Generating Boundary Conditions for Integrable Field Theories using Defects

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Abstract

In this thesis we examine the construction and characteristics of generalised reflection matrices, within the $a_1^{(1)}$, $a_2^{(1)}$ and $a_2^{(2)}$ integrable affine Toda field theories. In doing so, we generalise the existing finite-dimensional reflection matrices because our construction involves the dressing of an integrable boundary with a defect. Within this framework, an integrable defect's ability to store an unlimited amount of topological charge is exploited, therefore all generalised solutions are intrinsically infinite-dimensional and exhibit interesting features. Overall, further evidence of the rich interplay between integrable defects and boundaries is provided. It is hoped that the generalised solutions presented in this thesis are potential quantum analogues of more general classical integrable boundary conditions.

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Declaration

The work presented in this thesis is based on research carried out in the Department of Mathematics, University of York. This work has not previously been presented for an award at this, or any other, University.

Chapter 1 provides of a review of the background material underlying the research presented in this thesis, and is referenced appropriately. The main results contained in chapters 2, 3 and 4 are my own, unless referenced otherwise in the text.

Chapter 1

Aspects of Integrability

In this introductory chapter we aim to provide the necessary background material for all subsequent chapters. In the process, we will cover several important aspects of quantum integrability, markedly: the Yang-Baxter [1, 2, 3, 4], boundary Yang-Baxter [6, 7] and transmission Yang-Baxter equations [8, 9, 10], as well as explore their history and significance. The analogous classical framework of integrable boundaries [11, 12, 13, 14] and defects [15, 16] is also discussed. As a matter of course, it will become apparent how the ideas and theories of integrability have grown over the years. Eventually, this will lead us to the main focus and aim of this thesis: to understand the interplay between integrable boundaries and defects at the quantum level within the affine Toda field theories (ATFTs). We will achieve this by dressing integrable boundaries with an integrable defect. The results of [17], where infinite-dimensional reflection matrices were constructed for the sine-Gordon model, are extended by constructing generalised solutions for the $a_2^{(1)}$ and $a_2^{(2)}$ affine Toda models in chapters three and four. In chapter four, the behaviour of infinite-dimensional solutions suggests that the defect has an intrinsic purpose within the $a_2^{(1)}$ model, which is discussed in detail. To further emphasise the fundamental connections between boundaries and defects, the way in which an integrable defect fits into an algebraic framework proposed by Delius and MacKay [18] is documented in chapter two.

1.1 Factorised Scattering and the Yang-Baxter equation

An excellent starting point within the realm of two-dimensional quantum integrability is the exact scattering matrix, as to each such matrix there should correspond an integrable field theory. In two-dimensions the S-matrices possess the property of factorised scattering [3], which means that it is possible to reduce many particle scattering processes and simply consider two-particle processes.

The scattering matrices can be described by an associative algebra, known as the Faddeev-Zamolodchikov algebra [3, 4]. In this algebraic setting, the particles of a theory having rapidity, θ_i, are represented by the non-commutative symbols $A_i(\theta_i)$. Then one considers the scattering of this particle with another, $A_j(\theta_j)$. Importantly, the products are arranged in terms of decreasing rapidity to reflect their spatial order. For example, if $\theta_i > \theta_j$, we arrange the corresponding in-state as: $A_i(\theta_i)A_j(\theta_j)$. As time evolves the particles will meet and must scatter, because $\theta_i > \theta_j$. As a result of this, out-states are arranged in order of increasing rapidity so that as time continues to evolve the faster particle continues to move away and there is no further interaction. The S-matrix describes the processes involved during the particles' interaction:

$$A_i(\theta_i)A_j(\theta_j) = S_{ij}^{kl}(\theta_i - \theta_j)A_k(\theta_j)A_l(\theta_i). \tag{1.1}$$

From this we see that the subscript indices label incoming particles, while the superscripts label outgoing particles. By considering the incoming particles, and respecting the fact that there is no particle production in the integrable theories [19], one can label each non-zero entry of the scattering matrix. To find the following: $S_{ii}^{ii}(\theta_i - \theta_j)$, for two particles of the same type and different rapidities, $S_{ij}^{ji}(\theta_i - \theta_j)$ for the process $ij \to ij$ and finally $S_{ij}^{ij}(\theta_i - \theta_j)$, when two particles reflect, $ij \to ji$. In algebraic terms, the S-matrix acts as an intertwining map on the vector spaces associated to each particle in the two-particle scattering process:

$$S(\theta_a - \theta_b) : V_a \otimes V_b \to V_b \otimes V_a.$$

A comprehensive introduction to the theory of S-matrices can be found in [19].

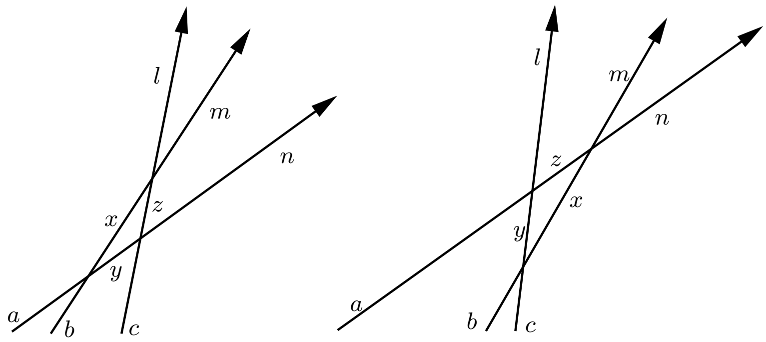

Figure 1.1: Diagrammatic representation of the Yang-Baxter equation.

We will now detail some of the properties of S-matrices. Arguably, the most important and significant property for integrability is the Yang-Baxter equation, which ensures consistency between all three-particle scattering processes [1, 2]:

$$S_{ab}^{fg}(\theta_a-\theta_b)S_{gc}^{hn}(\theta_a-\theta_c)S_{fh}^{lm}(\theta_b-\theta_c)=S_{bc}^{hf}(\theta_b-\theta_c)S_{ah}^{lg}(\theta_a-\theta_c)S_{gf}^{mn}(\theta_a-\theta_b). \tag{1.2}$$

The above equation can be obtained by following the particle trajectories in figure (1.1), where we sum over repeated indices in the above equation. Equation (1.2) has a great algebraic significance and deep connections within quantum groups, many results can be found in [20, 21]. By solving the Yang-Baxter equation, which is a difficult task, one can obtain the S-matrix entries. However, the equation does not constrain the S-matrix scalar prefactor, $\rho(\theta)$, as it simply cancels throughout the equation. There exist further properties that restrict the prefactor, they are unitarity:

$$S_{ij}^{kl}(\theta_i-\theta_j)S_{kl}^{mn}(\theta_j-\theta_i)=\delta_i^m\delta_j^n, \tag{1.3}$$

and crossing symmetry:

$$S_{ij}^{kl}(\theta_i-\theta_j)=S_{\bar{k}i}^{l\bar{j}}(i\pi-(\theta_i-\theta_j)), \tag{1.4}$$

where the barred indices label the appropriate anti-particle. The relation due to crossing symmetry is obtained by reversing one particle's trajectory in the two-particle scattering process, and considering following the new process. The S-matrix should also satisfy the bootstrap relation - alluding to soliton fusing relations that we do not require in this thesis, details are given in [19].

In due course we will see several S-matrices for various affine Toda models, which have been studied heavily in the past - for example, in the $a_n^{(1)}$ affine Toda models see [22]. Interestingly, a total classification of all solutions to the Yang-Baxter equation does not yet exist, although particular cases have been studied providing partial classification [23] - [26]. Our main focus will be the construction of more general objects, compatible with the S-matrix, that use the theory of factorisable scattering in the presence of boundaries and defects.

1.2 The Reflection equation

Over the years, as more results surrounding the Yang-Baxter equation and factorised scattering were discovered, the next logical step was to consider factorised scattering in the presence of a boundary, and if it is possible to maintain integrability. The boundary restricts the theory to the half-line $-\infty < x < 0$, where scattering far away from the boundary is still described by the S-matrix. A new object is needed to describe any reflection off the boundary. Moreover, it must be compatible with the S-matrix, so that the theory remains integrable. Processes of reflection are characterised by reflection ($R-$)matrices and the FZ-algebra can be extended to include them:

$$A_j(\theta_j)B = R_j^k(\theta_j)A_k(-\theta_j)B, \tag{1.5}$$

where B denotes the boundary at $x = 0$ and we assume summation over repeated indices. Overall, this means an incident particle, A_j, eventually encounters the boundary and then reflects as A_k with negative rapidity. From the above, we see that it is possible for a particle to change during reflection off the boundary. For example, the particle could reflect as itself, or its antiparticle:

$$R_j^k(\theta_j) : V_j \to V_j, \tag{1.6}$$

$$R_j^{\bar{k}}(\theta_j) : V_j \to V_{\bar{k}}, \tag{1.7}$$

where the spaces V_j and $V_{\bar{k}}$ correspond to the particle and antiparticle respectively. For the time being, we will deal with the particle preserving case unless otherwise stated. Processes of reflection and consequently R-matrices are defined by the reflection equation - also known as the boundary Yang-Baxter equation:

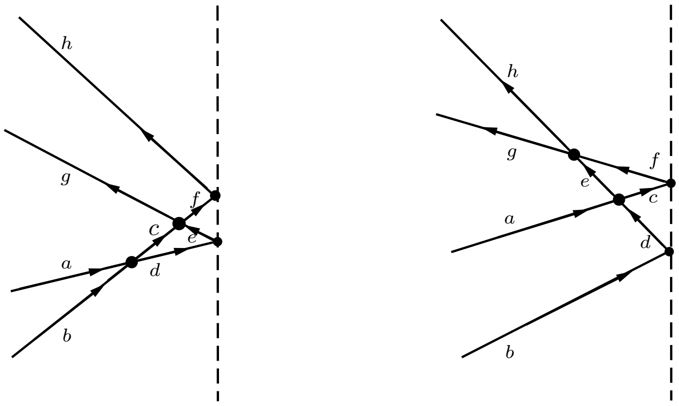

Figure 1.2: Diagrammatic representation of the reflection equation.

$$\begin{aligned} S_{ab}^{cd}(\theta_a - \theta_b) R_d^e(\theta_a) S_{ce}^{gf}(\theta_a + \theta_b) R_f^h(\theta_b) \\ = R_b^d(\theta_b) S_{ad}^{ec}(\theta_a + \theta_b) R_c^f(\theta_a) S_{ef}^{gh}(\theta_a - \theta_b). \end{aligned} \tag{1.8}$$

Equation (1.8) was first introduced by Cherednik [6] and later studied with regard to the quantum inverse scattering method in integrable theories with a boundary [7, 11, 12, 27]. Also, one can derive the index equation (1.8) by following particle trajectories in figure (1.2). Importantly, it is the associativity condition of the algebra (1.5) and within the vector space description of the particles, a completely analogous tensor product version of (1.8) exists:

$$S(\theta_a - \theta_b) R_1(\theta_a) S(\theta_a + \theta_b) R_2(\theta_b) = R_2(\theta_b) S(\theta_a + \theta_b) R_1(\theta_a) S(\theta_a - \theta_b). \tag{1.9}$$

In the above, $R_1 = R \otimes \mathbb{1}$ and $R_2 = \mathbb{1} \otimes R$. As one expects, equations (1.8) and (1.9) are equivalent. We have seen that the scattering matrix must satisfy certain properties, and the reflection matrix is no exception. The prefactor is not restricted by equation (1.8), but is restricted by the boundary cross-unitarity condition proposed by Ghoshal and Zamolodchikov [13]:

$$R_i^{\bar{j}}\left(\frac{i\pi}{2} - \theta\right) = S_{ji}^{kl}(2\theta) R_l^{\bar{k}}\left(\frac{i\pi}{2} + \theta\right). \tag{1.10}$$

Our investigations do not require the use of such prefactors, further details are found in [13].

The reflection, or boundary Yang-Baxter, equation is a difficult equation to solve. A popular method is employed in [5, 28, 29] to determine the entries of the reflection matrix. Firstly, a normalisation is specified - Kim divides through by matrix entry 2,2 when considering the three-by-three reflection matrices of $a_2^{(2)}$ [28]. Following this, to tackle the problem of the different rapidity dependences, the derivative is taken with respect to rapidity θ_b, after which θ_b is set to zero - these quantities then label the free parameters. The result gives functional equations of the remaining rapidity, θ_a. An assumption is then made, the reflection matrix should be proportional to the identity when $\theta_a = 0$. This procedure is applied to all equations, that are acquired by the expansion of indices in (1.8). As the dimension of the matrix increases, the procedure becomes more difficult.

1.2.1 An Algebraic construction

Some years later, in 2003, Delius and MacKay proposed an algebraic framework relying on an underlying quantum group symmetry. Reflection matrices are then defined as the solutions of a linear intertwining equation. Consequently, the difficulty of the problem is substantially reduced, although there is one caveat. We will now detail the method, first outlined in [18].

The construction begins by considering the representations of the particles of a $U_q(g)$ quantum algebra. The solitons of the particular theory, in multiplet μ with rapidity θ, span a vector space V_θ^μ. The solitons are then described by appropriate representations $\pi_\theta^\mu : U_q(g) \to \mathrm{End}(V_\theta^\mu)$. It is then questioned whether it is possible to regard the reflection matrix - now labelled $K^\mu(\theta)$ - as an intertwiner of the representations:

$$K^\mu(\theta)\pi_\theta^\mu(Q) = \pi_{-\theta}^{\bar{\mu}}(Q)K^\mu(\theta), \tag{1.11}$$

for all $Q \in U_q(g)$. Of course, it is not possible for all generators of $U_q(g)$ because the boundary breaks the quantum group symmetry. However, a remnant of the quantum symmetry might survive despite the inclusion of a boundary within the theory. Therefore, equation (1.11) may hold for all $Q \in \mathcal{B}$, where $\mathcal{B}$ is a symmetry-preserving subalgebra of $U_q(g)$. Moreover, the subalgebra, $\mathcal{B}$, must be a left coideal of $U_q(g)$ meaning that:

$$\Delta(Q) \in U_q(g) \otimes \mathcal{B}, \qquad \forall Q \in \mathcal{B},$$

hence, the boundary's processes are encoded in the second slot (as the boundary is on the right) and the usual bulk theory is described within the first slot. This is vital because the coproduct, Δ, must represent the action of multiparticle-states, therefore it must include both $U_q(g)$ and $\mathcal{B}$ to ensure compatibility throughout. The object of interest is now the subalgebra, $\mathcal{B}$, and its generators. There is an alternative framework where the generators of the boundary subalgebra are calculated using boundary conformal perturbation theory - the details are found in [18] - but the algebraic method is most relevant for our purposes. The caveat associated with the construction is that it requires prior knowledge of a solution of the reflection equation. Supposing that we do know a particular reflection matrix, we can move on to define the corresponding $L-$operators that are compatible with the vector spaces, V_θ^μ, and $U_q(g)$. The universal R-matrix, $\mathcal{R}$, of $U_q(g)$ is used to define the operators, full definitions are found in [30, 31]. For our purposes, all we need to know is that the universal R-matrix is related to the S-matrix that we are familiar with, as follows [18]:

$$S^{\mu\nu}(\theta-\theta') \propto PR^{\mu\nu}(\theta-\theta'), \qquad R^{\mu\nu}(\theta-\theta') = (\pi_\theta^\mu \otimes \pi_{\theta'}^\nu)(\mathcal{R}),$$

where P is the permutation operator, interchanging the tensor factors. The L_θ^μ-operators are then defined [32]:

$$\begin{aligned} L_\theta^\mu &= (\pi_\theta^\mu \otimes \mathbb{1})(\mathcal{R}) \in \mathrm{End}(V_\theta^\mu) \otimes U_q(g), \\ \bar{L}_{-\theta}^{\bar{\mu}} &= (\pi_{-\theta}^{\bar{\mu}} \otimes \mathbb{1})(\mathcal{R}^{\mathrm{op}}) \in \mathrm{End}(V_{-\theta}^{\bar{\mu}}) \otimes U_q(g), \end{aligned} \tag{1.12}$$

where $\mathcal{R}^{op}$ is the opposite universal R-matrix that is formed by swapping the two tensor factors. The generators of the boundary subalgebra are then constructed as follows [18]:

$$B_\theta^\mu = \bar{L}_\theta^{\bar{\mu}}(K^\mu(\theta) \otimes \mathbb{1}) L_\theta^\mu \in \mathrm{Hom}(V_\theta^\mu, V_{-\theta}^{\bar{\mu}}) \otimes U_q(g),$$

or equivalently in terms of matrix indices:

$$(B_\theta^\mu)^a{}_b = (\bar{L}_\theta^{\bar{\mu}})^a{}_c (K^\mu(\theta))^c{}_d (L_\theta^\mu)^d{}_b,$$

assuming the usual sum over repeated indices. Since the construction of the generators has been detailed, it remains to check if they satisfy the linear equation (1.11), together with known reflection matrix, $K^{\mu}(\theta)$,

$$K^{\nu}(\theta') \circ \pi^{\nu}_{\theta'}((B^{\mu}_{\theta})^{a}{}_{b}) = \pi^{\bar{\nu}}_{-\theta'}((B^{\mu}_{\theta})^{a}{}_{b}) \circ K^{\nu}(\theta') \tag{1.13}$$

or equivalently:

$$(\mathbb{1} \otimes K^{\nu}(\theta')) \circ (\mathbb{1} \otimes \pi^{\nu}_{\theta'})(B^{\mu}_{\theta}) = (\mathbb{1} \otimes \pi^{\bar{\nu}}_{-\theta'})(B^{\mu}_{\theta}) \circ (\mathbb{1} \otimes K^{\nu}(\theta')). \tag{1.14}$$

The beauty of this construction is that the properties of the universal R-matrix are exploited to show that equation (1.14) naturally leads to the reflection equation. Using the defining properties of the L-operators alongside the natural properties of the universal R-matrix it is found that [18]:

$$(\mathbb{1} \otimes \pi^{\nu}_{\theta'})(L^{\mu}_{\theta}) = (\pi^{\mu}_{\theta} \otimes \pi^{\nu}_{\theta'})(\mathcal{R}) = R^{\mu\nu}(\theta - \theta'), \tag{1.15}$$

$$(\mathbb{1} \otimes \pi^{\nu}_{\theta'})(\bar{L}^{\bar{\mu}}_{\theta}) = (\pi^{\bar{\mu}}_{-\theta} \otimes \pi^{\nu}_{\theta'})(\mathcal{R}^{\mathrm{op}}) = PR^{\nu\bar{\mu}}(\theta + \theta')P, \tag{1.16}$$

$$(\mathbb{1} \otimes \pi^{\bar{\nu}}_{-\theta'})(L^{\mu}_{\theta}) = (\pi^{\mu}_{\theta} \otimes \pi^{\bar{\nu}}_{-\theta'})(\mathcal{R}) = R^{\mu\bar{\nu}}(\theta + \theta'), \tag{1.17}$$

$$(\mathbb{1} \otimes \pi^{\bar{\nu}}_{-\theta'})(\bar{L}^{\bar{\mu}}_{\theta}) = (\pi^{\bar{\mu}}_{-\theta} \otimes \pi^{\bar{\nu}}_{-\theta'})(\mathcal{R}^{\mathrm{op}}) = PR^{\bar{\nu}\bar{\mu}}(\theta - \theta')P. \tag{1.18}$$

Straightforwardly, by substituting all definitions (of the generators of the boundary subalgebra and L-operators) into equation (1.14) and identifying S as PR, one is able to obtain that the reflection equation is satisfied. Chiefly, this guarantees compatibility between any reflection matrix and the generators of the boundary subalgebra, providing this construction is adhered to. Lastly, it remains to show that when the coproduct is applied to the generators B^{μ}_{θ} that the result is left coideal. This must hold as the generators are assumed to belong to the boundary subalgebra. As before, one simply follows the defining relations together with the properties that originate from the $\mathcal{R}$-matrix [18]:

$$\Delta(L^{\mu}_{\theta}) = L^{\mu}_{\theta} \otimes L^{\mu}_{\theta}, \qquad \Delta(\bar{L}^{\bar{\mu}}_{\theta}) = L^{\bar{\mu}}_{\theta} \otimes L^{\bar{\mu}}_{\theta},$$

to discover that:

$$\Delta(B^{\mu}_{\theta}) = \bar{L}^{\bar{\mu}}_{\theta} L^{\mu}_{\theta} \otimes B^{\mu}_{\theta},$$

and so, this does indeed belong to $U_q(g) \otimes \mathcal{B}$ as required.

In [18] the sine-Gordon is given as an example, and the generators of the boundary subalgebra are calculated by means of boundary conformal perturbation theory. Nonetheless, solving equation (1.11) is much simpler than employing the usual method. The boundary subalgebra, for the sine-Gordon model, contains two generators $Q_\pm$ that are represented as follows:

$$\pi_\theta(Q_+) = \begin{pmatrix} \hat{\epsilon}_+ q & cx \\ cx^{-1} & \hat{\epsilon}_+ q^{-1} \end{pmatrix}, \qquad \pi_\theta(Q_-) = \begin{pmatrix} \hat{\epsilon}_- q^{-1} & cx^{-1} \\ cx & \hat{\epsilon}_- q \end{pmatrix}, \tag{1.19}$$

with $x = e^{\theta/\gamma}$, $\gamma = \beta^2/(8\pi - \beta^2)$ and $q = e^{8\pi^2 i/\beta^2}$ and $c = \sqrt{\lambda\gamma^2(q^2-1)/2\pi i}$, where $\hat{\epsilon}_\pm$ are parameters associated with the boundary condition and λ originates from the perturbative conformal theory approach used to calculate the generators. Using this representation Delius and MacKay reproduce the reflection matrix first presented in [13]:

$$K(\theta) = k(\theta) \begin{pmatrix} \frac{q-q^{-1}}{c}(\hat{\epsilon}_+ x + \hat{\epsilon}_- x^{-1}) & x^2 - x^{-2} \\ x^2 - x^{-2} & \frac{q-q^{-1}}{c}(\hat{\epsilon}_- x + \hat{\epsilon}_+ x^{-1}), \end{pmatrix} \tag{1.20}$$

where $k(\theta)$ is the prefactor restricted by crossing and unitarity.

The problem of fully classifying all solutions to the reflection equation is reduced to the classification of boundary subalgebras, which may provide further insight. Up to this point, our recount of the reflection equation and its theory is finite-dimensional. In particular, its solutions do not depend on the topological charge. However, it seems natural that charge may be exchanged with the boundary, thus enabling solutions to become infinite-dimensional. If charge is included within the reflection matrix entries, it becomes yet more difficult to solve the reflection equation and even the linear process would require modification. As a result, one might begin to think that there is room for generalisation. There is no reason why a solution should not depend on the topological charge, but its inclusion causes difficulties.

In the physical world, there are many parallels between the classical and quantum scenarios. In the case of reflection matrices this is no different. To each reflection matrix there should exist a corresponding classically integrable boundary condition (IBC) of the specific theory in question. Reasonably, one might think that the potential to generalise reflection matrices will lead to more general IBCs. With this, we must now consider the theory of integrable boundary conditions and finally introduce the integrable models that we will work with.

1.3 Classically Integrable Boundary conditions

We have now briefly discussed the aspects of quantum integrability that are most relevant to the work in this thesis. Let us move on to consider the associated classical framework. In this thesis we primarily consider the affine Toda field theories (ATFTs). They are very special, not only because they are integrable, but because they showcase a great connection between mathematics and physics. The connection is that to each affine Dynkin diagram one can associate a $(1+1)$-dimensional ATFT, hence, a field theory is constructed from the root data of a particular algebra [33]. The theory without a boundary (the bulk) is described by the Lagrangian [34]:

$$\mathcal{L} = \frac{1}{2}\left(u_t \cdot u_t - u_x \cdot u_x\right) - U(u), \tag{1.21}$$

with potential:

$$U(u) = \frac{m^2}{\beta^2} \sum_{j=0}^{n} n_j \left(e^{\beta \alpha_j \cdot u} - 1\right), \tag{1.22}$$

where m concerns the mass scale, β is the coupling constant and n is the rank of the underlying algebra. The simple roots are labelled by α_j, j = 1,$\ldots, n$ and the marks, n_j, are a feature of the underlying algebra. The additional lowest root, α_0, is defined:

$$\alpha_0 = -\sum_{j=0}^{n} n_j \alpha_j, \tag{1.23}$$

with the convention $n_0 = 1$. Accordingly, the above Lagrangian describes a multi-component affine Toda field, $u = (u_1, \ldots, u_n)$, existing in the underlying algebra's root space. It is the additional root that differentiates between the massive and massless non-affine Toda field theories. The massive ATFTs are integrable, and possess the hallmarks of integrability such as: infinitely many conserved charges and a Lax pair representation, to name just two; more details are found in [35, 36].

Of particular importance are the $a_n^{(1)}$ ATFTs. They possess the most symmetric root/weight spaces and many results are known regarding their integrability. The marks of this algebra all take the same value, $n_j = 1$ and the roots are all of equal length, a conventional choice is $|\alpha_i| = \sqrt{2}$. If the coupling constant, β, is real and the fields are also restricted to be real, then upon quantisation the ATFT describes n interacting scalars - fundamental Toda particles, of mass:

$$m_a = 2m\sin\left(\frac{\pi a}{h}\right), \qquad a = 1, 2, \ldots, n,$$

with $h = n+1$ the Coxeter number of the algebra. To obtain classical soliton solutions, within the ATFTs, the fields must be complex [37]. The complex case is also described by the Lagrangian (1.21) where the coupling constant, β, is replaced by $i\beta$. In turn, this changes the nature of the potential in the Lagrangian (1.21). Indeed, it will vanish for particular constant values of the field, namely when:

$$u = \frac{2\pi w}{\beta}, \qquad \alpha_j \cdot w \in \mathbb{Z},$$

which means that w belongs to the weight lattice of the particular Lie algebra a_n. Such values of the field correspond to the stationary points of the affine Toda potential. The complex soliton solutions smoothly interpolate between these points along the whole x-axis. The solutions are then characterised by their topological charge, Q, which is defined:

$$Q = \frac{\beta}{2\pi}\int_{-\infty}^{\infty} dx\, u_x = \frac{\beta}{2\pi}\left(u(\infty, t) - u(-\infty, t)\right).$$

All of the topological charges belong to the weight lattice of the a_n Lie algebra. If it is assumed that $u(-\infty, t)$ is zero, one can obtain static solutions where the remaining contribution belongs to a subset of the weight lattice. Furthermore, these solutions can be described using the Hirota formalism [37, 38]:

$$u_a = \frac{im^2}{\beta}\sum_{j=0}^{n}\alpha_j \ln(1 + E_a\omega^{aj}), \tag{1.24}$$

$$E_a = e^{a_a x + b_a t + \xi_a}, \quad (a_a, b_a) = m_a(\cosh\theta, \sinh\theta), \quad \omega = e^{2\pi i/h},$$

where ξ_a is a complex constant and θ is the soliton rapidity. Hollowood uncovered that solutions of this type, even though they are complex, have real energy and momentum [37]. Furthermore, their masses are proportional to the mass parameters of the real theory:

$$M_a = \frac{2h}{\beta^2} m_a, \qquad a = 1, 2, \ldots, n.$$

In the subsequent chapters we will deal with the $a_n^{(1)}$, $n = 1, 2$ ATFTs - where $a_1^{(1)}$ is the famous sine-Gordon model, and the slightly different $a_2^{(2)}$ case. Previously, we have hinted that the work contained within subsequent chapters investigates a more general framework for the solitons of these theories. Before considering the ingredients and mechanism of the generalisation, we must recount the way in which the ATFTs are modified by a boundary.

If an ATFT is restricted to the half-line $-\infty < x < 0$ with a boundary at $x = 0$, the Lagrangian (1.21) is modified as follows:

$$\mathcal{L}' = \theta(-x)\mathcal{L} - \delta(x)\mathcal{B}, \tag{1.25}$$

where $\mathcal{L}$ is the usual Lagrangian (1.21), $\delta(x)$ is the Dirac delta function and $\mathcal{B}$ is a functional of the field u only, this is not totally necessary as it could also depend on the field's derivatives, but this restriction simplifies the discussion. The equations of motion are therefore:

$$u_{tt} - u_{xx} = -\frac{m^2}{\beta}\sum_{j=0}^{n} n_j \alpha_j e^{\beta\alpha_j\cdot u}, \qquad x < 0, \tag{1.26}$$

$$u_x = -\frac{\partial\mathcal{B}}{\partial u}, \qquad x = 0. \tag{1.27}$$

The boundary condition must be chosen carefully, so that it satisfies the above property in an attempt to maintain integrability. For the a, d and e series of ATFTs, the boundary contribution takes the form [14, 39, 40]:

$$\mathcal{B}(u) = \frac{m}{\beta^2}\sum_{j=0}^{n} A_j e^{(\beta/2)\alpha_j\cdot u}. \tag{1.28}$$

The above boundary conditions are very constrained. This is because the coefficients must take one of two forms, [14, 39, 40] either:

$$A_i = 0, \quad \forall i = 0, 1, \ldots, n \qquad \text{or} \qquad |A_i| = 2\sqrt{n_i}, \quad \forall i = 0, 1, \ldots, n,$$

to maintain integrability. The only exception is the sine-Gordon model

($n = 1$), where the two coefficients are free. Under these conditions the theory is shown to be integrable by addressing the conservation of several charges, of different spin, and also by a Lax pair approach [11, 12, 39].

For the $a_2^{(2)}$ ATFT, when the theory is restricted to be real, a similarly constrained boundary density occurs. The existing Lagrangian (1.21) is modified slightly, as it is built from a different root space. Specifically, the $a_2^{(2)}$ model is a member of the non-simply laced theories, which means that all roots do not have the same length and the potential (1.22) takes the form:

$$U(u) = \frac{m^2}{\beta^2}\left(e^{\sqrt{2}\beta u} + 2e^{-\beta u/\sqrt{2}} - 3\right),$$

The condition to maintain integrability is given by [39]:

$$\mathcal{B} = A_1 e^{u} + A_0 e^{-u/2}, \tag{1.29}$$

after rescaling the field suitably, where the coefficients satisfy

$$A_0(A_1^2 - 2) = 0.$$

Soon after, a new class of integrable boundary conditions for the ATFTs were calculated by Delius [41]. The discovery was prompted by allowing the ATFT to be complex and considering a 'method of mirror images' that uses parity reversal [41]. This method pairs a soliton in the region $x < 0$ with another soliton in region $x > 0$. The boundary condition is a combination of a Dirichlet condition on the imaginary component of the field, and a Neumann condition on the real part:

$$\left.\frac{\mathrm{Im}(u)}{2\pi}\right|_{x=0} \in \text{ coweight lattice of g}, \qquad \left.\partial_x \mathrm{Re}(u)\right|_{x=0} = 0, \tag{1.30}$$

where the coweight lattice is a set of weights, $\{l\}$, such that $l \cdot \alpha \in \mathbb{Z}$, for all roots in the root system of the underlying Lie algebra g [42]. The above condition still appears restrictive as there are no free parameters. However, there is increased freedom because the imaginary part of the field can be any element of the coweight lattice.

The curious integrable boundary conditions of the ATFTs, from the existing literature, have been documented. Yet again, one might hope to generalise the existing situation since they take such a severely restricted form, or at least, one might hope to uncover the mechanism that prescribes such restrictive conditions.

We will now consider a reflection matrix and its analogue in the classical setting, to give further evidence that points to the rich interplay between the classical and quantum set-ups. Diagonal reflection matrices are of particular importance for our further work, hence we will examine the sine-Gordon's only diagonal reflection factor [13]:

$$R_d(x) = \rho_R(\theta)\begin{pmatrix} (\frac{1}{rx} + rx) & 0 \\ 0 & (\frac{r}{x} + \frac{x}{r}) \end{pmatrix}, \tag{1.31}$$

where $\rho_R(\theta)$ is the prefactor restricted by crossing and unitarity, it is found in [13]. We can see that this matrix possesses one free parameter, r. The diagonal nature of the matrix suggests that the corresponding boundary condition takes a simple form. In fact, reflection matrix (1.31) corresponds to a soliton preserving Dirichlet condition. Moreover, if the parameter r is set to unity the boundary condition becomes $u(0,t) = 0$ and the matrix becomes a multiple of the identity.

Another interesting case within the sine-Gordon model is the non-diagonal reflection matrix first calculated by Zamolodchikov and Ghoshal [13]. It was calculated subsequently via the algebraic technique of Delius and MacKay [18] and is documented in matrix (1.20). In particular, it contains two parameters $\hat{\epsilon}_\pm$ that are associated with the boundary condition, this matches the known real integrable boundary condition which also contains two free parameters. In fact, the condition is presented as:

$$u_x = \frac{a}{\beta}\sin\beta\left(\frac{u-u_0}{2}\right)\Big|_{x=0}, \tag{1.32}$$

where a and u_0 are arbitrary constants and β is the sine-Gordon coupling. The above boundary condition is obtained from equation (3.18), in Delius and MacKay's paper [18], by making the equalities:

$$\hat{\epsilon}_+ = ae^{i\beta u_0/2}, \qquad \hat{\epsilon}_- = ae^{-i\beta u_0/2},$$

together with the normalisation that they adopt throughout [18].

We will now move on to recount another aspect of integrability, namely defects. Later, we will observe how they are used to generalise reflection matrices and therefore provide possible candidates for more general boundary conditions.

1.4 Integrable Defects

Following the logical progression of ideas, as the vast literature concerning the Yang-Baxter and reflection equations, as well as integrable boundary conditions developed, it is natural to next consider models with an internal boundary: also known as an impurity or defect. The foundations for defects within two-dimensional integrable theories were first established by Delfino, Mussardo and Simonetti [8, 9]. Their results showed that theories including a defect can only remain integrable if the defect is either purely reflecting (an integrable boundary) or purely transmitting. Alternative frameworks do exist, whereby an impurity both reflects and transmits particle content. In one case, the defect must possess internal degrees of freedom [43]. Another alternative was investigated by Mintchev et al. [44], however, their results rely on S-matrices that do not depend on the rapidity difference, $S(\theta_a - \theta_b)$, but on each rapidity separately, $S(\theta_a, \theta_b)$.

In the real world defects are commonplace. Within a two-dimensional integrable quantum field theory one can regard defects as a juncture of two bulk regions. Special defects conditions must hold to ensure that the two bulk regions are 'stitched' together. This will become evident in the following section, where we detail the Lagrangian formalism developed by Bowcock, Corrigan and Zambon [15], first for the sine-Gordon model. The results of the Lagrangian approach, in the sine-Gordon case, matched Konik and LeClair's earlier findings for the quantum sine-Gordon model [10]. For other models we will see, again, how the classical and quantum theories influence one another. This work was extended to include $a_n^{(1)}$ defects in [16]. Later work lead to the discovery of another type of defect, supported by the $a_n^{(1)}$ theories and the Tzitzéica model - $a_2^{(2)}$ affine Toda [45, 46, 47]. The sine-Gordon, $a_2^{(1)}$ and $a_2^{(2)}$ models are of particular importance for this work.

Furthermore, we will see how the classical framework provides insight into the quantum scenario, which is most important for the work contained herein, especially when we begin to investigate the rich interplay between integrable defects and integrable boundaries. The defect's intrinsic features are exploited to generalise a particular type of integrable boundary .

1.4.1 The Classical Picture

Throughout this section, we will detail the classical theory that is relevant to later work contained in the coming chapters. In particular, we will consider the $a_n^{(1)}$ affine Toda models, which includes $a_2^{(1)}$ that we will examine in chapter (4) and it generalises earlier work concerning the sine-Gordon model.

1.4.1.1 Type - I defects

The addition of a defect, at the point $x = 0$, to a theory is very natural. One simply needs to account for the two fields, u and v, either side of the defect, and the defect itself. Therefore, a single defect placed at $x = 0$ is described by the Lagrangian [16]:

$$\begin{aligned}\mathcal{L}_d =&\theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v \\ &+ \delta(x)\left(\frac{1}{2}\left(u\cdot Bu_t + u\cdot Dv_t - u_t\cdot Dv + v\cdot Bv_t\right) - \mathcal{D}(u,v)\right),\end{aligned} \tag{1.33}$$

where $\theta(x)$ is the Heaviside step function, to signify the bulk regions and $\delta(x)$ is the usual delta function, to signify where the defect is defined. The matrix B i s an anti-symmetric, n by n matrix (where the algebra has rank n) and the matrix D is defined as: $D = 1 - B$. The defect potential, $\mathcal{D}(u, v)$, is defined [16]

$$\mathcal{D}(u,v) = -\frac{m}{\beta^2}\sum_{j=0}^{n}\left(\sigma e^{\frac{i\beta}{2}\alpha_j\cdot(D^Tu+Dv)} + \frac{1}{\sigma}e^{\frac{i\beta}{2}\alpha_j\cdot D(u-v)}\right), \tag{1.34}$$

where: m is the usual mass parameter, β is the coupling, α_j, $j = 1, \ldots, n$ are the simple roots and α_0 is the additional root (1.23) of the particular affine

Toda model. Importantly, the matrix D, satisfies the constraints:

$$\alpha_k \cdot D\alpha_j = \begin{cases} 2, & \text{if } k = j, \\ -2, & \text{if } k = \pi(j), \\ 0, & \text{otherwise.} \end{cases} \qquad D + D^T = 2, \tag{1.35}$$

The matrix D is of central importance, as it depends on a permutation of the simple roots, $\pi(j)$. Consequently, there is associated with D a choice of the clockwise or anti-clockwise permutation. The permutation will specify the defect and we will see its importance in the quantum setting later in chapter (4). For now, let us choose the clockwise permutation, following [48],

$$\alpha_{\pi(j)} = \alpha_{j-1}, \quad j = 1, \ldots, n, \qquad \alpha_{\pi(0)} = \alpha_n.$$

The constraints are satisfied when D is chosen to take the form

$$D = 2\sum_{i=1}^{n} l_a (l_a - l_{a+1})^T, \tag{1.36}$$

where the vectors l_a, $a = 1, \ldots, n$ are the fundamental highest weights of the Lie algebra a_n. In particular, they satisfy: $l_i \cdot \alpha_j = \delta_{ij}$ for $i, j = 1, \ldots, n$, and $l_0 \equiv l_{n+1} = 0$. If the anti-clockwise permutation of simple roots is used, one simply replaces D, defined in (1.36), with its transpose.

Using the Lagrangian (1.33) we can obtain the equations of motion, as well as the defect conditions specified at $x = 0$ [16],

$$u_{tt} - u_{xx} = \frac{im^2}{\beta}\sum_{j=0}^{n} \alpha_j e^{i\beta\alpha_j \cdot u}, \qquad x < 0, \tag{1.37}$$

$$v_{tt} - v_{xx} = \frac{im^2}{\beta}\sum_{j=0}^{n} \alpha_j e^{i\beta\alpha_j \cdot v}, \qquad x > 0, \tag{1.38}$$

$$u_x - Bu_t - Dv_t + \mathcal{D}_u = 0, \qquad x = 0, \tag{1.39}$$

$$v_x - D^T u_t + Bv_t - \mathcal{D}_v = 0, \qquad x = 0. \tag{1.40}$$

It should be mentioned that the defect conditions in fact take the form of a Bäcklund transformation that is frozen at $x = 0$ [16]. This feature is interesting as such transformations are a hallmark of integrability. It is also reasonable, given that the defect is purely transmitting and solitons usually re-emerge from the defect, but it cannot be applied in the usual sense to obtain multi-soliton solutions. Furthermore, the bulk and defect potentials are both invariant under particular translations of the fields, u and v:

$$u \to u + \frac{2\pi b}{\beta}, \qquad v \to v + \frac{2\pi c}{\beta},$$

for b, c any two elements of the root lattice. As a result of this invariance, one can obtain other solutions, that have the same energy and momentum. Some comments regarding energy and momentum, in the presence of the defect are in order. The energy and momentum contributions from the bulk regions are given by the usual formulae, respectively:

$$\begin{aligned} E = &\int_{-\infty}^{0} dx \left(\frac{1}{2}(u_x \cdot u_x) + \frac{1}{2}(u_t \cdot u_t) + U(u)\right) \\ &+ \int_{0}^{\infty} dx \left(\frac{1}{2}(v_x \cdot v_x) + \frac{1}{2}(v_t \cdot v_t) + V(v)\right), \end{aligned} \tag{1.41}$$

$$P = \int_{-\infty}^{0} dx (u_x \cdot u_t) + \int_{0}^{\infty} dx (v_x \cdot v_t).$$

In order to contemplate their conservation we must consider their time derivatives. By differentiating the above with respect to time, as well as using the equations of motion appropriately together with the assumption that the fields provide no contribution at spatial infinity one finds [49]

$$\dot{E} = u_x \cdot u_t \,|_{x=0} - v_x \cdot v_t \,|_{x=0}, \tag{1.42}$$

$$\begin{aligned} \dot{P} = &\left(\frac{1}{2}(u_x \cdot u_x) + \frac{1}{2}(u_t \cdot u_t) - U(u)\right)\Bigg|_{x=0} \\ &- \left(\frac{1}{2}(v_x \cdot v_x) + \frac{1}{2}(v_t \cdot v_t) - V(v)\right)\Bigg|_{x=0}. \end{aligned} \tag{1.43}$$

To enable the exchange of energy and momentum at the defect, the above quantities must be expressed as time derivatives of functions of the fields at the defect location. For the conservation of momentum, we can use the defect conditions (1.39) and (1.40) to eliminate the spatial derivatives of the fields in the above equation:

$$\dot{P} = \left(\frac{1}{2}(Bu_t + Dv_t - \mathcal{D}_u)^2 + \frac{1}{2}(u_t \cdot u_t) - U(u)\right)$$
$$- \left(\frac{1}{2}(D^T u_t - Bv_t + \mathcal{D}_v)^2 + \frac{1}{2}(v_t \cdot v_t) - V(v)\right). \tag{1.44}$$

It is readily observed that the above contains only time derivatives of the fields, as well as the potentials: $U(u)$, $V(v)$, and derivatives of the defect potential. It is possible to arrange $\dot{P}$ as a total time derivative providing several relations hold:

$$u_t \cdot (1 + B^T B - DD^T)u_t = 0, \tag{1.45}$$

$$v_t \cdot (1 - D^T D - B^T B)v_t = 0, \tag{1.46}$$

$$u_t \cdot (B^T D + DB)v_t = 0, \tag{1.47}$$

$$\frac{1}{2}\mathcal{D}_u \cdot \mathcal{D}_u - \frac{1}{2}\mathcal{D}_v \cdot \mathcal{D}_v = U(u) - V(v). \tag{1.48}$$

Relations (1.45)-(1.47) are verified easily using the defining properties of the matrices B and D stated earlier. The final relation is not so easily verified. In [16], the final relation is verified by running the defect potential through (1.48), after which the properties of matrix D are again used to rearrange the result. The matrix D and its connection with the permutation of roots is crucial to obtain the end result [16]:

$$\dot{P} = \dot{\mathcal{U}}, \quad \mathcal{U} = -\frac{m}{\beta^2}\sum_{j=0}^{n}\left(\sigma e^{\frac{i\beta}{2}\alpha_j \cdot (D^T u + Dv)} - \frac{1}{\sigma} e^{\frac{i\beta}{2}\alpha_j \cdot D(u-v)}\right), \tag{1.49}$$

hence, the defect supplies a contribution, $\mathcal{U}$, to the momentum so that the total momentum, $\mathcal{P} = P + \mathcal{U}$, is conserved. Overall, this is result is quite remarkable because the defect's construction appears to account for its breaking of the space translational invariance. However, the defect does not disrupt time translational invariance and so the defect potential simply contributes to the total energy. Specifically, the equation (1.42) for $\dot{E}$ becomes equal to $\dot{\mathcal{D}}$, which means the total energy $\mathcal{E} = E + \mathcal{D}$ is conserved.

In fact, it is shown in [49] that if momentum is conserved the theory with a defect is classically integrable. A generalised Lax pair is also employed to show that the theory including a defect remains integrable. Consequently, the conservation of momentum is an important facet of the defect theory. Previously, the constant fields

$$u = \frac{2\pi b}{\beta}, \qquad v = \frac{2\pi c}{\beta},$$

were mentioned in section (1.3). However, some of their features are modified when a defect is included within the theory. Usually, the constant fields, where b and c are weights, belonging to the same representation in this instance, have zero energy. As we have seen, the defect adds its own contributions of energy and momentum so that their energy and momentum are now given by [48]:

$$\mathcal{E}_0 = -\frac{2mh}{\beta^2}\cosh\eta, \qquad \mathcal{P}_0 = \frac{2mh}{\beta^2}\sinh\eta,$$

where η is related to the defect parameter σ in the following way: $\sigma = e^{-\eta}$. This concludes our consideration of energy and momentum. We will now progress to review the way in which solitons transmit through a defect, so that we can work towards the quantum picture.

