The hunt for Skewes' number

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Abstract

We study the regions where the function $\pi(x) - \mathrm{li}(x)$ is positive, the first such point being known as Skewes' number. We prove a new theorem which, after extensive numerical calculations, allows us to obtain a new lowest value where $\pi(x) - \mathrm{li}(x)$ is positive, under the assumption of the Riemann Hypothesis. This new lowest value is $1.397166161527 \times 10^{316}$. Our new theorem builds on previous work, but is different in that it does not estimate a particular constant, instead keeping it exact. This simplifies some of the calculations, permitting the error terms to be analysed more easily.

Contents

List of Tables

List of Figures

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Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

1 Introduction

1.1 Notation

The symbols ϑ and ϑ_i are often used throughout this paper, for some $i \in \mathbb{N}$, they denote complex numbers such that $|\vartheta| \leq 1$ and $|\vartheta_i| \leq 1$. Note that the actual value can be different at each occurrence and it may also hold dependence on certain parameters.

Because there are a lot of compounded fractions in this paper, effort has gone into making all of the equations as legible as possible, this includes the layout of fractional indices. For instance, $e^{1/32\alpha}$ represents the value of e raised to the $\frac{1}{32\alpha}$ index, rather than $\frac{\alpha}{32}$ or even $\frac{2\alpha}{3}$.

1.2 History

1.2.1 Prime counting functions

A popular area of mathematics is prime numbers. The prime counting function, denoted by $\pi(x)$, counts all prime numbers p less than a given number x:

$$\pi(x) = \sum_{p \leq x} 1.$$

Legendre[10], in 1798, conjectured that

$$\pi(x) \sim \frac{x}{A\log(x) + B},$$

or, equivalently,

$$\lim_{x\to\infty} \frac{\pi(x)(A\log(x) + B)}{x} = 1,$$

for constants A and B, and where $\log(x)$ denotes the natural logarithm. In 1808[11], he refined this conjecture to

$$\pi(x) = \frac{x}{\log(x) - A(x)},$$

with $\lim_{x\to\infty} A(x) = 1.08366\ldots$. Over time, better estimates for $\pi(x)$ were discovered. In 1849, Gauss explained in a letter written to Enke that from 1791, he knew the logarithmic integral function, denoted $\mathrm{li}(x)$, was a better estimate than all before. Gauss stated that

$$\pi(x) \sim \mathrm{li}(x).$$

This is the statement of the Prime Number Theorem (PNT), proven independently in 1896 by Hadamard[8] and de la Vallée-Poussin[6]. The function $\mathrm{li}(x)$ is defined as

$$\mathrm{li}(x) = \int_0^x \frac{1}{\log(t)}dt = \lim_{\epsilon\to 0}\left(\int_0^{1-\epsilon} + \int_{1+\epsilon}^x\right)\frac{1}{\log(t)}dt.$$

Gauss showed that for all $2 < x < 3,000,000$, the inequality $\pi(x) < \mathrm{li}(x)$ held, and it was believed for over half a century that this was true for all x. At the time of writing, it has been shown by Büthe[4] that $\pi(x) < \mathrm{li}(x)$ holds for all x up to 10^{19}. It is required that $x \geq 2$ otherwise $\pi(x) = 0$.

It was not until Littlewood[13] in 1914, that we had proof that the logarithmic integral function *underestimates* the prime counting function for some large x. He showed that there were infinitely many crossover points, where the value of $\pi(x)-\mathrm{li}(x)$ became positive rather than negative. He also showed that there exists a positive number K, for which the function

$$\frac{\{\pi(x)-\mathrm{li}(x)\}\log(x)}{\sqrt{x}\log\log\log(x)}$$

is greater than K and less than $-K$ for arbitrarily large values of x. This was an existence proof, his paper did not state where the first (or indeed any) crossover point lies.

The first bound on the location of the first crossover point was given by Skewes, and hence this first crossover value was dubbed as Skewes' number. In his 1933 paper[22] he bounds the first crossover by $10^{10^{10^{34}}}$, this bound assumed that the Riemann Hypothesis held true[1]. In 1955, Skewes[23] gave a bound which did not require the assumption of the Riemann Hypothesis: $10^{10^{10^{964}}}$.

One of Riemann's well known contributions to mathematics is the prime power counting function $\Pi(x)$:

$$\Pi(x)=\sum_{p^n\leq x}1=\sum_{n=1}^{\infty}\frac{\pi(x^{1/n})}{n}=\pi(x)+\frac{1}{2}\pi(x^{1/2})+\frac{1}{3}\pi(x^{1/3})+\cdots. \tag{1}$$

In 1859, Riemann[17] formed a relationship between $\pi(x)$ and $\mathrm{li}(x)$ using this prime power counting function:

$$\Pi(x)=\mathrm{li}(x)-\sum_{\rho}\mathrm{li}(x^{\rho})-\log(2)+\int_x^{\infty}\frac{dt}{(t^2-1)t\log(t)}\quad\text{for}\quad x>1, \tag{2}$$

where the sum over ρ denotes the sum over the zeros of the Riemann zeta function, see below.

To evaluate the logarithmic integral of a complex argument, the following result was defined for $z=u+iv\in\mathbb{C}$ and $v\neq 0$,

$$\mathrm{li}(e^z)=\int_{-\infty+iv}^{u+iv}\frac{e^t}{t}dt.$$

We can perform integration by parts on this integral to obtain an expansion which we will use frequently.

$$\begin{aligned}\mathrm{li}(e^z)&=\frac{e^z}{z}+\frac{\vartheta e^z}{z^2} &(3)\\ &=\frac{e^z}{z}+\frac{e^z}{z^2}+\frac{2\vartheta e^z}{z^3}. &(4)\end{aligned}$$

1.2.2 Riemann zeta function

Another popular area of mathematics has been that of infinite series. One of the most well known infinite series is defined by the Riemann zeta function $\zeta(s)$. This is expressed

[1] In this paper we define Skewes' number to be the smallest x for which $\pi(x)>\mathrm{li}(x)$, rather than this first bound that Skewes' discovered.

as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

This series converges for all $s > 1$, and diverges for all $s \leq 1$. If we were to let s represent a complex number, that is; let $s = \sigma + it$, where $\sigma, t \in \mathbb{R}$, then we have the result that the Dirichlet series for $\zeta(s)$ converges for all $\sigma = \Re(s) > 1$, and diverges for all $\sigma = \Re(s) \leq 1$. Thanks to analytic continuation, however, we can extend the domain of $\zeta(s)$ to $\mathbb{C} \setminus \{1\}$, that is, to the entire complex plane, apart from the point $s = 1$.

There are infinitely many solutions to the equation $\zeta(s) = 0$; there are infinitely many trivial solutions, which are at the points where $s = -2n$, for $n \in \mathbb{N}$. There are also infinitely many non-trivial solutions – these are the ones in which we are interested. The Riemann Hypothesis states that all of the non-trivial solutions to the Riemann zeta function lie on the line $s = \frac{1}{2} + it$. It has been shown that all non-trivial zeros ρ have the property that $0 < \Re(\rho) < 1$, this region of the complex plane is known as the critical strip, and the line $\frac{1}{2} + it$ is known as the critical line. To this day, billions of non-trivial solutions have been found on this line. We will be making use of them in this paper.

Whilst the imaginary parts of the non-trivial zeros of the Riemann zeta function do not appear to follow an obvious pattern, the number of zeros before a given point T in the critical strip can be estimated by the following function, given by Backlund [2]:

$$N(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi e}\right) + Q(T), \tag{5}$$

where $Q(T) = O(\log(T))$. One should take note that this is not the same as the commonly used $S(T)$.

1.3 Previous work

The first improvement on Skewes' bounds was given by Lehman in 1966 [12]. His paper laid the foundations upon which nearly every paper written since has been based. He let $\pi(x) - \mathrm{li}(x)$ be integrated against a Gaussian kernel over a finite interval, bounded any error terms that arose, and performed numerical calculations on the sums of Riemann zeta zeros to obtain a region where $\pi(x) - \mathrm{li}(x)$ is positive. He was able to find three regions where it looked likely that a crossover point occurs, these regions were discovered by identifying where the result of the sum

$$-\sum_{0<|\gamma|\leq T} \frac{e^{i\gamma\omega}}{\rho}$$

is somewhat larger than 1, where T is the largest imaginary part of a zero on the critical strip used in the calculation, and $\gamma = \Im(\rho)$. The parameter e^{ω} is the value of x in the region being checked. These regions are in the vicinity of 1.398×10^{316}, 6.663×10^{370} and 1.593×10^{1165}, respectively. Lehman showed that there was a string of at least 10^{500} consecutive integers between 1.53×10^{1165} and 1.65×10^{1165} where $\pi(x) - \mathrm{li}(x)$ is positive. This method did not assume the Riemann Hypothesis, but did contain an additional error term in case it was shown not to be true!

The next improvement on the upper bound for Skewes' number came from te Riele[24], in 1987. He lowered the bound to a region near 6.687×10^{370}, confirming the second of Lehman's interval predictions.

In 1999, Bays and Hudson[3] succeeded in verifying the last of Lehman's regions, as they were the next to lower the bound again to 1.398244×10^{316}. To this day, the region first verified by Bays and Hudson has been the subject of many papers, and it is believed that this is the first region where a crossover takes place. Chao and Plymen[5] sharpened the region given by Bays and Hudson in 2006. In 2010, Saouter and Demichel[20] brought the bound down to 1.397199×10^{316}, which was improved on a few years later by Saouter, Trudgian and Demichel[21] to 1.397167×10^{316}. It is this last paper on which we base a lot of our work in this paper.

2 Lehman's theorem

As mentioned in the introduction, Lehman was the first to improve on the upper bound for Skewes' number. In this section we state some useful results and prove his theorem, which we will later adapt to further improve the result.

2.1 Some results concerning $\pi(x)$ and $\mathrm{li}(x)$

We have established the link between the two functions $\pi(x)$ and $\mathrm{li}(x)$, via the prime power counting function $\Pi(x)$. We can combine the results from (1) and (2) to obtain, for $x > 1$.

$$\pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots \\ = \mathrm{li}(x) - \sum_{\rho} \mathrm{li}(x^{\rho}) - \log(2) + \int_x^{\infty} \frac{dt}{(t^2-1)t\log(t)}. \tag{6}$$

We can show the number of terms in the LHS of the expansion above is finite. Let $\pi\left(x^{1/k}\right)/k$ be the last term in the expansion, then we want

$$\pi(x^{1/k}) \geq 1 \iff x^{1/k} \geq 2,$$

and via algebraic manipulation, we get

$$\begin{aligned} x^{1/k} &\geq 2, \\ \frac{1}{k}\log(x) &\geq \log(2), \\ \frac{\log(x)}{\log(2)} &\geq k, \end{aligned} \tag{7}$$

So we can state that there are at most $k = \left\lfloor \frac{\log(x)}{\log(2)} \right\rfloor$ terms in the LHS of (6), where $\lfloor x \rfloor$ is the integer part of x.

Rosser and Schoenfeld[19] showed that, for $x > 1$

$$\pi(x) = \frac{x}{\log(x)} + \frac{3\vartheta x}{2\log^2(x)}. \tag{8}$$

Lehman used this result, along with the estimate $\pi(x) < \frac{2x}{\log(x)}$, to obtain the following:

$$\frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots = \frac{x^{1/2}}{\log(x)} + \vartheta\left\{\frac{3x^{1/2}}{\log^2(x)} + \frac{2x^{1/3}}{\log(x)}\left(\frac{\log(x)}{\log(2)}\right)\right\}.$$

We can also estimate the integral in (6), for $x \geq e$,

$$0 < \int_x^\infty \frac{dt}{(t^2-1)t\log(t)} < 2\int_x^\infty \frac{dt}{t^3} = \frac{1}{x^2} < \log(2),$$

and, since $\frac{2}{\log(2)} + \log(2) < 4$, we obtain the result

$$\pi(x) - \mathrm{li}(x) = -\frac{x^{1/2}}{\log(x)} - \sum_\rho \mathrm{li}(x^\rho) + \vartheta\left(\frac{3x^{1/2}}{\log^2(x)} + 4x^{1/3}\right), \tag{9}$$

for $x \geq e$.

We are now in a position to state the theorem.

2.2 Statement of Lehman's theorem

Theorem 2.1. *Let A be a positive number such that $\beta = \frac{1}{2}$ for all zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$ for which $0 < \gamma \leq A$. Let α, η and ω be positive numbers such that $\omega - \eta > 1$ and the conditions*

$$\frac{4A}{\omega} \leq \alpha \leq A^2 \quad \text{and} \quad \frac{2A}{\omega} \leq \eta < \frac{\omega}{2} \tag{10}$$

hold. Let

$$K(y) := \sqrt{\frac{\alpha}{2\pi}}e^{-\alpha y^2/2}.$$

Then for $2\pi e < T \leq A$,

$$\begin{aligned} I(\omega,\eta) &:= \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\frac{u\{\pi(e^u) - \mathrm{li}(e^u)\}}{e^{u/2}}du \\ &= -1 - \sum_{0<|\gamma|\leq T} \frac{e^{i\gamma\omega}}{\rho}e^{-\gamma^2/2\alpha} + R. \end{aligned} \tag{11}$$

Where

$$|R| \leq s_1 + s_2 + s_3 + s_4 + s_5 + s_6$$

and

$$\begin{aligned} s_1 &= \frac{3.05}{\omega - \eta} \\ s_2 &= 4(\omega+\eta)e^{-(\omega-\eta)/6} \\ s_3 &= \frac{2e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\eta} \\ s_4 &= 0.08\sqrt{\alpha}e^{-\alpha\eta^2/2} \\ s_5 &= e^{-T^2/2\alpha}\left(\frac{\alpha}{\pi T^2}\log\frac{T}{2\pi} + \frac{8\log T}{T} + \frac{4\alpha}{T^3}\right) \\ s_6 &= A\log(A)e^{-A^2/2\alpha+(\omega+\eta)/2}(4\alpha^{-1/2} + 15\eta). \end{aligned}$$

If the Riemann Hypothesis holds, we can ignore the term s_6 and the inequalities given by (10).

There are a couple of things to note before we prove this theorem. Firstly, the s_6 error term is the aforementioned "extra error term" on page 11, in the scenario that the Riemann Hypothesis does not hold. Second, there are a lot of parameters involved in this theorem, effort has been put in to keep the same parameter labels unchanged since Lehman first used them, so we display each parameter and its role in the problem in Table 1:

Table 1: Parameters

ω	is the exponent of the centre of the interval along the real axis being checked,
η	is the radius of the interval,
α	is a parameter relating to the Gaussian kernel being used,
A	is the magnitude on the imaginary axis for which we know the Riemann Hypothesis holds,
T	is the largest imaginary part of a non-trivial Riemann zero being used in our numerical calculations.

2.3 Preliminary proofs

Before we prove Lehman's theorem, we state and prove some results which will help us along the way.

Lemma 2.2. *Let $\varphi(t)$ be a continuous function which is positive and decreasing monotonically for $2\pi e \leq T_1 \leq t \leq T_2$, then*

$$\sum_{T_1 \leq t \leq T_2} \varphi(\gamma) = \frac{1}{2\pi}\int_{T_1}^{T_2} \varphi(t)\log\left(\frac{t}{2\pi}\right)dt + \vartheta\left\{4\varphi(T_1)\log(T_1) + 2\int_{T_1}^{T_2}\frac{\varphi(t)}{t}dt\right\}.$$

Proof. We utilise Stieltjes integrals to prove this lemma.

$$\sum_{T_1 < \gamma \leq T_2} \varphi(\gamma) = \int_{T_1}^{T_2} \varphi(t)dN(t) = \frac{1}{2\pi}\int_{T_1}^{T_2} \varphi(t)\log\left(\frac{t}{2\pi}\right)dt + \int_{T_1}^{T_2}\varphi(t)dQ(t),$$

we have, by (5),

$$\begin{aligned}\left|\int_{T_1}^{T_2}\varphi(t)dQ(t)\right| &= |\varphi(T_2)Q(T_2) - \varphi(T_1)Q(T_1)| - \int_{T_1}^{T_2} Q(t)d\varphi(t),\\ &\leq 2\varphi(T_2)\log(T_2) + 2\varphi(T_1)\log(T_1) - 2\int_{T_1}^{T_2}\log(t)d\varphi(t),\\ &\leq 4\varphi(T_1)\log(T_1) + 2\int_{T_1}^{T_2}\varphi(t)d\log(t),\end{aligned}$$

where $Q(t) = \vartheta(2\log(t))$. Note this bound has been improved both by Trudgian[25] and then by Platt and Trudgian[16]. $\square$

Lemma 2.3. *If $T \geq 2\pi e$, then*

$$\sum_{\gamma > T} \frac{1}{\gamma^n} < T^{1-n}\log(T), \quad \textit{for} \quad n = 2, 3, \ldots$$

Proof. Using Lemma 2.2, we get

$$\begin{aligned}
\sum_{\gamma > T} \frac{1}{\gamma^n} &= \frac{1}{2\pi}\int_T^\infty t^{-n}\log\left(\frac{t}{2\pi}\right)dt + \vartheta T^{-n}\left(4\log(T) + \frac{2}{n}\right) \\
&= \frac{T^{1-n}}{2\pi}\left(\frac{\log(T/2\pi)}{n-1} + \frac{1}{(n-1)^2}\right) + \vartheta T^{-n}\left(4\log(T) + \frac{2}{n}\right) \\
&\leq T^{1-n}\log(T)\left(\frac{1}{2\pi} + \frac{1}{2\pi\log(T)} + \frac{4}{T} + \frac{1}{T\log(T)}\right) \\
&< T^{1-n}\log(T).
\end{aligned}$$

One should note that for large T, we can improve this bound to $T^{1-n}\log(T)/(2\pi)$, however for what we require the result in the proof is sufficient. □

Lemma 2.4. *We have*

$$\sum_{\gamma > 0} \frac{1}{\gamma^2} < 0.025.$$

Proof. See Appendix A. □

It should be noted that this bound can be reduced to 0.0231055, however for the sake of Lehman's paper the lemma is sufficient.

