{X})(\langle A, X\rangle, \langle C, (C \lhd B) \times (Z/Y)\rangle) \\
&= (\mathcal{C} \rtimes \mathcal{X})(\langle A, X\rangle, \langle C, Z\rangle / \langle B, Y\rangle)
\end{aligned}$$

□

However, the skew monoidal category $\mathcal{C} \rtimes \mathcal{X}$ is not, in general, left closed, as we will now show. In any left closed skew monoidal category, tensoring on the left with a fixed object has a right adjoint, and thus preserves the initial object. So, if $\mathcal{C} \rtimes \mathcal{X}$ were a left closed skew monoidal category, then we would necessarily have isomorphisms of the following form.

$$\langle C, X\rangle \otimes 0_{\mathcal{C} \rtimes \mathcal{X}} \cong 0_{\mathcal{C} \rtimes \mathcal{X}}$$

Evaluating each side of this equation gives the following.

$$\langle C, X \otimes 0_{\mathcal{X}} \rangle \cong \langle 0_{\mathcal{C}}, 0_{\mathcal{X}} \rangle$$

So, we would necessarily have isomorphisms of the following forms.

$$C \cong 0_{\mathcal{C}} \qquad X \otimes 0_{\mathcal{X}} \cong 0_{\mathcal{X}}$$

The second may exist, but the first will not, unless $\mathcal{C}$ is trivial.

We will now give some examples of semidirect product monoidal categories which are right closed but not left closed.

Example 4.7.3. Let $\mathcal{X}$ be a right closed monoidal category with finite products and finite coproducts, in which the tensor product preserves coproducts in both variables. Let $\mathcal{C}$ be $\mathcal{X}$, with binary coproducts as the tensor product and the initial object as the monoidal unit. Then there is a strong action of $\mathcal{X}$ on $\mathcal{C}$ using the original tensor product, as follows.

$$B^X = X \otimes B$$

Let $(- \lhd -)$ be the original internal hom, as follows.

$$C \lhd B = C/B$$

The isomorphism of hom sets

$$\mathcal{C}(B^X, C) \cong \mathcal{X}(X, C \lhd B)$$

is then just the usual hom-tensor adjunction.

$$\mathcal{X}(X \otimes B, C) \cong \mathcal{X}(X, C/B)$$

The resulting semidirect product $\mathcal{C} \rtimes \mathcal{X}$ is a right closed monoidal structure on $\mathcal{X} \times \mathcal{X}$, with tensor product and internal hom defined as follows.

$$\langle A, X \rangle \otimes \langle B, Y \rangle = \langle A + (X \otimes B), X \otimes Y \rangle$$

$$\langle C, Z \rangle / \langle B, Y \rangle = \langle C, (C/B) \times (Z/Y) \rangle$$

Example 4.7.4. Let $\mathcal{X}$ be SET, the category of sets. Let $\mathcal{C}$ be a category with small coproducts, with binary coproducts as the tensor product and the initial object as the monoidal unit. Then there is a strong action of $\mathcal{X}$ on $\mathcal{C}$ by copowers, as follows.

$$B^X = \coprod_{x \in X} B$$

The notation we have been using so far agrees with the notation usually used for powers, rather than copowers; this is unfortunate, but hopefully not too confusing. Let $(- \lhd -)$ be the usual hom-functor, as follows.

$$C \lhd B = \mathcal{C}(B, C)$$

The isomorphism of hom sets

$$\mathcal{C}(B^X, C) \cong \mathcal{X}(X, C \lhd B)$$

is then just the universal property of the copower.

$$\mathcal{C}(\coprod_{x \in X} B, C) \cong \prod_{x \in X} \mathcal{C}(B, C) \cong \textsc{Set}(X, \mathcal{C}(B, C))$$

The resulting semidirect product $\mathcal{C} \rtimes \mathcal{X}$ is a right closed monoidal structure on $\mathcal{C} \times \textsc{Set}$, with tensor product and internal hom defined as follows.

$$\langle A, X \rangle \otimes \langle B, Y \rangle = \langle A + (\coprod_{x \in X} B), X \times Y \rangle$$

$$\langle C, Z \rangle / \langle B, Y \rangle = \langle C, \mathcal{C}(B, C) \times \textsc{Set}(Y, Z) \rangle$$

Example 4.7.5. As a specific case of the previous example, let $\mathcal{C}$ be a complete lattice, considered as a preorder. In this category, the coproduct of a and b is their join, or least upper bound, denoted $a \vee b$. The resulting semidirect product $\mathcal{C} \rtimes \mathcal{X}$ is a right closed monoidal structure on $\mathcal{C} \times \textsc{Set}$, with tensor product and internal hom defined as follows.

$$\langle a, X \rangle \otimes \langle b, Y \rangle \cong \begin{cases} \langle a \vee b, X \times Y \rangle & \text{if } X \text{ is non-empty} \\ \langle a, \emptyset \rangle & \text{if } X \text{ is empty} \end{cases}$$

$$\langle c, Z \rangle / \langle b, Y \rangle \cong \begin{cases} \langle c, \textsc{Set}(Y, Z) \rangle & \text{if } b \leq c \\ \langle c, \emptyset \rangle & \text{if } b > c \end{cases}$$

Chapter 5

Coherence for Monoidal Adjunctions Between Closed Monoidal Categories

5.1 Introduction

The monoidal adjunctions between closed monoidal categories which we will be considering in this chapter are a fragment of Grothendieck's 'six operations' formalism. For an introduction to this, see [5]. In full generality, Grothendieck's 'six operations' consists of assigning to every suitable 'space' a closed symmetric monoidal category and to every suitable morphism of spaces f a monoidal adjunction $f^\star \dashv f_\star$ and an adjunction $f_! \dashv f^!$ between these categories. These four operations, together with the tensor product and internal hom, constitute the six operations.

There is a large amount of data involved in such a situation: associators and unitors for the monoidal structures, units and counits for the closed structures, units and counits for each of the adjunctions $f^\star \dashv f_\star$ and $f_! \dashv f^!$, structure maps for each of the strong monoidal functors $f^\star$, and additional pseudofunctoriality data for the four assignments $f \mapsto f^\star$, $f \mapsto f_\star$, $f \mapsto f_!$ and $f \mapsto f^!$. There are many diagrams which can be constructed from such data, and various authors have expressed the desire for a coherence theorem for at least part of this structure (for example, [5] §1 or [7] §6).

The fragment of this which we will consider consists of the monoidal adjunctions $f^\star \dashv f_\star$ between closed monoidal categories. Thus we will not be considering the symmetry of the tensor product or the adjunctions $f_! \dashv f^!$. The general method we use is based on Kelly and Mac Lane's coherence theorem for closed symmetric monoidal categories [12].

In §5.2, we will state the coherence theorem. This will involve constructing a category, denoted $\textsc{Gr}_{\mathfrak{G}}$, whose objects are diagrams of monoidal adjunctions between closed monoidal categories, and describing a notion of freely generated objects in this category, which we will denote $\textsc{Shp}_G$. Such freely generated diagrams of monoidal adjunctions between closed monoidal categories will involve certain closed monoidal categories whose objects we call shapes and whose morphisms we call allowable morphisms, which we will denote $\textsc{Shp}_G(\mathcal{C})$. It is these allowable morphisms which the coherence theorem will apply to.

In §5.3, we will prove a preliminary coherence theorem involving a certain class of invertible allowable morphisms, which we call central isomorphisms.

In §5.4, we will describe an alternate characterisation of the allowable morphisms, via a number of constructions which will turn out to be more convenient to work with than our original definition.

In §5.5, we will define an object of $\textsc{Gr}_{\mathfrak{G}}$, which we will denote $\mathbb{Z}\textsc{Rel}$, as well as a morphism of $\textsc{Gr}_{\mathfrak{G}}$ of the following form

$$\Omega \colon \textsc{Shp}_G \to \mathbb{Z}\textsc{Rel}$$

In §5.6, we will prove the main result of this chapter: that the functors

$$\Omega_{\mathcal{C}} \colon \textsc{Shp}_G(\mathcal{C}) \to \mathbb{Z}\textsc{Rel}(\mathcal{C})$$

which make up the morphism Ω are all faithful.

In §5.7, we will provide some example applications of the coherence theorem.

5.2 Shapes and Allowable Morphisms

In this section, we will define the machinery necessary to formally state the coherence theorem. We are going to prove a coherence theorem for arbitrary diagrams of monoidal adjunctions between closed monoidal categories. First, we will define a category whose objects are such diagrams. Then, we will describe the free objects of this category; these are the objects which the coherence theorem will apply to. Finally, we will describe the coherence theorem which is the main result of this section.

Choose a directed graph, $\mathfrak{G}$, which will determine the shape of the diagrams we will prove the coherence theorem for. We will describe a category, which we call the category of 'Grothendieck contexts', denoted $\textsc{Gr}_{\mathfrak{G}}$, parameterised by $\mathfrak{G}$. Informally, an object of $\textsc{Gr}_{\mathfrak{G}}$ is a $\mathfrak{G}$-shaped diagram of small closed monoidal categories and monoidal adjunctions, and a morphism of $\textsc{Gr}_{\mathfrak{G}}$ is like a natural transformation whose components are strict closed monoidal functors.

Definition 5.2.1 (CMC$_{\mathrm{ma}}$)**.** We will use the notation CMC$_{\mathrm{ma}}$ to refer to a 2-category whose objects are closed monoidal categories and whose morphisms are monoidal adjunctions, defined as follows.

- A 0-cell $\mathcal{C}$ of CMC$_{\mathrm{ma}}$ is a small closed monoidal category.
- A 1-cell $\Phi\colon \mathcal{C} \to \mathcal{D}$ of CMC$_{\mathrm{ma}}$ is a monoidal adjunction of the following form.

$$\mathcal{D} \underset{\Phi_\star}{\overset{\Phi^\star}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} \mathcal{C}$$

- A 2-cell $\gamma\colon \Phi \to \Psi$ of CMC$_{\mathrm{ma}}$ is a monoidal natural transformation $\Phi^\star \Rightarrow \Psi^\star$.

Definition 5.2.2 ($\textsc{Gr}_{\mathfrak{G}}$)**.** An object X of $\textsc{Gr}_{\mathfrak{G}}$ is a pseudofunctor from the free category generated by the graph $\mathfrak{G}$ to CMC$_{\mathrm{ma}}$. Explicitly, this consists of the following.

- For each vertex $\mathcal{C}$ of $\mathfrak{G}$, a small closed monoidal category, $X(\mathcal{C})$.

- For each path $\Gamma\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, a monoidal adjunction of the following form.

$$X(\mathcal{D}) \underset{\Gamma_\star}{\overset{\Gamma^\star}{\rightleftarrows}} X(\mathcal{C}) \qquad (\Gamma_\star \dashv \Gamma^\star)$$

- For each pair of paths $\Gamma\colon \mathcal{C} \to \mathcal{D}$ and $\Delta\colon \mathcal{D} \to \mathcal{E}$ in $\mathfrak{G}$, an invertible monoidal natural transformation $\kappa^{\Gamma,\Delta}\colon \Gamma^\star\Delta^\star \Rightarrow (\Delta\Gamma)^\star$.

- For each vertex $\mathcal{C}$ of $\mathfrak{G}$, an invertible monoidal natural transformation $\hat{\kappa}^{\mathcal{C}}\colon \mathrm{id}_{X(\mathcal{C})} \Rightarrow (\mathrm{id}_{\mathcal{C}})^\star$, where $\mathrm{id}_{X(\mathcal{C})}$ is the identity functor on $X(\mathcal{C})$, and $\mathrm{id}_{\mathcal{C}}$ is the empty path in $\mathfrak{G}$ starting and ending at $\mathcal{C}$.

This is subject to the commutativity of the following diagrams.

$$\begin{array}{ccc}
\Gamma^\star\Delta^\star\Lambda^\star & = & \Gamma^\star\Delta^\star\Lambda^\star \\
{\scriptstyle \kappa^{\Gamma,\Delta}\Lambda^\star}\downarrow & & \downarrow{\scriptstyle \Gamma^\star\kappa^{\Delta,\Lambda}} \\
(\Delta\Gamma)^\star\Lambda^\star & & \Gamma^\star(\Lambda\Delta)^\star \\
{\scriptstyle \kappa^{\Delta\Gamma,\Lambda}}\downarrow & & \downarrow{\scriptstyle \kappa^{\Gamma,\Lambda\Delta}} \\
(\Lambda\Delta\Gamma)^\star & = & (\Lambda\Delta\Gamma)^\star
\end{array} \tag{5.1}$$

$$\begin{array}{ccccc}
 & & \Gamma^\star & & \\
 & /\!/ & & \backslash\!\backslash & \\
\mathrm{id}\,\Gamma^\star & & & & (\Gamma\,\mathrm{id})^\star \\
 & {\scriptstyle \hat{\kappa}^{\mathcal{C}}\Gamma^\star}\searrow & & \nearrow{\scriptstyle \kappa^{\mathrm{id},\Gamma}} & \\
 & & \mathrm{id}^\star\Gamma^\star & &
\end{array} \tag{5.2}$$

$$\begin{array}{ccccc}
 & & \Gamma^\star\mathrm{id}^\star & & \\
 & {\scriptstyle \Gamma^\star\hat{\kappa}^{\mathcal{D}}}\nearrow & & \searrow{\scriptstyle \kappa^{\Gamma,\mathrm{id}}} & \\
\Gamma^\star\,\mathrm{id} & & & & (\mathrm{id}\,\Gamma)^\star \\
 & \backslash\!\backslash & & /\!/ & \\
 & & \Gamma^\star & &
\end{array} \tag{5.3}$$

A morphism $\Omega\colon X \to Y$ of $\textsc{Gr}_{\mathfrak{G}}$ consists of, for each vertex $\mathcal{C}$ of $\mathfrak{G}$, a functor $\Omega_{\mathcal{C}}\colon X(\mathcal{C}) \to Y(\mathcal{C})$ which strictly preserves everything. Explicitly, the following equalities must hold.

- For each vertex $\mathcal{C}$ of $\mathfrak{G}$,
$$\Omega_{\mathcal{C}}(\mathcal{I}) = \mathcal{I}.$$
- For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each pair of objects A and B of $X(\mathcal{C})$,
$$\Omega_{\mathcal{C}}(A \otimes B) = \Omega_{\mathcal{C}}(A) \otimes \Omega_{\mathcal{C}}(B).$$
- For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each pair of objects A and B of $X(\mathcal{C})$,
$$\Omega_{\mathcal{C}}(A \backslash B) = \Omega_{\mathcal{C}}(A) \backslash \Omega_{\mathcal{C}(B)}.$$
- For each path $\Gamma\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, for each object A of $X(\mathcal{D})$,
$$\Omega_{\mathcal{C}}\Gamma^{\star}(A) = \Gamma^{\star}\Omega_{\mathcal{D}}(A).$$
- For each path $\Gamma\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, for each object A of $X(\mathcal{C})$,
$$\Omega_{\mathcal{D}}\Gamma_{\star}(A) = \Gamma_{\star}\Omega_{\mathcal{C}}(A).$$
- For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each triple of objects A, B and C of $X(\mathcal{C})$,
$$\Omega_{\mathcal{C}}(\alpha_{A,B,C}) = \alpha_{\Omega_{\mathcal{C}}(A),\Omega_{\mathcal{C}}(B),\Omega_{\mathcal{C}(C)}}.$$
- For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each object A of $X(\mathcal{C})$,
$$\Omega_{\mathcal{C}}(\lambda_A) = \lambda_{\Omega_{\mathcal{C}}(A)}.$$
- For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each object A of $X(\mathcal{C})$,
$$\Omega_{\mathcal{C}}(\rho_A) = \rho_{\Omega_{\mathcal{C}}(A)}.$$
- For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each pair of objects A and B of $X(\mathcal{C})$,
$$\Omega_{\mathcal{C}}(\eta^A_B) = \eta^{\Omega_{\mathcal{C}}(A)}_{\Omega_{\mathcal{C}}(B)}.$$
- For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each pair of objects A and B of $X(\mathcal{C})$,
$$\Omega_{\mathcal{C}}(\varepsilon^A_B) = \varepsilon^{\Omega_{\mathcal{C}}(A)}_{\Omega_{\mathcal{C}}(B)}.$$
- For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each pair of morphisms f and g of $X(\mathcal{C})$,
$$\Omega_{\mathcal{C}}(f \otimes g) = \Omega_{\mathcal{C}}(f) \otimes \Omega_{\mathcal{C}}(g).$$

- For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each pair of morphisms f and g of $X(\mathcal{C})$,

$$\Omega_{\mathcal{C}}(f\backslash g) = \Omega_{\mathcal{C}}(f)\backslash\Omega_{\mathcal{C}(g)}.$$

- For each path $\Gamma\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, for each pair of objects A and B of $X(\mathcal{D})$,

$$\Omega_{\mathcal{C}}(\varphi^{\Gamma}_{A,B}) = \varphi^{\Gamma}_{\Omega_{\mathcal{D}}(A),\Omega_{\mathcal{D}}(B)}.$$

- For each path $\Gamma\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$,

$$\Omega_{\mathcal{C}}(\hat{\varphi}^{\Gamma}) = \hat{\varphi}^{\Gamma}.$$

- For each path $\Gamma\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, for each object D of $X(\mathcal{D})$,

$$\Omega_{\mathcal{D}}(\theta^{\Gamma}_{D}) = \theta^{\Gamma}_{\Omega_{\mathcal{D}}(D)}.$$

- For each path $\Gamma\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, for each object C of $X(\mathcal{C})$,

$$\Omega_{\mathcal{C}}(\zeta^{\Gamma}_{C}) = \zeta^{\Gamma}_{\Omega_{\mathcal{C}}(C)}.$$

- For each pair of paths $\Gamma\colon \mathcal{C} \to \mathcal{D}$ and $\Delta\colon \mathcal{D} \to \mathcal{E}$ in $\mathfrak{G}$, for each object E of $X(\mathcal{E})$,

$$\Omega_{\mathcal{C}}(\kappa^{\Gamma,\Delta}_{E}) = \kappa^{\Gamma,\Delta}_{\Omega_{\mathcal{E}}(E)}.$$

- For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each object A of $X(\mathcal{C})$,

$$\Omega_{\mathcal{C}}(\hat{\kappa}^{\mathcal{C}}_{A}) = \hat{\kappa}^{\mathcal{C}}_{\Omega_{\mathcal{C}}(A)}.$$

- For each path $\Gamma\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, for each morphism f of $X(\mathcal{D})$,

$$\Omega_{\mathcal{C}}\Gamma^{\star}(f) = \Gamma^{\star}\Omega_{\mathcal{D}}(f).$$

- For each path $\Gamma\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, for each morphism f of $X(\mathcal{C})$,

$$\Omega_{\mathcal{D}}\Gamma_{\star}(f) = \Gamma_{\star}\Omega_{\mathcal{C}}(f).$$

Example 5.2.3. Let $\mathfrak{G}$ be the graph $\{\mathcal{C}\}$, then a Grothendieck context $X \in \operatorname{ob}\textsc{Gr}_{\mathfrak{G}}$ consists of a small closed monoidal category $X(\mathcal{C})$.

Example 5.2.4. Let $\mathfrak{G}$ be the graph $\{\mathcal{C} \xrightarrow{\Phi} \mathcal{D}\}$, then a Grothendieck context $X \in \operatorname{ob}\textsc{Gr}_{\mathfrak{G}}$ consists of small closed monoidal categories $X(\mathcal{C})$ and $X(\mathcal{D})$ and a monoidal adjunction of the following form.

$$X(\mathcal{D}) \underset{\Phi_{\star}}{\overset{\Phi^{\star}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} X(\mathcal{C})$$

Example 5.2.5. Let $\mathfrak{G}$ be the graph $\{\mathcal{C} \xrightarrow{\Phi} \mathcal{D} \xrightarrow{\Psi} \mathcal{E}\}$, then a Grothendieck context $X \in \operatorname{ob} \textsc{Gr}_{\mathfrak{G}}$ consists of small closed monoidal categories $X(\mathcal{C})$, $X(\mathcal{D})$ and $X(\mathcal{E})$ and monoidal adjunctions of the following form.

$$X(\mathcal{E}) \underset{\Psi_\star}{\overset{\Psi^\star}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} X(\mathcal{D}) \underset{\Phi_\star}{\overset{\Phi^\star}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} X(\mathcal{C})$$

Example 5.2.6. Let $\mathfrak{G}$ be the disconnected graph $\{\mathcal{C} \xrightarrow{\Phi} \mathcal{D} \qquad \mathcal{E} \xrightarrow{\Psi} \mathcal{F}\}$, then a Grothendieck context $X \in \operatorname{ob} \textsc{Gr}_{\mathfrak{G}}$ consists of small closed monoidal categories $X(\mathcal{C})$, $X(\mathcal{D})$, $X(\mathcal{E})$ and $X(\mathcal{F})$ and monoidal adjunctions of the following form.

$$X(\mathcal{F}) \underset{\Psi_\star}{\overset{\Psi^\star}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} X(\mathcal{E}) \qquad X(\mathcal{D}) \underset{\Phi_\star}{\overset{\Phi^\star}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} X(\mathcal{C})$$

Next, we will describe a forgetful functor out of $\textsc{Gr}_{\mathfrak{G}}$ and then construct a free functor which is left adjoint to this forgetful functor. This will give us a notion of freely generated Grothendieck contexts in $\textsc{Gr}_{\mathfrak{G}}$, which are the objects the coherence theorem will apply to.

Denote by $\mathfrak{G}_0$ the set of vertices of the graph $\mathfrak{G}$. Consider the category $\textsc{Set}^{|\mathfrak{G}_0|}$. An object G of $\textsc{Set}^{|\mathfrak{G}_0|}$ is a collection of sets indexed by the vertices of $\mathfrak{G}$; denote the set corresponding to the vertex $\mathcal{C}$ by $G(\mathcal{C})$.

There is a forgetful functor $U\colon \textsc{Gr}_{\mathfrak{G}} \to \textsc{Set}^{|\mathfrak{G}_0|}$. Given a Grothendieck context $X \in \operatorname{ob} \textsc{Gr}_{\mathfrak{G}}$, the image under U of X, denoted U_X, is the $\mathfrak{G}_0$-indexed collection of sets of the form $\operatorname{ob}(X(\mathcal{C}))$.

$$U_X(\mathcal{C}) = \operatorname{ob}(X(\mathcal{C}))$$

Now we will define a left adjoint to the functor U, which we will denote $\textsc{Shp}\colon \textsc{Set}^{|\mathfrak{G}_0|} \to \textsc{Gr}_{\mathfrak{G}}$. Given a $\mathfrak{G}_0$-indexed collection of sets $G \in \operatorname{ob} \textsc{Set}^{|\mathfrak{G}_0|}$, the image under $\textsc{Shp}$ of G, denoted $\textsc{Shp}_G$, is the freely generated Grothendieck context in $\textsc{Gr}_{\mathfrak{G}}$ defined as follows.

For each vertex $\mathcal{C}$ of $\mathfrak{G}$, the objects of the closed monoidal category $\textsc{Shp}_G(\mathcal{C})$ we call $\mathcal{C}$-shapes.

Definition 5.2.7 (shapes)**.** The shapes are defined by the following rules.

type ($\mathcal{I}$) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, there is a $\mathcal{C}$-shape

$$\mathcal{I}.$$

type (G) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each element X of $G(\mathcal{C})$, there is a $\mathcal{C}$-shape

$$X.$$

type ($\otimes$) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each pair of $\mathcal{C}$-shapes A and B, there is a $\mathcal{C}$-shape

$$A \otimes B.$$

type ($\backslash$) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each pair of $\mathcal{C}$-shapes A and B, there is a $\mathcal{C}$-shape

$$A\backslash B.$$

type ($(-)^\star$) For each path $\Gamma\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, for each $\mathcal{D}$-shape A, there is a $\mathcal{C}$-shape

$$\Gamma^\star(A).$$

type ($(-)_\star$) For each path $\Gamma\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, for each $\mathcal{C}$-shape A, there is a $\mathcal{D}$-shape

$$\Gamma_\star(A).$$

For each vertex $\mathcal{C}$ of $\mathfrak{G}$, the morphisms of the closed monoidal category $\text{S}\small{\text{HP}}_G(\mathcal{C})$ we call allowable $\mathcal{C}$-morphisms.

Definition 5.2.8 (allowable morphisms)**.** The allowable morphisms are defined by the following rules.

type (id) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each $\mathcal{C}$-shape A, there is an allowable $\mathcal{C}$-morphism

$$\mathrm{id}_A\colon A \to A.$$

type (α) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each triple of $\mathcal{C}$-shapes A, B and C, there is an invertible allowable $\mathcal{C}$-morphism

$$\alpha_{A,B,C}\colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C.$$

type (λ) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each $\mathcal{C}$-shape A, there is an invertible allowable $\mathcal{C}$-morphism

$$\lambda_A\colon A \to \mathcal{I} \otimes A.$$

type (ρ) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each $\mathcal{C}$-shape A, there is an invertible allowable $\mathcal{C}$-morphism

$$\rho_A\colon A \otimes \mathcal{I} \to A.$$

type (η) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each pair of $\mathcal{C}$-shapes A and B, there is an allowable $\mathcal{C}$-morphism

$$\eta^A_B\colon B \to A\backslash(A \otimes B).$$

type (ε) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each pair of $\mathcal{C}$-shapes A and B, there is an allowable $\mathcal{C}$-morphism

$$\varepsilon^A_B \colon A \otimes (A \backslash B) \to B.$$

type ($\otimes$) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each pair of allowable $\mathcal{C}$-morphisms $f \colon A \to B$ and $g \colon C \to D$, there is an allowable $\mathcal{C}$-morphism

$$f \otimes g \colon A \otimes C \to B \otimes D.$$

type ($\backslash$) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each pair of allowable $\mathcal{C}$-morphisms $f \colon A \to B$ and $g \colon C \to D$, there is an allowable $\mathcal{C}$-morphism

$$f \backslash g \colon B \backslash C \to A \backslash D.$$

type (φ) For each path $\Gamma \colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, for each pair of $\mathcal{D}$-shapes A and B, there is an invertible allowable $\mathcal{C}$-morphism

$$\varphi^\Gamma_{A,B} \colon \Gamma^\star(A \otimes B) \to \Gamma^\star(A) \otimes \Gamma^\star(B).$$

type ($\hat{\varphi}$) For each path $\Gamma \colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, there is an invertible allowable $\mathcal{C}$-morphism

$$\hat{\varphi}^\Gamma \colon \Gamma^\star(\mathcal{I}) \to \mathcal{I}.$$

type (θ) For each path $\Gamma \colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, for each $\mathcal{D}$-shape A, there is an allowable $\mathcal{D}$-morphism

$$\theta^\Gamma_A \colon A \to \Gamma_\star \Gamma^\star(A).$$

type (ζ) For each path $\Gamma \colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, for each $\mathcal{C}$-shape A, there is an allowable $\mathcal{C}$-morphism

$$\zeta^\Gamma_A \colon \Gamma^\star \Gamma_\star(A) \to A.$$

type (κ) For each pair of paths $\Gamma \colon \mathcal{C} \to \mathcal{D}$ and $\Delta \colon \mathcal{D} \to \mathcal{E}$ in $\mathfrak{G}$, for each $\mathcal{E}$-shape A, there is an invertible allowable $\mathcal{C}$-morphism

$$\kappa^{\Gamma,\Delta}_A \colon \Gamma^\star \Delta^\star(A) \to (\Delta\Gamma)^\star(A).$$

type ($\hat{\kappa}$) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each $\mathcal{C}$-shape A, there is an invertible allowable $\mathcal{C}$-morphism

$$\hat{\kappa}^{\mathcal{C}}_A \colon A \to (\mathrm{id}_{\mathcal{C}})^\star(A).$$

type $((-)^\star)$ For each path $\Gamma\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, for each allowable $\mathcal{D}$-morphism $f\colon A \to B$, there is an allowable $\mathcal{C}$-morphism

$$\Gamma^\star(f)\colon \Gamma^\star(A) \to \Gamma^\star(B).$$

type $((-)_\star)$ For each path $\Gamma\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, for each allowable $\mathcal{C}$-morphism $f\colon A \to B$, there is an allowable $\mathcal{D}$-morphism

$$\Gamma_\star(f)\colon \Gamma_\star(A) \to \Gamma_\star(B).$$

type $(\circ)$ For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each pair of allowable $\mathcal{C}$-morphisms $f\colon A \to B$ and $g\colon B \to C$, there is an allowable $\mathcal{C}$-morphism

$$g \circ f\colon A \to C.$$

Definition 5.2.9 ($\textsc{Shp}_G(\mathcal{C})$)**.** For each vertex $\mathcal{C}$ of $\mathfrak{G}$, the closed monoidal category $\textsc{Shp}_G(\mathcal{C})$ is the category whose objects are $\mathcal{C}$-shapes and whose morphisms are allowable $\mathcal{C}$-morphisms, quotiented out by all of the necessary relations:

- associativity and unitality of composition;
- functoriality of $(- \otimes -)$, $(-\backslash-)$, $\Gamma^\star$ and $\Gamma_\star$;
- (extra)naturality of α, λ, ρ, η, ε, φ^Γ, θ^Γ, ζ^Γ, $\kappa^{\Gamma,\Delta}$ and $\hat{\kappa}^{\mathcal{C}}$;
- invertibility of α, λ, ρ, φ^Γ, $\hat{\varphi}^\Gamma$, $\kappa^{\Gamma,\Delta}$ and $\hat{\kappa}^{\mathcal{C}}$;
- the identities 2.1, 2.2, 2.3, 2.4 and 2.5 for each $\textsc{Shp}_G(\mathcal{C})$ being a monoidal category;
- the identities 2.6 and 2.7 for each $\textsc{Shp}_G(\mathcal{C})$ being a closed monoidal category;
- the identities 2.8, 2.9 and 2.10 for each $\Gamma^\star$ being a strong monoidal functor;
- the identities 2.11 and 2.12 for each $\kappa^{\Gamma,\Delta}$ and $\hat{\kappa}^{\mathcal{C}}$ being monoidal natural transformations;
- the triangle identities 2.13 and 2.14 for the adjunctions $\Gamma^\star \dashv \Gamma_\star$;
- and the identities 5.1, 5.2 and 5.3 for $\textsc{Shp}_G$ being a pseudofunctor.

Definition 5.2.10 ($\textsc{Shp}_G(\Phi)$)**.** For each edge $\Phi\colon \mathcal{C} \to \mathcal{D}$ of $\mathfrak{G}$, the monoidal adjunction $\textsc{Shp}_G(\Phi)$ is defined in the obvious way:

- The left adjoint, $\Phi^\star\colon \textsc{Shp}_G(\mathcal{D}) \to \textsc{Shp}_G(\mathcal{C})$, takes each $\mathcal{D}$-shape A to the $\mathcal{C}$-shape $\Phi^\star(A)$ and each allowable $\mathcal{D}$-morphism $f\colon A \to B$ to the allowable $\mathcal{C}$-morphism $\Phi^\star(f)\colon \Phi^\star(A) \to \Phi^\star(B)$.
- The right adjoint, $\Phi_\star\colon \textsc{Shp}_G(\mathcal{C}) \to \textsc{Shp}_G(\mathcal{D})$, takes each $\mathcal{C}$-shape A to the $\mathcal{D}$-shape $\Phi_\star(A)$ and each allowable $\mathcal{C}$-morphism $f\colon A \to B$ to the allowable $\mathcal{D}$-morphism $\Phi_\star(f)\colon \Phi_\star(A) \to \Phi_\star(B)$.
- The unit, $\theta^\Phi\colon \mathrm{id}_{\textsc{Shp}_G(\mathcal{D})} \Rightarrow \Phi_\star\Phi^\star$, has components of the form θ^Φ_A.
- The counit, $\zeta^\Phi\colon \Phi^\star\Phi_\star \Rightarrow \mathrm{id}_{\textsc{Shp}_G(\mathcal{C})}$, has components of the form ζ^Φ_A.

It should be clear that this defines a left adjoint to the functor U, and thus that there is a natural isomorphism of hom-sets of the following form.

$$\textsc{Gr}_{\mathfrak{G}}(\textsc{Shp}_G, X) \cong \textsc{Set}^{|\mathfrak{G}_0|}(G, U_X)$$

We should think of an object of $\textsc{Set}^{|\mathfrak{G}_0|}$ as providing a set of generating objects for each vertex of $\mathfrak{G}$ and the functor $\textsc{Shp}$ as freely generating a Grothendieck context in $\textsc{Gr}_{\mathfrak{G}}$ from this generating data. Such freely generated Grothendieck contexts in $\textsc{Gr}_{\mathfrak{G}}$ are the particular $\mathfrak{G}$-shaped diagrams of closed monoidal categories and monoidal adjunctions which the coherence theorem will apply to.

Explicitly, given a vertex $\mathcal{C}$ of $\mathfrak{G}$, and a pair of parallel morphisms s and t in the closed monoidal category $\textsc{Shp}_G(\mathcal{C})$, we wish to find a simple method of determining whether $s = t$. We will do this by constructing another object of $\textsc{Gr}_{\mathfrak{G}}$, which we will denote $\mathbb{Z}\textsc{Rel}$, and a morphism in $\textsc{Gr}_{\mathfrak{G}}$ of the following form.

$$\Omega\colon \textsc{Shp}_G \to \mathbb{Z}\textsc{Rel}$$

Such a morphism will have components which are functors between closed monoidal categories of the following form.

$$\Omega_{\mathcal{C}}\colon \textsc{Shp}_G(\mathcal{C}) \to \mathbb{Z}\textsc{Rel}(\mathcal{C})$$

The object $\mathbb{Z}\textsc{Rel}$ will have been chosen in such a way that it is easy to determine whether $\Omega_{\mathcal{C}}(s) = \Omega_{\mathcal{C}}(t)$. Thus, if the functor $\Omega_{\mathcal{C}}$ is faithful, we will have a simple method of determining whether $s = t$. The coherence theorem itself is precisely the statement that, for each vertex $\mathcal{C}$ of $\mathfrak{G}$, the functor $\Omega_{\mathcal{C}}$ is faithful.

5.3 Central Isomorphisms

In this section, we will prove a preliminary coherence theorem, which we will use extensively in the remaineder of this section. First, we will define a class of invertible allowable morphisms called the central isomorphisms. Then, we will prove a coherence theorem for the central isomorphisms. Finally, we will prove some technical lemmas involving the central isomorphisms.

5.3.1 Definitions

We will define the central isomorphisms themselves in several stages.

Definition 5.3.1 (primitive central isomorphisms)**.** The primitive central isomorphisms are defined by the following rules.

(α) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each triple of $\mathcal{C}$-shapes A, B and C,

$$\alpha_{A,B,C} \colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C$$

is a primitive central $\mathcal{C}$-isomorphism.

(λ) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each $\mathcal{C}$-shape A,

$$\lambda_A^{-1} \colon \mathcal{I} \otimes A \to A$$

is a primitive central $\mathcal{C}$-isomorphism.

(ρ) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each $\mathcal{C}$-shape A,

$$\rho_A \colon A \otimes \mathcal{I} \to A$$

is a primitive central $\mathcal{C}$-isomorphism.

($\bar{\lambda}$) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each $\mathcal{C}$-shape A,

$$\bar{\lambda}_A \colon \mathcal{I}\backslash A \xrightarrow{\lambda_{\mathcal{I}\backslash A}} \mathcal{I} \otimes (\mathcal{I}\backslash A) \xrightarrow{\varepsilon^{\mathcal{I}}_A} A$$

is a primitive central $\mathcal{C}$-isomorphism.

($\otimes_L$) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each $\mathcal{C}$-shape C and each primitive central $\mathcal{C}$-isomorphism $s \colon A \to B$,

$$s \otimes C \colon A \otimes C \to B \otimes C$$

is a primitive central $\mathcal{C}$-isomorphism.

($\otimes_R$) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each $\mathcal{C}$-shape C and each primitive central $\mathcal{C}$-isomorphism $s \colon A \to B$,

$$C \otimes s \colon C \otimes A \to C \otimes B$$

is a primitive central $\mathcal{C}$-isomorphism.

($\backslash_R$) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each $\mathcal{C}$-shape C and each primitive central $\mathcal{C}$-isomorphism $s \colon A \to B$,

$$C\backslash s \colon C\backslash A \to C\backslash B$$

is a primitive central $\mathcal{C}$-isomorphism.

($\backslash_L$) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each $\mathcal{C}$-shape C and each primitive central $\mathcal{C}$-isomorphism $s\colon A \to B$,

$$s^{-1}\backslash C\colon A\backslash C \to B\backslash C$$

is a primitive central $\mathcal{C}$-isomorphism.

(φ) For each path $\Gamma\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, for each pair of $\mathcal{D}$-shapes A and B,

$$\varphi^{\Gamma}_{A,B}\colon \Gamma^{\star}(A \otimes B) \to \Gamma^{\star}(A) \otimes \Gamma^{\star}(B)$$

is a primitive central $\mathcal{C}$-isomorphism.

($\hat{\varphi}$) For each path $\Gamma\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$,

$$\hat{\varphi}^{\Gamma}\colon \Gamma^{\star}(\mathcal{I}) \to \mathcal{I}$$

is a primitive central $\mathcal{C}$-isomorphism.

($(-)^{\star}$) For each path $\Gamma\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$, for each primitive central $\mathcal{D}$-isomorphism $s\colon A \to B$,

$$\Gamma^{\star}(s)\colon \Gamma^{\star}(A) \to \Gamma^{\star}(B)$$

is a primitive central $\mathcal{C}$-isomorphism.

(κ) For each pair of paths $\Gamma\colon \mathcal{C} \to \mathcal{D}$ and $\Delta\colon \mathcal{D} \to \mathcal{E}$ in $\mathfrak{G}$, for each $\mathcal{E}$-shape A,

$$\kappa^{\Gamma,\Delta}_{A}\colon \Gamma^{\star}\Delta^{\star}(A) \to (\Delta\Gamma)^{\star}(A)$$

is a primitive central $\mathcal{C}$-isomorphism.

($\hat{\kappa}$) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each $\mathcal{C}$-shape A,

$$(\hat{\kappa}^{\mathcal{C}}_{A})^{-1}\colon (\mathrm{id}_{\mathcal{C}})^{\star}(A) \to A$$

is a primitive central $\mathcal{C}$-isomorphism.

($\bar{\kappa}$) For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each $\mathcal{C}$-shape A,

$$\bar{\kappa}^{\mathcal{C}}_{A}\colon (\mathrm{id}_{\mathcal{C}})_{\star}(A) \xrightarrow{\hat{\kappa}^{\mathcal{C}}_{(\mathrm{id}_{\mathcal{C}})_{\star}(A)}} (\mathrm{id}_{\mathcal{C}})^{\star}(\mathrm{id}_{\mathcal{C}})_{\star}(A) \xrightarrow{\zeta^{\mathrm{id}_{\mathcal{C}}}_{A}} A$$

is a primitive central $\mathcal{C}$-isomorphism.

Definition 5.3.2 (reduced shape)**.** A reduced shape is a shape which is not the source of any primitive central isomorphism.

Definition 5.3.3 (partial central isomorphisms)**.** A partial central isomorphism is a composite of primitive central isomorphisms.

Definition 5.3.4 (central isomorphisms)**.** A central isomorphism is a composite of primitive central isomorphisms and their inverses.

5.3.2 Coherence

Now, we will prove a coherence theorem for the central isomorphisms. The method used is based on Mac Lane's original coherence theorem for monoidal categories [18].

We will prove the coherence theorem for the central isomorphisms by induction on a quantity which we call the rank of a shape.

Definition 5.3.5 (rank of a shape)**.** The rank of a shape A, denoted $\|A\|$, is a positive integer, defined as follows.

- For a shape of type ($\mathcal{I}$),
$$\|\mathcal{I}\| = 1.$$
- For a shape of type (G),
$$\|X\| = 1.$$
- For a shape of type ($\otimes$),
$$\|A \otimes B\| = \|A\| + 1 + 2 \cdot \|B\|.$$
- For a shape of type ($\backslash$),
$$\|A\backslash B\| = 2 \cdot \|A\| + 1 + \|B\|.$$
- For a shape of type ($(-)^\star$),
$$\|\Gamma^\star(A)\| = 2 \cdot \|A\|.$$
- For a shape of type ($(-)_\star$),
$$\|\Gamma_\star(A)\| = 2 \cdot \|A\|.$$

The definition of rank has been chosen in such a way that the following lemma holds.

Lemma 5.3.6. *Given a primitive central isomorphism $s\colon P \to Q$, it follows that*
$$\|P\| > \|Q\|.$$

Corollary 5.3.7. *Given a shape A, it follows that there exists a reduced shape $A_\star$ such that there exists a partial central isomorphism $s\colon A \to A_\star$.*

Proof. We will prove this by induction on the type of the primitive central isomorphism s.

- Consider the case where s is of type (α).

$$\alpha_{A,B,C}\colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C$$

Note the following.

$$\begin{aligned}\|A \otimes (B \otimes C)\| &= \|A\| + 1 + 2 \cdot \|B \otimes C\| \\ &= \|A\| + 1 + 2 \cdot \|B\| + 2 + 4 \cdot \|C\| \\ &> \|A\| + 1 + 2 \cdot \|B\| + 1 + 2 \cdot \|C\| \\ &= \|A \otimes B\| + 1 + 2 \cdot \|C\| \\ &= \|(A \otimes B) \otimes C\|\end{aligned}$$

- Consider the case where s is of type (λ).

$$\lambda_A^{-1}\colon \mathcal{I} \otimes A \to A$$

Note the following.

$$\|\mathcal{I} \otimes A\| = \|\mathcal{I}\| + 1 + 2 \cdot \|A\| = 1 + 2 \cdot \|A\| > \|A\|$$

- Consider the case where s is of type (ρ).

$$\rho_A\colon A \otimes \mathcal{I} \to A$$

Note the following.

$$\|A \otimes \mathcal{I}\| = \|A\| + 1 + 2 \cdot \|\mathcal{I}\| = \|A\| + 1 + 2 \cdot 1 > \|A\|$$

- Consider the case where s is of type $(\bar{\lambda})$.

$$\bar{\lambda}_A\colon \mathcal{I} \backslash A \to A$$

Note the following.

$$\|\mathcal{I} \backslash A\| = 2 \cdot \|\mathcal{I}\| + 1 + \|A\| = 2 \cdot 1 + 1 + \|A\| > \|A\|$$

- Consider the case where s is of type $(\otimes_L)$.

$$s' \otimes C\colon A \otimes C \to B \otimes C$$

Note the following.

$$\|A \otimes C\| = \|A\| + 1 + 2 \cdot \|C\| > \|B\| + 1 + 2 \cdot \|C\| = \|B \otimes C\|$$

- Consider the case where s is of type $(\otimes_R)$.

  $$C \otimes s' \colon C \otimes A \to C \otimes B$$

  Note the following.

  $$\|C \otimes A\| = \|C\| + 1 + 2 \cdot \|A\| > \|C\| + 1 + 2 \cdot \|B\| = \|C \otimes B\|$$

- Consider the case where s is of type $(\backslash_R)$.

  $$C\backslash s' \colon C\backslash A \to C\backslash B$$

  Note the following.

  $$\|C\backslash A\| = 2 \cdot \|C\| + 1 + \|A\| > 2 \cdot \|C\| + 1 + \|B\| = \|C\backslash B\|$$

- Consider the case where s is of type $(\backslash_L)$.

  $$s'^{-1}\backslash C \colon A\backslash C \to B\backslash C$$

  Note the following.

  $$\|A\backslash C\| = 2 \cdot \|A\| + 1 + \|C\| > 2 \cdot \|B\| + 1 + \|C\| = \|B\backslash C\|$$

- Consider the case where s is of type (φ).

  $$\varphi^{\Gamma}_{A,B} \colon \Gamma^\star(A \otimes B) \to \Gamma^\star(A) \otimes \Gamma^\star(B)$$

  Note the following.

  $$\begin{aligned}\|\Gamma^\star(A \otimes B)\| &= 2 \cdot \|A \otimes B\| \\ &= 2 \cdot \|A\| + 2 + 4 \cdot \|B\| \\ &> 2 \cdot \|A\| + 1 + 4 \cdot \|B\| \\ &= \|\Gamma^\star(A)\| + 1 + 2 \cdot \|\Gamma^\star(B)\| \\ &= \|\Gamma^\star(A) \otimes \Gamma^\star(B)\|\end{aligned}$$

- Consider the case where s is of type $(\hat{\varphi})$.

  $$\hat{\varphi}^{\Gamma} \colon \Gamma^\star(\mathcal{I}) \to \mathcal{I}$$

  Note the following.

  $$\|\Gamma^\star(\mathcal{I})\| = 2 \cdot \|\mathcal{I}\| > \|\mathcal{I}\|$$

- Consider the case where s is of type $((-)^\star)$.

$$\Gamma^\star(s')\colon \Gamma^\star(A) \to \Gamma^\star(B)$$

Note the following.

$$\|\Gamma^\star(A)\| = 2 \cdot \|A\| > 2 \cdot \|B\| = \|\Gamma^\star(B)\|$$

- Consider the case where s is of type (κ).

$$\kappa_A^{\Gamma,\Delta}\colon \Gamma^\star\Delta^\star(A) \to (\Delta\Gamma)^\star(A)$$

Note the following.

$$\|\Gamma^\star\Delta^\star(A)\| = 2 \cdot \|\Delta^\star(A)\| = 4 \cdot \|A\| > 2 \cdot \|A\| = \|(\Delta\Gamma)^\star(A)\|$$

- Consider the case where s is of type $(\hat{\kappa})$.

$$(\hat{\kappa}_A^{\mathcal{C}})^{-1}\colon \mathrm{id}^\star(A) \to A$$

Note the following.

$$\|\mathrm{id}^\star(A)\| = 2 \cdot \|A\| > \|A\|$$

- Consider the case where s is of type $(\bar{\kappa})$.

$$\bar{\kappa}_A\colon \mathrm{id}_\star(A) \to A$$

Note the following.

$$\|\mathrm{id}_\star(A)\| = 2 \cdot \|A\| > \|A\|$$

□

The first, and largest, step in proving the coherence theorem for the central isomorphisms is to prove that every shape is isomorphic to a unique reduced shape via a unique partial central isomorphism.

Lemma 5.3.8. *For any pair of partial central isomorphisms*

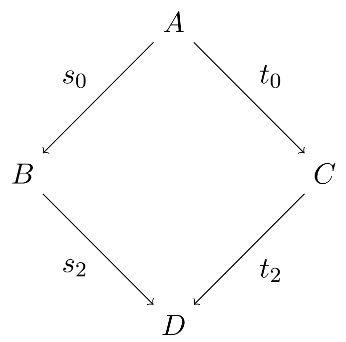
$$s\colon A \to B_\star \qquad \textit{and} \qquad t\colon A \to C_\star,$$

where $B_\star$ and $C_\star$ are reduced shapes, it follows that $B_\star = C_\star$ and $s = t$.

Proof. We will prove this lemma by induction on the rank of A, as follows. If A is a reduced shape, then $B_\star = A$ and $s = \mathrm{id}_A$ and $C_\star = A$ and $t = \mathrm{id}_A$, so the result follows. Otherwise, s and t are of the following forms, where s_0 and t_0 are primitive central isomorphisms and s_1 and t_1 are partial central isomorphisms.

$$s\colon A \xrightarrow{s_0} B \xrightarrow{s_1} B_\star \qquad \text{and} \qquad t\colon A \xrightarrow{t_0} C \xrightarrow{t_1} C_\star$$

Our general strategy will be to find a commutative diagram of the following form, where s_2 and t_2 are partial central isomorphisms.



By Corollary 5.3.7, we know that there is a reduced shape $D_\star$, and a partial central isomorphism $u\colon D \to D_\star$. By induction, we can conclude that $B_\star = D_\star$ and $s_1 = u \circ s_2$, since $\|A\| > \|B\|$. Similarly, we can conclude that $C_\star = D_\star$ and $t_1 = u \circ t_2$, since $\|A\| > \|C\|$. Then we can conclude with the following equalities.

$$s = s_1 \circ s_0 = u \circ s_2 \circ s_0 = u \circ t_2 \circ t_0 = t_1 \circ t_0 = t$$

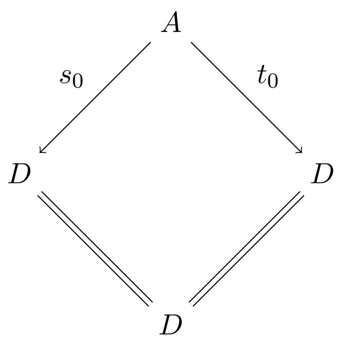
The remainder of the proof will be spent analysing the forms of s_0 and t_0 and, in each possible case, constructing such a commutative diagram.

Consider the type of the shape A. There is no primitive central isomorphism whose source is a shape of type (G) or $(\mathcal{I})$, so A must be of type $(\otimes)$, $(\backslash)$, $((-)^\star)$ or $((-)_\star)$.

Consider the case where A is of type $(\otimes)$. In this case, each of s_0 and t_0 must be of type (α), (λ), (ρ), $(\otimes_L)$ or $(\otimes_R)$. The table below lists all of the possibilities. We number each possible case, then consider each case separately. Note that the table is symmetric about its diagonal. The cases marked with a '(!)' are impossible, due to incompatible source shapes. The cases marked with a '(?)' we consider in more detail after the other cases have been analysed.

	α	λ	ρ	$\otimes_L$	$\otimes_R$
α	1	2	(!)	4	(?)
λ	2	1	3	(!)	5
ρ	(!)	3	1	6	(!)
$\otimes_L$	4	(!)	6	7	9
$\otimes_R$	(?)	5	(!)	9	8

1. If s_0 and t_0 are both of type (α), both of type (λ), or both of type (ρ), then they are equal. We can then form the following commutative diagram.

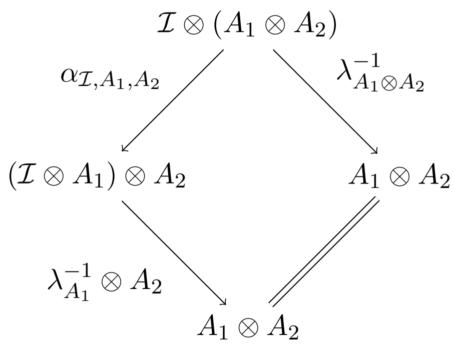


2. Consider the case where s_0 is of type (α) and t_0 is of type (λ).

$$s_0\colon A = \mathcal{I} \otimes (A_1 \otimes A_2) \xrightarrow{\alpha_{\mathcal{I},A_1,A_2}} (\mathcal{I} \otimes A_1) \otimes A_2 = B$$

$$t_0\colon A = \mathcal{I} \otimes (A_1 \otimes A_2) \xrightarrow{\lambda^{-1}_{A_1 \otimes A_2}} A_1 \otimes A_2$$

We can then form the following diagram, which commutes by 2.2.

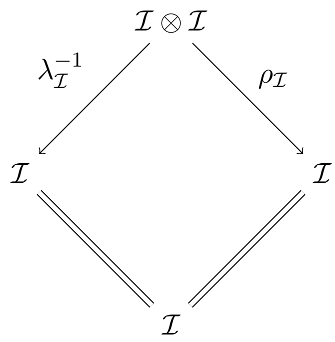


3. Consider the case where s_0 is of type (λ) and t_0 is of type (ρ).

$$s_0\colon A = \mathcal{I} \otimes \mathcal{I} \xrightarrow{\lambda^{-1}_{\mathcal{I}}} \mathcal{I} = B$$

$$t_0\colon A = \mathcal{I} \otimes \mathcal{I} \xrightarrow{\rho_{\mathcal{I}}} \mathcal{I} = C$$

We can then form the following diagram, which commutes by 2.5.

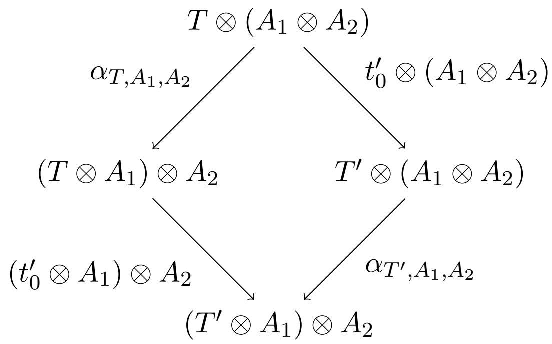


4. Consider the case where s_0 is of type (α) and t_0 is of type $(\otimes_L)$.

$$s_0 \colon A = T \otimes (A_1 \otimes A_2) \xrightarrow{\alpha_{T,A_1,A_2}} (T \otimes A_1) \otimes A_2 = B$$

$$t_0 \colon A = T \otimes (A_1 \otimes A_2) \xrightarrow{t_0' \otimes (A_1 \otimes A_2)} T' \otimes (A_1 \otimes A_2) = C$$

We can then form the following diagram, which commutes by naturality of α.

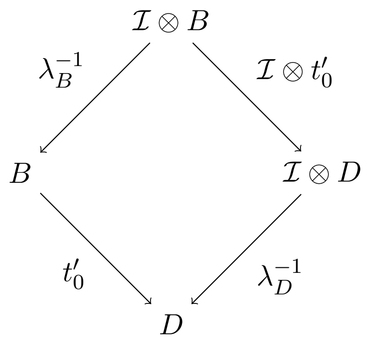


5. Consider the case where s_0 is of type (λ) and t_0 is of type $(\otimes_R)$.

$$s_0 \colon A = \mathcal{I} \otimes B \xrightarrow{\lambda_B^{-1}} B$$

$$t_0 \colon A = \mathcal{I} \otimes B \xrightarrow{\mathcal{I} \otimes t_0'} \mathcal{I} \otimes D = C$$

We can then form the following diagram, which commutes by naturality of λ.



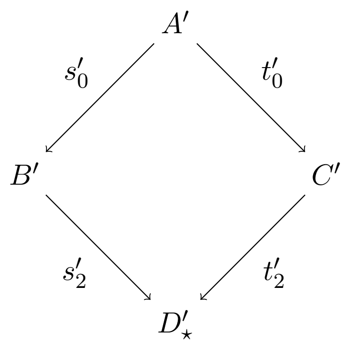
6. Consider the case where s_0 is of type (ρ) and t_0 is of type $(\otimes_L)$. This case is similar to the case where s_0 is of type (λ) and t_0 is of type $(\otimes_R)$. We can then form a diagram which commutes by naturality of ρ.

7. Consider the case where s_0 and t_0 are both of type $(\otimes_L)$.

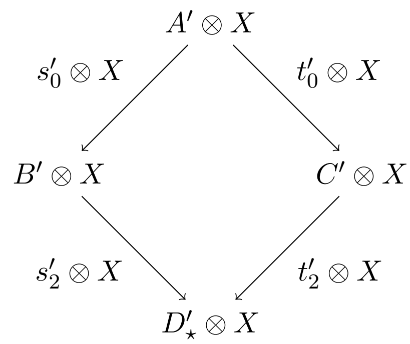
$$s_0\colon A = A' \otimes X \xrightarrow{s_0' \otimes X} B' \otimes X = B$$

$$t_0\colon A = A' \otimes X \xrightarrow{t_0' \otimes X} C' \otimes X = C$$

By induction, we can find a commutative diagram of the following form, where s_2' and t_2' are partial central isomorphisms and $D_\star'$ is a reduced shape, since $\|A\| = \|A' \otimes X\| = \|A'\| + 1 + 2 \cdot \|X\| > \|A'\|$.



We can then form the following commutative diagram.



