Restriction Semigroups: Structure, Varieties and Presentations

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Abstract

Classes of (left) restriction semigroups arise from partial transformation monoids and form a wider class than inverse semigroups.

Firstly, we produce a presentation of the Szendrei expansion of a monoid, which is a left restriction monoid, using a similar approach to Exel's presentation for the Szendrei expansion of a group. Presentations for the Szendrei expansion of an arbitrary left restriction semigroup and of an inverse semigroup are also found.

For our second set of results we look at structure theorems, or P-theorems, for proper restriction semigroups and produce results in a number of ways. Initially, we generalise Lawson's approach for the proper ample case, in which he adapted the one-sided result for proper left ample semigroups. The awkwardness of this approach illustrates the need for a symmetrical two-sided result. Creating a construction from partial actions, based on the idea of a double action, we produce structure theorems for proper restriction semigroups. We also consider another construction based on double actions which yields a structure theorem for a particular class of restriction semigroups. In fact, this was our first idea, but the class of proper restriction semigroups it produces is not the whole class.

For our final topic we consider varieties of left restriction semigroups. Specifically, we shall show that the class of (left) restriction semigroups having a cover over a variety of monoids is a variety of (left) restriction semigroups. We do this in two ways. Generalising results by Gomes and Gould on graph expansions, we consider the graph expansion of a monoid and obtain our result for the class of left restriction monoids. Following the same approach as Petrich and Reilly we produce the result for the class of left restriction semigroups and for the class of restriction semigroups.

Contents

Preface

Left restriction semigroups have appeared in the literature under various names including *function semigroups* in [57] in the work of Trokhimenko, *type SL2 γ-semigroups* in the work of Batbedat in [4] and [5], *twisted LC-semigroups* in the work of Jackson and Stokes in [33], *guarded semigroups* in the work of Manes in [39] and more recently as *weakly left E-ample semigroups*. Restriction semigroups are believed to have first appeared as *function systems* in the work of Schweizer and Sklar [54] in the 1960s. We shall look at these appearances in more detail in Chapter 2. We shall provide an abstract definition of left restriction semigroups and look at how they are precisely the $(2,1)$-subalgebras of partial transformation monoids, along with examples. We shall provide another definition for left restriction semigroups as a class of algebras defined by identities. In Chapter 2 we shall also give an introduction to weakly (left) ample and (left) ample semigroups. We shall look at the natural partial order on (left) restriction semigroups and the least congruence identifying the distingished semilattice of idempotents associated with the (left) restriction semigroup. We shall look at proper (left) restriction semigroups and proper covers in subsequent chapters, but provide a brief introduction in Chapter 2.

In Chapters 1 and 2 we provide background definitions and results from universal algebra. In particular, we look at different types of algebras, generating sets, morphisms and congruences. We also look at free objects, categories and varieties. We present some results, which are generalisations from the weakly ample case, for which the proofs are essentially the same.

After the introductory chapters we look at three different, but related, topics. Our first, presentations of Szendrei expansions, will be looked at in Chapter 3. Across Chapters 4, 5, 6, 7 and 8 we look at structure theorems, our second topic. Our final topic, varieties, shall be considered in Chapters 9 and 10 where we use two different approaches to prove the same result.

The Szendrei expansion is one of two types of expansions we consider. Expansions are used to produce a global action from a partial action, but we shall not study this directly. As well as Szendrei expansions, we also consider graph expansions. We use graph expansions in Chapter 9 as a tool to obtain the result that the class of left restriction monoids having a proper cover over a variety of monoids is itself a variety of left restriction monoids. In Chapter 3 we look at presentations of the Szendrei expansion of various algebras. Looking first at the Szendrei expansion of a group, which coincides with the Birget-Rhodes expansion (as pointed out in [56]), we consider the "expansion" of a group which Exel described, via generators and relations, in [11]. Kellendonk and Lawson later proved in [35] that Exel's expansion is isomorphic to the Szendrei expansion. We therefore have a presentation of the Szendrei expansion of a group, which involves factoring a free semigroup by the congruence generated by certain relations inspired by the definition of premorphism for groups. By looking at the relevant definition of a premorphism, we obtain a presentation for the Szendrei expansion of a monoid by factoring the free left restriction semigroup by a congruence generated by certain relations. Similarly we produce presentations of the Szendrei expansion of a left restriction semigroup and inverse semigroup by factoring the free left restriction semigroup and free inverse semigroup respectively by congruences determined by premorphisms.

Looking at our second topic, we provide mainly background material in Chapters 4 and 5. In Chapter 4 we present McAlister's covering theorem from [42] which states that every inverse semigroup has an E-unitary cover, which is the important point behind his P-theorem. The P-theorem from [43] is a structure theorem which states that every E-unitary inverse semigroup is isomorphic to a *P-semigroup*, a structure consisting of the ingredients of a group, a semilattice and a partially ordered set, and conversely that every such P-semigroup is an E-unitary inverse semigroup.

In Chapter 5 we present covering theorems and structure theorems that were prompted by McAlister's work, for proper left ample, proper weakly left ample and proper left restriction semigroups. In particular, we look at a structure theorem from [12] for proper left ample semigroups based on a structure $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$, where $\mathscr{X}$ is a partially ordered set, $\mathscr{Y}$ is a subsemilattice of $\mathscr{X}$ and T is a left cancellative monoid acting on the right of $\mathscr{X}$, all subject to certain conditions. This was originally defined in [12] with an alternative description of this structure presented in [36] where it was named an *M-semigroup*. We look at a structure $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$, similar to an M-semigroup, presented in [19], known as a *strong M-semigroup*. We present structure theorems for proper left restriction semigroups [7] and for proper weakly left ample semigroups [19] involving strong M-semigroups where we take T to be a monoid and a unipotent monoid respectively. We also demonstrate that if you restrict T to be a right cancellative monoid, we obtain a structure theorem for proper left ample semigroups and how if we take T to be a group and alter the definition of a strong M-semigroup we obtain a structure theorem for proper inverse semigroups.

In Chapter 6 we start to consider two-sided results. We begin by looking at how the one-sided structure theorem for proper left ample semigroups was adapted to obtain the two-sided result for proper ample semigroups in [36]. We use this approach to obtain two-sided results for proper restriction and proper weakly ample semigroups by adapting the one-sided results from Chapter 5. However, the conditions imposed on the strong M-semigroups are even more complicated, and like the structure considered in the proper ample case, they do not reflect the natural symmetry within the proper restriction, proper weakly ample and proper ample semigroups. In Chapters 7 and 8 we present two attempts at providing such a symmetrical structure theorem.

Based on the idea of a double action and adapted from a strong M-semigroup, we present a structure $\mathscr{M}(T, \mathscr{X}, \mathscr{X}', \mathscr{Y})$, where T is a monoid, $\mathscr{X}$ and $\mathscr{X}'$ are semilattices and $\mathscr{Y}$ is a subsemilattice of both $\mathscr{X}$ and $\mathscr{X}'$. We show that it is proper restriction, and it can be made proper weakly ample or proper ample by imposing conditions on the monoid. We explain the approaches to try to prove the converse and present the case that suggested that the converse was not necessarily always true. We present work by Gould on the class of restriction semigroups which are isomorphic to some $\mathscr{M}(T, \mathscr{X}, \mathscr{X}', \mathscr{Y})$, which are now known as *extra proper* restriction semigroups.

In Chapter 8 we present another structure based on partial actions that was adapted from the previous structure and still features symmetry. The construction $\mathscr{M}(T, \mathscr{Y})$, where T is a monoid and $\mathscr{Y}$ is a semilattice, we believe is analogous to that of Petrich and Reilly in the inverse case [48] and Lawson in the ample case [36]. We present proofs for a structure theorem for proper restriction semigroups based on $\mathscr{M}(T, \mathscr{Y})$, both using the one-sided results and also directly. We also consider the relationship between the partial actions and the original actions. By imposing conditions on the monoid we also obtain structure theorems for proper weakly ample and proper ample semigroups.

Moving on to our third topic, we look at the class of restriction semigroups that have a proper cover over a variety of monoids. As proved by Petrich and Reilly in [47], the class of inverse monoids having a proper cover over a variety of groups, $\mathcal{V}$, is a variety of inverse monoids, which is determined by

$$\Sigma = \{\bar{u}^2 \equiv \bar{u} : \bar{u} \equiv 1 \text{ is a law in } \mathcal{V}\}.$$

In the left ample case problems were encounted when trying to use Petrich and Reilly's approach due to left ample semigroups forming a quasivariety rather than a variety. A different approach was used by Gould in [23] for left ample monoids which involved graph expansions. It is proved in [23] that the class of left ample monoids having a cover over $\mathcal{V}$ forms a quasivariety, where $\mathcal{V}$ is a subquasivariety of the quasivariety $\mathcal{RC}$ of right cancellative monoids defined (within $\mathcal{RC}$) by equations. Weakly (left) ample semigroups also form a quasivariety, so similar difficulties would be encountered when trying to produce such a theorem, but as (left) restriction semigroups form a variety we were able to use Petrich and Reilly's method. We also use the graph expansions approach in Chapter 9 to prove that the class of left restriction monoids having a proper cover over a variety of monoids is a variety of left restriction monoids, where this variety is determined by

$$\Sigma = \{\bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u} : \bar{u} \equiv \bar{v} \text{ is a law in } \mathcal{V}\}.$$

Many of the proofs provided in this chapter are essentially the same as the original results presented for left ample monoids in [20], [22] and [23], but we are able to shorten a few due to the fact that left restriction monoids form a variety.

In Chapter 10 we use Petrich and Reilly's method to prove that the class of left restriction semigroups having a proper cover over a variety of monoids, $\mathcal{V}$, is a variety of left restriction semigroups. Combining this result with its dual we obtain that the class of restriction semigroups having a proper cover over a variety of monoids, $\mathcal{V}$, is a variety determined by

$$\Sigma = \{\bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u}, \bar{u}\bar{v}^* \equiv \bar{v}\bar{u}^* : \bar{u} \equiv \bar{v} \text{ is a law in } \mathcal{V}\}.$$

We also prove results on subhomomorphisms in the process and as an addition to the original aim we present results for (left) restriction semigroups having E-unitary proper covers over a variety of monoids. We explain why we cannot obtain the result that these semigroups form a variety of (left) restriction semigroup in the way we have deduced for the proper case.

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Author's declaration

Chapters 1, 2 and 4 mostly consist of definitions and results by other authors. Chapter 1 is a survey of definitions and results from universal algebra, mainly from [44] and [8]. Background information and definitions in Chapter 2 are presented from [21], [29] and original papers. Examples provided are mainly 'folklore', but a few are original. A couple of results are generalised from [22], the proofs for which are virtually identical to the originals. In Chapter 4 we provide background information on McAlister's P-theorem with most information from the original papers [42] and [43].

Chapter 3 consists of material that will be made into a joint paper with Victoria Gould. Some results are stated and used from [38], [24] and [28], which are referenced accordingly.

In Chapter 5, results by Fountain [12] and Lawson [36] on structure theorems for proper left ample semigroups are presented, which is followed by a structure theorem in Chapter 6 from [36] for proper ample semigroups. Results from [19] and [7] are also presented in Chapter 5 on structure theorems for proper weakly left ample and proper left restriction semigroups respectively.

Content in Chapters 7 and 8 will appear in a joint paper [10] with Victoria Gould, with results presented in Section 7.3 being due to Gould. The main result of Chapter 8 is we believe is analogous to that of Petrich and Reilly in the inverse case [48] and Lawson in the ample case [36], but the proofs are new.

In Chapter 9 we generalise results from [20], [22] and [23]. Many of the proofs are similar and in parts identical and the existing work has been used as a template with the authors' permission. Some results in Chapter 10 have been proved previously in the left ample case by Fountain in [15], but have been duly referenced. All other work is my own.

All things are possible to those who believe.
Mark 9:23

Chapter 1

Universal Algebra

We shall require some ideas from universal algebra. In this chapter we present algebras and signatures, free objects, categories and varieties. Our definitions and results are taken from [44] and [8].

1.1 Inverse semigroups

As restriction semigroups are a generalisation of inverse semigroups, we shall introduce them first.

Let S be a semigroup. An element $a \in S$ is *regular* if there exists $x \in S$ such that $a = axa$ and we say S is *regular* if every element of S is regular.

An element $a' \in S$ is an *inverse* of $a \in S$ if $a = aa'a$ and $a' = a'aa'$. If each element $a \in S$ has exactly one inverse in S, then S is an *inverse semigroup.* If $e \in E(S)$, i.e. an idempotent of S, and S is an inverse semigroup, then clearly $e' = e$. Inverse semigroups, which have interesting structural properties, were first studied by Vagner in 1952 and Preston in 1954 and have been used in many areas, for example, they have been used to represent partial symmetries [37].

If S is an inverse semigroup, then it is clearly regular, but the converse is not necessarily true. The following alternative characterisation for inverse semigroups, which can be found in [37], provides us with a useful alternative definition of an inverse semigroup.

Theorem 1.1.1. *A semigroup S is inverse if and only if S is regular and the idempotents of S commute.*

A generalisation of inverse semigroups could be found in many ways. As we shall see, this can be done by relaxing the regularity condition but still insisting that a given subset of idempotents must still commute. Left/right restriction semigroups are the generalisation of inverse semigroups that we shall mainly consider, but we shall also look at left/right weakly ample and left/right ample semigroups.

Green's relations, originally defined by J.A. Green in [25], are equivalence relations of mutual divisibility and a major tool in the study of regular and inverse semigroups. Green's relations for a general semigroup can be found in [26], but we shall only require them for inverse semigroups.

Green's relation $\mathcal{R}$ is defined on a semigroup by the rule that for $a, b \in S$,

$$a \,\mathcal{R}\, b \text{ if and only if } aS^1 = bS^1.$$

This is a left congruence. We have the following alternative definitions when we consider inverse semigroups.

Lemma 1.1.2. *Let $a, b \in S$ where S is an inverse semigroup. The following statements are equivalent:*
i) $a \,\mathcal{R}\, b$;
ii) $a = bt$ and $b = as$ for some $s, t \in S$;
iii) $a = bb'a$ and $b = aa'b$;
iv) $aa' = bb'$.

The second of Green's relations is the $\mathcal{L}$-relation, which is a right congruence. For $a, b \in S$ where S is a semigroup,

$$a \,\mathcal{L}\, b \text{ if and only if } S^1a = S^1b.$$

Alternative definitions of $\mathcal{L}$ for inverse semigroups are obtained dually to Lemma 1.1.2. In particular,

$$a \,\mathcal{L}\, b \Leftrightarrow a'a = b'b.$$

If S is an inverse semigroup, then the *natural partial order* relation $\leq$ is defined on S by

$$a \leq b \Leftrightarrow a = eb \text{ for some } e \in E(S).$$

As it coincides with the usual partial order on $E(S)$, and is compatible with multiplication, it is described as 'natural'. We shall be considering a natural partial order on restriction semigroups in Section 2.6.

We also have the Vagner-Preston representation theorem, which is the analogue of Cayley's theorem in group theory:

Theorem 1.1.3. *Let S be an inverse semigroup. Then there exists a symmetric inverse semigroup $\mathcal{I}_X$ and a one-to-one morphism*

$$\phi : S \to \mathcal{I}_X.$$

A symmetric inverse semigroup $\mathcal{I}_X$ is the analogue of the symmetric group $\mathcal{S}_X$. It consists of one-to-one mappings between subsets of X, under composition of partial maps. We shall explore partial mappings in Section 2.3.

1.2 Types of Algebras

We shall consider semigroups, monoids and inverse semigroups as examples of universal algebras. We shall also consider morphisms, congruences and generators in this context. We begin by looking at different types of operations.

Definition 1.2.1. Let B, C be sets. A *function* f from B to C, denoted by $f : B \to C$, is a subset of $B \times C$ such that for each $b \in B$, there is exactly one $c \in C$ such that $(b, c) \in f$. Let $n \in \mathbb{N}^0$ and let A be a set. An *operation of rank* n, or *arity* n, on A is a function from A^n to A, where A^n denotes the set of all n-tuples of elements of A and A^0 is a one-element set.

We are familar with *binary operations*, such as addition and multiplication, which are operations of rank 2. Operations of rank 1 are called *unary operations* such as the operation of taking inverses when studying inverse semigroups. Operations of rank 0 are called *nullary* and effectively a nullary operation f is determined by a constant, $f(A^0)$. In the examples we consider the nullary operation refers to the identity.

Before defining universal algebras, we need to say what is meant by a *signature type*.

Definition 1.2.2. Let I be a set and $\rho : I \to \mathbb{N}^0$ be a function; we write $i\rho$ as ρ_i. Then $(\rho_i)_{i \in I}$ is a *signature type*. If I is finite, say $I = \{1, \cdots, n\}$, we may write $(\rho_1, \cdots, \rho_n)$ for the signature type.

An algebra is a set equipped with a collection of operations:

Definition 1.2.3. Let A be a non-empty set and let $F = \{F_i : i \in I\}$ be a set of operations on A. Then $\mathbf{A} = (A, F)$ is called an *algebra*, which we shall also write as

$$\mathbf{A} = (A, F_i : i \in I).$$

Let $\rho : I \to \mathbb{N}^0$ be given by $i \mapsto \rho_i$, where ρ_i is the arity of F_i. Then $(\rho_i)_{i \in I}$ is the *signature type* of $\mathbf{A}$. If $\rho_i = 0$, then $F_i : A^0 \to A$ and so can be associated with some $a_i \in A$.

So, an algebra has signature type (2) if it has a binary operation, $(2, 1)$ if it also has a unary operation, $(2, 1, 0)$ if it also has an identity and so on.

We shall refer to each F_i as a 'basic' or 'fundamental' operation and I as the 'index set' of $\mathbf{A}$. Any operation t on A made up from the basic operations, projections and composition, is a *term function* of $\mathbf{A}$.

A semigroup is an algebra of signature type (2) which can be written

$$\mathbf{S} = (S, \cdot),$$

where the 2 refers to the binary operation. A monoid is an algebra

$$\mathbf{M} = (M, \cdot, 1),$$

with signature type $(2, 0)$, where the 0 refers to the identity nullary operation and $(M, \cdot)$ is a semigroup. Similarly we can describe inverse semigroups as algebras

$$\mathbf{I} = (I, \cdot, '),$$

with signature type $(2, 1)$ where the 1 represents the inverse unary operation and we note that an inverse semigroup can also be regarded as an algebra of signature type (2). A group is an algebra with signature type $(2, 1, 0)$, written as

$$\mathbf{G} = (G, \cdot, ^{-1}, 1)$$

where the 1 in the signature type in this case refers to the group inverse.

An algebra of a certain signature type can be considered as an algebra of another signature type. For example, a monoid can be considered as a semigroup.

Definition 1.2.4. Let Y be a non-empty set, X a subset of Y and F an operation of rank r on Y. Then X is *closed with respect to* F if and only if

$$F(a_0, a_1, ..., a_{r-1}) \in X$$

for all $a_0, a_1, ..., a_{r-1} \in X$.

Definition 1.2.5. Let $\mathbf{A} = (A, F_i : i \in I)$ and $\mathbf{B} = (B, G_i : i \in I)$ be algebras. Then $\mathbf{A}$ is a *subalgebra* of $\mathbf{B}$ if and only if $\mathbf{A}$ and $\mathbf{B}$ have the same rank function, B is a subset of A which is closed under each fundamental operation of $\mathbf{A}$ and for each $i \in I$, G_i is the restriction of F_i to B.

We also have different types of generators. Let $\mathbf{A} = (A, F_i : i \in I)$ be an algebra and X be a subset of A. The *subalgebra generated by* X, denoted by $\langle X \rangle$, is the smallest subalgebra containing X. If $A = \langle X \rangle$ we say that X is a *generating set* for A. Clearly $\langle X \rangle$ exists and is the intersection of all subalgebras of $\mathbf{A}$ containing X. It can be shown that $\langle X \rangle$ is the set of all elements that can be formed from elements of X by applications of the basic operations, that is, $\langle X \rangle$ is the value of all the term functions of $\mathbf{A}$ applied to the elements of X.

In particular, if X is a subset of a semigroup S, then X is a generating set of type (2) if

$$S = \{x_1 x_2 ... x_n : n \in \mathbb{N}, x_i \in X \text{ for } i \in \{1, 2, ..., n\}\}.$$

If X is a subset of a monoid M, then X is a generating set of type $(2, 0)$ if

$$M = \{x_1 x_2 ... x_n : n \in \mathbb{N}^0, x_i \in X \text{ for } i \in \{1, 2, ..., n\}\}.$$

We shall denote this by $M = \langle X \rangle_{(2,0)}$ and make use of this kind of notation in subsequent chapters.

We now introduce the notion of a morphism between two algebras of the same signature type.

Definition 1.2.6. Let $\mathbf{A} = (A, F_i : i \in I)$ and $\mathbf{B} = (B, G_i : i \in I)$ be algebras of the same signature type. Let f be a function from A to B. Then f is a *morphism* if for any $i \in I$ with $\rho_i = n$,

$$(F_i(a_1, a_2, ..., a_n))f = G_i(a_1 f, a_2 f, ..., a_n f).$$

Let $\mathbf{A}$ and $\mathbf{B}$ be algebras of type $(\rho_i)_{i\in I}$ and $t(x_1, ..., x_n)$ be a term function. Suppose $\theta : A \to B$ is a morphism and $a_1, ..., a_n \in A$. Then

$$t(a_1, ..., a_n)\theta = t(a_1\theta, ..., a_n\theta).$$

From now on, when considering morphisms, we shall assume they are between algebras of the same type. A morphism $\theta : S \to T$, where S and T are inverse semigroups, is a $(2,1)$-morphism if

(i) $(a\theta)(b\theta) = (ab)\theta$;

(ii) $a'\theta = (a\theta)'$,

for $a, b \in S$.

Lemma 1.2.7. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be algebras. If* $\theta : A \to B$ *is a morphism and* $A = \langle X \rangle$*, then*

$$A\theta = \langle X \rangle\theta = \langle X\theta \rangle.$$

Similarly we can define different types of congruences, where a congruence on an algebra $\mathbf{A} = (A, F_i : i \in I)$ has to preserve each of the operations. Let μ be an equivalence relation on A. Then μ is a congruence if for each $i \in I$, if $\rho_i = n$, $a_1, \cdots, a_n, b_1, \cdots b_n \in A$ and $a_j \,\mu\, b_j$, then

$$F_i(a_1, \cdots, a_n) \,\mu\, F_i(b_1, \cdots, b_n).$$

For example, a $(2,1)$-congruence μ on an inverse semigroup S must satisfy:

(i) $(a\mu)(b\mu) = (ab)\mu$;

(ii) $a'\mu = (a\mu)'$,

for $a, b \in S$. In fact, in this particular instance, (ii) follows from (i).

If μ is a congruence on an algebra $\mathbf{A} = (A, F_i : i \in I)$, we make A/μ into an algebra of the same signature type as $\mathbf{A}$ by defining operations $\bar{F}_i$, for $i \in I$, by $\rho_i = n$,

$$\bar{F}_i([a_1], \cdots, [a_n]) = [F_i(a_1, \cdots, a_n)]$$

and if $\rho_i = 0$, the constant associated with $\bar{F}_i$ is $[a]$, where a is the constant associated with F_i.

Definition 1.2.8. Let S be a semigroup and E_S a semilattice of idempotents of S. Then congruences ρ and μ on S *have the same trace* on E_S if $\rho = \mu$ on E_S, i.e. $\rho \cap (E_S \times E_S) = \mu \cap (E_S \times E_S)$.

Here we present a few definitions and results that will be used in subsequent chapters.

Let S be an algebra and suppose ρ is a congruence on S. Then we can define a morphism $\rho^\natural : S \to S/\rho$ by

$$s\rho^\natural = s\rho.$$

We have the following corollary of Lemma 1.2.7:

Corollary 1.2.9. *Let* $\mathbf{S}$ *be an algebra such that* $S = \langle Y \rangle$ *and let* ρ *be a congruence on* S*. Then* $S/\rho = \langle Y\rho^\natural \rangle$*.*

Proof. As $\rho^\natural : S \to S/\rho$ is a morphism, by Lemma 1.2.7 we have

$$S\rho^\natural = \langle Y \rangle \rho^\natural = \langle Y\rho^\natural \rangle.$$

As $\rho^\natural$ is clearly onto,

$$S\rho^\natural = S/\rho$$

and the result follows. $\square$

Proposition 1.2.10. *Let* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$ *be algebras of the same type. Let* $\theta : A \to B$ *and* $\psi : A \to C$ *be morphisms where* ψ *is onto and ker* $\psi \subseteq$ *ker* θ*. Then there exists a unique morphism* $\varphi : C \to B$ *such that the following diagram commutes:*

$$\begin{tikzcd} A \arrow[r, "\theta"] \arrow[d, "\psi"'] & B \\ C \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Proof. As ψ is onto, all the elements of C are of the form $a\psi$ for $a \in A$. Let us define $\varphi : C \to B$ by

$$(a\psi)\varphi = a\theta$$

for $a\psi \in C$.

The function φ is well-defined since

$$\begin{aligned} a\psi = b\psi &\Rightarrow (a,b) \in \text{ker } \psi \\ &\Rightarrow (a,b) \in \text{ker } \theta \\ &\Rightarrow a\theta = b\theta \\ &\Rightarrow (a\psi)\varphi = (b\psi)\varphi. \end{aligned}$$

It is also a morphism since for any n-ary function t and $a_1, ..., a_n \in A$, we have

$$\begin{aligned} t(a_1\psi, ..., a_n\psi)\varphi &= (t(a_1, ..., a_n)\psi)\varphi \\ &= t(a_1, ..., a_n)\theta \\ &= t(a_1\theta, ..., a_n\theta) \\ &= t(a_1\psi\varphi, ..., a_n\psi\varphi). \end{aligned}$$

As $\psi\varphi = \theta$, the diagram commutes and it remains to show that φ is unique. Suppose that $\mu : C \to B$ is another morphism such that $\psi\mu = \theta$. Then $\psi\varphi = \psi\mu$ and considering $a \in A$, we have

$$(a\psi)\varphi = (a\psi)\mu.$$

As every element of C is of the form $a\psi$, $\mu = \varphi$ and so we have uniqueness. $\square$

Using the previous result, we have the following:

Corollary 1.2.11. *Let* $\mathbf{S}$ *and* $\mathbf{T}$ *be algebras of the same type,* $\psi : S \to T$ *a morphism and* ρ *a congruence such that* $\rho \subseteq Ker\ \theta$*. Then there exists a unique morphism* $\varphi : S/\rho \to T$ *such that the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "\rho^\natural"] \arrow[d, "\theta"'] & S/\rho \arrow[dl, "\varphi"] \\ T & \end{tikzcd}$$

Proposition 1.2.12. *Let* X *be a set,* $\mathbf{M}$ *and* $\mathbf{N}$ *be algebras of the same type,* $f : X \to M$ *and* $g : X \to N$ *be maps,* $M = \langle Xf \rangle$ *and* θ *a morphism such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[dl, "\theta"] \\ N & \end{tikzcd}$$

Then θ *is unique.*

Proof. Take $t(x_1 f, \cdots, x_n f) \in M$. Note that all elements of M are of this form as it is generated by Xf. Suppose $\psi : M \to N$ is another morphism making the diagram above commute, i.e. $f\psi = g$. Then

$$\begin{aligned} t(x_1 f, ..., x_n f)\psi &= t(x_1 f\psi, ..., x_n f\psi) \\ &= t(x_1 g, ..., x_n g) \\ &= t(x_1 f\theta, ..., x_n f\theta) \\ &= t(x_1 f, ..., x_n f)\theta. \end{aligned}$$

Hence $\theta = \psi$ and consequently θ is unique.

□

1.3 Varieties

In Chapters 9 and 10 we consider varieties of (left) restriction semigroups.

Definition 1.3.1. Let $\mathbf{A_j} = (A_j, F_i^j : i \in I)$ be algebras of a given type, where $j \in J$ for some indexing set J. Let $A = \prod_{j\in J} A_j$ be the cartesian product, where we denote an arbitrary element by (a_j). For each $i \in I$ with $\rho_i = n$ we define an n-ary operation F_i on A by

$$F_i((a_j^1), ..., (a_j^n)) = (F_i^j(a_j^1, ..., a_j^n)).$$

Let $F = \{F_i : i \in I\}$. Then $\mathbf{A} = (A, F)$ is the *direct product* of algebras $\mathbf{A_j}$, where $j \in J$.

Definition 1.3.2. A *variety* is a non-empty class of algebras of a certain type which is closed under taking subalgebras, homomorphic images and direct products.

The class of inverse semigroups forms a variety of type $(2, 1)$, as do the classes of groups and monoids. We also have another definition of a variety, which we shall make use of in Chapters 9 and 10, which is provided by the HSP Theorem below. First we need a few definitions.

Definition 1.3.3. Let X be a countably infinite set and let $(\rho_i)_{i\in I}$ be a signature type. Let $\{f_i : i \in I\}$ be a set of symbols. The set $T(X)$ of *terms* of type $(\rho_i)_{i\in I}$ over X is the smallest set such that

(i) $X \cup C \subseteq T(X)$, where $C = \{f_i : \rho_i = 0\}$;

(ii) if $u_1, \cdots, u_n \in T(X)$ and $\rho_i = n$, then $f_i(u_1, \cdots, u_n) \in T(X)$.

We emphasise that elements of $T(X)$ are formal strings of symbols. However, each element of $T(X)$ has a natural interpretation as a term function in any algebra $\mathbf{A} = (A, F_i : i \in I)$ with signature type $(\rho_i)_{i\in I}$, where each f_i is interpreted as F_i.

Definition 1.3.4. An *identity* or *law* of type $(\rho_i)_{i\in I}$ over X is an expression of the form $p \equiv q$, where $p, q \in T(X)$. Let $\mathbf{A} = (A, F_i : i \in I)$ be an algebra of type $(\rho_i)_{i\in I}$. Then $\mathbf{A}$ *satisfies the identity*

$$p(x_1, \cdots, x_n) \equiv q(x_1, \cdots, x_n)$$

if
$$p^{\mathbf{A}}(a_1,\cdots,a_n)=q^{\mathbf{A}}(a_1,\cdots,a_n)$$
for every $a_1,\cdots,a_n\in A$, where $p^{\mathbf{A}}$ and $q^{\mathbf{A}}$ are the interpretations of p and q as term functions of $\mathbf{A}$. If Σ is a set of identities, we say a class of algebras K satisfies Σ if each member of K satisfies $p\equiv q$ for every identity $p\equiv q$ of Σ. We shall denote this by
$$K\models\Sigma.$$

HSP Theorem. *The following are equivalent for a non-empty class of algebras $\mathscr{V}$:*

(i) $\mathscr{V}$ is a variety;

(ii) $\mathscr{V}$ is defined by a set of identities.

A *quasi-identity* is an identity of the form
$$(p_1\equiv q_1\wedge\ldots\wedge p_n\equiv q_n)\rightarrow p\equiv q.$$
An algebra $\mathbf{A}$ satisfies the above quasi-identity if for every $a_1,\ldots,a_m\in A$ such that
$$p_i^{\mathbf{A}}(a_1,\cdots,a_m)=q_i^{\mathbf{A}}(a_1,\cdots,a_m)$$
for all $i\in\{1,\ldots,n\}$, then
$$p^{\mathbf{A}}(a_1,\cdots,a_m)=q^{\mathbf{A}}(a_1,\cdots,a_m).$$
A *quasivariety* is a class of algebras of a certain type defined by quasi-identities and identities. Such a class is closed under taking isomorphisms and subalgebras, but are not necessarily closed under homomorphic images.

In Chapter 10 we prove analogous results on restriction semigroups to Petrich and Reilly's results in [47] for inverse semigroups, which work due to the fact that restriction semigroups form a variety. However, left ample semigroups, which we shall define in Chapter 2, form a quasivariety and problems were encountered in the left ample case due to this, so a different approach was used by Gould in [23] for left ample monoids which involved graph expansions. Although this still did not produce the originally desired analogous result, we apply the graph expansion method in Chapter 9 for left restriction monoids. This provides us with the same result that we go on to prove in Chapter 10. Although many of the proofs are essentially the same as in the left ample case, in Chapter 9 we are able to shorten and alter some of them using the fact that we have closure under taking homomorphic images when considering left restriction monoids.

We now present a general result about varieties for use in later chapters.

We say that $\mathscr{V}$ is generated by $\mathscr{U}\subseteq\mathscr{V}$ if $\mathscr{V}$ is the smallest variety containing $\mathscr{U}$. This is equivalent to every member of $\mathscr{V}$ being obtainable from algebras in $\mathscr{U}$ via a sequence of taking homomorphic images, subalgebras and direct products (H, S and P).

Theorem 1.3.5. *A variety $\mathscr{V}$ is generated by $\mathscr{U} \subseteq \mathscr{V}$ if and only if every $A \in \mathscr{V}$ is in HSP($\mathscr{U}$), i.e. there exist $U_\alpha \in \mathscr{U}$ and $T \in \mathscr{V}$, which is a subalgebra of $\prod_{\alpha\in\Lambda} U_\alpha$ (where Λ is an indexing set), and an onto morphism $\varphi : T \to A$.*

1.4 Free objects

We shall require the definition of a free object on a set X. First we shall look at the general definition:

Definition 1.4.1. Let K be a class of algebras and X be a set. Then F_X is a *free object* on X for K if $F_X \in K$ and there exists a map $\iota : X \to F_X$, and for any $T \in K$ and map $\theta : X \to T$, there exists a unique morphism $\bar{\theta} : F_X \to T$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[d, "\theta"'] & F_X \arrow[dl, "\bar{\theta}"] \\ T & \end{tikzcd}$$

In particular, in Chapter 3 we shall consider the free inverse semigroup on a set X:

Definition 1.4.2. Let X be a set. Then F_X is the *free inverse semigroup* on X if F_X is an inverse semigroup and there exists a map $\iota : X \to F_X$, and for any inverse semigroup T and map $\theta : X \to T$, there exists a unique morphism $\bar{\theta} : F_X \to T$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[d, "\theta"'] & F_X \arrow[dl, "\bar{\theta}"] \\ T & \end{tikzcd}$$

Theorem 1.4.3. *In a non-trivial variety or quasivariety, i.e. one that contains algebras with more than one element, there is a free object on X for each set X.*

Further, we have the following result:

Proposition 1.4.4. *Let $\mathscr{V}$ be a variety and let $\mathscr{U}$ consist of the free objects of $\mathscr{V}$. Then $\mathscr{V}$ is generated by $\mathscr{U}$.*

Proof. Suppose $\mathscr{V}$ is a variety and $\mathscr{U}$ consist of the free objects of $\mathscr{V}$. Let A be an algebra of $\mathscr{V}$ and F_A, along with map $\iota : A \to F_A$, be the free algebra in $\mathscr{V}$ on A. Then for any $B \in \mathscr{V}$ and map $\theta : A \to B$, there is a unique morphism $\overline{\theta} : F_A \to B$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d, "\theta"'] & F_A \arrow[dl, "\overline{\theta}"] \\ B & \end{tikzcd}$$

In particular, let $\varphi : F_A \to A$ be the unique morphism making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d, "I_A"'] & F_A \arrow[dl, "\varphi"] \\ A & \end{tikzcd}$$

We have

$$a = aI_A = a\iota\varphi \in \mathrm{Im}\varphi.$$

So $\varphi : F_A \to A$ is an onto morphism. Using Theorem 1.3.5, $\mathscr{V}$ is generated by $\mathscr{U}$. $\square$

We note the following result:

Lemma 1.4.5. *Let K be a non-trivial variety or quasivariety of algebras of a given signature type and let F_X, along with the map $\iota : X \to F_X$, be a free object on X for K. Then*

(1) $F_X = \langle X\iota \rangle$;

(2) ι is one-to-one.

Suppose X is a set and F_X, along with $\iota : X \to F_X$, is a free object on X. For $x \in X$, let $x\iota = [x]$ and $\overline{X} = \{[x] : x \in X\}$. By part 1 of Lemma 1.4.5, $F_X = \langle \overline{X} \rangle$ and by part 2, $\overline{X}$ has the same number of elements as X. Now considering the inclusion map $i : \overline{X} \to F_X$, we have the following diagram, where T is any algebra of the same kind as F_X :

$$\begin{tikzcd} \overline{X} \arrow[r, "i"] \arrow[d, "\psi"'] & F_X \arrow[dl, "\overline{\psi}"] \\ T & \end{tikzcd}$$

Clearly ψ is the unique morphism making this diagram commute, so F_X is also a free object the subset $\overline{X}$ of F_X.

1.5 Categories

Discovered by Eilenberg and MacLane in the early 1940s, category theory allows us to compare many mathematical structures. These definitions are from [55], [34] and [3].

Categories consist of *objects* and *arrows*, or *morphisms*, between these objects. Unlike other areas of algebra, within category theory, arrows between the objects are given equal importance to the objects themselves.

Definition 1.5.1. Let $\mathbf{C}$ consist of a class of objects, $\mathrm{Ob}(\mathbf{C})$, a class of arrows, $\mathrm{Mor}(\mathbf{C})$, and two assignments, $\mathbf{d}$ and $\mathbf{r}$, from $\mathrm{Mor}(\mathbf{C})$ to $\mathrm{Ob}(\mathbf{C})$. For $f \in \mathrm{Mor}(\mathbf{C})$ we indicate by

$$A \xrightarrow{f} B$$

that $\mathbf{d}(f) = A$ and $\mathbf{r}(f) = B$. Let $\mathrm{Mor}_{\mathbf{C}}(A, B)$ denote the set of arrows between $A, B \in \mathrm{Ob}(\mathbf{C})$.

Suppose that for each $A \in \mathrm{Ob}(\mathbf{C})$, there exists an arrow $I_A \in \mathrm{Mor}_{\mathbf{C}}(A, A)$. Also, suppose that for $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ there exists a composite arrow $f \circ g \in \mathrm{Mor}_{\mathbf{C}}(A, C)$.

If the following two axioms are satisfied for $A, B, C, D \in \mathrm{Ob}(\mathbf{C})$, then $\mathbf{C}$ is a *category*:

(i) if $f \in \mathrm{Mor}_{\mathbf{C}}(A, B), g \in \mathrm{Mor}_{\mathbf{C}}(B, C)$ and $h \in \mathrm{Mor}_{\mathbf{C}}(C, D)$, then

$$f \circ (g \circ h) = (f \circ g) \circ h;$$

(ii) if $f \in \mathrm{Mor}_{\mathbf{C}}(A, B)$, then $I_A \circ f = f$ and $f \circ I_B = f$.

The arrows in general can be a number of things, such as continuous maps, partial functions or linear transformations between the objects, or even more abstract entities, depending of course on what the objects themselves are. However, in subsequent chapters, we shall only require arrows to be actual (homo)morphisms between objects. The two assignments will correspond to the domain and range of the morphisms.

Definition 1.5.2. Let $\mathbf{C}$ be a category. Then an object T of $\mathbf{C}$ is a *terminal object* if there is a unique morphism from A to T for any object A in $\mathbf{C}$.

An initial object is defined dually.

Passing between two categories, functors are useful as they allow us to compare categories.

Definition 1.5.3. Let $\mathbf{B}$ and $\mathbf{C}$ be categories. Let F consist of the following two assignments:

(i) $F : \text{Ob}(\mathbf{B}) \to \text{Ob}(\mathbf{C})$, where

$$A \mapsto AF;$$

(ii) $F : \text{Mor}(\mathbf{B}) \to \text{Mor}(\mathbf{C})$, where

$$f \mapsto fF,$$

which maps an element of $\text{Mor}_{\mathbf{B}}(G, H)$ to $\text{Mor}_{\mathbf{C}}(GF, HF)$:

$$\begin{tikzcd} G \arrow[d, "f"'] & \cdots\cdots & GF \arrow[d, "fF"] \\ H & \cdots\cdots & HF \end{tikzcd}$$

If F satisfies the following two axioms, then F is a *functor*:

(1) if $g \circ f$ is defined in $\mathbf{B}$, then $(g \circ f)F = (gF) \circ (fF)$;

(2) for $A \in Ob(\mathbf{B})$, $1_A F = 1_{AF}$.

Another idea in category theory is that of an adjunction, which consists of two functors and two assignments subject to certain conditions. An adjunction describes a relationship between two functors and is a type of generalised inverse.

Definition 1.5.4. Let $\mathbf{C}$ and $\mathbf{D}$ be categories and $F : \mathbf{C} \to \mathbf{D}$ and $U : \mathbf{D} \to \mathbf{C}$ be functors. Then F is a *left adjoint* of U if for any objects $C \in \mathbf{C}$ and $D \in \mathbf{D}$, there is a bijection

$$\lambda_{C,D} : \ \text{Mor}_{\mathbf{D}}(CF, D) \to \ \text{Mor}_{\mathbf{C}}(C, DU)$$

such that for $\phi \in \text{Mor}_{\mathbf{C}}(C', C)$ and $\theta \in \text{Mor}_{\mathbf{D}}(D, D')$, the square

$$\begin{tikzcd} \text{Mor}_{\mathbf{D}}(CF, D) \arrow[r, "\lambda_{C,D}"] \arrow[d, "\text{Mor }(\phi F, \theta)"'] & \text{Mor}_{\mathbf{C}}(C, DU) \arrow[d, "\text{Mor }(\phi, \theta U)"] \\ \text{Mor}_{\mathbf{D}}(C'F, D') \arrow[r, "\lambda_{C',D'}"'] & \text{Mor}_{\mathbf{C}}(C', D'U) \end{tikzcd}$$

is commutative, where

$$\text{Mor }(\phi F, \theta) : \ \text{Mor}_{\mathbf{D}}(CF, D) \to \ \text{Mor}_{\mathbf{D}}(CF, D')$$

is given by

$$\psi \text{ Mor } (\phi F, \theta) = (\phi F)\psi\theta$$

and

$$\text{Mor } (\phi, \theta U): \ \text{Mor}_{\mathbf{C}}(C, DU) \to \ \text{Mor}_{\mathbf{C}}(C, D'U)$$

is given by

$$\psi \text{ Mor } (\phi, \theta U) = \phi\psi(\theta U).$$

Chapter 2

Restriction, Weakly Ample and Ample semigroups

We shall define, and provide background information and basic results for, restriction, weakly ample and ample semigroups. We shall provide an abstract definition of restriction semigroups, look at when they were first considered and highlight other names they have gone by. Left restriction semigroups arise very naturally from partial transformation monoids in a similar way to that in which inverse semigroups arise from symmetric inverse monoids. We shall see that left restriction semigroups are precisely the $(2,1)$-subalgebras of some partial transformation monoid on a set X.

2.1 Background

Here we provide some background for left restriction semigroups, using information from [21] and [29] as well as original papers. We shall give a careful definition of left restriction semigroups in Section 2.2.

The terminology *weakly E-ample semigroup* was first used in [32]. In the abstract definition of 'ample' the relations $\mathcal{R}^*$ and $\mathcal{L}^*$ were replaced by the generalised relations $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$, where E denotes a subsemilattice of idempotents of the semigroup in question. The term *weakly ample* was used to refer to the special case when E is taken to be the entire set of idempotents of the semigroup under consideration.

The terminology *restriction semigroup* has been adopted due to the connections between semigroup theory and category theory:

Definition 2.1.1. [9] A *restriction category* is a category $\mathbf{X}$ such that for every arrow $f : A \to B$, there exists $\overline{f} : A \to A$ such that the following conditions hold (where composition is left to right):

(1) $\overline{f}f = f$ for all f;

(2) $\overline{f}\overline{g} = \overline{g}\overline{f}$ when $\mathrm{dom}g = \mathrm{dom}f$;

(3) $\overline{\overline{f}g} = \overline{f}\overline{g}$ when $\mathrm{dom}f = \mathrm{dom}g$;

(4) $f\overline{g} = \overline{fg}f$ when cod$f$ = dom$g$.

Restriction semigroups, as studied by Cockett and Lack, were influenced by the importance of categories of partial maps in theoretical computer science and the work done to develop the theory of these categories.

As a relatively new topic, it is believed that restriction semigroups first appeared as *function systems* in [54] in the 1960s. Throughout a series of papers, [51], [52] and [53], Schweizer and Sklar studied systems of functions in an attempt to characterise a class of algebras, before defining a function system in [54]. The structure consisted of a non-empty set A, an associative binary operation $\circ$ and two unary operations, denoted by L and R, such that the following conditions hold for all $a, b \in A$:

(1) $L(R(a)) = R(a)$;

(2) $R(L(a)) = L(a)$;

(3) $L(a) \circ a = a = a \circ R(a)$;

(4) $L(a \circ b) = L(a \circ L(b))$;

(5) $R(a \circ b) = R(R(a) \circ b)$;

(6) $L(a) \circ R(b) = R(b) \circ L(a)$;

(7) $R(a) \circ b = b \circ R(a \circ b)$.

Left restriction semigroups first appeared as a class in their own right as *function semigroups* in [57] by Trokhimenko in the early 1970s. A function semigroup is a set S with a binary operation $\circ$ and unary operation R such that S under $\circ$ is a semigroup and for $x, y \in S$:

(1) $R(x) \circ x = x$;

(2) $R(R(x) \circ y) = R(x) \circ R(y)$;

(3) $R(x) \circ R(y) = R(y) \circ R(x)$;

(4) $R(x \circ y) = R(x \circ R(y))$;

(5) $x \circ R(y) = R(x \circ y) \circ x$.

Also, the representation theory of left restriction semigroups by partial functions was first considered in this paper, specifically, the result that allows us to conclude that the left restriction semigroups are precisely the $(2, 1)$-subalgebras of some $\mathcal{PT}_X$, where $\mathcal{PT}_X$ is the partial transformation monoid on a set X. This is a concept which we shall look at in detail in Section 2.3.

Left restriction semigroups have also appeared as *type SL2 γ-semigroups* in the work of Batbedat in [4] and [5] in the late 1970s to early 1980s. They are a generalisation of inverse semigroups where the operation $x \mapsto xx'$ was replaced by a mapping $\gamma : S \to S$ for a semigroup S. For the semigroup S to be a type SL2 γ-semigroup, γ needs to satisy the following conditions:

(1) $\gamma(S)$ is a subsemilattice of S;

(2) for each $s \in S$, $\gamma(s)$ is the smallest γ-element a such that $as = s$, where a γ-element is an element of $\gamma(S)$;

(3) $x\gamma(y) = \gamma(xy)x$ for $x, y \in S$.

It is proved in [29] that every left restriction semigroup is a type SL2 γ-semigroup and conversely that every type SL2 γ-semigroup is a left restriction semigroup with distinguished semilattice $\gamma(S)$.

Left restriction semigroups have arisen in the work of Jackson and Stokes in 2001 in [33] as *twisted LC-semigroups* via a generalisation of closure operations on a semilattice. Motivated by examples, they defined an *LC-semigroup* to be a semigroup S with an additional unary operation C such that for $a, b \in S$:

(1) $C(a)a = a$;

(2) $C(a)C(b) = C(b)C(a)$;

(3) $C(C(a)) = C(a)$;

(4) $C(a)C(ab) = C(ab)$.

A LC-semigroup S is called *twisted* if in addition the following condition holds:

(5) $aC(b) = C(ab)a$.

As proved in [29], a left restriction semigroup S is a twisted LC-semigroup with $C(a) = a^+$ for $a \in S$ and conversely a twisted LC-semigroup S is a left restriction semigroup with distinguished semilattice $C(S)$, where $C(S) = \{C(a) : a \in S\}$.

More recently, left restriction semigroups appeared in the work of Manes as *guarded semigroups* in [39]. After the generalisation of inverse semigroups to left ample and weakly left ample semigroups, Manes generalised inverse semigroups to guarded semigroups. The definition was obtained by adapting the axioms for a restriction category. A guarded semigroup is a semigroup with a unary operation $x \mapsto \overline{x}$ such that the following conditions hold

(1) $\overline{x}\,x = x$;

(2) $\overline{x}\,\overline{y} = \overline{y}\,\overline{x}$;

(3) $\overline{\overline{x}\,y} = \overline{x}\,\overline{y}$;

(4) $x\,\overline{y} = \overline{xy}\,x$.

2.2 Restriction and weakly ample semigroups

We wish to apply the techniques of inverse semigroup theory to wider classes of semigroups having a semilattice of idempotents, but which need not be regular. We shall introduce relations, which will provide us with useful techniques to study non-regular semigroups. The definitions and results in the remainder of this chapter have been compiled using [21], [1], [16] and [36].

As mentioned, left/right restriction semigroups stem from studying partial transformation monoids, but throughout this thesis we shall use the abstract definitions which we shall present in this section.

Definition 2.2.1. Suppose S is a semigroup and E a set of idempotents of S. Let $a, b \in S$. Then the relation $\widetilde{\mathcal{R}}_E$ is defined by the rule that $a \,\widetilde{\mathcal{R}}_E\, b$ if and only if for all $e \in E$, $ea = a$ if and only if $eb = b$.

It can easily be seen that $\widetilde{\mathcal{R}}_E$ is an equivalence relation. Now let E be a *subsemilattice* of a semigroup S, i.e. a commutative subsemigroup of S consisting entirely of idempotents. We note that we can consider the case when $E = E(S)$, but E does not necessarily have to be the whole of $E(S)$. Note that if $E = E(S)$, we use $\widetilde{\mathcal{R}}$ instead of $\widetilde{\mathcal{R}}_E$.

Notation 2.2.2. If there is potential for ambiguity, we shall denote E by E_S to indicate that E is a subset of the semigroup S. However, if we are only considering one such semigroup we shall omit the subscript.

Proposition 2.2.3. *Let $\widetilde{\mathcal{R}}_E$ be defined on a semigroup S, where E be a subsemilattice of S. Then*

$$\mathcal{R} \subseteq \widetilde{\mathcal{R}}_E,$$

and

$$\mathcal{R} = \widetilde{\mathcal{R}}$$

if S is regular.

Proof. Let S be a semigroup and E be a subsemilattice of S. Suppose $a \,\mathcal{R}\, b$ for $a, b \in S$. Then $a = bt$ and $b = as$ for some $s, t \in S^1$. Then, for $e \in E$,

$$ea = a \Rightarrow eas = as \Rightarrow eb = b$$

and dually $eb = b$ implies $ea = a$. Therefore $a \,\widetilde{\mathcal{R}}_E\, b$.

Now suppose that S is regular and $a \,\widetilde{\mathcal{R}}\, b$ for $a, b \in S$. Then, for $e \in E(S)$,

$$ea = a \Leftrightarrow eb = b.$$

As S is regular, $a = axa$ and $b = byb$ for some $x, y \in S$. Since $ax, by \in E(S)$, $b = axb$ and $a = bya$. Therefore $a \,\mathcal{R}\, b$. $\square$

Proposition 2.2.4. *Let $a \in S$ and $e \in E$, where S is a semigroup and E is a subsemilattice of S. Then $a \,\widetilde{\mathcal{R}}_E\, e$ if and only if $ea = a$ and for all $f \in E$, if $fa = a$, then $fe = e$.*

Proof. Suppose that $a \mathrel{\widetilde{\mathcal{R}}_E} e$. We have for all $f \in E$,

$$fa = a \Rightarrow fe = e$$

and since $e \in E$,

$$ee = e \Rightarrow ea = a.$$

Conversely, suppose that $ea = a$ and for all $f \in E$,

$$fa = a \Rightarrow fe = e.$$

Suppose $fe = e$. Then

$$fa = fea = ea = a$$

and so $a \mathrel{\widetilde{\mathcal{R}}_E} e$. $\square$

It turns out that if an element a of a semigroup S is $\widetilde{\mathcal{R}}_E$-related to an element of a subsemilattice E of S, then that element of E is unique.

Proposition 2.2.5. *Let S be a semigroup and E be a semilattice of S. Then $a \in S$ is $\widetilde{\mathcal{R}}_E$-related to at most one idempotent in E.*

Proof. Suppose for $e, f \in E$, $a \mathrel{\widetilde{\mathcal{R}}_E} e$ and $a \mathrel{\widetilde{\mathcal{R}}_E} f$, so $e \mathrel{\widetilde{\mathcal{R}}_E} f$. As $ee = e$ and $ff = f$, then $ef = f$ and $fe = e$ and so

$$e = fe = ef = f.$$

$\square$

Definition 2.2.6. Let S be a semigroup and E be a semilattice of S. If every element of S is $\widetilde{\mathcal{R}}_E$-related to an idempotent in E, then S is *weakly left E-adequate*. If $E = E(S)$, then S is called *weakly left adequate*.

Let S be an inverse semigroup where $a = axa$ for $a, x \in S$. Then $a \mathrel{\widetilde{\mathcal{R}}} ax$, using Proposition 2.2.4 as $ax \in E(S)$. So, if S is an inverse semigroup, each $\widetilde{\mathcal{R}}$-class always contains an idempotent and S is therefore weakly left adequate. However, if S is a non-regular semigroup, then there may be a $\widetilde{\mathcal{R}}$-class that does not contain an idempotent.

Notation 2.2.7. In the case where S is weakly left E-adequate, each element $a \in S$ is $\widetilde{\mathcal{R}}_E$-related to one idempotent in the subsemilattice E by Proposition 2.2.5, which we shall denote by a^+. Note that for $e \in E$, $e^+ = e$.

Suppose that S is weakly left E-adequate and let $a \in S$. Then by Proposition 2.2.4, $a^+a = a$. We also have the following alternative description of $\widetilde{\mathcal{R}}_E$.

Lemma 2.2.8. *Let S be a weakly left E-adequate semigroup and let $a, b \in S$. Then $a \mathrel{\widetilde{\mathcal{R}}_E} b$ if and only if $a^+ = b^+$.*

Definition 2.2.9. Let S be a semigroup and E be a set of idempotents of S. Then S satisfies the *left congruence condition* with respect to E if $\widetilde{\mathcal{R}}_E$ is a left congruence.

The following proposition provides a useful alternative description of the left congruence condition.

Proposition 2.2.10. *Let $a, b \in S$, where S is a weakly left E-adequate semigroup. Then S satisfies the left congruence condition if and only if $(ab)^+ = (ab^+)^+$.*

Proof. Suppose that S satisfies the left congruence condition. As $b \, \widetilde{\mathcal{R}}_E \, b^+$, we have $ab \, \widetilde{\mathcal{R}}_E \, ab^+$. By Lemma 2.2.8, $(ab)^+ = (ab^+)^+$.