Lemma 2.5. *If $\alpha > 0$, and $\varphi(t)$ is positive and decreasing monotonically for $t \geq T > 0$, then*

$$\int_T^\infty \varphi(t)e^{-t^2/2\alpha}dt < \frac{\alpha}{T}\varphi(T)e^{-T^2/2\alpha}.$$

Proof. We have that

$$\frac{d}{dt}\left\{\frac{\alpha e^{-t^2/2\alpha}}{t}\right\} = -\frac{\alpha e^{-t^2/2\alpha}}{t^2} - e^{-t^2/2\alpha},$$

and so, we have

$$\int_T^\infty \varphi(t)e^{-t^2/2\alpha}dt < \int_T^\infty \varphi(t)\frac{d}{dt}\left(-\frac{\alpha e^{-t^2/2\alpha}}{t}\right)dt \leq \frac{\alpha}{T}\varphi(T)e^{-T^2/2\alpha}.$$

□

The proofs of Lemma 2.2, Lemma 2.3 and Lemma 2.5 are taken directly from Lehman's paper [12], Lemma 2.4 is both stated and proven by Rosser [18], and we have verified the calculation in Appendix A.

Lemma 2.6. *Let $\alpha > 0$ and let $K(y)$ be as stated in Lehman's theorem, then*

$$\int_{-\infty}^\infty K(y)e^{i\gamma y}dy = e^{-\gamma^2/2\alpha}.$$

Proof. This is simply a proof using algebraic manipulation.

$$\begin{aligned}
\int_{-\infty}^{\infty} K(y)e^{i\gamma y}dy &= \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha y^2/2}e^{i\gamma y}dy \\
&= \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha y^2/2+i\gamma y}dy \\
&= \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha y^2/2+i\gamma y+\gamma^2/2\alpha-\gamma^2/2\alpha}dy \\
&= e^{-\gamma^2/2\alpha}\sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha(y^2-2i\gamma y/\alpha-\gamma^2/\alpha^2)/2}dy \\
&= e^{-\gamma^2/2\alpha}\sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha(y-i\gamma/\alpha)^2/2}dy.
\end{aligned}$$

At this stage, let $t = \sqrt{\alpha}(y - i\gamma/\alpha)$, then we get

$$\begin{aligned}
\int_{-\infty}^{\infty} K(y)e^{i\gamma y}dy &= e^{-\gamma^2/2\alpha}\sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha(y-i\gamma/\alpha)^2/2}dy \\
&= \frac{e^{-\gamma^2/2\alpha}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}dt \\
&= \frac{e^{-\gamma^2/2\alpha}}{\sqrt{2\pi}}\sqrt{2\pi} \\
&= e^{-\gamma^2/2\alpha}
\end{aligned}$$

the last evaluation being the well-known Gaussian integral, thus completing the proof. □

Lemma 2.7. *Let $\alpha > 0$ and let $K(y)$ be as stated in Lehman's theorem, then*

$$\int_{-\infty}^{\infty} K(y)dy = 1.$$

Proof. Once again, we prove using algebraic manipulation, and using the change of variables $t = \sqrt{\alpha}y$:

$$\begin{aligned}
\int_{-\infty}^{\infty} K(y)dy &= \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha y^2/2}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sqrt{\alpha}e^{-\alpha y^2/2}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}dt \\
&= \frac{1}{\sqrt{2\pi}}\sqrt{2\pi} \\
&= 1.
\end{aligned}$$

□

2.4 Proof of Lehman's theorem assuming the Riemann Hypothesis

2.4.1 The s_1 and s_2 error terms

The proof of Lehman's theorem is rather long, we have reworded certain sections and we have amended some notation to fit with the other papers we are working with.

Proof. From (9), we let $x = e^u$, and multiply throughout by the function $ue^{-u/2}$, so that we are working with smaller, "less massive" terms. So we have, for $u > 1$,

$$\frac{u\{\pi(e^u) - \mathrm{li}(e^u)\}}{e^{u/2}} = -1 - \sum_\rho \frac{u\,\mathrm{li}(e^{\rho u})}{e^{u/2}} + \frac{3\vartheta_1}{u} + \frac{4u\vartheta_2}{e^{u/6}}. \tag{12}$$

This is what we will integrate against the Gaussian kernel K. We can split the integral into three parts:

$$\begin{aligned}&\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\frac{u\{\pi(e^u) - \mathrm{li}(e^u)\}}{e^{u/2}}du \\ &= -\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)du - \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\sum_\rho \frac{u\,\mathrm{li}(e^{\rho u})}{e^{u/2}}du \\ &+ \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\left(\frac{3\vartheta_1}{u} + \frac{4u\vartheta_2}{e^{u/6}}\right)du.\end{aligned} \tag{13}$$

We first evaluate the third term in (13). Since the kernel K is always positive, we have

$$\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\left(\frac{3}{u} + \frac{4u}{e^{u/6}}\right)du \leq \frac{3}{\omega-\eta} + 4(\omega+\eta)e^{-\frac{\omega-\eta}{6}}. \tag{14}$$

This result corresponds to s_2, and part of the s_1 term in the statement of Lehman's theorem.

2.4.2 The -1 term and s_3 error term

By symmetry, we have

$$\begin{aligned}\int_{-\infty}^{\omega-\eta} K(u-\omega)du = \int_{\omega+\eta}^{\infty} K(u-\omega)du &= \int_\eta^\infty K(y)dy \\ &= \sqrt{\frac{\alpha}{2\pi}}\int_\eta^\infty e^{-\alpha y^2/2}dy \\ &= \frac{1}{\sqrt{2\pi\alpha}}\int_{\eta\alpha}^\infty e^{-t^2/2\alpha}dt\end{aligned}$$

by letting $t = \alpha y$. By Lemma 2.5 we get

$$\begin{aligned}\frac{1}{\sqrt{2\pi\alpha}}\int_{\eta\alpha}^\infty e^{-t^2/2\alpha}dt &< \frac{1}{\sqrt{2\pi\alpha}}\frac{\alpha}{\eta\alpha}e^{-\eta^2\alpha^2/2\alpha} \\ &= \frac{e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\eta}.\end{aligned}$$

From this, and Lemma 2.7, we can evaluate the first term in (13):

$$
\begin{aligned}
-\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)du &= -\left(\int_{-\infty}^{\infty} K(u-\omega)du - 2\int_{\omega+\eta}^{\infty} K(u-\omega)du\right) \\
&= -1 + 2\int_{\omega+\eta}^{\infty} K(u-\omega)du \\
&= -1 + 2\int_{\eta}^{\infty} K(y)dy \\
&< -1 + \frac{2e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\eta}.
\end{aligned} \tag{15}
$$

So we have both the -1 term in (11), and

$$
s_3 = \frac{2e^{-\alpha\eta^2/2}}{\sqrt{2\pi\alpha}\eta}.
$$

2.4.3 The sum and s_4 and s_5 error terms

By (3), and letting $z = \rho u - t$, we have

$$
\mathrm{li}(e^{\rho u}) = \int_{-\infty+i\gamma u}^{\rho u} \frac{e^z}{z}dz = e^{\rho u}\int_0^{\infty} \frac{e^{-t}}{\rho u - t}dt. \tag{16}
$$

We can integrate this last integrand by parts to obtain

$$
\int_0^{\infty} \frac{e^{-t}}{\rho u - t}dt = \frac{1}{\rho u} + \int_0^{\infty} \frac{e^{-t}}{(\rho u - t)^2}dt = \frac{1}{\rho u} + \int_0^{\infty} \frac{\vartheta e^{-t}}{(\gamma u)^2}dt,
$$

and so

$$
\mathrm{li}(e^{\rho u}) = \frac{e^{\rho u}}{\rho u} + \frac{\vartheta e^{\beta u}}{\gamma^2 u^2}.
$$

We now look at the last term of (13). Let A be as defined in Table 1, as the height up the critical strip for which the Riemann Hypothesis holds, so $\beta = \frac{1}{2}$ for $|\gamma| \leq A$. We have

$$
-\sum_{\rho}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\frac{u\,\mathrm{li}(e^{\rho u})}{e^{u/2}}du = S_1 + S_2 + S_3, \tag{17}
$$

where

$$
\begin{aligned}
S_1 + S_2 &= -\sum_{|\gamma|\leq A}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\frac{u\,\mathrm{li}(e^{\rho u})}{e^{u/2}}du \\
S_3 &= -\sum_{|\gamma|> A}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\frac{u\,\mathrm{li}(e^{\rho u})}{e^{u/2}}du.
\end{aligned}
$$

We can obtain S_1 and S_2 explicitly:

$$
\begin{aligned}
S_1 + S_2 &= -\sum_{|\gamma|\le A} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) \frac{u \operatorname{li}(e^{\rho u})}{e^{u/2}} du \\
&= -\sum_{|\gamma|\le A} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) \frac{u}{e^{u/2}} \left(\frac{e^{\rho u}}{\rho u} + \frac{\vartheta e^{\beta u}}{\gamma^2 u^2} \right) du \\
&= -\sum_{|\gamma|\le A} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) \left(\frac{e^{i\gamma u}}{\rho} + \frac{\vartheta}{\gamma^2 u} \right) du \\
&= -\sum_{|\gamma|\le A} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) \frac{e^{i\gamma u}}{\rho} du - \sum_{|\gamma|\le A} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) \frac{\vartheta}{\gamma^2 u} du.
\end{aligned}
$$

So we have:

$$
\begin{aligned}
S_1 &= -\sum_{|\gamma|\le A} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) \frac{e^{i\gamma u}}{\rho} du \\
|S_2| &\le \sum_{|\gamma|\le A} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) \frac{\vartheta}{\gamma^2 u} du.
\end{aligned}
$$

We begin by looking at the S_1 term, by Lemma 2.6,

$$
\begin{aligned}
S_1 &= -\sum_{|\gamma|\le A} \frac{e^{i\omega\gamma}}{\rho} \int_{-\eta}^{\eta} K(y) e^{i\gamma y} dy \\
&= -\sum_{|\gamma|\le A} \frac{e^{i\omega\gamma}}{\rho} \left(\int_{-\infty}^{\infty} K(y) e^{i\gamma y} dy - 2\Re \left(\int_{\eta}^{\infty} K(y) e^{i\gamma y} dy \right) \right) \\
&= -\sum_{|\gamma|\le A} \frac{e^{i\omega\gamma}}{\rho} e^{-\gamma^2/2\alpha} + 2\Re \left(\sum_{|\gamma|\le A} \frac{e^{i\omega\gamma}}{\rho} \int_{\eta}^{\infty} K(y) e^{i\gamma y} dy \right) \\
&= -\sum_{|\gamma|\le A} \frac{e^{i\omega\gamma}}{\rho} e^{-\gamma^2/2\alpha} + 2\vartheta \left| \sum_{|\gamma|\le A} \frac{e^{i\omega\gamma}}{\rho} \right| \left| \int_{\eta}^{\infty} K(y) e^{i\gamma y} dy \right| \\
&= -\sum_{|\gamma|\le A} \frac{e^{i\omega\gamma}}{\rho} e^{-\gamma^2/2\alpha} + 4\vartheta \sum_{0<\gamma\le A} \frac{1}{\gamma} \left| \int_{\eta}^{\infty} K(y) e^{i\gamma y} dy \right|.
\end{aligned}
$$

We can now perform integration by parts:

$$
\begin{aligned}
\int_{\eta}^{\infty} K(y) e^{i\gamma y} dy &= \left[\frac{K(y) e^{i\gamma y}}{i\gamma} \right]_{\eta}^{\infty} - \int_{\eta}^{\infty} \frac{K'(y) e^{i\gamma y}}{i\gamma} dy \\
&= -\frac{K(\eta) e^{i\gamma\eta}}{i\gamma} - \int_{\eta}^{\infty} \frac{K'(y) e^{i\gamma y}}{i\gamma} dy \\
&= \frac{e^{i\gamma\eta}}{i\gamma} \int_{\eta}^{\infty} K'(y) dy - \int_{\eta}^{\infty} \frac{K'(y) e^{i\gamma y}}{i\gamma} dy \\
&= \int_{\eta}^{\infty} K'(y) \frac{e^{i\gamma\eta} - e^{i\gamma y}}{i\gamma} dy.
\end{aligned}
$$

Now we use the fact that $K(y)$ is monotonically decreasing for $y > 0$ to show

$$\left|\int_{\eta}^{\infty} K(y)e^{i\gamma y}dy\right| \leq \frac{2}{\gamma}\int_{\eta}^{\infty} |K'(y)|dy = \frac{2}{\gamma}K(\eta) = \frac{2}{\gamma}\sqrt{\frac{\alpha}{2\pi}}e^{-\alpha\eta^2/2}.$$

Then, by applying Lemma 2.4 and the inequality $(2\pi)^{-\frac{1}{2}} < 0.4$, we get

$$\begin{aligned} S_1 &= -\sum_{|\gamma|\leq A} \frac{e^{i\omega\gamma}}{\rho}e^{-\gamma^2/2\alpha} + 8\vartheta\sqrt{\frac{\alpha}{2\pi}}e^{-\alpha\eta^2/2}\sum_{0<\gamma\leq A}\frac{1}{\gamma^2} \\ &= -\sum_{|\gamma|\leq A} \frac{e^{i\omega\gamma}}{\rho}e^{-\gamma^2/2\alpha} + 0.08\vartheta\sqrt{\alpha}e^{-\alpha\eta^2/2}. \end{aligned} \tag{18}$$

To make the numerical calculations simpler, we can calculate

$$-\sum_{0<|\gamma|\leq A} \frac{e^{i\omega\gamma}}{\rho}e^{-\gamma^2/2\alpha}$$

only for the zeta zeros up to T, at the cost of having an extra error term. Using Lemma 2.2:

$$\begin{aligned} \left|\sum_{T<|\gamma|\leq A} \frac{e^{i\omega\gamma}}{\rho}e^{-\gamma^2/2\alpha}\right| \leq& 2\left(\sum_{T<\gamma\leq A}\frac{e^{-\gamma^2/2\alpha}}{\gamma}\right) \\ \leq& \left(\int_T^{\infty}\frac{e^{-t^2/2\alpha}}{\pi t}\log\left(\frac{t}{2\pi}\right)dt + \frac{8e^{-T^2/2\alpha}\log(T)}{T}\right. \\ &\left. + 4\int_T^{\infty}\frac{e^{-t^2/2\alpha}}{t^2}dt\right), \end{aligned}$$

provided that $T \geq 2\pi e$. Applying Lemma 2.5 to estimate the integrals, we obtain for $2\pi e \leq T \leq A$

$$\left|\sum_{T<|\gamma|\leq A} \frac{e^{i\omega\gamma}}{\rho}e^{-\gamma^2/2\alpha}\right| < e^{-T^2/2\alpha}\left(\frac{\alpha}{\pi T^2}\log\left(\frac{T}{2\pi}\right) + \frac{8\log(T)}{T} + \frac{4\alpha}{T^3}\right). \tag{19}$$

Combining the results of (18) and (19), we get the sum to calculate, plus results for s_4 and s_5:

$$\begin{aligned} S_1 &= -\sum_{0<|\gamma|\leq T} \frac{e^{i\omega\gamma}}{\rho}e^{-\gamma^2/2\alpha} + s_4 + s_5, \quad \text{with} \\ s_4 &= 0.08\sqrt{\alpha}e^{-\alpha\eta^2/2}, \\ s_5 &= e^{-T^2/2\alpha}\left(\frac{\alpha}{\pi T^2}\log\left(\frac{T}{2\pi}\right) + \frac{8\log(T)}{T} + \frac{4\alpha}{T^3}\right). \end{aligned} \tag{20}$$

By Lemma 2.4, we can estimate S_2. We have

$$\begin{aligned}|S_2| &= \sum_{0<|\gamma|\le A}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\frac{\vartheta}{\gamma^2 u}du \\ &\le \sum_{0<|\gamma|\le A}\frac{1}{\gamma^2}\int_{-\eta}^{\eta}\frac{K(y)}{\omega+y}dy \\ &\le \frac{0.05}{\omega-\eta}.\end{aligned} \tag{21}$$

This gives the remainder of our s_1 term, when combined with the result from (14):

$$s_1 = \frac{3.05}{\omega-\eta}.$$

If we were to assume the truth of the Riemann Hypothesis, then we could combine the results of (14), (15), (17), (20) and (21), and then let $A \to \infty$ to obtain the conclusion of the theorem, with the estimate for s_6 omitted. $\square$

2.5 Proof without the Riemann Hypothesis: the s_6 error term

Thus far we have not made use of the conditions stated in (10). Here, we bound the S_3 term.