The way in which a soliton transmits through the defect is of greater relevance to us. The classical behaviour provides insight into the defect transmission matrices of the quantum setting, which our work relies upon heavily. The main goal is to understand all possible behaviour exhibited by a right-moving soliton when it encounters the defect. One can again use Hirota's formulation [37, 38], where the $a_n^{(1)}$ single soliton is given by:

$$u_a = \frac{im^2}{\beta}\sum_{j=0}^{n}\alpha_j \ln(1 + E_a\omega^{aj}), \tag{1.50}$$

where the parameters are defined as before (1.24). Eventually, the soliton will encounter the defect at $x = 0$ and it typically emerges; albeit slightly modified

$$v_a = \frac{im^2}{\beta}\sum_{j=0}^{n}\alpha_j \ln(1 + z_a E_a\omega^{aj}). \tag{1.51}$$

The additional factor, z_a, encodes all possible processes a soliton can undergo during transmission. It first appeared in [16] where it was calculated for the anti-clockwise permutation of simple roots. Later, in [48], it was calculated for the clockwise permutation by sending the a^{th} soliton of the expression in [16] to the $(h-a)^{\text{th}}$ soliton. Specifically, the soliton is delayed by the defect and the delay parameter z_a with the clockwise permutation is:

$$z_a = \left(\frac{e^{-(\theta-\eta)} + ie^{-i\gamma_a}}{e^{-(\theta-\eta)} + ie^{i\gamma_a}}\right), \qquad \gamma_a = \frac{\pi a}{h}, \tag{1.52}$$

where h is the Coxeter number of the particular $a_n^{(1)}$ theory. Upon quick inspection we see that the delay is usually complex. However, the exceptions to this are the self-conjugate solitons where $a = h/2$ (with n odd), in this case the delay is real and is equal to the delay experienced by sine-Gordon solitons [15]:

$$z = \left(\frac{e^{\eta-\theta}+1}{e^{\eta-\theta}-1}\right) = \coth\left(\frac{\eta-\theta}{2}\right). \tag{1.53}$$

Let us now examine the delays experienced by the solitons. Firstly, consider the behaviour encoded within (1.53). There are several possible configurations of the parameters that return interesting results. If the overall argument is negative: $\eta - \theta < 0$, then $z < 0$ and the incoming soliton converts to an anti-soliton. This occurs when $\eta < 0 < \theta$ and again for $\eta < \theta, \eta > 0$. By anti-soliton we mean a soliton of the same mass, but with opposite topological charge. Topological charge is described by the imaginary part of ξ_a appearing in (1.24). Therefore, for large enough values of θ, enough topological charge is exchanged with the defect to cause the imaginary part of ξ_a to change, so that an anti-soliton re-emerges. If $\eta > 0$ and $\theta < \eta$, the argument remains positive and the soliton is simply delayed. Interestingly, if $\theta = \eta$, the argument becomes zero and the soliton never leaves the defect, it is subsumed by the defect. The delay expressed in (1.52) acts somewhat differently. Following [16, 48] it is instructive to consider the argument of the phase delay (1.52):

$$\tan(\arg z_a) = -\frac{\sin 2\gamma_a}{e^{-2(\theta-\eta)} + \cos 2\gamma_a}. \tag{1.54}$$

The above is very interesting because the phase shift appears able to vary between zero, provided $\theta \to -\infty$, and $-2\gamma_a$, provided $\theta \to \infty$. Furthermore, this interval coincides with the separation of the different topological charge sectors [50], in terms of the parameter ξ_a appearing in the soliton's definition (1.50) - specifically, its imaginary part. Thus, a soliton can convert to one of its neighbours when the rapidity is sufficiently large. This behaviour is most significant, because its repercussions are felt within the quantum theory. The transmission matrices must mirror the severely restricted processes, and indeed they do, as some contain zeroes that replicate the restricted process. The full details are discussed in chapter (4).

One final remark concerning (1.52) is required. Owing to the definition of γ_a, one readily observes that the delay diverges for a specific value of the rapidity:

$$\theta = \eta + \frac{i\pi}{2}\left(1 - \frac{2a}{h}\right).$$

Overall, except for the self-conjugate solitons where $a = h/2$, this means that the defect cannot absorb any soliton possessing real rapidity. This concludes our recount of type - I integrable defects. Let us now move on to the yet more interesting type - II defects.

1.4.1.2 Type - II defects

The fact that the type - I defect conditions, for the $a_n^{(1)}$ Toda models, take the form of a Bäcklund transformation frozen at the defect location is entirely consistent with the work of Fordy and Gibbons, where the sine-Gordon Bäcklund transformation was generalised to encompass the $a_n^{(1)}$ models [51]. However, the constraints placed on the theory, and appropriate defect potential, appear too stringent. As a result, it seemed that none of the other affine Toda models could support a defect.

To make progress, and discover defects in other affine Toda models, a generalisation was proposed by Corrigan and Zambon, whereby the defect possesses its own degree of freedom [47]. The added parameter induces extra freedom and one can similarly show that both energy and momentum are conserved. However, energy conservation is no surprise because the time translation invariance is not violated. On the whole, despite more complicated defect potentials, the type - II scenario is treated analogously and is shown to be compatible with the bulk theory. In particular, it is well-defined by the consideration of conservation of momentum, reference [47] contains full details. The framework is sufficiently more general to allow the inclusion of a defect within the Tzitzéica model. However, one must consider the conservation of different charges to show that the theory is integrable, as a generalised Lax pair methodology does not yet exist in this framework.

The Lagrangian framework is modified to include the defect's extra degree of freedom, λ, at $x = 0$, it takes the form [45, 46, 47]:

$$\mathcal{L}_{II} = \theta(-x)\mathcal{L}_u + \theta(x)\mathcal{L}_v + \delta(x)\left(2q \cdot \lambda_t - \mathcal{D}(u, v, \lambda)\right), \tag{1.55}$$

with $q = (u(0,t) - v(0,t))/2$ and as usual the fields u and v are defined on the left and right of the defect, respectively. The defect degree of freedom is restricted to $x = 0$. It is useful to consider the system in terms of:

$$p = \frac{u+v}{2}, \quad q = \frac{u-v}{2},$$

then as well as the typical bulk equations of the motion, the Lagrangian (1.55) supplies the defect conditions [46]:

$$2q_x = -\mathcal{D}_p, \quad 2p_x - 2\lambda_t = -\mathcal{D}_q, \quad 2q_t = -\mathcal{D}_\lambda. \tag{1.56}$$

Corrigan and Zambon went on to find that the defect potential must satisfy two further conditions:

$$\begin{aligned} \mathcal{D} &= f(p+\lambda, q) + g(p-\lambda, q), \\ \nabla_\lambda f \cdot \nabla_\lambda g - \nabla_\lambda g \cdot \nabla_q f &= U(u) - V(v). \end{aligned} \tag{1.57}$$

The second constraint above is most powerful, as it links the features of the defect to the potential difference across the defect. Moreover, the right-hand side of the second constraint does not contain any dependence on λ, further signifying its powerful nature.

We will now detail the defect potential for the $a_2^{(2)}$ affine Toda model. The fields u and v are single component scalar fields that have bulk potentials:

$$U(u) = -\frac{m^2}{\beta^2}\left(e^{i\beta u\sqrt{2}} + 2e^{-i\beta u/\sqrt{2}} - 3\right),$$

$$V(v) = -\frac{m^2}{\beta^2}\left(e^{i\beta v\sqrt{2}} + 2e^{-i\beta v\sqrt{2}} - 3\right),$$

where m represents the mass scale and β is the real coupling constant. The associated type - II defect potential, fulfilling conditions (1.57) is [45, 47]:

$$\begin{aligned}\mathcal{D}(p,q,\lambda) = {}& \frac{\sqrt{2}m\sigma}{\beta^2}\left(e^{i(p+\lambda)\beta/\sqrt{2}} + e^{-i(p+\lambda)\beta/2\sqrt{2}}\left(e^{iq\beta/\sqrt{2}} + e^{-iq\beta/\sqrt{2}}\right)\right)\\ &+ \frac{\sqrt{2}m}{2\beta^2\sigma}\left(8e^{-i(p-\lambda)\beta/2\sqrt{2}} + e^{i(p-\lambda)\beta/\sqrt{2}}\left(e^{iq\beta/\sqrt{2}} + e^{-iq\beta/\sqrt{2}}\right)^2\right),\end{aligned} \tag{1.58}$$

and we see that the usual defect parameter σ appears alongside λ that suitably generalises the framework. Again, we would like to consider the soliton's transmission through the defect; within the new framework. The Hirota formalism [37, 38] is used, again, to express the soliton solutions of the bulk theory:

$$\begin{gathered} e^{iu} = \frac{(1+E)^2}{(1-4E+e^2)}, \qquad e^{iv} = \frac{(1+zE)^2}{(1-4zE+z^2e^2)}, \\ E = e^{ax+bt+c}, \qquad a = \sqrt{6}\cosh\theta, \qquad b = -\sqrt{6}\sinh\theta. \end{gathered} \tag{1.59}$$

where z signifies the delay experienced by the soliton. We have, in this instance, dropped the dependence on the mass scale and coupling. Also, one must remember that within this model the soliton is complex, and the constant e^c is selected such that the expressions for the soliton are non-singular for all real x and t. Typically, we expect the soliton to exhibit standard behaviour when passing through the defect, even in the presence of the defect 'field', λ. The delay, of most relevance to us, experienced by a soliton is:

$$z = \coth\left(\frac{\theta-\eta}{2} - \frac{i\pi}{12}\right)\coth\left(\frac{\theta-\eta}{2} + \frac{i\pi}{12}\right), \qquad \sigma = \sqrt{2}e^{-\eta}. \tag{1.60}$$

The above is calculated in [47], where the full details are provided. The main point is that the defect field, λ, must take a particular value when calculating the possible delays. More importantly, this delay exhibits the same behaviour as the type - I sine-Gordon defect. Namely, if η is real, thus matching the rapidity that is always real, the delay is real and positive. In this instance, the soliton simply transmits through the defect and is delayed. It cannot convert to its anti-soliton. Finally, for specific choices of complex η, the soliton can either convert to its anti-soliton or the defect can absorb it. Such behaviour is important for our purposes, because it helps to lead investigations in the quantum picture where suitable transmission matrices are calculated. The auxiliary field possessed by the defect, in the classical picture, will translate into the presence of (related) free functions of the topological charge in the quantum transmission matrix. Consequently, this shows that the type - II defect can also store an unlimited amount of topological charge.

For completeness, we will now include the classical framework for the type - II defect in the $a_n^{(1)}$ theories [45], as we will consider the type - II transmission matrices of $a_2^{(1)}$ in chapter (4). The potentials of the bulk fields u and v are:

$$U(u) = -\frac{m^2}{\beta^2}\sum_{j=0}^{n}(e^{i\beta\alpha_j\cdot u} - 1), \qquad V(v) = -\frac{m^2}{\beta^2}\sum_{j=0}^{n}(e^{i\beta\alpha_j\cdot v} - 1),$$

and as usual, α_j, j=1,$\ldots,n$ are the simple roots, where $\alpha_0 = -\sum_{j=1}^{n}\alpha_j$ is defined as the lowest root. The Lagrangian for this type - II set-up takes the same form as (1.55). The main difference is that there are two possible defect potentials, referred to as setting A and setting B in [45]. The defect potential in setting A is given by:

$$\mathcal{D}(p,q,\lambda) = \frac{m}{\beta^2}\sum_{j=0}^{n}\left(\sigma e^{i\beta\alpha_j\cdot(p+\lambda)/2}A_j(q) + \frac{1}{\sigma}e^{i\beta\alpha_j\cdot(p-\lambda)/2}A_{j+1}(q)\right), \qquad (1.61)$$

with

$$A_j(q) = \gamma e^{i\beta\alpha_j\cdot Gq/2} + \frac{1}{\gamma}e^{-i\beta\alpha_j\cdot Gq/2},$$

where p and q are defined as before. Additionally, the constant matrix, G, is defined as:

$$G = 2\sum_{a=1}^{n}(\omega_a - \omega_{a+1})\omega_a^T, \qquad \alpha_i\cdot\omega_j = \delta_{ij}, \quad i,j = 1,\ldots,n,$$

where ω_i are the fundamental weights of the $a_n^{(1)}$ Lie algebra, also, σ,γ are the two defect parameters. To obtain the defect potential in setting B, simply make the substitutions $p \to -p$ and $G \to -G$. Let us recall that the classical type - I defect also contains two possible defect potentials, originating from a choice of permutation that defines the matrix D: hence, it is unsurprising that the type - II setting also contains two possibilities. Ultimately, we are most interested in the type - II behaviour within the quantum framework, and its impact on the transmission matrices. In the $a_2^{(1)}$ case, many parameters are included within a type - II T-matrix. Yet more significantly, there is a definitive relation between the type - I and type - II matrices. Altogether, this exemplifies the strong relationship between type - I and type - II defects within the same theory and it makes for interesting behaviour.

Finally, before the quantum setting is reviewed, a comment concerning type - II defects is required. In the case of the sine-Gordon model, one can view the type - II defect as two fused type - I defects [46]. If two defects are fused together, the presence of an auxiliary field within the resulting defect is entirely natural. The extra freedom could be a result of something formed or trapped during the fusion process. However, we will view type - II defects in their own right and do not need to consider the fusion process any further.

1.4.2 The Quantum Picture

In the quantum setting, the defect is described by an exact transmission (T-) matrix. All information concerning the interaction between soliton and defect is encoded within the T-matrix. The early results of Delfino, Mussardo and Simonetti [8, 9] showed that an integrable defect must be purely transmitting. Furthermore, the defect can be included in the FZ-algebra, where it is represented by the operator D_α, to denote a defect carrying topological charge α. The process of a soliton (labelled with the usual operators $A_i(\theta)$) transmitting through the defect from the left is given by:

$$A_i(\theta)D_\alpha = T_{i\alpha}^{j\beta}(\theta)D_\beta A_j(\theta).$$

The role of the T-matrix is clear from the above: the incident right-moving soliton, labelled subscript i, encounters the defect with initial charge α. During its interaction the soliton can exchange topological charge with the defect, and potentially convert to another soliton or anti-soliton. Consequently, the emerging soliton is labelled subscript j and it continues to move away from the defect, with final charge β. To account for topological charge conservation, each entry in the T-matrix includes a Kronecker-delta operator-like object: δ_α^β, $\delta_\alpha^{\beta\pm1}$, $\delta_\alpha^{\beta\pm2}$ for example. There exists an algebraic framework, developed by Weston [52, 53], wherein each matrix entry includes appropriate raising and lowering operators $a_i, a_i^\dagger$ satisfying algebraic relations, thus replacing the Kronecker-deltas. We will now give details of both theories.

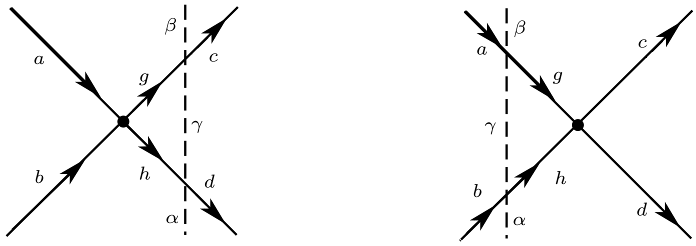

Figure 1.3: Diagrammatic representation of the transmission Yang-Baxter equation, where the dashed line represents the defect.

1.4.2.1 The Quadratic equation

Ultimately, we see that T-matrices are the crucial component. We will now consider the ways in which they are defined. In particular, we will describe two methods by which they are constructed: one involves solving the transmission Yang-Baxter equation and the other employs quantum groups.

The former ensures that the defect is compatible with the S-matrix, the process is illustrated in figure (1.3) and the equation is given by:

$$S_{ab}^{gh}(\theta_a - \theta_b)T_{h\alpha}^{d\gamma}(\theta_a)T_{g\gamma}^{c\beta}(\theta_b) = T_{b\alpha}^{h\gamma}(\theta_b)T_{a\gamma}^{g\beta}(\theta_a)S_{gh}^{cd}(\theta_a - \theta_b), \tag{1.62}$$

where the repeated indices are summed over. The above index approach is helpful because the charge dependence of the system is explicit, however one can reformulate equation (1.62) in terms of tensor products - as in the case of the reflection equation -

$$S(\theta_a - \theta_b)T_1(\theta_a)T_2(\theta_b) = T_2(\theta_b)T_1(\theta_a)S(\theta_a - \theta_b), \tag{1.63}$$

where $T_1 = T \otimes \mathbb{1}$ and $T_2 = \mathbb{1} \otimes T$. Of course, the two are equivalent, and we will use the latter in the subsequent section. The transmission Yang-Baxter equation does not constrain the T-matrix prefactor, as it simply cancels throughout the equation. However, the T-matrix must satisfy analogues of the crossing and unitarity conditions which do constrain the prefactor. For completeness, it should be noted that the bootstrap procedure also restricts the prefactor and is considered in [46]. However, the work in this thesis does not require the transmission matrices' prefactors, and so we will not discuss them further. Crossing symmetry imposes the condition

$$T_{\bar{a}\alpha}^{\bar{b}\gamma}(\theta) = T_{b\alpha}^{a\gamma}(i\pi - \theta), \tag{1.64}$$

where the barred indices represent the anti-particle of the particular soliton.

The unitarity condition is

$$T_{a\alpha}^{b\gamma}(\theta)\tilde{T}_{b\gamma}^{c\beta}(-\theta) = \delta_a^c \delta_\alpha^\beta. \tag{1.65}$$

The matrix, $\tilde{T}$, describes the transmission of a soliton from the right to the left. We expect that it is defined differently, because the defect breaks the parity invariance of the theory. In fact, it is defined as

$$\tilde{T}(\theta) = T^{-1}(-\theta)$$

and one can rearrange equations (1.62)/(1.63) to obtain its defining equation:

$$S_{dc}^{hg}(\theta_a - \theta_b)\tilde{T}_{h\alpha}^{b\beta}(\theta_b)\tilde{T}_{g\gamma}^{a\beta}(\theta_a) = \tilde{T}_{d\alpha}^{h\gamma}(\theta_a)\tilde{T}_{c\gamma}^{g\beta}(\theta_b)S_{hg}^{ba}(\theta_a - \theta_b).$$

Equation (1.65) highlights an important feature of any transmission matrix, namely, it must be invertible. Previously, in [46], it was shown that there are some solutions to the transmission Yang-Baxter equation that are not invertible - clearly, these solutions are not viable T-matrices as they can never satisfy (1.65). We will see, throughout the coming chapters, that inversion is a vital property. Without it, one cannot construct generalised reflection matrices. Usually, one can construct general formulae for the inverse transmission matrix. They coincide with the standard rules of inversion, but their constituent matrix entries must be shifted to account for changes in topological charge. This is most likely a consequence of the underlying quantum group structure. To give a simple example of this, let us consider a general sine-Gordon T-matrix

$$T_{a\alpha_{SG}}^{b\beta}(\theta) = \begin{pmatrix} T_{11}(\alpha,x)\delta_\alpha^\beta & T_{12}(\alpha,x)\delta_\alpha^{\beta-2} \\ T_{21}(\alpha,x)\delta_\alpha^{\beta+2} & T_{22}(\alpha,x)\delta_\alpha^\beta \end{pmatrix}. \tag{1.66}$$

The inverse T-matrix should possess the same Kronecker-deltas, in the same positions, together with slightly modified entries. The inversion formulae are readily obtained by forming and rearranging equations resulting from the multiplication of (1.66) by a matrix, T^{-1}, whose product returns $\mathbb{1}\cdot\delta_\alpha^\beta$, to find

$$T_{a\alpha_{SG}}^{b\beta^{-1}}(\theta) = \begin{pmatrix} \frac{Y_{11}(\alpha,x)}{\Delta(\alpha,x)}\delta_\alpha^\beta & \frac{Y_{12}(\alpha,x)}{\Delta(\alpha,x)}\delta_\alpha^{\beta-2} \\ \frac{Y_{21}(\alpha,x)}{\Delta(\alpha-2,x)}\delta_\alpha^{\beta+2} & \frac{Y_{22}(\alpha,x)}{\Delta(\alpha-2,x)}\delta_\alpha^\beta \end{pmatrix},$$

with entries:

$$\begin{aligned} Y_{11}(\alpha,x) &= T_{22}(\alpha+2,x), & Y_{12}(\alpha,x) &= -T_{12}(\alpha,x), \\ Y_{21}(\alpha,x) &= -T_{21}(\alpha,x), & Y_{22}(\alpha,x) &= T_{11}(\alpha-2,x), \end{aligned} \tag{1.67}$$

and determinant

$$\Delta(\alpha,x) := T_{11}(\alpha,x)T_{22}(\alpha+2,x) - T_{12}(\alpha,x)T_{21}(\alpha+2,x). \tag{1.68}$$

Immediately, the parallels between the standard inversion formulae of an arbitrary two-by-two matrix are clear. The determinant must be non-zero to guarantee that the matrix is invertible. This provides us with the true constraint, from which we can invert the matrix and go on to show that it is satisfies the unitarity condition. However, the determinant seems to depend upon the topological charge, α. In most cases, the determinant is independent of the topological charge and we will see several instances of this in chapters three and four.

The sine-Gordon model possesses the simplest T-matrices. We have seen that they are easily inverted and contain simple Kronecker-deltas that track exchanges of topological charge. In particular, if the first row/column is regarded as an incoming/outgoing soliton and the second row/column is viewed as an incoming/outgoing anti-soliton, we see that entries: 1,1 and 2,2 must include δ_α^β as no charge is exchanged. Therefore, we see that entry 1,2 (incoming soliton with outgoing anti-soliton) must include $\delta_\alpha^{\beta-2}$ to mirror this process. In the case of the sine-Gordon, the defect simply counts topological charge in units of $\pm 2\pi$; thus emulating the fact that a soliton could emerge from its interaction with the defect as an anti-soliton or vice-versa. However, the defect does not behave like this for all other models. We will now introduce the transmission matrices of the sine-Gordon model, which we will use later. The model is special in that it supports both type - I and type - II defects. The type - I transmission matrix was first calculated by Konik and LeClair

[10]. Some years later, their solution was generalised by Corrigan and Zambon [17]. The new solution includes both the type - I and type - II transmission factors via a specialisation of its parameters, the solution noted in [17] is:

$$T_{a\alpha}^{b\beta}(\theta) = \rho(\theta)\begin{pmatrix}(a_+q^{-\alpha/2}x^{-1} + a_-q^{\alpha/2}x)\delta_\alpha^\beta & \mu(\alpha)\delta_\alpha^{\beta-2} \\ \lambda(\alpha)\delta_\alpha^{\beta+2} & (d_+q^{-\alpha/2}x + d_-q^{\alpha/2}x^{-1})\delta_\alpha^\beta\end{pmatrix}, \tag{1.69}$$

where the functions $\mu(\alpha), \lambda(\alpha)$ satisfy:

$$\mu(\alpha)\lambda(\alpha-2) - \mu(\alpha-2)\lambda(\alpha) = (q-q^{-1})(a_-d_-q^\alpha - a_+d_+q^{-\alpha}). \tag{1.70}$$

The above relation implies the following:

$$\mu(\alpha-2)\lambda(\alpha) = a_-d_-q^{\alpha-1} + a_+d_+q^{-\alpha+1} + \gamma.$$

A solution to the constraint is given in [46], where the functions are chosen as follows:

$$\mu(\alpha) = b_+q^{-\alpha/2} + b_-q^{\alpha/2}, \quad \lambda(\alpha) = c_+q^{-\alpha/2} + c_-q^{\alpha/2}, \quad a_\pm d_\pm - b_\pm c_\pm = 0,$$

for complex constants $a_\pm, b_\pm, c_\pm, d_\pm$. If we substitute the functions into (1.69) and set $Q = q^{-1/2}$, we find:

$$T_{a\alpha}^{b\beta}(\theta) = \rho(\theta)\begin{pmatrix}(a_+Q^\alpha x^{-1} + a_-Q^{-\alpha}x)\delta_\alpha^\beta & (b_+Q^\alpha + b_-Q^{-\alpha})\delta_\alpha^{\beta-2} \\ (c_+Q^\alpha + c_-Q^{-\alpha})\delta_\alpha^{\beta+2} & (d_+Q^\alpha x + d_-Q^{-\alpha}x^{-1})\delta_\alpha^\beta\end{pmatrix}. \tag{1.71}$$

By carefully choosing the parameters in matrix (1.71) we can recover both the type - I and type - II defects. First, set

$$a_- = d_+ = c_- = b_+ = 0, \quad b_- = c_+ \text{ and extract a factor of } (a_+d_-)^{1/2}x^{-1},$$

to recover the type - I transmission matrix [10]:

$$T_{I\ a\alpha}^{\ b\beta} = \rho_I(\theta)\begin{pmatrix}\nu^{-1/2}Q^\alpha\delta_\alpha^\beta & \epsilon x Q^{-\alpha}\delta_\alpha^{\beta-2} \\ \epsilon x Q^\alpha \delta_\alpha^{\beta+2} & \nu^{1/2}Q^{-\alpha}\delta_\alpha^\beta\end{pmatrix}, \tag{1.72}$$

with the identification, $\nu^{1/2} = (d_-/a_+)^{1/2}, \ \ \epsilon = b_-(a_+d_-)^{-1/2}$.

Similarly, by setting,

$$a_+ = 1 = d_-, \qquad a_- = -b_-\bar{b}_+, \qquad d_+ = -b_+\bar{b}_-Q^{-4},$$
$$c_+ = -\bar{b}_-Q^{-4}, \qquad c_- = -\bar{b}_+,$$

as well as extracting a factor of x^{-1} from (1.71), to ensure the recovery of the type - II transmission matrix [17, 46]:

$$T_{II}{}^{b\beta}_{a\alpha} = \rho_{II}(\theta)\begin{pmatrix} (Q^{\alpha} - b_-\bar{b}_+Q^{-\alpha}x^2)\delta^{\beta}_{\alpha} & x(b_+Q^{\alpha} + b_-Q^{-\alpha})\delta^{\beta-2}_{\alpha} \\ -x(\bar{b}_-Q^{\alpha-4} + \bar{b}_+Q^{-\alpha})\delta^{\beta+2}_{\alpha} & (Q^{-\alpha} - b_+\bar{b}_-Q^{\alpha-4}x^2)\delta^{\beta}_{\alpha} \end{pmatrix}. \tag{1.73}$$

As we are aware, the sine-Gordon model is quite specialised and it is known that defects within the $a_n^{(1)}$, $n > 1$, ATFTs behave differently. Let us now document the general structure of an arbitrary $a_2^{(1)}$ transmission matrix, we will work with matrices of this form in chapter (4):

$$T^{b\beta}_{a\alpha} = \begin{pmatrix} T_{11}\delta^{\gamma}_{\alpha} & T_{12}\delta^{\gamma-\alpha_1}_{\alpha} & T_{13}\delta^{\gamma+\alpha_0}_{\alpha} \\ T_{21}\delta^{\gamma+\alpha_1}_{\alpha} & T_{22}\delta^{\gamma}_{\alpha} & T_{23}\delta^{\gamma-\alpha_2}_{\alpha} \\ T_{31}\delta^{\gamma-\alpha_0}_{\alpha} & T_{32}\delta^{\gamma+\alpha_2}_{\alpha} & T_{33}\delta^{\gamma}_{\alpha} \end{pmatrix}. \tag{1.74}$$

The topological charge is no longer counted in units of 2π, therefore, exchanges of charge are no longer tracked by objects of the form: $\delta^{\beta\pm j}_{\alpha}$, $j \in \mathbb{Z}$. The charges are now represented by the weights of the solitonic/anti-solitonic representations, labelled by α, β, and a soliton can move around the corresponding weight lattice by depositing root-like charges at the defect. Consequently, exchanges of topological charge are described by objects of the form: $\delta^{\beta\pm\alpha_i}_{\alpha}$, where α_1, α_2 are the simple roots of $a_2^{(1)}$ and α_0 is the additional lowest root. Such transmission matrices are also invertible, as they should be, and an infinite-dimensional analogue of Cramer's rule is easily derived.

1.4.2.2 The Linear equation

Presently, we have only considered T-matrices as solutions to the quadratic equation (1.62), which is not easily solved. Let us now move on to inspect the algebraic framework developed by Weston, in [53], where quantum groups are cleverly utilised. The importance of placing the defect in this larger algebraic context is to connect its theory with generalised oscillator algebras and Q-operators [52, 53]. We will now examine and recount the alternative method within [53] that reproduces the sine-Gordon type - I and type - II transmission matrices. In this framework, the S-matrix is replaced by the R'-matrix that acts as an intertwiner of the finite-dimensional representation V_x of $U_q(a_1^{(1)})$:

$$R'(x_1/x_2) : V_{x_1} \otimes V_{x_2} \to V_{x_1} \otimes V_{x_2},$$

satisfying the linear intertwining condition:

$$R'\Delta(a) = \Delta'(a)R', \tag{1.75}$$

for all a, where a is a generator of $U_q(a_1^{(1)})$. Likewise, the T-matrix is viewed as a specialisation of a more general intertwiner, $\mathcal{L}$,

$$\mathcal{L}(z/x) : W_z^{(\underline{r})} \otimes V_x \to W_z^{(\underline{r})} \otimes V_x.$$

We can see that $\mathcal{L}$ intertwines an infinite-dimensional space, $W_z^{(\underline{r})}$, and the finite-dimensional space, V_x. In fact, $W_z^{(\underline{r})}$ is a representation of a Borel subalgebra that is parametrised by a complex vector $(\underline{r}) = (r_0,\ r_1,\ r_2)$, and a rapidity-like parameter, z. The operator, $\mathcal{L}$, satisfies a linear intertwining condition

$$\mathcal{L}\Delta(b) = \Delta'(b)\mathcal{L}, \tag{1.76}$$

where b is any generator of the Borel subalgebra. In terms of the defect transmission matrix, (1.76) is now the defining equation to solve. As the result of the calculation is a more general object, one must choose the parameters $(\underline{r})$ precisely to obtain the sine-Gordon type - I and type - II defects that we are familiar with.

The main constituents of this construction are the infinite-dimensional Borel subalgebra and the finite-dimensional representation of the underlying quantum group. To detail the construction we require the defining relations of the quantum affine algebra $U_q(sl_2)$. It is generated by six generators $\{E_i, F_i, K_i\}$, i = 0, 1, satisfying the following relations:

$$[E_i, F_i] = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}}, \tag{1.77}$$

$$K_iE_iK_i^{-1} = q^2E_i, \qquad K_iE_jK_i^{-1} = q^{-2}E_j, \quad i \neq j, \tag{1.78}$$

$$K_iF_iK_i^{-1} = q^{-2}F_i, \qquad K_iF_jK_i^{-1} = q^2F_j, \quad i \neq j, \tag{1.79}$$

$$E_iE_j^3 - [3]E_jE_iE_j^2 + [3]E_j^2E_iE_j - E_j^3E_i = 0, \quad i \neq j, \tag{1.80}$$

$$F_iF_j^3 - [3]F_jF_iF_j^2 + [3]F_j^2F_iF_j - F_j^3F_i = 0, \quad i \neq j, \tag{1.81}$$

where the notation

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}},$$

is used. The construction relies upon the coproduct, Δ, which is defined as:

$$\begin{gathered} \Delta(E_i) = E_i \otimes \mathbb{1} + K_i \otimes E_i, \qquad \Delta(F_i) = F_i \otimes K_i^{-1} + \mathbb{1} \otimes F_i, \\ \Delta(K_i) = K_i \otimes K_i, \end{gathered} \tag{1.82}$$

for i = 0, 1. The opposite coproduct, Δ', is obtained from the above by applying the permutation operator, P, that interchanges the factors. Following Weston's methodology, we will now begin to construct the infinite-dimensional Borel subalgebra, $U_q(b_+)$. It is generated by the elements E_i, K_i i $= 0, 1$, hence only relations (1.78) and (1.80) must be satisfied by the subalgebra.

The next step in [53] is to define the generalised oscillator algebra that will be used to construct the representation of the Borel subalgebra. The generalised oscillator algebra, $\mathcal{O}^{(r_1,r_2)}$, for complex numbers r_1 and r_2, is generated by the operators: $a, a^\dagger, q^{\pm N}$ satisfying relations

$$\begin{aligned} &q^N a^\dagger q^{-N} = q a^\dagger, \quad q^N a q^{-N} = q^{-1} a, \\ &a a^\dagger = (r_1 + q^{-2N})(r_2 + q^{2N}) = F(N), \\ &a^\dagger a = (r_1 + q^{2-2N})(r_2 + q^{2N-2}) = F(N-1). \end{aligned} \tag{1.83}$$

One can regard the operators: $a, a^\dagger$, as raising and lowering operators that account for the defect's ability to store charge. As a result, clearly, there is no need for a ground state because the defect can store an unlimited amount of topological charge. In order to realise this the $\mathcal{O}^{(r_1,r_2)}$ module $W^{(r_1,r_2)} = \oplus_{j\in\mathbb{Z}}\mathbb{C}\left|j\right\rangle$ is considered, because it acts on the infinite-dimensional space as follows:

$$\begin{gathered} a\left|j\right\rangle = \left|j-1\right\rangle, \quad a^\dagger\left|j\right\rangle = (r_1 + q^{-2j})(r_2 + q^{2j})\left|j+1\right\rangle, \\ q^{\pm N}\left|j\right\rangle = q^{\pm j}\left|j\right\rangle. \end{gathered} \tag{1.84}$$

By setting either $r_1 = 0$ or $r_2 = 0$ one retrieves the usual q-oscillator algebra relations [54]

$$
\begin{aligned}
a^\dagger a - q^2 a a^\dagger &= (1-q^2), \text{ when } r_1 = 0, \\
a a^\dagger - q^2 a^\dagger a &= (1-q^2), \text{ when } r_2 = 0.
\end{aligned}
\tag{1.85}
$$

Armed with this knowledge, we must choose the generators of the Borel subalgebra such that they mimic the above behaviour and act on the infinite-dimensional space appropriately. Again, following [53] we will document the action of the $U_q(b_+)$ module, $W_z^{(\underline{r})}$, where $\underline{r} = (r_0, r_1, r_2) \in \mathbb{C}^3$ and the space is spanned by $|j\rangle \otimes z^n \in W^{(r_1,r_2)} \otimes \mathbb{C}[[z, z^{-1}]]$:

$$
\begin{aligned}
E_0(|j\rangle \otimes z^n) &= \frac{1}{(q-q^{-1})} a^\dagger |j\rangle \otimes z^{n+1}, \\
E_1(|j\rangle \otimes z^n) &= \frac{1}{(q-q^{-1})} a\, |j\rangle \otimes z^{n+1},
\end{aligned}
\tag{1.86}
$$
$$
K_1(|j\rangle \otimes z^n) = r_0 q^{-2N} |j\rangle \otimes z^n, \quad K_0(|j\rangle \otimes z^n) = \frac{1}{r_0} q^{2N} |j\rangle \otimes z^n.
$$

It is easy to verify that this action satisfies the relevant relations (1.78) and (1.80). Furthermore, in terms of constructing the defect transmission matrix, it is clear from the relations (1.86) that we should choose the generators of the infinite-dimensional Borel subalgebra in the following way:

$$
E_0 \propto a^\dagger, \quad E_1 \propto a, \quad K_0 \propto q^{2N}, \quad K_1 \propto q^{-2N}. \tag{1.87}
$$

One can include constants and functions of N in the above, provided that they satisfy the necessary algebraic relations. Any functions and/or constants included are then constrained by the linear intertwining equation.

Before detailing Weston's solution to equation (1.76), further comments regarding the generalised oscillator algebra are necessary. Looking back at Weston's definitions (1.84) we see that infinite-dimensional space is truncated for particular values of r_1 and r_2. For example, let either $r_1 = -q^{-2n}$ or $r_2 = -q^{2n}$ for an integer, n, then $a^\dagger |n\rangle = 0$. Of course, this is not the only possible truncation. In [45] Corrigan and Zambon consider a truncation of the infinite-dimensional space in order to find the scattering matrix, R', embedded within the T'-matrix. This was achieved by considering the action of the raising and lowering operators on the state space. The work contained in this thesis does not require a truncation of the infinite-dimensional space, further details are found in [45, 52, 53].

We are now ready to consider the linear intertwining equation (1.76). To evaluate, and then substitute, the coproduct into the linear equation one must also use the fundamental two-dimensional representation of the quantum affine algebra. The solution given in [53] is:

$$\mathcal{L}^{(\underline{r})}(z,q) = \begin{pmatrix} q^N + r_2 z^2 q^{2-N} & -r_0 z q^{-N+1} a^\dagger \\ -z q^{N+1} a & r_0 q^{-N} + r_0 r_1 z^2 q^N \end{pmatrix}. \tag{1.88}$$

The power of the algebraic approach is clear from the solution. The parameters r_0, r_1, r_2 and z add a lot of freedom, they can be chosen to take certain values that in turn affect the function $F(N)$, and the space over which the operators act. If the $\mathcal{L}$-matrix given in (1.88) is related to defect transmission matrices, then it must satisfy the relevant properties.

First of all, we can calculate the inverse matrix entries using the general formulae:

$$\mathcal{L}^{-1}_{11}(N) = \frac{\mathcal{L}_{22}(N-1)}{\Delta(N-1,z)}, \qquad \mathcal{L}^{-1}_{12}(N) = \frac{-\mathcal{L}_{12}(N)}{\Delta(N-1,z)},$$

$$\mathcal{L}^{-1}_{21}(N) = \frac{-\mathcal{L}_{21}(N)}{\Delta(N,z)}, \qquad \mathcal{L}^{-1}_{22}(N) = \frac{\mathcal{L}_{11}(N+1)}{\Delta(N,z)},$$

where the determinant is defined as:

$$\Delta(N,z) := \mathcal{L}_{11}(N+1)\mathcal{L}_{22}(N) - \mathcal{L}_{12}(N+1)\mathcal{L}_{21}(N)F(N);$$

hence, the inverse is:

$$(\mathcal{L}^{(\underline{r})})^{-1}(z,q) = \frac{1}{\Delta} \begin{pmatrix} r_0 q^{-N+1} + r_0 r_1 z^2 q^{N-1} & r_0 z q^{-N+1} a^\dagger \\ z q^{N+1} a & q^{N+1} + r_2 z^2 q^{1-N} \end{pmatrix}. \tag{1.89}$$

Note that we have dropped the N-dependence of the determinant, because it takes the form:

$$\Delta(z) = r_0 q (z^2 - 1)(r_1 r_2 z^2 - 1).$$

Therefore, the unitarity property is satisfied. Weston also states that the $\mathcal{L}$-matrix satisfies the crossing relation [53]:

$$\mathcal{L}^{-1}(-zq, q) = \frac{1}{\Delta(-zq)}(\sigma^x \mathcal{L}(z)\sigma^x)^{t_2},$$

where σ^x is the Pauli matrix and t_2 denotes the transposition of the two-dimensional space. Finally, the Yang-Baxter relation:

$$R'(x_1/x_2)\mathcal{L}(z/x_2)\mathcal{L}(z/x_1) = \mathcal{L}(z/x_1)\mathcal{L}(z/x_2)R'(x_1/x_2)$$

holds, it is a natural consequence of this construction, where both sides of the equation act on the space $W^{(\underline{r})}(z) \otimes V_{x_1} \otimes V_{x_2}$. As the above properties are satisfied, it appears that the general algebraic object, $\mathcal{L}$, fits into the usual defect framework. However, it is also connected to Baxter's Q-operator, see [55]-[58] for details of Baxter's Q-operator and [52, 53, 59, 60] for details surrounding the connection. The Q-operator can be identified as an appropriately regularised trace of the $\mathcal{L}$-matrix over the infinite-dimensional space [59], and this was first proposed by Bazhanov et al. in [60]. The construction of the general object, $\mathcal{L}$ - in both [52, 60] - is slightly different but remains very algebraic. In a similar fashion to Delius and MacKay, [18], the universal R-matrix is used together with particular evaluation representations. Bazhanov et al. go on to define the Q-operator in terms of this regularised trace, because they show that it satisfies similar functional relations to the Q-operator. On the whole, the workings required to showcase the connection fully are very involved and it is not suitable to document them here. Nonetheless, it is important to highlight that this connection does exist and ties the theory of integrable defects to a wider algebraic setting. We do not require any further details concerning this topic, for information please see references [52, 53, 59, 60].

It still remains to recover the sine-Gordon transmission matrices (1.72), (1.73) from the more general $\mathcal{L}$-matrix. To achieve this one must assign values to all parameters z and r_i, for all i and apply similarity transformations as required. Firstly, we will retrieve the type - I matrix by adopting the method detailed in [53]

$$T_{I_{SG}}(\theta, \eta) = \nu^{-1/2} U_I(\nu) \mathcal{L}^{(r_0=\nu, r_1=0, r_2=0)}(z = ie^{\gamma(\theta-\eta)}, q) U_I^{-1}(\nu), \tag{1.90}$$

where the similarity transformation U_I is:

$$U_I(\nu) = \begin{pmatrix} 1 & 0 \\ 0 & -\nu^{1/2}q^{-N-1/2} \end{pmatrix}.$$

The result of this calculation contains extra factors of i in the off-diagonal entries. This is a result of the different notations adopted by Corrigan and Zambon in [46] and Weston in [53]. When executing the substitution of parameters, one should identify $e^{\gamma(\theta-\eta)} := \epsilon x$. Interestingly, the type - I specialisation forces the oscillator algebra's function $F(N)$ to collapse to unity. This behaviour is expected, since r_0 appears to be the most significant parameter in the type - I case and it has no bearing on the function $F(N)$.

The recovery of the type - II T-matrix is similar, but more involved because all available parameters are given a value:

$$T_{II_{SG}}(\theta, b_-, b_+) = U_{II}\mathcal{L}^{(r_0=1, r_1=\bar{b}_- q^2/\bar{b}_+, r_2=b_-/q^2 b_+)}(z = ie^{\gamma\theta}|b_+|, q)U_{II}^{-1}, \quad (1.91)$$

where the similarity transformation U_{II} is:

$$U_{II}(b_-, b_+) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{i}{|b_+|}(b_+ + b_- q^{-2N-2}) \end{pmatrix}.$$

Performing this calculation returns a type - II T-matrix of the form:

$$T_{II_{SG}}(\theta, b_\pm) = \begin{pmatrix} q^N - b_-\bar{b}_+ q^{-N}x^2 & -\left(\frac{xb_+\bar{b}_+}{b_+q^{N-1}+b_-q^{-N-1}}\right)a^\dagger \\ x(b_+q^{N+1} + b_-q^{-N-1})a & q^{-N} - b_+\bar{b}_- x^2 q^{N+2} \end{pmatrix}, \quad (1.92)$$

and this does not exactly match the previously documented type - II transmission matrix (1.73), satisfying the quadratic equation, regardless of differing conventions. However, the off-diagonal entries satisfy a constraint that is similar to (1.70):

$$\begin{aligned} &\mu(N)\lambda(N-1) - \mu(N-1)\lambda(N) \\ &\qquad = \frac{qx^2 b_+\bar{b}_+}{b_+q^{N-1} + b_-q^{-N+1}}(q - q^{-1})(b_+q^N - b_-q^{-N}), \end{aligned} \quad (1.93)$$

where $\mu(N)$ and $\lambda(N)$ are entries 1,2 and 2,1 respectively. More significantly, we find in this limit that the function $F(N)$ becomes

$$\begin{aligned} F(N) &= 1 + \frac{b_-\bar{b}_-}{b_+\bar{b}_+} + \frac{b_-}{b_+}q^{-2N-2} + \frac{\bar{b}_-}{\bar{b}_+}q^{2N+2} \\ &= \frac{b_-}{b_+}\left(\frac{\bar{b}_-}{\bar{b}_+} + q^{-2N-2}\right) + \frac{\bar{b}_-}{\bar{b}_+}\left(q^{2N+2} + \frac{\bar{b}_+}{\bar{b}_-}\right). \end{aligned} \tag{1.94}$$

At this point, the difference between the type - I and type - II defects of the linear intertwining approach is clear. For the type - I defect, $F(N)$, is merely a constant. Whereas, for the type - II case, $F(N)$ contains two non-trivial parameters: b_+, b_-, thus resulting in a richer transmission matrix. Restricting the general object, $\mathcal{L}$, is illuminating because of the way that the parameters are handled. In the type - I case, r_0, is the only parameter of interest while the others, r_1, r_2, are simply 'switched off'. As one expects, when dealing with the type - II defect, r_1 and r_2 are 'switched on', and there is no need to include the type - I parameter; hence it is 'switched off'. Moreover, one should note that this behaviour is expected given the classical setting of the type - II defect. The extra field associated with the defect translates into the appearance of more parameters in the T-matrix. Of course, the sine-Gordon model is quite specialised and it is fortunate enough to admit both type - I and type - II defects. However, the Tzitzéica model only supports a type - II defect. If one were to calculate the general $\mathcal{L}$-operator for that model, we would see that the parameters r_i are related and one cannot simply 'switch them off' [45].