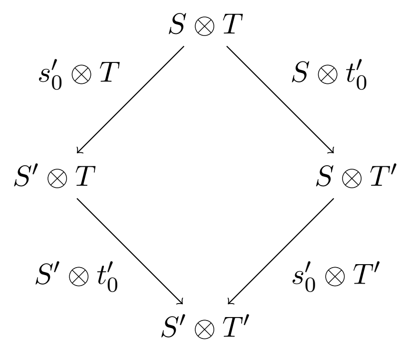
8. Consider the case where s_0 and t_0 are both of type $(\otimes_R)$. This case is similar to the case where s_0 and t_0 are both of type $(\otimes_L)$.

9. Consider the case where s_0 is of type $(\otimes_L)$ and t_0 is of type $(\otimes_R)$.

$$s_0 \colon A = S \otimes T \xrightarrow{s'_0 \otimes T} S' \otimes T = B$$

$$t_0 \colon A = S \otimes T \xrightarrow{S \otimes t'_0} S \otimes T' = C$$

We can then form the following diagram, which commutes by functoriality of $\otimes$.



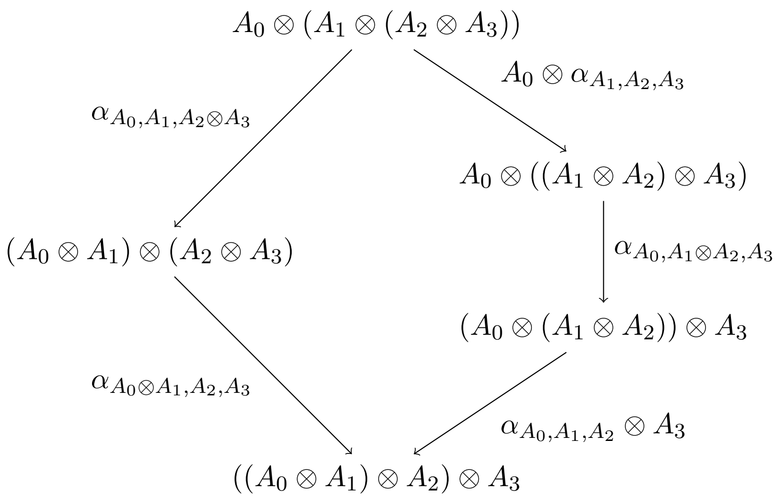
The only other possible remaining case is where s_0 is of type (α) and t_0 is of type $(\otimes_R)$, $t_0 = A_0 \otimes t'_0$. In this case, t'_0 must be of type (α), (λ), (ρ), $(\otimes_L)$ or $(\otimes_R)$.

- Consider the case where t'_0 is of type (α).

$$s_0 \colon A = A_0 \otimes (A_1 \otimes (A_2 \otimes A_3)) \xrightarrow{\alpha_{A_0, A_1, A_2 \otimes A_3}} (A_0 \otimes A_1) \otimes (A_2 \otimes A_3) = B$$

$$t_0 \colon A = A_0 \otimes (A_1 \otimes (A_2 \otimes A_3)) \xrightarrow{A_0 \otimes \alpha_{A_1, A_2, A_3}} A_0 \otimes ((A_1 \otimes A_2) \otimes A_3) = C$$

We can then form the following diagram, which commutes by 2.1.



- Consider the cases where t'_0 is of type (λ) or (ρ). These cases are similar to the case where s_0 is of type α and t_0 is of type λ. We can then form a diagram which commutes by 2.3 or 2.4.
- Consider the cases where t'_0 is of type $(\otimes_L)$ or $(\otimes_R)$. These cases are similar to the case where s_0 is of type α and t_0 is of type $\otimes_L$. We can then form a diagram which commutes by naturality of α.

Consider the case where A is of type $(\backslash)$. In this case, each of s_0 and t_0 must be of type $(\bar{\lambda})$, $(\backslash_R)$ or $(\backslash_L)$. The table below lists all of the possibilities. We number each possible case, then consider each case separately. Note that the table is symmetric about its diagonal. The cases marked with a '(!)' are impossible, due to incompatible source shapes.

	$\bar{\lambda}$	$\backslash_R$	$\backslash_L$
$\bar{\lambda}$	1	2	(!)
$\backslash_R$	2	3	5
$\backslash_L$	(!)	5	4

1. If s_0 and t_0 are both of type $(\bar{\lambda})$, then they are equal.
2. Consider the case where s_0 is of type $(\bar{\lambda})$ and t_0 is of type $(\backslash_R)$.

$$s_0 \colon A = \mathcal{I}\backslash B \xrightarrow{\bar{\lambda}_B} B$$

$$t_0 \colon A = \mathcal{I}\backslash B \xrightarrow{\mathcal{I}\backslash t'_0} \mathcal{I}\backslash D = C$$

This case is similar to the case where s_0 is of type (λ) and t_0 is of type $(\otimes_R)$. We can then form a diagram which commutes by naturality of $\bar{\lambda}$.

3. Consider the case where s_0 and t_0 are both of type $(\backslash_R)$. This case is similar to the case where s_0 and t_0 are both of type $(\otimes_L)$.

4. Consider the case where s_0 and t_0 are both of type $(\backslash_L)$. This case is similar to the case where s_0 and t_0 are both of type $(\otimes_L)$.

5. Consider the case where s_0 is of type $(\backslash_R)$ and t_0 is of type $(\backslash_L)$. This case is similar to the case where s_0 is of type $(\otimes_L)$ and t_0 is of type $(\otimes_R)$. We can then form a diagram which commutes by functoriality of $\backslash$.

Consider the case where A is of type $((-)^\star)$. In this case, each of s_0 and t_0 must be of type (φ), $(\hat{\varphi})$, $((-)^\star)$, (κ) or $(\hat{\kappa})$. The table below lists all of the possibilities. We number each possible case, then consider each case separately. Note that the table is symmetric about its diagonal. The cases marked with a '(!)' are impossible, due to incompatible source shapes. The cases marked with a '(?)' we consider in more detail after the other cases have been analysed.

	φ	$\hat{\varphi}$	$(-)^\star$	κ	$\hat{\kappa}$
φ	1	(!)	(?)	(!)	3
$\hat{\varphi}$	(!)	1	(!)	(!)	4
$(-)^\star$	(?)	(!)	2	(?)	5
κ	(!)	(!)	(?)	1	(!)
$\hat{\kappa}$	3	4	5	(!)	1

1. If s_0 and t_0 are both of type (φ), both of type $(\hat{\varphi})$, both of type (κ), or both of type $(\hat{\kappa})$, then they are equal.

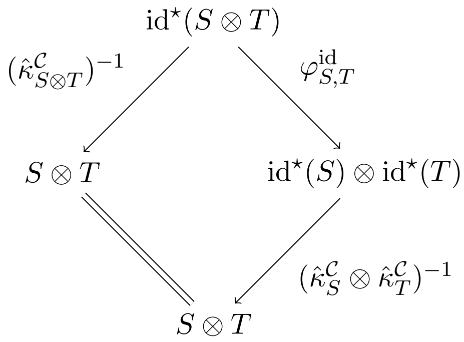
2. Consider the case where s_0 and t_0 are both of type $((-)^\star)$. This case is similar to the case where s_0 and t_0 are both of type $(\otimes_L)$.

3. Consider the case where s_0 is of type $(\hat{\kappa})$ and t_0 is of type (φ).

$$s_0 \colon A = \mathrm{id}^\star(S \otimes T) \xrightarrow{(\hat{\kappa}^{\mathcal{C}}_{S\otimes T})^{-1}} S \otimes T = B$$

$$t_0 \colon A = \mathrm{id}^\star(S \otimes T) \xrightarrow{\varphi^{\mathrm{id}}_{S,T}} \mathrm{id}^\star(S) \otimes \mathrm{id}^\star(T) = C$$

We can then form the following diagram, which commutes by 2.11.

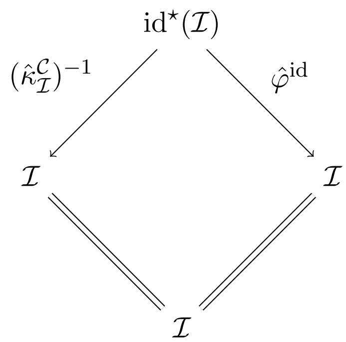


4. Consider the case where s_0 is of type $(\hat{\kappa})$ and t_0 is of type (φ).

$$s_0 \colon A = \mathrm{id}^\star(\mathcal{I}) \xrightarrow{(\hat{\kappa}^{\mathcal{C}}_{\mathcal{I}})^{-1}} \mathcal{I} = B$$

$$t_0 \colon A = \mathrm{id}^\star(\mathcal{I}) \xrightarrow{\hat{\varphi}^{\mathrm{id}}} \mathcal{I} = C$$

We can then form the following diagram, which commutes by 2.12.

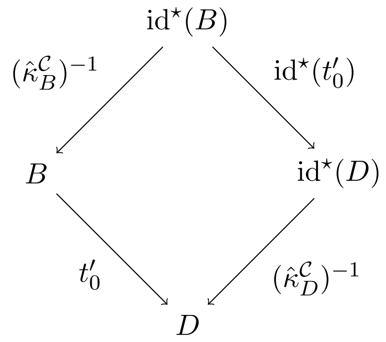


5. Consider the case where s_0 is of type $(\hat{\kappa})$ and t_0 is of type $((-)^\star)$.

$$s_0 \colon A = \mathrm{id}^\star(B) \xrightarrow{(\hat{\kappa}^{\mathcal{C}}_B)^{-1}} B = B$$

$$t_0 \colon A = \mathrm{id}^\star(B) \xrightarrow{\mathrm{id}^\star(t_0')} \mathrm{id}^\star(D) = C$$

We can then form the following diagram, which commutes by naturality of $\hat{\kappa}^{\mathcal{C}}$.



The only other possible remaining cases are where s_0 is of type (φ) or (κ) and t_0 is of type ($(-)^\star$).

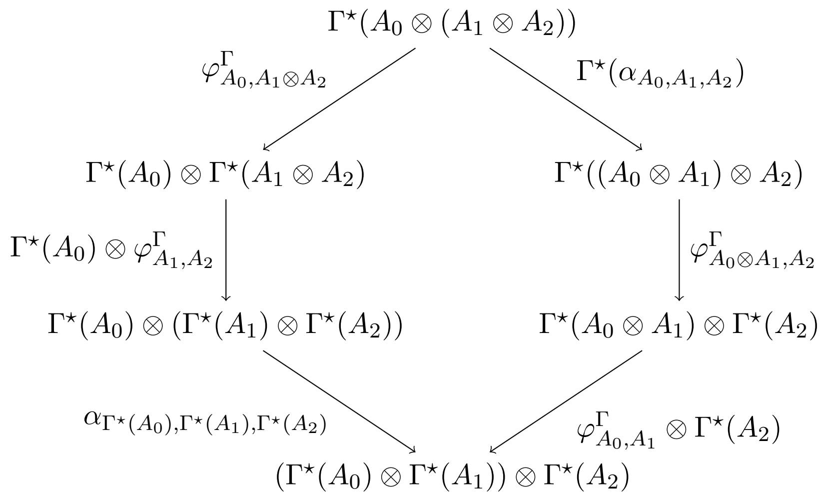
Consider the case where s_0 is of type (φ) and t_0 is of type ($(-)^\star$), $t_0 = \Gamma^\star(t'_0)$. In this case, we must further consider the type of t'_0.

- Consider the case where t'_0 is of type (α).

$$s_0\colon A = \Gamma^\star(A_0 \otimes (A_1 \otimes A_2)) \xrightarrow{\varphi_{A_0, A_1 \otimes A_2}} \Gamma^\star(A_0) \otimes \Gamma^\star(A_1 \otimes A_2) = B$$

$$t_0\colon A = \Gamma^\star(A_0 \otimes (A_1 \otimes A_2)) \xrightarrow{\Gamma^\star(\alpha_{A_0,A_1,A_2})} \Gamma^\star((A_0 \otimes A_1) \otimes A_2) = C$$

We can then form the following diagram, which commutes by 2.8.

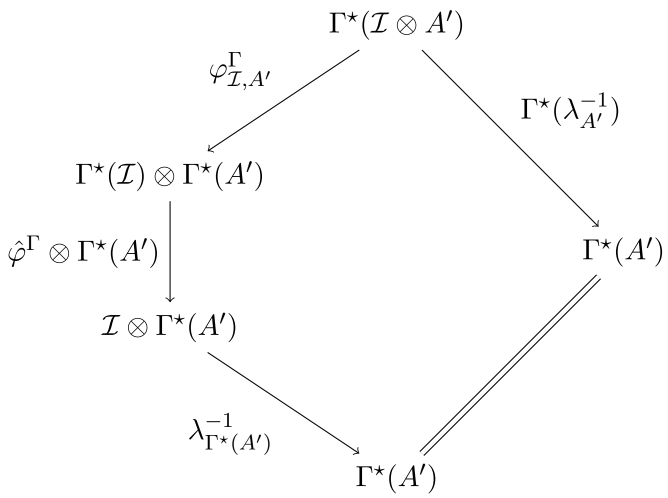


- Consider the case where t'_0 is of type (λ).

$$s_0\colon A = \Gamma^\star(\mathcal{I} \otimes A') \xrightarrow{\varphi^{\Gamma}_{\mathcal{I},A'}} \Gamma^\star(\mathcal{I}) \otimes \Gamma^\star(A') = B$$

$$t_0\colon A = \Gamma^\star(\mathcal{I} \otimes A') \xrightarrow{\Gamma^\star(\lambda^{-1}_{A'})} \Gamma^\star(A') = C$$

We can then form the following diagram, which commutes by 2.9.

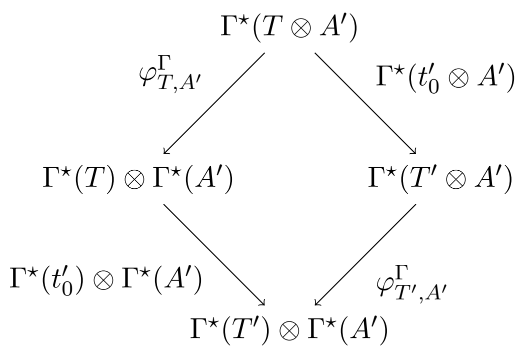


- Consider the case where t_0' is of type (ρ). This is similar to the case where t_0' is of type (λ). We can then form a diagram which commutes by 2.10.
- Consider the case where t_0' is of type $(\otimes_L)$.

$$s_0 \colon A = \Gamma^\star(T \otimes A') \xrightarrow{\varphi^\Gamma_{T,A'}} \Gamma^\star(T) \otimes \Gamma^\star(A') = B$$

$$t_0 \colon A = \Gamma^\star(T \otimes A') \xrightarrow{\Gamma^\star(t_0' \otimes A')} \Gamma^\star(T' \otimes A') = C$$

We can then form the following diagram, which commutes by naturality of φ^Γ.



- Consider the case where t_0' is of type $(\otimes_R)$. This case is similar to the case where t_0' is of type $(\otimes_L)$. We can then form a diagram which commutes by naturality of φ^Γ.

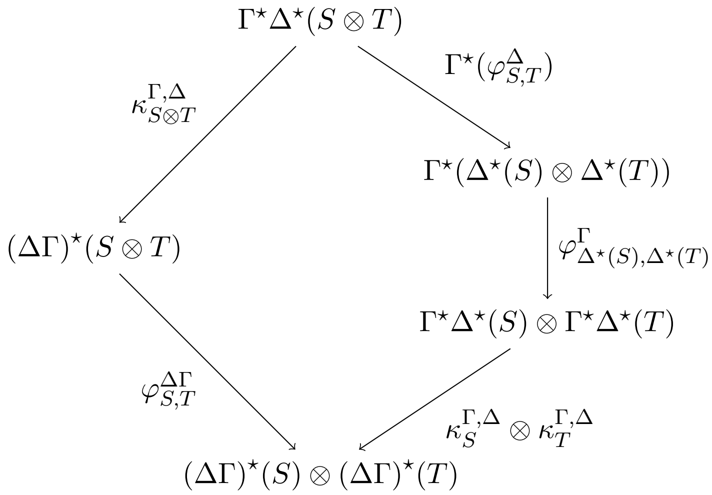
Consider the case where s_0 is of type (κ) and t_0 is of type $((-)^\star)$, $t_0 = \Gamma^\star(t_0')$. In this case, we must further consider the type of t_0'.

- Consider the case where t'_0 is of type (φ).

$$s_0\colon A = \Gamma^\star\Delta^\star(S \otimes T) \xrightarrow{\kappa^{\Gamma,\Delta}_{S\otimes T}} (\Delta\Gamma)^\star(S \otimes T) = B$$

$$t_0\colon A = \Gamma^\star\Delta^\star(S \otimes T) \xrightarrow{\Gamma^\star(\varphi^{\Delta}_{S,T})} \Gamma^\star(\Delta^\star(S) \otimes \Delta^\star(T)) = C$$

We can then form the following diagram, which commutes by 2.11.

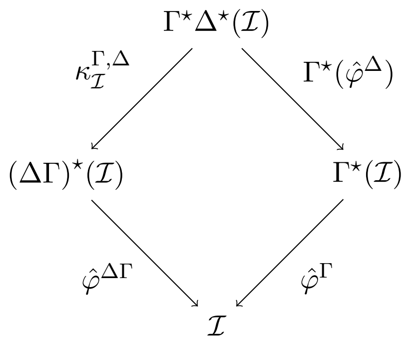


- Consider the case where t'_0 is of type $(\hat{\varphi})$.

$$s_0\colon A = \Gamma^\star\Delta^\star(\mathcal{I}) \xrightarrow{\kappa^{\Gamma,\Delta}_{\mathcal{I}}} (\Delta\Gamma)^\star(\mathcal{I}) = B$$

$$t_0\colon A = \Gamma^\star\Delta^\star(\mathcal{I}) \xrightarrow{\Gamma^\star(\hat{\varphi}^{\Delta})} \Gamma^\star(\mathcal{I}) = C$$

We can then form the following diagram, which commutes by 2.12.

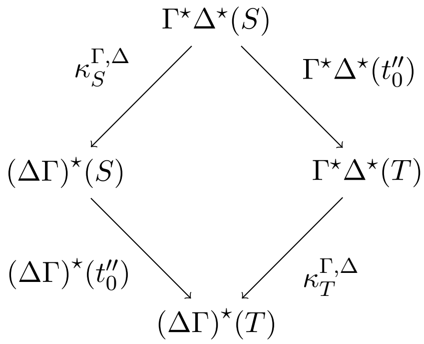


- Consider the case where t'_0 is of type $((-)^\star)$.

$$s_0\colon A = \Gamma^\star\Delta^\star(S) \xrightarrow{\kappa^{\Gamma,\Delta}_{S}} (\Delta\Gamma)^\star(S) = B$$

$$t_0\colon A = \Gamma^\star\Delta^\star(S) \xrightarrow{\Gamma^\star\Delta^\star(t_0'')} \Gamma^\star\Delta^\star(T) = C$$

We can then form the following diagram, which commutes by naturality of $\kappa^{\Gamma,\Delta}$.

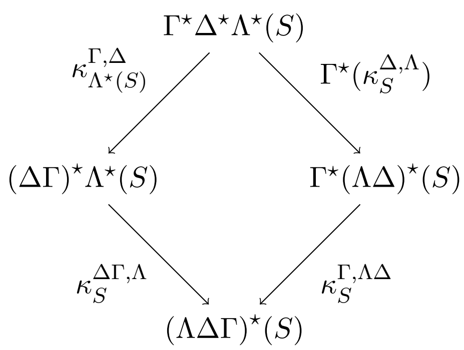


- Consider the case where t_0' is of type (κ).

$$s_0\colon A = \Gamma^\star\Delta^\star\Lambda^\star(S) \xrightarrow{\kappa^{\Gamma,\Delta}_{\Lambda^\star(S)}} (\Delta\Gamma)^\star\Lambda^\star(S) = B$$

$$t_0\colon A = \Gamma^\star\Delta^\star\Lambda^\star(S) \xrightarrow{\Gamma^\star(\kappa_S^{\Delta,\Lambda})} \Gamma^\star(\Lambda\Delta)^\star(S) = C$$

We can then form the following diagram, which commutes by 5.1.

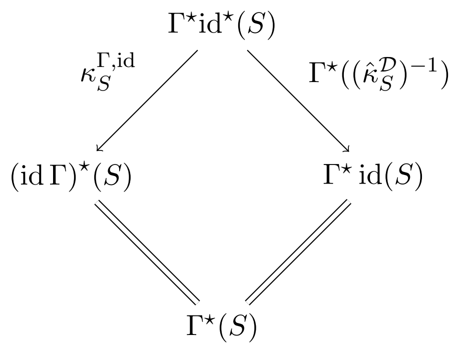


- Consider the case where t_0' is of type $(\hat{\kappa})$.

$$s_0\colon A = \Gamma^\star\mathrm{id}^\star(S) \xrightarrow{\kappa_S^{\Gamma,\mathrm{id}}} (\mathrm{id}\,\Gamma)^\star(S) = B$$

$$t_0\colon A = \Gamma^\star\mathrm{id}^\star(S) \xrightarrow{\Gamma^\star((\hat{\kappa}_S^{\mathcal{C}})^{-1})} \Gamma^\star\,\mathrm{id}(S) = B$$

We can then form the following diagram, which commutes by 5.3.



Consider the case where A is of type $((-)_\star)$. In this case, each of s_0 and t_0 must be of type $(\bar{\kappa})$. In this case, they are equal.

This covers every possible case, and thus ends the proof. □

We can now prove the coherence theorem for the central isomorphisms.

Theorem 5.3.9. *For each vertex $\mathcal{C}$ of $\mathfrak{G}$, for each pair of $\mathcal{C}$-shapes A and B, for each pair of central $\mathcal{C}$-isomorphisms $s, t\colon A \to B$, it follows that $s = t$.*

Proof. The central isomorphism s is of the following form, where each s_i is either a primitive central isomorphism or the inverse of a primitive central isomorphism.

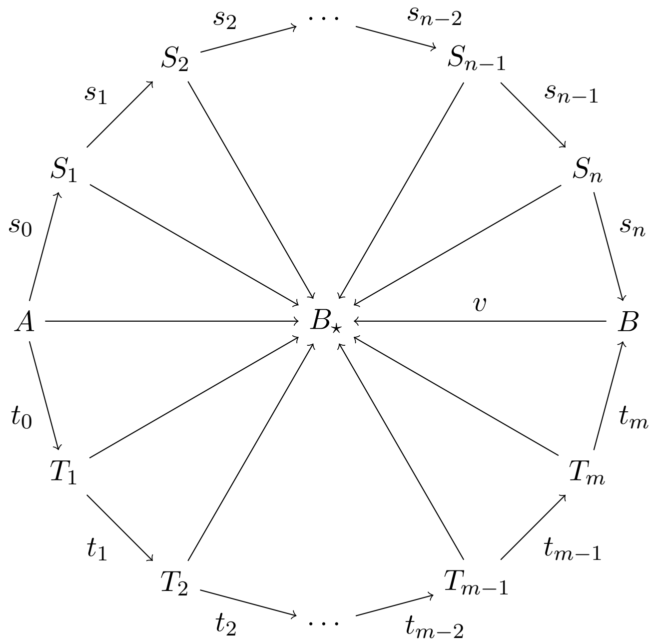
$$A = S_0 \xrightarrow{s_0} S_1 \xrightarrow{s_1} \cdots \xrightarrow{s_{n-1}} S_n \xrightarrow{s_n} S_{n+1} = B$$

The central isomorphism t is of the following form, where each t_j is either a primitive central isomorphism or the inverse of a primitive central isomorphism.

$$A = T_0 \xrightarrow{t_0} T_1 \xrightarrow{t_1} \cdots \xrightarrow{t_{m-1}} T_m \xrightarrow{T_m} T_{m+1} = B$$

By Corollary 5.3.7, there is a reduced shape $B_\star$ and a partial central isomorphism $v\colon B \to B_\star$. For each shape S_i, there is a partial central isomorphism $S_i \to C$. For each shape T_j, there is a partial central isomorphism $T_j \to C$.

Putting all of this data together, we can form the following diagram.



We claim that each triangle in this diagram commutes. To see this, consider a triangle with vertices S_i, S_{i+1} and $B_\star$. If s_i is a primitive central isomorphism, then the triangle describes two partial central isomorphisms $S_i \to B_\star$; whereas if s_i is the inverse of a primitive central isomorphism, then the triangle describes two partial central isomorphisms $S_{i+1} \to B_\star$. In either case, the triangle commutes, by Lemma 5.3.8. Therefore, each individual triangle commutes. Therefore, $v \circ s = v \circ t$. Therefore, since v is invertible, it follows that $s = t$. □

5.3.3 Prime Factorisations

Now, we will introduce the notions of prime shapes and prime factorisations of shapes, and prove some technical lemmas involving them, which we will make use of later.

A prime shape is a shape which cannot be written, up to central isomorphism, as a tensor product of other, simpler shapes, nor as a strong monoidal functor applied to a another, simpler shape The prime factorisation of a shape is then a canonical way of writing that shape, up to central isomorphism, as a tensor product of terms, each of which takes the form of a strong monoidal functor applied to a prime shape. For example, the shapes

$$\Delta^\star((A\backslash B) \otimes \Gamma^\star((\mathcal{I}\backslash C) \otimes D)) \qquad \text{and} \qquad \Delta^\star(A\backslash B) \otimes (\Gamma\Delta)^\star(C \otimes \mathrm{id}_\star(D))$$

are both centrally isomorphic to the following shape.

$$\Delta^\star(A\backslash B) \otimes (\Gamma\Delta)^\star(C) \otimes (\Gamma\Delta)^\star(D)$$

By giving each shape a canonical form like this, we can easily tell whether two shapes are centrally isomorphic.

Definition 5.3.10 (trivial shape)**.** A shape A is called a trivial shape if there is a central isomorphism of the form $A \cong \mathcal{I}$.

Definition 5.3.11 (prime shape)**.** The prime shapes are defined by the following rules.

- Let X be an element of $G(\mathcal{C})$. Then the $\mathcal{C}$-shape X of type (G) is a prime $\mathcal{C}$-shape.
- Let A be a non-trivial $\mathcal{C}$-shape. Let B be a $\mathcal{C}$-shape. Then the $\mathcal{C}$-shape $A \backslash B$ of type $(\backslash)$ is a prime $\mathcal{C}$-shape.
- Let $\Gamma \colon \mathcal{C} \to \mathcal{D}$ be a non-empty path in $\mathfrak{G}$. Let A be a $\mathcal{C}$-shape. Then the $\mathcal{D}$-shape $\Gamma_\star(A)$ of type $((-)_\star)$ is a prime $\mathcal{D}$-shape.

Explicitly, the prime factorisation of a $\mathcal{C}$-shape P is an ordered list of pairs of the form $(\Pi_i; P_i)$, such that a central isomorphism of the following form exists.

$$P \cong \Pi_0^\star(P_0) \otimes \cdots \otimes \Pi_{p-1}^\star(P_{p-1})$$

Definition 5.3.12. The prime factorisation of a $\mathcal{C}$-shape P, denoted $\mathfrak{p}(P)$, is an ordered list of pairs of the form $(\Pi_i; P_i)$, with each $\Pi_i \colon \mathcal{C} \to \mathcal{D}_i$ a path in $\mathfrak{G}$ and each P_i a prime $\mathcal{D}_i$-shape, defined as follows.

- For a shape of type $(\mathcal{I})$,
$$\mathfrak{p}(\mathcal{I}) = \{\}.$$
- For a shape of type (G),
$$\mathfrak{p}(X) = \{(\mathrm{id}; X)\}.$$
- For a shape of type $(\otimes)$,
$$\mathfrak{p}(A \otimes B) = \mathfrak{p}(A) \otimes \mathfrak{p}(B),$$
where the notation used is defined as follows.
$$\begin{aligned}&\{(\Gamma_0; A_0), \ldots, (\Gamma_{a-1}; A_{a-1})\} \otimes \{(\Delta_0; B_0), \ldots, (\Delta_{b-1}; B_{b-1})\} \\ &= \{(\Gamma_0; A_0), \ldots, (\Gamma_{a-1}; A_{a-1}), (\Delta_0; B_0), \ldots, (\Delta_{b-1}; B_{b-1})\}\end{aligned}$$
- For a shape of type $(\backslash)$,
$$\mathfrak{p}(A \backslash B) = \begin{cases} \{(\mathrm{id}; A \backslash B)\} & \text{if } A \text{ is non-trivial} \\ \mathfrak{p}(B) & \text{if } A \text{ is trivial} \end{cases}.$$

- For a shape of type $((-)^\star)$,

$$\mathfrak{p}(\Lambda^\star(A)) = \Lambda^\star(\mathfrak{p}(A)),$$

where the notation used is defined as follows.

$$\Lambda^\star\{(\Gamma_0; A_0), \ldots, (\Gamma_{a-1}; A_{a-1})\}$$
$$= \{(\Gamma_0\Lambda; A_0), \ldots, (\Gamma_{a-1}\Lambda; A_{a-1})\}$$

- For a shape of type $((-)_\star)$,

$$\mathfrak{p}(\Gamma_\star(A)) = \begin{cases} \{(\mathrm{id}; \Gamma_\star(A))\} & \text{if } \Gamma \text{ is non-empty} \\ \mathfrak{p}(A) & \text{if } \Gamma \text{ is empty} \end{cases}.$$

The main result about prime factorisations is that two shapes are centrally isomorphic if and only if they have the same prime factorisations. We will prove this result (Proposition 5.3.15) using the following two lemmas.

Lemma 5.3.13. *Let P be a shape with a prime factorisation of the following form.*

$$\mathfrak{p}(P) = \{(\Pi_0; P_0), \ldots, (\Pi_{p-1}; P_{p-1})\}$$

Then there is a central isomorphism of the following form.

$$P \cong \bigotimes_{0 \le i < p} \Pi_i^\star(P_i)$$

Proof. Consider the type of the shape P.

- Consider the case where P is of type $(\mathcal{I})$.

$$P = \mathcal{I}$$

Then P has a prime factorisation of the following form.

$$\mathfrak{p}(P) = \{\}$$

The result follows.

- Consider the case where P is of type (G).

$$P = X$$

Then P has a prime factorisation of the following form.

$$\mathfrak{p}(P) = \{(\mathrm{id}; X)\}$$

The result follows.

- Consider the case where P is of type $(\otimes)$.

  $$P = A \otimes B$$

  Let A and B have prime factorisations of the following forms.

  $$\mathfrak{p}(A) = \{(\Gamma_0; A_0), \ldots, (\Gamma_0; A_{a-1})\}$$

  $$\mathfrak{p}(B) = \{(\Delta_0; B_0), \ldots, (\Delta_0; B_{b-1})\}$$

  Then P has a prime factorisation of the following form.

  $$\mathfrak{p}(P) = \{(\Gamma_0; A_0), \ldots, (\Gamma_0; A_{a-1}), (\Delta_0; B_0), \ldots, (\Delta_0; B_{b-1})\}$$

  By induction, there are central isomorphisms of the following forms.

  $$A \cong \bigotimes_{0 \leq i < a} \Gamma_i^\star(A_i) \qquad B \cong \bigotimes_{0 \leq i < b} \Delta_i^\star(B_i)$$

  The result follows.

- Consider the case where P is of type $(\backslash)$.

  $$P = A \backslash B$$

  Let B have a prime factorisation of the following form.

  $$\mathfrak{p}(B) = \{(\Delta_0; B_0), \ldots, (\Delta_{b-1}; B_{b-1})\}$$

  Consider A.

  Consider the case where A is trivial. Then P has a prime factorisation of the following form.

  $$\mathfrak{p}(P) = \{(\Delta_0; B_0), \ldots, (\Delta_{b-1}; B_{b-1})\}$$

  By induction, there is a central isomorphism of the following form.

  $$B \cong \bigotimes_{0 \leq i < b} \Delta_i^\star(B_i)$$

  The following central isomorphism exists.

  $$A \backslash B \cong \mathcal{I} \backslash B \cong B \cong \bigotimes_{0 \leq i < b} \Delta_i^\star(B_i)$$

  Consider the case where A is non-trivial. Then P has a prime factorisation of the following form.

  $$\mathfrak{p}(P) = \{(\mathrm{id}; A \backslash B)\}$$

  The result follows.

- Consider the case where P is of type $((-)^\star)$.

$$P = \Lambda^\star(A)$$

Let A have a prime factorisation of the following form.

$$\mathfrak{p}(A) = \{(\Gamma_0; A_0), \ldots, (\Gamma_{a-1}; A_{a-1})\}$$

Then P has a prime factorisation of the following form.

$$\mathfrak{p}(P) = \{(\Gamma_0\Lambda; A_0), \ldots, (\Gamma_{a-1}\Lambda; A_{a-1})\}$$

By induction, there is a central isomorphism of the following form.

$$A \cong \bigotimes_{0 \le i < a} \Gamma_i^\star(A_i)$$

The result follows.

- Consider the case where P is of type $((-)_\star)$.

$$P = \Lambda_\star(A)$$

Let A have a prime factorisation of the following form.

$$\mathfrak{p}(A) = \{(\Gamma_0; A_0), \ldots, (\Gamma_{a-1}; A_{a-1})\}$$

Consider Λ.

Consider the case where Λ is empty. Then P has a prime factorisation of the following form.

$$\mathfrak{p}(P) = \{(\Gamma_0; A_0), \ldots, (\Gamma_{a-1}; A_{a-1})\}$$

By induction, there is a central isomorphism of the following form.

$$A \cong \bigotimes_{0 \le i < a} \Gamma_i^\star(A_i)$$

The following central isomorphism exists.

$$\Lambda_\star(A) \cong \mathrm{id}_\star(A) \cong A \cong \bigotimes_{0 \le i < a} \Gamma_i^\star(A_i)$$

Consider the case where Λ is non-empty. Then P has a prime factorisation of the following form.

$$\mathfrak{p}(P) = \{(\mathrm{id}; \Lambda_\star(A))\}$$

The result follows.

□

Lemma 5.3.14. *Let P and Q be shapes with prime factorisations of the following forms.*

$$\mathfrak{p}(P) = \{(\Pi_0; P_0), \dots, (\Pi_{p-1}; P_{p-1})\}$$

$$\mathfrak{p}(Q) = \{(\Sigma_0; Q_0), \dots, (\Sigma_{q-1}; Q_{q-1})\}$$

Let there be a primitive central isomorphism $s\colon P \to Q$. Then $p = q$ and, for each i, $\Pi_i = \Sigma_i$ and $P_i \cong Q_i$.

Proof. Consider the type of the primitive central isomorphism s.

- Consider the case where s is of type (α).

  $$\alpha_{A,B,C}\colon P = A \otimes (B \otimes C) \to (A \otimes B) \otimes C = Q$$

  Then P and Q have the same prime factors.

  $$\mathfrak{p}(P) = \mathfrak{p}(A) \otimes (\mathfrak{p}(B) \otimes \mathfrak{p}(C)) = (\mathfrak{p}(A) \otimes \mathfrak{p}(B)) \otimes \mathfrak{p}(C) = \mathfrak{p}(Q)$$

  The result follows.

- Consider the case where s is of type (λ).

  $$\lambda_A^{-1}\colon P = \mathcal{I} \otimes A \to A = Q$$

  Then P and Q have the same prime factors.

  $$\mathfrak{p}(P) = \mathfrak{p}(\mathcal{I}) \otimes \mathfrak{p}(A) = \mathfrak{p}(A) = \mathfrak{p}(Q)$$

  The result follows.

- Consider the case where s is of type (ρ).

  $$\rho_A\colon P = A \otimes \mathcal{I} \to A = Q$$

  Then P and Q have the same prime factors.

  $$\mathfrak{p}(P) = \mathfrak{p}(A) \otimes \mathfrak{p}(\mathcal{I}) = \mathfrak{p}(A) = \mathfrak{p}(Q)$$

  The result follows.

- Consider the case where s is of type $(\bar{\lambda})$.

  $$\bar{\lambda}_A\colon P = \mathcal{I}\backslash A \to A = Q$$

  Then P and Q have the same prime factors.

  $$\mathfrak{p}(P) = \mathfrak{p}(A) = \mathfrak{p}(Q)$$

  The result follows.

- Consider the case where s is of type $(\otimes_L)$.

$$s' \otimes C \colon P = A \otimes C \to B \otimes C = Q$$

Then P and Q have prime factorisations of the following forms.

$$\mathfrak{p}(P) = \mathfrak{p}(A) \otimes \mathfrak{p}(C)$$

$$\mathfrak{p}(Q) = \mathfrak{p}(B) \otimes \mathfrak{p}(C)$$

By induction, the result follows.

- Consider the case where s is of type $(\otimes_R)$.

$$C \otimes s' \colon P = C \otimes A \to C \otimes B = Q$$

Then P and Q have prime factorisations of the following forms.

$$\mathfrak{p}(P) = \mathfrak{p}(C) \otimes \mathfrak{p}(A)$$

$$\mathfrak{p}(Q) = \mathfrak{p}(C) \otimes \mathfrak{p}(B)$$

By induction, the result follows.

- Consider the case where s is of type $(\backslash_R)$.

$$C \backslash s' \colon P = C \backslash A \to C \backslash B = Q$$

Consider C.

Consider the case where C is trivial. Then P and Q have prime factorisations of the following forms.

$$\mathfrak{p}(P) = \mathfrak{p}(A)$$

$$\mathfrak{p}(Q) = \mathfrak{p}(B)$$

By induction, the result follows.

Consider the case where C is non-trivial. Then P and Q have prime factorisations of the following forms.

$$\mathfrak{p}(P) = \{(\mathrm{id}; C \backslash A)\}$$

$$\mathfrak{p}(Q) = \{(\mathrm{id}; C \backslash B)\}$$

The result follows.

- Consider the case where s is of type $(\backslash_L)$.

$$s'^{-1}\backslash C\colon P = A\backslash C \to B\backslash C = Q$$

By assumption, A is trivial if and only if B is trivial. Consider A and B.

Consider the case where A and B are trivial. Then P and Q have the same prime factors.

$$\mathfrak{p}(P) = \mathfrak{p}(C) = \mathfrak{p}(Q)$$

The result follows.

Consider the case where A and B are non-trivial. Then P and Q have prime factorisations of the following forms.

$$\mathfrak{p}(P) = \{(\mathrm{id}; A\backslash C)\}$$

$$\mathfrak{p}(Q) = \{(\mathrm{id}; B\backslash C)\}$$

The result follows.

- Consider the case where s is of type (φ).

$$\varphi^{\Gamma}_{A,B}\colon P = \Gamma^{\star}(A \otimes B) \to \Gamma^{\star}(A) \otimes \Gamma^{\star}(B) = Q$$

Then P and Q have the same prime factors.

$$\mathfrak{p}(P) = \Gamma^{\star}(\mathfrak{p}(A) \otimes \mathfrak{p}(B)) = \Gamma^{\star}(\mathfrak{p}(A)) \otimes \Gamma^{\star}(\mathfrak{p}(B)) = \mathfrak{p}(Q)$$

The result follows.

- Consider the case where s is of type $(\hat{\varphi})$.

$$\hat{\varphi}^{\Gamma}\colon P = \Gamma^{\star}(\mathcal{I}) \to \mathcal{I} = Q$$

Then P and Q have the same prime factors.

$$\mathfrak{p}(P) = \Gamma^{\star}(\mathfrak{p}(\mathcal{I})) = \mathfrak{p}(\mathcal{I}) = \mathfrak{p}(Q)$$

The result follows.

- Consider the case where s is of type $((-)^{\star})$.

$$\Gamma^{\star}(s')\colon P = \Gamma^{\star}(A) \to \Gamma^{\star}(B) = Q$$

Then P and Q have prime factorisations of the following forms.

$$\mathfrak{p}(P) = \Gamma^{\star}(\mathfrak{p}(A))$$

$$\mathfrak{p}(Q) = \Gamma^{\star}(\mathfrak{p}(B))$$

By induction, the result follows.

- Consider the case where s is of type (κ).

$$\kappa_A^{\Gamma,\Delta} \colon P = \Gamma^\star \Delta^\star(A) \to (\Delta\Gamma)^\star(A) = Q$$

Then P and Q have the same prime factors.

$$\mathfrak{p}(P) = \Gamma^\star \Delta^\star(\mathfrak{p}(A)) = (\Delta\Gamma)^\star(\mathfrak{p}(A)) = \mathfrak{p}(Q)$$

The result follows.

- Consider the case where s is of type $(\hat{\kappa})$.

$$(\hat{\kappa}_A^{\mathcal{C}})^{-1} \colon P = \mathrm{id}^\star(A) \to A = Q$$

Then P and Q have the same prime factors.

$$\mathfrak{p}(P) = \mathrm{id}^\star(\mathfrak{p}(A)) = \mathfrak{p}(A) = \mathfrak{p}(Q)$$

The result follows.

- Consider the case where s is of type $(\bar{\kappa})$.

$$\bar{\kappa}_A \colon P = \mathrm{id}_\star(A) \to A = Q$$

Then P and Q have the same prime factors.

$$\mathfrak{p}(P) = \mathfrak{p}(A) = \mathfrak{p}(Q)$$

The result follows.

□

Proposition 5.3.15. *Let P and Q be shapes with prime factorisations of the following forms.*

$$\mathfrak{p}(P) = \{(\Pi_0; P_0), \ldots, (\Pi_{p-1}; P_{p-1})\}$$

$$\mathfrak{p}(Q) = \{(\Sigma_0; Q_0), \ldots, (\Sigma_{q-1}; Q_{q-1})\}$$

The following are equivalent.

- *There is a central isomorphism $P \cong Q$.*
- *$p = q$ and, for each i, $\Pi_i = \Sigma_i$ and $P_i \cong Q_i$.*

Proof. This follows from Lemma 5.3.13 and Lemma 5.3.14. □

We end this section with some technical lemmas about central isomorphisms between shapes of certain forms. Each of these lemmas follows from Proposition 5.3.15.

Lemma 5.3.16. *Let $\Lambda\colon \mathcal{C} \to \mathcal{D}$ be a path in $\mathfrak{G}$. Let A and B be $\mathcal{D}$-shapes. The following are equivalent.*

- *There is a central $\mathcal{D}$-isomorphism of the following form.*

$$A \cong B$$

- *There is a central $\mathcal{C}$-isomorphism of the following form.*

$$\Lambda^\star(A) \cong \Lambda^\star(B)$$

Proof. Let A and B have prime factorisations of the following forms,

$$\mathfrak{p}(A) = \{(\Gamma_0; A_0), \ldots, (\Gamma_{a-1}; A_{a-1})\}$$

$$\mathfrak{p}(B) = \{(\Delta_0; B_0), \ldots, (\Delta_{b-1}; B_{b-1})\}$$

By Proposition 5.3.15, there is a central $\mathcal{D}$-isomorphism $A \cong B$ if and only if $a = b$ and, for each i, $\Gamma_i = \Delta_i$ and $A_i \cong B_i$. By Proposition 5.3.15, there is a central $\mathcal{C}$-isomorphism $\Lambda^\star(A) \cong \Lambda^\star(B)$ if and only if $a = b$ and, for each i, $\Gamma_i\Lambda = \Delta_i\Lambda$ and $A_i \cong B_i$. Since Γ_i, Δ_i and Λ are paths in a graph, these conditions are equivalent. □

Lemma 5.3.17. *Let $\Gamma\colon \mathcal{X} \to \mathcal{C}$ and $\Delta\colon \mathcal{X} \to \mathcal{D}$ be paths in $\mathfrak{G}$. Let A be a non-trivial $\mathcal{C}$-shape. Let B be a non-trivial $\mathcal{D}$-shape. Let there be a central $\mathcal{X}$-isomorphism of the following form.*

$$\Gamma^\star(A) \cong \Delta^\star(B)$$

Then, either there is a path $\Lambda\colon \mathcal{C} \to \mathcal{D}$ of $\mathfrak{G}$ such that $\Delta = \Lambda\Gamma$, or there is a path $\Lambda\colon \mathcal{D} \to \mathcal{C}$ of $\mathfrak{G}$ such that $\Gamma = \Lambda\Delta$.

Proof. Let A and B have prime factorisations of the following forms.

$$\mathfrak{p}(A) = \{(\Gamma_0; A_0), \ldots, (\Gamma_{a-1}; A_{a-1})\}$$

$$\mathfrak{p}(B) = \{(\Delta_0; B_0), \ldots, (\Delta_{b-1}; B_{b-1})\}$$

By Proposition 5.3.15, for each i, $\Gamma_i\Gamma = \Delta_i\Delta$. Since Γ_i, Γ, Δ_i and Δ are paths in a graph, it must be the case that either there is a Λ such that $\Delta = \Lambda\Gamma$ and $\Gamma_i = \Delta_i\Lambda$, or there is a Λ such that $\Gamma = \Lambda\Delta$ and $\Delta_i = \Gamma_i\Lambda$. □

Lemma 5.3.18. *Let $\Gamma\colon \mathcal{X} \to \mathcal{C}$ and $\Delta\colon \mathcal{X} \to \mathcal{D}$ be paths in $\mathfrak{G}$. Let A be a $\mathcal{C}$-shape. Let B be a $\mathcal{D}$-shape with at least one prime factor of the form $(\mathrm{id}; B_i)$. Let there be a central $\mathcal{X}$-isomorphism of the following form.*

$$\Gamma^\star(A) \cong \Delta^\star(B)$$

Then there is a path $\Lambda\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$ such that $\Delta = \Lambda\Gamma$.

Proof. Let A and B have prime factorisations of the following forms.

$$\mathfrak{p}(A) = \{(\Gamma_0; A_0), \ldots, (\Gamma_{a-1}; A_{a-1})\}$$

$$\mathfrak{p}(B) = \{(\Delta_0; B_0), \ldots, (\Delta_{b-1}; B_{b-1})\}$$

By Proposition 5.3.15, for each i, $\Gamma_i\Gamma = \Delta_i\Delta$. By assumption, there is some i for which Δ_i is empty. Choose Λ to be Γ_i. □

Lemma 5.3.19. *Let $\Gamma\colon \mathcal{X} \to \mathcal{C}$ and $\Delta\colon \mathcal{X} \to \mathcal{D}$ be paths in $\mathfrak{G}$. Let $\Phi\colon \mathcal{D} \to \mathcal{E}$ be a non-empty path in $\mathfrak{G}$. Let A be a $\mathcal{C}$-shape. Let B be a $\mathcal{D}$-shape. Let there be a central $\mathcal{X}$-isomorphism of the following form.*

$$\Gamma^\star(A) \cong \Delta^\star\Phi^\star\Phi_\star(B)$$

Then, either there is a path $\Lambda\colon \mathcal{C} \to \mathcal{D}$ in $\mathfrak{G}$ such that $\Delta = \Lambda\Gamma$ and $A \cong \Lambda^\star\Phi^\star\Phi_\star(B)$ or there is a non-empty path $\Lambda\colon \mathcal{D} \to \mathcal{C}$ and a path $\Psi\colon \mathcal{C} \to \mathcal{E}$ in $\mathfrak{G}$ such that $\Gamma = \Lambda\Delta$, $\Phi = \Psi\Lambda$ and $A \cong \Psi^\star\Phi_\star(B)$.

Proof. Let A have a prime factorisation of the following form.

$$\mathfrak{p}(A) = \{(\Gamma_0; A_0), \ldots, (\Gamma_{a-1}; A_{a-1})\}$$

By Proposition 5.3.15, $a = 1$, $\Gamma_0\Gamma = \Phi\Delta$ and $A_0 \cong \Phi_\star(B)$. Since Γ_0, Γ, Φ and Δ are paths in a graph, it must be the case that either there is a Λ such that $\Delta = \Lambda\Gamma$ and $\Gamma_0 = \Phi\Lambda$, or there is a Λ such that $\Gamma = \Lambda\Delta$ and $\Phi = \Gamma_0\Lambda$. If Λ is empty, then both cases are equivalent, so, without loss of generality, we may assume that Λ is non-empty in the second case.

If there is a Λ such that $\Delta = \Lambda\Gamma$ and $\Gamma_0 = \Phi\Lambda$, then $A \cong \Gamma_0^\star(A_0) \cong \Lambda^\star\Phi^\star\Phi_\star(B)$

If there is a non-empty Λ such that $\Gamma = \Lambda\Delta$ and a $\Phi = \Gamma_0\Lambda$, then, taking Ψ to be Γ_0, $A \cong \Gamma_0^\star(A_0) \cong \Psi^\star\Phi_\star(B)$. □

5.4 Constructible Morphisms

In this section, we will define a class of allowable morphisms called the constructible morphisms. These will be defined in a way which makes them more convenient to work with than the allowable morphisms. In particular, their definition will not explicitly mention composition. This will allow us to prove results about the constructible morphisms without mentioning composition.

In fact, we will show that the class of constructible morphisms consists of all of the allowable morphisms. This will allow us to prove results about the allowble morphisms without mentioning composition.

5.4.1 Definitions

First, we will introduce some concise notation for morphisms of certain forms which will appear frequently throughout this section and the rest of this chapter. Choose an object X of $\textsc{Gr}_{\mathfrak{G}}$.

Definition 5.4.1. Let $\mathcal{C}$ be a vertex of $\mathfrak{G}$. Given a morphism

$$f\colon I \otimes A \to B$$

in $X(\mathcal{C})$, we will use the notation $\langle I, f\rangle_\eta$ to denote the following morphism in $X(\mathcal{C})$.

$$A \xrightarrow{\eta^I_A} I\backslash(I \otimes A) \xrightarrow{I\backslash f} I\backslash B$$

Definition 5.4.2. Let $\Gamma\colon \mathcal{C} \to \mathcal{D}$ be a path in $\mathfrak{G}$. Given morphisms

$$f\colon B \to C$$

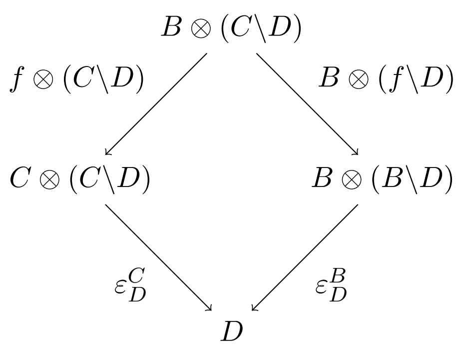
in $X(\mathcal{D})$ and

$$g\colon A \otimes \Gamma^\star(D) \otimes E \to F$$

in $X(\mathcal{C})$, we will use the notation $\langle \Gamma, f, g\rangle_\varepsilon$ to denote the following morphism in $X(\mathcal{C})$.

$$A \otimes \Gamma^\star(B \otimes (C\backslash D)) \otimes E \xrightarrow{A\otimes\Gamma^\star(\varepsilon^f_D)\otimes E} A \otimes \Gamma^\star(D) \otimes E \xrightarrow{g} F$$

Here, we are using the notation ε^f_D to refer to the diagonal of the following square, which commutes by extranaturality of ε.



Definition 5.4.3. Let $\Phi\colon \mathcal{C} \to \mathcal{D}$ be a path in $\mathfrak{G}$. Given a morphism

$$f\colon \Phi^\star(A) \to B$$

in $X(\mathcal{C})$, we will use the notation $\langle \Phi, f\rangle_\theta$ to denote the following morphism in $X(\mathcal{D})$.

$$A \xrightarrow{\theta^\Phi_A} \Phi_\star\Phi^\star(A) \xrightarrow{\Phi_\star(f)} \Phi_\star(B)$$

Definition 5.4.4. Let $\Gamma\colon \mathcal{B} \to \mathcal{C}$ be a path in $\mathfrak{G}$. Let $\Phi\colon \mathcal{C} \to \mathcal{D}$ be a path in $\mathfrak{G}$. Given a morphism

$$f\colon A \otimes \Gamma^\star(B) \otimes C \to D,$$

in $X(\mathcal{B})$, we will use the notation $\langle \Gamma, \Phi, f\rangle_\zeta$ to denote the following morphism in $X(\mathcal{B})$.

$$A \otimes \Gamma^\star\Phi^\star\Phi_\star(B) \otimes C \xrightarrow{A\otimes\Gamma^\star(\zeta^\Phi_B)\otimes C} A \otimes \Gamma^\star(B) \otimes C \xrightarrow{f} D$$

Definition 5.4.5. Let $\Gamma\colon \mathcal{C} \to \mathcal{D}$ be a path in $\mathfrak{G}$. Given morphisms

$$f\colon B \to C$$

in $X(\mathcal{D})$ and

$$g\colon A \otimes \Gamma^\star(C) \otimes D \to E$$

in $X(\mathcal{C})$, we will use the notation $\langle \Gamma, f, g\rangle_\circ$ to denote the following morphism in $X(\mathcal{C})$.

$$A \otimes \Gamma^\star(B) \otimes D \xrightarrow{A\otimes\Gamma^\star(f)\otimes D} A \otimes \Gamma^\star(C) \otimes D \xrightarrow{g} E$$

With these definitions, we are able to define the constructible morphisms.

Definition 5.4.6 (constructible morphisms)**.** The constructible morphisms are defined by the following rules.

type ($\cong$) Let $\mathcal{C}$ be a vertex of $\mathfrak{G}$. Every central $\mathcal{C}$-isomorphism is a constructible $\mathcal{C}$-morphism.

type ($\otimes$) Let $\mathcal{C}$ be a vertex of $\mathfrak{G}$. Given constructible $\mathcal{C}$-morphisms

$$f\colon A \to B \qquad \text{and} \qquad g\colon C \to D$$

such that B and D are non-trivial $\mathcal{C}$-shapes, and central $\mathcal{C}$-isomorphisms

$$P \cong A \otimes C \qquad \text{and} \qquad B \otimes D \cong Q,$$

the following is a constructible $\mathcal{C}$-morphism.

$$P \cong A \otimes C \xrightarrow{f\otimes g} B \otimes D \cong Q$$

type (η) Let $\mathcal{C}$ be a vertex of $\mathfrak{G}$. Given a constructible $\mathcal{C}$-morphism

$$f\colon I \otimes P \to A$$

such that I is a non-trivial $\mathcal{C}$-shape, and a central $\mathcal{C}$-isomorphism

$$I \backslash A \cong Q,$$

the following is a constructible $\mathcal{C}$-morphism.

$$P \xrightarrow{\langle I,f\rangle_\eta} I\backslash A \cong Q$$

type (ε) Let $\Gamma\colon \mathcal{C} \to \mathcal{D}$ be a path in $\mathfrak{G}$. Given a constructible $\mathcal{D}$-morphism

$$f\colon B \to C$$

such that C is a non-trivial $\mathcal{D}$-shape, a constructible $\mathcal{C}$-morphism

$$g\colon A \otimes \Gamma^\star(D) \otimes E \to Q$$

and a central $\mathcal{C}$-isomorphism

$$P \cong A \otimes \Gamma^\star(B \otimes (C\backslash D)) \otimes E,$$

the following is a constructible $\mathcal{C}$-morphism.

$$P \cong A \otimes \Gamma^\star(B \otimes (C\backslash D)) \otimes E \xrightarrow{\langle \Gamma,f,g\rangle_\varepsilon} Q$$

type ($(-)^\star$) Let $\Gamma\colon \mathcal{C} \to \mathcal{D}$ be a non-empty path in $\mathfrak{G}$. Given a constructible $\mathcal{D}$-morphism

$$f\colon A \to B$$

such that B is a non-trivial $\mathcal{D}$-shape, and central $\mathcal{C}$-isomorphisms

$$P \cong \Gamma^\star(A) \qquad \text{and} \qquad \Gamma^\star(B) \cong Q,$$

the following is a constructible $\mathcal{C}$-morphism.

$$P \cong \Gamma^\star(A) \xrightarrow{\Gamma^\star(f)} \Gamma^\star(B) \cong Q$$

type (θ) Let $\Phi\colon \mathcal{C} \to \mathcal{D}$ be a non-empty path in $\mathfrak{G}$. Given a constructible $\mathcal{C}$-morphism

$$f\colon \Phi^\star(P) \to A$$

and a central $\mathcal{D}$-isomorphism

$$\Phi_\star(A) \cong Q,$$

the following is a constructible $\mathcal{D}$-morphism.

$$P \xrightarrow{\langle \Phi,f\rangle_\theta} \Phi_\star(A) \cong Q$$

type (ζ) Let $\Gamma\colon \mathcal{B} \to \mathcal{C}$ be a path in $\mathfrak{G}$. Let $\Phi\colon \mathcal{C} \to \mathcal{D}$ be a non-empty path in $\mathfrak{G}$. Given a constructible $\mathcal{B}$-morphism

$$f\colon A \otimes \Gamma^\star(B) \otimes C \to Q$$

and a central $\mathcal{B}$-isomorphism

$$P \cong A \otimes \Gamma^\star\Phi^\star\Phi_\star(B) \otimes C,$$

the following is a constructible $\mathcal{B}$-morphism.

$$P \cong A \otimes \Gamma^\star\Phi^\star\Phi_\star(B) \otimes C \xrightarrow{\langle\Gamma,\Phi,f\rangle_\zeta} Q$$

5.4.2 Allowable Morphisms

In this section, we will show that the class of constructible morphisms contains all of the allowable morphisms.