Proof. We begin with the function

$$f_\rho(s) = \rho s e^{-\rho s}\,\mathrm{li}(e^{\rho s})e^{-\alpha(s-\omega)^2/2}$$

in the region $-\frac{\pi}{4} \le \arg(s) \le \frac{\pi}{4}$. The inequality $\frac{5\pi}{12} \le |\arg(\rho)| \le \frac{\pi}{2}$ holds for every zero ρ because $0 < \beta < 1$ and $|\gamma| > 14$. It follows from (3) that $f_\rho(s)$ is a regular analytic function in the region since $\frac{\pi}{6} < |\arg(\rho s)| < \frac{3\pi}{4}$. Also, by (16), we have

$$\begin{aligned}|f_\rho(s)| &= \left|\rho s e^{-\rho s}\,\mathrm{li}(e^{\rho s})e^{-\alpha(s-\omega)^2/2}\right| \\ &= \left|\rho s e^{-\rho s}\left(e^{\rho s}\int_0^\infty \frac{e^{-t}}{\rho s - t}dt\right)e^{-\alpha(s-\omega)^2/2}\right| \\ &= \left|\rho s e^{-\alpha(s-\omega)^2/2}\int_0^\infty \frac{e^{-t}}{\rho s - t}dt\right| \\ &\le \frac{|\rho s|\left|e^{-\alpha(s-\omega)^2/2}\right|}{|\Im(\rho s)|}\int_0^\infty e^{-t}dt \\ &\le 2\left|e^{-\alpha(s-\omega)^2/2}\right|.\end{aligned} \tag{22}$$

This is because:

$$\rho s = |\rho s|e^{i\arg(\rho s)} = |\rho s|\cos(\arg(\rho s)) + i|\rho s|\sin(\arg(\rho s)),$$

and

$$|\Im(\rho s)| = |\rho s|\sin(\arg(\rho s)) \ge |\rho s|\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}|\rho s|,$$

since the angle $\frac{\pi}{6}$ is "closer to the real axis" than the angle $\frac{3\pi}{4}$. In the sum

$$\begin{aligned} S_3 &= -\sum_{|\gamma|>A} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) \frac{u \operatorname{li}(e^{\rho u})}{e^{u/2}} du \\ &= \sqrt{\frac{\alpha}{2\pi}} \sum_{|\gamma|>A} \frac{1}{\rho} \int_{\omega-\eta}^{\omega+\eta} e^{u(\rho-1/2)} f_\rho(u) du, \end{aligned}$$

we perform integration by parts:

$$\begin{aligned} &\int_{\omega-\eta}^{\omega+\eta} e^{u(\rho-1/2)} f_\rho(u) du \\ =& \frac{e^{(u\rho-1/2)}}{\rho-1/2} f_\rho(u) \Big|_{\omega-\eta}^{\omega+\eta} - \int_{\omega-\eta}^{\omega+\eta} \frac{e^{u(\rho-1/2)}}{\rho-1/2} f_\rho'(u) du \\ =& \frac{e^{(\omega+\eta)(\rho-1/2)}}{\rho-1/2} f_\rho(\omega+\eta) - \frac{e^{(\omega-\eta)(\rho-1/2)}}{\rho-1/2} f_\rho(\omega-\eta) - \int_{\omega-\eta}^{\omega+\eta} \frac{e^{u(\rho-1/2)}}{\rho-1/2} f_\rho'(u) du. \end{aligned}$$

We perform integration by parts, again, to obtain

$$\begin{aligned} &\int_{\omega-\eta}^{\omega+\eta} e^{u(\rho-1/2)} f_\rho(u) du \\ =& \frac{e^{\omega(\rho-1/2)}}{\rho-1/2} \left(e^{\eta(\rho-1/2)} f_\rho(\omega+\eta) - e^{-\eta(\rho-1/2)} f_\rho(\omega-\eta) \right) - \frac{e^{(u\rho-1/2)}}{(\rho-1/2)^2} f_\rho'(u) \Big|_{\omega-\eta}^{\omega+\eta} \\ &- \int_{\omega-\eta}^{\omega+\eta} \frac{e^{u(\rho-1/2)}}{(\rho-1/2)^2} f_\rho''(u) du \\ =& \frac{e^{\omega(\rho-1/2)}}{\rho-1/2} \left(e^{\eta(\rho-1/2)} f_\rho(\omega+\eta) - e^{-\eta(\rho-1/2)} f_\rho(\omega-\eta) \right) \\ &- \frac{e^{\omega(\rho-1/2)}}{(\rho-1/2)^2} \left(e^{\eta(\rho-1/2)} f_\rho'(\omega+\eta) - e^{-\eta(\rho-1/2)} f_\rho'(\omega-\eta) \right) \\ &+ \int_{\omega-\eta}^{\omega+\eta} \frac{e^{u(\rho-1/2)}}{(\rho-1/2)^2} f_\rho''(u) du \end{aligned}$$

This, in turn, gives us

$$\begin{aligned} &\int_{\omega-\eta}^{\omega+\eta} e^{u(\rho-1/2)} f_\rho(u) du \\ &\qquad = \sum_{n=0}^{N-1} \frac{(-1)^n e^{\omega(\rho-1/2)}}{\left(\rho-\frac{1}{2}\right)^{n+1}} \left(e^{\eta(\rho-1/2)} f_\rho^{(n)}(\omega+\eta) - e^{-\eta(\rho-1/2)} f_\rho^{(n)}(\omega-\eta) \right) \\ &\qquad + \frac{(-1)^N}{\left(\rho-\frac{1}{2}\right)^N} \int_{\omega-\eta}^{\omega+\eta} e^{u(\rho-1/2)} f_\rho^{(N)}(u) du, \end{aligned}$$

where N is a positive integer, which we fix later.

We estimate $f_\rho^{(n)}(u)$ for $\omega-\eta \le u \le \omega+\eta$ with a contour integral around a circle of radius $r \le \frac{\omega}{4}$ about the point u. If s is on this circle, then Re$(s) \ge \omega-\eta-\frac{\omega}{4}$ because of (10), and Im$(s) \le \frac{\omega}{4}$. Thus the circle lies in the sector $|\arg s| \le \frac{\pi}{4}$ where $f_\rho(s)$ is regular and satisfies (22). Therefore, for $\omega-\eta \le u \le \omega+\eta$, we have

$$f_\rho^{(n)}(u) = \frac{n!}{2\pi i} \oint \frac{f_\rho(s)}{(s-u)^{n+1}} ds,$$

and hence

$$\left|f_\rho^{(n)}(u)\right| \le \frac{2n!}{r^n} \max_{|s-u|=r} \left|e^{-\alpha(s-\omega)^2/2}\right|.$$

If $s = \sigma + it$, then on the circle $(\sigma - u)^2 + t^2 = r^2$

$$\left|e^{-\alpha(s-\omega)^2/2}\right| = e^{\alpha(t^2-(\sigma-\omega)^2)/2} = e^{\alpha(r^2-(\sigma-u)^2-(\sigma-\omega)^2)/2} \le e^{\alpha r^2/2}.$$

If $N \le \frac{\alpha\omega^2}{16}$, then we can let $r = \sqrt{\frac{N}{\alpha}}$ and obtain

$$\left|f_\rho^{(N)}(u)\right| \le 2N!N^{-N/2}\alpha^{N/2}e^{N/2} = 2N!\left(\frac{\alpha e}{N}\right)^{N/2} \tag{23}$$

for $\omega - \eta \le u \le \omega + \eta$. To estimate the derivatives at $\omega \pm \eta$, let $r = \frac{\eta}{2}$, which is less than $\frac{\omega}{4}$, by (10). On the circle $|s - (\omega \pm \eta)| = r$ we have

$$\left|e^{-\alpha(s-\omega)^2/2}\right| = e^{\alpha(\eta^2/4-(\sigma-(\omega\pm\eta))^2-(\sigma-\omega)^2)/2} \le e^{-\alpha\eta^2/8},$$

and thus

$$\left|f_\rho^{(n)}(\omega \pm \eta)\right| \le 2n!\left(\frac{\eta}{2}\right)^{-n} e^{-\alpha\eta^2/8}. \tag{24}$$

Using (23) and (24) and the fact that all of the zeros lie in $0 < \beta < 1$, we obtain

$$|S_3| \le 2\sqrt{\frac{\alpha}{2\pi}}e^{(\omega+\eta)/2} \sum_{\gamma>A} \left(\frac{4e^{-\alpha\eta^2/8}}{\gamma^2} \sum_{n=0}^{N-1} \frac{n!}{(\gamma\eta/2)^2} + \frac{4\eta N!}{\gamma^{N+1}}\left(\frac{\alpha e}{N}\right)^{N/2}\right),$$

provided $1 \le N \le \frac{\alpha\omega^2}{16}$. We now let $N = \left[\frac{A^2}{\alpha}\right]$. By (10), we have $1 \le N \le \frac{A^2}{\alpha} \le \frac{\alpha\omega^2}{16}$, as required. Applying Lemma 2.3 and observing that, by (10), we obtain

$$\begin{aligned}
\sum_{\gamma>A} \frac{4e^{-\alpha\eta^2/8}}{\gamma^2} \sum_{n=0}^{N-1} \frac{n!}{(\gamma\eta/2)^2} &\le 4e^{-\alpha\eta^2/8}\log(A) \sum_{n=0}^{N-1} \frac{N^n}{(\eta/2)^2 A^{N+1}} \\
&\le 4e^{-\alpha\eta^2/8}\frac{N\log(A)}{A} \\
&\le 4e^{-\alpha\eta^2/8}\frac{A\log(A)}{\alpha}.
\end{aligned}$$

Also, since $\frac{A^2}{\alpha} - 1 < N < \frac{A^2}{\alpha}$, we have

$$\begin{aligned}
\sum_{\gamma>A} \frac{4\eta N!}{\gamma^{N+1}}\left(\frac{\alpha e}{N}\right)^{N/2} &\le 4\eta e^{1-N}N^{N+1/2}\left(\frac{\alpha e}{N}\right)^{N/2} A^{-N}\log(A) \\
&\le 4\eta e^{1-N/2}N^{1/2}\left(\frac{A^2}{N\alpha}\right)^{-N/2} \log(A) \\
&\le 4e^{3/2}\eta e^{-A^2/2\alpha}A\alpha^{-1/2}\log(A).
\end{aligned}$$

Since $(2\pi)^{-1/2} < 0.4$ and $e^{3/2} < 4.5$ it follows from (10) that

$$\begin{aligned}
|S_3| &\le 4\alpha^{-1/2}A\log(A)e^{-\alpha\eta^2/8+(\omega+\eta)/2} + 15\eta A\log(A)e^{-A^2/2\alpha+(\omega+\eta)/2} \\
&\le A\log(A)e^{-A^2/2\alpha+(\omega+\eta)/2}(4\alpha^{-1/2} + 15\eta).
\end{aligned}$$

Thus giving us the value of s_6, completing the proof! $\square$

3 Improvements to Lehman's theorem

3.1 Changes due to time

As referenced before, several improvements have been made in closing the bound on Skewes' number. When Lehman published his paper in 1966, he had his value of ω set around 2862.9768, the largest of his three estimates for a crossover point. His other parameters were $T = 12,000$, $\alpha = 10^7$, $A = 170,000$ and $\eta = 0.034$, he used approximately 12,520 zeta zeros.

Without any extra work we can immediately improve on these parameters; the number of zeta zeros we have access to has increased by several orders of magnitude, which means we can increase our value of T, which results in a decrease of the s_4 error term. With similar reasoning, A can be increased since we have verified the Riemann Hypothesis to a much higher point up the critical line, causing a decrease in the s_5 error term.

These increases do not come for free, unfortunately, as we have to abide by the restrictions placed in (10). We can note that an increase in A justifies potential increases in α and η. An increase in α results in the graph of the Gaussian kernel having a "narrower bell-curve", which, in turn, would mean we could work with a smaller η to give us a narrower interval to integrate against. This means we need to bear in mind not to increase α to the point where we have to increase η, as well as not letting the s_5 error term get too large.

3.2 Improvements in previous papers

It was shown by te Riele[24] that there was a crossover point in the vicinity of 6.663×10^{370}. He followed Lehman's theorem, but used improved parameters, he also used the first 50,000 zeta zeros, the first 15,000 to an accuracy of around 28 digits, and the remainder to around 14 digits.

Bays and Hudson[3] did similar with their paper showing a crossover point around 1.398×10^{316}. They used 1,000,000 zeros, 20 times that of te Riele.

Chao and Plymen[5] were the first to make modifications to Lehman's theorem. They reduced the constant in the leading s_1 error term from 3.05 to 2.1611 (their paper incorrectly states 2.1111 as the improved error term). Using their improvement, and with 2,000,000 zeta zeros at their disposal, they tightened the interval which Bays and Hudson discovered.

Saouter and Demichel[20] further bound the s_1 error term by using a different estimate for $\pi(x)$, by Dusart[7], as an improvement as the one given in (8). Dusart's estimate is

$$\pi(x) \leq \frac{x}{\log(x)}\left(1 + \frac{1}{\log(x)} + \frac{2.51}{\log^2(x)}\right),$$

for $x \geq 355,991$. They also greatly increased the number of zeta zeros used to 22,000,000.

Saouter, Trudgian and Demichel[21] improved further by modifying the weight function in the integral against the Gaussian kernel, that is, their integral was

$$\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\frac{u\{\pi(e^u)-\mathrm{li}(e^u)\}}{e^{u/2}\left(1+\frac{2}{u}+\frac{10.04}{u^2}\right)}du,$$

they also derived an improved version of Lehman's theorem and utilised 525,000,000 zeta zeros.

4 New theorem

Here we state our theorem, which we will prove in Section 6.

Theorem 4.1. *Let α, η and ω be positive numbers such that $\omega-\eta>43.7$, and let*

$$K(y):=\sqrt{\frac{\alpha}{2\pi}}e^{-\alpha y^2/2}. \tag{25}$$

Then for $2\pi e<M\leq T$,

$$\begin{aligned} I(\omega,\eta):=&\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\frac{u\{\pi(e^u)-\mathrm{li}(e^u)\}}{e^{u/2}}du \\ &> C-\sum_{0<|\gamma|\leq T}\frac{e^{i\gamma\omega}}{\rho}e^{-\gamma^2/2\alpha}-\sum_{0<|\gamma|\leq M}\frac{e^{i\gamma\omega}}{\rho^2\omega}e^{-\gamma^2/2\alpha}+E. \end{aligned} \tag{26}$$

Where C is equal to the integral

$$-\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\frac{u\,\mathrm{li}(e^{u/2})}{2e^{u/2}}du, \tag{27}$$

and

$$|E|\leq R_1+R_2+R_3+R_4+R_5+R_6$$

and

$$\begin{aligned} R_1=&\frac{1.812(\omega+\eta)}{e^{(\omega-\eta)/6}}, \\ R_2=&0.024e^{-(\omega-\eta)/4}\left(1+\frac{4}{\omega-\eta}\right)+e^{1/32\alpha-\omega/4}(1.301+0.04\alpha) \\ &+\frac{2e^{1/32\alpha-\omega/4}}{\alpha\eta-\frac{1}{4}}\left(\log^2\left(\frac{4\alpha\eta-1}{2\pi}\right)+\log(4\alpha\eta-1)+0.9321\right), \\ R_3=&\frac{\alpha e^{-T^2/2\alpha}}{2\pi}\log\left(\frac{T}{2\pi}\right)\left(\frac{1}{T^3}+\frac{2}{T^2}\right) \\ R_4=&\frac{K(\eta)}{\alpha\eta}\left(\frac{1}{\pi}\log^2\left(\frac{\alpha\eta}{\pi}\right)+4\log(2\alpha\eta)+4.52\right) \\ R_5=&\frac{2\alpha e^{-M^2/2\alpha}\log\left(\frac{M}{2\pi}\right)}{2\pi\omega M^3}+\frac{K(\eta)}{\alpha\omega\eta}\left(0.047+\frac{1}{\alpha\eta}\right)+\frac{0.019}{\sqrt{\alpha}\omega^2} \\ R_6=&\frac{2.92\times10^{-3}}{(\omega-\eta)^2}. \end{aligned}$$

It should be noted that the Riemann Hypothesis is assumed for this theorem.

Before we prove this theorem, one note is that by immediate comparison, our theorem looks a lot "messier" than Lehman's. This is because in trying to improve accuracy wherever we can, some of our error terms have been bounded more strictly.

Another thing to note is the addition of the parameter M. To make the calculation process easier, we are going to calculate the first term of the expansion of the logarithmic integral (4) with all T zeta zeros at our disposal. Whereas we are going to calculate using the second term only up to M zeros.

One should take note of the restriction that $\omega - \eta > 43.7$, this is set because it has been shown that $\pi(x) < \mathrm{li}(x)$ for all $x < e^{43.7}$, so it is pointless to consider lesser x.

We next state a corollary to our theorem, which utilises optimised parameters, which we shall prove in Section 7

Corollary 4.1 (Optimisation). *Let T, ω and E be positive numbers such that $M, T > 10^9$, $\omega > 400$ and $4.15 \times 10^{-6} < E < 1$. If we let*

$$\alpha = \frac{T^2}{2W\left(\frac{\log\left(\frac{T}{2\pi}\right)\left(2+\frac{1}{T}\right)}{4\pi E - 3.3\times 10^{-7})}\right)},$$

and

$$\eta = \sqrt{\frac{2}{\alpha}\log\left(\frac{0.00019\sqrt{\alpha}}{E^2}\right)},$$

where $W(x)$ is the Lambert W-function[2], and let $K(y)$ be as in (25), $I(\omega,\eta)$ be as in (26) and C be as in (27). Then, for $2\pi e < M \le T$,

$$I(\omega,\eta) = C - \sum_{0<|\gamma|\le T} \frac{e^{i\gamma\omega}}{\rho} e^{-\gamma^2/2\alpha} - \sum_{0<|\gamma|\le M} \frac{e^{i\gamma\omega}}{\rho^2\omega} e^{-\gamma^2/2\alpha} + \vartheta E.$$

It should be noted that the Riemann Hypothesis is assumed for this corollary.

It is worth noting that the purpose of this corollary is not to identify *the best* options for both α and η, simply because there is no best value for both simultaneously. This corollary aims to identify an estimate for the best possible α for the numerical calculations, followed by a corresponding η which keeps the total error terms below a specified value E.

5 Prerequisite results

Here we state a few results we will need to know in order to complete the proof of our theorem. First of all, we state and prove the following lemma regarding sums of reciprocals of powers of the imaginary parts of zeta zeros:

[2]The Lambert W-function is defined as the multivalued inverse function of $f(x) = xe^x$ [27]

Lemma 5.1. *We have*

$$\begin{aligned}
\sum_{\gamma>0} \frac{1}{\gamma^2} &< 0.0231055 \\
\sum_{\gamma>0} \frac{1}{\gamma^3} &< 0.00072955 \\
\sum_{\gamma>0} \frac{1}{\gamma^4} &< 0.0000371727.
\end{aligned} \tag{28}$$

Proof. We know the results of these sums for $0 < \gamma < 42,653,550$, from Saouter, Trudgian and Demichel[21] (Lemma 2.9), we then apply Lemma 2.3 for the remaining zeros.[3] □

This final lemma will prove useful a few times in our proof:

Lemma 5.2. *Let $K(x)$ be defined as in Theorem 4.1 equation (25), then for t, $\epsilon > 0$, we have*

$$\left| \int_\epsilon^\infty K(x) e^{ixt} dx \right| < K(\epsilon) \min \left\{ \frac{1}{\alpha\epsilon}, \frac{2}{t} \right\}. \tag{29}$$

Proof. This proof has two parts, one for each bound:

$$\begin{aligned}
\left| \int_\epsilon^\infty K(x) e^{ixt} dx \right| \leq & \int_\epsilon^\infty \sqrt{\frac{\alpha}{2\pi}} e^{-\alpha x^2/2} dx \\
= & \int_{\alpha\epsilon}^\infty \frac{1}{\sqrt{2\pi\alpha}} e^{-y^2/2\alpha} dy \\
< & \frac{1}{\sqrt{2\pi\alpha}} \frac{\alpha}{\alpha\epsilon} e^{-(\alpha\epsilon)^2/2\alpha} \\
= & \frac{1}{\sqrt{2\pi\alpha}\epsilon} e^{-\alpha\epsilon^2/2} \\
= & \frac{K(\epsilon)}{\alpha\epsilon},
\end{aligned}$$

by Lemma 2.5. We also have

$$\begin{aligned}
\int_\epsilon^\infty K(x) e^{ixt} dx = & \left[\frac{K(x) e^{ixt}}{it} \right]_\epsilon^\infty - \int_\epsilon^\infty \frac{K'(x) e^{ixt}}{it} dx \\
= & - \frac{K(\epsilon) e^{i\epsilon t}}{it} - \int_\epsilon^\infty \frac{K'(x) e^{ixt}}{it} dx \\
= & \frac{e^{i\epsilon t}}{it} \int_\epsilon^\infty K'(x) dx - \int_\epsilon^\infty \frac{K'(x) e^{ixt}}{it} dx \\
= & \int_\epsilon^\infty K'(x) \frac{e^{i\epsilon t} - e^{ixt}}{it} dx,
\end{aligned}$$

from which we get

$$\left| \int_\epsilon^\infty K'(x) \frac{e^{i\epsilon t} - e^{ixt}}{it} dx \right| \leq \frac{2}{t} \int_\epsilon^\infty |K'(x)| dx = \frac{2K(\epsilon)}{t}.$$

Taking the minimum of these two upper bounds completes the proof. □

[3] One should note that the proof in Appendix A is just for the proof of Lemma 2.4, Wolfram Mathematica 10 only has access to 10^7 zeta zeros, which results in less than a desired accuracy.