Conversely, suppose that $(ab)^+ = (ab^+)^+$ and let $a \, \widetilde{\mathcal{R}}_E \, b$ for $a, b \in S$. We wish to show that $ca \, \widetilde{\mathcal{R}}_E \, cb$ for $c \in S$. As $a \, \widetilde{\mathcal{R}}_E \, b$, $a^+ = b^+$. Using our assumptions and Lemma 2.2.8,

$$(ca)^+ = (ca^+)^+ = (cb^+)^+ = (cb)^+,$$

i.e. $ca \, \widetilde{\mathcal{R}}_E \, cb$. □

We note the following useful result:

Lemma 2.2.11. *Let S be a weakly left E-adequate semigroup such that the left congruence condition holds. Then*

$$a^+ b \, \widetilde{\mathcal{R}}_E \, b^+ a$$

for $a, b \in S$.

Proof. For any $a, b \in S$,

$$a^+ b \, \widetilde{\mathcal{R}}_E \, a^+ b^+ = b^+ a^+ \, \widetilde{\mathcal{R}}_E \, b^+ a$$

since $\widetilde{\mathcal{R}}_E$ is a left congruence. □

We are now in a position to provide the definition of a left restriction semigroup. Taking the semilattice under consideration to consist of all the idempotents of the semigroup, we have the same definition for a weakly left ample semigroup.

Definition 2.2.12. Suppose a weakly left E-adequate semigroup S satisfies the left congruence condition with respect to E. Suppose that it also satisfies the *left ample condition* that for all $a \in S$ and $e \in E$,

$$ae = (ae)^+ a.$$

Then S is *left restriction* (formerly *weakly left E-ample*) and if $E = E(S)$, then S is *weakly left ample.*

In other words, to check that a semigroup S is left restriction with respect to $E \subseteq E(S)$, we need to check:

(i) E is a subsemilattice of S;

(ii) every element $a \in S$ is $\widetilde{\mathcal{R}}_E$-related to an element of E (denoted by a^+);

(iii) $\widetilde{\mathcal{R}}_E$ is a left congruence;

(iv) the left ample condition holds.

We shall refer to E as the *distinguished semilattice* associated with the left restriction semigroup S. We have dual definitions and results if we consider the relation $\widetilde{\mathcal{L}}_E$ on a semigroup S. We can define right restriction semigroups and weakly right ample semigroups, where the unique idempotent in the $\widetilde{\mathcal{L}}_E$-class of $a \in S$ is denoted by a^*. A semigroup is *restriction* if it is both left and right restriction with respect to some distinguished semilattice E, and weakly ample if it is both weakly left and weakly right ample.

A left restriction or weakly left ample semigroup is an algebra with signature type $(2, 1)$, written as

$$\mathbf{S} = (S, \cdot, ^+),$$

where $^+$ is the unary operation. Dually a right restriction or weakly right ample semigroup is an algebra with signature type $(2, 1)$, written as

$$\mathbf{S} = (S, \cdot, ^*).$$

A restriction or weakly ample semigroup is an algebra with signature type $(2, 1, 1)$, written as

$$\mathbf{S} = (S, \cdot, ^+, ^*)$$

and a restriction or weakly ample monoid is an algebra with signature type $(2, 1, 1, 0)$, written as

$$\mathbf{S} = (S, \cdot, ^+, ^*, 1).$$

Let S and T be left restriction semigroups. A $(2, 1)$-morphism $\theta : S \to T$ preserves both the binary operation and the unary operation $^+$. For $e \in E_S$, we have $e\theta \in E_T$ as

$$e\theta = e^+\theta = (e\theta)^+.$$

Writing the binary operation as juxtaposition, left restriction semigroups are algebras defined by the following identities presented in [27], which first appeared in the work of Jackson and Stokes in [33]:

(i) $(xy)z = x(yz)$;

(ii) $x^+x = x$;

(iii) $x^+y^+ = y^+x^+$;

(iv) $(x^+y)^+ = x^+y^+$;

(v) $xy^+ = (xy)^+x$.

Therefore the class of all left restriction semigroups is a variety of algebras. We note that these identities imply $x^+x^+ = x^+$ and $(x^+)^+ = x^+$ and so these identities are not required in the definition.

In Section 2.3, we shall also provide the alternative definition of left restriction semigroups as precisely the $(2,1)$-subalgebras of the partial transformation monoid on a set.

We note that weakly left ample semigroups are a special type of left restriction semigroups, which are defined by the addition quasi-identity

$$x^2 = x \to x = x^+$$

and so weakly left ample semigroups form a quasivariety.

From Definition 2.2.12 we can see that an inverse semigroup is a weakly ample semigroup, but we shall see later from examples that a weakly ample semigroup need not be inverse.

Proposition 2.2.13. *Let S be an inverse semigroup. Then S is a weakly ample semigroup, where*

$$a^+ = aa'$$

and

$$a^* = a'a$$

for $a \in S$.

Proof. As S is an inverse semigroup, $E(S)$ is a semilattice. We have $aa'a = a$ and for $f \in E(S)$,

$$fa = a \Rightarrow faa' = aa'.$$

So $a \,\widetilde{\mathcal{R}}\, aa'$. Dually, $a \,\widetilde{\mathcal{L}}\, a'a$. It follows from Proposition 2.2.3 and the fact that $\mathcal{R}$ is a left congruence, $\widetilde{\mathcal{R}}$ is a left congruence. Dually $\widetilde{\mathcal{L}}$ is a right congruence.

Take $a \in S$ and $e \in E(S)$. We have

$$\begin{aligned}(ae)^+a &= (ae)(ae)'a \\ &= aeea'a \\ &= aa'aee \\ &= ae.\end{aligned}$$

Dually the left ample condition holds. $\square$

An inverse semigroup is an algebra of type $(2,1)$ where the unary operation is taking an inverse, but considered as a weakly ample semigroup it is an algebra of type $(2,1)$ where the unary operation is $^+$. When considering a congruence on an inverse semigroup, it transpires that we do not need to be too careful about the signature. To show this we need a consequence of Lallement's lemma:

Lallement's Lemma. *Let ρ be a congruence on a regular semigroup S and $a\rho$ be an idempotent in S/ρ. Then there exists an idempotent e in S such that $e\rho = a\rho$.*

Corollary 2.2.14. *Let S be an inverse semigroup. Then the following are equivalent:*

(i) *μ is a semigroup congruence on S;*

(ii) *μ is a $(2,1)$-congruence on S, where 1 corresponds to taking an inverse;*

(iii) *μ is a $(2,1)$-congruence on S, where 1 corresponds to the $^+$ unary operation.*

Proof. (i) $\Rightarrow$ (ii): As μ is a semigroup congruence and S is an inverse semigroup, we have

$$(a\mu)(a'\mu)(a\mu) = (aa'a)\mu = a\mu$$

so that S/μ is regular, and

$$(a'\mu)(a\mu)(a'\mu) = (a'aa')\mu = a'\mu.$$

It remains to show that the idempotents of S/μ commute. If $b\mu, c\mu \in E(S/\mu)$, then by Lallement's lemma,

$$b\mu = e\mu \text{ and } c\mu = f\mu$$

for some $e, f \in E(S)$. As S is an inverse semigroup, its idempotents commute and so

$$(b\mu)(c\mu) = (e\mu)(f\mu) = (ef)\mu = (fe)\mu = (f\mu)(e\mu) = (c\mu)(b\mu).$$

Therefore S/μ is an inverse semigroup and $(a\mu)' = a'\mu$.

(ii) $\Rightarrow$ (iii): By Proposition 2.2.13, an inverse semigroup can be considered as a weakly ample semigroup where $a^+ = aa'$ for each element of the inverse semigroup. We have

$$(a\mu)^+ = (a\mu)(a\mu)' = (a\mu)(a'\mu) = (aa')\mu = a^+\mu.$$

(ii) $\Rightarrow$ (iii): Clear.

$\square$

The following result gives us a standard form for elements of a left restriction monoid with a given set of generators. This is a generalisation of Lemma 4.1 from [22] for left ample monoids with a set of generators. The proof is the same as the use of the ample condition is key in both cases but we provide it for completeness.

Lemma 2.2.15. *Let M be a left restriction monoid and suppose $M = \langle Y \rangle$. Then any $a \in M$ can be written as*

$$a = (x_1^1 \dots x_{p(1)}^1)^+ \dots (x_1^m \dots x_{p(m)}^m)^+ y_1 \dots y_n$$

for some $m, n \in \mathbb{N}^0$ where $x_j^i, y_k \in Y, 1 \le i \le m, 1 \le j \le p(i), 1 \le k \le n$.

Proof. The elements of Y are of the required form. We make the inductive assumption that $q \in \mathbb{N}^0$ and all elements of M obtained from the elements of Y by less than q applications of fundamental operations have the required form. Suppose that $a \in M$ is obtained from Y by q applications of fundamental operations.

We need to consider 3 possibilities for a:

(i) Suppose $a = 1$. Putting $m = n = 0$, a has the required form.

(ii) Suppose $a = b^+$ where b is obtained from Y in $q - 1$ steps. By the inductive hypothesis,

$$b = (x_1^1 \dots x_{p(1)}^1)^+ \dots (x_1^m \dots x_{p(m)}^m)^+ y_1 \dots y_n$$

for some $m, n \in \mathbb{N}^0, x_j^i, y_k \in Y, 1 \le i \le m, 1 \le j \le p(i), 1 \le k \le n$. Now $b^+ = (ey_1 \dots y_n)^+$ where $e = (x_1^1 \dots x_{p(1)}^1)^+ \dots (x_1^m \dots x_{p(m)}^m)^+$ is an element of E_M, so that by Proposition 2.2.10, $a = b^+ = e(y_1 \dots y_n)^+$ and a has the required form.

(iii) Suppose $a = bc$ where b and c are obtained from Y in fewer than q steps. By the inductive hypothesis

$$b = (x_1^1 \dots x_{p(1)}^1)^+ \dots (x_1^m \dots x_{p(m)}^m)^+ y_1 \dots y_s$$

and

$$c = (z_1^1 \dots z_{q(1)}^1)^+ \dots (z_1^n \dots z_{q(n)}^n)^+ w_1 \dots w_t$$

for some $m, n, s, t \in \mathbb{N}^0$ where $x_j^i, y_k \in Y, 1 \le i \le m, 1 \le j \le p(i), 1 \le k \le s$ and $z_j^i, w_k \in Y, 1 \le i \le n, 1 \le j \le q(i), 1 \le k \le t$.

If $s = 0$ or $n = 0$ then $a = bc$ has the required form as there is no "last part" of b or "first part" of c. Suppose that $s \ne 0$ and $n \ne 0$. Put $y = y_1 \dots y_s$ and for $1 \le i \le n$ put $e_i = (z_1^i \dots z_{q(i)}^i)^+$. As M is left restriction we have

$$ye_1 \dots e_n = (ye_1)^+ ye_2 \dots e_n = \dots = (ye_1)^+ \dots (ye_n)^+ y.$$

Now for any $i \in \{1, \dots, n\}$,

$$(ye_i)^+ = (y(z_1^i \dots z_{q(i)}^i)^+)^+ = (yz_1^i \dots z_{q(i)}^i)^+,$$

using Proposition 2.2.10. It follows that $a = bc$ has the required form.

We therefore have the result by induction. □

We shall require the definition of the free left restriction semigroup and free left restriction monoid on a set X in subsequent chapters:

Definition 2.2.16. Let X be a set. Then F_X is the *free left restriction semigroup (monoid)* on X if F_X is a left restriction semigroup (monoid) and there exists a map $\iota : X \to F_X$, and for any left restriction semigroup (monoid) T and map $\theta : X \to T$, there exists a unique morphism $\bar{\theta} : F_X \to T$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[d, "\theta"'] & F_X \arrow[dl, "\bar{\theta}"] \\ T & \end{tikzcd}$$

2.3 Partial transformation monoids

We shall consider how left restriction semigroups arise very naturally from partial transformation monoids in a similar way to how inverse semigroups arise from symmetric inverse monoids.

A *partial transformation* on a set X is a function from A to B where $A, B \subseteq X$. We let

$$\mathcal{PT}_X = \{\theta : \theta : A \to B, A, B \subseteq X\}.$$

We can compose $\alpha, \beta \in \mathcal{PT}_X$ by taking

$$\mathrm{dom}(\alpha\beta) = [\mathrm{im}(\alpha) \cap \mathrm{dom}(\beta)]\alpha^{-1},$$

where α^{-1} is the preimage under α, and

$$x(\alpha\beta) = (x\alpha)\beta$$

for $x \in \mathrm{dom}(\alpha\beta)$. The set $\mathcal{PT}_X$, under this composition, is a monoid known as the *partial transformation monoid* on X.

Let us consider a set of idempotents of $\mathcal{PT}_X$, namely

$$E_{\mathcal{PT}_X} = \{I_Z : Z \subseteq X\},$$

i.e. those idempotents which are identities on their domains. We note that $\mathcal{PT}_X$ may have other idempotents. We define the unary operation $^+$ on $\mathcal{PT}_X$ by

$$\alpha^+ = I_{\mathrm{dom}(\alpha)}$$

for $\alpha \in \mathcal{PT}_X$. It is proved in [21] that $\mathcal{PT}_X$ is left restriction with distinguished semilattice $E_{\mathcal{PT}_X}$. As left restriction semigroups form a variety they are closed under taking subalgebras. Consequently, every $(2,1)$-subalgebra of $\mathcal{PT}_X$ is also left restriction.

Conversely, if S is left restriction, then we have the analogue of the Vagner-Preston representation theorem.

Theorem 2.3.1. *Let S be a left restriction semigroup. Then there exists a partial transformation monoid $\mathcal{PT}_X$ and a one-to-one morphism*

$$\phi : S \to \mathcal{PT}_X.$$

As proved in [21], we take the partial transformation monoid $\mathcal{PT}_S$ and $\phi : S \to \mathcal{PT}_S$ is given by $s\phi = \rho_S$, where dom $\rho_S = Ss^+$ and $x\rho_S = xs$ for all $x \in$ dom ρ_S.

To obtain the definition of a right restriction semigroup we need to consider the partial transformation monoid $\mathcal{PT}_X$ with composition from right to left.

2.4 Ample semigroups

(Left) ample semigroups also generalise inverse semigroups. They are weakly (left) ample (and hence (left) restriction) semigroups, but there are weakly (left) ample semigroups that are not (left) ample. We shall now introduce some more relations, which are relations of mutual cancellability.

Definition 2.4.1. Let S be a semigroup and let $a, b \in S$. Then $a\,\mathcal{R}^*\,b$ if and only if for all $x, y \in S^1$,

$$xa = ya \Leftrightarrow xb = yb.$$

If S is a left restriction semigroup, then we can just check the above condition for $x, y \in S$ rather than for $x, y \in S^1$.

Proposition 2.4.2. *Let S be a left restriction semigroup with distinguished semilattice E and suppose that $a\,\widetilde{\mathcal{R}}_E\,b$ for $a, b \in S$. If*

$$xa = ya \Leftrightarrow xb = yb$$

for $x, y \in S$, then $a\,\mathcal{R}^*\,b$.

Proof. Let $xa = ya \Leftrightarrow xb = yb$ for $x, y \in S$. Then

$$\begin{aligned} a = ya &\Rightarrow a^+a = ya \\ &\Rightarrow a^+b = yb \\ &\Rightarrow b^+b = yb \\ &\Rightarrow b = yb. \end{aligned}$$

Hence $xa = ya \Leftrightarrow xb = yb$ for $x, y \in S^1$. $\square$

Proposition 2.4.3. *Let S be a semigroup and suppose that $a \in S$ and $e \in E(S)$. Then $a\,\mathcal{R}^*\,e$ if and only if $ea = a$ and for all $x, y \in S^1$,*

$$xa = ya \Rightarrow xe = ye.$$

Proof. Let us first suppose that $a\,\mathcal{R}^*\,e$. It is immediate that for all $x, y \in S^1$,

$$xa = ya \Rightarrow xe = ye.$$

We also have

$$xe = ye \Rightarrow xa = ya$$

for all $x, y \in S^1$. We have

$$1e = ee \Rightarrow 1a = ea \Rightarrow ea = a.$$

Conversely, if $ea = a$ and for all $x, y \in S^1$,

$$xa = ya \Rightarrow xe = ye,$$

it remains to show that

$$xe = ye \Rightarrow xa = ya$$

for all $x, y \in S^1$. Suppose that $xe = ye$ and so

$$xa = xea = yea = ya.$$

□

Proposition 2.4.4. *Let $a \in S$ and $E(S)$ be a subsemilattice of S. Then a is $\mathcal{R}^*$-related to at most one idempotent.*

Proof. Suppose for $e, f \in E(S)$, $a\,\mathcal{R}^*\,e$ and $a\,\mathcal{R}^*\,f$, so $e\,\mathcal{R}^*\,f$. As $ef = e$ and $fe = f$, we have

$$e = ef = fe = f.$$

□

Definition 2.4.5. Let S be a semigroup. If $E(S)$ is a subsemilattice of S and each $\mathcal{R}^*$-class contains an idempotent, then S is *left adequate*.

By Proposition 2.4.4, if S is left adequate, then every $\mathcal{R}^*$-class contains a unique idempotent. For $a \in S$ we shall denote the unique idempotent in the $\mathcal{R}^*$-class of a by a^+.

We have the following alternative description of $\mathcal{R}^*$:

Lemma 2.4.6. *Let S be a left adequate semigroup and let $a, b \in S$. Then $a\,\mathcal{R}^*\,b$ if and only if $a^+ = b^+$.*

Proposition 2.4.7. *Let S be a semigroup and let E be a subsemilattice of S. Then*

$$\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}} \subseteq \widetilde{\mathcal{R}}_E.$$

Proof. Suppose $a \mathcal{R} b$ for $a, b \in S$. Then, $a = bt$ and $b = as$ for some $s, t \in S^1$. Suppose that for some $x, y \in S^1$, $xa = ya$. Now, $xb = xas = yas = yb$. Dually, $xb = yb$ implies $xa = ya$, so $a \mathcal{R}^* b$. Now suppose that $a \mathcal{R}^* b$ and $e \in E(S)$. Then by letting $x = e$ and $y = 1$ in the definition of $\mathcal{R}^*$, we see that $a \widetilde{\mathcal{R}} b$. Now suppose $a \widetilde{\mathcal{R}} b$, i.e. $ea = a$ if and only if $eb = b$ for $e \in E(S)$. If E is any subsemilattice of S, then this condition will hold for $e \in E$ and so $a \widetilde{\mathcal{R}}_E b$. $\square$

We note that the proposition above holds for any semigroup S. Following from Proposition 2.2.3, $\mathcal{R} = \mathcal{R}^* = \widetilde{\mathcal{R}}$ for an inverse semigroup. The relations $\mathcal{R}^*$ and $\widetilde{\mathcal{R}}$ also turn out to be equal on a left adequate semigroup.

Proposition 2.4.8. *If S is left adequate, then*

$$\mathcal{R}^* = \widetilde{\mathcal{R}}.$$

Proof. From Proposition 2.4.7,

$$\mathcal{R}^* \subseteq \widetilde{\mathcal{R}}.$$

If a^+ and b^+ are the unique idempotents in the $\mathcal{R}^*$-classes of a and b respectively and $a \widetilde{\mathcal{R}} b$, then

$$a^+ \mathcal{R}^* a \widetilde{\mathcal{R}} b \mathcal{R}^* b^+.$$

So $a^+ \widetilde{\mathcal{R}} b^+$. Therefore $a^+ = b^+$ and hence $a \mathcal{R}^* b$. $\square$

As a consequence of the previous result, a left adequate semigroup is a weakly left adequate semigroup. We shall now provide the definition of a left ample semigroup:

Definition 2.4.9. Let S be left adequate. For $a \in S$, let a^+ denote the unique idempotent in the $\mathcal{R}^*$-class of a (as it coincides with the unique idempotent in the $\widetilde{\mathcal{R}}$-class of a). If S satisfies the left ample condition that for all $a \in S$ and $e \in E(S)$,

$$ae = (ae)^+ a,$$

then S is a *left ample* (formerly, *left type A*) *semigroup.*

We note that we do not need to show that $\mathcal{R}^*$ is a left congruence as we can easily show that it is a left congruence regardless of the semigroup we are considering. The definition of $\mathcal{L}^*$ on a semigroup S and the results obtained are dual, allowing us to define right ample semigroups in the same way. A semigroup is *ample* if it is both left and right ample. Examples can be found in [29].

We note that a left ample semigroup is a weakly left ample semigroup and have the following useful connection between weakly left ample and left ample semigroups:

Corollary 2.4.10. *[21] Let S be a weakly left ample semigroup. Then S is left ample if and only if $\mathcal{R}^* = \widetilde{\mathcal{R}}$.*

A left ample semigroup is an algebra with signature type $(2,1)$, written as

$$\mathbf{S} = (S, \cdot, {}^{+}),$$

where $^{+}$ is the unary operation. Dually a right ample semigroup is an algebra with signature type $(2,1)$, written as

$$\mathbf{S} = (S, \cdot, {}^{*}).$$

An ample semigroup is an algebra with signature type $(2,1,1)$, written as

$$\mathbf{S} = (S, \cdot, {}^{+}, {}^{*})$$

and an ample monoid is an algebra with signature type $(2,1,1,0)$, written as

$$\mathbf{S} = (S, \cdot, {}^{+}, {}^{*}, 1).$$

The *symmetric inverse semigroup* on a set X, $\mathcal{I}_X$, is a left ample semigroup and it is deduced in [21] that left ample semigroups are precisely the $(2,1)$-subalgebras of some $\mathcal{I}_X$.

2.5 Examples

As left restriction semigroups are precisely the $(2,1)$-subalgebras of some $\mathcal{PT}_X$, we can find examples of left restriction semigroups by considering subsets of $\mathcal{PT}_X$ for some set X that are closed under the binary and unary operations.

Example 2.5.1. Let S be the subset of $\mathcal{PT}_{\{1,2\}}$ given by

$$S = \{\alpha, \alpha^+, \beta, \beta^+, \varepsilon\},$$

where ε is the empty transformation and α and β are given by

$$\alpha = \begin{pmatrix} 1 & 2 \\ \times & 1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 2 \\ 2 & \times \end{pmatrix}.$$

Then

$$\alpha^+ = \begin{pmatrix} 1 & 2 \\ \times & 2 \end{pmatrix} \text{ and } \beta^+ = \begin{pmatrix} 1 & 2 \\ 1 & \times \end{pmatrix}.$$

It can be easily seen that the multiplication table of S is

	α	β	α^+	β^+	ε
α	ε	α^+	ε	α	ε
β	β^+	ε	β	ε	ε
α^+	α	ε	α^+	ε	ε
β^+	ε	β	ε	β^+	ε
ε	ε	ε	ε	ε	ε

Clearly S is closed under composition and $^{+}$, so S is a $(2,1)$-subalgebra of $\mathcal{PT}_{\{1,2\}}$. As discussed in Section 2.3, S is a left restriction semigroup with distinguished semilattice

$$E_S = \{\alpha^+, \beta^+, \varepsilon\}.$$

Example 2.5.2. Let S be the subset of $\mathcal{PT}_{\{1,2,3,4\}}$ given by

$$S = \{\alpha, \alpha^+, \beta, \gamma, \gamma^+, \delta, \delta^+, \varepsilon\},$$

where ε is the empty transformation,

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & \times & \times \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \times & 2 & 3 & 4 \end{pmatrix},$$

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & \times & \times & \times \end{pmatrix} \text{ and } \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \times & 1 & \times & \times \end{pmatrix}.$$

We have $\beta^+ = \beta$,

$$\alpha^+ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & \times & \times \end{pmatrix}, \gamma^+ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & \times & \times & \times \end{pmatrix} \text{ and } \delta^+ = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \times & 2 & \times & \times \end{pmatrix}.$$

It can be easily seen that the multiplication table of S is

	α	β	α^+	γ	γ^+	δ	δ^+	ε
α	α^+	γ	α	δ^+	δ	γ^+	γ	ε
β	δ	β	δ^+	ε	ε	δ	δ^+	ε
α^+	α	δ^+	α^+	γ	γ^+	δ	δ^+	ε
γ	γ^+	γ	γ	ε	ε	γ^+	γ	ε
γ^+	γ	ε	γ^+	γ	γ^+	ε	ε	ε
δ	δ^+	ε	δ	δ^+	δ	ε	ε	ε
δ^+	δ	δ^+	δ^+	ε	ε	δ	δ^+	ε
ε	ε	ε	ε	ε	ε	ε	ε	ε

As S is closed under composition and $^+$, S is a $(2,1)$-subalgebra of $\mathcal{PT}_{\{1,2,3,4\}}$ and so S is a left restriction semigroup with distinguished semilattice

$$E_S = \{\alpha^+, \beta, \gamma^+, \delta^+, \varepsilon\}.$$

We shall now look at examples which are not directly derived as $(2,1)$-subalgebras of partial transformation monoids. As we have seen in Proposition 2.2.13, inverse semigroups are weakly ample semigroups. However, there is another familar type of semigroups that are not obviously restriction semigroups. The following example is key for many of the major results in later chapters:

Example 2.5.3. Let M be a monoid with identity 1 and $E = \{1\}$. Let

$$m^+ = 1 \text{ and } m^* = 1$$

for all $m \in M$. Then M is a restriction semigroup with distinguished semilattice of idempotents E. We shall refer to such a restriction semigroup as *reduced* restriction.

As a monoid may be regarded as a left restriction semigroup it can be regarded as an algebra of type $(2,1)$ or $(2,1,0)$. It follows that we have the following result generalised from [22]:

Lemma 2.5.4. *Let S be an arbitrary monoid. A subset X of S is a set of generators of S as an algebra of type* $(2,0)$ *if and only if it is a set of generators of S as an algebra of type* $(2,1,0)$.

If M and N are monoids and $\varphi : M \to N$ is a monoid morphism, then φ is a $(2,1,0)$-morphism. This is because

$$m^+\varphi = 1\varphi = 1 = (m\varphi)^+$$

for $m \in M$.

Before looking at our next example, we need a few definitions.

Definition 2.5.5. Let X be a set and S be a monoid. Then S *acts on X on the left* if there exists a map $S \times X \to X$, $(s,y) \mapsto s \cdot y$ such that $\forall y \in X$ and $\forall s,t \in S$ we have

$$1 \cdot y = y \text{ and}$$

$$st \cdot y = s \cdot (t \cdot y).$$

If in addition, X is a semigroup and

$$s \cdot ab = (s \cdot a)(s \cdot b)$$

for $s \in S$ and $a,b \in X$, then we say S *acts by morphisms* on X.

An action via morphisms on a semilattice X is order preserving, but the converse is not necessarily true.

We note that in the case when we are considering a group G acting on the left of a set X, which is defined in the same way, an element g of the group acts by a bijection and the inverse bijection maps $x \in X$ to $g^{-1} \cdot x \in X$.

Definition 2.5.6. Let X be a semilattice and let S be a monoid such that S acts by morphisms on X via $\cdot$. A binary operation is defined on $X \times S$ by

$$(x,s)(y,t) = (x(s \cdot y), st)$$

for $(x,s),(y,t) \in X \times S$. Then $X * S$, with underlying set $X \times S$ and binary operation as described, is the *semidirect product* of X and S.

Example 2.5.7. Let $X * S$ be the semidirect product of a semilattice X and monoid S. Then $X * S$ is a left restriction semigroup with $(x,s)^+ = (x,1)$ for all $(x,s) \in X * S$.

Firstly, let $(x,s),(y,t) \in X * S$. Then as $s \cdot y \in X$, we have $x(s \cdot y) \in X$. Therefore the binary operation is closed.

For $(x, s), (y, t), (z, u) \in X * S$,

$$\begin{aligned}(x, s)[(y, t)(z, u)] &= (x, s)(y(t \cdot z), tu) \\ &= (x(s \cdot (y(t \cdot z))), s(tu)) \\ &= (x(s \cdot y)(s \cdot (t \cdot z)), s(tu)) \\ &= (x(s \cdot y)(st \cdot z), (st)u) \\ &= (x(s \cdot y), st)(z, u) \\ &= [(x, s)(y, t)](z, u)\end{aligned}$$

using the fact that S acts on X via morphisms. Hence $X * S$ is a semigroup.

We wish to show that $X * S$ is a left restriction semigroup with distinguished semilattice

$$E = \{(e, 1) : e \in X\}.$$

It can easily be seen that each element of E is an idempotent and

$$\begin{aligned}(e, 1)(f, 1) &= (e(1 \cdot f), 1) \\ &= (ef, 1) \\ &= (fe, 1) \\ &= (f(1 \cdot e), 1) \\ &= (f, 1)(e, 1)\end{aligned}$$

since X is a semilattice. Therefore E is a subsemilattice of $X * S$.

We claim that

$$(e, s) \, \widetilde{\mathcal{R}}_E \, (e, 1)$$

for $(e, s) \in X * S$. We have

$$\begin{aligned}(e, 1)(e, s) &= (e(1 \cdot e), s) \\ &= (e, s),\end{aligned}$$

and for $(f, 1) \in E$,

$$\begin{aligned}(f, 1)(e, s) = (e, s) &\Rightarrow (f(1 \cdot e), s) = (e, s) \\ &\Rightarrow (fe, s) = (e, s) \\ &\Rightarrow fe = e \\ &\Rightarrow (f, 1)(e, 1) = (fe, 1) = (e, 1).\end{aligned}$$

Therefore

$$(e, s) \, \widetilde{\mathcal{R}}_E \, (e, 1)$$

for $(e, s) \in X * S$.

We wish to show $\widetilde{\mathcal{R}}_E$ is a left congruence. First we note that for $(e, s), (f, t) \in$

$X * S$,

$$\begin{aligned}(e,s)\,\widetilde{\mathcal{R}}_E\,(f,t) &\Leftrightarrow (e,s)^+ = (f,t)^+ \\ &\Leftrightarrow (e,1) = (f,1) \\ &\Leftrightarrow e = f.\end{aligned}$$

For $(e,s),(f,t),(g,u) \in X * S$, we have

$$\begin{aligned}(e,s)\,\widetilde{\mathcal{R}}_E\,(f,t) &\Rightarrow e = f \\ &\Rightarrow g(u \cdot e) = g(u \cdot f) \\ &\Rightarrow (g(u \cdot e), us)\,\widetilde{\mathcal{R}}_E\,(g(u \cdot f), ut) \\ &\Rightarrow (g,u)(e,s)\,\widetilde{\mathcal{R}}_E\,(g,u)(f,t).\end{aligned}$$

So $\widetilde{\mathcal{R}}_E$ is a left congruence.

It remains to show that the left ample condition holds. Take $(e,s) \in X*S$ and $(f,1) \in E$. Then

$$\begin{aligned}[(e,s)(f,1)]^+(e,s) &= (e(s \cdot f), s)^+(e,s) \\ &= (e(s \cdot f), 1)(e,s) \\ &= (e(s \cdot f)e, s) \\ &= (e(s \cdot f), s) \\ &= (e,s)(f,1).\end{aligned}$$

Therefore $X * S$ is a left restriction semigroup with distinguished semilattice E.

The following example is a special case of Definition 2.5.6:

Example 2.5.8. Let S be a left restriction monoid. We shall show that S acts by morphisms on E_S via $s \cdot e = (se)^+$. We have

$$1 \cdot e = (1e)^+ = e^+ = e$$

for $e \in E_S$. Using Proposition 2.2.10,

$$s \cdot (t \cdot e) = s \cdot (te)^+ = (s(te)^+)^+ = (ste)^+ = st \cdot e$$

and

$$s \cdot ef = (sef)^+ = ((se)^+sf)^+ = (se)^+(sf)^+ = (s \cdot e)(s \cdot f)$$

for $s,t \in S$ and $e,f \in E_S$.

The semidirect product $T = E_S * S$, with binary operation

$$\begin{aligned}(e,s)(f,t) &= (e(s \cdot f), st) \\ &= (e(sf)^+, st)\end{aligned}$$

where $(e,s),(f,t) \in E_S \times S$, is therefore a left restriction semigroup with distinguished semilattice

$$E_T = \{(e,1) : e \in E_S\}.$$

Here we present a special case of Example 2.7.3 in [29]:

Example 2.5.9. Let M be a monoid, I a non-empty set and P the $I \times I$ identity matrix. Let

$$\mathcal{M} := \mathcal{M}^0(M; I, I; P)$$

be a Rees matrix semigroup, i.e. a *Brandt semigroup* $\mathcal{B}^0(M; I)$, which consists of the set

$$S = (I \times M \times I) \cup \{0\}$$

and binary operation defined by

$$(i,a,\lambda)(j,b,\mu) = \begin{cases} (i,ab,\mu) & \text{if } \lambda = j \\ 0 & \text{if } \lambda \neq j. \end{cases}$$

and

$$(i,a,\lambda)0 = 0(i,a,\lambda) = 00 = 0$$

for $(i,a,\lambda),(j,b,\mu) \in S$.

Idempotents of $\mathcal{M}$ are of the form (i,e,i), where $i \in I$ and $e \in E(M)$. As M is a monoid, it is a restriction semigroup with distinguished semilattice of idempotents $\{1\}$. It follows that $\mathcal{M}$ is a restriction semigroup with distinguished semilattice of idempotents

$$E_S = \{(i,1,i) : i \in I\} \cup \{0\}.$$

Our next example is the *Bruck-Reilly extension* of a monoid determined by a morphism:

Example 2.5.10. Suppose M is a monoid and $\theta : M \to H_1$ is a monoid morphism, where H_1 is the group of units of M. We shall let θ^n denote n applications of θ and θ^0 denote the identity map. Let $S = BR(M;\theta)$ consist of set

$$S = \mathbb{N}^0 \times M \times \mathbb{N}^0$$

with binary operation defined by

$$(a,m,b)(c,n,d) = (a-b+t, m\theta^{t-b}n\theta^{t-c}, d-c+t),$$

where $t = \max\{b,c\}$, for $(a,m,b),(c,n,d) \in S$. As proved in [31], $BR(M;\theta)$ is a semigroup and the idempotents are of the form (a,e,a), where $a \in \mathbb{N}^0$ and $e \in E(M)$.

Let us consider

$$E_S = \{(a,1,a) : a \in \mathbb{N}^0\}.$$

We wish to show that $BR(M;\theta)$ is a restriction semigroup with distinguished semilattice of idempotents E_S. We shall show that $BR(M;\theta)$ is left restriction, with the proof that it is right restriction being dual.

The elements of E_S commute as, for $a, b \in \mathbb{N}^0$, we have

$$\begin{aligned}(a,1,a)(b,1,b) &= (t, 1\theta^{t-a}1\theta^{t-b}, t), \text{ where } t = \max\{a,b\}\\ &= (t,1,t)\\ &= (b,1,b)(a,1,a).\end{aligned}$$

We wish to show $(a,m,b)\,\widetilde{\mathcal{R}}_{E_S}\,(a,1,a)$ for $(a,m,b) \in S$. We have

$$\begin{aligned}(a,1,a)(a,m,b) &= (t, m\theta^{t-a}, b-a+t), \text{ where } t = \max\{a,a\} = a\\ &= (a, m\theta^0, b)\\ &= (a,m,b).\end{aligned}$$

For $(c,1,c) \in E_S$,

$$\begin{aligned}(c,1,c)(a,m,b) = (a,m,b) &\Rightarrow (t, m\theta^{t-a}, b-a+t) = (a,m,b),\\ &\qquad \text{where } t = \max\{a,c\}\\ &\Rightarrow t = a\\ &\Rightarrow (c,1,c)(a,1,a) = (a,1,a).\end{aligned}$$

So $(a,m,b)\,\widetilde{\mathcal{R}}_{E_S}\,(a,1,a)$ and we shall let $(a,m,b)^+ = (a,1,a)$.

Now we wish to show that $\widetilde{\mathcal{R}}_{E_S}$ is a left congruence. For $(a,m,b), (c,n,d) \in S$,

$$\begin{aligned}(a,m,b)\,\widetilde{\mathcal{R}}_{E_S}\,(c,n,d) &\Leftrightarrow (a,m,b)^+ = (c,n,d)^+\\ &\Leftrightarrow (a,1,a) = (c,1,c)\\ &\Leftrightarrow a = c.\end{aligned}$$

So

$$\begin{aligned}(a,m,b)\,\widetilde{\mathcal{R}}_{E_S}\,(c,n,d) &\Rightarrow a = c\\ &\Rightarrow \max\{l,a\} = \max\{l,c\}, \text{ for } l \in \mathbb{N}^0\\ &\Rightarrow k - l + \max\{l,a\} = k - l + \max\{l,c\}\\ &\qquad \text{for any } k, l \in \mathbb{N}^0\\ &\Rightarrow [(k,p,l)(a,m,b)]^+ = [(k,p,l)(c,n,d)]^+\\ &\qquad \text{for any } (k,p,l) \in S\\ &\Rightarrow (k,p,l)(a,m,b)\,\widetilde{\mathcal{R}}_{E_S}\,(k,p,l)(c,n,d)\\ &\qquad \text{for any } (k,p,l) \in S.\end{aligned}$$

Therefore $\widetilde{\mathcal{R}}_{E_S}$ is a left congruence.

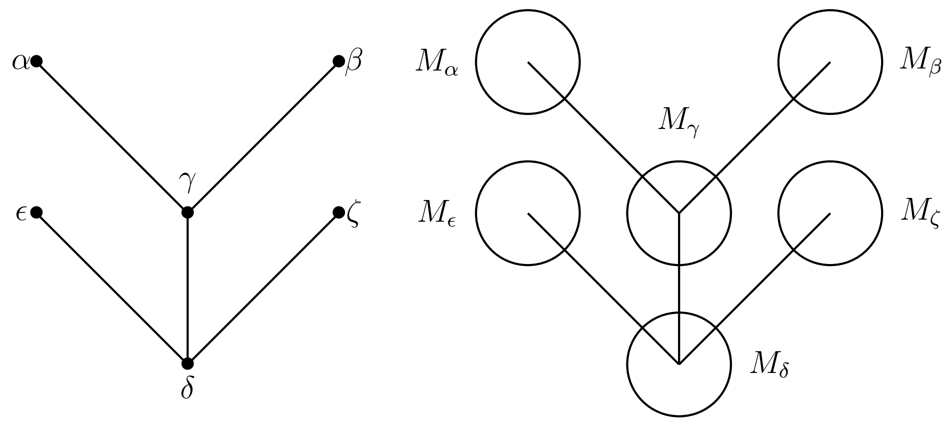
It remains to show that the left ample condition holds. We have for $(a, m, b) \in S$ and $(c, 1, c) \in E_S$,

$$\begin{aligned}
[(a, m, b)(c, 1, c)]^+(a, m, b) &= (a - b + t, m\theta^{t-b}, t)^+(a, m, b) \\
&\qquad \text{where } t = \max\{b, c\} \\
&= (a - b + t, 1, a - b + t)(a, m, b) \\
&= (s, m\theta^{s-a}, b - a + s) \\
&\qquad \text{where } s = \max\{a - b + t, a\} = a - b + t \\
&= (a - b + t, m\theta^{t-b}, t) \\
&= (a, m, b)(c, 1, c).
\end{aligned}$$

As the left ample condition holds, $BR(M; \theta)$ is a left restriction semigroup with distinguished semilattice of idempotents E_S.

Before we look at our next example, we need the definition of a strong semilattice of monoids, which we have adapted from the definition for semigroups in [30].

Definition 2.5.11. Let S be a semigroup which is a disjoint union of monoids M_α where the indices α form a semilattice Y suppose that for all $\alpha, \beta \in Y$, $M_\alpha M_\beta \subseteq M_{\alpha\beta}$. Then S is called a *semilattice Y of monoids M_α where $\alpha \in Y$*. This can be represented in the following way.

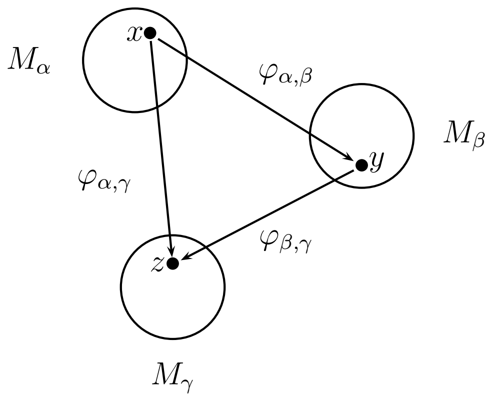


Now consider $\alpha, \beta \in Y$ where $\alpha \geq \beta$. Let $\varphi_{\alpha,\beta} : M_\alpha \to M_\beta$ be a monoid morphism such that:

(1) $\varphi_{\alpha,\alpha} = I_{M_\alpha}$ for all $\alpha \in Y$;

(2) for $\alpha, \beta, \gamma \in Y$, where $\alpha \geq \beta \geq \gamma$, $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$.

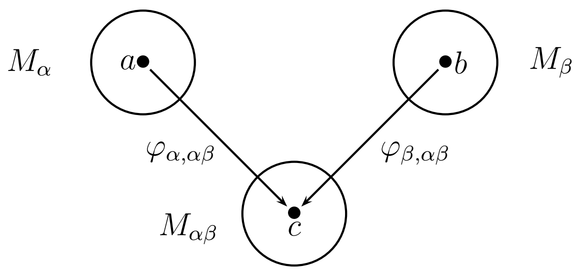
Then $\varphi_{\alpha,\beta}$ is called a *connecting morphism*. The second condition can be illustrated as follows:



Let us consider the set $S = \dot{\bigcup\limits_{\alpha \in Y}} M_\alpha$ and define a binary operation $*$ on S by

$$a * b = (a\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta}),$$

where $a \in M_\alpha$ and $b \in M_\beta$. These morphisms are illustrated in the diagram below, where we shall let $c = (a\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta})$.



Then $S = [Y; M_\alpha; \varphi_{\alpha,\beta}]$ is called a *strong semilattice Y of monoids M_α where $a \in Y$ with connecting morphisms $\varphi_{\alpha,\beta}$.*

It can be proved that S, along with binary operation $*$, forms a semigroup and that $a * b = ab$ where $a, b, ab \in M_\alpha$ and we write the original binary operation in the monoid M_α as juxtaposition. Moreover, S is a semilattice Y of monoids, M_α, where $\alpha \in Y$. We shall show that if we take S to be a semilattice of monoids such that their identities form a subsemilattice, then $S = [Y; M_\alpha; \varphi_{\alpha,\beta}]$ is a restriction semigroup.

Example 2.5.12. Let S be a strong semilattice of monoids $[Y; M_\alpha; \varphi_{\alpha,\beta}]$. We shall denote each identity by 1_α and its corresponding monoid by M_α. We shall show that S is left restriction, with the proof that it is right restriction being dual. We put $E_S = \{1_\alpha : \alpha \in Y\}$. Notice that $E_S \subseteq E(S)$.

The elements of E_S commute under the binary operation $*$ as, for $1_\alpha, 1_\beta \in E_S$, we have

$$\begin{aligned} 1_\alpha * 1_\beta &= (1_\alpha\varphi_{\alpha,\alpha\beta})(1_\beta\varphi_{\beta,\alpha\beta}) \\ &= 1_{\alpha\beta}1_{\alpha\beta} = 1_{\alpha\beta} \\ &= (1_\beta\varphi_{\beta,\alpha\beta})(1_\alpha\varphi_{\alpha,\alpha\beta}) \\ &= 1_\beta * 1_\alpha. \end{aligned}$$

We wish to show $a\,\widetilde{\mathcal{R}}_{E_S}\,1_\alpha$ for $a \in M_\alpha$. We have

$$1_\alpha * a = 1_\alpha a = a$$

as $1_\alpha, a \in M_\alpha$. For $1_\beta \in E_S$,

$$\begin{aligned} 1_\beta * a = a &\Rightarrow (1_\beta\varphi_{\beta,\beta\alpha})(a\varphi_{\alpha,\beta\alpha}) = a \\ &\Rightarrow \alpha\beta = \alpha \\ &\Rightarrow 1_\beta * 1_\alpha = 1_{\alpha\beta} = 1_\alpha \end{aligned}$$

as above. Therefore $a\,\widetilde{\mathcal{R}}_{E_S}\,1_\alpha$ for $a \in M_\alpha$ and we shall let $a^+ = 1_\alpha$.

Now we wish to show that $\widetilde{\mathcal{R}}_{E_S}$ is a left congruence. For $a, b \in S$, where $a \in M_\alpha$ and $b \in M_\beta$,

$$\begin{aligned} a\,\widetilde{\mathcal{R}}_{E_S}\,b &\Leftrightarrow a^+ = b^+ \\ &\Leftrightarrow 1_\alpha = 1_\beta \\ &\Leftrightarrow \alpha = \beta. \end{aligned}$$

So, the $\widetilde{\mathcal{R}}_{E_S}$-classes are the semigroups M_α, where $\alpha \in Y$. It is then clear that $\widetilde{\mathcal{R}}_{E_S}$ is a left congruence.

It remains to show that the left ample condition holds. We have, for $a \in M_\alpha$ and $1_\beta \in E_S$,

$$\begin{aligned} (a * 1_\beta)^+ * a &= [(a\varphi_{\alpha,\alpha\beta})(1_\beta\varphi_{\beta,\alpha\beta})]^+ * a \\ &= [(a\varphi_{\alpha,\alpha\beta})1_{\alpha\beta}]^+ * a \\ &= (a\varphi_{\alpha,\alpha\beta})^+ * a \\ &= 1_{\alpha\beta} * a \\ &= (1_{\alpha\beta}\varphi_{\alpha\beta,\alpha\beta\alpha})(a\varphi_{\alpha,\alpha\beta\alpha}) \\ &= 1_{\alpha\beta}(a\varphi_{\alpha,\alpha\beta}) \\ &= a\varphi_{\alpha,\alpha\beta} \\ &= a * 1_\beta. \end{aligned}$$

As the left ample condition holds, $S = [Y; M_\alpha; \varphi_{\alpha,\beta}]$ is a left restriction semigroup with distinguished semilattice of idempotents E_S.

2.6 The natural partial order

As in inverse semigroup theory we shall define the relation $\leq$. Let S be a left restriction semigroup with distinguished semilattice of idempotents E. We define $\leq$ on S by

$$a \leq b \Leftrightarrow a = eb \text{ for some } e \in E.$$

If S is a right restriction semigroup with distinguished semilattice of idempotents E, we define $\leq$ on S by

$$a \leq b \Leftrightarrow a = bf \text{ for some } f \in E.$$

However, we note that if S is a restriction semigroup, then these two definitions are in fact equivalent by the ample conditions.

In fact, we can be more specific about the idempotents e and f in the above definitions:

Proposition 2.6.1. *Let S be a restriction semigroup. Then for $a, b \in S$, where S is a left restriction semigroup,*

$$a \leq b \Leftrightarrow a = a^+ b$$

and if S is right restriction,

$$a \leq b \Leftrightarrow a = ba^*.$$

It can be easily checked that these relations are partial orders. For a left restriction semigroup S with distinguished semilattice of idempotents E, the relation $\leq$ is clearly right compatible with the multiplication of S and it can be seen that it is left compatible using the left ample condition. Dually, $\leq$ is compatible with multiplication of a right restriction semigroup.

When considering a left restriction semigroup with distinguished semilattice of idempotents E, we have for $a, b \in S$,

$$a^+ \leq b^+ \Leftrightarrow a^+ = a^+ b^+,$$

which is the usual order on E, and

$$a \leq b \Rightarrow a = a^+ b \Rightarrow a^+ = a^+ b^+ \Rightarrow a^+ \leq b^+.$$

We note the following useful lemma, the proof for which in [14] for left adequate semigroups can be easily adapted for left restriction semigroups.

Lemma 2.6.2. *Let S be a left restriction semigroup. Then*
(1) $(ab)^+ = (ab^+)^+$ for all $a, b \in S$;
(2) $(ea)^+ = ea^+$ for all $a \in S$ and $e \in E$;
(3) $(ab)^+ \leq a^+$ for all $a, b \in S$.

We return to the partial transformation monoid on a set X. All left restriction semigroups are embeddable into some $\mathcal{PT}_X$. The natural partial order defined on $\mathcal{PT}_X$ is

$$\alpha \leq \beta \Leftrightarrow \alpha = \beta\,|_{\mathrm{dom}(\alpha)},$$

which restricts to the usual partial order on idempotents in the distinguished semilattice.

2.7 The least congruence identifying E

In this section we introduction the relation σ on a semigroup S and consider σ on left/right restriction, weakly ample, ample and inverse semigroups.

Definition 2.7.1. [18] Let S be a semigroup and E be a set of idempotents contained in S. Then for $a, b \in S$, the relation σ_E is defined to be the least (semigroup) congruence on S identifying the elements of E.

If $E = E(S)$, then we may write σ for σ_E and if S is either left or right restriction we shall denote σ_{E_S} by σ_S, where E_S is the distinguished semilattice of S. Notice that if S is left restriction, then σ_S is actually a $(2,1)$-congruence, hence the least $(2,1)$-congruence identifying the elements of E_S.

The left ample condition ensures the following result, as proved in [21]:

Lemma 2.7.2. *Let S be a left restriction semigroup with distinguished semilattice E. Then for all $a, b \in S$,*

$$a \,\sigma_S\, b \Leftrightarrow ea = eb \textit{ for some } e \in E.$$

Proposition 2.7.3. *Let S be a left restriction semigroup with distinguished semilattice E. Then for all $a, b \in S$,*

$$a \,\sigma_S\, b \Leftrightarrow ea = fb \textit{ for some } e, f \in E.$$

Proof. If $a \,\sigma_S\, b$, then clearly $ea = fb$ for some $e, f \in E$. Conversely suppose that $ea = fb$ for some $e, f \in E$. Then

$$(ef)a = eefa = ef(ea) = ef(fb) = (ef)b.$$

So $a \,\sigma_S\, b$. □

Proposition 2.7.4. *If S and T are left restriction semigroups and $\theta : S \to T$ is a $(2,1)$-morphism, then*

$$a \,\sigma_S\, b \Rightarrow a\theta \,\sigma_T\, b\theta$$

for $a, b \in S$.

Proof. We have

$$\begin{aligned} a \,\sigma_S\, b &\Rightarrow ea = eb \text{ for some } e \in E_S \\ &\Rightarrow (ea)\theta = (eb)\theta \text{ for some } e \in E_S \\ &\Rightarrow (e\theta)(a\theta) = (e\theta)(b\theta) \text{ for some } e\theta \in E_T. \end{aligned}$$

□

The following result is straightforward and is dual to the results in Lemma 2.7.2 and Proposition 2.7.3.

Lemma 2.7.5. *Let S be a right restriction semigroup with distinguished semilattice E. Then for $a, b \in S$ the following are equivalent:*

(i) $a \,\sigma_S\, b$;

(ii) $af = bf$ *for some* $f \in E$;

(iii) $ae = bf$ *for some* $e, f \in E$;

Proposition 2.7.6. *Let S be a restriction semigroup with distinguished semilattice E. Then for $a, b \in S$ the following are equivalent:*

(i) $a \,\sigma_S\, b$;

(ii) $ea = eb$ *for some* $e \in E$;

(iii) $af = bf$ *for some* $f \in E$;

(iv) $ae = bf$ *for some* $e, f \in E$;

(v) $ea = fb$ *for some* $e, f \in E$;

(vi) $ea = bf$ *for some* $e, f \in E$.

Proof. As we know (i), (ii), (iii), (iv) and (v) are equivalent, we shall show that (ii) implies (vi) and that (iv) implies (v).

Suppose $ea = eb$ for some $e \in E$. As $eb = b(eb)^*$, we have

$$ea = b(eb)^*,$$

where $e, (eb)^* \in E$. Now suppose $ea = bf$ for some $e, f \in E$. Using the same argument,

$$ea = (bf)^+ b.$$

As $e, (bf)^+ \in E$, all the statements are equivalent. $\square$

If S is a left restriction monoid, S/σ_S is a monoid that can be regarded as an algebra of type $(2, 1, 0)$ by the comment preceding Lemma 2.5.4.

Lemma 2.7.7. *Let S be a left restriction semigroup. Then $e \,\sigma_S\, f$ for all $e, f \in E$. Further, if $e \in E$, then $e\sigma_S$ is the identity on $T = S/\sigma_S$.*

Proof. We shall consider a left restriction semigroup S with distinguished semilattice E. If $e \in E$ and $x\sigma_S \in T$, then $e \,\sigma_S\, x^+$. So

$$\begin{aligned}(e\sigma_S)(x\sigma_S) &= (x^+\sigma_S)(x\sigma_S)\\ &= (x^+x)\sigma_S\\ &= x\sigma_S\end{aligned}$$

and

$$\begin{aligned}(x\sigma_S)(e\sigma_S) &= (x\sigma_S)(e\sigma_S)\\ &= ((xe)^+x)\sigma_S\\ &= ((xe)^+\sigma_S)(x\sigma_S)\\ &= (x^+\sigma_S)(x\sigma_S)\\ &= x\sigma_S.\end{aligned}$$

□

Corollary 2.7.8. *If S is a left restriction monoid, then $\sigma_S^\natural : S \to S/\sigma_S$ defined by*

$$s\sigma_S^\natural = s\sigma_S$$

is a $(2,1,0)$-morphism.

Proof. It can easily be seen that $\sigma_S^\natural : S \to S/\sigma_S$ is a monoid morphism. For $s \in S$, we have

$$s^+\sigma_S^\natural = s^+\sigma_S = 1_{S/\sigma_S} = (s\sigma_S^\natural)^+.$$

□

The following results are shown in [45] and [37]:

Proposition 2.7.9. *If S is an inverse semigroup, then $T = S/\sigma$ is a group, so that σ is the least group congruence on S.*

Proof. From Lemma 2.7.7, $e\sigma$ is the identity of T for any $e \in E(S)$. We wish to show the existence of inverses in $T = S/\sigma$. Let $s \in S$. We know $(ss')\sigma = (s's)\sigma$ is the identity of T and clearly

$$(s\sigma)(s'\sigma) = (ss')\sigma = (s's)\sigma = (s'\sigma)(s\sigma).$$

Hence, $T = S/\sigma$ is a group and $(s\sigma)^{-1} = s'\sigma$. □

The dual of the following result is shown in [12] in the case where S is a monoid.

