Complexities of Proof-Theoretical Reductions

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Submitted in accordance with the requirements for the degree of Doctor of Philosophy

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September 2016

Abstract

The present thesis is a contribution to a project that is carried out by Michael Rathjen and Andreas Weiermann to give a general method to study the proof-complexity of Σ^0_1-sentences. This general method uses the generalised ordinal-analysis that was given by Buchholz, Rüede and Strahm in [5] and [44] as well as the generalised characterisation of provable-recursive functions of $PA + TI(\prec \alpha)$ that was given by Weiermann in [60]. The present thesis links these two methods by giving an explicit elementary bound for the proof-complexity increment that occurs after the transition from the theory $\widehat{ID}^i_\omega + TI(\prec \alpha)$, which was used by Rüede and Strahm, to the theory $PA + TI(\prec \alpha)$, which was analysed by Weiermann.

Contents

Introduction

The main purpose of the present thesis is to provide a technical lemma for a research project of Michael Rathjen and Andreas Weiermann that was outlined in Michael Rathjen's talk at the *Bertinoro International Center for Informatics* in 2011. Rathjen's and Weiermann's work on Kruskal's theorem in [41] foreshadows a general method to bound the complexity of deductions of Σ^0_1-sentences as well as a general method to prove Π^0_2-conservativity for arithmetical theories. Regarding the complexity of deductions of Σ^0_1-sentences, the idea is rather simple:

> If the witnesses of a Σ^0_1-sentence are very big, then the complexity of its deduction must lie above a certain limit.

The basic methods to make this precise are well known. In the case of a Σ^0_1-sentence σ that is deducible in a theory T with a deduction of complexity n, in symbols

$$T \vdash^n \sigma,$$

one tries to give an ordinal-analysis of T. Under the assumption that this ordinal-analysis gives the ordinal α one can usually prove that T is Π^0_2-conservative over $PA + TI(\prec \alpha)$:

$$T \equiv_{\Pi^0_2} PA + TI(\prec \alpha).$$

Since the theories $PA + TI(\prec \alpha)$ are well-studied, one can use classical subrecursion theory to bound the witnesses of σ by a Hardy function $H_{\beta_n}(0)$, where β_n is smaller than α and depends on the complexity of the deduction of σ in T. Hence, if the witnesses of σ are bigger than $H_{\beta_n}(0)$, then the complexity of any deduction of σ must be bigger than n. Generalising this argument faces two issues even in the case when a natural ordinal notation system is used in the ordinal-analysis. First, one has to find a recipe to prove that $T \equiv_{\Pi^0_2} PA + TI(\prec \alpha)$ from an ordinal-analysis that is general enough to draw general conclusions from it. Here we face the problems that one has to unify the methods of formalising an ordinal-analysis and that there are ordinal-analyses in the literature that include methods that hinder proving conservativity in the standard ways, e.g. the Ω-rule. Second, one has to find a general approach in subrecursion theory to bound the witnesses of a deducible Σ^0_1-sentence. Here we face the problem that the assignment of fundamental sequences to an ordinal notation system must be generalised in order to define a version of the Hardy hierarchy that is independent enough from the particular ordinal notation system, which was used in the ordinal-analysis, to give meaningful bounds.

Both of these issues have been already resolved. Buchholz discovered in [5] that an intuitionistic version of the theory of inductive definitions is Π^0_2-conservative over PA and can therefore be used to replace PA in $PA + TI(\prec \alpha)$. Since all infinite deduction systems that are used in ordinal-analysis are inductively defined, this solves the first problem for a large number of ordinal-analyses that are present in the literature. In [44] Rüede and Strahm strengthened Buchholz's result by showing that Π^0_2-conservativity over PA is preserved, when certain inductive definitions are iterated; hence also the missing ordinal-analyses, those which include the Ω-rule, are covered as well. The second issue was solved by Weiermann in [60] by drawing on Buchholz's, Cichon's and Weiermann's work, which is presented in [7]. Using Cichon's notion of a norm that can be defined on all ordinal notation systems which are present in the literature, in [60] Weiermann was able to generalise the Hardy functions in a way that supports the previously given aims.

However it remained open how these two solutions work together; in order to use Weiermann's approach of subrecursion theory in the case of the Π^0_2-conservativity that was proved for Rüede's and Strahm's theories of inductive definitions to study the complexity of deductions of Σ^0_1-sentences, one has to ensure that the function H_{β_n}, that is found by Weiermann's approach when working directly with $PA + TI(\prec \alpha)$, is not too far away from H_{β_m}, that is found by Weiermann's approach when working indirectly via theories of inductive definitions. In other words: one has to ensure that the proof-theoretical reduction that Rüede and Strahm used to prove their Π^0_2-conservativity result gives a rather weak speed-up for the theories of inductive definitions over $PA+TI(\prec \alpha)$.

This is done in the present thesis.

The plan of the thesis is as follows. **Chapter 1** has no connection to the rest of the thesis. It is a philosophical paper that I tried to publish during my PhD. I developed a method for philosopy inspired by proof theory by following a Carnapian line, in contrast to American pragmatism. I hoped that it might be a philosophically fruitful alternative to the contemporary idealists and pragmatists in modern inferentialism. I tried this by solving a central problem in proof-theoretical semantic by working out the differences between the requirement for harmony and that for compositionality, and focusing on the latter without ignoring the former.[1] I hoped that a purely syntax-based semantic, that is also an alternative to the commonly used holistic syntax-based semantic, would show the community of philosophy of logic that there is a fruitful way of doing philosophy syntactically without relying on common human behaviour, because common human behaviour seems to be a bad starting point, when a highly academic field like logic is the subject of philosophical theorising. For a certain amount of "educational brain-washing" has to take place before people consider studying logic in a certain manner, which usually differs from ordinary human behaviour.[2]

Chapter 2 starts with most of the bookkeeping that has to be done; hence most of the standard notions that are used in metamathematics are defined. Also the most common notions of theory reduction are presented and compared. Primarily the notion of proof-theoretical reduction is introduced and it is emphasised that most of the techniques that are used in proof-theory, including some other notions of theory reduction, fall under this notion. Since there was a dispute between Niebergall [35] and Feferman [18] about such matters, we use this dispute as a narrative for our presentation of proof-theoretical reduction.

Chapter 3 introduces the technique of ordinal-analysis and emphasises how it can be

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[1] Luckily, therefore my approach is not targeted by Rumfitt's critique as formulated in [45].

[2] I was very pleased that Dutilh Novaes realised this point in her book [14] as well, where she argues for it on a level of scrutiny that I am not yet able to provide.

used to prove conservativity results of the form

$$T \equiv_{\Pi^0_2} PA + TI(\prec \alpha).$$

Moreover it introduces fix-point free intuitionistic theories of iterated inductive definitions and presents Rüede's and Strahm's proof-theoretical reduction that proves that these theories are Π^0_2-conservative over PA. The chapter finishes with the proof that this Π^0_2-conservativity is preserved when axioms for transfinite induction are present.

Chapter 4 introduces the Ω-rule and sketches how these theories of iterated inductive definitions can be used to prove Π^0_2-conservativity for theories whose ordinal-analysis uses the Ω-rule. This is done on a particular case that is also an example for a situation that is discussed in Chapter 5.

Chapter 5 starts by introducing the notions of deduction complexity and speed-up. It continues by explaining Weiermann's general approach from subrecursion theory and applies this approach to the deduction complexity of Σ^0_1-sentences.

In **Chapter 6** the actual work is done. We analyse the relevant proofs from Chapter 2 and Chapter 3, which define a proof-theoretical reduction, to how the operations that are in use in these proofs increase the complexity of certain deductions.

In **Appendix A** I extend my Master Thesis, which I wrote at the University of Vienna. Here I show that Gentzen's method to prove the consistency of PA as presented in [57] is elementary in one use of an ε_0-descent recursion.

Appendix B fails to solve a problem in that I was interested for quite a while: to give an ordinal-analysis of a weak system like EA. The proofs are correct but the results are not new. The issue here was that I believed that transfinite induction for Δ^0_0-formulas and elementary-recursive well-foundedness have the same Π^0_2-consequences over EA. For trivial reasons this is not the case, as Arnold Beckmann told me after I submitted my Thesis for examination. However, Appendix B presents my thoughts about this topic.

Appendix C is just a list of the deduction systems that are used throughout the thesis.

Chapter 1

Justifying the use of Multi-Conclusion Sequents

1.1 Introduction

Proof-theoretic semantics goes back to Dag Prawitz's efforts to clarify Gerhard Gentzen's comments on the calculi that were developed from the latter and got its current form by Dummett's book *The Logical Basis of Metaphysics* published in 1991 (see [47]).[1] The original aim of proof-theoretic semantics was to give a theory of meaning for a logical vocabulary through introduction and elimination rules that were taken from the calculus of natural deduction, which was viewed as a model of ordinary language use. The least metaphysically loaded semantics, in these terms, was aimed at serving as a method of justification for logical inferences to form a logic which does not presuppose the solutions of great philosophical problems to which it is applied to (see [13]). Dummett's own explanatory example is the dispute about quantum mechanical realism. The distributive law

$$[\varphi \wedge (\psi_1 \vee \psi_2)] \leftrightarrow [(\varphi \wedge \psi_1) \vee (\varphi \wedge \psi_2)]$$

[1]The name "proof-theoretic semantics" was introduced by Schroeder-Heister much later.

is banned from quantum logic, but is used in validity proofs using truth tables to justify inferences of classical logic. If a quantum mechanical realist refuses to accept the distributive law, a validity proof using truth tables is of no use for convincing her. For she could object that, the use of two-valued semantics already presupposes a so-and-so-being of the world which is in contradiction to the quantum world. But according to Dummett, if one can justify the distributive law in terms of ordinary language use, then she has to change her opinion in one way or another.

One can find in the literature on proof-theoretic semantics the view that the character of intuitionistic logic is related to anti-realism while classical logic shows its own connection to realism. This is not meant as a statistically significant correlation between an anti-realistic position and the preference for intuitinistic logic in philosophers' stances, but as a inherent claim about those logics. In the context of proof-theoretic semantics the correlation is based on the possibility of giving intuitionistic logic a proof-theoretic semantics while for classical logic such an account fails.

> "According to Dummett, the logical position of intuitionism corresponds to the philosophical position of anti-realism. [...] Following Dummett, major parts of proof-theoretic semantics are associated with anti-realism." [47]

To count a proof-theoretic semantics as anti-realistic is based on its preference for syntactically given deduction rules, which are inspired by ordinary language use, to truth assignments and satisfiability by a model.

Unfortunately the literature is very imprecise in this respect. The proof-theoretic semanticists do not argue that those two diametrically opposed metaphysical theories correspond to the logics per se. But they do so, if one favours a compositional meaning theory, as a proof-theoretical semantics is as well. In Dummett's own account a logic is viewed as a set of schematically specified inferences without a presupposed semantics, since he tries to justify certain inferences by ordinary language use (through stipulated introduction and elimination rules) without relying on the pictures that are based in a metaphysics and usually govern a semantics. Subsequently Dummett diverts the possible meaning theories for these logics into compositional and holistic ones. In the latter a justification of a particular inference is not possible, since the meaning of the logical vocabulary is partly given through the possibility of this inference and not vice versa. Therefore a compositional character of proof-theoretic semantics is demanded. But, as Dummett explains in [13, p. 225], if we adopt a holistic meaning theory, an anti-realistic position that essentially includes classical logic is arguable. However one might grant that, since the context in which proof-theoretic semanticists argue is justification related and a holistic position lacks the possibility to justify anyway, such things do not have to be announced explicitly.

In this paper it will be argued that a compositional meaning theory can be given through a proof-theoretic semantics which is able to justify all inferences of classical propositional logic and does not diverge more from ordinary language use than that given by Dummett himself. By extending an account given by Franz von Kutschera in 1968 using sequents with multiple conclusions, a proof-theoretic semantics for classical logic will be constructed. An analysis of the general use of free variables in proof-theoretic approaches to first-order logic will expose that the arguments for a refusal of using multiple conclusions are disproportionate. For it will be shown that the use of free variables is as problematic for Dummett's account as that of sequents with multiple conclusions. This allows us to extend von Kutschera's account in such a way that it justifies classical propositional logic, because a general refusal of free variables is impossible, when the used inference rules have finitely many premisses.

The argument presented above is essentially different from the justification of sequents with multiple conclusion which is given by Greg Restall in [42]. Restall gave a pragmatic interpretation of sequents which is a possible (and maybe better) alternative to Dummett's, whereas we are not concerned here with the actual way sequents can be embedded into or interpreted through a framework of pragmatism. On the contrary we shall not make an issue of interpretations that might be given by pragmatists. If some connections are mentioned, then they are intended to help a hypothetical pragmatist, who reads the present text, to see how the argument would work after an embedding into his wider view. Also pragmatic views will only enter the argumentation through the so far established techniques of proof-theoretic semantics. But since those techniques are in the end just reasonable formal considerations that are heavily used in proof theory anyway or merely loosely connected with the particular pragmatic stance without referring to its conceptual specializations, these techniques can just be seen as syntactic considerations with an unspecific intuition supporting them.[2] The arguments presented here are therefore independent of the pragmatic stance which is adopted, in particular because of their technical nature. Moreover a syntactical orientated philosopher may not have to adopt a pragmatic view at all to follow the argumentation which is presented here.

The main consequence of the consideration given here is that any logic can be seen as metaphysically neutral, or may be considered as free from it, even for those who demand a compositional semantics.

1.2 A Gentzen-semantics

The sequent calculi developed by Gerhard Gentzen use sequents as the smallest operative unit. This separates them from all the other calculi commonly in use, since their operations are defined over formulas. A sequent is a syntactical object of the form $(\varphi_1, ..., \varphi_n \Rightarrow \psi)$, where $\varphi_1, ...\varphi_n, \psi$ are formulas of a presupposed language $\mathcal{L}$ while $\Rightarrow$ is not a primitive symbol of $\mathcal{L}$, but a symbol particularly attached to the calculus. There are many ways of interpreting a sequent. Here we restrict our self to a reading which interprets $(\varphi_1, ..., \varphi_n \Rightarrow \psi)$ as an argument with $\varphi_1, ..., \varphi_n$ as premises and the conclusion ψ. A sequent calculus can therefore be seen as a meta calculus of another

[2]For instance, the assertion that in general only finitely many terms can be put into a justification of a general statement is not in need of a fancy analysis of humans' use of substantives. Also such intuitions have been guiding proof-theory from the beginning on without being justified by a theory.

calculus K in the sense that, if K deduces from the formulas $\varphi_1, ..., \varphi_n$ the formula ψ, notated as $\varphi_1, ..., \varphi_n \vdash_K \psi$, then the associated sequent calculus deduces the sequent $(\varphi_1, ..., \varphi_n \Rightarrow \psi)$. Therefore a sequent can be seen as the syntactic counterpart of the logical entailment relation $\varphi_1, ..., \varphi_n \models \psi$ that was introduced by Tarski via the notion of satisfiability in models. This makes a proof-theoretic semantics formulated via sequents a possible alternative to Tarski's semantics. Since in this part of the paper we only aim to give a semantics for propositional logic, we restrict our self to a language of propositional logic. This collapses Tarski's notion to assignments of truth values to formulas.

Franz von Kutschera described in his paper from 1968 [30] a semantics, which can be seen as a proof-theoretic semantics (see [47]). This semantics, which was called Gentzen-semantics by von Kutschera, suits our purposes of giving a proof-theoretic semantics for classical logic very well. For it is already formulated via sequents, which avoids the complications that the common formulations via natural deduction offer to classical logic.[3] Before this semantics can be introduced a language $\mathcal{L}_0$ has to be defined, whose primitive symbols include the propositional constants $A_0, A_1, A_2, A_3, ...$ and finitely many connectives $F_0^{m_0}, ..., F_n^{m_n}$, where m_i denotes the arity of the connective. $\mathcal{L}_0$ is inductively defined so that any A_i is a formula and, if $\varphi_1, ..., \varphi_{m_i}$ have been realised as formulas, every $F_i^{m_i}(\varphi_1, ..., \varphi_{m_i})$ is a formula. The meaning of a connective is subsequently stipulated through the introduction rules into the left and right side of the sequent. To make sense of these stipulations one has to clarify how to treat sequents independently from the connectives. Therefore the following stipulations have to be made:

[3]This problems occur through particularities of the formulations for negation rules in natural deduction.

Initial Sequents: $\varphi \Rightarrow \varphi$

Weakening Left: $$\frac{\Delta \Rightarrow \varphi}{\Delta, \psi \Rightarrow \varphi}$$

Weakening Right: $$\frac{\Delta \Rightarrow}{\Delta \Rightarrow \varphi}$$

Cut: $$\frac{\Delta \Rightarrow \psi \quad \Delta, \psi \Rightarrow \varphi}{\Delta \Rightarrow \varphi}$$

Exchange: $$\frac{\Gamma, \psi, \varphi, \Delta \Rightarrow \sigma}{\Gamma, \varphi, \psi, \Delta \Rightarrow \sigma}$$

Contraction: $$\frac{\Gamma, \varphi, \varphi, \Rightarrow \psi}{\Gamma, \varphi, \Rightarrow \psi}$$

Here φ, ψ, σ are formulas of $\mathcal{L}_0$ and Δ, Γ are sequences of such formulas. A sequent of the form $(\varphi_1, ..., \varphi_n \Rightarrow)$ is, in accordance with Hilbert's generalisation of a contradictory system as a trivial one and the presence of weakening right, interpreted as the inconsistency of $\{\varphi_1, ..., \varphi_n\}$. Using this as a basis, the meaning of a connective can be stipulated by the introduction rules. But since we are aiming at a compositional theory of meaning, some restrictions have to be demanded.

First, one has to demand that there is only one occurrence of a single connective and no other connective occurs in the schema by which the rule is stipulated. Moreover every explicitly occurring formula in the schema must be a subformula of the formula in which the connective occurs. These restrictions ensure that the meanings of connectives are mutually independent of each other and that the semantics is compositional, i.e. the meaning of a formula is determined by the meaning of its subformulas and the meaning of the connectives it is formed of. As it is shown in [30], the initial sequents may therefore only be formed by propositional constants, and the introduction rules for a F^n on the right side of a sequent must be of the following form:

$$\frac{(\Delta, \Delta_{11} \Rightarrow \Omega_{11}), ..., (\Delta, \Delta_{1S_t} \Rightarrow \Omega_{1S_t})}{\Delta \Rightarrow F^n(\varphi_1, ..., \varphi_n)}$$

$$\vdots$$

$$\frac{(\Delta, \Delta_{t1} \Rightarrow \Omega_{t1}), ..., (\Delta, \Delta_{tS_t} \Rightarrow \Omega_{tS_t})}{\Delta \Rightarrow F^n(\varphi_1, ..., \varphi_n)}.$$

Here the Δ_{ij} and Ω_{ij} include only subformulas of $\varphi_1, ..., \varphi_n$, with the additional restriction that Ω_{ij} does not enumerate more than one formula.

As a second requirement, von Kutschera demands, by presence of the cut-rule, that the introduction rules on the left side for F^n have to have a form which ensures conservativity of the system over its fragment which does not include any rule for F^n, i.e. every provable sequent which does not include F^n must be provable without using an F^n introduction rule. As shown in [30], this requires the following form of the introduction rules on the left side:

$$\frac{(\Delta, \Delta_{11} \Rightarrow \Omega), ..., (\Delta, \Delta_{1,S_1} \Rightarrow \Omega)}{\Delta, F(\varphi_1, ..., \varphi_n) \Rightarrow \Omega}$$

$$\vdots$$

$$\frac{(\Delta, \Delta_{t1} \Rightarrow \Omega), ..., (\Delta, \Delta_{t,S_t} \Rightarrow \Omega)}{\Delta, F(\varphi_1, ..., \varphi_n) \Rightarrow \Omega}$$

As before Δ_{ij} and Ω only include subformulas of $\varphi_1, ..., \varphi_n$ and Ω does not enumerate more than one formula.

Clearly the second requirement is founded in the required compositional character of the semantics, since it separates the meaning of the connectives from each other. In a more modern reading however this requirement ensures an additional one. Since the introduction rules on the left side correlate with the elimination rules in natural deduction by the presence of the cut-rule, the second requirement guarantees intrinsic and total harmony in the sense of Dummett (see [13]). According to Dummett's harmony criterion the verificational and pragmatic aspects of a theory of meaning must be in harmony, i.e. depending on which is favoured the aspects of the other must be justified in terms of it. Thereby are the verificational aspects of a theory of meaning which determine when a sentence can be asserted as true while the pragmatic aspects settle what conclusions can be drawn from a given sentence. In the case of a theory of meaning that is given through a proof-theoretic semantics which is built via natural deduction, Dummett assigns the introduction rules to the verificational and the elimination rules to the pragmatic aspects of this meaning theory. Since here the interpretation of a sequent as a representation of an argument is chosen, the harmony criterion becomes the harmony between introduction rule on the right side to introduction rules on the left side. For, in contrast to Dummett, a sequent calculus is not understood through a natural deduction calculus (see [13, p. 185, p. 248]) here, but as a metacalculus for any other kind of calculus. Therefore the pragmatic aspect of a connective in a logic becomes the provability of a sequent where the connective in question appears on the left side. Hence the part an elimination rule plays in a proof-theoretic semantics that is formulated via natural deduction correlates to those of an introduction rule on the left side in a formulation via sequents. Consequently elimination rules for a sequent style formulation of a proof-theoretic semantics do not carry any additional meaning. Here intrinsic harmony correlates therefore with the possibility of justifying the introduction rules on the left side by the previously stipulated introduction rules on the right side by presence of the cut-rule or vice-versa.

The notion of total harmony is satisfied if the calculus formed by the stipulated rules is conservative over its parts where exactly one connective is removed (see [13, p. 250]).

As von Kutschera proved in [30], every connective which can be formulated by rules satisfying the two requirements of the Gentzen-semantics are already explicitly definable by the connectives $\neg, \vee, \wedge$ and $\rightarrow$ of intuitionistic logic.[4] This is founded on the fact that

[4]The proof uses generalised sequents where the symbol $\Rightarrow$ appears finitely nested, e.g. $\Gamma \Rightarrow (\Delta \Rightarrow \Lambda)$, but since the proof is not of any interest for our considerations here we will not have a closer look into it.

the usual introduction rules for $\neg, \vee, \wedge$ and $\rightarrow$ are generating intuitionistic logic when they are restricted to sequents with only one formula on their right side (see [57]), and satisfy the pre-given Gentzen-semantics, as one can easily see.[5]

Introduction Right	**Introduction Left**
$\dfrac{\varphi, \Delta \Rightarrow}{\Delta \Rightarrow \neg\varphi}$	$\dfrac{\Delta \Rightarrow \varphi}{\neg\varphi, \Delta \Rightarrow}$
$\dfrac{\Delta \Rightarrow \varphi \quad \Delta \Rightarrow \psi}{\Delta \Rightarrow \varphi \wedge \psi}$	$\dfrac{\varphi, \Delta \Rightarrow \Omega}{\varphi \wedge \psi, \Delta \Rightarrow \Omega}$
	$\dfrac{\psi, \Delta \Rightarrow \Omega}{\varphi \wedge \psi, \Delta \Rightarrow \Omega}$
$\dfrac{\Delta \Rightarrow \varphi}{\Delta \Rightarrow \varphi \vee \psi}$	$\dfrac{\varphi, \Delta \Rightarrow \Omega \quad \psi, \Delta \Rightarrow \Omega}{\varphi \vee \psi, \Delta \Rightarrow \Omega}$
$\dfrac{\Delta \Rightarrow \psi}{\Delta \Rightarrow \varphi \vee \psi}$	
$\dfrac{\varphi, \Delta \Rightarrow \psi}{\Delta \Rightarrow \varphi \rightarrow \psi}$	$\dfrac{\Delta \Rightarrow \varphi \quad \psi, \Delta \Rightarrow \Omega}{\varphi \rightarrow \psi, \Delta \Rightarrow \Omega}$

Therefore von Kutschera's result excludes such a semantics for classical logic. Von Kutschera's result is however not limited to only giving a division of intuitionistic and classical logic (this would have been clear according to the conservativity criterion).[6] Its importance is exhibited by its characterisation of intuitionistic logic as the maximal

[5] As explained before, we view intuitionistic logic as the set of all intuitionistically valid formulas.

[6] If one is choosing the common natural deduction calculus that is equivalent to the previously given one in sequent style, then extending this calculus by classical negation new $\neg$-free formulas can be proved. This is explained in the section about conservativity below.

logic satisfying the very natural requirements given above. This is of particular interest, since it characterises intuitionistic logic purely in terms of logic (as a discipline) without referring to a metaphysical notion of constructivism or to algebraic structures (such as tree-like ordered models).

If a similar semantics for classical logic, that keeps some compositional character, is aimed at, then, beside dropping the conservativity claim, the requirements on the schematic form of the introduction rules have to be changed. The first possibility is to drop the demand that F^n must occur exactly once in the schematic representation of the rule. This allows us to introduce rules of the form

$$\frac{\Delta \Rightarrow \neg\neg\varphi}{\Delta \Rightarrow \varphi} \qquad \frac{\Delta, \varphi \Rightarrow \psi}{\Delta, \neg\neg\varphi \Rightarrow \psi}.$$

By the Gödel-Gentzen-translation of intuitionistic into classical logic (see [58]) and the classically valid formula

$$\varphi \leftrightarrow \neg\neg\varphi,$$

extending the intuitionistic sequent calculus by those rules would yield classical logic. However, as Dummett notes in [13], such a move gives double negation a separate meaning from ordinary $\neg$, which would disturb our efforts to give a compositional semantics for the usual connectives. Another possibility is to drop the restriction to sequents having only one formula on the right side, i.e. allowing sequents of the form $(\varphi_1, ..., \varphi_n \Rightarrow \psi_1, ..., \psi_m)$, which are called multi-conclusion sequents in the following. In this case one has to add the rule of weakening right

$$\frac{\Delta \Rightarrow \Gamma}{\Delta \Rightarrow \Gamma, \varphi}$$

to the calculus as well. Using multi-conclusion sequents has the advantage that conservativity is still assured and we will have a closer look into it in the following.

1.3 Multi-Conclusion Sequents

Arguments against the use of multi-conclusion sequents in the literature of proof-theoretic semantics are mostly versions or direct citations of Dummett's argument in [13, p. 187], which refers to ordinary language use. According to Dummett, the use of sequents with only one formula on the right side for purposes of semantics is justified, since one does not have to have an understanding of a logical connective to understand the sequent.[7] As said before, a sequent $(\varphi_1, ..., \varphi_n \Rightarrow \psi)$ can be viewed as an argument with the premises $\varphi_1, ..., \varphi_n$ and conclusion ψ. But such an interpretation is not possible in cases of multi-conclusion sequents like $(\varphi_1, ..., \varphi_n \Rightarrow \psi_1, ..., \psi_n)$, because there are no occurrences of arguments drawing more than one conclusion in ordinary language use. Consequently such sequents must be interpreted through another sequent of the form $(\varphi_1, ..., \varphi_n \Rightarrow \psi_1 \vee ... \vee \psi_n)$ and therefore presuppose an understanding of $\vee$. This is not symmetric to the use of sequents with multiple premises as Dummett explains. In terms of valid arguments there is indeed no difference in using $\varphi_1 \wedge ... \wedge \varphi_n$ instead of $\varphi_1, ..., \varphi_n$, but by contrast with the case of multi-conclusions one does not have to do this. For assuming many premises is a primitive part of ordinary language use and can therefore be stipulated without harm.[8]

However, even when one is mainly interested in a theory of meaning for propositional connectives, the methods that are used must be somehow extendible to first-order logic, if the theory of meaning is aimed at being a proper alternative to those given by a model-theoretic semantics. But after an extension to some kind of quantification, most syntactic methods have in common that some syntactic entities enter in a close relationship

[7] Avoiding a circularity here is crucial for Dummett's aims, since he wants to give a justification for logical inferences (see [13]).

[8] Restall avoids being a target of this argument by changing the interpretation of a inference rule. While Dummett interprets sequents through the warrant to assert the sentence on the right side in [13], Restall sees a sequent as a state in a dispute where the $\varphi_1, ..., \varphi_n$ are accepted and the $\psi_1, ..., \psi_m$ are rejected by an opponent.

(according to their usage) with those quantifiers. Consequently formulas including an occurrence of such an entity without a related occurrence of one of their associated quantifiers obstruct, at least partly, an interpretation. For instance if one wants to prove in a finite calculus K the following

$$\varphi_1, ..., \varphi_n \vdash_K \forall x\psi(x),$$

she has to use a rule of the following form.[9]

Natural Deduction	**Sequent Calculus**
$\dfrac{\varphi(y)}{\forall x\varphi(x)}$	$\dfrac{\Delta \Rightarrow \varphi(y)}{\Delta \Rightarrow \forall x\varphi(x)}$

Here one has to take some restrictions on the occurrences of y into account. In natural deduction y must not occur in any open assumption and in the case of a sequent calculus y must not occur in $\Delta \Rightarrow \forall x\varphi(x)$. By following the approach that introduction and elimination rules stipulate the meaning of the logical vocabulary and presupposing total interpretability of every entity in use one faces the problem of not being able to interpret free variables without presupposing an understanding of the $\forall$-quantifier. For, since free variables do not occur in ordinary language use, we are prevented from giving an obvious interpretation. Also on the basis of the rules which are given so far only the quantifier rules take free variables into account. Consequently, without a fancy theory of meaning having been presupposed, one has to interpret $(\psi_1(x), ..., \psi_n(x) \Rightarrow \varphi(x))$ through the sequent $(\forall x\psi_1(x), ..., \forall x\psi_n(x) \Rightarrow \forall x\varphi(x))$[10] or treat the first sequent as uninterpreted. In the rest of this section we adopt the so far rather *naive* view of taking sequents including free variables as essentially lacking an interpretation. The next section

[9] A calculus is called finite, when all its rules have finitely many premises.

[10] Such a treatment is common in formal logic, but presupposes an understanding of the $\forall$-quantifier.

will include responses to the obvious objections against such a view.

Dummett does not recognise these difficulties, because he reacts to a consequence of the problem just described by dividing his justification methods for logical inferences into three kinds. A proof-theoretic justification of the first kind is to prove a logical principle from other principles that are seen as more fundamental and therefore treated as axioms. A proof-theoretic semantics is first used in a justification of the second kind. If the focus lies on the introduction rules of natural deduction, then the formulas in the logical inference that are to be justified are analysed according to their assumptions. This is exemplified by a justification of the inference

$$\frac{(\varphi \to \psi_1) \vee (\varphi \to \psi_2)}{\varphi \to (\psi_1 \vee \psi_2)} \quad .$$

By starting with the premise, the meaning of $\vee$, which is stipulated by the rules

$$\frac{\phi_1}{\phi_1 \vee \phi_2} \quad \frac{\phi_2}{\phi_1 \vee \phi_2},$$

leads us either to the formula $(\varphi \to \psi_1)$ or $(\varphi \to \psi_2)$.[11] Since the procedure of the justification is symmetric for the two cases, we continue with $(\varphi \to \psi_1)$. According to the rule

$$\frac{\begin{array}{c}[\phi_1] \\ \vdots \\ \phi_2\end{array}}{\phi_1 \to \phi_2},$$

which governs the meaning of $\to$, one gets φ.

Next the conclusion comes into focus. Since the main connective is $\to$, again we get φ through the meaning stipulated by the same rule. The conclusion needs the same assumptions as the premises, therefore the inference is justified by the meaning stipulated

[11] That one can use the rule for the main connective of a formula here is called the fundamental assumption in proof-theoretic semantics, i.e. that the rule last used was an introduction rule for the main connective of the formula (see [13]).

by the introduction rules. In the case of a formulation via sequents this corresponds with ending up to a sequent of the form $\varphi \Rightarrow \varphi$, because in a formulation with sequents an inference, which must be justified, it can be expressed as the sequent

$$(\varphi \to \psi_1) \vee (\varphi \to \psi_2) \Rightarrow \varphi \to (\psi_1 \vee \psi_2).$$

As Dummett points out (see [13, p. 259-264]), the justification method cannot work that way in the case of a $\forall$-quantifier, since it would justify the inference

$$\frac{\forall x[\varphi(x) \vee \psi(x)]}{[\forall x \varphi(x)] \vee [\forall x \psi(x)]},$$

which is generally seen as invalid. The reason for this is connected with the variable restrictions on the quantification rules. Since the justification goes through a rule from the conclusion to a premise, the restrictions for free variables, which are top-down formulated, cannot be taken into account. Dummett is therefore forced to substitute complex terms for free variables. Consequently if he reaches a $\forall x \phi(x)$ he has to proceed to a $\phi(t)$, where t is not a free variable, instead of a $\varphi(y)$ as the rule would say. Dummett admits this methodological gap by calling quantification-involving proof-theoretic justifications as of the third kind. Since the methodology has a gap here anyway according to the justification procedure, the problem of how to interpret formulas or sequences with free variables is not explicitly addressed.

But the justification method only uses the meaning stipulated by the rules and does not replace it. Therefore Dummett's method does not clarify how the possible or actual occurrence of a sequent including free variables in a rule, that stipulates the meaning of connectives or quantifiers, can be interpreted. Moreover it does not clarify how this lack of interpretation is differentiated from the missing primitive interpretation in the case of multi-conclusion sequents, which he himself described earlier as problematic. Hence he does not give an answer to the issues identified here.

If we are aiming for a proof-theoretic semantics in terms of rules governing only

$\neg, \vee, \wedge, \rightarrow, \exists$ and $\forall$ for first-order logic, then we cannot avoid the occurrence of syntactic entities which cannot be interpreted. This undermines the repudiation of multi-conclusion sequents, since they can be seen as just another artefact lacking an interpretation. Therefore a rule of the form

$$\frac{\Delta \Rightarrow \psi_1, \psi_2}{\Delta \Rightarrow \psi_1 \vee \psi_2}$$

is not more problematic than an $\forall$-quantifier introduction into the right side. This view correlates with proof-theoretic practice, where sequents are just taken seriously, when they have only one formula on the right side.[12]

Also, the extension to multi-conclusion sequents keeps the requirements of a compositional semantics satisfied in the same sense as von Kutschera formulated them in [30] and previously explained. For the satisfaction of the first demand can be easily seen by examination of the rules extended to multi-conclusion sequents. The second requirement will be explained below in the section about conservativity.

Thus a proof-theoretic semantics for classical logic, which does not differ more from ordinary language use than Dummett's, is constructed.

1.4 Comparison with other Positions and Responses to Objections

The first possible objection we want to scrutinise is that free variables do not need to be interpreted separately, since they have a primitive interpretation through the possibility of substituting terms for them. According to the aims of proof-theoretic semantics this cannot be understood as a closure property for initial sequents or deductions under substitution, since this would disturb the compositional character of the semantics. For in this case $\Delta(x) \Rightarrow \Gamma(x)$ can only be understood through the set of all sequents of the form

[12] If a sequent calculus is used for classical logic, then a provability predicate for a theory $T \vdash \varphi$ is defined as the provability of $\Delta \Rightarrow \varphi$, where $\Delta \subset T$, in the sequent calculus used.

$\Delta(t) \Rightarrow \Gamma(t)$ for any t, which is a significantly big part of the language.[13] However, it is possible to avoid running into holisms by formulating a rule of the form

$$\frac{\Delta(x) \Rightarrow \varphi(x)}{\Delta(x)[x/t] \Rightarrow \varphi(x)[x/t]},$$

where $\varphi(x)[x/t]$ denotes the formula one gets by substituting t for all occurrences of x in φ. A rule of this form still ensures a compositional character of the semantics, since $\varphi(x)[x/t]$ is understandable on the basis of φ and t. However the rule needs some variable restrictions to function properly in a setting of logic, which are similar to those governing $\forall$-quantification. Otherwise one might draw conclusions of the form $\forall x[\varphi(x) \to \varphi(f(x))]$ from it, which are generally considered as not purely logical.[14] Therefore one must demand in addition that, x does not occur in $\Delta(x)[x/t] \Rightarrow \varphi(x)[x/t]$ and no free variable in t freely occurs in $\Delta(x) \Rightarrow \varphi(x)$ either. But similarly as in the case of double negation, discussed earlier, such a stipulating rule is not just a separate stipulation governing free variables. The rule

$$\frac{\Delta(x) \Rightarrow \varphi(x)}{\Delta(x)[x/t] \Rightarrow \varphi(x)[x/t]},$$

together with the variable restrictions discussed above, expresses a meaning of free variables which is already given through the harmony of the $\forall$-quantifier rules. For instance the variable restrictions on the substitution rule allow us to form the following proof-figure for the special case where x does not occur in Δ, since these restrictions match those of the $\forall$-quantifier rule on the right side:

$$\dfrac{\dfrac{\Delta \Rightarrow \varphi(x)}{\Delta \Rightarrow \forall y \varphi(y)} \qquad \dfrac{\begin{array}{c}\varphi(t) \Rightarrow \varphi(t) \\ \vdots \\ \varphi(t), \Delta \Rightarrow \varphi(t)\end{array}}{\forall y \varphi(y), \Delta \Rightarrow \varphi(t)}}{\Delta \Rightarrow \varphi(t)}$$

[13] Here the term t must be substituted for any occurrence of x. Otherwise one might get from an initial sequent of the form $\varphi(x) \Rightarrow \varphi(x)$ to one of the form $\varphi(f(c_1)) \Rightarrow \varphi(f(c_2))$. The latter could be translated as "If Julia's father wears a hat, then Bob's father wears a hat.", which is generally seen as logically invalid.

[14] Following the previous example that might be translated as "When ever someone wears a hat, this person's father wears a hat too".

Moreover, this gives a theoretical basis to the previous mentioned naive view that variables are artefacts lacking an interpretation differentiable from the meaning of the $\forall$-quantifier, in the approach of proof-theoretic semantics.
Consequently it seems to be suitable to view substitution as a operation which gets its necessity through the lack of interpretation for free variables instead of interpreting variables through their possibility to get substituted.[15]

An objection similar to the one already discussed may be that free variables are some how primitive through their use as general or arbitrary names for the object to which the language is applied. But such a move disposes of a proof-theoretic semantics in favour of a model theoretical one following Tarski or Kripke. For such an objection presupposes a theory of meaning that surmounts the credo of proof-theoretic semantics, i.e. stipulating the meaning of the logical vocabulary through introduction and elimination rules.[16] If the use or meaning of free variables in a theory of meaning which is purely given through introduction and elimination rules does not differ from a primitive usage of names, then the inferences which are justifiable in terms of rules formulated via names should not differ from the set justifiable by rules formulated via free variables. For in proof-theoretic semantics this is the main demarcation criterion for separating two logics. In case of the $\forall$-quantifier this would be the indistinguishability of the following formulations according to the justification procedure.

[15] In this respect a compositional syntactic approach stands diametrically opposite to a holistic syntactic approach for which Carnap argues (see [11, pp. 142]).

[16] Given the close connection that variables have with respect to the quantifies, they should be considered as part of the logical vocabulary.

Formulation via free Variables	**Formulations via Names**
$\dfrac{\varphi(y)}{\forall x \varphi(x)}$	$\dfrac{\varphi(c) \text{ for any } c \text{ in } \mathcal{L}}{\forall x \varphi(x)}$

If a formulation via names has been chosen, the following differences can be found: In the case when $\mathcal{L}$ includes only one name the $\exists$-quantifier coincides with the $\forall$-quantifier, since the inference

$$\frac{\exists x \varphi(x)}{\forall x \varphi(x)}$$

is justifiable. If $\mathcal{L}$ includes more than one but finitely many names $c_1, ..., c_n$, then inferences of the form

$$\frac{\forall x_1 ... \forall x_i ... \forall x_j ... \forall x_{n+1} \varphi(x_1, ..., x_{n+1})}{\forall x_1 ... \forall x_j ... \forall x_i ... \forall x_{n+1} \varphi(x_1, ..., x_{n+1})},$$

where $x_1, ..., x_{n+1}$ are pairwise disjoint, are not justifiable any more. One might say that such inferences are in fact justifiable in the formulation with names, if a justification proceeds from a formula $\forall x \varphi(x)$ to a formula $\varphi(t)$, where t is any term, instead to a $\varphi(c_i)$. But according to the fundamental assumption as is explained in [13, pp. 274], doing so would be covertly using the formulation via free variables. Dummett faces serious complications, when it comes to a proof-theoretic justification of inferences including $\forall$-quantifiers, because he has to talk about an *adequate representative sample* $t_1, ..., t_n$, from which the t is taken, to make the step from $\forall x \varphi(x)$ to a $\varphi(t)$ plausible. But the notion of an adequate representative sample is very vague and sensitive to the particularities of an individual case. However in the case of a formulation of $\forall$-quantification via finitely many names the adequate representative sample is naturally given through the formulation of the rule itself. This becomes especially crucial in cases of a very small set of names, say $c_1, ..., c_5$, since in such a case the example of a $\forall$-quantifier permutation given above is particularly inconvenient while the adequate representative sample of $c_1, ..., c_5$ is highly natural.

In the case of infinitely (but countably) many names the two formulations coincide. Since the samples are in both cases potentially unbounded, the justification can be chosen to proceed identically under both formulations. However, beside the fact that it barely happens that ordinary language use operates with infinitely many names, it seems counter-intuitive that logical principles change according to the number of objects one wants to talk about.

The final objection which is discussed here criticises the actual way a justification must proceed in the previous given semantics for classical logic. In the case of

$$\frac{\neg[\exists x \neg\varphi(x)]}{\forall x\varphi(x)}$$

a justification would look as follows:

The sequent $\neg[\exists x \neg\varphi(x)] \Rightarrow \forall x\varphi(x)$ gives $\Rightarrow \forall x\varphi(x), \exists x\neg\varphi(x)$ through the meaning of $\neg$, which is stipulated by the $\neg$-introduction rule on the right side. The rule of $\exists$-introduction gives at least one t such that $\Rightarrow \forall x\varphi(x), \neg\varphi(t)$. As Dummett explains in [13], the fundamental assumption according to $\forall$-introduction gives $\Rightarrow \varphi(t), \neg\varphi(t)$. By using the meaning of $\neg$ again one gets $\varphi(t) \Rightarrow \varphi(t)$, which ends the justification.

The justification just performed necessarily uses multi-conclusion sequents while free variables do not occur, since one can choose t such that no free variable occurs. But as explained before, the cases under issue, which proceed from $\Delta \Rightarrow \forall x\varphi(x)$ to $\Delta \Rightarrow \varphi(x)$, were not excluded for lacking an interpretation, but for justifying an inference which is generally seen as invalid. Moreover the element stipulating the meaning, the rule, is necessarily formulated via free variables, which lack an interpretation if one does not want to interpret them by the $\forall$-quantifier. This raises the question: Why is the use of harmless sequents which lack an interpretation forbidden, while harmful ones, which have the same property, are used in the elements stipulating the meaning of a semantics? Moreover, since quantification rules appear nested in the rules for connectives during a justification, it is not possible to separate principles of first-order from propositional logic in first-order logic. Therefore an attempt, which tries to separate the use of sequents with free variables from those using multi-conclusions, to argue against the use of the latter but keeps the former, must be fruitless. However, in the same manner as for the fundamental assumption, one might try to establish an assumption, which separates the justification for first-order principles beforehand in a quantificational and a propositional part by the midsequent theorem. The midsequent theorem states that, in a calculus for which cut-elimination[17] has been established, the proof of a sequent which includes only formulas in prenex-normal-form can be transformed into a proof where the propositional part is done before any quantification takes place (see [57, p. 29]). However finding a midsequent (or a Herbrand disjunction) is, depending on the formal system that is in use, a hard task and it is not always clear how crucial the assumption that all formulas are in prenex-normal-form is. In any case the issue seems to be that in finding such a midsequent (or Herbrand disjunction) many inferences that one might want to justify are already in use. Hence one cannot separate first-order logic from proposition logic at the beginning of building a proof-theoretic semantics for first-order logic.

Following the arguments given earlier, using a sequent calculus in proof-theoretic semantics seems to be privileged. For an approach which operates on deductive relations, which are formulated as sequents here, might be distanced enough from ordinary language use to accept syntactic entities which lack an interpretation, but still close enough to ordinary language use by the way that introduction rules are formulated. In contrast, in a natural deduction formulation it might seem unnatural that entities which lack an interpretation through the occurrence of free variables appear in places where the well known concept of assuming a sentence is usually taking place.

[17] A property calculi, which are logical in nature, usually have.

1.5 Conservativity

As explained before, von Kutschera demands in his Genzen-semantics (given in [30]) conservativity for each individual connective over the others, which forces, together with the cut-rule, a particular form of the introduction rules on the left side. To explain what this means, the connection between the introduction rules for $\neg$ is given in the following.

$$\dfrac{\begin{array}{cc} \dfrac{\Delta', \varphi \Rightarrow}{\Delta' \Rightarrow \neg\varphi} & \dfrac{\Delta'' \Rightarrow \varphi}{\neg\varphi, \Delta'' \Rightarrow} \\ \vdots & \vdots \\ \Delta \Rightarrow \neg\varphi & \neg\varphi, \Delta \Rightarrow \end{array}}{\Delta \Rightarrow}$$

It is easy to see that $\Delta \Rightarrow$ can be obtained without using any rule for $\neg$, by using an application of the cut-rule at an earlier stage. So one can easily translate the given proof to the following one.

$$\dfrac{\begin{array}{cc} \Delta'' \Rightarrow \varphi & \Delta', \varphi \Rightarrow \\ \vdots & \vdots \\ \Delta \Rightarrow \varphi & \varphi, \Delta \Rightarrow \end{array}}{\Delta \Rightarrow} \quad .$$

Such translations can be found for every connective in a sequent calculus and are summarised in the so called cut-elimination algorithm.[18] Using this algorithm it is possible to show that, for each connective the calculus formed by the previous given rules for $\neg, \vee, \wedge$, and $\rightarrow$ is conservative over its fragments that are obtained by deleting the rules for exactly one of the connectives. Since side formulas do not essentially disturb this process, this property is preserved after extending the calculus by multi-conclusion sequents. Consequently all criteria for a compositional semantics are satisfied by the previous given proof-theoretical semantics for classical logic, because a formula can be fully understood on the basis of its subformulas together with the connective used

[18]This is a rather unfortunately chosen name, since it covers the syntactic nature of cut-elimination by constructive aspects. For as mentioned in [58], there are many cut-elimination algorithms in systems where the cut-rule is dispensable which are not syntactic in nature.

and the meaning of each connective is strictly separated from those of the others by the semantics. Moreover local and total harmony, in the sense given above, are fully established.

However, it was mentioned before that, in some sense, classical negation is not conservative over the other logical connectives, while intuitionistic negation is. This will be clarified in the following. The natural deduction calculus formed by the usual introduction and elimination rules for $\vee, \wedge$ and $\rightarrow$ proves the $\neg$-free fragment of intuitionistic logic. If one extends this calculus by $\neg$-rules giving intuitionistic negation, then no new $\neg$-free formulas can be proved in the calculus given by these rules while the $\neg$-rule that is usually taken as classical can. In this sense classical negation is not in total harmony with the rest of the connectives according to the proof-theoretic semantics given through the natural deduction rules. Moreover one might say that, the semantics given by the natural deduction rules for classical logic is not compositional in nature, by the lack of conservativity over the other connectives.[19] However, the reason for this is the way functional $\neg$-rules must be formulated in natural deduction. For in natural deduction $\neg$-rules, capturing intuitionistic negation is extremely different in their schematic form from those capturing classical negation. By contrast, in common sequent formulation the rules are very similar and only differ in allowing side formulas on the right side.

[19]Dummett explains in [13] that, according to his proof-theoretic semantics, which is given by natural deduction rules, classical negation refers to the consistency of the logic justifiable by his semantics. One might say this makes it holistic in nature.

Intuitionistic Negation	**Classical Negation**
$\dfrac{\varphi, \Delta \Rightarrow}{\Delta \Rightarrow \neg\varphi}$	$\dfrac{\varphi, \Delta \Rightarrow \Gamma}{\Delta \Rightarrow \Gamma, \neg\varphi}$
$\dfrac{\neg\varphi, \Delta \Rightarrow}{\Delta \Rightarrow \varphi}$	$\dfrac{\neg\varphi, \Delta \Rightarrow \Gamma}{\Delta \Rightarrow \Gamma, \varphi}$

However it must be stressed here that conservativity is used differently in these two cases. Since natural deduction operates on formulas, the conservativity notion is defined via provability of formulas, while in a sequent formulation it is the conservativity over deducible sequents, which are interpreted as arguments here.

As Schröder-Heister correctly asserts in [47], the calculus of natural deduction seems to have a strong connection with intuitionistic logic. This might not be very surprising in view of the fact that it is operating on formulas. For, this gives a semantics that is highly focussed on particular sentences rather than on inferential-relations. This circumstance coincides with the idea of intuitionism as a very rigorous justification system for particular sentences explained by the Brouwer-Heyting-Kolmogorov-interpretation, or its even more rigorous formal counterpart, realisability.

1.6 Conclusion

The present paper argues for the possibility of a proof-theoretic semantics for classical logic that uses multi-conclusion sequents, which is no further from ordinary language use than the one given by Dummett. Consequently, the correlation of logics with metaphysical stances is undermined. However, this was made possible by a paradigm shift; instead of primarily focusing on formulas and their use as such, sentences are understood through their function in an argument while avoiding a holistic meaning theory. For instead of using formulas as smallest operative unit we use sequents. The appearance of new syntactic objects which lack an interpretation in this process should not be worrying, because their appearance comes with baggage when a syntactic approach is systematised.

Chapter 2

Theory Reduction

To reduce a theory to another one has been a central concept of modern logic which is included in many approaches since the very beginning. For Hilbert already gave one in his *Grundlagen der Geometrie* from 1903 [27].[1] Recently Michael Rathjen used a reductive approach to formulate a notion of ordinal analysis in [40]. Since the aim of this thesis is to show that Rathjen's notion can be made even more convenient without any metamathematical or foundational costs by using particular theories of inductive definitions, we take a closer look at the commonly used notions of reduction in this chapter. Such a survey was already done by Karl-Georg Niebergall in [35], which therefore seems to be a good starting point. However Niebergall's aims differ from ours. For he wants to evaluate the different notions to find "the best". Also since he sees his work as a first step towards ontological reduction, his argumentation is flavoured by a branch of analytic metaphysics.[2] Consequently Niebergall's analysis is based on the philosophical intuitions of this tradition and phrased in a terminology which is influenced

[1] It is common in the logic community to view Hilbert's methods as an early model-theoretic construction. However Patricia Blanchette convincingly argued for a syntactic reading at the Logic Colloquium 2014.

[2] One should indicate here that the field of analytical metaphysics is the target of many objections. See [20] and T. Mohrmann's introduction in [12] for an extensive critique.

by a model-theoretic perspective on logic, as is common for more recent discussions in analytic philosophy. We therefore give a rather different presentation. We will argue for our preference of proof-theoretical reduction to interpretability in accordance with the aims of this thesis.[3] We will also have a closer look at realisability. For it is the reduction method which is heavily used and scrutinised in Chapter 3 and Chapter 6.

2.1 Preliminaries

Before we can talk about theory reductions we have to give some central definitions and state some important proof theoretical results. Unfortunately the concept of elementary recursive functions is not as well known as that of primitive recursive functions, therefore we will give a definition here.

Definition 2.1.1 *The class of* elementary (recursive) functions $\mathcal{ERF}$ *is defined as:*

1. $\mathcal{ERF}$ *contains: The constant* 0 *function, successor* S*, projection, addition* $+$*, multiplication* $\cdot$*, modified subtraction* $\dot{-}$ *and exponentiation* 2^x*.*
2. $\mathcal{ERF}$ *is closed under composition.*
3. $\mathcal{ERF}$ *is closed under limited recursion: Assume that* $h, g, k \in \mathcal{ERF}$ *and for all* $\vec{x}, y$

$$f(\vec{x}, 0) = g(\vec{x}),$$

$$f(\vec{x}, y+1) = h(\vec{x}, y, f(\vec{x}, y)),$$

$$\text{and } f(\vec{x}, y) \leq k(\vec{x}, y),$$

   then $f \in \mathcal{ERF}$*.*

[3]Niebergall favours interpretability and is challenged by Feferman in [18].

The class of primitive recursive functions $\mathcal{PR}$ *is defined similarly, but without the bounding condition in the third clause.*
By closing off the primitive recursive functions under the μ*-operation we get the (partial)* recursive functions $\mathcal{R}$*. That is, if* $g \in \mathcal{R}$ *is a* k*-ary function and*

$$f(x_0, ..., x_{k-1}) = \mu y[g(x_0, ...x_{k-1}, y) = 0] \text{ (the smallest such } y),$$

then $f \in \mathcal{R}$.[4]

It is possible to code the formation of a recursive k-ary function f into a natural number e_f, by inductively going through Definition 2.1.1, and to give an elementary predicate[5] $\mathcal{T}^k$ as well as an elementary function $\mathcal{U}$ such that $f(\vec{n}) = m$ is equivalent to $\mathcal{U}(\mu y[\mathcal{T}^k(e_f, \vec{n}, y)]) = m$. The latter is often abbreviated by $\{e_f\}^k(\vec{n}) = m$, which is called the Kleene brackets. For more details see [50].