Throughout this section and its discussion of the linear intertwining approach we have adopted Weston's notation and followed his work closely, [53]. Nevertheless, another notation is perhaps more intuitive, whereby the raising and lowering operators shift the topological charge in units of ± 2, which allows us to easily recognise the processes concerning the sine-Gordon soliton and anti-soliton. Indeed, such a change in notational convention is permitted, providing consistency is maintained ubiquitously. In chapter (2) the type - II function $F(N)$ will reappear, although with this slightly different convention.

Furthermore, some comments regarding the spaces over which the transmission matrices act are required. Let us recall that the transmission matrices defined as intertwiners of the infinite-dimensional Borel subalgebra and finite-dimensional representation act over the spaces:

$$T'(z/x) : \mathcal{V}_z \otimes V_x \to \mathcal{V}_z \otimes V_x,$$

and satisfy the linear intertwining equation. As a consequence of their construction, they naturally satisfy a transmission Yang-Baxter equation [52, 53]:

$$R'(x_1/x_2)T'(z/x_2)T'(z/x_1) = T'(z/x_1)T'(z/x_2)R'(x_1/x_2).$$

Both sides of this equation act on the space:

$$W' = \mathcal{V}_z \otimes V_{x_1} \otimes V_{x_2}.$$

Similarly, if we consider the transmission matrices obtained from the quadratic equation:

$$S_{ab}^{gh}(\theta_a - \theta_b)T_{h\alpha}^{d\gamma}(\theta_a)T_{g\gamma}^{c\beta}(\theta_b) = T_{b\alpha}^{h\gamma}(\theta_b)T_{a\gamma}^{g\beta}(\theta_a)S_{gh}^{cd}(\theta_a - \theta_b),$$

we see that this equation acts on the space

$$W = V_{x_1} \otimes V_{x_2} \otimes \mathcal{V}_z.$$

On the whole, this simply amounts to monitoring notational convention carefully. This does affect generalised reflection matrices slightly, and details are provided in the coming sections. In particular, we will see the effects of these considerations in chapter four, when relating transmission matrices from the two slightly different backgrounds.

We have now explored the two constructions, as well as the features, of transmission matrices. Overall, in the quantum story, it is easier to deal with the linear intertwining equation (1.76). Due to its linear nature the resulting equations are more accessible and easier to manipulate than the quadratic transmission Yang-Baxter relation (1.62); especially, when dealing with type - II defects and their added complexity. Let us now continue to introduce generalised reflection matrices, they are realised by dressing a particular type of reflection matrix. Chiefly, their importance lies in the fact that they combine the theories of boundaries and defects in such a way to produce interesting results.

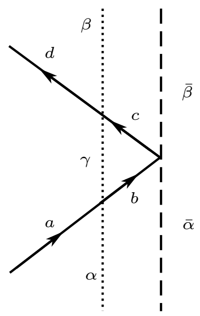

Figure 1.4: Dressing a boundary (represented by the dashed line) with a defect (represented by the dotted line).

1.5 Generalised Reflection Matrices

In light of our knowledge regarding reflection and transmission matrices, we can consider the way in which the two are combined to form a more general object, the generalised reflection matrix, satisfying a suitably generalised reflection equation. The new, more general, object describes a new process: one where an integrable defect is placed near the boundary.

Constructing solutions via the dressing procedure is important for several reasons. Firstly, within this framework one can easily exploit the defect's ability to store charge, therefore generalised reflection matrices depend upon the topological charge and are intrinsically infinite-dimensional. The generalised matrix is endowed with a transmission matrix-like structure, in that it includes either the Kronecker-deltas (with appropriate labels) or the raising/lowering operators $a_i, a_i^\dagger$ to track exchanges of topological charge, exemplifying the infinite-dimensional nature. Introducing the topological charge via the defect circumvents the additional complexity of including it within the original reflection equations. Secondly, the defect can add additional parameters: we will see the importance of these parameters in the coming chapters. Ultimately, it is hoped that this perspective provides new evidence that may help generalise existing integrable boundary conditions.

We will now detail the construction of generalised reflection matrices, the complete process is illustrated in figure (1.4). We have already seen that a reflection matrix should carry its own charge labels: $\bar{\alpha}, \bar{\beta}$, denoting the initial and final charge on the boundary, respectively. When the boundary is dressed

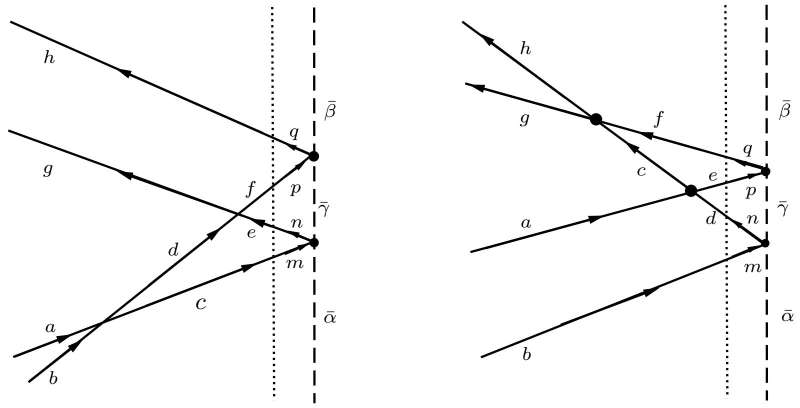

Figure 1.5: Diagrammatic representation of the generalised reflection equation. The dotted and dashed lines represent the defect and boundary respectively.

with a defect, we must include a new set of labels that account for the charge carried on the defect [12, 17, 61]. By consulting figure (1.4) and following the soliton's trajectory, it is clear that the soliton transmits through the defect - from left to right - before reflecting from the boundary and then transmits through the defect - this time from right to left. Let us now take a diagonal reflection matrix, $R^{c\bar{\beta}}_{b\bar{\alpha}}(\theta)$ and dress it with a defect, to obtain the modified matrix:

$$\tilde{R}^{d\beta\bar{\beta}}_{a\alpha\bar{\alpha}} = T^{b\gamma}_{a\alpha}(\theta)R^{c\bar{\beta}}_{b\bar{\alpha}}(\theta)\tilde{T}^{d\beta}_{c\gamma}(\theta), \tag{1.95}$$

where $\tilde{T}(\theta)$ is defined as before: $T^{-1}(-\theta)$, to describe the soliton's transmission through the defect from right to left. The matrix $\tilde{R}(\theta)$ satisfies a modified reflection equation, illustrated in figure (1.5):

$$\begin{aligned}S^{dc}_{ab}(\theta_a-\theta_b)\tilde{R}^{e\gamma\bar{\gamma}}_{c\alpha\bar{\alpha}}(\theta_a)S^{gf}_{de}(\theta_a+\theta_b)\tilde{R}^{h\beta\bar{\beta}}_{f\gamma\bar{\gamma}}(\theta_b)&\\ = \tilde{R}^{d\gamma\bar{\gamma}}_{b\alpha\bar{\alpha}}(\theta_b)S^{ce}_{ad}(\theta_a+\theta_b)\tilde{R}^{f\beta\bar{\beta}}_{e\gamma\bar{\gamma}}(\theta_a)S^{gh}_{cf}(\theta_a-\theta_b)&.\end{aligned} \tag{1.96}$$

In order to show that equation (1.96) is satisfied by the modified $\tilde{R}$-matrix, we must use the following facts: the original diagonal matrix, $R_d(\theta)$, satisfies the reflection equation, as well as a unitarity relation, and the $T/\tilde{T}$-matrices both satisfy the transmission Yang-Baxter equation. To prove this, it is most instructive to use the tensor product formulation of the equations, to avoid confusion involving the indices. Before we move on to the proof, let us detail one key property of the diagonal reflection matrix that is also exhibited by the generalised reflection matrix:

$$R_d(\theta) \cdot R_d(-\theta) = f(\theta) \cdot \mathbb{1}, \tag{1.97}$$

where $f(\theta)$ is a scalar function. In order to show that the $\tilde{R}$-matrix possesses this property we simply substitute the definition of the generalised reflection matrix and evaluate the products:

$$\begin{aligned}
\tilde{R}(\theta) \cdot \tilde{R}(-\theta) &= T(\theta)R_d(\theta)T^{-1}(-\theta) \cdot T(-\theta)R_d(-\theta)T^{-1}(\theta) \\
&= T(\theta)R_d(\theta)\mathbb{1}R_d(-\theta)T^{-1}(\theta) \\
&= f(\theta) \cdot \mathbb{1}
\end{aligned} \tag{1.98}$$

We will now begin with the generalised reflection equation (1.96) written in the tensor product language:

$$S(\theta_a - \theta_b)\tilde{R}_1(\theta_a)S(\theta_a + \theta_b)\tilde{R}_2(\theta_b) = \tilde{R}_2(\theta_b)S(\theta_a + \theta_b)\tilde{R}_1(\theta_a)S(\theta_a - \theta_b), \tag{1.99}$$

where, again, $\tilde{R}_1 = \tilde{R} \otimes \mathbb{1}$ and $\tilde{R}_2 = \mathbb{1} \otimes \tilde{R}$. The next step involves replacing $\tilde{R}$ with its definition throughout the left hand side of the equation, to find:

$$S(\theta_a - \theta_b) \cdot T_1(\theta_a)R_{d_1}(\theta_a)T_1^{-1}(-\theta_a) \cdot S(\theta_a + \theta_b) \cdot T_2(\theta_b)R_{d_2}(\theta_b)T_2^{-1}(-\theta_b). \tag{1.100}$$

To make progress, we must now isolate a particular section of the above and use the transmission Yang-Baxter equation, to find

$$T_1^{-1}(-\theta_a)S(\theta_a + \theta_b)T_2(\theta_b) = T_2(\theta_b)S(\theta_a + \theta_b)T_1^{-1}(-\theta_a).$$

The above relation is readily acquired by reversal of a particle trajectory (associated with rapidity θ_a) in figure (1.3), together with the $\tilde{T}$-matrix that denotes a soliton's transmission through the defect from right to left. Following this, we place the relation in (1.100) to achieve

$$S(\theta_a - \theta_b)T_1(\theta_a)R_{d_1}(\theta_a)\Big[T_2(\theta_b)S(\theta_a + \theta_b)T_1^{-1}(-\theta_a)\Big]R_{d_2}(\theta_b)T_2^{-1}(-\theta_b). \tag{1.101}$$

At this point, it is helpful to remember that R_d is a finite-dimensional diagonal matrix, and therefore the tensor products, R_{d_i}, are also diagonal. It is important that the original diagonal matrix is finite-dimensional. Specifically, its entries do not contain any dependence on topological charge, hence they are unaffected by the operators appearing in the transmission matrix. Consequently, the R_{d_i} will commute with all matrices in the product and we can make use of this in the above expression (1.101)

$$S(\theta_a - \theta_b)T_1(\theta_a)T_2(\theta_b)R_{d_1}(\theta_a)S(\theta_a + \theta_b)R_{d_2}(\theta_b)T_1^{-1}(-\theta_a)T_2^{-1}(-\theta_b). \quad (1.102)$$

Immediately, we recognise half of the transmission Yang-Baxter equation, and replace it to find:

$$T_2(\theta_b)T_1(\theta_a)\Big[S(\theta_a - \theta_b)R_{d_1}(\theta_a)S(\theta_a + \theta_b)R_{d_2}(\theta_b)\Big]T_1^{-1}(-\theta_a)T_2^{-1}(-\theta_b). \quad (1.103)$$

After this, we can use the reflection equation to modify the bracketed term to obtain:

$$T_2(\theta_b)T_1(\theta_a)\Big[R_{d_2}(\theta_b)S(\theta_a + \theta_b)R_{d_1}(\theta_a)S(\theta_a - \theta_b)\Big]T_1^{-1}(-\theta_a)T_2^{-1}(-\theta_b). \quad (1.104)$$

At this moment, we can only use a transmission Yang-Baxter equation to rearrange the terms on the far right of the above

$$T_2(\theta_b)T_1(\theta_a)R_{d_2}(\theta_b)S(\theta_a + \theta_b)R_{d_1}(\theta_a)T_2^{-1}(-\theta_b)T_1^{-1}(-\theta_a)S(\theta_a - \theta_b), \quad (1.105)$$

and then rearrange the factors of R_{d_i}, as they commute with all matrices,

$$T_2(\theta_b)R_{d_2}(\theta_b)\Big[T_1(\theta_a)S(\theta_a + \theta_b)T_2^{-1}(-\theta_b)\Big]R_{d_1}(\theta_a)T_1^{-1}(-\theta_a)S(\theta_a - \theta_b). \quad (1.106)$$

This time, to obtain the bracketed term, we reverse the other particle trajectory - that associated with θ_b and examine the new process to find:

$$T_2(\theta_b)R_{d_2}(\theta_b)\Big[T_2^{-1}(-\theta_b)S(\theta_a + \theta_b)T_1(\theta_a)\Big]R_{d_1}(\theta_a)T_1^{-1}(-\theta_a)S(\theta_a - \theta_b). \quad (1.107)$$

Finally, we can identify the generalised reflection matrices to discover the equality

$$\tilde{R}_{d_2}(\theta_b)S(\theta_a+\theta_b)\tilde{R}_{d_1}(\theta_a)S(\theta_a-\theta_b) = S(\theta_a-\theta_b)\tilde{R}_1(\theta_a)S(\theta_a+\theta_b)\tilde{R}_2(\theta_b) \quad (1.108)$$

which shows that the generalised reflection matrices do indeed satisfy a generalised reflection equation. As we have noted, a transmission matrix can track exchanges of topological charge via Kronecker-delta operator-like objects, or raising/lowering operators associated with an infinite-dimensional Borel subalgebra. The above procedure specialises to the framework where Kronecker-deltas are used within the transmission matrix. An analogous process exists for the transmission matrix originating from the linear intertwining equation. As we are aware, transmission matrices from different defining equations act over slightly different spaces. We will now consider their effect on the generalised reflection matrices.

While the general construction works irrespective of the language used to describe the T-matrices, the generalised reflection matrices are themselves affected by this difference. The upshot is that the generalised solutions act on slightly different spaces, depending on the construction of the T-matrix. Previously, we have discussed the slightly different spaces that the transmission matrices, from the linear and quadratic equation, act over. Let us recall that any T-matrix calculated from the linear intertwining equation (1.76) acts on the space

$$W' = \mathcal{V}_z \otimes V_x,$$

whereas, the T-matrices calculated from the quadratic equation (1.62) act on

$$W = V_x \otimes \mathcal{V}_z.$$

And so, when we calculate generalised reflection matrices using a diagonal (finite-dimensional) reflection matrix, acting on the space:

$$R_d(x) : V_x \to V_x,$$

they will act over the same space as the transmission matrix that is used in its calculation. Most importantly, this does not distort or alter our results but one complication can arise. Namely, when one would like to relate generalised solutions from the two backgrounds. As the defect has the power to introduce several parameters of particular significance in their own framework: either from the representations that are intertwined, or from each matrix entry when considering the transmission Yang-Baxter equation, it is not clear how the various parameters are related. This matter is most likely solved by using similarity transformations, but this is quite a difficult task. In chapter (4) we will encounter this complication.

To demonstrate the nature and features of generalised reflection matrices we will examine the well-studied sine-Gordon model, this work was first documented in [17]. As our starting point, we will take the diagonal reflection matrix introduced earlier (1.31) [13]

$$R_d(x) = \begin{pmatrix} R_{11}(x)\delta_{\bar{\alpha}}^{\bar{\beta}} & 0 \\ 0 & R_{22}(x)\delta_{\bar{\alpha}}^{\bar{\beta}} \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{rx} + xr\right)\delta_{\bar{\alpha}}^{\bar{\beta}} & 0 \\ 0 & \left(\frac{r}{x} + \frac{x}{r}\right)\delta_{\bar{\alpha}}^{\bar{\beta}} \end{pmatrix}. \tag{1.109}$$

First of all, let us consider the resulting multiplication of the dressing procedure, where any defect of the form:

$$T_{\alpha}^{\gamma}(x) = \begin{pmatrix} T_{11}\delta_{\alpha}^{\gamma} & T_{12}\delta_{\alpha}^{\gamma-2} \\ T_{21}\delta_{\alpha}^{\gamma+2} & T_{22}\delta_{\alpha}^{\gamma} \end{pmatrix} \tag{1.110}$$

is used. Of course, the above framework specialises to the sine-Gordon defect transmission matrices and other cases are considered in the coming chapters. Furthermore, the general formulae for the inversion of such a matrix were provided earlier. The $\tilde{T}$-matrix is labelled in the same fashion:

$$\tilde{T}_{\alpha}^{\gamma}(x) = \begin{pmatrix} \tilde{T}_{11}\delta_{\alpha}^{\gamma} & \tilde{T}_{12}\delta_{\alpha}^{\gamma-2} \\ \tilde{T}_{21}\delta_{\alpha}^{\gamma+2} & \tilde{T}_{22}\delta_{\alpha}^{\gamma} \end{pmatrix}. \tag{1.111}$$

The corresponding generalised solutions (for the sine-Gordon model) are calculated by evaluating the product (1.95), specifically:

$$\tilde{R}_{\alpha\bar{\alpha}}^{\beta\bar{\beta}}(x,\alpha) = \begin{pmatrix} \tilde{R}_{11}(x,\alpha)\delta_{\alpha}^{\beta}\delta_{\bar{\alpha}}^{\bar{\beta}} & \tilde{R}_{12}(x,\alpha)\delta_{\alpha}^{\beta-2}\delta_{\bar{\alpha}}^{\bar{\beta}} \\ \tilde{R}_{21}(x,\alpha)\delta_{\alpha}^{\beta+2}\delta_{\bar{\alpha}}^{\bar{\beta}} & \tilde{R}_{22}(x,\alpha)\delta_{\alpha}^{\beta}\delta_{\bar{\alpha}}^{\bar{\beta}} \end{pmatrix}, \tag{1.112}$$

with entries:

$$\begin{aligned} \tilde{R}_{11}(x,\alpha) &= T_{11}(\alpha)R_{11}(x)\tilde{T}_{11}(\gamma=\alpha) + T_{12}(\alpha)R_{22}(x)\tilde{T}_{21}(\gamma=\alpha+2), \\ \tilde{R}_{12}(x,\alpha) &= T_{11}(\alpha)R_{11}(x)\tilde{T}_{12}(\gamma=\alpha) + T_{12}(\alpha)R_{22}(x)\tilde{T}_{22}(\gamma=\alpha+2), \end{aligned} \tag{1.113}$$

$$\begin{aligned} \tilde{R}_{21}(x,\alpha) &= T_{21}(\alpha)R_{11}(x)\tilde{T}_{11}(\gamma=\alpha-2) + T_{22}(\alpha)R_{22}(x)\tilde{T}_{21}(\gamma=\alpha), \\ \tilde{R}_{22}(x,\alpha) &= T_{21}(\alpha)R_{11}(x)\tilde{T}_{12}(\gamma=\alpha-2) + T_{22}(\alpha)R_{22}(x)\tilde{T}_{22}(\gamma=\alpha). \end{aligned} \tag{1.114}$$

In the above, note that, the sum over the charge on the defect is evaluated by tracking the labels of the Kronecker-deltas. The boundary adds an additional $\delta_{\bar{\alpha}}^{\bar{\beta}}$, however its presence is optional because there is no exchange of charge at the boundary and in future work it will be disregarded. Overall, this provides a good example of the methodology used to obtain generalised solutions. For other models, the transmission matrices are typically higher in dimension and the defects are able to store different quantities of charge; described appropriately, as we will see. We will now detail the sine-Gordon generalised solutions and examine their characteristics.

As we have documented, the purpose of this construction is to obtain new solutions to the suitably generalised reflection equation. One might hope that the resulting solution corresponds to an integrable boundary condition lying outside of the known results. When considering the sine-Gordon model, promising results are obtained by dressing the boundary with a type - I transmission matrix (1.72). Primarily, we see that the original diagonal process is greatly modified. The defect appears to generalise the diagonal boundary process (corresponding to a simple Dirichlet condition), returning a generalised Zamolodchikov-Ghoshal type solution of shape (1.112) with entries:

$$\begin{aligned}
\tilde{R}_{11}(x,\alpha) &= x\left(\frac{r}{Q^2}-\frac{\epsilon^2}{r}\right)+\frac{1}{x}\left(\frac{1}{rQ^2}-r\epsilon^2\right),\\
\tilde{R}_{12}(x,\alpha) &= \frac{\epsilon}{r\nu^{1/2}}Q^{\alpha}\left(x^2-\frac{1}{x^2}\right),\\
\tilde{R}_{21}(x,\alpha) &= r\epsilon\nu^{1/2}Q^{-\alpha}\left(x^2-\frac{1}{x^2}\right),\\
\tilde{R}_{22}(x,\alpha) &= x\left(\frac{1}{rQ^2}-r\epsilon^2\right)+\frac{1}{x}\left(\frac{r}{Q^2}-\frac{\epsilon^2}{r}\right).
\end{aligned} \tag{1.115}$$

The results of dressing the simple diagonal matrix with a type - II defect are yet more striking. Again, the resulting generalised solution was first calculated in [17], but is expressed differently below. It has the shape (1.112) with entries

$$\begin{aligned}
\tilde{R}_{11}(x,\alpha) &= \frac{x}{r}b_+\bar{b}_-Q^{2\alpha-2}(1-x^{-4})+\frac{r}{x}b_-\bar{b}_+Q^{-2\alpha-2}(1-x^4)\\
&\quad+Q^{-2}\left(\frac{1}{rx}+xr\right)\left(1+|b_+|^2|b_-|^2\right)+Q^{-2}\left(\frac{x}{r}+\frac{r}{x}\right)\left(|b_+|^2-|b_-|^2\right),\\
\tilde{R}_{12}(x,\alpha) &= \frac{x^4-1}{rx^2}\left(b_++b_-Q^{-2\alpha}\right)\left(Q^{2\alpha}+r^2b_-\bar{b}_+\right),
\end{aligned} \tag{1.116}$$

$$\tilde{R}_{21}(x,\alpha) = -\frac{x^4-1}{rx^2}\left(\bar{b}_+ + \bar{b}_- Q^{2\alpha-4}\right)\left(b_+\bar{b}_- Q^{-4} + r^2 Q^{-2\alpha}\right),$$

$$\begin{aligned}\tilde{R}_{22}(x,\alpha) &= \frac{b_+\bar{b}_-}{rx}Q^{2\alpha-6}(1-x^4) + xrb_-\bar{b}_+ Q^{-2\alpha+2}(1-x^{-4}) \\ &+ Q^{-2}\left(\frac{1}{rx}+xr\right)\left(|b_+|^2-|b_-|^2\right) + Q^{-2}\left(\frac{x}{r}+\frac{r}{x}\right)\left(1+|b_+|^2|b_-|^2\right).\end{aligned} \tag{1.117}$$

In this instance, the dressing procedure produces a new type of solution, lying outside of the Zamolodchikov-Ghoshal class and containing more free parameters. As we might hope, a generalised reflection matrix that lies outside the known class of solutions could correspond to a new type of integrable boundary condition. Corrigan and Zambon proposed a possible Lagrangian density that might correspond to this more general solution, it takes the form [17]:

$$B(u,\lambda) = e^{\lambda/2}f(u) + e^{-\lambda/2}g(u), \tag{1.118}$$

where the functions $f(u)$ and $g(u)$ satisfy:

$$f(u)g(u) = h_+e^{u/2} + h_-e^{-u/2} + 2\left(e^u + e^{-u}\right) + h_0,$$

the functions can be arranged as:

$$f(u) = f_0 + \sqrt{2}\left(be^{u/2} + b^{-1}e^{-u/2}\right), \quad g(u) = g_0 + \sqrt{2}\left(b^{-1}e^{u/2} + be^{-u/2}\right).$$

The density provided above contains more freedom due to the type - II defect's auxiliary field, λ, and free constant parameters:

$$h_0 = g_0 f_0, \quad h_+ = \sqrt{2}\left(f_0 b^{-1} + g_0 b\right), \quad h_- = \sqrt{2}\left(f_0 b + g_0 b^{-1}\right).$$

It appears that this condition is not as heavily restricted, as the original known boundary conditions.

The sine-Gordon model, despite being the simplest ATFT, exhibits very interesting and pleasing results. It also supplies potential evidence as to how a defect can produce generalised solutions that might correspond to a generalised boundary condition. Additionally, one would hope that this phenomenon is exhibited by other ATFTs.

1.6 In summary

Integrability is a vast and wide-ranging area of research: hopefully, we have supplied some evidence of this by considering scattering, reflections/boundary processes and defects. The way in which they all 'marry' together, at the quantum level, is quite clear when one considers generalised reflection matrices. The coming chapters provide further evidence of the interplay between integrable boundaries and defects.

Firstly, in chapter 2, we will consider how sine-Gordon's generalised reflection matrices fit into the algebraic framework of [18]. In particular, we will investigate the way in which boundary subalgebras are modified to include the defect. The results are promising, given that the modified boundary subalgebra associated with the type - I generalised reflection factor appears to generalise the finite-dimensional case. Parallels between the finite and infinite-dimensional subalgebras are readily apparent. When a type - II defect is introduced the resulting subalgebra is yet more general, as one expects. The complex sine-Gordon model is not considered. A thorough examination of boundaries, defects and dressed boundaries within the complex sine-Gordon is contained in [62, 63, 64]. The dressing procedure also places an integrable defect near the boundary, and similar results are obtained; the dressed boundary appears to generalise the original one parameter boundary condition [62] by adding two more parameters [64]. All details are found in [62, 63, 64] and the references contained therein.

In chapter 3, we review the known finite-dimensional reflection matrices of the quantum $a_2^{(2)}$ affine Toda model, as well as the known transmission matrix. We then form several generalised solutions and attempt to relate them to the known finite-dimensional cases. The generalised solutions appear to develop the known finite-dimensional results naturally and embody some of their structures. Unfortunately, due to the complicated nature of the previously known reflection matrices, it did not prove possible to recover them all within new generalised reflection factors. However, we must recognise, it is possible that not all finite-dimensional solutions can be recovered. The original boundary that we dress with an integrable defect takes a particular form and is one that possesses a diagonal reflection factor. And so, it is possible that all existing solutions cannot be reached by dressing such a boundary.

In chapter 4, we will discuss the quantum $a_2^{(1)}$ affine Toda model. We will review all known results concerning reflection and transmission matrices. As we are aware, the model admits several type - I defects within the classical picture and this enriches the quantum setting as it contains several defect transmission matrices. Several classes of generalised solution are then constructed and peculiar behaviour arises. The generalised solutions contain several patterns of zeroes, which at first sight seem peculiar. However, when one recalls the strict selection rule that the defect can impose, the behaviour is not only reasonable but natural. It is illuminating to describe the new generalised solutions pictorially, and several such diagrams are found throughout the chapter. A strange limiting process is utilised to recover the structures of the finite-dimensional solutions; all details are found in chapter 4. The more complicated type - II defect T-matrix is used to construct generalised solutions, which prove to be the most general in that they do not contain any zero entries.

Chapter 5 summarises the findings contained within this thesis and discusses some possible areas of future work.

Appendix A contains the remaining diagrams that describe the $a_2^{(1)}$ generalised reflection matrices.

Finally, appendix B contains the defining relations of the determinant for an $a_2^{(1)}$-transmission matrix that satisfies the linear intertwining equation.

Chapter 2

Coideal Boundary Subalgebras and Defects

Integrable boundaries accommodate integrable defects naturally, however, the proof in chapter one (1.108) concerns the reflection equations originating from the tensor product (and equivalent index) approach. We will now provide further evidence: showing that integrable defects are totally consistent in the coideal boundary subalgebra framework, developed by Delius and MacKay in 2003 [18]. By including a defect, the algebraic framework is generalised to account for exchanges of topological charge and therefore, generalised reflection matrices as well. We specialise to the sine-Gordon model, building on Delius and MacKay's earlier results, where we will study the algebraic framework using both type - I and type - II defects. The close interplay between defects, boundaries and boundary conditions is this chapter's overriding theme.

2.1 Generalising the Framework using a Type - I Defect

At this point, we are familiar with the reflection equation and its importance, as well as its different likenesses: whether that be the tensor product/index approach or the linear intertwining relation,

$$R^{\mu}(\theta)\pi^{\mu}_{\theta}(Q) = \pi^{\bar{\mu}}_{-\theta}(Q)R^{\mu}(\theta), \tag{2.1}$$

for all $Q \in \mathcal{B} \subset U_q(g)$, where $\mathcal{B}$ is the remnant boundary subalgebra of the quantum group. The above equation enables us to view the reflection matrix as an intertwiner of the particle representations. The computation in [18] focusses on the sine-Gordon model and the generators of the boundary subalgebra corresponding to the Zamolodchikov-Ghoshal reflection matrix [13]. If a defect is to be added into this picture, we should be able to take a generalised reflection matrix containing dependence on the topological charge and calculate its modified representation of the subalgebra. The new representation will be more general, as it will include dependence on the topological charge. Once this is known, we can compare it with the known results of Delius and MacKay to see exactly how the defect has expanded on their results. The type - I modified representation is similar to that appearing in [18] and it is apparent how the defect generalises the generators of the boundary subalgebra. The type - II case produces a much more complex object. We will now begin by reviewing the required steps to generalise this particular approach.

Firstly, we will substitute the reflection matrix with a generalised reflection matrix, of the usual form $\tilde{R}(\theta) = T(\theta)R_d(\theta)\tilde{T}(\theta)$, into equation (2.1):

$$T^{\mu}(\theta)R^{\mu}(\theta)\tilde{T}^{\mu}(\theta)\pi^{\mu}_{\theta}(Q) = \pi^{\bar{\mu}}_{-\theta}(Q)T^{\mu}(\theta)R^{\mu}(\theta)\tilde{T}^{\mu}(\theta),$$

noting that the original equation is now changed dramatically. In order to rectify this imbalance, we must include the defect's effects on the representation and specialise to the soliton preserving (SP) case where $\mu = \bar{\mu}$, to give the full equation:

$$\begin{aligned} T^{\mu}(\theta)R^{\mu}_{d}(\theta)T^{\mu^{-1}}(-\theta)T^{\mu}(-\theta)\pi^{\mu}_{\theta}(Q)T^{\mu^{-1}}(-\theta) \\ = T^{\mu}(\theta)\pi^{\mu}_{-\theta}(Q)T^{\mu^{-1}}(\theta)T^{\mu}(\theta)R^{\mu}_{d}(\theta)T^{\mu^{-1}}(-\theta), \end{aligned} \tag{2.2}$$

where the definition, $\tilde{T}(\theta) = T^{-1}(-\theta)$, was used in the above. To demonstrate how this process generalises Delius and MacKay's findings we will specialise to the case of the sine-Gordon model, that has a single two-dimensional soliton multiplet spanned by the soliton and anti-soliton. As we are considering the SP case, we will drop the index μ from now on. Consequently, we see that the above equation is exactly the same as (2.1) if we multiply by $T^{\mu^{-1}}(\theta)$ on the left and $T^{\mu}(-\theta)$ on the right. Let us begin by introducing the type - I transmission matrix, first calculated in [10], presented in a different form:

$$T_I(\theta_i) = \rho_I(\theta) \begin{pmatrix} \frac{\alpha}{x_i} Q^N & \beta a \\ \beta a^\dagger & \frac{1}{\alpha x_i} Q^{-N} \end{pmatrix},$$

with:

$$x_i = e^{\gamma \theta_i}, \qquad q = e^{\frac{-4\pi^2 i}{\beta^2}}, \qquad \gamma = \frac{4\pi}{\beta^2} - 1, \qquad Q^{-2} = q,$$

and free parameters α, β. The coefficient ensures that the transmission matrix satisfies the analogues of crossing and unitarity, but is not required for our purposes. The Kronecker-deltas have been replaced by the annihilation and creation operators, acting on the infinite-dimensional space as follows:

$$a\,|j\rangle = F(j)\,|j-2\rangle\,, \qquad a^\dagger\,|j\rangle = |j+2\rangle\,, \qquad N\,|j\rangle = j\,|j\rangle\,, \qquad j \in \mathbb{Z}.$$

One must also recall the way in which the operators multiply together and act on functions of the number operator:

$$aa^\dagger = F(N+2), \qquad a^\dagger a = F(N),$$

$$aH(N) = H(N+2)a, \qquad a^\dagger H(N) = H(N-2)a^\dagger.$$

The above set-up is simplified for the type - I defect as $F(N) = f$, where f is a constant. The type - I transmission matrix satisfies the transmission Yang-Baxter equation (1.62) with the sine-Gordon S-matrix:

$$S(\Theta) = \rho_S(\Theta) \begin{pmatrix} Q^{-2}x - Q^2x^{-1} & 0 & 0 & 0 \\ 0 & Q^{-2} - Q^2 & x - x^{-1} & 0 \\ 0 & x - x^{-1} & Q^{-2} - Q^2 & 0 \\ 0 & 0 & 0 & Q^{-2}x - Q^2x^{-1} \end{pmatrix},$$

all definitions of the above parameters remain the same, where we now define $x = x_1/x_2$. Again, the coefficient ρ_S ensures that the S-matrix satisfies both unitarity and crossing properties, but it is not needed in this case. If we are to form a generalised solution with this transmission matrix we must calculate $\tilde{T}$. This is easily done using the following inversion formulae, recall:

$$Y_{11} := \frac{T_{22}(N+2,x)}{\Delta(N)}, \qquad Y_{12} = -\frac{T_{12}(N,x)}{\Delta(N)},$$

$$Y_{21} := -\frac{T_{21}(N,x)}{\Delta(N-2)}, \qquad Y_{22} := \frac{T_{11}(N-2,x)}{\Delta(N-2)},$$

with determinant

$$\Delta(N) := T_{11}(N,x)T_{22}(N+2,x) - T_{12}(N,x)T_{21}(N+2,x)F(N+2),$$

one quickly observes that the formulae are the standard ones required to invert a two-by-two matrix, but now they include the necessary shifts. For the T_I-matrix we find the determinant is: $\Delta := Q^{-2}x^{-2} - \beta^2 f$, and this is simplified slightly by the following identification $\beta^2 f = f_0$ which will be used from now - one could view this as a particular rescaling of the operators a and $a^\dagger$. With this information we can calculate the $\tilde{T}$-matrix:

$$\tilde{T}_I(\theta) = \frac{\tilde{\rho_I}(\theta)}{\tilde{\Delta}} \begin{pmatrix} \frac{x}{\alpha}Q^{-N-2} & -\beta a \\ -\beta a^\dagger & \alpha x Q^{N-2} \end{pmatrix},$$

where $\tilde{\Delta}(x) = \Delta(x^{-1})$ and we have dropped the subscript on x for simplicity. Let us now take a diagonal reflection matrix, corresponding to a simple Dirichlet boundary condition, of the form [13]:

$$R_d = \begin{pmatrix} (\frac{1}{rx} + rx) & 0 \\ 0 & (\frac{r}{x} + \frac{x}{r}) \end{pmatrix},$$

where x is still the same rapidity dependent parameter, and r is a free parameter. We can now calculate a type - I generalised solution by evaluating the multiplication (1.95) to obtain:

$$\tilde{R} = \frac{1}{\Delta} \begin{pmatrix} kx + lx^{-1} & \frac{\beta\alpha}{r}Q^N(x^2 - x^{-2})a \\ \frac{r\beta}{\alpha}Q^{-N}(x^2 - x^{-2})a^\dagger & lx + kx^{-1} \end{pmatrix}, \tag{2.3}$$

with

$$k = \left(\frac{r}{Q^2} - \frac{\beta^2 f}{r}\right), \qquad l = \left(\frac{1}{rQ^2} - \beta^2 fr\right).$$

The generalised solution (2.3) demonstrates the interplay between defects and boundary conditions very neatly. The result of dressing a simple diagonal reflection matrix (corresponding to a Dirichlet condition in this case) with a type - I defect is an infinite-dimensional generalisation of a Zamolodchikov-Ghoshal type reflection matrix containing dependence on topological charge and defect parameters. The solution can be simplified by setting $\alpha = r$, without losing generality, then we see that the defect has added the parameter $f_0 := \beta^2 f$. Returning to our original focus, we wish to see how this more general Zamolodchikov-Ghoshal (ZG) type solution expands upon the results of Delius and MacKay. We know that the finite-dimensional ZG solution, expressed in the language of [18]:

$$R(\theta) = \rho_R(\theta) \begin{pmatrix} \frac{q-q^{-1}}{c}(\epsilon_+ x + \epsilon_- x^{-1}) & (x^2 - x^{-2}) \\ (x^2 - x^{-2}) & \frac{q-q^{-1}}{c}(\epsilon_- x + \epsilon_+ x^{-1}) \end{pmatrix},$$

was found to possess the following representation of the boundary subalgebra, $\mathcal{B}$, with generators Q_+ and Q_- [18]:

$$\pi_\theta(Q_\pm) = \begin{pmatrix} \epsilon_\pm q^{\pm 1} & c x^{\pm 1} \\ c x^{\mp 1} & \epsilon_\pm q^{\mp 1} \end{pmatrix}, \tag{2.4}$$

where $\epsilon_\pm$ are parameters associated with the boundary condition and c is a constant, defined in [18]. The representation is easily broken down into a sum of generators of the original sine-Gordon algebra,

$$\pi_\theta(Q_\pm) = c x^{\pm 1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c x^{\mp 1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \epsilon_\pm \begin{pmatrix} q^{\pm 1} & 0 \\ 0 & q^{\mp 1} \end{pmatrix}.$$

It will be useful to bear this expression in mind as we try to calculate the analogue of $\pi_\theta(Q_\pm)$ for the type - I generalised solution, we will denote it by $\tilde{\pi}_\theta(\tilde{Q}_\pm)$. To begin, we will use the following ansatz:

$$\Pi_\theta(\tilde{Q}_\pm) = \begin{pmatrix} A(N,x) & B(N,x)a \\ C(N,x)a^\dagger & D(N,x) \end{pmatrix}, \tag{2.5}$$

where $\Pi_\theta(\tilde{Q}_\pm) := T(-\theta)\tilde{\pi}_\theta(\tilde{Q}_\pm)T^{-1}(-\theta)$. Substituting this in equation (2.1) and evaluating the multiplication returns the following four equations:

$$\begin{aligned} &(kx + lx^{-1})A(N,x) + \frac{\beta\alpha}{r}Q^N(x^2 - x^{-2})fC(N+2,x) \\ &\quad = A(N,x^{-1})(kx + lx^{-1}) + B(N,x^{-1})\frac{r\beta}{\alpha}Q^{-N-2}(x^2 - x^{-2})f, \end{aligned} \tag{2.6}$$

$$\begin{aligned}(kx+lx^{-1})B(N,x)a&+\frac{\beta\alpha}{r}Q^N(x^2-x^{-2})D(N+2,x)a\\&=A(N,x^{-1})\frac{\beta\alpha}{r}Q^N(x^2-x^{-2})a+B(N,x^{-1})(lx+kx^{-1}),\end{aligned}\tag{2.7}$$

$$\begin{aligned}\frac{r\beta}{\alpha}Q^{-N}(x^2-x^{-2})&A(N-2,x)a^\dagger+(lx+kx^{-1})C(N,x)a^\dagger\\&=C(N,x^{-1})(kx+lx^{-1})a^\dagger+D(N,x^{-1})\frac{r\beta}{\alpha}Q^{-N}(x^2-x^{-2})a^\dagger,\end{aligned}\tag{2.8}$$

$$\begin{aligned}\frac{r\beta}{\alpha}Q^{-N}(x^2-x^{-2})&B(N-2,x)f+(lx+kx^{-1})D(N,x)\\&=C(N,x^{-1})\frac{\beta\alpha}{r}Q^{N-2}(x^2-x^{-2})f+D(N,x^{-1})(lx+kx^{-1}).\end{aligned}\tag{2.9}$$

In order to balance the topological charge dependence in equations (2.6) - (2.9), we will introduce it into the off-diagonal entries as follows:

$$B(N,x)=Q^N\hat{B}(x),\quad \text{and}\quad C(N,x)=Q^{-N}\hat{C}(x),$$

ensuring that: equations (2.6), (2.9) are independent of the charge, and $Q^{\pm N}$ appears as an overall in equations (2.7) and (2.8) respectively. With the equations appearing more balanced, we now want to determine relations among the entries. One such relation can be obtained from equations (2.6) and (2.9), by collecting the functions $A(N,x^{\pm 1})$ on one side, and then invert x to find:

$$\begin{aligned}(lx+kx^{-1})&(A(N,x^{-1})-A(N,x))\\&=\frac{\beta\alpha}{r}Q^{-2}f(x^2-x^{-2})\hat{C}(x^{-1})-\frac{r\beta}{\alpha}Q^{-2}f(x^2-x^{-2})\hat{B}(x).\end{aligned}$$

If we perform the same manipulations on equation (2.9), but without inverting x, we find that we can equate it with the above, resulting in the relation:

$$D(N,x)-D(N,x^{-1})=A(N,x^{-1})-A(N,x).$$

Continuing in this way, we obtain further identities concerning the diagonal entries. For instance, similar manoeuvres within the second equation (2.7) return:

$$D(N+2,x)-D(N+2,x^{-1})=A(N,x^{-1})-A(N,x),$$

and when taken with the first relation we see that D must be independent of N. Repeating this kind of procedure enables us to find analogous relations regarding the function A. Ultimately, they again show that A must also be independent of N. We must now exercise caution when turning our attention to the off-diagonal entries, specifically the functions $\hat{B}(x), \hat{C}(x)$, because we must remember that our matrix representation $\Pi_\theta(\tilde{Q}_\pm)$ is a linear combination of the generators, therefore, one needs to be able to distinguish the generators at all times. It is instructive to recall Delius and MacKay's results: the off-diagonal entries in their representations are simply proportional to the rapidity, $x^{\pm 1}$. Bearing this in mind, as well as the fact that we are dealing with a linear combination of generators, we will use ansatz of the form:

$$\hat{B}(x) = b_+ x + b_- x^{-1}, \qquad \hat{C}(x) = c_+ x + c_- x^{-1}.$$

Substituting the above ansatz throughout equations (2.6) - (2.9) provides us with more insight into the properties of the system. The most helpful relations come from equations (2.7) and (2.8), which read, respectively:

$$\frac{r}{\alpha\beta}\left(kb_+ - lb_-\right) = A(x^{-1}) - D(x), \text{ and} \tag{2.10}$$

$$\frac{\alpha}{r\beta}\left(kc_- - lc_+\right) = A(x) - D(x^{-1}). \tag{2.11}$$

The left hand sides of both (2.10) and (2.11) are independent of x, hence, we can invert x in the first equation and equate it with the latter, to find that:

$$c_- = \frac{r^2}{\alpha^2} b_+, \qquad c_+ = \frac{r^2}{\alpha^2} b_-.$$

At this point, one could set $\alpha = r$, without losing generality to neaten the expressions. Using the above identities in the remaining two equations that read:

$$\begin{aligned}(kx + lx^{-1})(A(x) - A(x^{-1})) =& \frac{\beta r}{\alpha} Q^{-2} f(x^2 - x^{-2})(b_+ x^{-1} + b_- x) \\ &- \frac{\beta\alpha}{r} Q^{-2} f(x^2 - x^{-2})(c_+ x + c_- x^{-1}),\end{aligned} \tag{2.12}$$

$$\begin{aligned}(lx + kx^{-1})(D(x) - D(x^{-1})) =& \frac{\beta\alpha}{r} Q^{-2} f(x^2 - x^{-2})(c_+ x^{-1} + c_- x) \\ &- \frac{r\beta}{\alpha} Q^{-2} f(x^2 - x^{-2})(b_+ x + b_- x^{-1}),\end{aligned} \tag{2.13}$$

enables us to see that they reduce to $A(x) - A(x^{-1}) = 0$ and $D(x) - D(x^{-1}) = 0$, meaning that both A and D could be any symmetric Laurent polynomial satisfying equation (2.10), which is equal to equation (2.11) under the relations between $b_\pm$ and $c_\pm$. Combining all of this information, we see that the object $\Pi_\theta(\tilde{Q}_\pm)$ takes the form:

$$\Pi_\theta(\tilde{Q}_\pm) = \begin{pmatrix} A & Q^N(b_+x + b_-x^{-1})a \\ \frac{r^2}{\alpha^2}Q^{-N}(b_-x + b_+x^{-1})a^\dagger & D \end{pmatrix}, \tag{2.14}$$

together with the difference relation:

$$A - D = \frac{r}{\alpha\beta}(kb_+ - lb_-). \tag{2.15}$$

In pursuance of our original goal, to calculate $\tilde{\pi}_\theta(\tilde{Q}_\pm)$, we must now 'unpick' $\Pi_\theta(\tilde{Q}_\pm)$. By unpick, we mean perform the following multiplication, obtained via the simple rearrangement of the definition of Π:

$$\tilde{\pi}_\theta(\tilde{Q}_\pm) = T^{-1}(-\theta)\Pi_\theta(\tilde{Q}_\pm)T(-\theta),$$

where T still refers to the type - I defect transmission matrix. When evaluating the above product it is very important to keep track of the operators and include any shifts that they induce. Despite the product's nature, one finds that it simplifies neatly to give the following matrix:

$$\tilde{\pi}_\theta(\tilde{Q}_\pm) = \begin{pmatrix} A + \frac{\beta f}{\alpha}(b_+ - r^2 b_-) & \left(\frac{x}{r} + \frac{r}{x}\right)\frac{r}{\alpha^2}b_+Q^{-N-2}a \\ \left(\frac{1}{rx} + rx\right)rb_-Q^{N-2}a^\dagger & D + \frac{\beta f}{\alpha}(r^2 b_- - b_+) \end{pmatrix}, \tag{2.16}$$

where A and D obey the difference relation. It is illuminating to exploit the difference property (2.15) and make the above matrix traceless. This is achieved by extracting a multiple of the identity, namely $(A + D).\mathbb{1}/2$. The result is the matrix:

$$\tilde{\pi}_\theta(\tilde{Q}_\pm) = \begin{pmatrix} \tilde{\pi}_{11} & \left(\frac{x}{r} + \frac{r}{x}\right)\frac{r}{\alpha^2}b_+Q^{-N-2}a \\ \left(\frac{1}{rx} + rx\right)rb_-Q^{N-2}a^\dagger & -\tilde{\pi}_{11} \end{pmatrix}, \tag{2.17}$$

where

$$\tilde{\pi}_{11} := \frac{1}{2}\left(\frac{r^2}{\alpha\beta}Q^{-2}b_+ + \frac{\beta f}{\alpha}b_+ - \frac{Q^{-2}}{\alpha\beta}b_- - r^2\frac{\beta f}{\alpha}b_-\right).$$

The purpose of the coefficients, $b_\pm$, is to help us identify the separate generators within the linear combination, which we now can by expressing the above as a sum of two pieces with one proportional to b_+ and the other b_-. We will need to use this traceless matrix in the next section, when we introduce the type - II defect. The traceless matrix (2.17) is the original algebraic representation that corresponds to the diagonal reflection matrix, R_d. By substituting the diagonal solution into equation (2.1) we can obtain a very general form of the representation:

$$\pi_\theta(Q_\pm) = \begin{pmatrix} F & \tilde{B}(\frac{x}{r} + \frac{r}{x})a \\ \tilde{C}(rx + \frac{1}{xr})a^\dagger & G \end{pmatrix},$$

where $F, G, \tilde{B}$ and $\tilde{C}$ are all symmetric Laurent polynomials. The matrix representation of the generators that we have calculated, (2.17), does indeed fit into this picture. However, the freedom introduced by the many symmetric Laurent polynomials is lost. Nevertheless, we have shown how the defect is totally compatible within this framework. Particularly, the manner in which it transforms the simple initial representation (of the diagonal reflection factor) into a representation of another more general reflection matrix. This truly evidences the strong interplay between boundaries and defects. To add further significance to this body of evidence we will now use the result, (2.17), and apply the type - II defect transmission matrix. The calculations become more complex owing to the nature of this transmission matrix and the way its operators combine. The ensuing formulae do reduce to the type - I case, when the type - I limit of the type - II matrix is invoked.