Proposition 2.7.10. *If S is a left ample semigroup, then $T = S/\sigma$ is right cancellative, so that σ is the least right cancellative congruence on S.*

Proof. We wish to show that T is right cancellative by showing for $a, b, c \in S$,

$$(b\sigma)(a\sigma) = (c\sigma)(a\sigma) \Rightarrow b\sigma = c\sigma, \text{ i.e. } b\,\sigma\,c.$$

Using the fact that $f\sigma$ acts as the identity of T for any $f \in E(S)$, we have

$$\begin{aligned}(b\sigma)(a\sigma) = (c\sigma)(a\sigma) &\Rightarrow (ba)\sigma = (ca)\sigma \\ &\Rightarrow (ba)\,\sigma\,(ca) \\ &\Rightarrow eba = eca \text{ for some } e \in E(S) \\ &\Rightarrow eba^+ = eca^+ \\ &\Rightarrow e(ba^+)^+b = e(ca^+)^+c \text{ by the ample condition} \\ &\Rightarrow b\sigma = c\sigma \text{ by Proposition 2.7.3.}\end{aligned}$$

Therefore, T is right cancellative. By Definition 2.7.1, σ is the least right cancellative congruence on S. $\square$

The first part of the following result is proved in [20] in the case where S is a weakly left ample monoid and the latter is from [2]. A *unipotent monoid* is a monoid with only one idempotent.

Proposition 2.7.11. *If S is a weakly left ample semigroup, then $T = S/\sigma$ is unipotent, so that σ is the least unipotent monoid congruence on S, i.e. a congruence on S such that S/σ is a unipotent monoid.*

Proof. Let $a\sigma \in E(T)$. Then $(a\sigma)(a\sigma) = (a\sigma)$, i.e. $a^2\sigma = a\sigma$, which implies $eaa = ea$ for some $e \in E(S)$. We have

$$\begin{aligned}(eae)^2 &= eaeeae \\ &= e(ae)ae \\ &= e((ae)^+a)ae \\ &= e(ae)^+aae \\ &= (ae)^+eaae \\ &= (ae)^+eae \text{ as } eaa = ea \\ &= e(ae)^+ae \\ &= eae.\end{aligned}$$

Therefore, $eae \in E(S)$.

Let $g = eae(ae)^+ \in E(S)$. We have

$$\begin{aligned}g(eae) &= eae(ae)^+eae \\ &= eaee(ae)^+e(ae)^+a \\ &= eae(ae)^+a \\ &= ga.\end{aligned}$$

Therefore $a\,\sigma\,eae$ and so T is unipotent. $\square$

Proposition 2.7.12. *When S is a left restriction semigroup with distinguished semilattice of idempotents E, σ_S is the least congruence on S such that its image is reduced left restriction.*

Proof. For any $x\sigma_S \in S/\sigma_S$,

$$(x\sigma_S)^+ = x^+\sigma_S = 1_{S/\sigma_S},$$

so S/σ_S is reduced. Conversely, let ρ be a $(2,1)$-congruence on S such that S/ρ is reduced. Then if $e, f \in E$, we have

$$e\rho = e^+\rho = (e\rho)^+ = (f\rho)^+ = f^+\rho = f\rho,$$

so $E \times E \subseteq \rho$. Hence $\sigma_S \subseteq \rho$. $\square$

2.8 Proper restriction semigroups

We shall provide a brief introduction to proper restriction semigroups. The background to this topic will be further explored in Chapters 4 and 5.

Definition 2.8.1. A left restriction semigroup S with distinguished semilattice of idempotents E is *proper* if and only if

$$\widetilde{\mathcal{R}}_E \cap \sigma_S = \iota$$

and dually a right restriction semigroup is *proper* if and only if

$$\widetilde{\mathcal{L}}_E \cap \sigma_S = \iota.$$

A restriction semigroup is *proper* if both these conditions hold.

Proper weakly ample semigroups are defined similarly.

Definition 2.8.2. A left ample semigroup is *proper* if and only if

$$\mathcal{R}^* \cap \sigma = \iota$$

and a right ample semigroup is *proper* if and only if

$$\mathcal{L}^* \cap \sigma = \iota.$$

An ample semigroup is *proper* if both these conditions hold.

Example 2.8.3. Let $X * S$ be the semidirect product of a semilattice X and monoid S. Then $X * S$ is a proper left restriction semigroup.

As in Example 2.5.7, $X*S$ is a left restriction semigroup and for $(e,s),(f,t) \in X * S$,

$$(e,s)\,\widetilde{\mathcal{R}}_E\,(f,t) \Leftrightarrow e = f.$$

We shall show

$$(e,s)\,\sigma_S\,(f,t) \Leftrightarrow s = t.$$

We have

$$\begin{aligned}(e,s)\,\sigma_S\,(f,t) &\Rightarrow (g,1)(e,s) = (g,1)(f,t) \text{ for some } (g,1) \in E\\ &\Rightarrow (ge,s) = (gf,t) \text{ for some } (g,1) \in E\\ &\Rightarrow s = t.\end{aligned}$$

Conversely, suppose that $s = t$. Then considering $(ef, 1) \in E$,

$$\begin{aligned}(ef,1)(e,s) &= (efe,s)\\ &= (ef,s)\\ &= (eff,t)\\ &= (ef,1)(f,t)\end{aligned}$$

and so $(e,s)\,\sigma_S\,(f,t)$.

We have

$$\begin{aligned}(e,s)\,(\widetilde{\mathcal{R}}_E \cap \sigma_S)\,(f,t) &\Rightarrow (e,s)\,\widetilde{\mathcal{R}}_E\,(f,t) \text{ and } (e,s)\,\sigma_S\,(f,t)\\ &\Rightarrow e = f \text{ and } s = t\\ &\Rightarrow (e,s) = (f,t).\end{aligned}$$

Therefore $X * S$ is a proper left restriction semigroup.

We have the following lemma for proper left restriction semigroups.

Lemma 2.8.4. *Let S be a proper left restriction semigroup with distinguished semilattice of idempotents E. If $a, b \in S$, then $a\,\sigma_S\,b$ if and only if $b^+a = a^+b$.*

Proof. Suppose that S is proper and $a, b \in S$. By Lemma 2.2.11, $a^+b\,\widetilde{\mathcal{R}}_E\,b^+a$. If $a\,\sigma_S\,b$ then, as $b^+\,\sigma_S\,a^+$, we have $b^+a\,\sigma_S\,a^+b$. Now $\sigma_S \cap \widetilde{\mathcal{R}}_E = \iota$ so that $b^+a = a^+b$. The converse is clear. □

The following result is a corollary of Lemma 2.7.7.

Corollary 2.8.5. *If S is a proper left or right restriction semigroup, then E_S is a σ_S-class.*

Proof. We shall consider a left restriction semigroup S with distinguished semilattice E, with the argument being dual for right restriction semigroups. Following Lemma 2.7.7, it remains to show that if $a \in S$ and $a\,\sigma_S\,e$, then $a \in E_S$. If $a \in S$ and $a\,\sigma_S\,e$, then $a\,\sigma_S\,a^+$ by Lemma 2.7.7. By definition, $a\,\widetilde{\mathcal{R}}_{E_S}\,a^+$ and since S is proper, $a = a^+$. □

Within a variety we have closure under taking subalgebras, homomorphic images and direct products with respect to the fundamental operations, but we see that if we consider a class of proper left restriction semigroups then subalgebras and direct products are also proper. The following two propositions are stated in [23] for left adequate monoids, the first originally appearing in [49], but we require the more general versions.

Proposition 2.8.6. *Let M be a weakly left E-adequate semigroup and let N be a subalgebra of M. Then*

(1) the subalgebra N is weakly left E-adequate and for all $a, b \in N$,

$$a\,\widetilde{\mathcal{R}}_{E_N}\,b \textit{ if and only if } a\,\widetilde{\mathcal{R}}_{E_M}\,b;$$

(2) if M is left restriction, then N is left restriction;

(3) if M is proper left restriction, then so is N and for $a, b \in N$,

$$a \, \sigma_N \, b \text{ if and only if } a \, \sigma_M \, b.$$

Proof. (1) For $a, b \in N$, we have

$$\begin{aligned} a \, \widetilde{\mathcal{R}}_{E_N} \, b &\Leftrightarrow a^+ = b^+ \text{ in } N \\ &\Leftrightarrow a^+ = b^+ \text{ in } M \\ &\Leftrightarrow a \, \widetilde{\mathcal{R}}_{E_M} \, b. \end{aligned}$$

(2) As left restriction semigroups form a variety, they are closed under taking subalgebras. Therefore, if M is a left restriction, then so is N.

(3) First, we note that $a \, \sigma_N \, b$ implies that $a \, \sigma_M \, b$ for $a, b \in N$ as

$$\begin{aligned} a \, \sigma_N \, b &\Rightarrow ea = eb \text{ for some } e \in E_N \\ &\Rightarrow ea = eb \text{ for some } e \in E_M \text{ as } E_N \subseteq E_M \\ &\Rightarrow a \, \sigma_M \, b. \end{aligned}$$

Along with (1), it follows that

$$\begin{aligned} a \, (\widetilde{\mathcal{R}}_{E_N} \cap \sigma_N) \, b &\Rightarrow a \, \widetilde{\mathcal{R}}_{E_N} \, b \text{ and } a \, \sigma_N \, b \\ &\Rightarrow a \, \widetilde{\mathcal{R}}_{E_M} \, b \text{ and } a \, \sigma_M \, b \\ &\Rightarrow a \, (\widetilde{\mathcal{R}}_{E_M} \cap \sigma_M) \, b \\ &\Rightarrow a = b, \end{aligned}$$

so that N is proper. Using Lemma 2.8.4, we have

$$a \, \sigma_M \, b \Leftrightarrow a^+ b = b^+ a \Leftrightarrow a \, \sigma_N \, b$$

for $a, b \in N$. $\square$

Proposition 2.8.7. *Let M_i, where $i \in I$ for some indexing set I, be proper left restriction semigroups. Let $M = \prod_{i \in I} M_i$. Then*

(1) for $(a_i), (b_i) \in M$,

$$(a_i) \, \widetilde{\mathcal{R}}_E \, (b_i) \text{ if and only if } a_i \, \widetilde{\mathcal{R}}_{E_{M_i}} \, b_i \text{ for all } i \in I,$$

where $E = \prod_{i \in I} E_{M_i}$;

(2) M is a left restriction semigroup;

(3) M is proper left restriction and for $(a_i), (b_i) \in M$,

$$(a_i) \, \sigma_M \, (b_i) \text{ if and only if } a_i \, \sigma_{M_i} \, b_i \text{ for all } i \in I.$$

Proof. (1) The result follows from $(a_i)^+ = (a_i^+)$ for $(a_i) \in M$.

(2) As left restriction semigroups form a variety, they are closed under taking direct products. Therefore, as each M_i is left restriction so is M.

(3) Using Lemma 2.8.4, we have

$$\begin{aligned}
(a_i)\,\sigma_M\,(b_i) &\Rightarrow (u_i)(a_i) = (u_i)(b_i) \text{ where } (u_i) = (u_i)^+ = (u_i^+)\\
&\Rightarrow u_i^+ a_i = u_i^+ b_i \text{ for all } i \in I\\
&\Rightarrow a_i\,\sigma_{M_i}\,b_i \text{ for all } i \in I\\
&\Rightarrow a_i^+ b_i = b_i^+ a_i \text{ for all } i \in I\\
&\Rightarrow (a_i)^+(b_i) = (b_i)^+(a_i)\\
&\Rightarrow (a_i)\,\sigma_M\,(b_i).
\end{aligned}$$

Consequently, along with part (1), we have

$$\begin{aligned}
(a_i)\,(\widetilde{\mathcal{R}}_E \cap \sigma_M)\,(b_i) &\Rightarrow (a_i)\,\widetilde{\mathcal{R}}_E\,(b_i) \text{ and } (a_i)\,\sigma_M\,(b_i)\\
&\Rightarrow a_i\,\widetilde{\mathcal{R}}_{E_{M_i}}\,b_i \text{ and } a_i\,\sigma_{M_i}\,b_i \text{ for all } i \in I\\
&\Rightarrow a_i\,(\widetilde{\mathcal{R}}_{E_{M_i}} \cap \sigma_{M_i})\,b_i \text{ for all } i \in I\\
&\Rightarrow a_i = b_i \text{ for all } i \in I.
\end{aligned}$$

□

We consider the semidirect product considered in Example 2.5.8

Example 2.8.8. If S is a left restriction monoid and S acts by morphisms on E_S via $s \cdot e = (se)^+$, then the semidirect product $E_S * S$ is a left restriction semigroup and by Example 2.8.3 $E_S * S$ is a proper left restriction semigroup.

Let

$$\hat{S} = \{(e, s) : e \leq s^+\} \subseteq E_S * S$$

with binary operation

$$\begin{aligned}
(e, s)(f, t) &= (e(s \cdot f), st)\\
&= (e(sf)^+, st)
\end{aligned}$$

where $(e, s), (f, t) \in \hat{S}$. We shall show that $\hat{S}$ is a proper left restriction monoid.

Take $(e, s), (f, t) \in \hat{S}$. Then $e \leq s^+$ and $f \leq t^+$. As $e(sf)^+ \in E$ and $st \in S$, it remains to show that

$$e(sf)^+ \leq (st)^+$$

to show the binary operation is closed. As the action of S on E_S is by morphisms it is order preserving. Hence

$$(sf)^+ \leq (st^+)^+ = (st)^+$$

and so

$$e(sf)^+ \leq (sf)^+ \leq (st)^+.$$

Since $e \leq 1$, $\hat{S}$ is closed under $^+$ and so $\hat{S}$ is a $(2,1)$-subalgebra of $E_S * S$. Therefore $\hat{S}$ is a left restriction semigroup with distinguished semilattice $E = \{(e,1) : e \in E_S\}$.

Take $(1,1) \in \hat{S}$ and $(e,s) \in \hat{S}$. Then $e \leq s^+$. We have

$$\begin{aligned}(e,s)(1,1) &= (e(s \cdot 1), s) \\ &= (es^+, s) \\ &= (e,s) \text{ as } e \leq s^+\end{aligned}$$

and

$$\begin{aligned}(1,1)(e,s) &= (1(1 \cdot e), s) \\ &= (e,s).\end{aligned}$$

So $\hat{S}$ is a monoid with identity (1,1).

As $\hat{S}$ is a $(2,1)$-subalgebra of $E_S * S$ and $E_S * S$ is proper, then $\hat{S}$ is proper by Proposition 2.8.6.

Definition 2.8.9. Let S be a left restriction semigroup with distinguished semilattice of idempotents E. A morphism, $\psi : M \to N$, is *E-separating* if for $e, f \in E$, we have

$$e\psi = f\psi \Rightarrow e = f.$$

A *proper left restriction cover* of S is a proper left restriction semigroup U together with an onto $(2,1,0)$-morphism $\psi : U \to S$, which is E-separating.

The techniques in the following theorem are 'folklore' and appear in several papers including [18]:

Theorem 2.8.10. *Let S be a left restriction monoid. Then S has a proper left restriction cover.*

Proof. Let

$$\hat{S} = \{(e,s) : e \leq s^+\} \subseteq E_S * S$$

be the proper left restriction monoid in Example 2.8.8. Suppose $\phi : \hat{S} \to S$ is defined by $(e,s)\phi = es$ for $(e,s) \in \hat{S}$. Taking $(e,s),(f,t) \in \hat{S}$, we have

$$\begin{aligned}
[(e,s)(f,t)]\phi &= (e(sf)^+, st)\phi \\
&= e(sf)^+st \\
&= esft \text{ using the left ample condition} \\
&= (e,s)\phi(f,t)\phi.
\end{aligned}$$

We also have

$$\begin{aligned}
[(e,s)\phi]^+ &= (es)^+ \\
&= es^+ \\
&= e \text{ as } e \leq s^+ \\
&= (e,1)\phi \\
&= (e,s)^+\phi
\end{aligned}$$

and clearly $(1,1)\phi = 1$. So ϕ is a $(2,1,0)$-morphism.

Considering $s \in S$, there exists $(s^+, s) \in \hat{S}$ such that

$$(s^+, s)\phi = s^+s = s.$$

Therefore ϕ is onto. If $(e,1),(f,1) \in E_{\hat{S}}$ such that $(e,1)\phi = (f,1)\phi$, then $e = f$. So ϕ is $E_{\hat{S}}$-separating. Therefore $\hat{S}$ is a proper left restriction cover of S. $\square$

Chapter 3

The Szendrei expansion

3.1 The Szendrei expansion of a monoid

3.1.1 Definitions and background

We shall begin this section by defining the Szendrei expansion of an arbitrary monoid. We shall summarise the background working, including the fact that the Szendrei expansion of a group coincides with the Birget-Rhodes expansion (as pointed out in [56]), and some universal properties.

Definition 3.1.1. Let M be a monoid and let $\mathscr{P}^1_f(M)$ denote the collection of finite subsets of M that contain the identity. We shall define the *Szendrei expansion* of M to be

$$Sz(M) = \{(A, g) : A \in \mathscr{P}^1_f(M), g \in A\}$$

together with the binary operation given by

$$(A, g)(B, h) = (A \cup gB, gh)$$

and unary operation

$$(A, g)^+ = (A, 1)$$

for $(A, g), (B, h) \in Sz(M)$. The action of $g \in G$ on a subset B is given by $gB = \{gb : b \in B\}$.

This is a subsemigroup of a semidirect product; note that the Szendrei expansion of a group coincides with the Birget-Rhodes expansion, as pointed out in [56]. Szendrei showed in [56] that this expansion had some universal properties, regarding *F-inverse semigroups*, which are inverse semigroups where every σ-class has a greatest element under the natural partial order. Here σ is the least group congruence as defined in Section 2.7.

For a monoid M with identity 1, the Szendrei expansion of M, $Sz(M)$, is also a monoid with identity $(\{1\}, 1)$ [16] and is left restriction [28] with distinguished semilattice

$$E = \{(A, 1) : A \in \mathscr{P}^1_f(M)\}.$$

A monoid with exactly one idempotent is called a *unipotent monoid*. If M is a unipotent monoid, $E(Sz(M)) = E$ and so $Sz(M)$ is a weakly left ample monoid [17]. Let M be a monoid and let $a, b, c \in M$. Then M is *right cancellative* if for $a, b, c \in M$,

$$ab = cb \Rightarrow a = c.$$

A right cancellative monoid is unipotent. If M is a right cancellative monoid, then $Sz(M)$ is a left ample monoid [16], and if M is a group, then $Sz(M)$ is an inverse monoid [6], [56].

In [11], Exel describes, via generators and relations, an "expansion" of a group G [11]. It was unknown to Exel that this presentation was one for the Szendrei expansion. In [35], Kellendonk and Lawson proved that Exel's construction was isomorphic to the Szendrei expansion.

Exel took a set of generators, $\overline{G} = \{[g] : g \in G\}$, and the following relations for $s, t \in G$:

(i) $[s^{-1}][s][t] \equiv [s^{-1}][st]$;

(ii) $[s][t][t^{-1}] \equiv [st][t^{-1}]$;

(iii) $[s][1] \equiv [s]$;

(iv) $[1][s] \equiv [s]$.

It was shown in [35] that the Szendrei expansion of G is isomorphic to the free semigroup on $\overline{G}$ factored by the congruence generated by these relations.

We shall construct similar presentations for the Szendrei expansions of other types of semigroups.

3.1.2 Premorphisms

We shall define the premorphisms that provided the inspiration for presentations, via generators and relations, for the Szendrei expansion of a monoid.

Definition 3.1.2. Let S and T be monoids, where T is left restriction. Then the function $\theta : S \to T$ is a *premorphism* if for $s, t \in S$,

(i) $(s\theta)(t\theta) \leq (st)\theta$;

(ii) $1\theta = 1$.

A premorphism is equivalent to a partial action [28], which we define as follows:

Definition 3.1.3. Let X be a set and T a monoid. Suppose we have a partial function $\bullet : X \times T \to X$, where $(x, t) \to x \bullet t$ and we write $\exists x \bullet t$ to mean that the action of t on x is defined. If the following conditions hold for $x \in X$ and $s, t \in T$:

(i) for all $x \in \mathscr{X}$, $\exists x \bullet 1$ and $x \bullet 1 = x$;

(ii) if $\exists x \bullet s$ and $\exists (x \bullet s) \bullet t$, then $\exists x \bullet st$ and $(x \bullet s) \bullet t = x \bullet st$;

then $\bullet$ is a *partial right action* of T on X. A *partial left action* of T on X is defined dually.

We shall consider this type of partial action in Chapter 8.

Definition 3.1.4. Let S and T be monoids, where T is left restriction. Then the function $\theta : S \to T$ is a *strong premorphism* if for $s, t \in S$,

(i) $(s\theta)(t\theta) = (s\theta)^+(st)\theta$;

(ii) $1\theta = 1$.

A strong premorphism is equivalent to a strong partial action [28], which we define as above, but with an alternative second condition:

(ii) if $\exists x \bullet s$, then $[\exists (x \bullet s) \bullet t$ if and only if $\exists x \bullet st]$, in which case $(x \bullet s) \bullet t = x \bullet st$;

The following result follows from Theorem 4.1 in [28] by putting $\bar{\theta} = I_{Sz(M)}$ in the latter part, but we shall prove it directly.

Proposition 3.1.5. *Let M be a monoid. The map $\iota : M \to Sz(M)$ given by*

$$m\iota = (\{1, m\}, m)$$

is a strong premorphism.

Proof. (i) For $s, t \in S$,

$$\begin{aligned}
(s\iota)(t\iota) &= (\{1, s\}, s)(\{1, t\}, t) \\
&= (\{1, s\} \cup s\{1, t\}, st) \\
&= (\{1, s, st\}, st) \\
&= (\{1, s\} \cup \{1, st\}, st) \\
&= (\{1, s\}, 1)(\{1, st\}, st) \\
&= (\{1, s\}, s)^+(\{1, st\}, st) \\
&= (s\iota)^+(st)\iota.
\end{aligned}$$

(ii) We have

$$\begin{aligned}
1\iota &= (\{1, 1\}, 1) \\
&= (\{1\}, 1).
\end{aligned}$$

$\square$

Proposition 3.1.6. *[28] The Szendrei expansion of a monoid M is generated as a $(2,1)$-algebra by elements of the form $m\iota$, where $\iota : M \to Sz(M)$ is defined above.*

The presentations we shall present in the next section were inspired by the fact that any strong premorphism $\theta : S \to T$, where S and T are left restriction semigroups, factors as $\iota\overline{\theta}$, where $\iota : S \to Sz(S)$ is a strong premorphism and $\overline{\theta}$ is a morphism uniquely determined by θ [24].

Proposition 3.1.7. *[28] Let S and T be monoids, where T is a left restriction monoid. For every strong premorphism $\theta : S \to T$ there is a unique $(2,1,0)$-morphism $\bar{\theta} : Sz(S) \to T$ such that the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[d, "\theta"'] & Sz(S) \arrow[dl, "\bar{\theta}"] \\ T & \end{tikzcd}$$

Conversely, if $\bar{\theta} : Sz(M) \to S$ is a $(2,1,0)$-morphism, for some monoid M, then $\theta = \iota\bar{\theta}$ is a strong premorphism.

3.1.3 Presentations via generators and relations

We shall exhibit a presentation for the Szendrei expansion of a monoid, using a similar approach to Exel's. Instead of factoring the free semigroup, we factor the free left restriction semigroup. Noting that the Szendrei expansion of a monoid is a left restriction monoid, our method is as follows.

Let M be a monoid and let F be the free left restriction semigroup on the set M. As discussed in Chapter 1, we can consider F as the free left restriction semigroup on $\overline{M}$, where $\overline{M} = \{[m] : m \in M\}$ is a set of generators having the same number of elements as M. We consider the following relations for $s, t \in M$:

(i) $[s][t] \equiv [s]^+[st]$;

(ii) $[s][1] \equiv [s]$;

(iii) $[1][s] \equiv [s]$;

(iv) $[1]^+ \equiv [1]$.

Let

$$\rho = \langle ([s][t], [s]^+[st]), ([s][1], [s]), ([1][s], [s]), ([1]^+, [1]) : s, t \in M \rangle, \qquad (*)$$

i.e. the congruence generated by the relations (i)-(iv). Note that we are using Exel's original notation where $[s]$ denotes a generator and $[s]\rho$ denotes the ρ-class of $[s]$.

Proposition 3.1.8. *Let M and T be monoids, where T is left restriction. Let F be the free left restriction semigroup on $\overline{M}$. Let $\theta : M \to T$ be a* *strong premorphism and* $\bar{\theta} : F \to T$ *be defined on the set of generators of* F *by*

$$[m]\bar{\theta} = m\theta.$$

Then $\bar{\theta}$ *is a morphism such that* $\rho \subseteq ker\ \bar{\theta}$.

Proof. The map $\bar{\theta}$ is a well-defined morphism since F is free on $\overline{M}$.

We have for $[s], [t] \in \overline{M}$,

$$\begin{aligned}([s][t])\bar{\theta} &= ([s]\bar{\theta})([t]\bar{\theta}) \\ &= (s\theta)(t\theta) \\ &= (s\theta)^+((st)\theta) \\ &= ([s]\bar{\theta})^+([st]\bar{\theta}) \\ &= ([s]^+[st])\bar{\theta}\end{aligned}$$

and

$$\begin{aligned}([1][s])\bar{\theta} &= ([1]\bar{\theta})([s]\bar{\theta}) \\ &= (1\theta)(s\theta) \\ &= s\theta \\ &= [s]\bar{\theta}.\end{aligned}$$

Similarly, $([s][1])\bar{\theta} = [s]\bar{\theta}$. We also have

$$[1]^+\bar{\theta} = ([1]\bar{\theta})^+ = (1\theta)^+ = 1 = 1\theta = [1]\bar{\theta}.$$

It follows that $\rho \subseteq \ker\ \bar{\theta}$. □

Omitting consideration of the identity element in the proof of Lemma 2.2.15 and comments at the end of Section 1.4 we have the following result.

Proposition 3.1.9. *Let* M *be a monoid and* F *the free left restriction semigroup on* $\overline{M}$. *Then any element of* F *is of the form*

$$\varepsilon_1^+ \dots \varepsilon_m^+ w_1 \dots w_n$$

where $\varepsilon_1^+ = ([x_1^1] \dots [x_{p(1)}^1])^+, \dots, \varepsilon_m^+ = ([x_1^m] \dots [x_{p(m)}^m])^+$ *and* $w_1 = [y_1]$, $\dots, w_n = [y_n]$ *for some* $m, n \in \mathbb{N}^0$ *where* $[x_j^i], [y_k] \in \overline{M}, 1 \leq i \leq m, 1 \leq j \leq p(i), 1 \leq k \leq n$.

Proposition 3.1.10. *Let* M *be a monoid and* F *be the free left restriction semigroup on* $\overline{M}$. *Then* F/ρ *is a monoid with identity* $[1]\rho$.

Proof. Due to the closure and associativity of the binary operation in F, F/ρ is a semigroup. It remains to show that it has identity $[1]\rho$. If $a \in F$, then it has the form

$$\varepsilon_1^+ \dots \varepsilon_m^+ w_1 \dots w_n,$$

as in Proposition 3.1.9. To show that $(a\rho)([1]\rho) = a\rho$, we need to consider 2 cases. Firstly, if $n \neq 0$, then

$$(w_n\rho)([1]\rho) = ([y_n]\rho)([1]\rho) = ([y_n][1])\rho = [y_n]\rho$$

and it follows that $(a\rho)([1]\rho) = a\rho$.

Secondly, if $n = 0$ and $m \neq 0$, then a is of the form

$$\varepsilon_1^+ \dots \varepsilon_m^+.$$

We have

$$\begin{aligned}
(\varepsilon_m^+\rho)([1]\rho) &= ((([x_1^m] \dots [x_{p(m)}^m])^+\rho)([1]\rho) \\
&= ((([x_1^m] \dots [x_{p(m)}^m])^+\rho)([1]^+\rho) \\
&= ((([x_1^m] \dots [x_{p(m)}^m])^+[1]^+)\rho \\
&= ((([1][x_1^m] \dots [x_{p(m)}^m])^+[1]^+)\rho \\
&= ([1][x_1^m] \dots [x_{p(m)}^m])^+\rho \text{ by part 3 of Lemma 2.6.2} \\
&= ([x_1^m] \dots [x_{p(m)}^m])^+\rho \\
&= \varepsilon_m^+\rho.
\end{aligned}$$

This implies that $(a\rho)([1]\rho) = a\rho$.

Using the fact that elements of E_F commute, we can similarly deduce that $([1]\rho)(a\rho) = a\rho$. If $m \neq 0$, then

$$([1]\rho)(\varepsilon_1^+ \dots \varepsilon_m^+)\rho = ([1]^+\rho)(\varepsilon_1^+ \dots \varepsilon_m^+)\rho = (\varepsilon_1^+ \dots \varepsilon_m^+)\rho([1]^+\rho)$$

and we can apply the argument above. If $m = 0$ and $n \neq 0$, then $([1]\rho)([y_1]\rho) = ([y_1]\rho)$ implies $([1]\rho)(w_1\rho) = (w_1\rho)$ and our result follows. Therefore F/ρ is a monoid with identity $[1]\rho$. $\square$

In fact, F/ρ is isomorphic to the Szendrei expansion of the monoid M:

Proposition 3.1.11. *Let M be a monoid and F be the free left restriction semigroup on $\overline{M}$. Then*

$$F/\rho \cong Sz(M),$$

where ρ is defined by $(*)$.

Proof. We note that $\overline{M}$ generates F.

Let $\tau : M \to F/\rho$ be given by $m\tau = [m]\rho$. Then,

(i) for $s, t \in M$,

$$\begin{aligned}(s\tau)(t\tau) &= ([s]\rho)([t]\rho) \\ &= ([s][t])\rho \\ &= ([s]^+[st])\rho \\ &= ([s]\rho)^+([st]\rho) \\ &= (s\tau)^+(st)\tau;\end{aligned}$$

(ii) we also have

$$1\tau = [1]\rho = [1]^+\rho = 1_{F/\rho}.$$

Therefore $\tau : M \to F/\rho$ is a strong premorphism. As F is the free left restriction monoid on $\overline{M}$, F/ρ is also a left restriction monoid. So, by Proposition 3.1.7, there is a $(2,1,0)$-morphism $\bar{\tau} : Sz(M) \to F/\rho$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[d, "\tau"'] & Sz(M) \arrow[dl, "\bar{\tau}"] \\ F/\rho & \end{tikzcd}$$

For the monoid M, $Sz(M)$ is a left restriction monoid and by Proposition 3.1.5, $\iota : M \to Sz(M)$ is a strong premorphism. By Proposition 3.1.8, $\bar{\iota} : F \to Sz(M)$ defined by

$$[m]\bar{\iota} = m\iota,$$

for $[m] \in \overline{M}$, is a morphism such that $\rho \subseteq \ker \bar{\iota}$.

As $\rho^\natural : F \to F/\rho$ is an onto morphism with kernel ρ, by Proposition 1.2.10, there exists a unique morphism $\bar{\bar{\iota}} : F/\rho \to Sz(M)$ such that the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "\bar{\iota}"] \arrow[d, "\rho^\natural"'] & Sz(M) \\ F/\rho \arrow[ur, "\bar{\bar{\iota}}"'] & \end{tikzcd}$$

We note that

$$([1]\rho)\bar{\bar{\iota}} = [1]\bar{\iota} = 1\iota = 1,$$

so $\bar{\bar{\iota}}$ is a monoid morphism. Considering M, we have the following diagram:

$$\begin{tikzcd}
F \arrow[r, "\bar{\iota}"] \arrow[d, "\rho^\natural"'] & Sz(M) \\
F/\rho \arrow[ur, "\bar{\bar{\iota}}"] & M \arrow[u, "\iota"'] \arrow[l, "\tau"']
\end{tikzcd}$$

We check that the lower triangle commutes. For $m \in M$, we have

$$\begin{aligned}
(m\tau)\bar{\bar{\iota}} &= ([m]\rho)\bar{\bar{\iota}} \\
&= [m]\bar{\iota} \\
&= m\iota,
\end{aligned}$$

so we have the following commutative diagrams:

$$\begin{tikzcd}
M \arrow[r, "\iota"] \arrow[d, "\tau"'] & Sz(M) \arrow[dl, "\bar{\tau}"] \\
F/\rho &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\iota"] \arrow[d, "\tau"'] & Sz(M) \\
F/\rho \arrow[ur, "\bar{\bar{\iota}}"'] &
\end{tikzcd}$$

where ι and τ are strong premorphisms and $\bar{\tau}$ and $\bar{\bar{\iota}}$ are morphisms.

For $m \in M$, we have

$$\begin{aligned}
(m\iota)\bar{\tau}\bar{\bar{\iota}} &= m(\iota\bar{\tau})\bar{\bar{\iota}} \\
&= m\tau\bar{\bar{\iota}} \\
&= m\iota.
\end{aligned}$$

As $m\iota$ generates $Sz(M)$ by Proposition 3.1.6, $\bar{\tau}\bar{\bar{\iota}}$ is the identity on $Sz(M)$, i.e.

$$\bar{\tau}\bar{\bar{\iota}} = I_{Sz(M)}.$$

As $m\tau = [m]\rho$, for $[m] \in \overline{M}$,

$$\begin{aligned}
([m]\rho)\bar{\bar{\iota}}\bar{\tau} &= (m\tau)\bar{\bar{\iota}}\bar{\tau} \\
&= m(\tau\bar{\bar{\iota}})\bar{\tau} \\
&= m\iota\bar{\tau} \\
&= m\tau \\
&= [m]\rho.
\end{aligned}$$

As $\{[m] : m \in M\}$ generates F as a semigroup, we have that $\{[m]\rho : m \in M\}$ generates F/ρ as a semigroup and so

$$\bar{\bar{\iota}}\bar{\tau} = I_{F/\rho}.$$

As $\bar{\bar{\iota}}$ and $\bar{\tau}$ are mutually inverse,

$$F/\rho \cong Sz(M).$$

$\square$

3.2 The Szendrei expansion of a left restriction semigroup

3.2.1 Definitions and background

We can extend the work in Section 3.1 by looking at the Szendrei expansion $Sz(S)$ of a left restriction semigroup S. Left restriction semigroups have a natural unary operation, denoted by $^+$, so we shall take the signature of left restriction semigroups to be $(2,1)$.

Definition 3.2.1. [24] Let S be a left restriction semigroup. The *Szendrei expansion* of S is the set

$$Sz(S) = \{(A,a) \in \mathscr{P}^f(S) \times S : a, a^+ \in A \text{ and } A \subseteq (\widetilde{R}_E)_a\},$$

where $\mathscr{P}^f(S)$ denotes the collection of all finite subsets of S and $(\widetilde{R}_E)_a$ denotes the $\widetilde{\mathcal{R}}_E$-class of a, together with binary operation

$$(A,a)(B,b) = ((ab)^+A \cup aB, ab)$$

and unary operation

$$(A,a)^+ = (A,a^+)$$

for $(A,a),(B,b) \in Sz(S)$.

We can also regard an arbitary monoid M as a left restriction semigroup by taking the semilattice of idempotents as $\{1\}$. The definition of the Szendrei expansion of a left restriction semigroup simplifies to the definition for a monoid presented in Section 3.1.

Proposition 3.2.2. *[24] If S is a left restriction semigroup with distinguished semilattice E, then $Sz(S)$ is a left restriction semigroup with distinguished semilattice*

$$E_{Sz(S)} = \{(F,f) \in Sz(S) : f \in E\}.$$

3.2.2 Premorphisms

As in Section 3.1, the definition of a premorphism provided inspiration for presentations.

Definition 3.2.3. Let S and T be left restriction semigroups. Then the function $\theta : S \to T$ is a *strong premorphism* if for $s,t \in S$,

(i) $(s\theta)(t\theta) = (s\theta)^+(st)\theta$;

(ii) $(s\theta)^+ \leq s^+\theta$,

where $\leq$ is the natural partial order on T.

Again, this type of strong premorphism is equivalent to a strong partial action [24]:

Definition 3.2.4. Let X be a set and T a left restriction semigroup. Suppose $\bullet : X \times T \to X$ is a partial function. If the following conditions hold for $x \in X$ and $s, t \in T$:

(i) if $\exists x \bullet s$, then $[\exists (x \bullet s) \bullet t$ if and only if $\exists x \bullet st]$, in which case $(x \bullet s) \bullet t = x \bullet st$;

(ii) for all $x \in \mathscr{X}, \exists x \bullet 1$ and $x \bullet 1 = x$;

then $\bullet$ is a *strong partial right action* of T on X.

The following result follows from Theorem 5.2 in [24] by putting $\bar{\theta} = I_{Sz(S)}$ in the latter part, but we shall prove it directly.

Proposition 3.2.5. *Let S be a left restriction semigroup. The map $\iota : S \to Sz(S)$ given by*

$$s\iota = (\{s^+, s\}, s)$$

is a strong premorphism.

Proof. (i) For $s, t \in S$,

$$\begin{aligned}
(s\iota)^+(st)\iota &= (\{s^+, s\}, s)^+(\{(st)^+, st\}, st) \\
&= (\{s^+, s\}, s^+)(\{(st)^+, st\}, st) \\
&= ((s^+st)^+\{s^+, s\} \cup s^+\{(st)^+, st\}, s^+st) \\
&= (\{(st)^+s^+, (st)^+s, s^+(st)^+, st\}, st) \\
&= (\{(st)^+s^+, (st)^+s, st\}, st) \\
&= (\{(st)^+s^+, (st)^+s, st^+, st\}, st) \text{ (as } st^+ = (st^+)^+s = (st)^+s) \\
&= ((st)^+\{s^+, s\} \cup s\{t^+, t\}, st) \\
&= (\{s^+, s\}, s)(\{t^+, t\}, t) \\
&= (s\iota)(t\iota).
\end{aligned}$$

(ii) We also have

$$\begin{aligned}
(s\iota)^+(s^+\iota) &= (\{s^+, s\}, s^+)(\{s^+\}, s^+) \\
&= ((s^+s^+)^+\{s^+, s\} \cup s^+\{s^+\}, s^+s^+) \\
&= (\{s^+s^+, s, s^+s^+\}, s^+) \\
&= (\{s^+, s\}, s^+) \\
&= (s\iota)^+.
\end{aligned}$$

Therefore $(s\iota)^+ \leq (s^+\iota)$ and hence ι is a strong premorphism.

□

Proposition 3.2.6. *[24] The Szendrei expansion of a left restriction semigroup S is generated as a $(2,1)$-algebra by elements of the form $s\iota$, where $\iota : S \to Sz(S)$ is defined above.*

Proposition 3.2.7. *[24] Let S and T be left restriction semigroups. For every strong premorphism $\theta : S \to T$ there is a unique $(2,1)$-morphism $\bar{\theta} : Sz(S) \to T$ such that the following diagram commutes:*

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[d, "\theta"'] & Sz(S) \arrow[dl, "\bar{\theta}"] \\ T & \end{tikzcd}$$

Conversely, if $\bar{\theta} : Sz(S) \to T$ is a $(2,1)$-morphism, then $\theta = \iota\bar{\theta}$ is a strong premorphism.

3.2.3 Presentations via generators and relations

We shall describe, via generators and relations, the Szendrei expansion of a left restriction semigroup S.

Let F be the free left restriction semigroup on $\overline{S}$, where $\overline{S} = \{[s] : s \in S\}$ is a set of generators of F having the same number of elements as S, as in Section 3.1. We take the following relations for $s, t \in S$:

(i) $[s][t] \equiv [s]^+[st]$;

(ii) $[s]^+ \equiv [s]^+[s^+]$.

Let

$$\delta = \langle ([s][t], [s]^+[st]), ([s]^+, [s]^+[s^+]) : s, t \in S \rangle. \qquad (\dagger)$$

Proposition 3.2.8. *Let S and T be left restriction semigroups. Let F be the free left restriction semigroup on $\overline{S}$. Let $\theta : S \to T$ be a strong premorphism and $\bar{\theta} : F \to T$ be defined on the set of generators of F by*

$$[s]\bar{\theta} = s\theta.$$

Then $\bar{\theta}$ is a morphism such that $\delta \subseteq \ker \bar{\theta}$.

Proof. The map $\bar{\theta}$ is a well-defined morphism since F is free on $\overline{S}$.

We have for $[s], [t] \in \overline{S}$,

$$\begin{aligned} ([s][t])\bar{\theta} &= [s]\bar{\theta}[t]\bar{\theta} \\ &= (s\theta)(t\theta) \\ &= (s\theta)^+(st)\theta \\ &= ([s]\bar{\theta})^+[st]\bar{\theta} \\ &= ([s]^+[st])\bar{\theta}. \end{aligned}$$

We also have

$$\begin{aligned}([s]^+)\bar{\theta} &= ([s]\bar{\theta})^+ \\ &= (s\theta)^+ \\ &= (s\theta)^+(s^+\theta) \\ &= ([s]\bar{\theta})^+([s^+]\bar{\theta}) \\ &= ([s]^+[s^+])\bar{\theta}.\end{aligned}$$

Hence $\delta \subseteq \ker \bar{\theta}$. □

Proposition 3.2.9. *Let S be a left restriction semigroup and F be the free left restriction semigroup on $\overline{S}$. Then*

$$F/\delta \cong Sz(S),$$

where δ is defined by (†).

Proof. We note that $\overline{S}$ generates F.

Let $\tau : S \to F/\delta$ be given by $s\tau = [s]\delta$. Then,

(i) for $s, t \in S$,

$$\begin{aligned}(s\tau)(t\tau) &= ([s]\delta)([t]\delta) \\ &= ([s][t])\delta \\ &= ([s]^+[st])\delta \\ &= ([s]\delta)^+([st]\delta) \\ &= (s\tau)^+(st)\tau;\end{aligned}$$

(ii) we also have

$$\begin{aligned}(s\tau)^+ &= ([s]\delta)^+ \\ &= [s]^+\delta \\ &= ([s]^+[s^+])\delta \\ &= ([s]\delta)^+([s^+]\delta) \\ &= (s\tau)^+(s^+\tau).\end{aligned}$$

Therefore $\tau : S \to F/\delta$ is a strong premorphism. As F is the free restriction semigroup on $\overline{S}$, F/δ is also a left restriction semigroup. So, by Proposition 3.2.7, there is a $(2,1)$-morphism $\bar{\tau} : Sz(S) \to F/\delta$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[d, "\tau"'] & Sz(S) \arrow[dl, "\bar{\tau}"] \\ F/\delta & \end{tikzcd}$$

For a left restriction semigroup S, $Sz(S)$ is a left restriction semigroup and by Proposition 3.2.5, $\iota : S \to Sz(S)$ is a strong premorphism. By Proposition 3.2.8, $\bar{\iota} : F \to Sz(S)$ defined by

$$[s]\bar{\iota} = s\iota,$$

for $[s] \in \overline{S}$, is a morphism such that $\delta \subseteq \ker \bar{\iota}$.

As $\delta^\natural : F \to F/\delta$ is an onto morphism with kernel δ, by Proposition 1.2.10, there exists a unique morphism $\bar{\bar{\iota}} : F/\delta \to Sz(S)$ such that the following diagram commutes:

$$\begin{tikzcd}
F \arrow[r, "\bar{\iota}"] \arrow[d, "\delta^\natural"'] & Sz(S) \\
F/\delta \arrow[ur, "\bar{\bar{\iota}}"'] &
\end{tikzcd}$$

Considering S, we have the following diagram:

$$\begin{tikzcd}
F \arrow[r, "\bar{\iota}"] \arrow[d, "\delta^\natural"'] & Sz(S) \\
F/\delta \arrow[ur, "\bar{\bar{\iota}}"'] & S \arrow[l, "\tau"'] \arrow[u, "\iota"']
\end{tikzcd}$$

We check that the lower triangle commutes. For $s \in S$, we have

$$\begin{aligned}
(s\tau)\bar{\bar{\iota}} &= ([s]\delta)\bar{\bar{\iota}} \\
&= [s]\bar{\iota} \\
&= s\iota,
\end{aligned}$$

so we have the following commutative diagrams:

$$\begin{tikzcd}
S \arrow[r, "\iota"] \arrow[d, "\tau"'] & Sz(S) \arrow[dl, "\bar{\tau}"] \\
F/\delta &
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[r, "\iota"] \arrow[d, "\tau"'] & Sz(S) \\
F/\delta \arrow[ur, "\bar{\bar{\iota}}"'] &
\end{tikzcd}$$

where ι and τ are strong premorphisms and $\bar{\tau}$ and $\bar{\bar{\iota}}$ are morphisms.

For $s \in S$, we have

$$\begin{aligned}(s\iota)\bar{\tau}\bar{\bar{\iota}} &= s(\iota\bar{\tau})\bar{\bar{\iota}} \\ &= s\tau\bar{\bar{\iota}} \\ &= s\iota.\end{aligned}$$

As $s\iota$ generates $Sz(S)$ by Proposition 3.2.6, $\bar{\tau}\bar{\bar{\iota}}$ is the identity on $Sz(S)$, i.e.

$$\bar{\tau}\bar{\bar{\iota}} = I_{Sz(S)}.$$

As $s\tau = [s]\delta$, for $[s] \in \overline{S}$,

$$\begin{aligned}([s]\delta)\bar{\bar{\iota}}\bar{\tau} &= (s\tau)\bar{\bar{\iota}}\bar{\tau} \\ &= s(\tau\bar{\bar{\iota}})\bar{\tau} \\ &= s\iota\bar{\tau} \\ &= s\tau \\ &= [s]\delta.\end{aligned}$$

As $\{[s] : s \in S\}$ generates F, we have that $\{[s]\delta : s \in S\}$ generates F/δ and so

$$\bar{\bar{\iota}}\bar{\tau} = I_{F/\delta}.$$

As $\bar{\bar{\iota}}$ and $\bar{\tau}$ are mutually inverse,

$$F/\delta \cong Sz(S).$$

□

3.3 The Szendrei expansion of an inverse semigroup

3.3.1 Definitions and background

We note that inverse semigroups are left restriction semigroups. We shall specialise to an inverse semigroup S and obtain a presentation for the Szendrei expansion of an inverse semigroup.

Note that the free inverse semigroup is an algebra of type $(2,1)$, but in this case the unary operation is $a \mapsto a'$ rather than $a \mapsto a^+$.

We have the following definition which is a special case of Definition 3.2.1.

Definition 3.3.1. [38] Let S be an inverse semigroup. The *Szendrei expansion* of S is the set

$$Sz(S) = \{(A,a) \in \mathscr{P}^f(S) \times S : a, aa' \in A \text{ and } A \subseteq R_a\},$$

where $\mathscr{P}^f(S)$ denotes the collection of all finite subsets of S and R_a

denotes the $\mathcal{R}$-class of a, with binary operation

$$(A, a)(B, b) = (abb'a'A \cup aB, ab)$$

for $(A, a), (B, b) \in Sz(S)$.

Proposition 3.3.2. *[38] If S is an inverse semigroup, then $Sz(S)$ is an inverse semigroup.*

3.3.2 Premorphisms

Considering inverse semigroups as left restriction semigroups, we adapt Definition 3.2.3 to obtain the following:

Definition 3.3.3. Let S and T be inverse semigroups. Then the function $\theta : S \to T$ is a *strong premorphism* if for $s, t \in S$,

(i) $(s\theta)(t\theta) = s\theta(s\theta)'(st)\theta$;

(ii) $(s\theta)(s\theta)' \leq (ss')\theta$,

where $\leq$ is the natural partial order on T.

We note that this is not the standard definition of a premorphism between inverse semigroups. The usual definition of a *premorphism*, which appears as a *dual prehomomorphism* in [38], between inverse semigroups S and T is a function $\theta : S \to T$ that satisfies the following conditions:

(i) $(s\theta)(t\theta) \leq (st)\theta$;

(ii) $(s\theta)' = s'\theta$,

for $s, t \in S$.

Proposition 3.3.4. *Let S be an inverse semigroup. The map $\iota : S \to Sz(S)$ given by*

$$s\iota = (\{ss', s\}, s)$$

is a strong premorphism.

Proof. Considering S as a left restriction semigroup where $s^+ = ss'$, we can deduce that ι is a strong premorphism using Proposition 3.2.5. $\square$

Proposition 3.3.5. *([38]) The Szendrei expansion of an inverse semigroup S is generated as a $(2, 1)$-algebra by elements of the form $s\iota$, where $\iota : S \to Sz(S)$ is defined above.*

Proof. As S and $Sz(S)$ are inverse semigroups, they are left restriction semigroups with distinguished semilattices

$$E_S = E(S)$$

and

$$E_{Sz(S)} = E(Sz(S)).$$

Notice that in any inverse semigroup T, we have $a^+ = aa'$ for $t \in T$. By Proposition 3.2.6, $Sz(S)$ is generated as a $(2,1)$- algebra by elements of the form

$$s\iota = (\{ss', s\}, s),$$

where 1 in the signature denotes the unary $^+$-operation.

However, by the remark above, we see that $Sz(S)$ is generated as a $(2,1)$-algebra by elements of the form $s\iota$, where 1 in the signature represents the unary operation of taking inverses.

□

3.3.3 Presentations via generators and relations

We shall describe, via generators and relations, the Szendrei expansion of an inverse semigroup S. Here our strategy is the same as in Section 3.2, but some of the details are different.

Let F be the free inverse semigroup on $\overline{S}$, where $\overline{S} = \{[s] : s \in S\}$ is a set of generators of F having the same number of elements as S as in Sections 3.1 and 3.2. We take the following relations for $s, t \in S$:

(i) $[s][t] \equiv [s][s]'[st]$;

(ii) $[s][s]' \equiv [s][s]'[ss']$.

Let

$$\mu = \langle ([s][t], [s][s]'[st]), ([s][s]', [s][s]'[ss']) : s, t \in S\rangle. \qquad (\ddagger)$$

Proposition 3.3.6. *Let S and T be inverse semigroups. Let F be the free inverse semigroup on $\overline{S}$. Let $\theta : S \to T$ be a strong premorphism and $\bar{\theta} : F \to T$ be defined on the set of generators of F by*

$$[s]\bar{\theta} = s\theta.$$

Then $\bar{\theta}$ is a morphism such that $\mu \subseteq$ ker $\bar{\theta}$.

Proof. The map $\bar{\theta}$ is a well-defined morphism since F is free on $\overline{S}$

We have for $[s], [t] \in \overline{S}$,

$$\begin{aligned}([s][t])\bar{\theta} &= [s]\bar{\theta}[t]\bar{\theta} \\ &= (s\theta)(t\theta) \\ &= (s\theta)(s\theta)'(st)\theta \\ &= ([s]\bar{\theta})([s]\bar{\theta})'([st]\bar{\theta}) \\ &= ([s][s]'[st])\bar{\theta}.\end{aligned}$$

We also have

$$\begin{aligned}([s][s]')\bar{\theta} &= [s]\bar{\theta}([s]\bar{\theta})' \\ &= (s\theta)(s\theta)' \\ &= (s\theta)(s\theta)'(ss'\theta) \\ &= ([s]\bar{\theta})([s]\bar{\theta})'([ss']\bar{\theta}) \\ &= ([s][s]'[ss'])\bar{\theta}.\end{aligned}$$

Hence $\mu \subseteq \ker \bar{\theta}$. □

Proposition 3.3.7. *Let S be an inverse semigroup and F be the free inverse semigroup on $\overline{S}$. Then*

$$F/\mu \cong Sz(S),$$

where μ is defined by (‡).

Proof. We note that $\overline{S}$ generates F.

Let $\tau : S \to F/\mu$ be given by $s\tau = [s]\mu$. Then,

(i) for $s, t \in S$,

$$\begin{aligned}(s\tau)(t\tau) &= ([s]\mu)([t]\mu) \\ &= ([s][t])\mu \\ &= ([s][s]'[st])\mu \\ &= ([s]\mu)([s]\mu)'([st]\mu) \\ &= (s\tau)(s\tau)'(st)\tau;\end{aligned}$$

(ii) we also have,

$$\begin{aligned}(s\tau)(s\tau)' &= ([s]\mu)([s]\mu)' \\ &= ([s][s]')\mu \\ &= ([s][s]'[ss'])\mu \\ &= ([s]\mu)([s]\mu)'([ss']\mu) \\ &= (s\tau)(s\tau)'(ss')\tau.\end{aligned}$$

Therefore $\tau : S \to F/\mu$ is a strong premorphism. As F is the free inverse semigroup on $\overline{S}$, F/μ is also an inverse semigroup. As inverse semigroups are left restriction semigroups and the Szendrei expansion of a left restriction semigroup is a generalisation of the Szendrei expansion of an inverse semigroup, we can use Proposition 3.2.7 to deduce that there is a $(2,1)$-morphism $\bar{\tau} : Sz(S) \to F/\mu$ such that the following diagram commutes:

$$
\begin{tikzcd}
S \arrow[r, "\iota"] \arrow[d, "\tau"'] & Sz(S) \arrow[dl, "\bar{\bar{\tau}}"] \\
F/\mu &
\end{tikzcd}
$$

By Proposition 3.2.7, the morphism $\bar{\tau}$ preserves the $^+$ unary operation, but by Corollary 2.2.14 it also preserves the unary operation of taking inverses. So we have a morphism of the correct type.

For an inverse semigroup S, $Sz(S)$ is an inverse semigroup and by Proposition 3.3.4, $\iota : S \to Sz(S)$ is a strong premorphism. By Proposition 3.3.6, $\bar{\iota} : F \to Sz(S)$ defined by

$$[s]\bar{\iota} = s\iota,$$

for $[s] \in \overline{S}$, is a morphism such that $\mu \subseteq \ker \bar{\iota}$.

