Quantum Fields on BTZ Black Holes

A thesis presented by

Leonardo Ortíz Hernández

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Abstract

In this thesis we investigate some aspects of Quantum Fields on BTZ black holes in 1+1 and 1+2 dimensions. More specifically, we study how a ground state in AdS spacetime becomes a thermal state when restricted to the exterior region of the BTZ black hole. This ground state is for the quantum linear real scalar field which satisfies the Klein-Gordon equation and also, in the 1+2 dimensional case, for a general QFT satisfying the axioms of Algebraic Quantum Field Theory in Curved Spacetime. We also study how these states map to states on the conformal boundary of AdS spacetime. In order to do this we use Algebraic Holography, Boundary-limit Holography and Pre-Holography. As a preparation, we give an exposition of AdS spacetime, we review the principal aspects of the 1+1 and 1+2 dimensional BTZ black holes and their mapping to the boundary of AdS spacetime. In both cases they map to part of the boundary of AdS spacetime. In the 1+1 case we find that the restriction of the equivalent global vacuum or the Poincaré vacuum on the boundary becomes a thermal state (KMS state) when restricted to the BTZ black hole. In the 1+2 case we find that the Poincaré vacuum becomes a thermal state too when restricted to the exterior of the BTZ black hole. In both cases the temperature of the state is $\mathsf{T} = \frac{\kappa}{2\pi}$, where κ is the surface gravity. In 1+2 dimensions we give as a concrete model the quantum linear real scalar field. When we study these states in the boundary of AdS spacetime we use standard techniques of Conformal Field Theory. One of the main conclusions we get from our investigation is that in certain sense the Hawking effect in the eternal BTZ black hole maps to the Unruh effect on the boundary of AdS spacetime. In the final part of this thesis we study the brick wall model for these black holes for the quantum linear real scalar field. In both cases we obtain the two point function for the vacuum and thermal states in the bulk and on the boundary. We also study the expectation value of the renormalized energy-momentum tensor in 1+1 dimensions for the conformal massless quantum real scalar field. Most of this thesis fits in the framework of AdS/CFT in QFT which is more limited than AdS/CFT correspondence in string theory.

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Preface

> "I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."
>
> Sir Issac Newton[1]

The present thesis contains essentially all the written work which I carried out as PhD student in the Department of Mathematics of The University of York during the last three and a quarter years. In one way or another it reflects a part of the change that my conception of physics and mathematics has undergone since I came to York. This change has been produced by external and internal sources. The external sources have been principally from mathematical physics thinking. While being in York I have been exposed to more mathematical physics than before. Principally because of the suggested reading from my supervisor, Dr. Bernard S. Kay, and through conversations with him. I have no doubt that by following his suggestions and advice I have grown up a lot as a mathematical physicist.

Along with this formative process, this thesis also reflects the conclusion of a project proposed by my supervisor which grew through interesting conversations. In order to carry out this task I tried to understand, some

[1] In D. Brewster, *Memoirs of the life, writings and discoveries of Sir Isaac Newton*, Edinburgh: Thomas Constable and Co., Hamilton, Adams and Co., London 1855 (reprint Johnson Reprint Corporation, New York and London, 1965 Vol. 2, p. 407).

times with success others without it, part of the theory and applications of Quantum Field Theory in Curved Spacetime (QFTCS).

In this work the reader will find some applications of QFTCS to BTZ black holes in 1+1 and 1+2 dimensions. As the reader will see, these applications lead naturally to studying QFTCS on AdS spacetime and also to relating quantum fields on BTZ backgrounds to the AdS/CFT correspondence.

Also, the reader could see aspects of the Unruh and Hawking effect treated in a pedagogical form.

The two first chapters are essentially geometric whereas the rest of the thesis contain mostly aspects of quantum field theory. In particular Chapter 3 can be considered as the background on quantum field theory to understand this thesis. The results are principally in the last three chapters and appendices although the Chapter 2 has also some new aspects, especially the Section 2.3. In Appendix F, we added an essay on the quantization of the real linear scalar field which could serve as a first reading in algebraic quantum field theory.

It just remains to say that any errors of any type are completely my responsibility and that I hope the reader will find this work useful and interesting.

Acknowledgments

First and foremost, I thank Dr. Bernard S. Kay for accepting being my supervisor and for suggesting to me a research project, the study of thermal states in AdS spacetime and their relation to thermal states in BTZ black holes. I also thank him for his guidance and helpful advice during this work. Without his guidance and encouragement it would have been impossible to carry out this work. During my time in York I have learnt a lot from him, not just in physics and mathematics but also about life.

I thank my advisory panel members Prof. Paul Busch and Dr. Chris Fewster for their confidence and several positive comments on my work.

Also I thank the Department of Mathematics for allowing me to carry out research in it and for its hospitality.

I would like to thank Dr. Michael P. Ryan, Dr. Hernando Quevedo and Dr. Angel Prieto for their support and confidence.

Special thanks go for Yerim, Marcelo and Uncle Colin for all their care, support and encouragement during my PhD. Also, Dra. Laura, Ulrica and Susan for taking care of my health. I could not omit to mention Marco and Rafael whose friendship through the years has been invaluable for me.

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Without the support of CONACYT Mexico it would have been impossible to start and complete my PhD studies.

Finally, I would like to thank my family for all their support during all my years as student. Specially I thank my Mother and Father for being emotionally with me, even though an ocean separates us physically. Besides they put me on the road which culminates in a PhD.

Author's declaration

I wrote this thesis on my own and it contains all the results of the research which I carried out as PhD student in the Department of Mathematics of The University of York. The chapters 4, 5, 6 and sections 1.2 1.3 and 2.3 will be submitted for publication. This thesis has not been submitted anywhere else for obtaining any degree.

This thesis is the conclusion of a project suggested to me by Dr. Bernard S. Kay. The analysis of the thermalization of the state in 1+1 dimensions and its generalization to 1+2 dimensions were suggested to me by him. The analysis of the brick wall model was suggested by him too. The mentioned analysis in 1+1 dimensions and the brick wall model were done in collaboration with Dr. Bernard S. Kay.

Introduction

In recent years there has been great interest in the so-called AdS/CFT correspondence. It seems fair to say that the majority of works on this topic belong to the string theory framework. However, with a more limited scope than in this framework, in the context of Quantum Field Theory (QFT), also some approaches to the AdS/CFT correspondence have been proposed. More precisely, in the context of Algebraic Quantum Field Theory (AQFT) there appeared Algebraic Holography (AH) [70]. Algebraic Holography relates a covariant quantum field theory in the bulk and a conformally covariant quantum field theory on the conformal boundary[2] of AdS spacetime. In this sense AH gives an AdS/CFT correspondence too. Also there appeared the boundary-limit holography [12] where there has been constructed a correspondence between n-point functions of a covariant quantum field theory in the bulk and a conformally covariant quantum field theory in the boundary of AdS spacetime. More recently, partly motivated by these works, there appeared Pre-Holography [57], which studied some aspects of the correspondence between linear field theories in the bulk and in the boundary of AdS spacetime by using the symplectic structure associated to the phase space of the Klein-Gordon operator. When we take into account these works, it is clear that even at the level of QFT there are interesting aspects of the AdS/CFT correspondence which deserve to be studied. Amongst the aspects of the AdS/CFT in QFT[3] which have been studied so far is, for example, how the global ground state in the bulk of AdS spacetime maps to a state in

[2]From here on, instead of writing conformal boundary of AdS spacetime we will just write boundary of AdS spacetime unless a confusion can arise.

[3]By AdS/CFT in QFT we mean AH, the boundary-limit holography and Pre-Holography since all them fit in the QFT framework.

its boundary [57].

In QFT, besides ground states, equilibrium thermal states are outstanding elements in the theory. So, after studying the mapping of a ground state it is very natural to ask how an equilibrium thermal state maps from the bulk to the boundary of AdS spacetime. This issue is more appealing when one knows that models of black holes can be obtained from AdS spacetime. In particular it is known that in 1+2 dimensions there exists a solution to the Einstein's field equations which can be considered as a model of a black hole, the BTZ black hole (BTZbh) [11]. This solution can be obtained directly from Einstein's field equations or by making identifications in a proper subset of AdS spacetime [11].

The main purpose of this work is to study how a ground state in AdS spacetime in 1+2 dimensions is related to an equilibrium thermal state in the BTZbh and how it maps to the boundary of AdS spacetime under AdS/CFT in QFT. One of the main conclusions we obtain from our investigation is that in a certain sense the Hawking effect for the eternal BTZbh maps to the Unruh effect in the boundary of AdS spacetime and vice versa.

In order to carry out our objectives, we are faced with the issue of studying quantum fields on BTZ black holes. Apart from the main purpose just mentioned we have also studied the brick wall model [67] in 1+1 and 1+2 dimensional BTZ black holes for the quantum linear real scalar field. We also study the expectation value of the renormalized energy-momentum tensor in the 1+1 dimensional BTZ black hole for the conformal massless quantum real scalar field. These two aspects of our work are complementary to our study of thermal states in AdS and BTZ backgrounds and also shed some light on the thermal properties of a field propagating on these backgrounds.

In our study, the symmetries of AdS spacetime are fundamental. The principal fact is that the group $SO_0(2,d)$ acts in AdS spacetime and in its conformal boundary. In AdS spacetime it acts as the isometry group and in its conformal boundary as the global conformal group. This is fundamental in making AH possible. Also, although indirectly, the Tomita-Takesaki and Bisognano-Wichmann theorems play a fundamental rôle. These mathematical aspects are relevant when addressing our problem in the abstract setting.

When we give a concrete heuristic example of this formalism we use the procedure given in [12] to obtain a two point function in the boundary of AdS spacetime from a two point function in its bulk.

This thesis is organized as follows: in Chapter 1, we give a self-contained introduction to the geometry of AdS spacetime emphasizing the aspects of AdS spacetime relevant for this work. We also discuss some issues for the particular cases in 1+1 and 1+2 dimensions. In both cases we introduce global coordinates and Poincaré coordinates in the 1+2 dimensional case. This chapter can be used as an introduction to the geometry of AdS spacetime. In Chapter 2, we discuss the BTZ black hole in 1+1 and 1+2 dimensions giving the necessary concepts we need in the later chapters. We also introduce a model for a 1+1 dimensional BTZ black hole by using just the geometry of AdS spacetime. Also, we explain how the BTZ black hole maps from the bulk of AdS to its conformal boundary. In Chapter 3, we introduce the necessary elements of Quantum Field Theory in Curved Spacetime. Especially we give the basic elements of canonical quantization. We also introduce the concept of *conformal vacuum* and the *KMS condition*. This condition is very relevant when we study thermal states in the later chapters. In Chapter 4, we give our principal results. We show that the global vacuum coincides with the Poincaré vacuum in the one dimensional boundary of AdS spacetime in 1+1 dimensions. We also show how this vacuum becomes a thermal state when we pass from the global vacuum or Poincaré vacuum to the BTZ coordinates. This is done by using some results from Pre-Holography [57]. These results are in agreement with previous work [77] where there has been shown that the global vacuum and the Poincaré vacuum are equivalent in the bulk of AdS spacetime in 1+1 dimensions. Later we show how a ground state in AdS spacetime in 1+2 dimensions maps to a ground state in the boundary of AdS spacetime. In order to carry this out we use AdS/CFT in QFT. The principal fact is that when we restrict the Poincaré vacuum to the exterior of BTZ black hole, it becomes a thermal state. This thermal state maps to the boundary in a clear way. Roughly speaking, this is tantamount to saying that the Hawking effect in the eternal BTZ black hole corresponds to the Unruh effect in the boundary of AdS spacetime. The Unruh effect takes place in the conformal boundary of AdS spacetime. In Chapter 5, we discuss the brick wall model [67] for the BTZ black hole in 1+1 and in 1+2 dimensions for the quantum linear real scalar field. In both cases we obtain two point functions in the boundary corresponding to a vacuum and a thermal state. In the 1+1 case the thermal two point function coincides with the thermal two point function obtained in Chapter 4 when the brick wall is removed. In Chapter 6, we discuss the expectation value of the renormalized energy-momentum tensor for the 1+1 dimensional BTZ black hole for the conformal massless quantum real scalar field. We obtain closed expressions for its expectation value in the Hartle-Hawking state. In Chapter 7, we give some final comments, in particular we argue that by restricting the global vacuum in 1+2 dimensions to the Poincaré chart we do not obtain a thermal state. However the proof of this conjecture is still open. In Chapter 8, we give our conclusions and perspectives. In Appendix A, we discuss the Unruh effect in 1+1 dimensions for the massless real linear scalar field. In Appendix B, we calculate the finite transformation of the global conformal group in 1+1 dimensions. We also introduce the usual complex coordinates in conformal field theory and give the Lie brackets between the generators of the AdS group. In Appendix C, we show that AdS spacetime has a Misner spacetime at infinity in an appropriate parametrization. In Appendix D, we calculate the two point function in AdS spacetime by using the conformal vacuum defined with respect to Poincaré time. In Appendix E, we show that for vanishing boundary condition at infinity there is no superradiance in the BTZ black hole. In Appendix F, we present an essay on the quantization of the real linear scalar field.

Chapter 1

Aspects of AdS spacetime

In this chapter, we introduce AdS spacetime and discuss some aspects of it relevant for this work. At some point we focus the discussion on the 1+1 and 1+2 dimensional cases due to their relevance for our purposes.

1.1 AdS spacetime in 1+d dimensions

AdS spacetime is a solution of the Einstein's field equations in vacuum with negative cosmological constant, Λ,

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{1.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the scalar curvature and $g_{\mu\nu}$ is the Lorentzian metric of the spacetime [46]. It is a spacetime with constant negative curvature[1]. In 1+d dimensions it can be introduced as follows: Let us consider $\mathbb{R}^{2+d}$ endowed with metric

$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu, \tag{1.2}$$

[1]See, for example, [79] for a discussion of AdS spacetime in four dimensions.

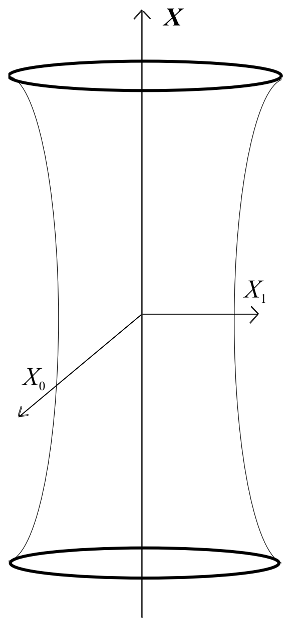


Figure 1.1: Schematic AdS spacetime in 1+d dimensions.

where $\eta_{\mu\nu} = \text{diag}(-1, -1, 1, 1, ...)$; the three dots denote $d-2$ entries with ones. This metric defines the scalar non positive definite inner product

$$X \cdot X' = \eta_{\mu\nu} X^\mu X'^\nu \tag{1.3}$$

between any two elements of $\mathbb{R}^{2+d}$. We are implicitly identifying $T_p(\mathbb{R}^{2+d}) \sim \mathbb{R}^{2+d}$ at each point p of $\mathbb{R}^{2+d}$. We denote the resulting non-Euclidian space as $\mathbb{R}^{2,d}$. AdS spacetime can be identified with the hypersurface in $\mathbb{R}^{2,d}$ defined by

$$-{X^0}^2 - {X^1}^2 + \sum_{\mu=2}^{1+d} {X^\mu}^2 = -l^2, \tag{1.4}$$

where $l^2 = \frac{d(1-d)}{2\Lambda}$ and with metric induced by the pull back of (1.2) to (1.4) under the inclusion map. The coordinates X^0 and X^1 are, let us say, timelike, whereas the others are spacelike. Note that the equation (1.4) is invariant under the identification $X \leftrightarrow -X$. AdS spacetime has the topology of $\mathbb{S}^1 \times \mathbb{R}^d$, see figure 1.1.

In this work we are interested in studying some aspects of AdS spacetime at infinity. Now we explain what we mean by AdS spacetime at infinity. Let us consider the quadratic equation in $\mathbb{R}^{3+d}$

$$Q(Y) = W^2 - X^{0^2} - X^{1^2} + \sum_{\mu=2}^{1+d} X^{\mu 2}, \tag{1.5}$$

where $Y = (W, X)$. We introduced the coordinate W in order to make our calculations more rigorously. Because of the homogeneity of Q, the locus of points where $Q(Y) = 0$ is invariant under $Y \to \lambda Y$. In particular we can always find a λ such that we recover (1.4) for any W. Also on $Q(Y) = 0$ we can fix W and let $X \to \infty$ and at the same time multiply all the equation by a small λ. In the limit of this procedure we get

$$X^{0^2} + X^{1^2} = \sum_{\mu=2}^{1+d} X^{\mu 2}. \tag{1.6}$$

We call (1.6) the 2+d-dimensional null cone and denote it by $\mathcal{C}^{2+d}$. This cone can be considered as the limit of (1.4) when $X \to \infty$ and is the compactification of a d-dimensional Minkowski spacetime. This can easily be visualized in the 1+2-dimensional case where the hyperboloid of one sheet defined by (1.4) is asymptotically $\mathcal{C}^{2+2}$ when $X \to \infty$. Now we can introduce a d-dimensional Minkowski spacetime in (1.6), see [42] for the 6-dimensional case. We define the coordinates of this Minkowski spacetime to be

$$\xi^\mu = \frac{X^\mu}{X^0 + X^{d+1}}, \qquad \mu = 1, 2, ..., d. \tag{1.7}$$

The metric on this spacetime is given by $\eta_{\mu\nu}$ with the entries 0 and $d+1$ deleted and non positive inner product

$$\xi \cdot \xi = (X^0 + X^{d+1})^{-1}(X^0 - X^{d+1}). \tag{1.8}$$

These coordinates do not cover all the manifold, points at infinity are left out. The null cone $\mathcal{C}^{2+d}$ is also invariant under $X \leftrightarrow -X$, hence AdS spacetime has a compactified Minkowski spacetime at infinity modulo this identification.

From (1.6) we can see that the topology of $\mathcal{C}^{2+d}$ is the topology of $\mathbb{S}\times\mathbb{S}^{d-1}/\pm I$.

The quadratic equations (1.4) and (1.6) have a common property: they are both invariant under the group $O(2,d)$. In particular they are invariant under the connected component of this group, namely $SO_0(2,d)$. However the action of this group has a different meaning when acting on these quadratic equations. On (1.4) it acts as rotations of $\mathbb{R}^{2+d}$ which preserve (1.4), and we will call it AdS group; whereas it acts on (1.6) as the global conformal group in d-dimensional Minkowski spacetime.

The generators of the AdS group are

$$\begin{aligned} J_{01} &= X^0\partial_{X^1} - X^1\partial_{X^0} \qquad & J_{0\mu} &= X^0\partial_{X^\mu} + X^\mu\partial_{X^0} \\ J_{\nu\mu} &= X^\nu\partial_{X^\mu} - X^\mu\partial_{X^\nu} \qquad & J_{1\mu} &= X^1\partial_{X^\mu} + X^\mu\partial_{X^1} \end{aligned} \tag{1.9}$$

where $\mu,\nu = 2,3,...1+d$. These generators form a base for the Lie algebra of the AdS group. The elements of the AdS group can be obtained by exponentiation of the elements of this algebra [87]. In Appendix B, we do this explicitly for $d=2$. From (1.9) we see that the metric of AdS spacetime in $1+d$ dimensions admits $\frac{(1+d)(1+d+1)}{2}$ killing vectors. Hence AdS spacetime is a maximally symmetric spacetime [88], [23]. Its Riemann tensor, Ricci tensor and Ricci scalar are given respectively by

$$R_{\mu\nu\alpha\delta} = -\frac{1}{l^2}\left(g_{\mu\alpha}g_{\nu\delta} - g_{\mu\delta}g_{\nu\alpha}\right), \tag{1.10}$$

$$R_{\mu\nu} = -\frac{d}{l^2}g_{\mu\nu}, \tag{1.11}$$

$$R = -\frac{d(d+1)}{l^2}. \tag{1.12}$$

1.2 AdS spacetime in 1+1 dimensions

In 1+1 dimensions the equation defining AdS spacetime is

$$-u^2 - v^2 + x^2 = -l^2. \tag{1.13}$$

We can parameterize this equation as

$$u = l \sec\rho \sin\lambda \qquad v = l \sec\rho \cos\lambda \qquad x = l \tan\rho, \tag{1.14}$$

where $\lambda \in [0, 2\pi)$ and $\rho \in (-\pi/2, \pi/2)$. These coordinates are known as global coordinates. With this parametrization the metric for AdS spacetime (AdS metric) is given by

$$ds^2 = l^2 \sec^2\rho \left(-d\lambda^2 + d\rho^2\right). \tag{1.15}$$

We can conformally map this metric to the strip $\rho \in [-\pi/2, \pi/2]$ of the two dimensional Minkowski spacetime with metric

$$d\hat{s}^2 = \Omega^2 ds^2 = -d\lambda^2 + d\rho^2, \tag{1.16}$$

where $\Omega = \frac{1}{l}\cos\rho$. We can draw a Penrose diagram for AdS spacetime in 1+1 dimensions by taking half of the strip $\rho \in [-\pi/2, \pi/2]$ as shown in figure 1.2.

So far we have not implemented the symmetry $X \leftrightarrow -X$. This symmetry can be implemented in global coordinates in AdS as follows. Let $P_2 : [-\pi, \pi) \times [-\pi/2, \pi/2] \to [-\pi, \pi) \times [-\pi/2, \pi/2]$ be the operator defined by $P_2(\lambda, \rho) = (\lambda, -\rho)$ and $T_2 : [-\pi, \pi) \times [-\pi/2, \pi/2] \to [-\pi, \pi) \times [-\pi/2, \pi/2]$ the operator defined by $T_2(\lambda, \rho) = (\lambda - \pi, \rho)$ for $\lambda \in [0, \pi)$ and $T_2(\lambda, \rho) = (\lambda + \pi, \rho)$ for $\lambda \in [-\pi, 0)$. Then $T_2 P_2(\lambda, -\pi/2) = P_2 T_2(\lambda, \pi/2)$. Hence under the simultaneous action of T_2 and P_2 we just need to consider the region with $\lambda \in [-\pi, 0)$ as $AdS/\pm I$.

For further reference we give the Killing vectors for AdS spacetime in 1+1 dimensions in global coordinates

$$J_{uv} = u\partial_v - v\partial_u = -\partial_\lambda, \tag{1.17}$$

$$J_{ux} = u\partial_x + x\partial_u = \cos\rho \sin\lambda \partial_\rho + \sin\rho \cos\lambda \partial_\lambda, \tag{1.18}$$

$$J_{vx} = v\partial_x + x\partial_v = \cos\rho \cos\lambda \partial_\rho - \sin\rho \sin\lambda \partial_\lambda. \tag{1.19}$$

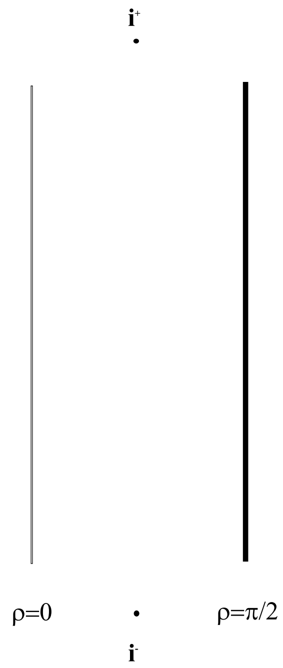


Figure 1.2: Penrose diagram of AdS spacetime. The two disjointed points, i^+ and i^-, represent future and past infinity respectively.

These vectors satisfy

$$[J_{\mu\nu}, J_{\alpha\beta}] = \eta_{\nu\alpha}J_{\mu\beta} - \eta_{\mu\alpha}J_{\nu\beta} - \eta_{\nu\beta}J_{\mu\alpha} + \eta_{\mu\beta}J_{\nu\alpha} \tag{1.20}$$

where $\eta_{uu} = \eta_{vv} = -\eta_{xx} = -1$ with all other entries zero.

1.3 AdS spacetime in 1+2 dimensions

The discussion of AdS spacetime in 1+2 dimensions follows similar lines to the 1+1 dimensional case, however there are some important differences. It can be identified with the hypersurface in $\mathbb{R}^{2,2}$ defined by

$$-u^2 - v^2 + x^2 + y^2 = -l^2. \tag{1.21}$$

Now let us introduce the global and the Poincaré charts. Global coordi-

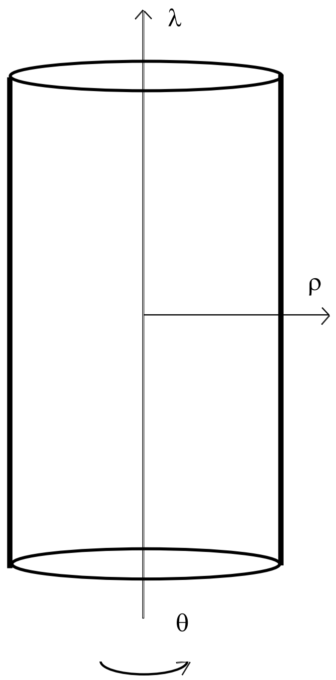


Figure 1.3: AdS spacetime in global coordinates.

nates (λ, ρ, θ) can be defined by

$$\begin{aligned} v &= l \sec\rho \cos\lambda \qquad & u &= l \sec\rho \sin\lambda \\ x &= l \tan\rho \cos\theta \qquad & y &= l \tan\rho \sin\theta, \end{aligned} \tag{1.22}$$

where $(\lambda, \rho, \theta) \in [-\pi, \pi) \times [0, \pi/2) \times [-\pi, \pi)$. In these coordinates the metric is

$$ds^2 = l^2 \sec^2\rho \left(-d\lambda^2 + d\rho^2 + \sin^2\rho d\theta^2\right). \tag{1.23}$$

In these coordinates we can represent AdS spacetime as a cylinder, see figure 1.3. From (1.22) we see that $\rho \to \pi/2$ corresponds to infinity. The metric (1.23) is not defined on this point, however we can define what is usually called an unphysical metric as $d\tilde{s}^2 = \Omega^2 ds^2$ with $\Omega = \frac{1}{l}\cos\rho$ and get

$$d\tilde{s}^2 = -d\lambda^2 + d\rho^2 + \sin^2\rho d\theta^2. \tag{1.24}$$

This metric is well defined for $\rho \in [0, \pi/2]$. When constructing a Penrose diagram for AdS spacetime this is the metric most commonly used [46][2]. This is the metric of the Einstein universe, but it cover just half of it since $\rho \in [0, \pi/2]$. Using this conformal mapping we can attach a boundary to AdS spacetime. This boundary is given by $\rho = \pi/2$ and its metric is

$$d\tilde{s}_b^2 = -d\lambda^2 + d\theta^2. \tag{1.25}$$

The topology of the boundary is $\mathbb{S}^1 \times \mathbb{S}^1$. In order to avoid close timelike curves it is customary to work with the covering space of AdS spacetime (CAdS), i.e., by letting λ to vary on $\mathbb{R}$, and then the boundary of CAdS is an infinitely long cylinder $\mathbb{R} \times \mathbb{S}^1$. It is again an Einstein universe but in 1+1 dimensions.

Poincaré coordinates (T, k, z) can be introduced as follows: define

$$m \equiv \frac{v+x}{l^2} \qquad n \equiv \frac{v-x}{l^2} \qquad T \equiv \frac{u}{lm} \qquad k \equiv \frac{y}{lm}. \tag{1.26}$$

Then (1.21) takes the form

$$mnl^4 + l^2m^2\left(T^2 - k^2\right) = l^2. \tag{1.27}$$

From (1.26) we have

$$v = \frac{l^2}{2}\left(m+n\right). \tag{1.28}$$

Now, using (1.27) in (1.28) we obtain

$$v = \frac{1}{2m}\left(1 + m^2\left(l^2 + k^2 - T^2\right)\right). \tag{1.29}$$

Defining $z = \frac{1}{m}$

$$v = \frac{1}{2z}\left(z^2 + l^2 + k^2 - T^2\right). \tag{1.30}$$

Doing analogously for x and using the definition of z in u and y we get

[2]However there are other possibilities. The interested reader can see [81] and references therein. For our purposes we attach to (1.24).

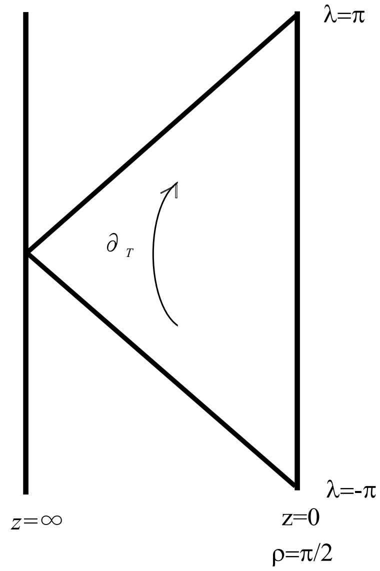


Figure 1.4: A transversal section of the Poincaré chart.

Poincaré coordinates

$$
\begin{aligned}
v &= \frac{1}{2z}\left(z^2 + l^2 + k^2 - T^2\right) \qquad & u &= \frac{lT}{z} \\
x &= \frac{1}{2z}\left(l^2 - z^2 + T^2 - k^2\right) \qquad & y &= \frac{lk}{z}.
\end{aligned} \tag{1.31}
$$

In these coordinates the metric is

$$
ds^2 = \frac{l^2}{z^2}\left(-dT^2 + dk^2 + dz^2\right), \tag{1.32}
$$

where $(T, k, z) \in (-\infty, \infty) \times (-\infty, \infty) \times (0, \infty)$. Actually Poincaré coordinates can also be defined for $z < 0$, in this work we use the chart with $z > 0$. From (1.31) we see that $z = 0$ corresponds to infinity. Hence from (1.32) we see that the boundary of AdS spacetime covered by Poincaré coordinates is conformal to a flat spacetime in 1+1 dimensions[3]. A section of the Poincaré chart is represented in figure 1.4.

AdS spacetime can be expressed also as a warp product [68]. By making

[3]From (1.32) we see that for $T =$ const. we have the metric of the Poincaré half-plane model for the hyperbolic plane. Hence the name of the coordinates (T, k, z).

$z = e^{-q}$ we have

$$ds^2 = l^2 e^{2q}\left(-dT^2 + dk^2\right) + l^2 dq^2 \tag{1.33}$$

which is the warp product $\mathbb{R} \times_{e^q} \mathbb{R}^{1,1}$, where $\mathbb{R}^{1,1}$ denotes a 1+1 dimensional Minkowski spacetime. This Minkowski spacetime is called a fiber or a 2-brane.

We point out that the exposition given so far in this section can be generalized to any dimension. See for example [9].

Now let us consider $AdS/\pm I$. In this case the identification $X \leftrightarrow -X$ can be implemented as follows. Let $P_3 : [-\pi,\pi) \times [0,\pi/2) \times [-\pi,\pi) \rightarrow [-\pi,\pi)\times[0,\pi/2)\times[-\pi,\pi)$ be the operator defined by $P_3(\lambda,\rho,\theta) = (\lambda,\rho,\theta-\pi)$ for $\theta \in [0,\pi)$ and $P_3(\lambda,\rho,\theta) = (\lambda,\rho,\theta+\pi)$ for $\theta \in [-\pi,0)$. Let $T_3 : [-\pi,\pi) \times [0,\pi/2) \times [-\pi,\pi) \rightarrow [-\pi,\pi) \times [0,\pi/2) \times [-\pi,\pi)$ be the operator defined by $T_3(\lambda,\rho,\theta) = (\lambda-\pi,\rho,\theta)$ for $\lambda \in [0,\pi)$ and $T_3(\lambda,\rho,\theta) = (\lambda+\pi,\rho,\theta)$ for $\lambda \in [-\pi,0)$. These operators can be extended to the boundary $\rho = \pi/2$ in the natural way.

In Poincaré coordinates the symmetry $X \leftrightarrow -X$ can be implemented by changing z for $-z$. Hence in $AdS/\pm I$ we just need one chart of Poincaré coordinates to cover the entire manifold.

Chapter 2

The BTZ Black Hole

In this chapter we give the generalities of the BTZ black hole and explain how it maps to the boundary of AdS spacetime. We begin with the 1+1 dimensional case and later introduce the 1+2 dimensional case.