Lemma 5.4.7. *Let $\mathcal{C}$ be a vertex of $\mathfrak{G}$. Given a constructible $\mathcal{C}$-morphism*

$$s\colon P \to Q$$

and central $\mathcal{C}$-isomorphisms

$$P' \cong P \qquad \text{and} \qquad Q \cong Q',$$

the following is a constructible $\mathcal{C}$-morphism.

$$P' \cong P \xrightarrow{s} Q \cong Q'$$

Proof. We will prove this by induction on the type of s.

- Consider the case where s is of type ($\cong$). In this case, the result is clear.

- Consider the case where s is of type ($\otimes$).

  $$s\colon P \cong A \otimes C \xrightarrow{f\otimes g} B \otimes D \cong Q$$

  The desired composite is the following constructible morphism.

  $$s'\colon P' \cong A \otimes C \xrightarrow{f\otimes g} B \otimes D \cong Q'$$

- Consider the case where s is of type (η).

$$s\colon P \xrightarrow{\langle I,f\rangle_\eta} I\backslash A \cong Q$$

Consider the following allowable morphism.

$$f'\colon I \otimes P' \cong I \otimes P \xrightarrow{f} A$$

By induction, f' is a constructible morphism. The desired composite is the following constructible morphism.

$$P' \xrightarrow{\langle A,f'\rangle_\eta} A\backslash C \cong Q'$$

- Consider the case where s is of type (ε).

$$s\colon P \cong A \otimes \Gamma^\star(B \otimes (C\backslash D)) \otimes E \xrightarrow{\langle \Gamma,f,g\rangle_\varepsilon} Q$$

Consider the following allowable morphism.

$$g'\colon A \otimes \Gamma^\star(D) \otimes E \xrightarrow{g} Q \cong Q'$$

By induction, g' is a constructible morphism. The desired composite is the following constructible morphism.

$$P' \cong A \otimes \Gamma^\star(B \otimes (C\backslash D)) \otimes E \xrightarrow{\langle \Gamma,f,g'\rangle_\varepsilon} Q'$$

- Consider the case where s is of type $((-)^\star)$.

$$s\colon P \cong \Gamma^\star(A) \xrightarrow{\Gamma^\star(f)} B^\star \cong Q$$

The desired composite is the following constructible morphism.

$$P' \cong \Gamma^\star(A) \xrightarrow{\Gamma^\star(f)} \Gamma^\star(B) \cong Q'$$

- Consider the case where s is of type (θ).

$$P \xrightarrow{\langle \Phi,f\rangle_\theta} \Phi_\star(A) \cong Q$$

Consider the following allowable morphism.

$$f'\colon \Phi^\star(P') \cong \Phi^\star(P) \xrightarrow{f} A$$

By induction, f' is a constructible morphism. The desired composite is the following constructible morphism.

$$P' \xrightarrow{\langle \Phi,f'\rangle_\theta} \Phi_\star(A) \cong Q'$$

- Consider the case where s is of type (ζ).

$$P \cong A \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes C \xrightarrow{\langle \Gamma, \Phi, f \rangle_\zeta} Q$$

  Consider the following allowable morphism.

$$f' \colon A \otimes \Gamma^\star(B) \otimes C \xrightarrow{f} Q \cong Q'$$

  By induction, f' is a constructible morphism. The desired composite is the following constructible morphism.

$$P' \cong A \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes C \xrightarrow{\langle \Gamma, \Phi, f' \rangle_\zeta} Q'$$

□

Lemma 5.4.8. *Let $\mathcal{C}$ be a vertex of $\mathfrak{G}$. Given constructible $\mathcal{C}$-morphisms*

$$f \colon A \to B \qquad \textit{and} \qquad g \colon C \to D,$$

the following is a constructible $\mathcal{C}$-morphism.

$$A \otimes C \xrightarrow{f \otimes g} B \otimes D$$

Proof. If B and D are non-trivial shapes, then the result follows. Assume that B is a trivial shape; the case where D is a trivial shape is similar. Consider the type of the constructible morphism f.

- Consider the case where f is of type $(\cong)$.

$$f \colon A \cong B$$

  Then $f \otimes g$ is the following constructible morphism.

$$A \otimes C \cong C \xrightarrow{g} D \cong B \otimes D$$

- Consider the case where f is of type $(\otimes)$.

$$f \colon A \cong A_1 \otimes A_2 \xrightarrow{f_1 \otimes f_2} B_1 \otimes B_2 \cong B$$

  Then neither B_1 nor B_2 is a trivial shape, which means that B cannot be a trivial shape. This contradicts our assumption.

- Consider the case where f is of type (η).

$$f \colon A \xrightarrow{\langle I, f' \rangle_\eta} I \backslash B' \cong B$$

  Then I is not a trivial shape, which means that B cannot be a trivial shape. This contradicts our assumption.

- Consider the case where f is of type (ε).

$$f\colon A \cong A_1 \otimes \Gamma^\star(A_2 \otimes (A_3 \backslash A_4)) \otimes A_5 \xrightarrow{\langle \Gamma, f_1, f_2 \rangle_\varepsilon} B$$

By induction,

$$A_1 \otimes \Gamma^\star(A_4) \otimes A_5 \otimes C \xrightarrow{f_2 \otimes g} B \otimes D$$

is a constructible morphism. Then $f \otimes g$ is the following constructible morphism of type (ε).

$$A \otimes C \cong A_1 \otimes \Gamma^\star(A_2 \otimes (A_3 \backslash A_4)) \otimes A_5 \otimes C \xrightarrow{\langle \Gamma, f_1, f_2 \otimes g \rangle_\varepsilon} B \otimes D$$

- Consider the case where f is of type $((-)^\star)$.

$$A \cong \Phi^\star(A') \xrightarrow{\Phi^\star(f')} \Phi^\star(B') \cong B$$

Then B' is not a trivial shape, which means that B cannot be a trivial shape. This contradicts our assumption.

- Consider the case where f is of type (θ).

$$A \xrightarrow{\langle \Phi, f' \rangle_\theta} \Phi_\star(B') \cong B$$

Then Φ is not an empty path, which means that B cannot be a trivial shape. This contradicts our assumption.

- Consider the case where f is of type (ζ).

$$A \cong A_1 \otimes \Gamma^\star \Phi^\star \Phi_\star(A_2) \otimes A_3 \xrightarrow{\langle \Gamma, \Phi, f' \rangle_\zeta} B$$

By induction,

$$A_1 \otimes \Gamma^\star(A_2) \otimes A_3 \otimes C \xrightarrow{f' \otimes g} B \otimes D$$

is a constructible morphism. Then $f \otimes g$ is the following constructible morphism of type (ζ).

$$A \otimes C \cong A_1 \otimes \Gamma^\star \Phi^\star \Phi_\star(A_2) \otimes A_3 \otimes C \xrightarrow{\langle \gamma, \Phi, f' \otimes g \rangle_\zeta} B \otimes D$$

□

Lemma 5.4.9. *Let $\mathcal{C}$ be a vertex of $\mathfrak{G}$. Given a constructible $\mathcal{C}$-morphism*

$$f\colon I \otimes A \to B,$$

the following is a constructible $\mathcal{C}$-morphism.

$$A \xrightarrow{\langle I, f \rangle_\eta} I \backslash B$$

Proof. If I is a non-trivial shape, then the result follows. Assume that I is a trivial shape. In this case, $\langle I, f\rangle_\eta$ is the following constructible morphism.

$$A \cong I \otimes A \xrightarrow{f} B \cong I\backslash B$$

□

Lemma 5.4.10. *Let $\Delta\colon \mathcal{C} \to \mathcal{D}$ be a path in $\mathfrak{G}$. Given a constructible $\mathcal{D}$-morphism*

$$f\colon B \to C$$

and a constructible $\mathcal{C}$-morphism

$$g\colon A \otimes \Delta^\star(D) \otimes E \to F,$$

the following is a constructible $\mathcal{C}$-morphism.

$$A \otimes \Delta^\star(B \otimes (C\backslash D)) \otimes E \xrightarrow{\langle \Delta, f, g\rangle_\varepsilon} F$$

Proof. If C is a non-trivial shape, then the result follows. Assume that C is a trivial shape. Consider the type of the constructible morphism f.

- Consider the case where f is of type ($\cong$).

  $$B \cong C$$

  Then $\langle \Delta, f, g\rangle_\varepsilon$ is the following constructible morphism.

  $$A \otimes \Delta^\star(B \otimes (C\backslash D)) \otimes E \cong A \otimes \Delta^\star(D) \otimes E \xrightarrow{g} F$$

- Consider the case where f is of type ($\otimes$).

  $$B \cong B_1 \otimes B_2 \xrightarrow{f_1 \otimes f_2} C_1 \otimes C_2 \cong C$$

  Then neither C_1 nor C_2 is a trivial shape, which means that C cannot be a trivial shape. This contradicts our assumption.

- Consider the case where f is of type (η).

  $$f\colon B \xrightarrow{\langle I, f'\rangle_\eta} I\backslash C' \cong C$$

  Then I is not a trivial shape, which means that C cannot be a trivial shape. This contradicts our assumption.

- Consider the case where f is of type (ε).

$$f\colon B \cong B_1 \otimes \Gamma^\star(B_2 \otimes (B_3\backslash B_4)) \otimes B_5 \xrightarrow{\langle \Gamma, f_1, f_2\rangle_\varepsilon} C$$

By induction,

$$A \otimes \Delta^\star(B_1 \otimes \Gamma^\star(B_4) \otimes B_5 \otimes (C\backslash D)) \otimes E \xrightarrow{\langle \Delta, f_2, g\rangle_\varepsilon} F$$

is a constructible morphism. Then $\langle \Delta, f, g\rangle_\varepsilon$ is the following constructible morphism of type (ε).

$$\begin{aligned}&A \otimes \Delta^\star(B \otimes (C\backslash D)) \otimes E\\ &\cong A \otimes \Delta^\star(B_1 \otimes \Gamma^\star(B_2 \otimes (B_3\backslash B_4)) \otimes B_5 \otimes (C\backslash D)) \otimes E \xrightarrow{\langle \Gamma\Delta, f_1, \langle \Gamma, f_2, g\rangle_\varepsilon\rangle_\varepsilon} F\end{aligned}$$

- Consider the case where f is of type $((-)^\star)$.

$$B \cong \Phi^\star(B') \xrightarrow{\Phi^\star(f')} \Phi^\star(C') \cong C$$

Then C' is not a trivial shape, which means that C cannot be a trivial shape. This contradicts our assumption.

- Consider the case where f is of type (θ).

$$B \xrightarrow{\langle \Phi, f'\rangle_\theta} \Phi_\star(C') \cong C$$

Then Φ is not an empty path, which means that C cannot be a trivial shape. This contradicts our assumption.

- Consider the case where f is of type (ζ).

$$B \cong B_1 \otimes \Gamma^\star\Phi^\star\Phi_\star(B_2) \otimes B_3 \xrightarrow{\langle \Gamma, \Phi, f'\rangle_\zeta} C$$

By induction,

$$A \otimes \Delta^\star(B_1 \otimes \Gamma^\star(B_2) \otimes B_3 \otimes (C\backslash D)) \otimes E \xrightarrow{\langle \Delta, f', g\rangle_\varepsilon} F$$

is a constructible morphism. Then $\langle \Delta, f, g\rangle_\varepsilon$ is the following constructible morphism of type (ζ).

$$\begin{aligned}&A \otimes \Delta^\star(B \otimes (C\backslash D)) \otimes E\\ &\cong A \otimes \Delta^\star(B_1 \otimes \Gamma^\star\Phi^\star\Phi_\star(B_2) \otimes B_3 \otimes (C\backslash D)) \otimes E \xrightarrow{\langle \Gamma\Delta, \Phi, \langle \Delta, f', g\rangle_\varepsilon\rangle_\zeta} F\end{aligned}$$

□

Lemma 5.4.11. *Let $\Delta\colon \mathcal{C} \to \mathcal{D}$ be a path in $\mathfrak{G}$. Given a constructible $\mathcal{D}$-morphism*

$$f\colon A \to B,$$

the following is a constructible $\mathcal{C}$-morphism.

$$\Delta^\star(A) \xrightarrow{\Delta^\star(f)} \Delta^\star(B)$$

Proof. If B is a non-trivial shape and Δ is a non-empty path, then the result follows. Assume that Δ is an empty path. In this case, $\Delta^\star(f)$ is the following constructible morphism.

$$\Delta^\star(A) \cong A \xrightarrow{f} B \cong \Delta^\star(B)$$

Assume that B is a trivial shape. Consider the type of the constructible morphism f.

- Consider the case where f is of type $(\cong)$.

  $$f\colon A \cong B$$

  Then $\Delta^\star(f)$ is the following constructible morphism of type $(\cong)$.

  $$\Delta^\star(A) \cong \Delta^\star(B)$$

- Consider the case where f is of type $(\otimes)$.

  $$f\colon A \cong A_1 \otimes A_2 \xrightarrow{f_1 \otimes f_2} B_1 \otimes B_2 \cong B$$

  Then neither B_1 nor B_2 is a trivial shape, which means that B cannot be a trivial shape. This contradicts our assumption.

- Consider the case where f is of type (η).

  $$f\colon A \xrightarrow{\langle I, f' \rangle_\eta} I \backslash B' \cong B$$

  Then I is not a trivial shape, which means that B cannot be a trivial shape. This contradicts our assumption.

- Consider the case where f is of type (ε).

  $$f\colon A \cong A_1 \otimes \Gamma^\star(A_2 \otimes (A_3 \backslash A_4)) \otimes A_5 \xrightarrow{\langle \Gamma, f_1, f_2 \rangle_\varepsilon} B$$

  By induction,

  $$\Delta^\star(A_1 \otimes \Gamma^\star(A_4) \otimes A_5) \xrightarrow{\Delta^\star(f_2)} \Delta^\star(B)$$

  is a constructible morphism. Then $\Delta^\star(f)$ is the following constructible morphism of type (ε).

  $$\Delta^\star(A) \cong \Delta^\star(A_1 \otimes \Gamma^\star(A_2 \otimes (A_3 \backslash A_4)) \otimes A_5) \xrightarrow{\langle \Gamma\Delta, f_1, \Delta^\star(f_2) \rangle_\varepsilon} \Delta^\star(B)$$

- Consider the case where f is of type $((-)^\star)$.

$$A \cong \Phi^\star(A') \xrightarrow{\Phi^\star(f')} \Phi^\star(B') \cong B$$

Then B' is not a trivial shape, which means that B cannot be a trivial shape. This contradicts our assumption.

- Consider the case where f is of type (θ).

$$A \xrightarrow{\langle \Phi, f' \rangle_\theta} \Phi_\star(B') \cong B$$

Then Φ is not an empty path, which means that B cannot be a trivial shape. This contradicts our assumption.

- Consider the case where f is of type (ζ).

$$A \cong A_1 \otimes \Gamma^\star \Phi^\star \Phi_\star(A_2) \otimes A_3 \xrightarrow{\langle \Gamma, \Phi, f' \rangle_\zeta} B$$

By induction,

$$\Delta^\star(A_1 \otimes \Gamma^\star(A_2) \otimes A_3) \xrightarrow{\Delta^\star(f')} \Delta^\star(B)$$

is a constructible morphism. Then $\Delta^\star(f)$ is the following constructible morphism of type (ζ).

$$\Delta^\star(A) \cong \Delta^\star(A_1 \otimes \Gamma^\star \Phi^\star \Phi_\star(A_2) \otimes A_3) \xrightarrow{\langle \Gamma\Delta, \Phi, \Delta^\star(f') \rangle_\zeta} \Delta^\star(B)$$

□

Lemma 5.4.12. *Let* $\Phi \colon \mathcal{C} \to \mathcal{D}$ *be a path in* $\mathfrak{G}$*. Given a constructible* $\mathcal{C}$*-morphism*

$$f \colon \Phi^\star(A) \to B,$$

the following is a constructible $\mathcal{D}$*-morphism.*

$$A \xrightarrow{\langle \Phi, f \rangle_\theta} \Phi_\star(B)$$

Proof. If Φ is a non-empty path, then the result follows. Assume that Φ is an empty path. In this case, $\langle \Phi, f \rangle_\theta$ is the following constructible morphism.

$$A \cong \Phi^\star(A) \xrightarrow{f} B \cong \Phi_\star(B)$$

□

Lemma 5.4.13. *Let* $\Gamma\colon \mathcal{B} \to \mathcal{C}$ *be a path in* $\mathfrak{G}$*. Let* $\Phi\colon \mathcal{C} \to \mathcal{D}$ *be a path in* $\mathfrak{G}$*. Given a constructible* $\mathcal{B}$*-morphism*

$$f\colon A \otimes \Gamma^\star(B) \otimes C \to D,$$

the following is a constructible $\mathcal{B}$*-morphism.*

$$A \otimes \Gamma^\star\Phi^\star\Phi_\star(B) \otimes C \xrightarrow{\langle\Gamma,\Phi,f\rangle_\zeta} D$$

Proof. If Φ is a non-empty path, then the result follows. Assume that Φ is an empty path. In this case, $\langle\Gamma, \Phi, f\rangle_\zeta$ is the following constructible morphism.

$$A \otimes \Gamma^\star\Phi^\star\Phi_\star(B) \otimes C \cong A \otimes \Gamma^\star(B) \otimes C \xrightarrow{f} D$$

□

The biggest challenge in showing that the class of constructible morphisms consists of all of the allowable morphisms is in showing that the class of constructible morphisms is closed under composition. In order to do this, we prove the following more general statement.

Lemma 5.4.14. *Let* $\Delta\colon \mathcal{C} \to \mathcal{D}$ *be a path in* $\mathfrak{G}$*. Given a constructible* $\mathcal{D}$*-morphism*

$$s\colon P \to Q$$

and a constructible $\mathcal{C}$*-morphism*

$$t\colon U \otimes \Delta^\star(Q) \otimes V \to R,$$

the following is a constructible $\mathcal{C}$*-morphism.*

$$U \otimes \Delta^\star(P) \otimes V \xrightarrow{\langle\Delta,s,t\rangle_\circ} R$$

Proof. We will prove this by induction on the types of s and t.

- If s or t is of type $(\cong)$, then the result follows from Lemma 5.4.7.
- Consider the case where s is of type $(\otimes)$.

$$s\colon P \cong A \otimes C \xrightarrow{f\otimes g} B \otimes D \cong Q$$

$$t\colon U \otimes \Delta^\star(Q) \otimes V \to R$$

  Consider the following constructible morphism.

$$t'\colon U \otimes \Delta^\star(B \otimes D) \otimes V \cong U \otimes \Delta^\star(Q) \otimes V \xrightarrow{t} R$$

Consider the following allowable morphism.

$$u\colon U \otimes \Delta^\star(B \otimes C) \otimes V \xrightarrow{\langle \Delta, g, t' \rangle_\circ} R$$

By induction, u is a constructible morphism. Consider the following allowable morphism.

$$v\colon U \otimes \Delta^\star(A \otimes C) \otimes V \xrightarrow{\langle \Delta, f, u \rangle_\circ} R$$

By induction, v is a constructible morphism. The desired composite is the following constructible morphism.

$$U \otimes \Delta^\star(P) \otimes V \cong U \otimes \Delta^\star(A \otimes C) \otimes V \xrightarrow{v} R$$

Note the following.

$$\langle \Delta, s, t \rangle_\circ \cong \langle \Delta, f \otimes g, t \rangle_\circ \cong \langle \Delta, f, \langle \Delta, g, t \rangle_\circ \rangle_\circ$$

- Consider the case where t is of type (η).

$$s\colon P \to Q$$

$$t\colon U \otimes \Delta^\star(Q) \otimes V \xrightarrow{\langle I, f \rangle_\eta} I \backslash A \cong R$$

Consider the following allowable morphism.

$$f'\colon I \otimes U \otimes \Delta^\star(P) \otimes V \xrightarrow{\langle \Delta, s, f \rangle_\circ} A$$

By induction, f' is a constructible morphism. The desired composite is the following constructible morphism of type (η).

$$U \otimes \Delta^\star(P) \otimes V \xrightarrow{\langle I, f' \rangle_\eta} I \backslash A \cong R$$

Note the following.

$$\langle \Delta, s, t \rangle_\circ \cong \langle \Delta, s, \langle I, f \rangle_\eta \rangle_\circ \cong \langle I, \langle \Delta, s, f \rangle_\circ \rangle_\eta$$

- Consider the case where s is of type (ε).

$$s\colon P \cong A \otimes \Gamma^\star(B \otimes (C \backslash D)) \otimes E \xrightarrow{\langle \Gamma, f, g \rangle_\varepsilon} Q$$

$$t\colon U \otimes \Delta^\star(Q) \otimes V \to R$$

Consider the following allowable morphism.

$$u\colon U \otimes \Delta^\star(A \otimes \Gamma^\star(D) \otimes E) \otimes V \xrightarrow{\langle \Delta, g, t \rangle_\circ} R$$

By induction, u is a constructible morphism. The desired composite is the following constructible morphism of type (ε).

$$U \otimes \Delta^\star(P) \otimes V \cong U \otimes \Delta^\star(A \otimes \Gamma^\star(B \otimes (C\backslash D)) \otimes E) \otimes V \xrightarrow{\langle \Gamma\Delta, f, u\rangle_\varepsilon} R$$

Note the following.

$$\langle \Delta, s, t\rangle_\circ \cong \langle \Delta, \langle \Gamma, f, g\rangle_\varepsilon, t\rangle_\circ \cong \langle \Gamma\Delta, f, \langle \Delta, g, t\rangle_\circ\rangle_\varepsilon$$

- Consider the case where s is of type $((-)^\star)$.

$$s\colon P \cong \Gamma^\star(A) \xrightarrow{\Gamma^\star(f)} \Gamma^\star(B) \cong Q$$

$$t\colon U \otimes \Delta^\star(Q) \otimes V \to R$$

Consider the following constructible morphism.

$$t'\colon U \otimes \Delta^\star\Gamma^\star(B) \otimes V \cong U \otimes \Delta^\star(Q) \otimes V \xrightarrow{t} R$$

Consider the following allowable morphism.

$$u\colon U \otimes \Delta^\star\Gamma^\star(A) \otimes V \xrightarrow{\langle \Gamma\Delta, f, t'\rangle_\circ} R$$

By induction, u is a constructible morphism. The desired composite is the following constructible morphism.

$$U \otimes \Delta^\star(P) \otimes V \cong U \otimes \Delta^\star\Gamma^\star(A) \otimes V \xrightarrow{u} R$$

Note the following.

$$\langle \Delta, s, t\rangle_\circ \cong \langle \Delta, \Gamma^\star(f), t\rangle_\circ \cong \langle \Gamma\Delta, f, t\rangle_\circ$$

- Consider the case where t is of type (θ).

$$s\colon P \to Q$$

$$t\colon U \otimes \Delta^\star(Q) \otimes V \xrightarrow{\langle \Phi, f\rangle_\theta} \Phi_\star(A) \cong R$$

Consider the following allowable morphism.

$$f'\colon \Phi^\star(U \otimes \Delta^\star(P) \otimes V) \xrightarrow{\langle \Delta\Phi, s, f\rangle_\circ} A$$

By induction, f' is a constructible morphism. The desired composite is the following constructible morphism of type (θ).

$$U \otimes \Delta^\star(P) \otimes V \xrightarrow{\langle \Phi, f'\rangle_\theta} \Phi_\star(A) \cong R$$

Note the following.

$$\langle \Delta, s, t\rangle_\circ \cong \langle \Delta, s, \langle \Phi, f\rangle_\theta\rangle_\circ \cong \langle \Phi, \langle \Delta\Phi, s, f\rangle_\circ\rangle_\theta$$

- Consider the case where s is of type (ζ).

$$s\colon P \cong A \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes C \xrightarrow{\langle \Gamma, \Phi, f\rangle_\zeta} Q$$

$$t\colon U \otimes \Delta^\star(Q) \otimes V \to R$$

Consider the following allowable morphism.

$$u\colon U \otimes \Delta^\star(A \otimes \Gamma^\star(B) \otimes C) \otimes V \xrightarrow{\langle \Delta, f, t\rangle_\circ} R$$

By induction, u is a constructible morphism. The desired composite is the following constructible morphism of type (ζ).

$$U \otimes \Delta^\star(P) \otimes V \cong U \otimes \Delta^\star(A \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes C) \otimes V \xrightarrow{\langle \Gamma\Delta, \Phi, u\rangle_\zeta} R$$

Note the following.

$$\langle \Delta, s, t\rangle_\circ \cong \langle \Delta, \langle \Gamma, \Phi, f\rangle_\zeta, t\rangle_\circ \cong \langle \Gamma\Delta, \Phi, \langle \Delta, f, t\rangle_\circ\rangle_\zeta$$

- Consider the case where s is of type (η) or (θ) and t is of type $(\otimes)$.

$$s\colon P \xrightarrow{\langle I, k\rangle_\eta} I\backslash K \cong Q \qquad \text{or} \qquad s\colon P \xrightarrow{\langle \Psi, l\rangle_\theta} \Psi_\star(L) \cong Q$$

$$t\colon U \otimes \Delta^\star(Q) \otimes V \cong A \otimes C \xrightarrow{f \otimes g} B \otimes D \cong R$$

In either case, Q has a single prime factor. It must be the case that U, Q, V, A and C have prime factorisations of the following forms, where $\Delta^\star(X'_a) = X_a$.

$$U \cong \bigotimes_{0 \le a < i} X_a \qquad Q \cong X'_i \qquad V \cong \bigotimes_{i < a < n} X_a$$

$$A \cong \bigotimes_{0 \le a < j} X_a \qquad C \cong \bigotimes_{j \le a < n} X_a$$

Compare i with j.

Consider the case where $0 \le i < j \le n$.

$$\underbrace{\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{U} \otimes \overbrace{X_i}^{\Delta^\star(Q)} \otimes X_{i+1} \otimes \cdots \otimes X_{j-1}}_{A} \otimes \underbrace{X_j \otimes \cdots \otimes X_{n-1}}_{C}, \qquad \overbrace{X_{i+1} \otimes \cdots \otimes X_{n-1}}^{V}$$

Define the following shape.

$$A \cap V = \bigotimes_{i < a < j} X_a$$

This shape has been chosen so that the following central isomorphisms exist.

$$V \cong (A \cap V) \otimes C$$

$$U \otimes \Delta^\star(Q) \otimes (A \cap V) \cong A$$

Consider the following constructible morphism.

$$f' \colon U \otimes \Delta^\star(Q) \otimes (A \cap V) \cong A \xrightarrow{f} B$$

Consider the following allowable morphism.

$$u \colon U \otimes \Delta^\star(P) \otimes (A \cap V) \xrightarrow{\langle \Delta, s, f' \rangle_\circ} B$$

By induction, u is a constructible morphism. The desired composite is the following constructible morphism of type $(\otimes)$.

$$U \otimes \Delta^\star(P) \otimes V \cong U \otimes \Delta^\star(P) \otimes (A \cap V) \otimes C \xrightarrow{u \otimes g} B \otimes D \cong R$$

Note the following.

$$\langle \Delta, s, t \rangle_\circ \cong \langle \Delta, s, f \otimes g \rangle_\circ \cong \langle \Delta, s, f \rangle_\circ \otimes g$$

Consider the case where $0 \leq j \leq i < n$.

$$\overbrace{\underbrace{X_0 \otimes \cdots \otimes X_{j-1}}_{A} \otimes \mathrlap{\underbrace{\phantom{X_j \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes X_{i+1} \otimes \cdots \otimes X_{n-1}}}_{C}} X_j \otimes \cdots \otimes X_{i-1}}^{U} \otimes \overbrace{X_i}^{\Delta^\star(Q)} \otimes \overbrace{X_{i+1} \otimes \cdots \otimes X_{n-1}}^{V}$$

This case is similar to the case where $0 \leq i < j \leq n$. Note the following.

$$\langle \Delta, s, t \rangle_\circ \cong \langle \Delta, s, f \otimes g \rangle_\circ \cong f \otimes \langle \Delta, s, g \rangle_\circ$$

- Consider the case where s is of type (η) or (θ) and t is of type (ε).

$$s \colon P \xrightarrow{\langle J,k \rangle_\eta} J \backslash K \cong Q \qquad \text{or} \qquad s \colon P \xrightarrow{\langle \Psi,l \rangle_\theta} \Psi_\star(L) \cong Q$$

$$t \colon U \otimes \Delta^\star(Q) \otimes V \cong A \otimes \Gamma^\star(B \otimes (C \backslash D)) \otimes E \xrightarrow{\langle \Gamma,f,g \rangle_\varepsilon} R$$

In either case, Q has a single prime factor. It must be the case that U, Q, V, A, B, $C \backslash D$ and E have prime factorisations of the following forms, where $\Delta^\star(X'_a) = X_a$ and $\Gamma^\star(X''_a) = X_a$.

$$U \cong \bigotimes_{0 \leq a < i} X_a \qquad Q \cong X'_i \qquad V \cong \bigotimes_{i < a < n} X_a$$

$$A \cong \bigotimes_{0 \leq a < j} X_a \qquad B \cong \bigotimes_{j \leq a < k} X''_a \qquad C \backslash D = X''_k \qquad E \cong \bigotimes_{k < a < n} X_a$$

Compare i with j and k.

Consider the case where $0 \leq i < j \leq k < n$.

$$\underbrace{\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{U} \otimes \overbrace{X_i}^{\Delta^\star(Q)} \otimes \rlap{$\overbrace{\phantom{X_{i+1} \otimes \cdots X_{j-1} \otimes X_j \otimes \cdots \otimes X_{k-1} \otimes X_k \otimes X_{k+1} \otimes \cdots \otimes X_{n-1}}}^{V}$} X_{i+1} \otimes \cdots X_{j-1}}_{A} \otimes \underbrace{X_j \otimes \cdots \otimes X_{k-1}}_{\Gamma^\star(B)} \otimes \underbrace{X_k}_{\Gamma^\star(C\backslash D)} \otimes \underbrace{X_{k+1} \otimes \cdots \otimes X_{n-1}}_{E}$$

Define the following shape.

$$A \cap V = \bigotimes_{i<a<j} X_a$$

This shape has been chosen so that the following central isomorphisms exist.

$$V \cong (A \cap V) \otimes \Gamma^\star(B \otimes (C\backslash D)) \otimes E$$
$$U \otimes \Delta^\star(Q) \otimes (A \cap V) \cong A$$

Consider the following constructible morphism.

$$g' \colon U \otimes \Delta^\star(Q) \otimes (A \cap V) \otimes \Gamma^\star(D) \otimes E \cong A \otimes \Gamma^\star(D) \otimes E \xrightarrow{g} R$$

Consider the following allowable morphism.

$$u \colon U \otimes \Delta^\star(P) \otimes (A \cap V) \otimes \Gamma^\star(D) \otimes E \xrightarrow{\langle \Delta, s, g' \rangle_\circ} R$$

By induction, u is a constructible morphism. The desired composite is the following constructible morphism of type (ε).

$$U \otimes \Delta^\star(P) \otimes V \cong U \otimes \Delta^\star(P) \otimes (A \cap V) \otimes \Gamma^\star(B \otimes (C\backslash D)) \otimes E \xrightarrow{\langle \Gamma, f, u \rangle_\varepsilon} R$$

Note the following.

$$\langle \Delta, s, t \rangle_\circ \cong \langle \Delta, s, \langle \Gamma, f, g \rangle_\varepsilon \rangle_\circ \cong \langle \Gamma, f, \langle \Delta, s, g \rangle_\circ \rangle_\varepsilon$$

Consider the case where $0 \leq j \leq i < k < n$.

$$\overbrace{\underbrace{X_0 \otimes \cdots X_{j-1}}_{A} \otimes \rlap{$\underbrace{\phantom{X_j \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes X_{i+1} \otimes \cdots \otimes X_{k-1}}}_{\Gamma^\star(B)}$} X_j \otimes \cdots \otimes X_{i-1}}^{U} \otimes \overbrace{X_i}^{\Delta^\star(Q)} \otimes \overbrace{X_{i+1} \otimes \cdots \otimes X_{k-1} \otimes \underbrace{X_k}_{\Gamma^\star(C\backslash D)} \otimes \underbrace{X_{k+1} \otimes \cdots \otimes X_{n-1}}_{E}}^{V}$$

Define the following shapes.

$$U \cap B = \bigotimes_{j \leq a < i} X''_a \qquad Q_B = X''_i \qquad B \cap V = \bigotimes_{i<a<k} X''_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$U \cong A \otimes \Gamma^\star(U \cap B) \quad \Delta^\star(Q) \cong \Gamma^\star(Q_B) \quad V \cong \Gamma^\star((B \cap V) \otimes (C\backslash D)) \otimes E$$

$$(U \cap B) \otimes Q_B \otimes (B \cap V) \cong B$$

By Lemma 5.3.18, there is a Λ such that $\Delta = \Lambda\Gamma$, since either $J\backslash K$ or $\Psi_\star(L)$ is a prime factor of Q. By Lemma 5.3.16, the following central isomorphism exists.

$$\Lambda^\star(Q) \cong Q_B$$

Consider the following constructible morphism.

$$f' \colon (U \cap B) \otimes \Lambda^\star(Q) \otimes (B \cap V) \cong B \xrightarrow{f} C$$

Consider the following allowable morphism.

$$u \colon (U \cap B) \otimes \Lambda^\star(P) \otimes (B \cap V) \xrightarrow{\langle \Lambda, s, f' \rangle_\circ} C$$

By induction, u is a constructible morphism. The desired composite is the following constructible morphism of type (ε).

$$\begin{aligned} &U \otimes \Delta^\star(P) \otimes V \\ &\cong A \otimes \Gamma^\star((U \cap B) \otimes \Lambda^\star(P) \otimes (B \cap V) \otimes (C\backslash D)) \otimes E \xrightarrow{\langle \Gamma, u, g \rangle_\varepsilon} R \end{aligned}$$

Note the following.

$$\langle \Delta, s, t \rangle_\circ \cong \langle \Lambda\Gamma, s, \langle \Gamma, f, g \rangle_\varepsilon \rangle_\circ \cong \langle \Gamma, \langle \Lambda, s, f \rangle_\circ, g \rangle_\varepsilon$$

Consider the case where $0 \leq j < i = k < n$,

$$\overbrace{\underbrace{X_0 \otimes \cdots \otimes X_{j-1}}_{A} \otimes \underbrace{X_j \otimes \cdots \otimes X_{i-1}}_{\Gamma^\star(B)}}^{U} \otimes \overbrace{\underbrace{X_i}_{\Gamma^\star(C\backslash D)}}^{\Delta^\star(Q)} \otimes \overbrace{\underbrace{X_{i+1} \otimes \cdots \otimes X_{n-1}}_{E}}^{V}$$

The following central isomorphisms exist.

$$U \cong A \otimes \Gamma^\star(B) \qquad \Delta^\star(Q) \cong \Gamma^\star(C\backslash D) \qquad V \cong E$$

It must be the case that s is of type (η), $\Delta = \Gamma$, $J = C$ and $K = D$. Consider the following allowable morphism.

$$u \colon A \otimes \Gamma^\star(C \otimes P) \otimes E \xrightarrow{\langle \Gamma, k, g \rangle_\circ} R$$

By induction, u is a constructible morphism. Consider the following allowable morphism.

$$v \colon A \otimes \Gamma^\star(B \otimes P) \otimes E \xrightarrow{\langle \Gamma, f, u \rangle_\circ} R$$

By induction, v is a constructible morphism. The desired composite is the following constructible morphism.

$$U \otimes \Delta^\star(P) \otimes V \cong A \otimes \Gamma^\star(B \otimes P) \otimes E \xrightarrow{v} R$$

Note the following.

$$\langle \Delta, s, t\rangle_\circ \cong \langle \Gamma, \langle J, k\rangle_\eta, \langle \Gamma, f, g\rangle_\varepsilon\rangle_\circ \cong \langle \Gamma, f, \langle \Gamma, k, g\rangle_\circ\rangle_\circ$$

Consider the case where $0 \le j \le k < i < n$.

$$\overbrace{\underbrace{X_0 \otimes \cdots X_{j-1}}_{A} \otimes \underbrace{X_j \otimes \cdots \otimes X_{k-1}}_{\Gamma^\star(B)} \otimes \underbrace{X_k}_{\Gamma^\star(C\backslash D)}}^{U} \otimes \underbrace{X_{k+1} \otimes \cdots \otimes X_{i-1} \otimes \overbrace{X_i}^{\Delta^\star(Q)} \otimes \overbrace{X_{i+1} \otimes \cdots \otimes X_{n-1}}^{V}}_{E}$$

This case is similar to the case where $0 \le i < j \le k < n$. Note the following.

$$\langle \Delta, s, t\rangle_\circ \cong \langle \Delta, s, \langle \Gamma, f, g\rangle_\varepsilon\rangle_\circ \cong \langle \Gamma, f, \langle \Delta, s, g\rangle_\circ\rangle_\varepsilon$$

- Consider the case where s is of type (η) or (θ) and t is of type $((-)^\star)$.

$$s\colon P \xrightarrow{\langle J,k\rangle_\eta} J\backslash K \cong Q \qquad \text{or} \qquad s\colon P \xrightarrow{\langle \Psi,l\rangle_\theta} \Psi_\star(L) \cong Q$$

$$t\colon U \otimes \Delta^\star(Q) \otimes V \cong \Phi^\star(A) \xrightarrow{\Phi^\star(f)} \Phi^\star(B) \cong R$$

In either case, Q has a single prime factor. It must be the case that U, Q, V and A have prime factorisations of the following forms, where $\Delta^\star(X'_a) = X_a$ and $\Phi^\star(X''_a) = X_a$.

$$U \cong \bigotimes_{0 \le a < i} X_a \qquad Q \cong X'_i \qquad V \cong \bigotimes_{i < a < n} X_a$$

$$A \cong \bigotimes_{0 \le a < n} X''_a$$

Define the following shapes.

$$U_A \cong \bigotimes_{0 \le a < i} X''_a \qquad Q_A \cong \bigotimes_{i \le a < j} X''_a \qquad V_A \cong \bigotimes_{j < a < n} X''_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$U \cong \Phi^\star(U_A) \quad \Delta^\star(Q) \cong \Phi^\star(Q_A) \quad V \cong \Phi^\star(V_A)$$

$$U_A \otimes Q_A \otimes V_A \cong A$$

By Lemma 5.3.18, there a Λ such that $\Delta = \Lambda\Phi$, since either $J\backslash K$ or $\Psi_\star(L)$ is a prime factor of Q. By Lemma 5.3.16, the following central isomorphism exists.

$$\Lambda^\star(Q) \cong Q_A$$

Consider the following constructible morphism.

$$f' \colon U_A \otimes \Lambda^\star(Q) \otimes V_A \cong A \xrightarrow{f} B$$

Consider the following allowable morphism.

$$u \colon U_A \otimes \Lambda^\star(P) \otimes V_A \xrightarrow{\langle \Lambda, s, f' \rangle_\circ} B$$

By induction, u is a constructible morphism. The desired composite is the following constructible morphism of type $((-)^\star)$.

$$U \otimes \Delta^\star(P) \otimes V \cong \Phi^\star(U_A \otimes \Lambda^\star(P) \otimes V_A) \xrightarrow{\Phi^\star(u)} \Phi^\star(B) \cong R$$

Note the following.

$$\langle \Delta, s, t \rangle_\circ \cong \langle \Lambda\Phi, s, \Phi^\star(f) \rangle_\circ \cong \Phi^\star(\langle \Lambda, s, f \rangle_\circ)$$

- Consider the case where s is of type (η) or (θ) and t is of type (ζ).

$$s \colon P \xrightarrow{\langle J,k \rangle_\eta} J\backslash K \cong Q \qquad \text{or} \qquad s \colon P \xrightarrow{\langle \Psi,l \rangle_\theta} \Psi_\star(L) \cong Q$$

$$t \colon U \otimes \Delta^\star(Q) \otimes V \cong A \otimes \Gamma^\star\Phi^\star\Phi_\star(B) \otimes C \xrightarrow{\langle \Gamma,\Phi,f \rangle_\zeta} R$$

In either case, Q has a single prime factor. It must be the case that U, Q, V, A, $\Phi_\star(B)$ and C have prime factorisations of the following forms, where $\Delta^\star(X'_a) = X_a$ and $\Gamma^\star\Phi^\star(X''_a) = X_a$.

$$U \cong \bigotimes_{0 \le a < i} X_a \qquad Q \cong X'_i \qquad V \cong \bigotimes_{i < a < n} X_a$$

$$A \cong \bigotimes_{0 \le a < j} X_a \qquad \Phi_\star(B) \cong X''_j \qquad C \cong \bigotimes_{j < a < n} X_a$$

Compare i with j.

Consider the case where $0 \le i < j < n$.

$$\overbrace{X_0 \otimes \cdots X_{i-1}}^{U} \otimes \overbrace{X_i}^{\Delta^\star(Q)} \otimes \overbrace{X_{i+1} \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes X_{j+1} \otimes \cdots \otimes X_{n-1}}^{V}$$

$$\underbrace{X_0 \otimes \cdots X_{i-1} \otimes X_i \otimes X_{i+1} \otimes \cdots \otimes X_{j-1}}_{A} \otimes \underbrace{X_j}_{\Gamma^\star\Phi^\star\Phi_\star(B)} \otimes \underbrace{X_{j+1} \otimes \cdots \otimes X_{n-1}}_{C}$$

Define the following shape.

$$A \cap V = \bigotimes_{i<a<j} X_a$$

This shape has been chosen so that the following central isomorphisms exist.

$$V \cong (A \cap V) \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes C$$
$$U \otimes \Delta^\star(Q) \otimes (A \cap V) \cong A$$

Consider the following constructible morphism.

$$f' \colon U \otimes \Delta^\star(Q) \otimes (A \cap V) \otimes \Gamma^\star(B) \otimes C \cong A \otimes \Gamma^\star(B) \otimes C \xrightarrow{f} R$$

Consider the following allowable morphism.

$$u \colon U \otimes \Delta^\star(P) \otimes (A \cap V) \otimes \Gamma^\star(B) \otimes C \xrightarrow{\langle \Delta, s, f' \rangle_\circ} R$$

By induction, u is a constructible morphism. The desired composite is the following constructible morphism of type (ζ).

$$U \otimes \Delta^\star(P) \otimes V \cong U \otimes \Delta^\star(P) \otimes (A \cap V) \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes C \xrightarrow{\langle \Gamma, \Phi, u \rangle_\zeta} R$$

Note the following.

$$\langle \Delta, s, t \rangle_\circ \cong \langle \Delta, s, \langle \Gamma, \Phi, f \rangle_\zeta \rangle_\circ \cong \langle \Gamma, \Phi, \langle \Delta, s, f \rangle_\circ \rangle_\zeta$$

Consider the case where $0 \leq i = j < n$.

$$\underbrace{\overbrace{X_0 \otimes \cdots X_{i-1}}^{U}}_{A} \otimes \underbrace{\overbrace{X_i}^{\Delta^\star(Q)}}_{\Gamma^\star \Phi^\star \Phi_\star(B)} \otimes \underbrace{\overbrace{X_{i+1} \otimes \cdots \otimes X_{n-1}}^{V}}_{C}$$

The following central isomorphisms exist.

$$U \cong A \qquad \Delta^\star(Q) \cong \Gamma^\star \Phi^\star \Phi_\star(B) \qquad V \cong C$$

It must be the case that s is of type (θ), $\Delta = \Phi\Gamma$, $\Psi = \Phi$ and $L = B$. Consider the following allowable morphism.

$$u \colon A \otimes \Gamma^\star \Phi^\star(P) \otimes C \xrightarrow{\langle \Gamma, l, f \rangle_\circ} R$$

By induction, u is a constructible morphism. The desired composite is the following constructible morphism.

$$U \otimes \Delta^\star(P) \otimes V \cong A \otimes \Gamma^\star \Phi^\star(P) \otimes C \xrightarrow{u} R$$

Note the following.

$$\langle \Delta, s, t \rangle_\circ \cong \langle \Phi\Gamma, \langle \Phi, l \rangle_\theta, \langle \Gamma, \Phi, f \rangle_\zeta \rangle_\circ \cong \langle \Gamma, l, f \rangle_\circ$$

Consider the case where $0 \leq j < i < n$.

$$\overbrace{\underbrace{X_0 \otimes \cdots X_{j-1}}_{A} \otimes \underbrace{X_j}_{\Gamma^\star\Phi^\star\Phi_\star(B)} \otimes X_{j+1} \otimes \cdots \otimes X_{i-1}}^{U} \otimes \underbrace{\overbrace{X_i}^{\Delta^\star(Q)} \otimes \overbrace{X_{i+1} \otimes \cdots \otimes X_{n-1}}^{V}}_{C}$$

This case is similar to the case where $0 \leq i < j < n$. Note the following.

$$\langle \Delta, s, t\rangle_\circ \cong \langle \Delta, s, \langle \Gamma, \Phi, f\rangle_\zeta\rangle_\circ \cong \langle \Gamma, \Phi, \langle \Delta, s, f\rangle_\circ\rangle_\zeta$$

□

Now the proof that every allowable morphism is a constructible morphism is not difficult.

Theorem 5.4.15. *Let $\mathcal{C}$ be a vertex of $\mathfrak{G}$. Let $s\colon A \to B$ be an allowable $\mathcal{C}$-morphism. Then s is a constructible $\mathcal{C}$-morphism.*

Proof. We will prove this by induction on the type of s

- If s is an allowable morphism of type (id), (α), (λ), (ρ), (φ), $(\hat{\varphi})$, (κ) or $(\hat{\kappa})$, then s is a constructible morphism of type $(\cong)$.
- If s is an allowable morphism of type (η),

  $$B \xrightarrow{\eta^A_B} A\backslash(A \otimes B),$$

  then s is the following allowable morphism, which is a constructible morphism, by Lemma 5.4.8 and Lemma 5.4.9.

  $$B \xrightarrow{\langle A, A\otimes B\rangle_\eta} A\backslash(A \otimes B)$$
- If s is an allowable morphism of type (ε),

  $$A \otimes (A\backslash B) \xrightarrow{\varepsilon^A_B} B,$$

  then s is the following allowable morphism, which is a constructible morphism, by Lemma 5.4.10.

  $$A \otimes (A\backslash B) \xrightarrow{\langle \mathrm{id}, A, B\rangle_\varepsilon} B$$
- If s is an allowable morphism of type $(\otimes)$,

  $$A \otimes C \xrightarrow{f\otimes g} B \otimes D,$$

  then s is the following allowable morphism, which is a constructible morphism, by Lemma 5.4.8.

  $$A \otimes C \xrightarrow{f\otimes g} B \otimes D,$$

- If s is an allowable morphism of type $(\backslash)$,

$$B\backslash C \xrightarrow{f\backslash g} A\backslash D,$$

then s is the following allowable morphism, which is a constructible morphism, by Lemma 5.4.10 and Lemma 5.4.9.

$$B\backslash C \xrightarrow{\langle A, \langle \mathrm{id}, f, g\rangle_\varepsilon\rangle_\eta} A\backslash D$$

- If s is an allowable morphism of type (θ),

$$A \xrightarrow{\theta^\Phi_A} \Phi_\star\Phi^\star(A),$$

then s is the following allowable morphism, which is a constructible morphism, by Lemma 5.4.11 and Lemma 5.4.12.

$$A \xrightarrow{\langle \Phi, \Phi^\star(A)\rangle_\theta} \Phi_\star\Phi^\star(A)$$

- If s is an allowable morphism of type (ζ),

$$\Phi^\star\Phi_\star(A) \xrightarrow{\zeta^\Phi_A} A,$$

then s is the following allowable morphism, which is a constructible morphism, by Lemma 5.4.13.

$$\Phi^\star\Phi_\star(A) \xrightarrow{\langle \mathrm{id}, \Phi, A\rangle_\zeta} A$$

- If s is an allowable morphism of type $((-)^\star)$,

$$\Phi^\star(A) \xrightarrow{\Phi^\star(f)} \Phi^\star(B),$$

then s is the following allowable morphism, which is a constructible morphism, by Lemma 5.4.11.

$$\Phi^\star(f) \colon \Phi^\star(A) \to \Phi^\star(B),$$

- If s is an allowable morphism of type $((-)_\star)$,

$$\Phi_\star(A) \xrightarrow{\Phi_\star(f)} \Phi_\star(B),$$

then s is the following allowable morphism, which is a constructible morphism, by Lemma 5.4.13 and Lemma 5.4.12

$$\Phi_\star(A) \xrightarrow{\langle \Phi, \langle \mathrm{id}, \Phi, f\rangle_\zeta\rangle_\theta} \Phi_\star(B)$$

- If s is an allowable morphism of type $(\circ)$,

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

  then s is the following allowable morphism, which is a constructible morphism, by Lemma 5.4.14.

$$A \xrightarrow{\langle \mathrm{id}, f, g\rangle_\circ} C$$

□

5.5 A Category of Relations

In this section, we will define a particular closed monoidal category, together with a collection of monoidal adjunctions indexed by the positive integers. We will then use this to define a particularly simple object of $\mathrm{GR}_{\mathfrak{G}}$, which we will denote $\mathbb{Z}\mathrm{REL}$, as well as a morphism of $\mathrm{GR}_{\mathfrak{G}}$ of the following form.

$$\Omega\colon \mathrm{SHP}_{\mathfrak{G}} \to \mathbb{Z}\mathrm{REL}$$

5.5.1 Notation and Terminology

Throughout this section, we will make use of predicates and relations.

By a predicate on a set A, we mean a function $P\colon A \to \{T, F\}$. Given such a predicate and an element $a \in A$, we say that $P(a)$ is true if $P(a) = T$ and that $P(a)$ is false if $P(a) = F$.

We will denote logical conjunction of $P(a)$ and $Q(a)$ by $P(a)Q(a)$; i.e. $P(a)Q(a)$ is true if and only if both $P(a)$ and $Q(a)$ are true.

By a relation between two sets A and B, we mean a predicate on the set $A \times B$. We denote such a relation by $R\colon A \dashrightarrow B$. Given such a relation and elements $a \in A$ and $b \in B$, we will usually denote $R(a, b)$ by R^a_b.

By the zero predicate on a set A, we mean the predicate on A which is constant at F. By the zero relation $A \dashrightarrow B$, we mean the zero predicate on the set $A \times B$.

Given a positive integer Φ, we will denote the set $\{0, 1, \ldots, \Phi - 1\}$ by $\underline{\Phi}$.

At times, it may be helpful to view a relation $A \dashrightarrow B$ as the corresponding matrix whose height is the cardinality of A and whose width is the cardinality of B. For example, the relation $R\colon \underline{3} \dashrightarrow \underline{4}$ defined by $R^a_b \iff a \le b$ can be represented by the following 3×4 matrix.

$$\begin{bmatrix} 0 \le 0 & 0 \le 1 & 0 \le 2 & 0 \le 3 \\ 1 \le 0 & 1 \le 1 & 1 \le 2 & 1 \le 3 \\ 2 \le 0 & 2 \le 1 & 2 \le 2 & 2 \le 3 \end{bmatrix} = \begin{bmatrix} T & T & T & T \\ F & T & T & T \\ F & F & T & T \end{bmatrix}$$

There is a canonical bijection $\underline{\Phi} \times \underline{\Psi} \cong \underline{\Phi\Psi}$ given by ordering the ordered pairs lexicographically. For example, the bijection $\underline{2} \times \underline{3} \cong \underline{6}$ is given by the following composite of order-preserving bijections.

$$\{0,1\} \times \{0,1,2\} \cong \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\} \cong \{0,1,2,3,4,5\}$$

We will use this bijection implicitly in order to represent relations with multiple indices as matrices. For example, the relation $R\colon \underline{2} \times \underline{3} \dashrightarrow \underline{3}$ defined by $R^{ab}_c \iff a + b \leq c$ can be represented by the following 6×3 matrix.

$$\begin{bmatrix} 0+0 \leq 0 & 0+0 \leq 1 & 0+0 \leq 2 \\ 0+1 \leq 0 & 0+1 \leq 1 & 0+1 \leq 2 \\ 0+2 \leq 0 & 0+2 \leq 1 & 0+2 \leq 2 \\ 1+0 \leq 0 & 1+0 \leq 1 & 1+0 \leq 2 \\ 1+1 \leq 0 & 1+1 \leq 1 & 1+1 \leq 2 \\ 1+2 \leq 0 & 1+2 \leq 1 & 1+2 \leq 2 \end{bmatrix} = \begin{bmatrix} T & T & T \\ F & T & T \\ F & F & T \\ F & T & T \\ F & F & T \\ F & F & F \end{bmatrix}$$

We will also use this to represent block matrices. For example, given a 2×2 matrix M^0_0, a 2×3 matrix M^0_1, a 4×2 matrix M^1_0, and a 4×3 matrix M^1_1, the 8×6 block matrix

$$\begin{bmatrix} M^0_0 & M^0_1 \\ M^1_0 & M^1_1 \end{bmatrix}$$

represents the relation $R\colon \underline{2} \times \underline{4} \dashrightarrow \underline{2} \times \underline{3}$ defined by $R^{ia}_{jb} \iff (M^i_j)^a_b$. The indices i and j determine the submatrix, the indices a and b determine the entry within the submatrix.

Given relations $R\colon A \dashrightarrow B$ and $S\colon B \dashrightarrow C$, we denote their composite relation by $R \bullet S\colon A \dashrightarrow C$, defined as follows.

$$(R \bullet S)^a_c \iff \exists b(R^a_b S^b_c)$$

Note that this agrees with the usual definition of matrix multiplication.

Given a set A, we will denote the identity relation on A by the Kronecker delta δ, defined as follows.

$$\delta^i_j = \begin{cases} T & \text{if } i = j \\ F & \text{if } i \neq j \end{cases}$$

We will typically denote the unique element of a 1-element set by $\star$.

5.5.2 Definitions

The closed monoidal category which we will describe in this section is a strict model of the category of representations of the group of integers under addition, valued in the category of finite sets and relations. Let $\mathbb{Z}$ be the group of integers under addition, interpreted as a 1-object groupoid. Let Rel be the category of finite sets and relations. Then the category which we will describe, and which we will denote by $[\mathbb{Z}, \textsc{Rel}]$, is equivalent to the category of functors from $\mathbb{Z}$ to $\textsc{Rel}$. For the sake of simplicity, we will assume that $\textsc{Rel}$ is a strict monoidal category, with tensor product given by cartesian product of sets.