As $\mu^\natural : F \to F/\mu$ is an onto morphism with kernel μ, by Proposition 1.2.10, there exists a unique morphism $\bar{\bar{\iota}} : F/\mu \to Sz(S)$ such that the following diagram commutes:

$$
\begin{tikzcd}
F \arrow[r, "\bar{\iota}"] \arrow[d, "\mu^\natural"'] & Sz(S) \\
F/\mu \arrow[ur, "\bar{\bar{\iota}}"'] &
\end{tikzcd}
$$

Considering S, we have the following diagram:

$$
\begin{tikzcd}
F \arrow[r, "\bar{\iota}"] \arrow[d, "\mu^\natural"'] & Sz(S) \\
F/\mu \arrow[ur, "\bar{\bar{\iota}}"'] & S \arrow[l, "\tau"'] \arrow[u, "\iota"']
\end{tikzcd}
$$

We need to check that the lower diagram commutes. For $s \in S$, we have

$$
\begin{aligned}
(s\tau)\bar{\bar{\iota}} &= ([s]\mu)\bar{\bar{\iota}} \\
&= [s]\bar{\iota} \\
&= s\iota,
\end{aligned}
$$

so we have the following commutative diagrams:

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[d, "\tau"'] & Sz(S) \arrow[dl, "\bar{\tau}"] \\ F/\mu & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "\iota"] \arrow[d, "\tau"'] & Sz(S) \\ F/\mu \arrow[ur, "\bar{\bar{\iota}}"'] & \end{tikzcd}$$

where ι and τ are strong premorphisms and $\bar{\tau}$ and $\bar{\bar{\iota}}$ are morphisms.

For $s \in S$, we have

$$\begin{aligned} (s\iota)\bar{\tau}\bar{\bar{\iota}} &= s(\iota\bar{\tau})\bar{\bar{\iota}} \\ &= s\tau\bar{\bar{\iota}} \\ &= s\iota. \end{aligned}$$

As $s\iota$ generates $Sz(S)$ by Proposition 3.3.5, $\bar{\tau}\bar{\bar{\iota}}$ is the identity on $Sz(S)$, i.e.

$$\bar{\tau}\bar{\bar{\iota}} = I_{Sz(S)}.$$

As $s\tau = [s]\mu$, for $[s] \in \overline{S}$,

$$\begin{aligned} ([s]\mu)\bar{\bar{\iota}}\bar{\tau} &= (s\tau)\bar{\bar{\iota}}\bar{\tau} \\ &= s(\tau\bar{\bar{\iota}})\bar{\tau} \\ &= s\iota\bar{\tau} \\ &= s\tau \\ &= [s]\mu. \end{aligned}$$

As $\{[s] : s \in S\}$ generates F, we have that $\{[s]\mu : s \in S\}$ generates F/μ and so

$$\bar{\bar{\iota}}\bar{\tau} = I_{F/\mu}.$$

As $\bar{\bar{\iota}}$ and $\bar{\tau}$ are mutually inverse,

$$F/\mu \cong Sz(S).$$

$\square$

Chapter 4

Background: McAlister's P-Theorem

McAlister's P-theorem [43] is a structure theorem for a class of inverse semigroups known as E-unitary, which are important due to McAlister's other major result that every inverse semigroup S has an E-unitary cover, i.e. there exists an E-unitary inverse semigroup U and an onto, idempotent-separating morphism $\phi : U \to S$ [42]. The P-theorem states that an object known as a P-semigroup, which is constructed from a group, semilattice and partially ordered set, is an E-unitary inverse semigroup and, conversely, that every E-unitary inverse semigroup is isomorphic to a P-semigroup. This theorem for E-unitary inverse semigroups provides us with a useful structure theorem as it determines the structure of all proper E-unitary inverse semigroups and it has many important consequences, such as O'Carroll's embedding theorem [46].

4.1 E-unitary inverse semigroups and McAlister's covering theorem

We shall begin the section by defining E-unitary inverse semigroups and E-unitary covers. We shall highlight the importance of E-unitary inverse semigroups by looking at McAlister's Covering Theorem [42].

An inverse semigroup S is *E-unitary* if for all $a \in S$ and all $e \in E(S)$, if $ae \in E(S)$, then $a \in E(S)$. We note that this definition is not one-sided due to the following proposition, which is true for a general semigroup, but we shall provide the result for inverse semigroups.

Proposition 4.1.1. *An inverse semigroup S is E-unitary if and only if $ea \in E(S)$ implies $a \in E(S)$ for $e \in E(S)$.*

Proof. Suppose that S is E-unitary and let $e, ea \in E(S)$. Then

$$ea = eaa'a = aa'ea,$$

where $a'ea \in E(S)$. This implies that $a \in E(S)$ by the assumption. The converse of the argument is dual. $\square$

The class of E-unitary inverse semigroups are important as many naturally arising inverse semigroups are E-unitary.

Example 4.1.2. Let $B = \mathbb{N}^0 \times \mathbb{N}^0$ and for $(a, b), (c, d) \in B$,

$$(a, b)(c, d) = (a - b + t, d - c + t), \text{ where } t = \max\{b, c\}.$$

Then B is a semigroup and is known as the *bicyclic semigroup*. It can be shown that B is an inverse semigroup and $E(B) = \{(a, a) : a \in \mathbb{N}^0\}$.

Let $(a, b) \in B$ and let $(c, c) \in E(B)$. Now suppose $(a, b)(c, c) \in E(B)$. Then

$$(a, b)(c, c) = (a - b + t, c - c + t) = (a - b + t, t),$$

where $t = \max\{b, c\}$. Since $(a, b)(c, c)$ is an idempotent, it must equal (u, u) for some $u \in \mathbb{N}^0$. This implies that $a - b + t = u$ and $t = u$, so $a-b+t = t$. Hence $a = b$ and therefore $(a, b) \in E(B)$. So B is E-unitary.

Definition 4.1.3. An inverse semigroup is *proper* if and only if $\mathcal{R} \cap \sigma = \iota$.

When we consider inverse semigroups, the definition of proper is equivalent to $\mathcal{L} \cap \sigma = \iota$, and to that of being E-unitary [31]. However, the analogous conditions for other classes of semigroups, such as restriction, are not necessarily equivalent.

Definition 4.1.4. A morphism, $\psi : M \to N$ say, is *idempotent-separating* if for $e, f \in E(M)$, we have

$$e\psi = f\psi \Rightarrow e = f.$$

Let S be an inverse semigroup. An *E-unitary cover* of S is an E-unitary inverse semigroup U together with an onto, idempotent-separating morphism $\psi : U \to S$.

For any inverse semigroup S, we can find an E-unitary inverse semigroup $\hat{S}$ and an onto morphism

$$\theta : \hat{S} \to S$$

where θ is one-to-one on the set of idempotents of $\hat{S}$ [42]. This is McAlister's Covering Theorem:

Covering Theorem. *Every inverse semigroup has an E-unitary cover.*

4.2 P-semigroups and McAlister's P-theorem

We shall define a P-semigroup and state McAlister's P-theorem [43], explaining its importance and stating consequences.

McAlister's P-theorem shows that every E-unitary inverse semigroup is isomorphic to a P-semigroup, the ingredients of which are groups, partially ordered sets and semilattices. This provides us with a structure for all E-unitary inverse semigroups. However, before defining a P-semigroup, we need to have a look at a couple of definitions and ideas.

Definition 4.2.1. A group G acts on a partially ordered set $\mathscr{X}$ by *order automorphisms* if G acts on $\mathscr{X}$ and for $a, b \in \mathscr{X}$,

$$a \leq b \Leftrightarrow g \cdot a \leq g \cdot b.$$

Definition 4.2.2. Let $\mathscr{X}$ be a partially ordered set and $\mathscr{Y}$ a semilattice which is a subset of $\mathscr{X}$. Then $\mathscr{Y}$ is an *order ideal* of $\mathscr{X}$ if for $a \in \mathscr{X}$ and $b \in \mathscr{Y}$,

$$a \leq b \Rightarrow a \in \mathscr{Y}.$$

We shall now define a McAlister triple and shall proceed to define a P-semigroup.

Definition 4.2.3. Let G be a group and let $(\mathscr{X}, \leq)$ be a partially ordered set where G acts on $\mathscr{X}$ by order automorphisms. Let $\mathscr{Y}$ be a subset of $\mathscr{X}$. Suppose that the following conditions are satisfied:
P1) $\mathscr{Y}$ is a semilattice under $\leq$;
P2) $G\mathscr{Y} = \mathscr{X}$;
P3) $\mathscr{Y}$ is an order ideal of $\mathscr{X}$;
P4) for all $g \in G$, $g\mathscr{Y} \cap \mathscr{Y} \neq \emptyset$.
Then $(G, \mathscr{X}, \mathscr{Y})$ is called a *McAlister triple*.

Definition 4.2.4. Let $(G, \mathscr{X}, \mathscr{Y})$ be a McAlister triple. The set

$$P(G, \mathscr{X}, \mathscr{Y}) = \{(A, g) \in \mathscr{Y} \times G : g^{-1}A \in \mathscr{Y}\},$$

along with the binary operation defined by

$$(A, g)(B, h) = (A \wedge gB, gh)$$

for $(A, g), (B, h) \in P(G, \mathscr{X}, \mathscr{Y})$, is called a *P-semigroup*.

We not only have that a P-semigroup is an E-unitary inverse semigroup, but for every E-unitary inverse semigroup we can find a P-semigroup which is isomorphic to it. This is McAlister's P-theorem:

P-Theorem. *[43] Let P be a P-semigroup. Then P is an E-unitary inverse semigroup. Conversely, any E-unitary inverse semigroup is isomorphic to a P-semigroup.*

This theorem has many important consequences such as O'Carroll's embedding theorem which states that every E-unitary inverse semigroup can be embedded into a much simpler structure than a P-semigroup.

Embedding Theorem. *[46] Let S be an inverse semigroup. Then S is E-unitary if and only if S can be embedded into the semidirect product of a semilattice and a group.*

Structure theorems are always desirable for a class of algebras, as providing a general structure for a wide class of semigroups is very useful and has many important consequences. Further, one of the main approaches to structure theory for inverse semigroups is to use E-unitary inverse semigroups. The P-theorem for E-unitary inverse semigroups prompted work on structure theorems for larger classes of semigroups, for example proper left ample semigroups.

Chapter 5

Background: One-sided P-theorems

There are theorems for proper left ample, proper weakly left ample and proper left restriction semigroups analogous to McAlister's covering theorem and P-theorem. In the proper left ample case, instead of a P-semigroup, a structure called an M-semigroup is introduced, and in the proper weakly left ample and proper left restriction cases, a structure known as a strong M-semigroup is considered.

5.1 Definitions and covering theorems

We shall remind the reader of the definition of 'proper' for various classes of semigroups and state covering theorems for left restriction [7], weakly left ample [18] and left ample semigroups [36].

Definition 5.1.1. A left restriction semigroup S is *proper* if and only if $\widetilde{\mathcal{R}}_E \cap \sigma_S = \iota$, a weakly ample semigroup is *proper* if and only if $\widetilde{\mathcal{R}}_E \cap \sigma = \iota$, and a left ample semigroup is *proper* if and only if $\mathcal{R}^* \cap \sigma = \iota$.

The definitions for proper right restriction, weakly right ample and right ample are defined dually.

Definition 5.1.2. Let S be a left restriction semigroup with distinguished semilattice of idempotents E. A *proper left restriction cover* of S is a proper left restriction semigroup U together with an onto morphism $\psi : U \to S$, which is E-separating.

The definitions for proper weakly left ample and proper left ample covers are defined similarly:

Definition 5.1.3. A *proper weakly left ample cover* of S is a proper weakly left ample semigroup U together with an onto, idempotent-separating morphism $\psi : U \to S$. A *proper left ample cover* of S is a proper left ample semigroup U together with an onto, idempotent-separating morphism $\psi : U \to S$.

Proper left ample and restriction semigroups are important due to the following theorems:

Theorem 5.1.4. *[12] Every left ample semigroup has a proper left ample cover.*

Theorem 5.1.5. *[19] Every weakly left ample semigroup has a proper weakly left ample cover.*

In fact, we have the following result.

Theorem 5.1.6. *[19] Every weakly left ample semigroup has a proper left ample cover.*

Theorem 5.1.7. *[7] Every left restriction semigroup has a proper left restriction cover.*

5.2 M-semigroups and P-theorem for proper left ample semigroups

We shall define an M-semigroup and state the structure theorem for proper left ample semigroups [12].

In [12], the idea of a P-semigroup is generalised to the case of a semigroup which is not necessarily regular but in which the idempotents commute. A structure $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ known as a *McAlister monoid* is presented, where $\mathscr{X}$ is a partially ordered set, $\mathscr{Y}$ is a subsemilattice of $\mathscr{X}$ and T is a left cancellative monoid acting on the right of $\mathscr{X}$, all subject to certain conditions. It is shown that such a structure is a proper right ample semigroup and conversely every proper right ample semigroup is isomorphic to a McAlister monoid.

We shall switch back to actions on the left. A different description of a McAlister monoid is provided in [36] and re-named as an *M-semigroup*, as we shall define. We note that a subset $\mathscr{Y}$ of a partially ordered set $\mathscr{X}$ is a subsemilattice of $\mathscr{X}$ if the meet of any two elements of $\mathscr{Y}$ exists.

Definition 5.2.1. Suppose that $\mathscr{X}$ is a partially ordered set, $\mathscr{Y}$ is a subsemilattice of $\mathscr{X}$ and there exists $\varepsilon \in \mathscr{X}$ such that $a \leq \varepsilon$ for all $a \in \mathscr{Y}$, i.e. ε is an upper bound of $\mathscr{Y}$. Let T be a right cancellative monoid which acts by order endomorphisms on the left of $\mathscr{X}$, i.e. T acts on the left of $\mathscr{X}$ and for all $a, b \in \mathscr{X}$ and $t \in T$,

$$a \leq b \Rightarrow t \cdot a \leq t \cdot b.$$

Suppose that the following hold:

(A) $T\mathscr{Y}^{\varepsilon} = \mathscr{X}$, where $\mathscr{Y}^{\varepsilon} = \mathscr{Y} \cup \{\varepsilon\}$;

(B) for all $t \in T$, $\exists b \in \mathscr{Y}$ such that $b \leq t \cdot \varepsilon$;

(C) if $a, b \in \mathscr{Y}$, and $a \leq t \cdot \varepsilon$, then $a \wedge t \cdot b \in \mathscr{Y}$;

(D) if $a, b, c \in \mathscr{Y}$ and $a \leq t \cdot \varepsilon$ and $b \leq u \cdot \varepsilon$, then

$$(a \wedge t \cdot b) \wedge tu \cdot c = a \wedge t \cdot (b \wedge u \cdot c).$$

We call the triple $(T, \mathscr{X}, \mathscr{Y})$ a *left admissible triple.*

Definition 5.2.2. Given $(T, \mathscr{X}, \mathscr{Y})$ as above, we define an *M-semigroup*

$$\mathscr{M}(T, \mathscr{X}, \mathscr{Y}) = \{(a, t) \in \mathscr{Y} \times T : a \leq t \cdot \varepsilon\},$$

with binary operation

$$(a, t)(b, u) = (a \wedge t \cdot b, tu)$$

for $(a, t), (b, u) \in \mathscr{M}(T, \mathscr{X}, \mathscr{Y})$.

We have the following structure theorem for proper left ample semigroups, with the theorem for proper right ample semigroups being dual.

Theorem 5.2.3. *[12] An M-semigroup, $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$, is proper left ample, where $(a, t)^+ = (a, 1)$ for $(a, t) \in \mathscr{M}(T, \mathscr{X}, \mathscr{Y})$. Conversely, a proper left ample semigroup is isomorphic to an M-semigroup for some left admissible triple $(T, \mathscr{X}, \mathscr{Y})$.*

Note that when we are considering a proper left ample semigroup S and we construct an M-semigroup, we take $T = S/\sigma$ and $\mathscr{Y}$ to be isomorphic to $E(S)$.

The existing result in the two-sided case involves that for the one-sided case with the addition of extra conditions. We have attempted to prove an alternative structure theorem for proper ample semigroups using the idea of a monoid acting doubly on a semilattice with identity [18], which we shall explore in Chapter 7.

5.3 Strong M-semigroups and P-theorems for proper left restriction and proper weakly left ample semigroups

We shall define strong M-semigroups and state the structure theorem for proper left restriction semigroups [7] and for proper weakly left ample semigroups [19]. We will also specialise this theorem to proper left ample semigroups and proper inverse semigroups.

Another similar structure to an M-semigroup, $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$, is presented in [19] and is called a *strong M-semigroup*, where T is a monoid, both $\mathscr{X}$ and $\mathscr{Y}$ are semilattices and the conditions are simplified, but a desirable property of $\mathscr{Y}$ is lost, namely Condition (A) in Definition 5.2.1. It is shown that every proper left restriction semigroup is isomorphic to a strong M-semigroup and conversely, a strong M-semigroup is a proper left restriction semigroup.

Definition 5.3.1. Let T be a monoid and let $\mathscr{X}$ be a semilattice. Then T *acts by morphisms* on the left of $\mathscr{X}$, via $\cdot$, if T acts on the left of $\mathscr{X}$ and for all $a, b \in \mathscr{X}$ and $t \in T$,

$$t \cdot (a \wedge b) = t \cdot a \wedge t \cdot b.$$

Dually, T *acts by morphisms* on the right of $\mathscr{X}$, via $\circ$, if T acts on the right of $\mathscr{X}$ and for all $a, b \in \mathscr{X}$ and $t \in T$,

$$(a \wedge b) \circ t = a \circ t \wedge b \circ t.$$

Lemma 5.3.2. *Let $\mathscr{X}$ be a semilattice and T a monoid, such that T acts on the left of $\mathscr{X}$ via morphisms. Then for $u, v \in \mathscr{Y}$ and $t \in T$,*

$$u \leq v \Rightarrow t \cdot u \leq t \cdot v.$$

Proof. Suppose T acts on $\mathscr{X}$ by morphisms. Letting $u, v \in \mathscr{X}$ and $t \in T$, we have

$$\begin{aligned} u \leq v &\Rightarrow u \wedge v = u \\ &\Rightarrow t \cdot (u \wedge v) = t \cdot u \\ &\Rightarrow t \cdot u \wedge t \cdot v = t \cdot u \\ &\Rightarrow t \cdot u \leq t \cdot v. \end{aligned}$$

□

The following structure has been taken from [19]:

Definition 5.3.3. Let $\mathscr{X}$ be a semilattice and $\mathscr{Y}$ a subsemilattice of $\mathscr{X}$. Suppose $\varepsilon \in \mathscr{X}$ is such that $a \leq \varepsilon$ for all $a \in \mathscr{Y}$. Let T be a monoid which acts by morphisms on the left of $\mathscr{X}$ via $\cdot$.

Suppose that the following also hold:

(A) for all $t \in T$, there exists $a \in \mathscr{Y}$ such that $a \leq t \cdot \varepsilon$;

(B) for all $a, b \in \mathscr{Y}$ and all $t \in T$,

$$a \leq t \cdot \varepsilon \Rightarrow a \wedge t \cdot b \text{ lies in } \mathscr{Y}.$$

The triple $(T, \mathscr{X}, \mathscr{Y})$ with the properties above shall be called a *strong left M-triple*.

Taking $\mathscr{X}$ to be a semilattice rather than just a partially ordered set means that we may no longer have $T\mathscr{Y}^{\varepsilon} = \mathscr{X}$ as in [36]. However, we gain something, as semilattices are easier to work with than partially ordered sets.

Definition 5.3.4. Given a strong left M-triple $(T, \mathscr{X}, \mathscr{Y})$, we define a *strong M-semigroup*,

$$\mathscr{M}(T, \mathscr{X}, \mathscr{Y}) = \{(a, t) \in \mathscr{Y} \times T : a \leq t \cdot \varepsilon\},$$

with binary operation

$$(a, t)(b, u) = (a \wedge t \cdot b, tu)$$

and unary operation

$$(a, t)^+ = (a, 1)$$

for $(a, t), (b, u) \in \mathscr{M}(T, \mathscr{X}, \mathscr{Y})$.

Dually, we can define a strong right M-triple $(T, \mathscr{X}', \mathscr{Y})$, where $\mathscr{X}'$ is a semilattice, $\mathscr{Y}$ a subsemilattice of $\mathscr{X}'$, ε' is a lower bound for $\mathscr{Y}$ and T acts on the right of $\mathscr{Y}$ via $\circ$. Its corresponding strong M-semigroup is

$$\mathscr{M}'(T, \mathscr{X}', \mathscr{Y}) = \{(t, a) \in T \times \mathscr{Y} : a \leq \varepsilon' \circ t\},$$

with binary operation

$$(t, a)(u, b) = (tu, a \circ u \wedge b)$$

and unary operation

$$(t, a)^* = (1, a)$$

for $(t, a), (u, b) \in \mathscr{M}'(T, \mathscr{X}', \mathscr{Y})$.

Proposition 5.3.5. *[7] If $(T, \mathscr{X}, \mathscr{Y})$ is a strong left M-triple, then the strong M-semigroup $\mathscr{M} = \mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ is a proper left restriction semigroup where*

$$(e, s)^+ = (e, 1) \text{ for } (e, s) \in \mathscr{M},$$

$$E_{\mathscr{M}} = \{(e, 1) : e \in \mathscr{Y}\} \cong \mathscr{Y} \text{ and } \mathscr{M}/\sigma_{\mathscr{M}} \cong T.$$

If T is a unipotent monoid, $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ is a proper weakly left ample semigroup [19] and if T is a right cancellative monoid, $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ is a proper left ample monoid.

Theorem 5.3.6. *[7] A semigroup is proper left restriction if and only if it is isomorphic to a strong M-semigroup for some strong left M-triple $(T, \mathscr{X}, \mathscr{Y})$.*

Corollary 5.3.7. *[19] A semigroup is proper weakly left ample if and only if it is isomorphic to a strong M-semigroup for some strong left M-triple $(T, \mathscr{X}, \mathscr{Y})$ where T is a unipotent monoid.*

As in the proper left ample case, when we consider a proper left restriction or proper weakly left ample semigroup S and construct a strong M-semigroup, we take $T = S/\sigma_S$ and $\mathscr{Y} \cong E$.

Before looking to find symmetrical structure theorems for proper weakly ample and proper ample semigroups, we wish to specialise Theorem 5.3.6 to proper left ample semigroups and proper inverse semigroups.

Corollary 5.3.8. *A proper left ample semigroup is isomorphic to a strong M-semigroup, $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ say, where T is right cancellative. Conversely, a strong M-semigroup $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$, where T is right cancellative, is proper left ample.*

Proof. Let S be a proper left ample semigroup. As S is a proper weakly left ample semigroup, S is isomorphic to a strong M-semigroup, say $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$, where T is a unipotent monoid and $T = S/\sigma$ as noted after Corollary 5.3.7. By Proposition 2.7.10, T is right cancellative.

Conversely, let $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ be a strong M-semigroup, where T is a right cancellative monoid. As T is right cancellative, it is a unipotent monoid. So $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ is a proper weakly left ample semigroup by Corollary 5.3.7.

It remains to show that $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ is left ample since Corollary 2.4.10 ensures that the proper condition holds. We shall show that

$$(a,t)\, \mathscr{R}^*\, (a,1)$$

for $(a,t) \in \mathscr{M}(T, \mathscr{X}, \mathscr{Y})$. As $(a,t)\, \widetilde{\mathscr{R}}\, (a,1)$,

$$(a,1)(a,t) = (a,t).$$

We have

$$\begin{aligned}
(x,y)(a,t) = (z,w)(a,t) &\Rightarrow (x \wedge y \cdot a, yt) = (z \wedge w \cdot a, wt) \\
&\Rightarrow x \wedge y \cdot a = z \wedge w \cdot a \text{ and } yt = wt \\
&\Rightarrow x \wedge y \cdot a = z \wedge w \cdot a \text{ and } y = w \\
&\qquad\qquad \text{as } T \text{ is right cancellative} \\
&\Rightarrow (x \wedge y \cdot a, y) = (z \wedge w \cdot a, w) \\
&\Rightarrow (x,y)(a,1) = (z,w)(a,1)
\end{aligned}$$

and

$$\begin{aligned}
(1,1)(a,t) = (z,w)(a,t) &\Rightarrow a = z \wedge w \cdot a \text{ and } t = wt \\
&\Rightarrow a = z \wedge w \cdot a \text{ and } w = 1 \\
&\Rightarrow (a,1) = (z \wedge w \cdot a, w) \\
&\Rightarrow (1,1)(a,1) = (z,w)(a,1).
\end{aligned}$$

Therefore, $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ is proper left ample. □

We shall also specialise the result to proper inverse semigroups, which requires more than insisting that T be a group.

Definition 5.3.9. Let $(T, \mathscr{X}, \mathscr{Y})$ be a strong left M-triple, where T is a group and Condition (A$'$) is satisfied:

(A$'$) for every $t \in T$, $\exists a \in \mathscr{Y}$ such that $a \leq t \cdot \varepsilon$ and $t^{-1} \cdot a \in \mathscr{Y}$.

Given $(T, \mathscr{X}, \mathscr{Y})$ as described, let us define

$$\mathscr{N}(T, \mathscr{X}, \mathscr{Y}) = \{(a, t) \in \mathscr{Y} \times T : a \leq t \cdot \varepsilon, t^{-1} \cdot a \in \mathscr{Y}\}.$$

Theorem 5.3.10. *A proper inverse semigroup is isomorphic to some $\mathscr{N}(T, \mathscr{X}, \mathscr{Y})$. Conversely, $\mathscr{N}(T, \mathscr{X}, \mathscr{Y})$ is a proper inverse semigroup, where*

$$(a, t)' = (t^{-1} \cdot a, t^{-1})$$

for $(a, t) \in \mathscr{N}(T, \mathscr{X}, \mathscr{Y})$.

Proof. Let S be a proper inverse semigroup. Then S is a proper left ample semigroup, so S is isomorphic to a strong M-semigroup, $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$, where T is a right cancellative monoid by Corollary 5.3.8. By Proposition 2.7.9, T is a group.

Let $t \in T$. We know there exists $a \in \mathscr{Y}$ such that $a \leq t \cdot \varepsilon$ since $(T, \mathscr{X}, \mathscr{Y})$ is a strong left M-triple. So $(a, t) \in \mathscr{M}(T, \mathscr{X}, \mathscr{Y})$. As S is an inverse semigroup, there exists $(b, s) \in \mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ such that

$$(a, t)(b, s)(a, t) = (a, t) \text{ and } (b, s)(a, t)(b, s) = (b, s),$$

i.e.

$$(a \wedge t \cdot b \wedge ts \cdot a, tst) = (a, t) \text{ and } (b \wedge s \cdot a \wedge st \cdot b, sts) = (b, s).$$

As T is a group,

$$tst = t \text{ and } sts = s \Rightarrow s = t^{-1}.$$

We also have

$$a \wedge t \cdot b \wedge tt^{-1} \cdot a = a \text{ and } b \wedge t^{-1} \cdot a \wedge t^{-1}t \cdot b = b,$$

i.e.

$$a \wedge t \cdot b = a \text{ and } b \wedge t^{-1} \cdot a = b,$$

i.e.

$$a \leq t \cdot b \text{ and } b \leq t^{-1} \cdot a.$$

We also have

$$\begin{aligned} a \leq t \cdot b &\Rightarrow t^{-1} \cdot a \leq t^{-1}t \cdot b \\ &\Rightarrow t^{-1} \cdot a \leq b \\ &\Rightarrow t^{-1} \cdot a = b \\ &\Rightarrow t^{-1} \cdot a \in \mathscr{Y}. \end{aligned}$$

So Condition (A′) is satisfied and $\mathscr{M}(T, \mathscr{X}, \mathscr{Y}) = \mathscr{N}(T, \mathscr{X}, \mathscr{Y})$.

Conversely, consider

$$\mathscr{N}(T, \mathscr{X}, \mathscr{Y}) = \{(a, t) \in \mathscr{Y} \times T : a \leq t \cdot \varepsilon, t^{-1} \cdot a \in \mathscr{Y}\}.$$

We wish to show that $\mathscr{N}(T, \mathscr{X}, \mathscr{Y})$ is a proper inverse semigroup. First, it is non-empty due to Condition (A$'$).

Take $(a, t), (b, s) \in \mathscr{N}(T, \mathscr{X}, \mathscr{Y})$. Then $a, b, t^{-1} \cdot a, s^{-1} \cdot b \in \mathscr{Y}$, $t, s \in T$, $a \leq t \cdot \varepsilon$ and $b \leq s \cdot \varepsilon$. By Condition (B), $a \wedge t \cdot k, b \wedge s \cdot k \in \mathscr{Y}$ for all $k \in \mathscr{Y}$. We wish to show

$$(a \wedge t \cdot b, ts) \in \mathscr{N}(T, \mathscr{X}, \mathscr{Y}).$$

We need only show that $(ts)^{-1} \cdot (a \wedge t \cdot b) \in \mathscr{Y}$. We have

$$\begin{aligned} (ts)^{-1} \cdot (a \wedge t \cdot b) &= s^{-1}t^{-1} \cdot a \wedge s^{-1}t^{-1}t \cdot b \\ &= s^{-1} \cdot (t^{-1} \cdot a) \wedge s^{-1} \cdot b \text{ where } t^{-1} \cdot a \in \mathscr{Y}. \end{aligned}$$

Therefore,

$$\begin{aligned} s^{-1} \cdot b \leq s^{-1} \cdot \varepsilon &\Rightarrow s^{-1} \cdot b \wedge s^{-1} \cdot k \in \mathscr{Y} \text{ for } k \in \mathscr{Y} \\ &\Rightarrow s^{-1} \cdot b \wedge s^{-1} \cdot (t^{-1} \cdot a) \in \mathscr{Y} \\ &\Rightarrow (ts)^{-1} \cdot (a \wedge t \cdot b) \in \mathscr{Y}. \end{aligned}$$

Let $(a, t) \in \mathscr{N}(T, \mathscr{X}, \mathscr{Y})$. We wish to show $(t^{-1} \cdot a, t^{-1}) \in \mathscr{N}(T, \mathscr{X}, \mathscr{Y})$. As $t^{-1} \cdot a \in \mathscr{Y}$ and $t^{-1} \cdot a \leq t^{-1} \cdot \varepsilon$, it remains to show $(t^{-1})^{-1} \cdot (t^{-1} \cdot a) \in \mathscr{Y}$, but $(t^{-1})^{-1} \cdot (t^{-1} \cdot a) = t \cdot (t^{-1} \cdot a) \in \mathscr{Y} = a \in \mathscr{Y}$. Therefore, $(t^{-1} \cdot a, t^{-1}) \in \mathscr{N}(T, \mathscr{X}, \mathscr{Y})$.

We see that

$$\begin{aligned} (a, t)(t^{-1} \cdot a, t^{-1})(a, t) &= (a \wedge t \cdot (t^{-1} \cdot a), tt^{-1})(a, t) \\ &= (a \wedge tt^{-1} \cdot a, 1)(a, t) \\ &= (a \wedge a, 1)(a, t) \\ &= (a, 1)(a, t) \\ &= (a \wedge a, t) \\ &= (a, t) \end{aligned}$$

and

$$\begin{aligned} (t^{-1} \cdot a, t^{-1})(a, t)(t^{-1} \cdot a, t^{-1}) &= (t^{-1} \cdot a, t^{-1})(a, 1) \\ &= (t^{-1} \cdot a, t^{-1}). \end{aligned}$$

As $E(\mathscr{N}) = \{(a, 1) : a \in \mathscr{Y}\} \cong \mathscr{Y}$, the idempotents of $\mathscr{N}$ commute and therefore $\mathscr{N}(T, \mathscr{X}, \mathscr{Y})$ is an inverse semigroup where $(a, t)' = (t^{-1} \cdot a, t^{-1})$.

We have for $(a,t),(b,u) \in \mathscr{N}(T,\mathscr{X},\mathscr{Y})$,

$$\begin{aligned}
(a,t)\,\mathscr{R}\,(b,u) &\Leftrightarrow (a,t)(a,t)' = (b,u)(b,u)' \\
&\Leftrightarrow (a,t)(t^{-1}\cdot a, t^{-1}) = (b,u)(u^{-1}\cdot b, u^{-1}) \\
&\Leftrightarrow (a \wedge tt^{-1}\cdot a, 1) = (b \wedge uu^{-1}\cdot b, 1) \\
&\Leftrightarrow (a,1) = (b,1) \\
&\Leftrightarrow a = b.
\end{aligned}$$

Considering $\mathscr{N}(T,\mathscr{X},\mathscr{Y})$ as a subalgebra of $\mathscr{M}(T,\mathscr{X},\mathscr{Y})$ that shares the same set of idempotents, we have

$(a,t)\,\sigma\,(b,u)$ in $\mathscr{N}(T,\mathscr{X},\mathscr{Y})$ if and only if $(a,t)\,\sigma\,(b,u)$ in $\mathscr{M}(T,\mathscr{X},\mathscr{Y})$.

Within $\mathscr{M}(T,\mathscr{X},\mathscr{Y})$, $(a,t)\,\sigma\,(b,u)$ if and only if $t=u$, so $\mathscr{N}(T,\mathscr{X},\mathscr{Y})$ is a proper inverse semigroup. $\square$

Chapter 6

Two-sided P-theorems

We shall explain how the one-sided structure theorem for proper left ample semigroups was adapted to obtain the two-sided result for proper ample semigroups [36]. Taking Theorem 5.2.3, the left-right dual of the theorem in [12], Lawson showed how to modify this (by his own admission, rather artificially) to get the two-sided result. He considered what additional conditions would be required on the triple to produce a structure theorem for proper ample semigroups.

6.1 Definitions and covering theorems

As in the one-sided case, we shall define 'proper' and state covering theorems for two-sided ample, weakly ample and restriction semigroups.

Definition 6.1.1. A restriction semigroup is *proper* if it is a proper left and proper right restriction semigroup. An ample semigroup is *proper* if it is a proper left and proper right ample semigroup.

Proper ample and restriction semigroups are important due to the following theorems:

Theorem 6.1.2. *[36] Every ample semigroup has a proper ample cover.*

In [18], the proof of the following result is given for monoids, but in Section 9 the authors explain how to deduce the corresponding proof for semigroups.

Theorem 6.1.3. *Every weakly ample semigroup has a proper weakly ample cover.*

As shown in [18], every restriction semigroup has a proper ample cover, which implies that we have the following result. We note that the proof is provided for monoids and it is later explained how to deduce the corresponding results for semigroups. In our joint paper, [10], we give a direct proof which we shall provide in Section 8.6.

Theorem 6.1.4. *Every restriction semigroup has a proper restriction cover.*

6.2 Structure theorem for proper ample semigroups

From Section 5.2, we know that given a left admissible triple $(T, \mathscr{X}, \mathscr{Y})$, its corresponding M-semigroup is proper left ample by Theorem 5.2.3.

Definition 6.2.1. Let $(T, \mathscr{X}, \mathscr{Y})$ be a left admissible triple and suppose $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ is an M-semigroup. The triple $(T, \mathscr{X}, \mathscr{Y})$ is called an *admissible triple* if T is a cancellative monoid and the following conditions hold:

(A) there is a (unique) element $[a, t] \in \mathscr{Y}$ for every $(a, t) \in \mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ such that $a \leq t \cdot [a, t]$ and $\forall c, d \in \mathscr{Y}$,

$$a \wedge t \cdot c = a \wedge t \cdot d \Rightarrow [a, t] \wedge c = [a, t] \wedge d;$$

(B) for $e \in \mathscr{Y}$ and $a \in \mathscr{Y}$ with $a \leq t \cdot \varepsilon$,

$$a \wedge e = a \wedge t \cdot [e \wedge a, t];$$

(C) for $a, b \in \mathscr{Y}$ with $a, b \leq t \cdot \varepsilon$, $[a, t] = [b, t] \Rightarrow a = b$.

With the extra conditions, we have the following theorem. However, having a more symmetrical two-sided structure theorem would be desirable.

Theorem 6.2.2. *[36] Let S be a proper ample semigroup. Then $S \cong \mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ for some admissible triple $(T, \mathscr{X}, \mathscr{Y})$. Conversely, every admissible triple gives rise to an M-semigroup which is proper ample.*

6.3 Structure theorem for proper restriction and proper weakly ample semigroups

We use Lawson's approach to obtain the analogous two-sided result for proper restriction semigroups from Theorem 5.3.6 and consequently the result for proper weakly ample semigroups. However, again the strategy is to modify the one-sided construction in a rather forced way; the result is not a construction that has a natural two-sided appearance.

Let $(T, \mathscr{X}, \mathscr{Y})$ now be a strong left M-triple and $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ its corresponding strong M-semigroup from Section 5.3. By Theorem 5.3.6, $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ is proper left restriction.

Lemma 6.3.1. *For $(a, t), (b, u) \in \mathscr{M}(T, \mathscr{X}, \mathscr{Y})$,*

$$(a, t) \widetilde{\mathscr{L}_E} (b, u) \Leftrightarrow \forall e \in \mathscr{Y}, [a \leq t \cdot e \Leftrightarrow b \leq u \cdot e].$$

Proof. We have

$$\begin{aligned}(a,t)\,\widetilde{\mathscr{L}}_E\,(b,u) &\Leftrightarrow \forall (e,1)\in E_{\mathscr{M}(T,\mathscr{X},\mathscr{Y})},\\ &\qquad [(a,t)(e,1)=(a,t)\Leftrightarrow (b,u)(e,1)=(b,u)]\\ &\Leftrightarrow \forall (e,1)\in E_{\mathscr{M}(T,\mathscr{X},\mathscr{Y})},\\ &\qquad [(a\wedge t\cdot e,t)=(a,t)\Leftrightarrow (b\wedge u\cdot e,u)=(b,u)]\\ &\Leftrightarrow \forall e\in\mathscr{Y},[a\wedge t\cdot e=a\Leftrightarrow b\wedge u\cdot e=b]\\ &\Leftrightarrow \forall e\in\mathscr{Y},[a\le t\cdot e\Leftrightarrow b\le u\cdot e].\end{aligned}$$

□

Considering elements in $E_{\mathscr{M}(T,\mathscr{X},\mathscr{Y})}$, we obtain the following lemma.

Lemma 6.3.2. *For $(a,t)\in\mathscr{M}(T,\mathscr{X},\mathscr{Y})$ and $(e,1)\in E_{\mathscr{M}(T,\mathscr{X},\mathscr{Y})}$,*

$$(a,t)\,\widetilde{\mathscr{L}}_E\,(e,1)\Leftrightarrow a\le t\cdot e \text{ and } \forall f\in\mathscr{Y},[a\le t\cdot f\Rightarrow e\le f].$$

Proof. We have

$$\begin{aligned}(a,t)\,\widetilde{\mathscr{L}}_E\,(e,1) &\Leftrightarrow (a,t)(e,1)=(a,t) \text{ and } \forall (f,1)\in E_{\mathscr{M}(T,\mathscr{X},\mathscr{Y})},\\ &\qquad [(a,t)(f,1)=(a,t)\Rightarrow (e,1)(f,1)=(f,1)]\\ &\Leftrightarrow (a\wedge t\cdot e,t)=(a,t) \text{ and } \forall (f,1)\in E_{\mathscr{M}(T,\mathscr{X},\mathscr{Y})},\\ &\qquad [(a\wedge t\cdot f,t)=(a,t)\Rightarrow (e\wedge f,1)=(f,1)]\\ &\Leftrightarrow a\wedge t\cdot e=a \text{ and } \forall f\in\mathscr{Y},[a\wedge t\cdot f=a\Rightarrow e\wedge f=f]\\ &\Leftrightarrow a\le t\cdot e \text{ and } \forall f\in\mathscr{Y},[a\le t\cdot f\Rightarrow e\le f].\end{aligned}$$

□

It follows from the lemma above that we can deduce a condition for a strong M-semigroup to be weakly right E-abundant, i.e. such that every element is $\widetilde{\mathscr{L}}_E$-related to an idempotent.

Proposition 6.3.3. *A strong M-semigroup $\mathscr{M}(T,\mathscr{X},\mathscr{Y})$ is weakly right E-abundant, where $E=\{(a,1):a\in\mathscr{Y}\}$, if and only if there is a (unique) element $[a,t]\in\mathscr{Y}$ for every $(a,t)\in\mathscr{M}(T,\mathscr{X},\mathscr{Y})$ such that*

A(i) $a\le t\cdot[a,t]$ *;*

A(ii) $\forall f\in\mathscr{Y}, a\le t\cdot f\Rightarrow [a,t]\le f$.

If the above conditions hold, $(a,t)\,\widetilde{\mathscr{L}}_E\,([a,t],1)$. Consequently, we have $(a,t)\,\widetilde{\mathscr{L}}_E\,(b,u)$ if and only if $[a,t]=[b,u]$ for $(a,t),(b,u)\in\mathscr{M}(T,\mathscr{X},\mathscr{Y})$.

In the proper ample case, $\mathscr{R}^*$ and $\mathscr{L}^*$ are left and right congruences respectively, but $\widetilde{\mathscr{R}}_E$ and $\widetilde{\mathscr{L}}_E$ are not necessarily so. Using Lemma 6.3.1, we have the following result.

Proposition 6.3.4. *The relation $\widetilde{\mathscr{L}}_E$ is a right congruence on the strong M-semigroup $\mathscr{M}(T,\mathscr{X},\mathscr{Y})$ if and only if for all $(a,t),(b,u),(x,y)\in\mathscr{M}(T,\mathscr{X},\mathscr{Y})$,*

(B)

$$\forall e \in \mathscr{Y}, [a \leq t \cdot e \Leftrightarrow b \leq u \cdot e]$$

implies that

$$\forall f \in \mathscr{Y}, [a \wedge t \cdot x \leq ty \cdot f \Leftrightarrow b \wedge u \cdot x \leq uy \cdot f].$$

Now we can consider which strong M-semigroups are right restriction.

Proposition 6.3.5. *Let $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ be a strong M-semigroup which satisfies Conditions (A(i)), (A(ii)) and (B). Then $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ is right restriction if and only if it satifies the following condition:*

(C) $\forall e \in \mathscr{Y}$ and $a \in \mathscr{Y}$ with $a \leq t \cdot \varepsilon$,

$$a \wedge t \cdot [e \wedge a, t] \leq a \wedge e.$$

Proof. Let $\mathscr{M} = \mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ be a strong M-semigroup satisfying Conditions (A(i)), (A(ii)) and (B). Then

$$\begin{aligned}
\mathscr{M} \text{ is right restriction} &\Leftrightarrow \text{ for all } (a,t),(e,1) \in \mathscr{M}, \\
&\qquad (e,1)(a,t) = (a,t)[(e,1)(a,t)]^* \\
&\Leftrightarrow \text{ for all } (a,t),(e,1) \in \mathscr{M}, \\
&\qquad (e \wedge a, t) = (a,t)(e \wedge a, t)^* \\
&\Leftrightarrow \text{ for all } (a,t),(e,1) \in \mathscr{M}, \\
&\qquad (e \wedge a, t) = (a,t)([e \wedge a, t], 1) \\
&\Leftrightarrow \text{ for all } (a,t),(e,1) \in \mathscr{M}, \\
&\qquad (e \wedge a, t) = (a \wedge t \cdot [e \wedge a, t], t) \\
&\Leftrightarrow \text{ for all } (a,t),(e,1) \in \mathscr{M}, \\
&\qquad e \wedge a = a \wedge t \cdot [e \wedge a, t].
\end{aligned}$$

Consider $(e,1)(a,t) \in \mathscr{M}$, i.e. $(e \wedge a, t) \in \mathscr{M}$. By Condition (A(i)), there is a unique element $([e \wedge a, t], 1) \in \mathscr{M}$ such that $e \wedge a \leq t \cdot [e \wedge a, t]$. Therefore

$$e \wedge a \leq a \wedge t \cdot [e \wedge a, t]$$

and so $\mathscr{M}$ is right restriction if and only if $a \wedge t \cdot [e \wedge a, t] \leq a \wedge e$. □

Proposition 6.3.6. *Let $\mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ be a strong M-semigroup satisfying Conditions (A(i)), (A(ii)), (B) and (C). Then $\mathscr{M} = \mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ is proper right restriction if and only if the following condition holds:*

(D) for $a, b \in \mathscr{Y}$ with $a, b \leq t \cdot \varepsilon$, $[a,t] = [b,t] \Rightarrow a = b$.

Proof. Suppose that Condition (D) holds. We wish to show

$$(a,t)\,(\widetilde{\mathscr{L}_E} \cap \sigma_{\mathscr{M}})\,(b,u) \Leftrightarrow (a,t) = (b,u).$$

We already know that $(a,t)\,\sigma_{\mathscr{M}}\,(b,u) \Leftrightarrow t = u$ as in Chapter 5. Suppose we have $(a,t)\,(\widetilde{\mathscr{L}_E} \cap \sigma_{\mathscr{M}})\,(b,u)$, which implies $(a,t)\,\widetilde{\mathscr{L}_E}\,(b,u)$ and $t =$

u. We can deduce that this implies $[a,t]=[b,t]$ and so $a=b$ by our assumption.

Conversely suppose $\mathscr{M}(T,\mathscr{X},\mathscr{Y})$ is a proper restriction semigroup. Let $a,b\in\mathscr{Y}$ with $a,b\leq t\cdot\varepsilon$ and suppose $[a,t]=[b,t]$. As $\mathscr{M}(T,\mathscr{X},\mathscr{Y})$ is proper,

$$(a,t)\,(\widetilde{\mathscr{L}}_E\cap\sigma_{\mathscr{M}})\,(b,u)\Rightarrow(a,t)=(b,u)$$

and so we have

$$\begin{aligned}[a,t]=[b,t]&\Rightarrow(a,t)\,(\widetilde{\mathscr{L}}_E\cap\sigma_{\mathscr{M}})\,(b,t)\\&\Rightarrow a=b.\end{aligned}$$

□

We shall call a triple $(T,\mathscr{X},\mathscr{Y})$ a *strong M-triple* if it is a strong left M-triple and Conditions (A(i)), (A(ii)), (B), (C) and (D) are satisfied. We can now generalise Theorem 6.2.2 for proper restriction semigroups.

Theorem 6.3.7. *Let S be a proper restriction semigroup. Then S is isomorphic to a strong M-semigroup $\mathscr{M}(T,\mathscr{X},\mathscr{Y})$ for some strong M-triple $(T,\mathscr{X},\mathscr{Y})$. Conversely, every strong M-triple $(T,\mathscr{X},\mathscr{Y})$ gives rise to a strong M-semigroup $\mathscr{M}(T,\mathscr{X},\mathscr{Y})$ that is proper restriction with distinguished semilattice*

$$E_{\mathscr{M}(T,\mathscr{X},\mathscr{Y})}=\{(e,1):e\in\mathscr{Y}\}.$$

Proof. Let S be a proper restriction semigroup. Since S is proper left restriction, $S\cong\mathscr{M}(T,\mathscr{X},\mathscr{Y})$ by Corollary 5.3.7, where $(T,\mathscr{X},\mathscr{Y})$ is a strong left M-triple. As S is proper right restriction, Conditions (A(i)), (A(ii)), (B), (C) and (D) hold. So $(T,\mathscr{X},\mathscr{Y})$ is a strong M-triple.

Conversely, $\mathscr{M}(T,\mathscr{X},\mathscr{Y})$, where $(T,\mathscr{X},\mathscr{Y})$ is a strong M-triple, is proper left restriction due to Corollary 5.3.7 and is proper right restriction due to Conditions (A(i)), (A(ii)), (B), (C) and (D). □

We can also produce a two-sided theorem for proper weakly ample semigroups:

Corollary 6.3.8. *Suppose that S is a proper weakly ample semigroup. Then $S\cong\mathscr{M}(T,\mathscr{X},\mathscr{Y})$ for some strong M-triple, where T is a unipotent monoid. Conversely, every strong M-triple, where T is a unipotent monoid, gives rise to a strong M-semigroup that is proper weakly ample.*

Proof. Let S be a proper weakly ample semigroup. Then S is a proper restriction semigroup and so $S\cong\mathscr{M}(T,\mathscr{X},\mathscr{Y})$ for some strong M-triple $(T,\mathscr{X},\mathscr{Y})$ by Theorem 6.3.7. By Proposition 2.7.11, T is a unipotent monoid.

Conversely, let $(T,\mathscr{X},\mathscr{Y})$ be a strong M-triple where T is unipotent. By Theorem 6.3.7, $\mathscr{M}(T,\mathscr{X},\mathscr{Y})$ is a proper restriction semigroup with distinguished semilattice $E_{\mathscr{M}(T,\mathscr{X},\mathscr{Y})}=\{(e,1):e\in\mathscr{Y}\}$.

Consider $(a,t) \in E(\mathscr{M}(T,\mathscr{X},\mathscr{Y}))$. Then $(a,t)(a,t) = (a,t)$, i.e. $(a \wedge ta, t^2) = (a,t)$. So $t^2 = t$. As T is a unipotent monoid, $t = 1$ and so $E_{\mathscr{M}(T,\mathscr{X},\mathscr{Y}))} = E(\mathscr{M}(T,\mathscr{X},\mathscr{Y}))$ when T is a unipotent monoid. So $\mathscr{M}(T,\mathscr{X},\mathscr{Y})$ is a proper weakly ample semigroup.

$\square$

Chapter 7

Construction based on double actions

Work in this chapter is taken from a joint paper [10].

7.1 Double actions

We shall provide the definition of a double action and explain why this idea was used in attempting to create symmetrical two-sided structure theorems.

The existing structure theorems for proper ample, proper weakly ample and proper restriction semigroups are, as we have remarked, artificial adaptations of those in the one-sided case. We wish to produce a structure theorem which is genuinely two-sided. Inspiration arose from the definition of a *double action* from [18] which consists of left and right actions of a monoid acting on a semilattice with identity along with compatibility conditions.

Definition 7.1.1. A monoid T *acts doubly* on a semilattice Y with identity 1 if T acts by morphisms on the left and right of Y and the *compatibility conditions* hold, that is,

(A) $(t \cdot e) \circ t = (1 \circ t)e$;

(B) $t \cdot (e \circ t) = e(t \cdot 1)$

for all $t \in T$ and $e \in \mathscr{Y}$.

It is proved in [18] that if T is a monoid acting doubly on a semilattice $\mathscr{Y}$, the set

$$S = \{(e,t) : e \leq t \cdot 1\} \subseteq \mathscr{Y} * T \tag{$\dagger$}$$

is a proper restriction monoid such that $(e,t)^+ = (e,1)$ and $(e,t)^* = (e \circ t, 1)$. It is also shown that if T is unipotent, S is proper weakly ample and if T is cancellative, then S is proper ample.

Let M be a proper ample monoid and U be a submonoid of M. Taking

$Y = E(M)$, we obtain that U acts on the left of Y by morphisms via

$$u \cdot e = (ue)^+$$

and on the right of Y by

$$e \circ u = (eu)^*.$$

Given that the free ample semigroup is proper and has a structure as in (†), this suggests that we could use the idea of a double action to produce a structure theorem for proper ample semigroups.

We would like symmetrical two-sided P-theorems for proper restriction, proper weakly ample and proper ample semigroups. We shall look at a construction $\mathscr{M}(T, \mathscr{X}, \mathscr{X}', \mathscr{Y})$ that is proper restriction.

7.2 Construction

We shall define the construction, $\mathscr{M}(T, \mathscr{X}, \mathscr{X}', \mathscr{Y})$, which is adapted from a strong M-semigroup as defined in Chapter 5, and show that it is proper restriction. We will obtain constructions that are proper weakly ample and proper ample by imposing further conditions.

Definition 7.2.1. Let $\mathscr{X}$ and $\mathscr{X}'$ be semilattices and $\mathscr{Y}$ be a subsemilattice of both $\mathscr{X}$ and $\mathscr{X}'$. Let $\varepsilon \in \mathscr{X}$ and $\varepsilon' \in \mathscr{X}'$ be such that $a \leq \varepsilon, \varepsilon'$ for all $a \in \mathscr{Y}$.

Let T be a monoid with identity 1, which acts by morphisms on the left of $\mathscr{X}$ via $\cdot$ and on the right of $\mathscr{X}'$, via $\circ$.

Suppose that $\forall t \in T$ and $\forall e \in \mathscr{Y}$, the following hold:

(A) $e \leq t \cdot \varepsilon \Rightarrow e \circ t \in \mathscr{Y}$;

(B) $e \leq \varepsilon' \circ t \Rightarrow t \cdot e \in \mathscr{Y}$;

(C) $e \leq t \cdot \varepsilon \Rightarrow t \cdot (e \circ t) = e$;

(D) $e \leq \varepsilon' \circ t \Rightarrow (t \cdot e) \circ t = e$;

(E) for all $t \in T$, there exists $a \in \mathscr{Y}$ such that $a \leq t \cdot \varepsilon$.

We shall call $(T, \mathscr{X}, \mathscr{X}', \mathscr{Y})$ a *strong M-quadruple.*

Definition 7.2.2. Given a strong M-quadruple $(T, \mathscr{X}, \mathscr{X}', \mathscr{Y})$, let us define

$$\mathscr{M} = \mathscr{M}(T, \mathscr{X}, \mathscr{X}', \mathscr{Y}) = \{(a, t) \in \mathscr{Y} \times T : a \leq t \cdot \varepsilon\},$$

with binary operation

$$(a, t)(b, u) = (a \wedge t \cdot b, tu)$$

for $(a, t), (b, u) \in \mathscr{M}(T, \mathscr{X}, \mathscr{X}', \mathscr{Y})$.

Let $t \in T$. Note that for $a, b \in \mathscr{X}$, if $a \leq b$, then $t \cdot a \leq t \cdot b$ by Lemma 5.3.2. By its dual, for $a, b \in \mathscr{X}'$, if $a \leq b$, then $a \circ t \leq b \circ t$. We also have the following propositions.

Proposition 7.2.3.
Let $(T, \mathscr{X}, \mathscr{X}', \mathscr{Y})$ be a strong M-quadruple. Then Condition (E) is equivalent to the following:

(F) for all $t \in T$, there exists $b \in \mathscr{Y}$ such that $b \leq \varepsilon' \circ t$.