2.1 1+1 dimensional BTZ black hole

The BTZ black hole in 1+1 dimensions has been considered before in [4] in the context of models of gravitational theories in 1+1 dimensions. We will follow a less ambitious approach and introduce a model of a BTZ black hole in 1+1 dimensions just by using the geometry of AdS spacetime in 1+1 dimensions. If we parameterize (1.13) by

$$v = l\left(\frac{r^2 - r_+^2}{r_+^2}\right)^{1/2} \sinh \kappa t \quad x = l\left(\frac{r^2 - r_+^2}{r_+^2}\right)^{1/2} \cosh \kappa t$$

$$u = -l\frac{r}{r_+}, \tag{2.1}$$

where $\kappa = \frac{r_+}{l^2}$, then we have

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2, \tag{2.2}$$

where

$$N^2 = -M + \frac{r^2}{l^2} \tag{2.3}$$

and $r_+ = l\sqrt{M}$. Using (1.14) and (2.1) we obtain

$$\tanh \kappa t = \frac{\cos\lambda}{\sin\rho} \qquad r = -r_+ \sec\rho \sin\lambda. \tag{2.4}$$

If we want to interpret r as a radial coordinate, then we should take $\lambda \in (-\pi, 0)$ in order to have $r > 0$. Clearly this region can not touch $\rho = 0$, however it approaches it asymptotically. When we conformally map CAdS in 1+1 dimensions to Minkowski spacetime it covers the interior of the strip $\rho \in (-\pi/2, \pi/2)$. Let us figure out which region of this strip is covered by the coordinates (t, r) given by (2.4). The line $\rho = 0$ can not be covered, for example. The lines $\lambda = \rho - \pi/2$ and $\lambda = -\rho - \pi/2$ give $r =$ const. These lines cross $\rho = 0$ where $t \to \pm\infty$. Also when $\rho \to \pi/2$ then $r \to \infty$ and $x \to \infty$. The line $\lambda = -\pi/2$ corresponds to $t = 0$. Taking into account that $\rho = 0$ can not be covered by the coordinates (t, r) and the fact that the metric (2.2) is singular at $r = r_+$ which in (λ, ρ) would corresponds to $\lambda = \rho - \pi/2$ and $\lambda = -\rho - \pi/2$ then we can conclude that (t, r) cover the region between these two lines with $\rho \in (0, \pi/2)$ and $\lambda \in (-\pi, 0)$. From (2.1) we see that we can introduce a mirror copy of this region with negative x and $\rho \in (-\pi/2, 0)$ and $\lambda \in (-\pi, 0)$. Clearly the coordinates (t, r) share the same features with Schwarzschild coordinates [65] and cover just a portion of CAdS spacetime. The singularity at $r = r_+$ is a singularity of these coordinates since CAdS spacetime is well behaved everywhere. When we conformally map CAdS in 1+1 dimensions to the strip $(-\pi/2, \pi/2)$ then we can consider the portion with $\lambda \in (-\pi, 0)$ as a maximally extended black hole, analogous to the maximally extended Schwarzschild black hole. The black hole corresponds to the region between the lines $\lambda = \rho - \pi/2$ and $\lambda = -\rho - \pi/2$ with $\lambda \in (-\pi/2, 0)$ and the white hole with $\lambda \in (-\pi, -\pi/2)$. See figure 2.1. We can ask ourselves why we have the right to just consider this portion of AdS spacetime in 1+1 dimensions as a black hole. The answer is given by the geometry. As we said in the previous chapter, the symmetry $X \leftrightarrow -X$ is fundamental when

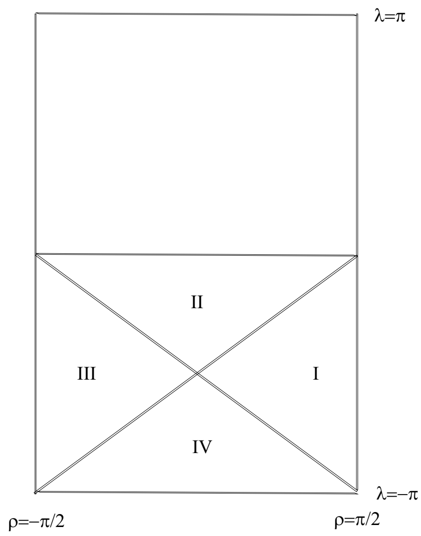


Figure 2.1: 1+1 dimensional BTZ black hole.

we want to identify infinity of AdS spacetime with Minkowski spacetime. In the present case a degenerated Minkowski spacetime of just one temporal dimension. If we apply this identification in the present case, with reflection with respect to the line $\rho = 0$ and antipodal identification in the circle $\mathbb{S}^1$ in the temporal dimension, then in the fundamental region of AdS, i.e. that with $\lambda \in (-\pi, \pi)$ we just need the region in $\lambda \in (-\pi, 0)$ in order to make sense of a maximally extended black hole. See figure 2.1. In this way by cutting out this region of AdS spacetime in 1+1 dimensions we have a model of 1+1 dimensional black hole. The singularities of this black hole are located at $\lambda = -\pi$ and $\lambda = 0$. Clearly these one dimensional regions are singularities in the sense that every time-like geodesic entering the black hole can not be extended beyond $\lambda = 0$; similarly a past directed time-like geodesic entering the white hole can not be extended beyond $\lambda = -\pi$. The region of CAdS spacetime representing the maximally extended black hole satisfies the requirements of the definition of a maximally analytic extension for the region which corresponds to the exterior of the black hole. This definition can be seen for example in [15]. Taking into account this analysis, the metric (2.2) can be considered a metric for a 1+1 dimensional black hole. We call it 1+1 dimensional BTZ black hole.

Let us consider additional geometrical properties of the 1+1 dimensional case. From figure 3.2 in [57] we can see that ∂_t is also the generator of the horizon. If we express this Killing vector in global coordinates we get

$$\partial_t = \kappa\left(x\partial_v + v\partial_x\right) = \kappa\cos\lambda\cos\rho\partial_\rho - \kappa\sin\rho\sin\lambda\partial_\lambda. \tag{2.5}$$

Hence at $(-\pi/2, 0)$, this Killing vector vanishes. This is the point where the past and future horizon of the maximally extended BTZ black hole intersect. Then using the definition of bifurcate killing horizon given in [55] and observing that this point is left invariant under the action of ∂_t we conclude that ∂_t generates this bifurcate killing horizon. This horizon divides locally the spacetime in four regions, R, L, F and P, following Kay and Wald notation. In [55] it was introduced a global definition of this regions for globally hyperbolic spacetimes. In the present case we do not have this property at hand, since AdS spacetime is not globally hyperbolic. We can still define these regions globally using the geometry of AdS spacetime as follows. Because of the identification $X \leftrightarrow -X$ we just need to define the regions above mentioned for half of AdS spacetime, for $\lambda \in (-\pi, 0)$. This definition is obvious, for example we can define R as the region invariant under ∂_t, analogously for the others, see figure 2.2. Hence we can see that the existence of the bifurcate Killing horizon plays a fundamental rôle in the thermal properties of AdS spacetime and the BTZ black hole.

2.2 The BTZ black hole

It is well known that there exists a solution to the Einstein's field equations in 1+2 dimensions which can be considered as a model of a black hole [11], better known as the BTZ black hole (BTZbh). The metric of this spacetime is given by

$$ds^2 = -f^2 dt^2 + f^{-2} dr^2 + r^2 (d\phi + N^\phi dt)^2, \tag{2.6}$$

where

$$f^2 = \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right), \quad N^\phi = -\frac{J}{2r^2}, \tag{2.7}$$

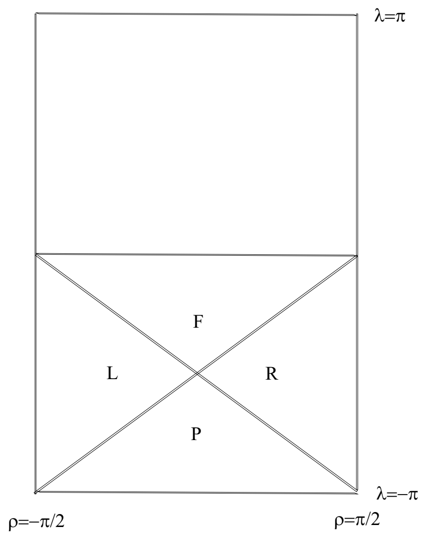


Figure 2.2: Penrose diagram of AdS spacetime in 1+1 dimensions where are indicated the identifications when making the global definitions of R, L, F and P.

with $|J| \leq Ml$. We call $(t, r, \phi) \in (-\infty, \infty) \times (0, \infty) \times [0, 2\pi)$ BTZ coordinates. The metric (2.6), which we call BTZ metric, has an inner and an outer horizon and also an ergosphere region analogously to the Kerr metric, however it is asymptotically AdS instead of asymptotically flat. This can be seen by letting $r \to \infty$. From (2.6) and (2.7) we see that in this limit

$$ds^2 \sim r^2 \left(-dt^2 + d\phi^2\right) . \tag{2.8}$$

Hence the BTZbh is asymptotically $\mathbb{R} \times \mathbb{S}^1$, an infinite long cylinder, as the boundary of the covering space of AdS spacetime. It is in this sense that the BTZbh is asymptotically AdS spacetime. The outer and inner horizon are given respectively by

$$r_\pm^2 = \frac{Ml^2}{2} \left(1 \pm \left(1 - \left(\frac{J}{Ml}\right)^2\right)^{1/2}\right) \tag{2.9}$$

This BTZ metric can be obtained directly from the Einstein's field equations by imposing time and axial symmetry [11], or it can be obtained as a quotient of AdS spacetime by a discrete subgroup of the AdS group. When expressed in BTZ coordinates (t, r, ϕ) the Killing vector which generates this subgroup turns out to be ∂_ϕ [11]. The second way of obtaining the metric (2.6) is the more suitable for the purposes of this work.

In terms of BTZ coordinates the three regions of the BTZbh are given by [11]: For $r_+ < r$

$$\begin{aligned} u &= \sqrt{B(r)}\sinh \tilde{t}\,(t,\phi) \qquad & v &= \sqrt{A(r)}\cosh \tilde{\phi}\,(t,\phi) \\ y &= \sqrt{B(r)}\cosh \tilde{t}\,(t,\phi) \qquad & x &= \sqrt{A(r)}\sinh \tilde{\phi}\,(t,\phi)\,, \end{aligned} \tag{2.10}$$

For $r_- < r < r_+$

$$\begin{aligned} u &= -\sqrt{-B(r)}\cosh \tilde{t}\,(t,\phi) \qquad & v &= \sqrt{A(r)}\cosh \tilde{\phi}\,(t,\phi) \\ y &= -\sqrt{-B(r)}\sinh \tilde{t}\,(t,\phi) \qquad & x &= \sqrt{A(r)}\sinh \tilde{\phi}\,(t,\phi)\,, \end{aligned} \tag{2.11}$$

For $0 < r < r_-$

$$\begin{aligned} u &= -\sqrt{-B(r)}\cosh \tilde{t}\,(t,\phi) \qquad & v &= \sqrt{-A(r)}\sinh \tilde{\phi}\,(t,\phi) \\ y &= -\sqrt{-B(r)}\sinh \tilde{t}\,(t,\phi) \qquad & x &= \sqrt{-A(r)}\cosh \tilde{\phi}\,(t,\phi)\,, \end{aligned} \tag{2.12}$$

where

$$A(r) = l^2\left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2}\right), \qquad B(r) = l^2\left(\frac{r^2 - r_+^2}{r_+^2 - r_-^2}\right) \tag{2.13}$$

and

$$\tilde{t} = \frac{1}{l}\left(r_+ t/l - r_-\phi\right), \qquad \tilde{\phi} = \frac{1}{l}\left(r_+\phi - r_- t/l\right). \tag{2.14}$$

In this parametrization the coordinate ϕ must be 2π-periodic in order to obtain the usual metric for the BTZbh.

In order to introduce the surface gravity we need to introduce the generator of the horizon, let us write it down [21]

$$\chi = \partial_v - N^\phi|_{r_+}\partial_\varphi, \tag{2.15}$$

where v and φ are Eddington-Finkelstein-like coordinates defined by

$$dv = dt + \frac{dr}{f^2}, \qquad d\varphi = d\phi - \frac{N^\phi}{f^2}dr. \tag{2.16}$$

Using (2.15), the surface gravity, κ, can be calculated and turns out to be

$$\kappa^2 = -\frac{1}{2}\nabla^\mu\chi^\nu\nabla_\mu\chi_\nu \qquad \kappa = \frac{r_+^2 - r_-^2}{l^2 r_+}. \tag{2.17}$$

This quantity will appear later in our study of thermal states in the BTZbh.

Now let us study the global structure of the non-rotating BTZbh. If we define r^* by $\frac{dr^*}{dr} = f^{-2}$ then the metric (2.6) takes the form

$$ds^2 = f^2(-dt^2 + dr^{*2}) + r^2 d\phi^2, \tag{2.18}$$

where we have made $J = 0$. If we solve for r^* we get

$$r^* = \frac{l^2}{2r_+}\ln\frac{|r - r_+|}{r + r_+}. \tag{2.19}$$

From this expression we see that[1] $-\infty < r^* < 0$.

Let us now introduce the analogue of Kruskal coordinates for the BTZbh. Defining null coordinates

$$\widetilde{u} = t - r^* \qquad \widetilde{v} = t + r^* \tag{2.20}$$

we obtain

$$ds^2 = -f^2 d\widetilde{u}d\widetilde{v} + r^2 d\phi^2, \tag{2.21}$$

where $r^2 = r^2\left(\widetilde{u}, \widetilde{v}\right)$. The relation between r, r^*, $\widetilde{u}$ and $\widetilde{v}$ is given by

$$r^* = \frac{l^2}{2r_+}\ln\frac{r - r_+}{r + r_+} = \frac{\widetilde{v} - \widetilde{u}}{2}. \tag{2.22}$$

[1]However the coordinate r^* also covers $0 \leq r < r_+$, it is only singular at $r = r_+$ where $r^* \to \infty$. In this work we shall restrict to $r > r_+$, so we shall omit the absolute value symbol in r^*.

From this expression we have

$$f^2 = \frac{r^2 - {r_+}^2}{l^2} = \frac{(r + r_+)^2}{l^2} e^{\frac{r_+}{l^2}(\tilde{v} - \tilde{u})}. \tag{2.23}$$

Substituting (2.23) in (2.21) we obtain

$$ds^2 = -\left(\frac{r + r_+}{l}\right)^2 e^{\frac{r_+}{l^2}(\tilde{v} - \tilde{u})} d\tilde{u} d\tilde{v} + r^2 d\phi^2. \tag{2.24}$$

Defining

$$u = -e^{-\frac{r_+}{l^2}\tilde{u}} \qquad v = e^{\frac{r_+}{l^2}\tilde{v}} \tag{2.25}$$

we have

$$\begin{aligned} ds^2 &= -\frac{l^4}{r_+^2}\left(\frac{r + r_+}{l}\right)^2 dudv + r^2 d\phi^2 \\ &= -\frac{4l^2}{(1 + uv)^2} dudv + r^2 d\phi^2, \end{aligned} \tag{2.26}$$

where we have used (2.22) and (2.25) to obtain the second equality. The range of u and v in (2.25) is $-\infty < u < 0$ and $0 < v < \infty$ respectively, however the metric (2.26) is not singular anymore at $r = r_+$ ($uv = 0$). Hence now we can extend both ranges from $-\infty$ to ∞ with the condition (2.28) which implies $uv < -1$. Defining $u = T - R$ and $v = T + R$ we have

$$ds^2 = \frac{4l^2}{(1 + T^2 - R^2)^2}\left(-dT^2 + dR^2\right) + r^2 d\phi^2. \tag{2.27}$$

The coordinates (T, R) are analogous to Kruskal coordinates in Schwarzschild spacetime. It is interesting to note that at $r = r_+$ the metric in Kruskal-like coordinates is essentially flat resembling the Schwarzschild case. Also we see that at this surface ∂_T becomes a killing vector. This implies that ∂_u and ∂_v are killing vectors on the past and on the future horizon respectively for $R > 0$. If we omit the angular part of the metric (2.27) we can draw a two dimensional Kruskal-like diagram for the BTZbh. From (2.22) and (2.25) we

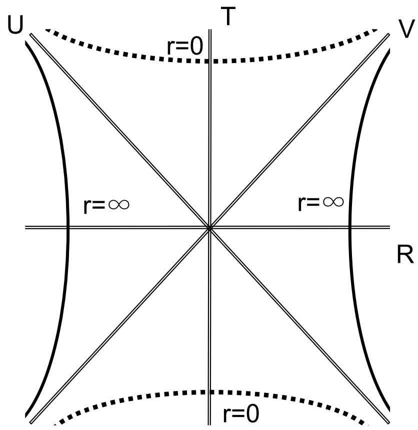


Figure 2.3: Kruskal diagram for the non-rotating BTZ black hole.

have

$$uv = -\frac{r - r_+}{r + r_+}. \tag{2.28}$$

Hence $uv \to -1$ when $r \to \infty$ and $uv \to 1$ when $r \to 0$ and we obtain the diagram 2.3. Each point of this diagram represents a circle. We define

$$\mathcal{R}_K \quad if \quad u < 0, v > 0 \qquad \mathcal{L}_K \quad if \quad u > 0, v < 0 \tag{2.29}$$

$$\mathcal{F}_K \quad if \quad u > 0, v > 0 \qquad \mathcal{P}_K \quad if \quad u < 0, v < 0 \tag{2.30}$$

with the appropriate bound given by $r = \infty$ and $r = 0$ respectively. If we foliate the Kruskal diagram with surfaces T =const. then we have an Einstein-Rosen-like throat bridge connecting two asymptotically AdS regions. The radius of this throat bridge is zero at $T = -1$ reaches its maximum at $r = r_+$ and is zero again at $T = 1$.

For later reference we express the Kruskal coordinates in terms of t and

r^*. In $\mathcal{R}_K$

$$T = e^{\frac{r_+}{l^2}r^*} \sinh\left(\frac{r_+}{l^2}t\right) \qquad R = e^{\frac{r_+}{l^2}r^*} \cosh\left(\frac{r_+}{l^2}t\right). \tag{2.31}$$

In $\mathcal{L}_K$

$$T = -e^{\frac{r_+}{l^2}r^*} \sinh\left(\frac{r_+}{l^2}t\right) \qquad R = -e^{\frac{r_+}{l^2}r^*} \cosh\left(\frac{r_+}{l^2}t\right). \tag{2.32}$$

In $\mathcal{F}_K$

$$T = e^{\frac{r_+}{l^2}r^*} \cosh\left(\frac{r_+}{l^2}t\right) \qquad R = e^{\frac{r_+}{l^2}r^*} \sinh\left(\frac{r_+}{l^2}t\right). \tag{2.33}$$

In $\mathcal{P}_K$

$$T = -e^{\frac{r_+}{l^2}r^*} \cosh\left(\frac{r_+}{l^2}t\right) \qquad R = -e^{\frac{r_+}{l^2}r^*} \sinh\left(\frac{r_+}{l^2}t\right). \tag{2.34}$$

From these expressions we see that the (T, R) and (t, r^*) coordinates are related as the Minkowski and Rindler coordinates in flat spacetime are.

In order to obtain the Penrose diagram we omit the angular part of the metric (2.27) and define

$$u = T - R = \tan\left(\frac{\lambda - \rho}{2}\right) \qquad v = T + R = \tan\left(\frac{\lambda + \rho}{2}\right). \tag{2.35}$$

Then we have

$$ds^2 = l^2 \sec^2 \rho \left(-d\lambda^2 + d\rho^2\right). \tag{2.36}$$

Hence, if we multiply (2.36) by the squared conformal factor $\Omega = l \cos \rho$ which is zero at the boundary ($\rho = \pm\pi/2$) we obtain as the unphysical metric

$$d\tilde{s}^2 = -d\lambda^2 + d\rho^2. \tag{2.37}$$

From (2.28) and (2.35) it follows that

$$uv = \frac{\tan^2 \lambda/2 - \tan^2 \rho/2}{1 - \tan^2 \lambda/2 \tan^2 \rho/2} = -\frac{r - r_+}{r + r_+}. \tag{2.38}$$

From this equation it follows that infinity, the singularity at $r = 0$ and the horizon are mapped to $\rho = \pm\pi/2$, $\lambda = \pm\pi/2$ and $\lambda = \pm\rho$ respectively, see figure 2.4.

For completeness we give the Penrose diagram for the rotating BTZ black

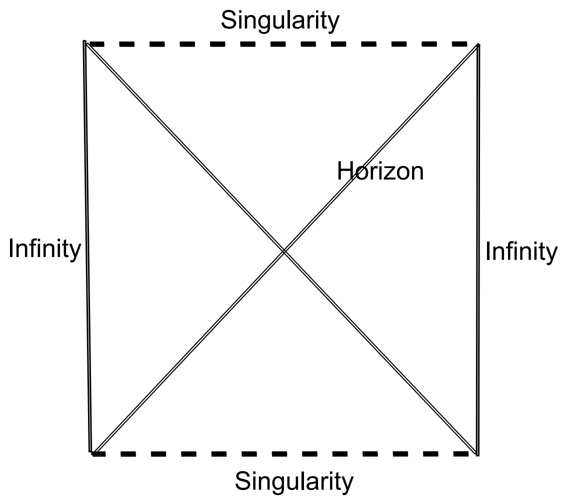


Figure 2.4: Penrose diagram for the non-rotating BTZ black hole.

hole [11] in figure 2.5.

2.3 The BTZ black hole in the boundary of AdS spacetime

We mentioned before that the BTZbh can be obtained by a quotient procedure from AdS spacetime. The quotient is made by a discrete subgroup of $SO_0(2,2)$. Because we want the resulting spacetime not to have closed timelike curves it is required the Killing vector which generates this subgroup be spacelike. It turns out that this criterion is not just necessary but also sufficient [11]. For the moment let us restrict ourselves to the non-rotating case. In this case, when expressed in embedding coordinates this generator turns out to be

$$\partial_\phi = \frac{r_+}{l}\left(x\partial_v + v\partial_x\right). \tag{2.39}$$

We are interested in studying quantum field theory on the boundary of AdS spacetime and consequently of the BTZbh. So it is useful to find out which regions on the boundary correspond to the covering space of the BTZbh. Because the maximally extended BTZbh have two exterior regions analogously

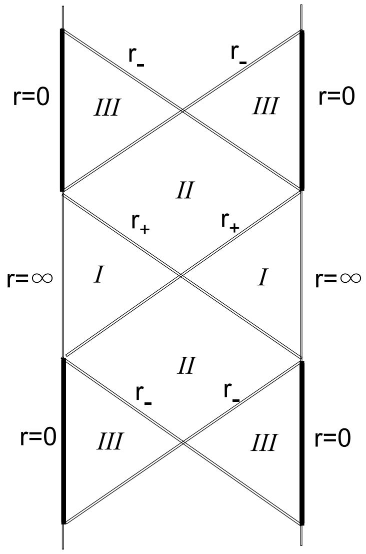


Figure 2.5: Penrose diagram for the rotating BTZ black hole.

to Schwarzschild spacetime, there will be two regions on the boundary which cover the maximally extended BTZbh. If we want to know what these regions are explicitly we can express ∂_ϕ in global coordinates, take the limit $\rho \to \pi/2$, and impose on it the condition of being spacelike in the metric (1.25).

Using

$$\sin\rho = \left(\frac{x^2+y^2}{u^2+v^2}\right)^{1/2} \qquad \tan\lambda = \frac{u}{v} \qquad \tan\theta = \frac{y}{x}, \tag{2.40}$$

we get

$$\frac{l}{r_+}\partial_\phi = -\sin\lambda\sin\rho\cos\theta\partial_\lambda + \cos\lambda\cos\rho\cos\theta\partial_\rho - \frac{\cos\lambda\sin\theta}{\sin\rho}\partial_\theta. \tag{2.41}$$

On the boundary

$$J_\phi \equiv \frac{l}{r_+}\partial_\phi|_{\rho=\pi/2} = -\sin\lambda\cos\theta\partial_\lambda - \cos\lambda\sin\theta\partial_\theta. \tag{2.42}$$

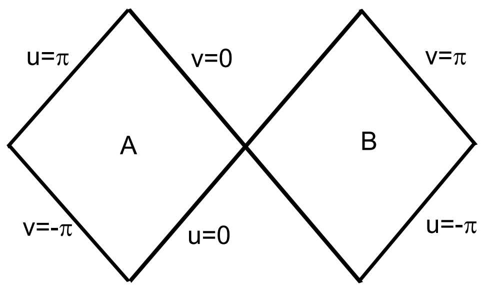


Figure 2.6: This figure represents the covering space of BTZ on the boundary of AdS spacetime in 1+2 dimensions.

Its norm is given by

$$||J_\phi||^2 = -\sin^2\lambda\cos^2\theta + \cos^2\lambda\sin^2\theta = -\sin u\sin v, \tag{2.43}$$

where we have introduced null coordinates

$$u = \lambda - \theta \qquad v = \lambda + \theta. \tag{2.44}$$

Clearly this vector can be timelike, spacelike or null. The regions where it is null are given by

$$u, v = n\pi \qquad n = 0, \pm1, \pm2... \tag{2.45}$$

Hence the covering space of the exterior BTZ black hole is the region inside the lines defined by, see figure 2.6,

$$u = \pi, 0, -\pi \qquad v = \pi, 0, -\pi. \tag{2.46}$$

Actually we can cover all the boundary of the covering space of AdS spacetime with regions like that defined by (2.46), but for our purposes we just need to consider one of them. The figure 2.6 has been given before in [2] and [19].

On the boundary the BTZ coordinates are (t, ϕ). We have found already the generator of translations in ϕ, let us see what the generator of translations in time is. In embedding coordinates

$$\partial_t = \frac{r_+}{l^2}\left(y\partial_u + u\partial_y\right). \tag{2.47}$$

In global coordinates

$$\frac{l^2}{r_+}\partial_t = \sin\lambda\cos\rho\sin\theta\partial_\rho + \cos\lambda\sin\rho\sin\theta\partial_\lambda + \frac{\sin\lambda\cos\theta}{\sin\rho}\partial_\theta. \tag{2.48}$$

On the boundary

$$J_t \equiv \frac{l^2}{r_+}\partial_t|_{\rho=\pi/2} = \cos\lambda\sin\theta\partial_\lambda + \sin\lambda\cos\theta\partial_\theta. \tag{2.49}$$

Clearly

$$J_t{}^\mu J_{\phi_\mu} = 0. \tag{2.50}$$

The norm of J_t is given by

$$||J_t||^2 = -\cos^2\lambda\sin^2\theta + \sin^2\lambda\cos^2\theta = \sin u\sin v. \tag{2.51}$$

From (2.51) it follows that when J_ϕ is spacelike J_t is timelike and viceversa.

Now let us see the region of AdS spacetime covered by the Poincaré chart defined before. Here we are going to consider just one fundamental region of CAdS spacetime, $\lambda \in [-\pi, \pi)$. From (1.22) and (1.31) it follows that

$$\cos\lambda = \frac{z^2 + l^2 + k^2 - T^2}{\sqrt{(z^2 + l^2 + k^2 - T^2)^2 + (2lT)^2}} \tag{2.52}$$

and

$$\cos\theta = -\frac{z^2 - l^2 + k^2 - T^2}{\sqrt{(z^2 + l^2 + k^2 - T^2)^2 + (2lT)^2 - (2lz)^2}}. \tag{2.53}$$

By following the analysis in [9] it can be shown that the equality $\cos\lambda = -\cos\theta$ can be satisfied on the surface $\rho = \pi/2$ and it corresponds to, let us say, the boundary of one Poincaré chart on this surface, see figure 2.7. From this we see that the covering region of the maximally extended BTZ black hole is half of the Poincaré chart. Let us see what the relation between BTZ coordinates and Poincaré coordinates is.

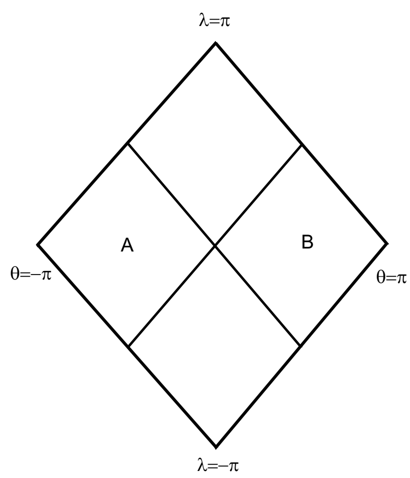


Figure 2.7: This figure shows how the covering space of the BTZbh on the boundary is related to the Poincaré chart. The big diamond is the Poincaré chart whereas the covering space of the BTZbh is the A and B small diamonds.

From (1.31) and (2.10) it follows that

$$\begin{aligned} \frac{u}{y} &= \tanh\left(\frac{r_+}{l^2}t\right) & \frac{v}{x} &= \coth\left(\frac{r_+}{l}\phi\right) \\ \frac{u}{y} &= \frac{T}{k} & \frac{v}{x} &= \frac{z^2+l^2+k^2-T^2}{l^2-z^2+T^2-k^2}. \end{aligned} \tag{2.54}$$

From these expressions it follows that at infinity (there is no dependence on r) $z = 0$ we have

$$\frac{T}{k} = \tanh\left(\frac{r_+}{l^2}t\right) \qquad \frac{l^2+k^2-T^2}{-l^2+k^2-T^2} = \coth\left(-\frac{r_+}{l}\phi\right). \tag{2.55}$$

Solving for k and T we obtain

$$T = le^{-\frac{r_+}{l}\phi}\sinh\left(\frac{r_+}{l^2}t\right) \qquad k = le^{-\frac{r_+}{l}\phi}\cosh\left(\frac{r_+}{l^2}t\right). \tag{2.56}$$

From (2.56) we can see that the relation between BTZ coordinates and Poincaré coordinates is analogous to the relation between Rindler and Minkowski coordinates. This suggests that some kind of Unruh effect is taking place on the boundary. In the next sections we shall show this is indeed the case.

For completeness, let us now find out the expressions for ∂_ϕ and ∂_t for the rotating black hole. From (2.10) and (2.14) it follows that

$$\partial_\phi = \frac{r_+}{l}\left(x\partial_v + v\partial_x\right) - \frac{r_-}{l}\left(y\partial_u + u\partial_y\right) \tag{2.57}$$

and

$$\partial_t = \frac{r_+}{l^2}\left(y\partial_u + u\partial_y\right) - \frac{r_-}{l^2}\left(x\partial_v + v\partial_x\right). \tag{2.58}$$

Now we can use the expressions (2.41) and (2.48), since these were obtained in general in global coordinates. Hence in global coordinates and at the boundary

$$\partial_\phi = -\frac{2r_+}{l}\left(\sin u\partial_u + \sin v\partial_v\right) - \frac{2r_-}{l}\left(\sin u\partial_u - \sin v\partial_v\right) \tag{2.59}$$

and

$$\partial_t = -\frac{2r_+}{l^2}\left(\sin u\partial_u - \sin v\partial_v\right) + \frac{2r_-}{l^2}\left(\sin u\partial_u + \sin v\partial_v\right) \tag{2.60}$$

where we have used the null coordinates u and v.

The vector fields of ∂_t and ∂_ϕ for the non-rotating and the rotating case are given in figure 2.8 and 2.9[2].

[2]Similar plots have been given in [2] and [1]

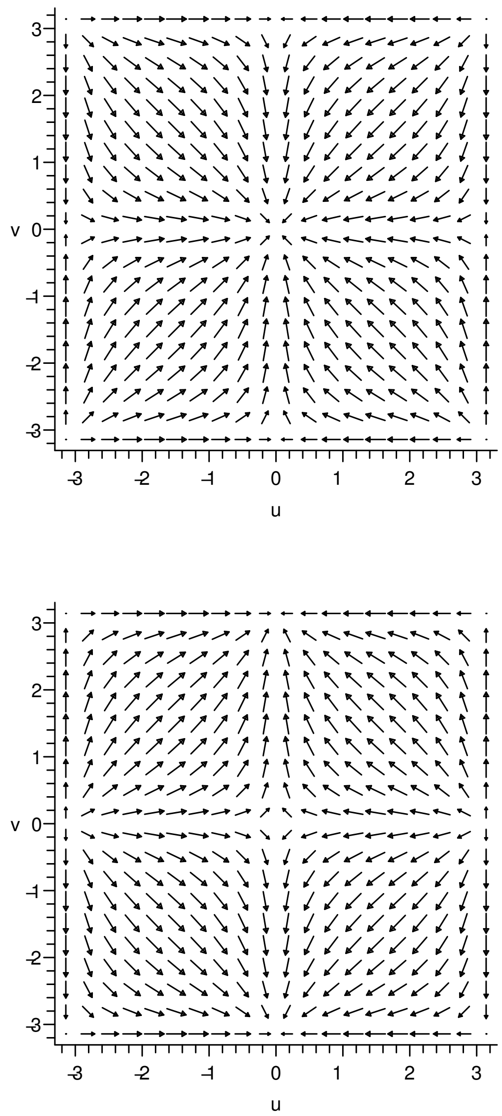
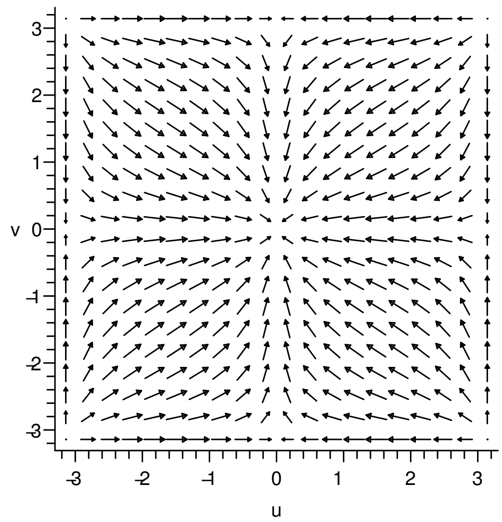


Figure 2.8: These figures show the vector fields ∂_ϕ and ∂_t respectively for the non-rotating BTZ black hole. These figures were made with Maple 10.