To fix notation we briefly talk about some standard definitions in logic (for more details see [16]). If τ is a set of symbols for predicates, functions and constants, then $\mathcal{L}(\tau)$ denotes the set of well formed formulas that are generated by τ. We call $\mathcal{L}(\tau)$ a language. We generally assume that τ is finite. The terms, formulas and sentences are defined inductively in the standard way for an n^{th}-order language.[6] If we want to specify the order of a language we denote it by a superscript, e.g. by $\mathcal{L}^n(\tau)$. If the superscript is suppressed the order is arbitrary. To denote symbols and expressions we use the following conventions: We use x, y, z for variables, $s, t, t_0, t_1, ..., t_n$ for terms and $\varphi, \psi, \phi, \varphi_1, ..., \varphi_n$ for formulas. Also we write $\langle t_0, t_1, t_2 \rangle$ for $\langle \langle t_0, t_1 \rangle, t_2 \rangle$. By a theory T we mean an elementarily definable set, if not specified otherwise, of formulas (called

[4] The functions in $\mathcal{R}$ are in general not total, since such a y, which is required by the μ-operation, might not exist.

[5] We identify predicates with their characteristic functions.

[6] We use the usual notational conventions concerning to logical connectives, brackets and identity, e.g. $x \neq y$ means $\neg(x = y)$.

the axioms) together with a logic. If it is not stated otherwise, the logic is the classical logic suitable to the order of the language. Other logics are specified by superscripts, e.g. in the case of intuitionistic logic by an i as in T^i.[7] The language of a theory T is denoted by $\mathcal{L}_T$ and is the $\mathcal{L}(\tau)$ whose τ contains only those symbols occurring in the axioms of T. In proofs we usually view the logic as given by a Hilbert or Gentzen system (see [58] for an introduction and Appendix C for the systems frequently used here). We denote provability in all contexts by $\vdash$. In proofs we will frequently switch back and forth between Hilbert and Gentzen systems. This practice is justified by the following theorem.

Theorem 2.1.2 *Let* $\varphi \in \mathcal{L}(\tau)$.
There is an elementary recursive function which assigns to every deduction $d_H \vdash \varphi$ *in a Hilbert system a deduction* $d_G \vdash \varphi$ *in a Gentzen system.*
There is an elementary recursive function which maps every deduction $d_G \vdash \varphi$ *in a Gentzen system to a deduction* $d_H \vdash \varphi$ *in a Hilbert system.*

The deductive closure of a theory T (the set of all formulas provable from T) is denoted by $T^{\vdash}$.
In the following we define important theories and notions that are used throughout in the text. All the following definitions use the first order language $\mathcal{L}^1_Q := \mathcal{L}^1(\{=, \leq, S, +, \cdot, \bar{0}\})$, where $S, +$ and $\cdot$ are read as successor, addition and multiplication respectively. We use $\bar{n}$ to denote the term $S(...S(\bar{0})...)$ with successor-depth n. A bounded quantifier is a quantifier in the following context

$$(\exists x \leq t)\varphi(x, y) :\equiv (\exists x)(x \leq y \wedge \varphi(x))$$

$$(\forall x \leq t)\varphi(x, y) :\equiv (\forall x)(x \leq t \rightarrow \varphi(x, y)),$$

[7] Since we are only interested in syntactical transformations of deduction systems in which the deductions of classical logic are a superset of those of other logics in use here, this is the most appropriate thing to do.

where x does not occur in t.

Definition 2.1.3 *1. The set of $\Sigma^0_0 = \Pi^0_0$ formulas contains all atomic formulas of a language $\mathcal{L}$ and is the least set that is closed under all logical connectives and bounded quantification.*

2. Σ^0_{n+1} contains Π^0_n and is closed under $\wedge$,$\vee$, bounded quantification and unbounded $\exists$-quantification.

3. Π^0_{n+1} contains Σ^0_n and is closed under $\wedge$,$\vee$, bounded quantification and unbounded $\forall$-quantification.

4. $\Pi^0_\infty := \bigcup_{n \in \mathbb{N}} \Pi^0_n$

5. We say a formula is Δ^0_n in a theory T $:\Leftrightarrow$ It is equivalent to a Σ^0_n as well as to a Π^0_n-formula in T.

6. A set is called Σ^0_n, when there is a Σ^0_n-formula defining it in the standard model.[8] We use an analogous notation for Π^0_n sets. A set is Δ^0_n, if the equivalences of its definition to a Σ^0_n and a Π^0_n-formula are both true in the standard model.

Note that being a Π^0_n- or Σ^0_n-formula is relative to the language in use.

Definition 2.1.4 *1. The theory Q contains, besides the usual axioms of identity, the following axioms:*

$$S(x) \neq \bar{0}$$

$$S(x) = S(y) \to x = y$$

$$x \neq \bar{0} \to (\exists y)(x = S(y))$$

$$x + \bar{0} = x$$

[8] The standard model of $\mathcal{L}^1_Q$ is $\langle \mathbb{N}, \leq^{\mathbb{N}}, S^{\mathbb{N}}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, \bar{0}^{\mathbb{N}} \rangle$.

$$x + S(y) = S(x + y)$$

$$x \cdot \bar{0} = \bar{0}$$

$$x \cdot S(y) = (x \cdot y) + x$$

$$x \leq y \equiv (\exists z)(z + x = y)$$

2. *The theory $I\Sigma_n$ is defined as Q together with the restricted induction schema:*

$$\varphi(0) \wedge (\forall x)[\varphi(x) \rightarrow \varphi(S(x))] \rightarrow (\forall x)\varphi(x),$$

where $\varphi \in \Sigma_n^0$.

3. *The theory $I\Pi_n$ is defined analogously.*

4. *Peano Arithmetic is defined as $PA := \bigcup_{n \in \mathbb{N}} I\Sigma_n$.*

5. *Elementary Arithmetic (EA) is defined as Q together with induction for all Σ_0^0 formulas and the following additional axioms (by extending the language by the function symbol 2)*

$$2^{\bar{0}} = 1$$

$$2^{S(x)} = 2^x \cdot \bar{2}.$$

6. *Primitive Recursive Arithmetic (PRA) is defined as Q together with axioms defining all primitive recursive functions and the schema of induction for atomic formulas of $\mathcal{L}^1_{PRA}$.*

7. *Heyting Arithmetic (HA) is defined as PRA^i together with induction for all formulas of $\mathcal{L}^1_{PRA}$.*

Remark 2.1.5 *In case the reader wonders how all primitive recursive functions can be introduced, she can consult Definition 6.2.1 and Definition 6.2.2. To improve readability we avoid those technicalities here. Moreover the exact way in which function symbols are introduced is only needed in the Lemmata following Definition 6.2.2. Therefore Chapter 6 seems to be the appropriate place.*

Consequently $\mathcal{L}^1_{EA}$ and $\mathcal{L}^1_{PRA}$ are richer languages than $\mathcal{L}^1_Q$. It is a well known fact (see [26] or [50]) that the graph of any recursive function can be represented in Q by a Σ_1 formula, e.g. by coding the Kleene brackets. This motivates the following definition.

Definition 2.1.6 *Let T be a theory such that $\mathcal{L}_Q \subset \mathcal{L}_T$. We say a function $f : \mathbb{N}^k \to \mathbb{N}$ is provably recursive in T $:\Leftrightarrow$ There is a Σ^0_1-formula $\varphi_f(\vec{x}, y)$ such that*

1. $f(\vec{n}) = m \Leftrightarrow \varphi_f(\vec{\bar{n}}, \bar{m})$ *is true (in the standard model);*

2. $T \vdash (\forall \vec{x})(\exists y)\varphi_f(\vec{x}, y)$*;*

3. $T \vdash \varphi_f(\vec{x}, y) \wedge \varphi_f(\vec{x}, y') \to y = y'$.

Remark 2.1.7 *It is standard to say that φ_f represents an f, if it satisfies* 1. *While φ_f is said to define a function f, if it also satisfies* 2 *and* 3.

The following theorem combines Parsons' and Parikh's Theorem, which are proved in [36] and [9, p. 87] respectively.

Theorem 2.1.8 *$I\Sigma_1$ and PRA prove the same Π^0_2 sentences. Therefore the provably recursive functions of $I\Sigma_1$ are the primitive recursive functions.*
The provably recursive functions of $I\Sigma_0$ all have polynomial growth rate.

It is obvious that elementary functions have super-polynomial growth rate. Hence to give a theory that has exactly the elementary recursive functions as provably recursive ones, it is not sufficient to weaken the induction schema. We also have to add a function of function of suitable growth rate. In fact it is enough to add the exponentiation function to $I\Sigma_0$. Let $\exp(x, y)$ be a Σ_1 formula defining the graph of exponentiation to the base 2, i.e. $x \mapsto 2^x$. We denote the totality statement for exponentiation by

$$\text{Exp} :\equiv (\forall x)(\exists y)\exp(x, y).$$

Theorem 2.1.9 *The provably recursive functions of $I\Sigma_0$+Exp and EA are the elementary recursive functions.*

Proof

There is a very nice proof of this result in [50, p. 122].□

It is well known that syntax can be coded by elementary functions (see [50]); we denote the code of a syntactic object by framing it with two corners, e.g. $\ulcorner\varphi\urcorner$ in case of a formula φ. Therefore one can give for a theory T in $I\Sigma_0 + \text{Exp}$ a standard provability predicate[9] $\text{Prov}_T(x, y)$ saying that x is a code of a deduction for the formula with the code y. The formula $\text{Prov}_T(x, y)$ is Σ^0_0, which makes provability $\text{Pr}_T(y)$, which is defined as $(\exists x)\,\text{Prov}_T(x, y)$, a Σ_1-formula. Consequently unprovability and therefore consistency is Π^0_1. One can just take an appropriate inconsistency[10] $(\bot)$ in T and define $\text{Con}(T) :\equiv (\forall x)\neg\,\text{Prov}_T(x, \ulcorner\bot\urcorner)$.[11] The following lemma states Σ^0_1-completeness and formalised Σ^0_1-completeness respectively and has many nice applications.

Lemma 2.1.10 *Assume that T is a theory such that $\mathcal{L}^1_Q \subset \mathcal{L}_T$. Then*

1. *$Q \vdash \varphi$ for any true Σ^0_1-sentence φ.*
2. *$I\Sigma_1 \vdash \varphi \to Pr_T(\ulcorner\varphi\urcorner)$ for any Σ^0_1-sentence φ.*

Proof

[9] By Theorem 2.1.2 we do not care about which deduction system is used and will switch between Hilbert and sequent style systems.

[10] If $Q \subset T$ we usually take $\bot$ to be $0 = 1$.

[11] As Fefermann elaborated in [17], one can get into trouble by not specifying a representation of a theory when coding syntax. However we always assume that we use a natural representation when defining $\text{Prov}_T(x, y)$; in the case of a finite theory just a finite disjunction. But we will never give a statement in a generality where such problems occur.

1. By convincing oneself that Q calculates correctly.

2. The only actual proof which I know of can be found in [24, pp. 69-72].

□

It is also possible to partially define truth in $I\Sigma_0 + \text{Exp}$ via syntax coding as the next theorem states. Here we use Feferman's dot-notation $\dot{x}$ to denote the function that maps x to the x^{th} numeral.[12]

Theorem 2.1.11 *For every $n \geq 1$ there is a Σ^0_n formula $\mathsf{T}_{\Sigma_n}(x)$ (where x is the only free variable) such that, if $\varphi(y_1, ..., y_m)$ is a Σ^0_n-formula, then*

$$I\Sigma_1 \vdash \varphi(y_1, ..., y_m) \leftrightarrow \mathsf{T}_{\Sigma^0_n}(\ulcorner \varphi(\dot{y}_1, ..., \dot{y}_m)\urcorner).$$

In the case of a Σ^0_0-formula $\varphi(y_1, ..., y_m)$, there is a Σ^0_1-formula $\mathsf{T}_{\Sigma^0_0}(x)$ such that

$$I\Sigma_0 + \textit{Exp} \vdash \varphi(y_1, ..., y_m) \leftrightarrow \mathsf{T}_{\Sigma^0_0}(\ulcorner \varphi(\dot{y}_1, ..., \dot{y}_m)\urcorner).^{13}$$

Analogous statements hold for Π^0_n-sentences.

---

[12]The dot-notation is closely related to the bar-notation, which is defined in the text above Definition 2.1.3 as

$$\bar{1} := S(\bar{0})$$

$$\overline{n+1} := S(\bar{n});$$

note that $\bar{0}$ is a primitive symbol of $\mathcal{L}_Q$. While the bar-notation is an abbreviation of terms at the meta-level, the dot-notation is an abbreviation of a formalisation of a function that outputs the codes of these terms, i.e. $\dot{x} := \ulcorner \bar{x} \urcorner$.

[13]Note that this truth-predicate cannot be used in EA, because the additional function symbol 2^x spoils the computational bound, that is required by the truth-predicate, on terms. In fact it is a deeper metamathematical insight that one either has a truth-predicate with a low syntactical-complexity for a Π_2-axiomatisation ($I\Delta_0 + Exp$) or has a Π_1-axiomatisation (EA) without such a truth-predicate.

Proof

1. See [26, p.59].

2. See [26, Corollary 5.5, p.365].

□

It is a well known fact that the usual Gentzen system for first order logic satisfies cut-elimination (see [58]) as well as that theories formulated within a Gentzen system have partial cut elimination[14] (see [9]). As it is shown in [26] (and can also be seen by the methods developed in Chapter A) the elimination of a single cut can be coded and carried out by an elementary function and is therefore available in $I\Sigma_0 + \text{Exp}$. However full cut-elimination cannot be done in $I\Sigma_0 + \text{Exp}$ and EA, since EA can be formulated by Σ^0_0-formulas (in $\mathcal{L}_{EA}$) and therefore would be able to prove its own consistency by using partial cut-elimination together with Theorem 2.1.11. Since the super-exponential function is primitive recursive and dominates the cut-elimination procedure, cut-elimination and partial cut-elimination can be fully carried out in $I\Sigma_1$ and PRA.

Another important proof-theoretical tool is reflection principles. We will only define the simplest form here and prove a very simple but handy fact about it. For more information one may consult [3].

Definition 2.1.12 *Assume that $\varphi \in \mathcal{L}^1_Q$ is a sentence. The local reflection schema for a*

[14]Roughly speaking, cut-elimination up to cuts whose cut-formula has not more alternating quantifiers than the axioms of the theories.

theory T (in symbols Rfn(T)) such that $\mathcal{L}^1_Q \subset \mathcal{L}^1_T$ is the schema

$$Pr_T(\ulcorner\varphi\urcorner) \to \varphi.$$

If the schema is restricted to a set of sentences Γ, then we denote it by Rfn$_\Gamma(T)$.

The following lemma is used when dealing with proof-theoretical reduction.

Lemma 2.1.13 $I\Sigma_0 + \mathit{Exp} \vdash \mathit{Con}(T) \leftrightarrow \mathit{Rfn}_{\Pi^0_1}(T)$

Proof

"$\Rightarrow$:" Assume that $\varphi \in \Pi^0_1$. Hence $\neg\varphi \in \Sigma^0_1$ and therefore

$$I\Sigma_0 + \text{Exp} \vdash \neg\varphi \to \text{Pr}_T(\ulcorner\neg\varphi\urcorner)$$

by Lemma 2.1.10.[15] Consequently the assumption $\neg\varphi$ in $I\Sigma_0 + \text{Exp}$ gives $\text{Pr}_T(\ulcorner\neg\varphi\urcorner)$. Since we can also assume that $\text{Con}(T)$ holds, we get $\neg\,\text{Pr}_T(\ulcorner\varphi\urcorner)$; which leads to

$$I\Sigma_0 + \text{Exp} + \text{Con}(T) \vdash \neg\varphi \to \neg\,\text{Pr}_T(\ulcorner\varphi\urcorner).$$

And we get the φ-instance of $\text{Rfn}_{\Pi^0_1}(T)$ by contraposition.

"$\Leftarrow$:" Assuming $\text{Rfn}_{\Pi^0_1}(T)$ and instantiating it by $\ulcorner 0 = 1\urcorner$ in $I\Sigma_0 + \text{Exp}$ gives by contraposition and the definition of $\text{Con}(T)$

$$I\Sigma_0 + \text{Exp} + \text{Rfn}_{\Pi^0_1}(T) \vdash 0 \neq 1 \to \text{Con}(T).$$

Since $I\Sigma_0 + \text{Exp} \vdash 0 \neq 1$, the theorem is proven.□

The notion of theory reduction is closely related to the notion of conservativity. To fix a notation we will define this well known concept in the following.

[15]The function defined in [24] is in fact elementary and, hence, the proof of Lemma 2.1.10 proves an analogous statement about $I\Sigma_0 + \text{Exp}$.

Definition 2.1.14 *Assume that $S \subset T$, $\mathcal{L}_S \subset \mathcal{L}_T$ and $\Gamma \subset \mathcal{L}_S$.*
The theory T is Γ-conservative over S (in symbols $T \subseteq_\Gamma S$) $:\Leftrightarrow$ Any $\varphi \in \Gamma$ that satisfies $T \vdash \varphi$ also satisfies $S \vdash \varphi$.

Some of the previous results can be reformulated via this notion. The final result of this section is a very useful conservativity result as well.

Lemma 2.1.15 *Assume that T is a theory such that $\mathcal{L}_Q \subset \mathcal{L}_T$ and $I\Delta_0 + Exp \subset T^{\vdash}$. Then $T \subseteq_{\Pi^0_1} I\Delta_0 + Exp + Con(T)$.*

Proof
Assume that $T \vdash \varphi$ for $\varphi \in \Pi^0_1$. Therefore $I\Delta_0 + \mathrm{Exp} \vdash \mathrm{Pr}_T(\ulcorner\varphi\urcorner)$ by Lemma 2.1.10. By Lemma 2.1.13 we get

$$I\Delta_0 + \mathrm{Exp} + \mathrm{Con}(T) \vdash \mathrm{Rfn}_{\Pi^0_1}(T).$$

Combining these two results gives $I\Delta_0 + \mathrm{Exp} + \mathrm{Con}(T) \vdash \varphi$.$\Box$

2.2 Some Syntactic Notions of Theory Reduction

As already mentioned above, our interest in theory reduction emerges from our wish to motivate the general notion of ordinal analysis that is given in [40]. Since some steps in an ordinal-analysis can be viewed as a syntactic reduction[16], we are mainly concerned with syntax here. However we will sketch some model-theoretical reductions as well, when they are related to a syntactic one that is under consideration.[17]

[16]This will be explained in chapter 3.

[17]There is no such thing as a semantic reduction that is in opposition to a syntactic reduction. The correct opposite to a syntactic reduction is a referential or model-theoretical reduction. This is because a

2.2.1 Interpretability

The notion of interpretability is arguably one of the most common forms of theory reduction in the logic-related literature. Roughly speaking, it can be viewed as a logic-preserving translation of the non-logical lexicon of one theory into another. In more model theoretical terms one can view interpretability as defining a model of one theory in a second theory. More precisely:

Definition 2.2.1 *1. Let $\mathcal{L}^1_1$ be a language and T a theory in the language $\mathcal{L}^1_2$.*[18] *An interpretability translation $*$ of $\mathcal{L}^1_1$ into T is given by:*

(a) A $\chi(x) \in \mathcal{L}^1_2$ such that $T \vdash (\exists x)\chi(x)$.

(b) For each constant c of $\mathcal{L}^1_1$, a formula $\psi_c(x)$, with all free variables exhibited, such that

$$T \vdash (\exists! x)(\chi(x) \wedge \psi_c(x)).$$

(c) For each n-ary predicate P of $\mathcal{L}^1_1$, a formula $\psi_P(x_1, ..., x_n) \in \mathcal{L}^1_2$ with all free variables exhibited.

(d) For each n-ary function symbol f of $\mathcal{L}^1_1$ a formula $\psi_f(x_1, ..., x_n, y) \in \mathcal{L}^1_2$, with all free variables exhibited, such that

$$T \vdash \bigwedge_{i=1}^{n} \chi(x_i) \rightarrow (\exists! y)(\chi(y) \wedge \psi_f(x_1, ..., x_n, y)).$$

The interpretation φ^* is then defined inductively over the definition of terms and formulas. At the atomic level we use (in the obvious way) the formulas given above

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"good syntactic reduction" is good, when semantically explanatory, and syntactic, when lacking referential concepts. Or, as Quine would put it, the reason why we talk about syntax and reference is because we are not able to directly grasp "semantic" by a reasonable methodology.

[18] Note that we assumed at the beginning of this chapter that the non-logical alphabet is finite.

for predicate, constant and function symbols respectively. During the induction step $*$ *distributes over the logical particles as expected:*

$$(\neg\varphi)^* \equiv \neg\varphi^*$$

$$(\varphi \circ \psi)^* \equiv \varphi^* \circ \psi^* \text{ for } \circ \in \{\wedge, \vee, \rightarrow\}$$

$$[(\exists x)(\varphi(x))]^* \equiv (\exists x)(\chi(x) \wedge \varphi(x)^*)$$

$$[(\forall x)(\varphi(x))]^* \equiv (\forall x)(\chi(x) \rightarrow \varphi(x)^*)$$

2. *Assume that* S *and* T *are two first-order theories in the languages* $\mathcal{L}^1_S$ *and* $\mathcal{L}^1_T$ *respectively.*
   Then S *is interpretable in a theory* T *(in symbols* $S \preceq T$*)* $:\Leftrightarrow$ *There is a relativized translation* $*$ *of* $\mathcal{L}^1_S$ *in* T *such that for every axiom* $\varphi(x_1, ..., x_n)$ *of* S*,*

$$T \vdash \bigwedge_{i=1}^{n} \chi(x_i) \rightarrow \varphi^*(x_1, ..., x_n).$$

3. *Let* S *and* T *be as before.* S *is locally interpretable in* T *(in symbols* $S \preceq_{loc} T$*)* $:\Leftrightarrow$ *For each axiom* φ *of* S*,* $\{\varphi\} \preceq T$*.*

This gives the following corollary.

Corollary 2.2.2 *If* $S \preceq T$ *via* $*$*, then for each* $\varphi(x_1, ..., x_n) \in \mathcal{L}^1_S$*, such that* $S \vdash \varphi(x_1, ..., x_n)$*,* $T \vdash \bigwedge \chi(x_i) \rightarrow \varphi^*(x_1, ..., x_n)$*; in particular, for each sentence* $\varphi \in \mathcal{L}^1_S$*,* $S \vdash \varphi$ *implies* $T \vdash \varphi^*$*.*

In the case of $\mathcal{L}^1_T \subset \mathcal{L}^1_S$ and when interpretability can be shown via the trivial translation, i.e. $\chi(x)$ is taken to be $x = x$ and the translation is the identity, one can obtain a conservativity result. For in such a case $\varphi^* \equiv \varphi$.
There are two notions of reduction that are known to be equivalent to interpretability and are model-theoretical in nature.

Definition 2.2.3 *Let S and T be two first-order theories in the languages $\mathcal{L}_S^1$ and $\mathcal{L}_T^1$ respectively.*
$S \lhd_1 T :\Leftrightarrow$ There is an $f : \mathcal{L}_S^1 \to \mathcal{L}_T^1$ which distributes over $\neg, \wedge, \forall, \exists$ (as in Definition 2.2.1) and for any model $\mathcal{A}$ of T there exists a model $\mathcal{B}$ of S such that for any formula $\varphi \in \mathcal{L}_S^1$ it holds that $\mathcal{B} \models \varphi$ iff $\mathcal{A} \models f(\varphi)$.

The equivalence to interpretability was established by Montague [34] . The other notion was proved equivalent to interpretability by Benthem and Pearce [4] and is defined as follows.

Definition 2.2.4 *Let S and T be as before. $S \lhd_2 T :\Leftrightarrow$ There is a function F : Mod$(T) \to$ Mod(S) which respects $\mathcal{L}_T^1$-isomorphisms, $\mathcal{L}_T^1$-ultraproducts and satisfies $|F(\mathcal{A})| \subset |\mathcal{A}|$ for each $\mathcal{A} \in$ Mod(T).*[19]

In the following we will give some examples of properties or meta-theorems about interpretability.

Theorem 2.2.5 *For all theories S,T and all formulas φ, ψ:*
If $S \preceq T + \varphi$ and $S \preceq T + \psi$, then $S \preceq T + (\varphi \vee \psi)$.

Proof
See [32, p. 82]□

Another important property is the following. Here $S \restriction k$ includes those elements of S with a code that is numerically smaller than k.

Theorem 2.2.6 *Assume that $PA \subset T$ is a theory with a Δ_1^0 definition and is formulated in $\mathcal{L}_Q^1$. If S is another theory with a Δ_1^0 definition, then the following are equivalent*

[19] By $|\mathcal{A}|$ we denote the domain of the model $\mathcal{A}$ and Mod(T) denotes the set of all models of a theory T.

1. $S \preceq T$

2. $S \preceq_{loc} T$

3. *For each* k, $T \vdash Con(S \restriction k)$

Proof

See [26, pp. 169-171]. □

2.2.2 Proof-Theoretical Reduction

The notion of Proof-Theoretical Reduction was introduced by Fefermann 1988 in [19] to extend Hilbert's Program by an approach of relativised consistency. While the consistency of a system enlarging arithmetic cannot be established from a purely finitistic perspective, relativised consistency proofs can often be done finitistically. This can be generalised to a group of questions of the form: can one justify the use of certain principles in terms of a less questionable set of principles by assuming that the set of those latter principles is already justified?

An example for such a task would be the justification of impredicative principles by predicative notions. Here justification of principles means showing that they are at least consistent with each other. But proof-theoretical reduction is much more than just consistency. It can be viewed as a strong version of formalised conservativity.

To improve readability of the following definition, we use abbreviations for formulas as follows. $f(x) = z$ denotes a Σ^0_1-formula $\varphi_f(x, z)$ that represents the graph of a function f in a natural way (see Remark 2.1.7).[20] To express totality we use

$$f \downarrow :\Leftrightarrow (\forall x)(\exists y)\varphi_f(x, y).$$

[20] If the recursive functions are coded via Definition 2.1.1 and not as Turing-machines, then one can use Kleene brackets here:

$$\varphi_f(x, z) :\equiv (\exists y)[\mathcal{T}(e_f, x, y) \land \mathcal{U}(y) = z].$$

Definition 2.2.7 *Let f be a recursive function. Assume that S and T are two theories in the languages $\mathcal{L}_S$ and $\mathcal{L}_T$ respectively such that $Q \subset T$ and that there is an elementary recursive set of formulas $\Gamma \subset \mathcal{L}_S \cap \mathcal{L}_T$.*
We say S is proof-theoretically reducible to T according to Γ by f (in symbols $f : S \leq_\Gamma T$) :⇔ There is a representation of f in $\mathcal{L}_T$ such that

$$T \vdash f \downarrow$$

$$T \vdash (\forall x)(\forall y)(\exists z)[\Gamma(y) \wedge \textit{Prov}_S(x, y) \rightarrow f(x) = z \wedge \textit{Prov}_T(z, y)].$$

We express the existence of such a function f, such that $f : S \leq_\Gamma T$, by $S \leq_\Gamma T$.
If $S \leq_\Gamma T$ and $T \leq_\Gamma S$ holds, then we write $S \equiv_\Gamma T$.

Note both requirements are Π^0_2-statements. But together the latter is also Δ^0_1 in T, since $\mathcal{U}$ is a function, which makes its output unique.

Corollary 2.2.8 *If S and T are as in Definition 2.2.7 and $S \leq_\Gamma T$ where $\bot \in \Gamma$ (e.g. S and T are arithmetical theories and $\bot$ is $0 = 1$), then*

$$T \vdash Con(T) \rightarrow Con(S).$$

Proof
Instantiate the second requirement in Definition 2.2.7 with $\ulcorner \bot \urcorner$ for y and take the contrapositive of this instantiation.□

It goes back to an observation by Kreisel that one can obtain a finitistic[21] relative consistency proof from most proof-theoretical reductions that have been given so far. This

The term "natural" means that we know the function f quite well from an informal-proof and that we have chosen a representation φ_f (or code e_f) that supports our formalisation of this proof.

[21]Finitistic means here, following Tait [56], that the proof can be carried out in PRA. But since Proposition 2.1.8 holds, one can also chose $I\Sigma_1$.

is the case when the function from the reduction is primitive recursive (and $I\Sigma_1 \subset T$). The following argument is partly taken from [40]:
Assuming that $f : S \leq_\Gamma T$, $\bot \in \Gamma$ and that f is primitive recursive. Hence we can choose a representation of f such that

$$I\Sigma_1 \vdash f \downarrow$$

$$(*) T \vdash (\forall x)[\text{Prov}_S(x, \ulcorner \bot \urcorner) \to (\exists y)(f(x) = y \wedge \text{Prov}_T(y, \ulcorner \bot \urcorner))].$$

Since $f(x) = z$ is a Σ^0_1-formula defining a function in $T \supset I\Sigma_1$, the succedent of the implication above is equivalent to the Π^0_1-formula

$$(\forall y)[f(x) = y \to \text{Prov}_T(y, \ulcorner \bot \urcorner)].$$

Since this makes $(*)$ equivalent to a Π^0_1-sentence, we get (by Lemma 2.1.15) $I\Sigma_1 + \text{Con}(T) \vdash (*)$. Which leads to

$$I\Sigma_1 + \text{Con}(T) \vdash \text{Con}(T) \to \text{Con}(S),$$

by the provable totality of f in $I\Sigma_1$ (see Theorem 2.1.8). Hence

$$I\Sigma_1 \vdash \text{Con}(T) \to \text{Con}(S).$$

Therefore, inspired by Hilbert's Programme, sometimes logicians use the term proof-theoretical reduction also in the weaker sense of relative consistency.

Definition 2.2.9 *Let S and T be theories. Then:*

$$S \lhd_{RC} T :\Leftrightarrow I\Sigma_1 \vdash Con(T) \to Con(S).$$

The order $\lhd_{RC}$ is therefore transitive. However, to come closer to the idea of the broader Definition 2.2.7, one may have a closer look at the following.

Definition 2.2.10 *Let S and T be theories. Then:*

$$S \blacktriangleleft_{RC} T :\Leftrightarrow T \vdash Con(T) \to Con(S).$$

Niebergall gives in [35] a counterexample to the claim that $\blacktriangleleft_{RC}$ is transitive.

Theorem 2.2.11 *There are arithmetical sound recursively enumerable theories S, T, U such that $S \blacktriangleleft_{RC} T$ and $T \blacktriangleleft_{RC} U$, but not $S \blacktriangleleft_{RC} U$.*

Proof

Take

$$\begin{aligned} S &:= I\Sigma_1 + \text{Con}(I\Sigma_1 + \text{Con}(PA)) \\ T &:= PA \\ U &:= I\Sigma_1 + \text{Con}(PA) \end{aligned}$$

so we have to prove the following to establish the claim of the theorem.

1. $PA \vdash \text{Con}(PA) \to \text{Con}(I\Sigma_1 + \text{Con}(I\Sigma_1 + \text{Con}(PA)))$
2. $I\Sigma_1 + \text{Con}(PA) \vdash \text{Con}(I\Sigma_1 + \text{Con}(PA)) \to \text{Con}(PA)$
3. $I\Sigma_1 + \text{Con}(PA) \nvdash \text{Con}(I\Sigma_1 + \text{Con}(PA)) \to \text{Con}(I\Sigma_1 + \text{Con}(I\Sigma_1 + \text{Con}(PA))$

1. We are working in $PA + \text{Con}(PA)$: Since $I\Sigma_1 + \text{Con}(I\Sigma_1 + \text{Con}(PA)) \subset \Pi^0_4$, we have partial cut-elimination down to Π^0_4 formulae.
   We proceed with a proof by contradiction: Assuming that $\neg\,\text{Con}(I\Sigma_1 + \text{Con}(I\Sigma_1 + \text{Con}(PA)))$, then there is a proof $p \vdash \ulcorner 0 = 1 \urcorner$ in $I\Sigma_1 + \text{Con}(I\Sigma_1 + \text{Con}(PA))$. Because of partial cut-freeness there is a proof p' which includes only Π^0_4 formulae. By Theorem 2.1.11 we have a truth predicate such that

   $$(*)\mathsf{T}_{\Pi^0_4}(\ulcorner\varphi\urcorner) \Leftrightarrow \varphi.$$

   By $(*)$ we can establish $\mathsf{T}_{\Pi^0_4}(\ulcorner\varphi\urcorner)$ for any φ that is an axiom of $I\Sigma_1$ and occurs in p'. This leaves us to establish the truth of $\text{Con}(I\Sigma_1 + \text{Con}(PA))$. Since $I\Sigma_1 + \text{Con}(PA) \subset \Pi^0_4$, we get from the assumption $\neg\,\text{Con}(I\Sigma_1 + \text{Con}(PA))$ a partial cut-free proof of $\ulcorner 0 = 1 \urcorner$. By using the assumption $\text{Con}(PA)$ together with the

truth of all axioms of $I\Sigma_1$ we get $0 = 1$ by $(*)$ and the truth-preservation of the proof system. This leads to Con$(I\Sigma_1 + \text{Con}(PA))$ and to its truth by $(*)$.
Since we know that all axioms of $I\Sigma_1 + \text{Con}(I\Sigma_1 + \text{Con}(PA))$ are true and there are only Π^0_4 formulas in p', we get by the truth preservation of the deduction system $\mathsf{T}_{\Pi^0_4}(\ulcorner 0 = 1 \urcorner)$. Hence $(*)$ gives $0 = 1$, which is the contradiction that we aimed at.

2. $I\Sigma_1 + \text{Con}(PA) \vdash \text{Con}(PA)$ implies
   $I\Sigma_1 + \text{Con}(PA) \vdash \varphi \to \text{Con}(PA)$ for every φ.

3. We proceed by a proof by contradiction. Assume that

$$I\Sigma_1 + \text{Con}(PA) \vdash$$

$$\text{Con}(I\Sigma_1 + \text{Con}(PA)) \to \text{Con}(I\Sigma_1 + \text{Con}(I\Sigma_1 + \text{Con}(PA)).$$

Obviously $I\Sigma_1 + \text{Con}(PA) \vdash \text{Con}(PA)$. Hence by Lemma 2.1.15 we get

$$I\Sigma_1 + \text{Con}(I\Sigma_1 + \text{Con}(PA)) \vdash \text{Con}(PA).$$

Which gives

$$I\Sigma_1 + \text{Con}(I\Sigma_1 + \text{Con}(PA)) \vdash$$

$$\text{Con}(I\Sigma_1 + \text{Con}(PA)) \to \text{Con}(I\Sigma_1 + \text{Con}(I\Sigma_1 + \text{Con}(PA)).$$

But this leads to

$$I\Sigma_1 + \text{Con}(I\Sigma_1 + \text{Con}(PA)) \vdash \text{Con}(I\Sigma_1 + \text{Con}(I\Sigma_1 + \text{Con}(PA)).$$

Hence we contradict the second incompleteness theorem.

□

The next theorem (taken from [35] as well) transports the intransitivity from $\blacktriangleleft_{RC}$ to proof-theoretical reduction. In the following Eq is the set of all closed equations.

Theorem 2.2.12 *Let S and T be first order theories with Σ^0_0 definitions, $I\Delta_0 + Exp \subset T$ and $Q \subset S$. Then*

$$I\Delta_0 + \mathit{Exp} \vdash \mathit{Con}(T) \to \mathit{Con}(S) \text{ implies } S \leq_{Eq} T.$$

Proof

[22]We work in T.

By $I\Delta_0 + \text{Exp} \subset T$ we get $\text{Con}(T) \to \text{Con}(S)$ by the assumption. We distinguish two cases. In both cases we will construct a T-deduction for a closed equation $t_1 = t_2$ from a given S-deduction.

1. $\neg\,\text{Con}(S)$: In this case also $\neg\,\text{Con}(T)$. Hence we can construct a T-deduction for $\ulcorner t_1 = t_2 \urcorner$ uniformly from the contradiction $\ulcorner 0 = 1 \wedge 0 \neq 1 \urcorner$.

2. $\text{Con}(S)$: We assume that there is a S-deduction for $\ulcorner t_1 = t_2 \urcorner$. Since the representation of S is Σ^0_0, we can prove $Q \subset S$ by Δ_0-induction. Moreover $\text{Con}(S)$ and the fact that Q proves all true closed equations (see Theorem 2.1.10) ensures that $\mathsf{T}_{Eq}(\ulcorner t_1 = t_2 \urcorner)$ holds. Note that the proof of Σ_1-completeness (see Theorem 2.1.10) proceeds by an elementary recursive function in [24, pp. 69-72].[23] Hence this proof-method is available, i.e. provably total, in $I\Delta_0 + \text{Exp}$ by Theorem 2.1.9. Therefore we can construct a T-deduction of $\ulcorner t_1 = t_2 \urcorner$.

□

By Theorem 2.2.12 we can lift the intransitivity of $\blacktriangleleft_{RC}$ to proof-theoretical reduction.

[22]Note this proof also establishes that $\text{Con}(T) \to \text{Con}(S)$ and formalised Eq-conservativity are equivalent in $I\Delta_0 + \text{Exp}$.

[23]The proof inductively proceeds on the length of a formula by a finite case distinction where all cases, except the atomic case, are schematic in the formulas used, hence can be bounded by a constant. Moreover the atomic case is about $\mathcal{L}_Q$ and, hence, its terms can be formally evaluated elementarily.

Theorem 2.2.13 *There are arithmetical sound recursively enumerable theories S, T, U such that $S \leq_{Eq} T$ and $T \leq_{Eq} U$, but not $S \leq_{Eq} U$.*

Proof

Take

$$\begin{aligned} S &:= I\Sigma_0 + \text{Exp} + \text{Con}(I\Sigma_0 + \text{Exp} + \text{Con}(I\Sigma_1)) \\ T &:= I\Sigma_1 \\ U &:= I\Sigma_0 + \text{Exp} + \text{Con}(I\Sigma_1) \end{aligned}$$

The following statements can be proved like those which are given in the proof of Theorem 2.2.11.

1. $I\Sigma_1 \vdash \text{Con}(I\Sigma_1) \rightarrow \text{Con}(I\Sigma_0 + \text{Exp} + \text{Con}(I\Sigma_0 + \text{Exp} + \text{Con}(I\Sigma_1)))$
2. $I\Sigma_0 + \text{Exp} + \text{Con}(I\Sigma_1) \vdash \text{Con}(I\Sigma_0 + \text{Exp} + \text{Con}(I\Sigma_1)) \rightarrow \text{Con}(I\Sigma_1)$
3. $I\Sigma_0 + \text{Exp} + \text{Con}(I\Sigma_1) \nvdash \text{Con}(I\Sigma_0 + \text{Exp} + \text{Con}(I\Sigma_1)) \rightarrow \text{Con}(I\Sigma_0 + \text{Exp} + \text{Con}(I\Sigma_0 + \text{Exp} + \text{Con}(I\Sigma_1))$

Consequently $S \leq_{Eq} T$ and $T \leq_{Eq} U$ can be obtained by 1 and 2 by Theorem 2.2.12. However, as we have assumed $S \leq_{Eq} U$, we can contradict 3.□

Therefore, one might change Definition 2.2.7 in order to make it transitive, by restricting the conditions

$$T \vdash f \downarrow$$

and

$$T \vdash (\forall x)(\forall y)(\exists z)[\Gamma(y) \wedge \text{Prov}_S(x, y) \rightarrow f(x) = z \wedge \text{Prov}_T(z, y)]$$

to

$$I\Sigma_1 \vdash f \downarrow$$ [24]

[24] That is sufficient to make f primitive recurisve, by Theorem 2.1.8.

and

$$I\Sigma_1 \vdash (\forall x)(\forall y)(\exists z)[\Gamma(y) \wedge \text{Prov}_S(x,y) \rightarrow f(x) = z \wedge \text{Prov}_T(z,y)].^{25}$$

Let's denote this version by $S \leq_{\Gamma}^{fin} T$. However this does not change much as the discussion that was given above Definition 2.2.9 shows.

Theorem 2.2.14 *Let S and T be theories that have a Σ_0^0 definitions, $I\Sigma_1 \subset T$ and $Q \subset S$. Then the following holds:*

$$f : S \leq_{Eq} T \textit{ and } f \textit{ is primitive recusive.} \Leftrightarrow S \leq_{Eq}^{fin} T.$$

Proof

"$\Rightarrow$:" Assume that $S \leq_{Eq} T$. By the argument given above Definition 2.2.9 we get

$$I\Sigma_1 \vdash \text{Con}(T) \rightarrow \text{Con}(S).$$

Which gives $S \leq_{Eq}^{fin} T$ by Theorem 2.2.12.
"$\Leftarrow$:" This direction follows from the definition, since $I\Sigma_1 \subset T$.□

Consequently transitivity is ensured in cases where the theories include $I\Sigma_1$ and the reduction is given in a primitive recursive way. But there is another reason why this lack of transitivity does not matter much to us. The centre of attention in proof-theoretical reduction (especially in relation to proof-theoretical ordinals) is the relation $S \leq_{\Pi_2^0} T$. But note that

$$\text{Con}(T) \rightarrow \text{Con}(S)$$

as well as

$$(\forall x)(\forall y)[\text{Prov}_S(x,y) \rightarrow (\exists z)(\{e_f\}(x) = z \wedge \text{Prov}_T(z,y))]$$

are both equivalent to Π_2^0-sentences. Therefore one gets the following theorem, which ensures transitivity in those cases.

[25] Here we follow [56] again.

Theorem 2.2.15 *Assume that S, T and U are theories such that $Q \subset T$ and $Q \subset U$.*

1. $S \blacktriangleleft_{RC} T$, $T \blacktriangleleft_{RC} U$ *and* $T \subseteq_{\Pi^0_2} U$
   implies $S \blacktriangleleft_{RC} U$.

2. $S \leq_\Gamma T$, $T \leq_{\Pi^0_2} U$ *and* $\Gamma \subset \Pi^0_2$ *has a* Δ^0_1 *definition (in* T*) implies* $S \leq_\Gamma U$.

2.2.3 Translation

According to Feferman [18] there is no useful general theory of translations which is different enough from proof-theoretical reduction to be explored.[26] But he gives three minimal assumptions a translation between two theories S and T should satisfy:[27]

1. $f : \mathcal{L}_S \to \mathcal{L}_T$ is recursive and total,

2. $S \vdash \varphi \Rightarrow T \vdash f(\varphi)$ and

3. $f(\neg\varphi) = \neg f(\varphi)$.

It is not possible to assume that f distributes on the other connectives or the quantifiers as well. For this would exclude translations between theories that are based on different logics, e.g. the Gödel-Gentzen translations does not satisfy such a notion (see [58]). Also Gödel's Dialectica interpretation does not distribute on quantifiers. But there is a stronger objection against a general approach of translations that supposes distribution on all logical connectives, as it was shown in [39].

[26]Most likely, Feferman either means that all cases that appear in the literature can be adequately handled by proof-theoretical reduction or that one would like to extend a translation to the deduction system and therefore gets something very close to proof-theoretical reduction.

[27]In addition several requirements have to be satisfied by the theories that are under consideration. However we have made them already in previous definitions: the language must be recursive, negation must be defined for every formula and provability must be given by axioms and rules.

Theorem 2.2.16 *For any consistent and recursively enumerable S that is based on classical logic, there is a primitive recursive translation f, which distributes with all logical connectives, into Q.*[28]

Moreover it is sometimes even suitable to drop distributivity with negation as well and substitute it by

$$3'. \quad f(\bot_S) = \bot_T,$$

where $\bot_S$ and $\bot_T$ are suitable atomic inconsistencies in S and T respectively. One may give the following definition.

Definition 2.2.17 $f : S \leq_{trans} T :\Leftrightarrow f$ *satisfies condition* 1, 2 *and* $3'$. *We write* $S \leq_{trans} T$ *iff there is such an* f.

It is easy to see that $S \leq_{trans} T$ insures relative consistency; for suitable f also in the sense of $S \blacktriangleleft_{RC} T$ or $S \lhd_{RC} T$. However one can get closer to proof-theoretical reduction by making the reasonable assumption that f can be extended to deductions, say to f^*. So one may strengthen property 2 by using

$$2'. \quad d \vdash_S \varphi \quad \Rightarrow \quad f^*(d) \vdash_T f(\varphi).$$

However in cases where the language intersects, e.g. in the case of arithmetical theories, one might be able to define a natural fragment of both languages so that the properties of a proof-theoretical reduction are satisfied. An example for such a translation is again Gödel-Gentzen as well as Friedman's negative translation (see [21]). Both translate PA into HA in a way which shows that their negative fragment and their provable Π^0_2-sentences coincide. Therefore one gets a proof-theoretical reduction according to these two sets of formulas.

[28] The reader should not be confused by this apparently strong statement, because the assumption that S is consistent allows to use unnatural representations of S in Q that encode various cheap tricks.

2.2.4 Realisability

Realisability is a method which is heavily used in the analysis of intuitionistic theories. Its most common form was defined via partial recursive functions by Kleene 1945 as a syntactic translation based on the intuition that is given by the Brouwer-Heyting-Kolmogorov-Interpretation (BHK). The idea was to view the BHK of implication and generalisation as grasped by the notion of partial recursive functions. Before we give a definition of such a realisability translation we contrast BHK with the idea of realisability. We are following here A. Troelstra's book [59] and omit most of the proofs because they are highly technical and long.

BHK	**Realisation**
A proof of $\phi \wedge \psi$ is a proof of ϕ and ψ.	$\phi \wedge \psi$ is realised by a pair of realisers for ϕ and ψ, $\langle n_\phi, n_\psi \rangle$
A proof of $\phi \vee \psi$ is either a proof of ϕ or ψ.	$\phi \vee \psi$ is realised by a pair $\langle m, n \rangle$ where m entails if the realizer n realises ϕ or ψ.
A proof of $\phi \rightarrow \psi$ is a procedure that transforms every proof of ϕ into a proof of ψ.	$\phi \rightarrow \psi$ is realised by a code of a partial recursive function that gives for any realizer of ϕ a realizer of ψ.
A proof of $(\exists y)\psi(y)$ is a pair of a witness m and a proof of $\psi(\bar{m})$.	$(\exists y)\psi(y)$ is realised by a pair $\langle m, n_{\phi(\bar{m})} \rangle$ of a witness m and a realiser of $\phi(\bar{m})$.

A proof of $(\forall y)\psi(y)$ is a procedure which gives for any n an evaluation of $\psi(\bar{n})$.	$(\forall y)\psi(y)$ is realised by the code of a partial recursive function which gives for any n a realiser of $\psi(\bar{n})$.

By using Kleene brackets one can formulate the implication and the generalisation cases of realisability in $\mathcal{L}^1_{HA}$. See [59] for details how to formalise $\mathcal{T}$ and $\mathcal{U}$ in HA. In the following we consider the easiest case where $\mathcal{L}^1_{HA}$ is interpreted in $\mathcal{L}^1_{HA}$. Also we use p_1, p_2 to denote the function symbols expressing the projection functions in $\mathcal{L}^1_{HA}$.

Definition 2.2.18 *Assume that x is a variable of $\mathcal{L}^1_{HA}$. The realisability translation $\mathrm{x}\boldsymbol{r}$: $\mathcal{L}^1_{HA} \to \mathcal{L}^1_{HA}$ is defined on the complexity of a $\varphi \in \mathcal{L}^1_{HA}$, where φ does not contain x free.*

1. *If φ is atomic, then $\mathrm{x}\boldsymbol{r}\,\varphi :\equiv \varphi$*
2. *If $\varphi \equiv \phi \wedge \psi$, then $\mathrm{x}\boldsymbol{r}\,\varphi :\equiv p_1(x)\boldsymbol{r}\phi \wedge p_2(x)\boldsymbol{r}\psi$*
3. *If $\varphi \equiv \phi \vee \psi$, then*
   $\mathrm{x}\boldsymbol{r}\,\varphi :\equiv [p_1(x) = 0 \to p_2(x)\boldsymbol{r}\phi] \wedge [p_1(x) \neq 0 \to p_2(x)\boldsymbol{r}\psi]$
4. *If $\varphi \equiv \phi \to \psi$, then*
   $\mathrm{x}\boldsymbol{r}\,\varphi :\equiv (\forall u)[u\boldsymbol{r}\phi \to (\exists v)[\mathcal{T}(x, u, v) \wedge \mathcal{U}(v)\boldsymbol{r}\psi]]$
5. *If $\varphi \equiv (\exists y)\psi(y)$, then $\mathrm{x}\boldsymbol{r}\,\varphi :\equiv p_2(x)\boldsymbol{r}\psi(p_1(x))$*
6. *If $\varphi \equiv (\forall y)\psi(y)$, then $\mathrm{x}\boldsymbol{r}\,\varphi :\equiv (\forall y)(\exists z)[\mathcal{T}(x, y, z) \wedge \mathcal{U}(z)\boldsymbol{r}\psi(y)]$*

The next theorem is called the *soundness theorem* for realisability and is central for the following investigations. We give the proof of the soundness theorem in full detail, since we will give an analysis of it in Chapter 6.

Theorem 2.2.19 *Assume that $\varphi \in \mathcal{L}^1_{HA}$ is a sentence and that $T := HA + \Gamma$, where Γ is a set of sentences of $\mathcal{L}^1_{HA}$, then:*

1. *If $HA \vdash \varphi$, then there is an $n \in \mathbb{N}$ such that $HA \vdash \bar{n}\boldsymbol{r}\varphi$.*

2. *If for any $\psi \in \Gamma$, $T \vdash (\exists x)(\mathrm{x}\boldsymbol{r}\,\psi)$, then*

$$T \vdash \varphi \Rightarrow T \vdash (\exists x)(\mathrm{x}\boldsymbol{r}\,\varphi).$$

3. *Assume that $T \equiv_{\Sigma^0_1} HA$ and that for any $\psi \in \Gamma$ there is an $n \in \mathbb{N}$ such that $T \vdash \bar{n}\boldsymbol{r}\psi$. Then*

$$T \vdash \varphi \text{ implies that there is an } n \in \mathbb{N} \text{ such that } T \vdash \bar{n}\boldsymbol{r}\varphi.$$

Proof

The following proof is taken from [59, pp. 190-192].

1. The proof proceeds by an induction over the length of a HA-deduction of φ. To do so we have to fix a deduction system. Since the realisability translation translates formulas, a Hilbert-system is favoured. We use a system going back to Gödel, which is defined in Appendix C. The proof uses p-terms[29] in an inessential way to denote functions whose codes serve as a realiser. Also, to improve readability, we write $t\mathbf{r}\varphi$, where t is a p-term, instead of $\ulcorner t \urcorner \mathbf{r}\varphi$.

   Since the theorem is stated about sentences we have to deal with the occurrence of free variables in the induction step before the induction hypothesis can be applied. Therefore we have to treat the free-variable case and the sentence case in parallel. But since every proof which ends in a formula with a free variable can be transformed into one ending in a sentence by $\forall$-introduction, that is a minor

[29] The set of p-terms is constructed by closing-off the set of terms of $\mathcal{L}_{HA}$ by λ-abstraction and Kleene-brackets. See [59] for details.

problem. However the constructed deduction for $\bar{n}\mathbf{r}\varphi$ becomes therefore longer; which will be an issue in Chapter 6. The following case-distinction follows the enumeration which is given in Appendix C.

(a) If the instance is a sentence, we take $0\mathbf{r}[\perp \to \varphi]$, otherwise we take the universal closure and realise it by a constant 0 function with appropriate arity $\lambda\vec{x}.0\mathbf{r}(\forall\vec{x})[\perp \to \varphi(\vec{x})]$.

In the following four cases we suppress the case where free variables occur, since in those the realiser is as well just the dummy-variable version of the one used in the sentence case.

(b) $\lambda x.p_2(x)\mathbf{r}[\varphi \vee \varphi \to \varphi]$
$\lambda x.\langle x, x\rangle\mathbf{r}[\varphi \to \varphi \wedge \varphi]$

(c) $\lambda x.\langle 0, x\rangle\mathbf{r}[\varphi \to \varphi \vee \psi]$
$\lambda x.p_1(x)\mathbf{r}[\varphi \wedge \psi \to \varphi]$

(d) $\lambda x.\langle 1\dot{-}p_1(x), p_2(x)\rangle\mathbf{r}[\varphi \vee \psi \to \psi \vee \varphi]$
$\lambda x.\langle p_2(x), p_1(x)\rangle\mathbf{r}[\varphi \wedge \psi \to \psi \wedge \varphi]$

(e) $\lambda y.\{y\}(t)\mathbf{r}[(\forall x)\varphi(x) \to \varphi(t)]$

(f) $\lambda y.\langle t, y\rangle\mathbf{r}[\varphi(t) \to (\exists x)\varphi(x)]$

(g) Assume that there is an application of Modus Ponens with premisses $\varphi(\vec{x})$ and $\varphi(\vec{x}) \to \psi(\vec{x})$. Hence by $\forall$-introduction we also have proofs of $(\forall\vec{x})\varphi(\vec{x})$ and $(\forall\vec{x})[\varphi(\vec{x}) \to \psi(\vec{x})]$. The induction hypothesis gives then an n and an m such that HA proves

$$\bar{n}\mathbf{r}(\forall\vec{x})\varphi(\vec{x})$$

and

$$\bar{m}\mathbf{r}(\forall\vec{x})[\varphi(\vec{x}) \to \psi(\vec{x})].$$

Therefore Definition 2.2.18 gives

$$\{n\}(x)\mathbf{r}\varphi(x)$$

and

$$\{\bar{m}\}(x)\mathbf{r}[\varphi(\vec{x}) \to \psi(\vec{x})].$$

By Modus Ponens and Definition 2.2.18 we therefore obtain

$$\{\{\bar{m}\}(\vec{x})\}(\{n\}(\vec{x}))\mathbf{r}\psi(\vec{x}).$$

Thus $\lambda\vec{x}.\{\{\bar{m}\}(\vec{x})\}(\{\bar{n}\}(\vec{x}))\mathbf{r}(\forall\vec{x})\psi(\vec{x})$.

For simplicity we only consider the cases without free variables for the other rules.