We next state a couple of results which we will be using a lot, which is summing expressions containing zeros in complex conjugate pairs:

$$\begin{aligned}\frac{e^{iu\gamma}}{\frac{1}{2}+i\gamma}+\frac{e^{-iu\gamma}}{\frac{1}{2}-i\gamma} &= \frac{\cos(u\gamma)+i\sin(u\gamma)}{\frac{1}{2}+i\gamma}+\frac{\cos(u\gamma)-i\sin(u\gamma)}{\frac{1}{2}-i\gamma} \\ &= \frac{\cos(u\gamma)+2\gamma\sin(u\gamma)}{\frac{1}{4}+\gamma^2}.\end{aligned} \tag{30}$$

and

$$\begin{aligned}\frac{e^{iu\gamma}}{(\frac{1}{2}+i\gamma)^2u}+\frac{e^{-iu\gamma}}{(\frac{1}{2}-i\gamma)^2u} &= \frac{\cos(u\gamma)+i\sin(u\gamma)}{(\frac{1}{2}+i\gamma)^2u}+\frac{\cos(u\gamma)-i\sin(u\gamma)}{(\frac{1}{2}-i\gamma)^2u} \\ &= \frac{\left(\frac{1}{2}-2\gamma^2\right)\cos(u\gamma)+2\gamma\sin(u\gamma)}{\left(\frac{1}{4}+\gamma^2\right)^2u}.\end{aligned} \tag{31}$$

We also note a result given by Saouter, Trudgian and Demichel [21], which is obtained by applying Lemma 2.2 when $\varphi(t)=\frac{1}{t}$:

$$\begin{aligned}\sum_{2\pi e<\gamma\leq T}\frac{1}{\gamma} &= \frac{1}{2\pi}\int_{2\pi e}^{T}\frac{1}{t}\log\left(\frac{t}{2\pi}\right)dt+\vartheta\left\{\frac{2\log(2\pi e)}{\pi e}+2\int_{2\pi e}^{T}\frac{dt}{t^2}\right\} \\ &= \frac{1}{2\pi}\left[\frac{1}{2}\log^2\left(\frac{t}{2\pi}\right)\right]_{2\pi e}^{T}+\vartheta\left\{\frac{2\log(2\pi e)}{\pi e}+2\left[-\frac{1}{t}\right]_{2\pi e}^{T}\right\} \\ &= \frac{1}{4\pi}\left(\log^2\left(\frac{T}{2\pi}\right)-1\right)+\vartheta\left\{\frac{2\log(2\pi e)+1}{\pi e}\right\} \\ &= \frac{1}{4\pi}\log^2\left(\frac{T}{2\pi}\right)+0.8614\vartheta.\end{aligned}$$

There is only one zeta zero which has imaginary part less than $2\pi e$, so we have

$$\sum_{\gamma\leq T}\frac{1}{\gamma}=\frac{1}{4\pi}\log^2\left(\frac{T}{2\pi}\right)+0.9321\vartheta, \tag{32}$$

this is the sum of the reciprocals of all zeta zeros up to T. As with the proof of Lemma 2.2, we also require that $Q(t)=\vartheta(2\log(t))$ for this result to hold.

6 Proof of our theorem (Theorem 4.1)

6.1 A better integrand

We begin our proof by restating a result used before; the equation relating $\pi(x)$ and $\mathrm{li}(x)$:

$$\begin{aligned}\pi(x)-\mathrm{li}(x) &= -\frac{1}{2}\pi(x^{1/2})-\frac{1}{3}\pi(x^{1/3})-\ldots \\ &\quad -\sum_{\rho}\mathrm{li}(x^{\rho})+\int_{x}^{\infty}\frac{dt}{(t^2-1)t\log(t)}-\log(2), \\ &\geq -\frac{1}{2}\pi(x^{1/2})-\frac{1}{3}\pi(x^{1/3})-\ldots-\sum_{\rho}\mathrm{li}(x^{\rho})-\log(2).\end{aligned} \tag{33}$$

We know that the integral above is positive, and for values of x greater than $e^{43.7}$, the value of the integral is less than 10^{-39}, so we can afford to ignore it in this inequality since some of our error terms will have a much larger magnitude.

Rosser and Schoenfeld[19] discovered an inequality regarding $\pi(x)$ which we will make use of[4], for all $x \geq 17$:

$$\pi(x) < \frac{1.25506x}{\log(x)}. \tag{34}$$

We can further bound the result in (33) with the following inequality, in a similar manner to (7):

$$-\frac{1}{3}\pi(x^{1/3}) - \frac{1}{4}\pi(x^{1/4}) - \ldots \geq -\frac{1}{3}\pi(x^{1/3})\left\lfloor \frac{\log(x)}{\log(2)} \right\rfloor,$$

which gives us

$$\begin{aligned}\pi(x) - \mathrm{li}(x) &\geq -\frac{1}{2}\pi(x^{1/2}) - \frac{1}{3}\pi(x^{1/3})\left\lfloor \frac{\log(x)}{\log(2)} \right\rfloor - \sum_\rho \mathrm{li}(x^\rho) - \log 2,\\ &> -\frac{1}{2}\pi(x^{1/2}) - \frac{1.25506x^{1/3}}{\log(2)} - \sum_\rho \mathrm{li}(x^\rho) - \log 2,\end{aligned}$$

by (34). At this stage, instead of estimating $\pi(x^{1/2})$ by an analytic function as has been done before, we substitute in an expression derived from the equalities for $\Pi(x)$, above:

$$\begin{aligned}\pi(x^{1/2}) + \frac{1}{2}\pi(x^{1/4}) + \frac{1}{3}\pi(x^{1/6}) + \ldots =& \mathrm{li}(x^{1/2}) - \sum_\rho \mathrm{li}(x^{\rho/2})\\ &+ \int_{x^{1/2}}^\infty \frac{dt}{(t^2-1)t\log(t)} - \log(2),\end{aligned}$$

we can then manipulate this to give:

$$\begin{aligned}\pi(x^{1/2}) =& -\frac{1}{2}\pi(x^{1/4}) - \frac{1}{3}\pi(x^{1/6}) - \ldots + \mathrm{li}(x^{1/2}) - \sum_\rho \mathrm{li}(x^{\rho/2})\\ &+ \int_{x^{1/2}}^\infty \frac{dt}{(t^2-1)t\log(t)} - \log(2).\end{aligned}$$

Then we can multiply through by $-\frac{1}{2}$:

$$\begin{aligned}-\frac{1}{2}\pi(x^{1/2}) =& \frac{1}{4}\pi(x^{1/4}) + \frac{1}{6}\pi(x^{1/6}) + \ldots - \frac{1}{2}\mathrm{li}(x^{1/2}) + \frac{1}{2}\sum_\rho \mathrm{li}(x^{\rho/2})\\ &- \frac{1}{2}\int_{x^{1/2}}^\infty \frac{dt}{(t^2-1)t\log(t)} + \frac{1}{2}\log(2),\\ \geq& -\frac{1}{2}\mathrm{li}(x^{1/2}) + \sum_\rho \frac{\mathrm{li}(x^{\rho/2})}{2}.\end{aligned}$$

Once again, the negligibility of the integral means it gets swallowed up in the inequality, as does that of the $x^{1/4}/\log(x)$ term and the $\frac{1}{2}\log(2)$ term[5], we can substitute this into

[4] We could find a much better inequality, however this simple one suits our needs, which will become apparent.

[5] As an example if we let $x = 10^{316}$ (which is in the region we are searching), the dominant term will

our result:

$$\begin{aligned}\pi(x)-\mathrm{li}(x) &> -\frac{1}{2}\,\mathrm{li}(x^{1/2})-\sum_{\rho}\mathrm{li}\left(x^{\rho}\right)+\sum_{\rho}\frac{\mathrm{li}\left(x^{\rho/2}\right)}{2}-\frac{1.25506x^{1/3}}{\log(2)}-\log(2),\\ &> -\frac{1}{2}\,\mathrm{li}(x^{1/2})-\sum_{\rho}\left(\mathrm{li}\left(x^{\rho}\right)-\frac{1}{2}\,\mathrm{li}(x^{\rho/2})\right)-\frac{1.25506x^{1/3}}{\log(2)}-\log(2).\end{aligned}$$

As we did previously, we replace x with e^u, and we also multiply through by the function $ue^{-u/2}$:

$$\begin{aligned}\frac{u\{\pi(e^u)-\mathrm{li}(e^u)\}}{e^{u/2}} > &-\frac{u\,\mathrm{li}\left(e^{u/2}\right)}{2e^{u/2}}-\sum_{\rho}\left(\frac{2u\,\mathrm{li}\left(e^{\rho u}\right)-u\,\mathrm{li}\left(e^{\frac{\rho u}{2}}\right)}{2e^{u/2}}\right)\\ &-\frac{1.25506u}{\log(2)e^{u/6}}-\frac{u\log(2)}{e^{u/2}}.\end{aligned}$$

One should note that we differ here from previous papers; most recently, Saouter, Trudgian and Demichel[21] multiplied their estimate for $\pi(x)-\mathrm{li}(x)$ by a function such that their first term to be integrated became exactly 1. However, it is the above expression we will integrate against the Gaussian kernel:

$$\begin{aligned}&\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)\frac{u\{\pi(e^u)-\mathrm{li}(e^u)\}}{e^{u/2}}du\\ &\qquad > -\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)\frac{u\,\mathrm{li}(e^{u/2})}{2e^{u/2}}du\\ &\qquad\quad -\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)\sum_{\rho}\left(\frac{2u\,\mathrm{li}(e^{\rho u})-u\,\mathrm{li}\left(e^{\frac{\rho u}{2}}\right)}{2e^{u/2}}\right)du\\ &\qquad\quad -\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)\left(\frac{1.25506u}{\log(2)e^{u/6}}+\frac{u\log(2)}{e^{u/2}}\right)du.\end{aligned}\tag{35}$$

We can evaluate the first integral with relative ease, this is our C term in Theorem 4.1. We can bound the third integral, which will be our R_1 error term:

$$\begin{aligned}\left|\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)\left(\frac{1.25506u}{\log(2)e^{u/6}}+\frac{u\log(2)}{e^{u/2}}\right)du\right| &\le \max_{u\in[\omega-\eta,\omega+\eta]}\left[\frac{1.25506u}{\log(2)e^{u/6}}+\frac{u\log(2)}{e^{u/2}}\right]\\ &\le \frac{1.811(\omega+\eta)}{e^{(\omega-\eta)/6}}+\frac{(\omega+\eta)\log(2)}{e^{(\omega-\eta)/2}}\\ &= \frac{1.811(\omega+\eta)}{e^{(\omega-\eta)/6}}+\frac{\log(2)}{e^{(\omega-\eta)/3}}\frac{\omega+\eta}{e^{(\omega-\eta)/6}}\\ &\le \frac{1.812(\omega+\eta)}{e^{(\omega-\eta)/6}},\end{aligned}$$

since $\log(2)/e^{(\omega-\eta)/3}\le 5\times 10^{-7}$ for all $\omega-\eta\ge 43.7$.

be $\mathrm{li}(x^{1/2})=\mathrm{li}(10^{158})\approx 10^{155}$, compared with $x^{1/4}/\log(x)=10^{79}/\log(10^{316})\approx 1.4\times 10^{76}$.

6.2 The R_2 error term

All of our attention is now focused on the second integral in (35). We begin by splitting the integral in two:

$$\begin{aligned}
&-\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\sum_{\rho}\left(\frac{2u\,\mathrm{li}(e^{\rho u})-u\,\mathrm{li}\left(e^{\frac{\rho u}{2}}\right)}{2e^{u/2}}\right)du \\
=&-\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\sum_{\rho}\left(\frac{u\,\mathrm{li}(e^{\rho u})}{e^{u/2}}\right)du+\frac{1}{2}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\sum_{\rho}\left(\frac{u\,\mathrm{li}\left(e^{\frac{\rho u}{2}}\right)}{e^{u/2}}\right)du.
\end{aligned} \tag{36}$$

The second integral in (36) is our second error term, R_2. We utilise the expansion of the logarithmic integral given in (3) and use the result from (30):

$$\begin{aligned}
&\frac{1}{2}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\sum_{\rho}\left(\frac{u\,\mathrm{li}\left(e^{\frac{\rho u}{2}}\right)}{e^{u/2}}\right)du \\
=&\frac{1}{2}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\frac{u}{e^{u/2}}\sum_{\rho}\left(\frac{e^{\rho u/2}}{\rho u/2}+\frac{\vartheta e^{\rho u/2}}{(\rho u/2)^2}\right)du \\
=&\frac{1}{2}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\sum_{|\gamma|>0}\left(\frac{2e^{-u/4+iu\gamma/2}}{\frac{1}{2}+i\gamma}+\frac{4\vartheta e^{-u/4+iu\gamma/2}}{\left(\frac{1}{2}+i\gamma\right)^2 u}\right)du \\
\leq&\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)e^{-u/4}\sum_{\gamma>0}\frac{\cos\left(\frac{u\gamma}{2}\right)+2\gamma\sin\left(\frac{u\gamma}{2}\right)}{\frac{1}{4}+\gamma^2}du \\
&+\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\frac{e^{-u/4}}{u}\sum_{\gamma>0}\left(\frac{4}{\gamma^2}+\frac{4}{\gamma^3}+\frac{1}{\gamma^4}\right)du.
\end{aligned} \tag{37}$$

There are two separate integrals here which we bound individually. We tackle the first one first. We can split this into two separate terms which we will bound separately. The cosine term is absolutely convergent, whilst the sine term is conditionally convergent.

$$\begin{aligned}
&\sum_{\gamma>0}\frac{1}{\frac{1}{4}+\gamma^2}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)e^{-u/4}\cos\left(\frac{u\gamma}{2}\right)du \\
&+\sum_{\gamma>0}\frac{2\gamma}{\frac{1}{4}+\gamma^2}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)e^{-u/4}\sin\left(\frac{u\gamma}{2}\right)du,
\end{aligned} \tag{38}$$

We take the first term, and bound the cosine term by 1:

$$\begin{aligned}
&\left|\sum_{\gamma>0}\frac{1}{\frac{1}{4}+\gamma^2}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)e^{-u/4}\cos\left(\frac{u\gamma}{2}\right)du\right| \\
\leq&\sum_{\gamma>0}\frac{1}{\gamma^2}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)e^{-u/4}du \\
\leq&\sum_{\gamma>0}\frac{1}{\gamma^2}\max_{u\in[\omega-\eta,\omega+\eta]} e^{-u/4} \\
\leq&0.024e^{-(\omega-\eta)/4}.
\end{aligned} \tag{39}$$