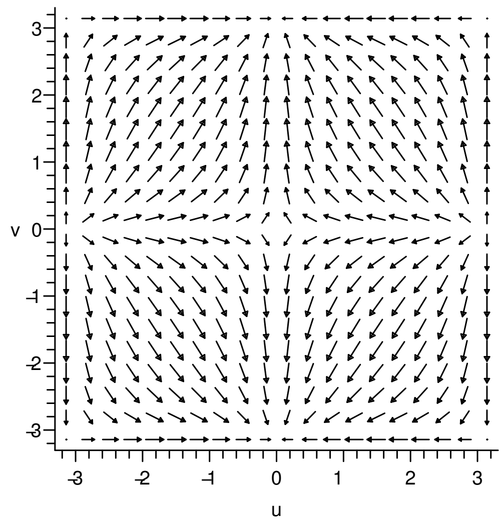


Figure 2.9: These figures show the vector fields ∂_ϕ and ∂_t for the rotating black hole. In the plotting we put $r_+ = 5r_-$. These figures were made with Maple 10.

Chapter 3

Elements of Quantum Field Theory

In this chapter we give the elements of Quantum Field Theory necessaries for this work. Our purpose is not to be exhaustive but just to give the necessary theory to understand the coming chapters.

3.1 Canonical quantization of the real linear scalar field

We consider the quantization of the real linear scalar field, ϕ, which obeys the Klein-Gordon equation

$$\left(\nabla_\mu \nabla^\mu - \xi R - m^2\right) \phi = 0, \tag{3.1}$$

where ξ is a coupling constant, R is the Ricci scalar and m can be considered as the mass of the field. This equation can be considered as the generalization of the Klein-Gordon equation in flat spacetime, i.e. when the second term does not appear. This term makes the equation conformally invariant when $m = 0$ and $\xi = \frac{1}{4}[(n-2)/(n-1)]$ [13].

The equation (3.1) is derived from the lagrangian density

$$\mathcal{L}(x) = -\frac{1}{2}\left[-g(x)\right]^{1/2}\left[g^{\mu\nu}(x)\phi(x)_{,\mu}\phi(x)_{,\nu} + \left[m^2 + \xi R(x)\right]\phi^2(x)\right] \tag{3.2}$$

by using the Euler-Lagrange equations [69]

$$\frac{\delta S}{\delta\phi} = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\phi_{,\mu})}\right), \tag{3.3}$$

where

$$S = \int \mathcal{L} d^n x \tag{3.4}$$

is the action of the field. In order to carry out the canonical quantization we need to define the momentum conjugate to ϕ. It is defined as

$$\pi = \frac{\partial\mathcal{L}}{\partial(\phi_{,0})}. \tag{3.5}$$

In the present case it turns out to be

$$\pi = (-g)^{1/2} g^{0\mu}\phi_{,\mu}. \tag{3.6}$$

Then we promote ϕ and π to operators and impose the canonical commutations relations [69], [36]

$$\begin{aligned} [\phi(t,\mathbf{x}), \phi(t,\mathbf{x}')] &= 0 \\ [\pi(t,\mathbf{x}), \pi(t,\mathbf{x}')] &= 0 \\ [\phi(t,\mathbf{x}), \pi(t,\mathbf{x}')] &= i\delta^{n-1}(\mathbf{x}-\mathbf{x}') \end{aligned} \tag{3.7}$$

where the delta function is defined by

$$\int d^{n-1}\mathbf{x}\,\delta^{n-1}(\mathbf{x}-\mathbf{x}')f(\mathbf{x}) = f(\mathbf{x}'). \tag{3.8}$$

Now we introduce an inner product for solutions to the equation (3.1)

$$(\phi_1, \phi_2) = -i\int_\Sigma (\phi_1\nabla_\mu\phi_2^* - \phi_2^*\nabla_\mu\phi_1)n^\mu\sqrt{\gamma}d^{n-1}x, \tag{3.9}$$

where Σ is a spacelike hypersurface, n^μ the normal to this hypersurface pointing to the future and $\gamma_{\mu\nu}$ is the metric induced in Σ. This inner product is independent of the hypersurface Σ if the field vanishes at infinity or vanishes on a timelike boundary the spacetime can have at infinity. This is a consequence of the fact that the covector

$$j_\mu = \phi_1 \nabla_\mu \phi_2^* - \phi_2^* \nabla_\mu \phi_1 \tag{3.10}$$

satisfies

$$\nabla^\mu j_\mu = 0 \tag{3.11}$$

for any two solutions of (3.1). By integrating this equation over a volume V of spacetime and using Stoke's theorem we obtain

$$\int_V dx^n \sqrt{|g|} \nabla^\mu j_\mu = \int_{\partial V} d^{n-1} y \sqrt{|\gamma|} n_\mu j^\mu = 0, \tag{3.12}$$

where $g = \det g_{\mu\nu}$ and $\gamma = \det \gamma_{\mu\nu}$ is the metric of spacetime V and the boundary ∂V of it respectively. If ∂V consists of two spacelike hypersurfaces Σ_1 and Σ_2 plus a hypersurface where the field vanishes then

$$\int_{\Sigma_1} d^{n-1} y \sqrt{\gamma} n_\mu j^\mu = \int_{\Sigma_2} d^{n-1} y \sqrt{\gamma} n_\mu j^\mu. \tag{3.13}$$

From this it follows that the inner product (3.9) does not depend of Σ.

When the spacetime is static, i.e., when the metric is diagonal and the metric is time independent then we can choose solutions harmonics in time. In this case the Lie derivative of these solutions is

$$\mathcal{L}_{K^\mu} u_i = -i\omega u_i, \tag{3.14}$$

where $K^\mu = (\partial_t)^\mu$ and t is the time function. The solutions satisfying (3.14) are called positive frequency modes. These modes can be normalized in such a way that they satisfy

$$(u_i, u_j) = \delta_{ij}. \tag{3.15}$$

The field operator then can be expanded in terms of these positive frequency modes as

$$\hat{\phi}(t,x) = \sum_i (u_i \hat{a}_i + u_i^* \hat{a}_i^\dagger), \tag{3.16}$$

where $*$ denotes complex conjugate. Imposing the conditions (3.7) we have the conditions on $\hat{a}$ and $\hat{a}^\dagger$ [23]

$$[\hat{a}_i, \hat{a}_j] = 0 \tag{3.17}$$

$$[\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \tag{3.18}$$

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}. \tag{3.19}$$

The vacuum is defined by

$$\hat{a}_i|0\rangle = 0 \quad \forall \quad i. \tag{3.20}$$

One should be careful with the word vacuum because strictly speaking the vacuum is defined in flat spacetime where we have Poincaré symmetry [41]. It is better to call the vector $|0\rangle$ in (3.20) ground state with respect to the time t. The literature on the quantization of the real scalar field is vast. Basic references are [43], [49], [56], [32], [51], [52] and [6]. General references on Quantum Field Theory in Curved Spacetime are [27], [33], [85] and [86].

The creation operators act on the vacuum creating states with certain number of particles. These states expand what is usually call the Fock space.

3.2 Conformal vacuum

As we said in the previous section, when $m = 0$ and $\xi = \frac{1}{4}[(n-2)/(n-1)]$ the Klein-Gordon operator is invariant under conformal transformations of the metric

$$g'_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x), \tag{3.21}$$

where $\Omega(x)$ is a scalar function of the spacetime. In this case if $\phi(x)$ is a solution of the Klein-Gordon operator with $g_{\mu\nu}$ then $\phi'(x) = \Omega^{(2-n)/2}\phi(x)$ is a solution of the Klein-Gordon equation with $g'_{\mu\nu}$. If we choose $g_{\mu\nu}$ as the flat metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, ...)$ and write

$$\eta_{\mu\nu} = \Omega^{-2} g_{\mu\nu}, \tag{3.22}$$

where we have made $g' = g$, then ϕ must satisfy

$$[\Box + \frac{1}{4}(n-2)R/(n-1)]\phi = 0 \tag{3.23}$$

and $\bar{\phi} = \Omega^{(n-2)/2}\phi$ must satisfy

$$\Box\bar{\phi} \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu\bar{\phi} = 0, \tag{3.24}$$

since $\bar{R} = 0$ in Minkowski. The equation (3.24) possesses the usual solutions

$$\bar{u}_{\mathbf{k}}(x) = [2\omega(2\pi)^{n-1}]^{-\frac{1}{2}} e^{ik\cdot x}, \quad k^0 = \omega. \tag{3.25}$$

Then the mode expansion for the field in the spacetime with metric $g_{\mu\nu}$ is

$$\phi(t, x) = \Omega^{(2-n)/2} \sum_i (a_{\mathbf{k}}\bar{u}_{\mathbf{k}} + a^\dagger_{\mathbf{k}}\bar{u}^*_{\mathbf{k}}), \tag{3.26}$$

where $\bar{u}_{\mathbf{k}}$ is the complete set of ortonormal solutions (3.25) of the Klein-Gordon operator. The vacuum defined with respect to the anihilation operators in (3.26) is the so-called conformal vacuum, i.e.

$$a_{\mathbf{k}}|0\rangle = 0 \qquad \forall \quad \mathbf{k}. \tag{3.27}$$

3.3 The KMS condition

The KMS condition reads

$$\langle BA_t \rangle = \langle A_{t-i\beta} B \rangle, \tag{3.28}$$

where

$$\langle A \rangle \equiv Z^{-1} \text{Tr}(e^{-\beta H} A), \tag{3.29}$$

$$Z = \mathrm{Tr} e^{-\beta H} \tag{3.30}$$

with temperature $\mathsf{T} = 1/\beta$. This condition is a consequence of the cyclic properties of the trace and can be seen as follows.

Define

$$\begin{aligned} G_+^\beta(t, A, B) &= \langle A_t B \rangle \\ &= Z^{-1}\mathrm{Tr}(e^{-\beta H} e^{itH} A e^{-itH} B) \\ &= Z^{-1}\mathrm{Tr}(e^{i(t+i\beta)H} A e^{-itH} B) \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} G_-^\beta(t, A, B) &= \langle B A_t \rangle \\ &= Z^{-1}\mathrm{Tr}(e^{-\beta H} B e^{itH} A e^{-itH}) \\ &= Z^{-1}\mathrm{Tr}(B e^{itH} A e^{-itH} e^{-\beta H}) \\ &= Z^{-1}\mathrm{Tr}(B e^{itH} A e^{-i(t-i\beta)H}). \end{aligned} \tag{3.32}$$

Then, by making $t = t - i\beta$ in (3.31) we have

$$G_+^\beta(t - i\beta, A, B) = G_-^\beta(t, A, B), \tag{3.33}$$

or

$$\langle B A_t \rangle = \langle A_{t-i\beta} B \rangle. \tag{3.34}$$

If the system satisfies CT then the KMS condition boils down to [37]

$$G_+^\beta(t - i\beta, A, B) = G_+^\beta(-t, A, B). \tag{3.35}$$

3.3.1 The simple quantum harmonic oscillator as an example

As an example of a system which satisfies the KMS condition we give the simple quantum harmonic oscillator.

The position operator is given by [72]

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(e^{-i\omega t}\hat{a} + e^{i\omega t}\hat{a}^\dagger\right), \tag{3.36}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the anihilation and creation operators respectively. The anihilation operator satisfies

$$\hat{a}|0\rangle = 0. \tag{3.37}$$

The two point function at zero temperature is given by

$$\langle 0|\hat{x}(t_1)\hat{x}(t_2)|0\rangle = \frac{\hbar}{2m\omega}e^{-i\omega(t_1-t_2)}. \tag{3.38}$$

The two point function at temperature T is given by

$$\langle \hat{x}(t_1)\hat{x}(t_2)\rangle_\beta = \frac{1}{Z}\mathrm{Tr}\left(e^{-\beta\hat{H}}\hat{x}(t_1)\hat{x}(t_2)\right), \tag{3.39}$$

where

$$Z = \left(1 - e^{-\beta\hbar\omega}\right)^{-1}. \tag{3.40}$$

In order to calculate this two point function we use the fact that

$$\mathrm{Tr}\left(e^{-\beta\hat{H}}\hat{a}^\dagger\hat{a}\right) = \frac{e^{-\beta\hbar\omega}}{\left(1 - e^{-\beta\hbar\omega}\right)^2} \tag{3.41}$$

and

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{3.42}$$

As a consequence of these two expression we have

$$\mathrm{Tr}\left(e^{-\beta\hat{H}}\hat{a}\hat{a}^\dagger\right) = \frac{1}{\left(1 - e^{-\beta\hbar\omega}\right)^2}. \tag{3.43}$$

From (3.36), (3.39), (3.40), (3.41) and (3.43) we have

$$\langle \hat{x}(t_1)\hat{x}(t_2)\rangle_\beta = \frac{\hbar}{2m\omega}\left(\frac{e^{-i\omega(t_1-t_2)} + e^{i\omega(t_1-t_2)}e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}}\right). \tag{3.44}$$

If we consider this expression as a function F of $t_1 - t_2 = \Delta t$, then we have

$$F(-\Delta t) = F(\Delta t - i\beta), \tag{3.45}$$

i.e., the KMS condition is satisfied.

In the previous characterization of the KMS condition we assumed that the trace which defines the Gibbs state (3.29) exists, however for infinitely extended systems this need not to be so. On the other hand, systems where the KMS condition is very relevant are those on which the thermodynamic limit is to be taken. These systems usually are infinitely extended. In these circumstances we face the problem of how to characterize these infinitely extended systems. It turn out that the KMS condition survives the thermodynamic limit [42] hence allowing us to use an algebraic characterization of thermal states. In this approach to the KMS condition we do not need to assume that the trace in (3.29) exists, we define a thermal equilibrium state as that which satisfies the KMS condition (3.28) in the algebraic definition of a state. In this case the translation in time is implemented by an automorphism of the algebra which defines time translation. With respect to this automorphism α_t, the thermal equilibrium state, ω, is invariant [42]

$$\omega(\alpha_t A) = \omega(A), \tag{3.46}$$

where A is an element of the algebra of observables. The advantage of the algebraic characterization is that it allows us to talk about the KMS conditions without referring to boxes, which otherwise are necessary in order to have a discrete spectrum of the hamiltonian.

3.3.2 The Tomita-Takesaki Theorem and the KMS condition

When the KMS condition is considered from a purely algebraic approach a very interesting connection of it with the theory of von Neumann algebras is possible. The purpose of this section is to sketch this connection.

In the theory of von Neumann algebras there exists the Tomita-Takesaki theorem which says [42]:

Let R be a von Neumann algebra in standard form, Ω a cyclic and separating vector and Δ, J, $U(t)$ as defined below. Then

$$JRJ = R',$$

$$U(t)RU^*(t) = R,$$

$$U(t)RU^*(t) = R'$$

for all real t.

The operators Δ and J are defined by $S = J\Delta^{1/2}$ where S is a closed operator which satisfies

$$SA\Omega = A^*\Omega, \ \ A \in R.$$

Here A^* is the adjoint operator to A. The operator $U(t)$ is defined by

$$U(t) = \Delta^{it}$$

and R' is the commutant of R[1].

The map σ_t defined by

$$\sigma_t A \equiv U(t)AU^*(t), \ \ A \in R$$

is an automorphism group of R. It is called the group of modular automorphisms of the state ω on the algebra R. Correspondingly J is called the modular conjugation and Δ the modular operator of (R, Ω).

Putting $\Delta = e^{-K}$ where K is a self adjoint operator we have $U(t) = e^{-itK}$. Then it can be shown [42] that the state Ω satisfies

$$\langle\Omega| \left(\sigma_t A\right) B|\Omega\rangle = \langle\Omega|B\sigma_{t-i}A|\Omega\rangle, \tag{3.47}$$

i.e., it satisfies the KMS condition with $\beta = -1$. Hence it is a thermal state

[1]For more details about these operators see [42] and [18].

with respect to the automorphism σ_t defined on the algebra R. If we put

$$\sigma_t = \alpha_{-\beta t}, \tag{3.48}$$

then

$$\langle \Omega | \left(\alpha_{t'} A \right) B | \Omega \rangle = \langle \Omega | B \sigma_{t' + i\beta} A | \Omega \rangle, \tag{3.49}$$

where $t' = -\beta t$, i.e., Ω satisfies the KMS condition with respect to t', the time in statistical mechanics. From the previous discussion we see the beautiful and powerful connection between mathematics and physics embodied in the Tomita-Takesaki theorem and its relation with the KMS condition. In the next section we shall see a consequence of this connection, the Unruh effect.

3.4 The Unruh effect

Quantum Field Theory in flat spacetime of the real linear scalar field is the *per excellence* example of a quantum theory of a field. In standard formulations of the theory the concept of particle plays a prominent rôle. This is done through the Fock representation. This concept is intimately related with the splitting into positive and negative frequency modes, see (3.16). The existence of this splitting is possible because the existence of a Killing vector. The positive modes are eigenfunctions of the Lie derivative with respect to this Killing vector. The usual introduction of the particle in field theory is by using the vector ∂_t as the Killing vector, which in no other thing than choosing translations in time t as the diffeomorphism associated with the symmetry. This choosing seems to be the more natural in Minkowski spacetime. However in a spacetime which does not have a natural symmetry this construction seems to be limited. This is indeed true and can be seen even in Minkowski spacetime. Let us see this with more detail. In an appropriate subset of Minkowski spacetime there is other Killing vector which is also natural. The usual metric in Minkowski spacetime is

$$ds^2 = -dt^2 + dx^2, \tag{3.50}$$

where we have restricted ourselves to the 1+1 dimensional case for simplicity in the argument. Under the coordinate transformation

$$t = a^{-1} e^{a\xi} \sinh a\eta \qquad x = a^{-1} e^{a\xi} \cosh a\eta, \tag{3.51}$$

where $a = \text{const.} > 0$, the metric takes the form

$$ds^2 = e^{2a\xi} \left(-d\eta^2 + d\xi^2 \right). \tag{3.52}$$

Because $x > 0$ and

$$x^2 - t^2 = a^{-2} e^{2a\xi} \tag{3.53}$$

the metric (3.52) is just defined in the region $x > |t|$, which is a wedge region in Minkowski spacetime. This region is known as Rindler wedge. From the form of (3.52) we see that ∂_η is also a killing vector in the Rindler wedge. Also we see that the Rindler metric is conformal to the Minkowski metric, hence using the conformal vacuum method we can expand the field not just with respect to Minkowski positive modes which satisfy

$$\frac{\partial}{\partial t} u_i = -i\omega u_i, \tag{3.54}$$

but also with respect to positive modes which satisfy

$$\frac{\partial}{\partial \eta} \tilde{u}_i = -i\omega \tilde{u}_i. \tag{3.55}$$

Using this alternative positive frequency splitting we can expand the field in terms of other set of creation and anihilation operators, more concretely we have

$$\hat{\phi}(t, x) = \sum_i (\tilde{u}_i \hat{\tilde{a}}_i + \tilde{u}_i^* \hat{\tilde{a}}_i^\dagger), \tag{3.56}$$

where now we define a vacuum

$$\hat{\tilde{a}}_i |0\rangle = 0 \quad \forall \quad i. \tag{3.57}$$

This vacuum will not coincide with the vacuum defined with respect to $\hat{a}_i$ [13]. The Rindler observer will see a thermal state at temperature $\mathsf{T} = \frac{a}{2\pi}$. This temperature can obtained from the analysis in Appendix A for $a = 1$.

3.4.1 The Bisognano-Wichmann theorem

In this section we shall sketch the Unruh effect in the context of axiomatic quantum field theory in the spirit of [78]. In order to do this we shall state the Bisognano-Wichmann theorem which belongs to this realm of quantum field theory and discuss its relation to the Unruh effect.

The mentioned theorem says [42]:

If $R(W)$ has a system of affiliated observable fields satisfying the axioms of axiomatic quantum field theory [78] then the modular conjugation[2] for the vacuum state is

$$J(W) = \Theta U\left(R_1(\pi)\right),$$

the modular operator is[3]

$$\Delta(W) = e^{-2\pi K},$$

the modular automorphism σ_t acts geometrically as the boost

$$\begin{aligned} x^0(s) &= x^0 \cosh s + x^1 \sinh s, \\ x^1(s) &= x^0 \sinh s + x^1 \cosh s, \\ x^r &= x^r \ \text{ for } \ r = 2, 3. \end{aligned} \tag{3.58}$$

Here x^μ are Minkowski coordinates belonging to the wedge W defined by

$$W = \{x \in M : x^1 > |x^0|; \ \ x^2, x^3 \text{arbitrary}\},$$

where M denotes Minkowski spacetime. Θ is the CPT-operator and $U(R_1(\pi))$ is the unitary representation of the rotation through an angle π around the

[2]This name comes from the Tomita-Takesaki theorem; also the name of modular operator. See section 3.3.2.

[3]The operator K is called the modular Hamiltonian and is a self adjoint operator whose spectrum will extend in general from $-\infty$ to ∞ and which has Ω as an eigenvector to eigenvalue zero. Here Ω is the vacuum vector restricted to the Rindler wedge.

1-axis, $R_1(\pi)$.

From (3.51) and (3.58) we see that for an observer at $\xi = 0$ its trajectory coincides with the boost (3.58) if initially $x^1 = \frac{1}{a}$. Hence the trajectory given by (3.58) for this value of x^1 can be interpreted as the trajectory of an accelerated observer with acceleration a. From the Tomita-Takesaki theorem and its relation with the KMS condition we know that the vacuum when restricted to the algebra $R(W)$ associated with the wedge W is a thermal state with respect to t at inverse temperature $\beta = -1$. Taking into account that the proper time of the accelerated observer is $\tau = s/a$ then we can conclude that for him the vacuum looks like a thermal state at temperature

$$\mathsf{T} = \frac{a}{2\pi}, \tag{3.59}$$

which is nothing else than the Unruh effect. We should notice that the time t for which the inverse temperature is $\beta = -1$ and the time τ for which the temperature is (3.59) are related as $\tau = -2\pi t/a$. From this discussion is clear that the Tomita-Takesaki theorem is one of elements of the mathematical machinery behind the Unruh effect.

3.5 The Hawking effect for an eternal black hole

One of the main predictions of Quantum Field Theory in curved spacetime is without doubt the Hawking effect [45]. This effect consists in the emission of thermal radiation by a black hole. In the more simple setting the black hole can be taken to be the Schwarzschild black hole formed by the collapse of a spherical distribution of matter. The story in this setting starts with a field and the distribution of matter in the past infinity. Then it is supposed that the matter collapses according to the laws of general relativity. At certain point during the collapse a black hole forms. Also it is supposed that initially the quantum field is in the vacuum state. As a consequence of the collapse particle pair production occurs around the horizon of the black hole. This

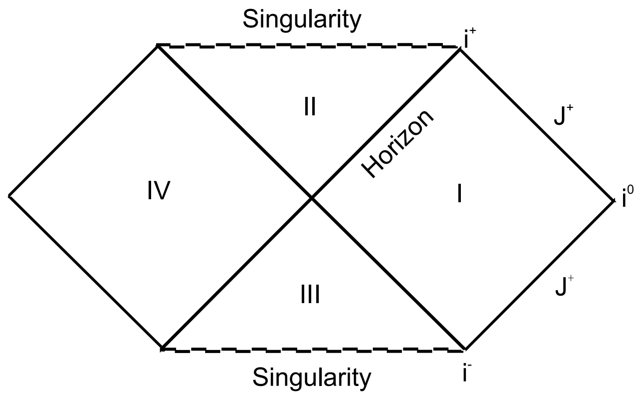


Figure 3.1: Penrose diagram for the Schwarzschild eternal black hole.

particle pair production causes that an observer at future infinity measures a flux of particles with thermal spectrum. The emission of this flux decreases the mass of the black hole, this process is known as black hole evaporation[4].

Related with black hole evaporation there exists other interesting phenomenon which involves the Schwarzschild black hole, whose metric is [23]

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 d\Omega^2, \tag{3.60}$$

where $d\Omega^2$ is the metric on the unit two-sphere

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{3.61}$$

In Section 2.2 we considered the maximal extension of the non-rotating BTZ black hole, its Penrose diagram is given in figure 2.4. An analogous maximal extension of the Schwarzschild black hole can be done and is given by [23]

$$ds^2 = \frac{32M^3}{r} e^{\frac{-r}{2M}} \left(-dT^2 + dR^2\right) + r^2 d\Omega^2 \tag{3.62}$$

where

$$T^2 - R^2 = \left(1 - \frac{r}{2M}\right) e^{\frac{r}{2M}}. \tag{3.63}$$

The Penrose diagram in this case is given in figure 3.1.

[4]The interested reader can see a detailed explanation of this effect in [35].

Let us consider a real scalar field on this background. Since defining positive frequency modes with respect to Kruskal time T does not give an interesting vacuum it is customary to define positive frequency modes with respect to the null coordinates along the past and future horizon, $u = T - R$ and $v = T + R$ [66]. The vacuum defined with respect to these modes is called the Kruskal vacuum or the Hartle-Hawking-Israel state [44], [50]. This vacuum is regular on the horizon [55] and is defined on all the background. If we restrict this state to the algebra of observables defined in the region I, then the resulting state is a thermal state with respect to the automorphisms of the algebra implementing translations in Schwarzschild time [55]. The temperature of this state is $\mathsf{T} = \frac{1}{8\pi M}$ where M is the mass of the black hole. This state represents a black hole which is emitting and absorbing radiation at the same rate and hence it is consistent only if it is placed in a thermal reservoir at temperature $\mathsf{T} = \frac{1}{8\pi M}$. This phenomenon is what we call Hawking effect for an eternal black hole.

The Unruh effect is the analogue of the Hawking effect for an eternal black hole. In this case the vacuum when restricted to the Rindler wedge becomes a thermal state with respect to translations in Rindler time [14], [83].

3.6 AdS/CFT in Quantum Field Theory

It is fair to say that most of works on the so called AdS/CFT correspondence fit into the string theory framework. However, partly motivated by this correspondence, recently have appeared some works, with a more limited scope, in the context of quantum field theory which address some issues related to the AdS/CFT correspondence. These works are the work by Rehren [70] which we call Algebraic Holography; the work of Bertola *et al.* [12] which we call Boundary-limit Holography and the work of Kay-Larkin [57] which we call Pre-Holography. We will refer to these three works as AdS/CFT in quantum field theory (AdS/CFT in QFT), since all three fit in the QFT framework. These three works in one sense or another give a correspondence between quantum field theories in the bulk of AdS spacetime and quantum field theories in its conformal boundary.

3.6.1 Algebraic Holography

In Algebraic Holography a correspondence between algebras in the bulk and the boundary of AdS spacetime is given. This correspondence relays heavily on the fact that the group $SO_0(2,d)$ acts in the bulk and in the boundary. In the first it acts as the group of isometry whereas in the second it acts as the global conformal group.

The correspondence between algebras is given as follows [70].

Lemma: Between the set of space-like wedge regions[5] in anti-deSitter space, $W \subset AdS_{1,s}$, and the set of double-cones in its conformal boundary space, $I \subset CM_{1,s}$, there is a canonical bijection $\alpha : W \to I = \alpha(W)$ preserving inclusions and causal complements, and interwining the actions of the anti-deSitter group $SO_0(2,d)$ and the conformal group $SO_0(2,d)$

$$\alpha(g(W)) = \dot{g}(\alpha(W)), \quad \alpha^{-1}(\dot{g}(I)) = g(\alpha^{-1}(I)), \tag{3.64}$$

where $\dot{g}$ is the restriction of the action of g to the boundary. The double-cone $I = \alpha(W)$ associated with a wedge W is the intersection of W with the boundary.

Given the lemma, the main algebraic result states that bulk observables localized in a wedge regions are identified with boundary observables localized in double-cone regions.

Corollary 1: The identification of local observables

$$B(W) := A(\alpha(W)), \quad A(I) := B(\alpha^{-1}(I)) \tag{3.65}$$

give rise to a 1:1 correspondence between isotonous causally covariant nets of algebras $I \to A(I)$ on $CM_{1,s-1}$ and isotonous causal anti-deSitter covariant nets of algebras $W \to B(W)$ on $AdS_{1,s}$.

[5]A wedge region in AdS spacetime can be defined as the region resulting from the intersection of the usual wedge region in the embedding spacetime with AdS spacetime. The wedge region in the embedding spacetime is the region defined by $X^i > |X^j|$ where X^i is a spatial coordinate and X^j is a timelike coordinate.

A very important result for our work is the following corollary.

Corollary 2: Under the identification of Corollary 1, a vacuum state on the net A corresponds to a vacuum state on the net B. Positive-energy representations on the net A correspond to positive-energy representations on the net B. The net A satisfies essential Haag duality[6] if and only if the net B does. The modular group and modular conjugation (in the sense of Tomita-Takesaki) of a wedge algebra $B(W)$ in a vacuum state act geometrically (by a subgroup of $SO_0(2,s)$ which preserves W and by a CPT reflection, respectively) if and only if the same holds for the double-cone algebras $A(I)$.

We should note that this Corollary is telling us that if the modular group acts in the boundary preserving a double-cone, then the correspondent modular group acts in the wedge which corresponds to this double-cone.

3.6.2 Boundary-limit Holography

The Boundary-limit Holography [12] relates a covariant quantum field theory in the bulk to a conformally covariant quantum field theory in the boundary of AdS spacetime. This is done for the real scalar field. This relation is set up in the rigorous framework of quantum field theory of Wightman formalism. The basic formalism can be seen in [78]. More explicitly, the Boundary-limit Holography gives a prescription for obtaining the n-point functions of a conformal quantum field theory in the boundary of AdS spacetime from the n-point functions of the covariant quantum field theory in the bulk. The exposition in [12] is rather technical and here we just will give the necessary information for our purposes. First we express an n-point function in Poincaré coordinates (T,k,z)

$$W_n\left(X_1,X_2,...X_n\right)=W_n\left(X_1(T_1,k_1,z_1),X_2(T_2,k_2,z_2),...X_n(T_n,k_n,z_n)\right),$$

where we have taken AdS spacetime in $1+2$ dimensions for being the case we are interested in, but this applies to any dimension. Here (T,k,z) are

[6] Definition: Let $O\to R(O)$ be a net of von Neumann algebras on a vacuum Hilbert space H_0. The dual net R^d of R is the assignment $O\to R(O')'$. Haag duality of R is equivalent to $R=R^d$, or $R(O')=R(O)'$.

Poincaré coordinates. Then we restrict the z coordinate to be the same for all X_i

$$W_n\left(X_1, X_2, ...X_n; z\right) = W_n\left(X_1(T_1, k_1, z), X_2(T_2, k_2, z), ...X_n(T_n, k_n, z)\right).$$

Finally we take the limit

$$W_n\left(X_i(T_i, k_i)\right) = \lim_{z\to 0} z^{n\Delta} W_n\left(X_1, X_2, ...X_n; z\right), \tag{3.66}$$

where $X_i(T_i, k_i)$ denotes all the n-coordinates and Δ is called the dimension scaling factor. In particular for the two point function we have

$$\begin{aligned} W_2\left(X_1(T_1, k_1), X_2(T_2, k_2)\right) &= \lim_{z\to 0} z^{2\Delta} \times \\ &\times\ W_2\left(X_1(T_1, k_1, z), X_2(T_2, k_2, z)\right), \end{aligned} \tag{3.67}$$

where the two point function on the left is defined in the boundary of AdS spacetime whereas the two point function in the right is defined in the bulk of AdS spacetime.

Strictly speaking all these equalities should be understood as equalities between distributions, however for our purposes we just need the limit procedure described before in order to obtain our results.

3.6.3 Pre-Holography

In Pre-Holography it was constructed a symplectic map from the classical solutions of the Klein-Gordon operator in the bulk of AdS spacetime to a certain space of functions in the boundary of AdS spacetime. Using this map it was possible to give new examples of Algebraic Holography and under certain conditions it was posible to make the Boundary-limit Holography a real duality. This was done for a massive real linear scalar field. It was shown that when

$$\Delta = \frac{d}{2} + \frac{1}{2}\left(d^2 + 4m^2\right)^{1/2} \tag{3.68}$$

is an integer or half-integer then the just mentioned duality holds. Also under these circumstances Algebraic Holography and the Boundary-limit Holography can be related.