First, we will give an explicit description of the category $[\mathbb{Z}, \textsc{Rel}]$. An object of this category consists of a set A_0 with an automorphism (that is, a bijection $A_0 \to A_0$). Given an automorphism $A\colon A_0 \dashrightarrow A_0$ and an automorphism $B\colon B_0 \dashrightarrow B_0$, a morphism $A \to B$ is relation $R\colon A_0 \dashrightarrow B_0$ such that $A \bullet R = R \bullet B$; we call such relations 'equivariant'. Composition of morphisms is defined by composition of relations in the obvious way; given morphisms $R\colon A \to B$ and $S\colon B \to C$, the composite $R \bullet S$ constitutes a morphism $A \to C$. Finally, the identity morphism on A is simply the identity relation on A_0. It can be easily seen that this composition is both associative and unital.

Now, we shall describe the strict monoidal structure on $[\mathbb{Z}, \textsc{Rel}]$, inherited pointwise from the strict monoidal structure on $\textsc{Rel}$.

Given an automorphism $A\colon A_0 \dashrightarrow A_0$ and an automorphism $B\colon B_0 \dashrightarrow B_0$, their tensor product, $A \otimes B$, is the automorphism $A_0 \times B_0 \dashrightarrow A_0 \times B_0$ defined as follows.

$$(A \otimes B)^{ab}_{a'b'} \iff A^a_{a'} B^b_{b'}$$

Given a morphism $R\colon A \to B$ and a morphism $S\colon C \to D$, their tensor product, $R \otimes S$, is the morphism $A \otimes C \to B \otimes D$ defined as follows.

$$(R \otimes S)^{ac}_{bd} \iff R^a_b S^c_d$$

To see that $R \otimes S$ is a morphism $A \otimes C \to B \otimes D$, note the following.

$$\begin{aligned}
((A \otimes C) \bullet (R \otimes S))^{ac}_{bd} &\iff \exists a'c'((A \otimes C)^{ac}_{a'c'}(R \otimes S)^{a'c'}_{bd}) \\
&\iff \exists a'c'(A^a_{a'} C^c_{c'} R^{a'}_b S^{c'}_d) \\
&\iff \exists a'(A^a_{a'} R^{a'}_b) \exists c'(C^c_{c'} S^{c'}_d) \\
&\iff (A \bullet R)^a_b (C \bullet S)^c_d \\
&\iff (R \bullet B)^a_b (S \bullet D)^c_d \\
&\iff \exists b'(R^a_{b'} B^{b'}_b) \exists d'(S^c_{d'} D^{d'}_d) \\
&\iff \exists b'd'(R^a_{b'} S^c_{d'} B^{b'}_b D^{d'}_d) \\
&\iff \exists b'd'((R \otimes S)^{ac}_{b'd'}(B \otimes D)^{b'd'}_{bd}) \\
&\iff ((R \otimes S) \bullet (B \otimes D))^{ac}_{bd}
\end{aligned}$$

The monoidal unit, $\mathcal{I}$, is the 1-element set $\{\star\}$ with the identity relation.

Now, we shall describe a pivotal structure on $[\mathbb{Z}, \textsc{Rel}]$, inherited pointwise from the pivotal structure on $\textsc{Rel}$; i.e. for each object, we shall construct a simultaneous left and right dual.

Given an automorphism $A\colon A_0 \dashrightarrow A_0$, the dual, $A^\vee$, is A. That is, each object is self-dual. Note the following identity.

$$\exists j(A^i_j(A^\vee)^{i'}_j) \iff \delta^i_{i'}$$

Given an automorphism $A\colon A_0 \dashrightarrow A_0$, the evaluation morphism

$$\mathrm{ev}_A\colon A\otimes A^\vee \to \mathcal{I}$$

is the relation $A_0 \times A_0 \dashrightarrow 1$ defined as follows.

$$(\mathrm{ev}_A)^{ii'}_\star \iff \delta^i_{i'}$$

To see that ev_A is a morphism $A\otimes A^\vee \to \mathcal{I}$, note the following.

$$\begin{aligned}
((A\otimes A^\vee)\bullet \mathrm{ev}_A)^{ii'}_\star &\iff \exists jj'((A\otimes A^\vee)^{ii'}_{jj'}(\mathrm{ev}_A)^{jj'}_\star)\\
&\iff \exists jj'(A^i_j(A^\vee)^{i'}_{j'}\delta^j_{j'})\\
&\iff \exists j(A^i_j(A^\vee)^{i'}_j)\\
&\iff \delta^i_{i'}\\
&\iff (\mathrm{ev}_A)^{ii'}_\star\\
&\iff (\mathrm{ev}_A)^{ii'}_\star\mathcal{I}^\star_\star\\
&\iff (\mathrm{ev}_A\bullet\mathcal{I})^{ii'}_\star
\end{aligned}$$

Given an automorphism $A\colon A_0 \dashrightarrow A_0$, the coevaluation morphism

$$\mathrm{coev}_A\colon \mathcal{I}\to A^\vee\otimes A$$

is the relation $1 \dashrightarrow A_0\times A_0$ defined as follows.

$$(\mathrm{coev}_A)^\star_{jj'} \iff \delta^j_{j'}$$

To see that coev_A is a morphism $\mathcal{I}\to A^\vee\otimes A$, note the following.

$$\begin{aligned}
(\mathcal{I}\bullet\mathrm{coev}_A)^\star_{kk'} &\iff \mathcal{I}^\star_\star(\mathrm{coev}_A)^\star_{kk'}\\
&\iff (\mathrm{coev}_A)^\star_{kk'}\\
&\iff \delta^k_{k'}\\
&\iff \exists j((A^\vee)^j_k A^j_{k'})\\
&\iff \exists jj'(\delta^j_{j'}(A^\vee)^j_k A^{j'}_{k'})\\
&\iff \exists jj'((\mathrm{coev}_A)^\star_{jj'}(A^\vee\otimes A)^{jj'}_{kk'})\\
&\iff (\mathrm{coev}_A\bullet(A^\vee\otimes A))^\star_{kk'}
\end{aligned}$$

The first triangle identity is that the following morphism is the identity on A.

$$A \xrightarrow{\mathrm{id}_A\otimes\mathrm{coev}_A} A\otimes A^\vee\otimes A \xrightarrow{\mathrm{ev}_A\otimes\mathrm{id}_A} A$$

To check this, note the following.

$$
\begin{aligned}
&((\mathrm{id}_A \otimes \mathrm{coev}_A) \bullet (\mathrm{ev}_A \otimes \mathrm{id}_A))^i_{k''}\\
&= \exists j j' j'' ((\mathrm{id}_A \otimes \mathrm{coev}_A)^i_{j j' j''} (\mathrm{ev}_A \otimes \mathrm{id}_A)^{j j' j''}_{k''})\\
&= \exists j j' j'' ((\mathrm{id}_A)^i_j (\mathrm{coev}_A)^\star_{j' j''} (\mathrm{ev}_A)^{j j'}_\star (\mathrm{id}_A)^{j''}_{k''})\\
&= \exists j j' j'' (\delta^i_j \delta^{j'}_{j''} \delta^j_{j'} \delta^{j''}_{k''})\\
&= \delta^i_{k''}\\
&= (\mathrm{id}_A)^i_{k''}
\end{aligned}
$$

The second triangle identity is that the following morphism is the identity on $A^\vee$.

$$A^\vee \xrightarrow{\mathrm{coev}_A \otimes \mathrm{id}_{A^\vee}} A^\vee \otimes A \otimes A^\vee \xrightarrow{\mathrm{id}_{A^\vee} \otimes \mathrm{ev}_A} A^\vee$$

To check this, note the following.

$$
\begin{aligned}
&((\mathrm{coev}_A \otimes \mathrm{id}_{A^\vee}) \bullet (\mathrm{id}_{A^\vee} \otimes \mathrm{ev}_A))^{i''}_k\\
&\iff \exists j j' j'' ((\mathrm{coev}_A \otimes \mathrm{id}_{A^\vee})^{i''}_{j j' j''} (\mathrm{id}_{A^\vee} \otimes \mathrm{ev}_A)^{j j' j''}_k)\\
&\iff \exists j j' j'' ((\mathrm{coev}_A)^\star_{j j'} (\mathrm{id}_{A^\vee})^{i''}_{j''} (\mathrm{id}_{A^\vee})^j_k (\mathrm{ev}_A)^{j' j''}_\star)\\
&\iff \exists j j' j'' (\delta^j_{j'} \delta^{i''}_{j''} \delta^j_k \delta^{j'}_{j''})\\
&\iff \delta^{i''}_k\\
&\iff (\mathrm{id}_{A^\vee})^{i''}_k
\end{aligned}
$$

Given a morphism $R\colon A \to B$, the morphism $R^\vee\colon B^\vee \to A^\vee$ is defined as follows.

$$R^\vee\colon B^\vee \xrightarrow{\mathrm{coev}_A \otimes \mathrm{id}_{B^\vee}} A^\vee \otimes A \otimes B^\vee \xrightarrow{\mathrm{id}_{A^\vee} \otimes R \otimes \mathrm{id}_{B^\vee}} A^\vee \otimes B \otimes B^\vee \xrightarrow{\mathrm{id}_{A^\vee} \otimes \mathrm{ev}_B} A^\vee$$

$$(R^\vee)^b_a \iff ((\mathrm{coev}_A \otimes \mathrm{id}_{B^\vee}) \bullet (\mathrm{id}_{A^\vee} \otimes R \otimes \mathrm{id}_{B^\vee}) \bullet (\mathrm{id}_{A^\vee} \otimes \mathrm{ev}_B)^b_a$$

Evaluating this yields the following.

$$(R^\vee)^b_a \iff R^a_b$$

Given a morphism $R\colon A \to B$, note that the following two morphisms are equal; this is extranaturality of ev.

$$A \otimes B^\vee \xrightarrow{R \otimes \mathrm{id}_{B^\vee}} B \otimes B^\vee \xrightarrow{\mathrm{ev}_B} \mathcal{I}$$

$$A \otimes B^\vee \xrightarrow{\mathrm{id}_A \otimes R^\vee} A \otimes A^\vee \xrightarrow{\mathrm{ev}_A} \mathcal{I}$$

$$
\begin{aligned}
((R \otimes \mathrm{id}_{B^\vee}) \bullet \mathrm{ev}_B)^{a\bar{b}}_\star &\iff \exists b' \bar{b}' ((R \otimes \mathrm{id}_{B^\vee})^{a\bar{b}}_{b'\bar{b}'} (\mathrm{ev}_B)^{b'\bar{b}'}_\star)\\
&\iff \exists b' \bar{b}' (R^a_{b'} (\mathrm{id}_{B^\vee})^{\bar{b}}_{\bar{b}'} (\mathrm{ev}_B)^{b'\bar{b}'}_\star)\\
&\iff \exists b' \bar{b}' (R^a_{b'} \delta^{\bar{b}}_{\bar{b}'} \delta^{b'}_{\bar{b}'})\\
&\iff R^a_{\bar{b}}
\end{aligned}
$$

$$\begin{aligned}((\mathrm{id}_A \otimes R^\vee) \bullet \mathrm{ev}_A)^{a\bar{b}}_\star &\iff \exists a'\bar{a}'((\mathrm{id}_A \otimes R^\vee)^{a\bar{b}}_{a'\bar{a}'}(\mathrm{ev}_A)^{a'\bar{a}'}_\star) \\ &\iff \exists a'\bar{a}'((\mathrm{id}_A)^a_{a'}(R^\vee)^{\bar{b}}_{\bar{a}'}(\mathrm{ev}_A)^{a'\bar{a}'}_\star) \\ &\iff \exists a'\bar{a}'(\delta^a_{a'}R^{\bar{a}'}_{\bar{b}}\delta^{a'}_{\bar{a}'}) \\ &\iff R^a_{\bar{b}}\end{aligned}$$

We will denote this morphism by $\mathrm{ev}_R\colon A \otimes B^\vee \to \mathcal{I}$.

Given a morphism $R\colon A \to B$, note that the following two morphisms are equal; this is extranaturality of coev.

$$\mathcal{I} \xrightarrow{\mathrm{coev}_A} A^\vee \otimes A \xrightarrow{\mathrm{id}_{A^\vee} \otimes R} A^\vee \otimes B$$

$$\mathcal{I} \xrightarrow{\mathrm{coev}_B} B^\vee \otimes B \xrightarrow{R^\vee \otimes \mathrm{id}_B} A^\vee \otimes B$$

$$\begin{aligned}(\mathrm{coev}_A \bullet (\mathrm{id}_{A^\vee} \otimes R))^\star_{\bar{a}b} &\iff \exists \bar{a}'a'((\mathrm{coev}_A)^\star_{\bar{a}'a'}(\mathrm{id}_{A^\vee} \otimes R)^{\bar{a}'a'}_{\bar{a}b}) \\ &\iff \exists \bar{a}'a'((\mathrm{coev}_A)^\star_{\bar{a}'a'}(\mathrm{id}_{A^\vee})^{\bar{a}'}_{\bar{a}}R^{a'}_b) \\ &\iff \exists \bar{a}'a'(\delta^{\bar{a}'}_{a'}\delta^{\bar{a}'}_{\bar{a}}R^{a'}_b) \\ &\iff R^{\bar{a}}_b\end{aligned}$$

$$\begin{aligned}(\mathrm{coev}_B \bullet (R^\vee \otimes \mathrm{id}_B))^\star_{\bar{a}b} &\iff \exists \bar{b}'b'((\mathrm{coev}_B)^\star_{\bar{b}'b'}(R^\vee \otimes \mathrm{id}_B)^{\bar{b}'b'}_{\bar{a}b}) \\ &\iff \exists \bar{b}'b'((\mathrm{coev}_B)^\star_{\bar{b}'b'}(R^\vee)^{\bar{b}'}_{\bar{a}}(\mathrm{id}_B)^{b'}_b) \\ &\iff \exists \bar{b}'b'(\delta^{\bar{b}'}_{b'}R^{\bar{a}}_{\bar{b}'}\delta^{b'}_b) \\ &\iff R^{\bar{a}}_b\end{aligned}$$

We will denote this morphism by $\mathrm{coev}_R\colon \mathcal{I} \to A^\vee \otimes B$.

Now, we will define, for each positive integer Φ, a strict monoidal endofunctor, denoted $\Phi^\star\colon [\mathbb{Z}, \textsc{Rel}] \to [\mathbb{Z}, \textsc{Rel}]$.

The action on objects is given by the following.

$$\Phi^\star(A) = A^\Phi = \overbrace{A \bullet \cdots \bullet A}^{\Phi}$$

The action on morphisms is given by $\Phi^\star(R) = R$; note that the underlying set of $\Phi^\star(A)$ is the same as the underlying set of A, so this makes sense. It can be easily seen that this defines a monoidal endofunctor.

Now, we shall describe, for each positive integer Φ, the right adjoint to $\Phi^\star$, denoted $\Phi_\star\colon [\mathbb{Z}, \textsc{Rel}] \to [\mathbb{Z}, \textsc{Rel}]$.

Given an automorphism $A\colon A_0 \dashrightarrow A_0$, the automorphism $\Phi_\star(A)\colon \underline{\Phi} \times A_0 \dashrightarrow \underline{\Phi} \times A_0$ is defined as follows.

$$\Phi_\star(A)^{ia}_{i'a'} \iff \begin{cases} A^a_{a'} & \text{if } i = 0 \text{ and } i' = \Phi - 1 \\ \delta^a_{a'} & \text{if } i = i' + 1 \\ 0 & \text{otherwise} \end{cases}$$

For example, $4_\star(A)$ can be represented by the following block matrix, where I represents the identity relation on a set of the same size as A_0 and 0 represents the zero relation on a set of the same size as A_0.

$$\begin{bmatrix} 0 & 0 & 0 & A \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}$$

Hopefully, the general pattern is clear.

Before going any further, we must prove some simple combinatorial properties involving $\Phi^\star$ and $\Phi_\star$.

Lemma 5.5.1. *Given an automorphism $A\colon A_0 \dashrightarrow A_0$, the following holds, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $(n)_\Phi$ denotes n modulo Φ,*

$$(\Phi_\star(A)^n)^{ia}_{i'a'} \iff \delta^i_{(i'+n)_\Phi}(A^{\lfloor \frac{i'+n}{\Phi} \rfloor})^a_{a'}$$

For example, the first few powers of $4_\star(A)$ can be represented by the following block matrices.

$$4_\star(A)^1 = \begin{bmatrix} 0 & 0 & 0 & A \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \qquad 4_\star(A)^2 = \begin{bmatrix} 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}$$

$$4_\star(A)^3 = \begin{bmatrix} 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \\ I & 0 & 0 & 0 \end{bmatrix} \qquad 4_\star(A)^4 = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{bmatrix}$$

$$4_\star(A)^5 = \begin{bmatrix} 0 & 0 & 0 & A^2 \\ A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \end{bmatrix} \qquad 4_\star(A)^6 = \begin{bmatrix} 0 & 0 & A^2 & 0 \\ 0 & 0 & 0 & A^2 \\ A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \end{bmatrix}$$

Hopefully, the general pattern is clear.

Proof. We will prove this by induction on n. The $\Phi_\star(A)^0$ case is clear. The $\Phi_\star(A)^1$ case follows from the definition. Assume the statement is true for

$\Phi_\star(A)^n$.

$$\begin{aligned}
(\Phi_\star(A)^{n+1})^{ia}_{i'a'} &\iff (\Phi_\star(A) \bullet \Phi_\star(A)^n)^{ia}_{i'a'} \\
&\iff \exists i''a''(\Phi_\star(A)^{ia}_{i''a''}(\Phi_\star(A)^n)^{i''a''}_{i'a'}) \\
&\iff \exists i''a''(\delta^i_{(i''+1)_\Phi}(A^{\lfloor\frac{i''+1}{\Phi}\rfloor})^a_{a''}\delta^{i''}_{(i'+n)_\Phi}(A^{\lfloor\frac{i'+n}{\Phi}\rfloor})^{a''}_{a'}) \\
&\iff \delta^i_{(i'+n+1)_\Phi}\exists a''((A^{\lfloor\frac{(i'+n)_\Phi+1}{\Phi}\rfloor})^a_{a''}(A^{\lfloor\frac{i'+n}{\Phi}\rfloor})^{a''}_{a'}) \\
&\iff \delta^i_{(i'+n+1)_\Phi}(A^{\lfloor\frac{(i'+n)_\Phi+1}{\Phi}\rfloor} \bullet A^{\lfloor\frac{i'+n}{\Phi}\rfloor})^a_{a'} \\
&\overset{(\star)}{\iff} \delta^i_{(i'+n+1)_\Phi}(A^{\lfloor\frac{i'+n+1}{\Phi}\rfloor})^a_{a'}
\end{aligned}$$

Step $(\star)$ follows from the following general equality.

$$\left\lfloor\frac{(k)_\Phi+1}{\Phi}\right\rfloor + \left\lfloor\frac{k}{\Phi}\right\rfloor = \left\lfloor\frac{k+1}{\Phi}\right\rfloor$$

□

Corollary 5.5.2. *Given an automorphism $A\colon A_0 \dashrightarrow A_0$, the following holds.*

$$\Gamma^\star\Phi^\star\Phi_\star(A)^{ia}_{i'a'} \iff \delta^i_{i'}\Gamma^\star(A)^a_{a'}$$

Corollary 5.5.3. *Given an automorphism $A\colon A_0 \dashrightarrow A_0$, the following holds.*

$$\forall j \in \underline{\Phi} \qquad (\Phi_\star(A)^j)^{ia}_{0a'} \iff \delta^i_j\delta^a_{a'}$$

Given a morphism $R\colon A \to B$, the morphism $\Phi_\star(R)\colon \Phi_\star(A) \to \Phi_\star(B)$ is the relation $\Phi_\star(R)\colon \underline{\Phi} \times A_0 \dashrightarrow \underline{\Phi} \times B_0$ defined as follows.

$$\Phi_\star(R)^{ia}_{jb} = \delta^i_j R^a_b$$

For example, $4_\star(R)$ can be represented by the following block matrix.

$$\begin{bmatrix} R & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{bmatrix}$$

Hopefully, the general pattern is clear. To see that $\Phi_\star(R)$ is a morphism

$\Phi_\star(A) \to \Phi_\star(B)$, note the following.

$$\begin{aligned}
(\Phi_\star(A) \bullet \Phi_\star(R))^{ia}_{jb} &\iff \exists i'a'(\Phi_\star(A)^{ia}_{i'a'}\Phi_\star(R)^{i'a'}_{jb}) \\
&\overset{5.5.1}{\iff} \exists i'a'(\delta^i_{(i'+1)_\Phi}(A^{\lfloor\frac{i'+1}{\Phi}\rfloor})^a_{a'}\delta^{i'}_j R^{a'}_b) \\
&\iff \delta^i_{(j+1)_\Phi}\exists a'((A^{\lfloor\frac{j+1}{\Phi}\rfloor})^a_{a'}R^{a'}_b) \\
&\iff \delta^i_{(j+1)_\Phi}(A^{\lfloor\frac{j+1}{\Phi}\rfloor} \bullet R)^a_b \\
&\iff \delta^i_{(j+1)_\Phi}(R \bullet B^{\lfloor\frac{j+1}{\Phi}\rfloor})^a_b \\
&\iff \delta^i_{(j+1)_\Phi}\exists b'(R^a_{b'}(B^{\lfloor\frac{j+1}{\Phi}\rfloor})^{b'}_b) \\
&\iff \exists j'b'(\delta^i_{j'}R^a_{b'}\delta^{j'}_{(j+1)_\Phi}(B^{\lfloor\frac{j+1}{\Phi}\rfloor})^{b'}_b) \\
&\overset{5.5.1}{\iff} \exists j'b'(\Phi_\star(R)^{ia}_{j'b'}\Phi_\star(B)^{j'b'}_{jb}) \\
&\iff (\Phi_\star(R) \bullet \Phi_\star(B))^{ia}_{jb}
\end{aligned}$$

For example, the previous calculation in matrix form, for $4_\star$, is the following.

$$\begin{bmatrix} 0 & 0 & 0 & A \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}\begin{bmatrix} R & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & AR \\ R & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & R & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & RB \\ R & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & R & 0 \end{bmatrix} = \begin{bmatrix} R & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & A \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}$$

To see that $\Phi_\star$ preserves composition, note the following.

$$\begin{aligned}
(\Phi_\star(R) \bullet \Phi_\star(S))^{ia}_{kc} &\iff \exists jb(\Phi_\star(R)^{ia}_{jb}\Phi_\star(S)^{jb}_{kc}) \\
&\iff \exists jb(\delta^i_j R^a_b \delta^j_k S^b_c) \\
&\iff \delta^i_k \exists b(R^a_b S^b_c) \\
&\iff \delta^i_k (R \bullet S)^a_c \\
&\iff \Phi_\star(R \bullet S)^{ia}_{kc}
\end{aligned}$$

To see that $\Phi_\star$ preserves identities, note the following.

$$\begin{aligned}
\Phi_\star(\mathrm{id}_A)^{ia}_{i'a'} &\iff \delta^i_{i'}(\mathrm{id}_A)^a_{a'} \\
&\iff \delta^i_{i'}\delta^a_{a'} \\
&\iff (\mathrm{id}_{\Phi_\star(A)})^{ia}_{i'a'}
\end{aligned}$$

The unit θ^Φ for the adjunction $\Phi^\star \dashv \Phi_\star$ is defined as follows. Given an automorphism $A\colon A_0 \dashrightarrow A_0$, the morphism

$$\theta^\Phi_A\colon A \to \Phi_\star\Phi^\star(A)$$

is the relation $A_0 \dashrightarrow \underline{\Phi} \times A_0$ defined as follows.

$$(\theta_A^\Phi)^a_{i'a'} \iff (A^{i'})^a_{a'}$$

For example, θ_A^4 can be represented by the following block matrix.

$$\begin{bmatrix} I & A & A^2 & A^3 \end{bmatrix}$$

Hopefully, the general pattern is clear. We will often omit the superscript, when it is clear from context. To see that θ_A^Φ is a morphism $A \to \Phi_\star\Phi^\star(A)$, note the following.

$$\begin{aligned}
(A \bullet \theta_A)^a_{i'a'} &\iff \exists a''(A^a_{a''}(\theta_A)^{a''}_{i'a'}) \\
&\iff \exists a''(A^a_{a''}(A^{i'})^{a''}_{a'}) \\
&\iff (A \bullet A^{i'})^a_{a'} \\
&\iff (A^{i'+1})^a_{a'} \\
&\iff (A^{(i'+1)_\Phi + \Phi\cdot\lfloor\frac{i'+1}{\Phi}\rfloor})^a_{a'} \\
&\iff (A^{(i'+1)_\Phi} \bullet A^{\Phi\cdot\lfloor\frac{i'+1}{\Phi}\rfloor})^a_{a'} \\
&\iff (A^{(i'+1)_\Phi} \bullet \Phi^\star(A)^{\lfloor\frac{i'+1}{\Phi}\rfloor})^a_{a'} \\
&\iff \exists a''((A^{(i'+1)_\Phi})^a_{a''}(\Phi^\star(A)^{\lfloor\frac{i'+1}{\Phi}\rfloor})^{a''}_{a'}) \\
&\iff \exists i''a''((A^{i''})^a_{a''}\delta^{i''}_{(i'+1)_\Phi}(\Phi^\star(A)^{\lfloor\frac{i'+1}{\Phi}\rfloor})^{a''}_{a'}) \\
&\overset{5.5.1}{\iff} \exists i''a''((\theta_A)^a_{i''a''}\Phi_\star\Phi^\star(A)^{i''a''}_{i'a'}) \\
&\iff (\theta_A \bullet \Phi_\star\Phi^\star(A))^a_{i'a'}
\end{aligned}$$

For example, the previous calculation in matrix form, for $\Phi = 4$, is the following.

$$A\begin{bmatrix} I & A & A^2 & A^3 \end{bmatrix} = \begin{bmatrix} A & A^2 & A^3 & A^4 \end{bmatrix} = \begin{bmatrix} I & A & A^2 & A^3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & A^4 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}$$

To see that θ is a natural transformation, consider a morphism $R\colon A \to B$, and note the following.

$$\begin{aligned}
(\theta_A \bullet \Phi_\star\Phi^\star(R))^a_{ib} &\iff \exists i'a'((\theta_A)^a_{i'a'}\Phi_\star\Phi^\star(R)^{i'a'}_{ib}) \\
&\iff \exists i'a'((A^{i'})^a_{a'}\delta^{i'}_i R^{a'}_b) \\
&\iff \exists a'((A^i)^a_{a'}R^{a'}_b) \\
&\iff (A^i \bullet R)^a_b \\
&\iff (R \bullet B^i)^a_b \\
&\iff \exists b'(R^a_{b'}(B^i)^{b'}_b) \\
&\iff \exists b'(R^a_{b'}(\theta_B)^{b'}_{ib}) \\
&\iff (R \bullet \theta_B)^a_{ib}
\end{aligned}$$

For example, the previous calculation in matrix form, for $\Phi = 4$, is the following.

$$\begin{bmatrix} I & A & A^2 & A^3 \end{bmatrix} \begin{bmatrix} R & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{bmatrix} = \begin{bmatrix} R & AR & A^2R & A^3R \end{bmatrix}$$

$$= \begin{bmatrix} R & RB & RB^2 & RB^3 \end{bmatrix} = R\begin{bmatrix} I & B & B^2 & B^3 \end{bmatrix}$$

We will denote this morphism by $\theta_R \colon A \to \Phi_\star\Phi^\star(B)$.

The counit ζ^Φ for the adjunction $\Phi^\star \dashv \Phi_\star$ is defined as follows. Given an automorphism $A \colon A_0 \dashrightarrow A_0$, the morphism

$$\zeta^\Phi_A \colon \Phi^\star\Phi_\star(A) \to A$$

is the relation $\underline{\Phi} \times A_0 \dashrightarrow A_0$ defined as follows.

$$(\zeta^\Phi_A)^{ia}_{a'} \iff \delta^i_0\delta^a_{a'}$$

For example, ζ^4_A can be represented by the following block matrix.

$$\begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hopefully, the general pattern is clear. We will often omit the superscript, when it is clear from context. To see that ζ_A is a morphism $\Phi^\star\Phi_\star(A) \to A$, note the following.

$$\begin{aligned}
(\Phi^\star\Phi_\star(A) \bullet \zeta_A)^{ia}_{a'} &\iff \exists i''a''(\Phi^\star\Phi_\star(A)^{ia}_{i''a''}(\zeta_A)^{i''a''}_{a'}) \\
&\overset{5.5.2}{\iff} \exists i''a''(\delta^i_{i''}A^a_{a''}\delta^{i''}_0\delta^{a''}_{a'}) \\
&\iff \delta^i_0 A^a_{a'} \\
&\iff \delta^i_0 \exists a''(\delta^a_{a''}A^{a''}_{a'}) \\
&\iff \exists a''((\zeta_A)^{ia}_{a''}A^{a''}_{a'}) \\
&\iff (\zeta_A \bullet A)^{ia}_{a'}
\end{aligned}$$

For example, the previous calculation in matrix form, for $\Phi = 4$, is the following.

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{bmatrix} \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} A$$

To see that ζ is a natural transformation, consider a morphism $R \colon A \to B$, and note the following.

$$\begin{aligned}
(\zeta_A \bullet R)^{ia}_b &\iff \exists a'((\zeta_A)^{ia}_{a'}R^{a'}_b) \\
&\iff \exists a'(\delta^i_0\delta^a_{a'}R^{a'}_b) \\
&\iff \delta^i_0 R^a_b \\
&\iff \exists i'b'(\delta^i_{i'}R^a_{b'}\delta^{i'}_0\delta^{b'}_b) \\
&\iff \exists i'b'(\Phi^\star\Phi_\star(R)^{ia}_{i'b'}(\zeta_B)^{i'b'}_b) \\
&\iff (\Phi^\star\Phi_\star(R) \bullet \zeta_B)^{ia}_b
\end{aligned}$$

For example, the previous calculation in matrix form, for $\Phi = 4$, is the following.

$$\begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} R = \begin{bmatrix} R \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R & 0 & 0 & 0 \\ 0 & R & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{bmatrix} \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We will denote this morphism by $\zeta_R \colon \Phi^\star\Phi_\star(A) \to B$.

The first triangle identity is that the following morphism is the identity on $\Phi^\star(A)$.

$$\Phi^\star(A) \xrightarrow{\Phi^\star(\theta_A)} \Phi^\star\Phi_\star\Phi^\star(A) \xrightarrow{\zeta_{\Phi^\star(A)}} \Phi^\star(A)$$

To see this, note the following.

$$\begin{aligned}
(\Phi^\star(\theta_A) \bullet \zeta_{\Phi^\star(A)})^a_{a'} &\iff \exists i''a''(\Phi^\star(\theta_A)^a_{i''a''}(\zeta_{\Phi^\star(A)})^{i''a''}_{a'}) \\
&\iff \exists i''a''((A^{i''})^a_{a''}\delta^{i''}_0\delta^{a''}_{a'}) \\
&\iff (A^0)^a_{a'} \\
&\iff \delta^a_{a'} \\
&\iff (\mathrm{id}_{\Phi^\star(A)})^a_{a'}
\end{aligned}$$

For example, the previous calculation in matrix form, for $\Phi = 4$, is the following.

$$\begin{bmatrix} I & A & A^2 & A^3 \end{bmatrix} \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} = I$$

The second triangle identity is that the following morphism is the identity on $\Phi_\star(A)$.

$$\Phi_\star(A) \xrightarrow{\theta_{\Phi_\star(A)}} \Phi_\star\Phi^\star\Phi_\star(A) \xrightarrow{\Phi_\star(\zeta_A)} \Phi_\star(A)$$

To see this, note the following.

$$\begin{aligned}
(\theta_{\Phi_\star(A)} \bullet \Phi_\star(\zeta_A))^{ja}_{i'a'} &\iff \exists i''j''a''((\theta_{\Phi_\star(A)})^{ja}_{i''j''a''}\Phi_\star(\zeta_A)^{i''j''a''}_{i'a'}) \\
&\iff \exists i''j''a''((\Phi_\star(A)^{i''})^{ja}_{j''a''}\delta^{i''}_{i'}\delta^{j''}_0\delta^{a''}_{a'}) \\
&\iff (\Phi_\star(A)^{i'})^{ja}_{0a'} \\
&\overset{5.5.3}{\iff} \delta^j_{i'}\delta^a_{a'} \\
&\iff (\mathrm{id}_{\Phi_\star(A)})^{ja}_{i'a'}
\end{aligned}$$

For example, the previous calculation in matrix form, for $\Phi = 3$, is the following.

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 & A & 0 & A & 0 \\ 0 & I & 0 & I & 0 & 0 & 0 & 0 & A \\ 0 & 0 & I & 0 & I & 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

5.5.3 The Ω Functors

We can now define a particularly simple object of $\textsc{Gr}_{\mathfrak{G}}$, which we denote $\mathbb{Z}\textsc{Rel}$, as follows.

- For each vertex $\mathcal{C}$ of $\mathfrak{G}$, the closed monoidal category $\mathbb{Z}\textsc{Rel}(\mathcal{C})$ is $[\mathbb{Z}, \textsc{Rel}]$.
- For each edge $\Phi \colon \mathcal{C} \to \mathcal{D}$ of $\mathfrak{G}$, the monoidal adjunction $\Phi^\star \dashv \Phi_\star$ is any of the previously described adjunctions of the form $k^\star \dashv k_\star$ for any integer $k \geq 2$. The choice of k is largely unimportant. A different k may be used for each edge, or they may all be the same.

Given an object G of $\textsc{Set}^{|\mathfrak{G}_0|}$, we have a natural isomorphism of hom-sets of the following form.

$$\textsc{Gr}_{\mathfrak{G}}(\textsc{Shp}_G, \mathbb{Z}\textsc{Rel}) \cong \textsc{Set}^{|\mathfrak{G}_0|}(G, U_{\mathbb{Z}\textsc{Rel}})$$

We can construct a morphism $G \to U_{\mathbb{Z}\textsc{Rel}}$ in $\textsc{Set}^{|\mathfrak{G}_0|}$ as follows.

- For each vertex $\mathcal{C}$ of $\mathfrak{G}$, the component

$$G(\mathcal{C}) \to U_{\mathbb{Z}\textsc{Rel}}(\mathcal{C}) = \mathrm{ob}([\mathbb{Z}, \textsc{Rel}])$$

  is any function whose image consists only of automorphisms on sets with at least 2 elements. The choice of sets and automorphisms is largely unimportant. A different set and automorphism may be used for each element of $G(\mathcal{C})$, or they may all be the same.

This then induces a morphism $\textsc{Shp}_G \to \mathbb{Z}\textsc{Rel}$ in $\textsc{Gr}_{\mathfrak{G}}$, which we denote Ω. For each vertex $\mathcal{C}$ of $\mathfrak{G}$, the component $\Omega_{\mathcal{C}}$ is a functor of the following form.

$$\Omega_{\mathcal{C}} \colon \textsc{Shp}_G(\mathcal{C}) \to \mathbb{Z}\textsc{Rel}(\mathcal{C}) = [\mathbb{Z}, \textsc{Rel}]$$

We will often omit the subscript, when it is clear from context.

5.6 The Coherence Theorem

In this section, we will prove the main result of this chapter: that the functors

$$\Omega_{\mathcal{C}} \colon \textsc{Shp}_G(\mathcal{C}) \to \mathbb{Z}\textsc{Rel}(\mathcal{C}) = [\mathbb{Z}, \textsc{Rel}]$$

are all faithful.

Before we can prove the coherence theorem, we must first prove four technical ‘recognition lemmas’, about recognising the form of a constructible morphism based on the form of its image under Ω. We will only state the four recognition lemmas here; proofs can be found in Appendix A.

Lemma 5.6.1. *Let $\mathcal{C}$ be a vertex of $\mathfrak{G}$. Let*

$$s \colon P \otimes R \to Q \otimes S$$

be a constructible $\mathcal{C}$-morphism. If there are morphisms

$$\sigma\colon \Omega_{\mathcal{C}}(P) \to \Omega_{\mathcal{C}}(Q) \qquad \tau\colon \Omega_{\mathcal{C}}(R) \to \Omega_{\mathcal{C}}(S)$$

such that $\Omega_{\mathcal{C}}(s)$ is of the form $\sigma \otimes \tau$, then there are constructible $\mathcal{C}$-morphisms

$$u\colon P \to Q \qquad v\colon R \to S$$

such that s is of the form $u \otimes v$.

Lemma 5.6.2. *Let $\Delta\colon \mathcal{C} \to \mathcal{D}$ be a path in $\mathfrak{G}$. Let*

$$s\colon P \otimes \Delta^\star(Q \otimes (R \backslash S)) \otimes T \to U$$

be a constructible $\mathcal{C}$-morphism. If there are morphisms

$$\sigma\colon \Omega_{\mathcal{D}}(Q) \to \Omega_{\mathcal{D}}(R) \qquad \tau\colon \Omega_{\mathcal{C}}(P \otimes \Delta^\star(S) \otimes T) \to \Omega_{\mathcal{C}}(U)$$

such that $\Omega_{\mathcal{C}}(s)$ is of the form $\langle \Delta, \sigma, \tau \rangle_\varepsilon$, then there is a constructible $\mathcal{D}$-morphism

$$u\colon Q \to R$$

and a constructible $\mathcal{C}$-morphism

$$v\colon P \otimes \Delta^\star(S) \otimes T \to U$$

such that s is of the form $\langle \Delta, u, v \rangle_\varepsilon$.

Lemma 5.6.3. *Let $\Delta\colon \mathcal{C} \to \mathcal{D}$ be a path in $\mathfrak{G}$. Let*

$$s\colon \Delta^\star(P) \to \Delta^\star(Q)$$

be a constructible $\mathcal{C}$-morphism. If there is a morphism

$$\sigma\colon \Omega_{\mathcal{D}}(P) \to \Omega_{\mathcal{D}}(Q)$$

such that $\Omega_{\mathcal{C}}(s)$ is of the form $\Delta^\star(\sigma)$, then there is a constructible $\mathcal{D}$-morphism

$$u\colon P \to Q$$

such that s is of the form $\Delta^\star(u)$.

Lemma 5.6.4. *Let $\Delta\colon \mathcal{C} \to \mathcal{D}$ and $\Phi\colon \mathcal{D} \to \mathcal{E}$ be paths in $\mathfrak{G}$. Let*

$$s\colon P \otimes \Delta^\star\Phi^\star\Phi_\star(Q) \otimes R \to S$$

be a constructible $\mathcal{C}$-morphism. If there is a morphism

$$\sigma\colon \Omega_{\mathcal{C}}(P \otimes \Delta^\star(Q) \otimes R) \to \Omega_{\mathcal{C}}(S)$$

such that $\Omega_{\mathcal{C}}(s)$ is of the form $\langle \Delta, \Phi, \sigma \rangle_\zeta$, then there is a constructible $\mathcal{C}$-morphism

$$u\colon P \otimes \Delta^\star(Q) \otimes R \to S$$

such that s is of the form $\langle \Delta, \Phi, u \rangle_\zeta$.

We will now prove our main theorem.

Theorem 5.6.5. *Let $\mathcal{C}$ be a vertex of $\mathfrak{G}$. Let $s, t\colon P \to Q$ be a pair of parallel constructible $\mathcal{C}$-morphisms. If $\Omega_{\mathcal{C}}(s) = \Omega_{\mathcal{C}}(t)$, then $s = t$.*

Proof. We will prove this by induction on the types of the constructible morphisms s and t.

- Consider the case where both s and t are of type ($\cong$). In this case, $s = t$, by Theorem 5.3.9.

- Consider the case where at least one of s or t, say s, is of type ($\otimes$).

$$s\colon P \cong A \otimes C \xrightarrow{f \otimes g} B \otimes D \cong Q$$

  Consider the following constructible morphism.

$$t'\colon A \otimes C \cong P \xrightarrow{t} Q \cong B \otimes D$$

  By Lemma 5.6.1, there are constructible morphisms

$$f'\colon A \to B \qquad g'\colon C \to D$$

  such that t' is of the form $f' \otimes g'$. By induction, $f = f'$ and $g = g'$. It follows that $s = t$.

- Consider the case where at least one of s or t, say s, is of type (η).

$$s\colon P \xrightarrow{\langle I, f\rangle_\eta} I\backslash A \cong Q$$

  Consider the following constructible morphism.

$$t'\colon P \xrightarrow{t} Q \cong I\backslash A$$

  Its adjunct under the adjunction $(I \otimes -) \dashv (I\backslash -)$ is a constructible morphism

$$f'\colon I \otimes P \to A$$

  such that t' is of the form $\langle A, f'\rangle_\eta$. By induction, $f = f'$. It follows that $s = t$.

- Consider the case where at least one of s or t, say s, is of type (ε).

$$s\colon P \cong A \otimes \Delta^\star(B \otimes (C\backslash D)) \otimes E \xrightarrow{\langle \Delta, f, g\rangle_\varepsilon} Q$$

  Consider the following constructible morphism.

$$t'\colon A \otimes \Delta^\star(B \otimes (C\backslash D)) \otimes E \cong P \xrightarrow{t} Q$$

  By Lemma 5.6.2, there are constructible morphisms

$$f'\colon B \to C \qquad g'\colon A \otimes \Delta^\star(D) \otimes E \to Q.$$

  such that t' is of the form $\langle \Delta, f', g'\rangle_\varepsilon$. By induction, $f = f'$ and $g = g'$. It follows that $s = t$.

- Consider the case where at least one of s or t, say s, is of type $((-)^\star)$.

  $$s\colon P \cong \Gamma^\star(A) \xrightarrow{\Gamma^\star(f)} \Gamma^\star(B) \cong Q$$

  Consider the following constructible morphism.

  $$t'\colon \Gamma^\star(A) \cong P \xrightarrow{t} Q \cong \Gamma^\star(B)$$

  By Lemma 5.6.3, there is a constructible morphism

  $$f'\colon A \to B$$

  such that t' is of the form $\Gamma^\star(f')$. By induction, $f = f'$. It follows that $s = t$.

- Consider the case where at least one of s or t, say s, is of type (θ).

  $$s\colon P \xrightarrow{\langle \Phi, f\rangle_\theta} \Phi_\star(A) \cong Q$$

  Consider the following constructible morphism.

  $$t'\colon P \xrightarrow{t} Q \cong \Phi_\star(A)$$

  Its adjunct under the adjunction $\Phi^\star \dashv \Phi_\star$ is a constructible morphism

  $$f'\colon \Phi^\star(P) \to A$$

  such that t' is of the form $\langle \Phi, f'\rangle_\theta$. By induction, $f = f'$. It follows that $s = t$.

- Consider the case where at least one of s or t, say s, is of type (ζ).

  $$s\colon P \cong A \otimes \Delta^\star\Phi^\star\Phi_\star(B) \otimes C \xrightarrow{\langle \Delta, \Phi, f\rangle_\zeta} Q$$

  Consider the following constructible morphism.

  $$t'\colon A \otimes \Delta^\star\Phi^\star\Phi_\star(B) \otimes C \cong P \xrightarrow{t} Q$$

  By Lemma 5.6.4, there is a constructible morphism

  $$f'\colon A \otimes \Delta^\star(B) \otimes C \to Q$$

  such that t' is of the form $\langle \Delta, \Phi, f'\rangle_\zeta$. By induction, $f = f'$. It follows that $s = t$.

□

5.7 Examples

We will end with some examples, illustrating the use of this coherence theorem.

5.7.1 Some Pentagons

Let $\mathcal{C}$ be a closed monoidal category. We have the natural transformation α.

$$\alpha_{A,B,C} \colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C$$

We can also define two other natural transformations ν and μ, denoted as follows.

$$\nu_{A,B,C} \colon (A\backslash B) \otimes C \to A\backslash(B \otimes C)$$

$$\mu_{A,B,C} \colon (B \otimes A)\backslash C \to A\backslash(B\backslash C)$$

To define the natural transformation ν, consider the associator α as a natural transformation of the following form.

$$A^\star \circ C_\star \Rightarrow C_\star \circ A^\star$$

Using the adjunction $A^\star \dashv A^!$, this natural transformation has a mate of the following form.

$$C_\star \circ A^! \Rightarrow A^! \circ C_\star$$

The component of this natural transformation at the object B is the morphism $\nu_{A,B,C}$. In the language of constructible morphisms, $\nu_{A,B,C}$ is defined as follows.

$$\langle A, \langle \mathrm{id}, A, B\rangle_\varepsilon \otimes C\rangle_\eta \colon (A\backslash B) \otimes C \to A\backslash(B \otimes C)$$

To define the natural transformation μ, consider the associator α as a natural transformation of the following form.

$$B^\star \circ A^\star \Rightarrow (B \otimes A)^\star$$

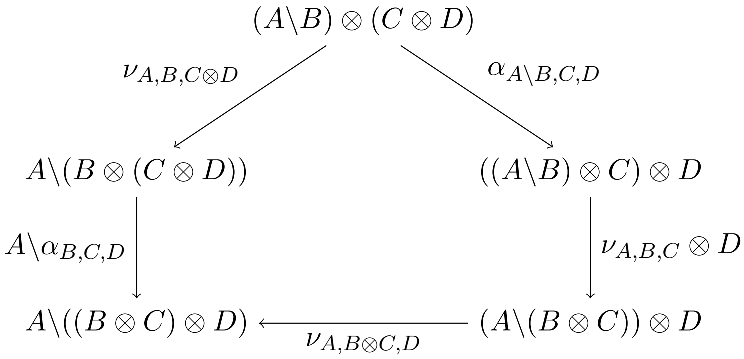
Using the adjunctions $A^\star \dashv A^!$, $B^\star \dashv B^!$ and $(B \otimes A)^\star \dashv (B \otimes A)^!$, this natural transformation has a mate of the following form.

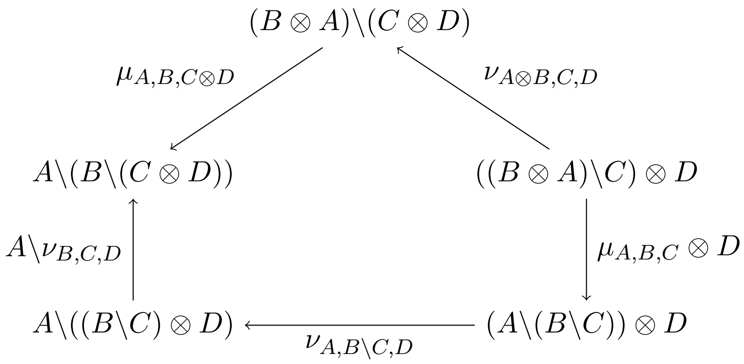
$$(B \otimes A)^! \Rightarrow A^! \circ B^!$$

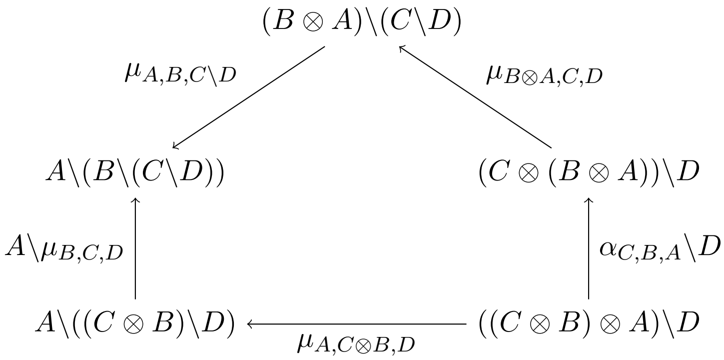
The component of this natural transformation at the object C is the morphism $\mu_{A,B,C}$. In the language of constructible morphisms, $\mu_{A,B,C}$ is defined as follows.

$$\langle A, \langle B, \langle \mathrm{id}, B \otimes A, C\rangle_\varepsilon\rangle_\eta\rangle_\eta \colon (B \otimes A)\backslash C \to A\backslash(B\backslash C)$$

As a consequence of the strictness of $\mathbb{Z}\textsc{Rel}$, the image under Ω of each of α, ν and μ is an identity. Therefore, any diagram constructed from these morphisms commutes. As examples of such diagrams, consider the following pentagon diagrams, each similar in form to 2.1.







5.7.2 Some Hexagons

Let $f^\star\colon \mathcal{D}\to\mathcal{C}$ be a strong monoidal functor between closed monoidal categories. Let $f_\star\colon \mathcal{C}\to\mathcal{D}$ be a right adjoint to $f^\star$. We have the natural transformations α, ν, μ and φ.

$$\alpha_{A,B,C}\colon A\otimes(B\otimes C)\to(A\otimes B)\otimes C$$

$$\nu_{A,B,C}\colon (A\backslash B)\otimes C\to A\backslash(B\otimes C)$$

$$\mu_{A,B,C}\colon (B\otimes A)\backslash C \to A\backslash(B\backslash C)$$

$$\varphi_{A,B}\colon f^\star(A\otimes B)\to f^\star(A)\otimes f^\star(B)$$

We can also define two other natural transformations ψ and χ, denoted as follows.

$$\psi_{A,B}\colon f_\star(A)\otimes f_\star(B)\to f_\star(A\otimes B)$$

$$\chi_{A,B}\colon f_\star(A\backslash B)\to f_\star(A)\backslash f_\star(B)$$

The natural transformation ψ has components defined as follows.

$$\begin{aligned}\psi_{A,B}\colon f_\star(A)\otimes f_\star(B) &\xrightarrow{\theta_{f_\star(A)\otimes f_\star(B)}} f_\star f^\star(f_\star(A)\otimes f_\star(B))\\ &\xrightarrow{f_\star(\varphi_{f_\star(A),f_\star(B)})} f_\star(f^\star f_\star(A)\otimes f^\star f_\star(B))\\ &\xrightarrow{f_\star(\zeta_A\otimes\zeta_B)} f_\star(A\otimes B)\end{aligned}$$

In the language of constructible morphisms, $\psi_{A,B}$ is defined as follows.

$$\psi_{A,B}=\langle f,\langle \mathrm{id},f,\langle \mathrm{id},f,\mathrm{id}_{A\otimes B}\rangle_\zeta\rangle_\zeta\rangle_\theta\colon f_\star(A)\otimes f_\star(B)\to f_\star(A\otimes B)$$

Evaluating this yields the following.

$$(\psi_{A,B})^{iajb}_{za'b'} \iff \delta^i_z\delta^j_z\delta^a_{a'}\delta^b_{b'}$$

The natural transformation χ has components defined as follows.

$$\begin{aligned}f_\star(A\backslash B) &\xrightarrow{\eta^{f_\star(A)}_{f_\star(A\backslash B)}} f_\star(A)\backslash(f_\star(A)\otimes f_\star(A\backslash B))\\ &\xrightarrow{f_\star(A)\backslash\psi_{A,A\backslash B}} f_\star(A)\backslash f_\star(A\otimes(A\backslash B))\\ &\xrightarrow{f_\star(A)\backslash f_\star(\varepsilon^A_B)} f_\star(A)\backslash f_\star(B)\end{aligned}$$

In the language of constructible morphisms, $\chi_{A,B}$ is defined as follows.

$$\langle f_\star(A),\langle f,\langle \mathrm{id},f,\langle \mathrm{id},f,\langle \mathrm{id},\mathrm{id}_A,\mathrm{id}_B\rangle_\varepsilon\rangle_\zeta\rangle_\zeta\rangle_\theta\rangle_\eta\colon f_\star(A\backslash B)\to f_\star(A)\backslash f_\star(B)$$

Evaluating this yields the following.

$$(\chi_{A,B})^{zab}_{i'a'j'b'} \iff \delta^{i'}_{j'}\delta^z_{j'}\delta^{a'}_a\delta^b_{b'}$$

We can use this to show that the following hexagon diagrams, similar in form to 2.8, commute.

Consider the following diagram, involving α.

$$
\begin{array}{ccc}
f_\star(A) \otimes (f_\star(B) \otimes f_\star(C)) & \xrightarrow{\alpha_{f_\star(A),f_\star(B),f_\star(C)}} & (f_\star(A) \otimes f_\star(B)) \otimes f_\star(C) \\
\Big\downarrow {\scriptstyle \mathrm{id}_{f_\star(A)} \otimes \psi_{B,C}} & & \Big\downarrow {\scriptstyle \psi_{A,B} \otimes \mathrm{id}_{f_\star(C)}} \\
f_\star(A) \otimes f_\star(B \otimes C) & & f_\star(A \otimes B) \otimes f_\star(C) \\
\Big\downarrow {\scriptstyle \psi_{A,B\otimes C}} & & \Big\downarrow {\scriptstyle \psi_{A\otimes B,C}} \\
f_\star(A \otimes (B \otimes C)) & \xrightarrow[f_\star(\alpha_{A,B,C})]{} & f_\star((A \otimes B) \otimes C)
\end{array}
$$

The clockwise path around this diagram evaluates as follows (ignoring α, since it becomes an identity).

$$
\begin{aligned}
&((\psi_{A,B} \otimes \mathrm{id}_{f_\star(C)}) \bullet \psi_{A\otimes B,C})^{iajbkc}_{z'a'b'c'} \\
&\iff \exists z''a''b''k''c''((\psi_{A,B})^{iajb}_{z''a''b''}(\mathrm{id}_{f_\star(C)})^{kc}_{k''c''}(\psi_{A\otimes B,C})^{z''a''b''k''c''}_{z'a'b'c'}) \\
&\iff \exists z''a''b''k''c''(\delta^i_{z''}\delta^j_{z''}\delta^a_{a''}\delta^b_{b''}\delta^k_{k''}\delta^c_{c''}\delta^{z''}_{z'}\delta^{k''}_{z'}\delta^{a''}_{a'}\delta^{b''}_{b'}\delta^{c''}_{c'}) \\
&\iff \delta^i_{z'}\delta^j_{z'}\delta^k_{z'}\delta^a_{a'}\delta^b_{b'}\delta^c_{c'}
\end{aligned}
$$

The anticlockwise path around this diagram evaluates as follows (ignoring α, since it becomes an identity).

$$
\begin{aligned}
&((\mathrm{id}_{f_\star(A)} \otimes \psi_{B,C}) \bullet \psi_{A,B\otimes C})^{iajbkc}_{z'a'b'c'} \\
&\iff \exists i''a''z''b''c''((\mathrm{id}_{f_\star(A)})^{ia}_{i''a''}(\psi_{B,C})^{jbkc}_{z''b''c''}(\psi_{A,B\otimes C})^{i''a''z''b''c''}_{z'a'b'c'}) \\
&\iff \exists i''a''z''b''c''(\delta^i_{i''}\delta^a_{a''}\delta^j_{z''}\delta^k_{z''}\delta^b_{b''}\delta^c_{c''}\delta^{i''}_{z'}\delta^{z''}_{z'}\delta^{a''}_{a'}\delta^{b''}_{b'}\delta^{c''}_{c'}) \\
&\iff \delta^i_{z'}\delta^j_{z'}\delta^k_{z'}\delta^a_{a'}\delta^b_{b'}\delta^c_{c'}
\end{aligned}
$$

Since these are equal, the diagram commutes.