2.2 Generalising the Framework using a Type - II Defect

Equipped with the workings of the previous section, we can now move on to consider the type - II sine-Gordon defect, first introduced in [46] and later [17], and its impact on the representation of the algebraic generators. The type - I case was particularly pleasing, as we know that a type - I generalised solution corresponds to a generalised Zamolodchikov-Ghoshal like reflection matrix and in turn, this has been studied extensively over time along with other integrable boundary conditions. The type - II defect gives an even more general solution, seemingly removed from the class of known integrable boundary conditions. Let us recall, from the introductory chapter, the proposed candidate for the boundary density (found in [17]) associated to the type - II generalised solution. The suggested boundary density is [17]:

$$B(u, \lambda) = e^{\lambda/2} f(u) + e^{-\lambda/2} g(u), \tag{2.18}$$

where the functions $f(u)$ and $g(u)$ satisfy:

$$f(u)g(u) = h_+e^{u/2} + h_-e^{-u/2} + 2\left(e^u + e^{-u}\right) + h_0,$$

and the functions can be arranged as:

$$f(u) = f_0 + \sqrt{2}\left(be^{u/2} + b^{-1}e^{-u/2}\right), \quad g(u) = g_0 + \sqrt{2}\left(b^{-1}e^{u/2} + be^{-u/2}\right).$$

This boundary density would lie outside the existing well-known results, due to the presence of the extra degree of freedom, λ, and free constant parameters:

$$h_0 = g_0 f_0, \quad h_+ = \sqrt{2}\left(f_0 b^{-1} + g_0 b\right), \quad h_- = \sqrt{2}\left(f_0 b + g_0 b^{-1}\right).$$

We will now examine the type - II generalised solution within the algebraic framework, hoping to identify similar parameters that could correspond to a boundary condition of this type.

To proceed let us detail the method that we will employ. Once the type - II transmission matrix and the corresponding $\tilde{T}$-matrix are introduced, it is possible to calculate the associated generalised solution. We can then take the matrix (2.17), simply apply the type - II defect matrices and generalised solution to equation (2.2). We do not need to perform the same working as in the previous section because we have already calculated the representation of the generators for the diagonal reflection matrix; via the unpicking procedure. This is sufficient to acquire the representation of the subalgebra generators in this case. With the help of Maple we can verify that the new object does indeed satisfy equation (2.2). The determinant of the type - II transmission matrix plays a key role in achieving this equality, as we will see.

The type - II defect transmission matrix is presented differently in (2.19) to that of [17, 46], as all Kronecker-deltas are replaced by the raising and lowering operators. It does satisfy the transmission Yang-Baxter relation (1.62), with the S-matrix introduced in the previous section and takes the form:

$$T_{II}(\theta) = \rho_{II}(\theta)\begin{pmatrix} a_+xQ^{-N} + a_-x^{-1}Q^N & a \\ a^\dagger & d_+xQ^N + d_-x^{-1}Q^{-N} \end{pmatrix}, \qquad (2.19)$$

with free parameters $a_\pm, d_\pm$ and $F(N) = f_0 + a_-d_+Q^{2N-2} + a_+d_-Q^{-2N+2}$. It is instantly obvious that this transmission matrix is more general, due to the extra free parameters and the form that $F(N)$ takes. We will now use the inversion formulae to invert the matrix, as well as reversing the rapidity to form the $\tilde{T}$-matrix:

$$\tilde{T}_{II}(\theta) = \frac{\tilde{\rho}_{II}(\theta)}{\tilde{\Delta}(\theta)} \begin{pmatrix} d_+ x^{-1} Q^{N+2} + d_- x Q^{-N-2} & -a \\ -a^\dagger & a_+ x^{-1} Q^{-N+2} + a_- x Q^{N-2} \end{pmatrix},$$

where the determinant is now,

$$\tilde{\Delta}(x) = \Delta(x^{-1}) = \frac{a_+ d_+ Q^2}{x^2} + \frac{a_- d_- x^2}{Q^2} - f_0.$$

It is useful to describe the limit of the type - II T-matrix to the type - I T-matrix, owing to the presence of all type - II objects. Simply prescribe these particular values to the type - II parameters:

$$a_+ = 0, \quad d_+ = 0, \quad a_- = \alpha, \quad d_- = \alpha^{-1}, \text{ and } f_0 = f, \tag{2.20}$$

where we have also simplified the type - I matrix by taking $\beta = 1$.

Armed with the type - II matrices, we will construct the associated generalised solution using the same diagonal reflection factor, R_d, returning a solution of the form:

$$\tilde{R}_{II} = \frac{1}{\Delta(x^{-1})} \begin{pmatrix} U(N,x) & L(N)(x^2 - x^{-2})a \\ M(N)(x^2 - x^{-2})a^\dagger & V(N,x) \end{pmatrix}, \tag{2.21}$$

with entries and coefficients:

$$\begin{aligned} U(N,x) \;&=\; \frac{r}{x} a_+ d_- Q^{-2N-2}(x^4 - 1) - f_0 \left(\frac{x}{r} - \frac{r}{x}\right) \\ &\quad + \frac{x}{r} a_- d_+ Q^{2N+2}(x^{-4} - 1) + (a_+ d_+ Q^2 + a_- d_- Q^{-2}) \left(rx + \frac{1}{xr}\right), \\ L(N) &= \left(\frac{a_-}{r} Q^N - r a_+ Q^{-N}\right), \\ M(N) &= \left(r d_- Q^{-N} - \frac{d_+}{r} Q^N\right), \\ V(N,x) \;&=\; \frac{a_- d_+}{rx} Q^{2N-2}(x^4 - 1) - f_0 \left(rx + \frac{1}{rx}\right) \\ &\quad + rx a_+ d_- Q^{-2N+2}(x^{-4} - 1) + (a_+ d_+ Q^2 + a_- d_- Q^{-2}) \left(\frac{x}{r} + \frac{r}{x}\right). \end{aligned} \tag{2.22}$$

We have yet more confirmation that a type - II object is manifestly more general, affirmed by the above generalised solution. Obviously, the algebraic representation associated with this reflection factor must be very comprehensive. Given our knowledge of the original algebraic representation (2.17) (corresponding to the original reflection factor, that accounts for topological charge) we can substitute the type - II transmission matrices into the following

$$\tilde{\Pi}_{\theta_{II}}(\tilde{Q}_{\pm}) := T_{II}(-\theta)\tilde{\pi}_{\theta}(\tilde{Q}_{\pm})T_{II}^{-1}(-\theta).$$

Yet again, this representation is a linear combination of the generators of the boundary subalgebra for the generalised $\tilde{R}_{II}$-matrix (2.21). Moreover, it is clear that it is not necessary to perform the same multiplication for the similar object appearing on the second line of equation (2.2) because the calculation is redundant. Simply expanding the above product and inverting the rapidity returns the required object. As a means of simplifying the calculation and its result, we have elected to split the traceless matrix (2.17) into the two separate generators:

$$\begin{aligned} \tilde{\pi}_{\theta}(\tilde{Q}_{+}) &= b_{+}\begin{pmatrix} \tilde{\pi}_{11_+} & \left(\frac{x}{r}+\frac{r}{x}\right)\frac{r}{\alpha^2}Q^{-N-2}a \\ 0 & -\tilde{\pi}_{11_+} \end{pmatrix}, \\ \tilde{\pi}_{\theta}(\tilde{Q}_{-}) &= b_{-}\begin{pmatrix} \tilde{\pi}_{11_-} & 0 \\ \left(\frac{1}{rx}+rx\right)rQ^{N-2}a^{\dagger} & -\tilde{\pi}_{11_-} \end{pmatrix}, \end{aligned} \tag{2.23}$$

with:

$$\tilde{\pi}_{II_+} = \frac{1}{2\alpha}\left(\frac{r^2}{\beta}Q^{-2}+\beta f\right), \qquad \tilde{\pi}_{II_-} = -\frac{1}{2\alpha}\left(\frac{Q^{-2}}{\beta}+r^2\beta f\right).$$

Once the matrices are multiplied many parameters will be contained within each matrix entry. With a view to simplify matters now, we will set $\beta = 1$ and $\alpha = r$, as the type - I parameters are not particularly important. It is also important to recognise that we can reformulate $\tilde{\pi}_{II_\pm}$ as follows, bearing in mind the specialisation of type - I parameters:

$$\begin{aligned} \tilde{\pi}_{II_+} &= \frac{1}{2}\left(rQ^{-2}+\frac{f}{r}\right) = \frac{1}{2}\left(k+\frac{2f}{r}\right), \\ \tilde{\pi}_{II_-} &= -\frac{1}{2}\left(\frac{Q^{-2}}{r}+fr\right) = \frac{1}{2}\left(l+2fr\right). \end{aligned} \tag{2.24}$$

The above manipulation is crucial for several reasons: we would like to identify new parameters or groups of parameters that the type - II matrix introduces. It is hoped that they are related to the extra degree of freedom appearing in the proposed classical boundary density (1.118); also, the parameters k and l appear linked to the Zamolodchikov-Ghoshal boundary condition. Identifying them within a type - II setting might possibly discern the way in which they are modified by a type - II matrix, therefore generalising the original condition. Moreover, when the representations are expressed in this way, one can quickly verify that only a diagonal matrix proportional to the original solution, R_d, commutes with them. With these slight modifications, let us now apply the type - II T-matrices and state the results. First, let us consider the generator $\tilde{Q}_+$. By evaluating the required product we obtain:

$$\tilde{\Pi}_{\theta_{II}}(\tilde{Q}_+) = \frac{b_+}{\Delta_{II}(x^{-1})} \begin{pmatrix} \tilde{\Pi}^+_{11} & \tilde{\Pi}^+_{12} a \\ \tilde{\Pi}^+_{21} a^\dagger & \tilde{\Pi}^+_{22}, \end{pmatrix} \tag{2.25}$$

and entries of the form:

$$\begin{aligned} \tilde{\Pi}^+_{11} &= \frac{1}{2}\left(k + \frac{2f}{r}\right)\left(\Delta_{II}(x^{-1}) + 2F(N+2)\right) \\ &\quad - \frac{Q^{-N-2}}{r}\left(\frac{x}{r} + \frac{r}{x}\right)\left(\frac{a_+}{x}Q^{-N} + xa_- Q^N\right) F(N+2), \\ \tilde{\Pi}^+_{12} &= \left(\frac{a_+}{x}Q^{-N} + xa_- Q^N\right)^2 \left(\frac{x}{r} + \frac{r}{x}\right)\frac{Q^{-N-2}}{r} a \\ &\quad - \left(k + \frac{2f}{r}\right)\left(\frac{a_+}{x}Q^{-N} + xa_- Q^N\right) a, \\ \tilde{\Pi}^+_{21} &= \left(k + \frac{2f}{r}\right)\left(\frac{d_+}{x}Q^{N} + xd_- Q^{-N}\right) a^\dagger \\ &\quad - \frac{Q^{-N}}{r}\left(\frac{x}{r} + \frac{r}{x}\right) F(N) a^\dagger, \\ \tilde{\Pi}^+_{22} &= -\frac{1}{2}\left(k + \frac{2f}{r}\right)\left(\Delta_{II}(x^{-1}) + 2F(N)\right) \\ &\quad + \frac{Q^{-N}}{r}\left(\frac{x}{r} + \frac{r}{x}\right)\left(\frac{a_+}{x}Q^{-N+2} + xa_- Q^{N-2}\right) F(N), \end{aligned}$$

where $\Delta_{II}, F(N)$ are those defined earlier for the type - II T-matrix. The results for the second generator $\tilde{Q}_-$ are similar:

$$\tilde{\Pi}_{\theta_{II}}(\tilde{Q}_-) = \frac{b_-}{\Delta_{II}(x^{-1})} \begin{pmatrix} \tilde{\Pi}^-_{11} & \tilde{\Pi}^-_{12} a \\ \tilde{\Pi}^-_{21} a^\dagger & \tilde{\Pi}^-_{22} \end{pmatrix}, \tag{2.26}$$

where the entries are now:

$$\begin{aligned}
\tilde{\Pi}^-_{11} &= -\frac{1}{2}\,(l+2fr)\left(\Delta_{II}(x^{-1})+2F(N+2)\right) \\
&\qquad + rQ^N\left(\frac{1}{rx}+rx\right)\left(\frac{d_+}{x}Q^{N+2}+xd_-Q^{-N-2}\right)F(N+2), \\
\tilde{\Pi}^-_{12} &= (l+2fr)\left(\frac{a_+}{x}Q^{-N}+xa_-Q^N\right)a \\
&\qquad\qquad - rQ^N\left(\frac{1}{rx}+rx\right)F(N+2)a, \\
\tilde{\Pi}^-_{21} &= -\,(l+2fr)\left(\frac{d_+}{x}Q^N+d_-xQ^{-N}\right)a^\dagger \\
&\qquad + \left(\frac{d_+}{x}Q^N+d_-xQ^{-N}\right)^2\left(\frac{1}{rx}+rx\right)rQ^{N-2}a^\dagger, \\
\tilde{\Pi}^-_{22} &= \frac{1}{2}\,(l+2fr)\left(\Delta_{II}(x^{-1})+2F(N)\right) \\
&\qquad - rQ^{N-2}\left(\frac{1}{rx}+rx\right)\left(\frac{d_+}{x}Q^N+xd_-Q^{-N}\right)F(N).
\end{aligned} \tag{2.27}$$

Unfortunately, the generators $\Pi_{\theta_{II}}(\tilde{Q}_\pm)$ do not factorise neatly, as in the type - I case where all factors of the determinant cancel. However, when expressed in this compact form, there are striking similarities between the representations of the generators. Another striking feature is that the trace is not preserved, this is due to the operators, a and $a^\dagger$, shifting the entries in a non-trivial way so that the trace cannot remain zero. Evaluation of the type - II to type - I limit, (2.20), returns the expected result of $\Pi_\theta(\tilde{Q}_\pm)$.

To illustrate the role that the determinant plays, in the type - II story, we will state how one can verify that the representations satisfy equation (2.2). This exercise was completed with the help of Maple. Begin with the type - II solution (2.21) and apply $\Pi_{\theta_{II}}(\tilde{Q}_\pm)$ on the right. Now, to form the remaining half of the equation, apply $\Pi_{-\theta_{II}}(\tilde{Q}_\pm)$ to the left of the same generalised solution. To simplify matters simply choose either $\tilde{Q}_+$ or $\tilde{Q}_-$, we will now specialise to the $\tilde{Q}_+$ case. Overall, the four equations read:

$$\begin{aligned}
&U(N,x)\tilde{\Pi}^+_{11}(N,x)+L(N)(x^2-x^{-2})\tilde{\Pi}^+_{21}(N+2,x)F(N+2) \\
&\quad = \tilde{\Pi}^+_{11}(N,x^{-1})U(N,x)+\tilde{\Pi}^+_{12}(N,x^{-1})M(N+2)(x^2-x^{-2})F(N+2),
\end{aligned}$$

$$
\begin{aligned}
&U(N,x)\tilde{\Pi}^+_{12}(N,x)a + L(N)(x^2-x^{-2})\tilde{\Pi}^+_{22}(N+2,x)a \\
&\quad = \tilde{\Pi}^+_{11}(N,x^{-1})L(N)(x^2-x^{-2})a + \tilde{\Pi}^+_{12}(N,x^{-1})V(N+2,x)a,
\end{aligned}
$$

$$
\begin{aligned}
&M(N)(x^2-x^{-2})\tilde{\Pi}^+_{11}(N-2,x)a^\dagger + V(N,x)\tilde{\Pi}^+_{21}(N,x)a^\dagger \\
&\quad = \tilde{\Pi}^+_{21}(N,x^{-1})U(N-2,x)a^\dagger + \tilde{\Pi}^+_{22}(N,x^{-1})M(N)(x^2-x^{-2})a^\dagger,
\end{aligned}
$$

$$
\begin{aligned}
&M(N)(x^2-x^{-2})\tilde{\Pi}^+_{12}(N-2,x)F(N) + V(N,x)\tilde{\Pi}^+_{22}(N,x) \\
&\quad = \tilde{\Pi}^+_{21}(N,x^{-1})L(N-2)(x^2-x^{-2})F(N) + \tilde{\Pi}^+_{22}(N,x^{-1})V(N,x).
\end{aligned}
$$

Any factors multiplying the generalised solutions will cancel throughout the four equations, however the same is not true for the representations $\tilde{\Pi}_{\theta_{II}}$, due to the inversion of the rapidity parameter, x. The representations are proportional to the type - II determinant, that depends on x and therefore cannot be cancelled. Effectively, to eliminate all denominators, the first/second line in each equation above is multiplied by $\Delta_{II}(x)/\Delta_{II}(x^{-1})$, respectively. This ensures that the equations are satisfied. In this case, the determinant is central to guarantee that the theory works, which is interesting, because it is not required in the type - I case where it appears as an overall factor. We know that T-matrices must be invertible. Therefore, they must have a non-zero determinant, it is one of their defining properties, but the determinant appears to have a wider significance in this algebraic framework.

2.3 Concluding Remarks

Throughout this chapter, the connection between boundaries and defects has been explored from an algebraic viewpoint. Existing finite-dimensional results, [18], have been generalised by including a defect. The defect's ability to store topological charge transforms the finite-dimensional to the infinite-dimensional. This is reflected in the representations of the boundary subalgebra by the dependence on the charge, N, and the presence of the operators a and $a^\dagger$. Interestingly, the known behaviour exhibited by the type - I defect, whereby a diagonal reflection matrix (corresponding to a Dirichlet boundary condition) is transformed to a generalised Zamolodchikov-Ghoshal (ZG) type solution, was explored within this framework. It was evident that the original freedom enjoyed by the diagonal matrix's representation was changed dramatically, to a more restricted object (1.14). However, despite the restrictions, it did indeed appear to match and develop the results of Delius and MacKay when they considered the (ZG) solution [18]. And so, because the type - I case works in this manner, we believe that its associated parameters, α, β and f are not significant in our quest to generalise the current framework. Therefore, we look to the type - II scenario where there are more parameters and it is thought to correspond to a more general boundary density in the classical Lagrangian description. To remind us, the type - II defect's parameters are $a_{\pm}, d_{\pm}$ and f_0. In the limit (2.20) we have observed that a_- and d_- are both related to the type - I parameter $\alpha^{\pm 1} = r^{\pm 1}$ and that f_0 is related to the type - I f. Also, f could be rescaled to a simple constant, even unity, by tweaking the operators a and $a^\dagger$. Consequently, we will not consider this a valuable parameter. However, parameters a_+, d_+ must be 'switched off' in the limit and set to zero. As a result of this behaviour, one can think of those two parameters as an addition, truly adding more generality to the system consisting of the parameter r. We believe that the three parameters r, a_+ and d_+ are the algebraic analogues of the constants: h_0, h_-, h_+, appearing in the proposed classical boundary density.

Despite constructing the representations of the modified boundary subalgebra and verifying that they satisfy the necessary properties, it is not clear how they are the fundamental objects that one would naturally look to first. Particularly, in the type - II case where the representation's entries are complicated and ungainly. Perhaps, the answer lies within the coideal framework. Unfortunately, we are unsure how these representations fit into that part of the story and what form the coproduct might take. This would be interesting to address in the future.

Nonetheless, defects can be introduced into this algebraic approach and they do generalise known results, as we expect. The results of this chapter show the versatility of the defect and help us to form a coherent picture of the interplay between defects and boundaries. As we have examined the generalised solutions of the sine-Gordon model within the algebraic setting, we will now move on to consider other models and their generalised solutions, investigating their features and ability to generalise finite-dimensional results.

Chapter 3

Generalised Reflection Matrices of the $a_2^{(2)}$ affine Toda model

As the previous chapter showed the way that generalised reflection matrices fit into an algebraic framework, we will now construct several generalised reflection factors for the $a_2^{(2)}$ affine Toda model (also known as the Tzitzéica, Bullough-Dodd or Mikhailov-Zhiber-Shabat model). Following this, we will detail their relation to the known finite-dimensional reflection matrices. Firstly, we will document the known finite-dimensional reflection factors calculated some years ago by Nepomechie and Mezincescu, Kim and Lima-Santos [27, 28, 29], as well as the type - II transmission matrix calculated by Corrigan and Zambon [45]. Subsequently, we will follow the construction of the generalised reflection matrices, detailed in chapter one: by evaluating the product $T(\theta)R_d(\theta)\tilde{T}(\theta)$. We will use all finite-dimensional diagonal reflection matrices, calculated by Nepomechie and Mezincescu [27], to form three generalised reflection matrices. The generalised solutions of this model are particularly well-ordered, as their structure naturally incorporates the finite-dimensional solutions. In this case, one can observe readily the way in which placing a defect near an integrable boundary develops the existing finite-dimensional theory.

3.1 Known Results

We will now chronologically detail the work within the literature concerning $a_2^{(2)}$ that has enabled the completion of this work. The literature heavily relies upon the R-matrix, calculated by Izergin and Korepin [65] in 1981, that is invariant under the action of the $U_q(a_2^{(2)})$ algebra. The R-matrix intertwines between two representations of the algebra, namely:

$$R(x_1/x_2, q) : V_{x_1} \otimes V_{x_2} \to V_{x_2} \otimes V_{x_1},$$

and is defined as:

$$R = (x^{-1} - 1)q^3 R_{12} + (1 - x)q^{-3} R_{21}^{-1} + q^{-5}(q^4 - 1)(q^6 + 1)P,$$

where $R_{21} = PR_{12}P$, with R_{12} the constant solution of the Yang-Baxter equations for the $U_q(sl_2)$ spin 1 representation and P is the permutation operator. Several years later, Smirnov built on this by calculating an appropriate S-matrix, defined as [66]:

$$S(\Theta) = \rho_S(\Theta) PR(x, q), \quad x = \frac{x_1}{x_2}, \quad x_i = q^{2\pi\theta_i/\xi}, \quad \xi = \frac{2}{3}\left(\frac{\pi\beta^2}{8\pi - \beta^2}\right), \tag{3.1}$$

in the above, R takes the form:

$$R = \left(\begin{array}{ccc|ccc|ccc}
c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b & 0 & \tilde{e} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d & 0 & \tilde{g} & 0 & \tilde{f} & 0 & 0 \\
\hline
0 & e & 0 & b & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & g & 0 & a & 0 & \tilde{g} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b & 0 & \tilde{e} & 0 \\
\hline
0 & 0 & f & 0 & g & 0 & d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e & 0 & b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c
\end{array}\right),$$

$$\begin{aligned}
a &= q^{-3}(1 - x) + q^3(x^{-1} - 1) - (q - q^{-1}) + (q^5 - q^{-5}), \\
b &= q^{-3}(1 - x) + q^3(x^{-1} - 1), \\
c &= q^{-5}(1 - x) + q^5(x^{-1} - 1) - (q - q^{-1}) + (q^5 - q^{-5}),
\end{aligned}$$

$$
\begin{aligned}
d &= q^{-1}(1-x) + q(x^{-1}-1),\\
e &= q^{-1}(x-1)(1-q^{-4}) - (q-q^{-1}) + (q^5-q^{-5}),\\
\tilde{e} &= q(x^{-1}-1)(q^4-1) - (q-q^{-1}) + (q^5-q^{-5}),\\
f &= (x-1)(1-q^{-4})(q^{-1}-q) - (q-q^{-1}) + (q^5-q^{-5}),\\
\tilde{f} &= (x^{-1}-1)(q^4-1)(q-q^{-1}) - (q-q^{-1}) + (q^5-q^{-5}).
\end{aligned}
$$

The advent of the S-matrix paved the way for others to calculate new and interesting objects within integrability, such as reflection and transmission matrices. At this point, it is worth noting how the index notation is used in this setting. The $a_2^{(2)}$-model exhibits a peculiar phenomenon: the quantum setting contains a fundamental particle that is represented by a three-component soliton, with charges $(-1, 0, +1)$, in contrast to the classical case where only two components appear. The indices within the reflection equation must account for all possible particle processes and we will label each charge $(-, 0, +)$ correspondingly. In total, we see that there are nine possible in-going and nine possible out-going configurations that return eighty one equations upon the evaluation of indices. All eighty one equations are listed in Kim's paper [28], and coincide with our equations following from (1.8) by expansion of indices - in the finite-dimensional case. As we are aware, when considering generalised reflection matrices it is important to include the charge dependence within the reflection matrix, this alters the equations in a very non-trivial fashion.

The first known solution to the reflection equation (1.8), for the Tzitzéica model, was calculated by Nepomechie and Mezincescu in 1991 [27]. Their paper has a different focus, that of integrable spin chains, therefore they consider the simplified case of diagonal solutions. They discovered three diagonal solutions: one of which is the identity, the remaining two are non-trivial diagonal matrices - that only differ by choice of sign - of the form [27]:

$$
R_a^b(x,q) = \begin{pmatrix} (q^3x^{-1} \pm i) & 0 & 0 \\ 0 & (q^3x \pm i) & 0 \\ 0 & 0 & x(q^3 \pm ix) \end{pmatrix},
$$

where the parameters are those defined earlier, except x now labels, $x = q^{2\pi\theta/\xi}$. In 1994, Kim developed the theory further by calculating several new reflection factors [28]. Kim utilised the same approach as Nepomechie and Mezincescu: whereby one forms functional equations from the original set by differentiating with respect to one variable, so to leave dependence on solely one variable [5]. This is the approach that was documented in the introductory chapter to solve the Yang-Baxter and Reflection equations. Kim successfully obtained three solutions satisfying the reflection equation. We shall denote them: Case I, Case II and Case III. Case I is as follows, [28]:

$$R_a^b(q,x) = \begin{pmatrix} B(q,x) & 0 & Z(q,x) \\ 0 & A(q,x) & 0 \\ \bar{Z}(q,x) & 0 & G(q,x) \end{pmatrix},$$

with entries:

$$\begin{aligned} Z(q,x) &= z_1(x^2-1), \\ \bar{Z}(q,x) &= \frac{b_1^2}{z_1}q^2(x^2-1), \\ B(q,x) &= 2x + b_1(q^2-1)(1-x), \\ A(q,x) &= 2x + b_1(q^2+x)(1-x), \\ G(q,x) &= 2x + b_1 x(1-q^2)(1-x), \end{aligned}$$

b_1, z_1 are two free parameters, they are introduced during the formation of the functional equations, and the parameters x, q are the same as before. In the above solution constraints reduce the number of free parameters to two, they are: $g_1 = q^2 b_1$ and $\bar{z}_1 = q^2 {b_1}^2/z_1$. The Case II solution takes the form, [28]:

$$R_a^b(q,x) = \begin{pmatrix} B(q,x) & X(q,x) & Z(q,x) \\ 0 & A(q,x) & Y(q,x) \\ 0 & 0 & G(q,x) \end{pmatrix},$$

with entries:

$$\begin{aligned} B(q,x) &= q^3x^{-1} \pm i, \\ A(q,x) &= q^3x \pm i, \\ G(q,x) &= x(q^3 \pm ix), \\ X(q,x) &= \frac{x_1}{2}(q^2 \pm iq - 1)(q \mp i)(x^2-1)x^{-1}, \\ Y(q,x) &= \frac{x_1}{2}(q^2 \pm iq - 1)(1 \pm iq)(x^2-1)q^{-2}, \\ Z(q,x) &= \frac{x_1^2(q^2 \pm iq - 1)^2(q \mp ix)(q \mp i)(1-x^2)q^{-3}x^{-1}}{4(q \pm i)}, \end{aligned}$$

this solution contains only one free parameter x_1, as all other parameters are related to it by the constraints:

$$y_1 = q^{-2}x_1, \qquad z_1 = -(q \pm i)(q^2 \pm iq - 1)x_1^2/2q^3(q \pm i).$$

The Case III solution takes the following form [28]:

$$R_a^b(q,x) = \begin{pmatrix} B(q,x) & 0 & 0 \\ \bar{X}(q,x) & A(q,x) & 0 \\ \bar{Z}(q,x) & \bar{Y}(q,x) & G(q,x) \end{pmatrix},$$

the entries in the above solution are exactly the same as those of Case II, providing that x_1, y_1, z_1 are replaced with $\bar{x}_1, \bar{y}_1, \bar{z}_1$. With the help of Maple one can verify that these solutions, despite their appearance, satisfy the relevant equations together with the S-matrix [65, 66].

In Kim's paper [28] it is stated that one can uncover a new diagonal solution from the Case I reflection matrix. Kim achieves this by evaluating a limit of the free parameters that are present in the solution. Kim asserts that if $b_1 \to 0$ then it follows that $g_1 \to 0$ also, which is correct. This then implies that either $z_1 \to 0$ or $\bar{z}_1 \to 0$. However, the paper incorrectly assumes that both $z_1, \bar{z}_1 \to 0$, which reduces the Case I solution to a diagonal form. Furthermore, if we try to verify this supposed diagonal solution, we find that it does not satisfy the relevant equations. Actually, Kim's limit corresponds to two new solutions, that we shall denote Case IV:

$$R_a^b(q,x) = \begin{pmatrix} B(q,x) & 0 & Z(q,x) \\ 0 & A(q,x) & 0 \\ 0 & 0 & G(q,x) \end{pmatrix},$$

$$R_a^b(q,x) = \begin{pmatrix} B(q,x) & 0 & 0 \\ 0 & A(q,x) & 0 \\ \bar{Z}(q,x) & 0 & G(q,x) \end{pmatrix},$$

where the entries are exactly those of Case I. Again, with the help of Maple one can verify that these solutions do indeed satisfy the reflection equation (1.8).

Kim's initial classification of solutions was extended by Lima-Santos in his papers [29]. All existing solutions were recovered, as well as another four solutions. The first two are denoted Case V, each contain a single free parameter and take the form:

$$R_a^b(q,x) = \begin{pmatrix} 1 & 0 & \frac{z_1}{2}(x-x^{-1}) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R_a^b(q,x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\bar{z}_1}{2}(x-x^{-1}) & 0 & 1 \end{pmatrix}.$$

The final two solutions, Case VI, are the most general of all the reflection factors as every matrix entry is non-zero:

$$R_a^b(q,x) = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix},$$

one solution has the following entries:

$$\begin{aligned} R_{11} := & \frac{q+q^{-1}}{4x}\left(\frac{\sqrt{x}}{q^3}+\frac{q^3}{\sqrt{x}}+i\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)\right) \\ & +\frac{q^3+q^{-3}}{4\sqrt{x}}\left(2\,\frac{{\beta_{12}}^2}{\beta_{13}\,(q^3+q)}+4\,\left(q^2+q^{-2}-i\left(q-q^{-1}\right)\right)^{-1}\right)\cdot \\ & \left(\frac{(q^2+q^{-2})}{4}\left(\frac{\sqrt{x}}{q}+\frac{q}{\sqrt{x}}\right)-\frac{i}{2}\left(\frac{1}{\sqrt{x}}+\frac{\sqrt{x}}{2}\left(q^2+q^{-2}\right)\right)\right)\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right), \end{aligned}$$

$$R_{12} := \frac{\beta_{21}\,(x-1)\,(x+1)\,(q^4-q^2+1)\,(q^2+1)^2}{4q^4x^{3/2}},$$

$$R_{13} := -\frac{\beta_{13}\,(x-1)\,(q^4-q^2+1)\,(x+1)\,(q^2+1)\,(-q^2+(x-1)\,iq-x)}{4q^4x^{3/2}},$$

$$R_{21} := \frac{\beta_{21}\,(x-1)\,(x+1)\,(q^4-q^2+1)\,(q^2+1)^2}{4q^4x^{3/2}},$$

$$R_{22} := \frac{(q+q^{-1})}{4}\left(\sqrt{x}q^3 + \frac{1}{\sqrt{x}q^3} - i\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\right)$$
$$+ \frac{1}{16}\left(\left(\frac{\sqrt{x}}{q^2} - \frac{q^2}{\sqrt{x}}\right)\left(\sqrt{x}q + \frac{1}{\sqrt{x}q}\right) + i\left(\frac{\sqrt{x}}{q} + \frac{q}{\sqrt{x}}\right)^2\right)\cdot$$
$$\left(2\,\frac{{\beta_{12}}^2}{\beta_{13}\,(q^3+q)} + 4\,\left(q^2+q^{-2} - i\left(q-q^{-1}\right)\right)^{-1}\right)\left(q^3+q^{-3}\right)\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right),$$

$$R_{23} := \frac{i\beta_{12}\,(x-1)\,(q^4-q^2+1)\,(x+1)\,(q^2+1)^2}{4q^6\sqrt{x}},$$

$$R_{31} := -\frac{{\beta_{21}}^2\beta_{13}\,(x-1)\,(q^4-q^2+1)\,(x+1)\,(q^2+1)\,(-q^2+(ix-i)\,q-x)}{4{\beta_{12}}^2x^{3/2}q^4},$$

$$R_{32} := \frac{i\beta_{21}\,(x-1)\,(q^4-q^2+1)\,(x+1)\,(q^2+1)^2}{4q^6\sqrt{x}},$$

$$R_{33} := \frac{x}{4}\left(q+q^{-1}\right)\left(\frac{\sqrt{x}}{q^3} + \frac{q^3}{\sqrt{x}} + i\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\right)$$
$$-\frac{\sqrt{x}}{4}\left(q^3+q^{-3}\right)\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\left(2\frac{{\beta_{12}}^2}{\beta_{13}\,(q^3+q)} + 4\left(q^2+q^{-2} - i\left(q-q^{-1}\right)\right)^{-1}\right)\cdot$$
$$\left(\frac{1}{4}\left(q^2+q^{-2}\right)\left(\frac{\sqrt{x}}{q} + \frac{q}{\sqrt{x}}\right) + \frac{i}{2}\left(\sqrt{x} + \frac{q^2+q^{-2}}{2\sqrt{x}}\right)\right).$$

Originally, the above solution contained many free parameters, β_{ij}. However, constraints are used to reduce the number of them significantly and only β_{12}, β_{13} remain. The constraints are:

$$\beta_{11} = -i\frac{\beta_{12}^2}{\beta_{13}}\left(\frac{q+q^{-1}+i}{q^2(q+q^{-1})}\right) - \frac{i}{q}\left(\frac{4-2(q-q^{-1})}{q+q^{-1}-i(q^2-q^{-2})}\right),$$
$$\beta_{33} = -i\frac{\beta_{12}^2}{\beta_{13}}\left(\frac{q+q^{-1}-i}{q^4(q+q^{-1})}\right) - iq\left(\frac{4-2(q-q^{-1})}{q+q^{-1}-i(q^2-q^{-2})}\right),$$
$$\beta_{21} = i\frac{\beta_{12}^3}{\beta_{13}^2}\left(\frac{1}{q^2(q+q^{-1})}\right) - i\frac{\beta_{12}}{\beta_{13}}\left(\frac{4-2(q-q^{-1})}{q+q^{-1}-i(q^2-q^{-2})}\right),$$
$$\beta_{23} = i\frac{\beta_{12}}{q^2}, \qquad \beta_{32} = i\frac{\beta_{21}}{q^2}, \qquad \beta_{31} = \frac{\beta_{21}^2\beta_{13}}{\beta_{12}^2},$$

and their complicated nature in turn influences the solution, making it difficult to manipulate. The second solution of this type is similar and differs by several sign choices within the entries and constraints.

It may seem surprising that the $a_2^{(2)}$-model possesses this multitude of solutions. However, one can argue this is precisely what we should expect because of their structure. To illustrate this argument let us examine a reflection matrix where each entry is replaced by its index notation, the index on the left/right refers to in/out-going particles respectively:

$$R_a^b = \begin{pmatrix} ++ & +0 & +- \\ 0+ & 00 & 0- \\ -+ & -0 & -- \end{pmatrix}.$$

The above matrix highlights the significance of each matrix entry and its associated boundary process. For example, the Case II (upper-triangular) solution represents a boundary process where the particles can remain as they are or lose one or two units of charge, if it is permitted. Similarly, the Case III (lower-triangular) solution concerns a boundary process where particles remain the same or gain charge. Collectively, the solutions cover all possible processes, whether it is Case II: a particle's charge decreases, or Cases IV and V, where only even units of charge may exchange. Adding an integrable defect near the boundary allows us to expand upon the $a_2^{(2)}$ story further and we will now review the details of a defect within $a_2^{(2)}$.

During the time that Kim and Lima-Santos calculated reflection matrices, interest in integrable defects began to stir. The early results of Delfino, Mussardo and Simonetti were expanded upon by many authors over the last twenty years, see [10, 45, 46, 47, 61] for example, leading us to 2011 when the $a_2^{(2)}$ transmission matrix was calculated in [45]. Before stating the transmission matrix we will briefly recount the details of the classical story reviewed in the introductory chapter, presented in [45, 47]. As mentioned in chapter one, the Tzitzéica model can only support type - II defects. In the Lagrangian description, this means that the additional defect contribution possesses an extra degree of freedom - an auxiliary field. The model's two complex soliton solutions, regarded as the soliton and anti-soliton, can experience: a delay when travelling through the defect and retain its particle type or convert to the anti-soliton, or the defect can absorb the soliton. Of course, the process taking place depends on the initial conditions of the fields u, v and λ, characterised in chapter one. With knowledge of the classical findings and the S-matrix one can study the transmission Yang-Baxter equation (1.62), as we know, to calculate suitable T-matrices. As we are aware, the transmission matrices' entries can be described by an index approach, whereby Kronecker-deltas keep track of charge or by annihilation and creation operators belonging to an infinite-dimensional representation of the Borel subalgebra. Throughout this chapter we will use the most general transmission matrix calculated in [45], expressed in the Kronecker-delta language for simplicity:

$$T^{b\beta}_{a\alpha}(\theta) = \rho(\theta)\begin{pmatrix} (\epsilon^2 q^{2\alpha} + q^{-2\alpha}\tau^2 x)\delta^{\beta}_{\alpha} & \epsilon\mu(\alpha)\delta^{\beta-1}_{\alpha} & M(\alpha)\delta^{\beta-2}_{\alpha} \\ \lambda(\alpha)\tau x\delta^{\beta+1}_{\alpha} & (\epsilon\tilde{\tau} + \tau\tilde{\epsilon}x)\delta^{\beta}_{\alpha} & \tilde{\tau}\mu(\alpha)q^{-2\alpha-1}\delta^{\beta-1}_{\alpha} \\ L(\alpha)x\delta^{\beta+2}_{\alpha} & \tilde{\epsilon}\lambda(\alpha)q^{2\alpha-1}x\delta^{\beta+1}_{\alpha} & (\tilde{\epsilon}^2 q^{2\alpha}x + \tilde{\tau}^2 q^{-2\alpha})\delta^{\beta}_{\alpha} \end{pmatrix},$$

containing the following parameters:

$$M(\alpha) = \mu(\alpha)\mu(\alpha+1)\frac{q^{-2\alpha-1}}{1+q^2}, \qquad L(\alpha) = \lambda(\alpha)\lambda(\alpha-1)\frac{q^{2\alpha-1}}{1+q^2}$$

with constraint

$$\mu(\alpha)\lambda(\alpha+1) = (q+q^{-1})(\tau\tilde{\tau}q^{-2\alpha-1} + \epsilon\tilde{\epsilon}q^{2\alpha+1}),$$

The T-matrix contains much freedom, because of the free parameters $\epsilon, \tilde{\epsilon}, \tau, \tilde{\tau}$ and one of the free functions $\mu(\alpha)$ or $\lambda(\alpha)$. Resultantly, the generalised solutions will possess more freedom. The coefficient $\rho(\theta)$ ensures that the T-matrix satisfies the analogues of unitarity and crossing symmetry, it can be found in [45], but it is not required for our purposes. The summary of known results is now complete, and charged with this knowledge we can construct generalised solutions.