Proof. Taking $t \in T$, by Condition (E), there exists $a \in \mathscr{Y}$ such that $a \leq t \cdot \varepsilon$. By Condition (A), $a \circ t \in \mathscr{Y}$ and by the above note, $a \leq \varepsilon'$ implies $a \circ t \leq \varepsilon' \circ t$. The converse is dual. □

Proposition 7.2.4. *Let $(T, \mathscr{X}, \mathscr{X}', \mathscr{Y})$ be a strong M-quadruple. Then $(T, \mathscr{X}, \mathscr{Y})$ is a strong left M-triple.*

Proof. It remains to show for $a \in \mathscr{Y}$ and $t \in T$,

$$a \leq t \cdot \varepsilon \Rightarrow a \wedge t \cdot b \in \mathscr{Y} \text{ for any } b \in \mathscr{Y}.$$

Suppose $a \leq t \cdot \varepsilon$. So $a \circ t \in \mathscr{Y}$ by Condition (A). Let $b \in \mathscr{Y}$. We have

$$b \wedge a \circ t \leq a \circ t \leq \varepsilon' \circ t,$$

so by Condition (B), we have

$$t \cdot (b \wedge (a \circ t)) \in \mathscr{Y}.$$

So $a \wedge t \cdot b \in \mathscr{Y}$ since

$$\begin{aligned} t \cdot (b \wedge (a \circ t)) &= t \cdot b \wedge t \cdot (a \circ t) \\ &= t \cdot b \wedge a \text{ by Condition (C).} \end{aligned}$$

□

Proposition 7.2.5. *Let $(T, \mathscr{X}, \mathscr{X}', \mathscr{Y})$ be a strong M-quadruple. If T is an arbitrary monoid, then $\mathscr{M} = \mathscr{M}(T, \mathscr{X}, \mathscr{X}', \mathscr{Y})$ is a proper restriction semigroup such that*

$$(e, t)^+ = (e, 1) \text{ and } (e, t)^* = (e \circ t, 1)$$

for $(e, t) \in \mathscr{M}$. Consequently,

$$E_{\mathscr{M}} = \{(e, 1) : e \in \mathscr{Y}\} \cong \mathscr{Y} \text{ and } \mathscr{M} / \sigma_{\mathscr{M}} \cong T.$$

Proof. If $(T, \mathscr{X}, \mathscr{X}', \mathscr{Y})$ is a strong M-quadruple, then by Proposition 7.2.4, $(T, \mathscr{X}, \mathscr{Y})$ is a strong left M-triple. So then by Proposition 5.3.5, $\mathscr{M}$ is a proper left restriction semigroup where $(e, s)^+ = (e, 1)$,

$$E_{\mathscr{M}} = \{(e, 1) : e \in \mathscr{Y}\} \cong \mathscr{Y} \text{ and } \mathscr{M} / \sigma_{\mathscr{M}} \cong T.$$

By Proposition 7.2.3 and the dual of Proposition 7.2.4, $(T, \mathscr{X}', \mathscr{Y})$ is a strong right M-triple. By the dual of Proposition 5.3.5,

$$\mathscr{M}' = \mathscr{M}'(T, \mathscr{X}', \mathscr{Y}) = \{(t, a) \in T \times \mathscr{Y} : a \leq \varepsilon' \circ t\},$$

with multiplication

$$(t, a)(s, b) = (ts, a \circ s \wedge b),$$

is a proper right restriction semigroup where

$$(s, e)^* = (1, e)$$

for $(s, e) \in \mathscr{M}'$ and $E_{\mathscr{M}'} = \{(1, e) : e \in \mathscr{Y}\}$.

We shall show that $\mathscr{M}$ is isomorphic to $\mathscr{M}'$. Let us define

$$\theta : \mathscr{M} \to \mathscr{M}' \text{ by } (e, s)\theta = (s, e \circ s)$$

for $(e, s) \in \mathscr{M}$. This is well-defined since if $(e, s) \in \mathscr{M}$, $e \circ s \in \mathscr{Y}$ by Condition (A) and $e \circ s \leq \varepsilon' \circ s$, so $(s, e \circ s) \in \mathscr{M}'$. It is clear that $\theta|_{E_{\mathscr{M}}} : E_{\mathscr{M}} \to E_{\mathscr{M}'}$ is an isomorphism where $(e, 1)\theta = (1, e)$.

Consider $(e, s), (f, t) \in \mathscr{M}$ and suppose $(e, s)\theta = (f, t)\theta$, i.e. $(s, e \circ s) = (t, f \circ t)$. Then clearly $s = t$ and $e \circ t = f \circ t$. As $e, f \leq t \cdot \varepsilon$, we have by Condition (C),

$$e = t \cdot (e \circ t) = t \cdot (f \circ t) = f.$$

So θ is one-one.

Consider $(u, g) \in \mathscr{M}'$. We have $g \leq \varepsilon' \circ u$, so by Condition (B), $u \cdot g \in \mathscr{Y}$ and as $u \cdot g \leq u \cdot \varepsilon$, we have $(u \cdot g, u) \in \mathscr{M}$. By Condition (D),

$$(u \cdot g, u)\theta = (u, (u \cdot g) \circ u) = (u, g).$$

So θ is onto, and hence a bijection.

To see that θ is an isomorphism, let us consider $(e, s), (f, t) \in \mathscr{M}$. Then

$$\begin{aligned} (e, s)\theta(f, t)\theta &= (s, e \circ s)(t, f \circ t) \\ &= (st, e \circ st \wedge f \circ t) \\ &= (st, (e \circ s \wedge f) \circ t). \end{aligned}$$

We have $e \circ s \wedge f \leq e \circ s \leq \varepsilon' \circ s$ and so

$$\begin{aligned} e \circ s \wedge f &= \big(s \cdot (e \circ s \wedge f)\big) \circ s \\ &= \big(s \cdot (e \circ s) \wedge s \cdot f\big) \circ s \\ &= (e \wedge s \cdot f) \circ s. \end{aligned}$$

We can now deduce that

$$\begin{aligned}(e,s)\theta(f,t)\theta &= (st,(e\circ s\wedge f)\circ t)\\ &= (st,((e\wedge s\cdot f)\circ s)\circ t)\\ &= (st,(e\wedge s\cdot f)\circ st)\\ &= (e\wedge s\cdot f,st)\theta\\ &= \big((e,s)(f,t)\big)\theta,\end{aligned}$$

so that θ is an isomorphism as required.

It follows that $\mathscr{M}$ is proper left restriction with distinguished semilattice E, hence proper restriction. Moreover, for any $(e,s)\in\mathscr{M}$,

$$(e,s)\theta = (s,e\circ s)\,\widetilde{\mathscr{L}}_{E_{\mathscr{M}'}}\,(1,e\circ s) = (e\circ s,1)\theta,$$

so that in $\mathscr{M}$,

$$(e,s)^* = (e\circ s,1).$$

$\square$

Considering T to be a unipotent monoid, we obtain the following corollary:

Corollary 7.2.6. *Let $(T,\mathscr{X},\mathscr{X}',\mathscr{Y})$ be a strong M-quadruple and T be a unipotent monoid. Then $\mathscr{M}=\mathscr{M}(T,\mathscr{X},\mathscr{X}',\mathscr{Y})$ is a proper weakly ample semigroup.*

Proof. By Proposition 7.2.5, $\mathscr{M}$ is a proper restriction semigroup with distinguished semilattice $E_{\mathscr{M}}=\{(e,1):e\in\mathscr{Y}\}$.

Using the same argument in Corollary 6.3.8, $E_{\mathscr{M}}=E(\mathscr{M})$ when T is a unipotent monoid. So $\mathscr{M}$ is a proper weakly ample semigroup. $\square$

We note that this result also follows from Corollary 5.3.7 by considering the strong quadruple $(T,\mathscr{X},\mathscr{X}',\mathscr{Y})$ as left and right strong M-triples. Similarly, restricting T to be a cancellative monoid, we can obtain the following proposition using Corollary 5.3.8, but we shall prove it directly.

Corollary 7.2.7. *Let $(T,\mathscr{X},\mathscr{X}',\mathscr{Y})$ be a strong M-quadruple and T be cancellative. Then $\mathscr{M}=\mathscr{M}(T,\mathscr{X},\mathscr{X}',\mathscr{Y})$ is a proper ample semigroup.*

Proof. As T is cancellative, it is unipotent and so by Corollary 7.2.6, $\mathscr{M}$ is a proper weakly ample semigroup.

As in the proof of Corollary 5.3.8,

$$(a,t)\,\mathscr{R}^*\,(a,1).$$

It can be deduced from the previous results that

$$(a,t)\,\mathscr{L}^*\,(a\circ t,1),$$

but we shall show it directly. Suppose $(a,t)(c,y) = (a,t)(d,w)$ for $(c,y),(d,w) \in \mathscr{M}$. Then

$$\begin{aligned}(a \wedge t \cdot c, ty) = (a \wedge t \cdot d, tw) &\Rightarrow a \wedge t \cdot c = a \wedge t \cdot d \text{ and } ty = tw\\ &\Rightarrow a \wedge t \cdot c = a \wedge t \cdot d \text{ and } y = w\end{aligned}$$

as T is left cancellative. By Condition (A), $a \circ t \in \mathscr{Y}$ and so

$$a \circ t \wedge c, a \circ t \wedge d \in \mathscr{Y}.$$

We also have

$$a \circ t \wedge c, a \circ t \wedge d \leq a \circ t \leq \varepsilon' \circ t.$$

We have

$$\begin{aligned}t \cdot (a \circ t \wedge c) &= t \cdot (a \circ t) \wedge t \cdot c\\ &= a \wedge t \cdot c\\ &= a \wedge t \cdot d\\ &= t \cdot (a \circ t \wedge d).\end{aligned}$$

Using Condition (D),

$$\begin{aligned}t \cdot (a \circ t \wedge c) = t \cdot (a \circ t \wedge d) &\Rightarrow [t \cdot (a \circ t \wedge c)] \circ t = [t \cdot (a \circ t \wedge d)] \circ t\\ &\Rightarrow a \circ t \wedge c = a \circ t \wedge d.\end{aligned}$$

So $(a,t)(c,y) = (a,t)(d,w)$ implies that $(a \circ t, 1)(c,y) = (a \circ t, 1)(d,w)$. Therefore $(a,t) \mathscr{L}^* (a \circ t, 1)$ and so $\mathscr{M}$ is an ample semigroup. It follows from Corollary 2.4.10, and its dual, that it is a proper ample semigroup.

Therefore $\mathscr{M}$ is a proper ample semigroup when T is a cancellative monoid.

$\square$

Corollary 7.2.8. *Let $(T, \mathscr{X}, \mathscr{X}', \mathscr{Y})$ be a strong M-quadruple and T be a group. Then $\mathscr{M} = \mathscr{M}(T, \mathscr{X}, \mathscr{X}', \mathscr{Y})$ is a proper inverse semigroup, where*

$$(a,t)' = (t^{-1} \cdot a, t^{-1})$$

for $(a,t) \in \mathscr{M}$.

Proof. As a group is a cancellative monoid, $\mathscr{M}$ is a proper ample semigroup by Corollary 7.2.7. Therefore the idempotents form a semilattice.

We wish to show that $\mathscr{M}$ is regular. Let $(a,t) \in \mathscr{M}$. We require $(t^{-1} \cdot a, t^{-1}) \in \mathscr{M}$. As $(a,t) \in \mathscr{M}$, $a \leq t \cdot \varepsilon$. We have

$$\begin{aligned}a \leq t \cdot \varepsilon &\Rightarrow a \circ t \in \mathscr{Y} \text{ and } t \cdot (a \circ t) = a\\ &\Rightarrow a \circ t \in \mathscr{Y} \text{ and } a \circ t = t^{-1} \cdot a\\ &\Rightarrow t^{-1} \cdot a \in \mathscr{Y}.\end{aligned}$$

Clearly, $t^{-1} \cdot a \leq t^{-1} \cdot \varepsilon$, so $(t^{-1} \cdot a, t^{-1}) \in \mathscr{M}$. As in the proof of Theorem

5.3.10,

$$(a,t)(t^{-1}\cdot a,t^{-1})(a,t)=(a,t).$$

So $\mathscr{M}$ is regular and hence is an inverse semigroup. As

$$\begin{aligned}(t^{-1}\cdot a,t^{-1})(a,t)(t^{-1}\cdot a,t^{-1}) &= (t^{-1}\cdot a,t^{-1})(a,1)\\ &= (t^{-1}\cdot a,t^{-1}),\end{aligned}$$

we have $(a,t)'=(t^{-1}\cdot a,t^{-1})$.

By Proposition 2.2.3, $\widetilde{\mathcal{R}}=\mathcal{R}$ as $\mathscr{M}$ is inverse. So $\mathscr{M}$ is a proper inverse semigroup. □

7.3 Converse to the structure theorem

We will explain why we do not necessarily have the converse to the structure theorem; our construction does not yield the whole class of proper restriction semigroups.

We would ideally like to show that every proper restriction semigroup is isomorphic to such a structure. We have tried the following approaches to prove the converse:

(Attempt 1) Starting from the one-sided constuction, let the partial right action of T on $\mathscr{Y}$ be defined by

$$\exists y\bullet m \Leftrightarrow y\leq m\cdot\varepsilon,\ \text{in which case, } y\bullet m=[y,m].$$

We have not managed to show that this is a strong partial right action, as defined in [29]. If it was, we would be able to globalise to obtain a right action. Instead, we can show that this action is a partial right action of T on $\mathscr{Y}$, where T preserves the partial order and the domain of each element of T is an order ideal. From this we can produce a structure theorem (see Chapter 8).

(Attempt 2) Take the left and right actions as defined for the proof of Theorem 5 from [12]. Conditions (C) and (D) hold provided (A) and (B) hold, but if S is finite, (A) and (B) hold if and only if S is a proper finite inverse semigroup. On the other hand, we know from the results of [18] that the free ample monoid will have this structure. This gave us the clue to prove the following (in which we do not assume that the actions are as in [12]).

Proposition 7.3.1. *Let S be a finite proper ample semigroup. Suppose that S is isomorphic to $\mathscr{M}=\mathscr{M}(T,\mathscr{X},\mathscr{X}',\mathscr{Y})$ for some $(T,\mathscr{X},\mathscr{X}',\mathscr{Y})$ where $T=S/\sigma$. Then S is inverse.*

Proof. As T is cancellative, it is a group. If we let $(a,t)\in\mathscr{M}$, then $a\leq t\cdot\varepsilon$ and so $a\circ t\in\mathscr{Y}$. As in Corollary 7.2.8, we have $a=t\cdot(a\circ t)$

and $a = t \cdot (t^{-1} \cdot a)$ and so

$$a \circ t = t^{-1} \cdot a.$$

As $t^{-1} \cdot a \leq t^{-1} \cdot \varepsilon$, $(t^{-1} \cdot a, t^{-1}) \in \mathscr{M}$ and

$$(a, t)(t^{-1} \cdot a, t^{-1})(a, t) = (a, 1)(a, t) = (a, t).$$

So $\mathscr{M}$ is regular and since $E(\mathscr{M})$ is a semilattice, $\mathscr{M}$ is an inverse semigroup. $\square$

We shall present precisely when a proper restriction semigroup S is isomorphic to some $\mathscr{M}(T, \mathscr{X}, \mathscr{X}', \mathscr{Y})$, where $T = S/\sigma_S$. These results from [10], are due to Gould, and so we do not give full proofs.

Definition 7.3.2. Let S be a restriction semigroup. Then S is *extra proper* if is satisfies (EP), which is the conjunction of $(\mathrm{EP})^r$ and its dual $(\mathrm{EP})^l$, where $(\mathrm{EP})^r$ is defined as follows:

$(\mathrm{EP})^r$: for all $s, t, u \in S$, if $s \,\sigma_S\, tu$ then there exists $v \in S$ with $t^+s = tv$ and $v \,\sigma_S\, u$.

Lemma 7.3.3. *Let S be an extra proper restriction semigroup such that E is a σ_S-class. Then S is proper.*

Proof. Let $a, b \in S$ and suppose that $a\,(\widetilde{\mathscr{R}}_E \cap \sigma_S)\,b$. Then $a \,\sigma_S\, bb^*$ so that with $a = s, b = t$ and $u = b^*$ in $(\mathrm{EP})^r$ we have that $b^+a = bv$ for some $v \in S$ with $v \,\sigma_S\, b^*$, so $v \in E$. But $b^+ = a^+$ and so $a = bv = (bv)^+b = a^+b = b$. Dually, $\widetilde{\mathscr{L}}_E \cap \sigma_S$ is trivial. $\square$

The proof of the following result can be found in [10].

Theorem 7.3.4. *Let S be a proper restriction semigroup. Then S is isomorphic to some $\mathscr{M}(S/\sigma_S, \mathscr{X}, \mathscr{X}', \mathscr{Y})$ if and only if S is extra proper.*

Example 7.3.5. Every inverse semigroup has (EP). For, if s, t, u are elements of an inverse semigroup S with $s \,\sigma\, tu$, then $t^+s = tt^{-1}s$ and $t^{-1}s \,\sigma\, tt^{-1}u \,\sigma\, u$.

Example 7.3.6. Every reduced restriction semigroup has (EP). For, if s, t, u are elements of a reduced restriction semigroup S with $s \,\sigma_S\, tu$, then $s = tu$ and $t^+s = s = tu$.

Less trivially, free ample monoids have (EP).

Example 7.3.7. Let $\mathcal{FRM}(X)$ be the free restriction monoid on a non-empty set X. We use the characterisation of $\mathcal{FRM}(X)$ as a submonoid of the free inverse monoid $\mathcal{FIM}(X)$ on X, given in [18].

Let $\mathcal{FG}(X)$ be the free group on X, and regard elements of $\mathcal{FG}(X)$ as reduced words over X. Let

$$\mathcal{Y} = \{A \subseteq \mathcal{FG}(X) : 1 \leq |A| < \infty, A \text{ is prefix closed}\}.$$

Then

$$\mathcal{FIM}(X) = \{(A, w) : A \in \mathcal{Y}, w \in A\}$$

with

$$(A, w)(B, v) = (A \cup wB, wv) \text{ and } (A, w)^{-1} = (w^{-1}A, w^{-1}).$$

From [18], $\mathcal{FRM}(X)$ is the submonoid of $\mathcal{FIM}(X)$ given by

$$\mathcal{FRM}(X) = \{(A, w) \in \mathcal{FIM}(X) : w \in X^*\}$$

and for any $(A, w), (B, v) \in \mathcal{FRM}(X)$, we have that

$$(A, w)^+ = (A, 1) \text{ and } (A, w)\, \sigma_{\mathcal{FRM}(X)}\, (B, v) \text{ if and only if } w = v.$$

Suppose that $(A, w), (B, v), (C, u) \in \mathcal{FRM}(X)$ with

$$(A, w)\, \sigma_{\mathcal{FRM}(X)}\, (B, v)(C, u).$$

Then $w = vu$ and

$$(B, v)^+(A, w) = (B, v)(B, v)^{-1}(A, w) = (B, v)(v^{-1}B, v^{-1})(A, w) =$$

$$= (B, v)(v^{-1}B \cup v^{-1}A, v^{-1}w) = (B, v)(v^{-1}B \cup v^{-1}A, u)$$

and as $(v^{-1}B \cup v^{-1}A, u) \in \mathcal{FRM}(X)$, Condition (EP)^r holds. Dually, (EP)^l holds.

Finally in this section we give an example of an infinite proper ample semigroup without (EP), also showing that a proper ample semigroup can be a $(2, 1, 1)$-subalgebra of a proper inverse semigroup, yet not itself be extra proper.

Example 7.3.8. Let X be a set with at least two elements, and let $X_i = \{x_i : x \in X\}$ for $i \in \{0, 1\}$ be sets in one-one correspondence with X. Let S be a strong semilattice $Y = \{1, 0\}$ of cancellative monoids $S_1 = X_1^*$ and $S_2 = \mathcal{FG}(X_0)$, with connecting morphism $\phi_{1,0}$ given by $x_1\phi_{1,0} = x_0$.

It follows from [13, Theorem 1], that S is ample, with $\mathcal{R}^* = \mathcal{L}^* = \mathcal{H}^*$-classes S_1 and S_0. As the connecting morphism is one-one, it is easy to see that S is proper.

Let x, y be distinct elements of X. Then

$$e_0x_1 = x_0 = y_0(y_0^{-1}x_0) = e_0y_1(y_0^{-1}x_0)$$

so that $x_1\, \sigma\, y_1(y_0^{-1}x_0)$. If $y_1^+x_1 = y_1w$ for some $w \in S$ we would have that $x_1 = y_1w$, which is impossible.

Chapter 8

Construction based on partial actions

8.1 Partial actions

To produce two-sided P-theorems for proper restriction, proper weakly ample and proper ample semigroups, we need to consider partial actions as in Definition 3.1.3.

When considering the partial right action, we say the domain of each $t \in T$ is an *order ideal* if the following condition holds for all $y, z \in X$ and $t \in T$:

(iii) if $y \leq z$ and $\exists z \bullet t$, then $\exists y \bullet t$.

The dual can be defined for partial left actions.

Suppose now that T acts partially on the right of a partially ordered set X. We say T *preserves the partial order* if the following condition holds for $y, z \in X$ and $t \in T$:

(iv) if $y \leq z$, $\exists y \bullet t$ and $\exists z \bullet t$, then $y \bullet t \leq z \bullet t$.

The definition for partial left actions is dual.

8.2 Construction based on partial actions

We shall define a construction, $\mathscr{M}(T, \mathscr{Y})$, which has been adapted from $\mathscr{M}(T, \mathscr{X}, \mathscr{X}', \mathscr{Y})$ and is based around partial actions. The construction is analogous to that of Petrich and Reilly in the inverse case [48] and Lawson in the ample case [36]. However, our proofs will be new. The structure we shall present is a proper restriction semigroup and conversely every proper restriction semigroup has its structure.

Definition 8.2.1. Let T be a monoid, acting partially on the right and left of a semilattice $\mathscr{Y}$, where $\circledcirc$ and $\odot$ are right and left partial actions of T on $\mathscr{Y}$ respectively. Suppose that T preserves the partial order and the domain of each $t \in T$ is an order ideal.

Suppose that for $e \in \mathscr{Y}$ and $a \in T$, the following hold:

(A) if $\exists e \circledcirc a$, then $\exists a \odot (e \circledcirc a)$ and $a \odot (e \circledcirc a) = e$;

(B) if $\exists a \odot e$, then $\exists (a \odot e) \circledcirc a$ and $(a \odot e) \circledcirc a = e$;

(C) for all $a \in T$, $\exists e \in \mathscr{Y}$ such that $\exists e \circledcirc a$.

Then $(T, \mathscr{Y})$ is called an *strong M-pair*.

Definition 8.2.2. Let us define

$$\mathscr{M}(T, \mathscr{Y}) = \{(e, a) \in \mathscr{Y} \times T : \exists e \circledcirc a\},$$

with binary operation given by

$$(e, a)(f, b) = (a \odot ((e \circledcirc a) \wedge f), ab)$$

for $(e, a), (f, b) \in \mathscr{M}(T, \mathscr{Y})$.

We show in the proof of Theorem 8.3.1 that the binary operation is closed.

Proposition 8.2.3. *If $(T, \mathscr{Y})$ is a strong M-pair, then for all $t \in T$, $\exists e \in \mathscr{Y}$ such that $\exists a \odot e$.*

Proof. By Condition (C), for all $t \in T$, there exists $e \in \mathscr{Y}$ such that $\exists e \circledcirc a$. By Condition (A), $\exists a \odot (e \circledcirc a)$, where $e \circledcirc a \in \mathscr{Y}$. □

Our main result requires use of the following proposition.

Proposition 8.2.4. *Let T be a monoid, $\circledcirc$ be a partial right action and $\odot$ be a partial left action of T on a semilattice $\mathscr{Y}$, such that T preserves the partial order and the domain of each $t \in T$ is an order ideal.*

(1) If $\exists e \circledcirc a$ and $\exists f \circledcirc a$, then $\exists (e \wedge f) \circledcirc a$ and

$$e \circledcirc a \wedge f \circledcirc a = (e \wedge f) \circledcirc a.$$

(2) If $\exists a \odot e$ and $\exists a \odot f$, then $\exists a \odot (e \wedge f)$ and

$$a \odot e \wedge a \odot f = a \odot (e \wedge f).$$

Proof. Suppose $\exists e \circledcirc a$ and $\exists f \circledcirc a$. As $\exists e \circledcirc a$ and $e \wedge f \leq e$, $\exists (e \wedge f) \circledcirc a$ since the domain of each element of T is an order ideal. It follows from $\circledcirc$ being order preserving that $(e \wedge f) \circledcirc a \leq e \circledcirc a$ and similarly we have $(e \wedge f) \circledcirc a \leq f \circledcirc a$. Therefore

$$(e \wedge f) \circledcirc a \leq e \circledcirc a \wedge f \circledcirc a.$$

As $\exists e \circledcirc a$, $\exists a \odot (e \circledcirc a)$ by Condition (A). As $e \circledcirc a \wedge f \circledcirc a \leq e \circledcirc a$ and

$\exists a \odot (e \circledcirc a)$, $\exists a \odot (e \circledcirc a \wedge f \circledcirc a)$ since the domain of each element of T is an order ideal. Since $\odot$ is order preserving,

$$a \odot (e \circledcirc a \wedge f \circledcirc a) \leq a \odot (e \circledcirc a),$$

i.e.

$$a \odot (e \circledcirc a \wedge f \circledcirc a) \leq e.$$

Similarly,

$$a \odot (e \circledcirc a \wedge f \circledcirc a) \leq f$$

and so

$$a \odot (e \circledcirc a \wedge f \circledcirc a) \leq e \wedge f.$$

We know $\exists a \odot (e \circledcirc a \wedge f \circledcirc a)$, so by Condition (B), $\exists [a \odot (e \circledcirc a \wedge f \circledcirc a)] \circledcirc a$ and

$$e \circledcirc a \wedge f \circledcirc a = [a \odot (e \circledcirc a \wedge f \circledcirc a)] \circledcirc a.$$

As $a \odot ((e \circledcirc a) \wedge (f \circledcirc a)) \leq e \wedge f$,

$$[a \odot (e \circledcirc a \wedge f \circledcirc a)] \circledcirc a \leq (e \wedge f) \circledcirc a$$

since $\circledcirc$ is order preserving. Hence $e \circledcirc a \wedge f \circledcirc a \leq (e \wedge f) \circledcirc a$ and so

$$e \circledcirc a \wedge f \circledcirc a = (e \wedge f) \circledcirc a.$$

The proof of (2) is dual. $\square$

8.3 Symmetrical two-sided structure theorem for proper restriction semigroups

We will prove how this construction allows us to produce a symmetrical two-sided P-theorem for proper restriction semigroups, proven from the one-sided results.

We have the following P-theorem for all proper restriction semigroups.

Theorem 8.3.1. *If $(T, \mathscr{Y})$ is a strong M-pair, $\mathscr{M} = \mathscr{M}(T, \mathscr{Y})$ is a proper restriction semigroup with distinguished semilattice*

$$E_{\mathscr{M}} = \{(e, 1) : e \in \mathscr{Y}\} \cong \mathscr{Y}$$

and $\mathscr{M}/\sigma_{\mathscr{M}} \cong T$, where

$$(e, a)^+ = (e, 1) \text{ and } (e, a)^* = (e \circledcirc a, 1)$$

for $(e, a) \in M$. Conversely, every proper restriction semigroup S, with distinguished semilattice $\mathscr{Y}$, is isomorphic to some $\mathscr{M}(T, \mathscr{Y})$ where

$$S/\sigma_S \cong T.$$

Proof. We shall first show that $\mathscr{M}(T, \mathscr{Y})$ is a proper restriction semigroup.

Let $(e, a), (f, b) \in \mathscr{M}$. We wish to show $(a \odot ((e \circledcirc a) \wedge f), ab) \in \mathscr{M}$. By Condition (A), $\exists a \odot (e \circledcirc a)$ since $\exists e \circledcirc a$. As $(e \circledcirc a) \wedge f \leq e \circledcirc a$ and the domain of each element of T is an order ideal, $\exists a \odot ((e \circledcirc a) \wedge f)$, i.e. $a \odot ((e \circledcirc a) \wedge f) \in \mathscr{Y}$. Clearly, $ab \in T$.

We wish to show that $\exists [a \odot ((e \circledcirc a) \wedge f)] \circledcirc ab$. We have $\exists a \odot ((e \circledcirc a) \wedge f)$, so by Condition (B), $\exists [a \odot ((e \circledcirc a) \wedge f)] \circledcirc a$ and $[a \odot ((e \circledcirc a) \wedge f)] \circledcirc a = (e \circledcirc a) \wedge f$. By showing $\exists ((e \circledcirc a) \wedge f) \circledcirc b$, we have $\exists ([a \odot ((e \circledcirc a) \wedge f)] \circledcirc a) \circledcirc b$. So, from Condition (ii) for a partial action, $\exists [a \odot ((e \circledcirc a) \wedge f)] \circledcirc ab$. Using the fact that the domain of each element of T is an order ideal, $(e \circledcirc a) \wedge f \leq f$ and $\exists f \circledcirc b$ imply that $\exists ((e \circledcirc a) \wedge f) \circledcirc b$. So $\exists [a \odot ((e \circledcirc a) \wedge f)] \circledcirc ab$. Therefore the binary operation is closed.

Suppose $(e, a), (f, b), (g, c) \in \mathscr{M}$. Then

$$\begin{aligned}(e, a)[(f, b)(g, c)] &= (e, a)(b \odot ((f \circledcirc b) \wedge g), bc) \\ &= (a \odot ((e \circledcirc a) \wedge b \odot ((f \circledcirc b) \wedge g)), a(bc)) \\ &= (a \odot ((e \circledcirc a) \wedge b \odot ((f \circledcirc b) \wedge g)), (ab)c).\end{aligned}$$

As $\exists b \odot ((f \circledcirc b) \wedge g)$, $\exists (b \odot ((f \circledcirc b) \wedge g)) \circledcirc b$ by Condition (B). We have

$$(e \circledcirc a) \wedge (b \odot ((f \circledcirc b) \wedge g)) \leq b \odot ((f \circledcirc b) \wedge g),$$

so

$$\exists ((e \circledcirc a) \wedge b \odot ((f \circledcirc b) \wedge g)) \circledcirc b$$

since the domain of each element of T is an order ideal. So, by Condition (A),

$$\exists b \odot (((e \circledcirc a) \wedge b \odot ((f \circledcirc b) \wedge g)) \circledcirc b)$$

and

$$b \odot (((e \circledcirc a) \wedge b \odot ((f \circledcirc b) \wedge g)) \circledcirc b) = (e \circledcirc a) \wedge b \odot ((f \circledcirc b) \wedge g).$$

So

$$\begin{aligned}(e, a)[(f, b)(g, c)] &= (a \odot ((e \circledcirc a) \wedge b \odot ((f \circledcirc b) \wedge g)), (ab)c) \\ &= (a \odot (b \odot (((e \circledcirc a) \wedge b \odot ((f \circledcirc b) \wedge g)) \circledcirc b)), (ab)c) \\ &= (ab \odot (((e \circledcirc a) \wedge (b \odot ((f \circledcirc b) \wedge g))) \circledcirc b), (ab)c)\end{aligned}$$

We also have $(f \circledcirc b) \wedge g \leq f \circledcirc b$, $\exists b \odot (f \circledcirc b)$ and, due to Condition (A), $b \odot (f \circledcirc b) = f$. So

$$b \odot ((f \circledcirc b) \wedge g) \leq b \odot (f \circledcirc b) = f,$$

since $\odot$ is order preserving and hence

$$b \odot ((f \circledcirc b) \wedge g) \leq f \Rightarrow (b \odot ((f \circledcirc b) \wedge g)) \wedge f = b \odot ((f \circledcirc b) \wedge g).$$

Using Proposition 8.2.4,

$$((e \circledcirc a) \wedge b \odot (((f \circledcirc b) \wedge g)) \circledcirc b) = ((e \circledcirc a) \wedge f \wedge (b \odot ((f \circledcirc b) \wedge g))) \circledcirc b =$$

$$((((e \circledcirc a) \wedge f) \circledcirc b) \wedge ((b \odot ((f \circledcirc b) \wedge g)) \circledcirc b) = ((e \circledcirc a) \wedge f) \circledcirc b \wedge (f \circledcirc b) \wedge g.$$

So

$$\begin{aligned} (e,a)[(f,b)(g,c)] &= (ab \odot (((e \circledcirc a) \wedge b \odot (((f \circledcirc b) \wedge g)) \circledcirc b)), (ab)c) \\ &= (ab \odot (((e \circledcirc a) \wedge f) \circledcirc b \wedge (f \circledcirc b) \wedge g), (ab)c). \end{aligned}$$

By Proposition 8.2.4,

$$\begin{aligned} (((e \circledcirc a) \wedge f) \circledcirc b) \wedge (f \circledcirc b) &= ((e \circledcirc a) \wedge f \wedge f) \circledcirc b \\ &= ((e \circledcirc a) \wedge f) \circledcirc b. \end{aligned}$$

As $(e,a)(f,b) \in \mathscr{M}$, $\exists a \odot ((e \circledcirc a) \wedge f)$ and so by Condition (B), $\exists (a \odot ((e \circledcirc a) \wedge f)) \circledcirc a$ and $(a \odot ((e \circledcirc a) \wedge f)) \circledcirc a = (e \circledcirc a) \wedge f$. So

$$\begin{aligned} (((e \circledcirc a) \wedge f) \circledcirc b) \wedge (f \circledcirc b) \wedge g &= (((e \circledcirc a) \wedge f) \circledcirc b) \wedge g \\ &= (((a \odot ((e \circledcirc a) \wedge f)) \circledcirc a) \circledcirc b) \wedge g \\ &= ((a \odot ((e \circledcirc a) \wedge f)) \circledcirc ab) \wedge g. \end{aligned}$$

Hence

$$\begin{aligned} (e,a)[(f,b)(g,c)] &= (ab \odot (((e \circledcirc a) \wedge f) \circledcirc b \wedge (f \circledcirc b) \wedge g), (ab)c) \\ &= (ab \odot (((a \odot ((e \circledcirc a) \wedge f)) \circledcirc ab) \wedge g), (ab)c) \\ &= (a \odot ((e \circledcirc a) \wedge f), ab)(g,c) \\ &= [(e,a)(f,b)](g,c). \end{aligned}$$

Therefore $\mathscr{M}$ is a semigroup.

Let $E = \{(e,1) : e \in \mathscr{Y}\}$. We note that for $e \in \mathscr{Y}$, $\exists e \circledcirc 1$ and

$$\begin{aligned} (e,1)(e,1) &= (1 \odot (e \circledcirc 1) \wedge e, 1) \\ &= (e \wedge e, 1) \\ &= (e,1). \end{aligned}$$

We also have for $(e,1),(f,1) \in E$,

$$(e,1)(f,1) = (e \wedge f, 1) = (f \wedge e, 1) = (f,1)(e,1),$$

so E is a semilattice, which is isomorphic to $\mathscr{Y}$.

We wish to show that for $(e,a) \in \mathscr{M}$,

$$(e,a)\, \widetilde{\mathcal{R}}_E\, (e,1).$$

First we have

$$(e,1)(e,a) = (1 \odot ((e \circledcirc 1) \wedge e), a) = (e,a).$$

We also have for all $(f,1) \in E$,

$$\begin{aligned}(f,1)(e,a) = (e,a) &\Rightarrow (1 \odot (f \wedge e), a) = (e,a)\\ &\Rightarrow (f \wedge e, a) = (e,a)\\ &\Rightarrow f \wedge e = e\\ &\Rightarrow (f,1)(e,1) = (e,1),\end{aligned}$$

as $\mathscr{Y}$ is isomorphic to E. So $(e,a)\, \widetilde{\mathcal{R}}_E\, (e,1)$ and we shall put

$$(e,a)^+ = (e,1).$$

We also wish to show that for $(e,a) \in \mathscr{M}$,

$$(e,a)\, \widetilde{\mathcal{L}}_E\, (e \circledcirc a, 1).$$

We have

$$\begin{aligned}(e,a)(e \circledcirc a, 1) &= (a \odot ((e \circledcirc a) \wedge (e \circledcirc a)), a)\\ &= (a \odot (e \circledcirc a), a)\\ &= (e,a)\end{aligned}$$

by Condition (A). Also, for all $(f,1) \in E$,

$$\begin{aligned}(e,a)(f,1) = (e,a) &\Rightarrow (a \odot ((e \circledcirc a) \wedge f), a) = (e,a)\\ &\Rightarrow a \odot ((e \circledcirc a) \wedge f) = e\\ &\Rightarrow (a \odot ((e \circledcirc a) \wedge f)) \circledcirc a = e \circledcirc a\\ &\Rightarrow (e \circledcirc a) \wedge f = e \circledcirc a \text{ by (B)}\\ &\Rightarrow ((e \circledcirc a) \wedge f, 1) = (e \circledcirc a, 1)\\ &\Rightarrow (e \circledcirc a, 1)(f,1) = (e \circledcirc a, 1).\end{aligned}$$

Therefore, $(e,a)\, \widetilde{\mathcal{L}}_E\, (e \circledcirc a, 1)$ and we put

$$(e,a)^* = (e \circledcirc a, 1).$$

We note that for $(e,a), (f,b) \in \mathscr{M}$,

$$(e,a)\, \widetilde{\mathcal{R}}_E\, (f,b) \Leftrightarrow e = f$$

and

$$(e,a)\,\widetilde{\mathcal{L}}_E\,(f,b) \Leftrightarrow e \circledcirc a = f \circledcirc b.$$

We wish to show that $\widetilde{\mathcal{R}}_E$ is a left congruence. Let $(e,a),(f,b),(g,c) \in \mathscr{M}$ and $(e,a)\,\widetilde{\mathcal{R}}_E\,(f,b)$. We wish to show

$$(g,c)(e,a)\,\widetilde{\mathcal{R}}_E\,(g,c)(f,b),$$

i.e.

$$(c \odot ((g \circledcirc c) \wedge e), ca)\,\widetilde{\mathcal{R}}_E\,(c \odot ((g \circledcirc c) \wedge f), cb),$$

which is equivalent to showing

$$c \odot ((g \circledcirc c) \wedge e) = c \odot ((g \circledcirc c) \wedge f).$$

But, as $(e,a)\,\widetilde{\mathcal{R}}_E\,(f,b)$, we have that $e = f$ and so it is clear that the above equation holds and hence $\widetilde{\mathcal{R}}_E$ is a left congruence.

We also require $\widetilde{\mathcal{L}}_E$ to be a right congruence. Suppose $(e,a)\,\widetilde{\mathcal{L}}_E\,(f,b)$, i.e. $e \circledcirc a = f \circledcirc b$. We wish to show

$$(e,a)(g,c)\,\widetilde{\mathcal{L}}_E\,(f,b)(g,c),$$

i.e.

$$(a \odot ((e \circledcirc a) \wedge g)) \circledcirc ac = (b \odot ((f \circledcirc b) \wedge g)) \circledcirc bc.$$

Since $\exists a \odot ((e \circledcirc a) \wedge g)$, it follows that $\exists (a \odot ((e \circledcirc a) \wedge g)) \circledcirc a$ and

$$(a \odot ((e \circledcirc a) \wedge g)) \circledcirc a = (e \circledcirc a) \wedge g.$$

Similarly,

$$(b \odot ((f \circledcirc b) \wedge g)) \circledcirc b = (f \circledcirc b) \wedge g.$$

As $e \circledcirc a = f \circledcirc b$,

$$(a \odot ((e \circledcirc a) \wedge g)) \circledcirc a = (b \odot ((f \circledcirc b) \wedge g)) \circledcirc b.$$

Notice that $e \circledcirc a \wedge g = f \circledcirc b \wedge g \leq g$ and $\exists g \circledcirc c$, so that $\exists (e \circledcirc a \wedge g) \circledcirc c$ and hence $\exists ((a \odot ((e \circledcirc a) \wedge g)) \circledcirc a) \circledcirc c$. By definition of a partial action,

$$(a \odot ((e \circledcirc a) \wedge g)) \circledcirc ac = ((a \odot ((e \circledcirc a) \wedge g)) \circledcirc a) \circledcirc c$$

and we deduce that

$$(a \odot ((e \circledcirc a) \wedge g)) \circledcirc ac = (b \odot ((f \circledcirc b) \wedge g)) \circledcirc bc.$$

Therefore $\widetilde{\mathcal{L}}_E$ is a right congruence.

We shall show that the ample conditions hold. For $(e,a) \in \mathscr{M}$ and

$(f,1) \in E$, using Condition (A) and Proposition 8.2.4 we have

$$\begin{aligned}
[(e,a)(f,1)]^+(e,a) &= (a \odot ((e \circledcirc a) \wedge f), a)^+(e,a) \\
&= (a \odot ((e \circledcirc a) \wedge f), 1)(e,a) \\
&= ((a \odot ((e \circledcirc a) \wedge f)) \wedge e, a) \\
&= ((a \odot ((e \circledcirc a) \wedge f)) \wedge a \odot (e \circledcirc a), a), \\
&= (a \odot ((e \circledcirc a) \wedge f \wedge (e \circledcirc a)), a), \\
&= (a \odot ((e \circledcirc a) \wedge f), a) \\
&= (e,a)(f,1)
\end{aligned}$$

and

$$\begin{aligned}
(e,a)[(f,1)(e,a)]^* &= (e,a)(f \wedge e, a)^* \\
&= (e,a)((f \wedge e) \circledcirc a, 1) \\
&= (a \odot ((e \circledcirc a) \wedge (f \wedge e) \circledcirc a), a) \\
&= (a \odot ((f \wedge e) \circledcirc a), a) \\
&= (f \wedge e, a) \\
&= (f,1)(e,a).
\end{aligned}$$

Therefore, $\mathscr{M}$ is a restriction semigroup with distinguished semilattice $E = E_{\mathscr{M}}$.

If $(e,a)\,\sigma_{\mathscr{M}}\,(f,b)$, there exists $(g,1) \in E_{\mathscr{M}}$ such that

$$(g,1)(e,a) = (g,1)(f,b),$$

i.e.

$$(g \wedge e, a) = (g \wedge f, b)$$

and so $a = b$.

We wish to show $(e,a)\sigma_{\mathscr{M}} = (f,b)\sigma_{\mathscr{M}}$ when $a = b$. Consider $(e \wedge f, 1) \in E_{\mathscr{M}}$. We have

$$\begin{aligned}
(e \wedge f, 1)(e,a) &= (e \wedge f \wedge e, a) \\
&= (e \wedge f, a) \\
&= (e \wedge f \wedge f, b) \\
&= (e \wedge f, 1)(f,b),
\end{aligned}$$

so $(e,a)\sigma_{\mathscr{M}} = (f,b)\sigma_{\mathscr{M}}$ when $a = b$. Hence

$$(e,a)\sigma_{\mathscr{M}}(f,b) \text{ if and only if } a = b.$$

Now, we wish to show that

$$(e,a)\,(\widetilde{\mathcal{R}}_{E_{\mathscr{M}}} \cap \sigma_{\mathscr{M}})\,(f,b) \text{ if and only if } (e,a) = (f,b)$$

and

$$(e,a)\,(\widetilde{\mathcal{L}}_{E_{\mathscr{M}}} \cap \sigma_{\mathscr{M}})\,(f,b) \text{ if and only if } (e,a) = (f,b).$$

We have

$$\begin{aligned}(e,a)\,(\widetilde{\mathcal{R}}_{E_{\mathscr{M}}} \cap \sigma_{\mathscr{M}})\,(f,b) &\Leftrightarrow (e,a)\,\widetilde{\mathcal{R}}_{E_{\mathscr{M}}}\,(f,b) \text{ and } (e,a)\,\sigma_{\mathscr{M}}\,(f,b)\\ &\Leftrightarrow e = f \text{ and } a = b\\ &\Leftrightarrow (e,a) = (f,b).\end{aligned}$$

We have

$$\begin{aligned}(e,a)\,(\widetilde{\mathcal{L}}_{E_{\mathscr{M}}} \cap \sigma_{\mathscr{M}})\,(f,b) &\Leftrightarrow (e,a)^* = (f,b)^* \text{ and } a = b\\ &\Leftrightarrow (e \circledcirc a, 1) = (f \circledcirc a, 1) \text{ and } a = b\\ &\Leftrightarrow e \circledcirc a = f \circledcirc a \text{ and } a = b\\ &\Leftrightarrow a \odot (e \circledcirc a) = a \odot (f \circledcirc a) \text{ and } a = b\\ &\Leftrightarrow e = f \text{ and } a = b\\ &\Leftrightarrow (e,a) = (f,b).\end{aligned}$$

Hence $\mathscr{M}$ is proper restriction.

We also wish to show $M/\sigma_{\mathscr{M}} \cong T$, where

$$M/\sigma_{\mathscr{M}} = \{(e,a)\sigma_{\mathscr{M}} : (e,a) \in \mathscr{M}\}.$$

We shall define $\theta : \mathscr{M}/\sigma_{\mathscr{M}} \to T$ by $[(e,a)\sigma_{\mathscr{M}}]\theta = a$ for $(e,a) \in \mathscr{M}$. Since $(e,a)\sigma_{\mathscr{M}}(f,b)$ if and only if $a = b$ for $(e,a),(f,b) \in \mathscr{M}$, it follows that θ is a well-defined, one-to-one morphism.

To show θ is also onto take $a \in T$. By Condition (C), $\exists e \in \mathscr{Y}$ such that $\exists e \circledcirc a$. So $(e,a) \in \mathscr{M}$ and $[(e,a)\sigma_{\mathscr{M}}]\theta = a$. So θ is an isomorphism and $M/\sigma_{\mathscr{M}} \cong T$.

Conversely, suppose we have a proper restriction semigroup S with distinguished semilattice E_S. By Theorem 5.3.6,

$$S \cong \mathscr{M}(T,\mathscr{X},\mathscr{Y}) = \{(e,a) \in \mathscr{Y} \times T : e \leq a \cdot \varepsilon\},$$

where T is a monoid with identity 1, $\mathscr{X}$ is a semilattice, $\mathscr{Y}$ is a subsemilattice of $\mathscr{X}$, ε is an element of $\mathscr{X}$ such that $a \leq \varepsilon$ for all $a \in \mathscr{Y}$ and T acts by morphisms on the left of $\mathscr{X}$ such that the triple $(T,\mathscr{X},\mathscr{Y})$ satisfies the following properties:

(A) for all $a \in T$, there exists $e \in \mathscr{Y}$ such that $e \leq a \cdot \varepsilon$;

(B) for all $e, f \in \mathscr{Y}$ and all $a \in T$,

$$e \leq a \cdot \varepsilon \Rightarrow e \wedge a \cdot f \text{ lies in } \mathscr{Y}.$$

Taking the left-right dual of the theorem,

$$S \cong \mathscr{M}'(T, \mathscr{X}', \mathscr{Y}) = \{(a, e) \in T \times \mathscr{Y} : e \leq \varepsilon' \circ a\},$$

where T is a monoid with identity 1, $\mathscr{X}'$ is a semilattice, $\mathscr{Y}$ is a subsemilattice of $\mathscr{X}'$, ε' is an element of $\mathscr{X}'$ such that $a \leq \varepsilon'$ for all $a \in \mathscr{Y}$ and T acts by morphisms on the left of $\mathscr{X}'$ such that $(T, \mathscr{X}', \mathscr{Y})$ satisfies the following properties:

(A) for all $a \in T$, there exists $e \in \mathscr{Y}$ such that $e \leq \varepsilon' \circ a$;

(B) for all $e, f \in \mathscr{Y}$ and all $a \in T$,

$$e \leq \varepsilon' \circ a \Rightarrow f \circ a \wedge e \text{ lies in } \mathscr{Y}.$$

Note that, from looking at the proof, there is no need for a separate monoid T' and subsemilattice $\mathscr{Y}'$ since they are taken to be the same in each of the left-right cases as $T \cong S/\sigma_S$ and $\mathscr{Y} \cong E_S$.

Proposition 8.3.2. *There is an isomorphism,*

$$\theta : \mathscr{M}(T, \mathscr{X}, \mathscr{Y}) \to \mathscr{M}'(T, \mathscr{X}', \mathscr{Y}),$$

such that $(e, a)\theta = (a, x)$*, for some* $x \in \mathscr{Y}$*, and* $(e, 1)\theta = (1, e)$ *for* $(e, a), (e, 1) \in \mathscr{M}(T, \mathscr{X}, \mathscr{Y})$*.*

Proof. As in the proof of Theorem 5.3.6, there is an isomorphism $\varphi : S \to \mathscr{M}(T, \mathscr{X}, \mathscr{Y})$ defined by

$$x\varphi = (x^+, x\sigma_S).$$

Taking the dual, there is an isomorphism $\phi : S \to \mathscr{M}'(T, \mathscr{X}', \mathscr{Y})$ defined by

$$x\phi = (x\sigma_S, x^*).$$

So there exists an isomorphism

$$\theta = \varphi^{-1}\phi : \mathscr{M}(T, \mathscr{X}, \mathscr{Y}) \to \mathscr{M}'(T, \mathscr{X}', \mathscr{Y}).$$

Let us consider $(k, x\sigma_S) \in \mathscr{M}(T, \mathscr{X}, \mathscr{Y})$. Then there exists an element of S, s say, such that

$$s\varphi = (k, x\sigma_S) = (s^+, s\sigma_S).$$

We have

$$\begin{aligned}(k, x\sigma_S)\theta &= (k, x\sigma_S)\varphi^{-1}\phi \\ &= (s^+, s\sigma_S)\varphi^{-1}\phi \\ &= s\phi \\ &= (s\sigma_S, s^*) \\ &= (x\sigma_S, s^*).\end{aligned}$$

Now let us consider $(e,1) \in E_{\mathscr{M}}$. We note that $1 = e\sigma_S$ for any $e \in E_S$. So, in particular,

$$e\varphi = (e,1)$$

and hence

$$\begin{aligned}(e,1)\theta &= (e,1)\varphi^{-1}\phi \\ &= e\phi \\ &= (1,e^*) \\ &= (1,e).\end{aligned}$$

$\square$

Similarly there is an isomorphism,

$$\psi = \theta^{-1} : \mathscr{M}'(T,\mathscr{X}',\mathscr{Y}) \to \mathscr{M}(T,\mathscr{X},\mathscr{Y}),$$

such that $(a,e)\psi = (x,a)$, for some $x \in \mathscr{Y}$, and $(1,e)\psi = (e,1)$.

Let us denote $\mathscr{M}(T,\mathscr{X},\mathscr{Y})$ by $\mathscr{M}$ and $\mathscr{M}'(T,\mathscr{X}',\mathscr{Y})$ by $\mathscr{M}'$.

As S is a left restriction semigroup, for each $(e,a) \in \mathscr{M}(T,\mathscr{X},\mathscr{Y})$, there is a unique idempotent in its $\widetilde{\mathcal{L}}_{E_{\mathscr{M}}}$-class, which we shall denote by

$$(e,a)^* = (e \circledcirc a, 1).$$

As S is also right restriction, for each $(a,e) \in \mathscr{M}'(T,\mathscr{X}',\mathscr{Y})$, there is a unique idempotent in its $\widetilde{\mathcal{R}}_{E_{\mathscr{M}}}$-class, which we shall denote by

$$(a,e)^+ = (1, a \odot e).$$

Proposition 8.3.3. *For* $(e,a) \in \mathscr{M}(T,\mathscr{X},\mathscr{Y})$,

$$(e,a)\theta = (a, e \circledcirc a),$$

and for $(a,e) \in \mathscr{M}'(T,\mathscr{X}',\mathscr{Y})$,

$$(a,e)\psi = (a \odot e, a).$$

Proof. Let $(e,a) \in \mathscr{M}(T,\mathscr{X},\mathscr{Y})$ and $(b,f) \in \mathscr{M}'(T,\mathscr{X}',\mathscr{Y})$. We have

$$(e,a)\,\widetilde{\mathcal{L}}_{E_{\mathscr{M}}}\,(e \circledcirc a, 1)$$

and

$$(b,f)\,\widetilde{\mathcal{L}}_{E_{\mathscr{M}'}}\,(1,f).$$

We know $(e,a)\theta = (a,x)$, for some $x \in \mathscr{Y}$ and

$$\begin{aligned}(e,a)\,\widetilde{\mathcal{L}}_{E_{\mathscr{M}}}\,(e \circledcirc a, 1) &\Rightarrow (e,a)\theta\,\widetilde{\mathcal{L}}_{E_{\mathscr{M}'}}\,(e \circledcirc a, 1)\theta \\ &\Rightarrow (a,x)\,\widetilde{\mathcal{L}}_{E_{\mathscr{M}'}}\,(1, e \circledcirc a) \\ &\Rightarrow (1,x) = (1, e \circledcirc a) \\ &\Rightarrow x = e \circledcirc a.\end{aligned}$$

Hence, $(e,a)\theta = (a, e \circledcirc a)$. Dually, $(b,f)\psi = (b \odot f, b)$. $\square$

As $(e,a)^+$ and $(b,f)^*$ are the unique idempotents in the $\widetilde{\mathcal{R}}_{E_{\mathscr{M}}}$-class of $(e,a) \in \mathscr{M}(T,\mathscr{X},\mathscr{Y})$ and the $\widetilde{\mathcal{L}}_{E_{\mathscr{M}'}}$-class of $(b,f) \in \mathscr{M}'(T,\mathscr{X}',\mathscr{Y})$, respectively, it follows that

$$\exists e \circledcirc a \text{ if and only if } e \leq a \cdot \varepsilon$$

and

$$\exists b \odot f \text{ if and only if } f \leq \varepsilon' \circ b.$$

One of the properties of the triple $(T,\mathscr{X},\mathscr{Y})$ is that for each $a \in T$, $\exists e \in \mathscr{Y}$ such that $e \leq a \cdot \varepsilon$. So for $t \in T$, $\exists e \in \mathscr{Y}$ such that $\exists e \circledcirc a$. Hence, Condition (C) is satisfied.

Proposition 8.3.4. *As defined above, $\circledcirc$ and $\odot$ are right and left partial actions on $\mathscr{Y}$ respectively such that T preserves the partial orders and the domain of each $t \in T$ is an order ideal.*

Proof. (i) We shall show that $\circledcirc$ is a right partial action on $\mathscr{Y}$. Let $y \in \mathscr{Y}$. As $y \leq \varepsilon$, then $\exists y \circledcirc 1$. We also have

$$(y,1)\,\widetilde{\mathcal{L}}_{E_{\mathscr{M}}}\,(y \circledcirc 1, 1) \Rightarrow y \circledcirc 1 = y$$

by uniqueness.