Consider the following diagram, involving ν.

$$
\begin{array}{ccc}
f_\star(A) \backslash (f_\star(B) \otimes f_\star(C)) & \xleftarrow{\nu_{f_\star(A),f_\star(B),f_\star(C)}} & (f_\star(A) \backslash f_\star(B)) \otimes f_\star(C) \\
\Big\downarrow {\scriptstyle \mathrm{id}_{f_\star(A)} \backslash \psi_{B,C}} & & \Big\uparrow {\scriptstyle \chi_{A,B} \otimes \mathrm{id}_{f_\star(C)}} \\
f_\star(A) \backslash f_\star(B \otimes C) & & f_\star(A \backslash B) \otimes f_\star(C) \\
\Big\uparrow {\scriptstyle \chi_{A,B\otimes C}} & & \Big\downarrow {\scriptstyle \psi_{A\backslash B,C}} \\
f_\star(A \backslash (B \otimes C)) & \xleftarrow[f_\star(\nu_{A,B,C})]{} & f_\star((A \backslash B) \otimes C)
\end{array}
$$

The clockwise path around this diagram evaluates as follows (ignoring ν, since it becomes an identity).

$$
\begin{aligned}
&(\psi_{A\backslash B,C} \bullet \chi_{A,B\otimes C})^{zabkc}_{i'a'z'b'c'} \\
&\iff \exists z''a''b''c''((\psi_{A\backslash B,C})^{zabkc}_{z''a''b''c''}(\chi_{A,B\otimes C})^{z''a''b''c''}_{i'a'z'b'c'}) \\
&\iff \exists z''a''b''c''(\delta^{z}_{z''}\delta^{k}_{z''}\delta^{a}_{a''}\delta^{b}_{b''}\delta^{c}_{c''}\delta^{i'}_{z'}\delta^{z''}_{z'}\delta^{a'}_{a''}\delta^{b''}_{b'}\delta^{c''}_{c'}) \\
&\iff \delta^{i'}_{z'}\delta^{z}_{z'}\delta^{k}_{z'}\delta^{a'}_{a}\delta^{b}_{b'}\delta^{c}_{c'}
\end{aligned}
$$

The anticlockwise path around this diagram evaluates as follows (ignoring ν, since it becomes an identity).

$$
\begin{aligned}
&((\chi_{A,B} \otimes \mathrm{id}_{f_\star(C)}) \bullet (\mathrm{id}_{f_\star(A)} \backslash \psi_{B,C}))^{zabkc}_{i'a'z'b'c'} \\
&\iff \exists i''a''j''b''k''c''((\chi_{A,B})^{zab}_{i''a''j''b''}(\mathrm{id}_{f_\star(C)})^{kc}_{k''c''}(\mathrm{id}^{\vee}_{f_\star(A)})^{i''a''}_{i'a'}(\psi_{B,C})^{j''b''k''c''}_{z'b'c'}) \\
&\iff \exists i''a''j''b''k''c''(\delta^{i''}_{j''}\delta^{z}_{j''}\delta^{a''}_{a}\delta^{b}_{b''}\delta^{k}_{k''}\delta^{c}_{c''}\delta^{i'}_{i''}\delta^{a'}_{a''}\delta^{j''}_{z'}\delta^{k''}_{z'}\delta^{b''}_{b'}\delta^{c''}_{c'}) \\
&\iff \delta^{i'}_{z'}\delta^{z}_{z'}\delta^{k}_{z'}\delta^{a'}_{a}\delta^{b}_{b'}\delta^{c}_{c'}
\end{aligned}
$$

Since these are equal, the diagram commutes.

Consider the following diagram, involving μ.

$$
\begin{array}{ccc}
f_\star(A)\backslash(f_\star(B)\backslash f_\star(C)) & \xleftarrow{\mu_{f_\star(A),f_\star(B),f_\star(C)}} & (f_\star(B)\otimes f_\star(A))\backslash f_\star(C) \\
\uparrow {\scriptstyle \mathrm{id}_{f_\star(A)}\backslash \chi_{B,C}} & & \uparrow {\scriptstyle \psi_{B,A}\backslash \mathrm{id}_{f_\star(C)}} \\
f_\star(A)\backslash f_\star(B\backslash C) & & f_\star(B\otimes A)\backslash f_\star(C) \\
\uparrow {\scriptstyle \chi_{A,B\backslash C}} & & \uparrow {\scriptstyle \chi_{B\otimes A,C}} \\
f_\star(A\backslash(B\backslash C)) & \xleftarrow[f_\star(\mu_{A,B,C})]{} & f_\star((B\otimes A)\backslash C)
\end{array}
$$

The clockwise path around this diagram evaluates as follows (ignoring μ, since it becomes an identity).

$$
\begin{aligned}
&(\chi_{A,B\backslash C} \bullet (\mathrm{id}_{f_\star(A)} \backslash \chi_{B,C}))^{zabc}_{i'a'j'b'k'c'} \\
&\iff \exists i''a''z''b''c''((\chi_{A,B\backslash C})^{zabc}_{i''a''z''b''c''}(\mathrm{id}^{\vee}_{f_\star(A)})^{i''a''}_{i'a'}(\chi_{B,C})^{z''b''c''}_{j'b'k'c'}) \\
&\iff \exists i''a''z''b''c''(\delta^{z''}_{k''}\delta^{z}_{z''}\delta^{a''}_{a}\delta^{b''}_{b}\delta^{c}_{c''}\delta^{i'}_{i''}\delta^{a'}_{a''}\delta^{j'}_{k'}\delta^{z''}_{k'}\delta^{b'}_{b''}\delta^{c''}_{c'}) \\
&\iff \delta^{i'}_{k'}\delta^{j'}_{k'}\delta^{z}_{k'}\delta^{a'}_{a}\delta^{b'}_{b}\delta^{c}_{c'}
\end{aligned}
$$

The anticlockwise path around this diagram evaluates as follows (ignoring

μ, since it becomes an identity).

$$
\begin{aligned}
&(\chi_{B\otimes A,C} \bullet (\psi_{B,A} \backslash \mathrm{id}_{f_\star(C)}))^{zabc}_{i'a'j'b'k'c'} \\
&\iff \exists z''a''b''k''c''((\chi_{B\otimes A,C})^{zabc}_{z''a''b''k''c''}(\psi^\vee_{B,A})^{z''a''b''}_{i'a'j'b'}(\mathrm{id}_{f_\star(C)})^{k''c''}_{k'c'}) \\
&\iff \exists z''a''b''k''c''(\delta^{z''}_{k''}\delta^{z}_{k''}\delta^{a''}_{a}\delta^{b''}_{b}\delta^{c}_{c''}\delta^{i'}_{z''}\delta^{j'}_{z''}\delta^{a'}_{a''}\delta^{b'}_{b''}\delta^{k''}_{k'}\delta^{c''}_{c'}) \\
&\iff \delta^{i'}_{k'}\delta^{j'}_{k'}\delta^{z}_{k'}\delta^{a'}_{a}\delta^{b'}_{b}\delta^{c}_{c'}
\end{aligned}
$$

Since these are equal, the diagram commutes.

5.7.3 The Projection Formula

Let $f^\star\colon \mathcal{D} \to \mathcal{C}$ be a strong monoidal functor between closed monoidal categories. Let $f_\star\colon \mathcal{C} \to \mathcal{D}$ be a right adjoint to $f^\star$. We have the natural transformations α, ν, μ, φ, ψ and χ.

$$\alpha_{A,B,C}\colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C$$

$$
\begin{aligned}
\nu_{A,B,C}&\colon (A\backslash B) \otimes C \to A\backslash(B \otimes C) \\
\mu_{A,B,C}&\colon (B \otimes A)\backslash C \to A\backslash(B\backslash C) \\
\varphi_{A,B}&\colon f^\star(A \otimes B) \to f^\star(A) \otimes f^\star(B) \\
\psi_{A,B}&\colon f_\star(A) \otimes f_\star(B) \to f_\star(A \otimes B) \\
\chi_{A,B}&\colon f_\star(A\backslash B) \to f_\star(A)\backslash f_\star(B)
\end{aligned}
$$

We can also define two other natural transformations π and σ, which we denote as follows.

$$\pi_{A,B}\colon f_\star(A) \otimes B \to f_\star(A \otimes f^\star(B))$$

$$\sigma_{A,B}\colon f_\star(f^\star(A)\backslash B) \to A\backslash f_\star(B)$$

The natural transformation π is known as the 'projection formula map' (e.g. in [5]). The natural transformation σ can be thought of as an internal version of the natural isomorphism of hom-sets defining the adjunction $f^\star \dashv f_\star$.

$$\mathcal{C}(f^\star(A), B) \cong \mathcal{D}(A, f_\star(B))$$

The natural transformation π has components defined as follows.

$$\pi_{A,B}\colon f_\star(A) \otimes B \xrightarrow{\mathrm{id}_{f_\star(A)} \otimes \theta_B} f_\star(A) \otimes f_\star f^\star(B) \xrightarrow{\psi_{A,f^\star(B)}} f_\star(A \otimes f^\star(B))$$

$$
\begin{aligned}
(\pi_{A,B})^{iab}_{z'a'b'} &\iff ((\mathrm{id}_{f_\star(A)} \otimes \theta_B) \bullet \psi_{A,f^\star(B)})^{iab}_{z'a'b'} \\
&\iff \exists i''a''j''b''((\mathrm{id}_{f_\star(A)})^{ia}_{i''a''}(\theta_B)^{b}_{j''b''}(\psi_{A,f^\star(B)})^{i''a''j''b''}_{z'a'b'}) \\
&\iff \exists i''a''j''b''(\delta^{i}_{i''}\delta^{a}_{a''}(B^{j''})^{b}_{b''}\delta^{i''}_{z'}\delta^{j''}_{z'}\delta^{a''}_{a'}\delta^{b''}_{b'}) \\
&\iff \delta^{i}_{z'}\delta^{a}_{a'}(B^{z'})^{b}_{b'}
\end{aligned}
$$

The natural transformation σ has components defined as follows.

$$\sigma_{A,B} \colon f_\star(f^\star(A)\backslash B) \xrightarrow{\chi_{f^\star(A),B}} f_\star f^\star(A)\backslash f_\star(B) \xrightarrow{\theta_A \backslash\, \mathrm{id}_{f_\star(B)}} A\backslash f_\star(B)$$

$$\begin{aligned}
(\sigma_{A,B})^{zab}_{a'j'b'} &\iff (\chi_{f^\star(A),B} \bullet (\theta_A \backslash\, \mathrm{id}_{f_\star(B)}))^{zab}_{a'j'b'} \\
&\iff \exists i''a''j''b''((\chi_{f^\star(A),B})^{zab}_{i''a''j''b''}(\theta^\vee_A)^{i''a''}_{a'}(\mathrm{id}_{f_\star(B)})^{j''b''}_{j'b'}) \\
&\iff \exists i''a''j''b''(\delta^{i''}_{j''}\delta^{z}_{j''}\delta^{a''}_{a}\delta^{b}_{b''}(A^{i''})^{a'}_{a''}\delta^{j''}_{j'}\delta^{b''}_{b'}) \\
&\iff \delta^{z}_{j'}(A^{z})^{a'}_{a}\delta^{b}_{b'}
\end{aligned}$$

We can use this to show that the following hexagon diagrams commute.

Consider the following diagram, involving the interaction between π and σ.

$$\begin{array}{ccc}
A\backslash(f_\star(B)\otimes C) & \xleftarrow{\nu_{A,f_\star(B),C}} & (A\backslash f_\star(B))\otimes C \\
\Big\downarrow{\scriptstyle \mathrm{id}_A \backslash \pi_{B,C}} & & \Big\uparrow{\scriptstyle \sigma_{A,B}\otimes \mathrm{id}_C} \\
A\backslash f_\star(B\otimes f^\star(C)) & & f_\star(f^\star(A)\backslash B)\otimes C \\
\Big\uparrow{\scriptstyle \sigma_{A,B\otimes f^\star(C)}} & & \Big\downarrow{\scriptstyle \pi_{f^\star(A)\backslash B,C}} \\
f_\star(f^\star(A)\backslash(B\otimes f^\star(C))) & \xleftarrow[f_\star(\nu_{f^\star(A),B,f^\star(C)})]{} & f_\star((f^\star(A)\backslash B)\otimes f^\star(C))
\end{array}$$

The clockwise path around this diagram evaluates as follows (ignoring ν, since it becomes an identity).

$$\begin{aligned}
&(\pi_{f^\star(A)\backslash B,C} \bullet \sigma_{A,B\otimes f^\star(C)})^{zabc}_{a'z'b'c'} \\
&\iff \exists z''a''b''c''((\pi_{f^\star(A)\backslash B,C})^{zabc}_{z''a''b''c''}(\sigma_{A,B\otimes f^\star(C)})^{z''a''b''c''}_{a'z'b'c'}) \\
&\iff \exists z''a''b''c''(\delta^{z}_{z''}\delta^{a''}_{a}\delta^{b}_{b''}(C^{z''})^{c}_{c''}\delta^{z''}_{z'}(A^{z''})^{a'}_{a''}\delta^{b''}_{b'}\delta^{c''}_{c'}) \\
&\iff \delta^{z}_{z'}(A^{z})^{a'}_{a}\delta^{b}_{b'}(C^{z'})^{c}_{c'}
\end{aligned}$$

The anticlockwise path around this diagram evaluates as follows (ignoring ν, since it becomes an identity).

$$\begin{aligned}
&((\sigma_{A,B}\otimes \mathrm{id}_C) \bullet (\mathrm{id}_A \backslash \pi_{B,C}))^{zabc}_{a'z'b'c'} \\
&\iff \exists a''j''b''c''((\sigma_{A,B})^{zab}_{a''j''b''}(\mathrm{id}_C)^{c}_{c''}(\mathrm{id}^\vee_A)^{a''}_{a'}(\pi_{B,C})^{j''b''c''}_{z'b'c'}) \\
&\iff \exists a''j''b''c''(\delta^{z}_{j''}(A^{z})^{a''}_{a}\delta^{b}_{b''}\delta^{c}_{c''}\delta^{a'}_{a''}\delta^{j''}_{z'}\delta^{b''}_{b'}(C^{z'})^{c''}_{c'}) \\
&\iff \delta^{z}_{z'}(A^{z})^{a'}_{a}\delta^{b}_{b'}(C^{z'})^{c}_{c'}
\end{aligned}$$

Since these are equal, the diagram commutes.

Consider the following diagram, involving $\pi_{A,B\otimes C}$.

$$\begin{array}{ccc}
f_\star(A)\otimes(B\otimes C) & \xrightarrow{\alpha_{f_\star(A),B,C}} & (f_\star(A)\otimes B)\otimes C \\
\downarrow{\scriptstyle \pi_{A,B\otimes C}} & & \downarrow{\scriptstyle \pi_{A,B}\otimes \mathrm{id}_C} \\
f_\star(A\otimes f^\star(B\otimes C)) & & f_\star(A\otimes f^\star(B))\otimes C \\
\downarrow{\scriptstyle f_\star(\mathrm{id}_A\otimes\varphi_{B,C})} & & \downarrow{\scriptstyle \pi_{A\otimes f^\star(B),C}} \\
f_\star(A\otimes(f^\star(B)\otimes f^\star(C))) & \xrightarrow[f_\star(\alpha_{A,f^\star(B),f^\star(C)})]{} & f_\star((A\otimes f^\star(B))\otimes f^\star(C))
\end{array}$$

The clockwise path around this diagram evaluates as follows (ignoring α, since it becomes an identity).

$$\begin{aligned}
&((\pi_{A,B}\otimes \mathrm{id}_C)\bullet \pi_{A\otimes f^\star(B),C})^{iabc}_{z'a'b'c'}\\
\iff\ & \exists z''a''b''c''((\pi_{A,B})^{iab}_{z''a''b''}(\mathrm{id}_C)^{c}_{c''}(\pi_{A\otimes f^\star(B),C})^{z''a''b''c''}_{z'a'b'c'})\\
\iff\ & \exists z''a''b''c''(\delta^i_{z''}\delta^a_{a''}(B^{z''})^b_{b''}\delta^c_{c''}\delta^{z''}_{z'}\delta^{a''}_{a'}\delta^{b''}_{b'}(C^{z''})^{c''}_{c'})\\
\iff\ & \delta^i_{z'}\delta^a_{a'}(B^{z'})^b_{b'}(C^{z'})^c_{c'}
\end{aligned}$$

The anticlockwise path around this diagram evaluates as follows (ignoring α and φ, since they become identities).

$$\begin{aligned}
(\pi_{A,B\otimes C})^{iabc}_{z'a'b'c'} &\iff \delta^i_{z'}\delta^a_{a'}((B\otimes C)^{z'})^{bc}_{b'c'}\\
&\iff \delta^i_{z'}\delta^a_{a'}(B^{z'}\otimes C^{z'})^{bc}_{b'c'}\\
&\iff \delta^i_{z'}\delta^a_{a'}(B^{z'})^b_{b'}(C^{z'})^c_{c'}
\end{aligned}$$

Since these are equal, the diagram commutes.

Consider the following diagram, involving $\sigma_{B\otimes A,C}$.

$$\begin{array}{ccc}
A\backslash(B\backslash f_\star(C)) & \xleftarrow{\mu_{A,B,f_\star(C)}} & (B\otimes A)\backslash f_\star(C) \\
\uparrow{\scriptstyle \mathrm{id}_A\backslash\sigma_{B,C}} & & \uparrow{\scriptstyle \sigma_{B\otimes A,C}} \\
A\backslash f_\star(f^\star(B)\backslash C) & & f_\star(f^\star(B\otimes A)\backslash C) \\
\uparrow{\scriptstyle \sigma_{A,f^\star(B)\backslash C}} & & \uparrow{\scriptstyle f_\star(\varphi_{B,A}\backslash\mathrm{id}_C)} \\
f_\star(f^\star(A)\backslash(f^\star(B)\backslash C)) & \xleftarrow[f_\star(\mu_{f^\star(A),f^\star(B),C})]{} & f_\star((f^\star(B)\otimes f^\star(A))\backslash C)
\end{array}$$

The clockwise path around this diagram evaluates as follows (ignoring μ, since it becomes an identity).

$$
\begin{aligned}
&(\sigma_{A,f^\star(B)\backslash C} \bullet (\mathrm{id}_A \backslash \sigma_{B,C}))^{zabc}_{a'b'k'c'} \\
&\iff \exists a''z''b''c''((\sigma_{A,f^\star(B)\backslash C})^{zabc}_{a''z''b''c''}(\mathrm{id}^\vee_A)^{a''}_{a'}(\sigma_{B,C})^{z''b''c''}_{b'k'c'}) \\
&\iff \exists a''z''b''c''(\delta^z_{z''}(A^z)^{a''}_a \delta^{b''}_b \delta^c_{c''} \delta^{a'}_{a''} \delta^{z''}_{k'} (B^{z''})^{b'}_{b''} \delta^{c''}_{c'}) \\
&\iff \delta^z_{k'}(A^z)^{a'}_a (B^z)^{b'}_b \delta^c_{c'}
\end{aligned}
$$

The anticlockwise path around this diagram evaluates as follows (ignoring μ and φ, since they become identities).

$$
\begin{aligned}
(\sigma_{B\otimes A,C})^{zabc}_{a'b'k'c'} &\iff \delta^z_{k'}((B\otimes A)^z)^{a'b'}_{ab}\delta^c_{c'} \\
&\iff \delta^z_{k'}(B^z \otimes A^z)^{a'b'}_{ab}\delta^c_{c'} \\
&\iff \delta^z_{k'}(A^z)^{a'}_a (B^z)^{b'}_b \delta^c_{c'}
\end{aligned}
$$

Since these are equal, the diagram commutes.

Consider the following diagram, involving $\pi_{A\otimes B,C}$.

$$
\begin{array}{ccc}
f_\star(A)\otimes(f_\star(B)\otimes C) & \xrightarrow{\alpha_{f_\star(A),f_\star(B),C}} & (f_\star(A)\otimes f_\star(B))\otimes C \\
\Big\downarrow{\scriptstyle \mathrm{id}_{f_\star(A)}\otimes\pi_{B,C}} & & \Big\downarrow{\scriptstyle \psi_{A,B}\otimes\mathrm{id}_C} \\
f_\star(A)\otimes f_\star(B\otimes f^\star(C)) & & f_\star(A\otimes B)\otimes C \\
\Big\downarrow{\scriptstyle \psi_{A,B\otimes f^\star(C)}} & & \Big\downarrow{\scriptstyle \pi_{A\otimes B,C}} \\
f_\star(A\otimes(B\otimes f^\star(C))) & \xrightarrow[f_\star(\alpha_{A,B,f^\star(C)})]{} & f_\star((A\otimes B)\otimes f^\star(C))
\end{array}
$$

The clockwise path around this diagram evaluates as follows (ignoring α, since it becomes an identity).

$$
\begin{aligned}
&((\psi_{A,B}\otimes\mathrm{id}_C)\bullet\pi_{A\otimes B,C})^{iajbc}_{z'a'b'c'} \\
&\iff \exists z''a''b''c''((\psi_{A,B})^{iajb}_{z''a''b''}(\mathrm{id}_C)^c_{c''}(\pi_{A\otimes B,C})^{z''a''b''c''}_{z'a'b'c'}) \\
&\iff \exists z''a''b''c''(\delta^i_{z''}\delta^j_{z''}\delta^a_{a''}\delta^b_{b''}\delta^c_{c''}\delta^{z''}_{z'}\delta^{a''}_{a'}\delta^{b''}_{b'}(C^{z'})^{c''}_{c'}) \\
&\iff \delta^i_{z'}\delta^j_{z'}\delta^a_{a'}\delta^b_{b'}(C^{z'})^c_{c'}
\end{aligned}
$$

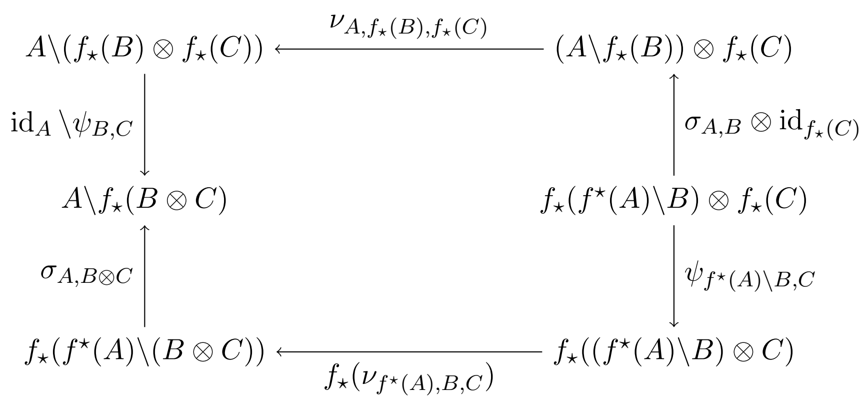
The anticlockwise path around this diagram evaluates as follows (ignoring

α, since it becomes an identity).

$$
\begin{aligned}
&((\mathrm{id}_{f_\star(A)} \otimes \pi_{B,C}) \bullet \psi_{A,B\otimes f^\star(C)})^{iajbc}_{z'a'b'c'} \\
&\iff \exists i''a''z''b''c''((\mathrm{id}_{f_\star(A)})^{ia}_{i''a''}(\pi_{B,C})^{jbc}_{z''b''c''}(\psi_{A,B\otimes f^\star(C)})^{i''a''z''b''c''}_{z'a'b'c'}) \\
&\iff \exists i''a''z''b''c''(\delta^{i}_{i''}\delta^{a}_{a''}\delta^{j}_{z''}\delta^{b}_{b''}(C^{z''})^{c}_{c''}\delta^{i''}_{z'}\delta^{z''}_{z'}\delta^{a''}_{a'}\delta^{b''}_{b'}\delta^{c''}_{c'}) \\
&\iff \delta^{i}_{z'}\delta^{j}_{z'}\delta^{a}_{a'}\delta^{b}_{b'}(C^{z'})^{c}_{c'}
\end{aligned}
$$

Since these are equal, the diagram commutes.

Consider the following diagram, involving $\sigma_{A,B\otimes C}$.



The clockwise path around this diagram evaluates as follows (ignoring ν, since it becomes an identity).

$$
\begin{aligned}
&(\psi_{f^\star(A)\backslash B,C} \bullet \sigma_{A,B\otimes C})^{zabkc}_{a'z'b'c'} \\
&\iff \exists z''a''b''c''((\psi_{f^\star(A)\backslash B,C})^{zabkc}_{z''a''b''c''}(\sigma_{A,B\otimes C})^{z''a''b''c''}_{a'z'b'c'}) \\
&\iff \exists z''a''b''c''(\delta^{z}_{z''}\delta^{k}_{z''}\delta^{a''}_{a}\delta^{b}_{b''}\delta^{c}_{c''}\delta^{z''}_{z'}(A^{z'})^{a'}_{a''}\delta^{b''}_{b'}\delta^{c''}_{c'}) \\
&\iff \delta^{z}_{z'}\delta^{k}_{z'}(A^{z})^{a'}_{a}\delta^{b}_{b'}\delta^{c}_{c'}
\end{aligned}
$$

The anticlockwise path around this diagram evaluates as follows (ignoring ν, since it becomes an identity).

$$
\begin{aligned}
&((\sigma_{A,B}\otimes \mathrm{id}_{f_\star(C)}) \bullet (\mathrm{id}_A \backslash \psi_{B,C}))^{zabkc}_{a'z'b'c'} \\
&\iff \exists a''j''b''k''c''((\sigma_{A,B})^{zab}_{a''j''b''}(\mathrm{id}_{f_\star(C)})^{kc}_{k''c''}(\mathrm{id}^{\vee}_{A})^{a''}_{a'}(\psi_{B,C})^{j''b''k''c''}_{z'b'c'}) \\
&\iff \exists a''j''b''k''c''(\delta^{z}_{j''}(A^{z})^{a''}_{a}\delta^{b}_{b''}\delta^{k}_{k''}\delta^{c}_{c''}\delta^{a'}_{a''}\delta^{j''}_{z'}\delta^{k''}_{z'}\delta^{b''}_{b'}\delta^{c''}_{c'}) \\
&\iff \delta^{z}_{z'}\delta^{k}_{z'}(A^{z})^{a'}_{a}\delta^{b}_{b'}\delta^{c}_{c'}
\end{aligned}
$$

Since these are equal, the diagram commutes.

Consider the following diagram, involving $\pi_{A\backslash B,C}$.

$$\begin{array}{ccc}
f_\star(A)\backslash(f_\star(B)\otimes C) & \xleftarrow{\nu_{f_\star(A),f_\star(B),C}} & (f_\star(A)\backslash f_\star(B))\otimes C \\
{\scriptstyle \mathrm{id}_{f_\star(A)}\backslash\pi_{B,C}}\downarrow & & \uparrow{\scriptstyle \chi_{A,B}\otimes\mathrm{id}_C} \\
f_\star(A)\backslash f_\star(B\otimes f^\star(C)) & & f_\star(A\backslash B)\otimes C \\
{\scriptstyle \chi_{A,B\otimes f^\star(C)}}\uparrow & & \downarrow{\scriptstyle \pi_{A\backslash B,C}} \\
f_\star(A\backslash(B\otimes f^\star(C))) & \xleftarrow[f_\star(\nu_{A,B,f^\star(C)})]{} & f_\star((A\backslash B)\otimes f^\star(C))
\end{array}$$

The clockwise path around this diagram evaluates as follows (ignoring ν, since it becomes an identity).

$$\begin{aligned}
&(\pi_{A\backslash B,C}\bullet\chi_{A,B\otimes f^\star(C)})^{zabc}_{i'a'z'b'c'} \\
&\iff \exists z''a''b''c''((\pi_{A\backslash B,C})^{zabc}_{z''a''b''c''}(\chi_{A,B\otimes f^\star(C)})^{z''a''b''c''}_{i'a'z'b'c'}) \\
&\iff \exists z''a''b''c''(\delta^{z}_{z''}\delta^{a''}_{a}\delta^{b}_{b''}(C^{z''})^{c}_{c''}\delta^{i'}_{z'}\delta^{z''}_{z'}\delta^{a'}_{a''}\delta^{b''}_{b'}\delta^{c''}_{c'}) \\
&\iff \delta^{i'}_{z'}\delta^{z}_{z'}\delta^{a'}_{a}\delta^{b}_{b'}(C^{z'})^{c}_{c'}
\end{aligned}$$

The anticlockwise path around this diagram evaluates as follows (ignoring ν, since it becomes an identity).

$$\begin{aligned}
&((\chi_{A,B}\otimes\mathrm{id}_C)\bullet(\mathrm{id}_{f_\star(A)}\backslash\pi_{B,C}))^{zabc}_{i'a'z'b'c'} \\
&\iff \exists i''a''j''b''c''((\chi_{A,B})^{zab}_{i''a''j''b''}(\mathrm{id}_C)^{c}_{c''}(\mathrm{id}^{\vee}_{f_\star(A)})^{i''a''}_{i'a'}(\pi_{B,C})^{j''b''c''}_{z'b'c'}) \\
&\iff \exists i''a''j''b''c''(\delta^{i''}_{j''}\delta^{z}_{j''}\delta^{a''}_{a}\delta^{b}_{b''}\delta^{c}_{c''}\delta^{i'}_{i''}\delta^{a'}_{a''}\delta^{j''}_{z'}\delta^{b''}_{b'}(C^{z'})^{c''}_{c'}) \\
&\iff \delta^{i'}_{z'}\delta^{z}_{z'}\delta^{a'}_{a}\delta^{b}_{b'}(C^{z'})^{c}_{c'}
\end{aligned}$$

Since these are equal, the diagram commutes.

Consider the following diagram, involving $\sigma_{A,B\backslash C}$.

$$\begin{array}{ccc}
A\backslash(f_\star(B)\backslash f_\star(C)) & \xleftarrow{\mu_{A,f_\star(B),f_\star(C)}} & (f_\star(B)\otimes A)\backslash f_\star(C) \\
{\scriptstyle \mathrm{id}_A\backslash\chi_{B,C}}\uparrow & & \uparrow{\scriptstyle \pi_{B,A}\backslash\mathrm{id}_{f_\star(C)}} \\
A\backslash f_\star(B\backslash C) & & f_\star(B\otimes f^\star(A))\backslash f_\star(C) \\
{\scriptstyle \sigma_{A,B\backslash C}}\uparrow & & \uparrow{\scriptstyle \chi_{B\otimes f^\star(A),C}} \\
f_\star(f^\star(A)\backslash(B\backslash C)) & \xleftarrow[f_\star(\mu_{f^\star(A),B,C})]{} & f_\star((B\otimes f^\star(A))\backslash C)
\end{array}$$

The clockwise path around this diagram evaluates as follows (ignoring μ, since it becomes an identity).

$$
\begin{aligned}
&(\sigma_{A,B\backslash C} \bullet (\mathrm{id}_A \backslash \chi_{B,C}))^{zabc}_{a'j'b'k'c'} \\
&\iff \exists a''z''b''c''((\sigma_{A,B\backslash C})^{zabc}_{a''z''b''c''}(\mathrm{id}^{\vee}_A)^{a''}_{a'}(\chi_{B,C})^{z''b''c''}_{j'b'k'c'}) \\
&\iff \exists a''z''b''c''(\delta^{z}_{z''}(A^z)^{a''}_{a}\delta^{b''}_{b}\delta^{c}_{c''}\delta^{a'}_{a''}\delta^{j'}_{k'}\delta^{z''}_{k'}\delta^{b'}_{b''}\delta^{c''}_{c'}) \\
&\iff \delta^{j'}_{k'}\delta^{z}_{k'}(A^z)^{a'}_{a}\delta^{b'}_{b}\delta^{c}_{c'}
\end{aligned}
$$

The anticlockwise path around this diagram evaluates as follows (ignoring μ, since it becomes an identity).

$$
\begin{aligned}
&(\chi_{B\otimes f^\star(A),C} \bullet (\pi_{B,A}\backslash \mathrm{id}_{f_\star(C)}))^{zabc}_{a'j'b'k'c'} \\
&\iff \exists z''a''b''k''c''((\chi_{B\otimes f^\star(A),C})^{zabc}_{z''a''b''k''c''}(\pi^{\vee}_{B,A})^{z''a''b''}_{a'j'b'}(\mathrm{id}_{f_\star(C)})^{k''c''}_{k'c'}) \\
&\iff \exists z''a''b''k''c''(\delta^{z''}_{k''}\delta^{z}_{k''}\delta^{a''}_{a}\delta^{b''}_{b}\delta^{c}_{c''}\delta^{j'}_{z''}(A^{z''})^{a'}_{a''}\delta^{b'}_{b''}\delta^{k''}_{k'}\delta^{c''}_{c'}) \\
&\iff \delta^{j'}_{k'}\delta^{z}_{k'}(A^z)^{a'}_{a}\delta^{b'}_{b}\delta^{c}_{c'}
\end{aligned}
$$

Since these are equal, the diagram commutes.

5.7.4 Some Non-Examples

Finally, we will provide some simple examples of pairs of parallel allowable morphisms which are not equal.

Let $\mathcal{C}$ be a closed monoidal category. We can define the following natural transformation.

$$A \otimes (A\backslash(A\otimes B)) \xrightarrow{\varepsilon^A_{A\otimes B}} A\otimes B \xrightarrow{\mathrm{id}_A \otimes \eta^A_B} A\otimes(A\backslash(A\otimes B))$$

$$
\begin{aligned}
(\varepsilon^A_{A\otimes B} \bullet (\mathrm{id}_A \otimes \eta^A_B))^{xyzb}_{x'y'z'b'} &\iff \exists a''b''((\varepsilon^A_{A\otimes B})^{xyzb}_{a''b''}(\mathrm{id}_A)^{a''}_{x'} \otimes \eta^A_B)^{b''}_{y'z'b'}) \\
&\iff \exists a''b''(\delta^x_y\delta^z_{a''}\delta^b_{b''}\delta^{a''}_{x'}\delta^{y'}_{z'}\delta^{b''}_{b'}) \\
&\iff \delta^x_y\delta^z_{x'}\delta^{y'}_{z'}\delta^b_{b'}
\end{aligned}
$$

We also have the identity natural transformation.

$$A \otimes (A\backslash(A\otimes B)) \xrightarrow{\mathrm{id}_{A\otimes(A\backslash(A\otimes B))}} A\otimes(A\backslash(A\otimes B))$$

$$(\mathrm{id}_{A\otimes(A\backslash(A\otimes B))})^{xyzb}_{x'y'z'b'} \iff \delta^x_{x'}\delta^{y'}_{y}\delta^z_{z'}\delta^b_{b'}$$

These relations are not equal, and so the natural transformations are not necessarily equal.

Let $f^\star\colon \mathcal{D} \to \mathcal{C}$ be a strong monoidal functor between closed monoidal categories. Let $f_\star\colon \mathcal{C} \to \mathcal{D}$ be a right adjoint to $f^\star$. We can define the following natural transformation.

$$f^\star f_\star f^\star(A) \xrightarrow{\zeta_{f^\star(A)}} f^\star(A) \xrightarrow{f^\star(\theta_A)} f^\star f_\star f^\star(A)$$

$$
\begin{aligned}
(\zeta_{f^\star(A)} \bullet f^\star(\theta_A))^{ia}_{i'a'} &\iff \exists a''((\zeta_{f^\star(A)})^{ia}_{a''} f^\star(\theta_A)^{a''}_{i'a'}) \\
&\iff \delta^i_0 \exists a''(\delta^a_{a''}(A^{i'})^{a''}_{a'}) \\
&\iff \delta^i_0 (A^{i'})^a_{a'}
\end{aligned}
$$

We also have the identity natural transformation.

$$
f^\star f_\star f^\star(A) \xrightarrow{\mathrm{id}_{f^\star f_\star f^\star(A)}} f^\star f_\star f^\star(A)
$$

$$
(\mathrm{id}_{f^\star f_\star f^\star(A)})^{ia}_{i'a'} \iff \delta^i_{i'} \delta^a_{a'}
$$

These relations are not equal, and so the natural transformations are not necessarily equal.

Appendix A

Recognition Lemmas

This appendix contains the proofs of the four recognition lemmas from §5.6. The content of this section is quite technical, repetitive and unenlightening. A thorough understanding of these proofs is not necessary to appreciate the results themselves.

In preparation, §A.1 contains some preliminary results regarding relations. The remainder of this appendix contains the proofs of the recognition lemmas themselves. Each of the recognition lemmas is proved by induction, based on the type of the constructible morphism. We will implicitly prove the four recognition lemmas by induction at the same time. For example, when proving a recognition lemma about a constructible morphism of the form $\langle \Gamma, f, g\rangle_\varepsilon$, we will assume that each recognition lemma holds for the constructible morphisms f and g.

A.1 Relation Lemmas

First, we will give explicit descriptions of morphisms of the forms used to define the constructible morphisms.

Given a morphism $\alpha\colon I \otimes A \to B$, we can form the following morphism.

$$\langle I, \alpha\rangle_\eta \colon A \xrightarrow{\mathrm{ev}_I \otimes \mathrm{id}_A} I^\vee \otimes I \otimes A \xrightarrow{\mathrm{id}_{I^\vee} \otimes \alpha} I^\vee \otimes B$$

Explicitly, this morphism has the following form.

$$\begin{aligned}
(\langle I, \alpha\rangle_\eta)^a_{\bar{i}b} &\iff ((\mathrm{ev}_I \otimes \mathrm{id}_A) \bullet (\mathrm{id}_{I^\vee} \otimes \alpha))^a_{\bar{i}b} \\
&\iff \exists \bar{i}' i' a' ((\mathrm{ev}_I \otimes \mathrm{id}_A)^a_{\bar{i}' i' a'} (\mathrm{id}_{I^\vee} \otimes \alpha)^{\bar{i}' i' a'}_{\bar{i}b}) \\
&\iff \exists \bar{i}' i' a' ((\mathrm{ev}_I)^\star_{\bar{i}' i'} (\mathrm{id}_A)^a_{a'} (\mathrm{id}_{I^\vee})^{\bar{i}'}_{\bar{i}} \alpha^{i'a'}_b) \\
&\iff \exists \bar{i}' i' a' (\delta^{\bar{i}'}_{i'} \delta^a_{a'} \delta^{\bar{i}'}_{\bar{i}} \alpha^{i'a'}_b) \\
&\iff \alpha^{\bar{i}a}_b
\end{aligned}$$

Given morphisms $\alpha\colon I \to J$ and $\beta\colon A \otimes \Gamma^\star(B) \otimes C \to D$, we can form the following morphism.

$$\langle \Gamma, \alpha, \beta\rangle_\varepsilon \colon A \otimes \Gamma^\star(I \otimes J^\vee \otimes B) \otimes C \xrightarrow{\mathrm{id}_A \otimes \Gamma^\star(\mathrm{coev}_\alpha \otimes \mathrm{id}_B) \otimes \mathrm{id}_C} A \otimes \Gamma^\star(B) \otimes C \xrightarrow{\beta} D$$

Explicitly, this morphism has the following form.

$$\begin{aligned}
(\langle \Gamma, \alpha, \beta\rangle_\varepsilon)^{aijbc}_d &\iff ((\mathrm{id}_A \otimes \Gamma^\star(\mathrm{coev}_\alpha \otimes \mathrm{id}_B) \otimes \mathrm{id}_C) \bullet \beta)^{aijbc}_d \\
&\iff \exists a'b'c' ((\mathrm{id}_A \otimes \Gamma^\star(\mathrm{coev}_\alpha \otimes \mathrm{id}_B) \otimes \mathrm{id}_C)^{aijbc}_{a'b'c'} \beta^{a'b'c'}_d) \\
&\iff \exists a'b'c' ((\mathrm{id}_A)^a_{a'} (\mathrm{coev}_\alpha)^{ij}_\star (\mathrm{id}_B)^b_{b'} (\mathrm{id}_C)^c_{c'} \beta^{a'b'c'}_d) \\
&\iff \exists a'b'c' (\delta^a_{a'} \alpha^i_j \delta^b_{b'} \delta^c_{c'} \beta^{a'b'c'}_d) \\
&\iff \alpha^i_j \beta^{abc}_d
\end{aligned}$$

To verify that $\langle\Gamma,\alpha,\beta\rangle_\varepsilon$ is a morphism $A\otimes\Gamma^\star(I\otimes J^\vee\otimes B)\otimes C\to D$, it can be shown that the following two identities hold.

$$\begin{aligned}&((A\otimes\Gamma^\star(I\otimes J^\vee\otimes B)\otimes C)\bullet\langle\Gamma,\alpha,\beta\rangle_\varepsilon)^{aijbc}_d\\ \iff\ &(\Gamma^\star(I)\bullet\Gamma^\star(\alpha)\bullet\Gamma^\star(J)^{-1})^i_j((A\otimes\Gamma^\star(B)\otimes C)\bullet\beta)^{abc}_d\end{aligned}$$

$$(\langle\Gamma,\alpha,\beta\rangle_\varepsilon\bullet D)^{aijbc}_d\iff\Gamma^\star(\alpha)^i_j(\beta\bullet D)^{abc}_d$$

Given a morphism $\alpha\colon\Phi^\star(A)\to B$, we can form the following morphism.

$$\langle\Phi,\alpha\rangle_\theta\colon A\xrightarrow{\theta_A}\Phi_\star\Phi^\star(A)\xrightarrow{\Phi_\star(\alpha)}\Phi_\star(B)$$

Explicitly, this morphism has the following form.

$$\begin{aligned}(\langle\Phi,\alpha\rangle_\theta)^a_{ib}&\iff(\theta_A\bullet\Phi_\star(\alpha))^a_{ib}\\&\iff\exists i'a'((\theta_A)^a_{i'a'}\Phi_\star(\alpha)^{i'a'}_{ib})\\&\iff\exists i'a'((A^{i'})^a_{a'}\delta^{i'}_i\alpha^{a'}_b)\\&\iff\exists a'((A^i)^a_{a'}\alpha^{a'}_b)\\&\iff(A^i\bullet\alpha)^a_b\end{aligned}$$

Given a morphism $\alpha\colon A\otimes\Gamma^\star(B)\otimes C\to D$, we can form the following morphism.

$$\langle\Gamma,\Phi,\alpha\rangle_\zeta\colon A\otimes\Gamma^\star\Phi^\star\Phi_\star(B)\otimes C\xrightarrow{\mathrm{id}_A\otimes\Gamma^\star(\zeta_B)\otimes\mathrm{id}_C}A\otimes\Gamma^\star(B)\otimes C\xrightarrow{\alpha}D$$

Explicitly, this morphism has the following form.

$$\begin{aligned}(\langle\Gamma,\Phi,\alpha\rangle_\zeta)^{aibc}_d&\iff((\mathrm{id}_A\otimes\Gamma^\star(\zeta_B)\otimes\mathrm{id}_C)\bullet\alpha)^{aibc}_d\\&\iff\exists a'b'c'((\mathrm{id}_A\otimes\Gamma^\star(\zeta_B)\otimes\mathrm{id}_C)^{aibc}_{a'b'c'}\alpha^{a'b'c'}_d)\\&\iff\exists a'b'c'((\mathrm{id}_A)^a_{a'}\Gamma^\star(\zeta_B)^{ib}_{b'}(\mathrm{id}_C)^c_{c'}\alpha^{a'b'c'}_d)\\&\iff\exists a'b'c'(\delta^a_{a'}\delta^i_0\delta^b_{b'}\delta^c_{c'}\alpha^{a'b'c'}_d)\\&\iff\delta^i_0\alpha^{abc}_d\end{aligned}$$

To verify that $\langle\Gamma,\Phi,\alpha\rangle_\zeta$ is a morphism $A\otimes\Gamma^\star\Phi^\star\Phi_\star(B)\otimes C\to D$, it can be shown that the following two identities hold.

$$((A\otimes\Gamma^\star\Phi^\star\Phi_\star(B)\otimes C)\bullet\langle\Gamma,\Phi,\alpha\rangle_\zeta)^{aibc}_d\iff\delta^i_0((A\otimes\Gamma^\star(B)\otimes C)\bullet\alpha)^{abc}_d$$

$$(\langle\Gamma,\Phi,\alpha\rangle_\zeta\bullet D)^{aibc}_d\iff\delta^i_0(\alpha\bullet D)^{abc}_d$$

Now, we will give a number of technical lemmas involving relations. We will use the above results implicitly, where necessary.

Lemma A.1.1. *Given non-zero predicates P, P', Q and Q' such that*

$$P(p)Q(q) \iff P'(p)Q'(q),$$

it follows that

$$P = P' \qquad \textit{and} \qquad Q = Q'.$$

Proof. To see that $P = P'$, consider an arbitrary p. If $P(p)$ holds, then choose q such that $Q(q)$ holds and note the following.

$$P(p)Q(q) \implies P'(p)Q'(q) \implies P'(p)$$

If $P'(p)$ holds, then choose q such that $Q'(q)$ holds and note the following.

$$P'(p)Q'(q) \implies P(p)Q(q) \implies P(p)$$

If neither $P(p)$ nor $P'(p)$ holds, then the claim is true, trivially. Thus, $P = P'$. The proof that $Q = Q'$ similar. □

Lemma A.1.2. *Given non-zero relations $\alpha\colon A \dashrightarrow B$ and $\beta\colon C \dashrightarrow D$ such that*

$$\alpha \otimes \beta\colon A \otimes C \dashrightarrow B \otimes D$$

is equivariant, it follows that

$$\alpha\colon A \dashrightarrow B$$

and

$$\beta\colon C \dashrightarrow D$$

are equivariant.

Proof. The relation $\alpha \otimes \beta$ being equivariant is equivalent to the following.

$$(A \bullet \alpha)^a_b (C \bullet \beta)^c_d \iff (\alpha \bullet B)^a_b (\beta \bullet D)^c_d$$

Thus, by Lemma A.1.1,

$$A \bullet \alpha = \alpha \bullet B$$

and

$$C \bullet \beta = \beta \bullet D.$$

□

Lemma A.1.3. *Given non-zero relations* $\alpha\colon I \dashrightarrow J$ *and* $\beta\colon A \otimes \Gamma^\star(B) \otimes C \dashrightarrow D$ *such that*

$$\langle \Gamma, \alpha, \beta \rangle_\varepsilon \colon A \otimes \Gamma^\star(I \otimes J^\vee \otimes B) \otimes C \dashrightarrow D$$

is equivariant, it follows that

$$\Gamma^\star(\alpha)\colon \Gamma^\star(I) \dashrightarrow \Gamma^\star(J)$$

and

$$\beta\colon A \otimes \Gamma^\star(B) \otimes C \dashrightarrow D$$

are equivariant.

Proof. The relation $\langle \Gamma, \alpha, \beta \rangle_\varepsilon$ being equivariant is equivalent to the following.

$$(\Gamma^\star(I) \bullet \Gamma^\star(\alpha) \bullet \Gamma^\star(J^{-1}))^i_j((A \otimes \Gamma^\star(B) \otimes C) \bullet \beta)^{abc}_d \iff \Gamma^\star(\alpha)^i_j(\beta \bullet D)^{abc}_d$$

Thus, by Lemma A.1.1,

$$\Gamma^\star(I) \bullet \Gamma^\star(\alpha) \bullet \Gamma^\star(J^{-1}) = \Gamma^\star(\alpha)$$

and

$$(A \otimes \Gamma^\star(B) \otimes C) \bullet \beta = \beta \bullet D.$$

□

Lemma A.1.4. *Given a relation* $\alpha\colon A \otimes \Gamma^\star(B) \otimes C \dashrightarrow D$ *such that*

$$\langle \Gamma, \Phi, \alpha \rangle_\zeta \colon A \otimes \Gamma^\star\Phi^\star\Phi_\star(B) \otimes C \dashrightarrow D$$

is equivariant, it follows that

$$\alpha\colon A \otimes \Gamma^\star(B) \otimes C \dashrightarrow D$$

is equivariant.

Proof. The relation $\langle \Gamma, \Phi, \alpha \rangle_\zeta$ being equivariant is equivalent to the following.

$$\delta^i_0((A \otimes \Gamma^\star(B) \otimes C) \bullet \alpha)^{abc}_d \iff \delta^i_0(\alpha \bullet D)^{abc}_d$$

Thus, by Lemma A.1.1,

$$(A \otimes \Gamma^\star(B) \otimes C) \bullet \alpha = \alpha \bullet D.$$

□

Lemma A.1.5. *Given non-zero predicates P, Q, R and S such that*

$$P(i,j)Q(k,l) \iff R(i,k)S(j,l),$$

there exist non-zero predicates A, B, C and D such that the following hold.

$$P(i,j) \iff A(i)B(j)$$
$$Q(k,l) \iff C(k)D(l)$$
$$R(i,k) \iff A(i)C(k)$$
$$S(j,l) \iff B(j)D(l)$$

Proof. Choose $\bar{i}$, $\bar{j}$, $\bar{k}$ and $\bar{l}$ such that $P(\bar{i},\bar{j})$, $Q(\bar{k},\bar{l})$, $R(\bar{i},\bar{k})$ and $S(\bar{j},\bar{l})$ hold. Define A, B, C and D as follows.

$$A(i) \iff P(i,\bar{j}) \iff R(i,\bar{k})$$
$$B(j) \iff P(\bar{i},j) \iff S(j,\bar{l})$$
$$C(k) \iff Q(k,\bar{l}) \iff R(\bar{i},l)$$
$$D(l) \iff Q(\bar{k},l) \iff S(\bar{j},l)$$

To see that $P(i,j) \iff A(i)B(j)$, note the following.

$$P(i,j) \iff P(i,j)Q(\bar{k},\bar{l}) \iff R(i,\bar{k})S(j,\bar{l}) \iff A(i)B(j)$$

To see that $Q(k,l) \iff C(k)D(l)$, note the following.

$$Q(k,l) \iff P(\bar{i},\bar{j})Q(k,l) \iff R(\bar{i},k)S(\bar{j},l) \iff C(j)D(l)$$

To see that $R(i,k) \iff A(i)C(k)$, note the following.

$$R(i,k) \iff R(i,k)S(\bar{j},\bar{l}) \iff P(i,\bar{j})Q(k,\bar{l}) \iff A(i)C(k)$$

To see that $S(j,l) \iff B(j)D(l)$, note the following.

$$S(j,l) \iff R(\bar{i},\bar{k})S(j,l) \iff P(\bar{i},j)Q(\bar{k},l) \iff B(j)D(l)$$

□

Lemma A.1.6. *Given non-zero predicates P, Q, R and S such that*

$$P(i,j)Q(l) \iff R(i)S(j,l),$$

there exists a non-zero predicate B such that

$$P(i,j) \iff R(i)B(j) \qquad \textit{and} \qquad S(j,l) \iff B(j)Q(l).$$

Proof. This follows from Lemma A.1.5. ☐

Lemma A.1.7. *Given non-zero morphisms*

$$\alpha\colon A \to B \qquad \bar{\alpha}\colon C \otimes E \to D \otimes F$$

$$\bar{\beta}\colon A \otimes C \to B \otimes D \qquad \beta\colon E \to F$$

such that

$$\alpha \otimes \bar{\alpha} = \bar{\beta} \otimes \beta\colon A \otimes C \otimes E \to B \otimes D \otimes F,$$

there is a non-zero morphism

$$\gamma\colon C \to D$$

such that the following hold.

$$\gamma \otimes \beta = \bar{\alpha}\colon C \otimes E \to D \otimes F$$

$$\alpha \otimes \gamma = \bar{\beta}\colon A \otimes C \to B \otimes D$$

Proof. Since $\alpha \otimes \bar{\alpha} = \bar{\beta} \otimes \beta$, it follows that

$$\alpha_b^a \bar{\alpha}_{df}^{ce} \iff \bar{\beta}_{bd}^{ac} \beta_f^e.$$

By Lemma A.1.6, there exists a non-zero relation $\gamma\colon C \dashrightarrow D$ such that $\gamma \otimes \beta = \bar{\alpha}$ and $\alpha \otimes \gamma = \bar{\beta}$. By Lemma A.1.2, γ is a morphism $C \to D$. ☐

Lemma A.1.8. *Given non-zero morphisms*

$$\alpha\colon I \to J \qquad \bar{\alpha}\colon A \otimes \Gamma^\star(B) \otimes C \otimes P \to D \otimes Q$$

$$\bar{\beta}\colon A \otimes \Gamma^\star(I \otimes J^\vee \otimes B) \otimes C \to D \qquad \beta\colon P \to Q$$

such that

$$\langle \Gamma, \alpha, \bar{\alpha} \rangle_\varepsilon = \bar{\beta} \otimes \beta\colon A \otimes \Gamma^\star(I \otimes J^\vee \otimes B) \otimes C \otimes P \to D \otimes Q,$$

there is a non-zero morphism

$$\gamma\colon A \otimes \Gamma^\star(B) \otimes C \to D$$

such that the following hold.

$$\gamma \otimes \beta = \bar{\alpha}\colon A \otimes \Gamma^\star(B) \otimes C \otimes P \to D \otimes Q$$

$$\langle \Gamma, \alpha, \gamma \rangle_\varepsilon = \bar{\beta}\colon A \otimes \Gamma^\star(I \otimes J^\vee \otimes B) \otimes C \to D$$

Proof. Since $\langle \Gamma, \alpha, \bar{\alpha} \rangle_\varepsilon = \bar{\beta} \otimes \beta$, it follows that

$$\alpha^i_j \bar{\alpha}^{abcp}_{dq} \iff \bar{\beta}^{aijbc}_d \beta^p_q.$$

By Lemma A.1.6, there exists a non-zero relation $\gamma\colon A \otimes \Gamma^\star(B) \otimes C \dashrightarrow D$ such that $\gamma \otimes \beta = \bar{\alpha}$ and $\langle \Gamma, \alpha, \gamma \rangle_\varepsilon = \bar{\beta}$. By Lemma A.1.2 or Lemma A.1.3, γ is a morphism $A \otimes \Gamma^\star(B) \otimes C \to D$. □

Lemma A.1.9. *Given non-zero morphisms*

$$\alpha\colon I \to J \qquad \bar{\alpha}\colon P \otimes A \otimes \Gamma^\star(B) \otimes C \to Q \otimes D$$

$$\beta\colon P \to Q \qquad \bar{\beta}\colon A \otimes \Gamma^\star(I \otimes J^\vee \otimes B) \otimes C \to D$$

such that

$$\langle \Gamma, \alpha, \bar{\alpha} \rangle_\varepsilon = \beta \otimes \bar{\beta}\colon P \otimes A \otimes \Gamma^\star(I \otimes J^\vee \otimes B) \otimes C \to Q \otimes D,$$

there is a non-zero morphism

$$\gamma\colon A \otimes \Gamma^\star(B) \otimes C \to D$$

such that the following hold.

$$\beta \otimes \gamma = \bar{\alpha}\colon P \otimes A \otimes \Gamma^\star(B) \otimes C \to Q \otimes D$$

$$\langle \Gamma, \alpha, \gamma \rangle_\varepsilon = \bar{\beta}\colon A \otimes \Gamma^\star(I \otimes J^\vee \otimes B) \otimes C \to D$$

Proof. The proof of this lemma is similar to the proof of Lemma A.1.8. □

Lemma A.1.10. *Given non-zero morphisms*

$$\bar{\alpha}\colon A \otimes \Gamma^\star(B) \otimes C \otimes P \to D \otimes Q$$

$$\bar{\beta}\colon A \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes C \to D \qquad \beta\colon P \to Q$$

such that

$$\langle \Gamma, \Phi, \bar{\alpha} \rangle_\zeta = \bar{\beta} \otimes \beta\colon A \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes C \otimes P \to D \otimes Q,$$

there is a non-zero morphism

$$\gamma\colon A \otimes \Gamma^\star(B) \otimes C \to D$$

such that the following hold.

$$\gamma \otimes \beta = \bar{\alpha}\colon A \otimes \Gamma^\star(B) \otimes C \otimes P \to D \otimes Q$$

$$\langle \Gamma, \Phi, \gamma \rangle_\zeta = \bar{\beta}\colon A \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes C \to D$$