3.2 Generalised Reflection Matrices for the $a_2^{(2)}$ affine Toda model

To form a generalised reflection matrix we must evaluate the product,

$$T(\theta)R_d(\theta)\tilde{T}(\theta),$$

where $\tilde{T}(\theta) := T^{-1}(-\theta)$ and care must be taken when tracking all shifts in the topological charge. We will dress diagonal solutions with the T-matrix detailed previously. Therefore, we require the inverse matrix $\tilde{T}(\theta)$. As we have mentioned previously, any transmission matrix must be invertible, it is one of its vital properties and is most likely a result of the underlining quantum group structure. Any inversion procedure must account for the presence of the Kronecker-deltas. Resultantly, the entries of the inverse matrix will contain various shifts. One can readily obtain the infinite-dimensional analogue of Cramer's rule that gives a formula for each entry in the inverse T-matrix. Those formulae are:

$$y_{11}(\alpha) = \frac{x_{22}(\alpha+1)x_{33}(\alpha+2) - x_{23}(\alpha+1)x_{32}(\alpha+2)}{\det_1}, \tag{3.2}$$

$$y_{21}(\alpha) = \frac{x_{31}(\alpha+1)x_{23}(\alpha) - x_{33}(\alpha+1)x_{21}(\alpha)}{\det_2}, \tag{3.3}$$

$$y_{31}(\alpha) = \frac{x_{32}(\alpha)x_{21}(\alpha-1) - x_{31}(\alpha)x_{22}(\alpha-1)}{\det_3}, \tag{3.4}$$

where

$$\begin{aligned} \det_1 = \quad & x_{11}(\alpha)[x_{22}(\alpha+1)x_{33}(\alpha+2) - x_{23}(\alpha+1)x_{32}(\alpha+2)] \\ & + x_{12}(\alpha)[x_{23}(\alpha+1)x_{31}(\alpha+2) - x_{21}(\alpha+1)x_{33}(\alpha+2)] \\ & + x_{13}(\alpha)[x_{21}(\alpha+1)x_{32}(\alpha+2) - x_{22}(\alpha+1)x_{31}(\alpha+2)], \end{aligned}$$

$\det_2, \det_3$ consist of the same entries but take different arguments,

$$y_{12}(\alpha) = \frac{x_{13}(\alpha)x_{32}(\alpha+2) - x_{12}(\alpha)x_{33}(\alpha+2)}{\det_4}, \tag{3.5}$$

$$y_{22}(\alpha) = \frac{x_{11}(\alpha-1)x_{33}(\alpha+1) - x_{13}(\alpha-1)x_{31}(\alpha+1)}{\det_5}, \tag{3.6}$$

$$y_{32}(\alpha) = \frac{x_{12}(\alpha-2)x_{31}(\alpha) - x_{11}(\alpha-2)x_{32}(\alpha)}{\det_6}, \tag{3.7}$$

$$y_{13}(\alpha) = \frac{x_{12}(\alpha)x_{23}(\alpha+1) - x_{13}(\alpha)x_{22}(\alpha+1)}{\det_7}, \tag{3.8}$$

$$y_{23}(\alpha) = \frac{x_{13}(\alpha-1)x_{21}(\alpha) - x_{11}(\alpha-1)x_{23}(\alpha)}{\det_8}, \tag{3.9}$$

$$y_{33}(\alpha) = \frac{x_{11}(\alpha-2)x_{22}(\alpha-1) - x_{12}(\alpha-2)x_{21}(\alpha-1)}{\det_9}. \tag{3.10}$$

Determinants $\det_5$ and $\det_9$ are formed by expanding the determinant around the second and third rows respectively. They are related to all other determinants via particular shifts of topological charge, for example:

$$\det_1(\alpha) = \det_5(\alpha+1), \qquad \det_1(\alpha) = \det_9(\alpha-2).$$

In fact, the determinants are all equal, and this might seem surprising given the T-matrix's dependence on the topological charge. However, when calculating $\det_1$, for instance, one finds that the determinant does not depend on the topological charge. Therefore, all determinants are equal, because they only differ by various shifts in the charge. Given that the topological charge is counted in integer units, similar to the sine-Gordon case, we might expect this behaviour. We can invert both the above matrix and the rapidity to obtain $\tilde{T}(\theta)$, which takes the form:

$$\tilde{T}_{a\alpha}^{b\beta}(\theta) = \frac{\tilde{\rho}(-\theta)}{\Delta(-\theta)} \begin{pmatrix} \tilde{T}_{11}(\alpha)\delta_\alpha^\beta & \mu(\alpha)\tilde{\tau}q^{-2\alpha-4}\delta_\alpha^{\beta-1} & -M(\alpha)q^{-2}\delta_\alpha^{\beta-2} \\ -\lambda(\alpha)q^{2\alpha}\tilde{\epsilon}x^{-1}\delta_\alpha^{\beta+1} & \tilde{T}_{22}(\alpha)\delta_\alpha^\beta & \epsilon\mu(\alpha)q^{-3}\delta_\alpha^{\beta-1} \\ L(\alpha)x^{-1}\delta_\alpha^{\beta+2} & -\lambda(\alpha)q\tau x^{-1}\delta_\alpha^{\beta+1} & \tilde{T}_{33}(\alpha)\delta_\alpha^\beta \end{pmatrix},$$

$$\begin{aligned} \tilde{T}_{11}(\alpha) &:= (q^{2\alpha+3}\tilde{\epsilon}^2x^{-1} - \tilde{\tau}^2q^{-2\alpha-3})q^{-1}, \\ \tilde{T}_{22}(\alpha) &:= q^{-4}(\tau\tilde{\epsilon}q^6x^{-1} - \epsilon\tilde{\tau}), \\ \tilde{T}_{33}(\alpha) &:= q^{-1}(\tau^2x^{-1}q^{-2\alpha+3} - \epsilon^2q^{2\alpha-3}), \end{aligned}$$

where the determinant $\Delta(-\theta) = (\tilde{\epsilon}\tau q^2x^{-1} - \epsilon\tilde{\tau})(\tilde{\epsilon}\tau x^{-1} + \epsilon\tilde{\tau}q^{-4})$. With ease one can invert the rapidity, in the above matrix, and verify that the inverse works as expected.

At this moment, all necessary components to form a generalised solution are available. Let us begin by choosing the identity solution, $R_d = \mathbb{1}_3$. Evaluating the product (1.95) returns a generalised reflection matrix of the form:

$$\hat{R}^{b\beta}_{a\alpha}(\theta) = \begin{pmatrix} X_{11}\delta^{\beta}_{\alpha} & X_{12}\delta^{\beta-1}_{\alpha} & X_{13}\delta^{\beta-2}_{\alpha} \\ X_{21}\delta^{\beta+1}_{\alpha} & X_{22}\delta^{\beta}_{\alpha} & X_{23}\delta^{\beta-1}_{\alpha} \\ X_{31}\delta^{\beta+2}_{\alpha} & X_{32}\delta^{\beta+1}_{\alpha} & X_{33}\delta^{\beta}_{\alpha}, \end{pmatrix},$$

with entries:

$$X_{11} = \frac{1}{\Delta}\left[\tilde{\epsilon}^2\tau^2 q^2 - \epsilon\tilde{\epsilon}\tau\tilde{\tau}x^{-1} + \epsilon\tilde{\epsilon}\tau\tilde{\tau}q^{-2}x^{-1} - \epsilon^2\tilde{\tau}^2 q^{-4} - \tau^2\tilde{\tau}^2 q^{-4\alpha-4}(x - x^{-1})\right],$$

$$X_{12} = \frac{1}{\Delta}\left[\mu(\alpha)\tau^2\tilde{\tau}q^{-4\alpha-4}(x - x^{-1})\right],$$

$$X_{13} = \frac{1}{\Delta}\left[-M(\alpha)\tau^2 q^{-2\alpha-2}(x - x^{-1})\right],$$

$$X_{21} = \frac{1}{\Delta}\left[-\lambda(\alpha)\tilde{\tau}^2\tau q^{-2\alpha-2}(x - x^{-1})\right],$$

$$X_{22} = \frac{1}{\Delta}\left[\tau^2\tilde{\tau}^2 q^{-4\alpha-2}(1+q^2)(x - x^{-1}) + \epsilon\tilde{\epsilon}\tau\tilde{\tau}q^{-2}x - \epsilon^2\tilde{\tau}^2 q^{-4} + \tilde{\epsilon}^2\tau^2 q^2 - \epsilon\tilde{\epsilon}\tau\tilde{\tau}x^{-1}\right],$$

$$X_{23} = \frac{1}{\Delta}\left[-\mu(\alpha)\tau^2\tilde{\tau}q^{-4\alpha-1}(x - x^{-1})\right],$$

$$X_{31} = \frac{1}{\Delta}\left[-L(\alpha)\tilde{\tau}^2 q^{-2\alpha}(x - x^{-1})\right],$$

$$X_{32} = \frac{1}{\Delta}\left[\lambda(\alpha)\tau\tilde{\tau}^2 q^{-2\alpha+1}(x - x^{-1})\right],$$

$$X_{33} = \frac{1}{\Delta}\left[\tilde{\epsilon}^2\tau^2 q^2 + \epsilon\tilde{\epsilon}\tau\tilde{\tau}q^{-2}x - \epsilon\tilde{\epsilon}\tau\tilde{\tau}x - \epsilon^2\tilde{\tau}^2 q^{-4} - \tau^2\tilde{\tau}^2 q^{-4\alpha+2}(x - x^{-1})\right].$$

The first generalised solution enjoys a seemingly compact form. To access more complicated solutions we must dress the non-trivial Nepomechie and Mezincescu diagonal solutions. Let us now dress the following solution with the defect:

$$R_a^b(x,q) = \begin{pmatrix} (q^3x^{-1}+i) & 0 & 0 \\ 0 & (q^3x+i) & 0 \\ 0 & 0 & x(q^3+ix) \end{pmatrix}.$$

Evaluating the multiplication provides another solution, that does not contain any zeroes, with entries:

$$\begin{aligned} X_{11} = \frac{1}{\Delta}\Big[&\epsilon^2\tilde{\epsilon}^2 q^{4\alpha+4}(i-qx^{-1})(x-x^{-1}) - \epsilon^2\tilde{\tau}^2 q^{-1}x^{-1} + \tilde{\epsilon}^2\tau^2 q^5 x^{-1} \\ &- i\epsilon^2\tilde{\tau}^2 q^{-4} + i\tilde{\epsilon}^2\tau^2 q^2 - \epsilon\tilde{\epsilon}\tau\tilde{\tau}q^3 - i\epsilon\tilde{\epsilon}\tau\tilde{\tau}q^2x^{-1} - i\epsilon\tilde{\epsilon}\tau\tilde{\tau}x^{-1} \\ &+ i\epsilon\tilde{\epsilon}\tau\tilde{\tau}q^2x + i\epsilon\tilde{\epsilon}\tau\tilde{\tau}q^{-2}x + \epsilon\tilde{\epsilon}\tau\tilde{\tau}q\Big], \end{aligned}$$

$$X_{12} = \frac{1}{\Delta}\left[-i\epsilon\mu(\alpha)q^{-1}(\tilde{\epsilon}\tau q^3 - i\epsilon\tilde{\tau})(x-x^{-1})\right],$$

$$X_{13} = \frac{1}{\Delta}\left[\epsilon^2 M(\alpha)q^{2\alpha}(q-ix)(x-x^{-1})\right],$$

$$X_{21} = \frac{1}{\Delta}\left[-\tilde{\epsilon}q^{2\alpha}\lambda(\alpha)(\tilde{\epsilon}\tau q^3 - i\epsilon\tilde{\tau})(x-x^{-1})\right],$$

$$\begin{aligned} X_{22} = \frac{1}{\Delta}\Big[&\epsilon\tilde{\epsilon}\tau\tilde{\tau}q + \epsilon\tilde{\epsilon}\tau\tilde{\tau}q^{-1} + i\epsilon\tilde{\epsilon}\tau\tilde{\tau}q^{-2}x + \tilde{\epsilon}^2\tau^2q^5x - \epsilon^2\tilde{\tau}^2q^{-1}x - \epsilon\tilde{\epsilon}\tau\tilde{\tau}q^{-1}x^2 \\ &+ i\epsilon\tilde{\epsilon}\tau\tilde{\tau}q^2x^{-1} + i\tilde{\epsilon}^2\tau^2q^2 - i\epsilon^2\tilde{\tau}^2q^{-4} - \epsilon\tilde{\epsilon}\tau\tilde{\tau}q^3 - i\epsilon\tilde{\epsilon}\tau\tilde{\tau}q^2x - i\epsilon\tilde{\epsilon}\tau\tilde{\tau}x\Big], \end{aligned}$$

$$X_{23} = \frac{1}{\Delta}\left[\epsilon\mu(\alpha)q^{-3}x(\tilde{\epsilon}\tau q^3 - i\epsilon\tilde{\tau})(x-x^{-1})\right],$$

$$X_{31} = \frac{1}{\Delta}\left[\tilde{\epsilon}^2 L(\alpha)q^{2\alpha}(ix-q)(x-x^{-1})\right],$$

$$X_{32} = \frac{1}{\Delta}\left[-i\tilde{\epsilon}\lambda(\alpha)q^{2\alpha-2}x(\tilde{\epsilon}\tau q^3 - i\epsilon\tilde{\tau})(x-x^{-1})\right],$$

$$\begin{aligned} X_{33} = \frac{1}{\Delta}\Big[&-\epsilon\tilde{\epsilon}\tau\tilde{\tau}q^3 - \epsilon\tilde{\epsilon}\tau\tilde{\tau}q^{-1} - i\epsilon\tilde{\epsilon}\tau\tilde{\tau}x + \epsilon^2\tilde{\epsilon}^2q^{4\alpha-4}(q-ix)(x^2-1) \\ &+ \epsilon\tilde{\epsilon}\tau\tilde{\tau}q^{-1}x^2 + \epsilon\tilde{\epsilon}\tau\tilde{\tau}qx^2 + i\epsilon\tilde{\epsilon}\tau\tilde{\tau}q^{-2}x + \tilde{\epsilon}^2\tau^2q^5x \\ &- \epsilon^2\tilde{\tau}^2q^{-1}x + i\tilde{\epsilon}^2\tau^2q^2x^2 - i\epsilon^2\tilde{\tau}^2q^{-4}x^2\Big]. \end{aligned}$$

The above generalised solution is not as simple as the former generalised reflection factor. This is expected due to the non-trivial diagonal matrix used in its construction. The third generalised solution, created by dressing the remaining Nepomechie and Mezincescu diagonal solution (with the negative sign), is almost identical to the above solution. The two solutions differ by multiple plus or minus signs within the matrix entries.

At first glance, the arrangement of the free parameters in particular matrix entries hints at an underlying structure. Almost immediately, one begins to see how the defect's ability to store charge, and introduce further degrees of freedom, generalises the finite-dimensional story. Let us now consider how they are related the solutions of Kim and Lima-Santos.

3.3 Relation to Existing Solutions

If we cast our eye over the first generalised solution, formed from the dressing of the identity solution. It becomes clear that if $\tau = 0$ all entries except the diagonal and entry 3,1 vanish. We believe this particular limit returns the infinite-dimensional analogue of Lima-Santos' Case V solution:

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\bar{z}_1}{2}(x - x^{-1}) & 0 & 1 \end{pmatrix}.$$

To illustrate this, take $\tau = 0$ and evaluate all matrix entries, including the determinant. The identity is now present on the diagonal, as the determinant cancels the remaining factor of $-\epsilon^2 \tilde{\tau}^2 q^{-4}$, and entry 3,1 can take the above form if for example:

$$L(0) \to 1, \qquad \epsilon^2 \to \frac{2q^4}{\bar{z}_1}.$$

Similarly, when $\tilde{\tau} = 0$, we find the infinite-dimensional analogue of the other Lima-Santos Case V solution:

$$R = \begin{pmatrix} 1 & 0 & \frac{z_1}{2}(x - x^{-1}) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

When taking this limit we find that there is an overall factor of x^2, originating from $\Delta_{\tilde{\tau}=0} = \tilde{\epsilon}^2\tau^2 q^2 x^{-2}$ and if we take, for instance, the following limits we will obtain the Case V solution exactly:

$$M(0) \to -1, \qquad\qquad \tilde{\epsilon}^2 \to \frac{2}{q^4 z_1}.$$

It is not immediately obvious if any other finite-dimensional solutions are hidden within this generalised solution and so we will now examine the second more complicated generalised solution.

Despite the appearance of the solution's entries, and their complicated structure, they do in fact possess natural upper and lower triangular structures. Setting $\epsilon = 0$ will give the solution a lower triangular structure, and similarly, $\tilde{\epsilon} = 0$ returns an upper triangular structure. We believe that the above limits represent the infinite-dimensional analogues of Kim's finite-dimensional lower and upper triangular matrices. Let us quickly recall Kim's upper triangular solution:

$$R = \begin{pmatrix} B(x) & X(x) & Z(x) \\ 0 & A(x) & Y(x) \\ 0 & 0 & G(x) \end{pmatrix},$$

with entries:

$$B(x) = (q^3 x^{-1} + i),$$

$$A(x) = (q^3 x + i),$$

$$G(x) = x^2(q^3 x^{-1} + i),$$

$$X(x) = \frac{x_1}{2}(q^3 + i)(x - x^{-1}),$$

$$Y(x) = \frac{x_1}{2}(1 - iq^3)(1 - x^2)q^{-2},$$

$$Z(x) = \frac{x_1^2(q^2 + iq - 1)(q - ix)(q^3 + i)(x^{-1} - x)}{4q^3(q + i)},$$

the above is presented slightly differently when compared to that of its first introduction in the Known Results section. Some brackets have been expanded and the uppermost sign was chosen, to achieve simplification and to help us relate the solutions. The above finite-dimensional solution is realised within our infinite-dimensional solution by taking the following limits:

$$\tilde{\epsilon} = 0, \qquad \alpha = 0, \qquad \mu(0) = 1, \qquad \tilde{\tau} \to \frac{2q^3}{x_1(q^3+i)}$$

with
$$\mu(1) \to \frac{(1+q^2)(q^2+iq-1)}{(q^3+i)(q+i)}.$$

Kim's lower triangular solution is simply the transpose of his upper triangular matrix, together with the conjugation of all free parameters. Somewhat similarly, we can realise the finite lower triangular solution by considering the following set up:

$$\epsilon = 0, \qquad \alpha = 0, \qquad \lambda(0) = 1, \qquad \tau \to \frac{-2q}{\bar{x}_1(q^3+i)}$$

and
$$\lambda(-1) \to \frac{q^2(1+q^2)(q^2+iq-1)}{(q^3+i)(q+i)}.$$

The equalities and limits used to realise the finite solutions are not unique and it is likely that there exists a nicer formulation. However, I believe that this set up illustrates how the infinite-dimensional solution, containing free functions of the charge and several free parameters, can collapse to return the finite-dimensional solutions.

Further scrutiny of the second generalised solution is most revealing, especially within entries: $X_{12}, X_{21}, X_{23}, X_{32}$. They all contain the factor $(\tilde{\epsilon}\tau q^3 - i\epsilon\tilde{\tau})$ and so the limit
$$\tilde{\epsilon}\tau \to i\epsilon\tilde{\tau}q^{-3}$$

returns a solution of the form:

$$\hat{R}^{b\tilde{\beta}}_{a\tilde{\alpha}}(\theta) = \begin{pmatrix} X'_{11}\delta^{\tilde{\beta}}_{\tilde{\alpha}} & 0 & X'_{13}\delta^{\tilde{\beta}-2}_{\tilde{\alpha}} \\ 0 & X'_{22}\delta^{\tilde{\beta}}_{\tilde{\alpha}} & 0 \\ X'_{31}\delta^{\tilde{\beta}+2}_{\tilde{\alpha}} & 0 & X'_{33}\delta^{\tilde{\beta}}_{\tilde{\alpha}}, \end{pmatrix},$$

where X'_{ii} and X'_{ij} represent the entries after implementing the limit. Given the above solution's shape, it is most likely an infinite-dimensional analogue of Kim's Case I solution. Unfortunately, due to the nature of the remaining entries, it is not clear how we can identify the free parameters of the finite-dimensional solution or reduce the solution so that it equals Kim's reflection factor exactly. It most likely requires a similarity transformation together with other specialisations of the leftover parameters.

Finally, it remains to recover Lima-Santos' most general solution within one of the new matrices. Unfortunately, this has escaped us so far. The identification of his constraints proves to be very difficult. Nevertheless, we believe that it is possible, because Lima-Santos obtains Kim's upper and lower triangular solutions [28] (cases II and III) from the most general solution (case VI) [29]. We know that the defect and boundary work together extremely well. The defect develops and generalises the simple reflection factor, and the resulting generalised reflection matrix satisfies a modified reflection equation as we have proved. Therefore, it is difficult to suppose that one might lose some solutions during the construction, but it might be the case. To achieve equality, we imagine that a similarity transformation is required, as well as giving many of the parameters a specific value. A promising observation is that the rapidity dependence of most off-diagonal entries appears to match, but this is as far as we can go at the moment.

The Tzitzéica model exhibits several peculiar features, which we have discussed. One further peculiarity is the fact that the identity matrix is a bona fide solution to the reflection equations. The result of dressing this reflection factor with an integrable defect is a generalised solution that has a compact form, $\hat{R} = T(\theta)\tilde{T}(\theta)$. However, this generalised solution is naturally related to the most simple non-diagonal Case V reflection matrices. Typically, some of the remaining (more complicated) solutions are related to the generalised matrices constructed from the non-trivial diagonal matrix. Moreover, this neatly shows that we can dress a boundary with a defect to extend the class of known solutions.

3.4 Future work concerning $a_2^{(2)}$

In the introductory chapter, the important role of integrable defects was described: including the conjecture that dressing a particular boundary (one having a diagonal reflection matrix) with an integrable defect enables the calculation of more general objects from which all known solutions are recovered. This chapter has focussed on the $a_2^{(2)}$- affine Toda model, which is the next logical step after the extensive study of the sine-Gordon, and we have provided generalised solutions that follow this construction. The recovery of all known solutions was then considered, but remains incomplete. This work supplies a large amount of support for the conjecture, and with a little more work, we will hopefully find that the conjecture is true in this case. Some of the difficulty may be attributed to the unusual phenomenon exhibited by this model, whereby the classical soliton spectrum does not match the quantum spectrum. In the future, the generalised solutions presented in this chapter require further examination. In particular, one should hope to discover their associated classical integrable boundary conditions. Given their nature, it is reasonable to expect an increase in the number of degrees of freedom. As a result, they would lie outside the known class of results (1.30) [41].

Chapter 4

Soliton Preserving Generalised Reflection Matrices of the $a_2^{(1)}$ affine Toda model

We have seen how defects and their associated generalised solutions fit, pleasingly, into an algebraic framework. In doing this, we also formed generalised solutions for the sine-Gordon ($a_1^{(1)}$) model, and commented on their curious properties. As a matter of course, it was natural to consider the Tzitzéica model next, where one is able to observe the way in which the defect effortlessly generalises the finite-dimensional solutions. It is now useful to study the $a_2^{(1)}-$ affine Toda model that admits both type - I and type - II defects. In this case, the type - I defect behaves in a very striking yet peculiar manner, but still generalises the finite-dimensional results of the literature. At the beginning of this chapter, we will state the known solutions to the reflection equation - concerning soliton preserving (SP) boundary conditions - as well as the known type - I transmission matrices. We will then construct the generalised reflection matrices via the dressing procedure and relate these solutions to the existing finite-dimensional reflection factors. The curious characteristics of the type - I generalised solutions are illustrated in several diagrams, some are found within this chapter and all others in an appendix. Following this we can study the type - II defect transmission matrices that build upon the results of the type - I case.

4.1 Review of Known Results in $a_2^{(1)}$

The S-matrix, given by Jimbo [24], enabled many to compute several reflection and transmission factors over the years and it takes the form:

$$S(\Theta_{12}) = \rho_S(\Theta_{12}) \left(\begin{array}{ccc|ccc|ccc} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & c^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & c^- & 0 & 0 \\ \hline 0 & c^- & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 & c^+ & 0 \\ \hline 0 & 0 & c^+ & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c^- & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \end{array} \right),$$

$$\begin{aligned} a &= qx_{12} - q^{-1}x_{12}^{-1}, \\ b &= x_{12} - x_{12}^{-1}, \\ c^{\pm} &= (q - q^{-1})x_{12}^{\pm 1/3}, \end{aligned}$$

$$\Theta_{12} = \theta_1 - \theta_2,\ x_i = e^{3\gamma\theta_i/2},\ i = 1, 2; \quad x_{12} = \frac{x_1}{x_2};\ q = -e^{-i\pi\gamma},\ \gamma = \frac{4\pi}{\beta^2} - 1.$$

We will now state the reflection matrices associated with SP boundary conditions, calculated in papers [67, 68, 69], where the diagonal solutions are of particular importance because we will use them to construct the generalised reflection matrices. The diagonal solutions are presented in such a way that they are compatible with the defect transmission matrices, which we will use to construct generalised solutions.

4.1.1 Reflection Matrices

The first step to classify all SP solutions to the reflection equation (1.8), in the $a_2^{(1)}$ case (actually for all $a_n^{(1)}$ models, $n > 1$), was taken by de Vega and Gonzalez-Ruiz in 1993, [67], where they calculate several diagonal solutions. They discovered three solutions of two different types: one containing no free parameters, and the other containing one parameter, υ. The solutions are of the form:

$$R_{d_0}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^{4/3} & 0 \\ 0 & 0 & x^{8/3} \end{pmatrix}, \tag{4.1}$$

$$R_{d_1}(x) = \begin{pmatrix} (x^2\upsilon - \upsilon^{-1}) & 0 & 0 \\ 0 & x^{4/3}(x^2\upsilon - \upsilon^{-1}) & 0 \\ 0 & 0 & x^{2/3}(\upsilon - x^2\upsilon^{-1}) \end{pmatrix}, \tag{4.2}$$

and

$$R_{d_2}(x) = \begin{pmatrix} (x^2\upsilon - \upsilon^{-1}) & 0 & 0 \\ 0 & x^{-2/3}(\upsilon - x^2\upsilon^{-1}) & 0 \\ 0 & 0 & x^{2/3}(\upsilon - x^2\upsilon^{-1}) \end{pmatrix}, \tag{4.3}$$

in the above $x = e^{3\gamma\theta/2}$ for a soliton of rapidity θ, and $\upsilon = e^{i\pi\xi}$ where ξ is the boundary parameter associated to all diagonal matrix entries. Note that the factors multiplying entries: 2,2 and 3,3, of the diagonal reflection matrices, are necessary to relate the different gradations of the S-matrix used here and in papers [67, 69].

Abad and Rios added to the literature by calculating a non-diagonal reflection matrix in 1995 [68]. Lima-Santos also rediscovered their solution in his 2002 paper [69], it takes the form:

$$R_{13}(u) = \begin{pmatrix} f_{11}(u) & 0 & \frac{1}{2}\beta_{13}(e^{2u}-1) \\ 0 & f_{11}(u) + f_{21}^{-}(u) & 0 \\ \frac{1}{2}\beta_{31}(e^{2u}-1) & 0 & e^{2u}f_{11}(-u) \end{pmatrix}, \tag{4.4}$$

$f_{11}(u) = \beta_{11}(e^u - 1) + 1$, $f_{21}^{-}(u) = \frac{1}{2}(\beta_{22} - \beta_{11})(e^{2u} - 1)$ where the β_{ij} are free parameters associated with the boundary, satisfying the condition $\beta_{13}\beta_{31} = (\beta_{22} + \beta_{11} - 2)(\beta_{22} - \beta_{11})$. Lima-Santos went on to calculate two more non-diagonal solutions [69]:

$$R_{12}(u) = \begin{pmatrix} f_{11}(u) & \frac{1}{2}\beta_{12}(e^{2u}-1) & 0 \\ \frac{1}{2}\beta_{21}(e^{2u}-1) & e^{2u}f_{11}(-u) & 0 \\ 0 & 0 & e^{2u}f_{11}(-u) + e^{u}f_{31}^{+}(u) \end{pmatrix}, \tag{4.5}$$

where: $f_{31}^{+}(u) = \frac{1}{2}(\beta_{33} + \beta_{11} - 2)(e^{2u} - 1)$ together with the constraint

$\beta_{12}\beta_{21} = (\beta_{33} - \beta_{11} - 2)(\beta_{33} + \beta_{11} - 2)$. And finally,

$$R_{23}(u) = \begin{pmatrix} f_{22}(-u) + e^{-u} f_{21}^{+}(u) & 0 & 0 \\ 0 & f_{22}(u) & \frac{1}{2}\beta_{23}(e^{2u} - 1) \\ 0 & \frac{1}{2}\beta_{32}(e^{2u} - 1) & e^{2u} f_{22}(-u) \end{pmatrix}, \tag{4.6}$$

with $f_{22}(u) = \beta_{22}(e^{u} - 1) + 1$, $f_{21}^{+}(u) = \frac{1}{2}(\beta_{22} + \beta_{11})(e^{2u} - 1)$ and constraint $\beta_{23}\beta_{32} = (\beta_{11} + \beta_{22})(\beta_{11} - \beta_{22})$. The non-diagonal solutions contain free parameters associated to each matrix entry. Specifically, we find that each non-diagonal solution includes three parameters: two β_{ii} parameters, as well as two β_{ij} parameters, which are related via a constraint. In order to relate the parameters β_{ii}, $i = 1, 2, 3$ to υ we must consider certain limits of Lima-Santos' non-diagonal solutions. All limits in this case should reduce solutions: $R_{13}(u), R_{12}(u), R_{23}(u)$ to diagonal form, without violating the constraint equations. The following limits [69]:

$$R_{12}(u) : \beta_{11} = \beta_{22} \text{ with } \beta_{13} = 0 = \beta_{31},$$

$$R_{13}(u) : \beta_{33} = -(\beta_{11} - 2) \text{ with } \beta_{12} = 0 = \beta_{21},$$

$$R_{23}(u) : \beta_{22} = -\beta_{11} \text{ with } \beta_{23} = 0 = \beta_{32},$$

force the non-diagonal solutions to take a diagonal form. Each matrix then depends on one free parameter that is identified with υ. The limits above allow us to recover the de Vega and Gonzalez-Ruiz solutions as well as two more:

$$R_{d_3}(x) = \begin{pmatrix} (x^{-2}\upsilon - \upsilon^{-1}) & 0 & 0 \\ 0 & x^{-8/3}(x^{2}\upsilon - \upsilon^{-1}) & 0 \\ 0 & 0 & x^{-4/3}(x^{-2}\upsilon - \upsilon^{-1}) \end{pmatrix}, \tag{4.7}$$

and

$$R_{d_4}(x) = \begin{pmatrix} (x^{2}\upsilon - \upsilon^{-1}) & 0 & 0 \\ 0 & x^{4/3}(x^{-2}\upsilon - \upsilon^{-1}) & 0 \\ 0 & 0 & x^{-4/3}(x^{2}\upsilon - \upsilon^{-1}) \end{pmatrix}, \tag{4.8}$$

further details regarding limits of the boundary parameters can be found in Appendix B of [69]. Furthermore, it can be shown that R_{d_4} is equal to R_{d_3}, by performing: $x \to x^{-1}$, extracting a factor of $x^{4/3}$ and applying the similarity transformation

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Consequently, we will not use R_{d_4} in the dressing procedure.

The notation, R_{ij}, adopted by Lima-Santos to label reflection matrices has an exact meaning. Essentially, it reflects his starting point and embodies a relation he observed within the reflection equations. In [69] it is noted that the relation:

$$\beta_{ij} r_{ji}(u) = \beta_{ji} r_{ij}(u), \qquad \forall i \neq j, \tag{4.9}$$

(r_{ij} label the entries of the reflection matrix) solves nine of the reflection equations. Specifically, the equations that correspond to processes where the in-going solitons are equal to the out-going solitons. These particular equations are obtained by equating the following indices: $a = g$ and $b = h$. This allowed Lima-Santos to choose a particular entry, r_{ij}, and assume that it is non-zero. Therefore, by the above condition r_{ji} is also non-zero. With this knowledge, one can then express the remaining entries of the reflection matrix in terms of this non-zero element. This is permitted providing further constraints hold [69]:

$$r_{pq}(u) = \begin{cases} e^{u} \frac{\beta_{pq}}{\beta_{ij}} r_{ij}(u), & \text{if} \quad p > i \quad \text{and} \quad q > j, \\ \frac{\beta_{pq}}{\beta_{ij}} r_{ij}(u), & \text{if} \quad p > i \quad \text{and} \quad q < j, \end{cases} \tag{4.10}$$

valid for $p \neq q$. By combining the above relations, Lima-Santos arrives at a very restrictive relation for the matrix entries r_{ij}:

$$r_{pq}(u) \neq 0 \implies \begin{cases} r_{pj}(u) = 0, & \text{for} \quad p \neq i, \\ r_{iq}(u) = 0, & \text{for} \quad q \neq j. \end{cases} \tag{4.11}$$

This is in fact why all three solutions R_{12}, R_{13}, R_{23} have four zero entries. In section 4.2 we will see how the diagonal solutions $R_{d_i}, i = 1, 2, 3$ are used to construct several new generalised reflection matrices that are characteristically different to R_{12}, R_{13}, R_{23}, because they appear to violate the constraint (4.9).

4.1.2 Transmission Matrices

A comprehensive account of the $a_2^{(1)}$ transmission matrices is detailed in [48] and in this article we will use the transmission matrices documented in Appendix B of [48] that satisfy the Transmission Yang-Baxter equation:

$$S_{ab}^{mn}(\theta_a-\theta_b)T_{n\alpha}^{t\beta}(\theta_a)T_{m\beta}^{s\gamma}(\theta_b)=T_{b\alpha}^{n\beta}(\theta_b)T_{a\beta}^{m\gamma}(\theta_a)S_{mn}^{st}(\theta_a-\theta_b).$$

As we know, all transmission matrices are infinite-dimensional and this mirrors the defect's ability to store the topological charge of solitons. The story for $a_2^{(1)}$ is slightly different, as the soliton and anti-soliton correspond to two different (though conjugate) representations: the solitonic representation consists of fundamental weights

$$l_1=\frac{1}{3}(2\alpha_1+\alpha_2),\quad l_2=-\frac{1}{3}(\alpha_1-\alpha_2),\quad l_3=-\frac{1}{3}(\alpha_1+2\alpha_2),$$

where α_1,α_2 are the two simple roots of the $a_2^{(1)}$ root system. The anti-solitonic representation has weights $-l_1,-l_2,-l_3$, all of which label the topological charges. When a soliton (anti-soliton) passes through the defect its topological charge can either remain the same, β for example, or it could become γ, another weight within the solitonic (anti-solitonic) representation. Consequently, the defect will not change the boundary condition: if a SP diagonal reflection matrix is used then the resulting generalised reflection matrix should correspond to another SP boundary condition. The generalised solutions will possess some curious features, because the defect allows movement within the weight space.

Before we state the transmission matrices of the $a_2^{(1)}$- model some remarks are required. In the classical picture there exist different defect Lagrangians. The difference originates from the choice of either a clockwise or anti-clockwise permutation of the extended simple roots, which includes the lowest root α_0. The permutation alters a matrix within the defect Lagrangian, thus to each permutation corresponds a different Lagrangian. In the quantum setting similar results appear. Transmission matrices can be calculated using either the solitonic or anti-solitonic representation and so we would expect at least two such matrices. However, we can also choose the clockwise or anti-clockwise permutation of the extended simple roots. Consequently, there are four transmission matrices corresponding to the solitonic/anti-solitonic representation with either choice of permutation [48].

We will now list the transmission matrices as they appear in [48]. The transmission matrix corresponding to the solitonic representation with the clockwise permutation is:

$$T_I = \begin{pmatrix} t_{11}q^{-\alpha\cdot l_1}\delta_\alpha^\gamma & \frac{t_{13}t_{32}}{t_{33}}x^{4/3}\delta_\alpha^{\gamma-\alpha_1} & t_{13}x^{2/3}q^{\alpha\cdot l_2}\delta_\alpha^{\gamma+\alpha_0} \\ t_{21}x^{2/3}q^{\alpha\cdot l_3}\delta_\alpha^{\gamma+\alpha_1} & t_{22}q^{-\alpha\cdot l_2}\delta_\alpha^\gamma & \frac{t_{21}t_{13}}{t_{11}}x^{4/3}\delta_\alpha^{\gamma-\alpha_2} \\ \frac{t_{32}t_{21}}{t_{22}}x^{4/3}\delta_\alpha^{\gamma-\alpha_0} & t_{32}x^{2/3}q^{\alpha\cdot l_1}\delta_\alpha^{\gamma+\alpha_2} & t_{33}q^{-\alpha\cdot l_3}\delta_\alpha^\gamma \end{pmatrix}, \tag{4.12}$$

The following T-matrix corresponds to the anti-solitonic representation with the anti-clockwise permutation:

$$T_{II} = \begin{pmatrix} q^{\alpha\cdot l_1}\delta_\alpha^\gamma & t_{12}x^{-2/3}q^{-\alpha\cdot l_3}\delta_\alpha^{\gamma-\alpha_1} & \frac{t_{12}t_{23}}{t_{22}}x^{-4/3}\delta_\alpha^{\gamma+\alpha_0} \\ \frac{t_{23}t_{31}}{t_{33}}x^{-4/3}\delta_\alpha^{\gamma+\alpha_1} & t_{22}q^{\alpha\cdot l_2}\delta_\alpha^\gamma & t_{23}x^{-2/3}q^{-\alpha\cdot l_1}\delta_\alpha^{\gamma-\alpha_2} \\ t_{31}x^{-2/3}q^{-\alpha\cdot l_2}\delta_\alpha^{\gamma-\alpha_0} & t_{31}t_{12}x^{-4/3}\delta_\alpha^{\gamma+\alpha_2} & t_{33}q^{\alpha\cdot l_3}\delta_\alpha^\gamma \end{pmatrix}. \tag{4.13}$$

The solitonic representation together with the anti-clockwise permutation returns a T-matrix of form:

$$T_{III} = \begin{pmatrix} q^{-\alpha\cdot l_1}\delta_\alpha^\gamma & 0 & t_{13}x^{2/3}\delta_\alpha^{\gamma+\alpha_0} \\ t_{21}x^{2/3}\delta_\alpha^{\gamma+\alpha_1} & t_{22}q^{-\alpha\cdot l_2}\delta_\alpha^\gamma & 0 \\ 0 & t_{32}x^{2/3}\delta_\alpha^{\gamma+\alpha_2} & t_{33}q^{-\alpha\cdot l_3}\delta_\alpha^\gamma \end{pmatrix}. \tag{4.14}$$

Finally, the anti-solitonic T-matrix with the clockwise permutation is:

$$T_{IV} = \begin{pmatrix} q^{\alpha\cdot l_1}\delta_\alpha^\gamma & t_{12}x^{-2/3}\delta_\alpha^{\gamma-\alpha_1} & 0 \\ 0 & t_{22}q^{\alpha\cdot l_2}\delta_\alpha^\gamma & t_{23}x^{-2/3}\delta_\alpha^{\gamma-\alpha_2} \\ t_{31}x^{-2/3}\delta_\alpha^{\gamma-\alpha_0} & 0 & t_{33}q^{\alpha\cdot l_3}\delta_\alpha^\gamma \end{pmatrix}. \tag{4.15}$$

Note that the Kronecker-deltas again allow us to track any exchange of charge during the soliton's interaction with the defect; albeit with slightly different labels: α, γ refer to the $a_2^{(1)}$ weights and $\alpha_1, \alpha_2, \alpha_0$ refer to the extended simple roots of $a_2^{(1)}$ with the lowest root $\alpha_0 = -(\alpha_1 + \alpha_2)$. It is helpful to rewrite the roots $\alpha_0, \alpha_1, \alpha_2$ and fundamental weights in terms of the standard orthonormal base vectors, $\{e_i\}$, $i = 1, 2, 3$, satisfying $e_i \cdot e_j = \delta_{ij}$:

$$l_1 = \frac{1}{3}(2e_1 - e_2 - e_3), \qquad l_2 = \frac{1}{3}(2e_2 - e_1 - e_3), \qquad l_3 = \frac{1}{3}(2e_3 - e_1 - e_2),$$

this allows us to evaluate any dot product of the simple roots easily, due to the constraint on the components $(\gamma_1, \gamma_2, \gamma_3)$ of the weights: $\gamma_1 + \gamma_2 + \gamma_3 = 0$, $\gamma \cdot l_i = \gamma_i$. These relations are very useful and ensure that we obtain compact expressions for the generalised reflection matrices, which we will see later.

4.1.3 Pictorial Representations of known Reflection and Transmission matrices

It is instructive to associate a pictorial representation to each known reflection and transmission matrix. In the following section, this allows us to observe the importance of the defect, and elucidate the generalised solutions. We have already detailed the way that transmission matrices track exchanges of topological charge and we will now view the boundary in a similar manner.

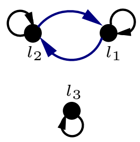

Figure 4.1: Pictorial representation of reflection matrix R_{12}.

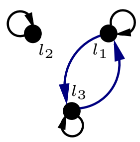

Figure 4.2: Pictorial representation of reflection matrix R_{13}.

As incoming solitons must reflect from the boundary as solitons, we see that the soliton can only remain as it is, or change to another weight within that same representation. The zeroes appearing in the reflection and transmission matrices represent the fact that movement between the weights of the solitonic/anti-solitonic representation is restricted. Figures (4.1), (4.2) and (4.3) illustrate this behaviour.

An example of this phenomenon is the T_{III}-matrix, it is illustrated in figure (4.4). However, some transmission matrices do not contain zeroes. In this

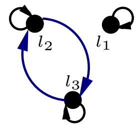

Figure 4.3: Pictorial representation of reflection matrix R_{23}.

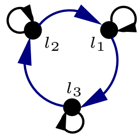

Figure 4.4: Pictorial representation of transmission matrix T_{III}.

case, the soliton possesses maximum freedom, one such example is the T_I-matrix and this is illustrated in figure (4.5). We will see the importance of these illustrations in the next section.

4.2 Construction of Type - I Generalised Reflection Matrices

We are now familiar with the construction of generalised reflection matrices. However, the solutions in this case appear different, because the topological charges are the fundamental weights: l_1, l_2 and l_3. The transmission matrices' dependence upon the topological charge, now denoted by the weights, is the crucial ingredient that transforms the diagonal reflection factor to an infinite-dimensional object. In this section, we will see that the generalised objects possess a different structure to the reflection matrices introduced in section 4.1.1. By using specific limits of the defect parameters, it is possible to reduce the generalised solutions so that the possess same shape as the finite-dimensional cases.

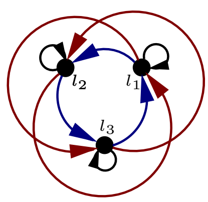

Figure 4.5: Pictorial representation of transmission matrix T_I.