(h) We can assume that there are proofs for $\bar{n}\mathbf{r}[\varphi \to \chi]$ and $\bar{m}\mathbf{r}[\chi \to \psi]$. With x$\mathbf{r}\,\varphi$ having been assumed, one gets $\{\bar{n}\}(x)\mathbf{r}\chi$ and hence $\{\bar{m}\}(\{\bar{n}\}(x))\mathbf{r}\psi$. By the deduction theorem and Definition 2.2.18 this gives

$$\lambda x.\{\bar{m}\}(\{\bar{n}\}(x))\mathbf{r}[\varphi \to \psi].$$

(i) Assume that there is a proof of $\bar{n}\mathbf{r}[\varphi \wedge \psi \to \chi]$. Also assume that x$\mathbf{r}\,\varphi$ and $y\mathbf{r}\psi$, which gives $\langle x, y\rangle\mathbf{r}[\varphi \wedge \psi]$. Together this gives $\{\bar{n}\}(\langle x, y\rangle)\mathbf{r}\chi$. Hence we get $\lambda x\lambda y.\{\bar{n}\}(\langle x, y\rangle)\mathbf{r}[\varphi \to (\psi \to \chi)]$.

(j) Assume that there is a proof of $\bar{n}\mathbf{r}[\varphi \to (\psi \to \chi)]$. To apply the deduction theorem, assume that x$\mathbf{r}[\varphi \wedge \psi]$. By Definition 2.2.18 and two applications of Modus Ponens we get $\{\{\bar{n}\}(p_1(x))\}(p_2(x))\mathbf{r}\chi$. Therefore $\lambda x.\{\{\bar{n}\}(p_1(x))\}(p_2(x))\mathbf{r}[\varphi \wedge \psi \to \chi]$ by Definition 2.2.18.

(k) Assume that $\bar{n}\mathbf{r}[\varphi \to \psi]$. To apply the deduction theorem, assume that x$\mathbf{r}[\chi \vee \varphi]$. By Definition 2.2.18 we get $p_1(x) = 0 \to (p_2(x)\mathbf{r}\chi)$ and $p_1(x) \neq 0 \to (p_2(x)\mathbf{r}\varphi)$. If $p_1(x) = 0$ is true, then

$$\text{x}\mathbf{r}[\chi \vee \psi];$$

otherwise we have a proof of $\{\bar{n}\}(p_2(x))\mathbf{r}\psi$ and therefore get

$$\langle p_1(x), \{\bar{n}\}(p_2(x))\rangle\mathbf{r}[\chi \vee \psi].$$

This gives

$$\lambda x.[(1\dot{-}p_1(x))x + \text{sgn}(p_1(x)) \cdot \langle p_1(x), \{\bar{n}\}(p_2(x))\rangle]\mathbf{r}[\chi \vee \psi].$$

(l) Assuming that there is a proof for $\psi(\vec{y}) \to \varphi(x, \vec{y})$, then we get a proof for $\bar{n}\mathbf{r}(\forall\vec{y}x)[\psi(\vec{y}) \to \varphi(x, \vec{y})]$ by induction hypothesis. To use the deduction theorem, assume that $z\mathbf{r}\psi(\vec{y})$, which leads to $\{\bar{n}\}(\vec{y}, x, z)\mathbf{r}\varphi(x, \vec{y})$. Hence we get

$$\lambda x.\{\bar{n}\}(\vec{y}, x, z)\mathbf{r}(\forall x)\varphi(x, y).$$

Therefore we get by Definition 2.2.18

$$\lambda\vec{y}\lambda z\lambda x.\{\bar{n}\}(\vec{y}, x, z)\mathbf{r}(\forall\vec{y})[\psi(\vec{y}) \to (\forall x)\varphi(x, \vec{y})].$$

(m) From a proof of $\varphi(x) \to \psi$ we get a proof of $\bar{n}\mathbf{r}(\forall x)[\varphi(x) \to \psi]$. To use the deduction theorem we assume that $u\mathbf{r}(\exists x)\varphi(x)$, which gives $p_2(u)\mathbf{r}\varphi(p_1(u))$. Therefore we get $\{\bar{n}\}(p_1(u), p_2(u))\mathbf{r}\psi$, hence we can obtain

$$\lambda u.\{\bar{n}\}(p_1(u), p_2(u))\mathbf{r}[(\exists x)\varphi(x) \to \psi].$$

We have completed all logical cases and will consider the axioms of HA in the following.

(a) We give explicit p-terms for the axioms of identity:

$$\lambda x.0\mathbf{r}(\forall x)[x = x]$$

$$\lambda x\lambda y\lambda z\lambda u.0\mathbf{r}(\forall xyz)[x = y \wedge y = z \to x = z]$$

$$\lambda x_1...\lambda x_n\lambda u.0\mathbf{r}(\forall x_1, ..., x_n)[x_i = y \to [\varphi(x_i) \leftrightarrow \varphi(y)]]$$

$$\lambda x\lambda u.0\mathbf{r}(\forall x)[S(x) \neq 0]$$

$$\lambda x\lambda y\lambda u.0\mathbf{r}(\forall xy)[S(x) = S(y) \to x = y]$$

(b) All axioms for primitive recursive functions are realised by a constant 0 function with same arity.

(c) Assume that $u\mathbf{r}[\varphi(0) \wedge (\forall x)[\varphi(x) \rightarrow \varphi(S(x))]]$. By the recursion theorem we can find a code for a partial recursive function

$$f(u, 0) \simeq p_1(u)$$

$$f(u, S(x)) \simeq \{p_2(u)\}(x, f(u, x)).$$

An easy induction shows that f is total. Hence we can obtain

$$\lambda u \lambda x. f(u, x)\mathbf{r}[\varphi(0) \wedge (\forall x)[\varphi(x) \rightarrow \varphi(S(x))] \rightarrow (\forall x)\varphi(x)].$$

2. Assume that $T \vdash \varphi$, where φ is a sentence. The deduction theorem gives us $HA \vdash \psi \rightarrow \varphi$, where ψ is a finite conjunction of sentences of Γ. Hence, by the assumptions of the theorem and the Modus Ponens step from above, we get the result.

3. As before the assumption that $T \vdash \varphi$, where φ is a sentence, gives a ψ such that $HA \vdash \psi \rightarrow \varphi$. This leads to $T \vdash \{\bar{n}\}(\bar{m}) \downarrow \wedge \{\bar{n}\}(\bar{m})\mathbf{r}\varphi$. Since $\{\bar{n}\}(\bar{m}) \downarrow$ is a Σ_1^0-sentence, we have $HA \vdash \{\bar{n}\}(\bar{m}) \downarrow$. By the fact that HA is Σ_1^0-correct, there is a suitable $m_0 = \{\bar{n}\}(\bar{m})$ such that $HA \vdash m_0\mathbf{r}\varphi$.

□

The proof of Theorem 2.2.19 proceeds by constructing in every case a realiser from given ones. Moreover the proof divides into a finite number of cases. Therefore the proof above gives a primitive recursive method to find such a realiser. Consequently one can formalise via primitive recursive functions f_1, f_2 a version of Theorem 2.2.19 as

$$HA \vdash \mathrm{Prov}_{HA}(x, \ulcorner \varphi \urcorner) \rightarrow \mathrm{Prov}_{HA}(f_1(x), \ulcorner \ulcorner t_{f_2(x)} \urcorner \mathbf{r} \varphi \urcorner).$$

However there is no primitive recursive universal realiser. As an diagonalisation (see [59] p. 192) shows, there are no provable total recursive functions f_1' and f_2', in HA, such that

$$HA \vdash \text{Prov}_{HA}(x, \ulcorner\varphi\urcorner) \to \text{Prov}_{HA}(f_1(x), \ulcorner\ulcorner\lambda x.t(x)\urcorner \mathbf{r}\varphi\urcorner),$$

such that the p-term t defines the function f_2.
Realisability does not effect formulas of a particular form (called almost negative formulas), which makes these formulas particularly interesting for conservativity results.

Definition 2.2.20 *An almost negative formula is an element of $\mathcal{L}^1_{HA}$ which does not contain $\vee$ and a $\exists$-quantifier can only occur in front of an equation (like $(\exists x)(t(x) = s(x))$).*

Theorem 2.2.21 *For any $\varphi \in \mathcal{L}^1_{HA}$ its realised version $\mathrm{x}\boldsymbol{r}\,\varphi$ is logically equivalent to an almost negative formula.*

Proof
See [59, p. 193].□

The proof of the next proposition is given in full detail, because it will be analysed in Chapter 6.

Theorem 2.2.22 *If φ is almost negative, then*

$$HA \vdash (\exists x)(\mathrm{x}\boldsymbol{r}\,\varphi) \leftrightarrow \varphi.$$

Proof
The proof goes by induction on the construction of φ.

1. In the case of an atomic φ the Lemma trivialises to $\varphi \leftrightarrow \varphi$ by Definition 2.2.18.

2. $\varphi \equiv (\exists y)[t(y) = s(y)]$:

   "⇒": We assume that $(\exists x)(\mathrm{x}\mathbf{r}\,\varphi)$, which is

   $$p_2(x)\mathbf{r}[t(p_1(x)) = s(p_1(x))].$$

   But this is just $t(p_1(x)) = s(p_1(x))$, hence we get $(\exists y)[t(y) = s(y)]$.

   "⇐": We assume that $(\exists y)[t(y) = s(y)]$. This is equivalent to

   $$(\exists y)[\langle y, 0\rangle \mathbf{r}(\exists y)[t(y) = s(y)]],$$

   by Definition 2.2.18 and $HA \vdash p_1(\langle y, 0\rangle) = y$. Hence we get $(\exists x)[x\mathbf{r}(\exists y)[t(y) = s(y)]]$ and are done.

3. $\varphi \equiv \psi_1 \to \psi_2$:

   "⇒": We assume that $(\exists x)(\mathrm{x}\mathbf{r}\,\varphi)$. This is

   $$(\exists x)(\forall u)[u\mathbf{r}\psi_1 \to (\exists v)[\mathcal{T}(x, u, v) \wedge \mathcal{U}(v)\mathbf{r}\psi_2]].$$

   The induction hypothesis

   $$(\exists w)[w\mathbf{r}\psi_2] \to \psi_2$$

   gives us

   $$(\forall u)[u\mathbf{r}\psi_1 \to \psi_2].$$

   By induction hypothesis

   $$\psi_1 \to (\exists u)[u\mathbf{r}\psi_1],$$

   which gives $\psi_1 \to \psi_2$.

   "⇐": We assume that $\psi_1 \to \psi_2$. By assuming $u\mathbf{r}\psi_1$, we get ψ_1 by induction hypothesis, and hence ψ_2. The induction hypothesis gives us therefore $(\exists v)[v\mathbf{r}\psi_2]$. Together this gives

   $$u\mathbf{r}\psi_1 \to (\exists v)[v\mathbf{r}\psi_2].$$

   Let e_v be the function that gives v for every input u. Hence $e_v\mathbf{r}[\psi_1 \to \psi_2]$ and are done.

4. $\varphi \equiv (\forall y)\psi(y)$:
"⇒": We assume that $(\exists x)(\mathrm{x}\mathbf{r}\,\varphi)$. This is

$$(\exists x)(\forall u)(\exists v)[\mathcal{T}(x,u,v) \wedge \mathcal{U}(v)\mathbf{r}\psi(u)]].$$

By induction hypothesis this gives $(\forall u)\psi(u)$. Hence, by renaming of variables, we are done.
"⇐": We assume that $(\forall y)\psi(y)$. By induction hypothesis we get $(\forall y)(\exists z)[z\mathbf{r}\psi(y)]$. Hence, by choosing a code for the function "$y \mapsto z$" as the realiser, we are done.

5. $\varphi \equiv \psi_1 \wedge \psi_2$:

$$\psi_1 \wedge \psi_2 \Leftrightarrow p_1(x)\mathbf{r}\psi_1 \wedge p_2(x)\mathbf{r}\psi_2 \Leftrightarrow \langle p_1(x), p_2(x)\rangle\mathbf{r}[\psi_1 \wedge \psi_2] \Leftrightarrow x\mathbf{r}[\psi_1 \wedge \psi_2]$$

□

The last two propositions easily give idempotence.

Theorem 2.2.23

$$HA \vdash (\exists x)(\mathrm{x}\boldsymbol{r}(\exists y)(y\boldsymbol{r}\varphi)) \leftrightarrow (\exists y)(y\boldsymbol{r}\varphi)$$

A classical application of realisability is the analysis of Church's Thesis. Therefore we will give two formulations of this thesis in the following.

Definition 2.2.24 *Let φ be almost negative and not contain y freely. The following schema is denoted by ECT_0 and is called extended Church's Thesis.*

$$(\forall x)[\varphi \rightarrow (\exists y)\psi(y)] \rightarrow (\exists u)(\forall x)[\varphi \rightarrow (\forall v)(\mathcal{T}(u,x,v) \wedge \varphi(\mathcal{U}(v)))]$$

Here ψ is just any formula of $\mathcal{L}^1_{HA}$.

By taking φ as $0 = 0$ one gets a formula equivalent to Church's Thesis (CT_0) itself, which is

$$(\forall x)(\exists y)\psi(y) \to (\exists u)(\forall x)[(\exists v)\mathcal{T}(u,x,v) \wedge \psi(\mathcal{U}(v))]^{30}.$$

In order to use Theorem 2.2.19 one has to establish the following lemma.

Lemma 2.2.25 *For any universal closure φ of an instance of ECT_0 there exists an $n \in \mathbb{N}$ such that*

$$HA \vdash \bar{n}\boldsymbol{r}\varphi.$$

Proof

See [59, p. 195]. □

In terms of ECT_0 one can characterise realisability as follows.

Theorem 2.2.26 *Assume that $T := HA + \Gamma$, where Γ are some sentences of $\mathcal{L}^1_{HA}$, such that for any $\psi \in \Gamma$*

$$T \vdash (\exists x)(\mathrm{x}\boldsymbol{r}\,\psi).$$

Then

- $T + ECT_0 \vdash \varphi \leftrightarrow (\exists x)(\mathrm{x}\boldsymbol{r}\,\varphi)$
- $T + ECT_0 \vdash \varphi$ *if and only if* $T \vdash (\exists x)(\mathrm{x}\boldsymbol{r}\,\varphi)$.

Proof

The first statement is shown by induction on the complexity of φ. For instance let's

[30] This can be read as "If a procedure is total, then there is already a total recursive function which executes it" or as "Every function is already a total recursive one".

assume that $\varphi :\equiv \phi \rightarrow \psi$;

$(\phi \rightarrow \psi)$	$\leftrightarrow$	(by induction hypothesis)
$((\exists x)[\mathrm{x}\mathbf{r}\,\phi] \rightarrow (\exists y)[y\mathbf{r}\psi])$	$\leftrightarrow$	(by prenexiation)
$(\forall x)([\mathrm{x}\mathbf{r}\,\phi] \rightarrow (\exists y)[y\mathbf{r}\psi])$	$\leftrightarrow$	(by ECT_0)
$(\exists z)(\forall x)([\mathrm{x}\mathbf{r}\,\phi] \rightarrow (\exists v)[\mathcal{T}(z,x,v) \wedge \mathcal{U}(v)\mathbf{r}\psi])$	$\leftrightarrow$	(by the definition of $\mathrm{x}\mathbf{r}$ according to implication)
$(\exists z)(z\mathbf{r}[\phi \rightarrow \psi])$		

For the second claim we firstly note that the implication from right to left is given by the first claim. To prove the implication from left to right we assume that $T + ECT_0 \vdash \varphi$. Hence $T \vdash \phi \rightarrow \varphi$, for a conjunction ϕ of universal closures of ECT_0-instances. Since Theorem 2.2.19 shows that $T \vdash \psi$ implies $T \vdash (\exists x)(\mathrm{x}\mathbf{r}\,\psi)$, we get $T \vdash (\exists x)[\mathrm{x}\mathbf{r}(\phi \rightarrow \varphi)]$. Also we get $T \vdash (\exists x)(\mathrm{x}\mathbf{r}\,\phi)$ by Lemma 2.2.25 and $HA \subset T$. By the implication case in Definition 2.2.18 and Modus Ponens, we conclude $T \vdash (\exists x)(\mathrm{x}\mathbf{r}\,\varphi)$. □

Since the proof of Theorem 2.2.26 is done by a uniform proof construction, this gives a proof-theoretical reduction $T + ECT_0 \leq_{alm.neg} T$ for the almost negative fragment of $\mathcal{L}^1_{HA}$ (Theorem 2.2.22).

For the purposes of ordinal analysis it is important to know that transfinite induction does not disturb Theorem 2.2.26. Therefore we first define this schema.

Definition 2.2.27 *Let $\langle A, \prec\rangle$ be such that A is a primitive recursive set and $\prec$ a primitive recursive order on A. The schema $TI(\prec)$ is then*

$$(\forall x)[[A(x) \wedge (\forall y \prec x)\varphi(y)] \rightarrow \varphi(x)] \rightarrow (\forall x)[A(x) \rightarrow \varphi(x)]$$

for $\varphi \in \mathcal{L}^1_{HA}$. Here $A(x)$ and $x \prec y$ are abbreviations for $f_A(x) = 1$ and $f_\prec(x,y) = 1$ respectively, where $f_A, f_\prec \in \mathcal{L}^1_{HA}$ are the function symbols denoting the characteristic

functions of the respective sets.
If we want to restrict the schema to a certain set of formulas Γ *we denote this by* $TI_{\Gamma}(\prec)$.

According to the side conditions on Γ, that are demanded by Theorem 2.2.26, we only need the following lemma to show that $TI(\prec)$ does not change the possibility of giving a proof-theoretical reduction.

Lemma 2.2.28 *For any closed instance* φ *of an* $TI(\prec)$ *it holds that there is an* $n \in \mathbb{N}$ *such that* $HA + TI(\prec) \vdash \bar{n}\boldsymbol{r}\varphi$.

Proof
See [59, p. 199].□

One easily gets the following corollary by putting Lemma 2.2.28 and Theorem 2.2.26 together.

Corollary 2.2.29 *For any primitive recursive* $\prec$
$HA + TI(\prec) + ECT_0 \vdash \varphi$ *if and only if* $HA + TI(\prec) \vdash (\exists x)(\mathrm{x}\boldsymbol{r}\,\varphi)$.

Even though the idea behind realisability is to interpret formulas by their computational content, it seems fair to say that it should be viewed as an ordinary translation and hence as a proof-theoretical reduction (as explained section 2.2.3).[31]

[31] As I said earlier, embracing semantics does not make a methodology less syntactic. On the contrary it characterises a good syntactic methodology to reveal the semantics without collapsing into a referentialistic point of view.

2.3 An axiomatization of Theory Reduction

In [35] Niebergall identifies certain features a good notion of theory reduction should have and are arguably accepted by most in the community[32]. He then presents a formalisation of these features in an unspecified base theory[33], that should be as weak as possible. The following axioms axiomatise the notation $S\rho T$ (read as "S is reducible to T"), where S, T, U, E, F range over theories as formalised objects in the unspecified base theory.

1. $\forall S, T(S \subset T \Rightarrow S\rho T)$
2. $\forall S, T, U(S\rho T \wedge T\rho U \Rightarrow S\rho U)$
3. $\forall S, T(S\rho T \Rightarrow (T \text{ is consistent } \Rightarrow S \text{ is consistent }))$
4. $\forall S, T(S\rho T \Rightarrow \forall\psi \in S\exists\varphi \in T\{\psi\}^{\vdash}\rho\{\varphi\}^{\vdash})$
5. $\forall E, F$ finitely axiomatisable

$$E\rho F \Rightarrow I\Sigma_1 \vdash \text{Con}(F) \to \text{Con}(E).$$

The most distinguishing characteristic of the axiomatic approach is, that it does not rest on the idea that a reduction has to be given by a function.[34] This accords with some people's view that reduction should not be a term-term, predicate-predicate or formula-formula relation, as stated in [35].

We do not wish to argue here for the axioms 1-4, in fact we doubt whether 1-3 would be subject to any objections anyway. Axiom 5 is chosen that way, because provable

[32]He calls them linguistic intuitions.

[33]In [35] on page 52 Niebergall explains that, since his reducibility logic (a modal logic like provability logic) is too weak to distinguish interpretability from proof-theoretical reduction, a "pure" axiomatization does not seem fruitful.

[34]Niebergall links the lack of a function to the question whether a reduction should be structure preserving or not. But, as we said before, we are not interested in this direction of his argumentation.

consistency-reduction is a wish of the community and the restriction to finite theories rules out non-standard representations of theories in arithmetic.[35] The subtheory relation, interpretability and local interpretability obviously satisfy the axioms above. Moreover a very surprising property of this axiom system is that it even almost defines interpretability.

Theorem 2.3.1 *If $S\rho T$ is governed by axiom 1-5, $I\Sigma_1 \subset T$ and T is reflexive*[36]*, then*

- *If $S\rho T$, then $S \preceq_{loc} T$.*
- *If $Q \subset S$, then: $S\rho T \Rightarrow S \subseteq_{\Pi^0_1} T$.*
- *If S is axiomatisable, then: if $S\rho T$, then $S \preceq T$.*

Proof

See [35, p. 41].□

Since $S \preceq_{loc} T$ satisfies axiom 1-5 we get the following corollary.

Corollary 2.3.2 *If $Q \subset S$, S is axiomatisable, $I\Sigma_1 \subset T$ and T is reflexive, then:*

- *$S \preceq_{loc} T \Rightarrow S \subseteq_{\Pi^0_1} T$ and $S \preceq T$.*
- *If E is finitely axiomatisable, then: $S\rho T$ and $S \vdash Con(E)$ implies $T \vdash Con(E)$.*

Moreover the close relationship with interpretability is preserved when axiom 5 is weakened to

$$6.\quad S\rho T \wedge S \vdash \text{Con}(E) \Rightarrow T \vdash \text{Con}(E),$$

where E is a finitely axiomatizable theory.

[35]For in the case of $E = \{e_0, ..., e_k\}$ one can chose the canonical representation $x = \ulcorner e_0 \urcorner \vee ... \vee x = \ulcorner e_k \urcorner$

[36]A theory T is called reflexive, if it proves the consistency of all its finite fragments.

Theorem 2.3.3 *If $S\rho T$ is governed by axiom 1-4 and 6, $I\Sigma_1 \subset T$ and S is a reflexive extension of Q, then*

- $S\rho T \Rightarrow S \subseteq_{\Pi_1^0} T$
- *If $S\rho T$ and S is axiomatisable, then $S \preceq T$.*

Proof

See [35, p. 42].□

However there are some differences with interpretability. For instance the property of interpretability given by Theorem 2.2.5 does not hold in general for a relation governed by axioms 1-5 as shown in [35, p. 44].

More impotently the above axiom system is not satisfied by proof-theoretical reduction, because proof-theoretical reduction fails to satisfy axiom 4. For instance it is a standard result of proof-theory that $ACA_0 \leq PA$. But if ρ is governed by axioms 1-5, then ACA_0 is not ρ-reduceable to PA, which is an easy result from the theorem following the definition of ACA_0.

Definition 2.3.4 *ACA_0 is formulated in $\mathcal{L}^2(\{0, =, S, +, \cdot\})$ with all axioms of Q together with the single second order induction axiom*

$$X(0) \wedge (\forall x)[X(x) \rightarrow X(S(x))] \rightarrow (\forall x)(X(x)),$$

the axiom of extensionality

$$X = Y \leftrightarrow (\forall x)(X(x) \leftrightarrow Y(x))$$

and arithmetical comprehension

$$(\exists X)(\forall x)(X(x) \leftrightarrow \varphi(x)),$$

where X and second-order quantification do not occur in φ.

Theorem 2.3.5 *1.* $ACA_0 \leq PA$

2. $ACA_0 \not\preceq PA$

Proof

$ACA_0 \leq PA$ can be easily seen by working in $I\Sigma_1$. If $ACA_0 \vdash \varphi$ for a sentence $\varphi \in \mathcal{L}_Q^1$, then there is a deduction in a Gentzen system for some sequent $\Gamma \Rightarrow \varphi$, where $\Gamma \subset ACA_0$. As explained in Section 2.1, cut-elimination can be done in $I\Sigma_1$, therefore we have a cut-free deduction of $\Gamma \Rightarrow \varphi$. By the subformula property all second order variables occurring in the deduction already occur in $\Gamma \Rightarrow \varphi$. Next we substitute for every second-order variable which occurs in an instance of a comprehension axiom of Γ, its comprehension formula. The comprehension axioms therefore becomes logical valid formulas, which can be cut out. For any remaining $X(y)$ we substitute $y = 0$. Note that there cannot be any second-order variables left after this procedure. Hence, since PA embraces induction for any formula of $\mathcal{L}_Q^1$, we get a sequent $\Delta \Rightarrow \varphi$ with $\Delta \subset PA$, which leads to $PA \vdash \varphi$.

It remains to show that $ACA_0 \not\preceq PA$. We only defined $\preceq$ for first order theories but it should be clear how one can extend it to second order. However in the second-order case Theorem 2.2.6 proceeds in the same way. Towards a contradiction let's assume that $ACA_0 \preceq PA$. Then by a version of Theorem 2.2.6 for any finite subtheory of ACA_0, let's call it E, $PA \vdash$ Con(E). But as we know from [26, p. 154], ACA_0 is finitely axiomatizable. Therefore $PA \vdash$ Con(ACA_0). But since $ACA_0 \leq PA$, this gives $PA \vdash$ Con(PA).□

2.4 Discussion

The disagreement of two notions of theory reduction in the case of natural theories, like PA and ACA_0, forces us to make a choice, when the "right","best" or most "appropriate" notion is aimed at. However even though we have to choose as well, our choice is much easier, because we merely ask for a notion which serves our well defined purposes. We want a notion that summarises all methods used in ordinal-analysis that have been used so far in the literature, for example translation, realisation and cut-elimination. Furthermore one of the most interesting outcomes of an ordinal-analysis is the characterisation of the provable-total functions of a theory. Therefore Π^0_2-conservativity is of particular interest. As a third point, we want to compare theories based on different logics. The fourth consideration is in fact a restriction: ordinal-analysts are mainly interested in strong theories far above PA.[37]

Hence we prefer proof-theoretical reduction. From a positive perspective, because it subsumes interlogical translations and therefore allows the comparision of intuitionistic and classical theories (something interpretabilitiy is lacking). Also from a negative perspective its lack of transitivity is not much of an issue for us, because Theorem 2.2.14 assures us that transitivity is satisfied, when the methods in use can be done primitively recursively and the theories are above $I\Sigma_1$. This is a condition which is easily met for ordinal-analysis that is done by transfinite cut-elimination (see [33]). Moreover, as Theorem 2.2.15 shows, even for those cases where the methods are not primitive recursive our aim for Π^0_2-conservativity ensures transitivity.

However I want to say something about the Niebergall-Feferman-dispute. The example that Niebergall gives in the proof of Theorem 2.2.13 is particularly misleading in view of the fact that those theories are rather weak. In the case of weak theories, where existence claims do not play much of a role because of their rather narrow ontological basis, the

[37]That is not only grounded in the personal interest of ordinal analysts. For weak systems an ordinal analysis is quite painful and not worth the effort since in most cases other methods work far better.

question of strength boils down to what a theory *can do*. Also, since fancy proof-methods are also missing in weak theories, the intuition behind *can do* boils down to whether a theory can perform a certain operation or not. Hence the issue here is the provability of Π_2-sentences. However, since by Theorem 2.2.15 Niebergall's example does not satisfy Π^0_2-conservativity, one might ask the question how this example is supposed to challenge our intuition on theory reduction. For it seems reasonable to assume that any reduction-relation should enjoy Π^0_2-conservativity when going up one of its chains in the realm of weak arithmetical theories.

Chapter 3

A better Base Theory for Ordinal-Analysis

The current chapter mainly serves as an introduction to the theories $\widehat{ID}^i_n$ and the results that are given by Buchholz, Rüede and Strahm in [5] and [44]. Our interest in these theories is based on an account which is given by Buchholz in [5] and concerns the interaction of intuitionistic logic with inductive definitions. According to Buchholz's results, these theories are not stronger than PA but offer an easy and nice definition for infinite systems that are used in ordinal-analysis. Hence one can substitute $\widehat{ID}^i_n$ for PA as a metatheory for formalised ordinal analysis. This prevents the analyst from drawing on formalised recursion theory in order to deal with infinite trees and operations upon them, which makes the actually presented proofs more readable and avoids hand waving. Since formalised ordinal-analysis is central for some results that one wants to obtain from an ordinal-analysis, this is a major success in methodology. Moreover there are ordinal-analyses that never have been formalised, since they use strong metamathematical principles like the Ω-rule. These can be straightforwardly formalised in $\widehat{ID}^i_n$. Hence these methodological advantages outline a recipe for an ordinal analysis that is general enough to draw certain general conclusions from it.

We will explain the role of formalised ordinal analysis in the first section of this chapter before we move on to study $\widehat{ID}^{i}_{n}$ in detail. Moreover the following discussion has a strong relation to Rathjen's generalised ordinal-analysis given in [40].

3.1 Ordinal-Analysis and Provable Conservativity

Ordinal-analysis is a proof-theoretical technique which goes back to Gentzen's work on the consistency of arithmetic in 1938 (see [23]).[1] By giving a reduction method that transforms a deduction of an inconsistency in PA into a deduction of a very weak subsystem S of PA, Gentzen's approach, as presented in [23], is very close to Definition 2.2.10. Moreover his proof can be extended to prove

$$PA \subseteq_{\Sigma_1} S$$

in the metatheory $EA + TI_{\Sigma^0_0}(\prec_{\varepsilon_0})^2$, where $\prec_{\varepsilon_0}$ is an elementary recursive well-ordering of order-type ε_0. Here ε_0 is defined as

$$\varepsilon_0 := \sup\{\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \omega^{\omega^{\omega^{\omega}}}, ...\}$$

and the ordering $\langle \varepsilon_0, \prec_{\varepsilon_0} \rangle$ is naturally given by a normal-form theorem of ordinal-arithmetic (for details see [57]). Besides the fact that the ordering is very natural, the proof has the advantage that it obviously can be done in PRA[3] at any step but one. For the system S is weak enough such that

$$PRA \vdash \text{Con}(S)$$

[1]The following considerations on ordinal analysis are ment to give a unifying framework. For a very recent recapitulation on ordinal analysis that gives most of the different aspects see [31].

[2]It should be clear how to adapt Definition 2.2.27 to the systems that are used in this chapter by substituting the defining formulas of the ordering for the function symbols that are used in Definitin 2.2.27.

[3]According to Chapter A even in EA, which is less obvious.

and every step in Gentzen's transformation can be done primitive recursively. Only to prove the transformation's totality one needs $TI_{\Delta_0^0}(\prec_{\varepsilon_0})$. Consequently Gentzen's proof can be seen in the light of Chapter 2 as showing that

$$PRA + TI_{\Sigma_0^0}(\prec_{\varepsilon_0}) \vdash \text{Con}(S) \to \text{Con}(PA),$$

which leads, together with the provable consistency of S, to

$$PRA + TI_{\Sigma_0^0}(\prec_{\varepsilon_0}) \vdash \text{Con}(PA).$$

Moreover Gentzen proved that for any proper initial segment of $\langle \varepsilon_0, \prec_{\varepsilon_0} \rangle$ this cannot be done. Consequently there is no shorter ordering that can prove the consistency in the range of all orderings provably comparable to $\langle \varepsilon_0, \prec_{\varepsilon_0} \rangle$. This motivates calling ε_0 the ordinal of PA and identifying ε_0 with the proof-theoretical strength of PA. The successors of Gentzen developed longer and more elaborate elementary recursive orderings, which we will call ordinal notation systems and denote by $OT(\beta)$, where β is the order type.[4] This motivates the following definition. Here $\prec_\beta |_\alpha$, for $\alpha \in OT(\beta)$, denotes the restriction of the ordering on $OT(\beta)$ to those elements of $OT(\beta)$ that are $\prec_{OT(\beta)}$-smaller than α.[5]

Definition 3.1.1 *Let T be a theory as defined in chapter 2 and fix an $OT(\beta)$. Assume that there is an $\alpha \in OT(\beta)$ such that*

$$PRA + TI_{\Sigma_0^0}(\prec_\beta |_\alpha) \vdash \mathit{Con}(T),$$

then we call the $\prec_\beta$-least of this α the proof theoretical ordinal of T and denote it by $\|T\|_{\mathit{Con}}$.

[4] An ordinal notation system in this sense is just an arbitrary elementary recursive set containing strings of symbols which are well-ordered. We denote these strings by small Greek letters to emphasise their close connection to ordinal numbers. In most cases we do not distinguish a string from its code, hence in many cases the elements of an $OT(\beta)$ are numbers.

[5] Note that $TI(\prec_\beta |_\alpha)$ states transfinite induction on α and not just for any ordinal smaller than α.

Definition 3.1.1 gets its foundational significance in metamathematics by trusting the well-foundness of a natural ordinal notation systems, like Gentzen's $\langle\varepsilon_0, \prec_{\varepsilon_0}\rangle$, more than the consistency of the theory in question. However note that this definition cannot be generalised by abstracting the preassumed ordinal notation system away, because there is no known definition of a natural ordinal notation system; and trying to give Definition 3.1.1 in terms of order types of arbitrary orderings collapses the notion to ω for every theory, as Kreisel emphasised.[6]

Another drawback of Definition 3.1.1 is that it obscures the actual benefits of a modern ordinal-analysis by narrowing it down to a consistency proof. But since the Schütte school developed their methods in the sixties much more can be achieved. Schütte gave a proof-theoretical analysis of several systems by using the ω-rule

$$\frac{\varphi(\bar{n}) \quad : \quad n \in \mathbb{N}}{\forall x \varphi(x)}$$

as a substitute for $\forall$-introduction and induction axioms. For instance PA_ω is defined as the system that embraces all true atomic and negated atomic sentences of $\mathcal{L}^1_{PRA}$ as axioms and is closed-off by the inference rules for $\wedge, \vee, \exists$, cut and the ω-rule; while negation is defined on the meta-level. If one views the deductions of PA_ω as infinite rooted trees and labels them by elements of an appropriate $OT(\alpha)$ in a suitable way, then cut-elimination for a subset of PA_ω's deductions, which depends on α, can be shown by a transfinite induction on α. This ensures the subformula property for this fragment of PA_ω, i.e. every formula occurring in the deduction is a subformula of the formula which is proved by the deduction. Moreover the subformula property has many successful applications.

It is an easy fact that restricting the application of the ω-rule in PA_ω to premisses that are elementarily recursively enumerable (we denote this system by PA^{el}_ω) does not change the extension of provability, i.e. $PA_\omega \equiv_{\Pi^0_\infty} PA^{el}_\omega$.[7] Therefore one can give a provability predicate of PA_ω in EA by using codes for the functions enumerating the premisses of

[6] There is a cheap trick to code the consistency of a theory in an elementary ordering of order type ω. For details see [29].

[7] Note that every step in the construction of a reduction tree (see [49, pp. 197-201] for a Definition) for

the ω-rule (see [51]). By enlarging the deduction system by repetition rules, e.g. rules of the following form

$$\frac{\varphi}{\varphi},$$

it is possible to do the cut-elimination procedure up to cut-freeness by an elementary recursive function as Minc shows in [33]. Note this from being not in conflict with Gödel's Theorems, because the presence of the repetition rule prohibits that EA is able to tell whether a particular cut-free object is a deduction or not, since the infinite repetition of a singular formula opens up the possibility of a correctly formed "proof"-tree that is not well-founded. However labelling the deductions by ordinals coming from the ordinal notation systems such that premisses have smaller ordinals than their conclusions ensures the property of being a deduction. Taking the ordinals into account means using some instance of a $TI(\alpha)$. The observation that every deduction of PA can be translated into PA_ω and that the latter system enjoys cut-freeness ensures consistency. Hence we are coming back to the view that is given by Definition 3.1.1.

The example of analysing PA via PA_ω is exemplary for ordinal-analysis in general. In many cases in the literature, where an ordinal-analysis of a theory T is given, it is customary to chose an infinite system in which the infinite rules are restricted to elementary enumerable premisses and some sort of cut-elimination can be done. In the following we denote such an infinite system for a theory T by T^∞. In addition we assume that the set of the axioms of T^∞ are a decidable, consistent and complete subset of the atomic sentences in the language $\mathcal{L}_{T^\infty}$.[8]

But what do we achieve by this method that is obscured by Definition 3.1.1? The answer is that we have just given a very uniform way to achieve conservativity results for T with

a formula in $\mathcal{L}_{PA_\omega}$ can be performed elementarily in an evaluation function for terms, which is of course not elementary, when $\mathcal{L}_{PA_\omega}$ has a symbol for any elementary function. Hence, since PA_ω enjoys cut-elimination, every deducible formula has a deduction that is enumerable by function that is elementary in an evalution function for terms.

[8]Consistent means that $P(\vec{t})$ and $\neg P(\vec{t})$ are not both axioms, when a negation operator is present. The requirements on the axioms of T^∞ ensure that we can use a "truth predicate" for any theory T.

$Q \subset T$. For classical theories it proceeds as follows.

First we fix an $\langle OT(\gamma), \prec_\gamma \rangle$ and assume that $\|T\|_{\text{Con}} = \alpha \in OT(\gamma)$. We also assume that the transformation f_∞ of T-deductions into T^∞ is primitive recursive, since in the literature this is always the case. Therefore we get

$$I\Sigma_1 \vdash \{e_{f_\infty}\} \downarrow$$

$$I\Sigma_1 \vdash \text{Prov}_T(x, y) \to \text{Prov}_{T^\infty}(\{e_{f_\infty}\}(x), y).$$

In the next step we eliminate any rule that might spoil cut-elimination. We assume that this can be done primitive recursively by a function f_r and ignore this step for now.[9] Since Minc's cut-elimination procedure f_c is elementary recursive we also have

$$I\Sigma_1 \vdash \{e_{f_c}\} \downarrow$$

$$I\Sigma_1 \vdash \text{Prov}_{T^\infty}(x, y) \to \text{Prov}_{T^\infty}(\{e_{f_c}\}(x), y).$$

Informal cut-elimination gives us $\|T\|_{\text{Con}} = \alpha$, which is the supremum of the ordinal heights of those T^∞ deductions that are relevant for T, i.e. the deductions that were obtained by cut-elimination on those that had been translated into T^∞ from T. Also, since cut-free deductions have the subformula property, a correctness proof by transfinite induction on a single deduction with ordinal hight β (for $\beta \prec_\gamma \alpha$) can be done by using an instance of $TI_{\Pi^0_n}(\prec_\gamma |_\beta)$ for a suitable n.[10] This gives for any $n \in \mathbb{N}$ and any $\beta \in OT(\gamma)$

$$I\Sigma_1 + TI_{\Pi^0_n}(\prec_\gamma |_\beta) \vdash \pi^0_n(y) \wedge \text{cutfree}(x, \beta) \wedge \text{Prov}_{T^\infty}(x, y) \to \mathsf{T}_{\Pi^0_n}(y),$$

where $\pi^0_n(y)$ and cutfree(x, β) are Σ^0_0-formulas formalising the property of being Π^0_n and cut-freeness with ordinal-hight β respectively.

[9] However this preliminary step to cut-elimination is the point where the hard work of ordinal-analysis is done. Consequently the assumption that this can be done primitive recursively most certainly excludes many interesting cases. But to keep our presentation simple we make this assumption.

[10] Not that in general the truth predicate has the same logical complexity as its grammatical subjects.

Next we can prove conservativity. To do so assume that $T \vdash \varphi$ and $\varphi \in \Pi^0_m$ for some $m \in \mathbb{N}$. By Lemma 2.1.10 we get

$$I\Sigma_1 \vdash \pi^0_m(\ulcorner\varphi\urcorner) \wedge \mathrm{Pr}_T(\ulcorner\varphi\urcorner),$$

which leads to

$$I\Sigma_1 + TI_{\Pi^0_m}(\prec_\gamma |_\beta) \vdash \mathsf{T}_{\Pi^0_m}(\ulcorner\varphi\urcorner)$$

for some $\beta \prec_\gamma \alpha$ by the results above. Consequently we get

$$I\Sigma_1 + TI_{\Pi^0_m}(\prec_\gamma |_\beta) \vdash \varphi$$

by Theorem 2.1.11.

Since the proof is uniform and m can be chosen primitive recursively, we get a proof-theoretical reduction

$$T \leq I\Sigma_1 + TI(\prec \alpha),$$

where $TI(\prec \alpha)$ is the set of all $TI(\prec_\gamma |_\beta)$ with $\beta \prec_\gamma \alpha$.

Conversely, since $T \vdash TI(\prec_\gamma |_\beta)$ for any $\beta \prec_\gamma \alpha$ by ordinal analysis, it is obvious that T proves all consequences of

$$I\Sigma_1 + TI(\prec \alpha).$$

Also since the deductions of $TI(\prec_\gamma |_\beta)$, that are given by the informal ordinal-analysis, are usually given uniformly, a proof-theoretical reduction is achieved as well.

By combining these two directions, we get

$$T \equiv I\Sigma_1 + TI(\prec \alpha).$$

However, if $\omega \prec \alpha$, then $TI(\alpha)$ proves all instances of induction. Hence $PA + TI(\prec \alpha) \equiv I\Sigma_1 + TI(\prec \alpha)$ and therefore we do not lose anything by going up to

$$T \equiv PA + TI(\prec \alpha).$$

An analogous argument gives

$$T^i \equiv HA + TI(\prec \alpha)$$

for intuitionistic theories.

For a fixed ordinal notation system $OT(\beta)$ and an $\alpha \in OT(\beta)$ we denote the theory $T + TI(\prec \alpha)$ by $[T]_\alpha$ in the following. The hierarchy of all $[T]_\alpha$ is denoted by $[T]_{OT(\beta)}$. Some times we will also write $[T]_\prec$.
The insights that are given above motivate the following definition (see [40]).

Definition 3.1.2 *Let T be a classical theory as defined in Chapter 2 and $OT(\beta)$ an ordinal notation system. Assume that there is an $\alpha \in OT(\beta)$ such that*

$$T \equiv [PA]_\alpha,$$

then we call α the reductive proof theoretical ordinal of T and denote it by $\|T\|_\mathcal{R}$. If T is intuitionistic, then we take

$$T \equiv [HA]_\alpha$$

to be the defining property.

However as Friedman proves in [21] $[HA]_\prec \equiv_{\Pi^0_2} [PA]_\prec$ for any primitive recursive ordering $\prec$. Therefore we can make Definition 3.1.2 smoother.

Definition 3.1.3 *Let T be a theory as defined in Chapter 2 and $OT(\beta)$ an ordinal notation system. Assume that there is an $\alpha \in OT(\beta)$ such that*

$$T \equiv_{\Pi^0_2} [PA]_\alpha,$$

then we call α the Π^0_2-proof-theoretical-ordinal of T and denote it by $\|T\|_{\Pi^0_2}$.

The advantages over Definition 3.1.1 are obvious. First, both Definitions give us provable conservativity by proof-theoretical reduction, which is relative to a hierarchy of theories that is uniformly given. $[PA]_{OT(\beta)}$ can therefore function as a scale. Hence an ordinal-analysis proves or refutes conservativity results for a large set of theories.[11] Moreover, since Definition 3.1.3 takes Π^0_2-conservativity into account and the $[PA]_{OT(\beta)}$ are well-studied theories, one gets a characterisation of the provable-total recursive functions of a theory almost free in therms of subrecursive-hierarchy theory (see Section 5.2.1 and Section 5.2.2). Second, the link between a proof-theoretical ordinal and proof-theoretical strength is explicitly given by mentioning the deducability relation in the definition.[12] Also we still grant the leading foundational-idea behind ordinal analysis, since $[PA]_{OT(\beta)}$ is a very natural hierarchy of theories as far as the ordering $\prec_\beta$ can be considered as natural. However to invoke Definition 3.1.3 one has to code T^∞ into PA or HA, which might be quite painful and long. Moreover the definitions of operations on these codes of infinite deductions require formalised versions of the recursion theorem, becouse infinite rules appear nested in these deductions and the function that enumerates their premisses must be explictly given by one of its codes when formalising the deduction in arithmetic.[13] Consequently, these proofs become very technical and to make them readable, ordinal-analysts use a lot of hand waving even in very technical and relatively complete texts like [51] and [38]. Also for some ordinal-analyses such proofs have never been given, e.g. the

[11] In Chapter 5 we connect an ordinal analysis to a function hierarchy. Therefore, when two theories are ordinal analiesed, one can extract totality statement (Π^0_2-sentences) which are provable in one but not in the other theory.

[12] By proof-theoretical strength we mean the strength of the deducibility relation and not necessarily the consistency strength, which might be different.

[13] E.g. one might chose to formalise an ω-rule application

$$\frac{\varphi(\bar{n}) \quad : \quad n \in \mathbb{N}}{(\forall x)\varphi(x)}\,,$$

where the premisses are enumerable by the elementary recursive function f, as $\langle e_f, \ulcorner(\forall x)\varphi(x)\urcorner\rangle$ with $\{e_f\}(n) := \ulcorner\varphi(\bar{n})\urcorner$.

Ω-rule using analysis given in [41]. But, since the T^∞ are, as any systems used in proof-theory, given by an inductive definition, moving on to theories that include this concept as a primitive notion seems a natural thing to do. We will discuss this possibility in the next section.

3.2 Theories of Inductive Definitions and Fixed points

In [1] Peter Aczel promoted his idea viewing an inductive definition of a set of natural numbers X as a set of pairs, called clauses, of the form $\langle A, b\rangle$, where the set $A \subset \mathbb{N}$ and $b \in \mathbb{N}$. Such a clause is read as:

If every $a \in A$ is an element of X , then $b \in X$.

Sometimes clauses are written more intuitively as $A \Rightarrow b$. We say that a set X satisfies a set of clauses $\mathcal{C}$, from $A \Rightarrow b \in \mathcal{C}$ with $A \subset X$ it follows that $b \in X$. Moreover we say that a set is inductively definable, if it is the least set (under set inclusion) satisfying such a $\mathcal{C}$. Theories of inductive definitions try to mimic the idea of the clauses by using certain formulas that are attached to names for a particular inductively definable set. It will be clear later that these formulas might be used to define a monotonic operator on the power set of $\mathbb{N}$ (denoted by $\mathcal{P}(\mathbb{N})$) whose smallest fixed point (again under set inclusion) is also the inductively definable set. Therefore the formulas below are called operator forms. However to define these formulas we have to define the language in use in more detail than we did in Chapter 2.

Definition 3.2.1 *The language $\mathcal{L}_{ID}$ is defined as usual as a one sorted and negation free*[14] *first order language including the constant $\bar{0}$ (zero), the binary predicate $<$ and a function symbol for every primitive recursive function in particular S (successor), $\langle .,. \rangle$ (pairing)*

[14] $\neg$ is not a primitive symbol.

and $(.)_1, (.)_2$ *(projection). As variables we use* $v_0, v_1, \ldots$.
We define $\mathcal{L}_{ID}(Q)$ *as* $\mathcal{L}_{ID}$ *with an additional unary predicate symbol* Q *and* $\mathcal{L}_{ID}(Q, P)$ *by adding the unary predicate symbol* P *in the same way to* $\mathcal{L}_{ID}(Q)$.

For technical reasons in the following definitions it is convenient to define negation $\neg\varphi$ as an abbreviation for $\varphi \to \bar{0} = \bar{1}$. Note that $\mathcal{L}_{ID}(Q)$ and $\mathcal{L}_{HA}$ have the same primitive symbols (up to the way negation is handled). For notational convenience we denote the objects defined in the following definitions by capital Latin letters.

Definition 3.2.2 *The set of strictly positive (in* P*) operator forms of* $\mathcal{L}_{ID}(Q, P)$ *is built up by the following clauses:*

1. *Every formula of* $\mathcal{L}_{ID}(Q)$ *is a strictly positive operator form.*
2. *For every term* t *the formula* $P(t)$ *is a strictly positive operator form.*
3. *The strictly positive operator forms are closed under* $\exists, \forall, \wedge$ *and* $\vee$.
4. *If* $A \in \mathcal{L}_{ID}(Q)$ *and* B *is a strictly positive operator form, then* $A \to B$ *is a strictly positive operator form.*

The set of strictly positive operators is the set of those strictly positive operator forms that contain at most v_0 *and* v_1 *free.*
An accessibility operator is a strictly positive operator that has the form

$$A \wedge \forall z[B(z) \to P(z)]$$

with $A, B \in \mathcal{L}(Q)$*. The set of positive operators (forms) is the set of strictly positive operators (forms) when it is restricted to* $\mathcal{L}_{ID}(P)$*. The set of strong positive operators (forms) is similarly defined as the set of positive operators (forms) but without including condition 4 into the definition.*

As suggested before, one idea behind positive operators is that each of them, say $A(P, x)$, can serve to define a monotone operator

$$\Phi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$$

by defining

$$\Phi(S) := \{n \in \mathbb{N} : A(S, n)\}$$

for $S \in \mathcal{P}(\mathbb{N})$. It is easy to prove monotonicity by using positivity.[15] Conversely it is well known that any monotone operator on $\mathcal{P}(\mathbb{N})$ can be defined by a positive operator form, where the other free variables serve as parameters (see [37] for details). Moreover it is well known that inductively definable sets coincide with the smallest fixed points of monotone operators. However it is possible to strengthen this notions by going transfinite; strictly positive operator forms can serve as a formalisation of transfinite iterations of inductively definable sets by using Q as a place holder for previously defined sets. In order to deal with inductively definable sets and their transfinite iterations in a first order system we have to give them explicit names, which can serve then as predicate constants. The next definition fixes a name for a set that is defined by an operator of $\mathcal{L}_{ID}(Q, P)$.

Definition 3.2.3 *The first order language $\mathcal{L}^*_{ID}(strict)$ is built from $\mathcal{L}_{ID}$ by adding for every strictly positive operator $A \in \mathcal{L}_{ID}(Q, P)$ a new unary predicate symbol P^A to the primitive symbols of $\mathcal{L}_{ID}$. The languages for accessible $\mathcal{L}^*_{ID}(acc)$, positive operators $\mathcal{L}^*_{ID}(pos)$ and strong positive operators $\mathcal{L}^*_{ID}(strong)$ are defined analogously.*

According to the question that arose before about a theory formalising the principles of inductive definitions one might suggest a theory that formalises all the principles that have been given in the presentation of inductive definitions above, i.e. formulating axioms expressing that P^A is the least fixed point of A. This theory is denoted by ID_1 and defined as follows.

[15] Φ is called monotone, if $S_1 \subseteq S_2$ implies $\Phi(S_1) \subseteq \Phi(S_2)$.

Definition 3.2.4 *The theory ID_1 is formulated in $\mathcal{L}^*_{ID}(pos)$ and includes all axioms of PRA, induction for all formulas of $\mathcal{L}^*_{ID}(pos)$ and the following two axioms governing the predicate symbols that are introduced by Definition 3.2.3:*

$$(\forall x)[P^A(x) \leftrightarrow A(P^A, x)]$$

$$(\forall x)[A(\varphi, x) \to \varphi(x)] \to (\forall x)[P^A(x) \to \varphi(x)],$$

where $A(P, v_1)$ is a positive operator and φ any formula of $\mathcal{L}^*_{ID}(pos)$ with only one free variable.[16]

But it causes problems to choose ID_1 as a base theory for a hierarchy, i.e. forming $[ID_1]_\prec$ as $[PA]_\prec$ was formed above. Even when proof-theorists do not have any more foundational issues with ID_1 as they have with PA, ID_1 is far to strong in order to make $[ID_1]_\prec$ useful. Since $\|ID_1\|_{\mathcal{R}} = \psi(\varepsilon_{\Omega+1})$[17] (see [38]), the hierarchy $[ID_1]_\prec$ already starts at a level where it cannot distinguish many interesting theories from one another.
It was Buchholz who discovered an interesting consequence of the fact that the axiom of choice is much weaker over intuitionistic logic than it is over classical. Because, as Buchholz proves in [5], ID_1^i without the second axiom that governs the predicates denoting inductively defined sets and the first restricted to strong positive operators (denoted by $\widehat{ID}_1^i(strong)$) is not stronger than HA.

Theorem 3.2.5 *$\widehat{ID}_1^i(strong)$ is conservative over HA with respect to almost negative formulas.*

The proof in fact gives $\widehat{ID}_1^i(strong) \leq_{alm.neg.} HA$ and can be lifted to

$$\widehat{ID}_1^i(strong) + TI(\prec) \leq_{alm.neg.} HA + TI(\prec)$$

[16]Here $A(\varphi, x)$ and $A(P^A, x)$ denote the formulas generated by substituting the formulas $\varphi(t), P^A(t)$ and the variable x for $P(t)$ and v_1 respectively in $A(P, v_1)$. Obviously, the term t varies from context to context.

[17]An ordinal far higher than ε_0.

for a primitive recursive ordering $\prec$. However, as Rüede and Strahm showed in [44], this can be strengthened to theories dealing with iterations of inductive definitions up to ω.[18] In order to formulate these theories in a readable way we have to introduce two conventions. In the following we identify

$$P_s^A(t) \text{ with } P^A(\langle t, s\rangle) \text{ and}$$

$$P_{<s}^A(t) \text{ with } t = \langle t_0, t_1\rangle \wedge t_1 < s \wedge P^A(t).$$

Definition 3.2.6 *The theory $\widehat{ID}^i_n(strict)$, which is formulated in $\mathcal{L}^*_{ID}(strict)$, is based on intuitionistic logic. It comprises the axioms of HA with induction being extended to $\mathcal{L}^*(strict)$ and includes for every strictly positive $A(P, Q, v_0, v_1)$ the fixed point axiom:*

$$(\forall y < n)(\forall x)[P_y^A(x) \leftrightarrow A(P_y^A, P_{<y}^A, x, y)],$$

where $A(P_y^A, P_{<y}^A, x, y)$ results from $A(P, Q, v_0, v_1)$ by substituting for $P(t)$ the atom $P_{v_1}^A(t)$, for $Q(t)$ the formula $P_{<v_1}^A(t)$, for v_0 the variable x and for v_1 the variable y.[19]
Also we define $\widehat{ID}^i_{<\omega}(strict) := \bigcup_{n\in\mathbb{N}} \widehat{ID}^i_n(strict)$.
The theories $\widehat{ID}^i_{<\omega}(acc)$ and $\widehat{ID}^i_n(acc)$ are the respective subsystems where only accessibility operators are used.