(ii) Suppose that $\exists y \circledcirc s$ and $\exists (y \circledcirc s) \circledcirc t$, where $s,t \in T$. So $(y \circledcirc s, t) \in \mathscr{M}(T,\mathscr{X},\mathscr{Y})$. We wish to show that $\exists y \circledcirc st$ and $y \circledcirc st = (y \circledcirc s) \circledcirc t$. We have

$$\begin{aligned}(y,s)\,\widetilde{\mathcal{L}}_{E_{\mathscr{M}}}\,(y \circledcirc s, 1) &\Rightarrow (y,s)(y \circledcirc s, 1) = (y,s) \\ &\Rightarrow (y \wedge s \cdot (y \circledcirc s), s) = (y,s) \\ &\Rightarrow y \wedge s \cdot (y \circledcirc s) = y.\end{aligned}$$

As $\widetilde{\mathscr{L}}_{E_{\mathscr{M}}}$ is a right congruence,

$$\begin{aligned}(y,s)\,\widetilde{\mathcal{L}}_{E_{\mathscr{M}}}\,(y \circledcirc s, 1) &\Rightarrow (y,s)(y \circledcirc s, t)\,\widetilde{\mathcal{L}}_{E_{\mathscr{M}}}\,(y \circledcirc s, 1)(y \circledcirc s, t) \\ &\Rightarrow (y \wedge s \cdot (y \circledcirc s), st)\,\widetilde{\mathcal{L}}_{E_{\mathscr{M}}}\,(y \circledcirc s \wedge y \circledcirc s, t) \\ &\Rightarrow (y, st)\,\widetilde{\mathcal{L}}_{E_{\mathscr{M}}}\,(y \circledcirc s, t) \\ &\Rightarrow (y \circledcirc st, 1)\,\widetilde{\mathcal{L}}_{E_{\mathscr{M}}}\,((y \circledcirc s) \circledcirc t, 1) \text{ since } \exists y \circledcirc st \\ &\Rightarrow y \circledcirc st = (y \circledcirc s) \circledcirc t.\end{aligned}$$

(iii) Now suppose that $y \leq z$ and $\exists z \circledcirc m$, i.e. $y \leq z$ and $z \leq m \cdot \varepsilon$. As $\leq$ is a partial order, $y \leq m \cdot \varepsilon$ and so $\exists y \circledcirc m$. So the domain of each $t \in T$ is an order ideal.

(iv) We now wish to show that T preserves this partial order. Let $y, z \in \mathscr{Y}$ and $m \in T$. Suppose that $y \leq z$, $\exists y \circledcirc m$ and $\exists z \circledcirc m$. We have

$$\begin{aligned}(y, m)(z \circledcirc m, 1) &= (y \wedge m \cdot (z \circledcirc m), m) \\ &= (y, m) \text{ since } y \leq z \leq m \cdot (z \circledcirc m) \text{ as in (ii).}\end{aligned}$$

So by the definition of $\widetilde{\mathcal{L}}_{E_{\mathscr{M}}}$,

$$(y \circledcirc m, 1) \leq (z \circledcirc m, 1),$$

which implies that $y \circledcirc m \leq z \circledcirc m$. Therefore, T preserves the partial order.

Dually we can show that $\odot$ is a left partial action on $\mathscr{Y}$. $\square$

We can see that the compatibility conditions (A) and (B) hold.

Proposition 8.3.5. *For $a \in T$ and $e \in \mathscr{Y}$,*

(A) if $\exists e \circledcirc a$, then $\exists a \odot (e \circledcirc a)$ and $a \odot (e \circledcirc a) = e$;

(B) if $\exists a \odot e$, then $\exists (a \odot e) \circledcirc a$ and $(a \odot e) \circledcirc a = e$.

Proof. (A) Suppose $\exists e \circledcirc a$. Then $e \leq a \cdot \varepsilon$ and hence $(e, a) \in \mathscr{M}(T, \mathscr{X}, \mathscr{Y})$. We have $(e, a)\theta \in \mathscr{M}'(T, \mathscr{X}', \mathscr{Y})$, i.e. $(a, e \circledcirc a) \in \mathscr{M}'(T, \mathscr{X}', \mathscr{Y})$. So $e \circledcirc a \leq \varepsilon' \circ a$, i.e. $\exists a \odot (e \circledcirc a)$. Also,

$$\begin{aligned}(e, a) &= (e, a)\theta\psi \\ &= (a, e \circledcirc a)\psi \\ &= (a \odot (e \circledcirc a), a),\end{aligned}$$

so $a \odot (e \circledcirc a) = e$. Dually, (B) holds. $\square$

We also have the following result.

Proposition 8.3.6. *Let $a \in T$ and $e \in \mathscr{Y}$. If $\exists e \circledcirc a$, then $a \odot (e \circledcirc a \wedge f) = e \wedge a \cdot f$ for $f \in \mathscr{Y}$.*

Proof. Suppose $\exists e \circledcirc a$ and $f \in \mathscr{Y}$. We have

$$(e, a)(f, 1) = (e \wedge a \cdot f, a),$$

but we also have

$$\begin{aligned}(e, a)(f, 1) &= [(e, a)(f, 1)]\theta\psi \\ &= [(a, e \circledcirc a)(1, f)]\psi \\ &= (a, e \circledcirc a \wedge f)\psi \\ &= (a \odot (e \circledcirc a \wedge f), a)\end{aligned}$$

and so
$$e \wedge a \cdot f = a \odot ((e \circledcirc a) \wedge f).$$

$\square$

We have shown that $\mathscr{M}(T, \mathscr{Y}) = \{(e, a) : \exists e \circledcirc a\}$ exists and that

$$S \cong \mathscr{M}(T, \mathscr{X}, \mathscr{Y}) = \{(e, a) : e \leq a \cdot \varepsilon\}$$

and as a set
$$\mathscr{M}(T, \mathscr{X}, \mathscr{Y}) = \{(e, a) : \exists e \circledcirc a\}.$$

Now, the binary operations defined on $\{(e, a) : e \leq a \cdot \varepsilon\}$ and $\{(e, a) : \exists e \circledcirc a\}$ are $(e, a)(f, b) = (e \wedge a \cdot f, ab)$ and $(e, a)(f, b) = (a \odot ((e \circledcirc a) \wedge f), ab)$ respectively. By Proposition 8.3.6,

$$e \wedge a \cdot f = a \odot ((e \circledcirc a) \wedge f)$$

when $\exists e \circledcirc a$ and so
$$S \cong \mathscr{M}(T, \mathscr{Y}).$$

Hence, every proper restriction semigroup is isomorphic to some $\mathscr{M}(T, \mathscr{Y})$.
$\square$

However, we believe this is the same structure presented in Section 4 of [36], but with a different proof and it elucidates the fact there are four actions at play. Although we did not obtain the desired result of having the converse of Proposition 7.2.5 concerning a double action, we showed that a proper restriction semigroup is isomorphic to a structure that does not involve a semilattice $\mathscr{X}$ and is based around partial actions, which have the following relationship with the original actions:

Proposition 8.3.7. *Let T, $\mathscr{Y}$, $\cdot$, $\circ$, $\odot$ and $\circledcirc$ be defined as in the converse proof of Theorem 8.3.1. For $a \in T$ and $e \in \mathscr{Y}$,*

(a) if $\exists e \circledcirc a$, then $e \circledcirc a \leq e \circ a$;

(b) if $\exists a \odot e$, then $a \odot e \leq a \cdot e$.

Proof. Let $a \in T$ and $e \in \mathscr{Y}$.

(a) Suppose $\exists e \circledcirc a$. Then

$$\begin{aligned}(e, 1)(e, a) = (e, a) &\Rightarrow (e, 1)\theta(e, a)\theta = (e, a)\theta \\ &\Rightarrow (1, e)(a, e \circledcirc a) = (a, e \circledcirc a) \\ &\Rightarrow (a, (e \circ a) \wedge (e \circledcirc a)) = (a, e \circledcirc a) \\ &\Rightarrow (e \circ a) \wedge (e \circledcirc a) = e \circledcirc a \\ &\Rightarrow e \circledcirc a \leq e \circ a.\end{aligned}$$

The proof of (b) is dual. $\square$

8.4 Structure theorems independent of one-sided results

We shall produce different proofs of the structure theorem in Section 8.3 reminiscent of that of Munn in the inverse case [45] that will not require use of the one-sided results in the proof.

Here we shall present an alternative proof of the converse of Theorem 8.3.1:

Theorem 8.4.1. *Every proper restriction semigroup S is isomorphic to some $\mathscr{M}(T, \mathscr{Y})$.*

Proof. Let S be a proper restriction semigroup. Take $\mathscr{Y} = E_S$ and $T = S/\sigma_S$. Then $\mathscr{Y}$ is a semilattice and T is a monoid. We shall define a partial action of T on the right of $\mathscr{Y}$ by

$$\exists e \circ m\sigma_S \Leftrightarrow \exists s \in S \text{ with } e = s^+ \text{ and } m\sigma_S = s\sigma_S,$$

in which case

$$e \circ m\sigma_S = s^+ \circ s\sigma_S = s^*.$$

This is well-defined since S is proper.

(i) For $s^+ \in \mathscr{Y}$, $\exists s^+ \circ 1$, i.e. $\exists s^+ \circ s^+\sigma_S$ and

$$s^+ \circ s^+\sigma_S = (s^+)^* = s^+.$$

(ii) Suppose $\exists s^+ \circ s\sigma_S$ and $\exists (s^+ \circ s\sigma_S) \circ t\sigma_S$ with

$$s^+ \circ s\sigma_S = s^*$$

and

$$(s^+ \circ s\sigma_S) \circ t\sigma_S = s^* \circ t\sigma_S.$$

So there exists $u \in S$ such that $s^* = u^+$ and $u\sigma_S = t\sigma_S$. Hence $\exists u^+ \circ u\sigma_S$ and

$$(s^+ \circ s\sigma_S) \circ t\sigma_S = u^+ \circ u\sigma_S = u^*.$$

We wish to show that $\exists s^+ \circ (st)\sigma_S$ and $s^+ \circ (st)\sigma_S = u^*$. We have $(su)^+ = (su^+)^+ = (ss^*)^+ = s^+$ and similarly $(su)^* = u^*$. So

$$u^* = (su)^* = (su)^+ \circ (su)\sigma_S = s^+ \circ (st)\sigma_S$$

as required. Therefore $\circ$ is a partial right action.

(iii) We shall show that the domain of each $t \in T$ is an order ideal. Take $y^+, z^+ \in \mathscr{Y}$ and $z\sigma_S \in T$. Suppose $y^+ \leq z^+$ and $\exists z^+ \circ z\sigma_S$, where $z^+ \circ z\sigma_S = z^*$. We wish to show $\exists y^+ \circ z\sigma_S$. We have

$$(y^+z)\sigma_S = (y^+\sigma_S)(z\sigma_S) = z\sigma_S$$

and

$$(y^+z)^+ = y^+z^+ = y^+$$

as $y^+ \leq z^+$. So $\exists (y^+z)^+ \circ (y^+z)\sigma_S$, i.e. $\exists y^+ \circ z\sigma_S$. Hence the domain of each $t \in T$ under $\circ$ is an order ideal.

(iv) We shall show that T preserves the partial order $\circ$. Let $y^+, z^+ \in \mathscr{Y}$ and $m\sigma_S \in T$ be such that $y^+ \leq z^+$, $\exists y^+ \circ m\sigma_S$ and $\exists z^+ \circ m\sigma_S$. So we can assume that $m\sigma_S = y\sigma_S = z\sigma_S$. We wish to show that

$$y^+ \circ m\sigma_S \leq z^+ \circ m\sigma_S,$$

i.e.

$$y^+ \circ y\sigma_S \leq z^+ \circ z\sigma_S,$$

i.e.

$$y^* \leq z^*.$$

Making use of Proposition 2.6.1, we have

$$\begin{array}{rcll} y\,\sigma_S\,z & \Rightarrow & y^+z = z^+y & \text{as } S \text{ is proper} \\ & \Rightarrow & y^+z = y & \text{as } z^+y = z^+y^+y = y^+y = y \\ & \Rightarrow & y \leq z & \\ & \Rightarrow & zy^* = y & \\ & \Rightarrow & y^* = (yz^*)^* = y^*z^* & \\ & \Rightarrow & y^* \leq z^*. & \end{array}$$

Hence T preserves $\circ$.

Let us define a partial left action of T on $\mathscr{Y}$ by

$$\exists m\sigma_S \cdot e \Leftrightarrow \exists s \in S \text{ with } e = s^* \text{ and } m\sigma_S = s\sigma_S,$$

in which case

$$m\sigma_S \cdot e = s\sigma_S \cdot s^* = s^+.$$

The proof that $\cdot$ is a partial left action where T preserves the partial order and the domain of each $t \in T$ is an order ideal, is dual to the right case.

(A) Suppose $\exists e \circ m\sigma_S$. Then $e = s^+$ and $m\sigma_S = s\sigma_S$ for some $s \in S$ and $e \circ m\sigma_S = s^+ \circ s\sigma_S = s^*$. Then $\exists m\sigma_S \cdot (e \circ m\sigma_S)$ and

$$m\sigma_S \cdot (e \circ m\sigma_S) = s\sigma_S \cdot (s^+ \circ s\sigma_S) = s\sigma_S \cdot s^* = s^+ = e.$$

Condition (B) holds dually.

(C) Let $m\sigma_S \in T$. Then $m \in S$ and as S is a restriction semigroup, $m^+ \in E_S$. So $\exists m^+ \circ m\sigma_S$. Hence $(T, \mathscr{Y})$ is a strong M-pair.

Let $\theta : S \to \mathscr{M}(T, \mathscr{Y})$ be defined by

$$s\theta = (s^+, s\sigma_S).$$

Then

$$\text{im } \theta = \{(s^+, m\sigma_S) : \exists s^+ \circ m\sigma_S\} = \mathscr{M}(T, \mathscr{Y}).$$

Taking $s, t \in S$, we have

$$\begin{aligned} s\theta t\theta &= (s^+, s\sigma_S)(t^+, t\sigma_S) \\ &= (s\sigma_S \cdot ((s^+ \circ s\sigma_S)t^+), s\sigma_S t\sigma_S) \\ &= (s\sigma_S \cdot (s^* t^+), (st)\sigma_S). \end{aligned}$$

To show that θ is a morphism, we need to show $(st)^+ = s\sigma_S \cdot (s^* t^+)$. Let $u = st^+$. Then

$$u^+ = (st^+)^+ = (st)^+,$$

$$u\sigma_S = (st^+)\sigma_S = s\sigma_S t^+\sigma_S = s\sigma_S$$

and

$$u^* = (st^+)^* = (s^* t^+)^* = s^* t^+.$$

As $u^+ = u\sigma_S \cdot u^*$, we have $(st)^+ = s\sigma_S \cdot (s^* t^+)$. Therefore θ is a morphism.

Note that θ is one-to-one since

$$\begin{aligned} s\theta = t\theta &\Rightarrow (s^+, s\sigma_S) = (t^+, t\sigma_S) \\ &\Rightarrow s^+ = t^+ \text{ and } s\sigma_S = t\sigma_S \\ &\Rightarrow s = t, \end{aligned}$$

since S is proper. Also,

$$s^+\theta = (s^+, s^+\sigma_S) = (s^+, 1) = (s^+, s\sigma_S)^+ = (s\theta)^+$$

and

$$s^*\theta = (s^*, s^*\sigma_S) = (s^*, 1) = (s^+ \circ s\sigma_S, 1) = (s^+, s\sigma_S)^* = (s\theta)^*.$$

Therefore $\theta : S \to \text{im } \theta$ is an isomorphism and so $S \cong \mathscr{M}(T, \mathscr{Y})$.

$\square$

8.5 Symmetrical two-sided structure theorems for proper weakly ample, proper ample and proper inverse semigroups

We will adapt the structure theorem for proper restriction semigroups to produce symmetrical structure theorems for proper weakly ample, proper ample and proper inverse semigroups.

Corollary 8.5.1. *If T is unipotent, $\mathscr{M} = \mathscr{M}(T, \mathscr{Y})$ is a proper weakly ample semigroup. Conversely, every proper weakly ample semigroup S is* *isomorphic to some $\mathscr{M}(T, \mathscr{Y})$, where T is unipotent.*

Proof. By Theorem 8.3.1, $\mathscr{M}(T, \mathscr{Y})$ is a proper restriction semigroup with distinguished semilattice

$$E_{\mathscr{M}} = \{(e, 1) : e \in \mathscr{Y}\}.$$

Considering $(e, a) \in E(\mathscr{M}(T, \mathscr{Y}))$,

$$\begin{aligned}(e, a)(e, a) = (e, a) &\Rightarrow (a \odot ((e \circledcirc a) \wedge e), a^2) = (e, a) \\ &\Rightarrow a^2 = a.\end{aligned}$$

Since T is unipotent, $a = 1$ and so $E(\mathscr{M}(T, \mathscr{Y})) = \{(e, 1) : e \in \mathscr{Y}\}$. Therefore $\mathscr{M}(T, \mathscr{Y})$ is a proper weakly ample semigroup.

Conversely, let S be a proper weakly ample semigroup. As S is a proper restriction semigroup, by Theorem 8.3.1,

$$S \cong \mathscr{M}(T, \mathscr{Y}),$$

where T is a monoid and $S/\sigma \cong T$. As in the proof of Corollary 6.3.8, T is a unipotent monoid. □

Restricting T to be a cancellative monoid, we also obtain a structure theorem for proper ample semigroups.

Theorem 8.5.2. *[36] If T is cancellative, $\mathscr{M} = \mathscr{M}(T, \mathscr{Y})$ is a proper ample semigroup. Conversely, every proper ample semigroup S is isomorphic to some $\mathscr{M}(T, \mathscr{Y})$, where T is cancellative.*

Proof. If T is cancellative, it is unipotent and so $\mathscr{M}(T, \mathscr{Y})$ is a proper weakly ample semigroup, where $\mathscr{M}/\sigma \cong T$, by Theorem 8.3.1.

We wish to show that for $(e, a) \in \mathscr{M}(T, \mathscr{Y})$,

$$(e, a) \mathscr{R}^* (e, 1).$$

We have for all $(x, c), (z, d) \in \mathscr{M}(T, \mathscr{Y})$,

$$\begin{aligned}(x, c)(e, a) = (z, d)(e, a) &\Rightarrow (c \odot (x \circledcirc c \wedge e), ca) = (d \odot (z \circledcirc d \wedge e), da) \\ &\Rightarrow c \odot (x \circledcirc c \wedge e) = d \odot (z \circledcirc d \wedge e) \text{ and } ca = da \\ &\Rightarrow c \odot (x \circledcirc c \wedge e) = d \odot (z \circledcirc d \wedge e) \text{ and } c = d \\ &\Rightarrow (c \odot (x \circledcirc c \wedge e), c) = (d \odot (z \circledcirc d \wedge e), d) \\ &\Rightarrow (x, c)(e, 1) = (z, d)(e, 1).\end{aligned}$$

Since $(e, a) \widetilde{\mathscr{R}}_E (e, 1)$, by Proposition 2.4.2 we have

$$(e, a) \mathscr{R}^* (e, 1).$$

We also wish to show that for $(e, a) \in \mathscr{M}(T, \mathscr{Y})$,

$$(e, a) \mathscr{L}^* (e \circledcirc a, 1).$$

For all $(x, c), (z, d) \in \mathscr{M}(T, \mathscr{Y})$,

$$\begin{aligned}
(e, a)(x, c) = (e, a)(z, d) &\Rightarrow (a \odot (e \circledcirc a \wedge x), ac) = (a \odot (e \circledcirc a \wedge z), ad) \\
&\Rightarrow a \odot (e \circledcirc a \wedge x) = a \odot (e \circledcirc a \wedge z) \text{ and } ac = ad \\
&\Rightarrow a \odot (e \circledcirc a \wedge x) = a \odot (e \circledcirc a \wedge z) \text{ and } c = d \\
&\Rightarrow [a \odot (e \circledcirc a \wedge x)] \circledcirc a = [a \odot (e \circledcirc a \wedge z)] \circledcirc a \\
&\qquad \text{and } c = d \\
&\Rightarrow e \circledcirc a \wedge x = e \circledcirc a \wedge z \text{ and } c = d \\
&\Rightarrow (e \circledcirc a \wedge x, c) = (e \circledcirc a \wedge z, d) \\
&\Rightarrow (e \circledcirc a, 1)(x, c) = (e \circledcirc a, 1)(z, d).
\end{aligned}$$

By the dual of Proposition 2.4.2, $(e, a) \mathscr{L}^* (e \circledcirc a, 1)$ as $(e, a) \widetilde{\mathscr{L}_E} (e \circledcirc a, 1)$. So $\mathscr{M}(T, \mathscr{Y})$ is ample and it follows from Corollary 2.4.10 and its dual that it is a proper ample semigroup.

Conversely, a proper ample semigroup S is isomorphic to some $\mathscr{M}(T, \mathscr{Y})$ where $T \cong S/\sigma$ due to Theorem 8.3.1. It follows from the fact that S is ample that T is cancellative as in Corollary 5.3.8.

□

Definition 8.5.3. A group G acts *partially* on the right of a set X if it acts partially as a monoid and if, in addition, for any $g \in G$ and $x \in X$, if $\exists x \circ g$, then $\exists (x \circ g) \circ g^{-1}$ and $(x \circ g) \circ g^{-1} = x$.

Whenever we talk explicitly of groups acting partially, we will assume the partial action is subject to this extra condition.

Theorem 8.5.4. *[48] If T is a group, $\mathscr{M} = \mathscr{M}(T, \mathscr{Y})$ is a proper inverse semigroup. Conversely, every proper inverse semigroup S is isomorphic to some $\mathscr{M}(T, \mathscr{Y})$, where T is a group.*

Proof. If T is a group, it is a cancellative monoid and so by Corollary 8.5.2, $\mathscr{M}(T, \mathscr{Y})$ is a proper ample semigroup and $M/\sigma \cong T$.

From Definition 8.5.3, $\exists (e \circledcirc a) \circledcirc a^{-1}$ as $\exists e \circledcirc a$, and so it follows that we have $(e \circledcirc a, a^{-1}) \in \mathscr{M}(T, \mathscr{Y})$.

We have

$$\begin{aligned}
(e, a)(e \circledcirc a, a^{-1})(e, a) &= (a \odot (e \circledcirc a \wedge e \circledcirc a), aa^{-1})(e, a) \\
&= (a \odot (e \circledcirc a), 1)(e, a) \\
&= (e, 1)(e, a) \\
&= (e, a).
\end{aligned}$$

So $\mathscr{M}(T, \mathscr{Y})$ is regular and as $E(S)$ is a semilattice, $\mathscr{M}(T, \mathscr{Y})$ is an inverse semigroup.

Before showing $(e \circledcirc a, a^{-1})$ is the inverse of (e, a), we note that

$$a \odot (e \circledcirc a) = e \text{ and } a \odot (a^{-1} \odot e) = e,$$

which exist since $\exists e \circledcirc a$ and $\exists 1 \odot e$. So

$$a \odot (e \circledcirc a) = a \odot (a^{-1} \odot e).$$

Hence $a^{-1}a \odot (e \circledcirc a) = a^{-1}a \odot (a^{-1} \odot e)$, i.e. $e \circledcirc a = a^{-1} \odot e$.

Therefore we have

$$\begin{aligned}(e \circledcirc a, a^{-1})(e, a)(e \circledcirc a, a^{-1}) &= (a^{-1} \odot (e \circledcirc aa^{-1} \wedge e), 1)(e \circledcirc a, a^{-1}) \\ &= (a^{-1} \odot e, 1)(e \circledcirc a, a^{-1}) \\ &= (a^{-1} \odot e \wedge e \circledcirc a, a^{-1}) \\ &= (e \circledcirc a, a^{-1}).\end{aligned}$$

So $(e, a)' = (e \circledcirc a, a^{-1})$. It follows from Proposition 2.2.3 and Corollary 8.5.2 that S is proper.

Conversely, let S be a proper inverse semigroup. Then S is a proper ample semigroup and so is isomorphic to some $\mathscr{M}(T, \mathscr{Y})$ where $T \cong S/\sigma$ is a cancellative monoid, by Corollary 8.5.2.

It remains to show that T is a group and T acts partially as a group on $E(S)$. Taking $e \in E(S)$,

$$(e\sigma)(a\sigma) = a\sigma = (a\sigma)(e\sigma),$$

for any $a \in S$. So $e\sigma = 1_T$ for any $e \in E(S)$. Taking $s \in S$, $s' \in S$ and

$$(s\sigma)(s'\sigma) = (ss'\sigma) = 1_T = (s's\sigma) = (s'\sigma)(s\sigma).$$

So $(s\sigma)^{-1} = (s'\sigma)$. Hence T is a group.

Notice that if $\exists t\sigma \cdot e$, then $t\sigma = s\sigma$ and $e = s^* = s^{-1}s$ for some $s \in S$. Now $t\sigma \cdot e = s\sigma \cdot s^* = s^+ = ss^{-1}$. We have $(t\sigma)^{-1} = (s\sigma)^{-1} = s^{-1}\sigma$, and $(s^{-1})^* = ss^{-1} = s^+$, so $\exists (t\sigma)^{-1} \cdot (t\sigma \cdot e) = s^{-1}\sigma \cdot (s^{-1})^* = (s^{-1})^+ = s^{-1}s = e$. The dual argument finishes the proof.

□

8.6 A covering theorem

Since every restriction semigroup has a proper restriction cover by Theorem 6.1.4, we can deduce the following result using Theorem 8.3.1. However we now give a direct proof.

Theorem 8.6.1. *Every restriction semigroup S has a proper restriction cover of the form $\mathscr{M}(T,\mathscr{Y})$, where $(T,\mathscr{Y})$ is a strong M-pair, and $\mathscr{Y} \cong E$.*

Proof. First we shall consider a restriction monoid S. We define 'partial' left and right actions of S on E by

$$\exists s \cdot e \text{ if and only if } e \leq s^*, \text{ in which case } s \cdot e = (se)^+$$

and

$$\exists e \circ s \text{ if and only if } e \leq s^+, \text{ in which case } e \circ s = (es)^*$$

for $e \in E$ and $s \in S$.

For $e \in E$, $\exists 1 \cdot e$ as $e \leq 1^* = 1$, and $1 \cdot e = (1e)^+ = e^+ = e$. Similarly $\exists e \circ 1$ and $e \circ 1 = e$. Let $s, t \in S$ and $e \in E$. Suppose $\exists s \cdot e$ and $\exists t \cdot (s \cdot e)$. So $e \leq s^*$, $s \cdot e = (se)^+$, $(se)^+ \leq t^*$ and $t \cdot (s \cdot e) = (t(se)^+)^+$. We wish to show $\exists ts \cdot e$ and $t \cdot (s \cdot e) = ts \cdot e$. We have

$$\begin{aligned}
(ts)^* e &= (tse)^* \\
&= (t^*(se)^+ se)^* \\
&= ((se)^+ se)^* && \text{as } (se)^+ \leq t^* \\
&= (se)^* \\
&= s^* e \\
&= e && \text{as } e^+ \leq s^*.
\end{aligned}$$

Hence $\exists ts \cdot e$. We also have

$$t \cdot (s \cdot e) = t \cdot (se)^+ = (t(se)^+)^+ = (tse)^+ = ts \cdot e.$$

Hence $\cdot$ is a partial left action and similarly, $\circ$ is a partial right action.

Let $e, f \in E$ and $s \in S$. Suppose $e \leq f$ and $\exists s \cdot f$, so $f \leq s^*$. We have $e \leq f \leq s^*$, so $\exists s \cdot e$. Similarly we have the dual for the partial right action, so the domain of each element of S is an order ideal.

Let $e, f \in E$ and $s \in S$. Suppose $e \leq f$, $\exists s \cdot f$ and $\exists s \cdot e$, so $f \leq s^*$ and $e \leq s^*$. We wish to show $s \cdot e \leq s \cdot f$, i.e. $(se)^+ \leq (sf)^+$. We have

$$(se)^+(sf)^+ = ((sf)^+ se)^+ = (sfe)^+ = (sf)^+.$$

Hence $s \cdot e \leq s \cdot f$. Similarly for the partial right action. So the action of S preserves the partial order in E.

Let $e \in E$ and $s \in S$. Suppose $\exists s \cdot e$. So $e \leq s^*$ and $s \cdot e = (se)^+$. We wish to show $\exists (s \cdot e) \circ s$, i.e. $(se)^+ \leq s^+$, and $(s \cdot e) \circ s = e$. We have

$$(se)^+ s^+ = ((se)^+ s^+)^+ = ((se)^+ s)^+ = (se)^+.$$

So $\exists (s \cdot e) \circ s$. We also have

$$(s \cdot e) \circ s = (se)^+ \circ s = ((se)^+ s)^* = (se)^* s^* e = e.$$

Dually Condition (A) holds.

For $s \in S$, $s^+ \in E$ which implies $\exists s^+ \circ s$, so (S, E) is a strong M-pair and we can construct the proper restriction semigroup

$$\mathscr{M} = \mathscr{M}(S, E) = \{(e, s) \in E \times S : \exists e \circ s\} = \{(e, s) \in E \times S : e \leq s^+\},$$

with binary operation given by

$$(e, s)(f, t) = (s \cdot ((e \circ s) \wedge f), st) = ((s(es)^* f)^+, st)$$

for $(e, s), (f, t) \in \mathscr{M}$.

Let us define $\theta : \mathscr{M} \to S$ by $(e, s)\theta = es$ for $(e, s) \in \mathscr{M}$. For any $s \in S$, $(s^+, s) \in \mathscr{M}$ and $(s^+, s)\theta = s$, so θ is onto. We also have

$$((e, s)(f, t))\theta = (s(es)^* f)^+ st = s(es)^* ft = esft = (e, s)\theta(f, t)\theta$$

for $(e, s), (f, t) \in \mathscr{M}$. For $(e, s) \in \mathscr{M}$,

$$(e, s)^+\theta = (e, 1)\theta = e = es^+ = (es)^+ = [(e, s)\theta]^+$$

and

$$(e, s)^*\theta = (e \circ s, 1)\theta = e \circ s = (es)^* = [(e, s)\theta]^*.$$

Clearly θ is $E_{\mathscr{M}}$-separating, so $\mathscr{M}$ is a proper cover of S.

Now consider a restriction semigroup S with distinguished semilattice E. As S^1 is a restriction monoid with distinguished semilattice E^1,

$$\mathscr{M}' = \mathscr{M}(S^1, E^1) = \{(e, s) \in E^1 \times S^1 : e \leq s^+\}$$

is a proper restriction monoid and $\theta : \mathscr{M}' \to S^1$, as defined above, is a covering morphism. Let

$$\mathscr{N} = \{(e, s) \in E \times S^1 : \exists e \circ s\} = \{(e, s) \in E \times S^1 : e \leq s^+\} \subseteq \mathscr{M}'.$$

Then $\mathscr{N}$ is a $(2, 1, 1)$-subalgebra of $\mathscr{M}'$ as, for $(e, s), (f, t) \in \mathscr{N}$,

$$(e, s)(f, t) = ((s(es)^* f)^+, st) \in \mathscr{N},$$

$$(e, s)^+ = (e, 1) \in \mathscr{N}$$

and

$$(e, s)^* = (e \circ s, 1) \in \mathscr{N}$$

as $e \circ s = (es)^* \in S$. Hence $\mathscr{N}$ is a restriction semigroup with distinguished semilattice $E_{\mathscr{N}} = \{(e, 1) : e \in E\}$. As $\mathscr{M}'$ is proper restriction, it follows that $\mathscr{N}$ is also proper. As θ restricted to $\mathscr{N}$ is a $(2, 1, 1)$-morphism and

$$s = (s^+, s)\theta \in \mathscr{N}\theta$$

for any $s \in S$, $\mathscr{N}$ is a proper cover for S. $\square$

Chapter 9

Graph expansions

In this chapter we generalise results from [20], [22] and [23]. Although many of the proofs are similar, we provide them for completeness, in parts using the existing work as a template. We shall show using graph expansions that the class of left restriction monoids having a proper cover over a variety of monoids is a variety of left restriction monoids. We shall also produce the results for restriction semigroups, as well as for left restriction semigroups, in Chapter 10 using Petrich and Reilly's approach.

9.1 Definitions

Taking a strong partial action, it is possible to produce an action through the *expansion* of a monoid [29]. We have already seen one such expansion of a monoid, the Szendrei expansion. We shall consider another known as a *graph expansion.* To define a graph expansion we require a number of steps.

Let X be a set, S a monoid and $f : X \to S$ such that Xf generates S as a monoid. Following the usual, but non-standard terminology in this area, we call (X, f, S) a *monoid presentation.*

We shall define the graph expansion of a monoid presentation, but first need some definitions.

We let $\Gamma = \Gamma(X, f, S)$ be the Cayley graph of (X, f, S). This has *vertices* S, denoted $V(\Gamma) = S$. The set of *edges* is denoted by $E(\Gamma)$ and consists of triples $(s, x, s(xf))$ where $s \in S$ and $x \in X$. The edge $(s, x, s(xf))$ has *initial vertex* s, *terminal vertex* $s(xf)$ and is represented pictorially by

$$\begin{tikzcd} \underset{s}{\bullet} \arrow[r, "x"] & \underset{s(xf)}{\bullet} \end{tikzcd}$$

We denote the initial and terminal vertices of an edge $e \in E(\Gamma)$ by $i(e)$ and $t(e)$ respectively.

Let Δ be a graph such that $V(\Delta) \subseteq V(\Gamma)$, $E(\Delta) \subseteq E(\Gamma)$ and the initial and terminal vertices of an edge in Δ are those of the edges in Γ. Then Δ is a *subgraph* of the Cayley graph Γ.

We say there is a *path* between $a, b \in V(\Gamma)$, where a is the initial vertex, if there is a sequence of edges, labelled by $x_1, x_2, ..., x_n$, such that

$$\begin{tikzcd} a \arrow[r, "x_1"] & a(x_1f) \arrow[r, "x_2"] & a(x_1f)(x_2f) \arrow[r, dotted, no head] & \bullet \arrow[r, "x_n"] & b \end{tikzcd}$$

where $b = a(x_1f)(x_2f)...(x_nf)$.

A subgraph Δ is said to be *a-rooted* if there is a path in Δ from the vertex $a \in S$ to every other vertex in the subgraph. In particular, a subgraph is *1-rooted* if there is a path from 1 to every other vertex in the subgraph; we shall denote a path from 1 to a, where a is an element of S, by P_a where it exists. Note that P_a is not necessarily uniquely determined by a.

A monoid S *acts* on a graph Γ on the left if S acts on $V(\Gamma)$ and $E(\Gamma)$ such that

$$i(se) = si(e) \text{ and } t(se) = st(e).$$

We shall define an action of the monoid S on Γ by $t \cdot v = tv$ for $t \in S$ and $v \in V(\Delta)$ and an edge $(s, x, s(xf))$ is taken to $(ts, x, ts(xf))$, i.e. the edge

$$\begin{tikzcd} s \arrow[r, "x"] & s(xf) \end{tikzcd}$$

becomes

$$\begin{tikzcd} ts \arrow[r, "x"] & ts(xf). \end{tikzcd}$$

Note that the action of S takes subgraphs to subgraphs and a-rooted subgraphs to sa-rooted subgraphs.

Let Δ and Σ be two finite subgraphs. Then their union is the subgraph created by taking vertices $V(\Delta \cup \Sigma) = V(\Delta) \cup V(\Sigma)$ and edges $E(\Delta \cup \Sigma) = E(\Delta) \cup E(\Sigma)$.

Definition 9.1.1. Let Γ_f be the set of finite 1-rooted subgraphs of Γ. Then the *graph expansion* of (X, f, S) is defined by

$$\mathcal{M} = \mathcal{M}(X, f, S) = \{(\Delta, s) : \Delta \in \Gamma_f, s \in V(\Delta)\},$$

with binary operation

$$(\Delta, s)(\Sigma, t) = (\Delta \cup s\Sigma, st)$$

and unary operation

$$(\Delta, s)^+ = (\Delta, 1)$$

for $(\Delta, s), (\Sigma, t) \in \mathcal{M}$.

Generalising results in [22] we can deduce the following result:

Proposition 9.1.2. *Let (X, f, S) be a monoid presentation. Then $\mathcal{M} = \mathcal{M}(X, f, S)$ is a proper left restriction monoid, where*

$$(\Delta, s)\,\widetilde{\mathcal{R}}_E\,(\Sigma, t) \Leftrightarrow \Delta = \Sigma$$

and

$$(\Delta, s)\,\sigma_{\mathcal{M}}\,(\Sigma, t) \Leftrightarrow s = t$$

for $(\Delta, s), (\Sigma, t) \in \mathcal{M}$.

Proof. It is shown in [22] that $\mathcal{M} = \mathcal{M}(X, f, S)$ is a monoid, but we shall verify this in detail.

Taking $(\Delta, s), (\Sigma, t) \in \mathcal{M}$, we know Δ and Σ are 1-rooted finite subgraphs of Γ where $s \in \Delta$ and $t \in \Sigma$. We wish to show that $(\Delta, s)(\Sigma, t) \in \mathcal{M}$, i.e. that $(\Delta \cup s\Sigma, st) \in \mathcal{M}$. Clearly $\Delta \cup s\Sigma$ is a finite subgraph of Γ. For $v \in V(\Delta)$ there is a path from 1 to v. For each $v \in V(s\Sigma)$ there is a path from s to v since Σ is 1-rooted as $s\Sigma$ is s-rooted. We also note there is a path in Δ from 1 to s since $s \in \Delta$ and Δ is 1-rooted. Therefore there is a path from 1 to v for each $v \in V(\Delta \cup s\Sigma)$ and so $\Delta \cup s\Sigma$ is 1-rooted. As $t \in V(\Sigma), st \in V(\Delta \cup s\Sigma)$. Hence the binary operation defined on $\mathcal{M}$ is closed.

By consideration of vertices and edges, we can see that

$$s(\Sigma \cup \Theta) = s\Sigma \cup s\Theta$$

for all $\Sigma, \Theta \in \Gamma_f$ and $s, t \in S$. The binary operation defined on $\mathcal{M}$ is associative since for $(\Delta, s), (\Sigma, t), (\Theta, u) \in \mathcal{M}$,

$$\begin{aligned}
[(\Delta, s)(\Sigma, t)](\Theta, u) &= (\Delta \cup s\Sigma, st)(\Theta, u) \\
&= ((\Delta \cup s\Sigma) \cup st\Theta, (st)u) \\
&= (\Delta \cup s(\Sigma \cup t\Theta), s(tu)) \\
&= (\Delta, s)(\Sigma \cup t\Theta, tu) \\
&= (\Delta, s)[(\Sigma, t)(\Theta, u)].
\end{aligned}$$

Taking $(\Delta, s) \in \mathcal{M}$, we can see the identity of $\mathcal{M}$ is $(\bullet_1, 1)$, where $\bullet_1$ is a subgraph consisting of only the vertex 1. Since 1 and s are vertices of Δ,

$$\begin{aligned}
(\bullet_1, 1)(\Delta, s) &= (\bullet_1 \cup \Delta, s) \\
&= (\Delta, s)
\end{aligned}$$

and

$$\begin{aligned}(\Delta, s)(\bullet_1, 1) &= (\Delta \cup s\bullet_1, s)\\ &= (\Delta \cup \bullet_s, s)\\ &= (\Delta, s).\end{aligned}$$

Therefore $\mathcal{M}$ is a monoid with identity $(\bullet_1, 1)$.

Let $E = \{(\Theta, 1) : (\Theta, 1) \in \mathcal{M}\}$, i.e.

$$E = \{(\Theta, 1) : \Theta \text{ is a finite 1-rooted subgraph of the Cayley graph}\}.$$

Every element of E is an idempotent, since for $(\Theta, 1) \in E$, we have

$$(\Theta, 1)(\Theta, 1) = (\Theta \cup \Theta, 1) = (\Theta, 1).$$

Taking $(\Theta, e) \in E(\mathcal{M})$, we have

$$\begin{aligned}(\Theta, e)(\Theta, e) = (\Theta, e) &\Rightarrow (\Theta \cup e\Theta, e^2) = (\Theta, e)\\ &\Rightarrow e^2 = e,\end{aligned}$$

but we cannot deduce that $e = 1$ without further restrictions on S.

As

$$\begin{aligned}(\Delta, 1)(\Sigma, 1) &= (\Delta \cup \Sigma, 1)\\ &= (\Sigma \cup \Delta, 1)\\ &= (\Sigma, 1)(\Delta, 1),\end{aligned}$$

we have that E is a semilattice.

For $(\Delta, s) \in \mathcal{M}$, we have $(\Delta, s)\,\widetilde{\mathcal{R}}_E\,(\Delta, 1)$ since

$$(\Delta, 1)(\Delta, s) = (\Delta \cup \Delta, s) = (\Delta, s)$$

and for all $(\Sigma, 1) \in E$,

$$\begin{aligned}(\Sigma, 1)(\Delta, s) = (\Delta, s) &\Rightarrow (\Sigma \cup \Delta, s) = (\Delta, s)\\ &\Rightarrow \Sigma \cup \Delta = \Delta\\ &\Rightarrow (\Sigma \cup \Delta, 1) = (\Delta, 1)\\ &\Rightarrow (\Sigma, 1)(\Delta, 1) = (\Delta, 1).\end{aligned}$$

Therefore, $(\Delta, s)\,\widetilde{\mathcal{R}}_E\,(\Delta, 1)$.

Before showing that $\widetilde{\mathcal{R}}_E$ is a left congruence, we note that

$$\begin{aligned}(\Delta, s)\,\widetilde{\mathcal{R}}_E\,(\Sigma, t) &\Leftrightarrow (\Delta, s)^+ = (\Sigma, t)^+\\ &\Leftrightarrow (\Delta, 1) = (\Sigma, 1)\\ &\Leftrightarrow \Delta = \Sigma.\end{aligned}$$

Let $(\Delta, s)\,\widetilde{\mathcal{R}}_E\,(\Sigma, t)$. Then $\Delta = \Sigma$. We wish to show that for any $(\Theta, u) \in \mathcal{M}$,

$$(\Theta, u)(\Delta, s)\,\widetilde{\mathcal{R}}_E\,(\Theta, u)(\Sigma, t),$$

i.e.

$$(\Theta \cup u\Delta, us)\,\widetilde{\mathcal{R}}_E\,(\Theta \cup u\Sigma, ut).$$

We have $\Theta \cup u\Delta = \Theta \cup u\Sigma$ since $\Delta = \Sigma$ and so by the previous result $\widetilde{\mathcal{R}}_E$ is a left congruence.

The left ample condition holds, since for $(\Delta, s) \in \mathcal{M}$ and $(\Sigma, 1) \in E$,

$$\begin{aligned}((\Delta, s)(\Sigma, 1))^+(\Delta, s) &= (\Delta \cup s\Sigma, s)^+(\Delta, s)\\ &= (\Delta \cup s\Sigma, 1)(\Delta, s)\\ &= (\Delta \cup s\Sigma \cup \Delta, s)\\ &= (\Delta \cup s\Sigma, s)\\ &= (\Delta, s)(\Sigma, 1).\end{aligned}$$

We conclude that $\mathcal{M}$ is indeed a left restriction monoid. As $(\Delta, s)\,\widetilde{\mathcal{R}}_E\,(\Sigma, t)$ implies $\Delta = \Sigma$, it remains to show that $(\Delta, s)\,\sigma_{\mathcal{M}}\,(\Sigma, t)$ if and only if $s = t$. We have

$$\begin{aligned}(\Delta, s)\,\sigma_{\mathcal{M}}\,(\Sigma, t) &\Leftrightarrow (\Theta, 1)(\Delta, s) = (\Theta, 1)(\Sigma, t) \text{ for some } (\Theta, 1) \in E\\ &\Leftrightarrow (\Theta \cup \Delta, s) = (\Theta \cup \Sigma, t)\\ &\qquad\quad \text{for some finite 1-rooted subgraph } \Theta\\ &\Rightarrow s = t.\end{aligned}$$

Conversely, if we suppose $s = t$ we obtain $(\Delta, s)\,\sigma_{\mathcal{M}}\,(\Sigma, t)$ by consideration of $(\Delta \cup \Sigma, 1) \in E$. Therefore, $\mathcal{M}$ is proper. $\square$

Imposing the condition that S is unipotent, gives us one direction of the following result:

Proposition 9.1.3. *[20] A graph expansion $\mathcal{M}(X, f, S)$ is a weakly left ample monoid if and only if S is a unipotent monoid.*

Restricting S further gives us one direction of the following result that gives us a necessary and sufficient condition for a graph expansion to be left ample.

Proposition 9.1.4. *[22] A graph expansion $\mathcal{M}(X, f, S)$ is a left ample monoid if and only if S is right cancellative.*

Let (X, f, G) be a *group presentation* of a group G as defined in [40]. The definition of $\mathcal{M}(X, f, G)$ is slightly different from that for monoids due to the consideration of inverses. An edge

$$g \xrightarrow{\;y\;} g(yf)$$

can also be considered as

$$g \xleftarrow{\;y^{-1}\;} g(yf).$$

As defined and proved in [40], $\mathcal{M}(X, f, G)$ is an inverse monoid.

9.2 The categories $\mathbf{PLR}(X)$ and $\mathbf{PLR}(X, f, S)$

We shall define the categories $\mathbf{PLR}(X)$ and $\mathbf{PLR}(X, f, S)$, where $\mathcal{PLR}$ is the class of proper left restriction monoids, and continue to generalise results by Gomes and Gould in [20], [22] and [23].

Definition 9.2.1. [22] Let X be a set and $\mathcal{A}$ a class of algebras of a given fixed type. Then $\mathbf{A}(X)$ is the category which has objects pairs (g, A) where $A \in \mathcal{A}, g : X \to A$ and $\langle Xg \rangle = A$; a morphism in $\mathbf{A}(X)$ from (g, A) to (h, B) is a morphism $\theta : A \to B$ such that

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[dr, "h"] & \\ A \arrow[rr, "\theta"'] & & B \end{tikzcd}$$

commutes.

Proposition 9.2.2. *In the category* $\mathbf{A}(X)$*, each morphism is unique and is onto.*

Proof. Let (g, A) and (h, B) be objects in $\mathbf{A}(X)$ and suppose θ and θ' are both morphisms from (g, A) to (h, B). Then

$$(xg)\theta = xh \text{ and } (xg)\theta' = xh$$

and so

$$(xg)\theta = (xg)\theta'.$$

Since $a\theta = a\theta'$ for all $a \in Xg$, where Xg is a set of generators for A, we have $\theta = \theta'$.

We wish to show that θ is onto. Consider $b \in B$. As $B = \langle Xh \rangle$,

$$b = t(x_1h, ..., x_nh)$$

and

$$t(x_1h, ..., x_nh) = t(x_1g\theta, ..., x_ng\theta) = t(x_1g, ..., x_ng)\theta \in A\theta.$$

So there exists $a \in A$ such that $a\theta = b$. $\square$

Definition 9.2.3. Let $\mathbf{M}(X)$ be the category where $\mathcal{M}$ is the class of monoids and let $\mathbf{PLR}(X)$ be the category where $\mathcal{PLR}$ is the class of proper left restriction monoids.

Suppose (X, f, S) is a monoid presentation of a fixed monoid S. As in [22], we shall define the subcategory $\mathbf{PLR}(X, f, S)$ of $\mathbf{PLR}(X)$. An object (g, M) of $\mathbf{PLR}(X)$ is an object in $\mathbf{PLR}(X, f, S)$ if the diagram

$$\begin{tikzcd} X \arrow[d, "g"'] \arrow[dr, "f"] & \\ M \arrow[r, "\sigma_M^\natural"'] & S \end{tikzcd}$$

commutes, where $\sigma_M^\natural$ is a morphism with kernel σ_M. As remarked above, $\sigma_M^\natural$ must be unique and onto. By Corollary 2.7.8, $\sigma_M^\natural$ is a $(2, 1, 0)$-morphism.

Proposition 9.2.4. *Let (X, f, S) be a monoid presentation of a monoid S. Then (f, S) is a terminal object in $\mathbf{PLR}(X, f, S)$.*

Proof. As S is a monoid, S can be regarded as proper reduced left restriction. Since we also have $f : X \to S$ and Xf generates S as either a monoid or left restriction monoid, (f, S) is an object of $\mathbf{PLR}(X)$. As (f, S) is an object of $\mathbf{PLR}(X)$, I_S is the unique morphism from S to S such that $fI_S = f$ in Definition 9.2.1. So $\sigma_S^\natural = I_S$ and clearly the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "f"] & \\ S \arrow[r, "\sigma_S^\natural"'] & S \end{tikzcd}$$

Therefore (f, S) is an object in $\mathbf{PLR}(X, f, S)$.

As $\sigma_M^\natural$ is a unique morphism from any object (g, M) of $\mathbf{PLR}(X, f, S)$ to (f, S), then (f, S) is a terminal object in $\mathbf{PLR}(X, f, S)$. $\square$

Lemma 9.2.5. *If $\theta \in Mor\,((g, M), (h, N))$ in the category $\mathbf{PLR}(X, f, S)$ then*

$$m\theta \in E_N \text{ implies } m \in E_M,$$

and all the triangles in the following diagram commute.

$$\begin{tikzcd}
& X \arrow[d, "g"] \arrow[ddl, "h"'] \arrow[ddr, "f"] & \\
& M \arrow[dl, "\theta"'] \arrow[dr, "\sigma_M^\natural"] & \\
N \arrow[rr, "\sigma_N^\natural"'] & & S
\end{tikzcd}$$

Proof. We shall begin by showing that all the triangles in the diagram commute. Due to the definition of $\mathbf{PLR}(X)$, $g\theta = h$ and due to the definition of $\mathbf{PLR}(X, f, S)$, $h\sigma_N^\natural = f$ and $g\sigma_M^\natural = f$. So it remains to show that the following diagram commutes:

$$\begin{tikzcd}
& M \arrow[dl, "\theta"'] \arrow[dr, "\sigma_M^\natural"] & \\
N \arrow[rr, "\sigma_N^\natural"'] & & S
\end{tikzcd}$$

Take $x \in X$. Then $xg \in M$. We have

$$(xg)\theta\sigma_N^\natural = xg\theta\sigma_N^\natural = xh\sigma_N^\natural = xf = xg\sigma_M^\natural = (xg)\sigma_M^\natural.$$

As $M = \langle Xg \rangle$ and $\theta\sigma_N^\natural$ and $\sigma_M^\natural$ are $(2,1,0)$-morphisms by Corollary 2.7.8, $\theta\sigma_N^\natural = \sigma_M^\natural$. Therefore all the triangles are commutative.

Let $m \in M$ and $m\theta \in E_N$. Since $\sigma_M^\natural$ is a morphism, $1_M\sigma_M^\natural = 1_S$. So we have

$$1_M\sigma_M^\natural = 1_S = m\theta\sigma_N^\natural = m\sigma_M^\natural$$

and hence $1_M \sigma_M m$. As M is proper, E_M is a σ_M-class by Corollary 2.8.5. So $m \in E_M$. $\square$

If (X, f, S) is a monoid presentation of a monoid S, let us define $\tau_{\mathcal{M}} : X \to \mathcal{M}(X, f, S)$ by

$$x\tau_{\mathcal{M}} = \left(\underset{1}{\bullet} \xrightarrow{\quad x \quad} \underset{xf}{\bullet}, xf \right)$$

for $x \in X$. Then we have the following proposition:

Proposition 9.2.6. *Let (X, f, S) be a monoid presentation of a monoid S. Putting $\mathcal{M} = \mathcal{M}(X, f, S)$ we have $\mathcal{M} = \langle X\tau_{\mathcal{M}} \rangle$ and $(\tau_{\mathcal{M}}, \mathcal{M})$ is an object in $\mathbf{PLR}(X, f, S)$.*

Proof. Let $(\Delta, s) \in \mathcal{M}$. If $\Delta = \bullet_1$, i.e. the trivial graph, then as s is a vertex of Δ, we have $s = 1$ and $(\Delta, s) = (\bullet_1, 1)$ is the identity of $\mathcal{M}$. Hence $(\Delta, s) \in \langle X\tau_{\mathcal{M}} \rangle$. Suppose now that Δ is not trivial. Then there is an edge

$$e = \underset{u}{\bullet} \xrightarrow{\quad x \quad} \underset{u(xf)}{\bullet}$$

in Δ. By definition, Δ is 1-rooted, so there is some path

$$\underset{1}{\bullet} \xrightarrow{\quad x_1 \quad} \underset{x_1 f}{\bullet} \xrightarrow{\quad x_2 \quad} \underset{(x_1 f)(x_2 f)}{\bullet} \cdots\cdots\cdots \bullet \xrightarrow{\quad x_n \quad} \underset{(x_1 f)\ldots(x_n f) = u}{\bullet}$$

from 1 to u in Δ, so that

$$\underset{1}{\bullet} \xrightarrow{\quad x_1 \quad} \underset{x_1 f}{\bullet} \xrightarrow{\quad x_2 \quad} \bullet \cdots\cdots\cdots \bullet \xrightarrow{\quad x_n \quad} \underset{u}{\bullet} \xrightarrow{\quad x \quad} \underset{u(xf)}{\bullet}$$

is a subgraph of Δ, which we shall denote by P_e. Note that

$$(P_e, 1) = ((x_1\tau_{\mathcal{M}})(x_2\tau_{\mathcal{M}}) \ldots (x_n\tau_{\mathcal{M}})(x\tau_{\mathcal{M}}))^+ \in \langle X\tau_{\mathcal{M}} \rangle$$

as we shall demonstrate. We have

$$x_1\tau_{\mathcal{M}} = \left(\underset{1}{\bullet} \xrightarrow{\quad x_1 \quad} \underset{x_1 f}{\bullet}, x_1 f \right)$$

and so

$$\begin{aligned}
(x_1\tau_{\mathcal{M}})(x_2\tau_{\mathcal{M}}) &= \left(\underset{1}{\bullet} \xrightarrow{\quad x_1 \quad} \underset{x_1 f}{\bullet}, x_1 f \right) \left(\underset{1}{\bullet} \xrightarrow{\quad x_2 \quad} \underset{x_2 f}{\bullet}, x_2 f \right) \\
&= \left(\underset{1}{\bullet} \xrightarrow{\quad x_1 \quad} \underset{x_1 f}{\bullet} \cup \underset{x_1 f}{\bullet} \xrightarrow{\quad x_2 \quad} \underset{(x_1 f)(x_2 f)}{\bullet}, (x_1 f)(x_2 f) \right) \\
&= \left(\underset{1}{\bullet} \xrightarrow{\quad x_1 \quad} \underset{x_1 f}{\bullet} \xrightarrow{\quad x_2 \quad} \underset{(x_1 f)(x_2 f)}{\bullet}, (x_1 f)(x_2 f) \right).
\end{aligned}$$

We can see by induction that $(x_1\tau_{\mathcal{M}})(x_2\tau_{\mathcal{M}})...(x_n\tau_{\mathcal{M}})$ is equal to

$$\left(\underset{1}{\bullet} \xrightarrow{\quad x_1 \quad} \underset{x_1 f}{\bullet} \xrightarrow{\quad x_2 \quad} \bullet \cdots\cdots\cdots \bullet \xrightarrow{\quad x_n \quad} \underset{u}{\bullet}, u \right),$$

where $u = (x_1 f)(x_2 f)...(x_n f)$. Therefore,

$$(x_1\tau_{\mathcal{M}})(x_2\tau_{\mathcal{M}})...(x_n\tau_{\mathcal{M}})(x\tau_{\mathcal{M}}) = (P_e, u(xf)) \in \langle X\tau_{\mathcal{M}} \rangle.$$

So, $(P_e, u(xf))^+ = (P_e, 1)$, i.e.

$$((x_1\tau_{\mathcal{M}})(x_2\tau_{\mathcal{M}})...(x_n\tau_{\mathcal{M}})(x\tau_{\mathcal{M}}))^+ = (P_e, 1).$$

Hence $(P_e, 1) \in \langle X\tau_{\mathcal{M}}\rangle$.