In order to construct the new solutions we need to calculate the $\tilde{T}$-matrices, which requires the inversion of all matrices $T_I, T_{II}, T_{III}, T_{IV}$. Inversion formulae are easily derived and they are very similar to the $a_2^{(2)}$ case, except the entries are now shifted by the extended simple roots. The $\tilde{T}$-matrices are as follows:

$$\tilde{T}_I(\theta) = \frac{\Sigma^1(x^{-1})}{\Delta(x^{-1})} \begin{pmatrix} \tilde{T}_{11}\delta_\gamma^\beta & 0 & \tilde{T}_{13}\delta_\gamma^{\beta+\alpha_0} \\ \tilde{T}_{21}\delta_\gamma^{\beta+\alpha_1} & \tilde{T}_{22}\delta_\gamma^\beta & 0 \\ 0 & \tilde{T}_{32}\delta_\gamma^{\beta+\alpha_2} & \tilde{T}_{33}\delta_\gamma^\beta \end{pmatrix}, \tag{4.16}$$

with determinant $\Delta(x) := \frac{q^2}{t_{11}t_{22}t_{33}}(t_{11}t_{22}t_{33} - t_{13}t_{32}t_{21}x^2q^{-1})^2$, coefficient $\Sigma^1(x) := (t_{11}t_{22}t_{33} - t_{13}t_{32}t_{21}x^2q^{-1})$, and entries:

$$\tilde{T}_{11} = \frac{q^{\gamma\cdot l_1+2}}{t_{11}}, \qquad \tilde{T}_{13} = -\frac{t_{13}}{t_{11}t_{33}}x^{-2/3}q, \qquad \tilde{T}_{21} = -\frac{t_{21}}{t_{11}t_{22}}x^{-2/3}q,$$
$$\tilde{T}_{22} = \frac{q^{\gamma\cdot l_2+2}}{t_{22}}, \qquad \tilde{T}_{32} = -\frac{t_{32}}{t_{22}t_{33}}x^{-2/3}q, \qquad \tilde{T}_{33} = \frac{q^{\gamma\cdot l_3+2}}{t_{33}}.$$

$$\tilde{T}_{II}(\theta) = \frac{\Sigma^2(x^{-1})}{\Delta(x^{-1})} \begin{pmatrix} \tilde{T}_{11}\delta_\gamma^\beta & \tilde{T}_{12}\delta_\gamma^{\beta-\alpha_1} & 0 \\ 0 & \tilde{T}_{22}\delta_\gamma^\beta & \tilde{T}_{23}\delta_\gamma^{\beta-\alpha_2} \\ \tilde{T}_{31}\delta_\gamma^{\beta-\alpha_0} & 0 & \tilde{T}_{33}\delta_\gamma^\beta \end{pmatrix}, \tag{4.17}$$

with the determinant $\Delta(x) := \frac{q^{-2}}{t_{22}t_{33}}(t_{22}t_{33} - t_{12}t_{23}t_{31}x^{-2}q)^2$, coefficient $\Sigma^2(x) := (t_{22}t_{33} - t_{12}t_{23}t_{31}x^{-2}q)$, and entries

$$\tilde{T}_{11} = q^{-\gamma\cdot l_1-2}, \qquad \tilde{T}_{12} = -\frac{t_{12}}{t_{22}}x^{2/3}q^{-1}, \qquad \tilde{T}_{22} = \frac{q^{-\gamma\cdot l_2-2}}{t_{22}},$$
$$\tilde{T}_{23} = -\frac{t_{23}}{t_{22}t_{33}}x^{2/3}q^{-1}, \quad \tilde{T}_{31} = -\frac{t_{31}}{t_{33}}x^{2/3}q^{-1}, \quad \tilde{T}_{33} = \frac{q^{-\gamma\cdot l_3-2}}{t_{33}}.$$

$$\tilde{T}_{III}(\theta) = \frac{1}{\Delta(x^{-1})} \begin{pmatrix} \tilde{T}_{11}\delta_\gamma^\beta & \tilde{T}_{12}\delta_\gamma^{\beta-\alpha_1} & \tilde{T}_{13}\delta_\gamma^{\beta+\alpha_0} \\ \tilde{T}_{21}\delta_\gamma^{\beta+\alpha_1} & \tilde{T}_{22}\delta_\gamma^\beta & \tilde{T}_{23}\delta_\gamma^{\beta-\alpha_2} \\ \tilde{T}_{31}\delta_\gamma^{\beta-\alpha_0} & \tilde{T}_{32}\delta_\gamma^{\beta+\alpha_2} & \tilde{T}_{33}\delta_\gamma^\beta \end{pmatrix}, \tag{4.18}$$

with determinant $\Delta(x) := t_{22}t_{33}q^2 + t_{13}t_{32}t_{21}x^2$, and entries:

$$\tilde{T}_{11} = t_{22}t_{33}q^{\gamma\cdot l_1+2}, \quad \tilde{T}_{12} = t_{13}t_{32}x^{-4/3}, \quad \tilde{T}_{13} = -t_{22}t_{13}x^{-2/3}q^{-\gamma\cdot l_2+1},$$
$$\tilde{T}_{21} = -t_{33}t_{21}x^{-2/3}q^{-\gamma\cdot l_3+1}, \quad \tilde{T}_{22} = t_{33}q^{\gamma\cdot l_2+2}, \quad \tilde{T}_{23} = t_{21}t_{13}x^{-4/3},$$

$$\tilde{T}_{31} = t_{32}t_{21}x^{-4/3}, \quad \tilde{T}_{32} = -t_{32}x^{-2/3}q^{-\alpha\cdot l_1+1}, \quad \tilde{T}_{33} = t_{22}q^{\gamma\cdot l_3+2}.$$

$$\tilde{T}_{IV}(\theta) = \frac{1}{\Delta(x^{-1})}\begin{pmatrix} \tilde{T}_{11}\delta_\gamma^\beta & \tilde{T}_{12}\delta_\gamma^{\beta-\alpha_1} & \tilde{T}_{13}\delta_\gamma^{\beta+\alpha_0} \\ \tilde{T}_{21}\delta_\gamma^{\beta+\alpha_1} & \tilde{T}_{22}\delta_\gamma^\beta & \tilde{T}_{23}\delta_\gamma^{\beta-\alpha_2} \\ \tilde{T}_{31}\delta_\gamma^{\beta-\alpha_0} & \tilde{T}_{32}\delta_\gamma^{\beta+\alpha_2} & \tilde{T}_{33}\delta_\gamma^\beta \end{pmatrix}, \tag{4.19}$$

with determinant $\Delta(x) := t_{22}t_{33}q^{-2} + t_{12}t_{23}t_{31}x^{-2}$ and entries:

$$\tilde{T}_{11} = t_{22}t_{33}q^{-\gamma\cdot l_1-2}, \quad \tilde{T}_{12} = -t_{33}t_{12}x^{2/3}q^{\gamma\cdot l_3-1}, \quad \tilde{T}_{13} = t_{12}t_{23}x^{4/3},$$
$$\tilde{T}_{21} = t_{23}t_{31}x^{4/3}, \quad \tilde{T}_{22} = t_{33}q^{-\gamma\cdot l_2-2}, \quad \tilde{T}_{23} = -t_{23}x^{2/3}q^{\gamma\cdot l_1-1},$$
$$\tilde{T}_{31} = -t_{31}t_{22}x^{2/3}q^{\gamma\cdot l_2-1}, \quad \tilde{T}_{32} = t_{31}t_{12}x^{4/3}, \quad \tilde{T}_{33} = t_{22}q^{-\gamma\cdot l_3-2}.$$

The transmission matrices are now used to dress the first diagonal reflection matrix, R_{d_0}. The generalised solutions are very peculiar, but the defect's significance is readily observed. We will label the most simple generalised solutions $\hat{R}_i$, $i = I, II, III, IV$, to denote the transmission matrices used in their construction. The solutions are as follows:

$$\hat{R}_I = \begin{pmatrix} \hat{r}_{11}\delta_\alpha^\beta & 0 & \hat{r}_{13}\delta_\alpha^{\beta+\alpha_0} \\ 0 & \hat{r}_{22}\delta_\alpha^\beta & \hat{r}_{23}\delta_\alpha^{\beta-\alpha_2} \\ 0 & 0 & \hat{r}_{33}\delta_\alpha^\beta \end{pmatrix}, \tag{4.20}$$

with entries:

$$\begin{aligned} \hat{r}_{11} &= \frac{1}{\Sigma^1(x^{-1})}\left(t_{11}t_{22}t_{33} - t_{13}t_{32}t_{21}x^2q^{-1}\right), \\ \hat{r}_{13} &= \frac{1}{\Sigma^1(x^{-1})}\left(t_{11}t_{22}t_{13}x^{-2/3}q^{\alpha\cdot l_1-1}(x^4-1)\right), \end{aligned} \tag{4.21}$$

$$\begin{aligned} \hat{r}_{22} &= \frac{1}{\Sigma^1(x^{-1})}\left(x^{4/3}\left(t_{11}t_{22}t_{33} - t_{13}t_{32}t_{21}x^2q^{-1}\right)\right), \\ \hat{r}_{23} &= \frac{1}{\Sigma^1(x^{-1})}\left(t_{22}t_{21}t_{13}q^{\alpha\cdot l_3-1}(x^4-1)\right), \\ \hat{r}_{33} &= \frac{1}{\Sigma^1(x^{-1})}\left(x^{2/3}\left(t_{11}t_{22}t_{33}x^2 - t_{13}t_{32}t_{21}q^{-1}\right)\right), \end{aligned} \tag{4.22}$$

$$\hat{R}_{II} = \begin{pmatrix} \hat{r}_{11}\delta_\alpha^\beta & 0 & 0 \\ \hat{r}_{21}\delta_\alpha^{\beta+\alpha_1} & \hat{r}_{22}\delta_\alpha^\beta & 0 \\ \hat{r}_{31}\delta_\alpha^{\beta-\alpha_0} & 0 & \hat{r}_{33}\delta_\alpha^\beta \end{pmatrix}, \tag{4.23}$$

with entries:

$$\begin{aligned}
\hat{r}_{11} &= \frac{1}{\Sigma^2(x^{-1})}\left(t_{22}t_{33} - t_{12}t_{23}t_{31}x^2 q\right), \\
\hat{r}_{21} &= -\frac{1}{\Sigma^2(x^{-1})}\left(t_{22}t_{23}t_{31}x^{-4/3}q^{-\alpha\cdot l_1+1}(x^4-1)\right), \\
\hat{r}_{22} &= \frac{1}{\Sigma^2(x^{-1})}\left(x^{-2/3}\left(t_{22}t_{33}x^2 - t_{12}t_{23}t_{31}q\right)\right), \\
\hat{r}_{31} &= -\frac{1}{\Sigma^2(x^{-1})}\left(t_{22}t_{33}t_{31}x^{-2/3}q^{\alpha\cdot l_3+1}(x^4-1)\right), \\
\hat{r}_{33} &= \frac{1}{\Sigma^2(x^{-1})}\left(x^{2/3}\left(t_{22}t_{33}x^2 - t_{12}t_{23}t_{31}q\right)\right),
\end{aligned} \tag{4.24}$$

$$\hat{R}_{III} = \begin{pmatrix} \hat{r}_{11}\delta_\alpha^\beta & \hat{r}_{12}\delta_\alpha^{\beta-\alpha_1} & \hat{r}_{13}\delta_\alpha^{\beta+\alpha_0} \\ 0 & \hat{r}_{22}\delta_\alpha^\beta & 0 \\ 0 & 0 & \hat{r}_{33}\delta_\alpha^\beta \end{pmatrix}, \tag{4.25}$$

with entries:

$$\begin{aligned}
\hat{r}_{11} &= \frac{1}{\Delta(x^{-1})}\left(t_{22}t_{33}q^2 + t_{13}t_{32}t_{21}x^2\right), \\
\hat{r}_{12} &= -\frac{1}{\Delta(x^{-1})}\left(t_{13}t_{32}x^{-4/3}q^{-\alpha\cdot l_1}(x^4-1)\right), \\
\hat{r}_{13} &= \frac{1}{\Delta(x^{-1})}\left(t_{22}t_{13}x^{-2/3}q^{\alpha\cdot l_3+1}(x^4-1)\right), \\
\hat{r}_{22} &= \frac{1}{\Delta(x^{-1})}\left(x^{-2/3}\left(t_{22}t_{33}q^2x^2 - t_{21}t_{13}t_{32}\right)\right), \\
\hat{r}_{33} &= \frac{1}{\Delta(x^{-1})}\left(x^{2/3}\left(t_{22}t_{33}q^2x^2 + t_{32}t_{21}t_{13}q\right)\right),
\end{aligned} \tag{4.26}$$

$$\hat{R}_{IV} = \begin{pmatrix} \hat{r}_{11}\delta_\alpha^\beta & 0 & 0 \\ 0 & \hat{r}_{22}\delta_\alpha^\beta & 0 \\ \hat{r}_{31}\delta_\alpha^{\beta-\alpha_0} & \hat{r}_{32}\delta_\alpha^{\beta+\alpha_2} & \hat{r}_{33}\delta_\alpha^\beta \end{pmatrix}, \tag{4.27}$$

with entries:

$$
\begin{aligned}
\hat{r}_{11} &= \frac{1}{\Delta(x^{-1})}\left(t_{22}t_{33}q^{-2} + t_{12}t_{23}t_{31}x^{2}\right), \\
\hat{r}_{22} &= \frac{1}{\Delta(x^{-1})}\left(x^{4/3}\left(t_{22}t_{33}q^{-2} - t_{23}t_{31}t_{12}x^{2}\right)\right), \\
\hat{r}_{31} &= -\frac{1}{\Delta(x^{-1})}\left(t_{22}t_{33}t_{31}x^{-2/3}q^{-\alpha\cdot l_2 - 1}(x^{4}-1)\right), \\
\hat{r}_{32} &= \frac{1}{\Delta(x^{-1})}\left(t_{33}t_{31}t_{12}q^{\alpha\cdot l_3}(x^{4}-1)\right), \\
\hat{r}_{33} &= \frac{1}{\Delta(x^{-1})}\left(x^{2/3}\left(t_{22}t_{33}q^{-2}x^{2} + t_{31}t_{12}t_{23}\right)\right).
\end{aligned}
\tag{4.28}
$$

Despite the very strange appearance of the above solutions, $\hat{R}_i$, an interesting pattern arises. If we label each entry of the generalised reflection matrix as follows:

$$
\hat{R}_i = \begin{pmatrix} ++ & +0 & +- \\ 0+ & 00 & 0- \\ -+ & -0 & -- \end{pmatrix},
$$

where $+$ corresponds to weight l_1, 0 corresponds to weight l_2, and so forth. The defect appears to modify a diagonal reflection factor, with no associated free parameter, by allowing the soliton to convert to a limited selection of the adjacent weights. Furthermore, note the similarity between $\hat{R}_I, \hat{R}_{IV}$, the first is built from the T_I matrix (solitonic representation with clockwise permutation), the second is built from the T_{IV} matrix (anti-solitonic representation with clockwise permutation). From the classical type - I set-up, we expect that the soliton and anti-soliton interact with the defect differently. Clearly,

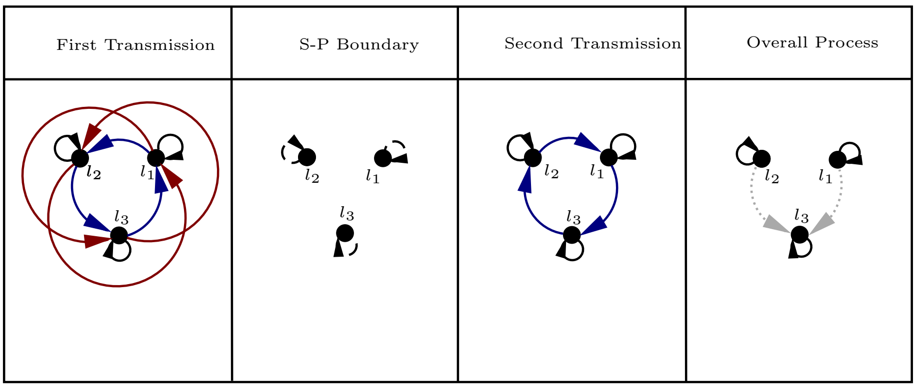

Figure 4.6: Pictorial representation of generalised solution $\hat{R}_I$.

this is an exhibition of that feature in the quantum theory. As always, the permutation specifies the defect, but the soliton and anti-soliton still interact with the defect differently. It is readily observed that $\hat{R}_I$ details a process where the incident solitons of charges: l_1 and l_2, can convert to their neighbour; of charge l_3. However, $\hat{R}_{IV}$ details a process where the incident soliton of charge l_3 can convert to either of the adjacent solitons. Similarly, for $\hat{R}_{II}$, any incident soliton is able to convert to its neighbour of charge l_1. And for $\hat{R}_{III}$, the incident soliton of charge l_1 is free to convert to either neighbour. Note that none of the solutions concern the soliton of charge l_2 - the classically forbidden particle. The first generalised solution is represented in figure (4.6), the remaining three are found in appendix (A).

We will now repeat the process for the diagonal reflection factors containing a boundary parameter, observing yet more interesting behaviour. Beginning with R_{d_1} (4.2) and dressing it with the T_I-matrices, we find a solution of the form:

$$\tilde{R}_I(\theta) = \begin{pmatrix} \tilde{r}_{11}\delta_\alpha^\beta & \tilde{r}_{12}\delta_\alpha^{\beta-\alpha_1} & \tilde{r}_{13}\delta_\alpha^{\beta+\alpha_0} \\ 0 & \tilde{r}_{22}\delta_\alpha^\beta & \tilde{r}_{23}\delta_\alpha^{\beta-\alpha_2} \\ 0 & \tilde{r}_{32}\delta_\alpha^{\beta+\alpha_2} & \tilde{r}_{33}\delta_\alpha^\beta \end{pmatrix}, \tag{4.29}$$

with entries:

$$\begin{aligned}
\tilde{r}_{11} &= \frac{1}{\Sigma^1(x^{-1})}\left((x^2\upsilon - \upsilon^{-1})(t_{11}t_{22}t_{33} - t_{13}t_{32}t_{21}x^2q^{-1})\right), \\
\tilde{r}_{12} &= \frac{1}{\Sigma^1(x^{-1})}\left(\upsilon t_{11}t_{13}t_{32}x^{2/3}q^{\alpha\cdot l_2-1}(x^4-1)\right), \\
\tilde{r}_{13} &= \frac{1}{\Sigma^1(x^{-1})}\left(\upsilon^{-1}t_{11}t_{13}t_{22}x^{-2/3}q^{-\alpha\cdot l_1-1}(1-x^4)\right), \\
\tilde{r}_{22} &= \frac{1}{\Sigma^1(x^{-1})}\left(x^{4/3}\left(t_{11}t_{22}t_{33}(x^2\upsilon-\upsilon^{-1}) - t_{21}t_{13}t_{32}(\upsilon - x^2\upsilon^{-1})q^{-1}\right)\right), \\
\tilde{r}_{23} &= \frac{1}{\Sigma^1(x^{-1})}\left(\upsilon^{-1}t_{22}t_{21}t_{13}q^{\alpha\cdot l_3-1}(1-x^4)\right), \\
\tilde{r}_{32} &= \frac{1}{\Sigma^1(x^{-1})}\left(\upsilon t_{11}t_{33}t_{32}q^{-\alpha\cdot l_3-1}(x^4-1)\right), \\
\tilde{r}_{33} &= \frac{1}{\Sigma^1(x^{-1})}\left(x^{2/3}\left(t_{11}t_{22}t_{33}(\upsilon - x^2\upsilon^{-1}) - t_{13}t_{32}t_{21}(x^2\upsilon-\upsilon^{-1})q^{-1}\right)\right),
\end{aligned}$$

with $\Sigma^1(x^{-1}) := (t_{11}t_{22}t_{33} - t_{13}t_{32}t_{21}x^{-2}q^{-1})$.

Secondly, if we dress R_{d_1} with the T_{II}-matrices we obtain a different solution,

$$\tilde{R}_{II}(\theta) = \begin{pmatrix} \tilde{r}_{11}\delta_\alpha^\beta & 0 & \tilde{r}_{13}\delta_\alpha^{\beta+\alpha_0} \\ \tilde{r}_{21}\delta_\alpha^{\beta+\alpha_1} & \tilde{r}_{22}\delta_\alpha^\beta & \tilde{r}_{23}\delta_\alpha^{\beta-\alpha_2} \\ \tilde{r}_{31}\delta_\alpha^{\beta-\alpha_0} & 0 & \tilde{r}_{33}\delta_\alpha^\beta \end{pmatrix}, \tag{4.30}$$

the entries are:

$$\begin{aligned}
\tilde{r}_{11} &= \frac{1}{\Sigma^2(x^{-1})}\left((x^2\upsilon - \upsilon^{-1})t_{22}t_{33} - t_{12}t_{23}t_{31}q(\upsilon - x^2\upsilon^{-1})\right), \\
\tilde{r}_{13} &= \frac{1}{\Sigma^2(x^{-1})}\left(\upsilon t_{12}t_{23}x^{-2/3}q^{-\alpha\cdot l_3+1}(1-x^4)\right), \\
\tilde{r}_{21} &= \frac{1}{\Sigma^2(x^{-1})}\left(\upsilon^{-1}t_{22}t_{23}t_{31}x^{-4/3}q^{-\alpha\cdot l_1+1}(x^4-1)\right), \\
\tilde{r}_{22} &= \frac{1}{\Sigma^2(x^{-1})}\left(x^{4/3}(x^2\upsilon - \upsilon^{-1})(t_{22}t_{33} - t_{12}t_{23}t_{31}x^{-2}q)\right), \\
\tilde{r}_{23} &= \frac{1}{\Sigma^2(x^{-1})}\left(\upsilon t_{22}t_{23}q^{\alpha\cdot l_2+1}(1-x^4)\right), \\
\tilde{r}_{31} &= \frac{1}{\Sigma^2(x^{-1})}\left(\upsilon^{-1}t_{22}t_{33}t_{31}x^{-2/3}q^{\alpha\cdot l_3+1}(x^4-1)\right), \\
\tilde{r}_{33} &= \frac{1}{\Sigma^2(x^{-1})}\left(x^{2/3}\left(t_{22}t_{33}(\upsilon - x^2\upsilon^{-1}) - t_{12}t_{23}t_{31}(x^2\upsilon - \upsilon^{-1})q\right)\right),
\end{aligned}$$

with $\Sigma^2(x^{-1}) := (t_{22}t_{33} - t_{12}t_{23}t_{31}x^2q)$.

Thirdly, by evaluating the multiplication (1.95) with the T_{III}-matrices we again discover a new solution with different characteristics:

$$\tilde{R}_{III}(\theta) = \begin{pmatrix} \tilde{r}_{11}\delta_\alpha^\beta & \tilde{r}_{12}\delta_\alpha^{\beta-\alpha_1} & \tilde{r}_{13}\delta_\alpha^{\beta+\alpha_0} \\ 0 & \tilde{r}_{22}\delta_\alpha^\beta & 0 \\ \tilde{r}_{31}\delta_\alpha^{\beta-\alpha_0} & \tilde{r}_{32}\delta_\alpha^{\beta+\alpha_2} & \tilde{r}_{33}\delta_\alpha^\beta \end{pmatrix}, \tag{4.31}$$

the entries are:

$$\begin{aligned}
\tilde{r}_{11} &= \frac{1}{\Delta(x^{-1})}\left((x^2\upsilon - \upsilon^{-1})t_{22}t_{33}q^2 + t_{13}t_{32}t_{21}(\upsilon - x^2\upsilon^{-1})\right), \\
\tilde{r}_{12} &= \frac{1}{\Delta(x^{-1})}\left(\upsilon^{-1}t_{13}t_{32}x^{-4/3}q^{-\alpha\cdot l_1}(x^4-1)\right), \\
\tilde{r}_{13} &= \frac{1}{\Delta(x^{-1})}\left(\upsilon^{-1}t_{13}t_{22}x^{-2/3}q^{\alpha\cdot l_3+1}(1-x^4)\right), \\
\tilde{r}_{22} &= \frac{1}{\Delta(x^{-1})}\left(x^{4/3}(x^2\upsilon - \upsilon^{-1})(t_{22}t_{33}q^2 + t_{13}t_{32}t_{21}x^{-2})\right),
\end{aligned}$$

$$\tilde{r}_{31} = \frac{1}{\Delta(x^{-1})}\left(\upsilon t_{33}t_{32}t_{21}x^{-2/3}q^{-\alpha\cdot l_3}(1-x^4)\right),$$
$$\tilde{r}_{32} = \frac{1}{\Delta(x^{-1})}\left(\upsilon t_{33}t_{32}q^{\alpha\cdot l_2+1}(x^4-1)\right),$$
$$\tilde{r}_{33} = \frac{1}{\Delta(x^{-1})}\left(x^{2/3}\left(t_{22}t_{33}q^2(\upsilon-x^2\upsilon^{-1})+t_{13}t_{32}t_{21}(x^2\upsilon-\upsilon^{-1})\right)\right).$$

Continuing in this fashion, now using the T_{IV}-matrices we find another new solution:

$$\tilde{R}_{IV}(\theta) = \begin{pmatrix} \tilde{r}_{11}\delta_\alpha^\beta & 0 & 0 \\ \tilde{r}_{21}\delta_\alpha^{\beta+\alpha_1} & \tilde{r}_{22}\delta_\alpha^\beta & \tilde{r}_{23}\delta_\alpha^{\beta-\alpha_2} \\ \tilde{r}_{31}\delta_\alpha^{\beta-\alpha_0} & \tilde{r}_{32}\delta_\alpha^{\beta+\alpha_2} & \tilde{r}_{33}\delta_\alpha^\beta \end{pmatrix}, \tag{4.32}$$

with entries:

$$\tilde{r}_{11} = \frac{1}{\Delta(x^{-1})}\left((x^2\upsilon-\upsilon^{-1})(t_{22}t_{33}q^{-2}+t_{12}t_{23}t_{31}x^2\right),$$
$$\tilde{r}_{21} = \frac{1}{\Delta(x^{-1})}\left(\upsilon t_{22}t_{23}t_{31}x^{2/3}q^{\alpha\cdot l_2}(x^4-1)\right),$$
$$\tilde{r}_{22} = \frac{1}{\Delta(x^{-1})}\left(x^{4/3}\left(t_{22}t_{33}q^{-2}(x^2\upsilon-\upsilon^{-1})+t_{23}t_{31}t_{12}(\upsilon-x^2\upsilon^{-1})\right)\right),$$
$$\tilde{r}_{23} = \frac{1}{\Delta(x^{-1})}\left(\upsilon t_{22}t_{23}q^{-\alpha\cdot l_3-1}(1-x^4)\right),$$
$$\tilde{r}_{31} = \frac{1}{\Delta(x^{-1})}\left(\upsilon^{-1}t_{22}t_{33}t_{31}x^{-2/3}q^{-\alpha\cdot l_1-1}(x^4-1)\right),$$
$$\tilde{r}_{32} = \frac{1}{\Delta(x^{-1})}\left(\upsilon^{-1}t_{33}t_{31}t_{12}q^{\alpha\cdot l_3}(1-x^4)\right),$$
$$\tilde{r}_{33} = \frac{1}{\Delta(x^{-1})}\left(x^{2/3}\left(t_{22}t_{33}q^{-2}(\upsilon-x^2\upsilon^{-1})+t_{23}t_{31}t_{12}(x^2\upsilon-\upsilon^{-1})\right)\right).$$

We will now form another collection of generalised solutions by dressing R_{d_2} with all T-matrices. The combination of the second diagonal solution and the T_I-matrices returns a solution, $\tilde{R}_V$, of the same shape as $\tilde{R}_{II}$, with entries:

$$\tilde{r}_{11} = \frac{1}{\Sigma^1(x^{-1})}\left(t_{11}t_{22}t_{33}(x^2\upsilon-\upsilon^{-1})-t_{13}t_{32}t_{21}q^{-1}(\upsilon-x^2\upsilon^{-1})\right),$$
$$\tilde{r}_{13} = \frac{1}{\Sigma^1(x^{-1})}\left(\upsilon^{-1}t_{11}t_{13}t_{22}x^{-2/3}q^{-\alpha\cdot l_1-1}(1-x^4)\right),$$
$$\tilde{r}_{21} = \frac{1}{\Sigma^1(x^{-1})}\left(\upsilon t_{22}t_{33}t_{21}x^{-4/3}q^{-\alpha\cdot l_2-1}(x^4-1)\right),$$
$$\tilde{r}_{22} = \frac{1}{\Sigma^1(x^{-1})}\left(x^{-2/3}\left(\upsilon-x^2\upsilon^{-1}\right)\left(t_{11}t_{22}t_{33}-t_{21}t_{13}t_{32}x^2q^{-1}\right)\right),$$

$$\tilde{r}_{23} = \frac{1}{\Sigma^1(x^{-1})}\left(\upsilon^{-1}t_{22}t_{21}t_{13}q^{\alpha\cdot l_3-1}(1-x^4)\right),$$
$$\tilde{r}_{31} = \frac{1}{\Sigma^1(x^{-1})}\left(\upsilon t_{33}t_{32}t_{21}x^{-2/3}q^{\alpha\cdot l_1-1}(x^4-1)\right),$$
$$\tilde{r}_{33} = \frac{1}{\Sigma^1(x^{-1})}\left(x^{2/3}\left(t_{11}t_{22}t_{33}(\upsilon-x^2\upsilon^{-1})-t_{13}t_{32}t_{21}(x^2\upsilon-\upsilon^{-1})q^{-1}\right)\right).$$

Another type of solution is uncovered when dressing R_{d_2} with the T_{II}-matrices, it has shape:

$$\tilde{R}_{VI}(\theta) = \begin{pmatrix} \tilde{r}_{11}\delta_\alpha^\beta & \tilde{r}_{12}\delta_\alpha^{\beta-\alpha_1} & 0 \\ \tilde{r}_{21}\delta_\alpha^{\beta+\alpha_1} & \tilde{r}_{22}\delta_\alpha^\beta & 0 \\ \tilde{r}_{31}\delta_\alpha^{\beta-\alpha_0} & \tilde{r}_{32}\delta_\alpha^{\beta+\alpha_2} & \tilde{r}_{33}\delta_\alpha^\beta \end{pmatrix}, \tag{4.33}$$

and entries:

$$\tilde{r}_{11} = \frac{1}{\Sigma^2(x^{-1})}\left(\left(t_{22}t_{33}(x^2\upsilon-\upsilon^{-1})-t_{31}t_{12}t_{23}q(\upsilon-x^2\upsilon^{-1})\right)\right),$$
$$\tilde{r}_{12} = \frac{1}{\Sigma^2(x^{-1})}\left(\upsilon t_{33}t_{12}x^{-4/3}q^{\alpha\cdot l_1+1}(1-x^4)\right),$$
$$\tilde{r}_{21} = \frac{1}{\Sigma^2(x^{-1})}\left(\upsilon^{-1}t_{22}t_{23}t_{31}x^{-4/3}q^{-\alpha\cdot l_1+1}(x^4-1)\right),$$
$$\tilde{r}_{22} = \frac{1}{\Sigma^2(x^{-1})}\left(x^{-2/3}\left(t_{22}t_{33}(\upsilon-x^2\upsilon^{-1})-t_{12}t_{23}t_{31}q(x^2\upsilon-\upsilon^{-1})\right)\right),$$
$$\tilde{r}_{31} = \frac{1}{\Sigma^2(x^{-1})}\left(\upsilon^{-1}t_{22}t_{33}t_{31}x^{-2/3}q^{\alpha\cdot l_3+1}(x^4-1)\right),$$
$$\tilde{r}_{32} = \frac{1}{\Sigma^2(x^{-1})}\left(\upsilon t_{33}t_{12}t_{31}x^{-2}q^{-\alpha\cdot l_2+1}(1-x^4)\right),$$
$$\tilde{r}_{33} = \frac{1}{\Sigma^2(x^{-1})}\left(x^{-4/3}(\upsilon-x^2\upsilon^{-1})\left(t_{22}t_{33}x^2-t_{12}t_{23}t_{31}q\right)\right).$$

Yet another new variety of solution, R_{VII}, is discovered by dressing R_{d_2} with the T_{III}-matrices:

$$\tilde{R}_{VII}(\theta) = \begin{pmatrix} \tilde{r}_{11}\delta_\alpha^\beta & \tilde{r}_{12}\delta_\alpha^{\beta-\alpha_1} & \tilde{r}_{13}\delta_\alpha^{\beta+\alpha_0} \\ \tilde{r}_{21}\delta_\alpha^{\beta+\alpha_1} & \tilde{r}_{22}\delta_\alpha^\beta & \tilde{r}_{23}\delta_\alpha^{\beta-\alpha_2} \\ 0 & 0 & \tilde{r}_{33}\delta_\alpha^\beta \end{pmatrix}, \tag{4.34}$$

its entries are:

$$\tilde{r}_{11} = \frac{1}{\Delta(x^{-1})}\left(t_{22}t_{33}q^2(x^2\upsilon-\upsilon^{-1})+t_{13}t_{32}t_{21}(\upsilon-x^2\upsilon^{-1})\right),$$

$$\tilde{r}_{12} = \frac{1}{\Delta(x^{-1})}\left(\upsilon^{-1}t_{13}t_{32}x^{-4/3}q^{-\alpha\cdot l_1}(x^4-1)\right),$$
$$\tilde{r}_{13} = \frac{1}{\Delta(x^{-1})}\left(\upsilon^{-1}t_{13}t_{22}x^{-2/3}q^{\alpha\cdot l_3+1}(1-x^4)\right),$$
$$\tilde{r}_{21} = \frac{1}{\Delta(x^{-1})}\left(\upsilon t_{22}t_{21}t_{33}x^{-4/3}q^{\alpha\cdot l_1+1}(x^4-1)\right),$$
$$\tilde{r}_{22} = \frac{1}{\Delta(x^{-1})}\left(x^{-2/3}\left(t_{22}t_{33}q^2(\upsilon-x^2\upsilon^{-1})+t_{21}t_{13}t_{32}(x^2\upsilon-\upsilon^{-1})\right)\right),$$
$$\tilde{r}_{23} = \frac{1}{\Delta(x^{-1})}\left(\upsilon t_{22}t_{21}t_{13}x^{-2}q^{-\alpha\cdot l_2}(1-x^4)\right),$$
$$\tilde{r}_{33} = \frac{1}{\Delta(x^{-1})}\left(x^{-4/3}(\upsilon-x^2\upsilon^{-1})(t_{22}t_{33}x^2q^2+t_{21}t_{13}t_{32})\right).$$

It now remains to dress R_{d_2} with the T_{IV}-matrices. The resulting solution, $\tilde{R}_{VIII}$ has the same shape as $\tilde{R}_{III}$ with entries:

$$\tilde{r}_{11} = \frac{1}{\Delta(x^{-1})}\left(t_{22}t_{33}q^{-2}(x^2\upsilon-\upsilon^{-1})+t_{12}t_{23}t_{31}(\upsilon-x^2\upsilon^{-1})\right),$$
$$\tilde{r}_{12} = \frac{1}{\Delta(x^{-1})}\left(\upsilon t_{33}t_{12}x^{-4/3}q^{-\alpha\cdot l_2-1}(1-x^4)\right),$$
$$\tilde{r}_{13} = \frac{1}{\Delta(x^{-1})}\left(\upsilon t_{12}t_{23}x^{-2/3}q^{\alpha\cdot l_1}(x^4-1)\right),$$
$$\tilde{r}_{22} = \frac{1}{\Delta(x^{-1})}\left(x^{-2/3}(\upsilon-x^2\upsilon^{-1})\left(t_{22}t_{33}q^{-2}+t_{12}t_{23}t_{31}x^2\right)\right),$$
$$\tilde{r}_{31} = \frac{1}{\Delta(x^{-1})}\left(\upsilon^{-1}t_{31}t_{22}t_{33}x^{-2/3}q^{-\alpha\cdot l_1-1}(x^4-1)\right),$$
$$\tilde{r}_{32} = \frac{1}{\Delta(x^{-1})}\left(\upsilon^{-1}t_{33}t_{31}t_{12}q^{\alpha\cdot l_3}(1-x^4)\right),$$
$$\tilde{r}_{33} = \frac{1}{\Delta(x^{-1})}\left(x^{2/3}\left(t_{22}t_{33}q^{-2}(\upsilon-x^2\upsilon^{-1})+t_{12}t_{23}t_{31}(x^2\upsilon-\upsilon^{-1})\right)\right).$$

We will now document the generalised solutions calculated by dressing the Lima-Santos diagonal solution, R_{d_3}. No new varieties of generalised reflection matrix are found, in fact we find that each structure: $\tilde{R}_I, \tilde{R}_{II}, \tilde{R}_{III}, \tilde{R}_{IV}, \tilde{R}_{VI}, \tilde{R}_{VII}$ appears twice. Dressing R_{d_3} with the T_I-matrices produces a generalised solution, $\tilde{R}_{IX}$, that has the same structure as $\tilde{R}_{VI}$, and entries:

$$\tilde{r}_{11} = \frac{1}{\Sigma^1(x^{-1})}\left(t_{11}t_{22}t_{33}(x^{-2}\upsilon-\upsilon^{-1})-t_{13}t_{32}t_{21}q^{-1}(\upsilon-x^{-2}\upsilon^{-1})\right),$$
$$\tilde{r}_{12} = \frac{1}{\Sigma^1(x^{-1})}\left(\upsilon t_{11}t_{13}t_{32}x^{-10/3}q^{\alpha\cdot l_2-1}(x^4-1)\right),$$

$$\tilde{r}_{21} = \frac{1}{\Sigma^1(x^{-1})}\left(\upsilon^{-1}t_{21}t_{22}t_{33}x^{-10/3}q^{-\alpha\cdot l_2-1}(1-x^4)\right),$$
$$\tilde{r}_{22} = \frac{1}{\Sigma^1(x^{-1})}\left(x^{-8/3}\left(t_{11}t_{22}t_{33}(x^2\upsilon-\upsilon^{-1})-t_{13}t_{32}t_{21}q^{-1}(\upsilon-x^2\upsilon^{-1})\right)\right),$$
$$\tilde{r}_{31} = \frac{1}{\Sigma^1(x^{-1})}\left(\upsilon^{-1}t_{33}t_{32}t_{21}x^{-8/3}q^{\alpha\cdot l_1-1}(1-x^4)\right),$$
$$\tilde{r}_{32} = \frac{1}{\Sigma^1(x^{-1})}\left(\upsilon t_{11}t_{33}t_{32}x^{-4}q^{-\alpha\cdot l_3-1}(x^4-1)\right),$$
$$\tilde{r}_{33} = \frac{1}{\Sigma^1(x^{-1})}\left(x^{-4/3}\left((x^{-2}\upsilon-\upsilon^{-1})(t_{11}t_{22}t_{33}-t_{13}t_{32}t_{21}x^2q^{-1})\right)\right).$$

Applying the T_{II}-matrices to R_{d_3} gives another solution, $\tilde{R}_X$, with structure $\tilde{R}_I$ and entries:

$$\tilde{r}_{11} = \frac{1}{\Sigma^2(x^{-1})}\left((x^{-2}\upsilon-\upsilon^{-1})(t_{22}t_{33}-t_{12}t_{23}t_{31}qx^{-2})\right),$$
$$\tilde{r}_{12} = \frac{1}{\Sigma^2(x^{-1})}\left(\upsilon^{-1}t_{12}t_{33}x^{-10/3}q^{\alpha\cdot l_1+1}(x^4-1)\right),$$
$$\tilde{r}_{13} = \frac{1}{\Sigma^2(x^{-1})}\left(\upsilon t_{12}t_{23}x^{-14/3}q^{-\alpha\cdot l_3+1}(1-x^4)\right),$$
$$\tilde{r}_{22} = \frac{1}{\Sigma^2(x^{-1})}\left(x^{-8/3}\left(t_{22}t_{33}(x^2\upsilon-\upsilon^{-1})-t_{23}t_{31}t_{12}x^2q(x^{-2}\upsilon-\upsilon^{-1})\right)\right),$$
$$\tilde{r}_{23} = \frac{1}{\Sigma^2(x^{-1})}\left(\upsilon t_{22}t_{23}x^{-4}q^{\alpha\cdot l_2+1}(1-x^4)\right),$$
$$\tilde{r}_{32} = \frac{1}{\Sigma^2(x^{-1})}\left(\upsilon^{-1}t_{33}t_{31}t_{12}x^{-4}q^{-\alpha\cdot l_2+1}(x^4-1)\right),$$
$$\tilde{r}_{33} = \frac{1}{\Sigma^2(x^{-1})}\left(x^{-4/3}\left(t_{22}t_{33}(x^{-2}\upsilon-\upsilon^{-1})-t_{31}t_{12}t_{23}qx^{-2}(x^2\upsilon-\upsilon^{-1})\right)\right).$$

Similarly, dressing R_{d_3} with the T_{III}-matrices we find a solution, $\tilde{R}_{XI}$ with the same shape as $\tilde{R}_{IV}$ and entries:

$$\tilde{r}_{11} = \frac{1}{\Delta(x^{-1})}\left((x^{-2}\upsilon-\upsilon^{-1})(t_{22}t_{33}q^2+t_{13}t_{32}t_{21}x^{-2}\right),$$
$$\tilde{r}_{21} = \frac{1}{\Delta(x^{-1})}\left(\upsilon^{-1}t_{22}t_{33}t_{21}x^{-10/3}q^{\alpha\cdot l_1+1}(1-x^4)\right),$$
$$\tilde{r}_{22} = \frac{x^{-8/3}}{\Delta(x^{-1})}\left(t_{22}t_{33}q^2(x^2\upsilon-\upsilon^{-1})+t_{21}t_{13}t_{32}x^2(x^{-2}\upsilon-\upsilon^{-1})\right),$$
$$\tilde{r}_{23} = \frac{1}{\Delta(x^{-1})}\left(\upsilon^{-1}t_{22}t_{21}t_{13}x^{-4}q^{-\alpha\cdot l_2}(x^4-1)\right),$$
$$\tilde{r}_{31} = \frac{1}{\Delta(x^{-1})}\left(\upsilon t_{33}t_{32}t_{21}x^{-14/3}q^{-\alpha\cdot l_3}(1-x^4)\right),$$

$$\tilde{r}_{32} = \frac{1}{\Delta(x^{-1})}\left(\upsilon t_{33}t_{32}x^{-4}q^{\alpha\cdot l_2+1}(x^4-1)\right),$$
$$\tilde{r}_{33} = \frac{x^{-4/3}}{\Delta(x^{-1})}\left(t_{22}t_{33}q^{2}(x^{-2}\upsilon-\upsilon^{-1})+t_{32}t_{21}t_{13}x^{-2}(x^{2}\upsilon-\upsilon^{-1})\right).$$

The final generalised reflection matrix, $\tilde{R}_{XII}$, is constructed by dressing R_{d_3} with the T_{IV}-matrices. In this case we find the solution has the same shape as $\tilde{R}_{VII}$ and entries:

$$\tilde{r}_{11} = \frac{1}{\Delta(x^{-1})}\left(t_{22}t_{33}q^{-2}(x^{-2}\upsilon-\upsilon^{-1})+t_{12}t_{23}t_{31}x^{-2}(x^{2}\upsilon-\upsilon^{-1})\right),$$
$$\tilde{r}_{12} = \frac{1}{\Delta(x^{-1})}\left(\upsilon^{-1}t_{12}t_{33}x^{-10/3}q^{-\alpha\cdot l_2-1}(x^4-1)\right),$$
$$\tilde{r}_{13} = \frac{1}{\Delta(x^{-1})}\left(\upsilon^{-1}t_{12}t_{23}x^{-8/3}q^{\alpha\cdot l_1}(1-x^4)\right),$$
$$\tilde{r}_{21} = \frac{1}{\Delta(x^{-1})}\left(\upsilon t_{22}t_{23}t_{31}x^{-10/3}q^{\alpha\cdot l_2}(x^4-1)\right),$$
$$\tilde{r}_{22} = \frac{x^{-8/3}}{\Delta(x^{-1})}\left(t_{22}t_{33}q^{-2}(x^{2}\upsilon-\upsilon^{-1})+t_{23}t_{31}t_{12}x^{2}(x^{-2}\upsilon-\upsilon^{-1})\right),$$
$$\tilde{r}_{23} = \frac{1}{\Delta(x^{-1})}\left(\upsilon t_{22}t_{23}x^{-4}q^{-\alpha\cdot l_3-1}(1-x^4)\right),$$
$$\tilde{r}_{33} = \frac{x^{-4/3}}{\Delta(x^{-1})}\left((x^{-2}\upsilon-\upsilon^{-1})(t_{22}t_{33}q^{-2}+t_{31}t_{12}t_{23}x^{2})\right).$$

Essentially, we have calculated six varieties of solution each containing two zeroes, which is in contrast to Lima-Santos' findings: three solutions each with four zeroes. The difference originates from the fact that our construction does not impose any conditions upon the entries of the reflection matrix at any stage. One can easily verify that Lima-Santos' constraint (4.9) does work for the same in-going and out-going processes, in the suitable generalisation of the reflection equation. However, this is because no overall shift in the charge occurs. Difficulties arise when one tries to express the remaining matrix elements in terms of a particular generalised reflection matrix entry, due to the shifts in the arguments of those entries. Factors are not easily extracted throughout all of the generalised reflection equations, and consequently the very restricting constraint (4.11) does not apply. However, in the limit $x \to 1$, all generalised solutions exhibit the same characteristics as the three finite-dimensional solutions. This limit reduces all generalised solutions to a multiple of the identity matrix.

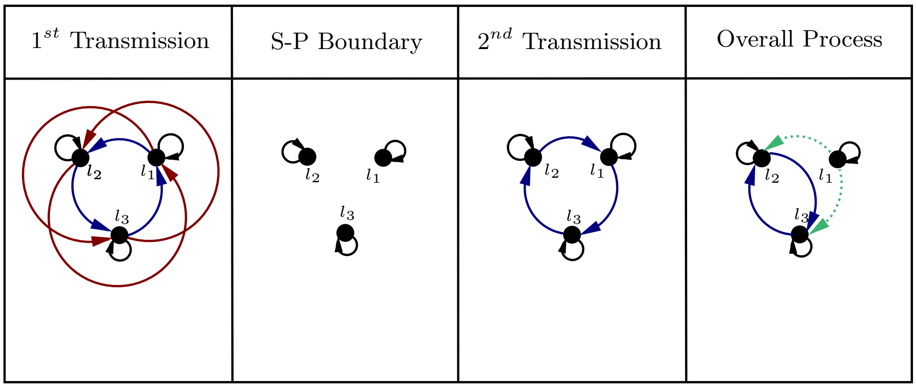

Figure 4.7: Pictorial representation of generalised solutions $\tilde{R}_{I,X}$.

Let us now scrutinise the generalised solutions and reason why they possess such a curious structure. Consider the shape of generalised solution $\tilde{R}_I$, and therefore $\tilde{R}_X$ as well. If we look at the processes involved in their creation - transmission, reflection and a second transmission - and inspect the pictorial representation, a pattern seems to emerge. Figure (4.7) depicts the processes that characterise this type of solution. Solutions of this form are a generalisation of R_{23}, depicted in figure (4.3) and the details are given in section four, figure (4.7) displays this very well. It appears that the defect adds extra freedom, allowing a soliton of weight l_1 to convert to either adjacent charge, indicated by the green dotted lines. Furthermore, a transmission matrix with the opposite permutation has the effect of reversing the direction of the new channels opened by the defect. For the opposite process see the illustrations of solutions $\tilde{R}_{IV}$ and $\tilde{R}_{XI}$ in appendix A - where all other diagrams are present.