3.3 Reducing $\widehat{ID}^i_{<\omega}(strict)$ to HA

As already suggested before, Rüede and Strahm showed the following theorem in [44].

Theorem 3.3.1 *(Rüede and Strahm)*

[18] This obviously is the best possible, since to make sense of transfinite iterations in a theory one has to add $TI(\prec)$ as well.

[19] in that order

$$\begin{array}{lll} \widehat{ID}^i_{<\omega}(strict) & \leq_{alm.neg.} & HA \\ \widehat{ID}^i_{<\omega}(strict) & \leq_{\Pi^0_2} & HA \end{array}$$

While Buchholz's proof of the special case with $n = 1$, which is stated in Theorem 3.2.5, is directly given via a nice coding and a realisability interpretation, Rüede's and Strahm's proof takes a little detour through a bunch of proof-theoretical results that have been previously given by other logicians. It starts with another result that goes back to Buchholz and is proved in [6].

Theorem 3.3.2 *(Buchholz)*
$ID^i_n(strict) \leq_{alm.neg.} ID^i_n(acc)$

The proof of this statement shows that the following stronger statement can be proved as well.

Theorem 3.3.3 $\widehat{ID}^i_n(strict) \leq_{alm.neg.} \widehat{ID}^i_n(acc)$

By the results that are stated in [44] one can get rid of the principles of inductive definitions in deductions by translating them into a second-order systems.

Theorem 3.3.4 *(Rüede and Strahm)*
$\widehat{ID}_n(acc) \leq_{\mathcal{L}_{HA}} ACA^-_n$

The missing links from ACA^-_n to PA and from PA to HA are straightforward constructions.[20]
Since the proofs are important for the result that will be given in Chapter 6, we give them in full detail.

[20]The theory ACA^-_n will be introduced in Definition 3.3.13

3.3.1 A Proof by Buchholz

The proof of Theorem 3.3.3 is given via a realisability translation that literally follows Definition 2.2.18. However, since the translation operates on $\mathcal{L}^*_{ID}(strict)$, we have to add how to proceed with the additional predicates P^A. This is not easy and already includes the general proof idea. Moreover we have to take a detour through realisability on $\mathcal{L}_{ID}(Q, P)$ in order to precisely state it. Since $\mathcal{L}_{ID}(Q, P)$ has the additional predicates Q and P, we have to add two respective cases to Definition 2.2.18.

Definition 3.3.5 *For every $\varphi \in \mathcal{L}_{ID}(Q, P)$ we define $\mathrm{x}\boldsymbol{r}\,\varphi$ inductively on the complexity of φ as we do in Definition 2.2.18 and add the following two cases:*

- *If $\varphi \equiv P(t)$, then $\mathrm{x}\boldsymbol{r}\,\varphi :\equiv P(\langle t, x\rangle)$.*
- *If $\varphi \equiv Q(t)$, then $\mathrm{x}\boldsymbol{r}\,\varphi :\equiv Q(\langle (t)_0, (x)_1, (t)_1\rangle)$.*

The central observation that is used to prove Theorem 3.3.3 is stated in the following lemma.

Lemma 3.3.6 *For any strictly positive operator form A there is a accessibility operator form B such that $\mathrm{x}\boldsymbol{r}\,A \leftrightarrow B$ is an intuitionistically valid tautology.*

Proof

We start by defining a set X, which acts like an alternative definition for strictly positive operator forms, by the following clauses.

1. $\mathcal{L}(Q) \subset X$.
2. If $\varphi \equiv P(x)$, then $\varphi \in X$.
3. X is closed under $\wedge$ and $\forall$.

4. If $\varphi \in \mathcal{L}(Q)$ and $\psi \in X$, then $\varphi \to \psi \in X$.

Note that, if a $A \in \mathcal{L}(Q, P)$ is a strictly positive operator form, then $x\mathbf{r}\, A \in X$. In the next step we give for any $\varphi \in X$ formulas $\psi_1, \psi_2 \in \mathcal{L}(Q)$ such that φ is intuitionistically equivalent to $\psi_1 \land (\forall x)[\psi_2(x) \to P(x)]$. The proof proceeds inductively on the definition of X.

1. $\varphi \in \mathcal{L}(Q)$: We take φ to be ψ_1 and $\bot$ to be ψ_2. It is easy to see that
$$\varphi \leftrightarrow \varphi \land (\forall x)[\bot \to P(x)]$$
is an intuitinistcally valid tautology.

2. $\varphi \equiv P(y)$: We take $\bar{0} = \bar{0}$ to be ψ_1 and $x = y$ to be ψ_2. As before the following formula is intuitionistically valid.
$$\varphi \leftrightarrow \bar{0} = \bar{0} \land (\forall x)[x = y \to P(x)]$$

3. $\varphi \equiv \varphi_1 \land \varphi_2$: By induction hypothesis we have $\psi_{i1}, \psi_{i2} \in \mathcal{L}(Q)$ for $i \in \{1, 2\}$ such that
$$\varphi_i \leftrightarrow \psi_{i1} \land (\forall x)[\psi_{i2}(x) \to P(x)]$$
is intuitionistically valid. By taking ψ_1 as $\psi_{11} \land \psi_{21}$ and ψ_2 as $\psi_{12} \lor \psi_{22}$ we can show the equivalence as follows.
$$\begin{aligned} \varphi \leftrightarrow \quad & \varphi_1 \land \varphi_2 \\ \leftrightarrow \quad & \psi_{11} \land (\forall x)[\psi_{12}(x) \to P(x)] \land \psi_{21} \land (\forall x)[\psi_{22}(x) \to P(x)] \\ \leftrightarrow \quad & \psi_{11} \land \psi_{21} \land (\forall x)[(\psi_{12} \to P(x)) \land (\psi_{22} \to P(x))] \\ \leftrightarrow \quad & \psi_{11} \land \psi_{21} \land (\forall x)[\psi_{12} \lor \psi_{22} \to P(x))]. \end{aligned}$$

4. $\varphi \equiv (\forall y)\chi(y)$: By induction hypothesis we have $\chi_1, \chi_2 \in \mathcal{L}(Q)$ such that
$$\chi(y) \leftrightarrow \chi_1(y) \land (\forall x)[\chi_2(y, x) \to P(x)]$$
is intuitionistically valid. We take ψ_1 as $(\forall y)\chi_1(y)$ and ψ_2 as $(\exists y)\chi_2(y, x)$. Hence we get the following.

$$\begin{aligned}\varphi \leftrightarrow\ & (\forall y)\chi(y) \\ \leftrightarrow\ & (\forall y)[\chi_1(y) \wedge (\forall x)[\chi_2(y,x) \to P(x)]] \\ \leftrightarrow\ & (\forall y)\chi_1(y) \wedge (\forall y)(\forall x)[\chi_2(y,x) \to P(x)] \\ \leftrightarrow\ & (\forall y)\chi_1(y) \wedge (\forall x)[(\exists y)\chi_2(y,x) \to P(x)].\end{aligned}$$

5. $\varphi \equiv \varphi_1 \to \varphi_2$: Here $\varphi_1 \in \mathcal{L}(Q)$. By induction hypothesis we have

$$\varphi_2 \leftrightarrow \psi_{21} \wedge (\forall x)[\psi_{22} \to P(x)].$$

We take ψ_1 as $\varphi_1 \to \psi_{21}$ and ψ_2 as $\varphi_1 \wedge \psi_{22}$. Therefore we can proceed as follows:

$$\begin{aligned}\varphi \leftrightarrow\ & \varphi_1 \to \varphi_2 \\ \leftrightarrow\ & \varphi_1 \to [\psi_{21} \wedge (\forall x)[\psi_{22} \to P(x)]] \\ \leftrightarrow\ & [\varphi_1 \to \psi_{21}] \wedge [\varphi_1 \to (\forall x)[\psi_{22} \to P(x)]] \\ \leftrightarrow\ & [\varphi_1 \to \psi_{21}] \wedge (\forall x)[\varphi_1 \wedge \psi_{22} \to P(x)]].\end{aligned}$$

□

Since the construction in the proof above does not change the occurrences of free variables, one can find for any strictly positive operator an accessibility operator. This makes the next definition possible.

Definition 3.3.7 *If $A(P,Q,v_0,v_1)$ is a strictly positive operator, then $A^{\mathbf{r}}(P,Q,v_0,v_1)$ is the accessibility operator that is constructed from*
x***r*** $A(P,Q,v_0,v_1)$ *by the method given in the proof of Lemma 3.3.6.*

We are now able to define the realisability translation for $\mathcal{L}^*_{ID}(strict)$ as promised above.

Definition 3.3.8 *For any $\varphi \in \mathcal{L}^*_{ID}(strict)$ we inductively define* x***r*** φ *on the complexity of φ as we did in Definition 2.2.18 and add the following case.*

- *If* $\varphi \equiv P^A(t)$ *for a strictly positive operator* A*,* *then* $\mathrm{x}\boldsymbol{r}\,\varphi :\equiv P^{A^r}(\langle (t)_0, x, (t)_1\rangle)$.[21]

It is easy to see that the translation of Definition 3.3.8 is a translation from $\mathcal{L}^*_{ID}(strict)$ into its fragment $\mathcal{L}^*_{ID}(acc)$ with the property that x is the only free variable that is added. The next lemma gives a technical property of the translation. Since the identity that is stated is an actual identity of formulas (not an equivalence), it does not affect the proof-complexity considerations in Chapter 6. We can therefore skip the proof.

Lemma 3.3.9 *For any* $\varphi \in \mathcal{L}_{ID}(Q, P)$*,*

$$\mathrm{x}\boldsymbol{r}\,\varphi(P^A_y, P^A_{<y}) = (\mathrm{x}\boldsymbol{r}\,\varphi)(P^{A^r}, P^{A^r}_{<y}).$$

Proof

See [6, p. 231]. □

We have to clarify one last technical fact before we are able to prove the actual theorem.

Lemma 3.3.10 *Assume that* $\varphi(z) \in \mathcal{L}_{ID}(Q, P)$ *is a strictly positive operator form and*

$$\psi(z) :\equiv \{v\}(y, w, (z)_0, (z)_1)\boldsymbol{r}\varphi((z)_0).$$

For any strictly positive operator form $A \in \mathcal{L}_{ID}(Q, P)$*, with at most* $z_1, ..., z_n$ *free, there is a realiser (expressed by a p-term*[22]*)* $p_A(x, v, w, y, z_1, ..., z_n)$ *such that*

$$HA \vdash (\mathrm{x}\boldsymbol{r}\,A)(\psi, P^{B^r}_{<y}) \leftrightarrow p_A(x, v, w, y, z_1, ..., z_n)\boldsymbol{r}A(\varphi, P^B_{<y}).$$

[21]The x has to be in the middle of the triple, since $\langle x, y, z\rangle = \langle\langle x, y\rangle, z\rangle$ and the conventions we made by formalising the iteration of inductive definitions.

[22]The set of p-terms is constructed by closing-off the set of terms of HA by λ-abstraction and Kleene-brackets.

Proof

The proof proceeds by induction on the complexity of A and is copied from [6, pp. 231-232]. In the following we abbreviate $p_A(x, v, w, y, z_1, ..., z_n)$ by $\mathrm{p_A(x, \bar{\bar{z}})}$.

1. $A \in \mathcal{L}_{ID}(Q)$: In this case $(\mathrm{x}\mathbf{r}\, A)(\psi, P^{B^{\mathbf{r}}}_{<y}) \equiv \mathrm{x}\mathbf{r}\, A(\varphi, P^{B}_{<y})$ and we take x to be $\mathrm{p_A(x, \bar{\bar{z}})}$.

2. $A \equiv P(u)$: Then $(\mathrm{x}\mathbf{r}\, A)(\psi, P^{B^{\mathbf{r}}}_{<y}) \equiv \psi(\langle u, x\rangle)$ and $\mathrm{x}\mathbf{r}\, A(\varphi, P^{B}_{<y}) \equiv \varphi(u)$. So by definition,
$$HA \vdash \psi(\langle u, x\rangle) \leftrightarrow \{v\}(y, w, u, x)\mathbf{r}\varphi(u).$$
Therefore we can use $\{v\}(y, w, u, x)$ as $\mathrm{p_A(x, \bar{\bar{z}})}$.

3. $A \equiv A_0 \wedge A_1$: Then,
$$(\mathrm{x}\mathbf{r}\, A)(\psi, P^{B^{\mathbf{r}}}_{<y}) \equiv ((x)_0\mathbf{r}A_0)(\psi, P^{B^{\mathbf{r}}}_{<y}) \wedge ((x)_1\mathbf{r}A_1)(\psi, P^{B^{\mathbf{r}}}_{<y}).$$
By induction hypothesis,
$$HA \vdash ((x)_i\mathbf{r}A_i)(\psi, P^{B^{\mathbf{r}}}_{<y}) \leftrightarrow p_{A_i}((x)_i, \bar{\bar{z}})\mathbf{r}A_i(\varphi, P^{B}_{<y})$$
for $i \in \{0, 1\}$. Hence,
$$HA \vdash (\mathrm{x}\mathbf{r}\, A)(\psi, P^{B^{\mathbf{r}}}_{<y}) \leftrightarrow \langle p_{A_0}((x)_0, \bar{\bar{z}}), p_{A_1}((x)_1, \bar{\bar{z}})\rangle\mathbf{r}A(\varphi, P^{B}_{<y}).$$

4. $A \equiv A_0 \vee A_1$: In this case $(\mathrm{x}\mathbf{r}\, A)(\psi, P^{B^{\mathbf{r}}}_{<y})$ is
$$[(x)_0 = 0 \rightarrow [((x)_1\mathbf{r}A_0)(\psi, P^{B^{\mathbf{r}}}_{<y})]] \wedge [(x)_0 \neq 0 \rightarrow [((x)_1\mathbf{r}A_1)(\psi, P^{B^{\mathbf{r}}}_{<y})]].$$
By induction hypothesis,
$$HA \vdash ((x)_1\mathbf{r}A_i)(\psi, P^{B^{\mathbf{r}}}_{<y}) \leftrightarrow p_{A_i}((x)_1, \bar{\bar{z}})\mathbf{r}A_i(\varphi, P^{B}_{<y}).$$
Therefore

$$\begin{aligned} HA \vdash (x\mathbf{r}\, A)(\psi, P^{B^{\mathbf{r}}}_{<y}) \leftrightarrow \quad & [(x)_0 = 0 \to [p_{A_0}((x)_1, \bar{\bar{z}})\mathbf{r}A_0(\varphi, P^{B}_{<y})]] \\ & \wedge[(x)_0 \neq 0 \to [p_{A_1}((x)_1, \bar{\bar{z}})\mathbf{r}A_1(\varphi, P^{B}_{<y})]] \\ \leftrightarrow \quad & \langle (x)_0, f((x)_0, p_{A_0}, p_{A_1})\rangle \mathbf{r}A(\varphi, P^{B}_{<y}), \end{aligned}$$

where f is a primitive recursive function that is defined by $f(0, y, z) = y$ and $f(x + 1, y, z) = z$.

5. $A \equiv (\forall u)C(u)$: This gives

$$(x\mathbf{r}\, A)(\psi, P^{B^{\mathbf{r}}}_{<y}) \equiv (\forall u)[(\{e\}(u)\mathbf{r}C)(\psi, P^{B^{\mathbf{r}}}_{<y})].$$

The induction hypothesis gives

$$HA \vdash (\{e\}(u)\mathbf{r}C)(\psi, P^{B^{\mathbf{r}}}_{<y}) \leftrightarrow p_C(\{e\}(u), \bar{\bar{z}}, u)\mathbf{r}C(u)(\varphi, P^{B}_{<y}).$$

Hence

$$\begin{aligned} HA \vdash (x\mathbf{r}\, A)(\psi, P^{B^{\mathbf{r}}}_{<y}) \leftrightarrow \quad & (\forall u)[p_C(\{e\}(u), \bar{\bar{z}}, u)\mathbf{r}C(u, \varphi, P^{B}_{<y})] \\ \leftrightarrow \quad & \lambda u.p_C(\{e\}(u), \bar{\bar{z}}, u)\mathbf{r}A(\varphi, P^{B}_{<y}). \end{aligned}$$

6. $A \equiv (\exists u)C(u)$: Then

$$(x\mathbf{r}\, A)(\psi, P^{B^{\mathbf{r}}}_{<y}) \equiv ((x)_1\mathbf{r}C((x)_0))(\psi, P^{B^{\mathbf{r}}}_{<y}).$$

By the induction hypothesis

$$HA \vdash ((x)_1\mathbf{r}C((x)_0))(\psi, P^{B^{\mathbf{r}}}_{<y}) \leftrightarrow p_C((x)_1, \bar{\bar{z}}, (x)_0)\mathbf{r}C((x)_0, \varphi, P^{B}_{<y}).$$

Consequently

$$\begin{aligned} HA \vdash (x\mathbf{r}\, A)(\psi, P^{B^{\mathbf{r}}}_{<y}) \leftrightarrow \quad & p_C((x)_1, \bar{\bar{z}}, (x)_0)\mathbf{r}C((x)_0, \varphi, P^{B}_{<y}) \\ \leftrightarrow \quad & \langle (x)_0, p_C((x)_1, \bar{\bar{z}}, (x)_0)\rangle \mathbf{r}A(\varphi, P^{B}_{<y}). \end{aligned}$$

7. $A \equiv C \to D$: In this case

$$(\mathrm{x}\mathbf{r}\, A)(\psi, P^{B^{\mathbf{r}}}_{<y}) \equiv (\forall u)[(u\mathbf{r}C)(\psi, P^{B^{\mathbf{r}}}_{<y}) \to (\{e\}(u)\mathbf{r}D)(\psi, P^{B^{\mathbf{r}}}_{<y})].$$

Since A is a strictly positive operator form, $C \in \mathcal{L}_{ID}(Q,P)$ and D is a strictly positive operator form as well. Consequently $(u\mathbf{r}C)(\psi, P^{B^{\mathbf{r}}}_{<y}) \equiv u\mathbf{r}C(\varphi, P^{B}_{<y})$. Also the induction hypothesis gives us

$$HA \vdash (\{e\}(u)\mathbf{r}D)(\psi, P^{B^{\mathbf{r}}}_{<y}) \leftrightarrow p_D(\{e\}(u), \bar{\bar{z}})\mathbf{r}D(\varphi, P^{B}_{<y}).$$

Which gives

$$\begin{aligned} HA \vdash (\mathrm{x}\mathbf{r}\, A)(\psi, P^{B^{\mathbf{r}}}_{<y}) \leftrightarrow \quad & (\forall u)[(u\mathbf{r}C)(\psi, P^{B^{\mathbf{r}}}_{<y}) \to (\{e\}(u)\mathbf{r}D)(\psi, P^{B^{\mathbf{r}}}_{<y})] \\ \leftrightarrow \quad & (\forall u)[(u\mathbf{r}C)(\psi, P^{B^{\mathbf{r}}}_{<y}) \to p_D(\{e\}(u), \bar{\bar{z}})\mathbf{r}D(\varphi, P^{B}_{<y})] \\ \leftrightarrow \quad & \lambda u.p_D(\{e\}(u), \bar{\bar{z}})\mathbf{r}A(\varphi, P^{B}_{<y}). \end{aligned}$$

□

In the following we want to rely on Theorem 2.2.19. Therefore we have to show that those axioms from $\widehat{ID}^{i}_{n}(strict)$ that are not axioms of HA are realisable in $\widehat{ID}^{i}_{n}(acc)$.

Lemma 3.3.11 *For any formula φ of the form*

$$(\forall y < n)(\forall x)[P^{A}_{y}(x) \leftrightarrow A(P^{A}_{y}, P^{A}_{<y}, x, y)],$$

where A is a strictly positive operator, there is an $m \in \mathbb{N}$ such that

$$\widehat{ID}^{i}_{n}(acc) \vdash \bar{m}\boldsymbol{r}\varphi.$$

Proof

Assume that φ is of the form $(\forall y < n)(\forall z)[P^{A}_{y}(z) \leftrightarrow A(P^{A}_{y}, P^{A}_{<y}, z, y)]$ for a strictly positive operator A. By Lemma 3.3.9

$$x\mathbf{r}A(P^{A}_{y}, P^{A}_{<y}, z, y) = (x\mathbf{r}A)(P^{A^{\mathbf{r}}}_{y}, P^{A^{\mathbf{r}}}_{<y}, z, y).$$

Therefore

$$HA \vdash x\mathbf{r}A(P_y^A, P_{<y}^A, z, y) \leftrightarrow A^{\mathbf{r}}(P_y^{A^{\mathbf{r}}}, P_{<y}^{A^{\mathbf{r}}}, x, y)$$

by Definition 3.3.7. By using this equivalence together with the fixed point axiom in $\widehat{ID}_n^i(acc)$, one gets

$$\widehat{ID}_n^i(acc) \vdash y < n \rightarrow [x\mathbf{r}A(P_y^A, P_{<y}^A, z, y) \rightarrow P_y^{A^{\mathbf{r}}}(\langle z, x\rangle)].$$

By Definition 3.3.5 this gives

$$\widehat{ID}_n^i(acc) \vdash y < n \rightarrow [x\mathbf{r}A(P_y^A, P_{<y}^A, z, y) \rightarrow x\mathbf{r}P_y^A(z)].$$

Therefore we can take $m_1 := \ulcorner \lambda yzx.x \urcorner$ as a realiser. The other direction of the axiom is proved in almost the same way, but with the antecedent and the succedent inverted. Hence we obtain an m_2 as we obtained m_1. Therefore we can take $m := \langle m_1, m_2\rangle$ such that

$$\widehat{ID}_n^i(acc) \vdash \bar{m}\mathbf{r}\varphi.$$

□

Now we are able to prove Theorem 3.3.3.

Proof

(of Theorem 3.3.3) Let's suppose for a sentence $\varphi \in \mathcal{L}_{ID}^*(strict)$ that $\widehat{ID}_n^i(strict) \vdash \varphi$. By Theorem 2.2.19 together with Lemma 3.3.11 there is an $m \in \mathbb{N}$ such that $\widehat{ID}_n^i(acc) \vdash \bar{m}\mathbf{r}\varphi$.

Which leads to

$$(*) \quad \widehat{ID}_n^i(acc) \vdash (\exists x)[x\mathbf{r}\varphi].$$

Moreover for almost negative φ Lemma 2.2.22 gives us

$$(**) \quad HA \vdash (\exists x)[x\mathbf{r}\varphi] \rightarrow \varphi.$$

Combining $(*)$ and $(**)$ by modus ponens, one gets

$$\widehat{ID}_n^i(acc) \vdash \varphi.$$

It should be clear to the reader that this can be done primitive recursively. □

Corollary 3.3.12 $\widehat{ID}^i_n(strict) \leq_{\Pi^0_2} \widehat{ID}^i_n(acc)$

Proof

Since $\mathcal{L}_{HA}$ includes all primitive recursive functions as primitive symbols, any formula in Δ_0 is equivalent to an equation. Therefore any Π^0_2 sentence $(\forall x)(\exists y)\varphi$ is equivalent to a $(\forall x)(\exists y)[t_1 = t_2]$ in HA. The latter however is an almost negative formula according to Definition 2.2.20. □

3.3.2 A Proof by Rüede and Strahm

As mentioned above, the aim is to prove that $\widehat{ID}_n(acc) \leq_{\mathcal{L}_{HA}} ACA^-_n$. Before this is done the missing theory ACA^-_n is defined. Even though ACA^-_n is viewed as a second order theory with parameter-free iterated arithmetic comprehension, the following definition introduces ACA^-_n as a first order system. However it should be clear that the present formulation is equivalent to the usual one by an obvious translation.

Definition 3.3.13 *The language $\mathcal{L}_{ACA^-_n}$ is defined as $\mathcal{L}_{ID}(Q)$ with an additional H^φ for any $\varphi(Q, x, y) \in \mathcal{L}_{ID}(Q)$.*[23]

ACA^-_n is based on classical logic, includes all axioms of HA and comprises the following first order comprehension axioms

$$(\forall y < n)(\forall x)[H^\varphi_y(x) \leftrightarrow \varphi(H^\varphi_{<y}, x, y)]$$

for any formula $\varphi(Q, x, y) \in \mathcal{L}_{ID}(Q)$.[24]

[23] Note that $\mathcal{L}_{ID}(Q)$ is the same as $\mathcal{L}_{HA}$ extended by a new predicate symbol Q and that $\neg\varphi$ is defined as $\varphi \rightarrow 0 = 1$.

[24] H^φ_y and $H^\varphi_{<y}$ are defined as before for the theories of inductive definitions.

In the next step we prove Theorem 3.3.4. Note that Theorem 3.3.4 talks about the classical versions of the considered theories, which is essential for the diagonalisation taking place in its proof.

Proof

(of Theorem 3.3.4)

We want to give a translation τ that assigns to a particular P^A a $H^{\tau(A)}$ such that the translations of the fixed point axioms are provable in ACA_n^-. Let $A(P,Q,x,y)$ be an accessibility operator. Hence there are $\varphi, \psi \in \mathcal{L}_{ID}(Q)$ such that

$$A(P,Q,x,y) \equiv \varphi \wedge (\forall z)[\psi(z) \to P(z)].$$

Next we choose an n such that φ and ψ are equivalent to a $\Pi^0_n(Q)$ and $\Sigma^0_n(Q)$ formula respectively.[25] We aim for a fixed point construction, therefore we consider a universal $\Pi^0_n(Q)$-formula for $\mathcal{L}_{ID}(Q)$, which is a formula $E_n(Q,u,x,y,z) \in \Pi^0_n(Q)$ such that for every $\varphi \in \Pi^0_n(Q)$

$$PA \vdash E_n(Q, \ulcorner\varphi\urcorner, x, y, z) \leftrightarrow \varphi.$$

Substituting $E_n(Q,u,u,z,y)$ for $P(z)$ in $A(P,Q,x,y)$, one gets

$$A(E_n(Q,u,u,z,y),Q,x,y).$$

It is easy to see that $A(E_n,Q,x,y)$ is equivalent to a $\Pi^0_n(Q)$-formula. Therefore there is a $k := \ulcorner A(E_n,Q,x,y)\urcorner$ such that

$$ACA_n^- \vdash E_n(Q,\bar{k},u,x,y) \leftrightarrow A(E_n(Q,u,u,z,y),Q,x,y).$$

Substituting $\bar{k}$ for u, we get

$$ACA_n^- \vdash E_n(Q,\bar{k},\bar{k},x,y) \leftrightarrow A(E_n(Q,\bar{k},\bar{k},z,y),Q,x,y).$$

[25]The definition of $\Pi^0_n(Q)$ literally follows the usual definition of Π^0_n (see Definition 2.1.3) but additionally considers $Q(t)$ to be a Δ^0_0 formula. $\Sigma^0_n(Q)$ is similarly defined.

We define $D(Q, x, y) :\equiv E_n(Q, \bar{k}, \bar{k}, x, y)$ for notational reasons.

The comprehension axiom gives

$$ACA_n^- \vdash (\forall y < n)(\forall x)[H_y^D(x) \leftrightarrow D(H_{<y}^D, x, y) \leftrightarrow E_n(H_{<y}^D, \bar{k}, \bar{k}, x, y)].$$

The definition of E_n gives:

$ACA_n^- \vdash$

$$(\forall y < n)(\forall x)[E_n(H_{<y}^D, \bar{k}, \bar{k}, x, y) \leftrightarrow A(E_n(H_{<y}^D, \bar{k}, \bar{k}, z, y), H_{<y}^D, x, y)].$$

Combining the two equivalences,

$$ACA_n^- \vdash (\forall y < n)(\forall x)[H_y^D(x) \leftrightarrow A(H_y^D, H_{<y}^D, x, y)].$$

It is therefore possible to reduce $ID_n(acc)$ to ACA_n^-, by relating P^A to its H^D through a translation τ as stated at the beginning of the proof. Note that, since this translation is uniform in the accessibility operators through E_n, the coding is primitive recursive and the deductions appearing in this proof are uniform in the accessibility operators through E_n, this translation gives a primitive recursive translation of $ID_n(acc)$-deductions into ACA_n^--deductions. Consequently we have a proof-theoretical reduction. □

It remains to reduce ACA_n^- to PA and PA to HA.

Theorem 3.3.14 $ACA_n^- \subset_{\mathcal{L}_{PA}} PA$

Proof

Let's assume that the logic is formulated via a sequent calculus. In this case $ACA_n^- \vdash \psi$ means $LK \vdash \Delta \Rightarrow \psi$, where $\Delta \subset ACA_n^-$. Since $LK \vdash \Delta \Rightarrow \psi$, there is a cut-free deduction d_0 of $\Delta \Rightarrow \psi$ in LK. Therefore, since $\psi \in \mathcal{L}_{PA}$, d_0 includes only those H^φ which occur in Δ, say $H^{\varphi_1}, ..., H^{\varphi_k}$. The only axioms of ACA_n^- including an H^{φ_i}

are those falling under the comprehension or induction schema. We temporarily ignore induction and consider formulas of the form

$$(\forall y < n)(\forall x)[H_y^{\varphi_i}(x) \leftrightarrow \varphi_i(H_{<y}^{\varphi_i}, x, y)].$$

By basic arithmetic these are equivalent to formulas of the form

$$\bigwedge_{j=1}^{n} (\forall x)[H_j^{\varphi_i}(x) \leftrightarrow \varphi_i(H_{<j}^{\varphi_i}, x, j)].$$

However, since a sequent

$$\bigwedge_{j=1}^{n} \chi_j, \Gamma \Rightarrow \psi$$

is equivalent to

$$\chi_1, ..., \chi_n, \Gamma \Rightarrow \psi,$$

we obtain from d_0 a deduction d_1 where the conjuncts of the comprehension formulas occur separately. (We continue to call the conjuncts of a conjunction that is equivalent to a comprehension formula the conjuncts of this particular comprehension formula.)
We fix an $m \leq n$ such that

$$(\forall x)[H_m^{\varphi_i}(x) \leftrightarrow \varphi_i(H_{<m}^{\varphi_i}, x, \bar{m})]$$

occurs in the sequent which is equivalent to $\Delta \Rightarrow \psi$ as explained above. According to the chosen notation, this gives that

$$H_m^{\varphi_i}(t) \text{ is the formula } H^{\varphi_i}(\langle t, \bar{m} \rangle)$$

and

$$H_{<m}^{\varphi_i}(t) \text{ is the formula } t = \langle t_0, t_1 \rangle \wedge t_1 < \bar{m} \wedge H^{\varphi_i}(t).$$

Basic arithmetic gives for $m = 0$ the equivalence

$$PA \vdash H_{<0}^{\varphi_i}(t) \leftrightarrow \bar{0} = \bar{1}$$

and for $m > 0$ the equivalence

$$PA \vdash H^{\varphi_i}_{<m}(t) \leftrightarrow H^{\varphi_i}(\langle t, \bar{0} \rangle) \vee ... \vee H^{\varphi_i}(\langle t, \overline{m-1} \rangle).$$

This justifies the substitution[26]

$$\begin{aligned}
\mathrm{s}(H^{\varphi_i}_{<0}(t)) &:\equiv \bar{0} = \bar{1} \\
\mathrm{s}(H^{\varphi_i}_{<m}(t)) &:\equiv H^{\varphi_i}(\langle t, \bar{0} \rangle) \vee ... \vee H^{\varphi_i}(\langle t, \overline{m-1} \rangle) \\
\mathfrak{s}(H^{\varphi_i}_{m}(t)) &:\equiv \varphi_i([\mathfrak{s}(\mathrm{s}(H^{\varphi_i}_{<m}(t)))], x, \bar{m}) \\
\mathfrak{s}(\varphi \circ \psi) &:\equiv \mathfrak{s}(\varphi) \circ \mathfrak{s}(\psi) && \circ \in \{\vee, \wedge, \rightarrow\} \\
\mathfrak{s}(\bot) &:\equiv \bot \\
\mathfrak{s}(Qz\varphi) &:\equiv Qz\mathfrak{s}(\varphi) && Q \in \{\exists, \forall\}
\end{aligned}$$

Using this substitution on d_1, we obtain a d_2 that does not include any occurrence of H^{φ_i}. Because, since $\mathfrak{s}(\varphi_i(H^{\varphi_i}_{<m}, x, \bar{m}))$ gives literally the same formula as $\mathfrak{s}(H^{\varphi_i}_{m}(x))$, the end sequent of d_2 contains formulas of the form

$$(\forall x)[\chi(x, \bar{m}) \leftrightarrow \chi(x, \bar{m})]$$

instead of comprehension axioms. Since these equivalences are valid formulas, they can be cut out from the final sequent. Consequently, and after substituting $t = 0$ into the remaining induction formulas for $H^{\varphi_i}(t)$, we get a deduction d_3 for a sequent $\Delta' \Rightarrow \psi$ such that $\Delta' \subset \Delta \cap PA$ and, hence, $PA \vdash \psi$. □

Using the translation Friedman gives in [21], we can easily reduce PA to HA for every Π^0_2-sentence.

[26] $\varphi_i([\chi(t)], x, \bar{0})$ is obtained from $\varphi_i(H^{\varphi_i}_{<m}, x, \bar{0})$ by substituting $\chi(t)$ for $H^{\varphi_i}_{<m}(t)$.

3.4 Conclusions

With Theorem 3.3.1 having been established, it is possible to rephrase Definition 3.1.3 without changing the ordering on theories that is induced by it. But before we can do so we have to ensure that the presence of transfinite induction does not change the situation. Hence we have to prove the following theorem.

Theorem 3.4.1

$$[PA]_{\prec} \equiv_{\Pi^0_2} [\widehat{ID}^i_{<\omega}(strict)]_{\prec}$$

Theorem 3.4.1 will follow easily from the following lemma.

Lemma 3.4.2 *Assume that $\prec$ is a primitive recursive relation on $\mathbb{N}$ and $TI(\prec)$ is the schema that is defined in Definition 2.2.27. For any closure φ of an instance of $TI(\prec)$ there is an $n \in \mathbb{N}$ such that*

$$HA + TI(\prec) \vdash \bar{n}\boldsymbol{r}\varphi.$$

Proof

As before we follow [59, p. 199] and give the proof in some detail, because we will need it in Chapter 6. But for simplicity we restrict ourselves to the case of a closed instance of $TI(\prec)$. Also we assume without loss of generality that any natural number is in the range of $\prec$; hence φ is of the form

$$(\forall x)[(\forall y \prec x)\psi(y) \to \psi(x)] \to (\forall x)\psi(x).$$

We assume that

$$w\mathbf{r}(\forall x)[(\forall y \prec x)\psi(y) \to \psi(x)];$$

hence

$$(\forall x)[\{w\}(x)\mathbf{r}[(\forall y \prec x)\psi(y) \to \psi(x)]]$$

by definition. Which expands to

$$(\forall x, z)[z\mathbf{r}(\forall y \prec x)\psi(y) \to \{w\}(x, z)\mathbf{r}\psi(x)].$$

Note that $z\mathbf{r}(\forall y \prec x)\psi(y)$ implies $(\forall y \prec x)[\{z\}(y, 0)\mathbf{r}\psi(y)]$, since $y \prec x$ is quantifier free in $\mathcal{L}_{HA}$ for a primitive recursive relation $\prec$. We define

$$f_c(u, x, y) :\simeq \begin{cases} 0 & : \quad x \succeq y \\ u & : \quad x \prec y \end{cases}.$$

It is easy to find a partial recursive function f such that

$$f(v, w, x) \simeq \{w\}(x, \lambda y \lambda u. f_c(\{v\}(w, y), y, x)).$$

The recursion theorem assures that there is an $\bar{n}$ such that

$$\{\bar{n}\}(w, x) \simeq \{w\}(x, \lambda y \lambda u. f_c(\{\bar{n}\}(w, y), y, x)).$$

Using $TI(\prec)$ with respect to x, we can easily prove the totality of $\{\bar{n}\}(w, x)$ and that $\{\bar{n}\}(w, x)\mathbf{r}\psi(x)$. For instance the latter follows from

$$(\forall x)(\forall y \prec x)[\{\bar{n}\}(w, y)\mathbf{r}\psi(y) \to \{\bar{n}\}(w, x)\mathbf{r}\psi(x)].$$

Hence $\bar{n}$ realises φ by Definition 2.2.18.□

With this lemma proved, the proof of Theorem 3.4.1 is almost trivial.

Proof

of Theorem 3.4.1.

The proof proceeds in three steps. To prove

$$[\widehat{ID}^i_n(strict)]_\prec \leq_{alm.neg.} [\widehat{ID}^i_n(acc)]_\prec$$

note that Lemma 3.4.2 provides the same resources for Theorem 3.4.1 as Lemma 3.3.11 provides for Theorem 3.3.3.

To prove

$$[\widehat{ID}_n(acc)]_\prec \leq_{\mathcal{L}_{HA}} [ACA^-_n]_\prec$$

note that the proof of Theorem 3.3.4 is given by a translation τ. Hence any translation of an instance of transfinite induction from $\mathcal{L}_{\widehat{ID}_n(acc)}$ is an instance of transfinite induction from $\mathcal{L}_{ACA_n^-}$.
Since Theorem 3.3.14 is proved by a translation as well, as an analogous argument gives

$$[ACA_n^-]_\prec \leq_{\mathcal{L}_{PA}} [PA]_\prec.$$

Combining these three steps, the theorem is proved. □
Consequently we can rephrase Definition 3.1.3 as follows.

Definition 3.4.3 *Let T be a theory as defined in chapter 2 and $OT(\beta)$ an ordinal notation system. If there is an $\alpha \in OT(\beta)$ such that*

$$T \equiv_{\Pi_2^0} [\widehat{ID}^i_{<\omega}(strict)]_\alpha,$$

then we call α the Π_2^0-proof-theoretical-ordinal of T and denote it by $\|T\|_{\Pi_2^0}$.

The natural way of formalising an ordinal-analysis in $[\widehat{ID}^i_{<\omega}(strict)]_\prec$, for an appropriate $\prec$, gives a recipe that is general enough for the purposes that are outlined in Chapter 5. But before we discuss these, we will justify this claim of naturalness in the very next chapter for the example of a system that includes the Ω-rule.

Chapter 4

A formulation of the Ω-Rule in $\widehat{ID}^{i}_{2}(strict)$

As explained in Section 3.1, it is of special interest to formalise the methods of ordinal-analysis in a theory of arithmetic in order to obtain conservativity results. In several places proof-theorists give such formalisations for infinite systems, e.g. systems that include the ω-rule (see [38] and [51]). There are two ways of formalising infinite deductions. The first is called the *local version*: the nodes of an infinite deduction-tree are coded as n-tuples that include a recursive function enumerating the direct predecessors of the node. The second is called the *global version*: the infinite deduction-trees are represented in arithmetic through a code of their characteristic (recursive) function. In both versions the codes of some recursive functions essentially appear in the representation of the deduction-trees. Consequently to define an operation upon these representations in an arithmetical theory one has to alter these codes in a recursive way, i.e. one has to rely on a formalised version of the recursion theorem that is accessible in this theory.[1] Hence the process of formalisation is not trivial and needs a lot of formalised recursion

[1]The most common non-formalised version of the recursion theorem is the following (see [24]): For any recursive function $f(\vec{x}, y)$ there exists an e such that $\{e\}(\vec{x}) \simeq f(\vec{x}, e)$.

theory. Moreover it lies in the nature of mathematical texts dealing with such matters that they include a lot of handwaving in order to make them readable.[2] Since the applications of formalised recursion theory are not trivial, this sometimes causes doubts in the community. In addition less commonly used concepts like the Ω-rule[3], which is given by Buchholz in [6], have never been properly formalised. This is not the case, because the formalisation would be to long or painful to write down. Rather in the case of the Ω-rule it is not clear what a formalisation in PA could look like. However in the following we show that using $\widehat{ID}^i_{<\omega}(strict)$ helps to avoid these issues (as was promised in Chapter 3). For in this theory the formalisation is straightforward. The present chapter tries to convince the reader of this claim by giving a formalisation of an infinite system, that comprises the Ω-rule, in $\widehat{ID}^i_2(strict)$. However the claim is not that the formalisation of ordinal-analysis in $\widehat{ID}^i_{<\omega}(strict)$ is less long, complex or painful than in PRA or PA. The aim of this chapter is to show that a formalisation of the Ω-rule can be given and that for most of the infinite systems, that are given in the literature, their formalisation is straightforward; for formalised recursion theory is not needed. The reader may skip the present chapter, when she autonomously sees how that might be done, because the content of this chapter is not used in any other part of the present thesis. However we will draw on the possibility of an Ω-rule formalisation later on.

4.1 A particular system including the Ω-rule

As was mentioned above, we aim to convince the reader that a formalisation of an ordinal-analysis that uses the Ω-rule can be done in $\widehat{ID}^i_{<\omega}(strict)$. This is done by formalising a provability predicate that includes the Ω-rule and is exemplary for those methods. We will formalise the one that is given by Rathjen in [41], since it is used in the ordinal-analysis

[2]However I personally believe that Kleene's [28] is a very nice piece of logical work. However not many would agree.

[3]See Definition 4.1.3 and its following discussion.

of theories that we will consider in Chapter 5. When this example is followed, it should be easy to apply the techniques to other systems as well.

Definition 4.1.1 *The language $\mathcal{L}^2_{T_\infty}$ is a second order language which uses $\in$ as an mediator between first-order terms and second order variables. Moreover $\mathcal{L}^2_{T_\infty}$ distinguishes between bound and free variables of any order and includes the logical connectives $\vee, \wedge, \rightarrow$ and $\neg$. Its primitive symbols are the same as those of $\mathcal{L}_{PRA}$. The atoms are $t_1 = t_2$, $t \in U$, $\neg(t_1 = t_2)$ and $\neg(t \in U)$. The formulas are built from the atoms through $\wedge, \vee, \forall x, \forall X, \exists x$ and $\exists X$. Hence we stipulate that formulas are in negation normal form. Moreover we call formulas arithmetical, Π^0_∞-formulas or Π^1_0-formulas, if they do not include bound second order variables. A formula is called weak, if it belongs to $\Pi^1_0 \cup \Pi^1_1$.*[4]

Free second-order variables and bounded second-order variables are denoted by U, V, W and X, Y, Z respectively on the meta-level. In [41] the complexity of a formula is measured by the following function.

Definition 4.1.2

1. *$gr(A) = 0$, if A is an atom.*[5]
2. *$gr(\forall X F(X)) = gr(\exists X F(X)) = \omega$, if $F(U)$ is arithmetical.*
3. *$gr(A \wedge B) = gr(A \vee B) = \max\{gr(A), gr(B)\} + 1$*
4. *$gr(\forall x F(x)) = gr(\exists x F(x)) = gr(F(0)) + 1$*
5. *$gr(\forall X F(X)) = gr(\exists X F(X)) = gr(F(U)) + 1$, if $F(X)$ is not arithmetical.*

The next definition gives the infinite deduction system that we want to formalise. We denote the system by T_∞ to emphasis that the system is related to a theory T from

[4] The set of Π^1_1-formulas includes the Π^1_0-formulas and is closed under second-order $\forall$-quantification.

[5] Note we defined $\neg(t_1 = t_2)$ to be atoms as well.

[41], which is not given here. Also we use the ordinal notation system of [41] merely like a black-box and denote it by $(OT(\theta), \prec, \lhd)$, where $\lhd$ is an additional ordering on $OT(\theta)$ that is defined by using $\prec$. In addition $\lhd$ is entangled with a notion of *fundamental functions*. However the actual definitions of $\lhd$ and *fundamental functions* are of no interest here. We just simply note that they can be formalised in arithmetic by an arithmetical formula. The following definition also lacks an explicit formulation of the logical rules, but these formulations can be found in Appendix C at item 4.

Definition 4.1.3 *Assume that $\alpha \in OT(\theta)$, $\rho \prec \omega + \omega$.*

1. *If A is a true atomic or negated-atomic sentence and $A \in \Gamma$, then $T_\infty \vdash_\rho^\alpha \Gamma$.*
2. *If Γ contains $A(s_1, ..., s_n)$ and $\neg A(t_1, ..., t_n)$ with grade 0 or ω, where s_i and t_i $(1 \leq i \leq n)$ are terms with equal value, then $T_\infty \vdash_\rho^\alpha \Gamma$.*
3. *Assume that $\langle \Gamma_1, ..., \Gamma_n : \Gamma \rangle$ is an instance of $(\wedge), (\vee), (\exists_1), (\forall_2)$ or (cut), where in the case of a cut the cut-formula is of a grade $\prec \rho$, and $\beta_i \lhd \alpha$ for any $1 \leq i \leq n$. If $T_\infty \vdash_\rho^{\beta_i} \Gamma_i$, then $T_\infty \vdash_\rho^\alpha \Gamma$.*
4. *If $T_\infty \vdash_\rho^\beta \Gamma, F(U)$ holds for some $\beta \lhd \alpha$ and $\omega \preceq gr(F(U))$, i.e. $F(U)$ is not arithmetical, then $T_\infty \vdash_\rho^\alpha \Gamma, \exists X F(U)$.*
5. *If for any $m \in \mathbb{N}$, both $T_\infty \vdash_\rho^{\beta_m} \Gamma, A(\bar{m})$ and $\beta_m \lhd \alpha$ holds, then $T_\infty \vdash_\rho^\alpha \Gamma, \forall x A(x)$.*
6. *Let f be a fundamental function. Moreover assume that*
   - *(a) $f(\Omega) \unlhd \alpha$,*
   - *(b) $T_\infty \vdash_\rho^{f(0)} \Gamma, \forall X F(X)$, where $\forall X F(X) \in \Pi_1^1$, and that*
   - *(c) $T_\infty \vdash_0^\beta \Delta, \forall X F(X)$ implies $T_\infty \vdash_0^{f(\beta)} \Delta, \Gamma$ for every set of weak formulas Δ and any $\beta \prec \Omega$.*

   In this case $T_\infty \vdash_\rho^\alpha \Gamma$ holds.

Clause 6 is called *the Ω-rule* and was given by Rathjen in [41]. Rathjen himself developed it from a natural-deduction version of an Ω-rule that is given by Buchholz in [6]. Buchholz justifies or explains the rule in the mode of the Brouwer-Heyting-Kolmogorov-interpretation (see [6]). In this interpretation the justification of an implication, say $A \to B$, is viewed as a transformation of a presupposed justification for A into a justification for B. Most of the contemporary intuitionistic literature identifies "justification" as a formal deduction in some constructive system. Consequently a deduction of $A \to B$ has to offer the possibility of transforming a deduction of A into a deduction of B, e.g. this is often considered as the idea behind $\to$-introduction in the calculus of natural deduction. In this mood, Buchholz argues as follows: one is allowed to conclude an implication $P^A(n) \to C$, where A is a strictly positive operator and C any formula, when one can constructively find for any deduction of $P^A(n)$ a deduction of C. He is therefore allowed to extend his formal system by the following rule.

> **Ω-rule:** *If for each direct*[6] *deduction X of $P^A(n)$ the deduction Y_X proves C, then one can form the deduction*

$$\frac{\begin{array}{c} Y_X \\ \vdots \\ C \end{array} \quad : \quad \textit{for any direct deduction } X \textit{ with} \quad \begin{array}{c} X \\ \vdots \\ P^A(n) \end{array}}{P^A(n) \to C}\,.$$

Note the differences between $\to$-introduction and the Ω-rule. While the Ω-rule explicitly and merely requires that any deduction for $P^A(n)$ can be transformed into a deduction of C, $\to$-introduction implicitly assures this constructibility requirement by presupposing a logical claim that implies the possibility of a transformation. Consequently the Ω-rule grasps the constructive character of the Brouwer-Heyting-Kolmogorov-interpretation more accurately than $\to$-introduction does.[7] Moreover the transformation requirement is

[6] Direct means that a deduction is cut-free or free of local maxima in a sequent calculus or in natural deduction respectively.

[7] In other words, if one takes the $\to$-introduction from natural deduction as the explication of the

enough to ensure the possibility of eliminating local maxima, since the information of how to return to a deduction of $P^A(n)$ is still available.

In Rathjen's formulation the Ω-rule is a little more. Only the third part of clause 6 represents the actual Ω-rule; the rest extends the rule by a cut and connects it to the fundamental functions. In order to understand Rathjen's restriction to Π^1_1-formulas, note that in Buchholz formulation $P^A(n)$ can be seen as a Π^1_1-formula, when P^A is considered as a free second-order variable. Moreover Rathjen's inclusion of a cut into clause 6 matches Buchholz's intentions as well. Since Buchholz wants to eliminate local maxima in his natural deduction formulation, he has to reduce an Ω-rule that is followed by an $\rightarrow$-elimination. However when one wants to reduce cuts in a sequent calculus, the analogous case to Buchholz's is an Ω-rule that is followed by a cut.

The issue of formalising such a rule should now be unravelled. The premiss quantifies over all deductions of a certain kind; hence the premiss presupposes a previously defined set whose elements are in the range of its metaquantifier. In other words, Definition 4.1.3 is a singularly iterated inductive definition. This cannot be covered (or at least not in an obvious way) by a set of infinite trees whose branches are elementarily enumerable as the deductions whose only infinite rule is the ω-rule can be. Therefore we cannot formalise Definition 4.1.3 in the language of PA as the literature does with deductions of PA_ω. But, since the definition is a singularly iterated inductive definition, the theory $\widehat{ID}^i_2(strict)$ is able to formalise it, as we will show in Section 4.3. In the next section however we formalise several easy concepts, like "being a formula", in order to have easy examples that show how such a formalisation work.

Brouwer-Heyting-Kolmogorov-interpretation, then it strongly undergenerates, i.e. the formal notion given does not cover everything the Brouwer-Heyting-Kolmogorov-interpretation seems to allow.

4.2 Working out concepts in $\widehat{ID}_1^i$

The present section shows how finite concepts of proof theory can be formulated in $\widehat{ID}_1^i(acc)$. In $\widehat{ID}_1^i(acc)$ inductive definitions are not iterated. Therefore we can work in the weaker language $\mathcal{L}(P)$ and with operators that have no occurrence of v_1. Consequently we only use fixed point axioms of the following form:

$$(\forall x)[P^{\varphi}(x) \leftrightarrow \varphi(P^{\varphi}, x)].$$

Since we have to work in arithmetic, a minimum of coding the syntax is necessary. Let's assume that we have already fixed a Gödel numbering. This allows us to express whether a symbol is a free or bound variable of a certain order ($FV^1(x), FV^2(x), BV^1(x), BV^2$), a constant ($C(x)$) or a function symbol ($F(x)$). Moreover we are able to define a function that gives the arity of a symbol ($arity(x)$) and to express that a string of symbols is an atom ($At(x)$). Since all primitive recursive functions are in the language, there is a length function for n-tuples $lg(x)$ and a substitution function $sub(x, y, z)$, which substitutes z for y in x, in the language. The reader should bear in mind that the following formulas are hard to read because of the special form they have to have. In order to reduce the number of symbols in a formula, projection onto the i-th entry of a tuple x is denoted by x_i. We start with a formulation for what it means to be a term ($Term(P, x)$):

$$FV^1(x) \vee C(x) \vee [lg(x) = arity(x_0) + 1 \wedge F(x_0)$$

$$\wedge \forall z((\exists i < lg(x))(i \neq 0 \wedge z = x_i) \rightarrow P(z))].$$

We denote the fixed point of this formula by P^T. Note that this formula is an accessibility operator.[8] In a similar way one can formulate the notion of formula as an accessibility operator $Form(P, x)$:

$$[At(x) \vee (lg(x) = 3 \wedge ((Q(x_0) \wedge (BV^1(x_1) \vee BV^2(x_1))) \vee PC(x_0)]$$

[8]Note that the chosen coding cannot view variables and constants as codes for pairs.

$$\wedge\forall z[lg(x) = 3 \wedge [(Q(x_0) \wedge (\exists y < f(lg(x)))(z = sub(x_2, x_1, y)))$$

$$\vee(PC(x_0) \wedge (z = x_1 \vee z = x_2)] \to P(z))].$$

Here $Q(x)$ is $x = \ulcorner\forall\urcorner \vee x = \ulcorner\exists\urcorner$ and $PC(x)$ is $x = \ulcorner\vee\urcorner \vee x = \ulcorner\wedge\urcorner$. The variable y can be bounded by $f(lg(x))$ for some elementary recursive function f, because the number of bound variables occurring in x is bounded by $lg(x)$. However, f depends on the coding.

In order to formulate the provability predicate of [41], we need the grade function on formulas. Therefore we reformulate $Form(P, x)$ so that any formula is a pair containing the actual formula and its grade. We achieve this by using the ordinal notation system from [41] as a black box $(OT, \prec)$.

$$[lg(x) = 2 \wedge OT(x_1) \wedge [[At(x_0) \wedge x_1 = \ulcorner 0\urcorner]\vee$$

$$[lg(x_0) = 3 \wedge [(Q(x_{0,0}) \wedge (BV^1(x_{0,1}) \vee BV^2(x_{0,1}))) \vee PC(x_{0,0})]]]\wedge$$

$$\forall z[lg(z) = 2 \wedge lg(x_0) = 3 \wedge [[Q(x_{0,0}) \wedge (\exists y < f(lg(x_0)))(z_0 = sub(x_{0,2,0}, x_{0,1}, y)\wedge$$

$$[(BV^1(x_{0,1}) \wedge x_1 = \ulcorner z_1 + 1\urcorner) \vee (BV^2(x_{0,1})\wedge$$

$$((z_1 \prec \ulcorner\omega\urcorner \to x_1 = \ulcorner\omega\urcorner) \vee (z_2 \succeq \ulcorner\omega\urcorner \to x_1 = \ulcorner z_2 + 1\urcorner))]\vee$$

$$[PC(x_{0,0}) \wedge (z_0 = x_{0,1} \vee z_0 = x_{0,2}) \wedge x_1 = \ulcorner\max(x_{0,1,1}, x_{0,2,1}) + 1\urcorner]$$

$$\to P(z)]$$

We denote the fixed point of this formula by P^F.