We now look at the second term in (38):

$$\sum_{\gamma>0} \frac{2\gamma}{\frac{1}{4}+\gamma^2} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)e^{-u/4} \sin\left(\frac{u\gamma}{2}\right) du,$$

we can estimate the integral using Lemma 2.6:

$$\begin{aligned}
&\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)e^{-u/4} \sin\left(\frac{u\gamma}{2}\right) du \\
=&\left(\int_{-\infty}^{\infty} - \int_{-\infty}^{\omega-\eta} + \int_{\omega+\eta}^{\infty}\right) K(u-\omega)e^{-u/4} \sin\left(\frac{u\gamma}{2}\right) du.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_{-\infty}^{\infty} K(u-\omega)e^{-u/4} \sin\left(\frac{u\gamma}{2}\right) du \\
=&\Im\left[\int_{-\infty}^{\infty} K(u-\omega)e^{-u/4}e^{iu\gamma/2} du\right] \\
=&\Im\left[\int_{-\infty}^{\infty} \sqrt{\frac{\alpha}{2\pi}}\,\mathrm{Exp}\left(-\frac{\alpha}{2}(u-\omega)^2 - \frac{u}{4} + \frac{iu\gamma}{2}\right) du\right] \\
=&\Im\left[\int_{-\infty}^{\infty} \sqrt{\frac{\alpha}{2\pi}}\,\mathrm{Exp}\left(-\frac{\alpha}{2}\left(u^2 - 2u\omega + \omega^2 + \frac{u}{2\alpha} - \frac{iu\gamma}{\alpha}\right)\right) du\right] \\
=&\Im\left[\int_{-\infty}^{\infty} \sqrt{\frac{\alpha}{2\pi}}\,\mathrm{Exp}\left(-\frac{\alpha}{2}\left(\left(u-\omega+\frac{1}{4\alpha} - \frac{i\gamma}{2\alpha}\right)^2\right.\right.\right. \\
&\left.\left.\left.+\ \frac{\omega}{2\alpha} - \frac{i\omega\gamma}{\alpha} + \frac{i\gamma}{4\alpha^2} - \frac{1}{16\alpha^2} + \frac{\gamma^2}{4\alpha^2}\right)\right) du\right] \\
=&\Im\left[\int_{-\infty}^{\infty} K\left(u-\omega+\frac{1}{4\alpha} - \frac{i\gamma}{2\alpha}\right) \mathrm{Exp}\left(-\frac{\omega}{4} + \frac{i\omega\gamma}{2} - \frac{i\gamma}{8\alpha} + \frac{1}{32\alpha} - \frac{\gamma^2}{8\alpha}\right) du\right] \\
=&\Im\left[\mathrm{Exp}\left(-\frac{\omega}{4} + \frac{i\omega\gamma}{2} - \frac{i\gamma}{8\alpha} + \frac{1}{32\alpha} - \frac{\gamma^2}{8\alpha}\right)\right] \\
=&e^{-\omega/4+1/32\alpha-\gamma^2/8\alpha} \sin\left(\frac{\gamma}{2}\left(\omega - \frac{1}{4\alpha}\right)\right),
\end{aligned}$$

since $\int_{-\infty}^{\infty} K(y)dy = 1$, by Lemma 2.7. We also have, via similar argument

$$\begin{aligned}
&\int_{\omega+\eta}^{\infty} K(u-\omega)e^{-u/4} \sin\left(\frac{u\gamma}{2}\right) du \\
=&\int_{\eta}^{\infty} K(x)e^{-(x+\omega)/4} \sin\left(\frac{(x+\omega)\gamma}{2}\right) dx \\
=&\Im\left[\int_{\eta}^{\infty} \sqrt{\frac{\alpha}{2\pi}}\,\mathrm{Exp}\left(-\frac{\alpha x^2}{2} - \frac{x+\omega}{4} + \frac{i(x+\omega)\gamma}{2}\right) dx\right] \\
=&\Im\left[\int_{\eta}^{\infty} \sqrt{\frac{\alpha}{2\pi}}\,\mathrm{Exp}\left(-\frac{\alpha}{2}\left(x^2 - \frac{x}{2\alpha}\right)\right) \mathrm{Exp}\left(\frac{ix\gamma}{2} - \frac{\omega}{4} + \frac{i\omega\gamma}{2}\right) dx\right] \\
=&\Im\left[\int_{\eta}^{\infty} \sqrt{\frac{\alpha}{2\pi}}\,\mathrm{Exp}\left(-\frac{\alpha}{2}\left(x-\frac{1}{4\alpha}\right)^2 - \frac{1}{16\alpha^2}\right) \mathrm{Exp}\left(\frac{ix\gamma}{2} - \frac{\omega}{4} + \frac{i\omega\gamma}{2}\right) dx\right] \\
=&\Im\left[\int_{\eta}^{\infty} K\left(x - \frac{1}{4\alpha}\right) e^{ix\gamma/2}\,\mathrm{Exp}\left(\frac{1}{32\alpha} - \frac{\omega}{4} + \frac{i\omega\gamma}{2}\right) dx\right].
\end{aligned}$$

We can bound this integral with Lemma 5.2, so we have

$$
\begin{aligned}
&= \Im\left[\int_{\eta}^{\infty} K\left(x-\frac{1}{4\alpha}\right) e^{ix\gamma/2} \operatorname{Exp}\left(\frac{1}{32\alpha}-\frac{\omega}{4}+\frac{i\omega\gamma}{2}\right) dx\right] \\
&\leq K\left(\eta-\frac{1}{4\alpha}\right) \min\left\{\frac{1}{\alpha\eta-\frac{1}{4}}, \frac{4}{\gamma}\right\} \left|\operatorname{Exp}\left(\frac{1}{32\alpha}-\frac{\omega}{4}+\frac{i\omega\gamma}{2}\right)\right| \\
&= K\left(\eta-\frac{1}{4\alpha}\right) \min\left\{\frac{1}{\alpha\eta-\frac{1}{4}}, \frac{4}{\gamma}\right\} e^{1/32\alpha-\omega/4}.
\end{aligned}
$$

We can combine our results to give

$$
\begin{aligned}
&\int_{\omega-\eta}^{\omega+\eta} K(u-\omega) e^{-u/4} \sin\left(\frac{u\gamma}{2}\right) du \\
=& e^{-\omega/4+1/32\alpha-\gamma^2/8\alpha} \sin\left(\frac{\gamma}{2}\left(\omega-\frac{1}{4\alpha}\right)\right) \\
&+ 2\vartheta K\left(\eta-\frac{1}{4\alpha}\right) \min\left\{\frac{1}{\alpha\eta-\frac{1}{4}}, \frac{4}{\gamma}\right\} e^{1/32\alpha-\omega/4} \\
\leq& e^{1/32\alpha-\omega/4-\gamma^2/8\alpha} + K\left(\eta-\frac{1}{4\alpha}\right) \min\left\{\frac{2}{\alpha\eta-\frac{1}{4}}, \frac{8}{\gamma}\right\} e^{1/32\alpha-\omega/4},
\end{aligned}
$$

and thus:

$$
\begin{aligned}
&\left|\sum_{\gamma>0} \frac{2\gamma}{\frac{1}{4}+\gamma^2} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) e^{-u/4} \sin\left(\frac{u\gamma}{2}\right) du\right| \\
\leq& 2e^{1/32\alpha-\omega/4} \left(\sum_{\gamma>0} \frac{e^{-\gamma^2/8\alpha}}{\gamma} + K\left(\eta-\frac{1}{4\alpha}\right) \sum_{\gamma>0} \frac{\min\left\{\frac{2}{\alpha\eta-1/4}, \frac{8}{\gamma}\right\}}{\gamma}\right).
\end{aligned}
$$

We have the result that $\min\left\{\frac{2}{\alpha\eta-1/4}, \frac{8}{\gamma}\right\}$ is $\frac{2}{\alpha\eta-1/4}$ for $\gamma < 4\alpha\eta-1$, and $\frac{8}{\gamma}$, otherwise, so our result becomes

$$
\begin{aligned}
&\left|\sum_{\gamma>0} \frac{2\gamma}{\frac{1}{4}+\gamma^2} \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) e^{-u/4} \sin\left(\frac{u\gamma}{2}\right) du\right| \\
\leq& 2e^{1/32\alpha-\omega/4} \left(\sum_{\gamma>0} \frac{e^{-\gamma^2/8\alpha}}{\gamma} + \sum_{0<\gamma\leq 4\alpha\eta-1} \frac{2}{\gamma\left(\alpha\eta-1/4\right)} + \sum_{\gamma>4\alpha\eta-1} \frac{8}{\gamma^2}\right).
\end{aligned}
$$

Each of these terms can be bounded with lemmas from Section 2.3; we have, letting $\delta = 4\alpha$

$$\begin{aligned}\sum_{\gamma>0}\frac{e^{-\gamma^2/8\alpha}}{\gamma} =&\frac{e^{-\gamma_1^2/2\delta}}{\gamma_1}+\sum_{\gamma>\gamma_1}\frac{e^{-\gamma^2/2\delta}}{\gamma}\\ \leq&\frac{e^{-\gamma_1^2/2\delta}}{\gamma_1}+\frac{1}{2\pi}\int_{\gamma_2}^{\infty}\frac{e^{-t^2/2\delta}\log\left(\frac{t}{2\pi}\right)}{t}dt+\frac{4e^{-\gamma_2^2/2\delta}\log(\gamma_2)}{\gamma_2}\\ &+2\int_{\gamma_2}^{\infty}\frac{e^{-t^2/2\delta}}{t^2}dt\\ <&\frac{e^{-\gamma_1^2/2\delta}}{\gamma_1}+\frac{1}{2\pi}\frac{\delta e^{-\gamma_2^2/2\delta}\log\left(\frac{\gamma_2}{2\pi}\right)}{\gamma_2^2}+\frac{4e^{-\gamma_2^2/2\delta}\log(\gamma_2)}{\gamma_2}\\ &+\frac{2\delta e^{-\gamma_2^2/2\delta}}{\gamma_2^2}\\ <&\frac{1}{\gamma_1}+\frac{4}{\gamma_2}\left(\frac{\alpha\log\left(\frac{\gamma_2}{2\pi}\right)}{\pi\gamma_2}+\log(\gamma_2)+\frac{2\alpha}{\gamma_2}\right).\end{aligned} \tag{40}$$

We also have

$$\begin{aligned}\sum_{0<\gamma\leq 4\alpha\eta-1}\frac{2}{\gamma\left(\alpha\eta-\frac{1}{4}\right)} =&\frac{2}{\alpha\eta-\frac{1}{4}}\sum_{0<\gamma\leq 4\alpha\eta-1}\frac{1}{\gamma}\\ \leq&\frac{2}{\alpha\eta-\frac{1}{4}}\left(\log^2\left(\frac{4\alpha\eta-1}{2\pi}\right)+0.9321\right).\end{aligned} \tag{41}$$

Finally, we have

$$\sum_{\gamma>4\alpha\eta-1}\frac{8}{\gamma^2}\leq\frac{8\log(4\alpha\eta-1)}{4\alpha\eta-1}=\frac{2\log(4\alpha\eta-1)}{\alpha\eta-\frac{1}{4}}. \tag{42}$$

Combining the results of (40), (41) and (42),

$$\begin{aligned}&\left|\sum_{\gamma>0}\frac{2\gamma}{\frac{1}{4}+\gamma^2}\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)e^{-u/4}\sin\left(\frac{u\gamma}{2}\right)du\right|\\ \leq&2e^{1/32\alpha-\omega/4}\left(\frac{1}{\gamma_1}+\frac{4}{\gamma_2}\left(\frac{\alpha\log\left(\frac{\gamma_2}{2\pi}\right)}{\pi\gamma_2}+\log(\gamma_2)+\frac{2\alpha}{\gamma_2}\right)\right.\\ &\left.+\frac{2}{\alpha\eta-\frac{1}{4}}\left(\log^2\left(\frac{4\alpha\eta-1}{2\pi}\right)+\log(4\alpha\eta-1)+0.9321\right)\right).\end{aligned} \tag{43}$$

We now look at the final term in (37); that is,

$$\begin{aligned}&\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)\frac{e^{-u/4}}{u}\sum_{\gamma>0}\left(\frac{4}{\gamma^2}+\frac{4}{\gamma^3}+\frac{1}{\gamma^4}\right)du\\ =&\sum_{\gamma>0}\left(\frac{4}{\gamma^2}+\frac{4}{\gamma^3}+\frac{1}{\gamma^4}\right)\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)\frac{e^{-u/4}}{u}du\\ \leq&(4(0.0231055)+4(0.00072955)+0.0000371727)\max_{u\in[\omega-\eta,\omega+\eta]}\frac{e^{-u/4}}{u}\\ \leq&\frac{0.096e^{-(\omega-\eta)/4}}{\omega-\eta}\end{aligned} \tag{44}$$

by Lemma 5.1. Combining (39), (43) and (44) gives us

$$
\begin{aligned}
&\left|\frac{1}{2}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\sum_{\rho}\left(\frac{u\,\mathrm{li}\left(e^{\frac{\rho u}{2}}\right)}{e^{u/2}}\right)du\right|\\
\leq & 0.024e^{-(\omega-\eta)/4}+\frac{0.096e^{-(\omega-\eta)/4}}{\omega-\eta}\\
&+2e^{1/32\alpha-\omega/4}\left(\frac{1}{\gamma_1}+\frac{4}{\gamma_2}\left(\frac{\alpha\log\left(\frac{\gamma_2}{2\pi}\right)}{\pi\gamma_2}+\log(\gamma_2)+\frac{2\alpha}{\gamma_2}\right.\right.\\
&\left.\left.+\frac{2}{\alpha\eta-\frac{1}{4}}\left(\log^2\left(\frac{4\alpha\eta-1}{2\pi}\right)+\log\left(4\alpha\eta-1\right)+0.9321\right)\right)\right)\\
\leq & 0.024e^{-(\omega-\eta)/4}\left(1+\frac{4}{\omega-\eta}\right)+e^{1/32\alpha-\omega/4}(1.301+0.04\alpha)\\
&+\frac{2e^{1/32\alpha-\omega/4}}{\alpha\eta-\frac{1}{4}}\left(\log^2\left(\frac{4\alpha\eta-1}{2\pi}\right)+\log\left(4\alpha\eta-1\right)+0.9321\right),
\end{aligned}
$$

our R_2 error term.

6.3 The R_3 error term

We now look back at equation (36). The second term was our previous error term, so we now look at the first. We split this integral by expanding out the logarithmic integral as in (4), the first three terms of this expansion give us the following integrals.

$$
\begin{aligned}
&-\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\sum_{\rho}\left(\frac{u\,\mathrm{li}(e^{\rho u})}{e^{u/2}}\right)du\\
=&-\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\sum_{\gamma}\left(\frac{e^{iu\gamma}}{\frac{1}{2}+i\gamma}\right)du-\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\sum_{\gamma}\left(\frac{e^{iu\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2u}\right)du \qquad (45)\\
&-\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\sum_{\gamma}\left(\vartheta\frac{2e^{iu\gamma}}{\left(\frac{1}{2}+i\gamma\right)^3u^2}\right)du.
\end{aligned}
$$

One should take note at this stage, that we could expand out the logarithmic integral further, and obtain a greater accuracy at the expense of further error terms. However we stick to three terms for the sake of saving computer processing power and time when it comes to numerical evaluations, and, as we shall see later on, the overall error obtained from three terms is sufficient for our needs.

Of the three terms in the RHS of (45), we split the first up and extract our R_3 and R_4 error terms, as well as something to numerically calculate. The second term gives us another term to calculate numerically and we get our R_5 error term. Finally the third term gives us our R_6 error term.

Our aim in this subsection is to bound the first integral in (45). We begin by splitting up the integral:

$$
\begin{aligned}
&-\int_{\omega-\eta}^{\omega+\eta} K(u-\omega) \sum_{\gamma} \left(\frac{e^{iu\gamma}}{\frac{1}{2}+i\gamma} \right) du \\
&= -\int_{-\infty}^{\infty} K(u-\omega) \sum_{|\gamma|\le T} \left(\frac{e^{iu\gamma}}{\frac{1}{2}+i\gamma} \right) du - \int_{-\infty}^{\infty} K(u-\omega) \sum_{|\gamma|>T} \left(\frac{e^{iu\gamma}}{\frac{1}{2}+i\gamma} \right) du \\
&+ \left(\int_{-\infty}^{\omega-\eta} + \int_{\omega+\eta}^{\infty} \right) K(u-\omega) \sum_{\gamma} \left(\frac{e^{iu\gamma}}{\frac{1}{2}+i\gamma} \right) du.
\end{aligned} \tag{46}
$$

The first term on the RHS of (46) is what will be calculated manually, see Section 8. The second term is what we evaluate in this Subsection, and the third term will become R_4 in the next Subsection.

We begin by combining each zeta zero γ with its complex conjugate $\bar{\gamma}$, as in (30):

$$
-\int_{-\infty}^{\infty} K(u-\omega) \sum_{|\gamma|>T} \left(\frac{e^{iu\gamma}}{\frac{1}{2}+i\gamma} \right) du = -\int_{-\infty}^{\infty} K(u-\omega) \sum_{\gamma>T} \left(\frac{\cos(u\gamma)+2\gamma\sin(u\gamma)}{\frac{1}{4}+\gamma^2} \right) du.
$$

The first term in the sum gives us absolute convergence, and the second has conditional convergence, so we can swap the order of summation and integration:

$$
\begin{aligned}
&\left| \int_{-\infty}^{\infty} K(u-\omega) \sum_{\gamma>T} \left(\frac{\cos(u\gamma)+2\gamma\sin(u\gamma)}{\frac{1}{4}+\gamma^2} \right) du \right| \\
&\qquad = \left| \sum_{\gamma>T} \frac{1}{\frac{1}{4}+\gamma^2} \int_{-\infty}^{\infty} K(u-\omega) \left(\cos(u\gamma)+2\gamma\sin(u\gamma)\right) du \right| \\
&\qquad = \left| \sum_{\gamma>T} \frac{e^{-\gamma^2/2\alpha}}{\frac{1}{4}+\gamma^2} \left(\cos(\omega\gamma)+2\gamma\sin(\omega\gamma)\right) \right| \\
&\qquad < \sum_{\gamma>T} \frac{e^{-\gamma^2/2\alpha}}{\gamma^2} + \sum_{\gamma>T} \frac{2e^{-\gamma^2/2\alpha}}{\gamma}.
\end{aligned}
$$

We can use Lemma 2.5 to give us

$$
\begin{aligned}
&\sum_{\gamma>T} \frac{e^{-\gamma^2/2\alpha}}{\gamma^2} + \sum_{\gamma>T} \frac{2e^{-\gamma^2/2\alpha}}{\gamma} \\
&\qquad < \frac{1}{2\pi} \int_{T}^{\infty} \frac{e^{-t^2/2\alpha}}{t^2} \log\left(\frac{t}{2\pi}\right) dt + \frac{1}{2\pi} \int_{T}^{\infty} \frac{2e^{-t^2/2\alpha}}{t} \log\left(\frac{t}{2\pi}\right) dt \\
&\qquad < \frac{\alpha e^{-T^2/2\alpha}}{2\pi} \log\left(\frac{T}{2\pi}\right) \left(\frac{1}{T^3} + \frac{2}{T^2} \right).
\end{aligned}
$$

This is our R_3 error term.