Also it was shown that the global vacuum in the bulk of AdS spacetime maps to a vacuum in the conformal boundary. Also it was studied some group theoretical aspects of AdS spacetime. In particular it was given the expressions for the Killing vectors in AdS spacetime in 1+1 dimensions. This is important because it was the starting point of our work. We studied how these killing vectors are related to the thermal properties of the state of the quantum field.

Chapter 4

Thermal States in AdS and BTZ spacetime

4.1 Pre-Holography and Thermal state on the boundary

In this section we shall use Pre-Holography [57], [60] to show the existence of a thermal state on the boundary of AdS spacetime in 1+1 dimensions for the massless real linear scalar field.

In [60], it has been shown that the symplectic map induces a two point function on the boundary given by

$$G\left(\lambda_1, \lambda_2\right) = \frac{1}{2\pi} \frac{1}{\cos\left(\lambda_1 - \lambda_2\right) - 1}, \tag{4.1}$$

where λ is the global time.

We want to relate this two point function to a two point function in Poincaré and in BTZ coordinates. Before we proceed, it is necessary to make some comments on these coordinate charts on AdS spacetime in 1+1 dimensions. The coordinates we are interested in are global (λ, ρ), Poincaré (T, z) and BTZ coordinates (t, r). These coordinates cover different regions of AdS spacetime, see figure 4.1. The two point function (4.1) corresponds to the vacuum defined with respect to global time. We can interpret it as

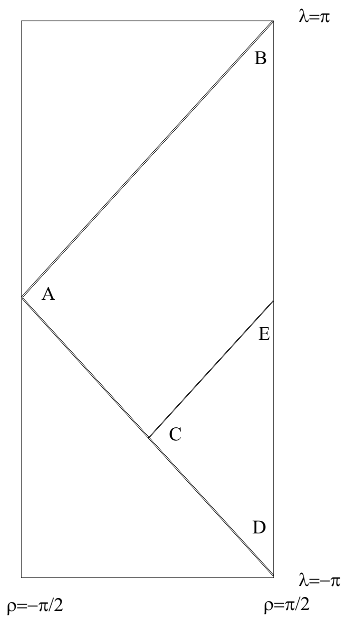


Figure 4.1: Regions in AdS spacetime in 1+1 dimensions covered by Poincaré and BTZ coordinates. The region ABD is covered by Poincaré coordinates whereas the region CED is covered by BTZ coordinates (exterior region). Global coordinates cover all the manifold.

a vacuum state on the boundary. Now, it has been shown [77] that global and Poincaré vacuums are the same in the bulk. If the same holds on the boundary then when we express (4.1) in Poincaré coordinates we expect to obtain a two point function on the boundary which corresponds to a vacuum state. Let us show that this is indeed the case. First, let us introduce Poincaré coordinates. We define

$$m \equiv \frac{v+x}{l^2} \qquad n \equiv \frac{v-x}{l^2} \qquad T \equiv \frac{u}{lm}, \tag{4.2}$$

where v, u and x are the coordinates in (1.13). Then (1.13) takes the form

$$-mnl^4 - m^2T^2l^2 = -l^2. \tag{4.3}$$

From (4.2) we have

$$v = \frac{l^2}{2}\left(m+n\right). \tag{4.4}$$

Now, using (4.3) in (4.4) we obtain

$$v = \frac{m}{2}\left(\frac{1}{m^2} + l^2 - T^2\right). \tag{4.5}$$

Defining $z = \frac{1}{m}$, the expression for the last equation is

$$v = \frac{1}{2z}\left(z^2 + l^2 - T^2\right). \tag{4.6}$$

Doing analogously for x and using the definition of z in u we get

$$x = \frac{1}{2z}\left(-z^2 + l^2 + T^2\right) \qquad u = \frac{lT}{z}. \tag{4.7}$$

The equations (4.6) and (4.7) define Poincaré coordinates (T, z). z plays the rôle of spacial coordinate and we can see from

$$\frac{1}{z} = \frac{v + x}{l^2} \tag{4.8}$$

that there are two charts $z > 0$ and $z < 0$, $z = 0$ belongs to the boundary [9]. Each chart covers half of AdS spacetime. It is standard to choose $z > 0$, we will attach to this convention. In these coordinates the expression for the metric is

$$ds^2 = \frac{l^2}{z^2}\left(-dT^2 + dz^2\right). \tag{4.9}$$

From this equation we see that the conformal boundary belongs to $z = 0$. Since we want to express (4.1) in Poincaré coordinates we need the relation between global time λ and Poincaré time T, let us see what this relation is. From (1.14) and (4.6)-(4.7) we have

$$\tan\lambda = \frac{2lT}{l^2 - T^2} \qquad \cos\lambda = \frac{l^2 - T^2}{l^2 + T^2} \qquad \sin\lambda = \frac{2lT}{l^2 + T^2}, \tag{4.10}$$

where we have made $z = 0$. The relation between λ and T can be written as

$$\frac{T}{l} = \tan\left(\frac{\lambda}{2}\right). \tag{4.11}$$

where we have used the tangent half-angle formula.

Now, expanding out $\cos(\lambda_1 - \lambda_2)$ and using (4.10) we obtain

$$\frac{1}{2\pi}\frac{1}{\cos(\lambda_1 - \lambda_2) - 1} = -\frac{1}{\pi}\frac{(l^2 + T_1^2)(l^2 + T_2^2)}{4l^2 (T_1 - T_2)^2}. \tag{4.12}$$

So far we just have made a naive and straight change of coordinates, however, according to AH [70], to the quantum field on the bulk of AdS spacetime corresponds a conformal quantum field on the boundary, and using Pre-Holography in [57] it has been given explicit examples of this correspondence. In particular the two point function (4.1) corresponds to a state on the boundary and from (4.11) it follows that the metric on the boundary in global and Poincaré coordinates are related by

$$d\lambda^2 = \frac{4l^2}{(l^2 + T^2)^2} dT^2. \tag{4.13}$$

The equation (4.13) expresses the same metric in two different coordinate systems. Let us do a bit of tensor analysis in order to obtain the relation between the components of the metric in the two different coordinate systems. If we express the metric in two different coordinate systems x^μ and x'^μ we have

$$g'_{\mu\nu}(x')dx'^\mu dx'^\nu = g_{\alpha\beta}(x)dx^\alpha dx^\beta. \tag{4.14}$$

If we use $dx'^\mu = \frac{\partial x'^\mu}{\partial x^\alpha} dx^\alpha$ in the left hand side of the last expression we get

$$g'_{\mu\nu}(x')\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} = g_{\alpha\beta}(x). \tag{4.15}$$

Then on the view of the coordinate transformation (4.11) we have

$$g_{TT}(T)\frac{dT}{d\lambda}\frac{dT}{d\lambda} = g_{\lambda\lambda} \tag{4.16}$$

which turns out to be

$$g_{\lambda\lambda} = \left(\frac{l^2 + T^2}{2l}\right)^2 g_{TT}(T). \tag{4.17}$$

On the other hand, in Conformal Field Theory (CFT) the starting point is to look for transformations $x \to x'$ of spacetime which satisfy [34]

$$g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x), \tag{4.18}$$

where x and x' represent generic points in spacetime and Ω is a smooth function. Hence the change from global to Poincaré coordinates can be analyzed in the context of CFT with

$$\Omega = \frac{l^2 + T^2}{2l}. \tag{4.19}$$

In order to do this, let us state some basic facts of CFT [34]. For quasi-primary fields the two point functions under conformal transformations are related as

$$\langle \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta_1}{d}} \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta_2}{d}} \langle \phi_1(\mathbf{x}'_1)\phi_2(\mathbf{x}'_2)\rangle, \tag{4.20}$$

where Δ_1 and Δ_2 are scaling dimensions. On other hand we have

$$\left|\frac{\partial x'}{\partial x}\right| = \Omega^{-d}. \tag{4.21}$$

Hence from (4.20) and (4.21) we have

$$\langle \phi_1(\mathbf{x}_1)\phi_2(\mathbf{x}_2)\rangle = \Omega(\mathbf{x}_1)^{-\Delta_1}\Omega(\mathbf{x}_2)^{-\Delta_2}\langle \phi_1(\mathbf{x}'_1)\phi_2(\mathbf{x}'_2)\rangle. \tag{4.22}$$

If we write (4.12) as

$$-\frac{1}{\pi}\frac{1}{(T_1 - T_2)^2} = \Omega^{-1}(T_1)\Omega^{-1}(T_2)\frac{1}{2\pi}\frac{1}{\cos(\lambda_1 - \lambda_2) - 1}, \tag{4.23}$$

where $\Delta_1 = \Delta_2 = 1$, then we can read off the two point function in Poincaré coordinates

$$G(T_1, T_2) = -\frac{1}{\pi}\frac{1}{(T_1 - T_2)^2}. \tag{4.24}$$

This two point function has the form, apart from a $1/4$ factor, of the two times differentiated vacuum two point function restricted to right or left movers[1]. Then we can say that also on the boundary the global and Poincaré vacuums coincide. This just strengthens the fact that (4.1) corresponds to a vacuum state on the boundary. It is important to remark that this function is defined on a complete (module identifications) boundary of AdS spacetime. So (4.1) and (4.24) corresponds to a vacuum defined on the whole boundary. Now let us consider the two point function expressed in BTZ coordinates.

We have seen that BTZ coordinates cover just half of one of the boundaries of AdS spacetime. Then the two point function (4.1) when expressed in these coordinates will correspond to a state defined on this part of the boundary. We expect this state to be a thermal state which corresponds to the state of a field on the BTZ black hole in 1+1 dimensions. We shall show that this is indeed the case.

Let us express (4.1) in BTZ coordinates. We obtain the same result if we express (4.24) in BTZ coordinates, however starting from (4.1) is more logical. From (1.14) and (2.1) it follows that on the boundary global time λ and BTZ time t are related by

$$\tanh \kappa t = \cos \lambda, \tag{4.25}$$

where $\kappa = \frac{r_+}{l^2}$. From this equation it follows that

$$\sin \lambda = \frac{1}{\cosh \kappa t}. \tag{4.26}$$

[1]One can ask why we compare the two-point function of a one-dimensional theory which is not differentiated to a twice differentiated two point function of a two-dimensional theory. The answer is in the similarity or analogy between the geometry and the expressions for the two point functions in both cases. The coordinate, U, on the null line and the coordinate u for the boost which leaves invariant the right wedge are related as $U = e^{au}$, where a is the acceleration of the Rindler observer, whereas the Poincaré time T and the BTZ time, t, are related as $T = le^{-\kappa T}$. Then there is an analogy between the null line in two dimensional Minkowski spacetime and the timelike boundary of AdS spacetime in 1+1 dimensions. So it is natural to compare the two point functions in both cases.

Then expanding out $\cos(\lambda_1 - \lambda_2)$ we have

$$\cos(\lambda_1 - \lambda_2) - 1 = \frac{1 - \cosh\kappa(t_1 - t_2)}{\cosh\kappa t_1 \cosh\kappa t_2} = -\frac{2\sinh^2\kappa\left(\frac{t_1-t_2}{2}\right)}{\cosh\kappa t_1 \cosh\kappa t_2}. \tag{4.27}$$

Hence the right side of (4.1) takes the form

$$\frac{1}{2\pi}\frac{1}{\cos(\lambda_1 - \lambda_2) - 1} = -\frac{1}{4\pi}\frac{\cosh\kappa t_1 \cosh\kappa t_2}{\sinh^2\left(\kappa\frac{t_1-t_2}{2}\right)}. \tag{4.28}$$

Now, from (4.25) it follows that the metric on the boundary in global and BTZ coordinates are related by

$$d\lambda^2 = \frac{\kappa^2}{\cosh^2\kappa t}dt^2. \tag{4.29}$$

From this equation it follows that

$$g_{\lambda\lambda}(\lambda) = \left(\frac{\cosh\kappa t}{\kappa}\right)^2 g_{tt}(t) \tag{4.30}$$

Hence both metrics are conformally related with

$$\Omega(t) = \frac{\cosh\kappa t}{\kappa}. \tag{4.31}$$

We can do the same analysis we did when we passed from global to Poincaré coordinates. If we write (4.28) as

$$-\frac{1}{4\pi}\frac{\kappa^2}{\sinh^2\left(\kappa\frac{t_1-t_2}{2}\right)} = \Omega^{-1}(t_1)\Omega^{-1}(t_2)\frac{1}{2\pi}\frac{1}{\cos(\lambda_1 - \lambda_2) - 1}, \tag{4.32}$$

where $\Delta_1 = \Delta_2 = 1$, we can read off the two point function in BTZ coordinates

$$G(t_1, t_2) = -\frac{1}{4\pi}\frac{\kappa^2}{\sinh^2\left(\kappa\frac{t_1-t_2}{2}\right)}. \tag{4.33}$$

This two point function has the form of the two times differentiated thermal two point function restricted to left movers. Then we can conclude that (4.33) corresponds to a thermal state on the boundary.

Now let us give an interpretation to what we have obtained so far. We have said that the Poincaré chart covers all the boundary of AdS spacetime whereas BTZ coordinates cover just half of it, see figure 4.1. On the other hand, the state induced on the boundary under the Pre-holography map has the form of a state at zero temperature, pure state, when expressed in Poincaré coordinates, whereas it has the form of a thermal or mixed state when expressed in BTZ coordinates and this chart covers just half of the system on the boundary. Hence it is natural to think the field on the boundary as pure system made of two subsystems, one for each half of the boundary, which are entangled.

Other interesting aspect of this problem is the, let us say, group theory aspect. The isometry Lie group of AdS 1+1 is the pseudo-orthogonal group SO(1,2) generated by

$$\begin{aligned} J_{uv} &= u\partial_v - v\partial_u \\ J_{ux} &= u\partial_x + x\partial_u \\ J_{vx} &= v\partial_x + x\partial_v. \end{aligned} \tag{4.34}$$

The global vacuum is invariant under the action of

$$J_{uv} = u\partial_v - v\partial_u = -\partial_\lambda, \tag{4.35}$$

which acts as translation in λ and indeed leaves invariant (4.1), whereas the vacuum defined with respect to BTZ time is invariant under the action of

$$\frac{1}{\kappa}\partial_t = v\partial_x + x\partial_v \tag{4.36}$$

which acts as translation in t and leaves invariant (4.33).

Let us now find out what elements of the global conformal group on the boundary of AdS spacetime corresponds to the elements generated by (4.34). First we introduce a representation of (4.34) in $\mathbb{R}^{2,1}$:

$$J_{uv} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.37}$$

$$J_{ux} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{4.38}$$

$$J_{vx} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.39}$$

Then we introduce a coordinate

$$\xi = \frac{u}{v+x}. \tag{4.40}$$

Now we introduce the combinations

$$A = J_{ux} - J_{uv} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{4.41}$$

and

$$C = J_{uv} + J_{ux} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.42}$$

These two matrices satisfy

$$A^3 = C^3 = 0. \tag{4.43}$$

Making the same calculation as in Appendix B we obtain that A generates translations in ξ and C generates special conformal transformations in time. Also we have

$$\xi = \frac{u}{v+x} = \frac{T}{l}. \tag{4.44}$$

Hence using (4.41), (4.42) and (4.44) we have that the translations in global time in the boundary is given as a linear combination of translations in T and special conformal transformations

$$\partial_\lambda = \frac{1}{2}(\partial_\xi - S_\xi), \tag{4.45}$$

where S_ξ is the generator of special conformal transformation in time. The generator (4.39) generates a boost in the plane xv, hence when this boost acts on a point $(u, v, x)^T$ we obtain that

$$\xi \to e^{-s}\xi, \tag{4.46}$$

where s is a parameter. Hence ∂_t generate dilations in the boundary.

4.2 Wedge regions in AdS spacetime

In the context of Algebraic Holography [70] wedge regions in AdS spacetime play a prominent rôle. In this section we will show that the exterior of the BTZbh is a wedge region.

Following [70] we define a wedge region in AdS spacetime as follows. Let us take two light-like vectors (e, f) in the embedding space $\mathbb{R}^{2,2}$ such that $e \cdot f > 0$, then a wedge region in AdS spacetime is defined by

$$\widetilde{W}(e, f) = \{x \in \mathbb{R}^{2+2} : x^2 = -l^2, e \cdot x < 0, f \cdot x < 0\}. \tag{4.47}$$

This region has two connected components. One where the resulting vector by acting on the tangent vector at the point $x \in \widetilde{W}(e, f)$ with the boost in the e-f plane is future directed and other where it is past directed. This is a consequence of the fact that the vector $\delta_{e,f}x = (f \cdot x)e - (e \cdot x)f$ is time-like, $(\delta_{e,f}x)^2 = -2(e \cdot f)(e \cdot x)(f \cdot x) < 0$. The wedge regions are defined in [70] as these regions modulo the identification $x \leftrightarrow -x$.[2] Hence in order to check that a region in AdS spacetime is a wedge region we just have to verify that it has these properties. Let us do this for the exterior of the BTZbh.

[2]It is worth noting that a more primitive definition of a wedge region in AdS spacetime is to define it as the intersection of AdS spacetime with a wedge region in the embedding spacetime.

4.2.1 The exterior of the BTZ black hole is a wedge region

Let us take $e = (1, 0, 0, -1)$ and $f = (-1, 0, 0, -1)$. These light-like vectors satisfy $e \cdot f > 0$. By using the parametrization (2.10) we get

$$e \cdot x = -\sqrt{B(r)}\left(\sinh \tilde{t} + \cosh \tilde{t}\right) < 0 \tag{4.48}$$

and

$$f \cdot x = \sqrt{B(r)}\left(\sinh \tilde{t} - \cosh \tilde{t}\right) < 0. \tag{4.49}$$

Hence the exterior of the rotating BTZ black hole is a wedge region. It is also true for the non-rotating case. As explained in [70] this wedge intersects the boundary in a double cone. In the previous section we have found this double cone explicitly. These regions are preserved by the action of the subgroup generated by the Killing vector ∂_t. This group acts on the wedge as a subgroup of the AdS group and as subgroup of the conformal group on the boundary.

4.2.2 The exterior of BTZ black hole and the Rindler wedge in Minkowski spacetime

We have found that the exterior of the BTZbh is invariant under the one-parameter subgroup of the AdS group generated by ∂_t. Let us calculate explicitly this one-parameter subgroup. The generator of this subgroup is given by (2.47), hence a matrix representation of it on the vector space $\mathbb{R}^{2,2}$ is given by

$$\partial_t = \kappa \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.50}$$

This matrix has the following recurrence properties

$$(\partial_t)^2 = \kappa^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (\partial_t)^3 = \kappa^3 \partial_t, \tag{4.51}$$

and so on. The matrix (4.50) is an element of the Lie algebra of $SO_0(2,2)$ and the one-parameter subgroup generated by it is given by

$$\Lambda(t) = e^{t\partial_t} = \sum_{n=0}^{\infty} \frac{(t\partial_t)^n}{n!}. \tag{4.52}$$

Splitting this sum in even and odd parts and using the recurrence properties (4.51) we obtain

$$\Lambda(t) = \begin{pmatrix} \cosh \kappa t & 0 & 0 & \sinh \kappa t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \kappa t & 0 & 0 & \cosh \kappa t \end{pmatrix}. \tag{4.53}$$

This one-parameter group acts on the vector space $\mathbb{R}^{2,2}$ leaving the exterior of the BTZbh invariant.

As we said in Section 2, AdS spacetime has a compactified Minkowski spacetime at infinity. Also we know that $SO_0(2,2)$ acts as the conformal group on this Minkowski spacetime [42]. Let us see to which element of the conformal group in 1+1 dimensions the element (4.53) corresponds.

Remembering the definition of the coordinates of Minkowski spacetime at infinity we have

$$\xi^1 = \frac{u}{v+x} \qquad \xi^2 = \frac{y}{v+x}. \tag{4.54}$$

If now we let $\Lambda(t)$ act on $x^T = (u, v, x, y)$ we obtain

$$\Lambda(t)x = \begin{pmatrix} u\cosh\kappa t + y\sinh\kappa t \\ v \\ x \\ u\sinh\kappa t + y\cosh\kappa t \end{pmatrix} = \begin{pmatrix} u' \\ v' \\ x' \\ y' \end{pmatrix}. \tag{4.55}$$

This transformation on the null cone, $\mathcal{C}^4$, induces a transformation on ξ^1 and ξ^2.

$$\xi^1 \to \xi'^1 = \frac{u'}{v' + x'} \qquad \xi^2 \to \xi'^2 = \frac{y'}{v' + x'}. \tag{4.56}$$

From (4.55) and (4.56) it follows that

$$\xi'^1 = \xi^1\cosh\kappa t + \xi^2\sinh\kappa t \qquad \xi'^2 = \xi^1\sinh\kappa t + \xi^2\cosh\kappa t. \tag{4.57}$$

Then the subgroup of the AdS group generated by ∂_t corresponds to a Lorentz boost in the 1+1 dimensional Minkowski spacetime. From this we can see that the Rindler wedge in this 1+1 dimensional Minkowski spacetime is invariant under the action of the subgroup of the global conformal group corresponding to the subgroup of the AdS group generated by ∂_t.

The correspondence between the others one-parameter subgroups can be found analogously. For example let us analyze the subgroup generated by ∂_ϕ.

The matrix representation of this generator on the vector space $\mathbb{R}^{2,2}$ is given by

$$\partial_\phi = \kappa l \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.58}$$

This matrix has also the recurrence properties (4.51). Hence the finite transformation is given by

$$\Lambda(\phi) = e^{\phi\partial_\phi} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\kappa l\phi & \sinh\kappa l\phi & 0 \\ 0 & \sinh\kappa l\phi & \cosh\kappa l\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.59}$$

If we let this transformation to act on $x^T = (u, v, x, y)$ we obtain

$$\Lambda(\phi)x = \begin{pmatrix} u \\ v\cosh\kappa l\phi + x\sinh\kappa l\phi \\ v\sinh\kappa l\phi + x\cosh\kappa l\phi \\ y \end{pmatrix} = \begin{pmatrix} u' \\ v' \\ x' \\ y' \end{pmatrix}. \tag{4.60}$$

This transformation induces a transformation on ξ^1 and ξ^2 given by

$$\xi'^1 = e^{-\kappa l\phi}\xi^1 \qquad \xi'^2 = e^{-\kappa l\phi}\xi^1. \tag{4.61}$$

Then the subgroup of AdS group generated by ∂_ϕ corresponds to the dilation group on the 1+1 dimensional Minkowski spacetime.

4.3 Thermal state in AdS spacetime and in the BTZ black hole

In this section we show that there exist an equilibrium thermal state in AdS spacetime in 1+2 dimensions and discuss its relation to an equilibrium thermal state in the BTZbh.

In the previous section, we showed that the exterior of the BTZbh is a wedge region. Now, also the Poincaré chart is likely to be a wedge region. If so we can associate a net of algebras to these regions. This can be done, for example, by adapting the formalism introduced in [28] to the present case. Due to the invariance of these wedge regions under the action of the subgroups generated by ∂_T and ∂_t we have

$$\omega(\alpha_T a) = \omega(a) \qquad a \in A(W_T) \tag{4.62}$$

and

$$\omega(\alpha_t a) = \omega(a) \qquad a \in A(W_t), \tag{4.63}$$

where W_T and W_t are the wedge regions associated to the Poincaré chart and the exterior of the BTZbh respectively. The symbol ω denote a state on these algebras, below we explain more about this state. The symbols α_T and α_t denote the automorphisms of the algebras associated to W_T and W_t respectively. We assume that these automorphisms satisfy

$$\alpha_T A(W_T) = A(\Lambda(T)W_T) \tag{4.64}$$

and

$$\alpha_t A(W_t) = A(\Lambda(t)W_t) \tag{4.65}$$

where $\Lambda(T)$ and $\Lambda(t)$ are the transformations which leave invariant W_T and W_t respectively. The last four expressions deserve some comments. The existence of $\Lambda(T)$ and $\Lambda(t)$ is a consequence of the existence of the Killing vectors ∂_T and ∂_t, which is a geometrical property of AdS spacetime. The equations (4.64) and (4.65) are part of the assumptions about the structure of the algebras. The equations (4.62) and (4.63) are a consequence of analogous expressions in the boundary assuming AH. Hence once we are in AdS spacetime and its geometry and we postulate the algebraic structure on its boundary these four equations should be valid. Now let us make some comments about the state ω. As we have said the last four equations have a bulk and boundary counterpart. In the boundary we have a Minkowski spacetime whereas in the bulk a spacetime with constant curvature. If we want to make quantum field theory on both and both should be equivalent there should be no preference for one of these two perspectives *a priori*. However, if we take into account that Quantum Field Theory in Minkowski spacetime has a well establish theory, it seems that we should go from the boundary to the bulk, because in this way we can use all the formalism at hand for

QFT in Minkowski spacetime and try to apply it to the bulk of AdS spacetime. It could be possible that some tools for Minkowski spacetime do not apply to AdS spacetime, however, we will notice this if we get a contradiction or an unphysical result. By taking this philosophy it seems that we should take the ground state on the boundary defined with respect to ∂_{ξ^1} which coincides with ∂_T, see Appendix B, as the vacuum of the theory on the boundary. If we do this, then in the bulk ω will be our vacuum, i.e., if we make the GNS construction [42] of this state then the unitary operator associated with translations in T has a self-adjoint generator operator, H_T, with spectrum $[0, \infty]$. This hamiltonian satisfies

$$H_T|\Psi_\omega\rangle = 0, \tag{4.66}$$

where $|\Psi_\omega\rangle$ is the cyclic vector associated with ω through the GNS construction. Also we have assumed that the unitary operator implementing translation in T is strongly continuos with respect to T. Hence in this case the von Neumann's theorem [71] assures the existence of H_T.

As was proven in [70], once we have set up this scenario we can apply the well-known theorems of Tomita-Takesaki and Bisognano-Wichmann to the theory on the boundary. From the previous discussion it is clear that ω is an equilibrium thermal state when restricted to the exterior of the BTZbh. Put in other way, it satisfies the KMS condition with respect to t. Let us find what the temperature of this state is.

From (4.55) we can see that the parameter of the boost is $t' = \kappa t$ with $\kappa = \frac{r_+}{l^2}$. Using theorem 4.1.1 (Bisognano-Wichmann theorem) in [42] we have that the parameter of the modular group which appears in the Tomita-Takesaki theorem is given by

$$\tau = -\frac{t'}{2\pi} = -\frac{\kappa}{2\pi}t. \tag{4.67}$$

Using theorem 2.1.1 (Tomita-Takesaki theorem) in [42] it is possible to prove that the state invariant under the modular group satisfies the KMS condition with $\beta = -1 = \frac{1}{\mathsf{T}}$, see [42]

$$\omega\left(\left(\alpha_\tau A\right) B\right) = \omega\left(B\left(\alpha_{\tau - i} A\right)\right). \tag{4.68}$$

Hence from (4.67) it follows the temperature of the thermal state with respect to t is

$$\mathsf{T} = \frac{\kappa}{2\pi}. \tag{4.69}$$

This is the so-called temperature of the black hole. The local temperature measured by an observer at constant radius r is

$$\mathsf{T}(r) = \frac{1}{(-g_{00})^{1/2}} \frac{\kappa}{2\pi}. \tag{4.70}$$

This is because the proper time of this observer, τ, and the time t are related as $\tau = (-g_{00})^{1/2} t$.

So far we have shown that ω satisfies the KMS condition on the covering space of one exterior of the BTZbh, however the exterior of the BTZbh is obtained after making ϕ 2π-periodic. This periodicity introduces new features because we have a non simply connected spacetime, a cylinder, instead of a simply connected spacetime, a plane. We shall assume that there is a way to construct the thermal state on the cylinder from one on the plane algebraically. Let us call the state on the covering space of one exterior region of the BTZbh ω_{AdS} and on the exterior region of the BTZbh ω_{BTZ}.

The state ω_{BTZ} is defined in $\mathbb{R}^1 \times \mathbb{S}^1$ which is one exterior region of the BTZbh whereas ω_{AdS} is defined on $\mathbb{R}^1 \times \mathbb{R}^1$ which is the covering space of one exterior region of the BTZbh. Put in this way, the state ω_{BTZ} is a thermal state on a black hole, i.e., the Hawking effect for an eternal black hole takes place and corresponds to the Unruh effect on AdS spacetime after making ϕ 2π-periodic. Put in this form we can say that the Hawking effect in the eternal BTZbh has its origin in the Unruh effect in the boundary of AdS spacetime and in the topological relation between $\mathbb{R}^1 \times \mathbb{R}^1$ and $\mathbb{R}^1 \times \mathbb{S}^1$. We point out that this result is in accordance with previous work on quantum field theory of the real scalar field in the BTZbh [62].

In the rotating case there is a tiny modification in the analysis. From the form of the parametrization of the exterior of BTZ black hole (2.10) we see that in the rotating case the subgroup of the AdS group which leaves invariant the wedge W is generated by $\partial_{\tilde{t}}$. Following the analysis of the previous section, now the parameter of the modular group which appears in Tomita-Takesaki theorem is related to the time $\tilde{t}$ as

$$\tau = -\frac{\tilde{t}}{2\pi}. \tag{4.71}$$

Hence the state is thermal with respect to $\tilde{t}$ at temperature $\mathsf{T} = \frac{1}{2\pi}$. Put in this form, because the state ω satisfies

$$\omega\left(\left(\alpha_{\tilde{t}-2\pi}A\right)B\right) = \omega\left(B\left(\alpha_{\tilde{t}}A\right)\right), \tag{4.72}$$

then there is a periodicity of the state in t and ϕ given by

$$(t,\phi) \rightarrow (t - i\beta, \phi + i\Omega_H\beta) \qquad \beta = \frac{1}{\mathsf{T}} = \frac{2\pi}{\kappa}. \tag{4.73}$$

where $\beta = \frac{2\pi}{\kappa}$. Hence again the black hole is hot at temperature $\mathsf{T} = \frac{\kappa}{2\pi}$.

4.3.1 Thermal state for a quantum real linear scalar field

In this section we show how a thermal state in the bulk of AdS spacetime maps to a thermal state on its boundary for a quantum real linear scalar field.

The equation the field satisfies is

$$\left(\nabla_\mu\nabla^\mu - \xi R - m^2\right)\phi = 0, \tag{4.74}$$

where ξ is a coupling constant, R is the Ricci scalar and m can be considered as the mass of the field. For the metric of AdS spacetime, $R = 6\Lambda = -\frac{6}{l^2}$. Hence the last equation can be written as

$$\left(\nabla_\mu\nabla^\mu - \widetilde{m}^2\right)\phi = 0, \tag{4.75}$$

where $\widetilde{m}^2 = m^2 - \frac{6\xi}{l^2}$. The operator $\nabla_\mu \nabla^\mu$ is given by

$$\nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \phi \right), \tag{4.76}$$

where $g = |g_{\mu\nu}|$. For our purposes it is convenient to use Poincaré coordinates. In these coordinates $g = -\frac{l^6}{z^6}$. Because ∂_T and ∂_k are Killing vectors we propose the ansatz $\phi(T, k, z) \propto e^{-i\omega T} e^{ink} f_{\omega n}(z)$. Then the equation (4.75) becomes a equation for $f(z)$:

$$z^2 \frac{d^2 f}{dz^2} - z \frac{df}{dz} + \left((\omega^2 - n^2) z^2 - l^2 \tilde{m}^2 \right) f = 0. \tag{4.77}$$

If we make $x = (\omega^2 - n^2)^{\frac{1}{2}} z$ and $y = \frac{f}{z}$ we obtain

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left(x^2 - 1 - l^2 \tilde{m}^2 \right) y = 0 \tag{4.78}$$

which is the Bessel equation with $p^2 = 1 + l^2 \tilde{m}^2$ [75]. Hence the solution of (4.77) is $f_{\omega n}(z) = z J_p(az)$ where $a = (\omega^2 - n^2)^{\frac{1}{2}}$ and J_p is a Bessel function. Following the standard procedure of canonical quantization [13] we normalize the modes $\phi(T, k, z)$ with the inner product

$$(f, g) = -i \int_{T=0} \left(f \dot{g}^* - g \dot{f}^* \right) \frac{l}{z} dk dz, \tag{4.79}$$

where $\dot{} \equiv \partial_T$ and $*$ means complex conjugate. The normalized modes turns out to be $F_{na}(T, k, z) = \left(\frac{a}{4 l \pi \omega} \right)^{\frac{1}{2}} e^{-i\omega T} e^{ink} z J_p(az)$, where we have used

$$\int_0^\infty J_p(az) J_p(a'z) z dz = \frac{1}{a} \delta(a - a'),$$

page 648 in [5]. These modes satisfy

$$(F_{na}, F_{n'a'}) = \delta_{nn'} \delta(a - a'). \tag{4.80}$$

Now we proceed to define the field operator

$$\hat{\phi}(T,k,z) = \int_{-\infty}^{\infty}\int_{0}^{\infty}\left(F_{na}\hat{a}_{na} + F^*_{na}\hat{a}^{\dagger}_{na}\right) dnda, \tag{4.81}$$

where

$$\hat{a}_{na}|0\rangle = 0 \qquad \forall \quad n, a \tag{4.82}$$

defines the Poincaré vacuum. The operator $\hat{\phi}$ acts on the Fock space constructed from the one particle Hilbert space spanned by $\{F_{na}\}$.