In the next section we will investigate the way in which generalised solutions are related to the existing ones, where we will see the importance of the clockwise/anti-clockwise permutation and explain the behaviour exhibited by the new solutions.

4.3 Relation to existing Solutions

A simple limiting procedure is adopted to give the new solutions the same structure (number of zeroes) as the three finite-dimensional solutions. We can then recognise the new solutions as a generalisation of the finite-dimensional reflection factors. Additional, previously unseen, solutions having the same structure as the $a_2^{(2)}$ reflection matrices [28, 29] are also found when certain defect parameters, t_{ij}, are set to zero. Some of the solutions gained in this way contain more than four zeroes and this is again because the constraint (4.9) no longer applies, details are supplied in the next section.

Ultimately, this shows that dressing a diagonal reflection factor with an integrable defect allows us to uncover many more reflection matrices, as well as revealing their potential power to help us classify the solutions to the reflection equation (1.8). Let us begin by considering generalised solution, $\tilde{R}_I$. To ensure that it has the same structure as the finite-dimensional solution, R_{23}, we require entries:

$$\tilde{r}_{12}, \tilde{r}_{13} \to 0.$$

After examining these entries we find that their only common parameter is: t_{13} and if this becomes zero entry $\tilde{r}_{23}$ will also vanish. To counteract this limit we relabel t_{21} as well. Now, we can relabel the necessary parameters in the following way:

$$t_{13} = p, \text{ and } t_{21} = \frac{A}{p},$$

such that when $p \to 0$ only entries $\tilde{r}_{12}, \tilde{r}_{13}$ vanish, for finite A. This limit, despite its strange appearance, works because the parameters involved also appear in other matrix entries as a product. Typcially, when parameters appear as a product, the limit does not harm them as it simply replaces them with a finite constant; A in the above case. Consequently, we can regard $\tilde{R}_I$ as an infinite-dimensional analogue of R_{23}. This procedure can be repeated for all remaining generalised reflection matrices. The parameters involved in the limit always appear elsewhere as a product, but providing the constants: A, B, C and others, remain finite we will obtain the required zeroes.

Solution $\tilde{R}_{II}$ is an infinite-dimensional generalisation of R_{13} after relabelling the following parameters:

$$t_{23} = r, \text{ and } t_{12} = \frac{C}{r}, \text{ for finite C},$$

and then take the limit $r \to 0$, to gain the required zeroes. Solution $\tilde{R}_{III}$ is an infinite-dimensional analogue of R_{13} also, after relabelling parameters:

$$t_{32} = \bar{r}, \text{ and } t_{21} = \frac{D}{\bar{r}}, \text{ for finite D},$$

and taking the limit $\bar{r} \to 0$. Solution $\tilde{R}_{IV}$ is another generalisation of R_{23}, upon relabelling:

$$t_{31} = \bar{p}, \text{ and } t_{12} = \frac{F}{\bar{p}}, \text{ for finite F},$$

and taking the limit $\bar{p} \to 0$. Solution $\tilde{R}_{V}$ is another infinite-dimensional analogue of R_{13}, via the procedure:

$$t_{21} = s \text{ and } t_{32} = \frac{G}{s}, \text{ for finite G},$$

and allowing $s \to 0$. Solution $\tilde{R}_{VI}$ is an infinite-dimensional version of R_{12}, realised by relabelling:

$$t_{31} = v \text{ and } t_{23} = \frac{H}{v}, \text{ for finite H},$$

together with the limit $v \to 0$. Solution $\tilde{R}_{VII}$ is also another infinite-dimensional generalisation of R_{12}, via:

$$t_{13} = \bar{v} \text{ and } t_{32} = \frac{J}{\bar{v}}, \text{ for finite J},$$

together with $\bar{v} \to 0$. Solution $\tilde{R}_{VIII}$ is another generalisation of R_{13}, shown by relabelling:

$$t_{12} = \bar{s} \text{ and } t_{23} = \frac{K}{\bar{s}}, \text{ for finite K},$$

and allowing $\bar{s} \to s$. Solution $\tilde{R}_{IX}$ is yet another generalisation of R_{12}, realised by taking:

$$t_{32} = v \text{ and } t_{13} = \frac{L}{v}, \text{ for finite L},$$

along with the limit $v \to 0$. Solution $\tilde{R}_{X}$ is another generalisation of R_{23}, acquired by relabelling:

$$t_{12} = x \text{ and } t_{31} = \frac{M}{x}, \text{ for finite M},$$

and evaluating the limit $x \to 0$. Solution $\tilde{R}_{XI}$ is another infinite-dimensional version of R_{23}, obtained by setting:

$$t_{21} = \bar{x} \text{ and } t_{13} = \frac{N}{\bar{x}}, \text{ for finite N},$$

and taking the limit $\bar{x} \to 0$. Finally, solution $\tilde{R}_{XII}$ is another generalisation of R_{12}, realised similarly by taking:

$$t_{23} = \bar{v} \text{ and } t_{31} = \frac{Q}{\bar{v}}, \text{ for finite Q},$$

and the limit $\bar{v} \to 0$.

Let us now discuss the importance of the permutation and its implications. For example, consider $\tilde{R}_I$ and $\tilde{R}_{IV}$, they are calculated from the matrices T_I and T_{IV}, both corresponding to the solitonic/anti-solitonic representations with the clockwise permutation. Any generalised solution constructed from these transmission matrices reduces to the same finite-dimensional reflection matrix. Curiously, note the relation between the parameters involved in the limits above. They appear to be related by the swapping of indices, reminiscent of a conjugation relation. Likewise, any generalised matrix calculated using T_{II} and T_{III} will reduce to the same finite-dimensional reflection matrix. In this case, the matrices come from different representations but possess the anti-clockwise permutation. It seems that the permutation is responsible for this structure and the reason why we have twelve generalised reflection factors of six different varieties. We can also group the generalised solutions according to their structure, for example, $\tilde{R}_{II}$ and $\tilde{R}_V$ both have zeroes in the same place. The first, $\tilde{R}_{II}$, is constructed from the T_{II} and $\tilde{T}_{II}$ matrices and they themselves correspond to the anti-solitonic representation with anti-clockwise and clockwise permutations respectively. The latter, $\tilde{R}_V$, is built from matrices T_I and $\tilde{T}_I$, that correspond to the solitonic representation with clockwise and anti-clockwise permutations respectively. If we label transmission matrices in 'conjugate pairs', whereby a T-matrix constructed from the solitonic representation with clockwise permutation has a conjugate constructed from the anti-solitonic representation with anti-clockwise permutation, then one observes that conjugate pairs return a generalised solution of the same structure, albeit with different entries.

One caveat concerning the limiting procedures: whatever limit is used must not reduce the determinant of the transmission matrix to zero. Such a limit would prevent the inversion of a transmission matrix, thus violating one of its crucial properties, as well as rendering the construction of generalised reflection matrices impossible. Moreover, the determinant is vital to ensure the correct function of the type - II defect generalisation of Delius and MacKay's algebraic framework, similarly any limit taking the determinant to zero destroys the construction.

4.4 Remarks concerning $a_2^{(1)}$ and the Type - I Defect

In this chapter we have calculated sixteen generalised reflection matrices (of ten different varieties) for the $a_2^{(1)}$ - affine Toda model by dressing all four diagonal reflection factors with defect transmission matrices. In doing so, we have provided further evidence that a defect placed near the boundary does indeed produce more general solutions to the suitably generalised reflection equation. Within the $a_2^{(1)}$ case, we see that the addition of a defect produces unexpected results. We believe that they originate from the extra choice associated with the clockwise/anti-clockwise permutation, which specifies the defect within the $a_2^{(1)}$ framework. Describing the solutions pictorially provided us with added insight. In particular, we can view the total process in terms of the weights of the solitonic/anti-solitonic representations and track the possible exchanges of topological charge. The non-diagonal finite-dimensional solutions appear to restrict the possible processes at the boundary, in that a soliton cannot convert to all of its neighbouring charges. When a defect is introduced, we find a multitude of new solutions and the dressed boundary is able to deal more effectively with the weights.

Interestingly, it is possible to obtain more solutions by 'switching off' different defect parameters within the type - I generalised solutions. Consequently, one can obtain matrices that have the same shape as the $a_2^{(2)}$ reflection matrices [28, 29]:

$$\text{Case I } = \begin{pmatrix} \tilde{r}_{11}\delta_\alpha^\beta & 0 & \tilde{r}_{13}\delta_\alpha^{\beta+\alpha_0} \\ 0 & \tilde{r}_{22}\delta_\alpha^\beta & 0 \\ 0 & 0 & \tilde{r}_{33}\delta_\alpha^\beta \end{pmatrix}, \tag{4.35}$$

$$\text{Case II} \;=\; \begin{pmatrix} \tilde{r}_{11}\delta_\alpha^\beta & 0 & 0 \\ 0 & \tilde{r}_{22}\delta_\alpha^\beta & 0 \\ \tilde{r}_{31}\delta_\alpha^{\beta-\alpha_0} & 0 & \tilde{r}_{33}\delta_\alpha^\beta \end{pmatrix}, \tag{4.36}$$

$$\text{Case III} \;=\; \begin{pmatrix} \tilde{r}_{11}\delta_\alpha^\beta & \tilde{r}_{12}\delta_\alpha^{\beta-\alpha_1} & \tilde{r}_{13}\delta_\alpha^{\beta+\alpha_0} \\ 0 & \tilde{r}_{22}\delta_\alpha^\beta & \tilde{r}_{23}\delta_\alpha^{\beta-\alpha_2} \\ 0 & 0 & \tilde{r}_{33}\delta_\alpha^\beta \end{pmatrix}, \tag{4.37}$$

$$\text{Case IV} \;=\; \begin{pmatrix} \tilde{r}_{11}\delta_\alpha^\beta & 0 & 0 \\ \tilde{r}_{21}\delta_\alpha^{\beta+\alpha_1} & \tilde{r}_{22}\delta_\alpha^\beta & 0 \\ \tilde{r}_{31}\delta_\alpha^{\beta-\alpha_0} & \tilde{r}_{32}\delta_\alpha^{\beta+\alpha_2} & \tilde{r}_{33}\delta_\alpha^\beta \end{pmatrix}. \tag{4.38}$$

The case I solutions are found in: $\hat{R}_I$, when $t_{21} = 0$, $\hat{R}_{III}$ when $t_{32} = 0$, $\tilde{R}_{III}$, when $t_{32} = 0$ and $\tilde{R}_V$, when $t_{21} = 0$ and also in $\tilde{R}_{VII}$, when $t_{33} = 0$.

Solutions that have the same structure as case II are found in: $\hat{R}_{II}$, when $t_{23} = 0$, $\hat{R}_{IV}$, when $t_{12} = 0$, $\tilde{R}_{II}$, when $t_{23} = 0$, $\tilde{R}_V$, when $t_{22} = 0$ and in $\tilde{R}_{VIII}$, when $t_{12} = 0$.

The solutions sharing the case III structure are: $\tilde{R}_I$, when $t_{33} = 0$ and $\tilde{R}_{VII}$, when $t_{33} = 0$ and also $\tilde{R}_X$, when $t_{31} = 0$.

Finally, the solutions sharing the case IV structure are: $\tilde{R}_{IX}$, when $t_{13} = 0$ and $\tilde{R}_{XI}$, when $t_{13} = 0$ and also $\tilde{R}_{XII}$, when $t_{31} = 0$.

As always, particular care must be taken to ensure that the ‘switching off’ of defect parameters does not cause the determinant to become zero. The above behaviour is not surprising given the root spaces of $a_2^{(1)}$ and $a_2^{(2)}$, since the former can be projected onto the latter. Overall, this again shows that generalised reflection matrices enjoy more freedom and are not as heavily constrained as the finite-dimensional solutions of Lima-Santos [69].

For this model, the type - I quantum setting has proved to be very fruitful so far. Especially, when we compare our findings of this chapter to those of the $a_2^{(2)}$- model, where the type - II defect naturally includes the finite-dimensional solutions in a clear cut fashion. We must now go on to consider the type - II defect, and investigate exactly how it develops the theory further.

4.5 Type - II Generalised solutions within $a_2^{(1)}$

We will now develop the $a_2^{(1)}$ case even further by using type - II defects to construct generalised solutions. Previous results indicate that the type - II defect will return more general solutions. In particular, we expect that the generalised solutions will not contain any zero entries, which means that the solitons are able to deposit charge freely. And so, after its interaction with the defect and boundary, it can possess any weight-like charge of the representation. In this section, a different notational convention is used, namely that of [45]. This allows us to state the type - II defect presented in [45], but we must alter the diagonal reflection factors $(R_{d_0}, R_{d_1}, R_{d_2}, R_{d_3})$ accordingly.

Before stating the type - II T'-matrix, we will detail the alterations introduced by the change of notation that concerns the way in which the generators and consequently, the matrices R' and T' act on 'in'-states [45]. Earlier results, including the type - I transmission matrices, are based on the S-matrix provided in section (4.1). In [45] the scattering matrix, R' labelled (5.5), satisfies the linear intertwining relation $R'\Delta(a) = \Delta'(a)R'$ where a is any generator of the $U_q(a_2^{(1)})$ algebra and is related to the R-matrix of [48], labelled (3.5), in the following way:

$$R'^T(q, x) = R(q^2, x).$$

To mirror this relation, q in the S-matrix of section (4.1) is replaced by q^2 and the transpose is taken. Consequently, all four diagonal reflection matrices are modified:

$$R_{d_0}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^{-4/3} & 0 \\ 0 & 0 & x^{-8/3} \end{pmatrix}, \tag{4.39}$$

$$R_{d_1}(x) = \begin{pmatrix} (x^2\upsilon - \upsilon^{-1}) & 0 & 0 \\ 0 & x^{-4/3}(x^2\upsilon - \upsilon^{-1}) & 0 \\ 0 & 0 & x^{-2/3}(\upsilon - x^2\upsilon^{-1}) \end{pmatrix}, \tag{4.40}$$

$$R_{d_2}(x) = \begin{pmatrix} (x^2\upsilon - \upsilon^{-1}) & 0 & 0 \\ 0 & x^{2/3}(\upsilon - x^2\upsilon^{-1}) & 0 \\ 0 & 0 & x^{-2/3}(\upsilon - x^2\upsilon^{-1}) \end{pmatrix}, \tag{4.41}$$

$$R_{d_3}(x) = \begin{pmatrix} (x^{-2}\upsilon - \upsilon^{-1}) & 0 & 0 \\ 0 & x^{8/3}(x^2\upsilon - \upsilon^{-1}) & 0 \\ 0 & 0 & x^{4/3}(x^{-2}\upsilon - \upsilon^{-1}) \end{pmatrix}. \tag{4.42}$$

We can now move on to detail the theory surrounding the type - II transmission matrices. Throughout this section, we will see the parallels between the Kronecker-deltas appearing in earlier transmission matrices satisfying the transmission Yang-Baxter equations and the raising/lowering operators of the infinite-dimensional Borel subalgebra, where transmission matrices satisfy the linear intertwining equation:

$$T'\Delta(b) = \Delta'(b)T',$$

where b is any generator of the Borel subalgebra. Specifically, there are two type - II T'-matrices that are each related to two type - I transmission matrices. Both are presented in the operator language. Despite the use of raising and lowering operators, the importance of the permutation is still apparent and this is used to relate the new solutions to the type - I solutions. As we know, any T'-matrix satisfying the linear equation acts as an intertwiner of the infinite-dimensional representation of space, $\mathcal{V}_z$, and the three-dimensional representation of space V_x:

$$T(z/x) : \mathcal{V}_z \otimes V_x \to \mathcal{V}_z \otimes V_x.$$

The representations must satisfy the $U_q(a_2^{(1)})$ algebra, consisting of nine generators, $\{X_i^{\pm}, K_i\}, i = 1, 2, 3$; with relations [70]:

$$[K_i, K_j] = 0, \qquad [X_i^{\pm}, X_j^{\mp}] = 0, \qquad [X^+, X_i^-] = \frac{K_i^2 - K_i^{-2}}{q^2 - q^{-2}},$$
$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \qquad K_i X_i^{\pm} K_i^{-1} = q^{\pm 2} X_i^{\pm}, \qquad K_i X_j^{\pm} K_i^{-1} = q^{\mp 1} X_j^{\pm},$$

for all $i, j = 1, 2, 3$. The coproducts Δ, Δ' are defined as follows:

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(X_i^{\pm}) = X_i^{\pm} \otimes K_i^{-1} + K_i \otimes X_i^{\pm},$$

$$\Delta'(K_i) = \Delta(K_i), \qquad \Delta'(X_i^{\pm}) = K_i^{-1} \otimes X_i^{\pm} + X_i^{\pm} \otimes K_i,$$

for all $i, j = 1, 2, 3$. As we are working within $a_2^{(1)}$, we must again differentiate the soliton from the anti-soliton. The three-dimensional soliton (first fundamental) representation was used to calculate the first type - II T'-matrix (5.10 in [45]):

$$\begin{gathered} K_1 = \begin{pmatrix} q & 0 & 0 \\ 0 & q^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^{-1} \end{pmatrix}, \quad K_3 = \begin{pmatrix} q^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix}, \\ X_1^+ = (X_1^-)^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2^+ = (X_2^-)^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ X_3^+ = (X_3^-)^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{gathered} \tag{4.43}$$

The representation is studied in the spin gradation, consequently certain generators are modified,

$$E_i = x^{2/3} X_i^+, \quad F_i = x^{-2/3} X_i^-, \quad i = 1, 2, 3.$$

Within the three-dimensional space the weight vectors are:

$$\lambda_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and with respect to the above representation λ_1 is the highest weight vector. Like before, these vectors match the topological charges defined previously:

$$l_1 = \frac{1}{3}(2\alpha_1 + \alpha_2), \quad l_2 = -\frac{1}{3}(\alpha_1 - \alpha_2), \quad l_3 = -\frac{1}{3}(\alpha_1 + 2\alpha_2),$$

expressed in terms of the simple roots of the $a_2^{(1)}$ algebra. Corrigan and Zambon use the following infinite-dimensional representation of the Borel subalgebra, which is most suitable for calculating the first type - II defect matrix [45]:

$$K_i = \kappa_i q^{N_i - N_{i+1}}, \qquad X_i^+ = a_i^\dagger a_{i+1}, \qquad i = 1, 2, 3. \tag{4.44}$$

The spin gradation affects the infinite-dimensional generators, X_i^+, in the following way:

$$E_i^+ = z^{2/3} X_i^+, \qquad i = 1, 2, 3.$$

The operators $a_i, a_i^\dagger$ are three independent sets of lowering and raising operators. All operators act in a direction given by one of the unit vectors $\{e_1, e_2, e_3\}$, therefore the way in which the operators model the exchange of charge and the soliton's movement within the weight lattice is evident. Of course, the choice of representation is not unique and others satisfying the algebraic relations could be used. Let us now detail the action of the operators on the infinite-dimensional space:

$$a_i \left| j_i \right\rangle = g_i(j_i) \left| j_i - 1 \right\rangle, \qquad a_i^\dagger \left| j_i \right\rangle = f_i(j_i) \left| j_i + 1 \right\rangle, \qquad N_i \left| j_i \right\rangle = j_i \left| j_i \right\rangle,$$

with

$$\begin{aligned} a_i a_i^\dagger &= F_i(j_i) \left| j_i \right\rangle = f_i(j_i) g_i(j_i + 1) \left| j_i \right\rangle, \\ a_i^\dagger a_i &= F_i(j_i - 1) \left| j_i \right\rangle = f_i(j_i - 1) g_i(j_i) \left| j_i \right\rangle, \quad i = 1, 2, 3. \end{aligned}$$

where the number functions are required to take the form:

$$F_i(N_i) = (f_i^+)^2 q^{2N_i} - (f_i^-)^2 q^{-2N_i}.$$

This is realised by choosing the auxiliary functions:

$$f_i(N_i) = (f_i^+ q^{N_i} + f_i^- q^{-N_i}), \quad g_i(N_i) = (f_i^+ q^{N_i - 1} - f_i^- q^{-N_i + 1}), \quad i = 1, 2, 3.$$

In fact, the type - II nature of the transmission matrix is signalled by the presence of both f_i^+ and f_i^-. This is exemplified when considering limits from the type - II to type - I matrices. Using the above representations, Corrigan and Zambon calculated a type - II T'-matrix [45]:

$$T' = \begin{pmatrix} A(N_1, N_2, N_3) & k q^{-N_3} a_1 a_2^\dagger & v q^{N_2} a_1 a_3^\dagger \\ j q^{N_3} a_2 a_1^\dagger & B(N_1, N_2, N_3) & m q^{-N_1} a_2 a_3^\dagger \\ w q^{-N_2} a_3 a_1^\dagger & l q^{N_1} a_3 a_2^\dagger & C(N_1, N_2, N_3) \end{pmatrix}, \tag{4.45}$$

with diagonal entries

$$A(N_1, N_2, N_3) = a' q^{-N_1 + N_2 + N_3 + 1} + a'' q^{N_1 - N_2 - N_3 - 1},$$

$$B(N_1,N_2,N_3) = b' q^{N_1-N_2+N_3+1} + b'' q^{-N_1+N_2-N_3-1},$$
$$C(N_1,N_2,N_3) = c' q^{N_1+N_2-N_3+1} + c'' q^{-N_1-N_2+N_3-1},$$

and coefficients

$$\begin{aligned}
a' &= \left(\frac{x}{z}\right)^{4/3} \frac{\kappa_3}{(1-q^4)^2}, & a'' &= \left(\frac{z}{x}\right)^{2/3} \kappa_3 (1-q^4)(f_1^+ f_2^- f_3^-)^2, \\
b' &= \left(\frac{x}{z}\right)^{4/3} \frac{\kappa_1}{\kappa_2 (1-q^4)^2}, & b'' &= \left(\frac{z}{x}\right)^{2/3} \frac{\kappa_1}{\kappa_2} (1-q^4)(f_1^- f_2^+ f_3^-)^2, \\
c' &= \left(\frac{x}{z}\right)^{4/3} \frac{1}{\kappa_3 (1-q^4)^2}, & c'' &= \left(\frac{z}{x}\right)^{2/3} \frac{(1-q^4)}{\kappa_3} (f_1^- f_2^- f_3^+)^2,
\end{aligned}$$
$$w = -(f_2^-)^2, \qquad k = -\frac{(f_3^-)^2}{\kappa_2}, \qquad m = -\kappa_1 (f_1^-)^2,$$
$$v = \left(\frac{x}{z}\right)^{2/3} \frac{1}{(1-q^4)}, \qquad j = \left(\frac{x}{z}\right)^{2/3} \frac{1}{\kappa_2 (1-q^4)}, \qquad l = \left(\frac{x}{z}\right)^{2/3} \frac{\kappa_1}{(1-q^4)}.$$

The constants, κ_i, satisfy the condition $(\kappa_1\kappa_2\kappa_3)^2 = 1$ and in order to obtain the above solution it is assumed that $\kappa_1\kappa_2\kappa_3 = 1$.

The second type - II T'-matrix is built from the following finite and infinite-dimensional representations of the $U_q(a_2^{(1)})$ algebra and Borel subalgebra, respectively:

$$K_1 = \begin{pmatrix} q^{-1} & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^{-1} & 0 \\ 0 & 0 & q \end{pmatrix}, \quad K_3 = \begin{pmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-1} \end{pmatrix},$$
$$X_1^+ = (X_1^-)^T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2^+ = (X_2^-)^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{4.46}$$
$$X_3^+ = (X_3^-)^T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$K_i = \kappa_i q^{-N_i+N_{i+1}}, \qquad X_i^+ = a_i a_{i+1}^\dagger, \qquad i = 1,2,3. \tag{4.47}$$

However, in this case, the spectral parameters are added to the two representations in the following way:

$$E_i = x^{-2/3} X_i^+, \qquad F_i = x^{2/3} X_i^-, \qquad i = 1,2,3.$$

The second type - II transmission matrix is obtained by combining the above representations with the action of the operators defined previously, it has the form:

$$T' = \begin{pmatrix} U(N_1,N_2,N_3) & rq^{N_3}a_1a_2^\dagger & gq^{-N_2}a_1a_3^\dagger \\ pq^{-N_3}a_2a_1^\dagger & V(N_1,N_2,N_3) & tq^{N_1}a_2a_3^\dagger \\ hq^{N_2}a_3a_1^\dagger & sq^{-N_1}a_3a_2^\dagger & W(N_1,N_2,N_3) \end{pmatrix}, \tag{4.48}$$

with diagonal entries:

$$\begin{aligned} U(N_1,N_2,N_3) &= u'q^{-N_1+N_2+N_3+1} + u''q^{N_1-N_2-N_3-1}, \\ V(N_1,N_2,N_3) &= v'q^{N_1-N_2+N_3+1} + v''q^{-N_1+N_2-N_3-1}, \\ W(N_1,N_2,N_3) &= w'q^{N_1+N_2-N_3+1} + w''q^{-N_1-N_2+N_3-1}, \end{aligned}$$

and coefficients

$$\begin{aligned} u' &= \left(\frac{z}{x}\right)^{4/3}\frac{1}{\kappa_3(1-q^4)^2}, & u'' &= \left(\frac{x}{z}\right)^{2/3}\frac{(1-q^4)}{\kappa_3}(f_1^+f_2^-f_3^-)^2, \\ v' &= \left(\frac{z}{x}\right)^{4/3}\frac{\kappa_2}{\kappa_1(1-q^4)^2}, & v'' &= \left(\frac{x}{z}\right)^{2/3}\frac{\kappa_2}{\kappa_1}(1-q^4)(f_1^-f_2^+f_3^-)^2, \\ w' &= \left(\frac{z}{x}\right)^{4/3}\frac{\kappa_3}{(1-q^4)^2}, & w'' &= \left(\frac{x}{z}\right)^{2/3}\kappa_3(1-q^4)(f_1^-f_2^-f_3^+)^2, \end{aligned}$$

$$p = -(f_3^-)^2\kappa_2, \qquad s = -\frac{(f_1^-)^2}{\kappa_1}, \qquad g = -(f_2^-)^2,$$

$$h = \left(\frac{z}{x}\right)^{2/3}\frac{1}{(1-q^4)}, \qquad r = \left(\frac{z}{x}\right)^{2/3}\frac{\kappa_2}{(1-q^4)}, \quad t = \left(\frac{z}{x}\right)^{2/3}\frac{1}{\kappa_1(1-q^4)}.$$

The constants, κ_i, remain constrained by $(\kappa_1\kappa_2\kappa_3)^2 = 1$, and to ensure that the above solution satisfies the linear intertwining equation it is again assumed that $\kappa_1\kappa_2\kappa_3 = 1$.

We readily observe that the type - II transmission matrices also contain nine parameters: $\kappa_i, f_i^+, f_i^-, i = 1,2,3$. However, the nine parameters, t_{ij}, appearing in the type - I T-matrix and the above type - II parameters do not share the same origin. The t_{ij} are a result of the transmission Yang-Baxter equation, where each matrix entry is assigned its own parameter and by solving the equations certain relations emerge. The type - II parameters are due to the infinite-dimensional representation that is used to construct the T'-matrix, together with the associated action of the operators. Consequently, the limiting process from type - II to type - I is somewhat intricate, and we will now move on to consider these limits before constructing type - II generalised reflection matrices.

4.5.1 Limits: From Type - II to Type - I

We will now delve into the details concerning the limits of type - II T'-matrices to type - I T-matrices. Following this, we will use some of their features to recover the structures of the earlier generalised solutions. In this section, we will also expand upon previous comments regarding the differences between type - II and type - I defect parameters.

An important aspect of these limits is the matching of raising/lowering operators to the Kronecker-deltas. We must bear in mind the essence of the operators, fundamentally they describe movement around the weight lattice (describing exchanges of charge) as each operator is associated with one of the orthonormal basis vectors, e_i. However, the states that the operators act on are restricted because any topological charge deposited on the defect must belong to the weight lattice [45]. If we express any weight, j, in terms of the orthonormal basis:

$$j = j_1 e_1 + j_2 e_2 + j_3 e_3,$$

then the following must hold:

$$j_1 + j_2 + j_3 = 0.$$

In light of this, during any limit from type - II to type - I, we will equate N_i with j_i and evaluate the above constraint. In the type - I case, l_i, are the fundamental weights and α, γ label the weights that obey the constraint $\gamma \cdot l_i = \gamma_i$, where $\gamma_1 + \gamma_2 + \gamma_3 = 0$. Before we investigate the limits, it is necessary to make one final remark. As explained earlier, in section (4.5), the notation adopted in the type - II case corresponds to a different action of the T'-matrix on the in-states. We have also observed earlier how the transmission Yang-Baxter equation and its analogue in the linear intertwining framework act on slightly different spaces. Consequently, we expect the following relation to hold between the type - II, T', and type - I, T:

$$T'^{T}(q, x) = T(q^{-2}, x).$$

Curiously, the transposition of the T'-matrix causes unusual behaviour. It is evident when considering the limits of the type - II generalised solutions, which we will detail in due course, but firstly let us specify the workings of the limits.

The first T'-matrix (4.45) is related to both T_{II} and T_{IV} type - I matrices. As noted earlier, the appearance of both parameters f_i^+ and f_i^- indicates that the T'-matrix is type - II. Therefore, to return to the type - I matrices we must 'switch off' either f_i^+ or f_i^- (for all $i = 1, 2, 3$). Switching off either set of parameters will return different structures, for instance, when $f_i^- = 0$ we obtain three zeroes. Subsequently, one expects that this limit concerns the T_{IV} type - I matrix. Likewise, if $f_i^+ = 0$, no zeroes appear and so we expect this limit to return the matrix T_{II}.

To obtain the type - I matrix, T_{II}, from (4.45) begin by setting: $f_i^+ = 0$, $\kappa_i = 1$, $N_i = j_i$ and evaluate $j_1 + j_2 + j_3 = 0$. Following this, extract a factor $q(1-q^4)^{-2}(x/z)^{4/3} =: A$ and take the transpose to find:

$$T'_{\to II} = A \begin{pmatrix} q^{-2j_1}\delta_j^k & \omega q^{j_3-1}\delta_j^{k-\alpha_1} & -\Omega_2 q^{-j_2-1}\delta_j^{k+\alpha_0} \\ -\Omega_3 q^{-j_3-1}\delta_j^{k+\alpha_1} & q^{-2j_2}\delta_j^k & \omega q^{j_1-1}\delta_j^{k-\alpha_1} \\ \omega q^{j_2-1}\delta_j^{k-\alpha_0} & -\Omega_1 q^{-j_1-1}\delta_j^{k+\alpha_2} & q^{-2j_3}\delta_j^k \end{pmatrix}, \quad (4.49)$$

with coefficients:

$$\omega = \left(\frac{z}{x}\right)^{2/3}(1-q^4), \quad \Omega_i = \left(\frac{z}{x}\right)^{4/3}(f_i^-)^2(1-q^4)^2, \text{ for all } i = 1, 2, 3.$$

Next, apply the following similarity transformation:

$$U = \mathrm{diag}(q^{-j_1}, q^{j_2}, q^{-2j_1-j_3}),$$

and set $(f_i^-)^2 = -q^{2j_i-1}$ - to ensure the powers of q match those of the type - I matrix. Finally, make the identification $\epsilon = z^{2/3}(1-q^4)q^{-1}$ and send q to $q^{-1/2}$ to find the matrix:

$$\hat{T}_{II} = A \begin{pmatrix} q^{j_1}\delta_j^k & \epsilon x^{-2/3}q^{-j_3}\delta_j^{k-\alpha_1} & \epsilon^2 x^{-4/3}\delta_j^{k+\alpha_0} \\ \epsilon^2 x^{-4/3}\delta_j^{k+\alpha_1} & q^{j_2} & \epsilon x^{-2/3}q^{-j_1}\delta_j^{k-\alpha_2} \\ \epsilon x^{-2/3}q^{-j_2}\delta_j^{k-\alpha_0} & \epsilon^2 x^{-4/3}\delta_j^{k+\alpha_2} & q^{j_3}\delta_j^k \end{pmatrix}.$$

The above matrix coincides with T_{II} when the following equalities are made:

$\alpha_i \equiv j_i$, $t_{ii} = 1$ and $t_{12} = t_{23} = t_{31} = \epsilon$.

To obtain T_{IV} a similar procedure is adopted, this time set $f_i^- = 0$, while f_i^+ is non-zero for all $i = 1, 2, 3$. As before, equate $\kappa_i = 1$, $N_i = j_i$ and evaluate $j_1 + j_2 + j_3 = 0$, extract the factor A and take the transpose so that the matrix becomes:

$$T'_{\to IV} = A \begin{pmatrix} q^{-2j_1}\delta_j^k & \omega q^{j_3-1}\delta_j^{k-\alpha_1} & 0 \\ 0 & q^{-2j_2}\delta_j^k & \omega q^{j_1-1}\delta_j^{k-\alpha_1} \\ \omega q^{j_2-1}\delta_j^{k-\alpha_0} & 0 & q^{-2j_3}\delta_j^k \end{pmatrix},$$

with ω as defined previously. Then, apply a slightly different similarity transformation:

$$U = \mathrm{diag}(q^{j_1}, q^{-j_2}, q^{2j_1+j_3}),$$

to remove all powers of q in the off-diagonal entries and identify ϵ in the same way. Finally, send q to $q^{-1/2}$ to acquire the matrix:

$$\hat{T}_{IV} = A \begin{pmatrix} q^{j_1}\delta_j^k & \epsilon x^{-2/3}\delta_j^{k-\alpha_1} & 0 \\ 0 & q^{j_2}\delta_j^k & \epsilon x^{-2/3}\delta_j^{k-\alpha_1} \\ \epsilon x^{-2/3}\delta_j^{k-\alpha_0} & 0 & q^{j_3}\delta_j^k \end{pmatrix},$$

which coincides with T_{IV} after setting $\alpha_i \equiv j_i$, $t_{ii} = 1$ and $t_{12} = t_{23} = t_{31} = \epsilon$.

To obtain the remaining type - I T-matrices, we follow very similar procedures. We can manipulate matrix (4.48) so that it resembles both T_I and T_{III} type - I matrices. We will begin with the T_I limit of (4.48), where we make the following equalities for all $i = 1, 2, 3$: $f_i^+ = 0$, $\kappa_i = 1$, $N_i = -j_i$ and evaluate $j_1 + j_2 + j_3 = 0$. Now, extract a factor $q(1-q^4)^{-2}(z/x)^{4/3} =: B$ and take the transpose to find:

$$T'_{\to I} = B \begin{pmatrix} q^{2j_1}\delta_j^k & -\Omega'_3 q^{j_3-1}\delta_j^{k-\alpha_1} & \omega' q^{-j_2-1}\delta_j^{k+\alpha_0} \\ \omega' q^{-j_3-1}\delta_j^{k+\alpha_1} & q^{2j_2}\delta_j^k & \Omega'_1 q^{j_1-1}\delta_j^{k-\alpha_1} \\ \Omega'_2 q^{j_2-1}\delta_j^{k-\alpha_0} & \omega' q^{-j_1-1}\delta_j^{k+\alpha_2} & q^{-2j_3}\delta_j^k \end{pmatrix}, \tag{4.50}$$

this time with slightly modified coefficients:

$$\omega' = \left(\frac{x}{z}\right)^{2/3}(1-q^4), \quad \Omega'_i = \left(\frac{x}{z}\right)^{4/3}(f_i^-)^2(1-q^4)^2, \text{ for all } i = 1, 2, 3.$$

Again, we require a similarity transformation to adjust the powers of q accordingly. In this case, we use the transformation:

$$U = \mathrm{diag}(q^{j_1}, q^{j_1-j_3}, q^{-j_3}),$$

and then set $(f_i^-)^2 = -q^{-2j_i-1}$, identify $\epsilon = z^{-2/3}(1-q^4)q^{-1}$ and finally send q to $q^{-1/2}$ to achieve:

$$\hat{T}_I = B\begin{pmatrix} q^{-j_1}\delta_j^k & \epsilon^2 x^{4/3}\delta_j^{k-\alpha_1} & \epsilon x^{2/3}q^{j_2}\delta_j^{k+\alpha_0} \\ \epsilon x^{2/3}q^{j_3}\delta_j^{k+\alpha_1} & q^{-j_2} & \epsilon^2 x^{4/3}q^{-j_1}\delta_j^{k-\alpha_2} \\ \epsilon^2 x^{4/3}q^{-j_2}\delta_j^{k-\alpha_0} & \epsilon x^{2/3}q^{j_1}\delta_j^{k+\alpha_2} & q^{-j_3}\delta_j^k \end{pmatrix}.$$

The above matrix coincides with T_I when the following equalities are made $\alpha_i \equiv j_i$, $t_{ii} = 1$ and $t_{13} = t_{21} = t_{32} = \epsilon$.

The final limit will reproduce the type - I matrix, T_{III} and we start this time by setting: $f_i^- = 0, \kappa_i = 1, N_i = -j_i$ and evaluate the constraint $j_1 + j_2 + j_3 = 0$. As before, extract the factor B and take the transpose to discover:

$$T'_{\to III} = B\begin{pmatrix} q^{2j_1}\delta_j^k & 0 & \omega' q^{-j_2-1}\delta_j^{k+\alpha_0} \\ \omega' q^{-j_3-1}\delta_j^{k+\alpha_1} & q^{2j_2}\delta_j^k & 0 \\ 0 & \omega' q^{-j_1-1}\delta_j^{k+\alpha_2} & q^{-2j_3}\delta_j^k \end{pmatrix}. \tag{4.51}$$

To ensure that the powers of q match the type - I case, we use the similarity transformation:

$$U = \mathrm{diag}(q^{-j_1}, q^{-j_1+j_3}, q^{+j_3}).$$

Then, as usual, identify $\epsilon = z^{-2/3}(1-q^4)q^{-1}$ and lastly send q to $q^{-1/2}$ so that the matrix becomes:

$$\hat{T}_{III} = B\begin{pmatrix} q^{-j_1}\delta_j^k & 0 & \epsilon x^{2/3}\delta_j^{k+\alpha_0} \\ \epsilon x^{2/3}\delta_j^{k+\alpha_1} & q^{-j_2} & 0 \\ 0 & \epsilon x^{2/3}\delta_j^{k+\alpha_2} & q^{-j_3}\delta_j^k \end{pmatrix}.$$

This matrix does indeed concur with T_{III} when $\alpha_i \equiv j_i$, $t_{ii} = 1$ and $t_{13} = t_{21} = t_{32} = \epsilon$.

We can extract more information about the type - I matrices from these limits. In particular, the parameter f_i^+ appears to be associated most closely to the type - I case. We know that the presence of both $f_i^\pm$ signals that a transmission matrix is of type - II, and we have seen how they collapse in different ways. However, for the type - I case, one would expect that either f_i^+ or f_i^- is present and the limits illustrate this fact because whenever f_i^- is non-zero a further condition is required to guarantee that the type - II matrix collapses as it should. Furthermore, the limits also highlight how difficult it is to recover type - I parameters, t_{ij}, within type - II parameters. A suitably elaborate similarity transformation might allow us to recover all parameters, but the limits noted here provide sufficient justification. In the next section, generalised reflection matrices are constructed using the intricate type - II matrices. Later, we will use certain aspects of the limits that will reveal the importance of transposition in the limiting process.

4.5.2 Construction of Type - II Generalised Solutions

As we are aware, generalised reflection matrices require the inversion of the transmission matrix and because the type - II matrices are expressed in terms of raising and lowering operators we must calculate new general formulae for the inverse matrix. It is indeed possible to invert the type - II matrix, but there is one very striking feature: a determinant is associated to each set of (independent) operators $a_i, a_i^\dagger$, which is dependent on N_i. This is in contrast to earlier examples, where the determinant does not contain any dependence on the topological charge. Ultimately, we will see that this is not a problem as they are all related via shifts in the N_i.