6.4 The R_4 error term

We now look at the second term of (46):

$$\begin{aligned}
&\left(\int_{-\infty}^{\omega-\eta}+\int_{\omega+\eta}^{\infty}\right)K(u-\omega)\sum_{\gamma}\left(\frac{e^{iu\gamma}}{\frac{1}{2}+i\gamma}\right)du\\
=&\left(\int_{-\infty}^{\omega-\eta}+\int_{\omega+\eta}^{\infty}\right)K(u-\omega)\sum_{\gamma>0}\frac{\cos(u\gamma)+2\gamma\sin(u\gamma)}{\frac{1}{4}+\gamma^2}du.
\end{aligned}$$

We perform the change of variables $x=u+\omega$, followed by applying Lemma 5.2:

$$\begin{aligned}
&\left|\left(\int_{-\infty}^{\omega-\eta}+\int_{\omega+\eta}^{\infty}\right)K(u-\omega)\sum_{\gamma>0}\frac{\cos(u\gamma)+2\gamma\sin(u\gamma)}{\frac{1}{4}+\gamma^2}du\right|\\
=&\left|\sum_{\gamma>0}\frac{1}{\frac{1}{4}+\gamma^2}\left(\int_{-\infty}^{-\eta}+\int_{\eta}^{\infty}\right)K(x)(\cos((x+\omega)\gamma)+2\gamma\sin((x+\omega)\gamma))dx\right|\\
<&2K(\eta)\sum_{\gamma>0}\frac{1+2\gamma}{\gamma^2}\min\left\{\frac{1}{\alpha\eta},\frac{2}{\gamma}\right\}\\
=&\frac{2K(\eta)}{\alpha\eta}\sum_{0<\gamma\le 2\alpha\eta}\left(\frac{1}{\gamma^2}+\frac{2}{\gamma}\right)+2K(\eta)\sum_{\gamma>2\alpha\eta}\left(\frac{2}{\gamma^3}+\frac{4}{\gamma^2}\right)
\end{aligned}$$

by Lemma 5.2. We can in turn bound this using Lemmas 2.3 and 5.1, and (32):

$$\begin{aligned}
&\frac{2K(\eta)}{\alpha\eta}\sum_{0<\gamma\le 2\alpha\eta}\left(\frac{1}{\gamma^2}+\frac{2}{\gamma}\right)+2K(\eta)\sum_{\gamma>2\alpha\eta}\left(\frac{2}{\gamma^3}+\frac{4}{\gamma^2}\right)\\
<&\frac{2K(\eta)}{\alpha\eta}\left(0.0231055+\frac{1}{2\pi}\log^2\left(\frac{\alpha\eta}{\pi}\right)+1.8642\right)+2K(\eta)\left(\frac{2\log(2\alpha\eta)}{(2\alpha\eta)^2}+\frac{4\log(2\alpha\eta)}{2\alpha\eta}\right)\\
<&\frac{2K(\eta)}{\alpha\eta}\left(\frac{1}{2\pi}\log^2\left(\frac{\alpha\eta}{\pi}\right)+1.888+\frac{\log(2\alpha\eta)}{2\alpha\eta}+2\log(2\alpha\eta)\right)\\
<&\frac{K(\eta)}{\alpha\eta}\left(\frac{1}{\pi}\log^2\left(\frac{\alpha\eta}{\pi}\right)+4\log(2\alpha\eta)+4.52\right),
\end{aligned}$$

since $\log(x)/x\le\frac{1}{e}$. This is our R_4 error term.

6.5 The R_5 error term

We now look at the second term in (45). Since the sum has a factor of γ^2 in the denominator we have absolute convergence everywhere so we can freely swap the orders of summation and integration:

$$
\begin{aligned}
&-\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\sum_{\gamma}\left(\frac{e^{iu\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2 u}\right)du\\
&=-\sum_{\gamma}\left(\frac{e^{i\omega\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2 \omega}\right)\int_{-\eta}^{\eta}\frac{K(x)e^{ix\gamma}}{1+\frac{x}{\omega}}dx\\
&=-\sum_{\gamma}\left(\frac{e^{i\omega\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2 \omega}\right)\left(\int_{-\eta}^{\eta}K(x)e^{ix\gamma}dx+\frac{\vartheta}{\omega}\int_{-\eta}^{\eta}|x|K(x)dx\right)\\
&=-\sum_{\gamma}\left(\frac{e^{i\omega\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2 \omega}\right)\left(\left(\int_{-\infty}^{\infty}-\int_{-\infty}^{-\eta}-\int_{\eta}^{\infty}\right)K(x)e^{ix\gamma}dx+\frac{2\vartheta}{\omega}\int_{0}^{\infty}xK(x)dx\right)\\
&=-\sum_{|\gamma|\le M}\left(\frac{e^{i\omega\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2 \omega}\right)\int_{-\infty}^{\infty}K(x)e^{ix\gamma}dx-\sum_{|\gamma|>M}\left(\frac{e^{i\omega\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2 \omega}\right)\int_{-\infty}^{\infty}K(x)e^{ix\gamma}dx\\
&\quad+\sum_{\gamma}\left(\frac{e^{i\omega\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2 \omega}\right)\left(\int_{-\infty}^{-\eta}+\int_{\eta}^{\infty}\right)K(x)e^{ix\gamma}dx\\
&\quad-\sum_{\gamma}\left(\frac{e^{i\omega\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2 \omega}\right)\frac{2\vartheta}{\omega}\int_{0}^{\infty}xK(x)dx.
\end{aligned}
\tag{47}
$$

In the RHS of (47), the first term is the second term which will be calculated numerically, whilst the remaining three become our R_5 error term.

We will look at the first of these three terms first:

$$-\sum_{|\gamma|>M}\left(\frac{e^{i\omega\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2 \omega}\right)\int_{-\infty}^{\infty}K(x)e^{ix\gamma}dx,$$

the integral can be evaluated by Lemma 2.6. We bound the sum as follows:

$$
\begin{aligned}
\left|\sum_{|\gamma|>M}\left(\frac{e^{i\omega\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2 \omega}\right)\int_{-\infty}^{\infty}K(x)e^{ix\gamma}dx\right| &= \left|\sum_{|\gamma|>M}\frac{e^{i\omega\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2 \omega}e^{-\gamma^2/2\alpha}\right|\\
&\le \sum_{\gamma>M}\frac{e^{-\gamma^2/2\alpha}}{\omega\gamma^2},
\end{aligned}
$$

this, in turn, can be bounded by Lemmas 2.2 and 2.5:

$$
\begin{aligned}
\sum_{\gamma>M}\frac{e^{-\gamma^2/2\alpha}}{\omega\gamma^2} \le&\frac{1}{2\pi\omega}\int_{M}^{\infty}\frac{e^{-t^2/2\alpha}}{t^4}\log\left(\frac{t}{2\pi}\right)dt+\frac{4e^{-M^2/2\alpha}\log(M)}{M^2}+2\int_{M}^{\infty}\frac{e^{-t^2/2\alpha}}{t^3}dt\\
<&\frac{1}{2\pi\omega}\frac{\alpha}{M}\frac{e^{-M^2/2\alpha}\log\left(\frac{M}{2\pi}\right)}{M^2}+\frac{4e^{-M^2/2\alpha}\log(M)}{M^2}+\frac{2\alpha}{M}\frac{e^{-M^2/2\alpha}}{M^3}\\
=&\frac{\alpha e^{-M^2/2\alpha}\log\left(\frac{M}{2\pi}\right)}{2\pi\omega M^3}+\frac{4e^{-M^2/2\alpha}\log(M)}{M^2}+\frac{2\alpha e^{-M^2/2\alpha}}{M^4}\\
<&\frac{2\alpha e^{-M^2/2\alpha}\log\left(\frac{M}{2\pi}\right)}{2\pi\omega M^3}.
\end{aligned}
$$

The last two terms here are absorbed by the first, giving us the first part of R_5. We next look at the next term in (47):

$$-\sum_{\gamma}\left(\frac{e^{i\omega\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2\omega}\right)\left(\int_{-\infty}^{-\eta}+\int_{\eta}^{\infty}\right)K(x)e^{ix\gamma}dx,$$

We can bound the sum as before, and we bound the integral using Lemmas 2.3, 5.1 and 5.2:

$$\begin{aligned}
&\left|\sum_{\gamma}\left(\frac{e^{i\omega\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2\omega}\right)\left(\int_{-\infty}^{-\eta}+\int_{\eta}^{\infty}\right)K(x)e^{ix\gamma}dx\right| \\
\le&\sum_{\gamma>0}\frac{2K(\eta)}{\omega\gamma^2}\min\left\{\frac{1}{\alpha\eta},\frac{2}{\gamma}\right\} \\
=&\frac{2K(\eta)}{\alpha\omega\eta}\sum_{0<\gamma\le 2\alpha\eta}\frac{1}{\gamma^2}+\frac{4K(\eta)}{\omega}\sum_{\gamma>2\alpha\eta}\frac{1}{\gamma^3} \\
\le&\frac{0.047K(\eta)}{\alpha\omega\eta}+\frac{4K(\eta)\log(2\alpha\eta)}{4\alpha^2\omega\eta^2} \\
\le&\frac{K(\eta)}{\alpha\omega\eta}\left(0.047+\frac{1}{\alpha\eta}\right),
\end{aligned}$$

our second term in R_5. Finally we tackle the last term in (47):

$$-\sum_{\gamma}\left(\frac{e^{i\omega\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2\omega}\right)\frac{2\vartheta}{\omega}\int_{0}^{\infty}xK(x)dx.$$

Once again, we bound the sum as previously, and explicitly evaluate the integral:

$$\begin{aligned}
&\sum_{\gamma}\left(\frac{e^{i\omega\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2\omega}\right)\frac{2\vartheta}{\omega}\int_{0}^{\infty}\sqrt{\frac{\alpha}{2\pi}}xe^{-\alpha x^2/2}dx \\
\le&\sum_{\gamma>0}\frac{2\vartheta}{\omega^2\gamma^2}\left[-\frac{K(x)}{\alpha}\right]_0^{\infty} \\
\le&\frac{\sqrt{2}}{\sqrt{\pi\alpha}\omega^2}\sum_{\gamma>0}\frac{1}{\gamma^2} \\
<&\frac{0.019}{\sqrt{\alpha}\omega^2}.
\end{aligned}$$

Combining the terms we have obtained gives us

$$\frac{2\alpha e^{-M^2/2\alpha}\log\left(\frac{M}{2\pi}\right)}{2\pi\omega M^3}+\frac{K(\eta)}{\alpha\omega\eta}\left(0.047+\frac{1}{\alpha\eta}\right)+\frac{0.019}{\sqrt{\alpha}\omega^2},$$

our R_5 error term.

6.6 The R_6 error term

The term we will look at next is the third and final term of (45):

$$-\int_{\omega-\eta}^{\omega+\eta}K(u-\omega)\sum_{\gamma}\left(\vartheta\frac{2e^{iu\gamma}}{\left(\frac{1}{2}+i\gamma\right)^3u^2}\right)du.$$

We have absolute convergence throughout this term, so we can rearrange accordingly,

$$\begin{aligned}\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\sum_{\gamma}\left|\vartheta\frac{2e^{iu\gamma}}{\left(\frac{1}{2}+i\gamma\right)^{3}u^{2}}\right|du \leq & \sum_{\gamma>0}\frac{4}{\gamma^{3}}\int_{\omega-\eta}^{\omega+\eta}\frac{K(u-\omega)}{u^{2}}du\\ \leq & 2.92\times 10^{-3}\min_{u\in[\omega-\eta,\omega+\eta]}\frac{1}{u^{2}}du\\ = & \frac{2.92\times 10^{-3}}{(\omega-\eta)^{2}}.\end{aligned}$$

Identifying this final error term completes the proof of Theorem 4.1.

7 Optimisation

In this section we prove Corollary 4.1, which deals with optimising the parameters. The parameters we are looking to optimise are α and η.

7.1 Initial bounds

Before we begin, we state a few loose restrictions on these parameters, they will help us tighten α and η later on. We will require that $\alpha\eta^2 > 20$, this is to assure that the term $e^{-\alpha\eta^2/2}$ (which crops up a lot) is sufficiently small.

Next, we state that

$$10^{15} < \alpha < 4.7\times 10^{20} \quad \text{and} \quad 2.07\times 10^{-10} < \eta < 6.4\times 10^{-6}. \tag{48}$$

The lower restriction on α and upper restriction on η comes from the fact that we are aiming for this to improve upon previous work, which relies on achieving an $\alpha > 10^{15}$ and $\eta < 6.4\times 10^{-6}$, as used by Saouter, Trudgian and Demichel [21]. The upper restriction on α comes from the requirement that $-\frac{T^2}{2\alpha} < -1$, and the lower restriction on η comes directly from $\alpha\eta^2 > 20$.

Finally, we will bound the total error from all terms, E, by 1.

We also make note that at the time of writing, we have access to just over 10^{11} zeros of the Riemann zeta function, corresponding to a maximum possible T of $30,610,046,000$.

We will now look at each error term in turn.

7.2 R_1

Our first error term

$$\frac{1.812(\omega+\eta)}{e^{(\omega-\eta)/6}}$$

needs little optimising, simply due to the fact that our previous restriction of $\omega-\eta \geq 43.7$, (see Theorem 4.1), means that our R_1 error term is never greater than 0.055. Furthermore, since we are mainly searching in regions where ω is greater than 700, we can rest assured that our first error term is comfortably less than 10^{-45}.

In their paper, Bays and Hudson [3] produced a plot ranging from 10^6 to 10^{400} of $\mathrm{li}(x)-\pi(x)$, the smallest crossover point they make any note of is in the vicinity of e^{405}, by setting our $\omega > 400$ we get both coverage of every area Bays and Hudson noted, and it means our R_1 error is bounded above by 10^{-26}. We create a similar plot in Appendix D.

7.3 R_2

Our second error term

$$0.024e^{-(\omega-\eta)/4}\left(1+\frac{4}{\omega-\eta}\right)+e^{1/32\alpha-\omega/4}(1.301+0.04\alpha)$$
$$+\frac{2e^{1/32\alpha-\omega/4}}{\alpha\eta-\frac{1}{4}}\left(\log^2\left(\frac{4\alpha\eta-1}{2\pi}\right)+\log\left(4\alpha\eta-1\right)+0.9321\right),$$

also needs little optimising. For $\omega > 400$, the first term is a lot less than 10^{-40}, so we are safe with there. Combining the second and third terms gives us a better idea of which terms are largest:

$$e^{1/32\alpha-\omega/4}\left[1.301+0.04\alpha+\frac{2\log^2\left(\frac{4\alpha\eta-1}{2\pi}\right)}{\alpha\eta-\frac{1}{4}}+\frac{2\log\left(4\alpha\eta-1\right)}{\alpha\eta-\frac{1}{4}}+\frac{1.8642}{\alpha\eta-\frac{1}{4}}\right].$$

We can easily deduce that the second term is the largest. Our condition that $\alpha\eta^2$ is greater than 20 ensures that the third and fourth terms do not grow anywhere near as large as the second.

We require little optimising here since $0.04\alpha e^{1/32\alpha-\omega/4}$ is a lot less than 10^{-20} for $\omega \geq 400$ and $\alpha \leq 5\times 10^{20}$.

7.4 R_5

The fifth error term is

$$\frac{2\alpha e^{-M^2/2\alpha}\log\left(\frac{M}{2\pi}\right)}{2\pi\omega M^3}+\frac{K(\eta)}{\alpha\omega\eta}\left(0.047+\frac{1}{\alpha\eta}\right)+\frac{0.019}{\sqrt{\alpha}\omega^2}.$$

We will split this up into the three terms and check each in turn. The first term

$$\frac{2\alpha e^{-M^2/2\alpha}\log\left(\frac{M}{2\pi}\right)}{2\pi\omega M^3}$$

is less than 7.51×10^{-9} for all $\alpha < 5\times 10^{20}$, $\omega > 400$ and $M > 10^9$, so we are safe here.

The second term

$$\frac{K(\eta)}{\alpha\omega\eta}\left(0.047+\frac{1}{\alpha\eta}\right)=\frac{e^{-\alpha\eta^2/2}}{\sqrt{2\pi}\omega\sqrt{\alpha\eta^2}}\left(0.047+\frac{1}{\alpha\eta}\right)$$

relies on the fact that $\alpha\eta^2 > 20$, from this we have that the term outside the brackets is bounded above by 1.02×10^{-8} for $\omega > 400$, and the term inside the brackets is bounded above by $0.047+4.84\times 10^{-6} < 0.048$, so the error for the whole term is less than 4.896×10^{-10}.

The final term

$$\frac{0.019}{\sqrt{\alpha}\omega^2}$$

is less than 3.8×10^{-15} for all $\alpha > 10^{15}$ and $\omega > 400$.

Combining these results means that R_5 never exceeds 8×10^{-9}.

7.5 R_6

The final error term is

$$\frac{2.92 \times 10^{-3}}{(\omega - \eta)^2},$$

which we can bound simply by

$$\frac{2.92 \times 10^{-3}}{\omega^2}.$$

This value is less than 1.825×10^{-8} for all $\omega > 400$, (and for $\omega = 728$, larger than any ω we will be checking, we have the result less than 5.51×10^{-9}). This is the largest error term which we will not be optimising.

7.6 Remaining terms

Until now we have been rather rash with our optimising, this is because the previous error terms are almost negligible. The remaining error terms are where we will have to be a little more careful, we have $f(\alpha) = R_3$ and $g(\alpha, \eta) = R_4$:

$$\begin{aligned} f(\alpha) &= \frac{\alpha e^{-T^2/2\alpha}}{2\pi} \log\left(\frac{T}{2\pi}\right)\left(\frac{1}{T^3} + \frac{2}{T^2}\right) \\ g(\alpha, \eta) &= \frac{K(\eta)}{\alpha\eta}\left(\frac{1}{\pi}\log^2\left(\frac{\alpha\eta}{\pi}\right) + 4\log(2\alpha\eta) + 4.52\right). \end{aligned}$$

The function f is increasing for α and g is a decreasing function for both α and η.

Combining all of our results thus far from this section, we can bound our total error E, we have, for $\omega > 400$:

$$f(\alpha) + g(\alpha, \eta) \leq E - 2.625 \times 10^{-8}.$$

We will find an upper bound for α; suppose we want f to be bounded by the RHS of this sum $E - 2.625 \times 10^{-8}$, we let $A = \frac{1}{2\pi}\log\left(\frac{T}{2\pi}\right)\left(\frac{1}{T^3} + \frac{2}{T^2}\right)$, for notational convenience:

$$\begin{aligned} A\alpha e^{-T^2/2\alpha} &\leq E - 2.625 \times 10^{-8} \\ \frac{1}{\alpha} e^{T^2/2\alpha} &\geq \frac{A}{E - 2.625 \times 10^{-8}} \\ \frac{T^2}{2\alpha} e^{T^2/2\alpha} &\geq \frac{T^2 A}{2E - 5.25 \times 10^{-8}} \\ \frac{T^2}{2\alpha} &\geq W\left(\frac{T^2 A}{2E - 5.25 \times 10^{-8}}\right) \\ \alpha &\leq \frac{T^2}{2W\left(\frac{T^2 A}{2E - 5.25 \times 10^{-8}}\right)}, \end{aligned}$$

where $W(x)$ is the Lambert W-function. We want $f(\alpha)$ to be as close to E as possible, so we turn this result into an equality, giving us:

$$\alpha = \frac{T^2}{2W\left(\frac{\log\left(\frac{T}{2\pi}\right)\left(2+\frac{1}{T}\right)}{4\pi E - 3.3\times 10^{-7})}\right)},$$

where

$$W(x) \sim \log(x) - \log\log(x),$$

We have a function which determines α and is dependent only on T and E, both of which we have full control over.