It follows that the two point function

$$F(T,k,z;T',k',z') = \langle 0|\hat{\phi}(T,k,z)\hat{\phi}(T',k',z')|0\rangle$$

is

$$\begin{aligned} F(\Delta T,\Delta k;z,z') &= \frac{zz'}{2\pi}\int_0^\infty dn \Bigg\{ \\ &\int_0^\infty da \frac{a\cos n\Delta k}{(a^2+n^2)^{1/2}} e^{-i(a^2+n^2)^{1/2}\Delta T} J_p(az)J_p(az')\Bigg\}, \end{aligned} \tag{4.83}$$

where $\Delta k = k - k'$ and $\Delta T = T - T'$. By making the integral over n we obtain

$$\begin{aligned} F(\Delta T,\Delta k;z,z') &= \frac{zz'}{2\pi}\lim_{\epsilon\to 0_+}\Bigg\{ \\ &\int_0^\infty daaK_0(a((\Delta k)^2 - (\Delta T + i\epsilon)^2)^{1/2})J_p(az)J_p(az')\Bigg\} \end{aligned} \tag{4.84}$$

where we have used the identity $\int_0^\infty dx \frac{\cos ax}{(\gamma^2+x^2)^{1/2}} e^{-\beta(\gamma^2+x^2)^{1/2}} = K_0(\gamma(a^2 + \beta^2)^{1/2})$, equation 3.961 2 in [40], with K_0 a zero order modified Bessel function. By integrating the last equation we obtain

$$F(\Delta T, \Delta k; z, z') = \frac{(zz')^{1+p}}{2\pi} \lim_{\epsilon \to 0_+} \left\{ \frac{F_4\left(\alpha, \alpha; \alpha, \alpha; -\frac{z^2}{(\Delta k)^2 - (\Delta T + i\epsilon)^2}, -\frac{z'^2}{(\Delta k)^2 - (\Delta T + i\epsilon)^2}\right)}{((\Delta k)^2 - (\Delta T + i\epsilon)^2)^{1+p}} \right\} \tag{4.85}$$

where $\alpha = 1 + p$, we have used the identity 6.578 2 in [40] and F_4 is a Hypergeometric function of two variables. Following [12], we now make $z = z'$ and multiply (4.85) by $z^{-2(1+p)}$ and take the limit $z \to 0$. We obtain[3]

$$\begin{aligned} F_b(\Delta T, \Delta k) &\equiv \lim_{z \to 0} z^{-2(1+p)} F(\Delta T, \Delta k; z, z') \\ &= \frac{1}{2\pi} \lim_{\epsilon \to 0_+} \frac{1}{((\Delta k)^2 - (\Delta T + i\epsilon)^2)^{1+p}}. \end{aligned} \tag{4.86}$$

Now let us analyze what happens when we restrict (4.86) to the exterior of BTZ black hole. By using (2.56) we get

$$F_b(\Delta t, \Delta\phi') = \frac{1}{2\pi} \frac{(2l^2 e^{-\kappa(\phi'_1 + \phi'_2)})^{-1-p}}{(\cosh \kappa\Delta\phi' - \cosh(\kappa\Delta t + i\epsilon))^{1+p}}, \tag{4.87}$$

where $\kappa = r_+/l^2$, $\phi' = l\phi$, $\Delta\phi' = \phi_1 - \phi_2$ and $\Delta t = t - t'$.

From (4.87) it follows that

$$F_b(-\Delta t, \Delta\phi') = F_b(\Delta t - i\beta, \Delta\phi') \tag{4.88}$$

where $\beta = \frac{1}{\mathsf{T}} = \frac{2\pi}{\kappa}$. From (4.88) it follows that the restricted two point function to the exterior of the BTZbh in the boundary satisfies the KMS condition [37]. Hence (4.85) is a thermal state at temperature $\mathsf{T} = \frac{\kappa}{2\pi}$ when restricted to the exterior of the BTZbh in accordance with [62]. From (4.85) we see that the thermal property does not change when we take the limit $z \to 0$, since $z = le^{-\kappa\phi'}\frac{r_+}{r}$, hence we can say the thermal state in the bulk of AdS spacetime maps to a thermal state on its boundary, and this state is

[3]This expression has been obtained before by Dr. Bernard S. Kay by using the Feynman propagator of the Klein-Gordon operator in Poincaré coordinates, private communication.

given by (4.87).

The jacobian of (2.56) is

$$\left|\frac{\partial(T,k)}{\partial(t,\phi')}\right| = l^2\kappa^2 e^{-2\kappa\phi'}. \tag{4.89}$$

If we consider (2.56) as a conformal transformation then from (4.89) and the relationship between two conformal metrics

$$g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x) \tag{4.90}$$

it follows that in the present case

$$\Omega(\phi') = \frac{1}{l\kappa e^{-\kappa\phi'}}. \tag{4.91}$$

If we want to consider F_b as a correlation function for a conformal field theory in the boundary of AdS spacetime and take into account that two correlation functions in a Conformal Field Theory are related by [34]

$$\Omega(x'_1)^{-\Delta_1}\Omega(x'_2)^{-\Delta_2}\langle\phi_1(x'_1)\phi_2(x'_2))\rangle = \langle\phi_1(x_1)\phi_2(x_2)\rangle, \tag{4.92}$$

where Δ_1 and Δ_2 are the scaling dimension of the field ϕ_1 and ϕ_2 respectively, then from the equality

$$\frac{1}{2\pi}\frac{\Omega(\phi'_1)^{-1-p}\Omega(\phi'_2)^{-1-p}}{((\Delta k)^2-(\Delta T+i\epsilon)^2)^{1+p}} = \frac{2^{-1-p}}{2\pi}\frac{\kappa^{2(1+p)}}{(\cosh\kappa\Delta\phi' - \cosh(\kappa\Delta t+i\epsilon))^{1+p}}$$

we finally have

$$F_b(\Delta t,\Delta\phi') \simeq \frac{1}{2\pi 2^{1+p}}\frac{\kappa^{2(1+p)}}{(\cosh\kappa\Delta\phi' - \cosh(\kappa\Delta t+i\epsilon))^{1+p}}. \tag{4.93}$$

According to standard Conformal Field Theory [34] this correlation function would correspond to a field with scaling dimension $\Delta = 1+p$.

From (4.93) we can obtain the two point correlation functions on the exterior of the BTZbh and its covering space respectively by using the image method. Let us explain briefly this method. The image sum method relates the two point function in a multiple connected and the two point function in a simply connected spacetime, see [10] or [24]. In this case we could apply the formula

$$F'_b(\Delta t, \Delta\phi) = \sum_{n\in\mathbb{Z}} e^{2\pi i n\alpha} F_b(\Delta t, \Delta\phi + 2\pi n), \tag{4.94}$$

where $F'_b(\Delta t, \Delta\phi)$ is the two point function in the covering space of the exterior of the BTZbh and α a parameter which is 0 for untwisted and $\frac{1}{2}$ for twisted fields. In this work we are interested in untwisted fields, so we take $\alpha = 0$. Applying this method to (4.93) we obtain

$$\begin{aligned} F'_b(\Delta t, \Delta\phi) &\equiv \sum_{n\in\mathbb{Z}} \frac{1}{2\pi 2^{1+p}} \times \\ &\times \frac{\kappa^{2(1+p)}}{(\cosh \frac{r_+}{l}(\Delta\phi + 2\pi n) - \cosh(\kappa\Delta t + i\epsilon))^{1+p}}. \end{aligned} \tag{4.95}$$

This correlation function would correspond to fields defined on the same covering space of one exterior region of the BTZbh. However the BTZ black hole has two exterior regions analogously to the Schwarzschild black hole. The parametrization of the other exterior region is given by changing the sign in (2.56). Hence in this case the correlation function is

$$\begin{aligned} F'_b(\Delta t, \Delta\phi) &\equiv \sum_{n\in\mathbb{Z}} \frac{1}{2\pi 2^{1+p}} \times \\ &\times \frac{\kappa^{2(1+p)}}{(\cosh \frac{r_+}{l}(\Delta\phi + 2\pi n) + \cosh(\kappa\Delta t + i\epsilon))^{1+p}}. \end{aligned} \tag{4.96}$$

From the previous analysis, it is clear that on the boundary we have a thermal state when we restrict the state defined on the Poincaré chart to the covering space of one exterior region of the BTZbh. Also because we have to make ϕ 2π-periodic then this covering space is a cylinder with spatial cross section. Now a massless field theory on a cylinder is equivalent to a 1+1 dimensional Conformal Field Theory [34], hence we can say that on the boundary of AdS we have two conformal theories related in such way that when we restrict a vacuum state to one of them it becomes a thermal state. We point out that the expressions (4.95) and (4.96) have been given in [63] and according to it they have been obtained in [59]. In [59] it was used the proposal given in [90] for the AdS/CFT correspondence. However we have just obtained them by using AdS/CFT in QFT which uses both Algebraic Holography and the Boundary-limit Holography. Hence we can say that the approach [90] and AdS/CFT in QFT are consistent at least for the present case. In our opinion they are consistent because both approaches are using in one way or in another techniques from quantum field theory in curved backgrounds. Taking into account this fact we conjecture that there are many aspects of the AdS/CFT correspondence which can be understood at this level and which just reflect the symmetry and group theoretical aspects of the AdS/CFT correspondence. It would be interesting to make a bit more realistic the correspondence in AdS/CFT in QFT for example by putting supersymmetry in it. It can be done for example by considering a supersymmetric model in BTZ black holes in 1+1 and 1+2 dimensions. Also this could be done in AdS spacetime itself.

So far in this section we have considered the non rotating BTZ black hole. However similar considerations apply to the rotating case. In the rotating case the relation between Poincaré and BTZ coordinates is

$$T = \pm l \left(\frac{r^2 - r_+^2}{r^2 - r_-^2} \right)^{1/2} e^{-\tilde{\phi}} \sinh \tilde{t} \qquad k = \pm l \left(\frac{r^2 - r_+^2}{r^2 - r_-^2} \right)^{1/2} e^{-\tilde{\phi}} \cosh \tilde{t}, \tag{4.97}$$

where $\tilde{t}$ and $\tilde{\phi}$ are defined in (2.14). Hence in the limit $r \to \infty$

$$T = \pm l e^{-\tilde{\phi}} \sinh \tilde{t} \qquad k = \pm l e^{-\tilde{\phi}} \cosh \tilde{t}. \tag{4.98}$$

The sign + corresponds to one exterior of the BTZbh and the sign $-$ to the other. Following the analysis for the non-rotating case we have

$$F_b(-\Delta\tilde{t}, \Delta\tilde{\phi}) = F_b(\Delta\tilde{t} - i2\pi, \Delta\tilde{\phi}) \tag{4.99}$$

if

$$(t, \phi) \to (t - i\beta, \phi + i\beta\Omega_H) \tag{4.100}$$

where $\beta = \frac{1}{\mathsf{T}}$ and Ω_H is the angular velocity of the horizon.

4.3.2 Further analysis of the thermal state

So far we have used the duality between the theories in the bulk and in the boundary. In order to do a simpler analysis we could just look at the boundary theory and try to get some conclusions.

Let us consider a massless real linear scalar free field in the boundary. The two point function in Poincaré coordinates is given by

$$\langle 0|\phi(U,V)\phi(U',V')|0\rangle = -\frac{1}{4\pi}\left[\ln(U-U'-i\epsilon)+\ln(V-V'-i\epsilon)\right], \quad (4.101)$$

where $U = T - k$ and $V = T + k$. Because the divergence of this two point function, it makes sense just after twice differentiated [54]. If we use (2.56) to express U and V in BTZ coordinates we obtain

$$U = -le^{-\kappa\tilde{u}} \qquad V = le^{\kappa\tilde{v}}, \quad (4.102)$$

where $\tilde{u} = t + \phi'$ and $\tilde{v} = t - \phi'$ with $\phi' = l\phi$. Hence in terms of $\tilde{u}$ and $\tilde{v}$, the resulting second derivative of (4.101) with respect, say, $\tilde{u}$ and $\tilde{u'}$ is

$$F(t,\phi') \equiv \frac{\partial^2}{\partial\tilde{u}\partial\tilde{u'}}\langle 0|\phi(\tilde{u},\tilde{v})\phi(\tilde{u'},\tilde{v'})|0\rangle = -\frac{1}{4\pi}\frac{k^2}{\left(e^{-\kappa\frac{\tilde{u}-\tilde{u}'}{2}} - e^{-\kappa\frac{\tilde{u}'-\tilde{u}}{2}} - i\epsilon\right)^2}, \quad (4.103)$$

which clearly satisfies the KMS condition $F(-t) = F(t - i\beta)$ [37], with $\beta = \frac{1}{\mathsf{T}} = \frac{2\pi}{\kappa}$. This implies that this twice differentiated two point function corresponds to a thermal state of the scalar field on the boundary. Now in order to obtain the covering space of the exterior of BTZ black hole we have to make the quotient procedure by making ϕ 2π-periodic. In this context we can make use of the image sum method again. Because each term in this sum satisfies the KMS condition then $F'(t,\phi')$ satisfies it too. Hence, in accordance with the previos subsection, we have a thermal state living on one exterior region the black hole.

From the previos discussion we can see that the vacuum state with respect to Poincaré time looks like a thermal state with respect to an observer moving along the integral curves of ∂_t. One might be worried about the vacuum with respect to global time, $|0_g\rangle$, so let us see how these two vacua are related. The two point function with respect to global time is

$$\langle 0_g|\hat{\phi}(\lambda,\theta)\hat{\phi}(\lambda',\theta')|0_g\rangle = -\frac{1}{4\pi}\ln\{\sin\frac{1}{2}(u-u'-i\epsilon)\sin\frac{1}{2}(v-v'-i\epsilon)\} \quad (4.104)$$

where $u = \lambda - \theta$ and $v = \lambda + \theta$. Hence

$$\frac{\partial^2\langle 0_g|\hat{\phi}(u,v)\hat{\phi}(u',v')|0_g\rangle}{\partial u\partial u'} = -\frac{1}{16\pi}\frac{1}{\sin^2\frac{1}{2}(u-u'-i\epsilon)}. \quad (4.105)$$

By using

$$T - k = l\tan\frac{\lambda-\theta}{2} \qquad T + k = l\tan\frac{\lambda+\theta}{2} \quad (4.106)$$

we get

$$\frac{\partial^2\langle 0_p|\hat{\phi}(w,z)\hat{\phi}(w',z')|0_p\rangle}{\partial u\partial u'} = -\frac{1}{16\pi}\frac{1}{\sin^2\frac{1}{2}(u-u'-i\epsilon)} \quad (4.107)$$

where $w = T - K$ and $z = T + k$. Hence global and Poincaré vacuum could be considered as the same, modulo some constant which we could be throwing away when differentiating the two point functions. Later we shall make some comments about the global and Poincaré vacuum. In figure 4.2, we show the integral curves of both vector fields, ∂_λ and ∂_T.

In passing we note that the previous analysis is in accordance with the classification of the conformal vacuum given in [13] if we identify the boundary of AdS spacetime with the Einstein universe and the Poincaré chart with Minkowski spacetime.

4.4 The dictionary

It is commonly seen in the AdS/CFT correspondence literature the phrase: the dictionary. By the dictionary people seems to mean a way to relate quantities in the bulk and quantities in the boundary of AdS spacetime. The word quantity in this context seems to be very broad. For example, we have just given the relation between a thermal state in the bulk and a thermal

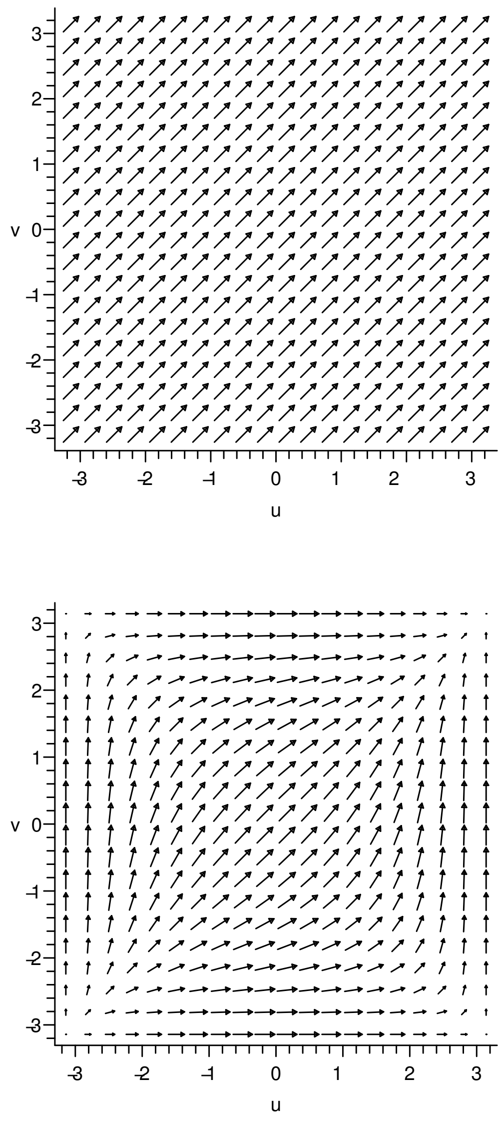


Figure 4.2: These figures show respectively the vector fields ∂_λ and ∂_T on the boundary. These figures were made with Maple 10.

state in the boundary, and as we said in the introduction a relation between the global ground state in the bulk and in the boundary of AdS spacetime has been given before in [57]. But even in the context of this work we have a relation between the mass of the field in the bulk and the scaling dimension, Δ, of the field in the boundary

$$\Delta = 1 + p, \tag{4.108}$$

where $p^2 = 1 + \tilde{m}^2$ and $\tilde{m}^2 = m^2 - \frac{6\xi}{l^2}$. It is clear that a state and a mass or scaling dimension have a different physical status, however glossing over these hierarchy of elements of the theory we can follow the fashion and give other entry of the dictionary. In [31] it has been shown how to obtain a field on the boundary of AdS spacetime from a field in its bulk and has been shown that the field on the boundary is a generalized free field. So to a field in the bulk corresponds a generalized free field in the boundary.

Another interesting entry in the dictionary could be the energy-momentum tensor in the bulk and its boundary limit. If we could know this entry then we could have an idea of how the energy and momentum in the boundary are distributed in terms of the distributions in the bulk. In quantum field theory one is not interested in the energy-momentum itself but on its expectation value on an appropriate state. In AdS/CFT in QFT we would be interested in this quantity too. It is well known that the calculation of the expectation value of the energy-momentum tensor is a very delicate problem and usually a method of regularization and renormalization is required [13]. In the present case we expect that closed expressions would result after these procedures. This is because AdS spacetime is a maximally symmetric spacetime. In the present work we do not address fully this problem although we work on it later in Chapter 6. We expect in the near future to work on it by using functional methods such as the effective action.

We conclude this section by saying that in our AdS/CFT in QFT we can have a dictionary too. It would be worth exploring more this dictionary with other models of fields.

Chapter 5

The Brick Wall model in BTZ black holes

In this chapter we shall consider the brick wall model [80] in BTZ black holes in 1+1 and 1+2 dimensions. It seems to be one of the natural extensions of the research carried so far. We will calculate states in the exterior of BTZ black holes and later map them to their boundary. We shall consider states at zero and non zero temperature. We will show that in both cases we can obtain states on the boundary of the BTZ black holes.

5.1 Thermal state with the brick wall model

So far we have been considering thermal states as arising from a restriction of a ground state. However thermal states also arise as heated up states on an appropriate Fock state representation. For example it is well known that the Hartle-Hawking state restricted to the exterior of Schwarzschild black hole is a thermal state with respect to Schwarzschild time, however the heated up Boulware state coincides with the state previously mentioned. After considering the arising of thermal states from a restriction of a ground state in AdS spacetime it seems natural and an instructive exercise to consider heated up states in a Fock representation. In this section we will do this by considering the vacuum with respect to BTZ time.

5.1.1 The brick wall model in 1+1 dimensions

The metric for a 1+1 dimensional BTZ black hole is given by (2.2). This metric can be written as

$$ds^2 = -N^2\left(dt^2 - dr^{*2}\right), \tag{5.1}$$

where $r^* = \frac{l^2}{2r_+}\ln\frac{r-r_+}{r+r_+}$. We want to solve the massless conformally coupled real linear scalar field in this geometry. Due to fact that this metric is conformally flat then we can use standard techniques from quantum field theory to solve our problem. This can be done by solving the Klein-Gordon operator with the flat metric in coordinates (t, r^*) and later mapping back the solutions to the geometry (5.1).

The coordinate r^* goes from $-\infty$ to 0, since we want to put a brick wall close to the horizon, let us say at $r_+ + \epsilon$ then $-B < r^* < 0$, where $B = -\frac{l^2}{2r_+}\ln\frac{\epsilon}{2r_+ + \epsilon}$. In this conditions our problem reduces to solve the equation

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^{*2}}\right)\phi(t, r^*) = 0 \tag{5.2}$$

with boundary conditions

$$\phi(t, -B) = 0 = \phi(t, 0). \tag{5.3}$$

If we use separation of variables and harmonic dependence in t then the solutions are

$$f_\omega(t, r^*) = \frac{1}{\sqrt{B\omega}}e^{-i\omega t}\sin(\omega r^*), \tag{5.4}$$

where

$$\omega = \frac{n\pi}{B} \qquad n = 1, 2, 3, ... \tag{5.5}$$

are the frequencies of the modes. At this point we can write the expression for the field operator

$$\hat{\phi}(t, r^*) = \sum_\omega \frac{1}{\sqrt{B\omega}}\left(\hat{a}_\omega f_\omega + \hat{a}^\dagger_\omega f^*_\omega\right), \tag{5.6}$$

with the vacuum defined by

$$\hat{a}_\omega|0\rangle = 0 \quad \forall \quad \omega. \tag{5.7}$$

5.1.2 Two point function on the boundary in 1+1 dimensions

In this subsection we obtain the two point function on the boundary for the vacuum and a thermal state.

The two point function for the vacuum is

$$\langle 0|\hat{\phi}(t,r^*)\hat{\phi}(t',r^{*\prime})|0\rangle = \sum_\omega \frac{1}{B\omega} e^{-i\omega(t-t')} \sin(\omega r^*)\sin(\omega r^{*\prime}). \tag{5.8}$$

If we multiply (5.8) by $1/r^*r^{*\prime}$ and take the limit $r^*, r^{*\prime} \to 0$, then we have

$$G(t,t') = \sum_\omega \frac{\omega}{B} e^{-i\omega(t-t')}, \tag{5.9}$$

where we have denoted by $G(t,t')$ the resulting limit. This function can be thought as a two point function on the boundary, $r^* = 0$, corresponding to the vacuum associated with the modes (5.4).

The two point function for a thermal state is defined as

$$\langle \phi(t,r^*)\phi(t,r^{*\prime})\rangle_\beta = \frac{\mathrm{Tr}\left(e^{-\beta\hat{H}}\phi(t,r^*)\phi(t,r^{*\prime})\right)}{Z}, \tag{5.10}$$

where $Z = \mathrm{Tr}e^{-\beta\hat{H}}$, $\hat{H}$ the hamiltonian of the system and $\beta = \frac{1}{\mathsf{T}}$ with T the temperature. In the present case

$$\begin{aligned}\langle \phi(t,r^*)\phi(t,r^{*\prime})\rangle_\beta \quad &= \quad \sum_\omega \frac{\sin(\omega r^*)\sin(\omega r^{*\prime})}{B\omega} \times \\ &\quad \times \left(\frac{e^{-i\omega(t-t')} + e^{i\omega(t-t')}e^{-\beta\omega}}{1-e^{-\beta\omega}}\right).\end{aligned} \tag{5.11}$$

We denote this function by $T(\Delta t; r^*, r^{*\prime})$ with $\Delta t = t - t'$. Clearly this function satisfies the KMS condition $T(-\Delta t) = T(\Delta t - i\beta)$. If we multiply

T by $1/r^* r^{*\prime}$ and take again the limit $r^*, r^{*\prime} \to 0$ then we obtain

$$T(\Delta t) = \sum_{\omega} \frac{\omega}{B} \left(\frac{e^{-i\omega(t-t')} + e^{i\omega(t-t')} e^{-\beta\omega}}{1 - e^{-\beta\omega}} \right). \tag{5.12}$$

We see that the KMS condition does not change when we take the limit, hence we can say that the thermal state maps to the boundary. Also we can see that in both cases where we have taken the limit, it just depends on the modes and the power for which we multiply the two point function.

The two point function (5.12) would correspond to a thermal state with the brick wall model and on the boundary. According to [67] the vacuum state in the brick wall model corresponds to the Boulware vacuum for Schwarzschild geometry. We would expect that the same happens in the present case and that the two point function (5.12) would correspond to the discrete limit of the two point function we found before (4.33). Now we show that this is indeed the case.

If we write (5.12) as

$$T(\Delta t) = \frac{1}{i} \frac{d}{d\Delta t} \frac{1}{B} \sum_{n \neq 0} \frac{e^{i \frac{n\pi}{B} (\Delta t + i\beta)}}{1 - e^{-\beta\omega}} \tag{5.13}$$

then we can approximate this sum by the integral

$$\frac{1}{B} \sum_{n \neq 0} \frac{e^{i \frac{n\pi}{B} (\Delta t + i\beta)}}{1 - e^{-\beta\omega}} \simeq \lim_{\epsilon_+ \to 0} \frac{1}{\pi\beta} \int_{-\infty}^{\infty} dx \frac{e^{-x(1 - \epsilon - i\frac{\Delta t}{\beta})}}{1 - e^{-x}} \tag{5.14}$$

where $x = \beta\omega$. Using 3.311 8 in [40], this integral can be evaluated and finally we obtain

$$T(\Delta t) \sim -\frac{1}{4\pi} \frac{\kappa^2}{\sinh^2(\kappa \frac{t-t'}{2})}. \tag{5.15}$$

The step (5.14) can be understood as taking $\Delta\omega = \frac{\pi}{B}$ infinitely small, which corresponds to $B \to \infty$, i.e., to removing the brick wall.

In this way we recover the expression we had obtained by restricting the global vacuum in AdS spacetime to the boundary and later to the exterior region of the BTZbh. So we have shown that the same conclusion obtained by [67] applies to the present case. The vacuum in the brick wall corresponds to the Boulware vacuum.

From the previous calculations we can consider the two point functions on the boundary (5.9) and (5.12) as two point functions for the brick wall on the boundary. In some sense these two point functions give us an idea how the near horizon geometry has an effect on the theory on the conformal boundary.

5.1.3 The brick wall model in 1+2 dimensions

In 1+2 dimensions the brick wall can be implemented too and an expression for a thermal state on the boundary can be obtained. The problem here is that the resulting expression for the frequencies involve implicitly the Gamma function and are too wild to give closed expressions, however due to the results in the 1+1 dimensional case we expect that the results can be interpreted similarly. We now give the expressions just mentioned.

In 1+2 dimensions the Klein-Gordon operator in the BTZ metric can be reduced to the hypergeometric equation if harmonic dependence in time and in the angular variable are assumed, see Appendix E. This equation has two linearly independent solutions around each regular singular point. It turns out that the three regular singular points of this equation correspond to the inner, outer horizon and infinity. If we impose vanishing boundary conditions at infinity just one of the solutions around this point satisfies this condition. This solution, for the non rotating case, is

$$f_{n\omega}(u) = M^{\alpha+\beta}(u-1)^{\alpha}u^{\beta-a}F(a, a-c+1; a-b+1, u^{-1}), \tag{5.16}$$

where $u = \frac{r^2}{r_+^2}$, $a = \alpha+\beta+\frac{1}{2}(1+\nu)$, $b = \alpha+\beta+\frac{1}{2}(1-\nu)$, $c = 2\beta+1$, $\nu^2 = 1+\widetilde{m}^2 l^2$, $\alpha^2 = -\frac{1}{4M^2}(r_+\omega)^2$ and $\beta^2 = -\frac{1}{4M^2}\left(\frac{r_+ n}{l}\right)^2$. Here $\widetilde{m}^2 = m^2 - \frac{6\xi}{l^2}$ with ξ the coupling factor in the Klein-Gordon operator and F is a hypergeometric function. Hence the positive frequency modes are

$$F_{n\omega} = \frac{1}{l}\sqrt{\frac{2}{cr_+\omega}}e^{-i\omega t}e^{in\phi}f_{n\omega}, \tag{5.17}$$

where c is a normalization constant [58] and n an integer.

In order to obtain the brick wall model we must impose vanishing boundary conditions close to the horizon, let us say at $u = 1 + \epsilon$. Using the linear relations between hypergeometric functions we can analytically continue the hypergeometric function in (5.16) to the horizon. After this continuation and close to the horizon we obtain

$$f_{n\omega} \propto e^{-i\theta}(u-1)^{-\alpha} + e^{i\theta}(u-1)^{\alpha}, \tag{5.18}$$

where $e^{2i\theta} = \frac{\Gamma(-\alpha-\beta+h_+)\Gamma(-\alpha+\beta+h_+)\Gamma(2\alpha)}{\Gamma(\alpha+\beta+h_+)\Gamma(\alpha-\beta+h_+)\Gamma(-2\alpha)}$ and $h_+ = \frac{1}{2}(1+\nu)$ with $\nu = \sqrt{1+\tilde{m}^2 l^2}$. Imposing that at the horizon this function should vanish and using that $u = 1 + \epsilon$ we have that

$$\theta(\omega, n) + \frac{\omega}{2\kappa} \ln \epsilon = \frac{2p+1}{2}\pi \tag{5.19}$$

with p an integer. At this point we can expand the field operator as we did in 1+1 dimensions and define as the vacuum the one associated with the positive modes (5.17).

5.1.4 Two point function on the boundary in 1+2 dimensions

Having solved the brick wall model now we proceed to give the two point functions for the vacuum and for a thermal state on the boundary.

The two point function for the vacuum state in the bulk is

$$\langle 0|\hat{\phi}(t,u,\phi)\hat{\phi}(t',u',\phi')|0\rangle = \sum_{n\omega} \frac{A}{\omega} e^{-i\omega(t-t')} e^{in(\phi-\phi')} f_{n\omega} f^*_{n\omega}, \tag{5.20}$$

where $A = \frac{2}{cr_+ l^2}$. If we multiply this expression by $(uu')^{\Delta/2}$ with $\Delta = 1+\nu$ and take the limit when $u, u' \to \infty$ we get

$$\langle 0|\hat{\phi}(t,\phi)\hat{\phi}(t',\phi')|0\rangle_b = \sum_{n\omega} \frac{A}{\omega} e^{-i\omega(t-t')} e^{in(\phi-\phi')}, \tag{5.21}$$

as the two point function on the boundary.

Using the definition (5.10) for the two point function of the thermal state we find that

$$\begin{aligned}\langle\phi(t,u,\phi)\phi(t,u',\phi')\rangle_\beta &= \sum_{n\omega} A\frac{f_{n\omega}f^*_{n\omega}}{\omega}e^{in(\phi-\phi')}\times \\ &\times\left(\frac{e^{-i\omega(t-t')}+e^{i\omega(t-t')}e^{-\beta\omega}}{1-e^{-\beta\omega}}\right).\end{aligned} \tag{5.22}$$

If we take the same limit as in the vacuum state we get on the boundary

$$\begin{aligned}\langle\phi(t,u,\phi)\phi(t,u',\phi')\rangle_\beta &= \sum_{n\omega} A\frac{e^{in(\phi-\phi')}}{\omega}\times \\ &\times\left(\frac{e^{-i\omega(t-t')}+e^{i\omega(t-t')}e^{-\beta\omega}}{1-e^{-\beta\omega}}\right).\end{aligned} \tag{5.23}$$

In this case we again see that the thermal state maps from the bulk, inside the exterior of the BTZbh, to the boundary. We expect that this two point function on the boundary in the limit in which the brick wall is removed should coincide with the two point function (4.95). However to proof this we have to make the sums over n and ω, and since ω depends implicitly on the Gamma function this can not be done in close form.

We just want to stress that the limits we have been taking when we go from the interior of AdS spacetime to its conformal boundary in all the cases so far studied give us two point functions on the boundary. This fact can be explained because the geometry of AdS spacetime. So in this sense the correspondence between the bulk and the boundary is just kinematical. We would expect that if we take for example the Dirac field the same limits can be taken. Also we expect to have the mapping of thermal states from the bulk to the boundary.

Other point we want to stress is the fact that by imposing the brick wall close to the horizon the thermal properties of the field are not modified, the brick wall just has the effect of making the spectrum discrete.