4.3 Formulating the Ω-rule

In this section we give a strictly positive operator $\mathcal{D}$ that can be used in a fixed point axiom and expresses the Ω-rule.

We do not use the predicates that were defined in the previous section here, since every

use of an inductive definition in another one increases the number of iterations. Instead we use a Δ_1-formula $Form(x)$ which gives formulas as pairs of the actual formula and their grade. We have to do this in order to keep the iteration of the inductive definitions below 2. $T_0(x)$ denotes the standard truth predicate for atoms; $\Pi^1_0(x)$ and $\Pi^1_1(x)$ are used to denote formulas which enumerate the codes of Π^1_0 and Π^1_1 formulas respectively. Sets of formulas $\{A_1, ..., A_n\}$ are viewed as sums of the form $2^{\ulcorner A_0 \urcorner} + ... + 2^{\ulcorner A_n \urcorner}$.[9] We use $FS(x)$ to denote $(\forall y)(\forall z)[x = 2^y + z \rightarrow Form(y)]$, which expresses that x is a set of formulas, and $y \in x$ to denote $(\exists z < x)[x = 2^y + z]$. Also we use $x \cup y$ to denote $x + y$, when x and y are viewed as sets. In this view, we define set-difference as
$$x \backslash u := \begin{cases} x - u & : \exists y < x[x = u + y] \\ x & : \text{otherwise} \end{cases}.$$
The dot-notation $\dot{x}$ denotes the function that gives the x-th numeral, e.g. $\dot{3} := \ulcorner S(S(S(\bar{0})))\urcorner$. We use $\ulcorner A \urcorner$ and $\ulcorner \Gamma \urcorner$ in formulas $B(\ulcorner A \urcorner)$ and $B(\ulcorner \Gamma \urcorner)$ as abbreviations for the formulas $Form(x) \wedge B(x)$ and $FS(x) \wedge B(x)$ respectively. Also we use this notation in cases where $\ulcorner A \urcorner$ and $\ulcorner \Gamma \urcorner$ are quantified by an $\exists$-quantifier. However in the case of a $\forall$-quantification we use the implication-analogue.[10] When we quantify over an ordinal α from the notation system, we use $\ulcorner \alpha \urcorner$ in a similar way. We also use $y \prec gr(\ulcorner A \urcorner)$ to denote $Form(x) \wedge y \prec (x)_1$.

We formalise the axioms of T_∞ by the following formula $Axiom(x)$
$$FS(x) \wedge [(\exists y < x)(y \in x \wedge At(y) \wedge T_0(x))) \vee$$
$$(\exists y, z < x)(y \in x \wedge z \in x \wedge neg(y) = z \wedge (At(y) \wedge gr(y) = \ulcorner \omega \urcorner))].$$

[9]Note that every natural number can be uniquely viewed as a sum of powers of 2.

[10]Which are $Form(x) \rightarrow B(x)$ and $FS(x) \rightarrow B(x)$ respectively. Consequently formulations like
$$(\forall \ulcorner A \urcorner)[...B(\ulcorner A \urcorner)...]$$
denote formulas like
$$(\forall x)[Form(x) \rightarrow ...B(x)...].$$
Also in some cases, where we build up formulas bit by bit, the actual formula is not specified before we say how we quantify. However that should not cause any confusion.

Here $neg(\ulcorner A \urcorner) := \ulcorner \neg A \urcorner$, which is primitive recursive.[11]

In the following we define a derivability predicate $\mathcal{D}(P, Q, x, y)$ for 5-tuples of the form $\langle \Gamma, \alpha, \rho, R_I, A \rangle$ in order to formalise T_∞-provability. Here Γ is a set of formulas that is provable with a cut-rank ρ in α many steps, R_I is the inference rule that was used last in the derivation and has A as its principal formula. Unfortunately, since we have to iterate the inductive definition, we talk about pairs of this 5-tuples with a natural number that counts the stage of iteration.

Since $\mathcal{D}(P, Q, x, y)$ has to be strictly positive, we have to ensure that no P is in the scope of a $\neg$. Also, since the Ω-rule is defined with respect to certain functions, which are not specified here, the following formulas are schemata in φ_f and Ψ, which act as place-holders for the functions and their properties respectively.

The formula $\mathcal{D}(P, Q, x, y)$ is defined as follows:

$$x = \langle \langle \ulcorner \Gamma \urcorner, \ulcorner \alpha \urcorner, \ulcorner \rho \urcorner, i, \ulcorner A \urcorner \rangle, y \rangle \wedge$$

$$[\mathcal{D}_{Ax} \vee \mathcal{D}_{\vee} \vee \mathcal{D}_{\wedge} \vee \mathcal{D}_{cut} \vee \mathcal{D}_{\exists_1} \vee \mathcal{D}_{\forall_2} \vee \mathcal{D}_{\exists_2} \vee \mathcal{D}_{\omega} \vee \mathcal{D}_{\Omega}],$$

where $\mathcal{D}_\Omega$ is the only formula that includes an occurrence of Q and i is a meta-notation for a first order variable. Note that, even x is not explicitly shown in the following metanotations of the subformulas of $\mathcal{D}$, it actually occurs in the formulas; it is covered by several formulations, e.g. "$i =$" and "$\ulcorner A \urcorner \in$".[12]

$\mathcal{D}_{Ax}(x)$ is

$$i = R_{Ax} \wedge Axiom(\ulcorner \Gamma \urcorner) \wedge \ulcorner A \urcorner \in \ulcorner \Gamma \urcorner$$

$\mathcal{D}_{\vee}(P, x, y)$ is

$$i = R_{\vee} \wedge \ulcorner A \urcorner = \ulcorner B \vee C \urcorner \wedge$$

[11] Note that this function does not simply place a $\neg$ in front of the formula, because the formulas of T_∞ are given in negation normal form.

[12] Note that these formulations denote formulas of $\mathcal{L}(Q, P)$, since the pair function is in the language.

$$(\exists z)[z = \langle\langle\ulcorner\Lambda\urcorner,\ulcorner\beta\urcorner,\ulcorner\gamma\urcorner, j,\ulcorner F\urcorner\rangle, y\rangle \wedge P(z)\wedge$$

$$[\ulcorner B\urcorner \in \ulcorner\Lambda\urcorner \vee \ulcorner C\urcorner \in \ulcorner\Lambda\urcorner] \wedge \ulcorner\Lambda\urcorner\backslash\ulcorner\{B, C\}\urcorner = \ulcorner\Gamma\urcorner\wedge$$

$$\ulcorner\beta\urcorner \lhd \ulcorner\alpha\urcorner \wedge \ulcorner\rho\urcorner = \ulcorner\gamma\urcorner].$$

$\mathcal{D}_\wedge(P, x, y)$ is

$$i = R_\wedge \wedge \ulcorner A\urcorner = \ulcorner B \wedge C\urcorner\wedge$$

$$(\exists z_1, z_2)[z_1 = \langle\langle\ulcorner\Lambda_1\urcorner,\ulcorner\beta_1\urcorner,\ulcorner\gamma_1\urcorner, j,\ulcorner F_1\urcorner\rangle, y\rangle\wedge$$

$$z_2 = \langle\langle\ulcorner\Lambda_2\urcorner,\ulcorner\beta_2\urcorner,\ulcorner\gamma_2\urcorner, j,\ulcorner F_2\urcorner\rangle, y\rangle \wedge P(z_1) \wedge P(z_2)\wedge$$

$$\ulcorner B\urcorner \in \ulcorner\Lambda_1\urcorner \wedge \ulcorner C\urcorner \in \ulcorner\Lambda_2\urcorner \wedge \ulcorner(\Lambda_1 \cup \Lambda_2)\backslash\{B, C\}\urcorner = \ulcorner\Gamma\urcorner\wedge$$

$$\ulcorner\beta_1\urcorner \lhd \ulcorner\alpha\urcorner \wedge \ulcorner\rho\urcorner = \ulcorner\gamma_1\urcorner \wedge \ulcorner\beta_2\urcorner \lhd \ulcorner\alpha\urcorner \wedge \ulcorner\rho\urcorner = \ulcorner\gamma_2\urcorner].$$

$\mathcal{D}_{\exists_1}(P, x, y)$ is

$$i = R_{\exists_1} \wedge \ulcorner A\urcorner = \ulcorner\exists v_l B(v_l)\urcorner\wedge$$

$$(\exists z)(\exists\ulcorner t\urcorner)[z = \langle\langle\ulcorner\Lambda\urcorner,\ulcorner\beta\urcorner,\ulcorner\gamma\urcorner, j,\ulcorner F\urcorner\rangle, y\rangle \wedge P(z)\wedge$$

$$\ulcorner B[t/v_l]\urcorner \in \ulcorner\Lambda\urcorner \wedge \ulcorner\Lambda\urcorner\backslash\ulcorner\{B[t/v_l]\}\urcorner = \ulcorner\Gamma\urcorner\wedge$$

$$\ulcorner\beta\urcorner \lhd \ulcorner\alpha\urcorner \wedge \ulcorner\rho\urcorner = \ulcorner\gamma\urcorner].$$

$\mathcal{D}_{\forall_2}(P, x, y)$ is

$$i = R_{\forall_2} \wedge \ulcorner A\urcorner = \ulcorner\forall X B(X)\urcorner\wedge$$

$$(\exists z)(\exists\ulcorner U\urcorner)[z = \langle\langle\ulcorner\Lambda\urcorner,\ulcorner\beta\urcorner,\ulcorner\gamma\urcorner, j,\ulcorner F\urcorner\rangle, y\rangle \wedge P(z)\wedge$$

$$\ulcorner B[U/X]\urcorner \in \ulcorner\Lambda\urcorner \wedge \ulcorner\Lambda\urcorner\backslash\ulcorner\{B[U/X]\}\urcorner = \ulcorner\Gamma\urcorner\wedge$$

$$\ulcorner\beta\urcorner \lhd \ulcorner\alpha\urcorner \wedge \ulcorner\rho\urcorner = \ulcorner\gamma\urcorner].$$

$\mathcal{D}_{\exists_2}(P, x, y)$ is

$$i = R_{\exists_2} \wedge \ulcorner A\urcorner = \ulcorner\exists X B(X)\urcorner\wedge$$

$$(\exists z)(\exists\ulcorner U\urcorner)[z = \langle\langle\ulcorner\Lambda\urcorner,\ulcorner\beta\urcorner,\ulcorner\gamma\urcorner, j,\ulcorner F\urcorner\rangle, y\rangle \wedge P(z)\wedge$$

$$\ulcorner B[U/X]\urcorner \in \ulcorner\Lambda\urcorner \wedge \ulcorner\Lambda\urcorner \backslash \ulcorner\{B[U/X]\}\urcorner = \ulcorner\Gamma\urcorner \wedge$$

$$\ulcorner\beta\urcorner \lhd \ulcorner\alpha\urcorner \wedge \ulcorner\rho\urcorner = \ulcorner\gamma\urcorner].$$

$\mathcal{D}_{cut}(P,x,y)$ is

$$i = R_{cut} \wedge (\exists z_1, z_2)[z_1 = \langle\langle\ulcorner\Lambda_1\urcorner, \ulcorner\beta_1\urcorner, \ulcorner\gamma_1\urcorner, j, \ulcorner F_1\urcorner\rangle, y\rangle \wedge$$

$$z_2 = \langle\langle\ulcorner\Lambda_2\urcorner, \ulcorner\beta_2\urcorner, \ulcorner\gamma_2\urcorner, j, \ulcorner F_2\urcorner\rangle, y\rangle \wedge P(z_1) \wedge P(z_2) \wedge$$

$$\ulcorner A\urcorner \in \ulcorner\Lambda_1\urcorner \wedge \ulcorner\neg A\urcorner \in \ulcorner\Lambda_2\urcorner \wedge \ulcorner(\Lambda_1 \cup \Lambda_2)\backslash\{A, \neg A\}\urcorner = \ulcorner\Gamma\urcorner \wedge$$

$$\ulcorner\beta_1\urcorner \lhd \ulcorner\alpha\urcorner \wedge \ulcorner\beta_2\urcorner \lhd \ulcorner\alpha\urcorner \wedge \ulcorner\rho\urcorner = \ulcorner\gamma_1\urcorner \wedge \ulcorner\rho\urcorner = \ulcorner\gamma_2\urcorner \wedge gr(\ulcorner A\urcorner) \prec \ulcorner\rho\urcorner].$$

$\mathcal{D}_{\omega}(P,x,y)$ is

$$i = R_{\omega} \wedge \ulcorner A\urcorner = \ulcorner\forall v_l B(v_l)\urcorner \wedge$$

$$(\forall w)(\exists z)[z = \langle\langle\ulcorner\Lambda\urcorner, \ulcorner\beta\urcorner, \ulcorner\gamma\urcorner, j, \ulcorner F\urcorner\rangle, y\rangle \wedge P(z) \wedge$$

$$\ulcorner B[\dot{w}/v_l]\urcorner \in \ulcorner\Lambda\urcorner \wedge \ulcorner\Lambda\urcorner \backslash \ulcorner\{B[\dot{w}/v_l]\}\urcorner = \ulcorner\Gamma\urcorner \wedge$$

$$\ulcorner\beta\urcorner \lhd \ulcorner\alpha\urcorner \wedge \ulcorner\rho\urcorner = \ulcorner\gamma\urcorner].$$

$\mathcal{D}_{\Omega}(Q,P,x,y)$ is

$$i = R_{\Omega} \wedge \ulcorner A\urcorner = \ulcorner\forall X B(X)\urcorner \wedge \Pi^1_1(\ulcorner A\urcorner) \wedge$$

$$(\exists e_f)[\Psi(e_f) \wedge \mathcal{D}_{\Omega,1} \wedge (\exists z)[\mathcal{D}_{\Omega,2} \wedge \mathcal{D}_{\Omega,3}]]].$$

Here $\Psi(e_f)$ is a formula expressing that e_f is the code of a recursive function and satisfies the definition of fundamental.[13] In the following we will formulate the subformulas $\mathcal{D}_{\Omega,1}$, $\mathcal{D}_{\Omega,2}$ and $\mathcal{D}_{\Omega,3}$. Note that only $\mathcal{D}_{\Omega,3}$ includes an occurrence of Q.

[13] The exact definition of fundamental does not concern us here, since it is a technicality of the proofs in [41]. But it can be found in [41].

$\mathcal{D}_{\Omega,1}$ is

$$(\exists w)[\{e_f\}(\ulcorner\Omega\urcorner) = w \wedge w \trianglelefteq \ulcorner\alpha\urcorner)].$$

$\mathcal{D}_{\Omega,2}(P, x, y)$ is

$$z = \langle\langle\ulcorner\Lambda\urcorner, \ulcorner\delta\urcorner, \ulcorner\gamma\urcorner, j, \ulcorner F\urcorner\rangle, y\rangle \wedge P(z) \wedge$$

$$\ulcorner\gamma\urcorner = \ulcorner\rho\urcorner \wedge \ulcorner\delta\urcorner = \{e_f\}(0) \wedge \ulcorner A\urcorner \in \ulcorner\Lambda\urcorner \wedge$$

$$\ulcorner\Gamma\urcorner = \ulcorner\Lambda\backslash\{A\}\urcorner.$$

$\mathcal{D}_{\Omega,3}(Q, P, x, y)$ is

$$(\forall v)(\exists w)[[v = \langle\langle\ulcorner\Delta\urcorner, \ulcorner\beta\urcorner, \ulcorner 0\urcorner, j, \ulcorner F\urcorner\rangle, y\rangle \wedge Q(v) \wedge Weak(\ulcorner\Delta\urcorner) \wedge$$

$$\ulcorner A\urcorner \in \ulcorner\Delta\urcorner \wedge \ulcorner\beta\urcorner \prec \ulcorner\Omega\urcorner] \rightarrow$$

$$[w = \langle\langle\ulcorner\Theta\urcorner, \ulcorner\mu\urcorner, \ulcorner 0\urcorner, j, \ulcorner G\urcorner\rangle, y\rangle \wedge P(w) \wedge$$

$$\ulcorner\Theta\urcorner = \ulcorner\Lambda \cup \Delta\backslash\{A\}\urcorner \wedge \ulcorner\mu\urcorner = \{e_f\}(\ulcorner\beta\urcorner)]].$$

As one can easily verify, $\mathcal{D}(P, Q, x, y)$ is a formula of $\mathcal{L}_{ID}(Q, P)$ which satisfies the definition of a strict operator form. Moreover the only part including an occurrence of Q is $\mathcal{D}_\Omega$. Consequently a system that includes the ω-rule as its only infinite rule can be formalised in $\widehat{ID}_1^i(strict)$.
Note that the Ω-rule cannot occur nested, since the rank of the deduction must be smaller than Ω in order to apply it and the rank of a deduction jumps above Ω after a single application. Therefore a single iteration of $\mathcal{D}(P, Q, x, y)$ models the notion of $T_\infty \vdash_\rho^\alpha \Gamma$, which can be done in $\widehat{ID}_2^i(strict)$.

4.4 Conclusion

Since derivability can be formalised in this straightforward way, the formalised proof follows the informal arguments in [41] almost literally. Hence, arguing in a similar way as in Chapter 3, one can establish

$$T \equiv \widehat{ID}_2^i(strict) + TI(\prec \alpha),$$

for a theory T, whose ordinal-analysis uses the Ω-rule, with $\alpha = \|T\|_{\Pi_2^0}$ in the ordinal notation system that was used in the analysis. Moreover by Theorem 3.4.1 we get all the benefits of the well-studied theories

$$PA + TI(\prec \alpha).$$

But, before putting these benefits into action in Chapter 5, we repeat the methodological advantages for any kind of ordinal analysis. The definition of $\mathcal{D}(P, Q, x, y)$ shows that there is no need for fancy formalised recursion theory and complex constructions. For instance, since $\mathcal{D}(P, Q, x, y)$ gives the previous step of a deduction through the the axiom

$$(\forall y < n)(\forall x)[P_y^{\mathcal{D}}(x) \leftrightarrow \mathcal{D}(P_y^{\mathcal{D}}, P_{<y}^{\mathcal{D}}, x, y)]$$

by an existential quantifier, there is no need for a function that enumerates the premisses of an infinite rule. Therefore no enumeration function has to be altered in order to operate on the derivability predicate.

Chapter 5

Proof-Complexity and the Deducibility of Σ^0_1-Sentences

The preceding chapters were mainly concerned with methodological considerations; in this chapter however we will connect the notions developed and explain how these can be put into action. We will introduce the notion of proof-complexity and connect it with the previously given notions of proof-theoretical reduction and ordinal analysis. This will lead to a straightforward method studying the proof-complexity of true Σ^0_1-sentences. The main idea is to connect the magnitude of a witness of a Σ^0_1-sentence with the length of its deduction.

5.1 Proof-Complexity

Kurt Gödel developed the notion of proof-complextiy in 1936 in his paper [25], where he introduced a notion of proof-complexity that counts the lines of a deduction in a Hilbert-calculus, to explain that the transition to a higher order system does not merely allow the deduction of new formulas, but also shortens the deductions of formulas that were already deducible in the lower-order system. However the notion of proof-complexity as deduction-length is from a metatheoretic as well as from a practical point of view inappropriate, because a short deduction might (in principle) still include monstrous formulas. Hence we prefer the following definition.

Definition 5.1.1 *Assume Σ is a countable set of symbols, which are called the primitive symbols, and Σ^* be the set of finite sequences of elements of Σ. A language $\mathcal{L}$ is an inductively defined subset of Σ^* whose elements are denoted by φ_i for $i \in \mathbb{N}$.*[1] *A set of rules $\mathcal{R}$ is a set of finite sequences of elements of $\mathcal{L}$. If $r \in \mathcal{R}$ has the form $r = \langle \varphi_1, ..., \varphi_n \rangle$, then we call $\varphi_1, ..., \varphi_{n-1}$ the premisses and φ_n the conclusion; an $r = \langle \varphi_1 \rangle$ is called an axiom. An $\mathcal{F} = \langle \Sigma, \mathcal{L}, \mathcal{R} \rangle$ is called a formal system. The set of $\mathcal{F}$-deductions $\mathcal{D}_\mathcal{F}$ comprises elements d of the form $\langle \varphi_1, ..., \varphi_n \rangle$ such that for any $1 \leq i \leq n$ there is an $r \in \mathcal{R}$ and $1 \leq i_1, ..., i_m < i$ with $r = \langle \varphi_{i_1}, ..., \varphi_{i_m}, \varphi_i \rangle$. Also we use $d \vdash \varphi$, if φ is the last entry of d.*

If lg is a length function for sequences and $d = \langle \varphi_1, ..., \varphi_n \rangle$ in $\mathcal{D}_\mathcal{F}$, we call $lg(d)$ the length of a deduction and

$$|d| := \sum_{i=1}^{lg(d)} lg(\varphi_i)$$

its complexity.[2] *We use $\mathcal{F} \vdash^n \varphi$, if $\mathcal{F} \vdash \varphi$ and*

$$\min\{|d| : d \in \mathcal{D}_\mathcal{F} \textit{ and } d \vdash \varphi\} \leq n.$$

It is a trivial consequence of this definition that, if $\max(d)$ denotes the length of the longest formula in d, then $|d| \leq lg(d) \cdot \max(d)$. We will silently use this fact throughout this and the next chapter; it is the reason why most of the bounds that are given in Chapter 6 are quadratic.

We can easily connect this notion of complexity with our notion of conservativity

[1] Note that a φ_i does not have to be a formula in the common sense, but can as well be a sequent of a Gentzen-calculus. Because the φ_i denote the objects for which a deduction system is defined.

[2] Hence the complexity of a deduction d is the total number of occurrences of primitive symbols in d.

(Definition 2.1.14) and therefore with proof-theoretical reduction (Definition 2.2.7) by the following notion of speed-up, which is a generalisation of those given in [10].

Definition 5.1.2 *Let $\langle \mathfrak{F}_\alpha \rangle_{\alpha\in\tau}$ be a hierarchy of sets of total recursive functions with length τ such that, if $\alpha < \beta$, then $\mathfrak{F}_\alpha \subsetneq \mathfrak{F}_\beta$.*[3]
Let $\mathcal{F}$ and $\mathcal{F}'$ be two formal systems that satisfy that there is a non-empty $\Gamma \subset \mathcal{L} \cap \mathcal{L}'$ such that for any $\varphi \in \Gamma$, if $\mathcal{F}' \vdash \varphi$, then $\mathcal{F} \vdash \varphi$.
$\mathcal{F}$ has a $\mathfrak{F}_\alpha$-speed-up over $\mathcal{F}'$ with respect to Γ, if there exists a sequence $\langle \varphi_i \rangle_{i\in\omega}$ of formulas from Γ such that, if $d_i^{\mathcal{F}'}$ and $d_i^{\mathcal{F}'}$ are the shortest proofs of φ_i in $\mathcal{F}$ and $\mathcal{F}'$ respectively, then:

1. *No $\mathfrak{F}_\beta$ with $\beta < \alpha$ includes a function f that satisfies $|d_i^{\mathcal{F}'}| < f(|d_i^{\mathcal{F}}|)$ for all $i \in \omega$.*
2. *There is a function $f \in \mathfrak{F}_\alpha$ such that for any $i \in \omega$ it holds that $|d_i^{\mathcal{F}'}| < f(|d_i^{\mathcal{F}}|)$.*

Definition 5.1.2 is only useful, when $\langle \mathfrak{F}_\alpha \rangle_{\alpha\in\tau}$ stratifies recursive functions in accordance with their growth rate. We give two examples of speed-ups in the following theorem.

Theorem 5.1.3 *1. $I\Sigma_1$ has a non-elementary speed-up over PRA.*

2. RCA has a polynomial speed-up over $I\Sigma_1$.[4]

Proof
These are the main results of [10]. □
A polynomial speed-up is usually seen as an insignificant speed-up. However we consider an elementary speed-up an insignificant one; why this is the case will become clear from the rest of this chapter.

[3]For a ω-long hierarchy see Definition B.4 for the Grzegorczyk Hierarchy. For a transfinite hierarchy one can use the Hardy Hierarchy, which is given by Definition 5.2.8, in a similar manner as the inducing functions of the Grzegorczyk Hierarchy.

[4]RCA is a second order system with comprehension for recursively definable sets and allows Σ_1-induction when second order parameters are present.

5.2 The Deducibility of Σ^0_1-Sentences

As stated in Theorem 2.1.10, any true Σ^0_1-sentence can be deduced in Q. However the complexity of the smallest possible deduction of some Σ^0_1-sentence can significantly differ between two theories extending Q. A good example is the Σ_1-sentence **Big**, which is a restriction of Kruskal's Tree Theorem (see [41]) and states the following.

> There exists an $n \leq 1$ such that, if $T_1, ..., T_n$ are finite trees whose vertices are labelled from $\{1, ..., 6\}$, where for all i, $|T_i| < i$, then there exist $i < j \leq n$ such that T_i is label preserving embeddable into T_j.

Note that the only quantifier that is not bounded is the one that states the existence of n. The rest of the sentence can be easily formulated by a Δ^0_0-formula in $\mathcal{L}_Q$. In order to motivate **Big** further, we would like to introduce the theories $\Pi^1_2 - BI$ and $\Pi^1_2 - BI_0$.

Definition 5.2.1 *1. Let $\langle A, \prec\rangle$ be such that A and $\prec$ are definable by Π^1_0-formulas. The (second order) formula $WF(\prec)$ is*

$$(\forall X)[(\forall x)[[A(x) \wedge (\forall y \prec x)[y \in X]] \rightarrow x \in X] \rightarrow (\forall x)[A(x) \rightarrow x \in X]].$$

2. The theory $\Gamma - BI_0$ is formulated in classical second order logic and contains the axioms of identity, the defining axioms of all primitive recursive functions, second order induction

$$(\forall X)[\bar{0} \in X \wedge (\forall x)[x \in X \rightarrow S(x) \in X] \rightarrow (\forall x)[x \in X]]$$

together with the following schema:

$$(\exists Y)(\forall x)[x \in Y \leftrightarrow \varphi(x)],$$

where $\varphi(x) \in \Pi^0_\infty$, and

$$WF(\prec) \rightarrow TI_\Gamma(\prec)$$

for any Π^1_0-definable relation $\prec$.

3. *The theory $\Gamma - BI$ is the theory $\Gamma - BI_0$ extended by the schema*

$$\varphi(0) \wedge (\forall x)[\varphi(x) \to \varphi(S(x))] \to (\forall x)\varphi(x)$$

for any $\varphi \in \mathcal{L}_{\Gamma-BI}$.

One can quickly deduce **Big** in $\Pi^1_2 - BI$. However any deduction of **Big** in $\Pi^1_2 - BI_0$ has to have a complexity of at least 2^1_{1000} symbols (see [52]).[5] How can such a statement be proved? The connections we drew in Chapter 3 between ordinal-analysis and proof-theoretical reduction can help here. Let $(\exists x)\varphi(x)$ be a Σ^0_1-sentence and assume that $\|T\|_{\Pi^0_2} = \alpha$. If $T \vdash^n (\exists x)\varphi(x)$, then $PA + TI(\prec \alpha) \vdash^{f(n)} (\exists x)\varphi(x)$. But the computational resources of $PA + TI(\prec \alpha)$ are well studied. Hence we know a hierarchy of functions $\langle H_\beta \rangle_{\beta \prec \alpha}$ such that, if $PA + TI(\prec \alpha) \vdash^{f(n)} (\exists x)\varphi(x)$, then there is a $\beta_{f(n)} \prec \alpha$ and an $m < H_{\beta_{f(n)}}(0)$ such that $\mathbb{N} \models \varphi(m)$. Hence, arguing indirectly, when the witnesses of a Σ^0_1-sentence are very big, then its deduction must have a certain length. Since the ordinal notation systems being used in ordinal-analysis differ in the way how function hierarchies can be defined, i.e. the assignment of fundamental sequences to limit ordinals differs,[6] one has to generalise the proofs of the step from $PA + TI(\prec \alpha)$ to a $\langle H_\beta \rangle_{\beta \prec \alpha}$ uniformly in order to develop a general method of studying the proof-complexity of Σ^0_1-sentences. This is done by Weiermann in [60]; we will present Weiermann's methods in the next section.

5.2.1 A general characterisation of the provable recursive functions of $PA + TI(\prec \alpha)$

We follow Weiermann's approach of dealing with ordinal-notation systems that is given in [60]. Instead of stating certain conditions that we require from an ordinal notation system

[5] Here $2^x_0 := x$ and $2^x_{y+1} := 2^{2^x_y}$.

[6] A fundamental sequence of a limit ordinal λ is a sequence $\langle \lambda_n \rangle_{n \in \omega}$ such that $\lambda = \lim_{n \to \infty} \lambda_n$.

$ON(\tau)$, e.g. Friedman and Sheard developed the notion of an *elementary recursive ordinal notation system* (ERONS) in [22], we define a norm N on an actual ordinal τ and require certain properties that a map

$$o : \mathbb{N} \to \tau$$

has to fulfil with respect to N. Also this saves us from defining ordinal arithmetic. In addition, as is common in the literature, we trust the assurances of Buchholz, Friedman, Rathjen and Weiermann that any ordinal notation system $ON(\tau)$ that was ever given in the literature is elementarily recursively definable and induces a function o satisfying these conditions.

Definition 5.2.2 *We say that τ has a norm $N : \tau \to \mathbb{N}$, if the following conditions are satisfied:*

- *τ is an ε-number.*[7]
- *N satisfies:*
  1. $N(0) = 0$
  2. $N(\alpha \sharp \beta) = N(\alpha) + N(\beta)$
  3. $N(\bar{\omega}^\alpha) = N(\alpha) + 1$, *where*

$$\bar{\omega}^\alpha := \begin{cases} \omega^{\alpha+1} & : \quad (\exists \alpha_0 < \alpha)(\exists n < \omega)[\omega^{\alpha_0} = \alpha_0 \land \alpha = \alpha_0 + n] \\ \omega^\alpha & : \quad \textit{otherwise} \end{cases}$$

  4. $\{\alpha < \tau : N(\alpha) < k\}$ *is finite for any* $k \in \mathbb{N}$.

- *For a variant of the Ackermann function*[8] $\Phi : \mathbb{N} \to \mathbb{N}$

[7]Hence for any $\alpha, \beta \in \tau$, $\alpha + \beta, \alpha\beta, \omega^\alpha \in \tau$.

[8]We use the Ackermann function just because we want to ensure that we are able to evaluate any primitive recursive function. In cases where the language includes fewer function symbols one might use a function with lower growth-rate.

1. *for any n, $2 \leq \Phi(n)$ and $\Phi(n) + \Phi(n) + 2 \leq \Phi(n+1)$*
2. *for any (n-ary) primitive recursive function f there is a $p \in \mathbb{N}$ such that*

$$f(k_1, ..., k_n) < \Phi(p + \max\{k_1, ..., k_n\})$$

for all $k_1, ...k_n \in \mathbb{N}$.

- *There is a binary primitive recursive relation $\prec$ on a primitive recursively definable $A \subset \mathbb{N}$ and a function $o : A \to \tau$ such that*[9]

1. *for any $n, m \in \mathbb{N}$, if $n \prec m$, then $o(n) < o(m)$.*
2. *for any $n \in \mathbb{N}$, $N(o(n)) \leq \Phi(n)$ and $n \leq \Phi(N(o(n)))$.*

We can use N to define a new ordering on a $\langle \tau, N \rangle$.

Definition 5.2.3 *Assume that $\langle \tau, N \rangle$ is an ordinal with a norm.*

$$\beta <_n^1 \alpha :\Leftrightarrow \beta < \alpha \text{ and } N(\beta) \leq \Phi(N(\alpha) + n)$$

$<_n$ is the transitive closure of $<_n^1$.

Note that $\beta <_n^1 \alpha$ and $n \leq m$ imply $\beta <_m^1 \alpha$.

Definition 5.2.4 *Assume that $\langle \tau, N \rangle$ is an ordinal with a norm and $\alpha, \beta \in \tau$.*

$$\Psi(\alpha) := \begin{cases} 0 & : \alpha = 0 \\ \max\{\Psi(\beta) + 1 : \beta <_0^1 \alpha\} & : \alpha > 0 \end{cases}$$

$$H_\alpha(x) := \max(\{x\} \cup \{H_\beta(x+1) : \beta < \alpha \wedge N(\beta) \leq 3^{N(\alpha)+x+1}\})$$

Note that, if we take $\Phi(x) = H_{\omega^\omega}(x)$, then the conditions that are demanded by Definition 5.2.2 are satisfied.

[9] $\langle A, \prec \rangle$ plays the role of an ordinal notation system.

Lemma 5.2.5 *Let $\langle \tau, N \rangle$ be an ordinal with a norm and $\Phi(x) = H_{\omega^\omega}(x)$. Then:*

1. *for any $m \in \mathbb{N}$ $\Psi(m) = m$*
2. *$\beta <_0 \alpha$ implies $\Psi(\beta) < \Psi(\alpha)$*
3. *$\Psi(\alpha) := \max\{k : \exists(\alpha_1, ..., \alpha_k)[\alpha_k <^1_0 ... <^1_0 \alpha_0 = \alpha]\}$*
4. *for any $n \in \mathbb{N}$, $\Psi(\alpha + n + 1) < H_{\omega^{\omega(\alpha+2)}}(n)$.*

Proof

See [60, pp. 54-55].□

Note that (3) roughly links the function Ψ to descent recursion from [22]. As section 5.2.2 will show, the following theorem is a general theorem for the several characterisations of the provable recursive functions of $PA + TI(\prec \alpha)$ that can be found in the literature for several particular $\prec$.

Theorem 5.2.6 *Assume that $\langle \tau, N \rangle$ is an ordinal with a norm. Let $\prec$ be the ordering that is demanded by Definition 5.2.2 and $TI(\prec \alpha)$ the schema that is defined as in the discussion above Definition 3.1.2, if we take for α the element n of $\prec$ such that $o(n) = \alpha = \omega^\omega \alpha_0 \leq \tau$, for $\alpha_0 > 0$. Also let $\varphi(x, y)$ be a Δ^0_0-formula of $\mathcal{L}_{PA}$ where only x and y occur freely.*

If $PA + TI(\prec \alpha) \vdash (\forall x)(\exists y)\varphi(x, y)$, then

$$(\exists \beta < \alpha)(\forall n \in \mathbb{N})(\exists k < \Psi(\beta + n + 1))[\mathbb{N} \models \varphi(\bar{n}, \bar{k})].$$

Proof

See [60, p. 59].□

Corollary 5.2.7 *Let $\varphi(y)$ be a Δ^0_0-formula of $\mathcal{L}_{PA}$ where only y occurs freely. If $PA + TI(\prec \alpha) \vdash (\exists y)\varphi(y)$, then*

$$(\exists \beta < \alpha)(\exists k < \Psi(\beta + 1))[\mathbb{N} \models \varphi(\bar{k})].$$

The proof in [60] is constructive, hence such a β can be explicitly given. Also the proof of Theorem 5.2.6 shows that such a β can be found by using the complexity of the deduction that deduces $(\forall x)(\exists y)\varphi(x, y)$ in T, because the complexity of the deduction in $PA + TI(\prec \alpha)$ depends on the complexity of the deduction in T; and the complexity of the deduction in $PA + TI(\prec \alpha)$ gives a good upper bound for the rank of a deduction in PA_ω, which is the infinite system from Chapter 3. Using cut-elimination on the latter, we get an explicit β for $(\forall x)(\exists y)\varphi(x, y)$.

5.2.2 The classical approach

For a reader who is wondering how Ψ and H_α are related to the classical approach using fundamental sequences, we discuss the common definition of the Hardy Hierarchy. Also we prove that Ψ and H_α are provable total in PA in the case of $\langle \varepsilon_0, N \rangle$ for an appropriate N. We start with the canonical definition of the Hardy Hierarchy.

Definition 5.2.8 *Let τ be an ordinal such that $(\exists \tau_0 > 0)[\tau = \omega^\omega \tau_0]$ and $\cdot[\cdot] : \tau \times \mathbb{N} \to \tau$. A pair $\langle \tau, \cdot[\cdot] \rangle$ is called a Bachmann system if*

1. $0[n] = 0$
   $(\alpha + 1)[n] = \alpha$
   $\alpha \in Lim \Rightarrow \alpha[n] < \alpha[n + 1] < \alpha$
   Such a $\cdot[\cdot]$ is called an assignment of a fundamental sequence to an ordinal.

2. *for any α, β, n,*

$$\alpha[n] < \beta < \alpha \Rightarrow \alpha[n] \leq \beta[0].$$

The Hardy Hierarchy $\langle H'_\alpha \rangle_{\alpha<\tau}$ of $\langle \tau, \cdot[\cdot] \rangle$ is defined as $H'_0(n) := n$ and $H'_\alpha(n) := H'_{\alpha[n]}(n+1)$.

The next theorem shows the promised connection with Definition 5.2.4.

Theorem 5.2.9 *Assume that $\langle \tau, N \rangle$ is a normed ordinal and that $p : \tau \to \mathbb{N}$ is a function such that for any α, $N(\alpha) \leq p(\alpha) + 1 \leq p(\alpha + 1)$. If we define $\cdot[\cdot]_p$ by $0[n]_p := 0$ and $\alpha[n]_p := \max\{\beta < \alpha : N(\beta) \leq p(\alpha + n)\}$, for $\alpha > 0$, and take $\langle H'_\alpha \rangle_{\alpha<\tau}$ as its Hardy Hierarchy, then*

1. *$\langle \tau, \cdot[\cdot]_p \rangle$ is a Bachmann system.*
2. *$H'_\alpha = \max\{H'_\beta(n+1) : \beta < \alpha \wedge N(\beta) \leq p(\alpha + n)\}$, for $\alpha > 0$.*

Proof

See [7, p. 9]. □

Corollary 5.2.10 *Assume that $\langle \tau, N \rangle$ is a normed ordinal and that $\langle \tau, \cdot[\cdot]_{3^{N(\cdot)+\cdot+1}} \rangle$ is defined as in Theorem 5.2.9. Then for any $n \in \mathbb{N}$*

$$H_\alpha(n) = H'_\alpha(n).$$

It is also proved in [7] that the $\langle H_\alpha \rangle_{\alpha<\varepsilon_0}$ that is induced by $\langle \varepsilon_0, N_1 \rangle$, where N_1 is defined as $N_1(0) := 0$ and $N_1(\omega^\alpha + \beta) := \max\{N_1(\alpha), N_1(\beta)\} + 1$, is an elementary variant of the most common textbook example of a Hardy Hierarchy, which is defined from $\langle \varepsilon_0, \cdot[\cdot]_{\text{textbook}} \rangle$ (see [50, p. 157-158] for a definition). This means that, when H_α is induced by $\langle \varepsilon_0, N_1 \rangle$ and H'_α is the hierarchy of $\langle \varepsilon_0, \cdot[\cdot]_{\text{textbook}} \rangle$, then

$$\mathcal{ERF}(H_\alpha) = \mathcal{ERF}(H'_\alpha),$$

where $\mathcal{ERF}(f)$ is the smallest set of functions that comprises all elementary functions, the function f and is closed under composition.[10] Such connections can be generally deduced, because all these ways of defining a Hardy Hierarchy have strong relations to naturally definable descent functions by properties like the one that is stated in Lemma 5.2.5 (3) (see [7, pp. 6-9]).

5.2.3 A bound for the witnesses of Σ^0_1-sentences

As we emphasised at the beginning of this section, there is a connection between the complexity of a deduction of a Σ^0_1-sentence and the size of its witnesses. We will make this precise by defining the following functions.[11] It should be easy to see how to adapt Definition 5.1.1 to the notion of a theory that was adopted in Section 2.1. Note that Theorem 2.1.2 shows that any theory that has a Hilbert system attached has at most an elementary speed-up over its version that has a Gentzen system attached and vice versa. Hence the following definition is invariant in the used deduction-system modulo elementary recursive speed-ups. Also we assume that for some $\alpha_0 > 0$, $\alpha = \omega^\omega \alpha_0$.

Definition 5.2.11 *1. For any $T \supset Q$ we define*

$$\chi_T(n) := \min\{k : \varphi \in \Delta^0_0 \wedge T \vdash^n (\exists y)\varphi(y) \Rightarrow (\exists m < k)[\mathbb{N} \models \varphi(\bar{m})]\}.$$

2.

$$\chi_\alpha(n) := \chi_{PA+TI(\prec\alpha)}(n)$$

We immediately get the following connections with the notion of a speed-up.

[10] Note that here the closure under composition implies the closure under limited recursion, since the bounded-μ-operator is an elementary function. The invariance of these hierarchies shown, when using the elementary functions as a basis, is the reason for considering an elementary speed-up as an insignificant one.

[11] These functions were defined by Michael Rathjen in a talk that was given at the Bertinoro International Center for Informatics in 2011.

Lemma 5.2.12 *1.* $\chi_\alpha(n) < \Psi(\alpha)$

2. If $T' \subset_{\Sigma_1^0} T$ and T has a $\mathfrak{F}_\alpha$-speed-up over T', then there is an $f \in \mathfrak{F}_{\alpha+1}$ such that

$$\chi_T(n) \leq \chi_{T'}(f(n)).$$

3. Assume $T \supset Q$ such that $\|T\|_{\Pi_2^0} = \alpha$ and T has at most an elementary speed-up over $PA + TI(\prec \alpha)$. Then there is an m_0 such that for any $n \in \mathbb{N}$

$$\chi_T(n) \leq \chi_\alpha(2^n_{m_0}).$$

Proof

1. Assuming that $PA + TI(\prec \alpha) \vdash^n (\exists y)\varphi(y)$, where $\varphi \in \Delta_0^0$, we can conclude

   $$(\exists \beta_n < \alpha)(\exists m < \Psi(\beta_n + 1))[\mathbb{N} \models \varphi(\bar{m})]$$

   by Corollary 5.2.7. Therefore

   $$\Psi(\beta_n + 1)) \in \{k : \varphi \in \Delta_0^0 \wedge T \vdash^n (\exists y)\varphi(y) \Rightarrow (\exists m < k)[\mathbb{N} \models \varphi(\bar{m})]\}.$$

   Hence, by the minimality of $\chi_\alpha(n)$, we get $\chi_\alpha(n) \leq \Psi(\beta_n + 1))$.
   Since α and β_n came from the cut-elimination that is given in [60], $\beta_n + 1 <_0 \alpha$ and, therefore, $\Psi(\beta_n + 1) < \Psi(\alpha)$.[12]

2. This is a trivial consequence of Definition 5.2.11.

3. This follows from (2) and the fact that for any elementary function f there is an m_0 such that $f(n) < 2^n_{m_0}$ for any $n \in \mathbb{N}$.

[12]Note that the infinite system that is defined in [60, p. 55] proceeds over ordinals which are governed by the relation $<_0$.

□

However here we face the problems that were raised in Chapter 3 and Chapter 4. As we said at the beginning of the present chapter, we are interested in the complexity of deductions in $\Pi^1_2 - BI$ and $\Pi^1_2 - BI_0$, especially for the particular sentence **Big**. But the ordinal-analysis of these systems is carried out by using the Ω-rule in [41]; in fact the infinite system that is used in [41] is as given in Definition 4.1.3 and formalised throughout Chapter 4. The result of this ordinal-analysis is

$$\|\Pi^1_2 - BI_0\|_{\Pi^0_2} = \theta\Omega^\omega 0,$$

which is also called the Ackermann ordinal or small Veblen ordinal. As was extensively explained in Chapter 3, for any primitive recursive $\prec$ on $\mathbb{N}$ we have

$$PA + TI(\prec) \equiv_{\Pi^0_2} \widehat{ID}^i_2(strict) + TI(\prec).$$

Hence Ψ and χ_α, when α is the ordinal-type of $\prec$, are still meaningful here. However, since the direction is as follows

$$\Pi^1_2 - BI_0 \subseteq_{\Sigma^0_1} \widehat{ID}^i_2(strict) + TI(\prec) \subseteq_{\Sigma^0_1} PA + TI(\prec),$$

the question whether $\widehat{ID}^i_2(strict)+TI(\prec)$ has an elementary speed-up over $PA+TI(\prec)$ for Π^0_2-sentences is of a special interest for the study of the proof-complexity of **Big**. Moreover, since we are aiming a general method of studying the proof-complexity of Σ^0_1-sentences, this question is of general interest, as is explained through Lemma 5.2.12 (2) and (3). We will give a positive answer in the next chapter.

Chapter 6

$\widehat{ID}^{i}_{<\omega}(strict)$ has at most an elementary speed-up over PA

In Chapter 2 I described the general framework of theory reduction, and compared it to associated contexts. Subsequently I used the notions of Chapter 2 to discuss ordinal-analysis in Chapter 3. There I explained why a "better" base theory for ordinal-analysis is needed and presented the arguments available in the literature supporting the theory $\widehat{ID}^{i}_{<\omega}(strict)$. Chapter 4 indicates the benefits of $\widehat{ID}^{i}_{<\omega}(strict)$ when formalising technically elaborate notions of provability, e.g. those which are constituted by the Ω-rule. In Chapter 5 I explained Rathjen's and Weiermann's idea concerning a general method to give a lower bound for the complexity of deductions of Σ^0_1-sentences. There we were left with the issue that, in order to make this general method work, an ordinal analysis has to be given and that in many cases this analysis uses the Ω-rule. Hence the transition from $\widehat{ID}^{i}_{<\omega}(strict) + TI(\prec \alpha)$ to $PA + TI(\prec \alpha)$ might increase the finite deduction of a Σ^0_1-sentence and, therefore, the method might pick a function that is higher up in the Hardy hierarchy than the ordinal α would have suggested. We will settle this issue by proving that $\widehat{ID}^{i}_{<\omega}(strict) + TI(\prec \alpha)$ has at most an elementary speed-up over $PA + TI(\prec \alpha)$. As in Chapter 3 we start by proving that $\widehat{ID}^{i}_{<\omega}(strict)$ has at most an elementary speed-up over PA and then show that this result can be easily extended to the cases where axioms for transfinite induction are present. We show this by bounding the length-increment of deductions, which is caused by the deduction-transformations that are used in Chapter 3.

6.1 Bounding Buchholz

We start with Buchholz, who heavily used Theorem 2.2.19 *(Soundness of Realisability)* in [6], which he took from [59]. The proof of Theorem 2.2.19 that is given in Chapter 2 is a literal copy from [59, pp. 190-192], but, since it relies on many facts from recursion theory that are not made explicit, it has many gaps which become crucial here. Moreover, since we are interested in upper bounds of the deduction-length-increment, the precise way of formalising recursion theory in HA plays a part; we want the facts of recursion theory to be accessible in full generality, because, if we are able to establish a formula $(\forall x)\varphi(x)$ in c many lines, then the deduction-length for an instance $\varphi(\bar{n})$ is bounded by $c + 1$ and, hence, independent from the particular instance. Therefore we are able of constructing deductions schematically in the instances. This gives a uniform way to construct deductions and it is, therefore, easy to extract a bound for their length. The canonical source for formalised recursion theory is Kleene's [28], which is the main reference of the following section.

6.1.1 Formalised Recursion Theory

To make this section more readable we occasionally use Kleene-brackets as we already did in Chapter 2. However, in proofs we try to avoid such abbreviations and give the actual formulas, because it might be easier for the reader to estimate the length of a formula and the shape of its deduction, when the full structure of the formula is revealed.

Theorem 6.1.1 *(S-m-n Theorem)*

For every m and n there is a primitive recursive function s^n_m such that

$$HA \vdash (\forall x_1, ..., x_{m+n})[\{z\}(x_1, ..., x_{m+n}) = \{s^n_m(z, x_1, ..., x_m)\}(x_{m+1}, ..., x_{m+n})]$$

Proof

See [28, p. 67]. □

Here we face our first difficulty. Since the *S-m-n Theorem* is in fact a set of theorems, i.e. one theorem for every pair $\langle n, m\rangle$, the length of its HA-deductions is a function in n and m. However the deductions are fairly regular (as it is described in [28, p. 67]) and their length can therefore be bounded by mc_n, where c_n is a constant depending on n. Since we are going to use the S-m-n Theorem only in cases where either n or m is fixed, its contribution to the length-increment of a deduction is only linear.
By using $\{e_1\}(x_1) \simeq \{e_2\}(x_2)$ as a meta-abbreviation for

$$(\exists v)\mathcal{T}(e_1, x_1, v) \to (\exists w)\mathcal{T}(e_2, x_2, w) \wedge$$

$$(\forall v)(\forall w)[\mathcal{T}(e_1, x_1, v) \wedge \mathcal{T}(e_2, x_2, w) \to \mathcal{U}(v) = \mathcal{U}(w)],$$

to imitate the notion of equality between two partial recursive functions, we can state the famous Normal Form Theorem.

Lemma 6.1.2 *(Formalised Normal Form Theorem)*

There is a $e_0 \in \mathbb{N}$ such that

$$HA \vdash (\forall x, y)[\{\bar{e_0}\}(x, y) \simeq \{x\}(y)]$$

Proof

See [28, p. 67] and note that $\{x\}(y)$ is an ordinary p-term in the sense of [28]. □

Since the S-m-n Theorem can be proved, it is possible to proof a version of the Recursion Theorem as well in HA.

Theorem 6.1.3 *(Formalised Recursion Theorem)*

$$HA \vdash (\forall x)(\exists y)(\forall z)[\{x\}(\langle y, z\rangle) \simeq \{y\}(z)]$$

Proof

See [28, p. 68] or [59, p. 27] for a less detailed proof. □

6.1.2 Deduction-Complexity Increment for Basic Moves in Logic

In [59] Troelstra used Gödel's Hilbert-style system for intuitionistic logic, which is described in Appendix C, to prove Theorem 2.2.19. We say that φ is deducible with complexity n in Gödel's system, in symbols $\vdash^n_G \varphi$, if there is a sequence of formulas containing at most n occurrences of symbols, which is build up in accordance with the axioms and rules of Gödel's system, and the formula occurs *somewhere* in the sequence (see Definition 5.1.1). However we will also talk about the length of a deduction as its number of lines, because the length-measurement is in many cases the stronger requirement for complexity. To extract deduction-complexity bounds from the proof of Theorem 2.2.19, we have to start by analysing the deduction-complexity increment that is caused by the usage of meta-theorems in a Hilbert-style system. The following Lemmata ensure that the deduction-complexity increment that is caused by the Deduction Theorem and $\forall$-quantification of free-variables is linearly bounded.

Lemma 6.1.4 *(Deduction Theorem)*

There is an $c_D \in \mathbb{N}$ *such that,*

$$T, \psi \vdash^n \varphi \Rightarrow T \vdash^{c_D n} \psi \to \varphi$$

for a ψ *not sharing any free variables with* φ.

Proof

We define $c_l := \max\{16, c_{dis} + 8, c_{ass} + 4\}$, where c_{dis} and c_{ass} are the length of the uniform deduction for distributivity and associativity respectively. We assume that there is a deduction $\langle \alpha_1, ..., \alpha_l \rangle$, where $\alpha_l = \varphi$, with $\sum_{i=1}^{l} lg(\varphi_i) \leq n$. The proof proceeds by induction on l. In every induction step we substitute into the original deduction for every α_i the formula $\psi \to \alpha_i$. Then we show that we can fill the gap between $\psi \to \alpha_{l-1}$ and $\psi \to \alpha_l$ by less than c_l formulas in order to get a deduction that proceeds from $\psi \to \alpha_{l-1}$ to $\psi \to \alpha_l$. Since the induction hypothesis ensures that this is possible for any index which is smaller than l, we establish a deduction of length $c_l l$. By taking $c_D := c_l c_F$, where c_F is the length of the longest schema that is used in the 9 cases below, we get $T \vdash^{c_D n} \psi \to \varphi$ (see the remark below Definition 5.1.1).

1. Assume that $l = 1$. Then φ is ψ. Hence $\psi \to \psi$ can be deduced as follows.

   1. $\vdash$ $\psi \to \psi \wedge \psi$ by (b).
   2. $\vdash$ $\psi \wedge \psi \to \psi$ by (c).
   3. $\vdash$ $\psi \to \psi$ by (h) from 1 and 3.

   Since $3 < c_l$, we are done.

2. Assume that $l = 1$ and that φ is an axiom of (a)-(f) or a formula in T. Then we can prove $\psi \to \varphi$ as follows.

   1. $\vdash$ $\varphi \wedge \psi \to \varphi$ by (c).
   2. $\vdash$ $\varphi \to (\psi \to \varphi)$ by (i) from 1.
   3. $\vdash$ φ by being an axiom or element of T.
   4. $\vdash$ $\psi \to \varphi$ by (g) from 2 and 3.

Since $4 < c_l$, we are done.

3. Assume that φ is derived by (g) from a γ and a $\gamma \to \varphi$. By induction hypothesis we therefore get a deduction of length $c_l(l-1)$ for $\psi \to \gamma$ and $\psi \to (\gamma \to \varphi)$. In the following we show how this deduction can be extended to a deduction $\psi \to \varphi$.