To ensure that α is defined, we must place a lower bound on E to prevent the argument of $W(x)$ from becoming negative, so we will state that

$$4\pi E - 3.3 \times 10^{-7} > 0 \qquad \Longleftrightarrow \qquad E > 4.15 \times 10^{-6}.$$

We now look at $g(\alpha, \eta)$, we now have our value of α, so this is essentially solely a function of η, which is decreasing:

$$\begin{aligned}\frac{K(\eta)}{\alpha\eta}\left(\frac{1}{\pi}\log^2\left(\frac{\alpha\eta}{\pi}\right) + 4\log(2\alpha\eta) + 4.52\right) &\leq K(\eta)(0.00019 + 0.00025 + 0.00003)\\ &\leq 0.00047K(\eta)\end{aligned}$$

by (48), and the fact that $\log^2(x)/x$ is decreasing for all $x > e^2$, and $\log(x)/x$ is decreasing for all $x > e$. We want to bound this from above by a value small in magnitude compared to E, so we choose E^2:

$$\begin{aligned}0.00047K(\eta) = \frac{0.00047\sqrt{\alpha}e^{-\alpha\eta^2/2}}{\sqrt{2\pi}} &\leq E^2\\ e^{-\alpha\eta^2/2} &\leq \frac{E^2\sqrt{2\pi}}{0.00047\sqrt{\alpha}}\\ -\alpha\eta^2/2 &\leq \log\left(\frac{E^2\sqrt{2\pi}}{0.00047\sqrt{\alpha}}\right)\\ \eta^2 &\geq \frac{2}{\alpha}\log\left(\frac{0.00047\sqrt{\alpha}}{E^2\sqrt{2\pi}}\right)\\ \eta &\geq \sqrt{\frac{2}{\alpha}\log\left(\frac{0.00019\sqrt{\alpha}}{E^2}\right)}.\end{aligned}$$

Once again, we turn this into an equality, to give our value for η, dependent on α and E only:

$$\eta = \sqrt{\frac{2}{\alpha}\log\left(\frac{0.00019\sqrt{\alpha}}{E^2}\right)}.$$

For $\alpha > 2.8 \times 10^7$ and $E < 1$, the value of this logarithm is positive, so η is well defined.

We now have our values for α and η, completing the proof.

7.7 Remark

At this stage, the reader may question why we put more effort into getting the largest α possible and finding a suitable cooperating η and not vice versa. We wished to use the largest α we could in an attempt to get the most accurate numeric results, in doing so we had to choose a larger η to assure our total error E did not exceed what we wanted. An increase in η provides negligible increase in C in (27).

If we had chosen to minimise η, we would have to choose a smaller α to compensate, which in turn would provide a less accurate numeric result.

8 Numerical evaluation

When we come to sum our zeta zeros, we will have to deal with round-off error; that is, the error from computing using each zeta zero to a finite accuracy. We apply Lemma 2.7, followed by equations (30) and (31) to the first terms of (46) and (47), respectively, to obtain the following:

$$-\int_{-\infty}^{\infty} K(u-\omega) \sum_{|\gamma|\leq T} \left(\frac{e^{iu\gamma}}{\frac{1}{2}+i\gamma} \right) du = -\sum_{|\gamma|\leq T} \frac{e^{i\omega\gamma}}{\frac{1}{2}+i\gamma} e^{-\gamma^2/2\alpha},$$

and

$$-\int_{-\infty}^{\infty} K(u-\omega) \sum_{|\gamma|\leq M} \left(\frac{e^{iu\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2 u} \right) du = -\sum_{|\gamma|\leq M} \frac{e^{i\omega\gamma}}{\left(\frac{1}{2}+i\gamma\right)^2 u} e^{-\gamma^2/2\alpha}.$$

These, in turn, give us

$$S_1 = -\sum_{0<\gamma\leq T} \left(\frac{e^{-\gamma^2/2\alpha}}{\frac{1}{4}+\gamma^2} (\cos(\omega\gamma) + 2\gamma \sin(\omega\gamma)) \right) \tag{49}$$

and

$$S_2 = -\sum_{0<\gamma\leq M} \left(\frac{e^{-\gamma^2/2\alpha}}{(\frac{1}{4}+\gamma^2)^2\omega} \left(\left(\frac{1}{2} - 2\gamma^2 \right) \cos(\omega\gamma) + 2\gamma \sin(\omega\gamma) \right) \right). \tag{50}$$

We let

$$t(\gamma) := e^{-\gamma^2/2\alpha} \frac{(\cos(\omega\gamma) + 2\gamma\sin(\omega\gamma))}{\frac{1}{4}+\gamma^2}$$

and

$$m(\gamma) := e^{-\gamma^2/2\alpha} \frac{\left((\frac{1}{2} - 2\gamma^2) \cos(\omega\gamma) + 2\gamma \sin(\omega\gamma) \right)}{(\frac{1}{4}+\gamma^2)^2\omega}.$$

We denote by S_1^* and S_2^* the sums which we calculate using our estimates for each zeta zero. We will let γ^* be our estimate for γ, we wish to find values for $|t(\gamma^*) - t(\gamma)|$ and $|m(\gamma^*) - m(\gamma)|$, which we can do by using the Mean Value Theorem, which states that

$$|t'(\bar{\gamma})| = \frac{|t(\gamma^*) - t(\gamma)|}{|\gamma^* - \gamma|} \iff |t(\gamma^*) - t(\gamma)| = |\gamma^* - \gamma| \cdot |t'(\bar{\gamma})|,$$

where $\bar{\gamma}$ is such that $|\bar{\gamma} - \gamma| < |\gamma^* - \gamma|$. The same equality holds with $m'(\bar{\gamma})$. We have

$$\begin{aligned} t'(\gamma) =& e^{-\gamma^2/2\alpha}\left[-\frac{\gamma}{\alpha}\left(\frac{(\cos(\omega\gamma)+2\gamma\sin(\omega\gamma))}{\frac{1}{4}+\gamma^2}\right) - \frac{(2\gamma)(\cos(\omega\gamma)+2\gamma\sin(\omega\gamma))}{(\frac{1}{4}+\gamma^2)^2}\right. \\ &\left. + \frac{(-\omega\sin(\omega\gamma)+2\sin(\omega\gamma)+2\omega\gamma\cos(\omega\gamma))(\frac{1}{4}+\gamma^2)}{(\frac{1}{4}+\gamma^2)^2}\right] \\ =& e^{-\gamma^2/2\alpha}\left(\frac{(2\omega\gamma-\gamma/\alpha)(\cos(\omega\gamma)+(2-\omega-2\gamma^2/\alpha)\sin(\omega\gamma))}{\frac{1}{4}+\gamma^2}\right. \\ &\left. - \frac{2\gamma\cos(\omega\gamma)+4\gamma^2\sin(\omega\gamma)}{(\frac{1}{4}+\gamma^2)^2}\right). \end{aligned}$$

We can bound the exponential and trigonometric terms by 1, and bound the absolute value of the derivative as such

$$\begin{aligned} |t'(\gamma)| \leq& \left(\frac{2\omega\gamma+\gamma/\alpha+2+\omega+2\gamma^2/\alpha)}{\frac{1}{4}+\gamma^2} + \frac{2\gamma+4\gamma^2}{(\frac{1}{4}+\gamma^2)^2}\right) \\ <& \left(\frac{2\omega}{\gamma}+\frac{1}{\alpha\gamma}+\frac{2}{\gamma^2}+\frac{\omega}{\gamma^2}+\frac{2}{\alpha}+\frac{2}{\gamma^3}+\frac{4}{\gamma^2}\right) \\ =& \frac{1}{\gamma}\left(2\omega+\frac{1}{\alpha}\right)+\frac{1}{\gamma^2}(6+\omega)+\frac{2}{\gamma^3}+\frac{2}{\alpha}. \end{aligned}$$

From this, we get

$$|S_1 - S_1^*| \leq \sum_{0<\gamma\leq T} |t(\gamma^*) - t(\gamma)| = \sum_{0<\gamma\leq T} |\gamma^* - \gamma| \cdot |t'(\bar{\gamma})|.$$

Our resource for zeta zeros comes from David Platt, and the L-Function and Modular Forms Database (LMFDB), where the accuracy of each zeta zero is known up to 30 decimal places, so we have that $|\gamma^* - \gamma| < 10^{-30}$:

$$\begin{aligned} |S_1 - S_1^*| \leq& 10^{-30}\sum_{0<\gamma\leq T}\left[\frac{1}{\gamma}\left(2\omega+\frac{1}{\alpha}\right)+\frac{1}{\gamma^2}(6+\omega)+\frac{2}{\gamma^3}+\frac{2}{\alpha}\right] \\ <& 10^{-30}\left[\left(2\omega+\frac{1}{\alpha}\right)\left(\frac{1}{4\pi}\log^2\left(\frac{T}{2\pi}\right)+0.9321\right)+0.02311\omega+\frac{2}{\alpha}+0.1401\right] \\ <& 10^{-30}\left[\frac{2\omega+\frac{1}{\alpha}}{4\pi}\log^2\left(\frac{T}{2\pi}\right)+1.888\omega+\frac{3}{\alpha}+0.15\right], \end{aligned}$$

by Lemma 5.1. We can bound the second sum's error in a similar manner.

$$\begin{aligned}
m'(\gamma) =& e^{-\gamma^2/2\alpha}\left[-\frac{\gamma}{\alpha}\left(\frac{\left(\left(\frac{1}{2}-2\gamma^2\right)\cos(\omega\gamma)+2\gamma\sin(\omega\gamma)\right)}{(\frac{1}{4}+\gamma^2)^2\omega}\right)\right.\\
&+\frac{(-\frac{\omega}{2}\sin(\omega\gamma)-4\gamma\cos(\omega\gamma)+2\omega\gamma^2\sin(\omega\gamma)+2\sin(\omega\gamma)+2\omega\gamma\cos(\omega\gamma))}{(\frac{1}{4}+\gamma^2)\omega}\\
&\left.-\frac{\left(\left(\frac{1}{2}-2\gamma^2\right)\cos(\omega\gamma)+2\gamma\sin(\omega\gamma)\right)\left(\omega\gamma+4\omega\gamma^3\right)}{(\frac{1}{4}+\gamma^2)^4\omega^2}\right]\\
=& e^{-\gamma^2/2\alpha}\left[\frac{\left(\frac{2\gamma^3}{\alpha}-\frac{\gamma}{2\alpha}-4\gamma+2\omega\gamma\right)\cos(\omega\gamma)+\left(2\omega\gamma^2-\frac{2\gamma^2}{\alpha}-\frac{\omega}{2}+2\right)\sin(\omega\gamma)}{(\frac{1}{4}+\gamma^2)^2\omega}\right.\\
&\left.-\frac{\left(\frac{\omega\gamma}{2}-8\omega\gamma^5\right)\cos(\omega\gamma)+\left(2\omega\gamma^2+8\omega\gamma^4\right)\sin(\omega\gamma)}{(\frac{1}{4}+\gamma^2)^4\omega^2}\right].
\end{aligned}$$

We can bound this as before:

$$\begin{aligned}
|m'(\gamma)| \le& \frac{\frac{2\gamma^3}{\alpha}+\frac{\gamma}{2\alpha}+4\gamma+2\omega\gamma+2\omega\gamma^2+\frac{2\gamma^2}{\alpha}+\frac{\omega}{2}+2}{(\frac{1}{4}+\gamma^2)^2\omega}\\
&+\frac{\frac{\omega\gamma}{2}+8\omega\gamma^5+2\omega\gamma^2+8\omega\gamma^4}{(\frac{1}{4}+\gamma^2)^4\omega^2}\\
<& \frac{2}{\alpha\omega\gamma}+\frac{1}{2\alpha\omega\gamma^3}+\frac{4}{\omega\gamma^3}+\frac{2}{\gamma^3}+\frac{2}{\gamma^2}+\frac{2}{\alpha\omega\gamma^2}\\
&+\frac{1}{2\gamma^4}+\frac{2}{\omega\gamma^4}+\frac{1}{2\omega\gamma^7}+\frac{8}{\omega\gamma^3}+\frac{2}{\omega\gamma^6}+\frac{8}{\omega\gamma^4}\\
=& \frac{1}{\gamma}\left(\frac{2}{\alpha\omega}\right)+\frac{1}{\gamma^2}\left(2+\frac{2}{\alpha\omega}\right)+\frac{1}{\gamma^3}\left(\frac{1}{2\alpha\omega}+\frac{12}{\omega}+2\right)\\
&+\frac{1}{\gamma^4}\left(\frac{1}{2}+\frac{10}{\omega}\right)+\frac{1}{\gamma^6}\left(\frac{2}{\omega}\right)+\frac{1}{\gamma^7}\left(\frac{1}{2\omega}\right).
\end{aligned}$$

We can now bound the error of our second sum

$$\begin{aligned}
|S_2-S_2^*| \le& 10^{-30}\sum_{0<\gamma\le M}\left[\frac{1}{\gamma}\left(\frac{2}{\alpha\omega}\right)+\frac{1}{\gamma^2}\left(2+\frac{2}{\alpha\omega}\right)+\frac{1}{\gamma^3}\left(\frac{1}{2\alpha\omega}+\frac{12}{\omega}+2\right)\right.\\
&\left.+\frac{1}{\gamma^4}\left(\frac{1}{2}+\frac{10}{\omega}\right)+\frac{1}{\gamma^6}\left(\frac{2}{\omega}\right)+\frac{1}{\gamma^7}\left(\frac{1}{2\omega}\right)\right]\\
<& 10^{-30}\left[\frac{1}{2\alpha\omega\pi}\log^2\left(\frac{M}{2\pi}\right)+\frac{0.05}{\alpha\omega}+\frac{0.01}{\omega}+0.048\right].
\end{aligned}$$

So our total numerical error δS is bounded by

$$\begin{aligned}
&10^{-30}\left[\frac{2\omega+\frac{1}{\alpha}}{4\pi}\log^2\left(\frac{T}{2\pi}\right)+\frac{1}{2\alpha\omega\pi}\log^2\left(\frac{M}{2\pi}\right)+1.888\omega+\frac{3}{\alpha}+\frac{0.05}{\alpha\omega}+\frac{0.01}{\omega}+0.2\right]\\
<&10^{-30}\left[\frac{2\omega+\frac{1}{\alpha}}{4\pi}\log^2\left(\frac{T}{2\pi}\right)+\frac{1}{2\alpha\omega\pi}\log^2\left(\frac{M}{2\pi}\right)+2\omega\right].
\end{aligned}$$

Since we have (at the time of writing) our largest T as $30,610,046,000$, and we have $43.7 < \omega < 728$, so

$$\begin{aligned}\delta S = |S_1 - S_1^*| + |S_2 - S_2^*| <& 10^{-30}\left(116\log^2\left(\frac{T}{2\pi}\right) + 4\times 10^{-3}\log^2\left(\frac{M}{2\pi}\right) + 1456\right)\\ <& 10^{-30}\left(57720.4 + 2 + 1456\right)\\ <& 6\times 10^{-26}.\end{aligned}$$

9 Numerical application

We are going to try and get the most accurate result we can, for which we require using as large a value of T as we can, so let $T = 30,610,046,000$. We do not want our total error to be too much larger than 10^{-5}, so we let $E = 10^{-5}$, which in turn gives us $\alpha = 4.48864 \times 10^{19}$, and $\eta = 1.28542 \times 10^{-9}$, see Appendix B. Since we intend for Corollary 4.1 to be a guideline for the parameters to calculate, we will let $\alpha = 4.5\times 10^{19}$ and $\eta = 1.3 \times 10^{-9}$, our error then is less than 1.029×10^{-5} for all $\omega > 400$.

All results are obtained with thanks to David Platt. All the obtained numerical data can be found in Appendix C. The following graph is plotted with $M = T$ and $\alpha = 4.5\times 10^{19}$ to calculate values the sum of (49) and (50) for ω between 727.95133539 and 727.95133542; the results are displayed in the following graph.