5.2 The entropy in the brick wall model in 1+1 dimensions

The principal motivation in [80] to introduce the brick wall model was to avoid divergences and to obtain a finite expression for the entropy of the Schwarzschild black hole. It was shown that by imposing vanishing boundary conditions on the field at a distance from the horizon of the size of the Planck length the resulting entropy is proportional to the area of the horizon. Later in [67] it was shown that if we take the entropy close to the horizon then by choosing adequately a parameter of normalization, which is proportional to the Planck length, the Bekenstein-Hawking entropy is recovered. Hence we can say that in this model the entropy of the black hole is distributed around the horizon. Let us see what happen in our model in 1+1 dimensions.

It is a fairly easy to obtain the entropy in 1+1 dimensions. The partition function for a gas of photons is

$$\ln Z(T,B) = -\sum_{r=1}^{\infty} \ln[1-\exp(-\beta E_r)], \tag{5.24}$$

where E_r is the energy of a photon and $\beta = \frac{1}{k\mathsf{T}}$. We can do the last sum as an integral by multiplying it for the correct factor $\frac{B}{\pi c}d\omega$ where B is the size of the one dimensional box. The result is

$$\ln Z(T,B) = -\int_0^{\infty} \frac{B}{\pi c} d\omega \ln[1-\exp(-\beta\omega\hbar)]. \tag{5.25}$$

In order to calculate the entropy we calculate first the free energy which is given by

$$\begin{aligned} F(T,B) &= -kT\ln Z \\ &= \frac{Bk^2T^2}{\pi c\hbar}\int_0^{\infty} dx \ln(1-e^{-x}) \\ &= -\frac{Bk^2T^2}{\pi c\hbar}\Gamma(2)\zeta(2). \end{aligned} \tag{5.26}$$

The entropy given by $S = -\left(\frac{\partial F}{\partial T}\right)_B$. The result is

$$S = \frac{Bk^2 \mathsf{T} \pi}{3\hbar c}, \tag{5.27}$$

where k is the Boltzmann's constant, $\hbar$ the Planck constant and c the speed of light. In order to obtain (5.27) we have used that $\Gamma(2) = 1$ and $\zeta(2) = \pi^2/6$. If we take $\mathsf{T} = \kappa/2\pi$ and make the constants equal to one then

$$S = -\frac{1}{12} \ln \frac{\epsilon}{2r_+ + \epsilon}. \tag{5.28}$$

In obtaining the last expression we have used that $\kappa = r_+/l^2$ and $B = -\frac{l^2}{2r_+} \ln \frac{\epsilon}{2r_+ + \epsilon}$. If we make $r_+ = 1$ then in order to have an entropy of the order of 2π we must have $\epsilon \sim 3.42 \times 10^{-33}$cm. This magnitude has the same order of the Planck length $l_p = 1.62 \times 10^{-33}$cm. However in 1+1 dimensions we do not know what the Planck length is, hence this result should be taken with care. Also we should note that in the present case the entropy is distributed over all the volume, not just outside the horizon. From this analysis we see that the brick wall model in 1+1 dimensions give also the correct entropy however differs physically from its counterpart 1+3 dimensional case[1]. By the correct entropy we mean an entropy proportional to 2π gives an ϵ similar to the Planck length, hence in this case also the brick wall model reproduces the features of the 1+3 dimensional case.

[1]For a calculation of the entropy of the 1+1 dimensional BTZ black hole by using the asymptotic symmetries of the 1+1 dimensional AdS spacetime and the conformal group in one dimension, see [20].

Chapter 6

The Energy-Momentum Tensor in the 1+1 dimensional BTZ black hole

It is well-known that the energy-momentum tensor in Quantum Field Theory in curved spacetime is a subtle issue. This is principally due to the divergences which ocurre when the expectation value of it in a certain state is calculated, see for example [13] for an extensive discussion. However, since it contains important physical information of the field it is worth trying to calculate it. It turns out that in 1+1 dimensions most of the difficulties can be removed and it is possible to obtain closed expressions for it [26]. In this chapter we will exploit this fact and will calculate this quantity for the 1+1 dimensional BTZ black hole.

6.1 The energy-momentum tensor in 1+1 dimensions

In 1+1 dimensions the energy-momentum tensor is almost determined by its trace. In what follows we give the basic formule for calculating this quantity.

Let us consider the following metric

$$ds^2 = C(-dt^2 + dx^2) = -C du dv, \tag{6.1}$$

where $u = t - x$ and $v = t + x$. Since every 1+1 dimensional metric is conformal to a 1+1 dimensional metric in Minkowski spacetime, the metric (6.1) is very general. The function C in general depends on both variables in the metric. In these circumstances the expectation value of the trace of the energy-momentum is [26]

$$< T^\mu_\mu >= -\frac{R}{24\pi} = \frac{1}{6\pi}\left(\frac{C_{uv}}{C^2} - \frac{C_u C_v}{C^3}\right), \tag{6.2}$$

where R is the Ricci scalar and $C_u = \frac{\partial}{\partial u}C$, etc. The last expression holds for the real scalar field. The expectation value of the components of the energy-momentum tensor in null coordinates is

$$\langle T_{uu}\rangle = -\frac{1}{12\pi}C^{1/2}\partial_u^2 C^{-1/2} + f(u) \tag{6.3}$$

$$\langle T_{vv}\rangle = -\frac{1}{12\pi}C^{1/2}\partial_v^2 C^{-1/2} + g(v) \tag{6.4}$$

where f and g are arbitrary functions of u and v respectively. These functions contain information about the state with respect to which the expectation value is taken. The mixed components are given by

$$\langle T_{uv}\rangle = \frac{CR}{96\pi}. \tag{6.5}$$

Now let us apply these formule to the 1+1 dimensional BTZ black hole.

6.1.1 The energy-momentum tensor in the 1+1 dimensional BTZ black hole

In the previos chapter we saw that the metric for the 1+1 dimensional BTZ black hole can be written in the form (6.1) with

$$C = N^2 = \left(-M + \frac{r^2}{l^2}\right) \tag{6.6}$$

and $r = x$. This function can be written as function of r^* or u and v as follows

$$C = \frac{M}{\sinh^2 \kappa r^*} = \frac{M}{\sinh^2 \kappa \frac{(v-u)}{2}}, \tag{6.7}$$

where $\kappa = \frac{r_+}{l^2}$ is the surface gravity.

Using the previos expression for C we obtain

$$\langle T_{uu} \rangle = -\frac{\kappa^2}{12\pi} + f(u) \tag{6.8}$$

and

$$\langle T_{vv} \rangle = -\frac{\kappa^2}{12\pi} + g(v). \tag{6.9}$$

Using that

$$T_{tt} = T_{uu} + 2T_{uv} + T_{vv}, \tag{6.10}$$

$$T_{xx} = T_{uu} - 2T_{uv} + T_{vv}, \tag{6.11}$$

and

$$T_{tx} = -T_{uu} + T_{vv} \tag{6.12}$$

we obtain

$$\langle T_{tt} \rangle = -\frac{\kappa^2}{6\pi} + \frac{CR}{48\pi} + f(u) + g(v), \tag{6.13}$$

$$\langle T_{xx} \rangle = -\frac{\kappa^2}{6\pi} - \frac{CR}{48\pi} + f(u) + g(v) \tag{6.14}$$

and

$$\langle T_{tx} \rangle = g(v) - f(u). \tag{6.15}$$

From the last three expressions it follows that

$$\langle T^{\mu}{}_{\nu}\rangle = A + B + D \tag{6.16}$$

where

$$A = \frac{\kappa^2}{6\pi C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.17}$$

$$B = -\frac{R}{48\pi}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.18}$$

$$D = \frac{1}{C}\begin{pmatrix} -f(u) - g(v) & f(u) - g(v) \\ g(v) - f(u) & f(u) + g(v) \end{pmatrix} \tag{6.19}$$

If we choose $f(u) = g(v) = 0$ we obtain the analogue of the Boulware state in Schwarzschild spacetime which is singular at the horizon ($C = 0$). However we can also obtain the analogous of the Hartle-Hawking state which is regular in both the future and the past horizons. The value of $f(u)$ and $g(v)$ can be obtained if we express the energy-momentum tensor in Kruskal like coordinates, U and V, see (2.25). In these coordinates the energy momentum tensor is

$$\langle T_{UU}\rangle = \frac{1}{U^2}\langle T_{uu}\rangle = \frac{1}{U^2}\left(\frac{f(u)}{\kappa^2} - \frac{1}{12\pi}\right), \tag{6.20}$$

$$\langle T_{VV}\rangle = \frac{1}{V^2}\langle T_{vv}\rangle = \frac{1}{V^2}\left(\frac{g(v)}{\kappa^2} - \frac{1}{12\pi}\right), \tag{6.21}$$

and

$$\langle T_{UV}\rangle = 0. \tag{6.22}$$

Hence we demand that $f(u) = g(v) = \frac{\kappa^2}{12\pi}$, although it is not the only possibility. We could choose functions which close to the horizon are have the values $\frac{\kappa^2}{12\pi}$ and a different value far from it. So f and g constants are not the only possibility. It is worth to point out that there is no natural analogue of the Unruh vacuum, since these would lead us to have no conservation of energy-momentum at infinity.

Also it is interesting to write this tensor in an orthonormal frame. This can be done by introducing two-beins. The appropriate orthonormal frame is given by

$$e^a{}_t = (N, 0) \tag{6.23}$$

and

$$e^b{}_r = (0, N), \tag{6.24}$$

where a and b are indexes associated with the orthonormal frame. In these circumstances the energy-momentum tensor is given by

$$\langle T^{ab} \rangle = E + F + G \tag{6.25}$$

where

$$E = -\frac{\kappa^2}{6\pi C} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.26}$$

$$F = \frac{R}{48\pi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.27}$$

$$G = \frac{1}{C} \begin{pmatrix} f(u) + g(v) & f(u) - g(v) \\ f(u) - g(v) & f(u) + g(v) \end{pmatrix} \tag{6.28}$$

From this expression we see that for the Hartle-Hawking state

$$\langle T^{ab} \rangle = \frac{R}{48\pi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.29}$$

Since in the present case $R = -\frac{2}{l^2} = 2\Lambda$, then the energy density and the pressure are given respectively by

$$\rho = \frac{\Lambda}{24\pi} \tag{6.30}$$

and

$$p = -\frac{\Lambda}{24\pi}. \tag{6.31}$$

Hence the cosmological constant determines the properties of the field in the 1+1 dimensional BTZ black hole.

It is also interesting to look at the semiclassical Einstein field equations. It is well known that in 1+1 dimensions the Einstein tensor vanishes identically, so there are no Einstein equations. In particular the Einstein field equations with cosmological constant in vacuum are imposible, however we now show that a kind of Einstein field equations with cosmological constant make sense when the right hand side of them is taken to be the expectation value of the energy-momentum tensor we have found.

If we write the semiclassical Einstein field equations as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \langle T_{\mu\nu} \rangle, \tag{6.32}$$

then in 1+1 dimensions the first two terms of the left hand side vanish identically and we are left with

$$\Lambda g_{\mu\nu} = \langle T_{\mu\nu} \rangle. \tag{6.33}$$

But according to our expressions for the energy-momentum tensor this equality can be satisfied if and only if

$$g_{\mu\nu} = \frac{C}{24\pi} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.34}$$

which is no other thing than the metric for the BTZ black hole in 1+1 dimensions, scaled by an overall factor and multiplied by minus one. This could be interpreted as the metric inside the horizon. Hence we could say that the back reaction shifted the horizon by making it bigger. It is interesting to note that if the sign would be opposite then there would not be change in the geometry. Since if we take this metric as the starting point for calculating the expectation value of the energy-momentum tensor we would find the same values as previously [8]. In this second scenario the BTZ metric would be stable under back reaction effects.

The previous discussion should be taken with care and just as an indication of the possible scenarios, since there is no Einstein equations in 1+1 dimensions. A more natural thing to do would be to plug in the expectation value of the energy-momentum we have found into a theory of gravity in 1+1 dimensions and see how it is modified.

Finally, if we consider the field in a box by putting the brick wall then we can calculate the expectation value of the energy-momentum tensor in a thermal state very simply. The expressions for the expectation value are the same plus a Casimir energy term [26]. Then if we want our thermal state to coincide with the Hartle-Hawking state we must have

$$\langle T^{\mu}{}_{\nu} \rangle = -\frac{R}{48\pi} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\pi}{6Cd^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.35}$$

where d is the size of the box corresponding to the brick wall conditions. The justification for obtaining the last expression is that when we remove the brick wall then the expectation values of the energy momentum tensor should be the same as the expectation value in the Hartle-Hawking state.

Chapter 7

Final Comments

In Section 4.3, in associating an algebra to the bulk of AdS spacetime we took the algebra associated with the Poincaré chart, however one can ask why not take the algebra associated to the global chart. Even one can think that the algebra associated to the global chart is the most natural since it covers all the spacetime. This property seems to be enough to give priority to the global chart over the Poincaré chart, however as we said the Poincaré chart is likely to be a wedge region and when seen from the boundary it is the natural candidate to work with. Also it is worth remarking that when we pass from the global chart to the Poincaré chart we are restricting the global ground state to a state on the Poincaré algebra. In this conditions the issue of thermalization of this state by restricting it to a subalgebra arises if we have in mind the Minkowski-Rindler story. However we claim this is not the case. As we saw in the last section by doing a little calculation we show that the two times differentiated two point functions coincide in both charts on the boundary, and then the ground states can differ at most by a constant. Now from the point of view of the bulk theory although the global chart and the Poincaré chart do not cover the same region they both share a common spacelike surface, $T = \lambda = 0$, hence the analogy to the Minkowski-Rindler case is not complete. Also related to this point is that the horizon defined by the Poincaré chart is not a bifurcate killing horizon as in the Minkowski-Rindler story. It would be worth while to study how the GNS representations of the global ground state and the Poincaré ground state are related. Our conjecture is that they are not disjoint and that the global ground state is not thermal when restricted to the Poincaré chart. Finally, we note that we have to do more work in order to construct rigorously the algebras we are talking about for example as was done in [28] for the real linear scalar field in a globally hyperbolic spacetime. The problem we have in our case is that AdS spacetime is not globally hyperbolic and there are not rigorous proofs on the existence of fundamental solutions to the Klein-Gordon operator. However in this work we have shown that by taking vanishing boundary conditions at infinity it is possible to construct an inner product conserved in time. Hence it would be worth while to explore this construction more rigorously.

Chapter 8

Conclusions and Perspectives

In this work we have presented the research we carried out on quantum field theory on AdS spacetime and on BTZ black holes in 1+1 and 1+2 dimensions. Also we presented how thermal states in the bulk of AdS spacetime are related to thermal states of a conformal field theory on the conformal boundary of AdS spacetime and of the BTZ black holes. In order to do this we used methods and ideas of Algebraic Holography, the Boundary-limit Holography and Pre-Holography. We called AdS/CFT in QFT these three approaches to the AdS/CFT correspondence. This part of the work fits naturally in the subject of AdS/CFT correspondence where just methods of quantum field theory are used.

We also studied the brick wall model in 1+1 and 1+2 dimensional BTZ black holes, and obtained two point functions in the bulk and in the boundary in both cases. This was done by using the Boundary-limit Holography. The 1+1 case is illustrative since it allowed us to obtain expressions which are interpreted easily.

While studying the brick wall model we came across the renormalized energy-momentum tensor. We studied it in the 1+1 dimensional BTZ black hole and obtained closed expressions for it. This was possible because in 1+1 dimensions the conformal anomaly fixes its form. In the Hartle-Hawking state the properties of the field are given in terms of the cosmological constant.

In the appendixes we presented complementary material to the body of the work.

A summary of the conclusions we have obtained is given now:

- There exist thermal states in the boundary of AdS spacetime in 1+1 and 1+2 dimensions.
- These thermal states result as a restriction of a ground state in the boundary. In 1+1 dimensions this ground state corresponds to the global and the Poincaré ground state. In 1+2 dimensions this ground state corresponds to the Poincaré ground state.
- The thermal properties of these states rely on the geometrical properties of AdS spacetime. More precisely on the existence of its killing vectors.
- The Hawking effect for the eternal BTZ black hole is a consequence of the Unruh effect on the boundary of AdS spacetime, since after making the identifications in ϕ the Unruh effect becomes the Hawking effect.
- Algebraic Holography, the Boundary-limit Holography and Witten's Holography are consistent when one studies a thermal state on the BTZ black hole.
- There are interesting aspects of AdS/CFT in QFT which deserve to be studied and reflects the geometrical and theoretical aspects of AdS spacetime, and perhaps of the AdS/CFT correspondence in string theory.
- The brick wall model gives a correct approximation for the BTZ black hole in 1+1 dimensions, however differs in the physical interpretation from its analogous 1+3 dimensional case. This is because in the 1+1 dimensional case the entropy is distributed in all the volume whereas in the 1+3 dimensional case the contribution to the entropy come from the entropy around the horizon and not from all the volume.

- There is no natural notion of Unruh vacuum on the 1+1 dimensional BTZ black hole. Also the properties of the field are fixed by the cosmological constant.

- There is no superradiance in the BTZ black hole in 1+2 dimensions for vanishing boundary conditions at infinity.

In general we can say that AdS/CFT in QFT makes sense as a theory where we ignore supersymmetry and on which the geometrical and theoretical aspects are fundamental. Our point of view is that AdS/CFT correspondence in the context of string theory is much more richer and complicated but in certain limit its predictions coincide with the predictions of AdS/CFT in QFT. It is worth emphasizing that the Boundary-limit Holography is supposed to be valid for interacting fields not just free fields which have been treated in this work.

As a future work we would like to explore more AdS/CFT in QFT but with more realistic models. For instance, we can work with supersymmetrical models in AdS spacetime in 1+1 and 1+2 dimensions and in BTZ black holes. Also it would be worth considering interacting fields in these spacetimes too. Other natural direction would be to generalize the present work to higher dimensions.

Related with these issues it would be worth exploring how the energy-momentum tensor around the horizon in the BTZ black holes enters in the theory in the boundary. In order to carry this out it would be interesting to use functional methods such as the effective action.

Appendix A

The Unruh effect in 1+1 dimensions

The Unruh effect can informally be stated as: a constant accelerated observer in Minkowski spacetime sees a thermal state that with respect to an inertial observer is a state at zero temperature. The existence of this effect can be shown by using the mode expansion of the field and the creation and anihilation operators, this can be seen, for example, in [13]. However there is another way to show it by using the expressions for the two point function, see [54]. In the sequel, we will follow the second way.

The Klein-Gordon equation in 1+1 dimensions on a flat background reduces to

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\right)\phi(t,x)=0 \tag{A.1}$$

which is nothing more than the wave equation in 1+1 dimensions. Defining null coordinates $u=t-x$ and $v=t+x$ we have

$$\frac{\partial^2}{\partial u \partial v}\phi(u,v)=0. \tag{A.2}$$

Hence a solution of this equation has the form

$$\phi(u,v)=F(u)+G(v)=F(t-x)+G(t+x). \tag{A.3}$$

It is clear that $F_\omega(t-x) = \exp(-i\omega(t-x))$ and $G_\omega(t+x) = \exp(-i\omega(t+x))$ are solutions of the wave equation. These solutions are called mode solutions and satisfy

$$\frac{\partial}{\partial t} F_\omega = -i\omega F_\omega, \tag{A.4}$$

similarly for G_ω. This equation shows that the mode solutions are eigenfunctions of the Lie derivative operator associated with the Killing vector ∂_t. The parameter ω is interpreted as the frequency of the mode and the mode functions are called positive with respect to t when they satisfy (A.4) with ω positive.

The quantum field operator can be expressed in terms of positive modes as

$$\begin{aligned}\phi(u,v) &= \frac{1}{(2\pi)^{1/2}}\int_0^\infty \frac{d\omega}{(2\omega)^{1/2}}\left[a_{u\omega}e^{-i\omega u} + a_{u\omega}{}^\dagger e^{i\omega u}\right] \\ &+ \frac{1}{(2\pi)^{1/2}}\int_0^\infty \frac{d\omega}{(2\omega)^{1/2}}\left[a_{v\omega}e^{-i\omega v} + a_{v\omega}{}^\dagger e^{i\omega v}\right]\end{aligned} \tag{A.5}$$

where † means the adjoint operation[1]. This expression can be found in many standard texts, for instance [66], where a detailed deduction of this expression is given by using the Hamiltonian formalism. The operators $a_{u\omega}$ and $a_{u\omega}{}^\dagger$ are the standard anihilation and creation operators respectively. The two point function is given by

$$\langle 0|\phi(u,v)\phi(u',v')|0\rangle = \frac{1}{2\pi}\left[\int_0^\infty \frac{d\omega}{2\omega}e^{-i\omega(u-u')} + \int_0^\infty \frac{d\omega}{2\omega}e^{-i\omega(v-v')}\right]. \tag{A.6}$$

In obtaining (A.6) we have used the definition of the anihilation operator $a_\omega|0\rangle = 0$ and $\left[a_\omega, a_{\omega'}{}^\dagger\right] = \delta_{\omega\omega'}$. In order to make the integration, let us take just one factor inside the square brackets. Defining $z \equiv u - u'$, we have the function

$$f(z) = \int_0^\infty \frac{d\omega}{2\omega}e^{-i\omega z}, \tag{A.7}$$

[1]It is standard to represent an operator with a letter with a hat on it, for instance $\hat{A}$, however for sake of simplicity in the notation we shall not use the hat.

then

$$\frac{\partial f}{\partial z} = -\frac{i}{2}\int_0^\infty d\omega e^{-i\omega z}. \tag{A.8}$$

If we write the last expression as

$$\frac{\partial f}{\partial z} = -\frac{i}{2}\int_0^\infty d\omega \lim_{\epsilon\to 0} e^{-i\omega(z-i\epsilon)} = -\frac{i}{2}\lim_{\epsilon\to 0}\int_0^\infty d\omega e^{-i\omega(z-i\epsilon)}, \tag{A.9}$$

we obtain

$$\frac{\partial f}{\partial z} = -\frac{1}{2}\lim_{\epsilon\to 0}\frac{1}{z-i\epsilon}. \tag{A.10}$$

Consequently

$$f(z) = -\frac{1}{2}\int \lim_{\epsilon\to 0}\frac{1}{z-i\epsilon} = -\frac{1}{2}\lim_{\epsilon\to 0}\ln(z-i\epsilon) + A, \tag{A.11}$$

where A is a constant of integration. Using (A.11) in (A.6) we finally obtain

$$\begin{aligned}\langle 0|\phi(u,v)\phi(u',v')|0\rangle &= -\frac{1}{4\pi}\left[\lim_{\epsilon\to 0}\ln(u-u'-i\epsilon)+\right.\\ &+ \left.\lim_{\epsilon'\to 0}\ln(v-v'-i\epsilon')\right], \end{aligned}\tag{A.12}$$

where we have omitted the constants of integration. It is interesting to note that the right (u) and left (v) modes do not interact between themselves, they behave as independent degrees of freedom of the system. Another important point about this two point function is that it is defined with respect to vacuum $|0\rangle$ which corresponds to the vacuum an inertial observer with proper time t sees. Actually, this vacuum will correspond also to an inertial observer related to the first by a Lorentz transformation, invariance of the vacuum state under Lorentz transformations.

The two point function (A.12) corresponds to a quantum field at zero temperature. Now, let us calculate the two point function of a quantum field at temperature different from zero. The state of this field can be described in the Fock space by

$$|n\rangle \equiv |n_1, n_2, ...\rangle = \left[\prod_s \frac{\left(a^\dagger_{\omega_s}\right)^{n_s}}{\sqrt{n_s!}}\right]|0\rangle. \tag{A.13}$$

This vector corresponds to the quantum state in which n_1 particles have frequency ω_1, n_2 particles have frequency ω_2, etc. After a bit long calculation we obtained

$$\frac{Tr\left(e^{-\beta H}\phi(u,v)\phi'(u',v')\right)}{Tr\left(e^{\beta H}\right)} = \int_0^\infty \frac{d\omega}{1-e^{-\beta\omega}} \frac{\left[e^{-i\omega(u-u')} + e^{i\omega((u-u')+i\beta)}\right]}{4\pi\omega} + \int_0^\infty \frac{d\omega}{1-e^{-\beta\omega}} \frac{\left[e^{-i\omega(v-v')} + e^{i\omega((v-v')+i\beta)}\right]}{4\pi\omega} \tag{A.14}$$

where $\beta = \frac{1}{\mathsf{T}}$ and T is the temperature. We shall show that this two point function corresponds to a thermal state, which is a state with finite temperature. A thermal state must satisfy [37]

$$F(t - i\beta) = F(-t), \tag{A.15}$$

where $F(t)$ is the two point function of the field for constant spatial coordinates. Let us consider the u-part of (A.14). If we keep x fixed and define $\tau = t - t'$ we have

$$F(\tau) = \int_0^\infty \frac{d\omega}{4\pi\omega} \frac{1}{1-e^{-\beta\omega}} \left[e^{-i\omega\tau} + e^{i\omega(\tau+i\beta)}\right]. \tag{A.16}$$

By making $\tau \to -\tau$ in (A.16) we have

$$F(-\tau) = \int_0^\infty \frac{d\omega}{4\pi\omega} \frac{1}{1-e^{-\beta\omega}} \left[e^{i\omega\tau} + e^{-i\omega(\tau-i\beta)}\right]. \tag{A.17}$$

Now, by making $\tau \to \tau - i\beta$ in (A.16) we have

$$F(\tau - i\beta) = \int_0^\infty \frac{d\omega}{4\pi\omega} \frac{1}{1-e^{-\beta\omega}} \left[e^{-i\omega(\tau-i\beta)} + e^{i\omega\tau}\right]. \tag{A.18}$$

Similarly for the v-part. Hence the two point function (A.14) corresponds to a thermal state at temperature $\beta = \frac{1}{\mathsf{T}}$. Let us calculate the integral (A.14). Let us just take the u-part. Defining $z = u - u'$ we have the integral

$$f(z) = \int_0^\infty \frac{d\omega}{4\pi\omega} \frac{1}{1-e^{-\beta\omega}} \left[e^{-i\omega z} + e^{i\omega(z+i\beta)}\right]. \tag{A.19}$$

Then

$$\frac{\partial f}{\partial z} = \frac{i}{4\pi}\int_0^\infty d\omega \frac{1}{1-e^{-\beta\omega}}\left[-e^{-i\omega z} + e^{i\omega(z+i\beta)}\right]. \tag{A.20}$$

After some algebra this integral can be written as

$$\frac{\partial f}{\partial z} = \frac{i}{4\pi}\int_{-\infty}^\infty d\omega \frac{e^{-\beta\omega}e^{i\omega z}}{1-e^{-\beta\omega}}. \tag{A.21}$$

If we define $y \equiv \beta\omega$ the we have

$$\frac{\partial f}{\partial z} = \frac{i}{4\pi}\frac{1}{\beta}\int_{-\infty}^\infty dy \frac{e^{-y(1-\frac{iz}{\beta})}}{1-e^{-y}} = \frac{i}{4\pi}\frac{1}{\beta}\int_{-\infty}^\infty dy \lim_{\epsilon\to 0}\frac{e^{-y(1-\epsilon-\frac{iz}{\beta})}}{1-e^{-y}}, \tag{A.22}$$

where in the last equality we have introduced an epsilon in order to use a known integral. Now, we have (3.311 8 in [40])

$$\int_{-\infty}^\infty dx \frac{e^{-\mu x}}{1-e^{-x}} = \pi\cot\pi\mu \qquad 0 < \mathrm{Re}\mu < 1. \tag{A.23}$$

Hence from (A.22) and (A.23) we obtain

$$\frac{\partial f}{\partial z} = \frac{i}{4\beta}\lim_{\epsilon\to 0}\cot\pi\left(1-\epsilon-\frac{iz}{\beta}\right) = \frac{i}{4\beta}\cot\pi\left(1-\frac{iz}{\beta}\right). \tag{A.24}$$

Finally

$$f(z) = -\frac{1}{4\pi}\ln\sin\pi\left(1-\frac{iz}{\beta}\right) + D, \tag{A.25}$$

where D is a constant of integration. After a bit of algebra and in terms of u and u' we obtain

$$f(u,u') = -\frac{1}{4\pi}\left[\ln\left(e^{\pi\frac{u-u'}{\beta}} - e^{-\pi\frac{u-u'}{\beta}}\right) + \ln\frac{i}{2}\right] + D. \tag{A.26}$$

If $\beta = 2\pi = \frac{1}{\mathsf{T}}$ we have

$$f(u,u') = -\frac{1}{4\pi}\left[\ln\left(e^{\frac{u-u'}{2}} - e^{-\frac{u-u'}{2}}\right) + \ln\frac{i}{2}\right] + D. \tag{A.27}$$

Hence the two point function (A.14) for $\beta = 2\pi$, modulo constants, is given by

$$\frac{Tr\left(e^{-\beta H}\phi(u,v)\phi'(u',v')\right)}{Tr\left(e^{\beta H}\right)} = -\frac{1}{4\pi}\left[\ln\left(e^{\frac{u-u'}{2}} - e^{-\frac{u-u'}{2}}\right) + \ln\left(e^{\frac{v-v'}{2}} - e^{-\frac{v-v'}{2}}\right)\right] \tag{A.28}$$

and corresponds to a thermal state at temperature $\mathsf{T} = 1/2\pi$. For our purposes we will need the second derivatives of this function, for instance

$$\frac{\partial^2\langle n|\phi(u,v)\phi(u',v')|n\rangle}{\partial v\partial v'} = -\frac{1}{4\pi}\frac{1}{\left(e^{\frac{v-v'}{2}} - e^{-\frac{v-v'}{2}}\right)^2}. \tag{A.29}$$

It is well known, see for instance [66], that in the right wedge of Minkowski spacetime we can introduce Rindler coordinates which correspond to a constant accelerated observer, see figure A.1. Let us call these coordinates (η, ξ). The relation between Minkowski and Rindler coordinates is given by

$$t = a^{-1}e^{a\xi}\sinh a\eta \qquad x = a^{-1}e^{a\xi}\cosh a\eta, \tag{A.30}$$

where a is the magnitude of the acceleration of the Rindler observer. The null coordinates are given by

$$u = -e^{-\widetilde{u}} \qquad v = e^{\widetilde{v}}, \tag{A.31}$$

where we have made $a = 1$ and $\widetilde{u} = \eta - \xi$, $\widetilde{v} = \eta + \xi$ are null with respect to Rindler coordinates. If we take the v-part of (A.12) and substitute (A.31) in it we obtain

$$\frac{\partial^2\langle 0|\phi(\widetilde{u},\widetilde{v})\phi(\widetilde{u}',\widetilde{v}')|0\rangle}{\partial\widetilde{v}\partial\widetilde{v}'} = -\frac{1}{4\pi}\frac{1}{\left(e^{\frac{\widetilde{v}-\widetilde{v}'}{2}} - e^{-\frac{\widetilde{v}-\widetilde{v}'}{2}}\right)^2}, \tag{A.32}$$

where we have taken the limit $\epsilon \longrightarrow 0$. Hence the second derivatives of the two point function $\langle 0|\phi(\widetilde{u},\widetilde{v})\phi(\widetilde{u}',\widetilde{v}')|0\rangle$ when expressed in Rindler coordinates have the same form as the second derivatives of the two point function of a thermal state. In conclusion, a constant accelerated observer sees a thermal

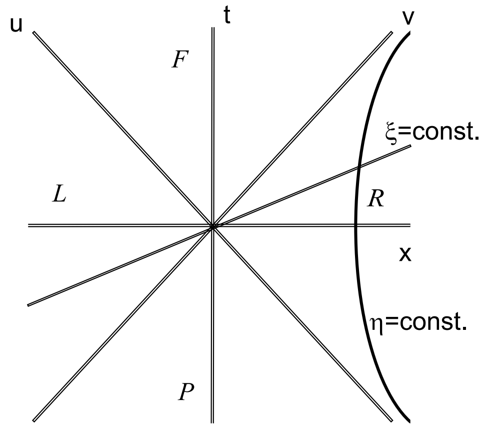


Figure A.1: 1+1 dimensional Minkowski and Rindler spacetime.

state whereas an inertial observer sees a state with zero temperature. This is the so-called Unruh effect [83][2].

As a historical curiosity it is interesting to note that the Unruh effect was discovered in an attempt to understand better the Hawking effect [83], however the Bisognano-Wichmann theorem was already available when Unruh made his discovery.

[2]An excellent review of this effect has recently appeared in [25].

Appendix B

Global conformal transformations in 1+1 dimensions

In Section 5, we found explicitly the subgroups of the conformal group in 1+1 dimensions induced by the generators J_{uy} and J_{vx}. In this appendix we give the others subgroups of this group.