Proof. Since $\langle \Gamma, \Phi, \bar{\alpha}\rangle_\zeta = \bar{\beta} \otimes \beta$, it follows that

$$\delta^i_0 \bar{\alpha}^{abcp}_{dq} \iff \bar{\beta}^{aibc}_d \beta^p_q.$$

By Lemma A.1.6, there exists a non-zero relation $\gamma\colon A \otimes \Gamma^\star(B) \otimes C \dashrightarrow D$ such that $\gamma \otimes \beta = \bar{\alpha}$ and $\langle \Gamma, \Phi, \gamma\rangle_\zeta = \bar{\beta}$. By Lemma A.1.2 or Lemma A.1.4, γ is a morphism $A \otimes \Gamma^\star(B) \otimes C \to D$. □

Lemma A.1.11. *Given non-zero morphisms*

$$\bar{\alpha}\colon P \otimes A \otimes \Gamma^\star(B) \otimes C \to Q \otimes D$$

$$\beta\colon P \to Q \qquad \bar{\beta}\colon A \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes C \to D$$

such that

$$\langle \Gamma, \Phi, \bar{\alpha}\rangle_\zeta = \beta \otimes \bar{\beta}\colon P \otimes A \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes C \to Q \otimes D,$$

there is a non-zero morphism

$$\gamma\colon A \otimes \Gamma^\star(B) \otimes C \to D$$

such that the following hold.

$$\beta \otimes \gamma = \bar{\alpha}\colon P \otimes A \otimes \Gamma^\star(B) \otimes C \to Q \otimes D$$

$$\langle \Gamma, \Phi, \gamma\rangle_\zeta = \bar{\beta}\colon A \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes C \to D$$

Proof. The proof of this lemma is similar to the proof of Lemma A.1.10. □

Lemma A.1.12. *Given non-zero morphisms*

$$\alpha\colon I \to J \qquad \bar{\alpha}\colon A \otimes \Gamma^\star(B) \otimes C \otimes \Delta^\star \Phi^\star \Phi_\star(D) \otimes E \to F$$

$$\bar{\beta}\colon A \otimes \Gamma^\star(I \otimes J^\vee \otimes B) \otimes C \otimes \Delta^\star(D) \otimes E \to F$$

such that

$$\langle \Gamma, \alpha, \bar{\alpha}\rangle_\varepsilon = \langle \Delta, \Phi, \bar{\beta}\rangle_\zeta\colon A \otimes \Gamma^\star(I \otimes J^\vee \otimes B) \otimes C \otimes \Delta^\star \Phi^\star \Phi_\star(D) \otimes E \to F,$$

there is a non-zero morphism

$$\gamma\colon A \otimes \Gamma^\star(B) \otimes C \otimes \Delta^\star(D) \otimes E \to F$$

such that the following hold.

$$\langle \Gamma, \alpha, \gamma\rangle_\varepsilon = \bar{\beta}\colon A \otimes \Gamma^\star(I \otimes J^\vee \otimes B) \otimes C \otimes \Delta^\star(D) \otimes E \to F$$

$$\langle \Delta, \Phi, \gamma\rangle_\zeta = \bar{\alpha}\colon A \otimes \Gamma^\star(B) \otimes C \otimes \Delta^\star \Phi^\star \Phi_\star(D) \otimes E \to F$$

Proof. Since $\langle \Gamma, \alpha, \bar{\alpha} \rangle_\varepsilon = \langle \Delta, \Phi, \bar{\beta} \rangle_\zeta$, it follows that

$$\alpha^i_j \bar{\alpha}^{abckde}_f \iff \delta^k_0 \bar{\beta}^{aijbcde}_f.$$

By Lemma A.1.6, there exists a non-zero relation $\gamma\colon A \otimes \Gamma^\star(B) \otimes C \otimes \Delta^\star(D) \otimes E \dashrightarrow F$ such that $\langle \Gamma, \alpha, \gamma \rangle_\varepsilon = \bar{\beta}$ and $\langle \Delta, \Phi, \gamma \rangle_\zeta = \bar{\alpha}$. By Lemma A.1.3 or Lemma A.1.4, γ is a morphism $A \otimes \Gamma^\star(B) \otimes C \otimes \Delta^\star(D) \otimes E \to F$. □

Lemma A.1.13. *Given non-zero morphisms*

$$\bar{\alpha}\colon A \otimes \Gamma^\star(B) \otimes C \otimes \Delta^\star(I \otimes J^\vee \otimes D) \otimes E \to F$$

$$\beta\colon I \to J \qquad \bar{\beta}\colon A \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes C \otimes \Delta^\star(D) \otimes E \to F$$

such that

$$\langle \Gamma, \Phi, \bar{\alpha} \rangle_\zeta = \langle \Delta, \beta, \bar{\beta} \rangle_\varepsilon \colon A \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes C \otimes \Delta^\star(I \otimes J^\vee \otimes D) \otimes E \to F,$$

there is a non-zero morphism

$$\gamma\colon A \otimes \Gamma^\star(B) \otimes C \otimes \Delta^\star(D) \otimes E \to F$$

such that the following hold.

$$\langle \Gamma, \Phi, \gamma \rangle_\zeta = \bar{\beta}\colon A \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes C \otimes \Delta^\star(D) \otimes E \to F$$

$$\langle \Delta, \beta, \gamma \rangle_\varepsilon = \bar{\alpha}\colon A \otimes \Gamma^\star(B) \otimes C \otimes \Delta^\star(I \otimes J^\vee \otimes D) \otimes E \to F$$

Proof. The proof of this lemma is similar to the proof of Lemma A.1.12. □

Lemma A.1.14. *Given non-zero morphisms*

$$\alpha\colon I \to J \qquad \bar{\alpha}\colon A \otimes \Gamma^\star(B) \otimes C \otimes \Delta^\star(K \otimes L^\vee \otimes D) \otimes E \to F$$

$$\beta\colon K \to L \qquad \bar{\beta}\colon A \otimes \Gamma^\star(I \otimes J^\vee \otimes B) \otimes C \otimes \Delta^\star(D) \otimes E \to F$$

such that

$$\langle \Gamma, \alpha, \bar{\alpha} \rangle_\varepsilon = \langle \Delta, \beta, \bar{\beta} \rangle_\varepsilon \colon A \otimes \Gamma^\star(I \otimes J^\vee \otimes B) \otimes C \otimes \Delta^\star(K \otimes L^\vee \otimes D) \otimes E \to F,$$

there is a non-zero morphism

$$\gamma\colon A \otimes \Gamma^\star(B) \otimes C \otimes \Delta^\star(D) \otimes E \to F$$

such that the following hold.

$$\langle \Gamma, \alpha, \gamma \rangle_\varepsilon = \bar{\beta}\colon A \otimes \Gamma^\star(I \otimes J^\vee \otimes B) \otimes C \otimes \Delta^\star(D) \otimes E \to F$$

$$\langle \Delta, \beta, \gamma \rangle_\zeta = \bar{\alpha}\colon A \otimes \Gamma^\star(B) \otimes C \otimes \Delta^\star(K \otimes L^\vee \otimes D) \otimes E \to F$$

Proof. The proof of this lemma is similar to the proof of Lemma A.1.12. □

Lemma A.1.15. *Given non-zero morphisms*

$$\bar{\alpha}\colon A \otimes \Gamma^\star(B) \otimes C \otimes \Delta^\star\Psi^\star\Psi_\star(D) \otimes E \to F$$

$$\bar{\beta}\colon A \otimes \Gamma^\star\Phi^\star\Phi_\star(B) \otimes C \otimes \Delta^\star(D) \otimes E \to F$$

such that

$$\langle \Gamma, \Phi, \bar{\alpha}\rangle_\zeta = \langle \Delta, \Psi, \bar{\beta}\rangle_\zeta\colon A \otimes \Gamma^\star\Phi^\star\Phi_\star(B) \otimes C \otimes \Delta^\star\Psi^\star\Psi_\star(D) \otimes E \to F,$$

there is a non-zero morphism

$$\gamma\colon A \otimes \Gamma^\star(B) \otimes C \otimes \Delta^\star(D) \otimes E \to F$$

such that the following hold.

$$\langle \Gamma, \Phi, \gamma\rangle_\zeta = \bar{\beta}\colon A \otimes \Gamma^\star\Phi^\star\Phi_\star(B) \otimes C \otimes \Delta^\star(D) \otimes E \to F$$

$$\langle \Delta, \Psi, \gamma\rangle_\zeta = \bar{\alpha}\colon A \otimes \Gamma^\star(B) \otimes C \otimes \Delta^\star\Psi^\star\Psi_\star(D) \otimes E \to F$$

Proof. The proof of this lemma is similar to the proof of Lemma A.1.12. □

Lemma A.1.16. *Given non-zero morphisms*

$$\alpha\colon C \to D \qquad \bar{\alpha}\colon A \otimes \Gamma^\star(B \otimes \Delta^\star(E) \otimes F \otimes G^\vee \otimes H) \otimes I \to J$$

$$\bar{\beta}\colon B \otimes \Delta^\star(C \otimes D^\vee \otimes E) \otimes F \to G \qquad \beta\colon A \otimes \Gamma^\star(H) \otimes I \to J$$

such that

$$\langle \Gamma\Delta, \alpha, \bar{\alpha}\rangle_\varepsilon = \langle \Gamma, \bar{\beta}, \beta\rangle_\varepsilon\colon A \otimes \Gamma^\star(B \otimes \Delta^\star(C \otimes D^\vee \otimes E) \otimes F \otimes G^\vee \otimes H) \otimes I \to J,$$

there is a non-zero morphism

$$\gamma\colon B \otimes \Delta^\star(E) \otimes F \to G$$

such that the following hold.

$$\langle \Gamma, \gamma, \beta\rangle_\varepsilon = \bar{\alpha}\colon A \otimes \Gamma^\star(B \otimes \Delta^\star(E) \otimes F \otimes G^\vee \otimes H) \otimes I \to J$$

$$\langle \Delta, \alpha, \gamma\rangle_\varepsilon = \bar{\beta}\colon B \otimes \Delta^\star(C \otimes D^\vee \otimes E) \otimes F \to G$$

Proof. Since $\langle \Gamma\Delta, \alpha, \bar{\alpha}\rangle_\varepsilon = \langle \Gamma, \bar{\beta}, \beta\rangle_\varepsilon$, it follows that

$$\alpha^c_d \bar{\alpha}^{abefghi}_j \iff \bar{\beta}^{bcdef}_g \beta^{ahi}_j.$$

By Lemma A.1.6, there exists a non-zero relation $\gamma\colon B \otimes \Delta^\star(E) \otimes F \dashrightarrow G$ such that $\langle \Gamma, \gamma, \beta\rangle_\varepsilon = \bar{\alpha}$ and $\langle \Delta, \alpha, \gamma\rangle_\varepsilon = \bar{\beta}$. By Lemma A.1.3, γ is a morphism $B \otimes \Delta^\star(E) \otimes F \to G$. □

Lemma A.1.17. *Given non-zero morphisms*

$$\bar{\alpha}\colon A \otimes \Gamma^\star(B \otimes \Delta^\star(C) \otimes D \otimes E^\vee \otimes F) \otimes G \to H$$

$$\bar{\beta}\colon B \otimes \Delta^\star\Phi^\star\Phi_\star(C) \otimes D \to E \qquad \beta\colon A \otimes \Gamma^\star(F) \otimes G \to H$$

such that

$$\langle \Delta\Gamma, \Phi, \bar{\alpha}\rangle_\zeta = \langle \Gamma, \bar{\beta}, \beta\rangle_\varepsilon \colon A \otimes \Gamma^\star(B \otimes \Delta^\star\Phi^\star\Phi_\star(C) \otimes D \otimes E^\vee \otimes F) \otimes G \to H,$$

there is a non-zero morphism

$$\gamma\colon B \otimes \Delta^\star(C) \otimes D \to E$$

such that the following hold.

$$\langle \Gamma, \gamma, \beta\rangle_\varepsilon = \bar{\alpha}\colon A \otimes \Gamma^\star(B \otimes \Delta^\star(C) \otimes D \otimes E^\vee \otimes F) \otimes G \to H$$

$$\langle \Delta, \Phi, \gamma\rangle_\zeta = \bar{\beta}\colon B \otimes \Delta^\star\Phi^\star\Phi_\star(C) \otimes D \to E$$

Proof. Since $\langle \Delta\Gamma, \Phi, \bar{\alpha}\rangle_\zeta = \langle \Gamma, \bar{\beta}, \beta\rangle_\varepsilon$, it follows that

$$\delta_0^i \bar{\alpha}_h^{abcdefg} \iff \bar{\beta}_e^{bicd}\beta_h^{afg}.$$

By Lemma A.1.6, there exists a non-zero relation $\gamma\colon B \otimes \Delta^\star(C) \otimes D \to E$ such that $\langle \Gamma, \gamma, \beta\rangle_\varepsilon = \bar{\alpha}$ and $\langle \Delta, \Phi, \gamma\rangle_\zeta = \bar{\beta}$. By Lemma A.1.4, γ is a morphism $B \otimes \Delta^\star(C) \otimes D \to E$. □

Lemma A.1.18. *Given non-zero morphisms*

$$\alpha\colon C \to D \qquad \bar{\alpha}\colon A \otimes \Gamma^\star(B \otimes E) \otimes F \to G$$

$$\bar{\beta}\colon B \otimes C \to D \qquad \beta\colon A \otimes \Gamma^\star(E) \otimes F \to G$$

such that

$$\langle \Gamma, \alpha, \bar{\alpha}\rangle_\varepsilon = \langle \Gamma, \bar{\beta}, \beta\rangle_\varepsilon \colon A \otimes \Gamma^\star(B \otimes C \otimes D^\vee \otimes E) \otimes F \to G,$$

there is a non-zero morphism

$$\gamma\colon B \to \mathcal{I}$$

such that the following hold.

$$\langle \Gamma, \gamma, \alpha\rangle_\varepsilon = \bar{\beta}\colon B \otimes C \to D$$

$$\langle \Gamma, \gamma, \beta\rangle_\varepsilon = \bar{\alpha}\colon A \otimes \Gamma^\star(B \otimes E) \otimes F \to G$$

Proof. Since $\langle \Gamma, \alpha, \bar{\alpha} \rangle_\varepsilon = \langle \Gamma, \bar{\beta}, \beta \rangle_\varepsilon$, it follows that

$$\alpha^c_d \bar{\alpha}^{abef}_g \iff \bar{\beta}^{bc}_d \beta^{aef}_g.$$

By Lemma A.1.6, there exists a non-zero relation $\gamma\colon B \to \mathcal{I}$ such that $\langle \Gamma, \gamma, \alpha \rangle_\varepsilon = \bar{\beta}$ and $\langle \Gamma, \gamma, \beta \rangle_\varepsilon = \bar{\alpha}$. By Lemma A.1.3, γ is a morphism $B \to \mathcal{I}$. □

Lemma A.1.19. *Given non-zero morphisms*

$$\bar{\alpha}_1\colon \Gamma^\star(A) \to \Gamma^\star(B) \qquad \bar{\alpha}_2\colon \Gamma^\star(C) \to \Gamma^\star(D)$$

$$\bar{\beta}\colon A \otimes C \to B \otimes D$$

such that

$$\bar{\alpha}_1 \otimes \bar{\alpha}_2 = \Gamma^\star(\bar{\beta})\colon \Gamma^\star(A \otimes C) \to \Gamma^\star(B \otimes D),$$

there are non-zero morphisms

$$\gamma_1\colon A \to B \qquad \gamma_2\colon C \to D$$

such that the following hold.

$$\Gamma^\star(\gamma_1) = \bar{\alpha}_1\colon \Gamma^\star(A) \to \Gamma^\star(B)$$

$$\Gamma^\star(\gamma_2) = \bar{\alpha}_2\colon \Gamma^\star(C) \to \Gamma^\star(D)$$

$$\gamma_1 \otimes \gamma_2 = \bar{\beta}\colon A \otimes C \to B \otimes D$$

Proof. Since $\bar{\alpha}_1 \otimes \bar{\alpha}_2 = \Gamma^\star(\bar{\beta})$, it follows that

$$(\bar{\alpha}_1)^a_b (\bar{\alpha}_2)^c_d \iff \bar{\beta}^{ac}_{bd}.$$

Define γ_1 and γ_2 as follows.

$$(\gamma_1)^a_b = (\bar{\alpha}_1)^a_b \qquad (\gamma_2)^c_d = (\bar{\alpha}_2)^c_d$$

To see that $\Gamma^\star(\gamma_1) = \bar{\alpha}_1$, note the following.

$$\Gamma^\star(\gamma_1)^a_b = (\gamma_1)^a_b = (\bar{\alpha}_1)^a_b$$

To see that $\Gamma^\star(\gamma_2) = \bar{\alpha}_2$, note the following.

$$\Gamma^\star(\gamma_2)^c_d = (\gamma_2)^c_d = (\bar{\alpha}_2)^c_d$$

To see that $\gamma_1 \otimes \gamma_2 = \bar{\beta}$, note the following.

$$(\gamma_1 \otimes \gamma_2)^{ac}_{bd} = (\gamma_1)^a_b (\gamma_2)^c_d = (\bar{\alpha}_1)^a_b (\bar{\alpha}_2)^c_d = \bar{\beta}^{ac}_{bd}$$

By Lemma A.1.2, γ_1 is a morphism $A \to B$. By Lemma A.1.2, γ_2 is a morphism $C \to D$. □

Lemma A.1.20. *Given non-zero morphisms*

$$\alpha\colon P \to Q \qquad \bar{\alpha}\colon \Gamma^\star(A \otimes \Delta^\star(B) \otimes C) \to \Gamma^\star(D)$$
$$\bar{\beta}\colon A \otimes \Delta^\star(P \otimes Q^\vee \otimes B) \otimes C \to D$$

such that

$$\langle \Delta\Gamma, \alpha, \bar{\alpha}\rangle_\varepsilon = \Gamma^\star(\bar{\beta})\colon \Gamma^\star(A \otimes \Delta^\star(P \otimes Q^\vee \otimes B) \otimes C) \to \Gamma^\star(D),$$

there is a non-zero morphism

$$\gamma\colon A \otimes \Delta^\star(B) \otimes C \to D$$

such that the following hold.

$$\Gamma^\star(\gamma) = \bar{\alpha}\colon \Gamma^\star(A \otimes \Delta^\star(B) \otimes C) \to \Gamma^\star(D)$$
$$\langle \Delta, \alpha, \gamma\rangle_\varepsilon = \bar{\beta}\colon A \otimes \Delta^\star(P \otimes Q^\vee \otimes B) \otimes C \to D$$

Proof. Since $\langle \Delta\Gamma, \alpha, \bar{\alpha}\rangle_\varepsilon = \Gamma^\star(\bar{\beta})$, it follows that

$$\alpha^p_q \bar{\alpha}^{abc}_d \iff \bar{\beta}^{apqbc}_d.$$

Define γ as follows.

$$\gamma^{abc}_d = \bar{\alpha}^{abc}_d$$

To see that $\Gamma^\star(\gamma) = \bar{\alpha}$, note the following.

$$\Gamma^\star(\gamma)^{abc}_d = \gamma^{abc}_d = \bar{\alpha}^{abc}_d$$

To see that $\langle \Delta, \alpha, \gamma\rangle_\varepsilon = \bar{\beta}$, note the following.

$$(\langle \Delta, \alpha, \gamma\rangle_\varepsilon)^{apqbc}_d = \alpha^p_q \gamma^{abc}_d = \alpha^p_q \bar{\alpha}^{abc}_d = \bar{\beta}^{apqbc}_d$$

By Lemma A.1.3, γ is a morphism $A \otimes \Delta^\star(B) \otimes C \to D$. □

Lemma A.1.21. *Given non-zero morphisms*

$$\bar{\alpha}\colon \Gamma^\star(A \otimes \Delta^\star(B) \otimes C) \to \Gamma^\star(D)$$
$$\bar{\beta}\colon A \otimes \Delta^\star\Phi^\star\Phi_\star(B) \otimes C \to D$$

such that

$$\langle \Delta\Gamma, \Phi, \bar{\alpha}\rangle_\zeta = \Gamma^\star(\bar{\beta})\colon \Gamma^\star(A \otimes \Delta^\star\Phi^\star\Phi_\star(B) \otimes C) \to \Gamma^\star(D),$$

there is a non-zero morphism

$$\gamma\colon A \otimes \Delta^\star(B) \otimes C \to D$$

such that the following hold.

$$\Gamma^\star(\gamma) = \bar{\alpha}\colon \Gamma^\star(A \otimes \Delta^\star(B) \otimes C) \to \Gamma^\star(D)$$
$$\langle \Delta, \Phi, \gamma\rangle_\zeta = \bar{\beta}\colon A \otimes \Delta^\star\Phi^\star\Phi_\star(B) \otimes C \to D$$

Proof. Since $\langle \Delta\Gamma, \Phi, \bar{\alpha}\rangle_\zeta = \Gamma^\star(\bar{\beta})$, it follows that

$$\delta_0^i \bar{\alpha}_d^{abc} \iff \bar{\beta}_d^{aibc}.$$

Define γ as follows.

$$\gamma_d^{abc} = \bar{\alpha}_d^{abc}$$

To see that $\Gamma^\star(\gamma) = \bar{\alpha}$, note the following.

$$\Gamma^\star(\gamma)_d^{abc} = \gamma_d^{abc} = \bar{\alpha}_d^{abc}$$

To see that $\langle \Delta, \Phi, \gamma\rangle_\zeta = \bar{\beta}$, note the following.

$$(\langle \Delta, \Phi, \gamma\rangle_\zeta)_d^{aibc} = \delta_0^i \gamma_d^{abc} = \delta_0^i \bar{\alpha}_d^{abc} = \bar{\alpha}_d^{aibc}$$

By Lemma A.1.4, γ is a morphism $A \otimes \Delta^\star(B) \otimes C \to D$. □

Lemma A.1.22. *Given non-zero morphisms*

$$\alpha_1 \colon A \otimes C \to B \qquad \beta_2 \colon C \otimes E \to F$$

$$\beta_1 \colon A \to B \otimes D \qquad \alpha_2 \colon E \to D \otimes F$$

such that

$$\alpha_1 \otimes \alpha_2 = \beta_1 \otimes \beta_2 \colon A \otimes C \otimes E \to B \otimes D \otimes F,$$

there are non-zero morphisms

$$\delta_1 \colon C \to \mathcal{I}$$

$$\gamma_1 \colon A \to B \qquad \gamma_2 \colon E \to F$$

$$\delta_2 \colon \mathcal{I} \to D$$

such that the following hold.

$$\gamma_1 \otimes \delta_1 = \alpha_1 \qquad \delta_1 \otimes \gamma_2 = \beta_2$$

$$\gamma_1 \otimes \delta_2 = \beta_1 \qquad \delta_2 \otimes \gamma_2 = \alpha_2$$

Proof. Since $\alpha_1 \otimes \alpha_2 = \beta_1 \otimes \beta_2$, it follows that

$$(\alpha_1)_b^{ac}(\alpha_2)_{df}^{e} \iff (\beta_1)_{bd}^{a}(\beta_2)_f^{ce}.$$

By Lemma A.1.5, there exist non-zero relations

$$\delta_1 \colon C \dashrightarrow \mathcal{I}$$

$$\gamma_1 \colon A \dashrightarrow B \qquad \gamma_2 \colon E \dashrightarrow F$$

$$\delta_2 \colon \mathcal{I} \dashrightarrow D$$

such that

$$\gamma_1 \otimes \delta_1 = \alpha_1 \qquad \delta_1 \otimes \gamma_2 = \beta_2$$

$$\gamma_1 \otimes \delta_2 = \beta_1 \qquad \delta_2 \otimes \gamma_2 = \alpha_2.$$

By Lemma A.1.2, δ_1 is a morphism $C \to \mathcal{I}$. By Lemma A.1.2, γ_1 is a morphism $A \to B$. By Lemma A.1.2, γ_2 is a morphism $E \to F$. By Lemma A.1.2, δ_2 is a morphism $\mathcal{I} \to D$. □

Lemma A.1.23. *Given non-zero morphisms*

$$\beta_1 \colon I \otimes J \to K$$

$$\alpha_1 \colon A \otimes \Gamma^\star(I) \to B \qquad \alpha_2 \colon \Gamma^\star(J \otimes K^\vee \otimes C) \otimes D \to E$$

$$\beta_2 \colon A \otimes \Gamma^\star(C) \otimes D \to B \otimes E$$

such that

$$\alpha_1 \otimes \alpha_2 = \langle \Gamma, \beta_1, \beta_2 \rangle_\varepsilon \colon A \otimes \Gamma^\star(I \otimes J \otimes K^\vee \otimes C) \otimes D \to B \otimes E,$$

there are non-zero morphisms

$$\delta_1 \colon I \to \mathcal{I} \qquad \gamma_2 \colon J \to K$$

$$\gamma_1 \colon A \to B \qquad \delta_2 \colon \Gamma^\star(C) \otimes D \to E$$

such that the following hold.

$$\delta_1 \otimes \gamma_2 = \beta_1$$

$$\langle \Gamma, \delta_1, \gamma_1 \rangle_\varepsilon = \alpha_1 \qquad \langle \Gamma, \gamma_2, \delta_2 \rangle_\varepsilon = \alpha_2$$

$$\gamma_1 \otimes \delta_2 = \beta_2$$

Proof. Since $\alpha_1 \otimes \alpha_2 = \langle \Gamma, \beta_1, \beta_2 \rangle_\varepsilon$, it follows that

$$(\alpha_1)^{ai}_b (\alpha_2)^{jkcd}_e \iff (\beta_1)^{ij}_k (\beta_2)^{acd}_{be}.$$

By Lemma A.1.5, there exist non-zero relations

$$\delta_1 \colon I \dashrightarrow \mathcal{I} \qquad \gamma_2 \colon J \dashrightarrow K$$

$$\gamma_1 \colon A \dashrightarrow B \qquad \delta_2 \colon \Gamma^\star(C) \otimes D \dashrightarrow E$$

such that

$$\delta_1 \otimes \gamma_2 = \beta_1$$

$$\langle \Gamma, \delta_1, \gamma_1 \rangle_\varepsilon = \alpha_1 \qquad \langle \Gamma, \gamma_2, \delta_2 \rangle_\varepsilon = \alpha_2$$

$$\gamma_1 \otimes \delta_2 = \beta_2.$$

By Lemma A.1.2, δ_1 is a morphism $I \to \mathcal{I}$. By Lemma A.1.2, γ_2 is a morphism $J \to K$. By Lemma A.1.2 or Lemma A.1.3, γ_1 is a morphism $A \to B$. By Lemma A.1.2 or Lemma A.1.3, δ_2 is a morphism $\Gamma^\star(C) \otimes D \to E$. □

Lemma A.1.24. *Given non-zero morphisms*

$$\alpha_1 \colon B \otimes C \to D \qquad \beta_1 \colon \Lambda^\star(C \otimes D^\vee \otimes E) \otimes F \to G$$

$$\beta_2 \colon A \otimes \Gamma^\star(\Lambda^\star(B) \otimes H) \otimes I \to J \qquad \alpha_2 \colon A \otimes \Gamma^\star(\Lambda^\star(E) \otimes F \otimes G^\vee \otimes H) \otimes I \to J$$

such that

$$\langle \Lambda\Gamma, \alpha_1, \alpha_2 \rangle_\varepsilon = \langle \Gamma, \beta_1, \beta_2 \rangle_\varepsilon \colon A \otimes \Gamma^\star(\Lambda^\star(B \otimes C \otimes D^\vee \otimes E) \otimes F \otimes G^\vee \otimes H) \otimes I \to J,$$

there are non-zero morphisms

$$\delta_1 \colon C \to D$$

$$\gamma_1 \colon B \to \mathcal{I} \qquad \gamma_2 \colon \Lambda^\star(E) \otimes F \to G$$

$$\delta_2 \colon A \otimes \Gamma^\star(H) \otimes I \to J$$

such that the following hold.

$$\langle \mathrm{id}, \gamma_1, \delta_1 \rangle_\varepsilon = \alpha_1 \qquad \langle \Lambda, \delta_1, \gamma_2 \rangle_\varepsilon = \beta_1$$

$$\langle \Lambda\Gamma, \gamma_1, \delta_2 \rangle_\varepsilon = \beta_2 \qquad \langle \Gamma, \gamma_2, \delta_2 \rangle_\varepsilon = \alpha_2$$

Proof. Since $\langle \Lambda\Gamma, \alpha_1, \alpha_2 \rangle_\varepsilon = \langle \Gamma, \beta_1, \beta_2 \rangle_\varepsilon$, it follows that

$$(\alpha_1)_d^{bc} (\alpha_2)_j^{aefghi} \iff (\beta_1)_g^{cdef} (\beta_2)_j^{abhi}.$$

By Lemma A.1.5, there exist non-zero relations

$$\delta_1 \colon C \dashrightarrow D$$

$$\gamma_1 \colon B \dashrightarrow \mathcal{I} \qquad \gamma_2 \colon \Lambda^\star(E) \otimes F \dashrightarrow G$$

$$\delta_2 \colon A \otimes \Gamma^\star(H) \otimes I \dashrightarrow J$$

such that

$$\langle \mathrm{id}, \gamma_1, \delta_1 \rangle_\varepsilon = \alpha_1 \qquad \langle \Lambda, \delta_1, \gamma_2 \rangle_\varepsilon = \beta_1$$

$$\langle \Lambda\Gamma, \gamma_1, \delta_2 \rangle_\varepsilon = \beta_2 \qquad \langle \Gamma, \gamma_2, \delta_2 \rangle_\varepsilon = \alpha_2.$$

By Lemma A.1.3, δ_1 is a morphism $C \to D$. By Lemma A.1.3, γ_1 is a morphism $B \to \mathcal{I}$. By Lemma A.1.3, γ_2 is a morphism $\Lambda^\star(E) \otimes F \to G$. By Lemma A.1.3, δ_2 is a morphism $A \otimes \Gamma^\star(H) \otimes I \to J$. □

Lemma A.1.25. *Given non-zero morphisms*

$$\alpha_1 \colon B \otimes \Lambda^\star(C) \to \mathcal{I} \qquad \beta_1 \colon C \otimes D \to E$$

$$\beta_2 \colon A \otimes \Gamma^\star(B \otimes \Lambda^\star(F)) \otimes G \to H \qquad \alpha_2 \colon A \otimes (\Lambda\Gamma)^\star(D \otimes E^\vee \otimes F) \otimes G \to H$$

such that

$$\langle \Gamma, \alpha_1, \alpha_2 \rangle_\varepsilon = \langle \Lambda\Gamma, \beta_1, \beta_2 \rangle_\varepsilon \colon A \otimes \Gamma^\star(B \otimes \Lambda^\star(C \otimes D \otimes E^\vee \otimes F)) \otimes G \to H,$$

there are non-zero morphisms

$$\delta_1 \colon C \to \mathcal{I}$$

$$\gamma_1 \colon B \to \mathcal{I} \qquad \gamma_2 \colon D \to E$$

$$\delta_2 \colon A \otimes (\Lambda\Gamma)^\star(F) \otimes G \to H$$

such that the following hold.

$$\langle \Lambda, \delta_1, \gamma_1 \rangle_\varepsilon = \alpha_1 \qquad \langle \mathrm{id}, \delta_1, \gamma_2 \rangle_\varepsilon = \beta_1$$

$$\langle \Gamma, \gamma_1, \delta_2 \rangle_\varepsilon = \beta_2 \qquad \langle \Lambda\Gamma, \gamma_2, \delta_2 \rangle_\varepsilon = \alpha_2$$

Proof. Since $\langle \Gamma, \alpha_1, \alpha_2 \rangle_\varepsilon = \langle \Lambda\Gamma, \beta_1, \beta_2 \rangle_\varepsilon$, it follows that

$$(\alpha_1)^{bc}_\star (\alpha_2)^{adefg}_h \iff (\beta_1)^{cd}_e (\beta_2)^{abfg}_h.$$

By Lemma A.1.5, there exist non-zero relations

$$\delta_1 \colon C \dashrightarrow \mathcal{I}$$

$$\gamma_1 \colon B \dashrightarrow \mathcal{I} \qquad \gamma_2 \colon D \dashrightarrow E$$

$$\delta_2 \colon A \otimes (\Lambda\Gamma)^\star(F) \otimes G \dashrightarrow H$$

such that

$$\langle \Lambda, \delta_1, \gamma_1 \rangle_\varepsilon = \alpha_1 \qquad \langle \mathrm{id}, \delta_1, \gamma_2 \rangle_\varepsilon = \beta_1$$

$$\langle \Gamma, \gamma_1, \delta_2 \rangle_\varepsilon = \beta_2 \qquad \langle \Lambda\Gamma, \gamma_2, \delta_2 \rangle_\varepsilon = \alpha_2.$$

By Lemma A.1.3, δ_1 is a morphism $C \to \mathcal{I}$. By Lemma A.1.3, γ_1 is a morphism $B \to \mathcal{I}$. By Lemma A.1.3, γ_2 is a morphism $D \to E$. By Lemma A.1.3, δ_2 is a morphism $A \otimes (\Lambda\Gamma)^\star(F) \otimes G \to H$. □

Lemma A.1.26. *Given non-zero morphisms*

$$\alpha_1 \colon \Lambda^\star(C) \to \mathcal{I} \qquad \beta_1 \colon B \otimes C \otimes D \to E$$

$$\beta_2 \colon A \otimes (\Lambda\Gamma)^\star(F) \otimes G \to H \qquad \alpha_2 \colon A \otimes (\Lambda\Gamma)^\star(B \otimes D \otimes E^\vee \otimes F) \otimes G \to H$$

such that

$$\langle \Gamma, \alpha_1, \alpha_2 \rangle_\varepsilon = \langle \Lambda\Gamma, \beta_1, \beta_2 \rangle_\varepsilon \colon A \otimes (\Lambda\Gamma)^\star(B \otimes C \otimes D \otimes E^\vee \otimes F) \otimes G \to H,$$

there are non-zero morphisms

$$\delta \colon C \to \mathcal{I}$$

$$\gamma \colon B \otimes D \to E$$

such that the following hold.

$$\Lambda^\star(\delta) = \alpha_1 \colon \Lambda^\star(C) \to \mathcal{I}$$

$$\langle \mathrm{id}, \delta, \gamma \rangle_\varepsilon = \beta_1 \colon B \otimes C \otimes D \to E$$

$$\langle \Lambda\Gamma, \gamma, \beta_2 \rangle_\varepsilon = \alpha_2 \colon A \otimes (\Lambda\Gamma)^\star(B \otimes D \otimes E^\vee \otimes F) \otimes G \to H$$

Proof. Choose $\bar{c}$ such that $(\alpha_1)^{\bar{c}}_\star$ holds and $\bar{a}$, $\bar{f}$, $\bar{g}$ and $\bar{h}$ such that $(\beta_2)^{\bar{a}\bar{f}\bar{g}}_{\bar{h}}$ holds. Since $\langle \Gamma, \alpha_1, \alpha_2 \rangle_\varepsilon = \langle \Lambda\Gamma, \beta_1, \beta_2 \rangle_\varepsilon$, it follows that

$$(\alpha_1)^c_\star (\alpha_2)^{\bar{a}bde\bar{f}\bar{g}}_{\bar{h}} \iff (\beta_1)^{bcd}_e$$

and

$$(\beta_1)^{b\bar{c}d}_e (\beta_2)^{afg}_h \iff (\alpha_2)^{abdefg}_h.$$

Define δ as follows.

$$\delta^c_\star \iff (\alpha_1)^c_\star$$

Define γ as follows.

$$\gamma^{bd}_e \iff (\alpha_2)^{\bar{a}bde\bar{f}\bar{g}}_{\bar{h}} \iff (\beta_1)^{b\bar{c}d}_e$$

To see that $\Lambda^\star(\delta) = \alpha_1$, note the following.

$$\Lambda^\star(\delta)^c_\star \iff \delta^c_\star \iff (\alpha_1)^c_\star$$

To see that $\langle \mathrm{id}, \delta, \gamma \rangle_\varepsilon = \beta_1$, note the following.

$$(\langle \mathrm{id}, \delta, \gamma \rangle_\varepsilon)^{bcd}_e \iff \delta^c_\star \gamma^{bd}_e \iff (\alpha_1)^c_\star (\alpha_2)^{\bar{a}bde\bar{f}\bar{g}}_{\bar{h}} \iff (\beta_1)^{bcd}_e$$

To see that $\langle \Lambda\Gamma, \gamma, \beta_2 \rangle_\varepsilon = \alpha_2$, note the following.

$$(\langle \Lambda\Gamma, \gamma, \beta_2 \rangle_\varepsilon)^{abdefg}_h \iff \gamma^{bd}_e (\beta_2)^{afg}_h$$

$$\iff (\beta_1)^{b\bar{c}d}_e (\beta_2)^{afg}_h \iff (\alpha_2)^{abdefg}_h$$

By Lemma A.1.3, δ is a morphism $C \to \mathcal{I}$. By Lemma A.1.3, γ is a morphism $B \otimes D \to E$. □

Lemma A.1.27. *Given non-zero morphisms*

$$\alpha_1 \colon \Lambda^\star(B) \to \mathcal{I} \qquad \alpha_2 \colon (\Lambda\Gamma)^\star(A \otimes C) \to (\Lambda\Gamma)^\star(D)$$

$$\beta \colon A \otimes B \otimes C \to D$$

such that

$$\langle \Gamma, \alpha_1, \alpha_2 \rangle_\varepsilon = (\Lambda\Gamma)^\star(\beta) \colon (\Lambda\Gamma)^\star(A \otimes B \otimes C) \to (\Lambda\Gamma)^\star(D),$$

there are non-zero morphisms

$$\gamma_1 \colon B \to \mathcal{I} \qquad \gamma_2 \colon A \otimes C \to D$$

such that the following hold.

$$\Lambda^\star(\gamma_1) = \alpha_1 \colon \Lambda^\star(B) \to \mathcal{I}$$

$$(\Lambda\Gamma)^\star(\gamma_2) = \alpha_2 \colon (\Lambda\Gamma)^\star(A \otimes C) \to (\Lambda\Gamma)^\star(D)$$

$$\langle \mathrm{id}, \gamma_1, \gamma_2 \rangle_\varepsilon = \beta \colon A \otimes B \otimes C \to D$$

Proof. Since $\langle \Gamma, \alpha_1, \alpha_2 \rangle_\varepsilon = (\Lambda\Gamma)^\star(\beta)$, it follows that

$$(\alpha_1)^b_\star (\alpha_2)^{ac}_d \iff \beta^{abc}_d.$$

Define (γ_1) as follows.

$$(\gamma_1)^b_\star = (\alpha_1)^b_\star$$

Define (γ_2) as follows.

$$(\gamma_2)^{ac}_d = (\alpha_2)^{ac}_d$$

To see that $\Lambda^\star(\gamma_1) = \alpha_1$, note the following.

$$\Lambda^\star(\gamma_1)^b_\star = (\gamma_1)^b_\star = (\alpha_1)^b_\star$$

To see that $(\Lambda\Gamma)^\star(\gamma_2) = \alpha$, note the following.

$$(\Lambda\Gamma)^\star(\gamma_2)^{ac}_d = (\gamma_2)^{ac}_d = (\alpha_2)^{ac}_d$$

To see that $\langle \mathrm{id}, \gamma_1, \gamma_2 \rangle_\varepsilon = \beta$, note the following.

$$(\langle \mathrm{id}, \gamma_1, \gamma_2 \rangle_\varepsilon)^{abc}_d = (\gamma_1)^b_\star (\gamma_2)^{ac}_d = (\alpha_1)^b_\star (\alpha_2)^{ac}_d = \beta^{abc}_d$$

By Lemma A.1.3, γ_1 is a morphism $B \to \mathcal{I}$. By Lemma A.1.3, γ_2 is a morphism $A \otimes C \to D$. □

Lemma A.1.28. *Given non-zero morphisms*

$$\alpha\colon A \otimes B \to A \qquad \beta\colon C \to B \otimes C$$

such that

$$\alpha \otimes \beta = \mathrm{id}_A \otimes \mathrm{id}_B \otimes \mathrm{id}_C,$$

the following holds.

$$B = \mathcal{I}$$

Proof. Since $\alpha \otimes \beta = \mathrm{id}$, it follows that

$$\alpha^{ab}_{a'} \beta^{c}_{b'c'} \iff \delta^{a}_{a'} \delta^{b}_{b'} \delta^{c}_{c'}.$$

Assume $B \neq \mathcal{I}$; choose $b_1 \neq b_2 \in B_0$. To derive a contradiction, note the following. Since $\alpha^{ab_1}_{a} \beta^{c}_{b_1 c} \iff \delta^{a}_{a} \delta^{b_1}_{b_1} \delta^{c}_{c}$ holds, it follows that $\alpha^{ab_1}_{a}$ holds. Since $\alpha^{ab_2}_{a} \beta^{c}_{b_2 c} \iff \delta^{a}_{a} \delta^{b_2}_{b_2} \delta^{c}_{c}$ holds, it follows that $\beta^{c}_{b_2 c}$ holds. But, since $\alpha^{ab_1}_{a} \beta^{c}_{b_2 c} \iff \delta^{a}_{a} \delta^{b_1}_{b_2} \delta^{c}_{c}$ does not hold, this is a contradiction. Thus $B = \mathcal{I}$. □

Lemma A.1.29. *Given non-zero morphisms*

$$\alpha\colon I \to J \qquad \beta\colon A \otimes \Gamma^{\star}(B) \otimes C \to A \otimes \Gamma^{\star}(I \otimes J^{\vee} \otimes B) \otimes C$$

such that

$$\langle \Gamma, \alpha, \beta \rangle_{\varepsilon} = \mathrm{id}_A \otimes \mathrm{id}_{\Gamma^{\star}(I)} \otimes \mathrm{id}_{\Gamma^{\star}(J^{\vee})} \otimes \mathrm{id}_{\Gamma^{\star}(B)} \otimes \mathrm{id}_C,$$

the following hold.

$$I = \mathcal{I} \qquad J = \mathcal{I}$$

Proof. Since $\langle \Gamma, \alpha, \beta \rangle_{\varepsilon} = \mathrm{id}$, it follows that

$$\alpha^{i}_{j} \beta^{abc}_{a'i'j'b'c'} \iff \delta^{a}_{a'} \delta^{i}_{i'} \delta^{j}_{j'} \delta^{b}_{b'} \delta^{c}_{c'}.$$

Assume $I \neq \mathcal{I}$; choose $i_1 \neq i_2 \in I_0$. To derive a contradiction, note the following. Since $\alpha^{i_1}_{j} \beta^{abc}_{ai_1jbc} \iff \delta^{a}_{a} \delta^{i_1}_{i_1} \delta^{j}_{j} \delta^{b}_{b} \delta^{c}_{c}$ holds, it follows that $\alpha^{i_1}_{j}$ holds. Since $\alpha^{i_2}_{j} \beta^{abc}_{ai_2jbc} \iff \delta^{a}_{a} \delta^{i_2}_{i_2} \delta^{j}_{j} \delta^{b}_{b} \delta^{c}_{c}$ holds, it follows that $\beta^{abc}_{ai_2jbc}$ holds. But, since $\alpha^{i_1}_{j} \beta^{abc}_{ai_2jbc} \iff \delta^{a}_{a} \delta^{i_1}_{i_2} \delta^{j}_{j} \delta^{b}_{b} \delta^{c}_{c}$ does not hold, this is a contradiction. Thus $I = \mathcal{I}$. Assume $J \neq \mathcal{I}$; choose $j_1 \neq j_2 \in J_0$. To derive a contradiction, note the following. Since $\alpha^{i}_{j_1} \beta^{abc}_{aij_1bc} \iff \delta^{a}_{a} \delta^{i}_{i} \delta^{j_1}_{j_1} \delta^{b}_{b} \delta^{c}_{c}$ holds, it follows that $\alpha^{i}_{j_1}$ holds. Since $\alpha^{i}_{j_2} \beta^{abc}_{aij_2bc} \iff \delta^{a}_{a} \delta^{i}_{i} \delta^{j_2}_{j_2} \delta^{b}_{b} \delta^{c}_{c}$ holds, it follows that $\beta^{abc}_{aij_2bc}$ holds. But, since $\alpha^{i}_{j_1} \beta^{abc}_{aij_2bc} \iff \delta^{a}_{a} \delta^{i}_{i} \delta^{j_1}_{j_2} \delta^{b}_{b} \delta^{c}_{c}$ does not hold, this is a contradiction. Thus $J = \mathcal{I}$. □

Lemma A.1.30. *Given a non-zero morphism*

$$\alpha\colon A \to A$$

such that

$$\Phi^\star(\alpha) = \mathrm{id}_{\Phi^\star(A)},$$

the following holds.

$$\alpha = \mathrm{id}_A$$

Proof.

$$\Phi^\star(\alpha) = \mathrm{id} \implies \alpha^a_{a'} = \delta^a_{a'}$$

□

Lemma A.1.31. *There is no non-zero morphism*

$$\alpha\colon A \otimes \Gamma^\star(B) \otimes C \to A \otimes \Gamma^\star\Phi^\star\Phi_\star(B) \otimes C$$

such that

$$\langle \Gamma, \Phi, \alpha\rangle_\zeta = \mathrm{id}_A \otimes \mathrm{id}_{\Gamma^\star\Phi^\star\Phi_\star(B)} \otimes \mathrm{id}_C$$

and

$$\Phi > 1.$$

Proof. Since $\langle \Gamma, \Phi, \alpha\rangle_\zeta = \mathrm{id}$, it follows that

$$\delta^i_0 \alpha^{abc}_{a'i'b'c'} \iff \delta^a_{a'}\delta^i_{i'}\delta^b_{b'}\delta^c_{c'}.$$

Choose $i_1 \neq i_2 \in \underline{\Phi}$. To derive a contradiction, note the following. Since $\delta^{i_1}_0 \alpha^{abc}_{ai_1bc} \iff \delta^a_a \delta^{i_1}_{i_1}\delta^b_b\delta^c_c$ holds, it follows that $\delta^{i_1}_0$ holds. Since $\delta^{i_2}_0 \alpha^{abc}_{ai_2bc} \iff \delta^a_a\delta^{i_2}_{i_2}\delta^b_b\delta^c_c$ holds, it follows that $\alpha^{abc}_{ai_2bc}$ holds. But, since $\delta^{i_1}_0\alpha^{abc}_{ai_2bc} \iff \delta^a_a\delta^{i_1}_{i_2}\delta^b_b\delta^c_c$ does not hold, this is a contradiction. □

Lemma A.1.32. *There are no non-zero morphisms*

$$\omega\colon \Gamma^\star(\Lambda^\star(A) \otimes B \otimes \Lambda^\star(C)) \to (\Lambda\Gamma)^\star(D)$$

$$\alpha\colon A \otimes \Psi^\star(\Psi\Lambda)_\star(B) \otimes C \to D$$

such that

$$\langle \Gamma, \Psi\Lambda, \omega\rangle_\zeta = (\Lambda\Gamma)^\star(\alpha)\colon (\Lambda\Gamma)^\star(A \otimes \Psi^\star(\Psi\Lambda)_\star(B) \otimes C) \to (\Lambda\Gamma)^\star(D)$$

and

$$\Lambda > 1.$$

Proof. Since $\langle \Gamma, \Psi\Lambda, \omega\rangle_\zeta = (\Lambda\Gamma)^\star(\alpha)$, it follows that

$$\delta_0^i \omega_d^{abc} \iff \alpha_d^{aibc}.$$

Since α is a morphism $A \otimes \Psi^\star(\Psi\Lambda)_\star(B) \otimes C \to D$, it follows that $(A \otimes \Psi^\star(\Psi\Lambda)_\star(B) \otimes C) \bullet \alpha = \alpha \bullet D$. Evaluating each side of this expression yields the following.

$$\begin{aligned}
&((A \otimes \Psi^\star(\Psi\Lambda)_\star(B) \otimes C) \bullet \alpha)_d^{aibc} \\
&\iff \exists a'i'b'c'((A \otimes \Psi^\star(\Psi\Lambda)_\star(B) \otimes C)_{a'i'b'c'}^{aibc} \alpha_d^{a'i'b'c'}) \\
&\iff \exists a'i'b'c'(A_{a'}^a \Psi^\star(\Psi\Lambda)_\star(B)_{i'b'}^{ib} C_{c'}^c \delta_0^{i'} \omega_d^{a'b'c'}) \\
&\iff \exists a'b'c'(A_{a'}^a \Psi^\star(\Psi\Lambda)_\star(B)_{0b'}^{ib} C_{c'}^c \omega_d^{a'b'c'}) \\
&\overset{(\star)}{\iff} \delta_\Psi^i \exists a'b'c'(A_{a'}^a \delta_{b'}^b C_{c'}^c \omega_d^{a'b'c'}) \\
&\iff \delta_\Psi^i \exists a'c'(A_{a'}^a C_{c'}^c \omega_d^{a'bc'})
\end{aligned}$$

Step $(\star)$ follows from the fact that $\Psi < \Psi\Lambda$, relying on our assumption that $\Lambda > 1$.

$$\begin{aligned}
(\alpha \bullet D)_d^{aibc} &\iff \exists d'(\alpha_{d'}^{aibc} D_d^{d'}) \\
&\iff \delta_0^i \exists d'(\omega_{d'}^{abc} D_d^{d'})
\end{aligned}$$

Thus, the following holds.

$$\delta_\Psi^i \exists a'c'(A_{a'}^a C_{c'}^c \omega_d^{a'bc'}) \iff \delta_0^i \exists d'(\omega_{d'}^{abc} D_d^{d'})$$

This implies that neither of the following holds.

$$\exists a'c'(A_{a'}^a C_{c'}^c \omega_d^{a'bc'}) \qquad \exists d'(\omega_{d'}^{abc} D_d^{d'})$$

But this contradicts our assumption that ω and α are non-zero. □

Lemma A.1.33. *There are no non-zero morphisms*

$$\omega\colon A \otimes \Gamma^\star(\Lambda^\star(B) \otimes C \otimes \Lambda^\star(D \otimes E^\vee \otimes F)) \otimes G \to H$$

$$\alpha\colon B \otimes \Psi^\star(\Psi\Lambda)_\star(C) \otimes D \to E \qquad \beta\colon A \otimes (\Lambda\Gamma)^\star(F) \otimes G \to H$$

such that

$$\langle \Gamma, \Psi\Lambda, \omega\rangle_\zeta = \langle \Lambda\Gamma, \alpha, \beta\rangle_\varepsilon\colon A \otimes (\Lambda\Gamma)^\star(B \otimes \Psi^\star(\Psi\Lambda)_\star(C) \otimes D \otimes E^\vee \otimes F) \otimes G \to H$$

and

$$\Lambda > 1.$$

Proof. Choose $\bar{a}$, $\bar{f}$, $\bar{g}$ and $\bar{h}$ such that $\beta_{\bar{h}}^{\bar{a}\bar{f}\bar{g}}$ holds. Since $\langle \Gamma, \Psi\Lambda, \omega \rangle_\zeta = \langle \Lambda\Gamma, \alpha, \beta \rangle_\varepsilon$, it follows that

$$\delta_0^i \omega_{\bar{h}}^{\bar{a}bcde\bar{f}\bar{g}} \iff \alpha_e^{bicd}.$$

Since α is a morphism $B \otimes \Psi^\star(\Psi\Lambda)_\star(C) \otimes D \to E$, it follows that $(B \otimes \Psi^\star(\Psi\Lambda)_\star(C) \otimes D) \bullet \alpha = \alpha \bullet E$. Evaluating each side of this expression yields the following.

$$\begin{aligned}
&((B \otimes \Psi^\star(\Psi\Lambda)_\star(C) \otimes D) \bullet \alpha)_e^{bicd} \\
&\iff \exists b'i'c'd'((B \otimes \Psi^\star(\Psi\Lambda)_\star(C) \otimes D)_{b'i'c'd'}^{bicd} \alpha_e^{b'i'c'd'}) \\
&\iff \exists b'i'c'd'(B_{b'}^b \Psi^\star(\Psi\Lambda)_\star(C)_{i'c'}^{ic} D_{d'}^d \delta_0^{i'} \omega_{\bar{h}}^{\bar{a}b'c'd'e\bar{f}\bar{g}}) \\
&\iff \exists b'c'd'(B_{b'}^b \Psi^\star(\Psi\Lambda)_\star(C)_{0c'}^{ic} D_{d'}^d \omega_{\bar{h}}^{\bar{a}b'c'd'e\bar{f}\bar{g}}) \\
&\overset{(\star)}{\iff} \delta_\Psi^i \exists b'c'd'(B_{b'}^b \delta_{c'}^c D_{d'}^d \omega_{\bar{h}}^{\bar{a}b'c'd'e\bar{f}\bar{g}}) \\
&\iff \delta_\Psi^i \exists b'd'(B_{b'}^b D_{d'}^d \omega_{\bar{h}}^{\bar{a}b'cd'e\bar{f}\bar{g}})
\end{aligned}$$

Step $(\star)$ follows from the fact that $\Psi < \Psi\Lambda$, relying on our assumption that $\Lambda > 1$.

$$\begin{aligned}
(\alpha \bullet E)_e^{bicd} &\iff \exists e'(\alpha_{e'}^{bicd} E_e^{e'}) \\
&\iff \delta_0^i \exists e'(\omega_{\bar{h}}^{\bar{a}bcde'\bar{f}\bar{g}} E_e^{e'})
\end{aligned}$$

Thus, the following holds.

$$\delta_\Psi^i \exists b'd'(B_{b'}^b D_{d'}^d \omega_{\bar{h}}^{\bar{a}b'cd'e\bar{f}\bar{g}}) \iff \delta_0^i \exists e'(\omega_{\bar{h}}^{\bar{a}bcde'\bar{f}\bar{g}} E_e^{e'})$$

This implies that neither of the following holds.

$$\exists b'd'(B_{b'}^b D_{d'}^d \omega_{\bar{h}}^{\bar{a}b'cd'e\bar{f}\bar{g}}) \qquad \exists e'(\omega_{\bar{h}}^{\bar{a}bcde'\bar{f}\bar{g}} E_e^{e'})$$

But this contradicts our assumption that ω and α are non-zero. □

A.2 Proof of Lemma 5.6.1

Proof. It must be the case that P, Q, R and S have prime factorisations of the following forms.

$$P \cong \bigotimes_{0 \leq a < i} X_a \qquad R \cong \bigotimes_{i \leq a < n} X_a$$

$$Q \cong \bigotimes_{0 \leq a < i'} Y_a \qquad S \cong \bigotimes_{i' \leq a < n'} Y_a$$

Consider the case where Q is trivial, so that $i' = 0$. In this case, the result follows from Lemma 5.6.2.

Consider the case where S is trivial, so that $i' = n'$. In this case, the result follows from Lemma 5.6.2.

Assume that both Q and S are non-trivial, so that $0 < i' < n'$. We will prove the result by induction on the type of s. Consider the type of the constructible morphism s.

- Consider the case where s is of type ($\cong$).

  $$P \otimes R \cong Q \otimes S$$

  By Proposition 5.3.15, it must be the case that $n = n'$ and, for each a, $X_a \cong Y_a$. Compare i with i'.

  Consider the case where $0 \leq i < i' \leq n$.

  $$\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{i'-1} \otimes X_{i'} \otimes \cdots \otimes X_{n-1}}^{R}$$
  $$\underbrace{X_0 \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes \cdots \otimes X_{i'-1}}_{Q} \otimes \underbrace{X_{i'} \otimes \cdots \otimes X_{n-1}}_{S}$$

  Define the following shape.

  $$Q \cap R = \bigotimes_{i \leq a < i'} X_a$$

  This shape has been chosen so that the following central isomorphisms exist.

  $$P \otimes (Q \cap R) \cong Q \qquad R \cong (Q \cap R) \otimes S$$

  By Lemma A.1.28, $\Omega(Q \cap R) = \mathcal{I}$. This contradicts our assumption that $i < i'$.