The inverse of a transmission matrix possessing the same structure as (4.45) and (4.48), takes the form:

$$T'^{-1} = \begin{pmatrix} \frac{T_{11}^{-1}(N_1,N_2,N_3)}{\Delta_1(N_1,N_2,N_3)} & \frac{T_{12}^{-1}(N_1,N_2,N_3)}{\Delta_2(N_1+1,N_2-1,N_3)}a_1a_2^\dagger & \frac{T_{13}^{-1}(N_1,N_2,N_3)}{\Delta_3(N_1+1,N_2,N_3-1)}a_1a_3^\dagger \\ \frac{T_{21}^{-1}(N_1,N_2,N_3)}{\Delta_1(N_1-1,N_2+1,N_3-1)}a_2a_1^\dagger & \frac{T_{22}^{-1}(N_1,N_2,N_3)}{\Delta_2(N_1,N_2,N_3)} & \frac{T_{23}^{-1}(N_1,N_2,N_3)}{\Delta_3(N_1,N_2+1,N_3-1)}a_2a_3^\dagger \\ \frac{T_{31}^{-1}(N_1,N_2,N_3)}{\Delta_1(N_1-1,N_2,N_3+1)}a_3a_1^\dagger & \frac{T_{32}^{-1}(N_1,N_2,N_3)}{\Delta_2(N_1,N_2-1,N_3+1)}a_3a_2^\dagger & \frac{T_{33}^{-1}(N_1,N_2,N_3)}{\Delta_3(N_1,N_2,N_3)} \end{pmatrix},$$

with coefficients:

$$\begin{aligned} T_{11}^{-1}(N_1,N_2,N_3) = \Big[&T_{22}(N_1+1,N_2-1,N_3)T_{33}(N_1+1,N_2,N_3-1) \\ &- T_{23}(N_1+1,N_2-1,N_3)T_{32}(N_1+1,N_2,N_3-1)F_2(N_2-1)F_3(N_3-1)\Big], \end{aligned}$$

$$
\begin{aligned}
T_{12}^{-1}(N_1,N_2,N_3) = \Big[&T_{13}(N_1,N_2,N_3)T_{32}(N_1+1,N_2,N_3-1)F_3(N_3-1) \\
&- T_{12}(N_1,N_2,N_3)T_{33}(N_1+1,N_2,N_3-1)\Big],
\end{aligned}
$$

$$
\begin{aligned}
T_{13}^{-1}(N_1,N_2,N_3) = \Big[&T_{12}(N_1,N_2,N_3)T_{23}(N_1+1,N_2-1,N_3)F_2(N_2-1) \\
&- T_{13}(N_1,N_2,N_3)T_{22}(N_1+1,N_2-1,N_3)\Big],
\end{aligned}
$$

$$
\begin{aligned}
T_{21}^{-1}(N_1,N_2,N_3) = \Big[&T_{23}(N_1,N_2,N_3)T_{31}(N_1,N_2+1,N_3-1)F_3(N_3-1) \\
&- T_{21}(N_1,N_2,N_3)T_{33}(N_1,N_2+1,N_3-1)\Big],
\end{aligned}
$$

$$
\begin{aligned}
T_{22}^{-1}(N_1,N_2,N_3) = \Big[&T_{11}(N_1-1,N_2+1,N_3)T_{33}(N_1,N_2+1,N_3-1) \\
&- T_{13}(N_1-1,N_2+1,N_3)T_{31}(N_1,N_2+1,N_3-1)F_1(N_1-1)F_3(N_3-1)\Big],
\end{aligned}
$$

$$
\begin{aligned}
T_{23}^{-1}(N_1,N_2,N_3) = \Big[&T_{13}(N_1-1,N_2+1,N_3)T_{21}(N_1,N_2,N_3)F_1(N_1-1) \\
&- T_{11}(N_1-1,N_2+1,N_3)T_{23}(N_1,N_2,N_3)\Big],
\end{aligned}
$$

$$
\begin{aligned}
T_{31}^{-1}(N_1,N_2,N_3) = \Big[&T_{21}(N_1,N_2-1,N_3+1)T_{32}(N_1,N_2,N_3)F_2(N_2-1) \\
&- T_{22}(N_1,N_2-1,N_3+1)T_{31}(N_1,N_2,N_3)\Big],
\end{aligned}
$$

$$
\begin{aligned}
T_{32}^{-1}(N_1,N_2,N_3) = \Big[&T_{12}(N_1-1,N_2,N_3+1)T_{31}(N_1,N_2,N_3)F_1(N_1-1) \\
&- T_{11}(N_1-1,N_2,N_3+1)T_{32}(N_1,N_2,N_3)\Big],
\end{aligned}
$$

$$
\begin{aligned}
T_{33}^{-1}(N_1,N_2,N_3) = \Big[&T_{11}(N_1-1,N_2,N_3+1)T_{22}(N_1,N_2-1,N_3+1) \\
&- T_{12}(N_1-1,N_2,N_3+1)T_{21}(N_1,N_2-1,N_3+1)F_1(N_1-1)F_2(N_2-1)\Big].
\end{aligned} \tag{4.52}
$$

The above formulae, despite their complex appearance, are an infinite generalisation of Cramer's rule that account for the presence of the operators. The defining relations for all determinants: $\Delta_1, \Delta_2, \Delta_3$ are found in Appendix (B). We will now state the inverse matrix entries for (4.45), whereupon inversion of the rapidity we will have all components necessary to construct the first four generalised solutions. The inverse of matrix (4.1) has diagonal entries:

$$
\begin{aligned}
T'^{-1}_{11} = &\left(\frac{x}{z}\right)^{2/3} \frac{\kappa_1^2 (f_1^-)^2}{(1-q^4)} F_2(N_2-1) F_3(N_3-1) + \Bigg[\left(\frac{x}{z}\right)^{4/3} \frac{\kappa_1 q^{N_1-N_2+N_3+3}}{\kappa_2 (1-q^4)^2} \\
&+ \left(\frac{z}{x}\right)^{2/3} \frac{\kappa_1 (f_1^- f_2^+ f_3^-)^2}{\kappa_2} (1-q^4) q^{-N_1+N_2-N_3-3} \Bigg] \cdot \Bigg[\left(\frac{x}{z}\right)^{4/3} \frac{q^{N_1+N_2-N_3+3}}{\kappa_3 (1-q^4)^2} \\
&\qquad + \left(\frac{z}{x}\right)^{2/3} \frac{(f_1^- f_2^- f_3^+)^2}{\kappa_3} (1-q^4) q^{-N_1-N_2+N_3-3} \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
T'^{-1}_{22} = &\left(\frac{x}{z}\right)^{2/3} \frac{\kappa_3^2 (f_2^-)^2}{(1-q^4)} F_1(N_1-1) F_3(N_3-1) + \Bigg[\left(\frac{x}{z}\right)^{4/3} \frac{\kappa_3 q^{-N_1+N_2+N_3+3}}{(1-q^4)^2} \\
&+ \left(\frac{z}{x}\right)^{2/3} (f_1^+ f_2^- f_3^-)^2 \kappa_3 (1-q^4) q^{N_1-N_2-N_3-3} \Bigg] \cdot \Bigg[\left(\frac{x}{z}\right)^{4/3} \frac{q^{N_1+N_2-N_3+3}}{\kappa_3 (1-q^4)^2} \\
&\qquad + \left(\frac{z}{x}\right)^{2/3} \frac{(f_1^- f_2^- f_3^+)^2}{\kappa_3} (1-q^4) q^{-N_1-N_2+N_3-3} \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
T'^{-1}_{33} = &\left(\frac{x}{z}\right)^{2/3} \frac{(f_3^-)^2}{\kappa_2^2 (1-q^4)} F_1(N_1-1) F_2(N_2-1) + \Bigg[\left(\frac{x}{z}\right)^{4/3} \frac{\kappa_3 q^{-N_1+N_2+N_3+3}}{(1-q^4)^2} \\
&+ \left(\frac{z}{x}\right)^{2/3} \kappa_3 (f_1^+ f_2^- f_3^-)^2 (1-q^4) q^{N_1-N_2-N_3-3} \Bigg] \cdot \Bigg[\left(\frac{x}{z}\right)^{4/3} \frac{\kappa_1 q^{N_1-N_2+N_3+3}}{\kappa_2 (1-q^4)^2} \\
&\qquad + \left(\frac{z}{x}\right)^{2/3} \frac{\kappa_1}{\kappa_2} (f_1^- f_2^+ f_3^-)^2 (1-q^4) q^{-N_1+N_2-N_3-3} \Bigg].
\end{aligned}
$$

Unfortunately, the above expressions do not factorise neatly, however the off-diagonal entries are less cumbersome:

$$
\begin{aligned}
T'^{-1}_{12} &= \kappa_1 \left(\frac{x}{z}\right)^{4/3} \frac{q^{N_1+N_2-1}}{(1-q^4)^2} \left(\kappa_3 (f_3^+)^2 q^{2N_3} - (f_3^-)^2 (\kappa_3 - 1) q^{-2N_3+4} \right) \\
&\qquad + \kappa_1 \left(\frac{z}{x}\right)^{2/3} (f_1^- f_2^- f_3^-)^2 (f_3^+)^2 (1-q^4) q^{-N_1-N_2-3}, \\
T'^{-1}_{13} &= -\frac{\kappa_1 \kappa_3}{\kappa_2} \left(\frac{x}{z}\right)^2 \frac{q^{N_1+N_3+3}}{(1-q^4)^3} - \frac{\kappa_1}{\kappa_2} (f_1^- f_2^- f_3^-)^2 q^{-N_1-2N_2-N_3+1} \\
&\qquad + \frac{\kappa_1}{\kappa_2} (1-\kappa_3) (f_1^- f_2^+ f_3^-)^2 q^{-N_1+2N_2-N_3-3}, \\
T'^{-1}_{21} &= \kappa_1 (f_1^- f_2^-)^2 q^{-N_1-N_2-3} \left((\kappa_3 - 1) (f_3^+)^2 q^{2N_3} - \kappa_3 (f_3^-)^2 q^{2N_3+4} \right) \\
&\qquad - \kappa_1 \left(\frac{x}{z}\right)^2 \frac{q^{N_1+N_2+3}}{(1-q^4)^3},
\end{aligned}
$$

$$T'^{-1}_{23} = \left(\frac{x}{z}\right)^{4/3} \frac{q^{N_2+N_3-1}}{\kappa_2(1-q^4)^2}\left(\kappa_3(f_1^+)^2 q^{N_1} + (f_1^-)^2 q^{-2N_1+4}(1-\kappa_3)\right) + \left(\frac{z}{x}\right)^{2/3}\frac{(f_1^- f_2^- f_3^-)^2}{\kappa_2}(f_1^+)^2(1-q^4)q^{-N_2-N_3-3},$$

$$T'^{-1}_{31} = \frac{\kappa_1}{\kappa_2}\left(\frac{x}{z}\right)^{4/3}\frac{q^{N_1+N_3-1}}{(1-q^4)^2}\left((f_2^+)^2 q^{2N_2} + (f_2^-)^2 q^{-2N_2+4}(\kappa_3 - 1)\right) + \frac{\kappa_1\kappa_3}{\kappa_2}\left(\frac{z}{x}\right)^{2/3}(f_1^- f_2^- f_3^-)^2 (f_2^+)^2(1-q^4)q^{-N_1-N_3-3},$$

$$T'^{-1}_{32} = \frac{\kappa_3}{\kappa_2}(f_2^- f_3^-)^2 q^{-N_2-N_3-3}\left((f_1^+)^2 q^{2N_1} - (f_1^-)^2 q^{-2N_1+4}\right) - \left(\frac{x}{z}\right)^2 \frac{q^{N_2+N_3+3}}{\kappa_2(1-q^4)^3} - \frac{(f_1^+ f_2^- f_3^-)^2}{\kappa_2}q^{2N_1-N_2-N_3-3}.$$

When a type - I limit is applied the way in which the inverse matrix collapses is very clear. In particular, we see that when $f_i^- = 0$ and $\kappa_1\kappa_2\kappa_3 = 1$ for all i=1,2,3, no zeroes appear, which is what we expect upon comparison with $\tilde{T}_{IV}$. If $f_i^+ = 0$ and $\kappa_1\kappa_2\kappa_3 = 1$, for all i=1,2,3, it is clear that entries: 1,2; 2,3; 3,1 become zero and once the transpose is taken we see that the modified matrix has the same structure as $\tilde{T}_{II}$.

The inverse of type - II matrix (4.48) is like the above and its entries are:

$$T'^{-1}_{11} = \left(\frac{z}{x}\right)^{2/3}\frac{(f_1^-)^2}{\kappa_1^2(1-q^4)}F_2(N_2-1)F_3(N_3-1) + \left[\left(\frac{z}{x}\right)^{4/3}\frac{\kappa_2 q^{N_1-N_2+N_3+3}}{\kappa_1(1-q^4)^2} + \left(\frac{x}{z}\right)^{2/3}\frac{\kappa_2(f_1^- f_2^+ f_3^-)^2}{\kappa_1}(1-q^4)q^{-N_1+N_2-N_3-3}\right]\cdot\left[\left(\frac{z}{x}\right)^{4/3}\frac{\kappa_3 q^{N_1+N_2-N_3+3}}{(1-q^4)^2} + \left(\frac{x}{z}\right)^{2/3}\kappa_3(f_1^- f_2^- f_3^+)^2(1-q^4)q^{-N_1-N_2+N_3-3}\right],$$

$$T'^{-1}_{22} = \left(\frac{z}{x}\right)^{2/3}\frac{(f_2^-)^2}{(1-q^4)}F_1(N_1-1)F_3(N_3-1) + \left[\left(\frac{z}{x}\right)^{4/3}\frac{q^{-N_1+N_2+N_3+3}}{\kappa_3(1-q^4)^2} + \left(\frac{x}{z}\right)^{2/3}\frac{(f_1^+ f_2^- f_3^-)^2}{\kappa_3}(1-q^4)q^{N_1-N_2-N_3-3}\right]\cdot\left[\left(\frac{z}{x}\right)^{4/3}\frac{\kappa_3 q^{N_1+N_2-N_3+3}}{(1-q^4)^2} + \left(\frac{x}{z}\right)^{2/3}\kappa_3(f_1^- f_2^- f_3^+)^2(1-q^4)q^{-N_1-N_2+N_3-3}\right],$$

$$T'^{-1}_{33} = \left(\frac{z}{x}\right)^{2/3} \frac{\kappa_2^2 (f_3^-)^2}{(1-q^4)} F_1(N_1-1)F_2(N_2-1) + \left[\left(\frac{z}{x}\right)^{4/3} \frac{q^{-N_1+N_2+N_3+3}}{\kappa_3(1-q^4)^2} + \left(\frac{x}{z}\right)^{2/3} \frac{(f_1^+ f_2^- f_3^-)^2}{\kappa_3}(1-q^4)q^{N_1-N_2-N_3-3} \right] \cdot \left[\left(\frac{z}{x}\right)^{4/3} \frac{\kappa_2 q^{N_1-N_2+N_3+3}}{\kappa_1(1-q^4)^2} + \left(\frac{x}{z}\right)^{2/3} \frac{\kappa_2(f_1^- f_2^+ f_3^-)^2}{\kappa_1}(1-q^4)q^{-N_1+N_2-N_3-3} \right].$$

The above expressions denoting the diagonal entries do not factorise nicely, however the off-diagonal entries are more manageable:

$$\begin{aligned}
T'^{-1}_{12} &= -\left(\frac{z}{x}\right)^{2} \frac{q^{N_1+N_2+3}}{\kappa_1(1-q^4)^3} - \frac{(f_1^- f_2^- f_3^-)^2}{\kappa_1} q^{-N_1-N_2-2N_3+1}, \\
T'^{-1}_{13} &= \left(\frac{z}{x}\right)^{4/3} \frac{\kappa_2 (f_2^+)^2}{\kappa_1(1-q^4)^2} q^{N_1+2N_2+N_3-1} \\
&\qquad - \left(\frac{x}{z}\right)^{2/3} \frac{\kappa_2}{\kappa_1}(f_1^- f_2^- f_3^-)^2 (f_2^+)^2 (1-q^4) q^{-N_1-N_3-3}, \\
T'^{-1}_{21} &= \left(\frac{z}{x}\right)^{4/3} \frac{(f_3^+)^2}{\kappa_1(1-q^4)^2} q^{N_1+N_2+2N_3-1} \\
&\qquad + \left(\frac{x}{z}\right)^{2/3} \frac{(f_1^- f_2^- f_3^-)^2}{\kappa_1} (f_3^+)^2 (1-q^4) q^{-N_1-N_2-3}, \\
T'^{-1}_{23} &= -\kappa_2 \left(\frac{z}{x}\right)^{2} \frac{q^{N_2+N_3+3}}{(1-q^4)^3} - \kappa_2 (f_1^- f_2^- f_3^-)^2 q^{-2N_1-N_2-N_3+1}, \\
T'^{-1}_{31} &= -\left(\frac{z}{x}\right)^{2} \frac{\kappa_2 q^{N_1+N_3+3}}{\kappa_1(1-q^4)^3} - \frac{\kappa_2 (f_1^- f_2^- f_3^-)^2}{\kappa_1} q^{-N_1-2N_2-N_3+1}, \\
T'^{-1}_{32} &= \kappa_2 \left(\frac{z}{x}\right)^{4/3} \frac{(f_1^+)^2}{(1-q^4)^2} q^{2N_1+N_2+N_3-1} \\
&\qquad + \kappa_2 \left(\frac{x}{z}\right)^{2/3} (f_1^- f_2^- f_3^-)^2 (f_1^+)^2 (1-q^4) q^{-N_2-N_2-3}.
\end{aligned}$$

In this case, different entries become zero in the various limits, but they display the desired characteristics. The non-diagonal entries have been simplified by using the constraint $\kappa_1\kappa_2\kappa_3 = 1$. Again, there are no zero entries when the limit requires $f_i^- = 0$, and this matches the structure of $\tilde{T}_{III}$. In contrast, if the limit requires $f_i^- = 0$ then three zeroes emerge in entries: 1,3; 2,1; 3,2 and by taking the transpose we find that the inverse matrix has the same shape as $\tilde{T}_I$.

Now, equipped with the inverse matrices we can invert their rapidities (to form

$\tilde{T}'$) and evaluate the product to obtain type - II generalised solutions. As a result of the involved nature of the transmission matrices it is more instructive to document the symbolic multiplication, where the action of the operators is most vivid, instead of listing all eight generalised solutions. In an attempt to maintain perspicuity, any label T_{ij} or $\tilde{T}_{ij}$ now represents $T_{ij}/\tilde{T}_{ij}(N_1, N_2, N_3)$ and only shifted variables are presented in the formulae. The generalised solutions' entries are obtained by substituting the relevant components into the following formulae:

$$\begin{aligned}
\tilde{R}_{11} = T_{11}R_{d_{i_{11}}}\tilde{T}_{11} \;+\; & T_{12}R_{d_{i_{22}}}\tilde{T}_{21}(N_1+1, N_2-1)F_1(N_1)F_2(N_2-1) \\
& + \; T_{13}R_{d_{i_{33}}}\tilde{T}_{31}(N_1+1, N_3-1)F_1(N_1)F_3(N_3-1), \\
\tilde{R}_{12} = T_{11}R_{d_{i_{11}}}\tilde{T}_{12} \;+\; & T_{12}R_{d_{i_{22}}}\tilde{T}_{22}(N_1+1, N_2-1) \\
& + \; T_{13}R_{d_{i_{33}}}\tilde{T}_{31}(N_1+1, N_3-1)F_3(N_3-1), \\
\tilde{R}_{13} = T_{11}R_{d_{i_{11}}}\tilde{T}_{13} \;+\; & T_{12}R_{d_{i_{22}}}\tilde{T}_{23}(N_1+1, N_2-1)F_2(N_2-1) \\
& + \; T_{13}R_{d_{i_{33}}}\tilde{T}_{33}(N_1+1, N_3-1), \\
\tilde{R}_{21} = T_{21}R_{d_{i_{11}}}\tilde{T}_{11}(N_1-1, N_2+1) \;+\; & T_{22}R_{d_{i_{22}}}\tilde{T}_{21} \\
& + \; T_{23}R_{d_{i_{33}}}\tilde{T}_{31}(N_2+1, N_3-1)F_3(N_3-1), \\
\tilde{R}_{22} = T_{21}R_{d_{i_{11}}}\tilde{T}_{12}(N_1-1, N_2+1)F_2(N_2)F_1(N_1-1) \;+\; & T_{22}R_{d_{i_{22}}}\tilde{T}_{22} \\
& + \; T_{23}R_{d_{i_{33}}}\tilde{T}_{32}(N_2+1, N_3-1)F_2(N_2)F_3(N_3-1), \\
\tilde{R}_{23} = T_{21}R_{d_{i_{11}}}\tilde{T}_{13}(N_1-1, N_2+1)F_1(N_1-1) \;+\; & T_{22}R_{d_{i_{22}}}\tilde{T}_{23} \\
& + \; T_{23}R_{d_{i_{33}}}\tilde{T}_{33}(N_2+1, N_3-1), \\
\tilde{R}_{31} = T_{31}R_{d_{i_{11}}}\tilde{T}_{11}(N_1-1, N_3+1) \;+\; & T_{32}R_{d_{i_{22}}}\tilde{T}_{21}(N_2-1, N_3+1)F_2(N_2-1) \\
& + \; T_{33}R_{d_{i_{33}}}\tilde{T}_{31}, \\
\tilde{R}_{32} = T_{31}R_{d_{i_{11}}}\tilde{T}_{12}(N_1-1, N_3+1)F_1(N_1-1) \;+\; & T_{32}R_{d_{i_{22}}}\tilde{T}_{22}(N_2-1, N_3+1) \\
& + \; T_{33}R_{d_{i_{33}}}\tilde{T}_{32}, \\
\tilde{R}_{33} = T_{31}R_{d_{i_{11}}}\tilde{T}_{13}(N_1-1, N_3+1)&F_1(N_1-1)F_3(N_3) \\
+ \; T_{32}R_{d_{i_{22}}}\tilde{T}_{23}(N_2-1, N_3+1)&F_2(N_2-1)F_3(N_3) \;+\; T_{33}R_{d_{i_{33}}}\tilde{T}_{33},
\end{aligned}$$

where $R_{d_{i_{jj}}}$ denotes each entry of the diagonal reflection matrices. At this point, we must remember to include the determinants that appear throughout the inverse matrix. For our purposes, we will consider the first row of the generalised reflection matrix where each entry depends upon (reading from entry 1,1 to 1,3):

$$\Delta_1^{-1}(N_1, N_2, N_3), \quad \Delta_2^{-1}(N_1+1, N_2-1, N_3), \quad \Delta_3^{-1}(N_1+1, N_2, N_3-1).$$

The second row depends upon the same determinants with the shifts: $N_1 \to N_1 - 1, N_2 \to N_2 + 1$ and the third row is shifted in the following way: $N_1 \to N_1 - 1, N_3 \to N_3 + 1$. Initially, this seems somewhat unusual but when Maple is used to calculate the shifted determinants we find several equalities:

$$\Delta_1(N_1, N_2, N_3) \ = \ \Delta_2(N_1+1, N_2-1, N_3) \ = \ \Delta_3(N_1+1, N_2-1, N_3).$$

And so, one quickly observes that each row contains the same overall factor, therefore it can be extracted via row operations. Furthermore, whenever a limit from type - II to type - I is applied, the determinants, Δ_i, collapse to achieve equality. And so, we can progress to consider all type - II generalised solutions, knowing that the determinants do not damage or further complicate the workings.

Unsurprisingly, all eight type - II generalised solutions do not contain any zero entries. Subsequently, this suggests that the type - II defect provides ultimate freedom within the system: by which we mean that any root like charge can be deposited at the defect, therefore generalising the type - I framework. However, we have not yet provided any details to explain how the type - II defect selects either the clockwise or anti-clockwise permutation of extended simple roots. Presently, the type - I limits of type - II defect matrices are known and through them we know that each matrix (4.48) and (4.45) is related to the solitonic/anti-solitonic representation respectively. As a consequence of the limits, one reasonably expects that type - II solutions should break down to reproduce the structure of a type - I solution. Undeniably, this does take place, but the emerging pattern leads us to connect the transposition of the solution with the permutation. Let us consider any generalised constructed from matrix (4.45) and apply the limit where $f_i^- = 0$. Due to the nature of this limit, matrix (4.45) will share the same structure as T_{IV}, we predict that zeroes will appear to give the solutions the same shape as: $\hat{R}_{IV}, \tilde{R}_{IV}, \tilde{R}_{VIII}, \tilde{R}_{XII}$. Actually, when $f_i^- = 0$ and the transpose is taken, we find the structures of solutions: $\hat{R}_{II}, \tilde{R}_{II}, \tilde{R}_{VI}, \tilde{R}_X$. Upon reflection, this discovery is not so shocking because we have taken the transpose of a product, specifically:

$$\left(T'_{\to IV} R_{d_i} \tilde{T}'_{\to IV}\right)^T = \left(\tilde{T}'_{\to IV}\right)^T R_{d_i} \left(T'_{\to IV}\right)^T.$$

If we now recognise the transposed matrices' structure, we actually have an object most like:

$$\left(\tilde{T}'_{\to IV}\right)^T R_{d_i} \left(T'_{\to IV}\right)^T \approx \hat{T}_{II} R_{d_i} \hat{\tilde{T}}_{II},$$

thus explaining why the other type - I structures appear. We can apply this argument to all limited type - II generalised solutions, to explain why they appear as they do. Therefore, we can reason that transposition of the solution is connected to the permutation of simple roots. All calculations of this sort, concerning the type - II defect matrices, were evaluated in Maple that can handle the elaborate matrix multiplication and impose limits easily. Furthermore, the pleasing behaviour exhibited by type - II limited solutions highlights the important relation between the $a_2^{(1)}$ type - I transmission matrices and their inverses.

Overall, we must remember that the defect does not change the original boundary condition. The diagonal reflection matrices used in this chapter are 'Soliton Preserving' (SP). We postulate that the new solutions correspond to a generalisation of that particular type of boundary condition (1.30). Such a generalisation might contain an extra free parameter, for example, which embodies the soliton's added freedom to exchange charge with the defect freely.

Chapter 5

Conclusions and Outlook

The main theme of this thesis concerns the generalisation of finite-dimensional reflection matrices and its associated mechanisms. The construction of generalised reflection matrices is known to provide new solutions to a suitably generalised reflection equation (1.96). In the case of the sine-Gordon model, the most general solution of this type lies outside the known classes of solution and is thought to correspond to a more general integrable boundary condition [17].

In chapter two, the role of sine-Gordon's generalised reflection matrices within the algebraic framework of [18] was documented. It was shown that the associated boundary subalgebra of a generalised type - I solution not only agrees with the results of Delius and MacKay, but generalises them by accounting for the topological charge at the boundary. As we already know, type - I generalised solutions are constructed from a diagonal reflection matrix that corresponds to an integrable Dirichlet boundary. The resulting solution has the same underlying structure as a Zamolodchikov-Ghoshal solution (2.3), but is infinite-dimensional due to dependence on the topological charge. The way in which the defect modifies the original boundary subalgbera of the diagonal reflection matrix is readily apparent. Calculations of this type might aid the classification of integrable boundary conditions, as each boundary condition possesses its own boundary subalgebra. Perhaps, in the future, other generalised solutions should be compared in the same way to discover if the new solution does indeed correspond to a more general boundary condition. However, when the same calculation is performed with the type - II defect, the resulting boundary subalgebra lies outside the known class of results. In this case, the determinant of the type - II T-matrix plays an important role, as it does not cancel throughout. Its presence guarantees that the construction works. Potentially, this might allude to some hidden algebraic significance, or it simply demonstrates the importance of transmission matrix inversion.

In chapters three and four, generalised solutions were calculated for the $a_2^{(2)}$ and $a_2^{(1)}$ ATFTs respectively. In both cases, the new solutions naturally generalise the finite-dimensional cases. Unfortunately, the results concerning the $a_2^{(2)}$ affine Toda model are incomplete, as it was not possible to recover the most general solution within a generalised solution, which we have discussed. For the $a_2^{(1)}$ theory, several intriguing results arise. However, this behaviour is unsurprising when the classical behaviour of the defect is considered. The specific way that solitons transmit through the defect clearly comes into play, and the significance of the original boundary parameter also becomes clear. Solutions constructed from a diagonal reflection factor that does not possess a free parameter, $\hat{R}$, mimicked the defect's selective behaviour - although, in a slightly different fashion. The behaviour is documented in several diagrams that explicitly illustrate the extra freedom added by a defect. The presence of a boundary parameter, υ, allowed further freedom within the solutions, $\tilde{R}$, and similar diagrams illustrate their processes. However, it still remains to associate these generalised solutions to integrable boundary conditions. We imagine that they generalise conditions of the form (1.30), whereby extra parameters may be included to describe the added movement around the weight lattice. In the future, it would be natural to investigate the $a_3^{(1)}$ affine Toda model to see if the same behaviour arises. If it does, then one might be able to generalise the framework to include all $a_n^{(1)}$ generalised reflection matrices. Ultimately, the generalised solutions that we have calculated and presented in this thesis show the strong relationship between defects and boundaries. For example, they naturally accommodate one another and generalised reflection matrices naturally satisfy a generalised reflection equation. Unfortunately, a full classification of solutions still escapes us, but generalised solutions definitely do generalise the finite-dimensional framework.

With the classification of solutions weighing heavily on our minds, we will briefly mention a recent classification of particular reflection matrices provided by Regelskis and Vlaar, [71]. The framework and classification that they propose associates a Satake diagram to every trigonometric reflection matrix corresponding to particular coideal subalgebras that are described by admissable pairs, the theory of which was developed by Letzter and Kolb [72, 73]. Their results also rely on the theory of quantum symmetric pairs, many properties of the quantum group and its subalgebras [72, 73]. However, their classification concerns only finite-dimensional solutions. It is natural to expect that a defect and its associated generalised solutions can be included in an appropriate infinite-dimensional framework. If this is possible, the classification of generalised solutions might become more simple and this could lead to advances regarding integrable boundary conditions.

Recently, Lima-Santos and Vieira obtained reflection matrices for the $D^{(2)}_{n+1}$ affine algebra [74]. This model appears to exhibit many interesting reflection factors, several of which possess different structures. In particular, several patterns of zeroes arise; much like the $a_2^{(1)}$ case. As defects and boundaries appear to marry together so naturally, one wonders whether the structure of the reflection matrices could indicate certain behaviours concerning a defect of the theory.

Finally, in chapter four, only soliton preserving solutions were considered. In the literature, concerning $a_2^{(1)}$, there exist several soliton non-preserving solutions, calculated by Gandenberger [75, 76]. If one allows a soliton to convert to an anti-soliton at the boundary, and change multiplet, the reflection matrix must act over the spaces:

$$R_a^{\bar{b}}(\theta) : V_a \to V_{\bar{b}},$$

$$R_{\bar{a}}^{b}(\theta) : V_{\bar{a}} \to V_b,$$

where the barred indices refer to anti-soliton and unbarred indices refer to the soliton. Such reflection matrices satisfy a reflection equation of the form [75, 76]:

$$\begin{aligned} S_{ij}^{kl}(\theta_i - \theta_j) R_l^{\bar{m}}(\theta_i) S_{k\bar{m}}^{\bar{p}n}(\theta_i + \theta_j) R_n^{\bar{r}}(\theta_j) \\ = R_j^{\bar{k}}(\theta_j) S_{i\bar{k}}^{\bar{l}m}(\theta + \theta') R_m^{\bar{n}}(\theta_i) S_{\bar{l}\bar{n}}^{\bar{p}\bar{r}}(\theta - \theta'). \end{aligned} \tag{5.1}$$

In the above equation one can use crossing symmetry, as follows:

$$S_{i\bar{j}}^{\bar{k}l}(\theta) = S_{ki}^{lj}(i\pi - \theta), \qquad S_{\bar{i}\bar{j}}^{\bar{k}\bar{l}}(\theta) = S_{lk}^{ji}(\theta),$$

to simplify equation (5.1). Unfortunately, when crossing symmetry is used and the indices are permuted, some of the resulting equations become unbalanced. As an example consider the following process,

incoming: $i = +, j = -$
outgoing: $p = +, r = -$

that returns the equation:

$$\begin{aligned} &S_{+-}^{+-}(\theta_i - \theta_j)R_{-}^{+}(\theta_i)S_{++}^{++}(\theta_i + \theta_j)R_{+}^{-}(\theta_j) \\ &+ S_{+-}^{-+}(\theta_i - \theta_j)R_{+}^{+}(\theta_i)S_{+-}^{-+}(\theta_i + \theta_j)R_{-}^{-}(\theta_j) \\ &+ S_{+-}^{-+}(\theta_i - \theta_j)R_{+}^{-}(\theta_i)S_{+-}^{+-}(\theta_i + \theta_j)R_{+}^{-}(\theta_j) \\ &= R_{-}^{+}(\theta_j)S_{++}^{++}(i\pi - (\theta_i + \theta_j))R_{+}^{-}(\theta_i)S_{-+}^{-+}(\theta_i - \theta_j) \\ &+ R_{-}^{+}(\theta_j)S_{-+}^{-+}(i\pi - (\theta_i + \theta_j))R_{-}^{+}(\theta_i)S_{-+}^{+-}(\theta_i - \theta_j) \\ &+ R_{-}^{-}(\theta_j)S_{-+}^{+-}(i\pi - (\theta_i + \theta_j))R_{+}^{+}(\theta_i)S_{-+}^{+-}(\theta_i - \theta_j). \end{aligned} \tag{5.2}$$

The solutions that Gandenberger supplies do indeed satisfy all eighty one equations [75, 76]. However, when one attempts to construct a generalised solution there is an immediate problem. Usually, the Kronecker-deltas and raising/lowering operators describing the exchange of topological charge appear unanimously throughout the whole equation. In the above, this is not the case. Four terms are proportional to δ_{α}^{β} but the remaining two contain $\delta_{\alpha}^{\beta+\alpha_0}$ and $\delta_{\alpha}^{\beta-\alpha_0}$, hence they will not cancel. Further investigation is required. However, one possible explanation is that the construction breaks down because one of the necessary steps in the proof (1.108) is violated. On the whole, this is strange because we know that the $a_2^{(1)}$ T-matrices are compatible with the bootstrap [48]. In the future, this problem should be addressed and perhaps the bootstrap can remedy the issues appearing here.

To conclude, several classes of generalised reflection matrix have been calculated and presented in this thesis. It is hoped that they offer some insight into potentially new integrable boundary conditions. Nevertheless, this is an exciting time within integrability, as new interest has been revived in quantum symmetric pairs [71, 72, 73] and the surrounding theory might provide further insight into outstanding problems. The author would like to thank the reader for their time and patience while reading this thesis, and also wishes them every success with their own mathematical endeavours.

Appendix A

Depiction of $a_2^{(1)}$ Generalised solutions

In this appendix all remaining diagrammatic representations of the generalised solutions can be found. The additional freedom added by the defect is shown by the dotted grey lines for all $\hat{R}$ solutions, where the diagonal reflection matrix contains no boundary parameter, represented by dashed lines. For the $\tilde{R}$ solutions, the dotted green lines denote the freedom added by the defect, when the original reflection matrix contains a boundary parameter, υ. As we have remarked earlier, the processes that the diagrams represent are one way of classifying the generalised solutions. It is not necessary to form similar diagrams for the type - II solutions, because the corresponding solutions possess ultimate freedom - the solitons can freely interact with the defect, depositing any allowable charge. Therefore, after this interaction the soliton's charge can convert to either neighbouring weight.

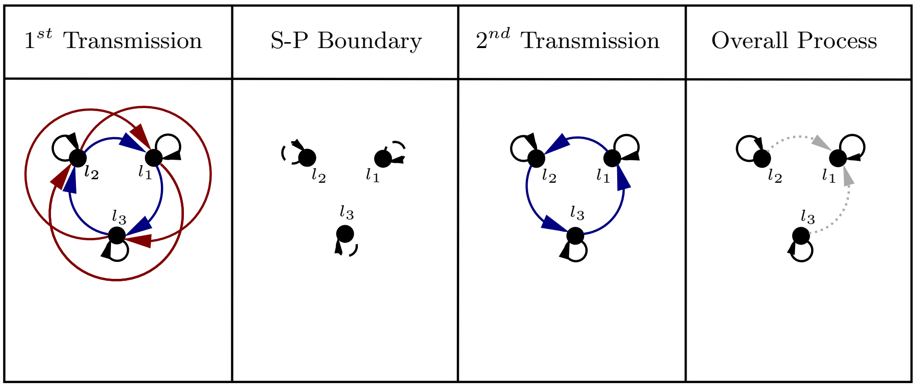

Figure A.1: Pictorial representation of generalised solution $\hat{R}_{II}$.

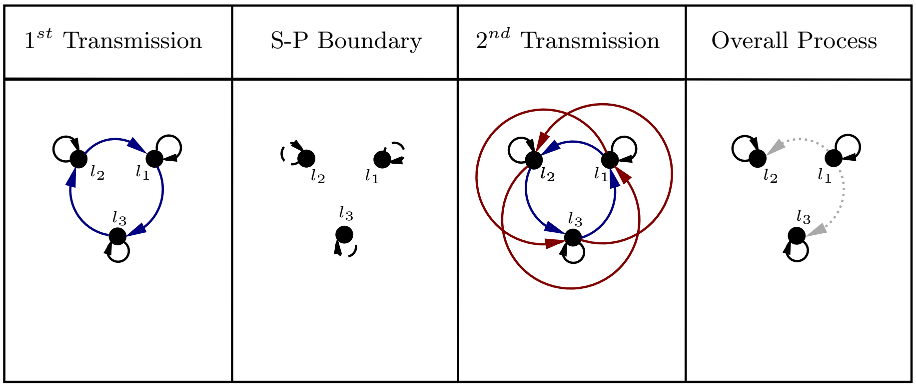

Figure A.2: Pictorial representation of generalised solution $\hat{R}_{III}$.

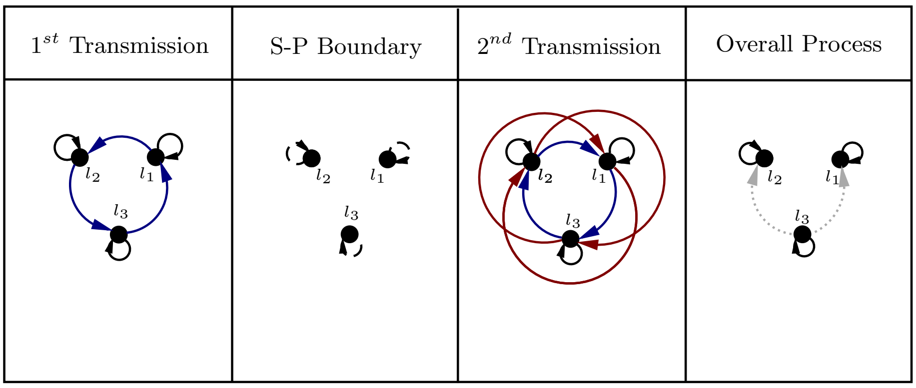

Figure A.3: Pictorial representation of generalised solution $\hat{R}_{IV}$.

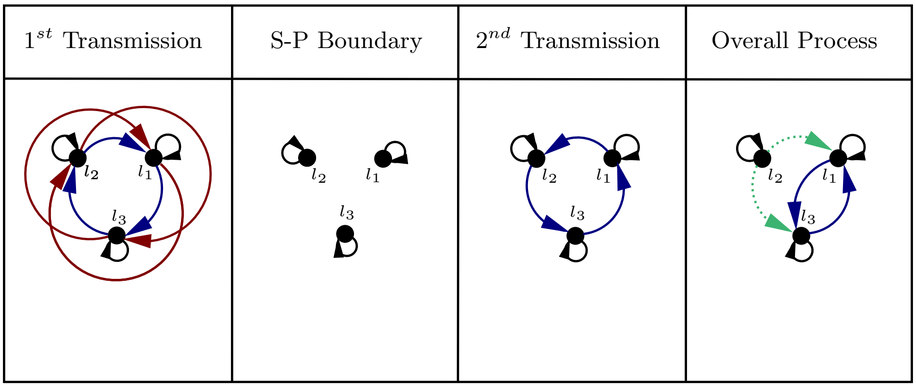

Figure A.4: Pictorial representation of generalised solutions $\tilde{R}_{IV,XI}$.

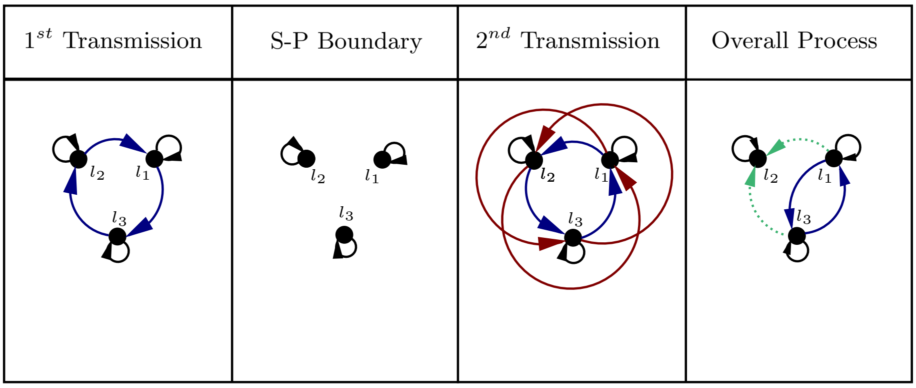

Figure A.5: Pictorial representation of generalised solutions $\tilde{R}_{II,V}$.

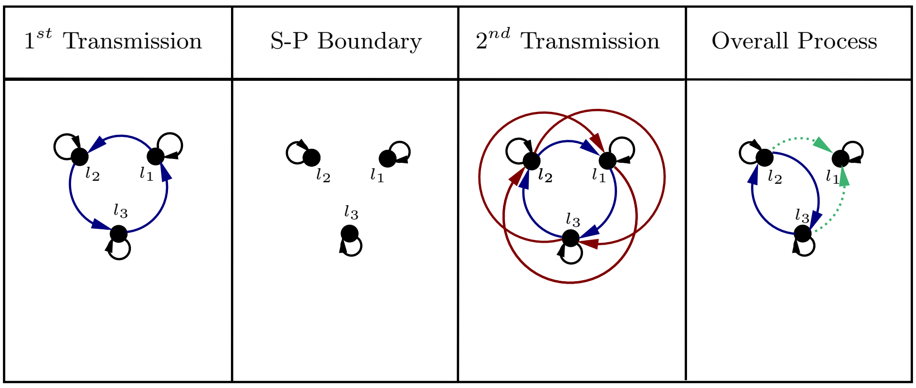

Figure A.6: Pictorial representation of generalised solutions $\tilde{R}_{III,VIII}$.

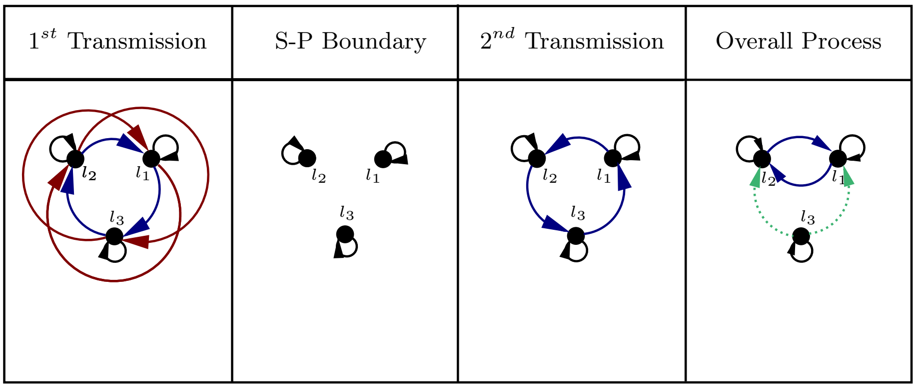

Figure A.7: Pictorial representation of generalised solutions $\tilde{R}_{VI,IX}$.

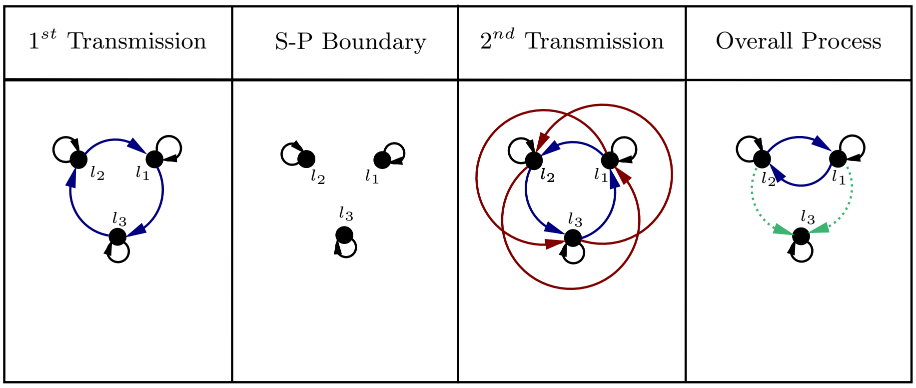

Figure A.8: Pictorial representation of generalised solutions $\tilde{R}_{VII,XII}$.

Appendix B

Determinant Formulae for the $a_2^{(1)}$ Transmission Matrix

Previously, several equalities among the determinants, $\Delta_1, \Delta_2, \Delta_3$, were stated and now we will present their defining equations. Yet again, we see that the formulae are an infinite-dimensional generalisation of the usual formulae to calculate a three-by-three matrix determinant, they are listed below where $\Delta_i := \Delta_i(N_1, N_2, N_3)$ for now:

$$\begin{aligned}
\Delta_1 := \quad & T_{11}(N_1, N_2, N_3)\big[T_{22}(N_1+1, N_2-1, N_3)T_{33}(N_1+1, N_2, N_3-1) \\
& - T_{23}(N_1+1, N_2-1, N_3)T_{32}(N_1+1, N_2, N_3-1)F_2(N_2-1)F_3(N_3-1)\big] \\
& + T_{12}(N_1, N_2, N_3)F_1(N_1)F_2(N_2-1)\cdot \\
& \qquad \big[T_{23}(N_1+1, N_2-1, N_3)T_{31}(N_1+1, N_2, N_3-1)F_3(N_3-1) \\
& \qquad\qquad - T_{21}(N_1+1, N_2-1, N_3)T_{33}(N_1+1, N_2, N_3-1)\big] \\
& + T_{13}(N_1, N_2, N_3)F_1(N_1)F_3(N_3-1)\cdot \\
& \qquad \big[T_{21}(N_1+1, N_2-1, N_3)T_{32}(N_1+1, N_2, N_3-1)F_2(N_2-1) \\
& \qquad\qquad - T_{22}(N_1+1, N_2-1, N_3)T_{31}(N_1+1, N_2, N_3-1)\big],
\end{aligned} \tag{B.1}$$

$$
\begin{aligned}
\Delta_2 :=\ & T_{21}(N_1,N_2,N_3)F_2(N_2)F_1(N_1-1)\cdot \\
& \big[T_{13}(N_1-1,N_2+1,N_3)T_{32}(N_1,N_2+1,N_3-1)F_3(N_3-1) \\
& \qquad - T_{12}(N_1-1,N_2+1,N_3)T_{33}(N_1,N_2+1,N_3-1)\big] \\
& + T_{22}(N_1,N_2,N_3)\big[T_{11}(N_1-1,N_2+1,N_3)T_{33}(N_1,N_2+1,N_3-1) \\
& - T_{13}(N_1-1,N_2+1,N_3)T_{31}(N_1,N_2+1,N_3-1)F_1(N_1-1)F_3(N_3-1)\big] \\
& + T_{23}(N_1,N_2,N_3)F_2(N_2)F_3(N_3-1)\cdot \\
& \quad \big[T_{12}(N_1-1,N_2+1,N_3)T_{31}(N_1,N_2+1,N_3-1)F_1(N_1-1) \\
& \qquad - T_{11}(N_1-1,N_2+1,N_3)T_{32}(N_1,N_2+1,N_3-1)\big],
\end{aligned} \tag{B.2}
$$

and finally,

$$
\begin{aligned}
\Delta_3 :=\ & T_{31}(N_1,N_2,N_3)F_3(N_3)F_1(N_1-1)\cdot \\
& \big[T_{12}(N_1-1,N_2,N_3+1)T_{23}(N_1,N_2-1,N_3+1)F_2(N_2-1) \\
& \qquad - T_{13}(N_1-1,N_2,N_3+1)T_{22}(N_1,N_2-1,N_3+1)\big] \\
& + T_{32}(N_1,N_2,N_3)F_3(N_3)F_2(N_2-1)\cdot \\
& \quad \big[T_{13}(N_1-1,N_2,N_3+1)T_{21}(N_1,N_2-1,N_3+1)F_1(N_1-1) \\
& \qquad - T_{11}(N_1-1,N_2,N_3+1)T_{23}(N_1,N_2-1,N_3+1)\big] \\
& + T_{33}(N_1,N_2,N_3)\big[T_{11}(N_1-1,N_2,N_3+1)T_{22}(N_1,N_2-1,N_3+1) \\
& - T_{12}(N_1-1,N_2,N_3+1)T_{21}(N_1,N_2-1,N_3+1)F_1(N_1-1)F_2(N_2-1)\big].
\end{aligned} \tag{B.3}
$$

The equalities:

$$\Delta_1(N_1,N_2,N_3) \ = \ \Delta_2(N_1+1,N_2-1,N_3) \ = \ \Delta_3(N_1+1,N_2-1,N_3),$$

are checked easily by shifting all N_i appropriately and identifying the like terms within the formulae.

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