First let us remember the elements of the Lie algebra of the AdS group. These elements are

$$\begin{aligned} J_{uv} &= u\partial_v - v\partial_u & J_{xy} &= x\partial_y - y\partial_x \\ J_{ux} &= u\partial_x + x\partial_u & J_{uy} &= u\partial_y + y\partial_u \\ J_{vx} &= v\partial_x + x\partial_v & J_{vy} &= v\partial_y + y\partial_v \end{aligned} \tag{B.1}$$

It is well known that in a representation on the vector space $\mathbb{R}^{2+2}$ these generators can be represented by the matrices

$$J_{uv} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad J_{xy} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{B.2}$$

$$J_{ux} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad J_{uy} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{B.3}$$

$$J_{vx} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad J_{vy} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{B.4}$$

For our purposes the relevant elements of the Lie algebra of the AdS group are

$$A = J_{ux} - J_{uv} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{B.5}$$

$$B = J_{xy} + J_{vy} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \tag{B.6}$$

$$C = J_{uv} + J_{ux} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{B.7}$$

$$D = J_{xy} - J_{vy} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \tag{B.8}$$

Let $\mathbf{a} = (a, b)$ be a two dimensional vector. Then

$$\Lambda(\mathbf{a}) = e^{aA+bB} = \begin{pmatrix} 1 & a & a & 0 \\ -a & 1+\frac{b^2-a^2}{2} & \frac{b^2-a^2}{2} & b \\ a & \frac{a^2-b^2}{2} & 1+\frac{a^2-b^2}{2} & -b \\ 0 & b & b & 1 \end{pmatrix}. \tag{B.9}$$

If we apply this transformation to $x^T = (u, v, x, y)$ we get

$$\begin{pmatrix} u' \\ v' \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} u + av + ax \\ -au + \left(1+\frac{b^2-a^2}{2}\right) v + \left(\frac{b^2-a^2}{2}\right) x + by \\ au + \left(\frac{a^2-b^2}{2}\right) v + \left(1+\frac{a^2-b^2}{2}\right) x - by \\ bv + vx + y \end{pmatrix}. \tag{B.10}$$

Using (4.54) we finally get

$$\xi'^1 = \xi^1 + a \qquad \xi'^2 = \xi^2 + b. \tag{B.11}$$

Hence $\Lambda(\mathbf{a})$ generate the translation subgroup on ξ^1 and ξ^2. The expressions (B.11) is true at all orders since it is the finite transformation generated by the linear combination of (B.5) and (B.6).

The special conformal transformations can be obtained in a similar way. Let $\mathbf{c} = (c, d)$ be a two dimensional vector. Then

$$\Lambda(\mathbf{c}) = e^{cC+dD} = \begin{pmatrix} 1 & -c & c & 0 \\ c & 1+\frac{d^2-c^2}{2} & \frac{c^2-d^2}{2} & -d \\ c & \frac{d^2-c^2}{2} & 1+\frac{c^2-d^2}{2} & -d \\ 0 & -d & d & 1 \end{pmatrix}. \tag{B.12}$$

Applying this transformation to $x^T = (u, v, x, y)$ we get

$$\begin{pmatrix} u' \\ v' \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} u - cv + cx \\ cu + \left(1 + \frac{d^2 - c^2}{2}\right) v + \left(\frac{c^2 - d^2}{2}\right) x - dy \\ cu + \left(\frac{d^2 - c^2}{2}\right) v + \left(1 + \frac{c^2 - d^2}{2}\right) x - dy \\ -dv + dx + y \end{pmatrix}. \tag{B.13}$$

Using again (4.54) we get

$$\xi'^1 = \frac{\xi^1 - c\,(\xi \cdot \xi)}{1 - 2\,(\xi \cdot \mathbf{c}) + (\mathbf{c} \cdot \mathbf{c})\,(\xi \cdot \xi)} \qquad \xi'^2 = \frac{\xi^2 - d\,(\xi \cdot \xi)}{1 - 2\,(\xi \cdot \mathbf{c}) + (\mathbf{c} \cdot \mathbf{c})\,(\xi \cdot \xi)},$$

where the inner product is with respect to the metric diag $= (-1, 1)$. Hence $\Lambda(\mathbf{c})$ generate the special conformal subgroup on ξ^1 and ξ^2.

For completeness let us write down the relationship between the metric expressed in ξ^1, ξ^2, and in global and Poincaré coordinates. On the boundary

$$\xi^1 = \frac{\sin\lambda}{\cos\lambda + \cos\theta} \qquad \xi^2 = \frac{\sin\theta}{\cos\lambda + \cos\theta}. \tag{B.14}$$

Hence

$$-d\xi^{1\,2} + d\xi^{2\,2} = \frac{1}{(\cos\lambda + \cos\theta)^2}\left(-d\lambda^2 + d\theta^2\right). \tag{B.15}$$

Also we have

$$\xi^1 = \frac{T}{l} \qquad \xi^2 = \frac{k}{l}. \tag{B.16}$$

Then

$$-d\xi^{1\,2} + d\xi^{2\,2} = \frac{1}{l^2}\left(-dT^2 + dk^2\right). \tag{B.17}$$

Before finishing this section let us see how are related the three generators of translation in time on the boundary, ∂_λ, ∂_T and ∂_{ξ^1}. We have chosen

$$\partial_\lambda = J_{vu}. \tag{B.18}$$

This can be seen from the expression for J_{vu} and the parametrization of AdS

(1.22). By using (B.5) and (B.7) we obtain that

$$\partial_\lambda = \frac{1}{2}\left(\partial_{\xi^1} - S_{\xi^1}\right), \tag{B.19}$$

where S_{ξ^1} denotes the generator of special conformal transformations in time (B.7). The last expression is obtained by solving for J_{vu} from (B.5) and (B.7) and taking into account that the first generates translations in time and the second special conformal transformation in time. We also have

$$\partial_T = \frac{1}{l}\partial_{\xi^1}. \tag{B.20}$$

B.1 Conformal field theory on the complex plane

In this section we will transform the coordinates ξ^1 and ξ^2 to complex coordinates on $\mathbb{C}^2$. From (B.14) we have

$$\xi^1 + \xi^2 = \tan\left(\frac{\lambda+\theta}{2}\right) \qquad \xi^1 - \xi^2 = \tan\left(\frac{\lambda-\theta}{2}\right). \tag{B.21}$$

If we define

$$z_1 \equiv \frac{1 + i(\xi^1 + \xi^2)}{1 - i(\xi^1 + \xi^2)} \qquad z_2 \equiv \frac{1 + i(\xi^1 - \xi^2)}{1 - i(\xi^1 - \xi^2)}, \tag{B.22}$$

then

$$z_1 = e^{i(\lambda-\theta)} \qquad z_2 = e^{i(\lambda+\theta)}. \tag{B.23}$$

If now we make $\tau = i\lambda$ then

$$z_1 = e^{-\tau}e^{-i\theta} \qquad z_2 = e^{-\tau}e^{i\theta}. \tag{B.24}$$

We can consider z_1 and z_2 as complex conjugate of each other and defined on the complex plane. However following the usual approach to Conformal Field Theory they can be considered as independent complex variables and define $\mathbb{C}^2$. At this point we could apply the standard Conformal Field Theory to our problem by using z_1 and z_2 as our complex variables.

B.2 Commutators for the Lie generators

For completeness we now give the Lie brackets for the Lie generators (B.1). These commutators are

$$[J_{uv}, J_{xy}] = 0 \qquad [J_{ux}, J_{uy}] = -J_{xy} \qquad [J_{uv}, J_{uy}] = J_{vy} \tag{B.25}$$

$$[J_{uv}, J_{vy}] = -J_{uy} \qquad [J_{ux}, J_{xy}] = -J_{uy} \qquad [J_{ux}, J_{vy}] = 0 \tag{B.26}$$

$$[J_{vx}, J_{xy}] = -J_{vy} \qquad [J_{vx}, J_{uy}] = 0 \qquad [J_{vx}, J_{vy}] = -J_{xy} \tag{B.27}$$

$$[J_{uv}, J_{ux}] = J_{vx} \qquad [J_{uv}, J_{vx}] = -J_{ux} \qquad [J_{ux}, J_{vx}] = -J_{uv} \tag{B.28}$$

$$[J_{xy}, J_{uy}] = -J_{ux} \qquad [J_{xy}, J_{vy}] = -J_{vx} \qquad [J_{uy}, J_{vy}] = J_{uv} \tag{B.29}$$

Appendix C

Misner space on the boundary

We shall show there is a Misner space on the boundary of AdS spacetime.

Let us parametrize a region of AdS spacetime as

$$\begin{aligned} v &= \sqrt{B(r)}\sinh\frac{r_+}{l^2}t \qquad & u &= \sqrt{A(r)}\cosh\frac{r_+}{l}\phi \\ x &= -\sqrt{B(r)}\cosh\frac{r_+}{l^2}t \qquad & y &= \sqrt{A(r)}\sinh\frac{r_+}{l}\phi \end{aligned} \tag{C.1}$$

where $A(r)$ and $B(r)$ are given as in (2.13) with $r_- = 0$. From (1.31) and (C.1) we get

$$T = le^{\frac{r_+}{l^2}t}\cosh\left(\frac{r_+}{l}\phi\right) \qquad k = le^{\frac{r_+}{l^2}t}\sinh\left(\frac{r_+}{l}\phi\right). \tag{C.2}$$

From (C.2) it follows that the metric on the boundary in Poincaré and BTZ coordinates is given by

$$ds^2 = -dT^2 + dk^2 = \frac{r_+^2}{l^2}e^{\frac{2r_+t}{l^2}}\left(-dt^2 + l^2 d\phi^2\right), \tag{C.3}$$

where we have introduced the surface gravity $\kappa = r_+/l^2$. The right side of this equation can be considered as the metric on the boundary of this region of AdS spacetime, which we will refer to as ds_M^2. Putting $y = e^{2\kappa t}$ and after rescaling it by $\frac{1}{4l^2\kappa^2}$ we get

$$ds_M^2 = l^4k^2\left(-y^{-1}dy^2 + yd\alpha^2\right) \tag{C.4}$$

where $\alpha = 2l\kappa\phi$. When α is 2π-periodic we have the Misner space metric [46]. It is important to note that the manifold and the Hausdorff property are still preserved after making the quotient [46] when $0 < y < \infty$.

Now we will show there is a thermal state with respect to t. First let us relate the M metric to a Rindler type metric. This can be done as follows

$$ds^2_M = e^{2\kappa t}\left(-dt^2 + d\phi'^2\right) = e^{2\kappa(t-\phi')} ds^2_{Rindler}, \tag{C.5}$$

where

$$ds^2_{Rindler} = e^{2\kappa\phi'}\left(-dt^2 + d\phi'^2\right) \tag{C.6}$$

and $\phi' = l\phi$. By defining

$$x_0 = \frac{1}{\kappa} e^{\kappa t}\cosh\kappa\phi' \qquad x_1 = \frac{1}{\kappa} e^{\kappa t}\sinh\kappa\phi' \tag{C.7}$$

we obtain

$$ds^2_M = -dx_0^2 + dx_1^2. \tag{C.8}$$

Similarly, by defining

$$x'_0 = \frac{1}{\kappa} e^{\kappa\phi'}\sinh\kappa t \qquad x'_1 = \frac{1}{\kappa} e^{\kappa\phi'}\cosh\kappa t \tag{C.9}$$

we obtain

$$ds^2_{Rindler} = -dx'^2_0 + dx'^2_1. \tag{C.10}$$

Hence if we consider x_μ and x'_μ with $\mu = 0, 1$ as charts covering 1+1 dimensional Minkowski spacetime we see that the M chart covers, F, the inside of the future light cone from the origin whereas the Rindler chart covers, as should be, the right wedge, R.

Now, ∂_t is a Killing vector of (C.6) and a conformal Killing vector [84] of (C.5) since it satisfies

$$\mathcal{L}_{(\partial_t)^\alpha} g_{\mu\nu} = 2\kappa g_{\mu\nu}, \tag{C.11}$$

where $\mathcal{L}_{(\partial_t)^\alpha}$ is the Lie derivative with respect to $(\partial_t)^\alpha$. This can be checked by using the Christoffel symbols $\Gamma^t_{tt} = \Gamma^\phi_{\phi t} = \kappa$ associated with the metric (C.5).

At this point we can use standard techniques [13] and write down the mode expansion for a field living on M spacetime

$$\phi(t,\phi') = \sum_k \left(a_k u_k + a_k^\dagger u_k^*\right), \tag{C.12}$$

where the normalized modes are given by

$$u_k(t,\phi') = \frac{1}{(4\omega\pi)^{\frac{1}{2}}} e^{-ik\cdot x} \qquad \omega = |k_R| \quad -\infty < k_R < \infty \tag{C.13}$$

and $x^\mu = (t,\phi')$. There is no factor before the sum because in general this factor should be $\Omega^{(n-2)/2}$ which is one in this case. The vacuum defined by

$$a_k|0_M\rangle = 0 \quad \forall \quad k \tag{C.14}$$

is the conformal vacuum which in this case coincides with the Rindler vacuum. Hence the vacuum with respect to positive Poincaré modes will be a thermal state with respect to positive modes with respect to t. The temperature of the state will be $\mathsf{T} = \frac{\kappa}{2\pi}$ too [13].

Appendix D

Conformal vacuum on AdS spacetime

As an exercise, in this appendix we will calculate the two point function associated with the conformal vacuum on AdS spacetime by starting with a set of normalized solutions of the Klein-Gordon operator with metric

$$ds^2 = -dT^2 + dk^2 + dz^2, \tag{D.1}$$

where the coordinates have the same range as Poincaré coordinates. One set of such solutions is

$$F'_{\omega na}(T, k, z) = \frac{1}{(2\pi\omega)^{1/2}} e^{-i\omega T} e^{ink} \sin(az), \tag{D.2}$$

where $\omega = \left(n^2 + a^2\right)^{1/2}$, $-\infty < n < \infty$ and $a > 0$. By choosing the coupling constant $\xi = \frac{1}{8}$ then a set of solutions of the Klein-Gordon operator on AdS spacetime in Poincaré coordinates is

$$F_{\omega na}(T, k, z) = \frac{l}{z^{1/2}} \frac{1}{(2\pi\omega)^{1/2}} e^{-i\omega T} e^{ink} \sin(az). \tag{D.3}$$

Then a real linear scalar field on AdS spacetime can be expanded as

$$\hat{\phi}(T, k, z) = \int_{-\infty}^{\infty} \int_{0}^{\infty} \left(F_{na}\hat{a}_{na} + F^*_{na}\hat{a}^\dagger_{na}\right) dnda, \tag{D.4}$$

where the conformal vacuum is defined by

$$\hat{a}_{na}|0\rangle = 0 \quad \forall \quad n, a. \tag{D.5}$$

From (D.4) and (D.5) it follows the two point function is given by

$$\begin{aligned}\langle 0|\hat{\phi}(T,k,z)\hat{\phi}(T',k',z')|0\rangle &= \frac{l^2}{\pi(zz')^{1/2}}\int_0^\infty\int_0^\infty \frac{dadn}{(a^2+n^2)^{1/2}}\times \\ &\times \sin(az)\sin(az')\cos(n\Delta k)e^{-i(a^2+n^2)^{1/2}\Delta T},\end{aligned} \tag{D.6}$$

where $\Delta T = T - T'$ and $\Delta k = k - k'$. After doing the integral over n we obtain

$$\begin{aligned}\langle 0|\hat{\phi}(T,k,z)\hat{\phi}(T',k',z')|0\rangle &= \frac{l^2}{(zz')^{1/2}\pi}\lim_{\epsilon_+\to 0}\int_0^\infty da\,\times \\ &\times K_0(a\left((\Delta k)^2-(\Delta T+i\epsilon)^2\right)^{1/2})\sin(az)\sin(az'),\end{aligned} \tag{D.7}$$

where we have used 3.961 2 in [40]. We can integrate this equation and obtain

$$\begin{aligned}\langle 0|\hat{\phi}(T,k,z)\hat{\phi}(T',k',z')|0\rangle &= \frac{l^2}{4(zz')^{1/2}}\times \\ &\times \lim_{\epsilon_+\to 0}\left(\frac{1}{((\Delta k)^2-(\Delta T+i\epsilon)^2+(z-z')^2)^{1/2}}-\right. \\ &- \left.\frac{1}{((\Delta k)^2-(\Delta T+i\epsilon)^2+(z+z')^2)^{1/2}}\right),\end{aligned} \tag{D.8}$$

where we have used that $2\sin(az)\sin(az') = \cos(a(z-z')) - \cos(a(z+z'))$ and the identity 6.671 6 in [40]. From this expression we see that we can not obtain a two point function on the boundary just by multiplying by some power of zz' and taking the limit $z, z' \to 0$. Hence our attempt to obtain a two point function on the boundary by using conformal techniques does not give a correct answer, however the dependence in T and k is quite similar to the correct one.

Appendix E

Superradiance

The superradiance phenomenon takes place when a wave is partially reflected and partially transmitted. It can be roughly defined by saying that: the reflected part has a bigger amplitude than the incident one. This phenomenon can be possible because energy is being taken from a source which might be for example a electrostatic field [64]. It turns out that this phenomenon also occurs on black hole geometries, particularly on the Kerr metric. However, in this metric the real scalar field present superradiance whereas a fermionic field not, see for example [82], [61]. As it has been said before, the BTZ metric has some similarities with the Kerr metric. Hence it seems natural to investigate what happen in the BTZ metric regarding superradiance. Apart from the interest in its own right, superradiance is closely related to quantum effects on black holes, for instance, pair particle creation.

Naively, we could expect that superradiance must occur on the BTZ metric for the scalar field whereas must be absent for the Dirac field. See for example [22] where it is argued that superradiance exist in the BTZ black hole for the scalar field. However, as we will show below, there is no superradiance for vanishing boundary conditions at infinity.

In order to achieve our goal we will use the known exact solutions of the Klein-Gordon operator in the BTZbh, [48] and [59]. Before doing this we will analyze some properties of the asymptotic solutions of the Klein-Gordon operator which shed some light on the peculiarities of the problem under consideration. We will do this by using some techniques borrowed from [27].

We shall assume that the scalar field φ satisfies the equation

$$\left(\nabla_\mu \nabla^\mu - \xi R - m^2\right)\varphi = 0, \tag{E.1}$$

where ξ is a coupling constant, R is the Ricci scalar and m can be considered as the mass of the field. For the BTZ metric

$$ds^2 = -f^2 dt^2 + f^{-2} dr^2 + r^2 (d\phi + N^\phi dt)^2, \tag{E.2}$$

where

$$f^2 = \left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right) = \frac{\left(r^2 - r_+^2\right)\left(r^2 - r_-^2\right)}{l^2 r^2} \tag{E.3}$$

and

$$N^\phi = -\frac{J}{2r^2} = -\frac{r_+ r_-}{l r^2} \tag{E.4}$$

where

$$r_\pm^2 = \frac{Ml^2}{2}\left(1 \pm \left(1 - \left(\frac{J}{Ml}\right)^2\right)^{1/2}\right), \tag{E.5}$$

with $|J| \leq Ml$, the Ricci scalar is $R = 6\Lambda = -\frac{6}{l^2}$. Hence the last equation can be written as

$$\left(\nabla_\mu \nabla^\mu - \widetilde{m}^2\right)\varphi = 0, \tag{E.6}$$

where $\widetilde{m}^2 = m^2 - \frac{6\xi}{l^2}$. Here $\widetilde{m}^2$ can be negative since we assume $m^2 \geq 0$. The operator $\nabla_\mu \nabla^\mu$ is given by

$$\nabla_\mu \nabla^\mu \varphi = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu \varphi\right), \tag{E.7}$$

where $g = |g_{\mu\nu}|$. For the BTZ metric with $J \neq 0$ we have $g = -r^2$, then $|g| = r^2$.

After a direct calculation, using (E.7) in (E.6), it is obtained

$$\frac{d^2 R}{dr^{*2}} + \{\left(\omega + nN^\phi\right)^2 - f^2[\frac{n^2}{r^2} + \widetilde{m}^2 + \frac{r^{1/2}}{2}\frac{d}{dr}\left(\frac{f^2}{r^{3/2}}\right) + \frac{f^2}{2r^2}]\}R = 0, \tag{E.8}$$

where it has been made the ansatz $\varphi(r,t,\phi) = e^{-i\omega t} e^{in\phi} \frac{R(r)}{\sqrt{r}}$ and $r = r\,(r^*)$. This equation has been written in [47]. However, there is a typo in the expression given in this reference, the factor $r^{1/2}$ in the third term in the square bracket is missing.

If (E.8) is rewritten as

$$\frac{d^2R}{dr^{*2}} + V\,(r^*)\,R = 0, \tag{E.9}$$

where

$$V\,(r^*) = \left(\omega + nN^\phi\right)^2 - f^2[\frac{n^2}{r^2} + \widetilde{m}^2 + \frac{r^{1/2}}{2}\frac{d}{dr}\left(\frac{f^2}{r^{3/2}}\right) + \frac{f^2}{2r^2}], \tag{E.10}$$

then

$$\frac{d}{dr^*}\left(R_1\frac{dR_2}{dr^*} - R_2\frac{dR_1}{dr^*}\right) = 0, \tag{E.11}$$

where R_1 and R_2 are solutions of (E.9). Hence

$$R_1\frac{dR_2}{dr^*} - R_2\frac{dR_1}{dr^*} = \text{const.} \tag{E.12}$$

The equation (E.9) is valid for all the region "outside"the horizon where r^* goes from $-\infty$ to 0. In analyzing superradiance in Kerr metric, the equation (E.12) is the starting point [27]. The idea is to use this equation for two asymptotic solutions of (E.9) where in the asymptotic regions the potential V is finite. In the present case at the horizon, $N = 0$

$$\frac{d^2R}{dr^{*2}} + \widetilde{\omega}^2 R = 0, \tag{E.13}$$

where $\widetilde{\omega} = \omega + nN^\phi$. From the last equation it follows that at the horizon the behavior of R is

$$R \propto e^{-i\widetilde{\omega}r^*} \qquad R \propto e^{i\widetilde{\omega}r^*}. \tag{E.14}$$

However, $V\,(r^*)$ goes to ∞ when $r^*(r)$ goes to $0\,(\infty)$. This is because at infinity the BTZbh is asymptotically AdS. So it seems that we can not proceed further in the analysis by this method. However from (E.9) it follows that if

V were finite at infinity then we would have

$$R \propto e^{-i\omega' r^*} \qquad R \propto e^{i\omega' r^*} \tag{E.15}$$

for some ω'. In this case we could analyze the superradiance phenomenon in the same lines as in the Kerr case. It turns out that V is finite at infinity when

$$\widetilde{m}^2 + \frac{3}{4l^2} = 0. \tag{E.16}$$

Hence, in this case (E.15) is true with

$$\omega' = \sqrt{\omega^2 - \frac{1}{l^2}\left(n^2 + \frac{M}{4}\right)}. \tag{E.17}$$

If at infinity there is a incident and a reflected wave $R_\infty \propto e^{-i\omega r^*} + Ae^{i\omega r^*}$ with A a complex constant, and at the horizon an incident wave $R_H \propto Be^{-i\widetilde{\omega} r^*}$ with B also a complex constant, then after substituting this solution and its complex conjugate in (E.12) it is obtained

$$1 - |A|^2 = \frac{\widetilde{\omega}}{\omega}|B|^2. \tag{E.18}$$

From this equation it follows that if $\widetilde{\omega} < 0$ or $\omega < n\Omega_H$ with $\Omega_H = -N^\phi$, the angular velocity of the horizon, then the reflected wave has a bigger amplitude than the incident one. At this stage it seems to exist superradiance when (E.16) is satisfied. However because of (E.17), it must be satisfied

$$\omega > \frac{1}{l}\sqrt{n^2 + \frac{M}{4}}. \tag{E.19}$$

Also because $\omega < n\Omega_H$, it must be satisfied

$$\omega < \frac{nJ}{2Ml^2}, \tag{E.20}$$

where we have used $\Omega_H = \frac{J}{2r_+^2}$ and $r_+ = l\sqrt{M}$. Because $|J| \leq lM$ and $\omega > 0$ both inequalities can not be satisfied at the same time. Hence the fact that

$\omega < n\Omega_H$ could happen does not imply that superradiance exists. In the next section we will show that it does not exist for vanishing boundary conditions at infinity. We point out that these boundary conditions are between the more natural ones since the BTZbh is asymptotically AdS spacetime, and it has been shown [7] that a well defined quantization scheme can be set up in AdS spacetime with these boundary conditions. Also related with this issue is the fact that in four dimensions in the Kerr-AdS black hole the existence of superradiance depends on the boundary conditions at infinity [89]. So it is expected that in the present case something analogous is happening.

E.1 No superradiance in the BTZ black hole

The discussion of this section follows closely the discussion in [48] and [59], however in those works no mention to superradiance is made.

If we assume harmonic dependence in t and ϕ, then the operator (E.7) reads

$$\nabla_\mu \nabla^\mu \varphi = -\frac{1}{f^2 r^2}\left(-\omega^2 r^2 + n^2\left(\frac{r^2}{l^2} - M\right) + n\omega J\right) + \frac{1}{r}\partial_r\left(r f^2 \partial_r\right). \quad \text{(E.21)}$$

Hence the equation (E.6) reduces to an equation in r for $f_{\omega n}$

$$\left[-\frac{1}{f^2 r^2}\left(-\omega^2 r^2 + n^2\left(\frac{r^2}{l^2} - M\right) + n\omega J\right) + \frac{1}{r}\frac{d}{dr}\left(r f^2 \frac{d}{dr}\right) - \widetilde{m}^2\right] f_{\omega n}(r) = 0.$$

If we make $v = \frac{r^2}{l^2}$, then after some algebra we get

$$\left(\frac{d^2}{dv^2} + \frac{\Delta'}{\Delta}\frac{d}{dv} + \frac{1}{4\Delta^2}\left(n\left(Mn - J\omega\right) - \widetilde{m}^2 l^2 \Delta - \left(n^2 - \omega^2 l^2\right) v\right)\right) f_{n\omega}(v) = 0,$$

where $\Delta = (v - v_+)(v - v_-)$ and $' \equiv \frac{d}{dv}$. If now we let

$$f_{n\omega} = (v - v_+)^\alpha (v - v_-)^\beta g_{n\omega} \quad \text{(E.22)}$$

we get

$$u(1-u)g''_{n\omega}(u) + (c - (a+b+1))\, g'_{n\omega}(u) - ab g_{n\omega}(u) = 0, \tag{E.23}$$

where $u = \frac{v-v_-}{v_+-v_-}$, $a = \alpha+\beta+\frac{1}{2}(1+\nu)$, $b = \alpha+\beta+\frac{1}{2}(1-\nu)$, $c = 2\beta+1$, $\nu^2 = 1+\widetilde{m}^2 l^2$, $\alpha^2 = -\frac{1}{4(v_+-v_-)^2}\left(r_+\omega - \frac{r_- n}{l}\right)^2$ and $\beta^2 = -\frac{1}{4(v_+-v_-)^2}\left(r_-\omega - \frac{r_+ n}{l}\right)^2$. The equation for $g_{n\omega}$ is the hypergeometric differential equation, its solutions are well known. This equation has three (regular) singular points at 0, 1, ∞ and two linear independent solutions in a neighborhood of these points. Any of these solutions can be analytically continued to another by using the so-called linear transformation formulas, we will use this property later. The solutions are divided in several cases depending on the values of some combinations of the coefficients a, b and c. Let us consider the case when none of c, $c-a-b$, $a-b$ is a integer.

The points $u = 0, 1, \infty$ correspond to the inner horizon, outer horizon and infinity respectively. Because of the timelike boundary of the BTZbh at infinity, we are interested in solutions which allows us to have predictability. Let us consider the two solutions at infinity. These solutions are given by

$$g_{n\omega} = u^{-a} F(a, a-c+1; a-b+1; u^{-1}) \tag{E.24}$$

and

$$g_{n\omega} = u^{-b} F(b, b-c+1; b-a+1; u^{-1}), \tag{E.25}$$

where $F(a,b;c;z)$ is the hypergeometric function with coefficients a, b and c. If we write (E.22) as a function of u we have

$$f_{n\omega}(u) = (v_+ - v_-)^{\alpha+\beta}\,(u-1)^{\alpha}\, u^{\beta} g_{n\omega}(u). \tag{E.26}$$

Using (E.26) in (E.24) and (E.25) we have two functions at infinity given by

$$f_{n\omega}(u) = (v_+ - v_-)^{\alpha+\beta}\,(u-1)^{\alpha}\, u^{\beta-a} F(a, a-c+1; a-b+1; u^{-1}) \tag{E.27}$$

and

$$f_{n\omega}(u) = (v_+ - v_-)^{\alpha+\beta}\,(u-1)^{\alpha}\,u^{\beta-b}F(b, b-c+1; b-a+1; u^{-1}). \quad \text{(E.28)}$$

The last two equations can be approximated as

$$f_{n\omega}(u) \sim (v_+ - v_-)^{\alpha+\beta}\,u^{-h_+}F(a, a-c+1; a-b+1; u^{-1}) \quad \text{(E.29)}$$

and

$$f_{n\omega}(u) \sim (v_+ - v_-)^{\alpha+\beta}\,u^{-h_-}F(b, b-c+1; b-a+1; u^{-1}), \quad \text{(E.30)}$$

where $h_+ = \frac{1}{2}(1+\nu)$, $h_- = \frac{1}{2}(1-\nu)$ with $\nu = \pm\sqrt{1+\widetilde{m}^2 l^2}$. If we take the positive square root then the first solution converges for any value of ν and the second solution converges for $0 \leq \nu < 1$ and diverges for $\nu \geq 1$. If we take the negative square root the situation is inverted. Let us take the positive square root and just the first solution. We can analytically continue this solution to a neighborhood of $u = 1$ using the following linear relation [3]

$$\begin{aligned} F(a,b;c;u) &= \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-u)^{c-a-b}u^{a-c} \times \\ &\times F(c-a, 1-a; c-a-b+1; 1-1/u) \\ &+ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}u^{-a} \times \\ &\times F(a, a-c+1; a+b-c+1; 1-1/u), \end{aligned} \quad \text{(E.31)}$$

where $\Gamma(x)$ is the gamma function. By letting $u = \frac{1}{u}$, $a = a$, $b = a-c+1$ and $c = a-b+1$ in the last equation we have

$$\begin{aligned} F(a, a-c+1; a-b+1; \frac{1}{u}) &= \frac{\Gamma(a-b+1)\Gamma(a+b-c)}{\Gamma(a)\Gamma(a-c+1)}\left(\frac{u-1}{u}\right)^{c-a-b} \times \\ &\times u^{1-b}F(1-b, 1-a; c-a-b+1; 1-u) \\ &+ \frac{\Gamma(a-b+1)\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(c-b)}u^{a} \times \\ &\times F(a, b; a+b-c+1; 1-u). \end{aligned} \quad \text{(E.32)}$$

Inserting (E.32) in (E.29), close the outer horizon, we have

$$\begin{aligned} f_{n\omega} &\sim \frac{\Gamma(a-b+1)\Gamma(a+b-c)}{\Gamma(a)\Gamma(a-c+1)}\,(u-1)^{-\alpha}\,u^{-\beta}\,\times \\ &\times\ F(1-b,1-a;-2\alpha+1;1-u) \\ &+\ \frac{\Gamma(a-b+1)\Gamma(c-a-b)}{\Gamma(1-b)\Gamma(c-b)}\,(u-1)^{\alpha}\,u^{\beta}F(a,b;2\alpha+1;1-u). \end{aligned} \tag{E.33}$$

The expression (E.33) can be expressed as

$$\begin{aligned} f_{n\omega} &\sim \frac{\Gamma(1+\nu)\Gamma(2\alpha)}{\Gamma(\alpha+\beta+h_+)\Gamma(\alpha-\beta+h_+)}(u-1)^{-\alpha}u^{-\beta}\times \\ &\times\ F(-\alpha-\beta+h_+,-\alpha-\beta+h_-;-2\alpha+1;1-u)+ \\ &+\ \frac{\Gamma(1+\nu)\Gamma(-2\alpha)}{\Gamma(-\alpha-\beta+h_+)\Gamma(-\alpha+\beta+h_+)}(u-1)^{\alpha}u^{\beta}\times \\ &\times\ F(\alpha+\beta+h_+,\alpha+\beta+h_-;2\alpha+1;1-u) \end{aligned} \tag{E.34}$$

From this expression we can see that the two coefficients in both terms are conjugate one of each other. Hence near $u=1$ we can write the last expression as

$$f_{n\omega} \sim e^{i\theta}(u-1)^{\alpha} + e^{-i\theta}(u-1)^{-\alpha} \tag{E.35}$$

where $e^{2i\theta} = \frac{\Gamma(-\alpha-\beta+h_+)\Gamma(-\alpha+\beta+h_+)\Gamma(2\alpha)}{\Gamma(\alpha+\beta+h_+)\Gamma(\alpha-\beta+h_+)\Gamma(-2\alpha)}$. We would like to write the last expression as a sum of two wave modes. In order to do this we introduce another variable [59]. First we notice that

$$\alpha = \pm\frac{i}{4\pi\mathsf{T}}(\omega-\Omega n), \tag{E.36}$$

where $\mathsf{T} = \frac{r_+^2-r_-^2}{2\pi l^2 r_+}$ and $\Omega = \frac{r_-}{lr_+}$. We now define $x = \frac{1}{4\pi\mathsf{T}}\ln(u-1)$. With this definition the equation (E.35) becomes

$$f_{n\omega} \sim e^{i\theta}e^{ix(\omega-\Omega n)} + e^{-i\theta}e^{-ix(\omega-\Omega n)}. \tag{E.37}$$

From here we conclude that the solution to the Klein-Gordon operator near the outer horizon goes like

$$\varphi \sim e^{-i\omega t} e^{in\phi} \left(e^{i\theta} e^{ix(\omega - \Omega n)} + e^{-i\theta} e^{-ix(\omega - \Omega n)} \right) . \tag{E.38}$$

From this expression we see that the mode near the outer horizon is a superposition of an ingoing and an outgoing wave, both with the same amplitude, hence cancelling each other. This is what we expected since at infinity this mode vanishes, hence the superradiance phenomenon does not appear.