  Consider the case where $0 \leq i = i' \leq n$.

  $$\underbrace{\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P}}_{Q} \otimes \underbrace{\overbrace{X_i \otimes \cdots \otimes X_{n-1}}^{R}}_{S}$$

  By Proposition 5.3.15, the following central isomorphisms exist.

  $$P \cong Q \qquad R \cong S$$

  Define u and v to be these central isomorphisms.

  Consider the case where $0 \leq i' < i \leq n$.

  $$\overbrace{X_0 \otimes \cdots \otimes X_{i'-1} \otimes X_{i'} \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{n-1}}^{R}$$
  $$\underbrace{X_0 \otimes \cdots \otimes X_{i'-1}}_{Q} \otimes \underbrace{X_{i'} \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes \cdots \otimes X_{n-1}}_{S}$$

This case is similar to the case where $0 \leq i < i' \leq n$.

- Consider the case where s is of type $(\otimes)$.

$$P \otimes R \cong A \otimes C \xrightarrow{f \otimes g} B \otimes D \cong Q \otimes S$$

It must be the case that A, B, C and D have prime factorisations of the following forms.

$$A \cong \bigotimes_{0 \leq a < j} X_a \qquad C \cong \bigotimes_{j \leq a < n} X_a$$

$$B \cong \bigotimes_{0 \leq a < j'} Y_a \qquad D \cong \bigotimes_{j' \leq a < n'} Y_a$$

Compare i with j and i' with j'.

Consider the case where $0 \leq i \leq j \leq n$ and $0 \leq i' \leq j' \leq n'$.

$$\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes \cdots \otimes X_{n-1}}^{R}$$
$$\underbrace{X_0 \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes \cdots \otimes X_{j-1}}_{A} \otimes \underbrace{X_j \otimes \cdots \otimes X_{n-1}}_{C}$$

$$\overbrace{Y_0 \otimes \cdots \otimes Y_{i'-1} \otimes Y'_i \otimes \cdots \otimes Y_{j'-1}}^{B} \otimes \overbrace{Y'_j \otimes \cdots \otimes Y_{n'-1}}^{D}$$
$$\underbrace{Y_0 \otimes \cdots \otimes Y_{i'-1}}_{Q} \otimes \underbrace{Y'_i \otimes \cdots \otimes Y_{j'-1} \otimes Y'_j \otimes \cdots \otimes Y_{n'-1}}_{S}$$

Define the following shapes.

$$A \cap R = \bigotimes_{i \leq a < j} X_a \qquad B \cap S = \bigotimes_{i' \leq a < j'} Y_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$R \cong (A \cap R) \otimes C$$
$$P \otimes (A \cap R) \cong A$$
$$B \cong Q \otimes (B \cap S)$$
$$(B \cap S) \otimes D \cong S$$

By Lemma A.1.7, there is a morphism

$$\upsilon \colon \Omega(A \cap R) \to \Omega(B \cap S)$$

such that the following hold.

$$\sigma \otimes \upsilon = \Omega(f)$$

$$v \otimes \Omega(g) = \tau$$

By induction, there are constructible morphisms

$$u \colon P \to Q \qquad f' \colon A \cap R \to B \cap S$$

such that f is of the form $u \otimes f'$. Define v to be the following constructible morphism.

$$v \colon R \cong (A \cap R) \otimes C \xrightarrow{f' \otimes g} (B \cap S) \otimes D \cong S$$

Note the following.

$$u \otimes v \cong u \otimes (f' \otimes g) \cong (u \otimes f') \otimes g \cong f \otimes g$$

Consider the case where $0 \leq i \leq j \leq n$ and $0 \leq j' \leq i' \leq n'$.

$$\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes \cdots \otimes X_{n-1}}^{R}$$
$$\underbrace{X_0 \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes \cdots \otimes X_{j-1}}_{A} \otimes \underbrace{X_j \otimes \cdots \otimes X_{n-1}}_{C}$$

$$\overbrace{Y_0 \otimes \cdots \otimes Y_{j'-1}}^{B} \otimes \overbrace{Y'_j \otimes \cdots \otimes Y_{i'-1} \otimes Y'_i \otimes \cdots \otimes Y_{n'-1}}^{D}$$
$$\underbrace{Y_0 \otimes \cdots \otimes Y_{j'-1} \otimes Y'_j \otimes \cdots \otimes Y_{i'-1}}_{Q} \otimes \underbrace{Y'_i \otimes \cdots \otimes Y_{n'-1}}_{S}$$

Define the following shapes.

$$A \cap R = \bigotimes_{i \leq a < j} X_a \qquad Q \cap D = \bigotimes_{j' \leq a < i'} Y_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$R \cong (A \cap R) \otimes C$$
$$P \otimes (A \cap R) \cong A$$
$$D \cong (Q \cap D) \otimes S$$
$$B \otimes (Q \cap D) \cong Q$$

By Lemma A.1.22, there are morphisms

$$\tau_0 \colon \Omega(A \cap R) \to \Omega(\mathcal{I})$$
$$\sigma_1 \colon \Omega(P) \to \Omega(B) \qquad \tau_1 \colon \Omega(C) \to \Omega(S)$$
$$\sigma_0 \colon \Omega(\mathcal{I}) \to \Omega(Q \cap D)$$

such that the following hold.

$$\sigma_1 \otimes \tau_0 = \Omega(f)$$

$$\tau_0 \otimes \tau_1 = \tau$$

$$\sigma_1 \otimes \sigma_0 = \sigma$$

$$\sigma_0 \otimes \tau_1 = \Omega(g)$$

By induction, there are constructible morphisms

$$f_1 \colon P \to B \qquad f_0 \colon A \cap R \to \mathcal{I}$$

such that f is of the form $f_1 \otimes f_0$. By induction, there are constructible morphisms

$$g_0 \colon \mathcal{I} \to Q \cap D \qquad g_1 \colon C \to S$$

such that g is of the form $g_0 \otimes g_1$. Define u and v to be the following constructible morphisms.

$$u \colon P \cong P \otimes \mathcal{I} \xrightarrow{f_1 \otimes g_0} B \otimes (Q \cap D) \cong Q$$

$$v \colon R \cong (A \cap R) \otimes C \xrightarrow{f_0 \otimes g_1} \mathcal{I} \otimes S \cong S$$

Note the following.

$$u \otimes v \cong (f_1 \otimes g_0) \otimes (f_0 \otimes g_1) \cong (f_1 \otimes f_0) \otimes (g_0 \otimes g_1) \cong f \otimes g$$

Consider the case where $0 \leq j \leq i \leq n$ and $0 \leq i' \leq j' \leq n'$.

$$\mathrlap{\overbrace{\phantom{X_0 \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes \cdots \otimes X_{i-1}}}^{P}}\underbrace{X_0 \otimes \cdots \otimes X_{j-1}}_{A} \otimes \underbrace{X_j \otimes \cdots \otimes X_{i-1} \otimes \overbrace{X_i \otimes \cdots \otimes X_{n-1}}^{R}}_{C}$$

$$\mathrlap{\overbrace{\phantom{Y_0 \otimes \cdots \otimes Y_{i'-1} \otimes Y'_i \otimes \cdots \otimes Y_{j'-1}}}^{B}}\underbrace{Y_0 \otimes \cdots \otimes Y_{i'-1}}_{Q} \otimes \underbrace{Y'_i \otimes \cdots \otimes Y_{j'-1} \otimes \overbrace{Y'_j \otimes \cdots \otimes Y_{n'-1}}^{D}}_{S}$$

This case is similar to the case where $0 \leq i \leq j \leq n$ and $0 \leq j' \leq i' \leq n'$.

Consider the case where $0 \leq j \leq i \leq n$ and $0 \leq j' \leq i' \leq n'$.

$$\mathrlap{\overbrace{\phantom{X_0 \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes \cdots \otimes X_{i-1}}}^{P}}\underbrace{X_0 \otimes \cdots \otimes X_{j-1}}_{A} \otimes \underbrace{X_j \otimes \cdots \otimes X_{i-1} \otimes \overbrace{X_i \otimes \cdots \otimes X_{n-1}}^{R}}_{C}$$

$$\rlap{$\underbrace{\phantom{Y_0 \otimes \cdots \otimes Y_{j'-1} \otimes Y'_j \otimes \cdots \otimes Y_{i'-1}}}_{Q}$}\overbrace{Y_0 \otimes \cdots \otimes Y_{j'-1}}^{B} \otimes \overbrace{Y'_j \otimes \cdots \otimes Y_{i'-1} \otimes \underbrace{Y'_i \otimes \cdots \otimes Y_{n'-1}}_{S}}^{D}$$

This case is similar to the case where $0 \leq i \leq j \leq n$ and $0 \leq i' \leq j' \leq n'$.

- Consider the case where s is of type (η).

$$P \otimes R \xrightarrow{\langle I,f\rangle_\eta} I\backslash A \cong Q \otimes S$$

Then I is non-trivial. By Proposition 5.3.15, either $Q \cong I\backslash A$ and $S \cong \mathcal{I}$, or $Q \cong \mathcal{I}$ and $S \cong I\backslash A$. Either way, this contradicts our assumption that both Q and S are non-trivial.

- Consider the case where s is of type (ε).

$$P \otimes R \cong A \otimes \Gamma^\star(B \otimes (C\backslash D)) \otimes E \xrightarrow{\langle \Gamma,f,g\rangle_\varepsilon} Q \otimes S$$

Then C is non-trivial. It must be the case that A, B, $C\backslash D$ and E have prime factorisations of the following forms, where $\Gamma^\star(X'_a) = X_a$.

$$A \cong \bigotimes_{0\leq a<j} X_a \qquad B \cong \bigotimes_{j\leq a<k} X'_a \qquad C\backslash D = X'_k \qquad E \cong \bigotimes_{k<a<n} X_a$$

Compare i with j and k.

Consider the case where $0 \leq i \leq j \leq k < n$.

$$\rlap{$\underbrace{\phantom{X_0 \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes \cdots \otimes X_{j-1}}}_{A}$}\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{j-1} \otimes \underbrace{X_j \otimes \cdots \otimes X_{k-1}}_{\Gamma^\star(B)} \otimes \underbrace{X_k}_{\Gamma^\star(C\backslash D)} \otimes \underbrace{X_{k+1} \otimes \cdots \otimes X_{n-1}}_{E}}^{R}$$

Define the following shape.

$$A \cap R = \bigotimes_{i\leq a<j} X_a$$

This shape has been chosen so that the following central isomorphisms exist.

$$R \cong (A \cap R) \otimes \Gamma^\star(B \otimes (C\backslash D)) \otimes E$$

$$P \otimes (A \cap R) \cong A$$

By Lemma A.1.9, there is a morphism

$$\upsilon \colon \Omega((A \cap R) \otimes \Gamma^\star(D) \otimes E) \to \Omega(S)$$

such that the following hold.

$$\sigma \otimes \upsilon = \Omega(g)$$

$$\langle \Gamma, \Omega(f), \upsilon \rangle_{\varepsilon} = \tau$$

By induction, there are constructible morphisms

$$u \colon P \to Q \qquad g' \colon (A \cap R) \otimes \Gamma^{\star}(D) \otimes E \to S$$

such that g is of the form $u \otimes g'$. Define v to be the following constructible morphism.

$$v \colon R \cong (A \cap R) \otimes \Gamma^{\star}(B \otimes (C \backslash D)) \otimes E \xrightarrow{\langle \Gamma, f, g' \rangle_{\varepsilon}} S$$

Note the following.

$$u \otimes v \cong u \otimes \langle \Gamma, f, g' \rangle_{\varepsilon} \cong \langle \Gamma, f, u \otimes g' \rangle_{\varepsilon} \cong \langle \Gamma, f, g \rangle_{\varepsilon}$$

Consider the case where $0 \leq j \leq i \leq k < n$.

$$\overbrace{\underbrace{X_0 \otimes \cdots \otimes X_{j-1}}_{A} \otimes X_j \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{k-1} \otimes \underbrace{X_k}_{\Gamma^{\star}(C \backslash D)} \otimes \underbrace{X_{k+1} \otimes \cdots \otimes X_{n-1}}_{E}}^{R}$$

where $X_j \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes \cdots \otimes X_{k-1}$ is $\Gamma^{\star}(B)$.

Define the following shapes.

$$P \cap B = \bigotimes_{j \leq a < i} X'_a \qquad B \cap R = \bigotimes_{i \leq a < k} X'_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$P \cong A \otimes \Gamma^{\star}(P \cap B) \qquad R \cong \Gamma^{\star}((B \cap R) \otimes (C \backslash D)) \otimes E$$

$$(P \cap B) \otimes (B \cap R) \cong B$$

By Lemma A.1.23, there are morphisms

$$\sigma_0 \colon \Omega(P \cap B) \to \Omega(\mathcal{I}) \qquad \tau_1 \colon \Omega(B \cap R) \to \Omega(C)$$

$$\sigma_1 \colon \Omega(A) \to \Omega(Q) \qquad \tau_2 \colon \Omega(\Gamma^{\star}(D) \otimes E) \to \Omega(S)$$

such that the following hold.

$$\sigma_0 \otimes \tau_1 = \Omega(f)$$

$$\langle \Gamma, \sigma_0, \sigma_1 \rangle_{\varepsilon} = \sigma$$

$$\langle \Gamma, \tau_1, \tau_2 \rangle_{\varepsilon} = \tau$$

$$\sigma_1 \otimes \tau_2 = \Omega(g)$$

By induction, there are constructible morphisms

$$f_0 \colon P \cap B \to \mathcal{I} \qquad f_1 \colon B \cap R \to C$$

such that f is of the form $f_0 \otimes f_1$. By induction, there are constructible morphisms

$$g_1 \colon A \to Q \qquad g_2 \colon \Gamma^\star(D) \otimes E \to S$$

such that g is of the form $g_1 \otimes g_2$. Define u and v to be the following constructible morphisms.

$$u \colon P \cong A \otimes \Gamma^\star((P \cap B) \otimes (\mathcal{I} \backslash \mathcal{I})) \xrightarrow{\langle \Gamma, f_0, g_1 \rangle_\varepsilon} Q$$

$$v \colon R \cong \Gamma^\star((B \cap R) \otimes (C \backslash D)) \otimes E \xrightarrow{\langle \Gamma, f_1, g_2 \rangle_\varepsilon} S$$

Note the following.

$$u \otimes v \cong \langle \Gamma, f_0, g_1 \rangle_\varepsilon \otimes \langle \Gamma, f_1, g_2 \rangle_\varepsilon \cong \langle \Gamma, f_0 \otimes f_1, g_1 \otimes g_2 \rangle_\varepsilon \cong \langle \Gamma, f, g \rangle_\varepsilon$$

Consider the case where $0 \leq j \leq k < i \leq n$.

$$\overbrace{\underbrace{X_0 \otimes \cdots \otimes X_{j-1}}_{A} \otimes \underbrace{X_j \otimes \cdots \otimes X_{k-1}}_{\Gamma^\star(B)} \otimes \underbrace{X_k}_{\Gamma^\star(C \backslash D)} \otimes X_{k+1} \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{n-1}}^{R}, \quad \underbrace{X_{k+1} \otimes \cdots \otimes X_{n-1}}_{E}$$

This case is similar to the case where $0 \leq i \leq j \leq k < n$.

- Consider the case where s is of type $((-)^\star)$.

$$P \otimes R \cong \Gamma^\star(A) \xrightarrow{\Gamma^\star(f)} \Gamma^\star(B) \cong Q \otimes S$$

It must be the case that A and B have prime factorisations of the following forms, where $\Gamma^\star(X'_a) = X_a$.

$$A \cong \bigotimes_{0 \leq a < n} X'_a$$

$$B \cong \bigotimes_{0 \leq a < n'} Y'_a$$

Define the following shapes.

$$P_A \cong \bigotimes_{0 \leq a < i} X'_a \qquad R_A \cong \bigotimes_{i \leq a < n} X'_a$$

$$Q_A \cong \bigotimes_{0 \leq a < i'} Y'_a \qquad S_A \cong \bigotimes_{i' \leq a < n'} Y'_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$P \cong \Gamma^\star(P_A) \qquad R \cong \Gamma^\star(R_A)$$

$$P_A \otimes R_A \cong A$$

$$B \cong Q_A \otimes S_A$$

$$\Gamma^\star(Q_A) \cong Q \qquad \Gamma^\star(S_A) \cong S$$

By Lemma A.1.19, there are morphisms

$$\sigma' \colon \Omega(P_A) \to \Omega(Q_A) \qquad \tau' \colon \Omega(R_A) \to \Omega(S_A)$$

such that the following hold.

$$\Gamma^\star(\sigma') = \sigma$$

$$\Gamma^\star(\tau') = \tau$$

$$\sigma' \otimes \tau' = \Omega(f)$$

By induction, there are constructible morphisms

$$f_1 \colon P_A \to Q_A \qquad f_2 \colon R_A \to S_A$$

such that f is of the form $f_1 \otimes f_2$. Define u and v to be the following constructible morphisms.

$$u \colon P \cong \Gamma^\star(P_A) \xrightarrow{\Gamma^\star(f_1)} \Gamma^\star(Q_A) \cong Q$$

$$v \colon R \cong \Gamma^\star(R_A) \xrightarrow{\Gamma^\star(f_2)} \Gamma^\star(S_A) \cong S$$

Note the following.

$$u \otimes v \cong \Gamma^\star(f_1) \otimes \Gamma^\star(f_2) \cong \Gamma^\star(f_1 \otimes f_2) \cong \Gamma^\star(f)$$

- Consider the case where s is of type (θ).

$$P \otimes R \xrightarrow{\langle \Phi, f\rangle_\theta} \Phi_\star(A) \cong Q \otimes S$$

Then Φ is non-empty. By Proposition 5.3.15, either $Q \cong \Phi_\star(A)$ and $S \cong \mathcal{I}$, or $Q \cong \mathcal{I}$ and $S \cong \Phi_\star(A)$. Either way, this contradicts our assumption that both Q and S are non-trivial.

- Consider the case where s is of type (ζ).

$$P \otimes R \cong A \otimes \Gamma^\star\Phi^\star\Phi_\star(B) \otimes C \xrightarrow{\langle \Gamma, \Phi, f\rangle_\zeta} Q \otimes S$$

Then Φ is non-empty. It must be the case that A, $\Phi_\star(B)$ and C have prime factorisations of the following forms, where $\Gamma^\star\Phi^\star(X'_a) = X_a$.

$$A \cong \bigotimes_{0 \leq a < j} X_a \qquad \Phi_\star(B) = X'_j \qquad C \cong \bigotimes_{j < a < n} X_a$$

Compare i with j.

Consider the case where $0 \leq i \leq j < n$.

$$\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes X_{j+1} \otimes \cdots \otimes X_{n-1}}^{R}$$

$$\underbrace{X_0 \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes \cdots \otimes X_{j-1}}_{A} \otimes \underbrace{X_j}_{\Gamma^\star\Phi^\star\Phi_\star(B)} \otimes \underbrace{X_{j+1} \otimes \cdots \otimes X_{n-1}}_{C}$$

Define the following shape.

$$A \cap R = \bigotimes_{i \leq a < j} X_a$$

This shape has been chosen so that the following central isomorphisms exist.

$$R \cong (A \cap R) \otimes \Gamma^\star\Phi^\star\Phi_\star(B) \otimes C$$
$$P \otimes (A \cap R) \cong A$$

By Lemma A.1.11, there is a morphism

$$\upsilon \colon \Omega((A \cap R) \otimes \Gamma^\star(B) \otimes C) \to \Omega(S)$$

such that the following hold.

$$\sigma \otimes \upsilon = \Omega(f)$$
$$\langle \Gamma, \Omega(\Phi), \upsilon\rangle_\zeta = \tau$$

By induction, there are constructible morphisms

$$u \colon P \to Q \qquad f' \colon (A \cap R) \otimes \Gamma^\star(B) \otimes C \to S$$

such that f is of the form $u \otimes f'$. Define v to be the following constructible morphism.

$$v \colon R \cong (A \cap R) \otimes \Gamma^\star\Phi^\star\Phi_\star(B) \otimes C \xrightarrow{\langle \Gamma, \Phi, f'\rangle_\zeta} S$$

Note the following.

$$u \otimes v \cong u \otimes \langle \Gamma, \Phi, f'\rangle_\zeta \cong \langle \Gamma, \Phi, u \otimes f'\rangle_\zeta \cong \langle \Gamma, \Phi, f\rangle_\zeta$$

Consider the case where $0 \leq j < i \leq n$.

$$\overbrace{X_0 \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes X_{j+1} \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{n-1}}^{R}$$
$$\underbrace{X_0 \otimes \cdots \otimes X_{j-1}}_{A} \otimes \underbrace{X_j}_{\Gamma^\star\Phi^\star\Phi_\star(B)} \otimes \underbrace{X_{j+1} \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes \cdots \otimes X_{n-1}}_{C}$$

This case is similar to the case where $0 \leq i \leq j < n$.

□

A.3 Proof of Lemma 5.6.2

Proof. It must be the case that P, Q, $R\backslash S$ and T have prime factorisations of the following forms, where $\Delta^\star(X'_a) = X_a$.

$$P \cong \bigotimes_{0 \leq a < i} X_a \qquad Q \cong \bigotimes_{i \leq a < j} X'_a \qquad R\backslash S = \bigotimes_{j \leq a < j'} X'_a \qquad T \cong \bigotimes_{j' \leq a < n} X_a$$

If R is trivial, then we may assume that S is also trivial, by replacing T with $\Delta^\star(S) \otimes T$ otherwise; in this case, $j' = j$. If R is non-trivial, then $R\backslash S$ is a prime shape; in this case $j' = j + 1$. In either case, there is no a with $j < a < j'$.

Consider the case where both Q and R are trivial, so that $i = j'$. In this case, we can simply define u and v to be the following central isomorphisms.

$$u \colon Q \cong R$$
$$v \colon P \otimes \Delta^\star(S) \otimes T \cong P \otimes \Delta^\star(Q \otimes (R\backslash S)) \otimes T \xrightarrow{s} U$$

Assume that Q are R not both trivial, so that $i < j'$. We will prove the result by induction on the type of s. Consider the type of the constructible morphism s.

- Consider the case where s is of type $(\cong)$.

  $$P \otimes \Delta^\star(Q \otimes (R\backslash S)) \otimes T \cong U$$

  By Lemma A.1.29, $\Omega(Q) = \mathcal{I}$ and $\Omega(R) = \mathcal{I}$. This contradicts our assumption that Q are R not both trivial.

- Consider the case where s is of type $(\otimes)$.

  $$P \otimes \Delta^\star(Q \otimes (R\backslash S)) \otimes T \cong A \otimes C \xrightarrow{f \otimes g} B \otimes D \cong U$$

  It must be the case that A and C have prime factorisations of the following forms.

  $$A \cong \bigotimes_{0 \leq a < k} X_a \qquad C \cong \bigotimes_{k \leq a < n} X_a$$

  Compare i and j with k.

  Consider the case where $0 \leq k \leq i \leq j \leq j' \leq n$.

$$\overbrace{\underbrace{X_0 \otimes \cdots \otimes X_{k-1}}_{A} \otimes X_k \otimes \cdots \otimes X_{i-1}}^{P} \otimes \underbrace{\overbrace{X_i \otimes \cdots \otimes X_{j-1}}^{\Delta^\star(Q)} \otimes \overbrace{X_j \otimes \cdots \otimes X_{j'-1}}^{\Delta^\star(R\backslash S)} \otimes \overbrace{X_{j'} \otimes \cdots \otimes X_{n-1}}^{T}}_{C}$$

Define the following shape.

$$P \cap C = \bigotimes_{k \le a < i} X_a$$

This shape has been chosen so that the following central isomorphisms exist.

$$P \cong A \otimes (P \cap C)$$

$$(P \cap C) \otimes \Delta^\star(Q \otimes (R\backslash S)) \otimes T \cong C$$

By Lemma A.1.9, there is a morphism

$$\upsilon \colon \Omega((P \cap C) \otimes \Delta^\star(S) \otimes T) \to \Omega(D)$$

such that the following hold.

$$\Omega(f) \otimes \upsilon = \tau$$

$$\langle \Delta, \sigma, \upsilon \rangle_\varepsilon = \Omega(g)$$

By induction, there are constructible morphisms

$$u \colon Q \to R \qquad g' \colon (P \cap C) \otimes \Delta^\star(S) \otimes T \to D$$

such that g is of the form $\langle \Delta, u, g' \rangle_\varepsilon$. Define v to be the following constructible morphism.

$$v \colon P \otimes \Delta^\star(S) \otimes T \cong A \otimes (P \cap C) \otimes \Delta^\star(S) \otimes T \xrightarrow{f \otimes g'} B \otimes D \cong U$$

Note the following.

$$\langle \Delta, u, v \rangle_\varepsilon \cong \langle \Delta, u, f \otimes g' \rangle_\varepsilon \cong f \otimes \langle \Delta, u, g' \rangle_\varepsilon \cong f \otimes g$$

Consider the case where $0 \le i \le k \le j \le j' \le n$.

$$\underbrace{\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes X_i \otimes \cdots \otimes X_{k-1}}_{A} \otimes \underbrace{X_k \otimes \cdots \otimes X_{j-1} \otimes \overbrace{X_j \otimes \cdots \otimes X_{j'-1}}^{\Delta^\star(R\backslash S)} \otimes \overbrace{X_{j'} \otimes \cdots \otimes X_{n-1}}^{T}}_{C}$$

(with $\Delta^\star(Q)$ spanning $X_i \otimes \cdots \otimes X_{j-1}$)

Define the following shapes.

$$A \cap Q = \bigotimes_{i \le a < k} X'_a \qquad Q \cap C = \bigotimes_{k \le a < j} X'_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$Q \cong (A \cap Q) \otimes (Q \cap C)$$

$$P \otimes \Delta^\star(A \cap Q) \cong A \qquad \Delta^\star((Q \cap C) \otimes (R\backslash S)) \otimes T \cong C$$

By Lemma A.1.23, there are morphisms

$$\sigma_0 \colon \Omega(A \cap Q) \to \Omega(\mathcal{I}) \qquad \sigma_1 \colon \Omega(Q \cap C) \to \Omega(R)$$

$$\tau_1 \colon \Omega(P) \to \Omega(B) \qquad \tau_2 \colon \Omega(\Delta^\star(S) \otimes T) \to \Omega(D)$$

such that the following hold.

$$\sigma_0 \otimes \sigma_1 = \sigma$$

$$\langle \Delta, \sigma_0, \tau_1 \rangle_\varepsilon = \Omega(f)$$

$$\langle \Delta, \sigma_1, \tau_2 \rangle_\varepsilon = \Omega(g)$$

$$\tau_1 \otimes \tau_2 = \tau$$

By induction, there are constructible morphisms

$$f_0 \colon A \cap Q \to \mathcal{I} \qquad f_1 \colon P \to B$$

such that f is of the form $\langle \Delta, f_0, f_1 \rangle_\varepsilon$. By induction, there are constructible morphisms

$$g_1 \colon Q \cap C \to R \qquad g_2 \colon \Delta^\star(S) \otimes T \to D$$

such that g is of the form $\langle \Delta, g_1, g_2 \rangle_\varepsilon$. Define u and v to be the following constructible morphisms.

$$u \colon Q \cong (A \cap Q) \otimes (Q \cap C) \xrightarrow{f_0 \otimes g_1} \mathcal{I} \otimes R \cong R$$

$$v \colon P \otimes \Delta^\star(S) \otimes T \xrightarrow{f_1 \otimes g_2} B \otimes D \cong U$$

Note the following.

$$\langle \Delta, u, v \rangle_\varepsilon \cong \langle \Delta, f_0 \otimes g_1, f_1 \otimes g_2 \rangle_\varepsilon \cong \langle \Delta, f_0, f_1 \rangle_\varepsilon \otimes \langle \Delta, g_1, g_2 \rangle_\varepsilon \cong f \otimes g$$

Consider the case where $0 \leq i \leq j \leq j' \leq k \leq n$.

$$\underbrace{\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{j-1}}^{\Delta^\star(Q)} \otimes \overbrace{X_j \otimes \cdots \otimes X_{j'-1}}^{\Delta^\star(R\backslash S)} \otimes \overbrace{X_{j'} \otimes \cdots \otimes X_{k-1}}^{} \otimes}_{A} \overbrace{\underbrace{X_k \otimes \cdots \otimes X_{n-1}}_{C}}^{T}$$

This case is similar to the case where $0 \leq k \leq i \leq j \leq j' \leq n$.

- Consider the case where s is of type (η).

$$P \otimes \Delta^\star(Q \otimes (R\backslash S)) \otimes T \xrightarrow{\langle I,f\rangle_\eta} I\backslash A \cong U$$

By induction, there are constructible morphisms

$$u\colon Q \to R \qquad f'\colon I \otimes P \otimes \Delta^\star(S) \otimes T \to A$$

such that f is of the form $\langle \Delta, u, f'\rangle_\varepsilon$. Define v to be the following constructible morphism.

$$v\colon P \otimes \Delta^\star(S) \otimes T \xrightarrow{\langle I,f'\rangle_\eta} I\backslash A \cong Q$$

Note the following.

$$\langle \Delta, u, v\rangle_\varepsilon \cong \langle \Delta, u, \langle I, f'\rangle_\eta\rangle_\varepsilon \cong \langle I, \langle \Delta, u, f'\rangle_\varepsilon\rangle_\eta \cong \langle I, f\rangle_\eta$$

- Consider the case where s is of type (ε).

$$P \otimes \Delta^\star(Q \otimes (R\backslash S)) \otimes T \cong A \otimes \Gamma^\star(B \otimes (C\backslash D)) \otimes E \xrightarrow{\langle \Gamma,f,g\rangle_\varepsilon} U$$

Then C is non-trivial. It must be the case that A, B, $C\backslash D$ and E have prime factorisations of the following forms, where $\Gamma^\star(X''_a) = X_a$.

$$A \cong \bigotimes_{0\le a<k} X_a \qquad B \cong \bigotimes_{k\le a<l} X''_a \qquad C\backslash D = X''_l \qquad E \cong \bigotimes_{l<a<n} X_a$$

Compare i and j with k and l.

Consider the case where $0 \le i \le j \le j' \le k \le l < n$.

$$\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{j-1}}^{\Delta^\star(Q)} \otimes \overbrace{X_j \otimes \cdots \otimes X_{j'-1}}^{\Delta^\star(R\backslash S)} \otimes \overbrace{X_{j'} \otimes \cdots \otimes X_{k-1} \otimes X_k \otimes \cdots \otimes X_{l-1} \otimes X_l \otimes X_{l+1} \otimes \cdots \otimes X_{n-1}}^{T}$$

$$\underbrace{X_0 \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes \cdots \otimes X_{j'-1} \otimes X_{j'} \otimes \cdots \otimes X_{k-1}}_{A} \otimes \underbrace{X_k \otimes \cdots \otimes X_{l-1}}_{\Gamma^\star(B)} \otimes \underbrace{X_l}_{\Gamma^\star(C\backslash D)} \otimes \underbrace{X_{l+1} \otimes \cdots \otimes X_{n-1}}_{E}$$

Define the following shape.

$$A \cap T = \bigotimes_{j'\le a<k} X_a$$

This shape has been chosen so that the following central isomorphisms exist.

$$T \cong (A \cap T) \otimes \Gamma^\star(B \otimes (C\backslash D)) \otimes E$$
$$P \otimes \Delta^\star(Q \otimes (R\backslash S)) \otimes (A \cap T) \cong A$$

By Lemma A.1.14, there is a morphism

$$v\colon \Omega(P \otimes \Delta^\star(S) \otimes (A \cap T) \otimes \Gamma^\star(D) \otimes E) \to \Omega(U)$$

such that the following hold.

$$\langle \Delta, \sigma, \upsilon \rangle_\varepsilon = \Omega(g)$$

$$\langle \Gamma, \Omega(f), \upsilon \rangle_\varepsilon = \tau$$

By induction, there are constructible morphisms

$$u \colon Q \to R \qquad g' \colon P \otimes \Delta^\star(S) \otimes (A \cap T) \otimes \Gamma^\star(D) \otimes E \to U$$

such that g is of the form $\langle \Delta, u, g' \rangle_\varepsilon$. Define v to be the following constructible morphism.

$$v \colon P \otimes \Delta^\star(S) \otimes T \cong P \otimes \Delta^\star(S) \otimes (A \cap T) \otimes \Gamma^\star(B \otimes (C \backslash D)) \otimes E \xrightarrow{\langle \Gamma, f, g' \rangle_\varepsilon} U$$

Note the following.

$$\langle \Delta, u, v \rangle_\varepsilon \cong \langle \Delta, u, \langle \Gamma, f, g' \rangle_\varepsilon \rangle_\varepsilon \cong \langle \Gamma, f, \langle \Delta, u, g' \rangle_\varepsilon \rangle_\varepsilon \cong \langle \Gamma, f, g \rangle_\varepsilon$$

Consider the case where $0 \le i \le k < j \le j' \le l < n$.

$$\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{k-1} \otimes X_k \otimes \cdots \otimes X_{j-1}}^{\Delta^\star(Q)} \otimes \overbrace{X_j \otimes \cdots \otimes X_{j'-1}}^{\Delta^\star(R \backslash S)} \otimes \overbrace{X_{j'} \otimes \cdots \otimes X_{l-1} \otimes X_l \otimes X_{l+1} \otimes \cdots \otimes X_{n-1}}^{T}$$

$$\underbrace{X_0 \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes \cdots \otimes X_{k-1}}_{A} \otimes \underbrace{X_k \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes \cdots \otimes X_{j'-1} \otimes X_{j'} \otimes \cdots \otimes X_{l-1}}_{\Gamma^\star(B)} \otimes \underbrace{X_l}_{\Gamma^\star(C \backslash D)} \otimes \underbrace{X_{l+1} \otimes \cdots \otimes X_{n-1}}_{E}$$

Define the following shapes.

$$A \cap Q = \bigotimes_{i \le a < k} X'_a \qquad B_Q = \bigotimes_{k \le a < j} X'_a$$

$$Q_B = \bigotimes_{k \le a < j'} X''_a \qquad B \cap T = \bigotimes_{j' \le a < l} X''_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$Q \cong (A \cap Q) \otimes B_Q \qquad T \cong \Gamma^\star((B \cap T) \otimes (C \backslash D)) \otimes E$$

$$P \otimes \Delta^\star(A \cap Q) \cong A \qquad Q_B \otimes (B \cap T) \cong B$$

$$\Delta^\star(B_Q \otimes (R \backslash S)) \cong \Gamma^\star(Q_B)$$

If $B_Q \otimes (R \backslash S)$ is trivial, then this contradicts our assumption that $k < j$. If B_Q is non-trivial, then, by Lemma 5.3.17, either there is a Λ such that $\Delta = \Lambda\Gamma$ or there is a Λ such that $\Gamma = \Lambda\Delta$. If $R \backslash S$ is non-trivial, then, by Lemma 5.3.18, there is a Λ such that $\Delta = \Lambda\Gamma$.

Consider the case where there is a Λ such that $\Delta = \Lambda\Gamma$. By Lemma 5.3.16, the following central isomorphism exists.

$$\Lambda^\star(B_Q \otimes (R \backslash S)) \cong Q_B$$

By Lemma A.1.24, there are morphisms

$$\sigma_1 \colon \Omega(B_Q) \to \Omega(R)$$

$$\sigma_0 \colon \Omega(A \cap Q) \to \Omega(\mathcal{I}) \qquad \tau_1 \colon \Omega(\Lambda^\star(S) \otimes (B \cap T)) \to \Omega(C)$$

$$\tau_2 \colon \Omega(P \otimes \Gamma^\star(D) \otimes E) \to \Omega(U)$$

such that the following hold.

$$\langle \mathrm{id}, \sigma_0, \sigma_1 \rangle_\varepsilon = \sigma$$

$$\langle \Lambda, \sigma_1, \tau_1 \rangle_\varepsilon = \Omega(f)$$

$$\langle \Lambda\Gamma, \sigma_0, \tau_2 \rangle_\varepsilon = \Omega(g)$$

$$\langle \Gamma, \tau_1, \tau_2 \rangle_\varepsilon = \tau$$

By induction, there are constructible morphisms

$$f_1 \colon B_Q \to R \qquad f_2 \colon \Lambda^\star(S) \otimes (B \cap T) \to C$$

such that f is of the form $\langle \Lambda, f_1, f_2 \rangle_\varepsilon$. By induction, there are constructible morphisms

$$g_0 \colon A \cap Q \to \mathcal{I} \qquad g_1 \colon P \otimes \Gamma^\star(D) \otimes E \to U$$

such that g is of the form $\langle \Delta, g_0, g_1 \rangle_\varepsilon$. Define u and v to be the following constructible morphisms.

$$u \colon Q \cong (A \cap Q) \otimes B_Q \xrightarrow{\langle \mathrm{id}, g_0, f_1 \rangle_\varepsilon} R$$

$$v \colon P \otimes \Delta^\star(S) \otimes T \cong P \otimes \Gamma^\star(\Lambda(S) \otimes (B \cap T) \otimes (C \backslash D)) \otimes E \xrightarrow{\langle \Gamma, f_2, g_1 \rangle_\varepsilon} U$$

Note the following.

$$\langle \Delta, u, v \rangle_\varepsilon \cong \langle \Lambda\Gamma, \langle \mathrm{id}, g_0, f_1 \rangle_\varepsilon, \langle \Gamma, f_2, g_1 \rangle_\varepsilon \rangle_\varepsilon$$

$$\cong \langle \Gamma, \langle \Lambda, f_1, f_2 \rangle_\varepsilon, \langle \Lambda\Gamma, g_0, g_1 \rangle_\varepsilon \rangle_\varepsilon \cong \langle \Gamma, f, g \rangle_\varepsilon$$

Consider the case where $R \backslash S$ is trivial and there is a Λ such that $\Gamma = \Lambda\Delta$. By Lemma 5.3.16, the following central isomorphism exists.

$$B_Q \cong \Lambda^\star(Q_B)$$

By Lemma A.1.25, there are morphisms

$$\sigma_1 \colon \Omega(Q_B) \to \Omega(\mathcal{I})$$

$$\sigma_2 \colon \Omega(A \cap Q) \to \Omega(\mathcal{I}) \qquad \tau_1 \colon \Omega(B \cap T) \to \Omega(C)$$

$$\tau_2 \colon \Omega(P \otimes (\Lambda\Delta)^\star(D) \otimes E) \to \Omega(U)$$

such that the following hold.

$$\langle \Lambda, \sigma_1, \sigma_2 \rangle_\varepsilon = \sigma$$

$$\langle \mathrm{id}, \sigma_1, \tau_1 \rangle_\varepsilon = \Omega(f)$$
$$\langle \Delta, \sigma_2, \tau_2 \rangle_\varepsilon = \Omega(g)$$
$$\langle \Lambda\Delta, \tau_1, \tau_2 \rangle_\varepsilon = \tau$$

By induction, there are constructible morphisms

$$f_1 \colon Q_B \to \mathcal{I} \qquad f_2 \colon B \cap T \to C$$

such that f is of the form $\langle \mathrm{id}, f_1, f_2 \rangle_\varepsilon$. By induction, there are constructible morphisms

$$g_0 \colon A \cap Q \to \mathcal{I} \qquad g_1 \colon P \otimes (\Lambda\Delta)^\star(D) \otimes E \to U$$

such that g is of the form $\langle \Delta, g_0, g_1 \rangle_\varepsilon$. Define u and v to be the following constructible morphisms.

$$u \colon Q \cong (A \cap Q) \otimes \Lambda^\star(Q_B) \xrightarrow{\langle \Lambda, f_1, g_0 \rangle_\varepsilon} \mathcal{I} \cong R$$

$$v \colon P \otimes \Delta^\star(S) \otimes T \cong P \otimes (\Lambda\Delta)^\star((B \cap T) \otimes (C \backslash D)) \otimes E \xrightarrow{\langle \Lambda\Delta, f_2, g_1 \rangle_\varepsilon} U$$

Note the following.

$$\langle \Delta, u, v \rangle_\varepsilon \cong \langle \Delta, \langle \Lambda, f_1, g_0 \rangle_\varepsilon, \langle \Lambda\Delta, f_2, g_1 \rangle_\varepsilon \rangle_\varepsilon$$

$$\cong \langle \Lambda\Delta, \langle \mathrm{id}, f_1, f_2 \rangle_\varepsilon, \langle \Delta, g_0, g_1 \rangle_\varepsilon \rangle_\varepsilon \cong \langle \Gamma, f, g \rangle_\varepsilon$$

Consider the case where $0 \le k \le i \le j \le j' \le l < n$.

$$\overbrace{X_0 \otimes \cdots \otimes X_{k-1} \otimes X_k \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{j-1}}^{\Delta^\star(Q)} \otimes \overbrace{X_j \otimes \cdots \otimes X_{j'-1}}^{\Delta^\star(R \backslash S)} \otimes \overbrace{X_{j'} \otimes \cdots \otimes X_{l-1} \otimes X_l \otimes X_{l+1} \otimes \cdots \otimes X_{n-1}}^{T}$$

$$\underbrace{X_0 \otimes \cdots \otimes X_{k-1}}_{A} \otimes \underbrace{X_k \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes \cdots \otimes X_{j'-1} \otimes X_{j'} \otimes \cdots \otimes X_{l-1}}_{\Gamma^\star(B)} \otimes \underbrace{X_l}_{\Gamma^\star(C \backslash D)} \otimes \underbrace{X_{l+1} \otimes \cdots \otimes X_{n-1}}_{E}$$

Define the following shapes.

$$P \cap B = \bigotimes_{k \le a < i} X''_a \qquad Q_B = \bigotimes_{i \le a < j'} X''_a \qquad B \cap T = \bigotimes_{j' \le a < l} X''_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$P \cong A \otimes \Gamma^\star(P \cap B) \qquad T \cong \Gamma^\star((B \cap T) \otimes (C \backslash D)) \otimes E$$

$$(P \cap B) \otimes Q_B \otimes (B \cap T) \cong B$$

$$\Delta^\star(Q \otimes (R\backslash S)) \cong \Gamma^\star(Q_B)$$

If $Q\otimes(R\backslash S)$ is trivial, then this contradicts our assumption that $i < j'$. If Q is non-trivial, then, by Lemma 5.3.17, either there is a Λ such that $\Delta = \Lambda\Gamma$ or there is a Λ such that $\Gamma = \Lambda\Delta$. If $R\backslash S$ is non-trivial, then, by Lemma 5.3.18, there is a Λ such that $\Delta = \Lambda\Gamma$.

Consider the case where $R\backslash S$ is trivial and there is a Λ such that $\Gamma = \Lambda\Delta$. By Lemma 5.3.16, the following central isomorphism exists.

$$Q \cong \Lambda^\star(Q_B)$$

By Lemma A.1.26, there are morphisms

$$\sigma'\colon \Omega(Q_B) \to \Omega(\mathcal{I})$$

$$\tau'\colon \Omega((P \cap B) \otimes (B \cap T)) \to \Omega(C)$$

such that the following hold.

$$\Lambda^\star(\sigma') = \sigma$$

$$\langle \mathrm{id}, \sigma', \tau' \rangle_\varepsilon = \Omega(f)$$

$$\langle \Gamma, \tau', \Omega(g) \rangle_\varepsilon = \tau$$

By induction, there are constructible morphisms

$$f_0\colon Q_B \to \mathcal{I} \qquad f_1\colon (P \cap B) \otimes (B \cap T) \to C$$

such that f is of the form $\langle \mathrm{id}, f_0, f_1 \rangle_\varepsilon$. Define u and v to be the following constructible morphism.

$$u\colon Q \cong \Lambda^\star(Q_B) \xrightarrow{\Lambda^\star(f_0)} \Lambda^\star(\mathcal{I}) \cong R$$

$$v\colon P\otimes\Delta^\star(S)\otimes T \cong A\otimes\Gamma^\star((P\cap B)\otimes(B\cap T)\otimes(C\backslash D))\otimes E \xrightarrow{\langle \Gamma, f_1, g\rangle_\varepsilon} U$$

Note the following.

$$\langle \Delta, u, v \rangle_\varepsilon \cong \langle \Delta, \Lambda^\star(f_0), \langle \Lambda\Delta, f_1, g \rangle_\varepsilon \rangle_\varepsilon$$

$$\cong \langle \Lambda\Delta, \langle \mathrm{id}, f_0, f_1 \rangle_\varepsilon, g \rangle_\varepsilon \cong \langle \Gamma, f, g \rangle_\varepsilon$$

Consider the case where there is a Λ such that $\Delta = \Lambda\Gamma$. By Lemma 5.3.16, the following central isomorphism exists.

$$\Lambda^\star(Q \otimes (R\backslash S)) \cong Q_B$$

By Lemma A.1.16, there is a morphism

$$\upsilon\colon \Omega((P \cap B) \otimes \Lambda^\star(S) \otimes (B \cap T)) \to \Omega(C)$$

such that the following hold.

$$\langle \Gamma, \upsilon, \Omega(g) \rangle_\varepsilon = \tau$$

$$\langle \Lambda, \sigma, \upsilon \rangle_\varepsilon = \Omega(f)$$

By induction, there are constructible morphisms

$$u\colon Q \to R \qquad f'\colon (P \cap B) \otimes \Lambda^\star(S) \otimes (B \cap T) \to C$$

such that f is of the form $\langle \Lambda, u, f' \rangle_\varepsilon$. Define v to be the following constructible morphism.

$$\begin{aligned} v\colon\, & P \otimes \Delta^\star(S) \otimes T \\ & \cong A \otimes \Gamma^\star((P \cap B) \otimes \Lambda^\star(S) \otimes (B \cap T) \otimes (C \backslash D)) \otimes E \xrightarrow{\langle \Gamma, f', g \rangle_\varepsilon} U \end{aligned}$$

Note the following.

$$\langle \Delta, u, v \rangle_\varepsilon \cong \langle \Lambda\Gamma, u, \langle \Gamma, f', g \rangle_\varepsilon \rangle_\varepsilon \cong \langle \Gamma, \langle \Lambda, u, f' \rangle_\varepsilon, g \rangle_\varepsilon \cong \langle \Gamma, f, g \rangle_\varepsilon$$

Consider the case where $0 \leq i \leq k \leq j = l < j' \leq n$.

$$\underbrace{\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes \rlap{$\overbrace{\phantom{X_i \otimes \cdots \otimes X_{k-1} \otimes X_k \otimes \cdots \otimes X_{j-1}}}^{\Delta^\star(Q)}$} X_i \otimes \cdots \otimes X_{k-1}}_{A} \otimes \underbrace{X_k \otimes \cdots \otimes X_{j-1}}_{\Gamma^\star(B)} \otimes \underbrace{\overbrace{X_j \otimes \cdots \otimes X_{j'-1}}^{\Delta^\star(R \backslash S)}}_{\Gamma^\star(C \backslash D)} \otimes \underbrace{\overbrace{X_{j'} \otimes \cdots \otimes X_{n-1}}^{T}}_{E}$$

The following central isomorphisms exist.

$$P \otimes \Delta^\star(Q) \cong A \otimes \Gamma^\star(B) \qquad \Delta^\star(R \backslash S) \cong \Gamma^\star(C \backslash D) \qquad T \cong E$$

It must be the case that $R = C$, $S = D$ and $\Delta = \Gamma$. Define the following shape.

$$A \cap Q = \bigotimes_{i \leq a < k} X'_a$$

This shape has been chosen so that the following central isomorphisms exist.

$$Q \cong (A \cap Q) \otimes B$$

$$P \otimes \Delta^\star(A \cap Q) \cong A$$

By Lemma A.1.18, there is a morphism

$$\upsilon\colon \Omega(A \cap Q) \to \Omega(\mathcal{I})$$

such that the following hold.

$$\langle \mathrm{id}, \upsilon, \Omega(f) \rangle_\varepsilon = \sigma$$

$$\langle \Gamma, \upsilon, \tau \rangle_\varepsilon = \Omega(g)$$

By induction, there are constructible morphisms

$$g' \colon A \cap Q \to \mathcal{I} \qquad v \colon P \otimes \Gamma^\star(D) \otimes T \to U$$

such that g is of the form $\langle \Gamma, g', v \rangle_\varepsilon$. Define u to be the following constructible morphism.

$$u \colon Q \cong (A \cap Q) \otimes B \xrightarrow{\langle \mathrm{id}, g', f \rangle_\varepsilon} C = R$$

Note the following.

$$\langle \Delta, u, v \rangle_\varepsilon \cong \langle \Delta, \langle \mathrm{id}, g', f \rangle_\varepsilon, v \rangle_\varepsilon \cong \langle \Gamma, f, \langle \Gamma, g', v \rangle_\varepsilon \rangle_\varepsilon \cong \langle \Gamma, f, g \rangle_\varepsilon$$

Consider the case where $0 \leq k \leq i \leq j = l < j' \leq n$.

$$\overbrace{X_0 \otimes \cdots \otimes X_{k-1} \otimes X_k \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{j-1}}^{\Delta^\star(Q)} \otimes \overbrace{X_j \otimes \cdots \otimes X_{j'-1}}^{\Delta^\star(R \backslash S)} \otimes \overbrace{X_{j'} \otimes \cdots \otimes X_{n-1}}^{T}$$
$$\underbrace{X_0 \otimes \cdots \otimes X_{k-1}}_{A} \otimes \underbrace{X_k \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes \cdots \otimes X_{j-1}}_{\Gamma^\star(B)} \otimes \underbrace{X_j \otimes \cdots \otimes X_{j'-1}}_{\Gamma^\star(C \backslash D)} \otimes \underbrace{X_{j'} \otimes \cdots \otimes X_{n-1}}_{E}$$

The following central isomorphisms exist.

$$P \otimes \Delta^\star(Q) \cong A \otimes \Gamma^\star(B) \qquad \Delta^\star(R \backslash S) \cong \Gamma^\star(C \backslash D) \qquad T \cong E$$

It must be the case that $R = C$, $S = D$ and $\Delta = \Gamma$. Define the following shape.

$$P \cap B = \bigotimes_{k \leq a < i} X''_a$$

This shape has been chosen so that the following central isomorphisms exist.

$$P \cong A \otimes \Gamma^\star(P \cap B)$$

$$(P \cap B) \otimes Q \cong B$$

By Lemma A.1.18, there is a morphism

$$\upsilon \colon \Omega(P \cap B) \to \Omega(\mathcal{I})$$

such that the following hold.

$$\langle \mathrm{id}, \upsilon, \sigma \rangle_\varepsilon = \Omega(f)$$

$$\langle \Delta, \upsilon, \Omega(g) \rangle_\varepsilon = \tau$$

By induction, there are constructible morphisms

$$f' \colon P \cap B \to \mathcal{I} \qquad u \colon Q \to R$$

such that f is of the form $\langle \mathrm{id}, f', u\rangle_\varepsilon$. Define v to be the following constructible morphism.

$$v\colon P \otimes \Delta^\star(S) \otimes T \cong A \otimes \Gamma^\star((P \cap B) \otimes D) \otimes E \xrightarrow{\langle \Gamma, f', g\rangle_\varepsilon} U$$

Note the following.

$$\langle \Delta, u, v\rangle_\varepsilon \cong \langle \Delta, u, \langle \Gamma, f', g\rangle_\varepsilon\rangle_\varepsilon \cong \langle \Gamma, \langle \mathrm{id}, f', u\rangle_\varepsilon, g\rangle_\varepsilon \cong \langle \Gamma, f, g\rangle_\varepsilon$$

Consider the case where $0 \le i \le k \le l < j \le j' \le n$.

$$\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{k-1} \otimes X_k \otimes \cdots \otimes X_{l-1} \otimes X_l \otimes X_{l+1} \otimes \cdots \otimes X_{j-1}}^{\Delta^\star(Q)} \otimes \overbrace{X_j \otimes \cdots \otimes X_{j'-1}}^{\Delta^\star(R\backslash S)} \otimes \overbrace{X_{j'} \otimes \cdots \otimes X_{n-1}}^{T}$$

$$\underbrace{X_0 \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes \cdots \otimes X_{k-1}}_{A} \otimes \underbrace{X_k \otimes \cdots \otimes X_{l-1}}_{\Gamma^\star(B)} \otimes \underbrace{X_l}_{\Gamma^\star(C\backslash D)} \otimes \underbrace{X_{l+1} \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes \cdots \otimes X_{j'-1} \otimes X_{j'} \otimes \cdots \otimes X_{n-1}}_{E}$$

Define the following shapes.

$$A \cap Q = \bigotimes_{i \le a < k} X'_a \qquad B_Q = \bigotimes_{k \le a \le l} X'_a \qquad Q \cap E = \bigotimes_{l < a < j} X'_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$Q \cong (A \cap Q) \otimes B_Q \otimes (Q \cap E)$$
$$P \otimes \Delta^\star(A \cap Q) \cong A \qquad \Delta^\star((Q \cap E) \otimes (R\backslash S)) \otimes T \cong E$$
$$\Delta^\star(B_Q) \cong \Gamma^\star(B \otimes (C\backslash D))$$

By Lemma 5.3.18, there is a Λ such that $\Gamma = \Lambda\Delta$. By Lemma 5.3.16, the following central isomorphism exists.

$$B_Q \cong \Lambda^\star(B \otimes (C\backslash D))$$

By Lemma A.1.16, there is a morphism

$$\upsilon\colon \Omega((A \cap Q) \otimes \Lambda^\star(D) \otimes (Q \cap E)) \to \Omega(R)$$

such that the following hold.

$$\langle \Delta, \upsilon, \tau\rangle_\varepsilon = \Omega(g)$$
$$\langle \Lambda, \Omega(f), \upsilon\rangle_\varepsilon = \sigma$$

By induction, there are constructible morphisms

$$g'\colon (A \cap Q) \otimes \Lambda^\star(D) \otimes (Q \cap E) \to R \qquad v\colon P \otimes \Delta^\star(S) \otimes T \to U$$

such that g is of the form $\langle \Delta, g', v\rangle_\varepsilon$. Define u to be the following constructible morphism.

$$u\colon Q \cong (A \cap Q) \otimes \Lambda^\star(B \otimes (C\backslash D)) \otimes (Q \cap E) \xrightarrow{\langle \Lambda, f, g'\rangle_\varepsilon} R$$

Note the following.

$$\langle \Delta, u, v\rangle_\varepsilon \cong \langle \Delta, \langle \Lambda, f, g'\rangle_\varepsilon, v\rangle_\varepsilon \cong \langle \Lambda\Delta, f, \langle \Delta, g', v\rangle_\varepsilon\rangle_\varepsilon \cong \langle \Gamma, f, g\rangle_\varepsilon$$

Consider the case where $0 \le k \le i \le l < j \le j' \le n$.

$$\overbrace{X_0 \otimes \cdots \otimes X_{k-1} \otimes X_k \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{l-1} \otimes X_l \otimes X_{l+1} \otimes \cdots \otimes X_{j-1}}^{\Delta^\star(Q)} \otimes \overbrace{X_j \otimes \cdots \otimes X_{j'-1}}^{\Delta^\star(R\backslash S)} \otimes \overbrace{X_{j'} \otimes \cdots \otimes X_{n-1}}^{T}$$

$$\underbrace{X_0 \otimes \cdots \otimes X_{k-1}}_{A} \otimes \underbrace{X_k \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes \cdots \otimes X_{l-1}}_{\Gamma^\star(B)} \otimes \underbrace{X_l}_{\Gamma^\star(C\backslash D)} \otimes \underbrace{X_{l+1} \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes \cdots \otimes X_{j'-1} \otimes X_{j'} \otimes \cdots \otimes X_{n-1}}_{E}$$

Define the following shapes.

$$P \cap B = \bigotimes_{k \leq a < i} X''_a \qquad Q_B = \bigotimes_{i \leq a < l} X''_a$$

$$B_Q = \bigotimes_{i \leq a \leq l} X'_a \qquad Q \cap E = \bigotimes_{l < a < j} X'_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$P \cong A \otimes \Gamma^\star(P \cap B) \qquad Q \cong B_Q \otimes (Q \cap E)$$

$$(P \cap B) \otimes Q_B \cong B \qquad \Delta^\star((Q \cap E) \otimes (R\backslash S)) \otimes T \cong E$$

$$\Delta^\star(B_Q) \cong \Gamma^\star(Q_B \otimes (C\backslash D))$$

By Lemma 5.3.18, there is a Λ such that $\Gamma = \Lambda\Delta$. By Lemma 5.3.16, the following central isomorphism exists.

$$B_Q \cong \Lambda^\star(Q_B \otimes (C\backslash D))$$

By Lemma A.1.24, there are morphisms

$$\sigma_1 \colon \Omega(Q_B) \to \Omega(C)$$

$$\sigma_0 \colon \Omega(P \cap B) \to \Omega(\mathcal{I}) \qquad \tau_1 \colon \Omega(\Lambda^\star(D) \otimes (Q \cap E)) \to \Omega(R)$$

$$\tau_2 \colon \Omega(A \otimes \Delta^\star(S) \otimes T) \to \Omega(U)$$

such that the following hold.

$$\langle \mathrm{id}, \sigma_0, \sigma_1 \rangle_\varepsilon = \Omega(f)$$

$$\langle \Lambda, \sigma_1, \tau_1 \rangle_\varepsilon = \sigma$$

$$\langle \Lambda\Delta, \sigma_0, \tau_2 \rangle_\varepsilon = \tau$$

$$\langle \Delta, \tau_1, \tau_2 \rangle_\varepsilon = \Omega(g)$$

By induction, there are constructible morphisms

$$f_0 \colon P \cap B \to \mathcal{I} \qquad f_1 \colon Q_B \to C$$

such that f is of the form $\langle \mathrm{id}, f_0, f_1 \rangle_\varepsilon$. By induction, there are constructible morphisms

$$g_1 \colon \Lambda^\star(D) \otimes (Q \cap E) \to R \qquad g_2 \colon A \otimes \Delta^\star(S) \otimes T \to U$$

such that g is of the form $\langle \Delta, g_1, g_2 \rangle_\varepsilon$. Define u and v to be the following constructible morphisms.

$$u\colon Q \cong \Lambda^\star(Q_B \otimes (C\backslash D)) \otimes (Q \cap E) \xrightarrow{\langle \Lambda, f_1, g_1 \rangle_\varepsilon} R$$

$$v\colon P \otimes \Delta^\star(S) \otimes T \cong A \otimes \Delta^\star(\Lambda^\star(P \cap B) \otimes S) \otimes T \xrightarrow{\langle \Lambda\Delta, f_0, g_2 \rangle_\varepsilon} U$$

Note the following.

$$\langle \Delta, u, v \rangle_\varepsilon \cong \langle \Delta, \langle \Lambda, f_1, g_1 \rangle_\varepsilon, \langle \Lambda\Delta, f_0, g_2 \rangle_\varepsilon \rangle_\varepsilon$$

$$\cong \langle \Lambda\Delta, \langle \mathrm{id}, f_0, f_1 \rangle_\varepsilon, \langle \Delta, g_1, g_2 \rangle_\varepsilon \rangle_\varepsilon \cong \langle \Gamma, f, g \rangle_\varepsilon$$

Consider the case where $0 \leq k \leq l < i \leq j \leq j' \leq n$.

$$\overbrace{\underbrace{X_0 \otimes \cdots \otimes X_{k-1}}_{A} \otimes \underbrace{X_k \otimes \cdots \otimes X_{l-1}}_{\Gamma^\star(B)} \otimes \underbrace{X_l}_{\Gamma^\star(C\backslash D)} \otimes X_{l+1} \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{j-1}}^{\Delta^\star(Q)} \otimes \overbrace{X_j \otimes \cdots \otimes X_{j'-1}}^{\Delta^\star(R\backslash S)} \otimes \overbrace{X_{j'} \otimes \cdots \otimes X_{n-1}}^{T}$$

(where E is the underbraced part $X_{l+1} \otimes \cdots \otimes X_{n-1}$)

This case is similar to the case where $0 \leq i \leq j \leq j' \leq k \leq l < n$.