0.	$\vdash$	$\psi \to \gamma$	by I.H.
0'.	$\vdash$	$\psi \to (\gamma \to \varphi)$	by I.H.
1.	$\vdash$	$\psi \wedge \gamma \to \varphi$	by (j) from 0'.
2.	$\vdash$	$\gamma \wedge \psi \to \psi \wedge \gamma$	by (d).
3.	$\vdash$	$\gamma \wedge \psi \to \varphi$	by (h) from 1 and 2.
4.	$\vdash$	$\gamma \to (\psi \to \varphi)$	by (i) from 3.
5.	$\vdash$	$\psi \to (\psi \to \varphi)$	by (h) from 0 and 4.
6.	$\vdash$	$\psi \wedge \psi \to \varphi$	by (j) from 5.
7.	$\vdash$	$\psi \to \psi \wedge \psi$	by (b).
8.	$\vdash$	$\psi \to \varphi$	by (h) from 6 and 7.

Since $c_l(l-1) + 8 < c_l l$, we are done.

4. Assume that φ is the formula $\gamma \to \delta$ and is derived by (h) from $\gamma \to \chi$ and $\chi \to \delta$. By induction hypothesis we have a deduction of length $c_l(l-1)$ of $\psi \to (\gamma \to \chi)$ and $\psi \to (\chi \to \delta)$. This deduction can be enlarged to prove $\psi \to (\gamma \to \delta)$ in the following way.

0.	$\vdash$	$\psi \to (\gamma \to \chi)$	by I.H.
0'.	$\vdash$	$\psi \to (\chi \to \delta)$	by I.H.
1.	$\vdash$	$\psi \wedge \gamma \to \chi$	by (j) from 0.
2.	$\vdash$	$\psi \wedge \chi \to \delta$	by (j) from 0'.
3.	$\vdash$	$\chi \wedge \psi \to \psi \wedge \chi$	by (d).
4.	$\vdash$	$\chi \wedge \psi \to \delta$	by (h) from 2 and 3.
5.	$\vdash$	$\chi \to (\psi \to \delta)$	by (i) from 4.
6.	$\vdash$	$\psi \wedge \gamma \to (\psi \to \delta)$	by (h) from 0' and 5.
7.	$\vdash$	$(\psi \wedge \gamma) \wedge \psi \to \delta$	by (j) from 6.
8.	$\vdash$	$\psi \wedge (\psi \wedge \gamma) \to (\psi \wedge \gamma) \wedge \psi$	by (d).
9.	$\vdash$	$\psi \wedge (\psi \wedge \gamma) \to \delta$	by (h) from 7 and 8.
10.	$\vdash$	$\psi \to [(\psi \wedge \gamma) \to \delta]$	by (i) from 9.
11.	$\vdash$	$\psi \wedge \gamma \to \psi$	by (c).
12.	$\vdash$	$\psi \wedge \gamma \to (\psi \wedge \gamma \to \delta)$	by (h) from 10 and 11.
13.	$\vdash$	$(\psi \wedge \gamma) \wedge (\psi \wedge \gamma) \to \delta$	by (j) from 12.
14.	$\vdash$	$\psi \wedge \gamma \to (\psi \wedge \gamma) \wedge (\psi \wedge \gamma)$	by (b).
15.	$\vdash$	$\psi \wedge \gamma \to \delta$	by (h) 13 and 14.
16.	$\vdash$	$\psi \to (\gamma \to \delta)$	by (i) from 15.

Since $c_l(l-1) + 16 \leq c_l l$, we are done.

5. Assume that φ is of the form $\gamma \to (\delta \to \chi)$ and is deduced from $\gamma \wedge \delta \to \chi$ by (i). By induction hypothesis we have $\vdash^{c(l-1)} \psi \to (\gamma \wedge \delta \to \chi)$. This deduction can be enlarged to prove $\psi \to [\gamma \to (\delta \to \chi)]$ in the following way.

0.	$\vdash$	$\psi \to (\gamma \wedge \delta \to \chi)$	by I.H.
1.	$\vdash$	$\psi \wedge (\gamma \wedge \delta) \to \chi$	by (j) from 0.
$\vdots$	$\vdots$	$\vdots$	The uniform deduction of
$c_{ass} + 1.$	$\vdash$	$(\psi \wedge \gamma) \wedge \delta \to \psi \wedge (\gamma \wedge \delta)$	associativity in c_{ass} many lines.
$c_{ass} + 2.$	$\vdash$	$(\psi \wedge \gamma) \wedge \delta \to \chi$	by (h) from 1 and $c_{ass} + 1$.
$c_{ass} + 3.$	$\vdash$	$\psi \wedge \gamma \to (\delta \to \chi)$	by (i) from $c_{ass} + 2$.
$c_{ass} + 4.$	$\vdash$	$\psi \to [\gamma \to (\delta \to \chi)]$	by (i) from $c_{ass} + 3$.

Since $c_l(l-1) + c_{ass} + 4 \leq c_l l$, we are done.

6. Assume that φ is of the form $\gamma \wedge \delta \to \chi$ and deduced from $\gamma \to (\delta \to \chi)$ by (j). By induction hypothesis we have $\vdash^{c(l-1)} \psi \to [\gamma \to (\delta \to \chi)]$.This deduction can be enlarged to prove $\psi \to (\gamma \wedge \delta \to \chi)$ in the following way.

0.	$\vdash$	$\psi \to [\gamma \to (\delta \to \chi)]$	by I.H.
1.	$\vdash$	$\psi \wedge \gamma \to (\delta \to \chi)$	by (j) from 0.
2.	$\vdash$	$(\psi \wedge \gamma) \wedge \delta \to \chi$	by (j) from 1.
$\vdots$	$\vdots$	$\vdots$	The uniform deduction of
$c_{ass} + 2.$	$\vdash$	$\psi \wedge (\gamma \wedge \delta) \to (\psi \wedge \gamma) \wedge \delta$	associativity in c_{ass} lines.
$c_{ass} + 3.$	$\vdash$	$\psi \wedge (\gamma \wedge \delta) \to \chi$	by (h) from 2 and $c_{ass} + 2$.
$c_{ass} + 4.$	$\vdash$	$\psi \to (\gamma \wedge \delta \to \chi)$	by (i) from $c_{ass} + 3$.

Since $c_l(l-1) + c_{ass} + 4 \leq c_l l$, we are done.

7. Assume that φ is of the form $\chi \vee \gamma \to \chi \vee \delta$ and deduced from $\gamma \to \delta$ by (k). By induction hypothesis, we have $\vdash^{c(l-1)} \psi \to (\gamma \to \delta)$.This deduction can be enlarged to prove $\psi \to (\chi \vee \gamma \to \chi \vee \delta)$ in the following way.

0.	$\vdash$	$\psi \to (\gamma \to \delta)$	by I.H.
1.	$\vdash$	$\psi \wedge \gamma \to \delta$	by (j) from 0.
2.	$\vdash$	$\chi \vee (\psi \wedge \gamma) \to \chi \vee \delta$	by (k) from 1.
$\vdots$	$\vdots$	$\vdots$	The uniform deduction of
$c_{dis}+2.$	$\vdash$	$(\chi \vee \psi) \wedge (\chi \vee \gamma) \to \chi \vee (\psi \wedge \gamma)$	distributivity in c_{dis} lines.
$c_{dis}+3.$	$\vdash$	$(\chi \vee \psi) \wedge (\chi \vee \gamma) \to \chi \vee \delta$	by (h) from 2 and $c_{dis}+2$.
$c_{dis}+4.$	$\vdash$	$\chi \vee \psi \to (\chi \vee \gamma \to \chi \vee \delta)$	by (i) from $c_{dis}+3$.
$c_{dis}+5.$	$\vdash$	$\psi \to \psi \vee \chi$	by (c).
$c_{dis}+6.$	$\vdash$	$\psi \vee \chi \to \chi \vee \psi$	by (d).
$c_{dis}+7.$	$\vdash$	$\psi \to \chi \vee \psi$	by (h) from $c_{dis}+5$ and $c_{dis}+6$.
$c_{dis}+8.$	$\vdash$	$\psi \to (\chi \vee \gamma \to \chi \vee \delta)$	by (h) from $c_{dis}+4$ and $c_{dis}+7$.

Since $c_l(l-1)+c_{dis}+8 \leq c_l l$, we are done.

8. Assume that φ is $\gamma \to (\forall x)\delta(x)$ and was deduced from $\gamma \to \delta(x)$ by (l). By induction hypothesis, we have $\vdash^{c(l-1)}_{G} \psi \to (\gamma \to \delta(x))$. In the following we show how this deduction has to be extended to deduce $\psi \to [\gamma \to (\forall x)\delta(x)]$.

0.	$\vdash$	$\psi \to [\gamma \to \delta(x)]$	by I.H.
1.	$\vdash$	$\psi \wedge \gamma \to \delta(x)$	by (j) from 0.
2.	$\vdash$	$\psi \wedge \gamma \to (\forall x)\delta(x)$	by (l) from 1. Note that φ and ψ don't share free variables.
3.	$\vdash$	$\psi \to [\gamma \to (\forall x)\delta(x)]$	by (i) from 2.

Since $c_l(l-1)+3 \leq c_l l$, we are done.

9. Assume that φ is $(\exists x)\gamma(x) \to \delta$ and deduced from $\gamma(x) \to \delta$ by (m). By induction hypothesis we have $\vdash^{c(l-1)}_{G} \psi \to (\gamma(x) \to \delta)$. In the following we show how this deduction has to be extended to deduce $\psi \to [(\exists x)\gamma(x) \to \delta]$.

0.	$\vdash$	$\psi \to [\gamma(x) \to \delta]$	by I.H.
1.	$\vdash$	$\psi \wedge \gamma(x) \to \delta$	by (j) from 0.
2.	$\vdash$	$\gamma(x) \wedge \psi \to \psi \wedge \gamma(x)$	by (b).
3.	$\vdash$	$\gamma(x) \wedge \psi \to \delta$	by (h) from 1 and 2.
4.	$\vdash$	$\gamma(x) \to (\psi \to \delta)$	by (i) from 3.
5.	$\vdash$	$(\exists x)\gamma(x) \to (\psi \to \delta)$	by (m) from 4. Note that φ and ψ don't share free variables.
6.	$\vdash$	$(\exists x)\gamma(x) \wedge \psi \to \delta$	by (j) from 5.
7.	$\vdash$	$\psi \wedge (\exists x)\gamma(x) \to (\exists x)\gamma(x) \wedge \psi$	by (d).
8.	$\vdash$	$\psi \wedge (\exists x)\gamma(x) \to \delta$	by (h) from 6 and 7.
9.	$\vdash$	$\psi \to [(\exists x)\gamma(x) \to \delta]$	by (i) from 8.

Since $c_l(l-1) + 9 \leq c_l l$, we are done.

□

Lemma 6.1.5 *(Generalisation)*

$$T \vdash^n_G \varphi(x) \Rightarrow T \vdash^{7n}_G (\forall x)\varphi(x).$$

Proof

Assume that $T \vdash^n_G \varphi(x)$, then by (c) and (i) we get $T \vdash^{n+2n}_G \varphi(x) \to [(\bot \to \bot) \to \varphi(x)]$. An application of (g) gives $T \vdash^{n+3n}_G (\bot \to \bot) \to \varphi(x)$. By (l) we get $T \vdash^{n+4n}_G (\bot \to \bot) \to (\forall x)\varphi(x)$, which leads to $T \vdash^{n+6n}_G (\forall x)\varphi(x)$ by (a) and (g). □

6.1.3 Realisability in HA

The next step is to extract a deduction-complexity-increment bound for realisability from the proof of Theorem 2.2.19, which is given in Chapter 2. However it would be an unnecessarily hard task to extract such a bound from the actual formulation, because Theorem 2.2.19 states that there is an explicitly given natural number which realises a particular formula and its proof shows how this natural number can be constructed. But after a closer examination of Buchholz's proof that is given in Chapter 3 the reader will recognise that only the existence statement is needed in order to prove Theorem 3.3.3. We therefore reformulate Theorem 2.2.19 to the weaker existential statement and, hence, avoid actual calculations in HA. This spares us from giving a complexity-analysis of basic calculations and the coding-machinery for recursive function.

The proofs of the following lemmata use the fact that the formula-length increment that is caused by the realisability translation can be linearly bounded:

$$|x\mathbf{r}\varphi| \leq 30|\varphi|,$$

which can be easily verified by Definition 2.2.18.[1]

Lemma 6.1.6 *Assume that* $\varphi(\vec{x}) \in \mathcal{L}^1_{HA}$ *and all free variables are shown. Then there is a* $c \in \mathbb{N}$ *such that*

$$HA \vdash^m_G \varphi(\vec{x}) \Rightarrow HA \vdash^{m^2 c}_G (\exists y)[y\boldsymbol{r}(\forall \vec{x})\varphi(\vec{x})].$$

Proof

The numeration of the cases follows Appendix C. In the case of an axiom, we assume at first that there are no free variables occurring in the formulas. However, after we

[1]The function $|\varphi|$ gives the total number of symbol occurrences at φ. Also note that Definition 2.2.18 expands a formula from the outside and adds only a constant number of symbols in every step. Hence for any symbol only a constant number of symbols is added; which leads to a linear bound.

have gone through all the axiom cases, we will show by a general argument how the presence of free variables affects the deduction-complexity increment. In the following we only count the lines of the new deduction. The reader however can easily check that no formula is longer than its respective formula in the original deduction times a constant c_{Form}. Consequently the constant c can be taken as the product of c_{Form} and the biggest constant appearing in the deduction-length increment-bound.

(a) The realiser of an axiom of the form $\bot \to \varphi$ is a code of the constant 0 function of the same variables as the axiom, which is $\ulcorner \lambda\vec{x}.0 \urcorner =: e_0$. Hence we have to find a deduction in HA of the formula $e_0\mathbf{r}[\bot \to \varphi]$.
We can deduce

$$\bot \to 0\mathbf{r}\varphi$$

in one line by (a). From this we can deduce in HA the formula

$$\bot \to (\exists v)[\mathcal{T}(e_0, u, v) \wedge U(v)\mathbf{r}\varphi].$$

Since $u\mathbf{r}\bot \equiv \bot$, we get

$$e_0\mathbf{r}[\bot \to \varphi]$$

by generalisation in, say, c_a many lines.

(b) • The realiser of an axiom with the form $\varphi \vee \varphi \to \varphi$ is a code of the projection function that projects to the second variable $\ulcorner \lambda x.p_2(x) \urcorner =: e_{p_2}$. We expand the realised formula according to Definition 2.2.18:

$$e_{p_2}\mathbf{r}[\varphi \vee \varphi \to \varphi]$$

$$(\forall u)[u\mathbf{r}\varphi \vee \varphi \to (\exists v)[\mathcal{T}(e_{p_2}, u, v) \wedge \mathcal{U}(v)\mathbf{r}\varphi]]$$

$$(\forall u)[[(p_1(u) = 0 \to p_2(u)\mathbf{r}\varphi) \wedge (p_1(u) \neq 0 \to p_2(u)\mathbf{r}\varphi)] \to$$

$$(\exists v)[\mathcal{T}(e_{p_2}, u, v) \wedge \mathcal{U}(v)\mathbf{r}\varphi]]$$

Since the computation of a single explicitly given elementary function can be primitive-recursively constructed, there is a primitive recursive function f, which is contained in $\mathcal{L}_{HA}$ as a function symbol, such that $HA \vdash p_2(u) = \mathcal{U}(f(u))$.[2] This gives us

$$HA \vdash (p_1(u) = 0 \rightarrow p_2(u)\mathbf{r}\varphi) \wedge (p_1(u) \neq 0 \rightarrow p_2(u)\mathbf{r}\varphi) \rightarrow$$

$$\mathcal{T}(e_{p_2}, u, f(u)) \wedge \mathcal{U}(f(u))\mathbf{r}\varphi,$$

where the deduction-length corresponds only to the particular coding that is used to formulate $\mathcal{T}$ and $\mathcal{U}$. Therefore the deduction.length is the same for every realised instance of the axiom schema (b). By logic alone we can quantify $f(u)$ by an existential quantifier, which gives us the realised version of the axiom. So let's say that the deduction is c_{b_1} lines long.

- The realiser of an axiom of the form $\varphi \rightarrow \varphi \wedge \varphi$ is a code of the pairing function that has only one input $\ulcorner \lambda x.\langle x, x\rangle \urcorner =: e_{\langle,\rangle}$. The realiser translates the formula as follow.

$$e_{\langle,\rangle}\mathbf{r}[\varphi \rightarrow \varphi \wedge \varphi]$$

$$(\forall u)[u\mathbf{r}\varphi \rightarrow (\exists v)[\mathcal{T}(e_{\langle,\rangle}, u, v) \wedge \mathcal{U}(v)\mathbf{r}[\varphi \wedge \varphi]]$$

$$(\forall u)[u\mathbf{r}\varphi \rightarrow (\exists v)[\mathcal{T}(e_{\langle,\rangle}, u, v) \wedge p_1(\mathcal{U}(v))\mathbf{r}\varphi \wedge p_2(\mathcal{U}(v))\mathbf{r}\varphi]]$$

As before, we can find a primitive recursive function f that outputs the computation of the pairing function, in accordance with $e_{\langle,\rangle}$, such that $HA \vdash \langle x, x\rangle = \mathcal{U}(f(x))$. However, since $u = p_i(\langle u, u\rangle) = p_i(\mathcal{U}(f(u)))$ this proof is even relatively simple, because it is essentially an application of (b) through $u\mathbf{r}\varphi \rightarrow u\mathbf{r}\varphi \wedge u\mathbf{r}\varphi$. So the proof-length is therefore uniform in $u\mathbf{r}\varphi$ and has, say, c_{b_2} lines.

We put those two constants together by taking the maximum $c_b := \max\{c_{b_1}, c_{b_2}\}$.

[2]By a *computation* of a recursive function f we mean a sequence or a tree that is inductively constructed by computing through f in accordance with one of its codes.

(c) • For the axioms $\varphi \rightarrow \varphi \vee \psi$ the obvious realiser is a code for the pair-0-with-x function $\ulcorner \lambda x.\langle 0, x\rangle \urcorner =: e_{\langle 0, \rangle}$. As before we expand according to the translation.

$$e_{\langle 0, \rangle}\mathbf{r}[\varphi \rightarrow \varphi \vee \psi]$$

$$(\forall u)[u\mathbf{r}\varphi \rightarrow (\exists v)[\mathcal{T}(e_{\langle 0, \rangle}, u, v) \wedge \mathcal{U}(v)\mathbf{r}[\varphi \vee \psi]]$$

$$(\forall u)[u\mathbf{r}\varphi \rightarrow (\exists v)[\mathcal{T}(e_{\langle 0, \rangle}, u, v) \wedge$$

$$(p_1(\mathcal{U}(v)) = 0 \rightarrow p_2(\mathcal{U}(v))\mathbf{r}\varphi) \vee (p_1(\mathcal{U}(v)) \neq 0 \rightarrow p_2(\mathcal{U}(v))\mathbf{r}\psi)]]$$

Again, since the representation of the function is explicitly given as a code, we can construct a primitive recursive function f, which is therefore present in the language as a function symbol, which gives us a computation for a given input, i.e. $HA \vdash \langle 0, x\rangle = \mathcal{U}(f(x))$. Since $u = p_2(\langle 0, u\rangle) = p_2(\mathcal{U}(f(u)))$ and $0 = p_1(\mathcal{U}(f(u)))$ are both deducible in HA, we can deduce the realised statement by taking v as $f(u)$. Note that this deduction works entirely schematically in $u\mathbf{r}\varphi$ and has therefore a constant number of lines, let's say c_{c_1} many, for any φ.

• In the case of an axiom $\varphi \wedge \psi \rightarrow \varphi$ the realiser is a code of the projection function to the first entry, i.e. $\ulcorner \lambda x.p_1(x) \urcorner =: e_{p_1}$. Moreover the realisation expands as follows.

$$e_{p_1}\mathbf{r}[\varphi \wedge \psi \rightarrow \varphi]$$

$$(\forall u)[u\mathbf{r}[\varphi \wedge \psi] \rightarrow (\exists v)[\mathcal{T}(e_{p_1}, u, v) \wedge \mathcal{U}(v)\mathbf{r}\varphi]]$$

$$(\forall u)[[p_1(u)\mathbf{r}\varphi \wedge p_2(u)\mathbf{r}\psi] \rightarrow (\exists v)[\mathcal{T}(e_{p_1}, u, v) \wedge \mathcal{U}(v)\mathbf{r}\varphi]]$$

As before, by constructing a primitive recursive function from e_{p_1} that gives the computation of e_{p_1} for an input, we get $HA \vdash p_1(x) = \mathcal{U}(f(x))$. So by instantiating v by $f(u)$ we get a proof which is uniform in $u\mathbf{r}\varphi$, let's say, in length c_{c_2}.

By taking the maximum of these two numbers $c_c := \max\{c_{c_1}, c_{c_2}\}$, we can bound the deductions for every use of axiom (c).

(d) • The realiser for an axiom of the form $\varphi \vee \psi \to \psi \vee \varphi$ is a code of the function that changes the first entry of a pair from 0 to something else and conversely, i.e. $\ulcorner \lambda x.\langle 1 \dot{-} p_1(x), p_2(x)\rangle \urcorner =: e_{\langle 1-,\rangle}$. This realiser expands the axiom as follows.

$$e_{\langle 1-,\rangle}\mathbf{r}[\varphi \vee \psi \to \psi \vee \varphi]$$

$$(\forall u)[u\mathbf{r}[\varphi \vee \psi] \to (\exists v)[\mathcal{T}(e_{\langle 1-,\rangle}, u, v) \wedge \mathcal{U}(v)\mathbf{r}[\psi \vee \varphi]]]$$

$$(\forall u)[(p_1(u) = 0 \to p_2(u)\mathbf{r}\varphi) \wedge (p_1(u) \neq 0 \to p_2(u)\mathbf{r}\psi) \to$$

$$(\exists v)[\mathcal{T}(e_{\langle 1-,\rangle}, u, v) \wedge$$

$$(p_1(\mathcal{U}(v)) = 0 \to p_2(\mathcal{U}(v))\psi) \wedge (p_1(\mathcal{U}(v)) \neq 0 \to p_2(\mathcal{U}(v))\varphi)]]$$

As before we can find a primitive recursive function f that gives the computation of $e_{\langle 1-,\rangle}$ for a given input x. Therefore we get $HA \vdash \langle 1 \dot{-} p_1(x), p_2(x)\rangle = \mathcal{U}(f(x))$. As before it is clear that this formula can be deduced by taking $f(u)$ as v in a way that is schematic in $p_2(u)\mathbf{r}\varphi$ and $p_2(u)\mathbf{r}\psi$. Therefore this deduction is constant in length for any substitution of this formula. Let's say that it has c_{d_1} many lines.

• A realiser for the axiom $\varphi \wedge \psi \to \psi \wedge \varphi$ is a code of the function that changes the order of the entries in a pair, i.e. $\ulcorner \lambda x.\langle p_2(x), p_1(x)\rangle \urcorner =: e_{swap}$. In according with the realisability translation, the formula expands as follows.

$$e_{swap}\mathbf{r}[\varphi \wedge \psi \to \psi \wedge \varphi]$$

$$(\forall u)[u\mathbf{r}[\varphi \wedge \psi] \to (\exists v)[\mathcal{T}(e_{swap}, u, v) \wedge \mathcal{U}(v)\mathbf{r}[\psi \wedge \varphi]]$$

$$(\forall u)[(p_1(u)\mathbf{r}\varphi \wedge p_2(u)\mathbf{r}\psi) \to$$

$$(\exists v)[\mathcal{T}(e_{swap}, u, v) \wedge p_1(\mathcal{U}(v))\mathbf{r}\psi \wedge p_2(\mathcal{U}(v))\mathbf{r}\varphi]$$

As before, by defining an appropriate function, the proof is schematic in $p_1(u)\mathbf{r}\varphi$ and $p_2(u)\mathbf{r}\psi$ and has a length of, say, c_{d_2} many lines.

To combine these two cases we take $c_d := \max\{c_{d_1}, c_{d_2}\}$.

(e) The realiser of an axiom which has the form $(\forall x)\varphi(x) \to \varphi(t)$ is a code, say $e_{\{.\}}$, for the function

$$y \mapsto \{y\}(t).$$

This gives for

$$\ulcorner \lambda y.\{y\}(t)\urcorner \mathbf{r}[(\forall x)\varphi(x) \to \varphi(t)]$$

the formula

$$(\forall u)[u\mathbf{r}(\forall x)\varphi(x) \to (\exists v)[\mathcal{T}(e_{\{.\}}, u, v) \wedge \mathcal{U}(v)\mathbf{r}\varphi(t)]].$$

By using Definition 2.2.18 again, we get

$$(\forall u)[(\forall x)(\exists z)[\mathcal{T}(u, x, z) \wedge \mathcal{U}(z)\mathbf{r}\varphi(x)] \to (\exists v)[\mathcal{T}(e_{\{.\}}, u, v) \wedge \mathcal{U}(v)\mathbf{r}\varphi(t)]].$$

To deduce this formula in HA, we assume that

$$(\forall x)(\exists z)[\mathcal{T}(u, x, z) \wedge \mathcal{U}(z)\mathbf{r}\varphi(x)].$$

By an application of (g) and (j) we get

$$(\exists z)[\mathcal{T}(u, t, z) \wedge \mathcal{U}(z)\mathbf{r}\varphi(t)].$$

By Theorem 6.1.2 we get the code e_0 for the function

$$\langle x, y\rangle \mapsto \{x\}(y)$$

in HA. Since substituting for y the term t into e_0 can be done primitive recursively, we can construct in HA the code $e_{\{.\}}$ from e_0 and $\ulcorner t \urcorner$ such that

$$(\exists v)[\mathcal{T}(e_{\{.\}}, u, v) \wedge \mathcal{U}(v)\mathbf{r}\varphi(t)]$$

is deducible. By using the deduction theorem and generalisation we get

$$e_0\mathbf{r}[(\forall x)\varphi(x) \rightarrow \varphi(t)]$$

by a deduction that is schematic in $x\mathbf{r}\varphi$. So for every instance of axiom (e) we get a deduction of

$$(\exists y)[y\mathbf{r}[(\forall x)\varphi(x) \rightarrow \varphi(t)]]$$

of constant length, say of c_e many lines.

(f) In the case of an axiom of the form $\varphi(t) \rightarrow (\exists x)\varphi(x)$ we choose the realiser to be the code of the pair-with-t function

$$y \mapsto \langle t, y \rangle.$$

By Definition 2.2.18 we get that

$$e_{\langle t,.\rangle}\mathbf{r}[\varphi(t) \rightarrow (\exists x)\varphi(x)]$$

translates into

$$(\forall u)[u\mathbf{r}\varphi(t) \rightarrow (\exists v)[\mathcal{T}(e_{\langle t,.\rangle}, u, v) \wedge \mathcal{U}(v)\mathbf{r}[(\exists x)\varphi(x)]]].$$

Using Definition 2.2.18 again, we get

$$(\forall u)[u\mathbf{r}\varphi(t) \rightarrow (\exists v)[\mathcal{T}(e_{\langle t,.\rangle}, u, v) \wedge p_1(\mathcal{U}(v))\mathbf{r}\varphi(p_2(\mathcal{U}(v)))]].$$

However it is clear that

$$HA \vdash (\forall x, y)[\mathcal{T}(e_{\langle t,.\rangle}, x, y) \rightarrow p_1(\mathcal{U}(y)) = t \wedge p_2(\mathcal{U}(y)) = x].$$

So assuming that $u\mathbf{r}\varphi(t)$, we can deduce in HA

$$(\exists v)[\mathcal{T}(e_{\langle t,.\rangle}, u, v) \wedge p_1(\mathcal{U}(v))\mathbf{r}\varphi(p_2(\mathcal{U}(v))]$$

schematically in $x\mathbf{r}\varphi(y)$. Consequently by using the deduction theorem and generalisation, we get

$$e_{\langle t,.\rangle}\mathbf{r}[\varphi(t) \rightarrow (\exists x)\varphi(x)]$$

in a constant number of lines, say c_f many.

We have seen so far that the length of a deduction that deduces a realised version of a free-variable free instance of any axiom can be bounded by a constant $c_{ax} := \max\{c_a, c_b, c_c, c_d, c_e, c_f\}$. In the case of an axiom $\varphi(\vec{x})$ with n free variables $x_1, ..., x_n$ we can deduce in the same way that there is an m such that $HA \vdash \bar{m}\mathbf{r}\varphi(\vec{x})$ and that the length of its deduction is bounded by c_{ax}. It is not possible to quantify the whole formula in order to introduce a $\forall$-quantifier at the right side of the realiser, since $(\forall x)[\bar{m}\mathbf{r}\varphi]$ is not the same as $\bar{m}\mathbf{r}(\forall x)\varphi$. Therefore we have to proceed by the following construction:
First we note that there is a primitive recursive function f such that

$$HA \vdash (\forall x, y, z)[\mathcal{T}(f(z), x, y) \wedge \mathcal{U}(y) = z], (*)$$

because f only gives the code for the constant-z-function with the additional input variable x. Also note that in this case HA deduces the totality of this function, since z is explicitly given. Since $(*)$ is provable by a fixed deduction, all its instances for z are deducible in the same number of lines, say c_{gen} many. Consequently we get in c_{gen} lines $(\forall x, y)[\mathcal{T}(f(\bar{m}), x, y) \wedge \mathcal{U}(y) = \bar{m}]$. This gives together with $\bar{n}\mathbf{r}\varphi(\vec{x})$ a deduction for

$$(\forall x_n)(\exists y_n)[T(f(\bar{m}), x_n, y_n) \wedge \mathcal{U}(y_n)\mathbf{r}\varphi(\vec{x})]$$

in, say, $c_{ax} + c_{gen} + c'$ many lines. This is

$$f(\bar{m})\mathbf{r}(\forall x_n)\varphi(\vec{x}).$$

If we repeat this process $n-1$ many times, then we get

$$f...f(\bar{m})\mathbf{r}(\forall\vec{x})\varphi(\vec{x})$$

in $c_{ax}+n\cdot(c_{gen}+c')$ many lines. Since the existence statement

$$(\exists y)[y\mathbf{r}(\forall\vec{x})\varphi(\vec{x})]$$

is only 2 more lines away, we can bound all axiom cases by $(n+1)\cdot c_{axiom}$ for

$$c_{axiom}:=\max\{(c_{ax}+2),(c_{gen}+c'+2)\}.$$

Now that we have dealt with all the cases of axioms, we continue with the rules that constitute the calculus.

(g) Assume that there is an application of Modus Ponens with the premisses $\varphi(\vec{x})$ and $\varphi(\vec{x})\rightarrow\psi(\vec{x})$. We also assume that only n mutually distinct free variables $x_1,...,x_n$ occur in these two formulas. By induction hypothesis we have deductions for

$$(\exists e_1)[e_1\mathbf{r}(\forall\vec{x})\varphi(\vec{x})]$$

$$(\exists e_2)[e_2\mathbf{r}(\forall\vec{x})[\varphi(\vec{x})\rightarrow\psi(\vec{x})]].$$

So we can assume that there are such e_1 and e_2 by adding the formulas

$$e_1\mathbf{r}(\forall\vec{x})\varphi(\vec{x})$$

$$e_2\mathbf{r}(\forall\vec{x})[\varphi(\vec{x})\rightarrow\psi(\vec{x})]$$

to the deduction.

According to Definition 2.2.18 the first formula can be expanded to

$$(\forall x_1)(\exists y_1)[\mathcal{T}(e_0,x_1,y_1)\wedge(\forall x_2)(\exists y_2)[\mathcal{T}(\mathcal{U}(y_1),x_2,y_2)\wedge$$

$$\mathcal{U}(y_2)\mathbf{r}(\forall x_3, ..., x_n)\varphi(\vec{x})]].$$

Using Kleene brackets, we can write this as

$$\{\{e_1\}(x_1)\}(x_2)\mathbf{r}(\forall x_3, ..., x_n)\varphi(\vec{x}).$$

By Theorem 6.1.2 we get

$$\{e_0\}(\{e_1\}(x_1), x_2)\mathbf{r}(\forall x_3, ..., x_n)\varphi(\vec{x}).$$

Since the composition of two codes can be done primitive recursively, there is a function symbol in the language such that

$$\{f(e_0, e_1)\}(x_1, x_2)\mathbf{r}(\forall x_3, ..., x_n)\varphi(\vec{x}).$$

Since f is a function symbol in $\mathcal{L}_{HA}$ and Theorem 6.1.2 is deducible in full generality in HA, this enlarges the deduction in a constant number of lines; let's say in c_{g_1} many. Repeating this process another $n-2$ times we get in $(n-1)c_{g_1}$ lines

$$\{f(e_0, ...f(e_0, f(e_0, e_1))...)\}(x_1, x_2, ..., x_n)\mathbf{r}\varphi(\vec{x}).$$

In the following we take t_1 to be $t_1 := f(e_0, ...f(e_0, f(e_0, e_1))...)$. Similarly proceeding for $e_2\mathbf{r}(\forall\vec{x})[\varphi(\vec{x}) \rightarrow \psi(\vec{x})]$, we get in $2(n-1)c_{g_1}$ many lines the formulas

$$(\forall\vec{x})(\exists y)[\mathcal{T}(t_1, \vec{x}, y) \wedge \mathcal{U}(y)\mathbf{r}\varphi(\vec{x})]$$

and

$$(\forall\vec{x})(\exists y)[\mathcal{T}(t_2, \vec{x}, y) \wedge \mathcal{U}(y)\mathbf{r}[\varphi(\vec{x}) \rightarrow \psi(\vec{x})]].$$

The latter is

$$(\forall\vec{x})(\exists y)[\mathcal{T}(t_2,\vec{x},y)\wedge(\forall u)[u\mathbf{r}\varphi(\vec{x})\rightarrow(\exists v)[\mathcal{T}(\mathcal{U}(y),u,v)\wedge\mathcal{U}(v)\mathbf{r}\psi(\vec{x})]]].$$

By a deduction which is uniformly constructed from $u\mathbf{r}\varphi(\vec{x})$ and $u\mathbf{r}\psi(\vec{x})$ these two formulas yield the formula

$$(\forall\vec{x})(\exists y,z)[\mathcal{T}(t_2,\vec{x},y)\wedge\mathcal{T}(t_1,\vec{x},z)\wedge(\exists v)[\mathcal{T}(\mathcal{U}(y),\mathcal{U}(z),v)\wedge\mathcal{U}(v)\mathbf{r}\psi(\vec{x})]].$$

Using Kleene brackets, we can denote this formula by

$$\{\{t_2\}(\vec{x})\}(\{t_1\}(\vec{x}))\mathbf{r}\psi(\vec{x}).$$

By Theorem 6.1.2 and the fact that concatenation can be done primitive recursively, we get

$$\{h(e_0,t_1,t_2)\}(\vec{x})\mathbf{r}\psi(\vec{x})$$

for some function symbol h. The last moves are schematic in $u\mathbf{r}\varphi(\vec{x})$ and $u\mathbf{r}\psi(\vec{x})$, hence the deduction is enlarged by a constant number of lines, say c_{g_2} many. Therefore we reach $2(n-1)c_{g_1}+c_{g_1}$ many lines. To quantify at the right side of the realiser we have to look at

$$(\forall\vec{x})(\exists v)[\mathcal{T}(h(e_0,t_1,t_2),\vec{x},v)\wedge\mathcal{U}(v)\mathbf{r}\psi(\vec{x})].$$

Using the S-m-n theorem, we get

$$S^1_{n-1}(h(e_0,t_1,t_2),x_1,...,x_{n-1})\mathbf{r}(\forall x_n)\psi(\vec{x}).$$

Since h,f and S^1_{n-1} are primitive recursive, we can find a primitive recursive function h' such that

$$(\forall x_1, ..., x_{n-1})(\exists v)[\mathcal{T}(h'(e_0, e_1, e_2, e_f, e_h, e_{S^1_{n-1}}), \langle x_1, ..., x_{n-1}\rangle, v)\wedge$$
$$\mathcal{U}(v)\mathbf{r}(\forall x_n)\psi(\vec{x})$$

can be deduced in c_{g_3} many lines. Repeating this procedure $n-1$ times, we get a term $t_3(e_1, e_2)$ such that

$$t_3(e_1, e_2)\mathbf{r}(\forall \vec{x})\mathbf{r}\psi(\vec{x})$$

is deducible in $2(n-1)c_{g_1} + c_{g_2} + n \cdot c_{g_3}$ many lines.

Hence, by adding two more lines, we get

$$(\exists y)[y\mathbf{r}(\forall \vec{x})\mathbf{r}\psi(\vec{x})].$$

In total we need less than $3nc_{\bar{g}}$ lines, where

$$c_{\bar{g}} := \max\{2, c_{g_1}, c_{g_2}, c_{g_3}\}.$$

Using the deduction theorem twice for the two assumptions respectively and some logic, we get a deduction of

$$(\exists y)[y\mathbf{r}(\forall \vec{x})\mathbf{r}\psi(\vec{x})]$$

with at most $3nc_{\bar{g}}c_D^2 + c_{g_4}$ many lines. Taking $c_g := \max\{3c_{\bar{g}}c_D^2, c_{g_4}\}$, we therefore get an upper bound for this case of

$$(n+1)c_g \text{ many lines.}$$

(h) By induction hypothesis we have a deduction for $(\exists e_1)[e_1\mathbf{r}(\forall \vec{x})[\varphi(\vec{x}) \rightarrow \chi(\vec{x})]]$ and $(\exists e_2)[e_2\mathbf{r}(\forall \vec{x})[\chi(\vec{x}) \rightarrow \psi(\vec{x})]]$. Moreover, in these two formulas, the variables $x_1, ..., x_n$ are the only freely occurring variables. We assume that

$$e_1\mathbf{r}(\forall \vec{x})[\varphi(\vec{x}) \rightarrow \chi(\vec{x})]$$

and

$$e_2\mathbf{r}(\forall \vec{x})[\chi(\vec{x}) \rightarrow \psi(\vec{x})].$$

By similar constructions as in (g) we get in $2 \cdot (n-1) \cdot c_{h_1}$ lines the formulas

$$(\exists v)[\mathcal{T}(t_1(e_1), \vec{x}, v) \wedge \mathcal{U}(v)\mathbf{r}[\varphi(\vec{x}) \rightarrow \chi(\vec{x})]]$$

and

$$(\exists w)[\mathcal{T}(t_2(e_2), \vec{x}, w) \wedge \mathcal{U}(w)\mathbf{r}[\chi(\vec{x}) \rightarrow \psi(\vec{x})]],$$

which are

$$(\exists v)[\mathcal{T}(t_1(e_1), \vec{x}, v) \wedge$$
$$(\forall u)[u\mathbf{r}\varphi(\vec{x}) \rightarrow (\exists v')[\mathcal{T}(\mathcal{U}(v), u, v') \wedge \mathcal{U}(v')\mathbf{r}\chi(\vec{x})]]]$$

and

$$(\exists w)[\mathcal{T}(t_2(e_2), \vec{x}, w) \wedge$$
$$(\forall u)[u\mathbf{r}\chi(\vec{x}) \rightarrow (\exists w')[\mathcal{T}(\mathcal{U}(w), u, w') \wedge \mathcal{U}(w')\mathbf{r}\psi(\vec{x})]]].$$

We enlarge this deduction by $y\mathbf{r}\varphi(\vec{x})$ and get after several lines

$$(\exists v)(\exists v')[\mathcal{T}(t_1(e_1), \vec{x}, v) \wedge \mathcal{T}(\mathcal{U}(v), y, v') \wedge \mathcal{U}(v')\mathbf{r}\chi(\vec{x})].$$

Using Theorem 6.1.2 and the fact that composing two codes is primitive recursive, we get

$$(\exists v)[\mathcal{T}(t_3(e_1), \langle \vec{x}, y \rangle, v) \wedge \mathcal{U}(v)\mathbf{r}\chi(\vec{x})].$$

Next we perform a similar construction with

$$(\exists w)[\mathcal{T}(t_2(e_2), \vec{x}, w) \wedge$$
$$(\forall u)[u\mathbf{r}\chi(\vec{x}) \rightarrow (\exists w')[\mathcal{T}(\mathcal{U}(w), u, w') \wedge \mathcal{U}(w')\mathbf{r}\psi(\vec{x})]]]$$

to get

$$(\exists w)[\mathcal{T}(t_4(e_1, e_2), \langle \vec{x}, y \rangle, w) \wedge \mathcal{U}(w)\mathbf{r}\psi(\vec{x})].$$

Using the S-m-n Theorem, we get

$$(\exists w)[\mathcal{T}(t_5(e_1, e_2, \vec{x}), y, w) \wedge \mathcal{U}(w)\mathbf{r}\psi(\vec{x})].$$

Since the recursion theory needed is available in HA in full generality and the constructions that have been given so far, are schematic in $u\mathbf{r}\varphi, u\mathbf{r}\chi$ and $u\mathbf{r}\psi$, we obtain the last result in a constant number of lines, say c_{h_2} many. Therefore we reach a length of $2(n-1)c_{h_1} + c_{h_2}$. Using the deduction theorem for $y\mathbf{r}\varphi(\vec{x})$, we get

$$y\mathbf{r}\varphi(\vec{x}) \rightarrow (\exists w)[\mathcal{T}(t_5(e_1, e_2, \vec{x}), y, w) \wedge \mathcal{U}(w)\mathbf{r}\psi(\vec{x})]$$

by $(n-1)2c_{h_1}c_D + c_{h_2}c_D$ many lines. By generalisation we get

$$t_5(e_1, e_2, \vec{x})\mathbf{r}[\varphi(\vec{x}) \rightarrow \psi(\vec{x})]$$

in $(n-1)2c_{h_1}c_D + c_{h_2}c_D + 4$ many lines. Using the constructions that are described in (g), we get

$$t_6(e_1, e_2)\mathbf{r}(\forall\vec{x})\varphi(\vec{x}) \rightarrow \psi(\vec{x})]$$

in $(n-1)2c_{h_1}c_D + c_{h_2}c_D + 4 + (n-1)c_{h_3}$ many lines. Using the deduction theorem twice and some logic, we get

$$(\exists e_3)[e_3\mathbf{r}(\forall\vec{x})\varphi(\vec{x}) \rightarrow \psi(\vec{x})]]$$

in $(n-1)2c_{h_1}c_D^3 + c_{h_2}c_D^3 + 4c_D^2 + (n-1)c_{h_3}c_D^2 + c_{h_4}$ many lines. Taking $c_h = \max\{4c_{h_1}c_D^3, 2c_{h_2}c_D^3, 8c_D^2, 2c_{h_3}c_D^2, 2c_{h_4}\}$, we can bound this case by

$$(n+1)c_h \text{ many lines.}$$

(i) We assume that there is a deduction of $(\exists e_1)[e_1\mathbf{r}(\forall\vec{x})[\varphi \wedge \psi \rightarrow \chi]]$. We choose such an e_1 by adding the formula $e_1\mathbf{r}(\forall\vec{x})[\varphi \wedge \psi \rightarrow \chi]$ to the deduction. We extend this deduction by two lines for $y\mathbf{r}\varphi(\vec{x})$ and $z\mathbf{r}\psi(\vec{x})$. Therefore we get $\langle y, z\rangle\mathbf{r}[\varphi \wedge \psi]$. From $e_1\mathbf{r}(\forall\vec{x})[\varphi \wedge \psi \to \chi]$ we get by the same procedure as in (g)

$$(\exists v)[\mathcal{T}(t_1(e_1), \vec{x}, v) \wedge \mathcal{U}(v)\mathbf{r}[\varphi \wedge \psi \to \chi]]$$

in $(n-1)c_{i_1}$ many lines. By a deduction that is schematic in $\langle y, z\rangle\mathbf{r}[\varphi \wedge \psi]$, we therefore get

$$(\exists v)(\exists w)[\mathcal{T}(t_1(e_1), \vec{x}, v) \wedge \mathcal{T}(\mathcal{U}(v), \langle y, z\rangle, w) \wedge \mathcal{U}(w)\mathbf{r}\chi(\vec{x})].$$

As in the earlier cases we get

$$(\exists w)[\mathcal{T}(t_2(e_1), \langle\vec{x}, y, z\rangle, w) \wedge \mathcal{U}(w)\mathbf{r}\chi(\vec{x})].$$

Using the S-m-n Theorem, we obtain

$$(\exists w)[\mathcal{T}(t_3(e_1, \vec{x}, y), z, w) \wedge \mathcal{U}(w)\mathbf{r}\chi(\vec{x})]$$

in $(n-1)c'_i + (n+1)c_{i_2}$ many lines. By using the deduction theorem for $z\mathbf{r}\psi(\vec{x})$, we get

$$t_3(e_1, \vec{x}, y)\mathbf{r}[\psi \to \chi]$$

in $(n-1)c_{i_1}c_D + (n+1)c_{i_2}c_D + 6$ many lines. Since t_3 is built up by symbols for primitive recursive functions, we can use Theorem 6.1.2 to get a t_4 such that

$$\{t_4(e_1)\}(\vec{x}, y)\mathbf{r}[\psi \to \chi].$$

Hence, using the S-m-n theorem, we get a t_5 such that

$$\{t_5(e_1, \vec{x})\}(y)\mathbf{r}[\psi \to \chi]$$

is deducible in $(n-1)c_{i_1}c_D + (n+1)c_{i_2}c_D + nc_{i_3}$ many lines. Using the Deduction Theorem for $y\mathbf{r}\varphi$, we can therefore deduce

$$t_5(e_1, \vec{x})\mathbf{r}[\varphi \to [\psi \to \chi]]$$

in $(n-1)c_{i_1}c_D^2+(n+1)c_{i_2}c_D^2+nc_{i_3}c_D+6$ many lines. By using similar constructions as in (g), we get in $(n-1)c_{i_1}c_D^2+(n+1)c_{i_2}c_D^2+nc_{i_3}c_D+(n-1)c_{i_4}$ many lines

$$t_6(e_1)\mathbf{r}(\forall\vec{x})[\varphi \to [\psi \to \chi]].$$

Using the Deduction Theorem for $e_1\mathbf{r}(\forall\vec{x})[\varphi \wedge \psi \to \chi]$ and some logic, we get

$$(\exists e_2)[e_2\mathbf{r}(\forall\vec{x})[\varphi \to [\psi \to \chi]]$$

in $(n-1)c_{i_1}c_D^3+(n+1)c_{i_2}c_D^3+nc_{i_3}c_D^2+(n-1)c_{i_4}c_D+c_{i_5}$ many lines. Choosing c_i in an appropriate way, we therefore get an upper bound of

$$(n+1)c_i \text{ many lines.}$$

(j) By the induction hypothesis we have a deduction for $(\exists e_1)[e_1\mathbf{r}(\forall\vec{x})[\varphi \to (\psi \to \chi)]]$. We assume that e_1 is a witness, by assuming the formula $e_1\mathbf{r}(\forall\vec{x})[\varphi \to (\psi \to \chi)]$. Using the same procedure as in (g), this gives a t_1 such that

$$(\exists v)[\mathcal{T}(t_1(e_1),\vec{x},v) \wedge \mathcal{U}(v)\mathbf{r}[\varphi \to (\psi \to \chi)]]]$$

is deducible in $(n-1)c_{j_1}$ many lines. We also assume that $y\mathbf{r}[\varphi\wedge\psi]$. Hence $p_1(y)\mathbf{r}\varphi$ and $p_2(y)\mathbf{r}\psi$ can be proved in a uniform way. From $p_1(y)\mathbf{r}\varphi$ we get

$$(\exists v_1)(\exists w_1)[\mathcal{T}(t_1(e_1),\vec{x},v_1) \wedge \mathcal{T}(\mathcal{U}(v_1),p_1(y),w_1) \wedge \mathcal{U}(w_1)\mathbf{r}[\psi \to \chi]].$$

Moreover Theorem 6.1.2 gives us

$$(\exists w_1)[\mathcal{T}(t_2(e_1),\langle\vec{x},p_1(y)\rangle,w_1) \wedge \mathcal{U}(w_1)\mathbf{r}[\psi \to \chi]].$$

Using $p_2(y)\mathbf{r}\psi$, we get

$$(\exists w)[\mathcal{T}(t_3(e_1),\langle\vec{x},p_1(y),p_2(y)\rangle,w) \wedge \mathcal{U}(w)\mathbf{r}\chi]$$

by Theorem 6.1.2. Consequently we reach a length of $(n-1)c_{j_1}+c_{j_2}$ lines.