It would be interesting to explore superradiance with other fields, for example, the Dirac field. Also with the real scalar field it would be interesting to study other boundary conditions and see what happen.

Appendix F

On the quantization of the real linear scalar field

The quantization of a classical field in a curved spacetime is a key issue in understanding fundamental processes in nature. The par excellence example of quantization of a classical field is the quantization of the real linear scalar field. The general (canonical) formalism for quantizing the real linear scalar field is the topic of this essay.

Basic elements in the quantization of a classical system

A classical field represents a physical system with infinite degrees of freedom. Hence it is natural to consider the quantization of this field as the generalization of the quantization of a classical system with finite degrees of freedom. Let us review the basic ideas in the canonical quantization[1] of a system with finite degrees of freedom. In the canonical quantization of this system one starts with the phase space of the system, which is the space of states of the system. Each point in this space represents a state of the system. The coordinates of this space are the canonical coordinates and comprise the (configuration) coordinates $\{q_i\}$ and the associated canonical conjugate mo-

[1]It is important to have in mind that there are other approaches to quantization of a classical system, for example, path integral quantization.

mentums $\{p_i\}$. For a system of $6N$ degrees of freedom the phase space is a $6N$-dimensional manifold $\mathcal{M}$. The classical observables are represented by functions from this manifold to the real line, $\mathcal{O} : \mathcal{M} \longrightarrow \mathbb{R}$. On the other hand the states of a quantum system are elements of a Hilbert space $\mathcal{H}$ and the observables $\hat{\mathcal{O}}$ are represented by Hermitian operators which act upon these states, $\hat{\mathcal{O}} : \mathcal{H} \longrightarrow \mathcal{H}$. The problem of the quantization of a classical system then is that we need to find an appropriate Hilbert space and a map $\widehat{\ }$ from classical observables to quantum observables, $\widehat{\ }: \mathcal{O} \longrightarrow \hat{\mathcal{O}}$ which allows the passage from the classical system to the quantum system. It is apparent that both mathematical structures (classical v.s. quantum) are completely different. However there is a similar algebraic structure which allows to make the desired transition for linear systems[2]. The algebraic classical structure is the Poisson bracket of two functions on $\mathcal{M}$ defined by

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right), \tag{F.1}$$

while the algebraic quantum structure is the commutator of two operators defined by

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}. \tag{F.2}$$

We demand the map $\widehat{\ }$ to satisfy[3]

$$\widehat{\{f, g\}} = i[\hat{f}, \hat{g}]. \tag{F.3}$$

It turns out that when the phase space is the cotangent bundle, $\mathcal{M} = TQ^*$, then it is possible to choose a Hilbert space $\mathcal{H}$ and the map $\widehat{\ }$ such that the

[2]In this work we just consider linear systems. A linear system is a system which satisfy: 1) The phase space has the structure of a vector space and 2) The Hamiltonian is a quadratic function on $\mathcal{M}$, and then the equations of motion are linear in the canonical coordinates. In this case the space of solutions has the structure of vector space. For more details see [85] and [73]. The quantization of non-linear systems seems to be much more complicated.

[3]The problem of finding the correct map $\widehat{\ }$ is known as the Dirac problem. See [51] for a discussion of this issue.

canonical coordinates satisfy [85]

$$[\hat{q}_\mu, \hat{q}_\nu] = 0, \quad [\hat{p}_\mu, \hat{p}_\nu] = 0, \tag{F.4}$$

$$[\hat{q}_\mu, \hat{p}_\nu] = i\widehat{\{q_\mu, p_\nu\}} = i\delta_{\mu\nu}\mathbf{I}. \tag{F.5}$$

It is well-known that on $\mathcal{M}$ a symplectic form can be defined [73]

$$\Omega = dq^i \wedge dp_i. \tag{F.6}$$

Furthermore, when the system is linear, $\mathcal{M}$ and its tangent space can be identified, i.e., there is an isomorphism between $\mathcal{M}$ and its tangent space on each point of $\mathcal{M}$. Then in this case $\mathcal{M}$ can be thought of as a symplectic vector space equipped with a symplectic form $\Omega : \mathcal{M} \times \mathcal{M} \longrightarrow \mathbb{R}$. Also it is well-known that on the integral curves of Hamilton equations, i.e., along the canonical flows, the symplectic inner product is conserved

$$\frac{d}{dt}\Omega\left(Y_1(t), Y_2(t)\right) = 0, \tag{F.7}$$

where $Y_1(t)$ and $Y_2(t)$ are solutions of the Hamilton equations [73]. In classical mechanics, this property of the symplectic inner product is used to define conserved quantities. We shall see that in the quantum theory of the real linear scalar field a symplectic inner product can also be defined together with a conserved current.

Because of the uniqueness of the solution of the Hamilton equations and the conservation of the symplectic inner product we can identify each point on $\mathcal{M}$ with a solution. Hence the basic structure on which the theory relays can be taken as $(\mathcal{S}, \Omega)$, i.e., the space of solutions $\mathcal{S}$ equipped with a symplectic structure. In terms of Ω, the Poisson brackets for the canonical coordinates can be written as

$$\{\Omega(Y_1, \ \ ), \Omega(Y_2, \ \ )\} = -\Omega(Y_1, Y_2) \tag{F.8}$$

and the corresponding quantum commutator as

$$[\hat{\Omega}(Y_1, \ \ ), \hat{\Omega}(Y_2, \ \ )] = -\Omega(Y_1, Y_2)\mathbf{I}. \tag{F.9}$$

This form of the quantum commutator is very important in the formal formulation of Quantum Field Theory in Curved Spacetime (QFTCS) [85].

Basic elements in the quantization of the real linear scalar field

Let us now proceed to sketch the general formalism for the quantization of the real linear scalar field in curved spacetime. In QFTCS a quantum field which propagates on a fixed classical spacetime is studied[4]. This theory is expected to be valid at scales where the expected quantum nature of the spacetime, whatever it could be, is not relevant for the phenomenon in consideration. In particular, it is expected to be valid on scales above Planck length.

The spacetime is modelled as a manifold M equipped with a pseudo-Riemannian metric $g_{\mu\nu}$. We shall assume that this manifold is well-behaved (smooth, paracompact, Hausdorff, etc.). Let us suppose we have a classical field φ defined on M which satisfies

$$F\varphi = 0, \tag{F.10}$$

where F is a linear partial differential operator. The first step in the quantization of this system is to solve this equation. The field in (F.10) can be bosonic or fermionic. Let us consider a real scalar field, i.e., a bosonic field. In this case we have

$$F = \Box - m^2 - \xi R, \tag{F.11}$$

where $\Box \equiv g^{\mu\nu}\nabla_\mu \partial_\nu (= (|\det(g)|)^{-\frac{1}{2}}\partial_\mu((|\det(g)|)^{\frac{1}{2}} g^{\mu\nu}\partial_\nu))$ is the Laplace-Beltrami

[4] In the back reaction problem the spacetime has dynamics but it is studied after the quantum field is known.

operator, m is a parameter of the theory and can be taken as a smooth function of spacetime[5], ξ is a coupling constant and R is the scalar curvature. Two particular choices of ξ are of interest: the minimal coupling $\xi = 0$ and the conformal coupling $\xi = \frac{1}{4}[(n-2)/(n-1)]$[6]. The equation can be obtained from the variation of the lagrangian density L with respect to φ where L is given by

$$L = -\frac{1}{2}\,[-g]^{\frac{1}{2}}\,\{g^{\mu\nu}\varphi_{,\mu}\,\varphi_{,\nu} + [m^2 + \xi R]\varphi^2\}. \tag{F.12}$$

In the quantization of the classical system with finite degrees of freedom we identified the space of solutions with the phase space. In the quantization of φ we would like to do the same, even though, in this case the phase space concept is less intuitive. The aforementioned identification was possible because the existence and uniqueness of the solution of the dynamical equations. Hence in the present case it is natural to demand that (F.10) has this property too. This is known as the well-posed initial value problem. It turns out that (F.10) has a well-posed initial value problem when M is *globally hyperbolic* [84]. A globally hyperbolic spacetime is one which has a *Cauchy surface*, which is a closed achronal set Σ with $D(\Sigma) = M$ where $D(\Sigma)$ means the domain of dependence of Σ. The domain of dependence represents the complete set of events for which all conditions should be determined by the knowledge of conditions on[7] Σ. Therefore if we know the initial data, position and momentum of the field, on a Cauchy surface, we can predict deterministically its behavior.

There is a theorem which states that *in a globally hyperbolic spacetime a global time function f can be chosen such that each surface of constant f is a Cauchy surface and consequently M can be foliated by Cauchy surfaces and the topology of M is $\mathbb{R} \times \Sigma$, where Σ denotes any Cauchy surface* [84]. Let us denote the global time coordinate by t and each Cauchy surface of constant t as Σ_t. Let n^μ be the unit normal vector field to Σ_t. The metric

[5]In flat spacetime this parameter corresponds to the mass of the field [49].

[6]For more details on the conformal coupling and conformal transformations see [13], sections 3.1 and 3.2.

[7]For an illustrative discussion of the importance of the hyperbolicity of the manifold in quantum field theory see [49], section 2.

$g_{\mu\nu}$ induces a metric $h_{\mu\nu}$ on each Σ_t by the formula[8]

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{F.13}$$

Let t^μ be a vector field on M satisfying $t^\mu \nabla_\mu t = 1$ and decompose it into a part tangential to Σ_t and a part tangential to n^μ. We write it as

$$t^\mu = N n^\mu + N^\mu, \tag{F.14}$$

where N and N^μ are known as the lapse and the shift function respectively. We may interpret the vector field t^μ as representing the flow of time throughout the spacetime. Then the 4-volume element is given by [65]

$$d^4x = N h^{1/2} dt d^3x, \tag{F.15}$$

where d^3x is the 3-volume element and h is the determinant of $h_{\mu\nu}$. The action defined by

$$S = \int L d^4x \tag{F.16}$$

then takes the form

$$S = \int L' dt \tag{F.17}$$

with

$$L' = \frac{1}{2} \int_{\Sigma_t} \{(n^\mu \nabla_\mu \varphi)^2 - h^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - [m^2 + \xi R] \varphi^2\} N h^{1/2} d^3x. \tag{F.18}$$

As we said before we want to associate a phase space to the real linear scalar field. We can take φ as the coordinate in the configuration space, but we need to construct the canonical momentum. It is defined by

$$\pi = \frac{\delta S}{\delta \dot{\varphi}}. \tag{F.19}$$

[8]In a fancier language, $h_{\mu\nu}$ is the pull back of $g_{\mu\nu}$ from M to Σ_t. See, for example, [23].

With the lagrangian density (F.18), the canonical momentum is

$$\pi = (n^\mu \nabla_\mu \varphi)\, h^{1/2}. \tag{F.20}$$

Then a point in the phase space is defined by giving the values of φ and π on a Cauchy surface, say Σ_0. Furthermore, there is a theorem [84] which states that *the equation (F.10) with the operator (F.11) has a well-posed initial value problem in a hyperbolic spacetime. More precisely, given arbitrary smooth initial data on a Cauchy surface there exists a unique solution to (F.10) with the operator (F.11).* Using this theorem and noting that we are considering linear fields, then we can make the same identification we did in the case of a system of finite degrees of freedom between the set of solutions[9] $\mathcal{S}$ and the phase space $\mathcal{M}$. Additionally, in order that all structures be mathematically well defined we should demand that φ and π have compact support, i.e., both belong to $\mathcal{C}_0^\infty(\mathcal{M})$.

The symplectic structure, Ω, on $\mathcal{M}$ is given by

$$\Omega([\varphi_1, \pi_1], [\varphi_2, \pi_2]) = \int_\Sigma (\pi_1 \varphi_2 - \pi_2 \varphi_1)\, d^3x. \tag{F.21}$$

The symplectic product (F.21) is conserved, i.e., if Σ_1 and Σ_2 are two different Cauchy surfaces then

$$\Omega([\varphi_1, \pi_1], [\varphi_2, \pi_2])_{\Sigma_1} = \Omega([\varphi_1, \pi_1], [\varphi_2, \pi_2])_{\Sigma_2}. \tag{F.22}$$

To prove this equation the four dimensional version of Gauss' law and the equation of motion are used [33]. Also it can be seen as consequence of the current conservation, $\nabla_\mu j^\mu = 0$, with

$$j^\mu = \varphi_1 \overleftrightarrow{\nabla}^\mu \varphi_2 = \varphi_2 n^\mu \nabla_\mu \varphi_1 - \varphi_1 n^\mu \nabla_\mu \varphi_2. \tag{F.23}$$

where φ_1 and φ_2 are two solutions. As in the case of finite degrees of freedom

[9] We shall use the same notation as in the case of the system with finite degrees of freedom.

we can define the Poisson bracket

$$\{\Omega([\varphi_1,\pi_1],\cdot),\Omega([\varphi_2,\pi_2],\cdot)\} = -\Omega([\varphi_1,\pi_1],[\varphi_2,\pi_2]) \tag{F.24}$$

and the quantum commutator

$$[\hat{\Omega}([\varphi_1,\pi_1],\cdot),\hat{\Omega}([\varphi_2,\pi_2],\cdot)] = -i\Omega([\varphi_1,\pi_1],[\varphi_2,\pi_2])\mathbf{I}. \tag{F.25}$$

Let μ be a positive, symmetric, bilinear map $\mu : \mathcal{S}\times\mathcal{S} \longrightarrow \mathbb{R}$ and let us demand that

$$\frac{1}{2}|\Omega\,(\varphi_1,\varphi_2)\,| \leq [\mu\,(\varphi_1,\varphi_1)]^{1/2}[\mu\,(\varphi_2,\varphi_2)]^{1/2}, \tag{F.26}$$

for all $\varphi_1,\ \varphi_2 \in \mathcal{S}$. There is a theorem [55] that states: *for any real vector space $\mathcal{S}$ on which are defined both a bilinear symplectic form, Ω, and a bilinear positive symmetric form, μ, satisfying (F.26), one can always find a complex Hilbert space $\mathcal{H}$ together with a real-linear map $K : \mathcal{S} \longrightarrow \mathcal{H}$ such that*

(i) the complexified range of $\mathcal{K}$, (i.e., $\mathcal{KS} + i\mathcal{KS}$) is dense in $\mathcal{H}$,
(ii) $\mu\,(\varphi_1,\varphi_2) = Re\langle\mathcal{K}\varphi_1,\mathcal{K}\varphi_2\rangle_{\mathcal{H}}\ \forall\ \varphi_1,\ \varphi_2 \in \mathcal{S}$,
(iii) $\Omega\,(\varphi_1,\varphi_2) = 2Im\langle\mathcal{K}\varphi_1,\mathcal{K}\varphi_2\rangle_{\mathcal{H}}\ \forall\ \varphi_1,\ \varphi_2 \in \mathcal{S}$.

Moreover, the pair $(\mathcal{K},\mathcal{H})$ will be uniquely determined up to a unitary transformation.

Hence it is possible to construct a Hilbert space using the space of solutions. Now we can proceed to define the anihilation operator, a, and creation operator, $a^\dagger$, on the Fock space $\mathcal{F}(\mathcal{H})$. We define the operator associated with $\Omega(\varphi,\cdot)$ by

$$\hat{\Omega}(\varphi,\cdot) = ia\left(\overline{\mathcal{K}\varphi}\right) - ia^\dagger\left(\mathcal{K}\varphi\right). \tag{F.27}$$

With the Fock space $\mathcal{F}\,(\mathcal{H})$ we could try to associate particles, however there is an ambiguity in the particle concept associated in this way since the product μ is arbitrary up to (F.26). Then there is an ambiguity in the particle concept in a curved spacetime, at least with this association through the Fock space construction, but probably it could be possible to find another construct which allows a well defined particle concept in a curved spacetime. It seems that, as far as we know, in the present status of the theory there is not such construct.

The operator (F.27) is in the, let us say, Schrödinger representation. We can introduce the Heisenberg operator through the evolution of the solution φ:

$$\hat{\Omega}_H(\varphi,\cdot) = ia(\overline{\mathcal{K}\varphi_t}) - ia^\dagger\left(\mathcal{K}\varphi_t\right), \tag{F.28}$$

where φ_t is the solution whose initial data at time t is the same as the initial data of φ at $t=0$. Using the advanced, E_A, and retarded, E_R, fundamental solutions of the operator (F.11) we can write (F.28) as

$$\hat{\varphi}(f) := \hat{\Omega}(Ef,\cdot) = ia(\overline{\mathcal{K}(Ef)}) - ia^\dagger\left(\mathcal{K}(Ef)\right), \tag{F.29}$$

where $E := E_A - E_R$ is a map from the $C_0^\infty(\mathcal{M})$ to $\mathcal{S}$. This operator is the Heisenberg operator smeared over all spacetime with the function f. This operator can be obtained from the corresponding operator in the heuristic approach integrating the latter over all spacetime with the weight function f.

Even though there is an ambiguity in the particle concept in a curved spacetime, there are cases when the particle concept has an unambiguous meaning. These are the static and the stationary spacetime. The Minkowski spacetime is the simplest static type, where as is well known, and every day verified, the particle concept is well defined. Another (curved) static spacetime could be one in which the shift vector N^μ is zero (see (F.14)) and hence we have an hypersurface orthogonal timelike killing vector. In such a spacetime it is always possible to construct an unambiguous Fock space and associate with each state of the field particles with determined energy and momentum. In the stationary case we no longer have an hypersurface orthogonal timelike Killing vector but just a globally timelike Killing vector[10],

[10]The rigorous mathematical analysis for a stationary spacetime has been given in [6] and [52].

K^μ. However in this case it is also always possible to give a particle meaning. In this case the mode decomposition of the field is given with respect to the time coordinate chosen along the integral curves of K^μ. In this case we have[11]

$$\mathcal{L}_{K^\mu} u_i = -i\omega u_i, \tag{F.30}$$

and

$$\mathcal{L}_{K^\mu} u_i^* = i\omega u_i^*, \tag{F.31}$$

where $\mathcal{L}_{K^\mu}$ denote the Lie derivative, $\{u_i, u_i^*\}$ is a complete set of solutions[12] and $\omega > 0$; the vacuum state, $|0\rangle$, is defined by

$$a|0\rangle = 0. \tag{F.32}$$

Some comments on the Energy-momentum tensor

A fundamental element in the theory we are dealing with is the energy-momentum tensor $T^{\mu\nu}$. Its importance is clear from a physical point of view. It contains the information about energy and momentum of the field. If we want to treat interacting fields then the currents formed with this tensor take a relevant role in the formalism, besides it is a fundamental element in the study of the back reaction problem through the semiclassical Einstein field equations

$$G_{\mu\nu} = \langle T_{\mu\nu} \rangle, \tag{F.33}$$

where $\langle T_{\mu\nu} \rangle$ means the expectation value of the energy-momentum tensor operator with respect to the state $\rangle$. The energy-momentum tensor is defined

[11]See, for example, [27].

[12]In this case the field can be written as $\varphi = \sum_i [a_i u_i + a_i^\dagger u_i^*]$. This expansion is identical to the expansion of the displacement function of a string in classical mechanics where the corresponding modes satisfy the harmonic oscillator equation, and hence the continuous system can be thought as an infinite collection of decoupled harmonic oscillators. In the case we are dealing with we can give the same meaning to this expansion. The modes u_i are called positive modes and u_j^* negative modes.

by

$$T^{\mu\nu} = \frac{2}{[-g]^{\frac{1}{2}}} \frac{\delta S}{\delta g^{\mu\nu}}, \tag{F.34}$$

the factor $(-g)^{-\frac{1}{2}}$ is introduced to give a tensor rather than a tensor density[13]. For the minimally coupled massless scalar field it is given by

$$T_{\mu\nu} = \varphi_{;\mu}\varphi_{;\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \varphi_{;\rho}\varphi_{;\sigma}. \tag{F.35}$$

In order to calculate the expectation value of this tensor we can just make it an operator and make the usual sandwich procedure, however, for example, the expectation with respect to the *vacuum* state is given by[14]

$$\langle 0|T_{\mu\nu}|0\rangle = \sum_{\mathbf{k}} T_{\mu\nu}[u_{\mathbf{k}}, u^*_{\mathbf{k}}], \tag{F.36}$$

which diverges. In flat Minkowski spacetime this divergence corresponds to the sum of the zero-point energies of the infinite collection of harmonic oscillators which form the field. Let us see another particular case of this divergence. Let us take as the spacetime a cylinder, which is locally isomorphic to the two dimensional Minkowski spacetime but topologically different. In this spacetime the expectation value of the energy density is

$$\langle 0|T_{tt}|0\rangle = (2\pi/L^2) \sum_{n=0}^{\infty} n \tag{F.37}$$

where L is the period in the compactified spacial dimension of the cylinder. This expression clearly diverges. Then it is necessary to find a procedure which allows us to obtain a finite energy density. In this case a heuristic careful subtraction of the infinite energy of the Minkowski is enough to cure this disease[15], however in a curved spacetime this is not enough and some regularization and renormalization techniques must be used[16]. It turns out that the expression (F.33) can not be given meaning for all states, but just

[13] [13], p. 87.

[14] $T_{\mu\nu}[\varphi, \varphi]$ denotes the bilinear expression (F.35).

[15] For a conceptual and illuminating discussion of this example see [53].

[16] See, for example, [13] for an introduction to these techniques.

for some, say, the physical ones. These states are the Hadamard states [55], [85].

Basic elements in the Algebraic Approach to QFTCS

We have seen that the specification of the map μ allows the construction of a Hilbert space on which the operators act. It turns out that in a general globally hyperbolic spacetime for different μ's the resulting quantum field structures are not unitary equivalents and then we have a problem: what is the correct one? It is worth mentioning that in a system with finite degrees of freedom the Stone-von Neumann theorem saves us from this problem, however this theorem does not hold in the case we are considering. It could seem that we can not develop further QFTCS consistently. Fortunately it is not so, and there is a formalism which allows us to circumvent this problem and formulate QFTCS in an unambiguous mathematical setting. This formalism is based on the algebraic structure generated by the field operators,[17] and we shall refer to it as Algebraic Approach to Quantum Field Theory in Curved Spacetime (AAQFTCS).

Let us see the basic elements of this algebraic approach. Let us consider the fields (F.27) and the Hilbert space where they act. We define the unitary operators[18]

$$\hat{W}(\varphi) = \exp[i\hat{\Omega}(\varphi, \cdot)], \tag{F.38}$$

which satisfy

$$\hat{W}(\varphi_1)\hat{W}(\varphi_2) = \exp[-i\Omega(\varphi_1, \varphi_2)/2]\hat{W}(\varphi_1, \varphi_2), \tag{F.39}$$

and

$$\hat{W}^{\dagger}(\varphi) = \hat{W}(-\varphi). \tag{F.40}$$

These relations are known as Weyl's relations and a system $(\mathcal{S}, \Omega, W)$ is known as Weyl system [17]. To the algebra generated by the Weyl operators

[17] This formalism can be thought as a generalization of the Segal's approach to Quantum Mechanics, see [74].

[18] It is also possible to work with the symplectically smeared field operator $\Omega(\hat{\varphi}, \varphi)$ [55].

can be given the structure of a C^*-algebra[19]. In the formulation of the AAQFTCS this C^*-algebra (Weyl algebra) is taken as the minimal algebra of observables $\mathcal{A}$. We say minimal algebra because there are other observables which do not belong to it, for instance the energy-momentum tensor. There are proposals to incorporate this tensor as part of the algebra[20], however it seems that most authors do not consider it as an element of $\mathcal{A}$. A fundamental property of the Weyl algebra is that even though two inner products μ_1 and μ_2 may give two inequivalent quantum field theories, the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ which arise are isomorphic.

The states are defined as positive linear functionals on $\mathcal{A}$, $\omega : \mathcal{A} \longrightarrow \mathbb{C}$ which satisfy[21]

$$\omega(\mathbf{I}) = 1. \tag{F.41}$$

Positive means

$$\omega(A^* A) \geq 0 \ \ \forall \ A \in \mathcal{A}. \tag{F.42}$$

The two point function is defined by

$$\lambda(\varphi_1, \varphi_2) = -\frac{\partial^2}{\partial s \partial t}\{\omega[W(s\varphi_1 + t\varphi_2)]e^{-ist\sigma(\varphi_1,\varphi_2)/2}\}_{s,t=0}. \tag{F.43}$$

The function in curled brackets is called the generating functional[22]. The other n-point functions are defined similarly.

There is a subclass of states known as quasifree states. These are specified as

$$\omega_\mu[W(\varphi)] = \exp[-\mu(\varphi, \varphi)/2]. \tag{F.44}$$

For these states the two point function is given by[23]

$$\lambda(\varphi_1, \varphi_2) = \mu(\varphi_1, \varphi_2) + \frac{1}{2} i\sigma(\varphi_1, \varphi_1). \tag{F.45}$$

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[19]See [38] for a definition of a C^*-algebra.

[20]See [86], p.12.

[21]A more detailed discussion of this issue can be seen in [41].

[22]See [17]. A complementary discussion of the generating functional concept can be seen in [43].

[23]For more details see [55].

It is interesting that the inner product μ is just the real part of the two point function.

The relation between the state in the algebraic approach and the familiar notion of a state as an element in a Hilbert space can be seen as follows: every vector or density matrix in $\mathcal{H}$ can be realized as an algebraic state when the expectation value of an operator is taken, which corresponds to the representation of an element of the algebra $\mathcal{A}$. Conversely, for each algebraic state ω we can obtain by the GNS construction a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ with a cyclic vector Ξ such that

$$\omega[A] = \langle \Xi, \pi(A)\Xi \rangle \;\; \forall A \in \mathcal{A}. \tag{F.46}$$

A vector Ξ is cyclic if $\pi(A)\Xi$ is dense in $\mathcal{H}$ [55][24].

Any quasifree state can be represented as a vacuum state

$$a\Xi^{\mathcal{F}} = 0, \tag{F.47}$$

where $\Xi^{\mathcal{F}}$ is the vacuum state in the Fock space constructed with the inner product μ. In this case an element of the Weyl algebra is represented as

$$\pi_\mu[W(\varphi)] = \exp\{-\overline{[a^\dagger(K\varphi) - a(K\varphi)]}\}, \tag{F.48}$$

where the bar means the closure of the operator. Then for all $A \in \mathcal{A}$ we have

$$\omega_\mu(A) = \langle \Xi^{\mathcal{F}}, \pi_\mu(A)\Xi^{\mathcal{F}} \rangle. \tag{F.49}$$

The last expression shows that to any quasifree state corresponds a vacuum state. It is worth mentioning that when the GNS representation is irreducible then ω_μ is a pure state [76][25]. A state ω is said to be mixed if it can be expressed in the form

$$\omega = c_1\omega_1 + c_2\omega_2 \tag{F.50}$$

[24] For the detail of this construction see, for example, [76].

[25] An irreducible representation, by definition, just leaves invariant the hole space and the identity element.

where c_1, $c_2 > 0$, otherwise the state is said to be pure.

The dynamics in the algebraic setting can be introduced as follows[26]. As we have seen the symplectic product is conserved. This is equivalent to the existence of a symplectic map, $T : \mathcal{S} \longrightarrow \mathcal{S}$ which satisfies

$$\Omega\left(T\varphi_1, T\varphi_2\right) = \Omega\left(\varphi_1, \varphi_2\right). \tag{F.51}$$

This symplectic map induces a *-automorphism α_T on the Weyl algebra such that [17]

$$\alpha_T\left(W(\varphi)\right) = W\left(T\varphi\right). \tag{F.52}$$

This *-automorphism sometimes is called Bogoliubov transformation. Explicitly, the last equation is

$$W\left(T\varphi\right) = V(t)W(\varphi)V(-t), \tag{F.53}$$

with

$$V(t)\Xi = \Xi. \tag{F.54}$$

The unitary group $V(t)$ is generated by the Hamiltonian of the system [52]. The equation (F.53) gives the evolution of W in the Heisenberg picture.

In addition to the symplectic map arising from the equation of motion there is a symplectic map which arises from the one-parameter group of isometries generated by a Killing vector. Let us denote this map as $\tau_t : M \longrightarrow M$ with $t \in \mathbb{R}$. This group of isometries acts on the space of solutions $\mathcal{S}$ through the map $\Upsilon(t) : \mathcal{S} \longrightarrow \mathcal{S}$ as follows

$$(\Upsilon(t)\varphi)(x) = \varphi(\tau_t(x)). \tag{F.55}$$

The symplectic form is invariant under this action and because of the group structure of $\Upsilon(t)$, which is inherited from the group structure of τ_t, then $\Upsilon(t)$ is a one-parameter group of symplectic transformations. Hence as before we

[26] A careful discussion of the introduction of the dynamics in the algebra can be seen in [43].

have an automorphism on the algebra $\mathcal{A}$:

$$\alpha(t)[W(\varphi)] = W(\Upsilon(t)\varphi). \tag{F.56}$$

Sometimes one is interested in states which satisfy

$$\omega[\alpha(t)A] = \omega(A) \quad \forall A \in \mathcal{A}. \tag{F.57}$$

For example, for a quasifree state ω_μ this will be satisfied if and only if[27]

$$\mu(\Upsilon(t)\varphi_1, \Upsilon(t)\varphi_2) = \mu(\varphi_1, \varphi_2) \quad \forall t \in \mathbb{R}, \ \ \forall \varphi_1, \varphi_2 \in \mathcal{S}. \tag{F.58}$$

In addition to the algebraic setting sketched above there is a slightly different approach[28] where to each bounded open region $\mathcal{B}$ of the spacetime it is associated an algebra $\mathcal{A}(\mathcal{B})$, generated by the operators defined on $\mathcal{B}$. These algebras are called local algebras and must satisfy certain axioms [**?**]. Perhaps between these axioms the two more outstanding for their apparent connection with the causality principle[29] are: if $\mathcal{B}' \subset \mathcal{B}$ then $\mathcal{A}' \subset \mathcal{A}$; and if $\mathcal{B}'$ and $\mathcal{B}$ are spacelike separated, then $\mathcal{A}'$ and $\mathcal{A}$ commute.[30] In this approach no reference is made at the outset to the Hilbert space concept and operators defined on it, however this construct can be recovered by the GNS construction [56].

Some comments on the quantization of the Dirac field

In order to quantize the Dirac field on a globally hyperbolic spacetime we could follow similar lines to the quantization of the scalar field, although there are some physical and mathematical differences which must be taken into account. The principal physical differences are the additional degree of

[27] For more details see [43] and [55].

[28] The basic reference for this approach is [28]. This algebraic approach seems to be a generalization to curved manifolds of the formalism introduced by Haag and Kastler [**?**] for Minkowski spacetime.

[29] We must remember that one of the cornerstones of modern physics is the locality principle which somehow is incorporated by the field concept.

[30] See, for example, [56].

freedom of the system: the *spin* and that fermion fields anti-commute. The spin gives a richer geometric structure and makes the mathematics somewhat harder. Let us consider some relevant mathematical elements which arise in a fermionic system[31].

The basic elements in a fermionic system are: a real Hilbert space, $\mathcal{H}$, with an inner product, V, together with an orthogonal transformation, T, which leaves invariant this product, and a C^*-algebra $\mathcal{A}$ over $\mathcal{H}$. In the quantization of the Dirac field in an electromagnetic field, $\mathcal{H}$ is constructed with solutions to the Dirac equation, hence in the case we are discussing we could do the same, since we know the initial value problem is well-posed for the Dirac equation in the spacetime we are considering [29]. The orthogonal transformation can be constructed using that the Hamiltonian is self-adjoint and the Stone theorem, as in the case of the electromagnetic field. This transformation can be used to introduce the dynamics in the algebra, as in the bosonic case. In the construction of the vacuum additional care must be taken since the Hamiltonian is not positive. It is worth noting that in this case we do not need to use the Weyl algebra since the operators (generators) are bounded. As an alternative procedure we can use the formalism given by Dimock [29] in terms of local algebras.

[31] For a further discussion see [16] and [17].

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