As Δ is 1-rooted, $\Delta = \bigcup_{e\in E(\Delta)} P_e$, where $E(\Delta)$ denotes the set of edges of Δ, and we have that

$$(\Delta, 1) = \prod_{e\in E(\Delta)} (P_e, 1),$$

so $(\Delta, 1) \in \langle X\tau_{\mathcal{M}}\rangle$. Thus if $s = 1, (\Delta, s) \in \langle X\tau_{\mathcal{M}}\rangle$. Suppose $s \neq 1$. Then as $s \in V(\Delta)$ and Δ is 1-rooted, there is some edge $e \in E(\Delta)$ with terminal vertex s. Then s is a vertex of P_e so that $(P_e, s) \in \langle X\tau_{\mathcal{M}}\rangle$. We have

$$(\Delta, 1)(P_e, s) = (\Delta \cup P_e, s) = (\Delta, s).$$

So $(\Delta, s) \in \langle X\tau_{\mathcal{M}}\rangle$ as required. Therefore $\mathcal{M} = \langle X\tau_{\mathcal{M}}\rangle$.

The above shows that $(\tau_{\mathcal{M}}, \mathcal{M})$ is an object in $\mathbf{PLR}(X)$. To show that $(\tau_{\mathcal{M}}, \mathcal{M})$ is an object in the subcategory $\mathbf{PLR}(X, f, S)$ we must show that

$$\begin{tikzcd} X \arrow[d, "\tau_{\mathcal{M}}"'] \arrow[dr, "f"] & \\ \mathcal{M} \arrow[r, "\sigma^{\sharp}_{\mathcal{M}}"'] & S \end{tikzcd}$$

commutes, where $\sigma^{\sharp}_{\mathcal{M}}$ is a morphism with kernel $\sigma_{\mathcal{M}}$. Defining $\sigma^{\sharp}_{\mathcal{M}} : \mathcal{M} \to S$ by $(\Delta, s)\sigma^{\sharp}_{\mathcal{M}} = s$ it is clear that $\sigma^{\sharp}_{\mathcal{M}}$ is a morphism. By Proposition 9.1.2, Ker $\sigma^{\sharp}_{\mathcal{M}} = \sigma_{\mathcal{M}}$; clearly $\tau_{\mathcal{M}}\sigma^{\sharp}_{\mathcal{M}} = f$. □

Theorem 9.2.7. *Let (X, f, S) be a monoid presentation of a monoid S. Then putting $\mathcal{M} = \mathcal{M}(X, f, S)$, the pair $(\tau_{\mathcal{M}}, \mathcal{M})$ is an initial object in $\mathbf{PLR}(X, f, S)$.*

Proof. We need to show that for any object (h, N) in $\mathbf{PLR}(X, f, S)$,

$$|\mathrm{Mor}((\tau_{\mathcal{M}}, \mathcal{M}), (h, N))| = 1.$$

From Proposition 9.2.2, this is equivalent to showing that

$$\mathrm{Mor}((\tau_{\mathcal{M}}, \mathcal{M}), (h, N)) \neq \emptyset.$$

Let (h, N) be an object in $\mathbf{PLR}(X, f, S)$. So $N = \langle Xh\rangle$ and

$$\begin{tikzcd} X \arrow[d, "h"'] \arrow[dr, "f"] & \\ N \arrow[r, "\sigma^{\sharp}_{N}"'] & S \end{tikzcd}$$

commutes, where $\sigma^{\sharp}_{N}$ is a morphism with kernel σ_N.

Let us define $\theta : \mathcal{M} \to N$ by

$$((x_1^1\tau_{\mathcal{M}} \dots x_{p(1)}^1\tau_{\mathcal{M}})^+ \dots (x_1^m\tau_{\mathcal{M}} \dots x_{p(m)}^m\tau_{\mathcal{M}})^+ y_1\tau_{\mathcal{M}} \dots y_s\tau_{\mathcal{M}})\theta$$

$$= (x_1^1h \dots x_{p(1)}^1h)^+ \dots (x_1^mh \dots x_{p(m)}^mh)^+ y_1h \dots y_sh$$

where $m, s \in \mathbb{N}^0, x_j^i, y_k \in X, 1 \le i \le m, 1 \le j \le p(i)$ and $1 \le k \le s$. As $\mathcal{M}$ is a left restriction monoid and $\mathcal{M} = \langle X\tau_{\mathcal{M}} \rangle$, by Lemma 2.2.15 all its elements are of the form above. Similarly for N as it is a left restriction monoid and $N = \langle Xh \rangle$. However, we need to show that θ is well-defined.

Suppose that

$$(x_1^1\tau_{\mathcal{M}} \dots x_{p(1)}^1\tau_{\mathcal{M}})^+ \dots (x_1^m\tau_{\mathcal{M}} \dots x_{p(m)}^m\tau_{\mathcal{M}})^+ y_1\tau_{\mathcal{M}} \dots y_s\tau_{\mathcal{M}}$$

$$= (z_1^1\tau_{\mathcal{M}} \dots z_{q(1)}^1\tau_{\mathcal{M}})^+ \dots (z_1^n\tau_{\mathcal{M}} \dots z_{q(n)}^n\tau_{\mathcal{M}})^+ w_1\tau_{\mathcal{M}} \dots w_t\tau_{\mathcal{M}} \ (*)$$

where $m, n, s, t \in \mathbb{N}^0, x_j^i, y_k \in X, 1 \le i \le m, 1 \le j \le p(i), 1 \le k \le s$ and $z_j^i, w_k \in X, 1 \le i \le n, 1 \le j \le q(i), 1 \le k \le t$.

We aim to show

$$(x_1^1h \dots x_{p(1)}^1h)^+ \dots (x_1^mh \dots x_{p(m)}^mh)^+ y_1h \dots y_sh$$

$$= (z_1^1h \dots z_{q(1)}^1h)^+ \dots (z_1^nh \dots z_{q(n)}^nh)^+ w_1h \dots w_th \ (**).$$

Note first that if $m = s = 0$ then the left hand side of $(*)$ is the identity $(\bullet_1, 1)$ of $\mathcal{M}$. It follows from the definition of $\tau_{\mathcal{M}}$, the multiplication in $\mathcal{M}$, and the description of $^+$ in Proposition 9.1.2, that also $n = t = 0$. Clearly $(**)$ holds in this case.

To continue, we need a result which we shall state as a lemma.

Lemma 9.2.8. *Let $a_1, \dots, a_s, b_1, \dots, b_t \in X$ (where s or t may be 0) and suppose that*

$$a_1f \dots a_sf = b_1f \dots b_tf,$$

where the empty product is taken to be 1. *Then*

$$(a_1h \dots a_sh) \ \sigma_N \ (b_1h \dots b_th).$$

Proof. If $s \neq 0$ and $t \neq 0$ then

$$\begin{aligned}(a_1h \dots a_sh)\sigma_N^\sharp &= a_1h\sigma_N^\sharp \dots a_sh\sigma_N^\sharp \\ &= a_1f \dots a_sf \\ &= b_1f \dots b_tf \\ &= b_1h\sigma_N^\sharp \dots b_th\sigma_N^\sharp \\ &= (b_1h \dots b_th)\sigma_N^\sharp,\end{aligned}$$

as $\sigma_N^\sharp$ is a morphism and $h\sigma_N^\sharp = f$. So the result is true in this case.

If $s \neq 0$ and $t = 0$ then

$$(a_1 h \dots a_s h)\sigma_N^\sharp = 1 = 1\sigma_N^\sharp$$

so that $a_1 h \dots a_s h \ \sigma_N \ 1$. It follows that the result is true in every case. $\square$

Returning to the proof of Theorem 9.2.7, suppose that m and s are *not both* 0 and n and t are *not both* 0, so we are not just considering the identity. Applying $\sigma_M^\sharp$ to $(*)$, we obtain

$$(y_1\tau_{\mathcal{M}} \dots y_s\tau_{\mathcal{M}}) \ \sigma_{\mathcal{M}} \ (w_1\tau_{\mathcal{M}} \dots w_t\tau_{\mathcal{M}})$$

as $x^+\sigma_{\mathcal{M}} = (x\sigma_{\mathcal{M}})^+ = 1$ for any $x \in M$. We have

$$\begin{aligned} y_1 f \dots y_s f &= y_1(\tau_{\mathcal{M}}\sigma_{\mathcal{M}}) \dots y_s(\tau_{\mathcal{M}}\sigma_{\mathcal{M}}) \\ &= (y_1\tau_{\mathcal{M}} \dots y_s\tau_{\mathcal{M}})\sigma_{\mathcal{M}} \\ &= (w_1\tau_{\mathcal{M}} \dots w_t\tau_{\mathcal{M}})\sigma_{\mathcal{M}} \\ &= (w_1\tau_{\mathcal{M}} \dots w_t\tau_{\mathcal{M}})\sigma_{\mathcal{M}} \\ &= w_1 f \dots w_t f. \end{aligned}$$

By Lemma 9.2.8,

$$(y_1 h \dots y_s h) \ \sigma_N \ (w_1 h \dots w_t h).$$

Now Lemma 2.8.4 gives us

$$(w_1 h \dots w_t h)^+ y_1 h \dots y_s h = (y_1 h \dots y_s h)^+ w_1 h \dots w_t h.$$

Let us write

$$y_1 = x_1^{m+1}, \dots, y_s = x_{p(m+1)}^{m+1}$$

and

$$w_1 = z_1^{n+1}, \dots, w_t = z_{q(n+1)}^{n+1}.$$

With the usual convention for empty products we put

$$E = (x_1^1 h \dots x_{p(1)}^1 h)^+ \dots (x_1^m h \dots x_{p(m)}^m h)^+,$$

$$Y = x_1^{m+1} h \dots x_{p(m+1)}^{m+1} h$$

$$F = (z_1^1 h \dots z_{q(1)}^1 h)^+ \dots (z_1^n h \dots z_{q(n)}^n h)^+,$$

and

$$W = z_1^{n+1} h \dots z_{q(n+1)}^{n+1} h.$$

We still aim to show $(**)$, which is now

$$EY = FW.$$

We have shown

$$W^+Y = Y^+W.$$

Our next aim is to show that $EY^+ = FW^+$.

Let $i \in \{1, \dots, n+1\}$ (where $q(i) \neq 0$) and write

$$z_1^i = z_1, \dots, z_{q(i)}^i = z_u.$$

Lemma 9.2.9. *With notation as above,*

$$EY^+ \leq (z_1 h \dots z_u h)^+.$$

Proof. If Δ denotes the graph that is the first coordinate of $(*)$, then from the expression for $(*)$ we know that

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "z_1"] & \underset{z_1f}{\bullet} \arrow[r, "z_2"] & \underset{(z_1f)(z_2f)}{\bullet} \arrow[r, dotted, no head] & \bullet \arrow[r, "z_u"] & \underset{(z_1f)\dots(z_uf)}{\bullet} \end{tikzcd}$$

is a subgraph of Δ for a particular i. It follows that there exist $i_1, \dots, i_u \in \{1, \dots, m+1\}$ and $j_1, \dots, j_u$ with $j_k \in \{1, \dots, p(i_k)\}$ for $k \in \{1, \dots, u\}$ such that

$$\begin{aligned}
z_1 &= x_{j_1}^{i_1} \text{ where } && x_1^{i_1} f \dots x_{j_1-1}^{i_1} f = 1 \\
z_2 &= x_{j_2}^{i_2} \text{ where } && x_1^{i_2} f \dots x_{j_2-1}^{i_2} f = z_1 f \\
z_3 &= x_{j_3}^{i_3} \text{ where } && x_1^{i_3} f \dots x_{j_3-1}^{i_3} f = z_1 f z_2 f \\
&\ \vdots \\
z_u &= x_{j_u}^{i_u} \text{ where } && x_1^{i_u} f \dots x_{j_u-1}^{i_u} f = z_1 f \dots z_{u-1} f.
\end{aligned}$$

From Lemma 9.2.8 we have

$$x_1^{i_1} h \dots x_{j_1-1}^{i_1} h \ \sigma_N \ 1.$$

Since N is proper, E_N is a σ_N-class by Proposition 2.8.5 and so we have $x_1^{i_1} h \dots x_{j_1-1}^{i_1} h \in E_N$. Using Lemma 2.6.2 we deduce that

$$EY^+ \leq (x_1^{i_1} h \dots x_{p(i_1)}^{i_1} h)^+ = ((x_1^{i_1} h \dots x_{j_1-1}^{i_1} h) z_1 h (x_{j_1+1}^{i_1} h \dots x_{p(i_1)}^{i_1} h))^+$$

$$\leq ((x_1^{i_1} h \dots x_{j_1-1}^{i_1} h) z_1 h)^+ = (x_1^{i_1} h \dots x_{j_1-1}^{i_1} h)(z_1 h)^+ \leq (z_1 h)^+.$$

Assume for induction that for $1 \leq v < u$,

$$EY^+ \leq (z_1 h \dots z_v h)^+$$

and put $t = v + 1$. We wish to show

$$EY^+ \leq (z_1h \dots z_th)^+.$$

We have

$$EY^+ \leq (x_1^{i_t}h \dots x_{p(i_t)}^{i_t}h)^+ = ((x_1^{i_t}h \dots x_{j_t-1}^{i_t}h)(z_th)(x_{j_t+1}^{i_t}h \dots x_{p(i_t)}^{i_t}h))^+$$

which together with Lemma 2.6.2 and the induction hypothesis gives

$$EY^+ \leq (z_1h \dots z_vh)^+((x_1^{i_t}h \dots x_{j_t-1}^{i_t}h)(z_th))^+.$$

We know that

$$z_1f \dots z_vf = x_1^{i_t}f \dots x_{j_t-1}^{i_t}f.$$

So

$$(z_1h \dots z_vh)\,\sigma_N\,(x_1^{i_t}h \dots x_{j_t-1}^{i_t}h)$$

by Lemma 9.2.8. By Lemma 2.8.4,

$$(x_1^{i_t}h \dots x_{j_t-1}^{i_t}h)^+z_1h \dots z_vh = (z_1h \dots z_vh)^+x_1^{i_t}h \dots x_{j_t-1}^{i_t}h.$$

Now as $\widetilde{\mathcal{R}}_{E_N}$ is a left congruence,

$$(z_1h \dots z_vh)^+((x_1^{i_t}h \dots x_{j_t-1}^{i_t}h)z_th)^+\;\widetilde{\mathcal{R}}_{E_N}\;(z_1h \dots z_vh)^+(x_1^{i_t}h \dots x_{j_t-1}^{i_t}h)z_th$$

$$= (x_1^{i_t}h \dots x_{j_t-1}^{i_t}h)^+z_1h \dots z_vhz_th\;\widetilde{\mathcal{R}}_{E_N}\;(x_1^{i_t}h \dots x_{j_t-1}^{i_t}h)^+(z_1h \dots z_th)^+.$$

As each $\widetilde{\mathcal{R}}_{E_N}$-class contains only one element of E_N,

$$(z_1h \dots z_vh)^+((x_1^{i_t}h \dots x_{j_t-1}^{i_t}h)(z_th))^+ = (x_1^{i_t}h \dots x_{j_t-1}^{i_t}h)^+(z_1h \dots z_th)^+.$$

So

$$EY^+ \leq (x_1^{i_t}h \dots x_{j_t-1}^{i_t}h)^+(z_1h \dots z_th)^+ \leq (z_1h \dots z_th)^+.$$

By finite induction,

$$EY^+ \leq (z_1h \dots z_uh)^+$$

as required. □

Since Lemma 9.2.9 holds for *any* $i \in \{1, \dots, n+1\}$ with $q(i) \neq 0$ we obtain $EY^+ \leq FW^+$. The dual argument gives $FW^+ \leq EY^+$ and so $EY^+ = FW^+$. Then

$$EY = EY^+Y = FW^+Y = FY^+W$$

since $W^+Y = Y^+W$. From $EY^+ = FW^+$ we also have that

$$Y^+FW^+ = Y^+EY^+ = EY^+ = FW^+.$$

So

$$EY = FY^+W^+W = FW^+W = FW.$$

Therefore θ is well-defined.

It remains to show that θ is a morphism. By definition, $1\theta = 1$ and from Lemma 2.6.2 we see $a^+\theta = (a\theta)^+$ for any $a \in M$. Take $b, d \in \mathcal{M}$ such that

$$b = (a_1^1 \dots a_{p(1)}^1)^+ \dots (a_1^m \dots a_{p(m)}^m)^+ b_1 \dots b_s$$

and

$$d = (c_1^1 \dots c_{q(1)}^1)^+ \dots (c_1^n \dots c_{q(n)}^n)^+ d_1 \dots d_t$$

where $m, n, s, t \in \mathbb{N}^0, a_j^i, b_k \in X, 1 \leq i \leq m, 1 \leq j \leq p(i), 1 \leq k \leq s$ and $c_j^i, d_k \in X, 1 \leq i \leq n, 1 \leq j \leq q(i), 1 \leq k \leq t$. We have, using the proof of Lemma 2.2.15,

$$bd = (a_1^1 \dots a_{p(1)}^1)^+ \dots (a_1^m \dots a_{p(m)}^m)^+ (b_1 \dots b_s c_1^1 \dots c_{q(1)}^1)^+$$

$$\dots (b_1 \dots b_s c_1^n \dots c_{q(n)}^n)^+ b_1 \dots b_s d_1 \dots d_t.$$

It is then clear that θ preserves multiplication and so is a $(2,1,0)$-morphism.

Finally, for any $x \in X$ we have

$$x\tau_{\mathcal{M}}\theta = xh$$

so that θ is the unique morphism in Mor $((\tau_{\mathcal{M}}, \mathcal{M}), (h, N))$. This completes the proof that $(\tau_{\mathcal{M}}, \mathcal{M})$ is an initial object in $\mathbf{PLR}(X, f, S)$. $\square$

Definition 9.2.10. A *graph morphism* φ from a graph Γ to Γ' consists of two functions, both denoted φ, such that

$$\varphi : V(\Gamma) \to V(\Gamma') \text{ and } \varphi : E(\Gamma) \to E(\Gamma'),$$

where for any $e \in E(\Gamma)$,

$$i(e)\varphi = i(e\varphi) \text{ and } t(e)\varphi = t(e\varphi).$$

Proposition 9.2.11. *Let (f, S) and (g, T) be objects in $\mathbf{M}(X)$ and suppose that $\theta \in Mor_{\mathbf{M}(X)}((f, S), (g, T))$. Then the map $\theta'' : \Gamma(X, f, S) \to \Gamma(X, g, T)$, given by actions*

$$s\theta'' = s\theta$$

and

$$(s, x, s(xf))\theta'' = (s\theta, x, s\theta xg)$$

on vertices and edges respectively, is a graph morphism.

Proof. Considering a typical edge of $\Gamma(X, f, S)$, say $(s, x, s(xf))$, i.e.

$$s \xrightarrow{\;x\;} s(xf),$$

we wish to show that this is mapped to $(s\theta, x, [s(xf)]\theta)$, i.e.

$$\underset{s\theta}{\bullet} \xrightarrow{\quad x \quad} \underset{[s(xf)]\theta.}{\bullet}$$

Under θ'', it is mapped to $(s\theta, x, s\theta(xg))$:

$$\underset{s\theta}{\bullet} \xrightarrow{\quad x \quad} \underset{s\theta(xg).}{\bullet}$$

We have

$$s\theta(xg) = s\theta(xf\theta) = [s(xf)]\theta$$

as $\theta \in \mathrm{Mor}_{\mathbf{M}(X)}((f,S),(g,T))$. Therefore, θ'' is a graph morphism.

$\square$

Proposition 9.2.12. *Let $\theta_X^e : \mathcal{M}(X,f,S) \to \mathcal{M}(X,g,T)$ be defined by*

$$(\Delta, s)\theta_X^e = (\Delta\theta'', s\theta).$$

Then θ_X^e is a $(2,1,0)$-morphism such that

$$\theta_X^e \in \ Mor_{\mathbf{PLR}(X)}((\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X,f,S)), (\tau_{\mathcal{M}(X,g,T)}, \mathcal{M}(X,g,T)).$$

Proof. Let $(\Delta, s), (\Sigma, t) \in \mathcal{M}(X,f,S)$. We have

$$\begin{aligned} [(\Delta,s)(\Sigma,t)]\theta_X^e &= (\Delta \cup s\Sigma, st)\theta_X^e \\ &= ((\Delta \cup s\Sigma)\theta'', (st)\theta) \\ &= (\Delta\theta'' \cup (s\Sigma)\theta'', s\theta t\theta) \end{aligned}$$

and

$$\begin{aligned} [(\Delta,s)\theta_X^e][(\Sigma,t)\theta_X^e] &= (\Delta\theta'', s\theta)(\Sigma\theta'', t\theta) \\ &= (\Delta\theta'' \cup s\theta\Sigma\theta'', s\theta t\theta). \end{aligned}$$

Looking at the definition of θ'',

$$(s\Sigma)\theta'' = s\theta\Sigma\theta''$$

and so θ'' is a morphism. This can be seen by considering an edge of Σ:

$$\underset{a}{\bullet} \xrightarrow{\quad x \quad} \underset{ax}{\bullet} \ \overset{\text{Action}}{\underset{s}{\cdots\cdots\to}} \ \underset{sa}{\bullet} \xrightarrow{\quad x \quad} \underset{sax}{\bullet} \ \overset{\theta''}{\cdots\cdots\to} \ \underset{(sa)\theta}{\bullet} \xrightarrow{\quad x \quad} \underset{(sax)\theta}{\bullet}$$

and alternatively

$$\underset{a}{\bullet} \xrightarrow{\quad x \quad} \underset{ax}{\bullet} \ \overset{\theta''}{\cdots\cdots\to} \ \underset{a\theta}{\bullet} \xrightarrow{\quad x \quad} \underset{(a\theta)(x\theta)}{\bullet} \ \overset{\text{Action}}{\underset{s\theta}{\cdots\cdots\to}} \ \underset{(s\theta)(a\theta)}{\bullet} \xrightarrow{\quad x \quad} \underset{(s\theta)(a\theta)(x\theta)}{\bullet}$$

$$= \underset{(sa)\theta}{\bullet} \xrightarrow{\quad x \quad} \underset{(sax)\theta}{\bullet} .$$

We have

$$\begin{aligned}
(\Delta, s)^+\theta_X^e &= (\Delta, 1)\theta_X^e \\
&= (\Delta\theta'', 1\theta) \\
&= (\Delta\theta'', 1) \\
&= (\Delta\theta'', s\theta)^+ \\
&= [(\Delta, s)\theta_X^e]^+
\end{aligned}$$

and

$$\begin{aligned}
(\bullet_1, 1)\theta_X^e &= (\bullet_1\theta'', 1\theta) \\
&= (\bullet_{1\theta}, 1) \\
&= (\bullet_1, 1),
\end{aligned}$$

so θ_X^e is a $(2, 1, 0)$-morphism.

It remains to show that the following diagram commutes:

$$\begin{tikzcd}
& X \arrow[dl, "\tau_{\mathcal{M}(X,f,S)}"'] \arrow[dr, "\tau_{\mathcal{M}(X,g,T)}"] & \\
\mathcal{M}(X, f, S) \arrow[rr, "\theta_X^e"'] & & \mathcal{M}(X, g, T)
\end{tikzcd}$$

Considering $x \in X$,

$$\begin{aligned}
x\tau_{\mathcal{M}(X,f,S)}\theta_X^e &= \left(\underset{1}{\bullet} \xrightarrow{\quad x \quad} \underset{xf}{\bullet}, xf \right) \theta_X^e \\
&= \left(\underset{1}{\bullet} \xrightarrow{\quad x \quad} \underset{xf}{\bullet}\theta'', (xf)\theta \right) \\
&= \left(\underset{1}{\bullet} \xrightarrow{\quad x \quad} \underset{xg}{\bullet}, xg \right) \quad \text{as } f\theta = g \\
&= x\tau_{\mathcal{M}(X,g,T)}.
\end{aligned}$$

So

$$\theta_X^e \in \operatorname{Mor}_{\mathbf{PLR}(X)}((\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X, f, S)), (\tau_{\mathcal{M}(X,g,T)}, \mathcal{M}(X, g, T))).$$

$\square$

We shall use the previous result to prove the following theorem on free left restriction monoids:

Theorem 9.2.13. *Let X be a set and let $\iota : X \to X^*$ be the canonical embedding. Let $\mathcal{M} = \mathcal{M}(X, \iota, X^*)$. Then $\tau_{\mathcal{M}} : X \to \mathcal{M}$ is an embedding* *and $\mathcal{M}$ is the free left restriction monoid on $X\tau_{\mathcal{M}}$.*

Proof. Consider $x, y \in X$ and let $x\tau_{\mathcal{M}} = y\tau_{\mathcal{M}}$. Then

$$\left(\underset{1}{\bullet} \xrightarrow{\quad x \quad} \underset{x}{\bullet}, x \right) = \left(\underset{1}{\bullet} \xrightarrow{\quad y \quad} \underset{y}{\bullet}, y \right)$$

and so $x = y$. Therefore $\tau_{\mathcal{M}} : X \to \mathcal{M}$ is an embedding.

Let M be a left restriction monoid and $g : X \to M$ a function. By Theorem 2.8.10, there is a proper left restriction monoid P and an onto morphism $\phi : P \to M$. For $x \in X$, there exists $xg \in M$ and as ϕ is onto there exists $p_x \in P$ such that $p_x\phi = xg$. So such a p_x exists for each $x \in X$. For each $x \in X$ choose $p_x \in P$ such that

$$p_x\phi = xg.$$

Let $h : X \to P$ be given by $xh = p_x$ and $Q = \langle Xh \rangle$. Then Q is a proper left restriction monoid as P is a proper left restriction monoid and Q is a subalgebra of P.

Let $S = Q/\sigma_Q$, so S is a monoid. We have

$$S = Q\sigma_Q^\natural = \langle Xh \rangle \sigma_Q^\natural = \langle Xh\sigma_Q^\natural \rangle$$

since $S = \{q\sigma_Q : q \in Q\} = \{q\sigma_Q^\natural : q \in Q\}$. Let us consider $(X, h\sigma_Q^\natural, S)$. We know X is a set, S is a monoid and $h\sigma_Q^\natural : X \to S$ is such that $S = \langle Xh\sigma_Q^\natural \rangle$. So $(X, h\sigma_Q^\natural, S)$ is a monoid presentation and $\mathcal{N} = \mathcal{M}(X, h\sigma_Q^\natural, S)$ is a proper left restriction monoid due to Proposition 9.1.2.

Let us extend $h\sigma_Q^\natural : X \to S$ to a morphism $\theta : X^* \to S$ in the usual way by

$$(x_1 \dots x_n)\theta = (x_1 h\sigma_Q^\natural) \dots (x_n h\sigma_Q^\natural)$$

for $x_1 \dots x_n \in X$. We have that $(\iota, X^*), (h\sigma_Q^\natural, S) \in \text{Ob } \mathbf{M}(X)$ since X^* and S are monoids, $\iota : X \to X^*$, $h\sigma_Q^\natural : X \to S$, $\langle X\iota \rangle = X^*$ and $\langle Xh\sigma_Q^\natural \rangle = S$. Also, θ is a morphism from (ι, X^*) to $(h\sigma_Q^\natural, S)$ as $\theta : X^* \to S$ and,

$$\begin{tikzcd} & X \arrow[dl, "\iota"'] \arrow[dr, "h\sigma_Q^\natural"] & \\ X^* \arrow[rr, "\theta"'] & & S \end{tikzcd}$$

commutes as $h\sigma_Q^\natural$ was extended to Q. From Proposition 9.2.12,

$$\theta_X^e \in \text{ Mor }_{\mathbf{PLR}(X)}((\tau_{\mathcal{M}}, \mathcal{M}), (\tau_{\mathcal{N}}, \mathcal{N}))$$

so that $\theta_X^e : \mathcal{M} \to \mathcal{N}$ is a $(2,1,0)$-morphism and the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "\tau_{\mathcal{M}}"'] \arrow[dr, "\tau_{\mathcal{N}}"] & \\ \mathcal{M} \arrow[rr, "\theta^{\mathrm{e}}_X"'] & & \mathcal{N} \end{tikzcd}$$

Now, (h, Q) is an object in $\mathbf{PLR}(X, h\sigma^{\natural}_Q, S)$. By Theorem 9.2.7, $(\tau_{\mathcal{N}}, \mathcal{N})$ is the initial object in this category. So there is a morphism $\psi : \mathcal{N} \to Q$ such that

$$\begin{tikzcd} & X \arrow[dl, "\tau_{\mathcal{N}}"'] \arrow[dr, "h"] & \\ \mathcal{N} \arrow[rr, "\psi"'] & & Q \end{tikzcd}$$

commutes. We note that $\psi : \mathcal{N} \to Q \subseteq P$ and that we can regard $\theta^{\mathrm{e}}_X\psi$ as a morphism from $\mathcal{M}$ to P. Hence $\theta^{\mathrm{e}}_X\psi\phi : \mathcal{M} \to M$ is a morphism and for any $x \in X$,

$$x\tau_{\mathcal{M}}\theta^{\mathrm{e}}_X\psi\phi = x\tau_{\mathcal{N}}\psi\phi = xh\phi = p_x\phi = xg.$$

As $\tau_{\mathcal{M}}\theta^{\mathrm{e}}_X\psi\phi = g$, the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "\tau_{\mathcal{M}}"'] \arrow[dr, "g"] & \\ \mathcal{M} \arrow[rr, "\theta^{\mathrm{e}}_X\psi\phi"'] & & M \end{tikzcd}$$

By a similar argument to that in the proof of Proposition 9.2.2, $\theta^{\mathrm{e}}_X\psi\phi$ is the unique morphism making this diagram commute since $\mathcal{M} = \langle X\tau_{\mathcal{M}}\rangle$. As $\mathcal{M} = \mathcal{M}(X, \iota, X^*)$ is a graph expansion of a monoid, it follows from Proposition 9.1.2 that $\mathcal{M}$ is a left restriction monoid. Hence $\mathcal{M}$ is the free left restriction monoid on $X\tau_{\mathcal{M}}$. $\square$

9.3 The Functors F^e, F^σ, F^e_X and F^σ_X

In this section, we shall generalise results from [20], [22] and [23] on functors between the categories we have been considering. These results are not required for the main aim of this chapter, but we shall provide them for completeness. Initially, we shall construct functors $F^e : \mathbf{M} \to \mathbf{PLR}$ and $F^\sigma : \mathbf{PLR} \to \mathbf{M}$. The functor F^e is not a left adjoint of F^σ, but we shall look at two different ways to adapt these functors to obtain a left adjunction.

We shall begin by constructing the functor $F^e : \mathbf{M} \to \mathbf{PLR}$. Suppose that S is an object of $\mathbf{M}$, so S is a monoid. The triple (S, I_S, S) is certainly a monoid presentation of S, where $I_S : S \to S$ is the identity map. We put

$$SF^e = \mathcal{M}(S, I_S, S).$$

By Proposition 9.1.2, $\mathcal{M}(S, I_S, S)$ is a proper left restriction monoid. So, F^e is a function from the objects of $\mathbf{M}$ to the objects of $\mathbf{PLR}$.

Let us consider objects S and T in $\mathbf{M}$ and let $\theta : S \to T$ be a morphism between S and T. Let us define a map

$$\theta' : \Gamma(S, I_S, S) \to \Gamma(T, I_T, T)$$

by

$$v\theta' = v\theta$$

for any vertex v of $\Gamma(S, I_S, S)$ and

$$(s, x, sx)\theta' = (s\theta, x\theta, s\theta x\theta)$$

for any edge (s, x, sx) of $\Gamma(S, I_S, S)$. Thus the edge

$$\underset{s}{\bullet} \xrightarrow{\quad x \quad} \underset{sx}{\bullet}$$

is mapped to

$$\underset{s\theta}{\bullet} \xrightarrow{\quad x\theta \quad} \underset{(sx)\theta,}{\bullet}$$

where $(s\theta)(x\theta) = (sx)\theta$. Clearly θ' is a graph morphism. So, θ' maps subgraphs to subgraphs and paths to paths. As $1\theta = 1$, θ' maps 1-rooted subgraphs to 1-rooted subgraphs. So we can define θF^e by

$$\theta F^e = \theta^e,$$

where $\theta^e : SF^e \to TF^e$ is given by

$$(\Delta, s)\theta^e = (\Delta\theta', s\theta).$$

For any subgraph Δ of $\Gamma(S, I_S, S)$ and $s \in S$, we have

$$(s\Delta)\theta' = s\theta\Delta\theta'.$$

Proposition 9.3.1. *For objects S and T in $\mathbf{M}$,*

$$\theta^e \in \operatorname{Mor}_{\mathbf{PLR}}(SF^e, TF^e).$$

Proof. Consider objects S and T in $\mathbf{M}$. Then SF^e and TF^e are objects in $\mathbf{PLR}$, i.e. $\mathcal{M}(S, I_S, S)$ and $\mathcal{M}(T, I_T, T)$ are objects in $\mathbf{PLR}$. Consider

$(\Delta, s), (\Sigma, t) \in \mathcal{M}(S, I_S, S)$. We have

$$\begin{aligned}
[(\Delta, s)(\Sigma, t)]\theta^e &= (\Delta \cup s\Sigma, st)\theta^e \\
&= ((\Delta \cup s\Sigma)\theta', (st)\theta) \\
&= (\Delta\theta' \cup (s\Sigma)\theta', (s\theta)(t\theta)) \\
&= (\Delta\theta' \cup (s\theta)(\Sigma\theta'), (s\theta)(t\theta)) \\
&= (\Delta\theta', s\theta)(\Sigma\theta', t\theta) \\
&= (\Delta, s)\theta^e(\Sigma, t)\theta^e.
\end{aligned}$$

We also have

$$\begin{aligned}
(\Delta, s)^+\theta^e &= (\Delta, 1)\theta^e \\
&= (\Delta\theta', 1\theta) \\
&= (\Delta\theta', 1) \\
&= (\Delta\theta', s\theta)^+ \\
&= [(\Delta, s)\theta^e]^+
\end{aligned}$$

and

$$(\bullet_1, 1)\theta^e = (\bullet_1\theta', 1\theta) = (\bullet_{1\theta}, 1) = (\bullet_1, 1).$$

Hence θ^e is a $(2, 1, 0)$-morphism and so

$$\theta^e \in \text{ Mor}_{\mathbf{PLR}}(\mathcal{M}(S, I_S, S), \mathcal{M}(T, I_T, T)),$$

i.e.

$$\theta^e \in \text{ Mor}_{\mathbf{PLR}}(SF^e, TF^e).$$

□

We note that F^e associates each object of $\mathbf{M}$ with an object of $\mathbf{PLR}$. It also associates a morphism of $\mathbf{M}$ with a morphism of $\mathbf{PLR}$. We have the following proposition:

Proposition 9.3.2. *As defined above, F^e is a functor from $\mathbf{M}$ to $\mathbf{PLR}$.*

Proof. Let us consider $I_S : S \to S$, where S is an object of $\mathbf{M}$. Then

$$I_S F^e = I_S^e.$$

Considering $(\Delta, s) \in SF^e$, i.e. $(\Delta, s) \in \mathcal{M}(S, I_S, S)$, we have

$$\begin{aligned}
(\Delta, s)I_S F^e &= (\Delta, s)I_S^e \\
&= (\Delta I_S', s\, I_S) \\
&= (\Delta, s).
\end{aligned}$$

So,

$$I_S F^e = I_{SF^e}.$$

Consider $\mu \in \text{ Mor}_{\mathbf{M}}(S, T)$ and $\delta \in \text{ Mor}_{\mathbf{M}}(T, U)$, where S, T and U are objects in $\mathbf{M}$. Consider $(\Delta, s) \in SF^e$, i.e. $(\Delta, s) \in \mathcal{M}(S, I_S, S)$. We have, using a symbol for composition of functions for the sake of clarity,

$$\begin{aligned}
(\Delta, s)(\mu \circ \delta)F^e &= (\Delta, s)(\mu \circ \delta)^e \\
&= (\Delta(\mu \circ \delta)', s(\mu \circ \delta)) \\
&= ((\Delta\mu')\delta', (s\mu)\delta) \\
&= (\Delta\mu', s\mu)\delta^e \\
&= [(\Delta, s)\mu^e]\delta^e \\
&= [(\Delta, s)\mu F^e]\delta F^e \\
&= (\Delta, s)(\mu F^e \circ \delta F^e).
\end{aligned}$$

Hence

$$(\mu \circ \delta)F^e = \mu F^e \circ \delta F^e$$

and F^e is a functor between **M** and **PLR**. □

In fact, F^e is an *expansion* in the sense of Birget-Rhodes [56]. We can regard **M** as a subcategory of **PLR**.

Definition 9.3.3. We say that a functor $F : \mathbf{M} \to \mathbf{PLR}$ is an *expansion* if for any object S of **M** there is an onto morphism

$$\eta_S \in \ \mathrm{Mor}_{\mathbf{PLR}}(SF, S)$$

such that

(i) for each $\theta \in \mathrm{Mor}_{\mathbf{M}}(S, T)$, the following square commutes

$$\begin{tikzcd}
SF \arrow[r, "\theta F"] \arrow[d, "\eta_S"'] & TF \arrow[d, "\eta_T"] \\
S \arrow[r, "\theta"'] & T
\end{tikzcd}$$

and

(ii) if $\theta \in \mathrm{Mor}_{\mathbf{M}}(S, T)$ is onto, then $\theta F \in \mathrm{Mor}_{\mathbf{PLR}}(SF, TF)$ is also onto.

Proposition 9.3.4. *The functor $F^e : \mathbf{M} \to \mathbf{PLR}$ is an expansion.*

Proof. Let S and T be objects of **M**. We wish to show there is an onto morphism $\eta_S \in \mathrm{Mor}_{\mathbf{PLR}}(SF^e, S)$ such that for each morphism θ between S and T, the square

$$\begin{tikzcd}
SF^e \arrow[r, "\theta^e"] \arrow[d, "\eta_S"'] & TF^e \arrow[d, "\eta_T"] \\
S \arrow[r, "\theta"'] & T
\end{tikzcd}$$

commutes, and if θ is onto then so is θF^e.

Let us define η_S by

$$(\Delta, s)\eta_S = s.$$

Clearly η_S is an onto monoid morphism and $(\Delta, s)^+\eta_S = ((\Delta, s)\eta_S)^+$. Hence

$$\eta_S \in \operatorname{Mor}_{\mathbf{PLR}}(SF^e, S).$$

Considering $\theta \in \operatorname{Mor}_{\mathbf{M}}(S, T)$ and $(\Delta, s) \in SF^e$, i.e. $(\Delta, s) \in \mathcal{M}(S, I_S, S)$, we have

$$(\Delta, s)\theta^e\eta_T = (\Delta\theta', s\theta)\eta_T = s\theta = (\Delta, s)\eta_S\theta.$$

So the square commutes.

Now suppose that θ is onto. By Proposition 9.2.6, $\mathcal{M} = \langle T\tau_{\mathcal{M}}\rangle$, where $\mathcal{M} = \mathcal{M}(T, I_T, T)$, i.e.

$$\left\{ \left(\underset{1}{\bullet} \xrightarrow{\quad t \quad} \underset{t}{\bullet}, t \right) : t \in T \right\}$$

generates TF^e. As θ is onto, for $t \in T$ there exists $s \in S$ such that $t = s\theta$. So,

$$\left(\underset{1}{\bullet} \xrightarrow{\quad t \quad} \underset{t}{\bullet}, t \right) = \left(\underset{1}{\bullet} \xrightarrow{\quad s\theta \quad} \underset{s\theta}{\bullet}, s\theta \right) = \left(\underset{1}{\bullet} \xrightarrow{\quad s \quad} \underset{s}{\bullet}, s \right)\theta^e.$$

So, as θ^e is a morphism by Proposition 9.3.1, θ^e is onto if θ is onto. $\square$

We shall define another functor

$$F^\sigma : \mathbf{PLR} \to \mathbf{M}$$

as follows. Let the action of F^σ on objects be given by $MF^\sigma = M/\sigma_M$ for an object M in **PLR**. As M/σ_M is a monoid, F^σ maps an object of **PLR** to an object of **M**. Considering $\theta \in \operatorname{Mor}_{\mathbf{PLR}}(M, N)$, we put

$$\theta F^\sigma = \theta^\sigma,$$

where

$$[m]\theta^\sigma = [m\theta].$$

Proposition 9.3.5. *For objects M and N in* **PLR**,

$$\theta^\sigma \in \operatorname{Mor}_{\mathbf{M}}(MF^\sigma, NF^\sigma).$$

Proof. First let us show that θ^σ is well defined. If $m, m' \in M$ such that

$$[m] = [m'],$$

then $m \,\sigma_M\, m'$ and so $em = em'$ for some $e \in E_M$. Therefore

$$(e\theta)(m\theta) = (e\theta)(m'\theta)$$

and hence $(m\theta)\,\sigma_N\,(m'\theta)$ as $e\theta \in E_N$. We have $[m\theta] = [m'\theta]$, i.e.

$$[m]\theta^\sigma = [m']\theta^\sigma.$$

So θ^σ is well-defined.

Let us consider $[m], [n] \in M/\sigma_M$. We note σ_M is a $(2,1,0)$-congruence. We have

$$\begin{aligned}([m][n])\theta^\sigma &= [mn]\theta^\sigma \\ &= [(mn)\theta] \\ &= [(m\theta)(n\theta)] \\ &= ([m]\theta^\sigma)([n]\theta^\sigma).\end{aligned}$$

We also have

$$\begin{aligned}[m]^+\theta^\sigma &= [m^+]\theta^\sigma \\ &= [m^+\theta] \\ &= [(m\theta)^+] \\ &= [m\theta]^+ \\ &= ([m]\theta^\sigma)^+\end{aligned}$$

and

$$1_{M/\sigma_M}\theta^\sigma = [1_M]\theta^\sigma = [1_M\theta] = [1_N] = 1_{N/\sigma_N}.$$

So θ^σ is a $(2,1,0)$-morphism. □

We note that θ^σ is a morphism within $\mathbf{M}$ and so we have the following proposition:

Proposition 9.3.6. *As defined above, F^σ is a functor from* **PLR** *to* **M**.

Proof. Let M be an object of **PLR** and consider MF^σ, i.e. M/σ_M. Taking $[m] \in M/\sigma_M$, we have

$$\begin{aligned}[m]I_MF^\sigma &= [m]I_M^\sigma \\ &= [m\,I_M] \\ &= [m].\end{aligned}$$

So,

$$I_MF^\sigma = I_{MF^\sigma}.$$

Now consider $\psi \in \mathrm{Mor}_{\mathbf{PLR}}(S,T)$ and $\varphi \in \mathrm{Mor}_{\mathbf{PLR}}(T,U)$, where S, T and U are objects in **PLR**. Considering $[m] \in M/\sigma_M$, we have the following. As in the proof of Proposition 9.3.2, we use the symbol $\circ$ to denote the composition of functions for clarity:

$$\begin{aligned}
[m](\psi \circ \varphi)F^\sigma &= [m](\psi \circ \varphi)^\sigma \\
&= [m(\psi \circ \varphi)] \\
&= [(m\psi)\varphi] \\
&= [m\psi]\varphi^\sigma \\
&= ([m]\psi^\sigma)\varphi^\sigma \\
&= ([m]\psi F^\sigma)\varphi F^\sigma \\
&= [m](\psi F^\sigma \circ \varphi F^\sigma).
\end{aligned}$$

Hence

$$(\psi \circ \varphi)F^\sigma = \psi F^\sigma \circ \varphi F^\sigma$$

and so F^σ is a functor between **PLR** and **M**. □

Ideally, we would like to show that F^e is a left adjoint of F^σ. However, this is not the case. We present two alternative approaches to give us the desired result. Our first method is analogous to that in [40] for X-generated proper inverse monoids and that in [22] for X-generated proper left ample monoids. Our second is analogous to that in [23] for proper left ample monoids, where we alter our functors to $F^e : \mathbf{M} \to \mathbf{PLR^0}$ and $F^\sigma : \mathbf{PLR^0} \to \mathbf{M}$, where $\mathbf{PLR^0}$ is the category of proper left restriction monoids equipped with an extra unary operation, which we shall define.

For the first method, we fix a set of generators for the monoids under consideration and define functors

$$F^e_X : \mathbf{M}(X) \to \mathbf{PLR}(X)$$

and

$$F^\sigma_X : \mathbf{PLR}(X) \to \mathbf{M}(X),$$

where $\mathbf{M}(X)$ and $\mathbf{PLR}(X)$ are defined in Section 9.2. We shall show that F^e_X is an expansion and a left adjoint of F^σ_X.

First we shall define $F^e_X : \mathbf{M}(X) \to \mathbf{PLR}(X)$. Suppose that (f, S) is an object in $\mathbf{M}(X)$. From Proposition 9.2.6, $(\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X, f, S))$ is an object in $\mathbf{PLR}(X)$. Let

$$(f, S)F^e_X = (\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X, f, S)).$$

If (g, T) is another object in $\mathbf{M}(X)$ and $\theta \in \mathrm{Mor}_{\mathbf{M}(X)}((f, S), (g, T))$, then we define a map, denoted by θ'', from $\Gamma(X, f, S)$ to $\Gamma(X, g, T)$ by the obvious action on vertices and action on edges given by

$$(s, x, s(xf))\theta'' = (s\theta, x, s\theta xg).$$

Then θ'' is a graph morphism as in Section 9.2. Hence, θ'' maps subgraphs to subgraphs and paths to paths. In particular, θ'' maps a 1-rooted subgraph of $\Gamma(X, f, S)$ to a 1-rooted subgraph of $\Gamma(X, g, T)$ and so we can define

$$\theta_X^e : \mathcal{M}(X, f, S) \to \mathcal{M}(X, g, T)$$

by

$$(\Delta, s)\theta_X^e = (\Delta\theta'', s\theta).$$

By Proposition 9.2.12,

$$\theta_X^e \in \text{ Mor}_{\mathbf{PLR}(X)}(\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X, f, S)), (\tau_{\mathcal{M}(X,g,T)}, \mathcal{M}(X, g, T)))$$

for objects (f, S) and (g, T) in $\mathbf{M}(X)$ and so

$$\theta_X^e \in \text{ Mor}_{\mathbf{PLR}(X)}((f, S)F_X^e, (g, T)F_X^e).$$

Let us put

$$\theta F_X^e = \theta_X^e.$$

We have the following result.

Proposition 9.3.7. *As defined above, F_X^e is a functor from $\mathbf{M}(X)$ to $\mathbf{PLR}(X)$.*

Proof. First we note that F_X^e associates each object of $\mathbf{M}(X)$ to an object of $\mathbf{PLR}(X)$ and a morphism of $\mathbf{M}(X)$ to a morphism of $\mathbf{PLR}(X)$.

Let (f, S) be an object of $\mathbf{M}(X)$ and let us consider $(f, S)F_X^e$, i.e. $(\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X, f, S))$. As in the proof of Proposition 9.3.2, considering $(\Delta, s) \in \mathcal{M}(X, f, S)$, we have

$$I_{(f,S)}F_X^e = I_{(f,S)F_X^e}.$$

Considering $\mu \in \text{ Mor}_{\mathbf{M}(X)}((f, S), (g, T))$ and $\delta \in \text{ Mor}_{\mathbf{M}(X)}((g, T), (h, U))$, where (f, S), (g, T) and (h, U) are objects in $\mathbf{M}(X)$, we have $\mu : S \to T$ and $\delta : T \to U$. As in the proof of Proposition 9.3.2, considering $(\Delta, s) \in \tau_{\mathcal{M}(X,f,S)}$,

$$(\mu \circ \delta)F_X^e = \mu F_X^e \circ \delta F_X^e$$

and F_X^e is a functor between $\mathbf{M}(X)$ and $\mathbf{PLR}(X)$.

□

As we did for Definition 9.3.3, we regard $\mathbf{M}(X)$ as a subcategory of $\mathbf{PLR}(X)$ and have the following definition:

Definition 9.3.8. We say that a functor $F : \mathbf{M}(X) \to \mathbf{PLR}(X)$ is an *expansion* if for any object (f, S) of $\mathbf{M}(X)$ there is an onto morphism

$$\eta_{(f,S)} \in \text{ Mor}_{\mathbf{PLR}(X)}((f, S)F, (f, S))$$

such that

(i) for each $\theta \in \text{Mor}_{\mathbf{M}(X)}((f, S), (g, T))$, the following square commutes

$$\begin{tikzcd}
(f,S)F \arrow[r, "\theta F"] \arrow[d, "\eta_{(f,S)}"'] & (g,T)F \arrow[d, "\eta_{(g,T)}"] \\
(f,S) \arrow[r, "\theta"'] & (g,T)
\end{tikzcd}$$

and

(ii) if $\theta \in \mathrm{Mor}_{\mathbf{M}(X)}((f,S),(g,T))$ is onto, then

$$\theta F \in \mathrm{Mor}_{\mathbf{PLR}(X)}((f,S)F,(g,T)F)$$

is also onto.

Proposition 9.3.9. *The functor* $F_X^e : \mathbf{M}(X) \to \mathbf{PLR}(X)$ *is an expansion.*

Proof. From Proposition 9.2.2, if $\psi : A \to B$ is a morphism in $\mathbf{PLR}(X)$, then ψ is unique and is onto if it exists. It is therefore enough to show that

$$\mathrm{Mor}_{\mathbf{PLR}(X)}((f,S)F_X^e,(f,S)) \neq \emptyset,$$

where (f,S) is an object of $\mathbf{M}(X)$, to show that F_X^e is an expansion. If this mapping exists, we shall denote it by $\eta_{(f,S)}$. We note that if this mapping exists, Condition (i) would hold as $\theta F_X^e \eta_{(g,T)}$ and $\eta_{(f,S)}\theta$ would both be morphisms from $(f,S)F_X^e$ to (g,T), and by uniqueness they would be equal.

Let us define

$$\eta_{(f,S)} : \mathcal{M}(X,f,S) \to S$$

by

$$(\Delta, s)\eta_{(f,S)} = s.$$

Then $\eta_{(f,S)}$ is a $(2,1,0)$-morphism and for $x \in X$,

$$x\tau_{\mathcal{M}(X,f,S)}\eta_{(f,S)} = \left(\underset{1}{\bullet} \xrightarrow{\quad x \quad} \underset{xf}{\bullet}, xf \right) \eta_{(f,S)} = xf.$$

So

$$\eta_{(f,S)} \in \ \mathrm{Mor}_{\mathbf{PLR}(X)} \left((\tau_{\mathcal{M}(X,f,S)}, \mathcal{M}(X,f,S)), (f,S) \right),$$

i.e.

$$\eta_{(f,S)} \in \ \mathrm{Mor}_{\mathbf{PLR}(X)}((f,S)F_X^e,(f,S)).$$

□

We now define the functor $F_X^\sigma : \mathbf{PLR}(X) \to \mathbf{M}(X)$ and show that F_X^e is a left adjoint of F_X^σ.

The action of F_X^σ on objects of $\mathbf{PLR}(X)$ is given by

$$(f,M)F_X^\sigma = (f\sigma_M^\natural, M/\sigma_M)$$

where $\sigma_M^\natural : M \to M/\sigma_M$ is the natural morphism. As (f, M) is an object of $\mathbf{PLR}(X)$, M is a proper left restriction monoid, $f : X \to M$ and $\langle Xf \rangle = M$. As M/σ_M is a monoid, $f\sigma_M^\natural : X \to M/\sigma_M$ and, by Corollary 1.2.9, $\langle Xf\sigma_M^\natural \rangle = M/\sigma_M$, so that $(f\sigma_M^\natural, M/\sigma_M)$ is an object in $\mathbf{M}(X)$. Suppose now that (f, M) and (g, N) are objects in $\mathbf{PLR}(X)$ and $\theta \in \mathrm{Mor}_{\mathbf{PLR}(X)}((f, M), (g, N))$ so that

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ M \arrow[rr, "\theta"'] & & N \end{tikzcd}$$

commutes. Let us define

$$\theta_X^\sigma : M/\sigma_M \to N/\sigma_N$$

by

$$[m]\theta_X^\sigma = [m\theta].$$

Proposition 9.3.10. *If (f, M) and (g, N) are objects in $\mathbf{PLR}(X)$ and we have $\theta \in \mathrm{Mor}_{\mathbf{PLR(X)}}((f, M), (g, N))$, then*

$$\theta_X^\sigma \in \mathrm{Mor}_{\mathbf{M(X)}}((f, M)F_X^\sigma, (g, N)F_X^\sigma).$$

Proof. As in the proof of Proposition 9.3.5, θ_X^σ is a $(2, 1, 0)$-morphism. For $x \in X$,

$$xf\sigma_M^\natural\theta_X^\sigma = [xf]\theta_X^\sigma = [xf\theta] = [xg] = xg\sigma_N^\natural.$$

So, $\theta_X^\sigma \in \mathrm{Mor}_{\mathbf{M}(X)}((f\sigma_M^\natural, M/\sigma_M), (g\sigma_N^\natural, N/\sigma_N))$, i.e.

$$\theta_X^\sigma \in \mathrm{Mor}_{\mathbf{M}(X)}((f, M)F_X^\sigma, (g, N)F_X^\sigma).$$

□

Let us put $\theta F_X^\sigma = \theta_X^\sigma$.