Therefore we get

$$(\exists w)[\mathcal{T}(t_4(e_1,\vec{x}),\langle p_1(y),p_2(y)\rangle,w)\wedge\mathcal{U}(w)\mathbf{r}\chi]$$

by the S-m-n Theorem. This leads to

$$(\exists w)[\mathcal{T}(t_4(e_1,\vec{x}),y,w)\wedge\mathcal{U}(w)\mathbf{r}\chi]$$

by $HA\vdash y=\langle p_1(y),p_2(y)\rangle$. Since the S-m-n Theorem is used, the deduction has a length of $(n-1)c_{j_1}+c_{j_2}+nc_{j_3}$ many lines. Using the Deduction theorem with $y\mathbf{r}[\varphi\wedge\psi]$ and generalisation, we therefore get

$$t_4(e_1,\vec{x})\mathbf{r}[\varphi\wedge\psi\rightarrow\chi]$$

in $(n-1)c_{j_1}c_D+c_{j_2}c_D+nc_{j_3}c_D+6$ many lines. By a similar construction as in the preceding cases, we get therefore

$$t_5(e_1)\mathbf{r}(\forall\vec{x})[\varphi\wedge\psi\rightarrow\chi]$$

in $(n-1)c_{j_1}c_D+c_{j_2}c_D+nc_{j_3}c_D+c_{j_4}$ many lines. Using the Deduction Theorem with $e_1\mathbf{r}(\forall\vec{x})[\varphi\rightarrow(\psi\rightarrow\chi)]$ and some logic, we get

$$(\exists e_2)[e_2\mathbf{r}(\forall\vec{x})[\varphi\wedge\psi\rightarrow\chi]]$$

in $(n-1)c_{j_1}c_D^2+c_{j_2}c_D^2+nc_{j_3}c_D^2+c_{j_4}c_D+c_{j_5}$ many lines. If c_j is chosen big enough, then it is possible to bound the deduction by

$$(n+1)c_j \text{ many lines.}$$

(k) By induction hypothesis we have a deduction for $(\exists e_1)[e_1\mathbf{r}(\forall\vec{x})[\varphi\rightarrow\psi]]$. We assume that e_1 is such a realiser, by assuming the formula $e_1\mathbf{r}(\forall\vec{x})[\varphi\rightarrow\psi]$. By a similar construction as in the previous cases, we get

$$(\exists v)[\mathcal{T}(t_1(e_1),\vec{x},v)\wedge\mathcal{U}(v)\mathbf{r}[\varphi\rightarrow\psi]]$$

in $(n-1)c_{k_1}$ many lines. In order to use the Deduction Theorem, we start by assuming that $y\mathbf{r}[\chi \vee \varphi]$. This is

$$(p_1(y) = 0 \rightarrow p_2(y)\mathbf{r}\chi) \wedge (p_2(y) \neq 0 \rightarrow p_2(y)\mathbf{r}\varphi).$$

We assume that $p_1(y) = 0$ and get, therefore, in two lines $p_2(y)\mathbf{r}\chi$. Hence, by a schematic deduction (we only need logic), we obtain

$$(p_1(y) = 0 \rightarrow p_2(y)\mathbf{r}\chi) \wedge (p_1(y) \neq 0 \rightarrow p_2(y)\mathbf{r}\psi).$$

Using the Deduction theorem, we get a deduction of

$$p_1(y) = 0 \rightarrow (p_1(y) = 0 \rightarrow p_2(y)\mathbf{r}\chi) \wedge (p_2(y) \neq 0 \rightarrow p_2(y)\mathbf{r}\psi),$$

which is

$$p_1(y) = 0 \rightarrow y\mathbf{r}[\chi \vee \psi].$$

Therefore the deduction reaches $(n_1)c_{k_1} + c_{k_2}c_D$ many lines.
Next, we assume that $p_1(y) \neq 0$. This gives $p_2(y)\mathbf{r}\psi$; which together with

$$(\exists v)[\mathcal{T}(t_1(e_1), \vec{x}, v) \wedge \mathcal{U}(v)\mathbf{r}[\varphi \rightarrow \psi]]$$

the formula

$$(\exists v)(\exists w)[\mathcal{T}(t_1(e_1), \vec{x}, v) \wedge \mathcal{T}(\mathcal{U}(v), p_2(y), w) \wedge \mathcal{U}(w)\mathbf{r}\psi].$$

By Theorem 6.1.2 and primitive recursive operations, we, therefore, get

$$(\exists w)[\mathcal{T}(t_2(e_1), \langle \vec{x}, p_2(y)\rangle, w) \wedge \mathcal{U}(w)\mathbf{r}\psi].$$

Hence we are able to obtain

$$(\exists w)[\mathcal{T}(t_2(e_1), \langle \vec{x}, p_2(y)\rangle, w) \wedge$$

$$(p_1(y) = 0 \rightarrow \mathcal{U}(w)\mathbf{r}\chi) \wedge (p_1(y) \neq 0 \rightarrow \mathcal{U}(w)\mathbf{r}\psi)].$$

Because, since we have assumed that $p_1(y) \neq 0$ holds, $(p_1(y) = 0 \to \mathcal{U}(w)\mathbf{r}\chi)$ is deducible and $(p_1(y) \neq 0 \to \mathcal{U}(w)\mathbf{r}\psi)$ is just a weakening of $\mathcal{U}(w)\mathbf{r}\psi$. By Definition 2.2.18 and facts about the $\langle .,. \rangle$ function, we therefore get

$$(\exists w)[\mathcal{T}(t_2(e_1), \langle \vec{x}, p_2(y) \rangle, w) \wedge \langle p_1(y), \mathcal{U}(w) \rangle \mathbf{r}[\chi \vee \psi]]$$

in $(n-1)c_{k_1} + c_{k_2}c_D + c_{k_3}$ many lines. Using the Deduction Theorem with $p_1(y) \neq 0$, we get a deduction of

$$p_1(y) \neq 0 \to (\exists w)[\mathcal{T}(t_2(e_1), \langle \vec{x}, p_2(y) \rangle, w) \wedge \langle p_1(y), \mathcal{U}(w) \rangle \mathbf{r}[\chi \vee \psi]],$$

in $(n-1)c_{k_1} + c_{k_2}c_D + c_{k_3}c_D$ many lines. Consequently both formulas,

$$p_1(x) = 0 \to y\mathbf{r}[\chi \vee \psi]$$

and

$$p_1(x) \neq 0 \to (\exists w)[\mathcal{T}(t_2(e_1), \langle \vec{x}, p_2(y) \rangle, w) \wedge \langle p_1(y), \mathcal{U}(w) \rangle \mathbf{r}[\chi \vee \psi]],$$

are included in one deduction of length $(n-1)c_{k_1} + c_{k_2}c_D + c_{k_3}c_D$. It is well known that arithmetic can combine logical connectives between two equations into one equation. Therefore we get a deduction for

$$(\exists w)[\mathcal{T}(t_2(e_1), \langle \vec{x}, p_2(y) \rangle, w) \wedge t\mathbf{r}[\chi \vee \psi]],$$

where $t = (1 \dot{-} p_1(y))y + sgn(p_1(y))) \cdot \langle p_1(y), \mathcal{U}(w) \rangle$. Since the composition of a recursive function with finitely many primitive recursive ones can be done primitive recursively, there is a function symbol f such that

$$HA \vdash (\forall \vec{x}, y, z)[(\mathcal{T}(f(e_1), \langle \vec{x}, y \rangle, z) \to$$

$$(\exists w)[\mathcal{T}(t_2(e_1), \langle \vec{x}, p_2(y) \rangle, w) \wedge \mathcal{U}(w) = t(y, \mathcal{U}(z))]].$$

Therefore we get

$$(\exists w)[\mathcal{T}(f(e_1), \langle \vec{x}, y \rangle, w) \wedge \mathcal{U}(w)\mathbf{r}[\chi \vee \psi]]$$

in $(n-1)c_{k_1}+c_{k_2}c_D+c_{k_3}c_D+c_{k_4}$ many lines. Using the S-m-n Theorem, we can obtain

$$(\exists w)[\mathcal{T}(t_3(e_1,\vec{x}),y,w)\wedge\mathcal{U}(w)\mathbf{r}[\chi\vee\psi]]$$

in $(n-1)c_{k_1}+c_{k_2}c_D+c_{k_3}c_D+c_{k_4}+nc_{k_5}$ many lines. By using the deduction theorem for $y\mathbf{r}[\chi\vee\varphi]$ and generalisation, we therefore get in $(n-1)c_{k_1}c_D+c_{k_2}c_D^2+c_{k_3}c_D^2+c_{k_4}c_D+nc_{k_5}c_D+6$ many lines

$$t_3(e_1,\vec{x})\mathbf{r}[\chi\vee\varphi\rightarrow\chi\vee\psi].$$

As in (g), we get from this

$$t_4(e_1)\mathbf{r}(\forall\vec{x})[\chi\vee\varphi\rightarrow\chi\vee\psi]$$

in $(n-1)c_{k_1}c_D+c_{k_2}c_D^2+c_{k_3}c_D^2+c_{k_4}c_D+nc_{k_5}c_D+6+(n-1)c_{k_6}$ many lines. Using the deduction theorem with $e_1\mathbf{r}[\varphi\rightarrow\psi]$, we obtain

$$(\exists e_2)[e_2\mathbf{r}(\forall\vec{x})[\chi\vee\varphi\rightarrow\chi\vee\psi]$$

in $(n-1)c_{k_1}c_D+c_{k_2}c_D^2+c_{k_3}c_D^2+c_{k_4}c_D+nc_{k_5}c_D+6+(n-1)c_{k_6}+c_{k_7}$ many lines. Choosing c_k big enough, we obtain a bound for this case of

$$(n+1)c_k.$$

(l) To stay in line with the notation that is used in the rest of the present proof, let's use y to denote the free variable that is quantified by the rule (l). Assume that we have a deduction for $(\exists e_1)[e_1\mathbf{r}[(\forall\vec{x},y)(\psi\rightarrow\varphi(y))]]$. Hence we assume that $e_1\mathbf{r}[(\forall\vec{x},y)(\psi\rightarrow\varphi(y))]$, which gives the formula

$$(\forall y)(\exists z)[\mathcal{T}(t_1(e_1),\langle\vec{x},y\rangle,z)\wedge(\forall u)[u\mathbf{r}\psi\rightarrow(\exists v)[\mathcal{T}(\mathcal{U}(z),u,v)\wedge\mathcal{U}(v)\mathbf{r}\varphi(y)]]$$

by the same constructions as in (g) and Definition 2.2.18. Using Theorem 6.1.2 and the fact that concatenation can be done primitive recursively, we get a primitive recursive function f_1 such that

$$(\forall\vec{x},y,u)[u\mathbf{r}\psi\rightarrow(\exists v)[\mathcal{T}(f_1(e_0,t_1(e_1)),\langle\vec{x},y,u\rangle,v)\wedge\mathcal{U}(v)\mathbf{r}\varphi(y)]].$$

Hence the deduction reaches $(n-1)c_{l_1} + c_{l_2}$ many lines. As in previous cases, we want to use the deduction theorem and assume therefore that $z\mathbf{r}\psi$. We can then conclude

$$(\forall \vec{x}, y)(\exists v)[\mathcal{T}(f_1(e_0, t_1(e_1)), \langle \vec{x}, y, z\rangle, v) \wedge \mathcal{U}(v)\mathbf{r}\varphi(y)].$$

Using the formalised S-m-n theorem, we get

$$(\forall y)(\exists v)[\mathcal{T}(t_2(e_1, \vec{x}, z), y, v) \wedge \mathcal{U}(v)\mathbf{r}\varphi(y)]$$

in $(n-1)c_{l_1} + c_{l_2} + nc_{l_3}$ many lines. This is

$$t_2(e_1, \vec{x}, z)\mathbf{r}[(\forall y)\varphi(y)].$$

Since t_2 is built up by function symbols for primitive recursive function, we can find a primitive recursive f_2 such that

$$HA \vdash (\forall \vec{x}, y, z)[\mathcal{T}(f_2(e_1, \vec{x}), y, z) \rightarrow t_2(e_1, \vec{x}, y) = \mathcal{U}(z)]$$

and

$$HA \vdash (\forall \vec{x}, y)(\exists z)\mathcal{T}(f_2(e_1, \vec{x}), y, z).$$

Hence we get

$$(\exists v)[\mathcal{T}(f_2(e_1, \vec{x}), z, v) \wedge \mathcal{U}(v)\mathbf{r}[(\forall y)\varphi(y)]].$$

The construction of this deduction is schematic in $x\mathbf{r}\varphi(x)$ and uses formalised recursion theory which is provable in full generality in HA. Consequently this extension of the deduction is bounded by $(n-1)c_{l_1} + c_{l_2} + nc_{l_3} + c_{l_4}$ many lines. Using the deduction theorem (with $z\mathbf{r}\psi$) and generalisation, we get

$$f_2(e_1, \vec{x})\mathbf{r}[\psi \rightarrow (\forall y)\varphi(y)].$$

Hence, using similar constructions as in the other cases, we get in $(n-1)c_{l_1} + c_{l_2} + nc_{l_3} + c_{l_4} + (n-1)c_{l_5}$ many lines the formula

$$t_3(e_1)\mathbf{r}(\forall \vec{x})[\psi \rightarrow (\forall y)\varphi(y)].$$

Using the deduction theorem with $e_1\mathbf{r}(\forall\vec{x}, y)[\psi \to \varphi(y)]$ and combining this with the induction hypothesis, we get

$$(\exists e_2)[e_2\mathbf{r}(\forall\vec{x})[\psi \to (\forall y)\varphi(y)]]$$

in $(n-1)c_{l_1}c_D + c_{l_2}c_D + nc_{l_3}c_D + c_{l_4}c_D + (n-1)c_{l_5}c_D + c_{l_6}$ many lines. Choosing c_l big enough, we can bound this case by

$$(n+1)c_l.$$

(m) By the induction hypothesis we have a deduction of

$$(\exists e_1)[e_1\mathbf{r}[(\forall\vec{x}, y)(\varphi(\vec{x}, y) \to \psi(\vec{x}))]].$$

So we assume that e_1 is such a witness by assuming the formula

$$e_1\mathbf{r}[(\forall\vec{x}, y)(\varphi(y) \to \psi)].$$

This formula is, according to Definition 2.2.18 and by similar constructions as in (g), equal to the formula

$$(\forall y)(\exists z)[\mathcal{T}(t_1(e_1), \langle\vec{x}, y\rangle, z) \wedge (\forall u)[u\mathbf{r}\varphi(y) \to (\exists v)[\mathcal{T}(\mathcal{U}(z), u, v) \wedge \mathcal{U}(v)\mathbf{r}\psi]]].$$

This leads to a deduction-length increment of $(n-1)c_{m_1}$ many lines. Using the same construction as in case (l), we get

$$(\forall y, u)[u\mathbf{r}\varphi(y) \to (\exists v)[\mathcal{T}(f(e_1), \langle\vec{x}, u, y\rangle, v) \wedge \mathcal{U}(v)\mathbf{r}\psi]].$$

In order to prove the conclusion of (m), we assume that $u\mathbf{r}(\exists y)\varphi(y)$. This is, according to Definition 2.2.18, equal to the formula $p_1(u)\mathbf{r}\varphi(p_2(u))$. Consequently we can deduce

$$(\exists v)[\mathcal{T}(f(e_1), \langle\vec{x}, \langle p_1(u), p_2(u)\rangle, y\rangle, v) \wedge \mathcal{U}(v)\mathbf{r}\psi]].$$

This leads to

$$(\exists v)[\mathcal{T}(f(e_1), \langle\vec{x}, u, y\rangle, v) \wedge \mathcal{U}(v)\mathbf{r}\psi]]$$

by using that

$$HA \vdash (\forall x)[\langle p_1(x), p_2(x)\rangle = x].$$

Using the Deduction Theorem, similar constructions as in (g) and generalisation, we get

$$t_2(e_1, \vec{x})\mathbf{r}[(\exists y)\varphi(y) \to \psi]$$

in $(n-1)c_{m_1}c_D + c_{m_2}$ many lines. Using Theorem 6.1.2 $n-1$ many times, we get a deduction for

$$t_3(e_1)\mathbf{r}(\forall \vec{x})[(\exists y)\varphi(y) \to \psi]$$

after $(n-1)c_{m_1}c_D + c_{m_2} + (n-1)c_{m_3}$ many lines. Using the Deduction Theorem and some logic, we obtain

$$(\exists e_2)[e_2\mathbf{r}(\forall \vec{x})[(\exists y)\varphi(y) \to \psi]]$$

by a deduction of length $(n+1)c_m$ for a suitable c_m, which was chosen as in previous cases.

This finishes the logical part of a HA-deduction. In the following cases we continue with the arithmetical part.

(n) (We exceptionally assume that we have m free variables, since unfortunately this is case (n).) Note that all axioms defining a primitive recursive function and the first identity-axiom ($x = x$) are of the form

$$t(\vec{x}) = s(\vec{x}),$$

i.e. they are atoms. Therefore they are their own realisations. Consequently, using for an m-ary function-axiom the code for the m-ary constant-0-function, we can deduce $\{e_0\}(\vec{x})\mathbf{r}t(\vec{x}) = s(\vec{x})$ in one line. Using the same constructions as we did for the logical axioms, we get

$$(\exists e_1)[e_1\mathbf{r}(\forall \vec{x})t(\vec{x}) = s(\vec{x})]$$

in $mc_n + c_n = (m+1)c_n$ many lines.

(o) The next group of axioms is realised by a constant-0-function with appropriate arity.

$$x = y \wedge y = z \to x = z$$

$$S(x) \neq 0$$

$$S(x) = S(y) \to x = y$$

These are three axioms with at most three variables. Hence the deduction of their realised versions can be bounded by the number c_o. The axiom schema

$$x_i = y \to [\varphi(x_1, ..., x_i, ..., x_n) \leftrightarrow \varphi(x_1, ..., y, ..., x_n)]$$

has n free variables. Hence we choose c_o big enough such that nc_o bounds the length of this axiom's deduction.

(p) The last step is the case of an induction axiom. To deduce the realised version of an induction axiom in HA, we assume that

$$u\mathbf{r}[\varphi(0, \vec{x}) \wedge (\forall y)[\varphi(y, \vec{x}) \to \varphi(S(y), \vec{x})]].$$

As is explained in the proof of Theorem 2.2.19, we can use the Formalised Recursion Theorem to obtain a realiser for $(\forall y)\varphi(y)$ that is also total. The length of the deduction that establishes totality is independent from φ, since it depends only on the realiser's construction by the Formalised Recursion Theorem. Consequently we get a t_1 such that

$$t_1(\vec{x})\mathbf{r}[[\varphi(0, \vec{x}) \wedge (\forall y)[\varphi(y, \vec{x}) \to \varphi(S(y), \vec{x})]] \to (\forall y)\varphi(y, \vec{x})]$$

is deducible in c_p many lines. Using a similar construction as in previous cases, we therefore get a deduction of

$$t_2\mathbf{r}(\forall \vec{x})[\varphi(0, \vec{x}) \wedge (\forall y)[\varphi(y, \vec{x}) \to \varphi(S(y), \vec{x})]] \to (\forall y)\varphi(y, \vec{x})],$$

with a constant term t_2, that has nc_p many lines.

We finish the proof by noting that the total number of occurrences of free variables in the deduction is bounded by m. □

At this stage the reader should recapitulate the extension of the realisability translation for $\mathcal{L}_{HA}$ to the language $\mathcal{L}_{ID}(Q, P)$, which is given by Definition 3.3.5. We continue our investigation by having a closer look at the complexity of deductions that are given by Lemma 3.3.6. In order to do this, we define a complexity function that ignores the length of formulas from $\mathcal{L}_{ID}(Q)$.

$$\ell(\varphi) := \begin{cases} 1 & : \ \varphi \in \mathcal{L}_{ID}(Q) \text{ or } \varphi \equiv P(x) \\ \ell(\psi_1) + \ell(\psi_2) + 1 & : \ \varphi \equiv \psi_1 \circ \psi_2 \text{ for } \circ \in \{\wedge, \vee, \rightarrow\} \\ \ell(\psi) + 1 & : \ \varphi \equiv \forall y \psi(y) \end{cases}$$

Note that ℓ grows linearly in the the code of φ.
In the following every constant that is used in the estimation of a deduction-complexity is denoted by c. Of course this c is always a different c. If we refer to previous results in a proof, then c always denotes the product of all constants that have been introduced so far.[3] With this convention we try to prevent an overuse of indices.

Lemma 6.1.7 *For any strictly positive operator form A there is a accessibility operator B and a $c \in \mathbb{N}$ such that*

$$\vdash_G^{\ell(xrA)^2 c} x\boldsymbol{r}A \leftrightarrow B.$$

Proof

Since the proof of Lemma 3.3.6 uses only logic, the length increment in every case is bounded by a constant (the constructions are very similar to those that were used to prove Theorem 4.1.3). By taking c as the maximum of these constants, we can bound the length

[3] By using the product rather than the maximum, we take into account that the length of a deduction times the length of its longest formula bounds this deduction's complexity.

of the deductions by $\ell(x\mathrm{r}A)c$, because the proof goes by induction on $\ell(x\mathrm{r}A)$. However, in addition the length of the schemata, which are used in the deductions, are constant in every case. Hence we can bound them by $\ell(x\mathbf{r}A)c$ as well.[4] $\square$

Before we are able to extend Lemma 6.1.6 to $\mathcal{L}_{ID}(strict)$, we have to ensure that some technicalities, which are proved in Lemma 3.3.10, do not enlarge the length of the deductions to much.

Lemma 6.1.8 *Assume that* $\varphi(z) \in \mathcal{L}_{ID}(Q, P)$ *is a strictly positive operator form and that*

$$\psi(z) :\equiv \{v\}(y, w, (z)_0, (z)_1)\boldsymbol{r}\varphi((z)_0).$$

There is a c *such that for any strictly positive operator form* $A \in \mathcal{L}_{ID}(Q, P)$*, where* $z_1, ..., z_n$ *are the only freely occurring variables, there is a realizer (which is expressed by a p-term)* $p_A(x, v, w, y, z_1, ..., z_n)$ *such that*

$$HA \vdash^{\ell(A)c} (\mathrm{x}\boldsymbol{r}\, A)(\psi, P^{B^r}_{<y}) \leftrightarrow p_A(x, v, w, y, z_1, ..., z_n)\boldsymbol{r}A(\varphi, P^{B}_{<y}).^5$$

Proof

First notice that $p_A(x, v, w, y, z_1, ..., z_n)\mathbf{r}A(\varphi, P^B_{<y})$ is neither a formula nor a translation of one by itself. It is an abbreviation which denotes a realised formula that might include an occurrence of the $\mathcal{T}$ formula, which is used in the coding of recursive functions.

The proof proceeds on the inductive definition of strictly positive operator forms. The cases are enumerated in accordance with the proof of Lemma 3.3.10. The cases 1-4 and 6 include only logic, identity axioms and general facts about primitive recursive functions

[4] We distinguish the length of the schema and the length of the formula that is an instance of this schema, e.g. $([(\varphi \wedge \psi) \wedge (\varphi \wedge \psi)] \rightarrow (\varphi \wedge \psi))$ has a length of 21 as a formula but a length of 9 as an instance of the schema $((\varphi \wedge \varphi) \rightarrow \varphi)$.

[5] The set of p-terms is constructed by closing the set of terms of HA by λ-abstraction and Kleene-brackets.

(like pairing, projection and the function f of case 4). Therefore these cases are schematic in φ, which makes the deductions that are given by these cases uniformly constructible. Hence the lengths of these deductions are obviously bounded by a constant. The only cases which remain problematic, because of they involve formalised recursion theory, are 5 and 7.

However the only problematic step in case 5 is the construction of a deduction for

$$(\forall u)[p_C(\{e\}(u), \bar{\bar{z}}, u)\mathbf{r}C(u, \varphi, P^B_{<y})] \leftrightarrow \lambda u.p_C(\{e\}(u), \bar{\bar{z}}, u)\mathbf{r}A(\varphi, P^B_{<y}).$$

But note that this is equivalent to

$$p_C(\{e\}(u), \bar{\bar{z}}, u)\mathbf{r}C(u, \varphi, P^B_{<y}) \leftrightarrow \lambda u.p_C(\{e\}(u), \bar{\bar{z}}, u)\mathbf{r}A(\varphi, P^B_{<y}),$$

which is just a generalisation at the right side of **r**. This can be done by a similar construction as in the proof of Lemma 6.1.6. Since all the resources of formalized recursion theory that are needed here are provable in full generality, the deductions given thus are length-bounded by a constant.

In case 7 the situation is simpler. The only problematic step here is

$$(\forall u)[(u\mathbf{r}C)(\psi, P^{B^\mathbf{r}}_{<y}) \rightarrow p_D(\{e\}(u), \bar{\bar{z}})\mathbf{r}D(\varphi, P^B_{<y})] \leftrightarrow \lambda u.p_D(\{e\}(u), \bar{\bar{z}})\mathbf{r}A(\varphi, P^B_{<y}).$$

This can be done by the S-m-n Theorem (for a fix m and n) and the fact that concatenation of two recursive functions can be done primitive recursively. Hence this case's length-increment is bounded by a constant as well.

Since the proof proceeds by an induction on $\ell(A)$, we can bound the length-increment by $\ell(A)c$, where c is the the maximum of all constants that are used in the present proof. To bound the formulas in the deductions we have a similar situation. In all cases the formulas are bounded by $\ell(A)c$ as well. □

Lemma 6.1.9 *There is a c such that for any formula φ following under the schema*

$$(\forall y < n)(\forall x)[P^A_y(x) \leftrightarrow A(P^A_y, P^A_{<y}, x, y)],$$

where A is a strictly positive operator, there is an $m \in \mathbb{N}$ such that

$$\widehat{ID}^i_n(acc) \vdash^{\ell(A)^2c} \bar{m}\boldsymbol{r}\varphi.$$

Proof

Assume that φ is of the form $(\forall y < n)(\forall x)[P^A_y(x) \leftrightarrow A(P^A_y, P^A_{<y}, x, y)]$ for a strictly positive operator A. First notice that, since A is a strictly positive operator, φ is a sentence. By Lemma 3.3.9 we know that $x\mathbf{r}A(P^A_y, P^A_{<y}, z, y)$ and $(x\mathbf{r}A)(P^{A^\mathbf{r}}_y, P^{A^\mathbf{r}}_{<y}, z, y)$ are identical formulas. Therefore we get

$$HA \vdash^{\ell(x\mathrm{r}A)^2c} x\mathbf{r}A(P^A_y, P^A_{<y}, z, y) \leftrightarrow A^\mathbf{r}(P^{A^\mathbf{r}}_y, P^{A^\mathbf{r}}_{<y}, \langle z, x\rangle, y)$$

by Definition 3.3.7 and Lemma 6.1.7. Using this equivalence together with the fixed point axiom in $\widehat{ID}^i_n(acc)$, one gets

$$\widehat{ID}^i_n(acc) \vdash y < n \to [x\mathbf{r}A(P^A_y, P^A_{<y}, z, y) \to P^{A^\mathbf{r}}_y(\langle z, x\rangle)].$$

The deduction is bounded by $\ell(x\mathrm{r}A)^2c+\ell(x\mathbf{r}A)^2c_1$, because this step is only using logical moves, which are schematic in the formulas used. By Definition 3.3.5 this gives

$$\widehat{ID}^i_n(acc) \vdash y < n \to [x\mathbf{r}A(P^A_y, P^A_{<y}, z, y) \to x\mathbf{r}P^A_y(z)].$$

Therefore we can take $m_1 := \ulcorner \lambda uyzx.x \urcorner$ and use it as a realiser. To prove

$$m_1\mathbf{r}[y < n \to [A(P^A_y, P^A_{<y}, z, y) \to P^A_y(z)]]$$

from this result one only needs a constant numbers of steps. For the deduction is uniform in A and A includes not more than three free variables. Note that the variable u appearing in m_1 comes from the fact that

$$m_1\mathbf{r}[y < n \to [A(P^A_y, P^A_{<y}, z, y) \to P^A_y(z)]]$$

is

$$(\forall u)[u\mathbf{r}(y < n) \to \{m\}(u)\mathbf{r}[A(P^A_y, P^A_{<y}, z, y) \to P^A_y(z)]].$$

This does not cause a problem, since $u\mathbf{r}(y < n)$ is identical to $y < n$ by the Definition of realisability.

The other direction of the axiom is provable in almost the same way, but with the antecedent and the succedent permuted. So we can take m_2 as the same as m_1. Since we can rely on the established equivalence

$$x\mathbf{r}A(P_y^A, P_{<y}^A, z, y) \leftrightarrow A^{\mathbf{r}}(P_y^{A^{\mathbf{r}}}, P_{<y}^{A^{\mathbf{r}}}, \langle z, x\rangle, y),$$

which was given in the previously shown direction, in a Hilbert-system, this needs only a constant number of steps. Hence by taking $m := \langle m_1, m_2\rangle$ we can deduce $\bar{m}\mathbf{r}\varphi$ in $2(\ell(x\mathrm{r}A)^2c + \ell(x\mathbf{r}A)^2c_2)$ many steps in $\widehat{ID}^i_n(acc)$. Moreover we know that there is a constant c_3 such that $\ell(x\mathrm{r}A) \leq c_3\ell(A)$. Therefore we get a bound for the complexity of the deduction by $\ell(A)^2c_4$, where c_4 is our new c. □

Before we are able to use those facts to extract a bound from the proof of Theorem 3.3.3 we have to go back to Chapter 2 one last time to have a closer look of Proposition 2.2.22.

Lemma 6.1.10 *If φ is almost negative, then there is a c such that*

$$HA \vdash^{c\cdot|\varphi|^2} (\exists x)(\mathrm{x}\boldsymbol{r}\,\varphi) \leftrightarrow \varphi.$$

Proof

The proof follows that of Proposition 2.2.22. It should not be hard to see that all cases lead to a uniformly constructed deduction and, hence, can be bounded by a constant. So by choosing the maximum of these constants we have found our c. As before we can bound the formulas by a linear function as well and therefore reach a quadratic bound. □

Remark 6.1.11 *Note that the bound given in Lemma 6.1.10 is possible, because case 2 in the proof of Proposition 2.2.22 is rather ambiguous. Usually one would want to give* *a uniform function which chooses the minimum of the y, whose existence is assumed (as is done in [59, p.193]). However choosing the minimum involves a computation of the terms t and s involved. Therefore the deduction cannot be uniformly constructed. Here the uniformity of choosing an instance is in tension with the uniformity of constructing a deduction for which this instance satisfies the formula.*

Theorem 6.1.12 *There is a constant c such that for any almost negative sentence $\varphi \in \mathcal{L}_{HA}$*

$$\widehat{ID}^{i}_{n}(strict) \vdash^{m} \varphi \Rightarrow \widehat{ID}^{i}_{n}(acc) \vdash^{(m+1)^{12}c} \varphi.$$

Proof

Assume that $\varphi \in \mathcal{L}_{HA}$ is an almost negative sentence with $\widehat{ID}^{i}_{n}(strict) \vdash^{m_1} \varphi$. Also assume that the deduction of φ includes only k many instances of the fixed point axiom $\psi_1, ..., \psi_k$ and that the biggest formula includes not more than l many symbols. Therefore $k, l < m$ and

$$HA \vdash \psi_1 \to ... \to \psi_k \to \varphi$$

in $m^2 c_D$ many steps, by k applications of the Deduction Theorem with formulas whose length is bounded by $(m+1)m$. Using Lemma 6.1.6, we get

$$HA \vdash^{((m+1)^4 c_D)^2 c} (\exists x)[x\mathbf{r}[\psi_1 \to ... \to \psi_k \to \varphi]].$$

By Lemma 6.1.9 we get

$$\widehat{ID}^{i}_{n}(acc) \vdash^{m^2 c} \bar{m_1}\mathbf{r}\psi_1$$

$$\vdots$$

$$\widehat{ID}^{i}_{n}(acc) \vdash^{m^2 c} \bar{m_k}\mathbf{r}\psi_k.$$

After combining these deductions we can apply a Modus Ponens construction as in case (g) from the proof of Lemma 6.1.6 and get

$$\widehat{ID}^{i}_{n}(acc) \vdash^{((m+1)^4 c_D)^2 c + 2km^2 c} (\exists x)[\{x\}(\bar{m_1}, ..., \bar{m_k})\mathbf{r}\varphi],$$

where the term $2km^2c$ emerges from k deductions of length m^2c and m applications of the modus ponens construction, which is bounded by c as defined in the proof of Lemma 6.1.6. Taking c_2 big enough so that $c_D^2c + c + c_1 < c_2$, where c_1 is the length of the logical deduction that is needed to introduce $(\exists y)$,[6] we get

$$\widehat{ID}^i_n(acc) \vdash^{2((m+1)^4)^3 c_2} (\exists y)[y\mathbf{r}\varphi].$$

By Lemma 6.1.10 we get

$$HA \vdash^{m^2 c_2} (\exists y)[y\mathbf{r}\varphi] \to \varphi.$$

This leads to

$$\widehat{ID}^i_n(acc) \vdash^{3(m+1)^{12} c_2} \varphi.$$

Hence, by taking $3c_2$ as our new c, we are done. □

6.2 Bounding Rüede Strahm

Since we gave a bound for $\widehat{ID}^i_n(strict) \leq_{\Pi^0_2} \widehat{ID}^i_n(acc)$, the next step is to give a bound for $\widehat{ID}^i_n(acc) \leq_{\Pi^0_2} ACA^-_n \leq_{\Pi^0_\infty} PA$. As is described in Chapter 3 we give a translation

$$\tau : \mathcal{L}^*_{ID} \to \mathcal{L}_{ACA_n},$$

which is induced by a mapping

$$P^A \mapsto H^{\tau(A)}$$

for every accessibility operator A. From the proof of Theorem 3.3.4 we can extract a bound for the proof-theoretic reductions given above. However, as the reader might have already noted, the proof of Theorem 3.3.4 includes the use of a formalised truth predicate for Π^0_n-formulas of $\mathcal{L}_{ID}(Q)$. To make this notion precise we have to alter

[6]This inequality is ensured by the definition of c. Note that c was defined by adding such constructions to some multiples of c_D by the arguments in this chapter.

Definition 2.1.3 and the Truth-predicate given in Theorem 2.1.11. Definition 2.1.3 is extended to the language with includes all primitive recursive functions, i.e. equations with terms including function symbols for primitive recursive functions are viewed as Δ_0-formulas, and we add the clause that $Q(t)$ is viewed as a Δ_0-formula. Since we want to keep the possibility of talking about Π_n-formulas in the original sense of Definition 2.1.3, we call this new notion the $\Pi_n(Q)$-formulas. Before we avoided the definition of a truth predicate by citing Theorem 2.1.11. Here however we have to describe the truth predicate that is in use in order to analyse how it acts in a formal deduction. Hence we sketch its definition in a way that allows us to estimate how many lines are needed to deduce basic properties of the truth predicate in our Hilbert-style system. It is possible to define an evaluation function that evaluates every term of $\mathcal{L}_{ID}(Q)$ by a Δ_1-formula φ_{ev}. The Δ_1-formula φ_{ev} states the existence of a formal computation of the term in question. Here, again, we have to use the so called dot-notation for free variables, where $\dot{x}$ denotes the x^{th} numeral. Hence, for a term $t(x)$, the evaluation function is formulated as $\varphi_{ev}(\ulcorner t(\dot{x})\urcorner, y)$, which is a formula where only x and y occur freely.

We can use the evaluation function to formulate a truth predicate for $\Pi_n(Q)$-formulas. According to the definition of $\Pi_n(Q)$-formulas, the atomic case is divided into the following two cases:

$$\mathsf{T}_{id}(\ulcorner t_1 = t_2\urcorner) \equiv (\forall x, y)[\varphi_{ev}(\ulcorner t_1\urcorner, x) \wedge \varphi_{ev}(\ulcorner t_2\urcorner, y) \rightarrow x = y],$$

which is in fact a Δ_1-formula, since φ_{ev} defines a function, and

$$\mathsf{T}_Q(\ulcorner Q(t)\urcorner) \equiv Q(t),$$

because Q is a predicate symbol that is not given a meaning by the axioms. Then $\mathsf{T}_{atom}(x)$ can be built up from $\mathsf{T}_{id}(x)$ and $\mathsf{T}_Q(x)$ in the obvious way.

In the following considerations it will be necessary to measure the deduction-length of deductions establishing the equivalence

$$\mathsf{T}_{\Pi_n(Q)}(\ulcorner\varphi\urcorner) \leftrightarrow \varphi.$$

For the atomic case the complexity of terms occurring in the formula will govern the length-increment of these deductions. It is therefore necessary to specify how all primitive recursive functions can be introduced by definitions like Definition 2.1.4 and Definition 3.2.1. For the following proofs it is convenient to introduce primitive recursive functions by axioms governing complex function symbols, i.e. the function symbols have a structure that reflects the axioms which govern them.

Definition 6.2.1 *The set of primitive recursive function symbols (p.r.f.s) is inductively defined as follows:*

1. *The symbol for the successor function S is a symbol for a p.r.f.s of arity* 1.
2. *The symbol for the constant-0-function of arity n, C^n, is a p.r.f.s of arity n.*
3. *The symbol for the projection into the k-th input out of n inputs, P^n_k, is a p.r.f.s of arity n.*
4. *If $f_1^{n_1}, ..., f_{n_1+1}^{n_{n_1+1}}$ are p.r.f.s with their arities shown, then $[Sub(f_1^{n_1}, ..., f_{n_1+1}^{n_{n_1+1}})]^n$, where $n := \max\{n_2, ..., n_{n_1+1}\}$, is a p.r.f.s of arity n.*[7]
5. *If $f_1^{n_1}, f_2^{n_2}$ are p.r.f.s with their arities shown, then $[Rec(f_1^{n_1}, f_2^{n_2})]^n$, where $n := \max\{n_1 + 1, n_2\}$, is a p.r.f.s of arity n.*

Definition 6.2.1 determines the way that the p.r.f.s have to be introduced into a theory, e.g. the theories that are defined in Definition 2.1.4 and Definition 3.2.1.

Definition 6.2.2 *The axioms of the p.r.f.s are the universal closures of the following formulas.*

[7] Note that n_1 function symbols have to be substituted into a function symbol with arity n_1.

1. *For* S*:*

$$\neg S(x) = \bar{0}$$

$$S(x) = S(y) \rightarrow x = y$$

2. *For a* C^n*:*

$$C^n(x_1, ..., x_n) = \bar{0}$$

3. *For a* P^n_k*:*

$$P^n_k(x_1, ..., x_n) = x_k$$

4. *For a* $f^n \equiv [Sub(f^{n_1}_1, ..., f^{n_{n_1+1}}_{n_1+1})]^n$*:*

$$f^n(x_1, ..., x_n) = f^{n_1}_1(f^{n_2}_2(x_1, ..., x_{n_2}), ..., f^{n_{n_1+1}}_{n_1+1}(x_1, ..., x_{n_{n_1+1}}))$$

5. *For a* $f^n \equiv [Rec(f^{n_1}_1, f^{n_2}_2)]^n$*:*

$$f^n(0, x_1, ..., x_{n-1}) = f^{n_1}_1(x_1, ..., x_{n_1})$$

$$f^n(S(y), x_1, ..., x_{n-1}) = f^{n_2}_2(f^n(y, x_2, ..., x_{n-1}), y, x_3, ..., x_{n_2})$$

Note that in the axioms of case 4 and 5 the right term's function symbols are less complex than those of the left term. However the depth of the right is less than the left term's. To deal with this in the proof of the next lemma, we introduce two complexity notions.

Definition 6.2.3 *Assume that* f *is a p.r.f.s. The degree of* f*, in symbols* $dg(f)$*, is defined as follows:*

1. *If* f *is* S, C^n *or* P^n_k*, then* $dg(f) = 1$.

2. *If* $f \equiv [Sub(f^{n_1}_1, ..., f^{n_{n_1+1}}_{n_1+1})]^n$*, then*

$$dg(f) = \sum_{i=0}^{n_1+1} dg(f_i) + 1.$$

3. *If* $f \equiv [Rec(f_1^{n_1}, f_2^{n_2})]^n$*, then*

$$dg(f) = dg(f_1) + dg(f_2) + 1.$$

The degree of a term $t \in \mathcal{L}_{ID}(Q)$ *is defined as the maximum of the degrees of all p.r.f.s occurring in* t*.*
The rank of t*,* $rk(t)$*, is defined as follows.*

1. *If* $t \equiv x$ *or* $t \equiv \bar{0}$*, then* $rk(t) = 0$*.*

2. *If* $t \equiv f(t_1, ..., t_n)$*, then* $rk(t) = n(\max\{rk(t_i) | 1 \leq i \leq n\} + 1)$*.*

The next lemma bounds the deduction-length of deductions whose existence is stated in Lemma 2.1.11 for the atomic case.

Lemma 6.2.4 [8] *If* φ *is an atomic formula of* $\mathcal{L}_{ID}(Q)$*, then*

$$PA \vdash^{c \cdot 2^{(2d+6)r}} \mathsf{T}_{atom}(Q, \ulcorner \varphi \urcorner) \leftrightarrow \varphi,$$

where $r := rk(\varphi)$ *and* $d := dg(\varphi)$*.*

Proof

In the case of $\varphi \equiv Q(t)$, there is a constant $c_{\to Q}$ bounding all deductions independently

[8] In the first version of my thesis, which was handed in for examination, this lemma demanded a super-exponential growth rate and was proven by a double induction on the rank and degree of a term. But I blocked the super-exponential growth rate by restricting the language to functions with a degree smaller than 4. However the external examiner, Georg Moser, realised that the induction basis of this double induction was not matching the induction step and announced his feelings that an exponential bound might be possible. When I tried to fix the mismatch, I realised that no induction on the rank is needed and came up with the proof that is given below. This also smoothens the rest of this chapter, because I do not have to discuss the super-exponential growth rate away.

from t.

In the case where φ is $t_1(\vec{x}) = t_2(\vec{x})$, we proceed by induction on

$$\text{dg}(\varphi) := \text{dg}(t_1) + \text{dg}(t_2)$$

.

- Induction base: Assume $\text{dg}(\varphi) = 0$. Hence φ is $x_1 = x_2$, $\bar{0} = x$, $x = \bar{0}$ or $\bar{0} = \bar{0}$. We only check the case where $x_1 = x_2$; the others are entirely analogous. The assumption $\mathsf{T}_{atom}(\ulcorner \dot{x}_1 = \dot{x}_2 \urcorner)$ gives

  $$(\forall x, y)[\varphi_{ev}(\dot{x}_1, x) \wedge \varphi_{ev}(\dot{x}_2, y) \rightarrow x = y].$$

  Therefore a very short deduction gives $x_1 = x_2$. The other direction follows easily from the fact that

  $$PA \vdash \varphi_{ev}(\dot{x}, x)$$

  for every x by the definition of φ_{ev}.

  Bound: All four cases are schematic in the variables and can therefore be bounded by a constant c. Also the maximal length of the formulas is bounded by a c.

- Induction step: Before we start with the argument we note that PA proves the following sentences

  $$(\forall x)\mathsf{T}_{atom}(\ulcorner \dot{x} = \dot{x} \urcorner)$$

  $$(\forall x, y)[\mathsf{T}_{atom}(\ulcorner \dot{x} = \dot{y} \urcorner) \rightarrow \mathsf{T}_{atom}(\ulcorner \dot{y} = \dot{x} \urcorner)]$$

  $$(\forall x, y, z)[\mathsf{T}_{atom}(\ulcorner \dot{x} = \dot{y} \urcorner) \wedge \mathsf{T}_{atom}(\ulcorner \dot{y} = \dot{z} \urcorner) \rightarrow \mathsf{T}_{atom}(\ulcorner \dot{x} = \dot{z} \urcorner)].$$

  Assume that $\text{dg}(\varphi) > 0$. In the proof we construct $\psi_1, ...\psi_m$ with $\text{dg}(\psi_i) < \text{dg}(\varphi)$ for any $1 \leq i \leq m$ on which the induction hypothesis can be applied. We know that either $\text{dg}(t_1) \leq \text{dg}(t_2)$ or $\text{dg}(t_2) \leq \text{dg}(t_1)$, and we pick the term with the higher degree (or t_1, when they have the same). Without loss of generality we pick t_1, which can be assumed to be of the form $f(s_1, ..., s_n)$. In the next step we introduce new variables $x_1, ..., x_n, y$ and reformulate the problem through the following equations

$$f(x_1, ..., x_n) = y$$

$$x_i = s_i \text{ for any } 1 \leq i \leq n$$

$$y = t_2.$$

We continue to do that until all formulas have the form $g(\vec{x}) = y$. If such a ψ_i has dg$(\psi_i) <$ dg(f), then we apply the induction hypothesis. For the rest we proceed as follows. According to Definition 6.2.1 we have to consider five cases:

1. f is S^1: This case is trivial and bounded by a constant, because $S(\dot{x})$ and $\dot{S(x)}$ give the same output.
2. f is a C^n: Here we note that dg$(\bar{0} = y) <$ dg$(f(\vec{x}) = y)$. Hence

   $$\mathsf{T}_{atom}(\ulcorner \bar{0} = y \urcorner) \leftrightarrow \bar{0} = y,$$

   by the induction hypothesis. Using the definition of φ_{ev}, we get

   $$\mathsf{T}_{atom}(\ulcorner C^n(\vec{x}) = y \urcorner) \leftrightarrow \mathsf{T}_{atom}(\ulcorner \bar{0} = y \urcorner).$$

   Using the axioms governing C^n and identity, we get a deduction for

   $$C^n(\vec{x}) = y \leftrightarrow \bar{0} = y.$$

   Combining this by logic, we get the desired implications.
   Bound: Here the length increment is bounded by a constant.
3. f is a P^n_k: This case is analogous to the case for C^n.
4. f is a $[Sub(f_1^{n_1}, ..., f_{n_1+1}^{n_{n_1+1}})]^n$: Using the definition of φ_{ev} and the axioms governing Sub, we get deductions that are schematic in the terms and function symbols in $f(\vec{x}) = y$ for the following formulas; for readability we abbreviate $f_1^{n_1}(f_2^{n_2}(x_1, ..., x_{n_2}), ..., f_{n_1+1}^{n_{n_1+1}}(x_1, ..., x_{n_{n_1+1}}))$ by t.

   $$\mathsf{T}_{atom}(\ulcorner [Sub(f_1^{n_1}, ..., f_{n_1+1}^{n_{n_1+1}})]^n(x_1, ...x_n) = y \urcorner) \leftrightarrow \mathcal{T}_{atom}(t = y \urcorner)$$

$$[Sub(f_1^{n_1}, ..., f_{n_1+1}^{n_{n_1+1}})]^n(x_1, ...x_n) = y \leftrightarrow t = y.$$

Note that

$$\text{dg}(t) < \text{dg}([Sub(f_1^{n_1}, ..., f_{n_1+1}^{n_{n_1+1}})]^n(x_1, ...x_n)).$$

Hence

$$\mathcal{T}_{atom}(\ulcorner t = y \urcorner) \leftrightarrow t = y$$

by the induction hypothesis for dg, which fills the gap.

Bound: Here the length increment is bounded by a constant as in the cases before, but we get $n < r$ many new formulas of the form $z_i = f_i^{n_i}(x_1, ..., x_{n_i})$ on which the induction hypothesis is applied.

5. f is a $[Rec(f_1^{n_1}, f_2^{n_2})]^n$: In this case we know that the first subterm of t_1 (which we denoted by s_1) is of the form $S(...S(0)...)$ (denoted by $S^m(0)$). From the definition of T_{atom} and the assumption

$$\mathsf{T}_{atom}(\ulcorner f(S^m(0), x_2, ..., x_n) = y \urcorner),$$

we get in a deduction of length mc the formulas:

$$\mathsf{T}_{atom}(\ulcorner f_2^{n_2}(f^n(S^{m-1}(0), x_2, ..., x_{n-1}), S^{m-1}(0), x_3, ..., x_{n_2}) = y \urcorner)$$

$$\mathsf{T}_{atom}(\ulcorner f^n(S^{m-1}(0), x_2, ..., x_{n-1}) =$$

$$f_2^{n_2}(f^n(S^{m-2}(0), x_2, ..., x_{n-1}), S^{m-2}(0), x_3, ..., x_{n_2})\urcorner)$$

$$\vdots$$

$$\mathsf{T}_{atom}(\ulcorner f^n(S(0), x_2, ..., x_{n-1}) = f_2^{n_2}(f_1^{n_1}(x_1, x_2, ..., x_{n-1}), 0, x_3, ..., x_{n_2})\urcorner).$$

The degree of all the formulas is smaller than that dg(f), therefore we can deduce their appropriate equivalences by the induction hypothesis for dg. Then we can combine these equivalences in a deduction of a length that is linear in m to establish the equivalence for $f(\vec{x}) = y$; and we are done.

Bound: Note that $m < r$ hence we have a bound on the length increment of $c \cdot r^2$ (m deduction with a length of at most $m \cdot c$ lines).

Now we summarise the restrictions on the bound. According to the preparation step that splits the formulas to those of the form $f(\vec{x}) = y$, we have at most r many formulas on which the induction hypothesis is applied to. In the five cases above the number of applications of the induction hypothesis is bounded by r. Hence the total number of applications is bounded by $2r$. In addition any of the five cases above adds at most $c \cdot r^2$ many lines to the deduction. Hence the length-bound is given by the following recursion

$$h(d+1) = 2r \cdot h(d) + r^2 \cdot c.$$

Calculations lead to

$$h(d) < c \cdot (2r)^{d+3}.$$

In order to bound the formulas, it should be clear that the length of the deduction grows stronger than the length of the formulas, because all formulas introduced are shorter than the previously given and substituted into schematic deductions. Hence by squaring the deduction length and choosing a new constant c we have a complexity bound of $c \cdot r^{2d+6}$, which can be bounded by $c \cdot 2^{(2d+6)r}$.

□

Lemma 6.2.5 *If φ is any Π^0_n-formula of $\mathcal{L}_{ID}(Q)$ whose function symbols do not have a degree bigger than g, then there is a truth-predicate T_n such that*

$$PA \vdash^{c \cdot |\varphi|^2 \cdot 2^{(2g+6)\cdot|\varphi|}} \mathsf{T}_n(Q, \ulcorner\varphi\urcorner) \leftrightarrow \varphi.$$

Proof

The proof uses the previous lemmata together with the fact that a standard truth predicate is inductively defined on the construction of a formula. Hence the bound for the deduction-length is obtained from the bound of the atomic case, which may appear $|\varphi|$

many times, and a linear factor that respects the complexity of the formula. In the same way the formula-length is bounded linearly as well. Also note that the rank of the term is bounded by the total length of the term and therefore by the total length of the formula. □

Lemma 6.2.6 *For any* $\varphi \in \mathcal{L}_{ID}(Q)$ *there is an* $n \in \mathbb{N}$ *and a* $\pi \in \Pi^0_n(Q)$ *such that*

$$PA \vdash^{c \cdot |\varphi|^2} \varphi \leftrightarrow \pi.$$

Proof

Reduction to prenex form is well known. The proof goes in both directions of the implication by induction on the construction of φ. According to the main connectives and the quantifiers there are several different cases. One of them has the longest deduction-length increment and another the longest formula-length increment; which therefore gives a constant. □

Lemma 6.2.7 *There is polynomial in natural coefficients* $P[X]$ *such that the length of a* ACA_n^-*-deduction for the translations of the fixed point axioms for the accessibility operator* A*, which only contains function symbols of degree smaller than* g*, can be bounded by* $P[2^{(2g+6)\cdot|A|}]$.

Proof

We work in ACA_n^- and follow the proof of Theorem 3.3.4. Assume that $A(P, Q, x, y)$ is an accessibility operator. Hence there are $\varphi, \psi \in \mathcal{L}_{ID}(Q)$ such that

$$A(P, Q, x, y) \equiv \varphi \wedge (\forall z)[\psi(z) \rightarrow P(z)].$$

We also assume that the degree of all functions occurring in these formulas is bounded by g. Then we choose an n such that φ and ψ are equivalent to a $\Pi_n(Q)$-formula $(\bar{\varphi})$

and a $\Sigma_n(Q)$-formula ($\bar{\psi}$) respectively. As in the proof of Theorem 3.3.4, we replace $P(z)$ in $A(P,Q,x,y)$ by a truth-predicate and therefore get a formula equivalent to a $\Pi_n(Q)$-formula. We denote this $\Pi_2(Q)$-formula by $\bar{A}$ and can deduce its equivalence to the original one with a $c \cdot |A|^2$-complex deduction. $\bar{A}$ has a Gödel number $\bar{k}$ and we can therefore prove by a deduction of complexity

$$|\bar{A}|^2 \cdot c \cdot [2^{(2g+6)\cdot|\bar{A}|} + 1]^2$$

the equivalence

$$\mathsf{T}_n(Q,\bar{k},u,x,y) \leftrightarrow \bar{A}(\mathsf{T}_n(Q,u,u,z,y),Q,x,y).$$

Adding a constant c to the already reached complexity, we can substitute u by $\bar{k}$ to get

$$\mathsf{T}_n(Q,\bar{k},\bar{k},x,y) \leftrightarrow \bar{A}(\mathsf{T}_n(Q,\bar{k},\bar{k},z,y),Q,x,y).$$

For notational reasons we abbreviate $E_n(Q,\bar{k},\bar{k},x,y)$ by $D(Q,x,y)$. Hence we get

$$(\forall y < n)(\forall x)[H^D_y(x) \leftrightarrow D(H^D_{<y},x,y)]$$

in one line by the comprehension Axiom with a formula of length $|\bar{A}| + c$.
Moreover, from the formula

$$\mathsf{T}_n(Q,\bar{k},\bar{k},x,y) \leftrightarrow \bar{A}(\mathsf{T}_n(Q,\bar{k},\bar{k},z,y),Q,x,y),$$

we can deduce

$$(\forall y < n)(\forall x)[\mathsf{T}_n(Q,\bar{k},\bar{k},x,y) \leftrightarrow \bar{A}(\mathsf{T}_n(Q,\bar{k},\bar{k},z,y),Q,x,y)]$$

in a constant number of lines. Hence we have reached a complexity of

$$|\bar{A}|^2 \cdot c \cdot [2^{(2g+6)\cdot|\bar{A}|} + 1]^2 + 2c^2 \cdot |\bar{A}|.$$

Note, since we are only talking about implications, the length-increment of formulas can be linearly bounded. Combining these two equivalences by another schematic deduction we get

$$(\forall y < n)(\forall x)[H^D_y(x) \leftrightarrow \bar{A}(\mathsf{T}_n(Q,\bar{k},\bar{k},z,y),Q,x,y)]$$

in a deduction with complexity

$$|\bar{A}|^2 \cdot c \cdot [2^{(2g+6)\cdot|\bar{A}|} + 1]^2 + 3c \cdot |A|.$$

However in order to arrive at the version with A instead of $\bar{A}$, we have to add the equivalence of these two formulas plus a schematic deduction which puts them together. Therefore we reach a complexity of

$$|\bar{A}|^2 \cdot c \cdot [2^{(2g+6)\cdot|\bar{A}|} + 1]^2 + |A|^2 \cdot c + 4c^2 \cdot |\bar{A}|.$$

To finish the proof note that $|\bar{A}| \leq c \cdot |A|$ by reduction to prenex form. $\Box$

Now we are able to bound the proof-theoretical reduction given by Theorem 3.3.4. For readability we introduce the notation $\vdash^m_g$, which means that there is a deduction of complexity m in which only function symbols occur that have a degree less than g.

Theorem 6.2.8 *There is a polynomial in natural coefficients $P[X]$ such that*

$$\widehat{ID}_n(acc) \vdash^m_g \Rightarrow ACA^-_n \vdash^{P[2^{(2g+6)m}]}_g \varphi.$$

Proof

The theorem follows easily by the previous Lemma and the fact that ACA^-_n and $\widehat{ID}_n(acc)$ differ only in one axiom schema. $\Box$

The next lemma gives the next link to the end-result.

Lemma 6.2.9 *There is a polynomial in natural coefficients $P[X]$ such that for any formula $\varphi \in \mathcal{L}_{PA}$*

$$ACA^-_n \vdash^m_g \varphi \Rightarrow PA \vdash^{P[m]}_g \varphi.$$

Proof

To prove the statement we use the substitution defined in the proof of Theorem 3.3.14. However we do not proceed as in that proof, because the proof of Chapter 3 uses cut-elimination which has super-exponential length increment. Instead we proceed inductively on the length of the deduction. Since the logic and most of the axioms are the same, the only interesting case is the basis case of a comprehension axiom

$$(\forall y < n)(\forall x)[H^{\varphi}_{y}(x) \leftrightarrow \varphi(H^{\varphi}_{<y}, x, y)],$$

which is equivalent to

$$\bigwedge_{k=0}^{n-1} (\forall x)[H^{\varphi}_{k}(x) \leftrightarrow \varphi(H^{\varphi}_{<k}, x, \bar{k})]$$

over basic arithmetic. However that might cost a deduction-length increment of $c \cdot n$ and an increment of formula-length of $m \cdot n$.

Then we can use $\mathfrak{s}$ from the proof of Theorem 3.3.14, which trivialises these formulas to tautologies and adds $c \cdot n$ many lines to the deduction in order to deduce them. However the formula-length increment after applying $\mathfrak{s}$ to one of the sides of the equivalence is $(k+1)! \cdot (|\varphi| \cdot 3)^{k+1}$, which we will prove next.[9] However this is still only a polynomial formula-length increment. For $(n+1)!$ is only a constant for a particular theory ACA^{-}_{n} and $k < n$. We prove the bound by induction on k.

Induction Base: The formula $\mathfrak{s}(H^{\varphi}_{0}(t))$ is $\varphi([\bar{0} = \bar{1}], x, \bar{0})$. Which gives

$$|\mathfrak{s}(H^{\varphi}_{0}(t))| \quad \leq \quad |\varphi| \cdot |\bar{0} = \bar{1}| \quad \leq \quad 1! \cdot (|\varphi| \cdot 3)^{1}.$$

Since $\mathfrak{s}(H^{\varphi}_{0}(t)) \equiv \mathfrak{s}(\varphi(H^{\varphi}_{<0}, x, \bar{0}))$, we are done with the base case.

Induction Step: As before we note that $\mathfrak{s}(H^{\varphi}_{k+1}(t))$ and $\mathfrak{s}(\varphi(H^{\varphi}_{<k+1}, x, \bar{0}))$ give the same formula. Therefore it is enough to take a closer look at only one of these formulas.

$$\begin{aligned} \mathfrak{s}(H^{\varphi}_{k+1}(t)) &\equiv \varphi([\mathfrak{s}(\mathrm{s}(H^{\varphi}_{<k+1}(t)))], x, \bar{k}) \\ &\equiv \varphi([\mathfrak{s}(H^{\varphi}_{0}(t) \vee ... \vee H^{\varphi}_{k}(t))], x, \bar{k}) \\ &\equiv \varphi([\mathfrak{s}(H^{\varphi}_{0}(t)) \vee ... \vee \mathfrak{s}(H^{\varphi}_{k}(t))], x, \bar{k}) \end{aligned}$$

[9] The function $n!$ is defined by $1! := 1$ and $(n+1)! := (n+1) \cdot n!$.