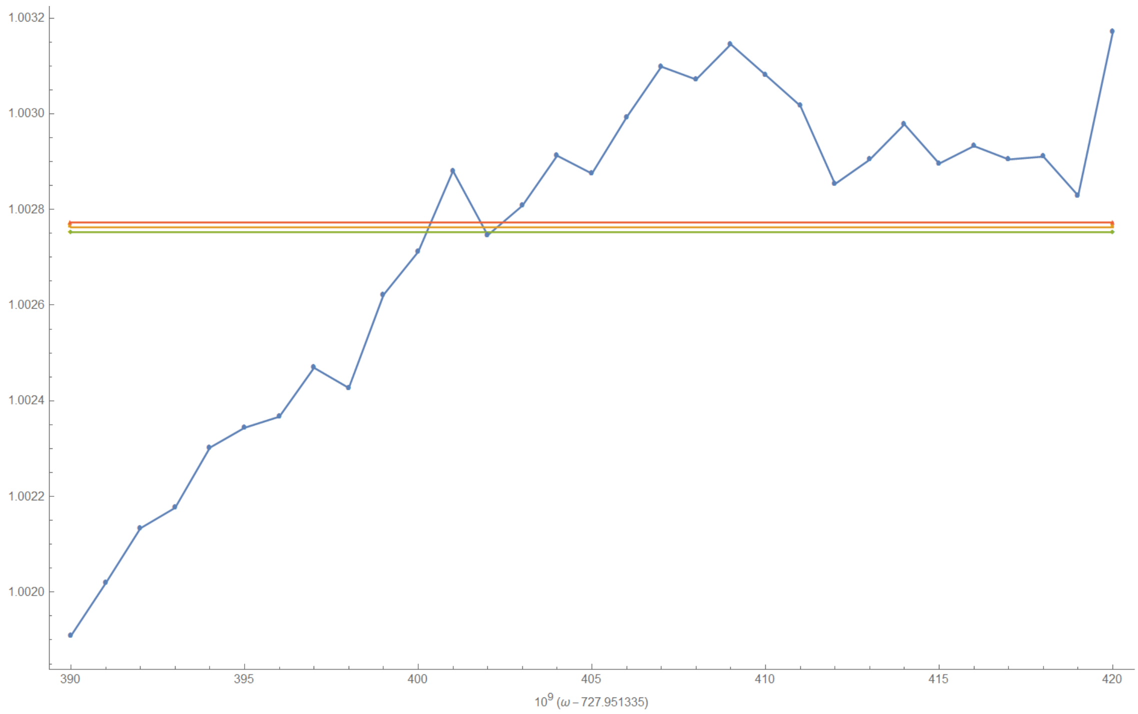

Figure 1: Sum of (49) and (50) over 103,800,788,359 zeros

There are three almost-parallel lines in the plot. The middle one corresponds to C, (27) in the statement of our theorem, it is actually decreasing, albeit very slowly. Our total error E corresponds to the two other "straight" lines in the plot; that is, the error with the given α and η, and the corresponding ω. We zoom in on the area around the crossover points:

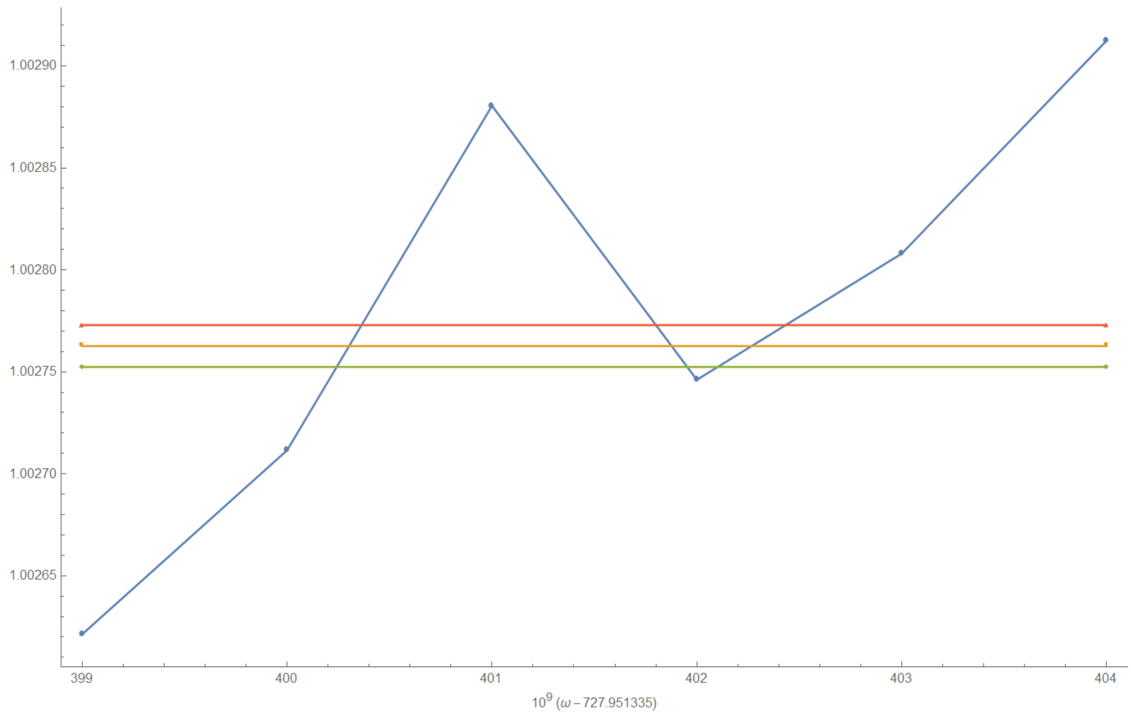

Figure 2: Sum of (49) and (50) over 103,800,788,359 zeros (zoomed in)

For $\omega = 727.951335401$ and $\eta = 1.3 \times 10^{-9}$, we have

$$-\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\frac{u\,\mathrm{li}(e^{u/2})}{2e^{u/2}}du = -1.002762659\ldots$$

and thus, $I(\omega,\eta) > 0.000107555$, which means that we have a new record for the smallest ω such that $\pi(x) > \mathrm{li}(x)$.

By comparison, for $\omega = 727.951335400$, we have $I(\omega,\eta) < -0.0000407$. From this we can deduce that for some $\omega \in (727.951335400, 727.951335401)$, we have $I(\omega,\eta) = 0$.

If we let $F(u) = ue^{-u/2}\{\pi(e^u) - \mathrm{li}(e^u)\}$, then we can say that, for $\omega = 727.951335401$,

$$\int_{\omega-\eta}^{\omega+\eta} K(u-\omega)F(u)du \geq \delta, \quad \text{where} \quad \delta = 1.07555 \times 10^{-4},$$

and, since $K(u-\omega)$ is a probability measure, we have

$$0 < \delta \leq \int_{\omega-\eta}^{\omega+\eta} K(u-\omega)\sup\{F(u)\}du < \sup\{F(u)\}.$$

Since F is continuous except at the points where u is of the form $\log(p)$, for some prime p, there exists at least one point $u \in (\omega-\eta, \omega+\eta)$ where $F(u) > \delta$; that is, where

$$\frac{u\{\pi(e^u) - \mathrm{li}(e^u)\}}{e^{u/2}} > \delta$$
$$\pi(e^u) - \mathrm{li}(e^u) > \frac{\delta e^{(\omega-\eta)/2}}{\omega-\eta} > 1.74643 \times 10^{151}.$$

If we let $b > 0$, then we know that $\mathrm{li}(x-b) < \mathrm{li}(x)$. We can also state, by the nature of the prime counting function, that $\pi(x-1) \geq \pi(x) - 1$, for $x > 1$. Combining this inequality with recurrence, we can come to the conclusion that $\pi(x-b)-\mathrm{li}(x-b) > 1.74643\times10^{151}-b$; that is, that the 1.74643×10^{151} integers preceding our chosen x satisfy $\pi(x) > \mathrm{li}(x)$.

Since we know that $\pi(x)$ is increasing, and we know how the function $-\mathrm{li}(x)$ decays, we can state the following theorem which gives us a narrower interval of positivity:

Theorem 9.1. *There is at least one value of x for which $\pi(x) > \mathrm{li}(x)$ holds in the interval* $[\exp(727.9513353997), \exp(727.9513354023)]$*. Furthermore, there is a sequence of at least* 1.27132×10^{154} *successive integers which begins in this interval where the inequality also holds.*

Proof. We first note that, if $x > 1$ and $y > 0$, then

$$\mathrm{li}(x+y) - \mathrm{li}(x) = \int_x^{x+y} \frac{dt}{\log(t)} < \frac{y}{\log(x)}.$$

Next, we note that, for $\pi(x) - \mathrm{li}(x) > A > 0$ and $y > 0$,

$$\begin{aligned}\pi(x+y) - \mathrm{li}(x+y) &= [\pi(x+y) - \pi(x)] + [\pi(x) - \mathrm{li}(x)] + [\mathrm{li}(x) - \mathrm{li}(x+y)] \\ &= A - \frac{y}{\log(x)}.\end{aligned}$$

So for $0 < y < A\log(x)$, we have that $\pi(x+y)-\mathrm{li}(x+y)$ is positive. For our case, we have $A = 1.74643 \times 10^{151}$ and $\log(x) > 727.9513353997$. Multiplying these two values gives the result in the theorem. $\square$

This result consists of an interval just over 150 times narrower than that of Saouter, Trudgian and Demichel[21], with over 17 times the verified positive integers.

10 After thoughts

There are a few comments to make regarding the results of this paper.

Firstly, the assumption of the Riemann Hypothesis can be removed, despite its not being likely to make a huge impact on the result, until it is proven (or disproven!) it should be included for accuracy.

Next, from the numerical analysis that has been performed, we can make a remark on how modifying the parameters changes the results. When observing ω over an interval of around 10^{-8}, as we have done here, changes in α and η of less than one order of magnitude will likely leave the underlying result unchanged, we verified this for different values of α, as shown in Appendix C.

We would be able to obtain a tighter bound for an improvement to Theorem 9.1 if we were to sharpen our interval in a similar manner to Section 9 of Saouter, Trudgian and Demichel[21].

Finally, when scouring for areas where crossovers could occur before the main region we have been searching, we also looked into an area suggested by Saouter, Trudgian and Demichel[21], around $\omega = 727.951332982$. They theorised that one would require 10^{12} zeros to fully identify whether or not a crossover occurs. Researching the area with our 10^{11} zeros *did* provide a crossover, however it was trumped by our total error, so we cannot state with certainty that a crossover point exists. The fact that our largest error term can be reduced with an increase in T backs up the claim that more zeros are required.

11 Appendices

A Proof of Lemma 2.4

Here we have the proof of Lemma 2.4. We have Wolfram Mathematica 10 manually calculate the sum of the reciprocals of the squares of the first 100,000 zeta zeros, and apply Lemma 2.3 to bound the remaining zeros:

```
In[6]:= Sum[1 / Im[ZetaZero[i]] ^2, {i, 1, 100000}] + Log[Im[ZetaZero[100001]]] / Im[ZetaZero[100001]] // N
Out[6]= 0.0232327
```

Figure 3: Sum over all γ with exponent -2

From this, we get the result that our sum is less than 0.025, as required.

B Defining our parameters

We show how we use Corollary 4.1 to obtain α and η:

```
In[1]:= T = 30 610 046 000. // N
        omega = 727 951 335 401. / 1 000 000 000. // N
        epsilon = 1. / 100 000. // N
        alpha = T^2. / (2. ProductLog[Log[T / (2. Pi)] (2. + 1. / T) / (4. Pi * epsilon - 3.3 * 10^-7)]) // N
        eta = Sqrt[2. / alpha Log[0.00019 Sqrt[alpha] / epsilon^2]] // N
Out[1]=
        3.061 × 10^10
Out[2]=
        727.951
Out[3]=
        0.00001
Out[4]=
        4.48864 × 10^19
Out[5]=
        1.28542 × 10^-9
```

Figure 4: α and η

C Results of computations

Here we state all of the results obtained by David Platt, who implemented our theorem, for computational specifications, see Appendix E. The following are all the results of summing

$$-\sum_{0<\gamma\leq T}\left(\frac{e^{-\gamma^2/2\alpha}}{\frac{1}{4}+\gamma^2}\left(\cos(\omega\gamma)+2\gamma\sin(\omega\gamma)+\frac{\left(\left(\frac{1}{2}-2\gamma^2\right)\cos(\omega\gamma)+2\gamma\sin(\omega\gamma)\right)}{(\frac{1}{4}+\gamma^2)\omega}\right)\right)$$

for $T = 30,610,046,000$. The first table was summed for ω between 727.951335402 and 727.951335422 for three different values of α:

Table 2: Change in ω

ω	α		
	1.64×10^{19}	3.59902×10^{19}	4.5×10^{19}
727.951335402	1.002748	1.002746	1.002746
727.951335404	1.002901	1.002911	1.002912
727.951335406	1.002998	1.002994	1.002993
727.951335408	1.003076	1.003072	1.003072
727.951335410	1.003096	1.003084	1.003081
727.951335412	1.002864	1.002855	1.002853
727.951335414	1.002948	1.002972	1.002978
727.951335416	1.002926	1.002931	1.002933
727.951335418	1.002899	1.002908	1.002910
727.951335420	1.003170	1.003172	1.003172

Our next table shows how changing the value of α effects the sum around two particular values of ω:

Table 3: Change in α

α	ω	
	727.951335402	727.951335404
1.64×10^{19}	1.002747564	1.002901612
3.59902×10^{19}	1.002745771	1.002910837
3.82475×10^{19}	1.002745843	1.002911317
3.83194×10^{19}	1.002745846	1.002911331
3.86149×10^{19}	1.002745856	1.002911389
3.92513×10^{19}	1.002745880	1.002911509

The final table shows with more depth the change in the sum as ω varies for $\alpha = 4.5 \times 10^{19}$, this is the data which is plotted in Figures 1 and 2. All values are correct to 20 decimal places.

Table 4: Sum over ω

ω	$S_1 + S_2$
727.951335390	1.00190866884011382864
727.951335391	1.00201854967684632649
727.951335392	1.00213347518371362329
727.951335393	1.00217629389659785571
727.951335394	1.00230198730807473534
727.951335395	1.00234359282628042873
727.951335396	1.00236680388663977650
727.951335397	1.00246904879740147398
727.951335398	1.00242685873369174199
727.951335399	1.00262137363445462331
727.951335400	1.00271158300323501821
727.951335401	1.00288049783641096204
727.951335402	1.00274612189791114819
727.951335403	1.00280798486275898499
727.951335404	1.00291240960630788449
727.951335405	1.00287492794724527094
727.951335406	1.00299281882170823702
727.951335407	1.00309848260820786820
727.951335408	1.00307170506180394891
727.951335409	1.00314523129500517202
727.951335410	1.00308122682380819222
727.951335411	1.00301671794161463288
727.951335412	1.00285314012033711954
727.951335413	1.00290411389339446590
727.951335414	1.00297796001400488544
727.951335415	1.00289589505461519867
727.951335416	1.00293260539679423985
727.951335417	1.00290456710554147644
727.951335418	1.00291046400991108639
727.951335419	1.00282888557072673681
727.951335420	1.00317163421701490261

D Searching for crossover points with smaller ω

As we mentioned in Section 7, Bays and Hudson [3] produced a plot of the function $\mathrm{li}(x) - \pi(x)$ from 10^6 to 10^{400} on a logarithmic scale. Their plot used the first 10^6 zeros.

We have created a similar plot, computed and plotted by David Platt. This plot is of our sum $S_1 + S_2$, for $\omega \in [137, 738]$ and $\alpha = 2.836 \times 10^{11}$. Our plot uses all the zeros with $|\Im(\rho)| < 2,546,000$; that is, the first 4,826,908 zeros.

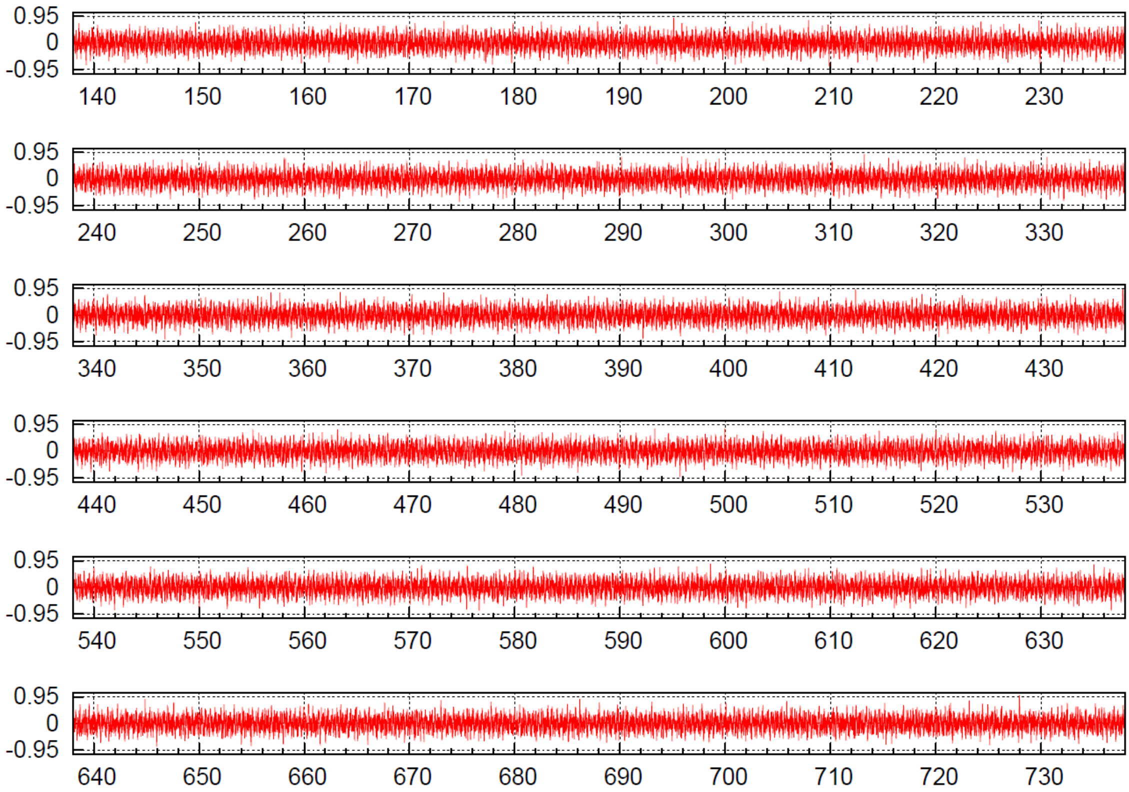

Figure 5: $S_1 + S_2$ for $\omega \in [137, 738]$

We can see immediately that the region around 728 is very positive, and it is the only point on the plot where the sum is greater than 0.95. There are a few other areas on the plot which look interesting, we have plotted a table of each of the four regions where the sum exceeds 0.9.

Table 5: $S_1 + S_2$ for intervals of ω

ω	$S_1 + S_2$
195.105	0.807711275587995987352550595097
195.106	0.863260249618894657279674785427
195.107	0.901574357267209432232451687928
195.108	0.859128204729691621487646363984
195.109	0.792968020610560854896084919455
412.390	0.753153454124809224184101977078
412.391	0.763379439898559464054571801391
412.392	0.918461113414711366613373113701
412.393	0.891552882555343037626367080424
412.394	0.887925204080240275520615004642
437.780	0.652260670147146642793044763734
437.781	0.755401412538619285463776001982
437.782	0.93312036945013591131580620296
437.783	0.886657145457510703708917185371
437.784	0.77388213347577176803915161091
727.950	0.891243815943253607920130574107
727.951	0.928398837586333834943594078605
727.952	1.00074565427825151175660792005
727.953	0.91402382005620585979261892379
727.954	0.885836734976227789559917330905

As we can see from the table, there exists a value for ω where the sum is greater than 1, which does not show on the plot, so the quality of the image does not show every detail from the data.

E Computing specifications

The program to compute the sum over the non-trivial zeros was written in 255 lines of C++ and used Fredrik Johansson's ARB [9] interval arithmetic package to manage rounding errors. The database of zeros used was computed as described in Platt's article [15] and all the computations were undertaken on the University of Bristol BlueCrystal Phase III cluster [1]. Each node of Phase III comprises two 8 core Intel(R) Xeon(R) E5-2670 CPUs clocked at 2.60GHz and we were able to use all 16 cores concurrently.

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