- Consider the case where s is of type $((-)^\star)$.

$$P \otimes \Delta^\star(Q \otimes (R\backslash S)) \otimes T \cong \Gamma^\star(A) \xrightarrow{\Gamma^\star(f)} \Gamma^\star(B) \cong U$$

It must be the case that A has prime factorisation of the following form, where $\Gamma^\star(X''_a) = X_a$.

$$A \cong \bigotimes_{0 \leq a < n} X''_a$$

Define the following shapes.

$$P_A \cong \bigotimes_{0 \leq a < i} X''_a \qquad Q_A \cong \bigotimes_{i \leq a < j'} X''_a \qquad T_A \cong \bigotimes_{j' \leq a < n} X''_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$P \cong \Gamma^\star(P_A) \qquad \Delta^\star(Q \otimes (R\backslash S)) \cong \Gamma^\star(Q_A) \qquad T \cong \Gamma^\star(T_A)$$

$$P_A \otimes Q_A \otimes T_A \cong A$$

If $Q \otimes (R\backslash S)$ is trivial, then this contradicts our assumption that $i < j'$. If Q is non-trivial, then, by Lemma 5.3.17, either there is a Λ such that $\Delta = \Lambda\Gamma$ or there is a Λ such that $\Gamma = \Lambda\Delta$. If $R\backslash S$ is non-trivial, then, by Lemma 5.3.18, there is a Λ such that $\Delta = \Lambda\Gamma$.

Consider the case where there is a Λ such that $\Delta = \Lambda\Gamma$ By Lemma 5.3.16, the following central isomorphism exists.

$$\Lambda^\star(Q \otimes (R\backslash S)) \cong Q_A$$

By Lemma A.1.20, there is a morphism

$$\upsilon\colon \Omega(P_A \otimes \Lambda^\star(S) \otimes T_A) \to \Omega(B)$$

such that the following hold.

$$\Gamma^\star(\upsilon) = \tau$$

$$\langle \Lambda, \sigma, \upsilon\rangle_\varepsilon = \Omega(f)$$

By induction, there are constructible morphisms

$$u\colon Q \to R \qquad f'\colon P_A \otimes \Lambda^\star(S) \otimes T_A \to B$$

such that f is of the form $\langle \Lambda, u, f'\rangle_\varepsilon$. Define v to be the following constructible morphism.

$$v\colon P \otimes \Delta^\star(S) \otimes T \cong \Gamma^\star(P_A \otimes \Lambda^\star(S) \otimes T_A) \xrightarrow{\Gamma^\star(f')} \Gamma^\star(B) \cong U$$

Note the following.

$$\langle \Delta, u, v\rangle_\varepsilon \cong \langle \Lambda\Gamma, u, \Gamma^\star(f')\rangle_\varepsilon \cong \Gamma^\star(\langle \Lambda, u, f'\rangle_\varepsilon) \cong \Gamma^\star(f)$$

Consider the case where $R\backslash S$ is trivial and there is a Λ such that $\Gamma = \Lambda\Delta$ By Lemma 5.3.16, the following central isomorphism exists.

$$Q \cong \Lambda^\star(Q_A)$$

By Lemma A.1.27, there are morphisms

$$\sigma'\colon \Omega(Q_A) \to \Omega(\mathcal{I}) \qquad \tau'\colon \Omega(P_A \otimes T_A) \to \Omega(B)$$

such that the following hold.

$$\Lambda^\star(\sigma') = \sigma$$

$$(\Lambda\Delta)^\star(\tau') = \tau$$

$$\langle \mathrm{id}, \sigma', \tau'\rangle_\varepsilon = \Omega(f)$$

By induction, there are constructible morphisms

$$f_0\colon Q_A \to \mathcal{I} \qquad f_1\colon P_A \otimes T_A \to B$$

such that f is of the form $\langle \mathrm{id}, f_0, f_1\rangle_\varepsilon$. Define u and v to be the following constructible morphisms.

$$u\colon Q \cong \Lambda^\star(Q_A) \xrightarrow{\Lambda^\star(f_0)} \Lambda^\star(\mathcal{I}) \cong R$$

$$v\colon P \otimes \Delta^\star(S) \otimes T \cong (\Lambda\Delta)^\star(P_A \otimes T_A) \xrightarrow{(\Lambda\Delta)^\star(f_1)} (\Lambda\Delta)^\star(B) \cong U$$

Note the following.

$$\begin{aligned}\langle \Delta, u, v\rangle_\varepsilon &\cong \langle \Delta, \Lambda^\star(f_0), (\Lambda\Delta)^\star(f_1)\rangle_\varepsilon \\ &\cong (\Lambda\Delta)^\star(\langle \mathrm{id}, f_0, f_1\rangle_\varepsilon) \cong (\Lambda\Delta)^\star(f)\end{aligned}$$

- Consider the case where s is of type (θ).

$$P \otimes \Delta^\star(Q \otimes (R\backslash S)) \otimes T \xrightarrow{\langle \Phi, f\rangle_\theta} \Phi_\star(A) \cong U$$

By induction, there are constructible morphisms

$$u\colon Q \to R \qquad f'\colon \Phi^\star(P \otimes \Delta^\star(S) \otimes T) \to A$$

such that f is of the form $\langle \Delta\Phi, u, f'\rangle_\varepsilon$. Define v to be the following constructible morphism.

$$v\colon P \otimes \Delta^\star(S) \otimes T \xrightarrow{\langle \Phi, f'\rangle_\theta} \Phi_\star(A) \cong Q$$

Note the following.

$$\langle \Delta, u, v\rangle_\varepsilon \cong \langle \Delta, u, \langle \Phi, f'\rangle_\theta\rangle_\varepsilon \cong \langle \Phi, \langle \Delta\Phi, u, f'\rangle_\varepsilon\rangle_\theta \cong \langle \Phi, f\rangle_\theta$$

- Consider the case where s is of type (ζ).

$$P \otimes \Delta^\star(Q \otimes (R\backslash S)) \otimes T \cong A \otimes \Gamma^\star\Phi^\star\Phi_\star(B) \otimes C \xrightarrow{\langle \Gamma, \Phi, f\rangle_\zeta} U$$

Then Φ is non-empty. It must be the case that A, $\Phi_\star(B)$ and C have prime factorisations of the following forms, where $\Gamma^\star\Phi^\star(X''_a) = X_a$.

$$A \cong \bigotimes_{0 \le a < k} X_a \qquad \Phi_\star(B) = X''_k \qquad C \cong \bigotimes_{k < a < n} X_a$$

Compare i and j with k.

Consider the case where $0 \le k < i \le j \le j' \le n$.

$$\overbrace{\underbrace{X_0 \otimes \cdots \otimes X_{k-1}}_{A} \otimes \underbrace{X_k}_{\Gamma^\star\Phi^\star\Phi_\star(B)} \otimes X_{k+1} \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{j-1}}^{\Delta^\star(Q)} \otimes \overbrace{X_j \otimes \cdots \otimes X_{j'-1}}^{\Delta^\star(R\backslash S)} \otimes \overbrace{X_{j'} \otimes \cdots \otimes X_{n-1}}^{T}$$

(where C is the underbrace spanning $X_{k+1} \otimes \cdots \otimes X_{n-1}$)

Define the following shape.

$$P \cap C = \bigotimes_{k<a<i} X_a$$

This shape has been chosen so that the following central isomorphisms exist.

$$P \cong A \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes (P \cap C)$$

$$(P \cap C) \otimes \Delta^\star(Q \otimes (R \backslash S)) \otimes T \cong C$$

By Lemma A.1.13, there is a morphism

$$\upsilon \colon \Omega(A \otimes \Gamma^\star(B) \otimes (P \cap C) \otimes \Delta^\star(S) \otimes T) \to \Omega(U)$$

such that the following hold.

$$\langle \Gamma, \Phi, \upsilon \rangle_\zeta = \tau$$

$$\langle \Delta, \sigma, \upsilon \rangle_\varepsilon = \Omega(f)$$

By induction, there are constructible morphisms

$$u \colon Q \to R \qquad f' \colon A \otimes \Gamma^\star(B) \otimes (P \cap C) \otimes \Delta^\star(S) \otimes T \to U$$

such that f is of the form $\langle \Delta, u, f' \rangle_\varepsilon$. Define v to be the following constructible morphism.

$$v \colon P \otimes \Delta^\star(S) \otimes T \cong A \otimes \Gamma^\star \Phi^\star \Phi_\star(B) \otimes (P \cap C) \otimes \Delta^\star(S) \otimes T \xrightarrow{\langle \Gamma, \Phi, f' \rangle_\zeta} U$$

Note the following.

$$\langle \Delta, u, v \rangle_\varepsilon \cong \langle \Delta, u, \langle \Gamma, \Phi, f' \rangle_\zeta \rangle_\varepsilon \cong \langle \Gamma, \Phi, \langle \Delta, u, f' \rangle_\varepsilon \rangle_\zeta \cong \langle \Gamma, \Phi, f \rangle_\zeta$$

Consider the case where $0 \leq i \leq k < j \leq j' \leq n$.

$$\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{k-1} \otimes X_k \otimes X_{k+1} \otimes \cdots \otimes X_{j-1}}^{\Delta^\star(Q)} \otimes \overbrace{X_j \otimes \cdots \otimes X_{j'-1}}^{\Delta^\star(R\backslash S)} \otimes \overbrace{X_{j'} \otimes \cdots \otimes X_{n-1}}^{T}$$
$$\underbrace{X_0 \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes \cdots \otimes X_{k-1}}_{A} \otimes \underbrace{X_k}_{\Gamma^\star \Phi^\star \Phi_\star(B)} \otimes \underbrace{X_{k+1} \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes \cdots \otimes X_{j'-1} \otimes X_{j'} \otimes \cdots \otimes X_{n-1}}_{C}$$

Define the following shapes.

$$A \cap Q = \bigotimes_{i \leq a < k} X'_a \qquad B_Q = X'_k \qquad Q \cap C = \bigotimes_{k<a<j} X'_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$Q \cong (A \cap Q) \otimes B_Q \otimes (Q \cap C)$$

$$P \otimes \Delta^\star(A \cap Q) \cong A \qquad \Delta^\star((Q \cap C) \otimes (R \backslash S)) \otimes T \cong C$$

$$\Delta^\star(B_Q) \cong \Gamma^\star\Phi^\star\Phi_\star(B)$$

By Lemma 5.3.19, either there is a Λ such that $\Gamma = \Lambda\Delta$ and $B_Q \cong \Lambda^\star\Phi^\star\Phi_\star(B)$, or there are a non-empty Λ and a Ψ such that $\Delta = \Lambda\Gamma$, $\Phi = \Psi\Lambda$ and $B_Q \cong \Psi^\star\Phi_\star(B)$.

Consider the case where there is a Λ such that $\Gamma = \Lambda\Delta$ and $B_Q \cong \Lambda^\star\Phi^\star\Phi_\star(B)$. By Lemma 5.3.16, the following central isomorphism exists.

$$B_Q \cong \Lambda^\star\Phi^\star\Phi_\star(B)$$

By Lemma A.1.17, there is a morphism

$$\upsilon\colon \Omega((A \cap Q) \otimes \Lambda^\star(B) \otimes (Q \cap C)) \to \Omega(R)$$

such that the following hold.

$$\langle \Delta, \upsilon, \tau \rangle_\varepsilon = \Omega(f)$$

$$\langle \Lambda, \Phi, \upsilon \rangle_\zeta = \sigma$$

By induction, there are constructible morphisms

$$f'\colon (A \cap Q) \otimes \Lambda^\star(B) \otimes (Q \cap C) \to R \qquad v\colon P \otimes \Delta^\star(S) \otimes T \to U$$

such that f is of the form $\langle \Delta, f', v \rangle_\varepsilon$. Define u to be the following constructible morphism.

$$u\colon Q \cong (A \cap Q) \otimes \Lambda^\star\Phi^\star\Phi_\star(B) \otimes (Q \cap C) \xrightarrow{\langle \Lambda, \Phi, f' \rangle_\zeta} R$$

Note the following.

$$\langle \Delta, u, v \rangle_\varepsilon \cong \langle \Delta, \langle \Lambda, \Phi, f' \rangle_\zeta, v \rangle_\varepsilon \cong \langle \Lambda\Delta, \Phi, \langle \Delta, f', v \rangle_\varepsilon \rangle_\zeta \cong \langle \Gamma, \Phi, f \rangle_\zeta$$

Consider the case where there is a non-empty Λ and a Ψ such that $\Delta = \Lambda\Gamma$, $\Phi = \Psi\Lambda$ and $B_Q \cong \Psi^\star\Phi_\star(B)$. By Lemma A.1.33, Λ is empty, which is a contradiction.

Consider the case where $0 \leq i \leq j = k < j' \leq n$.

$$\underbrace{\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{j-1}}^{\Delta^\star(Q)}}_{A} \otimes \underbrace{\overbrace{X_j \otimes \cdots \otimes X_{j'-1}}^{\Delta^\star(R\backslash S)}}_{\Gamma^\star\Phi^\star\Phi_\star(B)} \otimes \underbrace{\overbrace{X_{j'} \otimes \cdots \otimes X_{n-1}}^{T}}_{C}$$

This case cannot occur, since the following is impossible.

$$\Delta^\star(R\backslash S) = \Gamma^\star\Phi^\star\Phi_\star(B)$$

Consider the case where $0 \leq i \leq j \leq j' \leq k < n$.

$$\underbrace{\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i \otimes \cdots \otimes X_{j-1}}^{\Delta^\star(Q)} \otimes \overbrace{X_j \otimes \cdots \otimes X_{j'-1}}^{\Delta^\star(R\backslash S)} \otimes \rlap{\overbrace{\phantom{X_{j'} \otimes \cdots \otimes X_{k-1} \otimes X_k \otimes X_{k+1} \otimes \cdots \otimes X_{n-1}}}^{T}} X_{j'} \otimes \cdots \otimes X_{k-1}}_{A} \otimes \underbrace{X_k}_{\Gamma^\star\Phi^\star\Phi_\star(B)} \otimes \underbrace{X_{k+1} \otimes \cdots \otimes X_{n-1}}_{C}$$

This case is similar to the case where $0 \leq k < i \leq j \leq j' \leq n$.

$\square$

A.4 Proof of Lemma 5.6.3

Proof. It must be the case that P and Q have prime factorisations of the following forms, where $\Delta^\star(X'_a) = X_a$ and $\Delta^\star(Y'_a) = Y_a$.

$$P \cong \bigotimes_{0 \le a < n} X'_a$$

$$Q \cong \bigotimes_{0 \le a < n'} Y'_a$$

Consider the case where Δ is empty. In this case, we can simply define u to be the following constructible morphism.

$$u \colon P \cong \Delta^\star(P) \xrightarrow{s} \Delta^\star(Q) \cong Q$$

Consider the case where Q is trivial, so that $n' = 0$. In this case, the result follows from Lemma 5.6.2.

Assume that Δ is non-empty and Q is non-trivial, so that $n' > 0$. We will prove the result by induction on the type of s. Consider the type of the constructible morphism s.

- Consider the case where s is of type $(\cong)$.

  $$\Delta^\star(P) \cong \Delta^\star(Q)$$

  By Lemma 5.3.16, the following central isomorphism exists.

  $$P \cong Q$$

  Define u to be this central isomorphism.

- Consider the case where s is of type $(\otimes)$.

  $$\Delta^\star(P) \cong A \otimes C \xrightarrow{f \otimes g} B \otimes D \cong \Delta^\star(Q)$$

  It must be the case that A, B, C and D have prime factorisations of the following forms.

  $$A \cong \bigotimes_{0 \le a < i} X_a \qquad C \cong \bigotimes_{i \le a < n} X_a$$

  $$B \cong \bigotimes_{0 \le a < i'} Y_a \qquad D \cong \bigotimes_{i' \le a < n'} Y_a$$

  Define the following shapes.

  $$A_P \cong \bigotimes_{0 \le a < i} X'_a \qquad C_P \cong \bigotimes_{i \le a < n} X'_a$$

$$B_Q \cong \bigotimes_{0 \leq a < i'} Y'_a \qquad D_Q \cong \bigotimes_{i' \leq a < n'} Y'_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$P \cong A_P \otimes C_P$$

$$\Delta^\star(A_P) \cong A \qquad \Delta^\star(C_P) \cong C$$

$$C \cong \Delta^\star(C_P) \qquad D \cong \Delta^\star(D_P)$$

$$B_Q \otimes D_Q \cong Q$$

By Lemma A.1.19, there are morphisms

$$\upsilon_1 \colon \Omega(A_P) \to \Omega(B_Q) \qquad \upsilon_2 \colon \Omega(C_P) \to \Omega(D_Q)$$

such that the following hold.

$$\Delta^\star(\upsilon_1) = \Omega(f)$$

$$\Delta^\star(\upsilon_2) = \Omega(g)$$

$$\upsilon_1 \otimes \upsilon_2 = \sigma$$

By induction, there is a constructible morphism

$$f' \colon A_P \to B_Q$$

such that f is of the form $\Delta^\star(f')$. By induction, there is a constructible morphism

$$g' \colon C_P \to D_Q$$

such that g is of the form $\Delta^\star(g')$. Define u to be the following constructible morphism.

$$u \colon P \cong A_P \otimes C_P \xrightarrow{f' \otimes g'} B_Q \otimes D_Q \cong Q$$

Note the following.

$$\Delta^\star(u) \cong \Delta^\star(f' \otimes g') \cong \Delta^\star(f') \otimes \Delta^\star(g') \cong f \otimes g$$

- Consider the case where s is of type (η).

$$\Delta^\star(P) \xrightarrow{\langle I, f \rangle_\eta} I \backslash A \cong \Delta^\star(Q)$$

Then I is non-trivial. By assumption, Δ is non-empty. In this case, there is no central isomorphism of the following form.

$$I \backslash A \cong \Delta^\star(Q)$$

- Consider the case where s is of type (ε).

$$\Delta^\star(P) \cong A \otimes \Gamma^\star(B \otimes (C \backslash D)) \otimes E \xrightarrow{\langle \Gamma, f, g \rangle_\varepsilon} \Delta^\star(Q)$$

Then C is non-trivial. It must be the case that A, B, $C\backslash D$ and E have prime factorisations of the following forms, where $\Gamma^\star(X''_a) = X'_a$.

$$A \cong \bigotimes_{0 \le a < i} X_a \qquad B \cong \bigotimes_{i \le a < j} X''_a \qquad C\backslash D = X''_j \qquad E \cong \bigotimes_{j < a < n} X_a$$

Define the following shapes.

$$A_P \cong \bigotimes_{0 \le a < i} X'_a \qquad B_P \cong \bigotimes_{i \le a \le j} X'_a \qquad E_P \cong \bigotimes_{j < a < n} X'_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$P \cong A_P \otimes B_P \otimes E_P$$

$$\Delta^\star(A_P) \cong A \qquad \Delta^\star(B_P) \cong \Gamma^\star(B \otimes (C\backslash D)) \qquad \Delta^\star(E_P) \cong E$$

By Lemma 5.3.18, there is a Λ such that $\Gamma = \Lambda\Delta$. By Lemma 5.3.16, the following central isomorphism exists.

$$B_P \cong \Lambda^\star(B \otimes (C\backslash D))$$

By Lemma A.1.20, there is a morphism

$$\upsilon \colon \Omega(A_P \otimes \Lambda^\star(D) \otimes E_P) \to \Omega(Q)$$

such that the following hold.

$$\Delta^\star(\upsilon) = \Omega(g)$$

$$\langle \Lambda, \Omega(f), \upsilon \rangle_\varepsilon = \sigma$$

By induction, there is a constructible morphism

$$g' \colon A_P \otimes \Lambda^\star(D) \otimes E_P \to Q$$

such that g is of the form $\Delta^\star(g')$. Define u to be the following constructible morphism.

$$u \colon P \cong A_P \otimes \Lambda^\star(B \otimes (C\backslash D)) \otimes E_P \xrightarrow{\langle \Lambda, f, g' \rangle_\varepsilon} Q$$

Note the following.

$$\Delta^\star(u) \cong \Delta^\star(\langle \Lambda, f, g' \rangle_\varepsilon) \cong \langle \Lambda\Delta, f, \Delta^\star(g') \rangle_\varepsilon \cong \langle \Gamma, f, g \rangle_\varepsilon$$

- Consider the case where s is of type $((-)^{\star})$.

$$\Delta^{\star}(P) \cong \Gamma^{\star}(A) \xrightarrow{\Gamma^{\star}(f)} \Gamma^{\star}(B) \cong \Delta^{\star}(Q)$$

By Lemma 5.3.17, either there is a Λ such that $\Gamma = \Lambda\Delta$ or there is a Λ such that $\Delta = \Lambda\Gamma$.

Consider the case where there is a Λ such that $\Gamma = \Lambda\Delta$. By Lemma 5.3.16, the following central isomorphism exists.

$$P \cong \Lambda^{\star}(A) \qquad \Lambda^{\star}(B) \cong Q$$

Define u to be the following constructible morphism.

$$u \colon P \cong \Lambda^{\star}(A) \xrightarrow{\Lambda^{\star}(f)} \Lambda^{\star}(B) \cong Q$$

Note the following.

$$\Delta^{\star}(u) \cong (\Lambda\Delta)^{\star}(f) \cong \Gamma^{\star}(f)$$

Consider the case where there is a Λ such that $\Delta = \Lambda\Gamma$. By Lemma 5.3.16, the following central isomorphism exists.

$$\Lambda^{\star}(P) \cong A \qquad B \cong \Lambda^{\star}(Q)$$

By induction, there is a constructible morphism

$$u \colon P \to Q$$

such that f is of the form $\Lambda^{\star}(u)$. Note the following.

$$\Delta^{\star}(u) \cong (\Lambda\Gamma)^{\star}(u) \cong \Gamma^{\star}(f)$$

- Consider the case where s is of type (θ).

$$\Delta^{\star}(P) \xrightarrow{\langle\Phi,f\rangle_{\theta}} \Phi_{\star}(A) \cong \Delta^{\star}(Q)$$

Then Φ is non-empty. By assumption, Δ is non-empty. In this case, there is no central isomorphism of the following form.

$$\Phi_{\star}(A) \cong \Delta^{\star}(Q)$$

- Consider the case where s is of type (ζ).

$$\Delta^{\star}(P) \cong A \otimes \Gamma^{\star}\Phi^{\star}\Phi_{\star}(B) \otimes C \xrightarrow{\langle\Gamma,\Phi,f\rangle_{\zeta}} \Delta^{\star}(Q)$$

Then Φ is non-empty. It must be the case that A, $\Phi_\star(B)$ and C have prime factorisations of the following forms, where $\Gamma^\star\Phi^\star(X''_a) = X'_a$.

$$A \cong \bigotimes_{0 \leq a < i} X_a \qquad \Phi_\star(B) \cong X''_i \qquad C \cong \bigotimes_{i < a < n} X_a$$

Define the following shapes.

$$A_P \cong \bigotimes_{0 \leq a < i} X'_a \qquad B_P \cong X'_i \qquad C_P \cong \bigotimes_{i < a < n} X'_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$P \cong A_P \otimes B_P \otimes E_P$$

$$\Delta^\star(A_P) \cong A \qquad \Delta^\star(B_P) \cong \Gamma^\star\Phi^\star\Phi_\star(B) \qquad \Delta^\star(C_P) \cong C$$

By Lemma 5.3.19, either there is a Λ such that $\Gamma = \Lambda\Delta$ and $B_P \cong \Lambda^\star\Phi^\star\Phi_\star(B)$, or there is a non-empty Λ and a Ψ such that $\Delta = \Lambda\Gamma$, $\Phi = \Psi\Lambda$ and $B_P \cong \Psi^\star\Phi_\star(B)$.

Consider the case where there is a Λ such that $\Gamma = \Lambda\Delta$ and $B_P \cong \Lambda^\star\Phi^\star\Phi_\star(B)$. By Lemma 5.3.16, the following central isomorphism exists.

$$B_P \cong \Lambda^\star\Phi^\star\Phi_\star(B)$$

By Lemma A.1.21, there is a morphism

$$\upsilon \colon \Omega(A_P \otimes \Lambda^\star(B) \otimes C_P) \to \Omega(Q)$$

such that the following hold.

$$\Delta^\star(\upsilon) = \Omega(f)$$

$$\langle \Lambda, \Phi, \upsilon \rangle_\zeta = \sigma$$

By induction, there is a constructible morphism

$$f' \colon A_P \otimes \Lambda^\star(B) \otimes C_P \to Q$$

such that f is of the form $\Delta^\star(f')$. Define u to be the following constructible morphism.

$$u \colon P \cong A_P \otimes \Lambda^\star\Phi^\star\Phi_\star(B) \otimes C_P \xrightarrow{\langle \Lambda, \Phi, f' \rangle_\zeta} Q$$

Note the following.

$$\Delta^\star(u) \cong \Delta^\star(\langle \Lambda, \Phi, f' \rangle_\zeta) \cong \langle \Lambda\Delta, \Phi, \Delta^\star(f') \rangle_\zeta \cong \langle \Gamma, \Phi, f \rangle_\zeta$$

Consider the case where there is a non-empty Λ and a Ψ such that $\Delta = \Lambda\Gamma$, $\Phi = \Psi\Lambda$ and $B_P \cong \Psi^\star\Phi_\star(B)$. By Lemma A.1.32, Λ is empty, which is a contradiction.

□

A.5 Proof of Lemma 5.6.4

Proof. Consider the case where Φ is empty. In this case, we can simply define u to be the following constructible morphism.

$$u\colon P \otimes \Delta^\star(Q) \otimes R \cong P \otimes \Delta^\star\Phi^\star\Phi_\star(Q) \otimes R \xrightarrow{s} S$$

Assume that Φ is non-empty. It must be the case that P, $\Phi_\star(Q)$ and R have prime factorisations of the following forms, where $\Delta^\star\Phi^\star(X'_a) = X_a$.

$$P \cong \bigotimes_{0\leq a<i} X_a \qquad Q \cong X'_i \qquad R \cong \bigotimes_{i<a<n} X_a$$

We will prove the result by induction on the type of s. Consider the type of the constructible morphism s.

- Consider the case where s is of type $(\cong)$.

  $$P \otimes \Gamma^\star\Phi^\star\Phi_\star(Q) \otimes R \cong S$$

  By Lemma A.1.31, Φ is empty. This contradicts our assumption that Φ is non-empty.

- Consider the case where s is of type $(\otimes)$.

  $$P \otimes \Delta^\star\Phi^\star\Phi_\star(Q) \otimes R \cong A \otimes C \xrightarrow{f\otimes g} B \otimes D \cong S$$

  It must be the case that A and C have prime factorisations of the following forms.

  $$A \cong \bigotimes_{0\leq a<j} X_a \qquad C \cong \bigotimes_{j\leq a<n} X_a$$

  Compare i with j.

  Consider the case where $0 \leq i < j \leq n$.

  $$\underbrace{\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i}^{\Delta^\star\Phi^\star\Phi_\star(Q)} \otimes X_{i+1} \otimes \cdots \otimes X_{j-1}}_{A} \otimes \underbrace{X_j \otimes \cdots \otimes X_{n-1}}_{C}$$

  Define the following shape.

  $$A \cap R = \bigotimes_{i<a<j} X_a$$

  This shape has been chosen so that the following central isomorphisms exist.

  $$R \cong (A \cap R) \otimes C$$

$$P \otimes \Delta^\star \Phi^\star \Phi_\star(Q) \otimes (A \cap R) \cong A$$

By Lemma A.1.10, there is a morphism

$$\upsilon \colon \Omega(P \otimes \Delta^\star(Q) \otimes (A \cap R)) \to \Omega(B)$$

such that the following hold.

$$\upsilon \otimes \Omega(g) = \sigma$$

$$\langle \Delta, \Phi, \upsilon \rangle_\zeta = \Omega(f)$$

By induction, there is a constructible morphism

$$f' \colon P \otimes \Delta^\star(Q) \otimes (A \cap R) \to B$$

such that f is of the form $\langle \Delta, \Phi, f' \rangle_\zeta$. Define u to be the following constructible morphism.

$$u \colon P \otimes \Delta^\star(Q) \otimes R \cong P \otimes \Delta^\star(Q) \otimes (A \cap R) \otimes C \xrightarrow{f' \otimes g} B \otimes D \cong S$$

Note the following.

$$\langle \Delta, \Phi, u \rangle_\zeta \cong \langle \Delta, \Phi, f' \otimes g \rangle_\zeta \cong \langle \Delta, \Phi, f' \rangle_\zeta \otimes g \cong f \otimes g$$

Consider the case where $0 \leq j \leq i < n$.

$$\overbrace{X_0 \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i}^{\Delta^\star \Phi^\star \Phi_\star(Q)} \otimes \overbrace{X_{i+1} \otimes \cdots \otimes X_{n-1}}^{R}$$

$$\underbrace{X_0 \otimes \cdots \otimes X_{j-1}}_{A} \otimes \underbrace{X_j \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes X_{i+1} \otimes \cdots \otimes X_{n-1}}_{C}$$

This case is similar to the case where $0 \leq i < j \leq n$.

- Consider the case where s is of type (η).

$$P \otimes \Delta^\star \Phi^\star \Phi_\star(Q) \otimes R \xrightarrow{\langle I, f \rangle_\eta} I \backslash A \cong S$$

By induction, there is a constructible morphism

$$f' \colon I \otimes P \otimes \Delta^\star(Q) \otimes R \to A$$

such that f is of the form $\langle \Delta, \Phi, f' \rangle_\zeta$. Define u to be the following constructible morphism.

$$u \colon P \otimes \Delta^\star(Q) \otimes R \xrightarrow{\langle I, f' \rangle_\eta} I \backslash A \cong Q$$

Note the following.

$$\langle \Delta, \Phi, u \rangle_\zeta \cong \langle \Delta, \Phi, \langle I, f' \rangle_\eta \rangle_\zeta \cong \langle I, \langle \Delta, \Phi, f' \rangle_\zeta \rangle_\eta \cong \langle I, f \rangle_\eta$$

- Consider the case where s is of type (ε).

$$P \otimes \Delta^\star\Phi^\star\Phi_\star(Q) \otimes R \cong A \otimes \Gamma^\star(B \otimes (C\backslash D)) \otimes E \xrightarrow{\langle \Gamma, f, g\rangle_\varepsilon} S$$

Then C is non-trivial. It must be the case that A, B, $C\backslash D$ and E have prime factorisations of the following forms, where $\Gamma^\star(X''_a) = X_a$.

$$A \cong \bigotimes_{0 \leq a < j} X_a \qquad B \cong \bigotimes_{j \leq a < k} X''_a \qquad C\backslash D = X''_k \qquad E \cong \bigotimes_{k < a < n} X_a$$

Compare i with j and k.

Consider the case where $0 \leq i < j \leq k < n$.

$$\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i}^{\Delta^\star\Phi^\star\Phi_\star(Q)} \otimes \overbrace{X_{i+1} \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes \cdots \otimes X_{k-1} \otimes X_k \otimes X_{k+1} \otimes \cdots \otimes X_{n-1}}^{R}$$

$$\underbrace{X_0 \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes X_{i+1} \otimes \cdots \otimes X_{j-1}}_{A} \otimes \underbrace{X_j \otimes \cdots \otimes X_{k-1}}_{\Gamma^\star(B)} \otimes \underbrace{X_k}_{\Gamma^\star(C\backslash D)} \otimes \underbrace{X_{k+1} \otimes \cdots \otimes X_{n-1}}_{E}$$

Define the following shape.

$$A \cap R = \bigotimes_{i < a < j} X_a$$

This shape has been chosen so that the following central isomorphisms exist.

$$R \cong (A \cap R) \otimes \Gamma^\star(B \otimes (C\backslash D)) \otimes E$$

$$P \otimes \Delta^\star\Phi^\star\Phi_\star(Q) \otimes (A \cap R) \cong A$$

By Lemma A.1.13, there is a morphism

$$\upsilon \colon \Omega(P \otimes \Delta^\star(Q) \otimes (A \cap R) \otimes \Gamma^\star(D) \otimes E) \to \Omega(S)$$

such that the following hold.

$$\langle \Delta, \Phi, \upsilon\rangle_\zeta = \Omega(g)$$

$$\langle \Gamma, \Omega(f), \upsilon\rangle_\varepsilon = \sigma$$

By induction, there is a constructible morphism

$$g' \colon P \otimes \Delta^\star(Q) \otimes (A \cap R) \otimes \Gamma^\star(D) \otimes E \to S$$

such that g is of the form $\langle \Delta, \Phi, g'\rangle_\zeta$. Define u to be the following constructible morphism.

$$u \colon P \otimes \Delta^\star(Q) \otimes R \cong P \otimes \Delta^\star(Q) \otimes (A \cap R) \otimes \Gamma^\star(B \otimes (C\backslash D)) \otimes E \xrightarrow{\langle \Gamma, f, g'\rangle_\varepsilon} S$$

Note the following.

$$\langle \Delta, \Phi, u\rangle_\zeta \cong \langle \Delta, \Phi, \langle \Gamma, f, g'\rangle_\varepsilon\rangle_\zeta \cong \langle \Gamma, f, \langle \Delta, \Phi, g'\rangle_\zeta\rangle_\varepsilon \cong \langle \Gamma, f, g\rangle_\varepsilon$$

Consider the case where $0 \leq j \leq i < k < n$.

$$\rlap{$\overbrace{\phantom{X_0\otimes\cdots\otimes X_{j-1}\otimes X_j\otimes\cdots\otimes X_{i-1}}}^{P}\phantom{{}\otimes{}}\overbrace{}^{\Delta^\star\Phi^\star\Phi_\star(Q)}\phantom{{}\otimes{}}\overbrace{\phantom{X_{i+1}\otimes\cdots\otimes X_{k-1}\otimes X_k\otimes X_{k+1}\otimes\cdots\otimes X_{n-1}}}^{R}$}\underbrace{X_0\otimes\cdots\otimes X_{j-1}}_{A}\otimes\underbrace{X_j\otimes\cdots\otimes X_{i-1}\otimes X_i\otimes X_{i+1}\otimes\cdots\otimes X_{k-1}}_{\Gamma^\star(B)}\otimes\underbrace{X_k}_{\Gamma^\star(C\backslash D)}\otimes\underbrace{X_{k+1}\otimes\cdots\otimes X_{n-1}}_{E}$$

Define the following shapes.

$$P \cap B = \bigotimes_{j \le a < i} X''_a \qquad Q_B = X''_i \qquad B \cap R = \bigotimes_{i < a < k} X''_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$P \cong A \otimes \Gamma^\star(P \cap B) \qquad R \cong \Gamma^\star((B \cap R) \otimes (C \backslash D)) \otimes E$$

$$(P \cap B) \otimes Q_B \otimes (B \cap R) \cong B$$

$$\Delta^\star \Phi^\star \Phi_\star(Q) \cong \Gamma^\star(Q_B)$$

By Lemma 5.3.19, either there is a Λ such that $\Delta = \Lambda\Gamma$ and $Q_B \cong \Lambda^\star \Phi^\star \Phi_\star(Q)$, or there is a non-empty Λ and a Ψ such that $\Gamma = \Lambda\Delta$, $\Phi = \Psi\Lambda$ and $Q_B \cong \Psi^\star \Phi_\star(Q)$.

Consider the case where there is a Λ such that $\Delta = \Lambda\Gamma$ and $Q_B \cong \Lambda^\star \Phi^\star \Phi_\star(Q)$. By Lemma 5.3.16, the following central isomorphism exists.

$$\Lambda^\star \Phi^\star \Phi_\star(Q) \cong Q_B$$

By Lemma A.1.17, there is a morphism

$$\upsilon\colon \Omega((P \cap B) \otimes \Lambda^\star(Q) \otimes (B \cap R)) \to \Omega(C)$$

such that the following hold.

$$\langle \Gamma, \upsilon, \Omega(g) \rangle_\varepsilon = \sigma$$

$$\langle \Lambda, \Phi, \upsilon \rangle_\zeta = \Omega(f)$$

By induction, there is a constructible morphism

$$f'\colon (P \cap B) \otimes \Lambda^\star(Q) \otimes (B \cap R) \to C$$

such that f is of the form $\langle \Lambda, \Phi, f' \rangle_\zeta$. Define u to be the following constructible morphism.

$$\begin{aligned} &u\colon P \otimes \Delta^\star(Q) \otimes R \\ &\cong A \otimes \Gamma^\star((P \cap B) \otimes \Lambda^\star(Q) \otimes (B \cap R) \otimes (C \backslash D)) \otimes E \xrightarrow{\langle \Gamma, f', g \rangle_\varepsilon} S \end{aligned}$$

Note the following.

$$\langle \Delta, \Phi, u \rangle_\zeta \cong \langle \Lambda\Gamma, \Phi, \langle \Gamma, f', g \rangle_\varepsilon \rangle_\zeta \cong \langle \Gamma, \langle \Lambda, \Phi, f' \rangle_\zeta, g \rangle_\varepsilon \cong \langle \Gamma, f, g \rangle_\varepsilon$$

Consider the case where there is a non-empty Λ and a Ψ such that $\Gamma = \Lambda\Delta$, $\Phi = \Psi\Lambda$ and $Q_B \cong \Psi^\star\Phi_\star(Q)$. By Lemma A.1.33, Λ is empty, which is a contradiction.

Consider the case where $i = k$. This case cannot occur, since the following is impossible.

$$\Delta^\star\Phi^\star\Phi_\star(Q) = \Gamma^\star(C\backslash D)$$

Consider the case where $0 \leq j \leq k < i < n$.

$$\overbrace{\underbrace{X_0 \otimes \cdots \otimes X_{j-1}}_{A} \otimes \underbrace{X_j \otimes \cdots \otimes X_{k-1}}_{\Gamma^\star(B)} \otimes \underbrace{X_k}_{\Gamma^\star(C\backslash D)} \otimes X_{k+1} \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i}^{\Delta^\star\Phi^\star\Phi_\star(Q)} \otimes \overbrace{X_{i+1} \otimes \cdots \otimes X_{n-1}}^{R}$$

(where E underbraces $X_{k+1} \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes X_{i+1} \otimes \cdots \otimes X_{n-1}$)

This case is similar to the case where $0 \leq i < j \leq k < n$.

- Consider the case where s is of type $((-)^\star)$.

$$P \otimes \Delta^\star\Phi^\star\Phi_\star(Q) \otimes R \cong \Gamma^\star(A) \xrightarrow{\Gamma^\star(f)} \Gamma^\star(B) \cong S$$

It must be the case that A has prime factorisation of the following form, where $\Gamma^\star(X''_a) = X_a$.

$$A \cong \bigotimes_{0 \leq a < n} X''_a$$

Define the following shapes.

$$P_A \cong \bigotimes_{0 \leq a < i} X''_a \qquad Q_A \cong X''_i \qquad R_A \cong \bigotimes_{i < a < n} X''_a$$

These shapes have been chosen so that the following central isomorphisms exist.

$$P \cong \Gamma^\star(P_A) \qquad \Delta^\star\Phi^\star\Phi_\star(Q) \cong \Gamma^\star(Q_A) \qquad R \cong \Gamma^\star(R_A)$$

$$P_A \otimes Q_A \otimes R_A \cong A$$

By Lemma 5.3.19, either there is a Λ such that $\Delta = \Lambda\Gamma$ and $Q_A \cong \Lambda^\star\Phi^\star\Phi_\star(Q)$, or there is a non-empty Λ and a Ψ such that $\Gamma = \Lambda\Delta$, $\Phi = \Psi\Lambda$ and $Q_A \cong \Psi^\star\Phi_\star(Q)$.

Consider the case where there is a Λ such that $\Delta = \Lambda\Gamma$ and $Q_A \cong \Lambda^\star\Phi^\star\Phi_\star(Q)$. By Lemma 5.3.16, the following central isomorphism exists.

$$\Lambda^\star\Phi^\star\Phi_\star(Q) \cong Q_A$$

By Lemma A.1.21, there is a morphism

$$\upsilon\colon \Omega(P_A \otimes \Lambda^\star(Q) \otimes R_A) \to \Omega(B)$$

such that the following hold.

$$\Gamma^\star(\upsilon) = \sigma$$

$$\langle \Lambda, \Phi, \upsilon \rangle_\zeta = \Omega(f)$$

By induction, there is a constructible morphism

$$f'\colon P_A \otimes \Lambda^\star(Q) \otimes R_A \to B$$

such that f is of the form $\langle \Lambda, \Phi, f' \rangle_\zeta$. Define u to be the following constructible morphism.

$$u\colon P \otimes \Delta^\star(Q) \otimes R \cong \Gamma^\star(P_A \otimes \Lambda^\star(Q) \otimes R_A) \xrightarrow{\Gamma^\star(f')} \Gamma^\star(B) \cong S$$

Note the following.

$$\langle \Delta, \Phi, u \rangle_\zeta \cong \langle \Lambda\Gamma, \Phi, \Gamma^\star(f') \rangle_\zeta \cong \Gamma^\star(\langle \Lambda, \Phi, f' \rangle_\zeta) \cong \Gamma^\star(f)$$

Consider the case where there is a non-empty Λ and a Ψ such that $\Gamma = \Lambda\Delta$, $\Phi = \Psi\Lambda$ and $Q_A \cong \Psi^\star\Phi_\star(Q)$. By Lemma A.1.32, Λ is empty, which is a contradiction.

- Consider the case where s is of type (θ).

$$P \otimes \Delta^\star\Phi^\star\Phi_\star(Q) \otimes R \xrightarrow{\langle \Psi, f \rangle_\theta} \Psi_\star(A) \cong S$$

By induction, there is a constructible morphism

$$f'\colon \Psi^\star(P \otimes \Delta^\star(Q) \otimes R) \to A$$

such that f is of the form $\langle \Delta\Psi, \Phi, f' \rangle_\zeta$. Define u to be the following constructible morphism.

$$u\colon P \otimes \Delta^\star(Q) \otimes R \xrightarrow{\langle \Psi, f' \rangle_\theta} \Psi_\star(A) \cong Q$$

Note the following.

$$\langle \Delta, \Phi, u \rangle_\zeta \cong \langle \Delta, \Phi, \langle \Psi, f' \rangle_\theta \rangle_\zeta \cong \langle \Psi, \langle \Delta\Psi, \Phi, f' \rangle_\zeta \rangle_\theta \cong \langle \Psi, f \rangle_\theta$$

- Consider the case where s is of type (ζ).

$$P \otimes \Delta^\star\Phi^\star\Phi_\star(Q) \otimes R \cong A \otimes \Gamma^\star\Psi^\star\Psi_\star(B) \otimes C \xrightarrow{\langle\Gamma,\Psi,f\rangle_\zeta} S$$

Then Ψ is non-empty. It must be the case that A, $\Psi_\star(B)$ and C have prime factorisations of the following forms, where $\Gamma^\star\Psi^\star(X''_a) = X_a$.

$$A \cong \bigotimes_{0 \le a < j} X_a \qquad \Psi_\star(B) = X''_j \qquad C \cong \bigotimes_{j < a < n} X_a$$

Compare i and j.

Consider the case where $0 \le i < j < n$.

$$\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i}^{\Delta^\star\Phi^\star\Phi_\star(Q)} \otimes \overbrace{X_{i+1} \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes X_{j+1} \otimes \cdots \otimes X_{n-1}}^{R}$$

$$\underbrace{X_0 \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes X_{i+1} \otimes \cdots \otimes X_{j-1}}_{A} \otimes \underbrace{X_j}_{\Gamma^\star\Psi^\star\Psi_\star(B)} \otimes \underbrace{X_{j+1} \otimes \cdots \otimes X_{n-1}}_{C}$$

Define the following shape.

$$A \cap R = \bigotimes_{i < a < j} X_a$$

This shape has been chosen so that the following central isomorphisms exist.

$$R \cong (A \cap R) \otimes \Gamma^\star\Psi^\star\Psi_\star(B) \otimes C$$
$$P \otimes \Delta^\star\Phi^\star\Phi_\star(Q) \otimes (A \cap R) \cong A$$

By Lemma A.1.15, there is a morphism

$$\upsilon \colon \Omega(P \otimes \Delta^\star(Q) \otimes (A \cap R) \otimes \Gamma^\star(B) \otimes C) \to \Omega(S)$$

such that the following hold.

$$\langle\Delta, \Phi, \upsilon\rangle_\zeta = \Omega(f)$$
$$\langle\Gamma, \Psi, \upsilon\rangle_\zeta = \sigma$$

By induction, there is a constructible morphism

$$f' \colon P \otimes \Delta^\star(Q) \otimes (A \cap R) \otimes \Gamma^\star(B) \otimes C \to S$$

such that f is of the form $\langle\Delta, \Phi, f'\rangle_\zeta$. Define u to be the following constructible morphism.

$$u \colon P \otimes \Delta^\star(Q) \otimes R \cong P \otimes \Delta^\star(Q) \otimes (A \cap R) \otimes \Gamma^\star\Psi^\star\Psi_\star(B) \otimes C \xrightarrow{\langle\Gamma,\Psi,f'\rangle_\zeta} S$$

Note the following.

$$\langle\Delta, \Phi, u\rangle_\zeta \cong \langle\Delta, \Phi, \langle\Gamma, \Psi, f'\rangle_\zeta\rangle_\zeta \cong \langle\Gamma, \Psi, \langle\Delta, \Phi, f'\rangle_\zeta\rangle_\zeta \cong \langle\Gamma, \Psi, f\rangle_\zeta$$

Consider the case where $0 \le i = j < n$.

$$\underbrace{\overbrace{X_0 \otimes \cdots \otimes X_{i-1}}^{P}}_{A} \otimes \underbrace{\overbrace{X_i}^{\Delta^\star\Phi^\star\Phi_\star(Q)}}_{\Gamma^\star\Psi^\star\Psi_\star(B)} \otimes \underbrace{\overbrace{X_{i+1} \otimes \cdots \otimes X_{n-1}}^{R}}_{C}$$

The following central isomorphisms exist.

$$P \cong A \qquad \Delta^\star\Phi^\star\Phi_\star(Q) \cong \Gamma^\star\Psi^\star\Psi_\star(B) \qquad R \cong C$$

It must be the case that $Q = B$, $\Phi = \Psi$ and $\Delta = \Gamma$. Define u to be the following constructible morphism.

$$u\colon P \otimes \Delta^\star(Q) \otimes R \cong A \otimes \Gamma^\star(B) \otimes C \xrightarrow{f} S$$

Consider the case where $0 \leq j < i < n$.

$$\overbrace{X_0 \otimes \cdots \otimes X_{j-1} \otimes X_j \otimes X_{j+1} \otimes \cdots \otimes X_{i-1}}^{P} \otimes \overbrace{X_i}^{\Delta^\star\Phi^\star\Phi_\star(Q)} \otimes \overbrace{X_{i+1} \otimes \cdots \otimes X_{n-1}}^{R}$$

$$\underbrace{X_0 \otimes \cdots \otimes X_{j-1}}_{A} \otimes \underbrace{X_j}_{\Gamma^\star\Phi^\star\Phi_\star(B)} \otimes \underbrace{X_{j+1} \otimes \cdots \otimes X_{i-1} \otimes X_i \otimes X_{i+1} \otimes \cdots \otimes X_{n-1}}_{C}$$

This case is similar to the case where $0 \leq i < j < n$.

□

Appendix B

Glossary of Symbols

alpha The symbol α is used for the associator in a skew monoidal category.

$$\alpha_{A,B,C}\colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C$$

epsilon The symbol ε is used for the counit in a closed skew monoidal category.

$$\varepsilon^{A}_{B}\colon A \otimes (A\backslash B) \to A$$

zeta The symbol ζ is used for the counit of various adjunctions.

$$\zeta^{\Phi}_{A}\colon \Phi^{\star}\Phi_{\star}(A) \to A$$

$$\zeta_{A}\colon L \otimes (R \otimes A) \to A$$

eta The symbol η is used for the unit in a closed skew monoidal category.

$$\eta^{A}_{B}\colon B \to A\backslash(A \otimes B)$$

theta The symbol θ is used for the unit of various adjunctions.

$$\theta^{\Phi}_{A}\colon A \to \Phi_{\star}\Phi^{\star}(A)$$

$$\theta_{A}\colon A \to R \otimes (L \otimes A)$$

kappa The symbol κ is used for the structure maps for a pseudofunctor, and a related natural transformation.

$$\kappa^{\Gamma,\Delta}_{A}\colon \Gamma^{\star}\Delta^{\star}(A) \to (\Delta\Gamma)^{\star}(A) \qquad \hat{\kappa}^{\mathcal{C}}_{A}\colon A \to (\mathrm{id}_{\mathcal{C}})^{\star}(A)$$

$$\bar{\kappa}^{\mathcal{C}}_{A}\colon (\mathrm{id}_{\mathcal{C}})_{\star}(A) \to A$$

lambda The symbol λ is used for the left unitor in a skew monoidal category, and a related natural transformation.

$$\lambda_{A}\colon A \to \mathcal{I} \otimes A$$

$$\bar{\lambda}_{A}\colon \mathcal{I}\backslash A \to A$$

mu The symbol μ is used for the following natural transformation, similar in form to the associator α, in a closed skew monoidal category.

$$\mu_{A,B,C}\colon (B \otimes A)\backslash C \to A\backslash(B\backslash C)$$

nu The symbol ν is used for the following natural transformation, similar in form to the associator α, in a closed skew monoidal category.

$$\nu_{A,B,C}\colon (A\backslash B) \otimes C \to A\backslash(B \otimes C)$$

xi The symbol ξ is used for the following natural transformation, used in the definition of dual pairs in closed skew monoidal categories.

$$\xi_A \colon L\backslash A \to R \otimes A$$

pi The symbol π is used for the projection map for a monoidal adjunction.

$$\pi_{A,B} \colon \Phi_\star(A) \otimes B \to \Phi_\star(A \otimes \Phi^\star(B))$$

rho The symbol ρ is used for the right unitor in a skew monoidal category.

$$\rho_A \colon A \otimes \mathcal{I} \to A$$

sigma The symbol σ is used for the following internal version of a monoidal adjunction between closed categories.

$$\sigma_{A,B} \colon \Phi_\star(\Phi^\star(A)\backslash B) \to A\backslash\Phi_\star(B)$$

phi The symbol ϕ is used for the structure maps for various oplax monoidal functors.

$$\varphi^\Phi_{A,B} \colon \Phi^\star(A \otimes B) \to \Phi^\star(A) \otimes \Phi^\star(B) \qquad \hat{\varphi}^\Phi \colon \Phi^\star(\mathcal{I}) \to \mathcal{I}$$

$$\varphi^{X,Y}_C \colon C^{X\otimes Y} \to (C^X)^Y \qquad \hat{\varphi}_C \colon C^{\mathcal{I}} \to C$$

chi The symbol χ is used for the following natural transformation, similar in form to the structure map ψ^Φ, for a lax monoidal functor between closed categories.

$$\chi^\Phi_{A,B} \colon \Phi_\star(A\backslash B) \to \Phi_\star(A)\backslash\Phi_\star(B)$$

psi The symbol ψ is used for the structure maps for various lax monoidal functors.

$$\psi^\Phi_{A,B} \colon \Phi_\star(A) \otimes \Phi_\star(B) \to \Phi_\star(A \otimes B) \qquad \hat{\psi}^\Phi \colon \mathcal{I} \to \Phi_\star(\mathcal{I})$$

$$\psi^X_{A,B} \colon A^X \otimes B^X \to (A \otimes B)^X \qquad \hat{\psi}^X \colon \mathcal{I} \to \mathcal{I}^X$$

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