And for this formula we can see that

$$\begin{aligned} |\varphi([\mathfrak{s}(H_0^\varphi(t)) \vee ... \vee \mathfrak{s}(H_k^\varphi(t))], x, \bar{k})| &\leq |\varphi| \cdot k \cdot (k+1)! \cdot (|\varphi| \cdot 3)^{k+1} \\ &\leq (k+2)!(|\varphi| \cdot 3)^{k+2}. \end{aligned}$$

□

The last step is to translate PA into HA.

Lemma 6.2.10 *There is a polynomial in natural coefficients $P[X]$ such that for any Π^0_2-sentence φ*

$$PA \vdash^m_g \varphi \Rightarrow HA \vdash^{P[m]}_g \varphi$$

Proof

See the trick Friedman uses in [21], which is schematic in the formula that is in use, and can therefore be constructed uniformly. Since the proof relies on a logical trick, the deductions can be bounded by a polynomial. □

6.3 Sticking together and Conclusion

Sticking together Theorem 6.2.8, Lemma 6.2.10 and Lemma 6.2.9 gives the final result of this chapter.

Theorem 6.3.1 *There is a polynomial in natural coefficients $P[X]$ such that for any Π^0_2-sentence φ*

$$\widehat{ID}^i_n(acc) \vdash^m_g \varphi \Rightarrow HA \vdash^{P[2^{(2g+6)m}]}_g \varphi$$

Obviously this result confirms the claim from the beginning of this chapter that the proof-theoretic reduction gives at most an elementary speed-up for $\widehat{ID}^i_n(acc)$ over PA and HA. But in order to be useful for our aims we have to extended this result to cases where axioms for transfinite induction are present. However Theorem 6.3.1 can be easily extended to the cases with axioms for transfinite induction, because the proof of Lemma 3.4.2 uses the same constructions as the proof of Theorem 2.2.19 does and gives, therefore, also only a polynomial complexity increment; and Theorem 3.4.1 can be complexity-analysed in the same way as Theorem 3.3.1 was. For, following the proof of Theorem 3.4.1, in the realisability-step it makes no difference for an appeal to Theorem 6.1.6 what logical form a realisable axiom has. Also in the case of a step that uses one of Rüede's and Strahm's translations a one line deduction for a $TI(\prec \alpha)$ axiom in one language is translated into a one line deduction of a $TI(\prec \alpha)$ axiom in another language. Moreover Lemma 6.2.10 can be easily extended to the cases with transfinite induction so that they only give a polynomial deduction-complexity increment (as can be easily seen from [21]). Hence, using Theorem 6.3.1, we get the following theorem.

Theorem 6.3.2 *Assume that $\prec$ is a primitive recursive relation whose characteristic function is in $\mathcal{L}_{PA}$. Also assume that α is an element in the domain of $\prec$. Then there is a polynomial in natural coefficients $P[X]$ such that for any Π^0_2-sentence φ*

$$\widehat{ID}^i_n(acc) + TI(\prec \alpha) \vdash^m_g \varphi \Rightarrow HA + TI(\prec \alpha) \vdash^{P[2^{(2g+6)m}]} \varphi.$$

Appendices

A Gentzen's Method can be done Elementarily

As explained in Chapter 3, ordinal-analysis usually uses infinitary systems to establish cut-elimination. As Mint proved in [33], this cut-elimination can be done elementary-recursively. This raises the question whether those part of Gentzen's method that do not use the well-foundness of the ordinal-notation system $OT(\varepsilon_0)$ can be done elementarily as well. Or in other words whether Gentzen's method, which is usually viewed as establishing

$$PRA + PRWO(\prec_{\varepsilon_0}) \vdash \text{Con}(PA),$$

can also be used to prove

$$EA + ERWO(\prec_{\varepsilon_0}) \vdash \text{Con}(PA).$$

The answer to this questions is yes and we will show this in the present appendix.[10] A similar result was given by Hajek and Pudlak in their book [26, pp. 373-375]. However the proof of the result is less explicit by its shorter presentation.

[10]The schema $PRWO(\prec_{\varepsilon_0})$ is the set of all formulas of the form

$$(\forall \vec{x})(\exists y)[f(\vec{x}, y) \preceq_{\varepsilon_0} f(\vec{x}, y+1) \vee \text{“}f(\vec{x}, y) \text{ is not an element of } OT(\varepsilon_0)\text{''}],$$

where f is a function symbol of $\mathcal{L}_{PRA}$. $ERWO(\prec_{\varepsilon_0})$ is similarly defined as $PRWO(\prec_{\varepsilon_0})$ but is formulated with the defining formulas of elementary recursive functions.

The following presentation of Gentzen's method follows almost entirely that given in [57]. The crucial point in the reduction method that is given there is that, if one tags the proofs with ordinals in a certain way, the output of the reduction has a smaller ordinal than the input had. But, since we are not interested in the consistency proof itself, we skip the ordinal part entirely.[11] This has the disadvantage that we cannot emphasise why the reduction method is useful. Moreover we are not able to formulate the statement about the reduction as it is formulated in [57]. Instead we refer to the indicated lemma in [57] and define the reduction algorithms in the actual proof of its elementariness. This saves us from writing the definition of the reduction twice.

The consistence proof of PA operates on a sequent calculus (with a division of bounded and free variables as two separate sets of symbols) in the ordinary language of arithmetic with rules for all propositional connectives and both quantifiers (see [57, p. 9]). The meta-logical vocabulary for sequent calculi, which includes primary and auxiliary formulas, is defined as usual (see [57]). The logical axioms are of the form $A \Rightarrow A$ for atomic A and the mathematical axioms include the usual ones for equality as well as the following:

$$S(t_1) = S(t_2) \Rightarrow t_1 = t_2$$

$$S(t) = 0 \Rightarrow$$

$$\Rightarrow t + 0 = t$$

$$\Rightarrow t_1 + S(t_2) = S(t_1 + t_2)$$

$$\Rightarrow t \cdot 0 = 0$$

$$\Rightarrow t_1 \cdot S(t_2) = (t_1 \cdot t_2) + t_1.$$

The crucial point is the reformulation of the induction axiom as a rule:

$$\frac{\varphi(a), \Gamma \Rightarrow \Delta, \varphi(S(a))}{\varphi(0), \Gamma \Rightarrow \Delta, \varphi(t)}\ \text{IND}$$

[11] We can do that, because it is folklore that the standard ordinal notation systhem $OT(\varepsilon_0)$ is elementarily definable.

where t is an arbitrary term and a a free variable.

Note: All axioms are atoms, since induction is formulated as a rule.

In order to define the reduction method we have to give a bunch of definitions. It should be clear that these notions can be defined elementarily by using some sort of standard coding (like the one described later in this appendix).

Definition A.1 *1. Fix a deduction D. The successor of a formula φ is defined as follows:*

(a) If φ is a cut formula, then φ has no successor.

(b) If φ is an auxiliary formula of an deduction rule other than a cut or exchange, then the principal formula is the successor of φ.

(c) If φ is an auxiliary formula of exchange then in the lower sequent of exchange φ is the successor of φ.

(d) If φ is the k-th formula of Γ, Π, Δ or Λ in the upper sequence, then the k-th formula of Γ, Π, Δ or Λ in the lower sequent is the successor of φ.

2. A sequence of sequents will be called a thread in a deduction D if the following properties are satisfied:

(a) The sequence begin with an (logical or mathematical) axiom and ends with the end-sequent of D.

(b) Every sequent in the sequence except the last is an upper sequent of an inference, and is immediately followed by the lower sequent of this inference.

3. Assume that $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_3$ are sequents in a deduction D. The sequent $\mathcal{S}_1$ is above $\mathcal{S}_2$ (or $\mathcal{S}_2$ is below $\mathcal{S}_1$) iff there is a thread containing $\mathcal{S}_1$ and $\mathcal{S}_2$ and $\mathcal{S}_1$ appears *before $\mathcal{S}_2$. The sequent $\mathcal{S}_3$ is between $\mathcal{S}_1$ and $\mathcal{S}_2$ iff $\mathcal{S}_1$ is above $\mathcal{S}_3$ and $\mathcal{S}_2$ is below $\mathcal{S}_3$.*

4. *An inference is below a sequent $\mathcal{S}$ iff the lower sequent of the inference is below $\mathcal{S}$.*

5. *Let D be a deduction and $\mathcal{S}$ a sequent in D. We call D' the subdeduction of S in D iff D' is a deduction and contains exactly those sequents, which appear, in every thread of D, above $\mathcal{S}$.*

6. *A formula φ is called an axiom-formula or end-formula of D iff φ is contained in an axiom or the end-sequent of D.*

7. *A sequence of formulas is called a bundle iff it satisfies the following conditions:*

   (a) *The sequence begins with an axiom-formula or weakening-formula.*

   (b) *The sequence ends with an end-formula or cut-formula.*

   (c) *Every formula except the last in the sequence is immediately followed by its successor.*

8. *Assume that φ and ψ are formulas. The formula φ is called the ancestor of ψ and ψ is called the descendant of φ iff there is a bundle containing φ and ψ in which φ appears before ψ.*

9. *The notation of implicit and explicit:*

   (a) *A bundle is called explicit iff the the last formula in the bundle is an end-formula.*

   (b) *A bundle is called implicit iff the the last formula in the bundle is an cut-formula.*

   (c) *A formula in a deduction is called implicit or explicit iff the bundle which contains the formula is implicit or explicit.*

(d) *A sequent in a deduction is called implicit or explicit iff the sequent contains a formula which is implicit or explicit.*

(e) *A logical inference is called implicit or explicit iff the principal formula of the logical inference is implicit or explicit.*

10. *A part E of a deduction D is called the end-piece of D if it satisfied the following properties:*

    (a) *The end-sequent is in E.*

    (b) *The upper sequent of an inference other than an implicit logical inference is contained in E iff the lower sequent of the inference is in E.*

    (c) *The upper sequent of an implicit logical inference is not in E.*

    Or for short: An sequent in a deduction is in the end-piece iff there is no implicit logical inference below this sequent.

11. *An inference I is in the end-piece of a deduction iff the lower sequent of I is in the end-piece.*

12. *An inference I is a boundary iff the lower sequent of I is in the end-piece and the upper is not.*

13. *A cut in the end-piece is called suitable iff each cut formula of the cut has an ancestor which is the principal formula of a boundary.*

Definition A.2 *A deduction D is called regular iff D satisfies:*

1. *All eigenvariables in D are distinct.*

2. *If a is a free variable which occurs as a eigenvariable in a sequent $\mathcal{S}$ of D, then a occurs just in sequents above $\mathcal{S}$.*

In the following we define a coding and prove some facts about it. We have to scrutinise a particular coding, since the main result will be proved by analysing the numerical growth of codes under syntactical operations.

According to this task, we choose to code numbers and sequences by taking a detour over firstly translating these into finite 0-1-strings as in [9]. Consequently we can take the corresponding natural number of a basis-2-representation number-systhem as the code of this string. In the following we sketch how this idea proceeds. The details that are not mentioned here can be found in [9, pp. 89-94].

We start by defining functions, which allow us to deal with 0-1-strings in number theory:

$$|x| := \lceil \log_2(x+1) \rceil$$

$$\text{LenBit}(y, x) := \lfloor x/y \rfloor \text{mod } 2$$

$$\text{Bit}(i, x) := \text{LenBit}(2^i, x).$$

Here $|x|$ serves as a length function for the binary representation of a natural number, since it gives the least n such that 2^n is bigger than x. The function Bit(i, x) serves as a decoding function. Because, in the case where $y = 2^i$ is satisfied, LenBit(y, x) gives the $i-$th bit in the binary expansion of x. Here the least significant bit is by convention the zeroth bit. Consequently Bit(i, x) gives the $i-$th bit of the 0-1-string that is coded by x. Also LenBit(y, x) and Bit(i, x) are polynomial bounded.

To code sequences of natural numbers, we identify their entries with their binary representation. Consequently sequences can be viewed as strings of the symbols "0", "1" and ",". To code those we translate them to a full 0-1-string by the following translation rules:

$$, \mapsto 01$$

$$0 \mapsto 10$$

$$1 \mapsto 11.$$

For example the sequence $\langle 3, 0, 4\rangle$ is viewed as $, 11, , 100$ (we add a comma at the beginning to separate numbers form sequences). Hence, using the translation rules, we get 0111110101111010, which represents the number 32122. Consequently 32122 is the code of $\langle 3, 0, 4\rangle$. We use $\ulcorner\langle a_0, ..., a_n\rangle\urcorner$ to denote the code of $\langle a_0, ..., a_n\rangle$.

Using the functions that are defined so far, we can define the usual properties and, using a trick, also a decoding and length function

$$\beta(i, \langle a_0, ..., a_n\rangle) = a_i \text{ for } i \leq n$$

$$\mathrm{Lg}(\langle a_0, ..., a_n\rangle) = n + 1,$$

which are polynomial bounded. But, since we are primarily interested in elementary functions and substitution is necessarily exponential (as Proposition A.4 will show), we skip the definitions and refer to [9] instead.

The next proposition gives the crucial fact of our coding.

Theorem A.3 *For every sequence $\langle a_0, ..., a_n\rangle$ it holds that*

$$\ulcorner\langle a_0, ..., a_n\rangle\urcorner \leq 2^{2\cdot(\Sigma_{i=0}^{n}|a_i|+n+1)}$$

and

$$\ulcorner\langle a_0, ..., a_n\rangle\urcorner \leq 2^{2n\cdot(\max(|a_i|)+2)}.$$

Proof

Take an arbitrary sequence $\bar{a} = \langle a_0, ..., a_n\rangle$. According to our coding the binary representation of $\bar{a}$ is build up by the binary representation of the a_i and $n + 1$ commas. By the translation rules for comma, 0 and 1 the length doubles. Consequently we get $|\ulcorner\langle a_0, ..., a_n\rangle\urcorner| = 2 \cdot (\Sigma_{i=0}^{n}|a_i| + n + 1)$. Since $n \leq 2^{|n|}$ for any $n \in \mathbb{N}$, the first result follows. The second statement is an easy consequence of the first. □

An interesting and (of course) well known result is that metamathematics cannot be done with a class of functions that is properly contained in the elementary functions, presupposing some natural closure properties.

Theorem A.4 *The function Sub($\langle a_0, ..., a_n\rangle, a, b$), which gives the sequence that is obtained from $\langle a_0, ..., a_n\rangle$ by substituting for every occurrence of a the number b, is elementary and grows super-polynomially.*

Proof

That the definition can be carried out primitive recursively with elementary functions is easy to see, but it is also quite long to write down. Elementarity and super-polynomiality can be shown in one step by showing that this application of primitive recursion can be bounded by an elementary function.

Take a sequence $\bar{a} = \langle a_0, ..., a_n\rangle$ and assume that it has m occurrences of a. By the definition of our coding it is easy to see that $|\bar{a}|$ changes to $|\bar{a}| - 2m|a| + 2m|b|$ after substituting b for a. Without loss of generality we can assume that $|\bar{a}| \leq |b|$. Consequently we get $|\text{Sub}(\bar{a}, a, b)| \leq |b|^2$. Finally, using Proposition A.3, we get $\text{Sub}(\bar{a}, a, b) \leq 2^{|\text{Sub}(\bar{a},a,b)|} \leq 2^{|b|^2}$. Since that is obviously the best one can do and $2^{|b|^2}$ is elementary, we are done.□

From [9] by some results of [50] it is easy to see that all the usual syntactical operations on proofs and formulae are elementary; we will therefore take these results as given in the following. The following theorem shows that the additional operations, that Takeuti presents in [57], do not lead out of the realm of elementary functions. To show this we will use an argument that is similar to the proof of Proposition A.4. It will give an elementary bound on Takeuti's transformations, which is formulated with elementary functions. Since the functions used are elementary and it can be seen that the methods, which are described in the proof, can be defined by a single application of primitive recursion, the bound shows that the transformation is elementary according to Definition 2.1.1. Moreover Proposition A.4 shows that this is the best possible result according to the classes that are usually used in subrecursive hierarchy theory.

Theorem A.5 *The function Φ defined in the proof of [57, Lemma 12.8, p. 105] can be carried out elementarily.*

Proof

In the following we will proceed as in the proof of [57, Lemma 12.8, p. 105]. We iteratively define Φ on the codes of proofs. But instead of giving an analysis of how Φ lowers the ordinal rank, we estimate the numerical output of Φ. This will show that the function is dominated by an elementary function. Since Φ can be defined via elementary functions by primitive recursion, it follows from Definition 2.1.1 that Φ is elementary. For this purpose we code a proof as a sequence of sequents. Consequently the length of a deduction D, which is denoted by $\mathrm{Lg}(D)$, can be defined as the length of a sequence as was mentioned above. The rank of a formula, which is denoted by $\mathrm{rk}(\varphi)$, is the number of logical symbols in φ.

Φ operates on deductions D of a contradiction (denoted by $D \vdash\Rightarrow$) and gives a new deduction D' of a contradiction; we define Φ which outputs 0 for all the other deductions. Since being a deduction of a contradiction is an elementary predicate, we do not leave the realm of elementary functions.

Assume that D is a deduction of a contradiction. We can also assume that D is regular, since making a proof regular is an elementary operation by Proposition A.4. In the following we describe the function iteratively. Here it is important to note that Φ includes a case distinction of four cases. These four cases are described in step 2 to 5, while step 1 is only a preparation for step 2. We call the functions into which Φ is divided by the case distinction Φ_1, Φ_2, Φ_3 and Φ_4. If one of the Φ_i is divided by a case distinction as well, then we emphasis the distinction by an additional index, e.g. we use $\Phi_{3,1}$.

Step 1: Assume that D contains a free variable a that is not used as an eigenvariable. Replace a by the constant 0. This step applies as many times as possible.
Arbitrarily iterative substitution is not elementary; but here the length does not change anyway. Because 0 is a single symbol.

Step 2:(Φ_1) Assume that the end-piece of D contains some inferences that are applications of IND. Assume that the lowest one is $\mathcal{I}$ and that it is of the form

$$\dfrac{\begin{array}{c}\vdots\\ \varphi(a), \Gamma \Rightarrow \Delta, \varphi(S(a))\end{array}}{\begin{array}{c}\varphi(0), \Gamma \Rightarrow \Delta, \varphi(t)\\ \vdots\\ \Rightarrow\end{array}}\ \text{IND} \qquad .$$

We denote the part of the subproof above $\mathcal{I}$ by $P(a)$. When step 1 has been performed, t is closed. Assume that the evaluation of t is n.[12] Next we replace the subdeduction $P(a)$ by another deduction P' for $\varphi(0), \Gamma \Rightarrow \Delta, \varphi(t)$. We construct deductions P'_n inductively (on n) of $\varphi(0), \Gamma \Rightarrow \Delta, \varphi(n)$.
$\underline{n = 0}$: Here we replace $P(a)$ by a deduction for $\varphi(0), \Gamma \Rightarrow \Delta, \varphi(0)$. It is easy to see that this can be deduced by a purely logical deduction of length $\leq 4 \cdot \text{rk}(\varphi(0))^2 + \text{Lg}(\Gamma) + \text{Lg}(\Delta)$. Hence the deduction P'_0 satisfies

$$\text{Lg}(P'_0) \leq 4 \cdot \text{rk}(\varphi(0))^2 + \text{Lg}(\Gamma) + \text{Lg}(\Delta) + \text{Lg}(D) - \text{Lg}(P(a)).$$

We call this bound p_0.
$\underline{n \to n+1}$: Assume that we already have a deduction P'_n. Then we can combine it with $P(n)$ by a cut in order to get P'_{n+1} as follows:

$$\dfrac{\begin{array}{c}P'_n\\ \vdots\\ \varphi(0), \Gamma \Rightarrow \Delta, \varphi(n)\end{array} \quad \begin{array}{c}P(n)\\ \vdots\\ \varphi(n), \Gamma \Rightarrow \Delta, \varphi(S(n))\end{array}}{\varphi(0), \Gamma \Rightarrow \Delta, \varphi(S(n))}\ Cut \, .$$

[12]Note that t is formulated in $\mathcal{L}_Q$. Hence the evaluation function can be chosen as elementary.

An easy induction ensures the following:

$$\mathrm{Lg}(P'_n) \le \max\{(\mathrm{Lg}(P(0)) + n) \cdot n, p_0\}.$$

In order to get the conclusion of $\mathcal{I}$ one has to deduce $\varphi(n), \Gamma \Rightarrow \Delta, \varphi(t)$. It can be easily seen that one can do that by a deduction of length $\le 4 \cdot \mathrm{rk}(\varphi(n))^2 + \mathrm{Lg}(\Gamma) + \mathrm{Lg}(\Delta) + f(\ulcorner t \urcorner)$ (which we call p_t) for some uninteresting elementary function f. Hence, combining these deductions with P'_n as follows

$$\dfrac{\begin{matrix} P'_n \\ \vdots \\ \varphi(0), \Gamma \Rightarrow \Delta, \varphi(n) \end{matrix} \quad \begin{matrix} \vdots \\ \varphi(n), \Gamma \Rightarrow \Delta, \varphi(t) \end{matrix}}{\varphi(0), \Gamma \Rightarrow \Delta, \varphi(t)}\ Cut\ .$$

one gets the conclusion of $\mathcal{I}$. We call this deduction P'. Substituting P' for $P(a)$ and $\mathcal{I}$ into D, one gets the required D'. Since $\mathrm{Lg}(P(a)) \le \mathrm{Lg}(D)$, $p_0 \le p_t$ and because of the fact that below $\varphi(0), \Gamma \Rightarrow \Delta, \varphi(t)$ the deductions D' and D are identical, we get

$$\mathrm{Lg}(D') \le \max\{(\mathrm{Lg}(D) + 1 + n) \cdot n, p_t\},$$

where n is the evaluation of t from the original application of IND.
Since no sequent in D' has a code that is numerically bigger than the biggest in D, an application of Proposition A.3 gives

$$\ulcorner D' \urcorner = \Phi_1(\ulcorner D \urcorner) \le 2^{2 \cdot \max\{(d+1+n) \cdot n, p_t\} \cdot (d_{\max}+2)}$$

where $d = \mathrm{Lg}(D)$, $d_{\max}$ is the code of the maximal sequent in D and n, t are as before.

From now on we assume that there is no IND in D.

Step 3:(Φ_2) Assume that the end-piece of D contains a logical axiom, say $\varphi \Rightarrow \varphi$. Since $D \vdash \Rightarrow$ and there are no logical inferences in an end-piece, φ has to be cut out on both sides.

Case 1: The φ from the antecedent is cut out first.

Consider the subdeduction D_0 that ends with the cut

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

where φ (from the succedent) is in Λ. So we can deduce $\Gamma, \Pi \Rightarrow \Delta, \Lambda$ (we call it $\mathcal{S}$) from $\Gamma \Rightarrow \Delta, \varphi$, using weakening and exchange, and get a new deduction for $\mathcal{S}$ that is called D_0'. Replacing D_0 by D_0' in D, leads to a deduction D'.

Note that the sequent with the biggest code in D, say d_{max}, also dominates all the codes of sequents in D'. Consequently we get

$$\ulcorner D' \urcorner = \Phi_{2,1}(\ulcorner D \urcorner) \leq 2^{2 \cdot (\mathrm{Lg}(D) + \mathrm{Lg}(\Pi) + \mathrm{Lg}(\Lambda)) \cdot (d_{\max} + 2)}$$

by Proposition A.3.

Case 2: In the case where φ from the succedent is cut out first the argument is analogous.

Therefore we can assume that there is no application of IND and no logical axiom in the end-piece of D.

Step 4:(Φ_3) Assume that there is a weakening in the end-piece and let $\mathcal{R}$ be the lowest of these. Since $D \vdash \Rightarrow$, there must be a cut $\mathcal{C}$ below $\mathcal{R}$ such that the principal formula of $\mathcal{R}$ is the cut formula of $\mathcal{C}$, because the descendent of a formula has to be the formula itself in the end-piece of any deduction that ends in the empty sequence. So a part of D has the form:

$$\begin{array}{c} \dfrac{\Pi' \Rightarrow \Lambda'}{\varphi, \Pi' \Rightarrow \Lambda'}\ \mathcal{R} \\ \vdots \\ \dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\ \mathcal{C} \end{array} .$$

Case 1: If no contraction is applied to φ between $\mathcal{R}$ and $\mathcal{C}$, then reduce D to D' by replacing the considered part of D by the following:

$$\dfrac{\dfrac{\begin{array}{c}\Pi' \Rightarrow \Lambda' \\ \vdots \\ \Pi \Rightarrow \Lambda\end{array}}{\text{weakenings and exchanges}}}{\Gamma, \Pi \Rightarrow \Delta, \Lambda} \quad .$$

Since the part from $\Pi' \Rightarrow \Lambda'$ to $\Pi \Rightarrow \Lambda$ is as long as the corresponding part in D and the number of weakenings is bounded by the number of added formulas, we get

$$\ulcorner D' \urcorner = \Phi_{3,1}(\ulcorner D \urcorner) \leq 2^{2 \cdot (\mathrm{Lg}(D) + \mathrm{Lg}(\Gamma) + \mathrm{Lg}(\Delta)) \cdot (d_{\max}+2)},$$

where $d_{\max}$ is again the code of the sequent with the biggest code. Note that the biggest sequent of D and D' is the same object.

Case 2: Assume that case 1 does not hold. Then we consider the uppermost application of contraction in the range between $\mathcal{R}$ and $\mathcal{C}$ and replace it as described below to get a deduction D' from D.

$$\begin{array}{cc}
D & D' \\
\dfrac{\Pi' \Rightarrow \Lambda'}{\varphi, \Pi' \Rightarrow \Lambda'} & \Pi' \Rightarrow \Lambda' \\
\vdots & \\
\dfrac{\varphi, \varphi, \Pi'' \Rightarrow \Lambda''}{\varphi, \Pi'' \Rightarrow \Lambda''} & \begin{array}{c}\vdots \\ \varphi, \Pi'' \Rightarrow \Lambda''\end{array} \\
\vdots & \vdots \\
\varphi, \Pi \Rightarrow \Lambda & \varphi, \Pi \Rightarrow \Lambda
\end{array}$$

Obviously

$$\ulcorner D' \urcorner = \Phi_{3,2}(\ulcorner D \urcorner) \leq 2^{2 \cdot \mathrm{Lg}(D) \cdot (d_{\max}+2)},$$

where $d_{\max}$ is the sequent with the biggest code in D.

From now on we assume that there is no application of weakening in the end-piece of D.

Step 5:(Φ_4) Assume that there is a suitable cut in the end-piece of D. In fact one can prove that there is allays one, which shows that the case distinction is complete, but we do not pay attention to this here. The lowermost of these cuts, called $\mathcal{C}$, will be reduced.
Case 1: The cut formula of $\mathcal{C}$ has the form $\phi \land \psi$, so D has the form

$$
\begin{array}{c}
\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \Gamma' \Rightarrow \Theta', \varphi\end{array} \quad \begin{array}{c}\vdots\\ \Gamma' \Rightarrow \Theta', \psi\end{array}}{\Gamma' \Rightarrow \Theta', \varphi \land \psi}\,\mathcal{R}_1 \\ \vdots \\ \Gamma \Rightarrow \Theta, \varphi \land \psi\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \varphi, \Pi' \Rightarrow \Lambda'\end{array}}{\varphi \land \psi, \Pi' \Rightarrow \Lambda'}\,\mathcal{R}_2 \\ \vdots \\ \varphi \land \psi, \Pi \Rightarrow \Lambda\end{array}}{\Gamma, \Pi \Rightarrow \Theta, \Lambda}\,\mathcal{C} \\
\vdots \\
\Delta \Rightarrow \Xi \\
\vdots \\
\Rightarrow
\end{array}
$$

where $\Delta \Rightarrow \Xi$ is a sequent whose ordinal-tag satisfies some properties. Since there is an elementary ε_0 representation, $\Delta \Rightarrow \Xi$ can be found elementarily and this search does not affect the argument.

Consider the following deductions:
D_1':

$$
\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}\vdots \\ \Gamma' \Rightarrow \Theta', \varphi \\ \hline\hline \Gamma' \Rightarrow \varphi, \Theta'\end{array}}{\Gamma' \Rightarrow \varphi, \Theta', \varphi \land \psi}\,\text{weakening} \qquad}{} \\
\dfrac{\begin{array}{c}\vdots \\ \Gamma \Rightarrow \varphi, \Theta, \varphi \land \psi\end{array} \qquad \begin{array}{c}\vdots \\ \varphi \land \psi, \Pi \Rightarrow \Lambda\end{array}}{\Gamma, \Pi \Rightarrow \varphi, \Theta, \Lambda}\,\mathcal{R}_{3_1} \\
\vdots \\
\Delta \Rightarrow \varphi, \Xi \\ \hline\hline
\Delta \Rightarrow \Xi, \varphi
\end{array}
$$

D'_2:

$$
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Theta, \varphi \wedge \psi\end{array} \qquad \begin{array}{c} \dfrac{\genfrac{}{}{1.5pt}{}{\begin{array}{c}\vdots\\ \varphi, \Pi' \Rightarrow \Lambda'\end{array}}{\Pi', \varphi \Rightarrow \Lambda'}}{\varphi \wedge \psi, \Pi', \varphi \Rightarrow \Lambda'}\ \text{weakening}\\ \vdots\\ \varphi \wedge \psi, \Pi, \varphi \Rightarrow \Lambda\end{array}}{\Gamma, \Pi, \varphi \Rightarrow \Theta, \Lambda}\ \mathcal{R}_{3_2}\\
\vdots\\
\genfrac{}{}{1.5pt}{}{\Delta, \varphi \Rightarrow \Xi}{\varphi, \Delta \Rightarrow \Xi}
\end{array}
$$

Now D' is constructed from D'_1 and D'_2 by sticking them together using a cut, and this follows from $\Delta \Rightarrow \Xi$ as in $\hat{D}$.

$$
\begin{array}{c}
\dfrac{\dfrac{D'_1}{\Delta \Rightarrow \Xi, \varphi} \quad \dfrac{D'_2}{\varphi, \Delta \Rightarrow \Xi}}{\Delta \Rightarrow \Xi}\\
\vdots\\
\Rightarrow
\end{array} .
$$

To bound this function we denote the subproof of D that proves $\Delta \Rightarrow \Xi$ by P and the one ending in $\mathcal{R}_1$ by R.

Note that the subproof of D'_1 that ends with the new weakening has length $\leq \mathrm{Lg}(R) + \mathrm{Lg}(\Theta')$. Consequently

$$\mathrm{Lg}(D'_1) \leq \mathrm{Lg}(P) + \mathrm{Lg}(\Theta') + \mathrm{Lg}(\Xi).$$

Analogously

$$\mathrm{Lg}(D'_2) \leq \mathrm{Lg}(P) + \mathrm{Lg}(\Theta') + \mathrm{Lg}(\Delta).$$

Since $\mathrm{Lg}(P) \leq \mathrm{Lg}(D)$ and the new cut introduces 2 lines we get

$$\mathrm{Lg}(D') \leq 2 \cdot \mathrm{Lg}(D) + \mathrm{Lg}(\Theta') + \mathrm{Lg}(\Delta) + \mathrm{Lg}(\Xi) + 2.$$

In the following we will call this bound $h(D)$.

Unlike the cases so far, the codes of the sequents in D' are in general not bounded by the biggest of D (denoted by $d_{\max}$). Since the only formulas that are added to the sequents are φ, $\varphi \wedge \psi$ and concatenations of (old) formulas with these two, we get that all sequents in D' are bounded by $d_{\max} * \ulcorner \varphi \urcorner * \ulcorner \varphi \wedge \psi \urcorner$. The $*$ means concatenation, which is an elementary function as well.

Applying Proposition A.3 as before we get

$$\ulcorner D' \urcorner = \Phi_{4,1}(\ulcorner D \urcorner) \leq 2^{2 \cdot h(D) \cdot (d_{\max} * \ulcorner \varphi \urcorner * \ulcorner \varphi \wedge \psi \urcorner + 2)}.$$

Case 2: The cut formula of $\mathcal{C}$ has the form $\forall x \varphi(x)$, so D has the form

$$\dfrac{\begin{array}{c} \vdots \\ \dfrac{\Gamma' \Rightarrow \Theta', \varphi(a)}{\Gamma' \Rightarrow \Theta', \forall x \varphi(x)}\ \mathcal{R}_1 \\ \vdots \\ \Gamma \Rightarrow \Theta, \forall x \varphi(x) \end{array} \qquad \begin{array}{c} \vdots \\ \dfrac{\varphi(s), \Pi' \Rightarrow \Lambda'}{\forall x \varphi(x), \Pi' \Rightarrow \Lambda'}\ \mathcal{R}_2 \\ \vdots \\ \forall x \varphi(x), \Pi \Rightarrow \Lambda \end{array}}{\begin{array}{c} \Gamma, \Pi \Rightarrow \Theta, \Lambda \\ \vdots \\ \Delta \Rightarrow \Xi \\ \vdots \\ \Rightarrow \end{array}}\ \mathcal{C}$$

As in case 1 D' is constructed from two deductions D'_1 and D'_2 defined as follows.

D'_1 :

$$\dfrac{\begin{array}{c} \vdots \\ \dfrac{\dfrac{\Gamma' \Rightarrow \Theta', \varphi(s)}{\Gamma' \Rightarrow \varphi(s), \Theta'}}{\Gamma' \Rightarrow \varphi(s), \Theta', \forall x \varphi(x)}\ \text{weakening} \\ \vdots \\ \Gamma \Rightarrow \varphi(s), \Theta, \forall x \varphi(x) \end{array} \qquad \begin{array}{c} \vdots \\ \forall x \varphi(x), \Pi \Rightarrow \Lambda \end{array}}{\begin{array}{c} \Gamma, \Pi \Rightarrow \varphi(s), \Theta, \Lambda \\ \vdots \\ \dfrac{\Delta \Rightarrow \varphi(s), \Xi}{\Delta \Rightarrow \Xi, \varphi(s)} \end{array}}$$

where the deduction of $\Gamma' \Rightarrow \Theta', \varphi(s)$ comes from the deduction of $\Gamma' \Rightarrow \Theta', \varphi(a)$ by substitution.

D'_2 :

$$\begin{array}{c} \dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Theta, \forall x\varphi(x)\end{array} \qquad \begin{array}{c}\vdots\\ \dfrac{\dfrac{\varphi(s), \Pi' \Rightarrow \Lambda'}{\Pi', \varphi(s) \Rightarrow \Lambda'}}{\forall x\varphi(x), \Pi', \varphi(s) \Rightarrow \Lambda'}\ \text{weakening}\\ \vdots\\ \forall x\varphi(x), \Pi, \varphi(s) \Rightarrow \Lambda\end{array}}{\Gamma, \Pi, \varphi(s) \Rightarrow \Theta, \Lambda}\\ \vdots\\ \dfrac{\Delta, \varphi(s) \Rightarrow \Xi}{\varphi(s), \Delta \Rightarrow \Xi} \end{array}$$

From these two deductions, D' is defined as follows:

$$\begin{array}{c} \dfrac{\dfrac{D'_1}{\Delta \Rightarrow \Xi, \varphi(s)} \quad \dfrac{D'_2}{\varphi(s), \Delta \Rightarrow \Xi}}{\Delta \Rightarrow \Xi}\\ \vdots\\ \Rightarrow \end{array} .$$

The boundary can be found analogously to case 1.

For the rest of the cases the proof is very similar to the two which have been given.

Since any Φ_i for $1 \leq i \leq 4$, which build the case distinction of Φ, can be bounded by an elementary function, Φ is bounded by the sum of these bounds. Consequently $\Phi \in \mathcal{ERF}$ by Definition 2.1.1. $\Box$

B The Proof-Theoretic Ordinal of EA

The theory EA comprises the following axioms

1. $(\forall x)[S(x) \neq 0]$
2. $(\forall x, y)[S(x) = S(y) \to x = y]$
3. $(\forall x)[x + 0 = x]$
4. $(\forall x, y)[x + S(y) = S(x + y)]$
5. $(\forall x)[x * 0 = 0]$
6. $(\forall x, y)[x * S(y) = (x * y) + x]$
7. $2^0 = S(0)$
8. $(\forall x)[2^{S(x)} = 2^x + 2^x]$
9. $(\forall x)[x \leq 0 \leftrightarrow x = 0]$
10. $(\forall x, y)[x \leq S(y) \leftrightarrow [x \leq y \vee x = S(y)]]$
11. together with
$$\varphi(0) \wedge (\forall x)[\varphi(x) \to \varphi(S(x))] \to (\forall x)\varphi(x),$$
for every $\varphi(x) \in \Delta_0^0$ (for $\mathcal{L}_{EA}$).

We analyse EA in accordance to the following definition of proof-theoretic ordinal.

Definition B.1 *Let T be a theory and $OT(\varepsilon_0)$ the natural ordinal notation system for ordinals smaller than ε_0 (see [57]). Assume that there are some ordinals $\alpha, \beta \in OT(\varepsilon_0)$ such that*

$$T \vdash ERWF(\alpha),$$

but

$$T \nvdash ERWF(\beta).$$

Then we call the $\prec_{\varepsilon_0}$-least of the β the bounded proof-theoretical ordinal of T and denote it by $\|T\|_{\min}$.
Here $ERWF(\alpha)$ is the set of all formulas of the form

$$(\forall \vec{x})(\exists y)[f(\vec{x}, y+1) \not\prec f(\vec{x}, y) \vee f(\vec{x}, y) \not\prec \alpha]$$

where f is an elementary recursive function.[13]

Remark B.2 *One might want to define $\|T\|_{\min}$ via $TI_{\Delta_0^0}(\alpha)$ instead. I was sure that this should be equivalent, because I expected that*

$$EA + \{TI_{\Delta_0^0}(\beta) | \beta \prec \alpha\} \equiv_{\Pi_2^0} EA + \{ERWF(\beta) | \beta \prec \alpha\}$$

and $TI_{\Delta_0^0}(\alpha)$ is a Π_2^0-sentence. However this is not the case, because Sommer proved in [54]

$$EA + TI_{\Delta_0^0}(\omega^2) \equiv I\Sigma_1.$$

But $I\Sigma_1$ proves $ERWF(\beta)$ for any $\beta < \omega^\omega$ (see [22]).

The proof-theoretic ordinal of EA will turn out to be ω^3 and the proof uses mainly recursion theoretic techniques; it only uses proof-theory through the back-door by referring to Theorem 2.1.9.[14] We will use results about the well known Grzegorczyk Hierarchy, which is defined by using the following functions.

[13] Our formulation of the schema is of course an abbreviation, since we do not have all elementary functions as function symbols in the language. $f(y) = \gamma$ is used as an abbreviation of the natural Σ_1^0-formula that represents the graph of the function f in $\mathcal{L}_{EA}$.

[14] In fact I tried to give an ordinal-analysis by proof-theoretic methods and failed, because logical-complexity is too rough a measure below $I\Sigma_1$.

Definition B.3 *We define a sequence of primitive recursive functions $\langle E_n \rangle_{n\in\mathbb{N}}$ as follows.*

$$\begin{aligned}
E_0(x,y) &:= x+y \\
E_1(x) &:= x^2+2 \\
E_{n+2}(0) &:= 2 \\
E_{n+2}(x+1) &:= E_{n+1}(E_{n+2}(x))
\end{aligned}$$

By using this function we can define the *Grzegorczyk Hierarchy*.

Definition B.4 *The set of primitive recursive functions $\mathcal{E}^0$ includes the successor function S, the constant zero function 0 and for any n and m the projection function p^n_m. In addition $\mathcal{E}^0$ is closed under composition and limited recursion.*
$\mathcal{E}^{n+1}$ is a superset of $\mathcal{E}^n$, includes E_n and is closed under composition and limited recursion.

In the following $\mathcal{ERF}$ and $\mathcal{PRF}$ are the sets of elementary and primitive recursive functions respectively. We can prove the following facts about the Grzegorczyk Hierarchy.

Theorem B.5

1. *For any $n \in \mathbb{N}$, $\mathcal{E}^n \subsetneq \mathcal{E}^{n+1}$.*
2. *$\mathcal{E}^3 = \mathcal{ERF}$*
3. *$\bigcup_{n\in\omega} \mathcal{E}^n = \mathcal{PRF}$*

Proof

1. Easy.
2. See [43, p. 33].
3. See [43, p. 35].

□

The following definition regulates the connections between these sets and an ordering.

Definition B.6 *A function θ is called a predecessor function for an ordering $\langle \mathbb{N}, \prec \rangle$, if it satisfies the following conditions:*

$$\theta(0) = 0$$

$$x > 0 \textit{ implies } \theta(x) \prec x.$$

We say that f is defined from g and h by recursion on $\prec$, if

$$f(\vec{x}, y) := \begin{cases} g(\vec{x}) & : \quad y = 0 \\ h(\vec{x}, y, f(\vec{x}, \theta(\vec{x}, y))) & : \quad y > 0 \end{cases}$$

The following theorem is the main recursion-theoretic result for the ordinal analysis which follows.

Theorem B.7 *Assume that $\langle \mathbb{N}, \prec \rangle$ is a well-founded ordering with order-type ω^n. Let f be a function that is definable by functions from $\mathcal{E}^n$ by recursion on $\prec$. Then $f \in \mathcal{E}^{n+1}$.*

Proof

See [43, p. 59, Theorem 1.2]. □

By these two theorems we can easily conclude the following corollary.

Corollary B.8

$$\|EA\|_{\min} \preceq \omega^3$$

Proof

To prove the corollary we have to ensure that

$$EA \nvdash ERWF(\omega^3).$$

We will proceed indirectly. So lets assume that

$$EA \vdash ERWF(\omega^3).$$

Theorem 2.1.9 and Theorem B.5 (2) ensure that we can define all functions of $\mathcal{E}^3$ in EA. Consequently, using $ERWF(\omega^3)$, we can deduce the totality of any function f that is definable by elementary functions along the well-ordering $\prec_{\epsilon_0} |_{\omega^3}$. But the order type of $\prec_{\epsilon_0} |_{\omega^3}$ is ω^3. Hence f is in $\mathcal{E}^4$ by Theorem B.7; which is a contradiction to Theorem 2.1.9. □

In order to give an ordinal-analysis it remains to prove that

$$\omega^3 \preceq \|EA\|_{\min}.$$

However we start by showing that $\omega^2 \prec \|EA\|_{\min}$.

Lemma B.9

$$EA \vdash ERWO(\omega^2)$$

Proof

Assume f is an elementary recursive function. Then the function

$$g(\vec{x}) := \mu y[f(\vec{x}, y+1) \not\prec f(\vec{x}, y)]$$

is defined in terms of elementary recursive functions. However the μ-operator is not elementary, but the bounded-μ-operator is. Hence, finding an elementary $t(\vec{x})$ such that

$$g(\vec{x}) := \mu y < t(\vec{x})[f(\vec{x}, y+1) \not\prec f(\vec{x}, y)],$$

we can prove that g is elementary and can proceed as follows. Since g is elementarily, its totality can be deduced in EA by Theorem 2.1.9. Consequently $ERWO(\omega^2)$ is witnessed by g; and we are done.

Hence we have to find such a $t(\vec{x})$. To keep the following readable, we skip the free variables. Note that all the ordinals below ω^2 have the form $\omega k_1 + k_2$ and are usually coded as (or at least their codes are elementary transformable to) $\langle k_1, k_2 \rangle$. Hence, assuming the range of f is bounded by ω^2, the descending sequence that is defined by f looks like this:

$$\omega k_{f(0)} + k'_{f(0)} \succeq \omega k_{f(1)} + k'_{f(1)} ... \succeq \omega k_{f(n)} + k'_{f(n)} ...$$

If $k_{f(n)}$ stays the same, then there are only $k'_{f(n)}$ many possibilities to be strictly smaller. However there are only $k_{f(0)}$ many possibilities for strictly smaller $k_{f(n)}$. Hence

$$y < \sum_{i=0}^{p_1(f(0))} (\sum_{j=0}^{p_2(f(i))} j).$$

Since elementary functions are closed under bounded sums, projection is elementary and f is elementary by assumption, we have found our t. $\square$

Theorem B.10 *For any* $n, m \in \mathbb{N}$,

$$EA \vdash ERWF(\omega^m) \to ERWF(\omega^m n).$$

Proof

We prove the theorem by metainduction on n.

$\underline{n = 1}$: The induction basis is equivalent to an instance of the tautology $\psi \to \psi$.

$\underline{n \Rightarrow n + 1}$: We work in EA and assume the two formulas

$$ERWF(\omega^m n)$$

$$ERWF(\omega^m).$$

The first gives a number z for counting down an f along $\omega^m n$ and the latter gives a number w for counting down ω^m. Note that ordinal addition is (or is elementarily recursively transformable to)

$$\alpha + \beta := \{\langle x, \gamma \rangle | [x = 0 \to \gamma \prec \alpha] \wedge [x = 1 \to \gamma \prec \beta]\},$$

when it is ordered lexicographically. Using the same line of argument as in Lemma B.9, we can therefore bound y in a

$$(\forall \vec{x})(\exists y)[f(\vec{x}, y+1) \not\prec f(\vec{x}, y) \vee f(\vec{x}, y) \not\prec \omega^m n + \omega^m]$$

by $z + w + 1$, which is an elementary bound. Moreover this is equivalent to $ERWF(\omega^m(n+1))$. Hence we are done.

□

This leads to the following corollary.

Corollary B.11

$$\omega^3 \preceq \|EA\|_{\min}$$

Hence the following theorem can be proved.

Theorem B.12

$$\|EA\|_{\min} = \omega^3$$

Proof

By Corollary B.11 and Corollary B.8, we get

$$\omega^3 \preceq \|EA\|_{\min} \preceq \omega^3.$$

□

Remark B.13 *One might suspect that Takeuti's note in [57, p.29] that cut-elimination can be proved by ω^2-induction together with Minc's elementary cut-elimination from [33] implies cut-elimination in EA. This is not the case. It is true that a deduction in LK with* *cut-rank r and deduction height h can be seen as being labelled by $\omega r + h$. However there is an important difference between formalised finite and infinite cut-elimination. In continuous cut-elimination one has to decorate the well-formed quasi-deductions by ordinal labels in order to formally prove that they are in fact deductions; one has to ensure that all their branches end in an axiom and not in an infinite chain of repetition rules. Whether labelling is done during the cut-elimination procedure or afterwards is inessential. The labelling starts by labelling the formal deduction by the estimated result, which is taken from the informal proof. For instance, in the ordinal analysis of PA (see Chapter 3), a cut-elimination starting from $\vdash_r^\alpha \varphi$ would assign to the end-formula of the resulting cut-free deduction the ordinal ω_r^α and then rebuild the deduction accordingly. This can be done elementarily because the operation on the ordinal notation system*

$$(\alpha, \beta) \mapsto \omega_\beta^\alpha$$

is elementary (in this ordinal notation system). However in the finite case, labelling $LK \vdash_r^h \Gamma \Rightarrow \Delta$ by $\omega r + h$, Minc's cut-elimination has to assign 2_r^h to the resulting cut-free deduction in order to have enough room to construct a well-formed cut-free quasi-deduction. But

$$(n, m) \mapsto 2_m^n,$$

which is a function on numbers and not on ordinal terms, is not elementary. This problem does not occur in the finite fragment of the infinite system that is used in the ordinal-analysis of PA, because a finite deduction in this infinite system cannot include the ω-rule and, hence, its end-formula must be a Σ_1^0-formula. Consequently the cut-rank can be bounded in these cases.

C Deduction Systems

The following systems are all logical in nature and implicitly or explicitly used in the present thesis in various places.

1. Gödel's system for intuitionistic logic is a Hilbert style system defined by the following schemata for an arbitrary first-order language.

   (a) $\bot \to \varphi$

   (b) $\varphi \vee \varphi \to \varphi$ and $\varphi \to \varphi \wedge \varphi$

   (c) $\varphi \to \varphi \vee \psi$ and $\varphi \wedge \psi \to \varphi$

   (d) $\varphi \vee \psi \to \psi \vee \varphi$ and $\varphi \wedge \psi \to \psi \wedge \varphi$

   (e) $(\forall x)\varphi(x) \to \varphi(t)$

   (f) $\varphi(t) \to (\exists x)\varphi(x)$

   (g)
   $$\frac{\varphi \quad \varphi \to \psi}{\psi}$$

   (h)
   $$\frac{\varphi \to \chi \quad \chi \to \psi}{\varphi \to \psi}$$

   (i)
   $$\frac{\varphi \wedge \psi \to \chi}{\varphi \to (\psi \to \chi)}$$

   (j)
   $$\frac{\varphi \to (\psi \to \chi)}{\varphi \wedge \psi \to \chi}$$

   (k)
   $$\frac{\varphi \to \psi}{\chi \vee \varphi \to \chi \vee \psi}$$

(l)

$$\frac{\psi \to \varphi(x)}{\psi \to (\forall x)\varphi(x),}$$

where x does not occur freely in ψ.

(m)

$$\frac{\varphi(x) \to \psi}{(\exists x)\varphi(x) \to \psi,}$$

where x does not occur freely in ψ.

2. The sequent calculus LK for classical logic is defined as follows.

(a) Logical Axioms:

$$\varphi \Rightarrow \varphi$$

for every atomic $\varphi \in \mathcal{L}(\tau)$.

(b) Structural rules:

i. Weakening:

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \text{ (left)} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \text{ (right)}$$

The formula φ is called the weakening-formula.

ii. Contraction:

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \text{ (left)} \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi} \text{ (right)}$$

iii. Exchange:

$$\frac{\Gamma, \psi, \varphi, \Pi \Rightarrow \Delta}{\Gamma, \varphi, \psi, \Pi \Rightarrow \Delta} \text{ (left)} \quad \frac{\Gamma \Rightarrow \Delta, \psi, \varphi, \Lambda}{\Gamma \Rightarrow \Delta, \varphi, \psi, \Lambda} \text{ (right)}$$

iv. Cut:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

The formula φ is called the *cut-formula*. The rules (a)-(c) are called *weak structural rules*.

(c) Logical Rules:

i. Negation:

$$\frac{(\neg\varphi), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \text{ ($\neg$-left)} \quad \frac{\Gamma \Rightarrow \Delta, (\neg\varphi)}{\varphi, \Gamma \Rightarrow \Delta} \text{ ($\neg$-right)}$$

ii. Conjunction:

$$\frac{\varphi, \Gamma \Rightarrow \Delta}{(\varphi \wedge \psi), \Gamma \Rightarrow \Delta} \text{ ($\wedge$-left$_1$)} \quad \frac{\varphi, \Gamma \Rightarrow \Delta}{(\psi \wedge \varphi), \Gamma \Rightarrow \Delta} \text{ ($\wedge$-left$_2$)}$$

and

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, (\varphi \wedge \psi)} \text{ ($\wedge$-right)}$$

iii. Disjunction:

$$\frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{(\varphi \vee \psi), \Gamma \Rightarrow \Delta} \text{ ($\vee$-left)}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, (\varphi \vee \psi)} \text{ ($\vee$-right$_1$)} \quad \text{and} \quad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, (\varphi \vee \psi)} \text{ ($\vee$-right$_2$)}$$

iv. Implication:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Pi \Rightarrow \Lambda}{(\varphi \to \psi), \Gamma, \Pi \Rightarrow \Delta, \Lambda} \text{ ($\to$ -left)} \quad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, (\varphi \to \psi)} \text{ ($\to$ -right)}$$

v. Generalisation:

$$\frac{\varphi(t), \Gamma \Rightarrow \Delta}{\forall x \varphi(x), \Gamma \Rightarrow \Delta} \text{ ($\forall$-left)} \quad \frac{\Gamma \Rightarrow \Delta, \varphi(a)}{\Gamma \Rightarrow \Delta, \forall x \varphi(x)} \text{ ($\forall$-right)}$$

where t is a term and a is a free variable not occurring in the lower sequent, which is called the *eigenvariable*.

vi. Existence:

$$\frac{\varphi(a), \Gamma \Rightarrow \Delta}{\exists x \varphi(x), \Gamma \Rightarrow \Delta} \text{ ($\exists$-left)} \quad \frac{\Gamma \Rightarrow \Delta, \varphi(t)}{\Gamma \Rightarrow \Delta, \exists x \varphi(x)} \text{ ($\exists$-right)}$$

where t is a term and a is a free variable not occurring in the lower sequent, which is called the *eigenvariable*.

The Rules (a)-(d) are called propositional rules and (d) and (h) quantifier rules. $\forall$-right and $\exists$-left are called strong the other cases weak quantifier rules.

The formulas in the upper sequents that are used in the rule are called *auxiliary formulas* (a.f.) the formulas in the lower sequent that are modified are called *principal formulas* (p.f.) and formulas which are not used are called *side formulas* (s.f.).

3. LJ is defined as LK where the sequents that are in use are restricted to those which do not have more than one formula on the right side.

4. The system that is used in [41] is defined as follows.

$(\wedge)$	$\vdash \Gamma, A$ and $\vdash \Gamma, B$	$\Rightarrow$	$\vdash \Gamma, A \wedge B$
$(\vee)$	$\vdash \Gamma, A_i$ and $i \in \{1, 2\}$	$\Rightarrow$	$\vdash \Gamma, A_1 \vee A_2$
$(\forall_1)$	$\vdash \Gamma, F(a)$	$\Rightarrow$	$\vdash \Gamma, \forall x F(x)$
$(\exists_1)$	$\vdash \Gamma, F(t)$	$\Rightarrow$	$\vdash \Gamma, \exists x F(x)$
$(\forall_2)$	$\vdash \Gamma, F(U)$	$\Rightarrow$	$\vdash \Gamma, \forall X F(X)$
$(\exists_2)$	$\vdash \Gamma, F(U)$	$\Rightarrow$	$\vdash \Gamma, \exists X F(X)$
(cut)	$\vdash \Gamma, A$ and $\vdash \Gamma, \neg A$	$\Rightarrow$	$\vdash \Gamma$

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