Proposition 9.3.11. *As defined above, F_X^σ is a functor from $\mathbf{PLR}(X)$ to $\mathbf{M}(X)$.*

Proof. Let (f, M) be an object of $\mathbf{PLR}(X)$. Let us consider $(f, M)F_X^\sigma$, i.e. $(f\sigma_M^\natural, M/\sigma_M)$. Considering $[m] \in M/\sigma_M$, as in Proposition 9.3.6 we have

$$I_{(f,M)}F_X^\sigma = I_{(f,M)F_X^\sigma}.$$

Let $\psi \in \mathrm{Mor}_{\mathbf{PLR}(X)}((f, M), (g, N))$ and $\varphi \in \mathrm{Mor}_{\mathbf{PLR}(X)}((g, N), (h, P))$, where (f, M), (g, N) and (h, P) are objects in $\mathbf{PLR}(X)$, $\psi : M \to N$ and $\varphi : N \to P$. As in Proposition 9.3.6,

$$(\psi \circ \varphi)F_X^\sigma = \psi F_X^\sigma \circ \varphi F_X^\sigma$$

and so F_X^σ is a functor between $\mathbf{PLR}(X)$ and $\mathbf{M}(X)$. $\square$

Theorem 9.3.12. *The functor F_X^e is a left adjoint of the functor F_X^σ.*

Proof. We have to show that for any objects (f,S) in $\mathbf{M}(X)$ and (g,T) in $\mathbf{PLR}(X)$ there is a bijection

$$\lambda_{(f,S),(g,T)}:\ \mathrm{Mor}_{\mathbf{PLR}(X)}((f,S)F_X^e,(g,T)) \to\ \mathrm{Mor}_{\mathbf{M}(X)}((f,S),(g,T)F_X^\sigma)$$

so that for $\phi \in \mathrm{Mor}_{\mathbf{M}(X)}((f',S'),(f,S))$ and $\theta \in \mathrm{Mor}_{\ \mathbf{PLR}(X)}((g,T),(g',T'))$, the square

$$\begin{array}{ccc}
\mathrm{Mor}_{\mathbf{PLR}(X)}((f,S)F_X^e,(g,T)) & \xrightarrow{\lambda_{(f,S),(g,T)}} & \mathrm{Mor}_{\mathbf{M}(X)}((f,S),(g,T)F_X^\sigma) \\
{\scriptstyle \mathrm{Mor}\ (\phi_X^e,\theta)}\downarrow & & \downarrow{\scriptstyle \mathrm{Mor}\ (\phi,\theta_X^\sigma)} \\
\mathrm{Mor}_{\mathbf{PLR}(X)}((f',S')F_X^e,(g',T')) & \xrightarrow[\lambda_{(f',S'),(g',T')}]{} & \mathrm{Mor}_{\mathbf{M}(X)}((f',S'),(g',T')F_X^\sigma)
\end{array}$$

is commutative. Here

$$\mathrm{Mor}\ (\phi_X^e,\theta):\ \mathrm{Mor}_{\mathbf{PLR}(X)}((f,S)F_X^e,(g,T)) \to\ \mathrm{Mor}_{\mathbf{PLR}(X)}((f',S')F_X^e,(g',T'))$$

is given by

$$\psi\ \mathrm{Mor}\ (\phi_X^e,\theta) = (\phi_X^e)\psi\theta$$

and

$$\mathrm{Mor}\ (\phi,\theta_X^\sigma):\ \mathrm{Mor}_{\mathbf{M}(X)}((f,S),(g,T)F_X^\sigma) \to\ \mathrm{Mor}_{\mathbf{M}(X)}((f',S'),(g',T')F_X^\sigma)$$

is given by

$$\psi\ \mathrm{Mor}\ (\phi,\theta_X^\sigma) = \phi\psi(\theta_X^\sigma).$$

Let

(i) $\alpha \in \mathrm{Mor}_{\mathbf{PLR}(X)}((f,S)F_X^e,(g,T))$;

(ii) $\beta \in \mathrm{Mor}_{\mathbf{PLR}(X)}((f',S')F_X^e,(g',T'))$;

(iii) $\gamma \in \mathrm{Mor}_{\mathbf{M}(X)}((f,S),(g,T)F_X^\sigma)$;

(iv) $\delta \in \mathrm{Mor}_{\mathbf{M}(X)}((f',S'),(g',T')F_X^\sigma)$.

Then $\alpha : \mathcal{M}(X,f,S) \to T$, $\beta : \mathcal{M}(X,f',S') \to T'$, $\gamma : S \to T/\sigma_T$ and $\delta : S' \to T'/\sigma_{T'}$ are such that the following diagrams commute:

$$\begin{tikzcd} & X \arrow[dl, "\tau_{\mathcal{M}(X,f,S)}"'] \arrow[dr, "g"] & \\ \mathcal{M}(X,f,S) \arrow[rr, "\alpha"'] & & T \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "\tau_{\mathcal{M}(X,f',S')}"'] \arrow[dr, "g'"] & \\ \mathcal{M}(X,f',S') \arrow[rr, "\beta"'] & & T' \end{tikzcd}$$

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g\sigma_T^\natural"] & \\ S \arrow[rr, "\gamma"'] & & T/\sigma_T \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "f'"'] \arrow[dr, "g'\sigma_{T'}^\natural"] & \\ S' \arrow[rr, "\delta"'] & & T'/\sigma_{T'} \end{tikzcd}$$

The following diagrams also commute:

$$\begin{tikzcd} & X \arrow[dl, "\tau_{\mathcal{M}(X,f',S')}"'] \arrow[dr, "\tau_{\mathcal{M}(X,f,S)}"] & \\ \mathcal{M}(X,f',S') \arrow[rr, "\phi_X^e"'] & & \mathcal{M}(X,f,S) \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "g\sigma_T^\natural"'] \arrow[dr, "g'\sigma_{T'}^\natural"] & \\ T/\sigma_T \arrow[rr, "\theta_X^\sigma"'] & & T'/\sigma_{T'} \end{tikzcd}$$

We note that the morphism sets of $\mathbf{PLR}(X)$ and $\mathbf{M}(X)$ contain at most one element. As $\phi_X^e\alpha\theta : \mathcal{M}(X,f',S') \to T'$ is such that

$$\begin{tikzcd} & X \arrow[dl, "\tau_{\mathcal{M}(X,f',S')}"'] \arrow[dr, "g'"] & \\ \mathcal{M}(X,f',S') \arrow[rr, "\phi_X^e\alpha\theta"'] & & T' \end{tikzcd}$$

commutes, we have

$$\alpha \mathrm{Mor}\ (\phi_X^e, \theta) = \beta.$$

Similarly, as $\phi\gamma\theta_X^\sigma : S' \to T'/\sigma_{T'}$ is such that

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g'\sigma_{T'}^\natural"] & \\ S' \arrow[rr, "\phi\gamma\theta_X^\sigma"'] & & T'/\sigma_{T'} \end{tikzcd}$$

commutes, we have

$$\gamma \text{ Mor } (\phi, \theta_X^\sigma) = \delta.$$

It remains to show that for any objects (f, S) in $\mathbf{M}(X)$ and (g, T) in $\mathbf{PLR}(X)$,

$$\mathrm{Mor}_{\mathbf{PLR}(X)}((f, S)F_X^e, (g, T)) \neq \emptyset$$

if and only if

$$\mathrm{Mor}_{\mathbf{M}(X)}((f, S), (g, T)F_X^\sigma) \neq \emptyset.$$

First suppose that

$$\theta \in \mathrm{Mor}_{\mathbf{PLR}(X)}((f, S)F_X^e, (g, T)).$$

Since $F_X^\sigma : \mathbf{PLR}(X) \to \mathbf{M}(X)$ is a functor, we have

$$\theta F_X^\sigma = \theta_X^\sigma \in \mathrm{Mor}_{\mathbf{M}(X)}((f, S)F_X^e F_X^\sigma, (g, T)F_X^\sigma).$$

Now, writing $\mathcal{M} = \mathcal{M}(X, f, S)$, we have

$$(f, S)F_X^e F_X^\sigma = (\tau_{\mathcal{M}}, \mathcal{M})F_X^\sigma = (\tau_{\mathcal{M}}\sigma_{\mathcal{M}}^\natural, \mathcal{M}/\sigma_{\mathcal{M}})$$

and $(g, T)F_X^\sigma = (g\sigma_T^\natural, T/\sigma_T)$. So $\theta_X^\sigma : \mathcal{M}/\sigma_{\mathcal{M}} \to T/\sigma_T$ and the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "\tau_{\mathcal{M}}\sigma_{\mathcal{M}}^\natural"'] \arrow[dr, "g\sigma_T^\natural"] & \\ \mathcal{M}/\sigma_{\mathcal{M}} \arrow[rr, "\theta_X^\sigma"'] & & T/\sigma_T \end{tikzcd}$$

By Proposition 9.2.6, $(\tau_{\mathcal{M}}, \mathcal{M})$ is an object in $\mathbf{PLR}(X, f, S)$, so

$$\begin{tikzcd} X \arrow[d, "\tau_{\mathcal{M}}"'] \arrow[dr, "f"] & \\ \mathcal{M} \arrow[r, "\sigma_{\mathcal{M}}^\sharp"'] & S \end{tikzcd}$$

commutes, where $\sigma_{\mathcal{M}}^\sharp$ is a morphism with kernel $\sigma_{\mathcal{M}}$. As remarked in

Section 9.2 $\sigma^\sharp_{\mathcal{M}}$ is onto, so S is isomorphic to $\mathcal{M}/\sigma_{\mathcal{M}}$ and we can define

$$\beta : S \to \mathcal{M}/\sigma_{\mathcal{M}}$$

by

$$(m\sigma^\sharp_{\mathcal{M}})\beta = [m].$$

For any $m, m' \in \mathcal{M}$,

$$m\sigma^\sharp_{\mathcal{M}} = m'\sigma^\sharp_{\mathcal{M}} \text{ if and only if } m \ \sigma_{\mathcal{M}} \ m',$$

so β is well defined. It is easy to check that β is a $(2,0)$-morphism. For any $x \in X$,

$$xf\beta = x\tau_{\mathcal{M}}\sigma^\sharp_{\mathcal{M}}\beta = [x\tau_{\mathcal{M}}] = x\tau_{\mathcal{M}}\sigma^\sharp_{\mathcal{M}}$$

so the following diagram commutes. The interior triangles commute by the above. Therefore the exterior triangles commute also.

$$\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[d, "\tau_{\mathcal{M}}\sigma^\natural_{\mathcal{M}}"] \arrow[dr, "g\sigma^\natural_T"] & \\
S \arrow[r, "\beta"'] & \mathcal{M}/\sigma_{\mathcal{M}} \arrow[r, "\theta^\sigma_X"'] & T/\sigma_T
\end{tikzcd}$$

Therefore

$$\beta\theta^\sigma_X \in \operatorname{Mor}_{\mathbf{M}(X)}((f,S),(g,T)F^\sigma_X).$$

Conversely, suppose that

$$\psi \in \operatorname{Mor}_{\mathbf{M}(X)}((f,S),(g,T)F^\sigma_X).$$

Since $F^e_X : \mathbf{M}(X) \to \mathbf{PLR}(X)$ is a functor,

$$\psi F^e_X = \psi^e_X \in \operatorname{Mor}_{\mathbf{PLR}(X)}((f,S)F^e_X,(g,T)F^\sigma_X F^e_X).$$

Now $(g,T)F^\sigma_X = (g\sigma^\natural_T, T/\sigma_T)$ so that $(X, g\sigma^\natural_T, T/\sigma_T)$ is a monoid presentation of the monoid T/σ_T. Putting $\mathcal{M} = \mathcal{M}(X, g\sigma^\natural_T, T/\sigma_T)$, we have

$$(g,T)F^\sigma_X F^e_X = (\tau_{\mathcal{M}}, \mathcal{M}).$$

By Proposition 9.2.6, $(\tau_{\mathcal{M}}, \mathcal{M})$ is an object in $\mathbf{PLR}(X, g\sigma^\natural_T, T/\sigma_T)$. As

$$\begin{tikzcd}
X \arrow[d, "g"'] \arrow[dr, "g\sigma^\natural_T"] & \\
T \arrow[r, "\sigma^\natural_T"'] & T/\sigma_T
\end{tikzcd}$$

certainly commutes, (g,T) is an object in $\mathbf{PLR}(X, g\sigma^\natural_T, T/\sigma_T)$. By Theorem 9.2.7, $(\tau_{\mathcal{M}}, \mathcal{M})$ is an initial object in this category, so there is a morphism

$$\phi \in \operatorname{Mor}_{\mathbf{PLR}(X)}((\tau_{\mathcal{M}}, \mathcal{M}), (g, T)),$$

i.e.

$$\phi \in \operatorname{Mor}_{\mathbf{PLR}(X)}((g, T)F_X^{\sigma}F_X^{e}, (g, T)).$$

As $\psi_X^e \in \operatorname{Mor}_{\mathbf{PLR}(X)}((f, S)F_X^e, (g, T)F_X^{\sigma}F_X^e)$,

$$\psi_X^e \phi \in \operatorname{Mor}_{\mathbf{PLR}(X)}((f, S)F_X^e, (g, T))$$

and therefore $\operatorname{Mor}_{\mathbf{PLR}(X)}((f, S)F_X^e, (g, T)) \neq \emptyset$. Hence F_X^e is a left adjoint of the functor F_X^{σ}. $\square$

We shall now return to the functors F^e and F^{σ} and look at how to adapt these so that F^e is a left adjoint of F^{σ}. To do this, we shall consider proper left restriction monoids as having an extra unary operation. The category $\mathbf{PLR^0}$ has as objects proper left restriction monoids given an added unary operation $^{\circ}$ such that for any proper left restriction monoid S the following hold:

(i) $s \,\sigma_S\, s^{\circ}$ for all $s \in S$;

(ii) $\{s^{\circ} : s \in S\}$ is a cross-section of the σ_S-classes, i.e. each σ_S-class contains exactly one element of $\{s^{\circ} : s \in S\}$.

The morphisms of $\mathbf{PLR^0}$ are the morphisms between objects regarded as algebras of type $(2, 1, 1, 0)$.

For a proper left restriction monoid, there are many choices for $^{\circ}$. We note that if S is a monoid regarded as a reduced left restriction monoid with distinguished semilattice $E_S = \{1\}$, then for $a, b \in S$,

$$\begin{aligned} a \,\sigma_S\, b &\Leftrightarrow ea = eb \text{ for some } e \in E_S \\ &\Leftrightarrow a = b \text{ as } E_S = \{1\}. \end{aligned}$$

Hence the only way a monoid S can be made into an object of $\mathbf{PLR^0}$ is if $s^{\circ} = s$ for all $s \in S$. Let $\mathbf{M^0}$ denote the category of monoids with the extra unary operation $^{\circ}$. We know $F^e : \mathbf{M} \to \mathbf{PLR}$ is a functor. We shall choose $^{\circ}$ such that F^e is a functor between $\mathbf{M^0}$ and $\mathbf{PLR^0}$.

For a monoid S, let us define $^{\circ}$ on $SF^e = \mathcal{M}(S, I_S, S)$ by

$$(\Sigma, s)^{\circ} = \left(\underset{1}{\bullet} \xrightarrow{\quad s \quad} \underset{s}{\bullet}, s \right),$$

i.e. $(\Sigma, s)^{\circ} = s\tau_{\mathcal{M}}$ where $(\Sigma, s) \in \mathcal{M}(S, I_S, S)$. By Proposition 9.1.2, this definition satisfies Conditions (i) and (ii) and SF^e is an object in $\mathbf{PLR^0}$.

Let S and T be objects in $\mathbf{M^0}$ and $\theta \in \operatorname{Mor}_{\mathbf{M^0}}(S, T)$, so $\theta : S \to T$. We know

$$\theta^e \in \operatorname{Mor}_{\mathbf{PLR}}(SF^e, TF^e)$$

and we also have

$$
\begin{aligned}
(\Delta, s)^{\circ}\theta^{e} &= \left(\underset{1}{\bullet} \xrightarrow{\quad s \quad} \underset{s}{\bullet}, s \right) \theta^{e} \\
&= \left(\underset{1}{\bullet} \xrightarrow{\quad s\theta \quad} \underset{s\theta}{\bullet}, s\theta \right) \\
&= (\Delta\theta', s\theta)^{\circ} \\
&= [(\Delta, s)\theta^{e}]^{\circ}.
\end{aligned}
$$

So $\theta^e : SF^e \to TF^e$ is a $(2, 1, 1, 0)$-morphism and hence

$$\theta F^e = \theta^e \in \ \mathrm{Mor}_{\mathbf{PLR^0}}(SF^e, TF^e).$$

Therefore F^e is a functor from $\mathbf{M^0}$ to $\mathbf{PLR^0}$.

As $s^{\circ} = s$ for all $s \in S$, where S is an object in $\mathbf{M^0}$, θ^{σ} preserves $\circ$ and hence F^{σ} is a functor from $\mathbf{PLR^0}$ to $\mathbf{M^0}$. We have our desired result:

Theorem 9.3.13. *Regarded as functors between* $\mathbf{M^0}$ *and* $\mathbf{PLR^0}$*,* F^e *is a left adjoint of* F^{σ}*.*

Proof. We must prove that for any object T in $\mathbf{M^0}$ and object S in $\mathbf{PLR^0}$, there is a bijection

$$\alpha_{T:S} : \ \mathrm{Mor}_{\mathbf{PLR^0}}(TF^e, S) \to \ \mathrm{Mor}_{\mathbf{M^0}}(T, SF^{\sigma})$$

such that for any $T' \in \mathrm{Ob}\ \mathbf{M^0}$, $S' \in \mathrm{Ob}\ \mathbf{PLR^0}$, $\phi \in \mathrm{Mor}_{\mathbf{M^0}}(T', T)$ and $\theta \in \mathrm{Mor}_{\mathbf{PLR^0}}(S, S')$, the square

$$
\begin{tikzcd}
\mathrm{Mor}_{\mathbf{PLR^0}}(TF^e, S) \arrow[r, "\alpha_{T:S}"] \arrow[d, "\mathrm{Mor}\ (\phi^e, \theta)"'] & \mathrm{Mor}_{\mathbf{M^0}}(T, SF^{\sigma}) \arrow[d, "\mathrm{Mor}\ (\phi, \theta^{\sigma})"] \\
\mathrm{Mor}_{\mathbf{PLR^0}}(T'F^e, S') \arrow[r, "\alpha_{T':S'}"'] & \mathrm{Mor}_{\mathbf{M^0}}(T', S'F^{\sigma})
\end{tikzcd}
$$

commutes. Here

$$\mathrm{Mor}\ (\phi^e, \theta) : \ \mathrm{Mor}_{\mathbf{PLR^0}}(TF^e, S) \to \ \mathrm{Mor}_{\mathbf{PLR^0}}(T'F^e, S')$$

is given by

$$\psi\ \mathrm{Mor}\ (\phi^e, \theta) = \phi^e \psi \theta$$

and

$$\mathrm{Mor}\ (\phi, \theta^{\sigma}) : \ \mathrm{Mor}_{\mathbf{M^0}}(T, SF^{\sigma}) \to \ \mathrm{Mor}_{\mathbf{M^0}}(T', S'F^{\sigma})$$

is given by

$$\psi\ \mathrm{Mor}\ (\phi, \theta^{\sigma}) = \phi\psi\theta^{\sigma}.$$

Let T be an object in $\mathbf{M^0}$. In view of Proposition 9.1.2, we define an isomorphism $\sigma_T : T \to TF^e/\sigma$, where $\sigma = \sigma_{TF^e}$, by

$$t\sigma_T = \left[\left(\underset{1}{\bullet} \xrightarrow{\quad t \quad} \underset{t}{\bullet}\, , t \right)\right] = [(\Sigma, t)]$$

for any $(\Sigma, t) \in TF^e$. If S is an object in $\mathbf{PLR^0}$ then we define

$$\alpha_{T:S} : \ \mathrm{Mor}_{\mathbf{PLR^0}}(TF^e, S) \to \ \mathrm{Mor}_{\mathbf{M^0}}(T, SF^\sigma)$$

by

$$\psi\alpha_{T:S} = \sigma_T\psi^\sigma$$

as $\psi^\sigma : TF^e/\sigma \to SF^\sigma$, where $\sigma = \sigma_{TF^e}$.

We first show that the above diagram commutes. Let $\psi \in \mathrm{Mor}_{\mathbf{PLR^0}}(TF^e, S)$ and $t' \in T'$. Then

$$\begin{aligned} t'(\psi\alpha_{T:S}\ \mathrm{Mor}\ (\phi, \theta^\sigma)) &= t'(\phi\psi\alpha_{T:S}\theta^\sigma) \\ &= t'(\phi\sigma_T\psi^\sigma\theta^\sigma) \\ &= [(\Sigma, t'\phi)]\psi^\sigma\theta^\sigma \\ &= [(\Sigma, t'\phi)\psi\theta] \end{aligned}$$

for any $(\Sigma, t'\phi) \in TF^e$. We also have

$$\begin{aligned} t'(\psi\ \mathrm{Mor}\ (\phi^e, \theta)\alpha_{T':S'}) &= t'(\phi^e\psi\theta\alpha_{T':S'}) \\ &= t'(\sigma_{T'}(\phi^e\psi\theta)^\sigma) \\ &= [(\Delta, t')](\phi^e\psi\theta)^\sigma \\ &= [(\Delta, t')\phi^e\psi\theta] \\ &= [(\Delta\phi', t'\phi)\psi\theta] \end{aligned}$$

for any $(\Delta, t') \in T'F^e$. For any $(\Sigma, t'\phi) \in TF^e$ and $(\Delta, t') \in T'F^e$ we have $(\Sigma, t'\phi), (\Delta\phi', t'\phi) \in TF^e$ and by Proposition 9.1.2,

$$(\Sigma, t'\phi)\ \sigma_{TF^e}\ (\Delta\phi', t'\phi).$$

Then by Proposition 2.7.4,

$$(\Sigma, t'\phi)\psi\theta\ \sigma_T\ (\Delta\phi', t'\phi)\psi\theta.$$

So $\psi\alpha_{T:S}$ Mor (ϕ, θ^σ) and ψ Mor $(\phi^e, \theta)\alpha_{T':S'}$ are the same maps and hence the diagram commutes.

It remains to show that $\alpha_{T:S}$ is a bijection. We shall use earlier results to construct an inverse $\beta_{T:S}$ of $\alpha_{T:S}$.

First consider $\gamma : S/\sigma_S \to S$ defined by $[s]\gamma = s^\circ, s \in S$. Then for any $[s] \in S/\sigma_S$,

$$[s]\gamma\sigma_S^\natural = s^\circ\sigma_S^\natural = [s^\circ] = [s] = [s]I_{S/\sigma_S}$$

and hence the following diagram commutes:

$$\begin{tikzcd}
S/\sigma_S \arrow[d, "\gamma"'] \arrow[dr, "I_{S/\sigma_S}"] & \\
S \arrow[r, "\sigma_S^\natural"'] & S/\sigma_S
\end{tikzcd}$$

Let $K = \langle (S/\sigma_S)\gamma \rangle$. So K is the $(2,1,0)$-subalgebra, or equivalently, the $(2,1,1,0)$-subalgebra of S generated by $\{s^\circ : s \in S\}$. It follows from Proposition 2.8.6 that K is a proper left restriction monoid and so K is an object of $\mathbf{PLR^0}$. Certainly

$$\begin{tikzcd}
S/\sigma_S \arrow[d, "\gamma"'] \arrow[dr, "I_{S/\sigma_S}"] & \\
K \arrow[r, "\delta"'] & S/\sigma_S
\end{tikzcd}$$

commutes, where δ is the restriction of $\sigma_S^\natural$ to K. By Proposition 2.8.6, Ker $\delta = \sigma_K$. Therefore the pair (γ, K) is an object in the category $\mathbf{PLR}(S/\sigma_S, I_{S/\sigma_S}, S/_{\sigma_S})$. By Theorem 9.2.7, $(\tau_{\mathcal{M}}, \mathcal{M})$, where $\mathcal{M} = \mathcal{M}(S/\sigma_S, I_{S/\sigma_S}, S/\sigma_S)$, is an initial object in $\mathbf{PLR}(S/\sigma_S, I_{S/\sigma_S}, S/_{\sigma_S})$. Here $\tau_{\mathcal{M}} : S/\sigma_S \to \mathcal{M}$ is the map given by

$$[s]\tau_{\mathcal{M}} = \left(\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "{[s]}"] & \underset{[s]}{\bullet} \end{tikzcd}, [s] \right).$$

So there is a unique $(2,1,0)$-morphism $\pi : \mathcal{M} \to K$ such that

$$\begin{tikzcd}
S/\sigma_S \arrow[d, "\tau_{\mathcal{M}}"'] \arrow[dr, "\gamma"] & \\
\mathcal{M} \arrow[r, "\pi"'] & K
\end{tikzcd}$$

commutes.

We note $\mathcal{M} = (S/\sigma_S)F^e$. Regarding $\mathcal{M}$ and K as $(2,1,1,0)$-algebras we then have

$$(\Sigma, [s])^\circ \pi = \left(\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "{[s]}"] & \underset{[s]}{\bullet} \end{tikzcd}, [s] \right) \pi = [s]\tau_{\mathcal{M}}\pi = [s]\gamma = s^\circ$$

for any $(\Sigma, [s]) \in \mathcal{M}$. As π is a $(2,1,0)$-morphism,

$$(\Sigma, [s])\pi \ \sigma_K \ (\Sigma, [s])^\circ\pi,$$

so

$$((\Sigma, [s])\pi)^\circ = ((\Sigma, [s])^\circ)\pi^\circ = (s^\circ)^\circ = s^\circ = (\Sigma, [s])^\circ\pi.$$

Therefore π is a $(2,1,1,0)$-morphism.

We now define

$$\beta_{T:S} : \ \mathrm{Mor}_{\mathbf{M^0}}(T, SF^\sigma) \to \ \mathrm{Mor}_{\mathbf{PLR^0}}(TF^e, S)$$

by

$$\psi\beta_{T:S} = \psi^e\pi$$

where we regard π as a morphism from $\mathcal{M}$ to S.

We wish to show $\beta_{T:S}\alpha_{T:S}$ is the identity map in $\mathrm{Mor}_{\mathbf{M^0}}(T, SF^\sigma)$. Let $\psi \in \mathrm{Mor}_{\mathbf{M^0}}(T, SF^\sigma)$. From the definitions,

$$\psi\beta_{T:S}\alpha_{T:S} = \psi^e\pi\alpha_{T:S} = \sigma_T(\psi^e\pi)^\sigma.$$

Let $t \in T$. Then

$$\begin{aligned}
t\sigma_T(\psi^e\pi)^\sigma &= \left[\left(\underset{1}{\bullet} \xrightarrow{\quad t \quad} \underset{t}{\bullet}, t \right)\right](\psi^e\pi)^\sigma \\
&= \left[\left(\underset{1}{\bullet} \xrightarrow{\quad t \quad} \underset{t}{\bullet}, t \right)\psi^e\pi\right] \\
&= \left[\left(\underset{1}{\bullet} \xrightarrow{\quad t\psi \quad} \underset{t\psi}{\bullet}, t\psi \right)\pi\right].
\end{aligned}$$

We have $t\psi = [s]$ for some $s \in S$ and so

$$\begin{aligned}
t\psi\beta_{T:S}\alpha_{T:S} &= \left[\left(\underset{1}{\bullet} \xrightarrow{\quad [s] \quad} \underset{[s]}{\bullet}, [s] \right)\pi\right] \\
&= [[s]\tau_{\mathcal{M}}\pi] = [[s]\gamma] = [s^\circ] = [s] \\
&= t\psi.
\end{aligned}$$

Thus $\psi\beta_{T:S}\alpha_{T:S} = \psi$ and $\beta_{T:S}\alpha_{T:S}$ is the identity map in $\mathrm{Mor}_{\mathbf{M^0}}(T, SF^\sigma)$.

It remains to show $\alpha_{T:S}\beta_{T:S}$ is the identity map in $\mathrm{Mor}_{\mathbf{PLR^0}}(TF^e, S)$. Consider $\psi \in \mathrm{Mor}_{\mathbf{PLR^0}}(TF^e, S)$. Again from the definitions,

$$\psi\alpha_{T:S}\beta_{T:S} = \sigma_T\psi^\sigma\beta_{T:S} = (\sigma_T\psi^\sigma)^e\pi.$$

From Proposition 9.2.6, $TF^e = \langle T\tau_{\mathcal{M}(T,I_T,T)}\rangle$, so is generated as a $(2,1,0)$-

algebra by elements of the form $\left(\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t\right)$ where $t \in T$. Thus to show that $\psi\alpha_{T:S}\beta_{T:S} = \psi$, it is enough to show that for any $t \in T$,

$$\left(\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t\right)(\sigma_T\psi^\sigma)^e\pi = \left(\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t\right)\psi.$$

Let $t \in T$. We have

$$\left(\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t\right)(\sigma_T\psi^\sigma)^e\pi = \left(\underset{1}{\bullet}\xrightarrow{\quad [s] \quad}\underset{[s]}{\bullet}, [s]\right)\pi$$

where $[s] = t\sigma_T\psi^\sigma = \left[\left(\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t\right)\psi\right]$. Thus, for $\mathcal{M} = (S/\sigma_S)F^e$,

$$\begin{aligned}\left(\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t\right)(\sigma_T\psi^\sigma)^e\pi &= [s]\tau_{\mathcal{M}}\pi = [s]\gamma = s^\circ \\ &= \left(\left(\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t\right)\psi\right)^\circ \\ &= \left(\left(\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t\right)^\circ\right)\psi,\end{aligned}$$

using the fact that ψ is a $(2,1,1,0)$-morphism. As

$$\left(\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t\right)^\circ = \left(\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t\right),$$

we have

$$\left(\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t\right)(\sigma_T\psi^\sigma)^e\pi = \left(\underset{1}{\bullet}\xrightarrow{\quad t \quad}\underset{t}{\bullet}, t\right)\psi$$

as required. Therefore F^e is a left adjoint of F^σ when regarded as functors between $\mathbf{M^0}$ and $\mathbf{PLR^0}$. $\square$

9.4 A construction of $\mathcal{M}(X, f, S)$

We shall continue to generalise results in [23] in preparation for the next section.

Let us write $\tau_{\mathcal{M}}$ for $\tau_{\mathcal{M}(X,f,S)}$, F_X for $\mathcal{M}(X, \iota, X^*)$ and τ for $\tau_{\mathcal{M}(X,\iota,X^*)}$. By Theorem 9.2.13, F_X is the free left restriction monoid on $X\tau$. So there is a morphism $\theta : F_X \to \mathcal{M}(X, f, S)$, for any monoid presentation (X, f, S), such that $\tau\theta = \tau_{\mathcal{M}}$.

In this section, we shall find a congruence ρ on F_X such that the pair $(\tau\rho^\natural, F_X/\rho)$ is an initial object in the category $\mathbf{PLR}(X, f, S)$. As the pair $(\tau_{\mathcal{M}}, \mathcal{M}(X, f, S))$ is also an initial object in this category by Theorem 9.2.7, uniqueness will give us $\mathcal{M}(X, f, S) \cong F_X/\rho$ and Ker $\theta = \rho$.

Before looking at the congruence, we need a few small results concerning F_X. By Proposition 9.2.6, (τ, F_X) is an object in the category $\mathbf{PLR}(X, \iota, X^*)$ and so

$$\begin{tikzcd} X \arrow[d, "\tau"'] \arrow[dr, "\iota"] & \\ F_X \arrow[r, "\sigma^\sharp_{F_X}"'] & X^* \end{tikzcd}$$

commutes, where $\sigma^\sharp_{F_X}$ is the onto morphism with kernel σ_{F_X} given by $(\Sigma, \overline{x})\sigma^\sharp_{F_X} = \overline{x}$. We can lift the maps τ and ι to monoid morphisms $\overline{\tau} : X^* \to F_X$ and $\overline{\iota} = I_{X^*} : X^* \to X^*$ in the usual way.

Lemma 9.4.1. *c.f. [15] With $\overline{\tau}$ defined as above, the diagram*

$$\begin{tikzcd} X^* \arrow[d, "\overline{\tau}"'] \arrow[dr, "I_{X^*}"] & \\ F_X \arrow[r, "\sigma^\sharp_{F_X}"'] & X^* \end{tikzcd}$$

commutes and Im $\overline{\tau}$ is isomorphic to X^*. Further, if $e(\bar{x}\overline{\tau}) = g(\bar{y}\overline{\tau})$ for $e, g \in E_{F_X}$ and $\overline{x}, \overline{y} \in X^*$, then $\overline{x} = \overline{y}$.

Proof. Clearly $1\overline{\tau}\sigma^\sharp_{F_X} = 1I_{X^*}$. Now take $x_1x_2\ldots x_n \in X^*$. Then

$$\begin{aligned}(x_1x_2\ldots x_n)\overline{\tau}\sigma^\sharp_{F_X} &= (x_1\overline{\tau}\sigma^\sharp_{F_X})(x_2\overline{\tau}\sigma^\sharp_{F_X})\ldots(x_n\overline{\tau}\sigma^\sharp_{F_X}) \\ &= (x_1\iota)(x_2\iota)\ldots(x_n\iota) \\ &= x_1x_2\ldots x_n \\ &= (x_1x_2\ldots x_n)I_{X^*}.\end{aligned}$$

Therefore the diagram commutes. Now take $u, v \in X^*$ and suppose

$$u\overline{\tau} = v\overline{\tau}.$$

Then

$$u\overline{\tau}\sigma^\sharp_{F_X} = v\overline{\tau}\sigma^\sharp_{F_X}, \text{ i.e. } u = v$$

as $\overline{\tau}\sigma^\sharp_{F_X} = I_{X^*}$. So $\overline{\tau}$ is one-to-one on X^* and so $\overline{\tau}$ is an isomorphism between X^* and Im $\overline{\tau}$.

Suppose $e(\bar{x}\bar{\tau}) = g(\bar{y}\bar{\tau})$, $e, g \in E_{F_X}$ and $\overline{x}, \overline{y} \in X^*$. Then

$$(e(\bar{x}\bar{\tau}))\sigma^\sharp_{F_X} = (g(\bar{y}\bar{\tau}))\sigma^\sharp_{F_X}.$$

Using Proposition 2.8.5 and $\overline{\tau}\sigma^\sharp_{F_X} = I_{X^*}$, we have

$$\begin{aligned}(e(\bar{x}\bar{\tau}))\sigma^\sharp_{F_X} = (g(\bar{y}\bar{\tau}))\sigma^\sharp_{F_X} &\Rightarrow (e\sigma^\sharp_{F_X})(\bar{x}\bar{\tau}\sigma^\sharp_{F_X}) = (g\sigma^\sharp_{F_X})(\bar{y}\bar{\tau}\sigma^\sharp_{F_X}) \\ &\Rightarrow (\bar{x}\bar{\tau}\sigma^\sharp_{F_X}) = (\bar{y}\bar{\tau}\sigma^\sharp_{F_X}) \\ &\Rightarrow \bar{x} = \bar{y}.\end{aligned}$$

□

Proposition 9.4.2. *Any element of F_X can be written as $e(\bar{x}\bar{\tau})$ for some $e \in E_{F_X}$ and $\overline{x} \in X^*$.*

Proof. By Proposition 9.2.6, $F_X = \langle X\tau \rangle$. Then by Lemma 2.2.15, $a \in F_X$ can be written as

$$a = (x^1_1 \dots x^1_{p(1)})^+ \dots (x^m_1 \dots x^m_{p(m)})^+ y_1 \dots y_n$$

for some $m, n \in \mathbb{N}^0$ where $x^i_j, y_k \in X\tau, 1 \le i \le m, 1 \le j \le p(i), 1 \le k \le n$. As $y_1, y_2, \dots, y_n \in X\tau$, $y_1 y_2 \dots y_n \in X^*\overline{\tau}$. So $a = e(\bar{x}\bar{\tau})$, where $e \in E_{F_X}$ and $\overline{x} \in X^*$. □

For $a \in F_X$, we shall define the *positive part* $p(a)$ of $a \in F_X$ by $p(a) = \overline{x}$ where $a = e(\bar{x}\bar{\tau})$, where $e \in E_{F_X}$ and $\bar{x} \in X^*$.

Lemma 9.4.3. *The function $p : F_X \to X^*$ is a monoid morphism.*

Proof. Suppose $a \in F_X$ such that $a = e(\bar{x}\bar{\tau})$ and $a = h(\bar{z}\bar{\tau})$ where $e, h \in E_{F_X}$ and $\overline{x}, \overline{z} \in X^*$. Then by Lemma 9.4.1 $\overline{x} = \overline{z}$ and consequently $p(e(\bar{x}\bar{\tau})) = p(h(\bar{z}\bar{\tau}))$. Therefore the function p is well-defined.

Let $a, b \in F_X$ where $a = e(\bar{x}\bar{\tau})$ and $b = g(\bar{y}\bar{\tau})$ for $e, g \in E_{F_X}$ and $\overline{x}, \overline{y} \in X^*$. Using the ample condition,

$$ab = e(\bar{x}\bar{\tau})g(\bar{y}\bar{\tau}) = e(\bar{x}\bar{\tau}g)^+\bar{x}\bar{\tau}\bar{y}\bar{\tau} = e(\bar{x}\bar{\tau}g)^+(\bar{x}\,\bar{y})\bar{\tau}$$

so

$$p(a)p(b) = \bar{x}\,\bar{y} = p(ab).$$

Clearly, $p(1) = 1$, so p is a monoid morphism. □

Note that $p : F_X \to X^*$ can also be viewed as a $(2, 1, 0)$-morphism as X^* is cancellative.

Let (X, f, S) be a monoid presentation of a monoid S. Let $\bar{f}$ denote the extension of f to a monoid morphism from X^* to S. Let

$$H = H_{(X,f,S)} =$$

$$\{((\bar{u}\bar{\tau})^+\bar{v}\bar{\tau}, (\bar{v}\bar{\tau})^+\bar{u}\bar{\tau}) \in F_X \times F_X : \bar{u}, \bar{v} \in X^* \text{ and } \bar{u}\bar{f} = \bar{v}\bar{f}\}$$

and let $\rho = \rho_{(X,f,S)}$ be the $(2,1,0)$-congruence on F_X generated by H.

In the following result we use the fact that left restriction monoids form a variety. In [23], the analogous result is proved for the case when (X, f, S) is a monoid presentation of a right cancellative monoid S. Throughout that paper, left ample monoids are encountered. These form a quasivariety which as such is not closed under homomorphic images, a property of varieties which we shall use.

Proposition 9.4.4. *With ρ defined as above, F_X/ρ is a proper left restriction monoid.*

Proof. As left restriction monoids form a variety, we have closure under homomorphic images. So F_X/ρ is a left restriction monoid and it remains to show that F_X/ρ is proper.

Let $a\rho, b\rho \in F_X/\rho$ and suppose that

$$a\rho\ (\widetilde{\mathcal{R}}_{E_F} \cap \sigma_F)\ b\rho,$$

where $F = F_X/\rho$. So $a\rho\ \widetilde{\mathcal{R}}_{E_F}\ b\rho$ and $ha\ \rho\ hb$ for some $h \in E_{F_X}$ as

$$h\rho = (h\rho)^+ = h^+\rho$$

for $h\rho \in E_F$. From $a\rho\ \widetilde{\mathcal{R}}_{E_F}\ b\rho$ we obtain

$$a^+\rho = (a\rho)^+ = (b\rho)^+ = b^+\rho$$

so that

$$a^+\ \rho\ b^+. \tag{9.1}$$

We wish to show that $ha\ \rho\ hb$ implies that $p(a)\bar{f} = p(b)\bar{f}$. First we show that $H \subseteq \text{Ker } \sigma^\sharp_{F_X}\bar{f}$. Let $\bar{u}, \bar{v} \in X^*$ and $\bar{u}\bar{f} = \bar{v}\bar{f}$. Then by Lemma 9.4.1, taking $((\bar{u}\bar{\tau})^+\bar{v}\bar{\tau}, (\bar{v}\bar{\tau})^+\bar{u}\bar{\tau}) \in H$, we have $((\bar{u}\bar{\tau})^+\bar{v}\bar{\tau}, (\bar{v}\bar{\tau})^+\bar{u}\bar{\tau}) \in \text{Ker } \sigma^\sharp_{F_X}\bar{f}$ because

$$((\bar{u}\bar{\tau})^+\bar{v}\bar{\tau})\sigma^\sharp_{F_X}\bar{f} = \bar{v}\bar{\tau}\sigma^\sharp_{F_X}\bar{f} = \bar{v}\bar{f} = \bar{u}\bar{f} = \bar{u}\bar{\tau}\sigma^\sharp_{F_X}\bar{f} = ((\bar{v}\bar{\tau})^+\bar{u}\bar{\tau})\sigma^\sharp_{F_X}\bar{f}.$$

So $H \subseteq \text{Ker } \sigma^\sharp_{F_X}\bar{f}$. We note that $\sigma^\sharp_{F_X}$ is a $(2,1,0)$-morphism by Corollary 2.7.8. As $\bar{f} : X^* \to S$ is a monoid morphism it is a $(2,1,0)$-morphism by the comments after Lemma 2.5.4. It follows that $\sigma^\sharp_{F_X}\bar{f} : F_X \to S$ is a $(2,1,0)$-morphism. So $\rho \subseteq \text{Ker } \sigma^\sharp_{F_X}\bar{f}$.

Suppose $c, d \in F_X$ such that $c\ \rho\ d$. We wish to show that $p(c)\bar{f} = p(d)\bar{f}$. Since $c\ \rho\ d$, we have $(c, d) \in \text{Ker } \sigma^\sharp_{F_X}\bar{f}$ and hence

$$c\sigma^\sharp_{F_X}\bar{f} = d\sigma^\sharp_{F_X}\bar{f}.$$

As $c, d \in F_X$, then $c = e(\bar{x}\bar{\tau})$ and $d = g(\bar{y}\bar{\tau})$ for some $e, g \in E_{F_X}$ and

$\overline{x}, \overline{y} \in X^*$. So we have

$$[e(\bar{x}\bar{\tau})]\sigma^{\sharp}_{F_X}\bar{f} = [g(\bar{y}\bar{\tau})]\sigma^{\sharp}_{F_X}\bar{f}.$$

As any idempotent in E_{F_X} must be mapped to 1 in X^*,

$$\bar{x}\bar{\tau}\sigma^{\sharp}_{F_X}\bar{f} = \bar{y}\bar{\tau}\sigma^{\sharp}_{F_X}\bar{f}$$

and since $\sigma^{\sharp}_{F_X}\bar{f} = I_{X^*}$, we have

$$\bar{x}\bar{f} = \bar{y}\bar{f}.$$

Hence $c\ \rho\ d$ implies $p(c)\bar{f} = p(d)\bar{f}$.

In particular, if $h \in E_{F_X}$ such that $ha\ \rho\ hb$, $p(ha)\bar{f} = p(hb)\bar{f}$. As $a, b \in F_X$, then $a = j(\bar{w}\bar{\tau})$ and $b = k(\bar{z}\bar{\tau})$ for some $j, k \in E_{F_X}$ and $\overline{w}, \overline{z} \in X^*$. We have $ha = hj(\bar{w}\bar{\tau})$ and $hb = hk(\bar{z}\bar{\tau})$, where $hj, hk \in E_{F_X}$. So

$$p(ha) = p(a) \text{ and } p(hb) = p(b)$$

and therefore $p(a)\bar{f} = p(b)\bar{f}$.

It remains to show that $a\ \rho\ b$. We have

$$(p(b)\bar{\tau})^+p(a)\bar{\tau}\ \rho\ (p(a)\bar{\tau})^+p(b)\bar{\tau}.$$

As $a^+\ \rho\ b^+$, we have

$$\begin{aligned}
a\rho &= (a^+a)\rho \\
&= (b^+a)\rho \\
&= [k(\bar{z}\bar{\tau})^+j(\bar{w}\bar{\tau})]\rho \\
&= (kj)\rho[(\bar{z}\bar{\tau})^+(\bar{w}\bar{\tau})]\rho \\
&= (kj)\rho[(\bar{w}\bar{\tau})^+(\bar{z}\bar{\tau})]\rho \\
&= \cdots = (a^+b)\rho = (b^+b)\rho = b\rho.
\end{aligned}$$

Therefore F_X/ρ is a proper left restriction monoid. $\square$

Lemma 9.4.5. *The pair $(\tau\rho^{\natural}, F_X/\rho)$ is an object in $\mathbf{PLR}(X, f, S)$.*

Proof. First we note that F_X/ρ is a proper left restriction monoid by Proposition 9.4.4. Now $\tau : X \to F_X$ and $\rho^{\natural} : F_X \to F_X/\rho$, so $\tau\rho^{\natural} : X \to F_X/\rho$. We also have $F_X/\rho = \langle X\tau\rho^{\natural}\rangle$ as $F_X = \langle X\tau\rangle$ by Proposition 9.2.6 and $F_X\rho^{\natural} = \langle X\tau\rangle\rho^{\natural}$ implies $F_X/\rho = \langle X\tau\rho^{\natural}\rangle$ by Corollary 1.2.9. So $(\tau\rho^{\natural}, F_X/\rho)$ is an object in $\mathbf{PLR}(X, f, S)$.

Suppose $\eta : F_X/\rho \to S$ is defined by $(a\rho)\eta = a\sigma^{\sharp}_{F_X}\bar{f}$ for $a\rho \in F_X/\rho$. The function η is well-defined since $\rho \subseteq \operatorname{Ker}\ \sigma^{\sharp}_{F_X}\bar{f}$ as in the proof of Proposition 9.4.4. As $\sigma^{\sharp}_{F_X}\bar{f}$ is $(2,1,0)$-morphism, also discussed in the proof of Proposition 9.4.4, η is a $(2,1,0)$-morphism.

Let $x \in X$. Then

$$
\begin{aligned}
x\tau\rho^{\natural}\eta &= \left(\underset{1}{\bullet} \xrightarrow{\quad x \quad} \underset{x}{\bullet}, x \right) \rho^{\natural}\eta \\
&= \left[\left(\underset{1}{\bullet} \xrightarrow{\quad x \quad} \underset{x}{\bullet}, x \right) \rho \right] \eta \\
&= \left(\underset{1}{\bullet} \xrightarrow{\quad x \quad} \underset{x}{\bullet}, x \right) \sigma^{\sharp}_{F_X} \bar{f} \\
&= x\bar{f} \\
&= xf.
\end{aligned}
$$

So the following diagram commutes:

$$
\begin{tikzcd}
X \arrow[d, "\tau\rho^{\natural}"'] \arrow[dr, "f"] & \\
F_X/\rho \arrow[r, "\eta"'] & S
\end{tikzcd}
$$

It remains to show that Ker $\eta = \sigma_F$, where $F = F_X/\rho$. Consider $a, b \in F_X$ and suppose $a = e(\bar{x}\bar{\tau}), b = g(\bar{y}\bar{\tau}) \in F_X$ where $e, g \in E_{F_X}$ and $\bar{x}, \bar{y} \in X^*$. Let $a\rho \; \sigma_F \; b\rho$. Then $ha \; \rho \; hb$ for some $h \in E_{F_X}$. So by the proof of Proposition 9.4.4, $\bar{x}\bar{f} = \bar{y}\bar{f}$. As $\bar{\tau}\sigma^{\sharp}_{F_X} = I_{X^*}$, we have

$$\bar{x}\bar{\tau}\sigma^{\sharp}_{F_X}\bar{f} = \bar{y}\bar{\tau}\sigma^{\sharp}_{F_X}\bar{f}$$

and as $e\sigma^{\sharp}_{F_X} = g\sigma^{\sharp}_{F_X} = 1$, we have

$$[e(\bar{x}\bar{\tau})]\sigma^{\sharp}_{F_X}\bar{f} = [g(\bar{y}\bar{\tau})]\sigma^{\sharp}_{F_X}\bar{f},$$

i.e.

$$a\sigma^{\sharp}_{F_X}\bar{f} = b\sigma^{\sharp}_{F_X}\bar{f}.$$

By the definition of η, $(a\rho)\eta = (b\rho)\eta$ and hence $\sigma_F \subseteq$ Ker η.

Now suppose that $(a\rho)\eta = (b\rho)\eta$. By definition of η, $a\sigma^{\sharp}_{F_X}\bar{f} = b\sigma^{\sharp}_{F_X}\bar{f}$, i.e.

$$[e(\bar{x}\bar{\tau})]\sigma^{\sharp}_{F_X}\bar{f} = [g(\bar{y}\bar{\tau})]\sigma^{\sharp}_{F_X}\bar{f}.$$

Using the same arguments as above, we have $\bar{x}\bar{f} = \bar{y}\bar{f}$ and so

$$(\bar{x}\bar{\tau})^{+}\bar{y}\bar{\tau} \; \rho \; (\bar{y}\bar{\tau})^{+}\bar{x}\bar{\tau}.$$

Therefore $a\rho \; \sigma_F \; b\rho$ in F_X/ρ as $ha \, \rho \, hb$ for $h = eg(\bar{x}\bar{\tau})^{+}(\bar{y}\bar{\tau})^{+} \in E_{F_X}$. So Ker $\eta \subseteq \sigma_F$ and hence Ker $\eta = \sigma_F$. $\square$

Before our next result we note the following lemma:

Lemma 9.4.6. *Let (g, U) and (h, V) be objects in $\mathbf{M}(X)$ and θ a morphism such that*

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[dr, "h"] & \\ U \arrow[rr, "\theta"'] & & V \end{tikzcd}$$

commutes. If $\overline{g} : X^* \to U$ and $\overline{h} : X^* \to V$ denote extensions of g and h to monoid morphisms respectively, then the following diagram commutes:

$$\begin{tikzcd} & X^* \arrow[dl, "\overline{g}"'] \arrow[dr, "\overline{h}"] & \\ U \arrow[rr, "\theta"'] & & V \end{tikzcd}$$

Lemma 9.4.7. *The pair $(\tau\rho^\natural, F_X/\rho)$ is an initial object in $\mathbf{PLR}(X, f, S)$.*

Proof. We need to show that if (g, N) is an object in $\mathbf{PLR}(X, f, S)$, then

$$| \text{ Mor }_{\mathbf{PLR}}((\tau\rho^\natural, F_X/\rho), (g, N)) |= 1,$$

i.e. by Proposition 9.2.2, we need to show

$$\text{Mor }_{\mathbf{PLR}}((\tau\rho^\natural, F_X/\rho), (g, N)) \neq \emptyset.$$

If (g, N) is an object in $\mathbf{PLR}(X, f, S)$, then N is a proper left restriction monoid, $g : X \to N$, $\langle Xg \rangle = N$ and

$$\begin{tikzcd} X \arrow[d, "g"'] \arrow[dr, "f"] & \\ N \arrow[r, "\mu"] & S \end{tikzcd}$$

commutes, where $\mu : N \to S$ is a morphism with kernel σ_N. We must show that there is a $(2, 1, 0)$-morphism $\psi : F_X/\rho \to N$ such that

$$\begin{tikzcd} & X \arrow[dl, "\tau\rho^\natural"'] \arrow[dr, "g"] & \\ F_X/\rho \arrow[rr, "\psi"'] & & N \end{tikzcd} \qquad (*)$$

commutes.

Since F_X is the free left restriction monoid on $X\tau$ there is a $(2,1,0)$-morphism ϕ such that

$$\begin{tikzcd} & X \arrow[dl, "\tau"'] \arrow[dr, "g"] & \\ F_X \arrow[rr, "\phi"'] & & N \end{tikzcd}$$

commutes. We claim that $\rho \subseteq$ Ker ϕ. Let us extend g to a monoid morphism $\bar{g} : X^* \to N$, so that $\bar{g}\mu = \bar{f}$ and $\bar{\tau}\phi = \bar{g}$ by Lemma 9.4.6. Let $\bar{x}, \bar{y} \in X^*$ with $\bar{x}\bar{f} = \bar{y}\bar{f}$. Then $\bar{x}\bar{g}\mu = \bar{y}\bar{g}\mu$, so $(\bar{x}\bar{g}, \bar{y}\bar{g}) \in$ Ker μ. Since Ker $\mu = \sigma_N$ and N is proper, it follows from Lemma 2.8.4 that

$$(\bar{x}\bar{g})^+\bar{y}\bar{g} = (\bar{y}\bar{g})^+\bar{x}\bar{g}.$$

As $\bar{\tau}\phi = \bar{g}$,

$$(\bar{x}\bar{\tau}\phi)^+\bar{y}\bar{\tau}\phi = (\bar{y}\bar{\tau}\phi)^+\bar{x}\bar{\tau}\phi,$$

i.e.

$$[(\bar{x}\bar{\tau})^+\bar{y}\bar{\tau}]\phi = [(\bar{y}\bar{\tau})^+\bar{x}\bar{\tau}]\phi$$

as ϕ is a $(2,1,0)$-morphism. So

$$((\bar{x}\bar{\tau})^+\bar{y}\bar{\tau}, (\bar{y}\bar{\tau})^+\bar{x}\bar{\tau}) \in \text{ Ker } \phi.$$

Hence $H \subseteq$ Ker ϕ and so $\rho \subseteq$ Ker ϕ.

We can thus define a $(2,1,0)$-morphism $\psi : F_X/\rho \to N$ by $(a\rho)\psi = a\phi$, for $a \in F_X$, which is well-defined since $\rho \subseteq$ Ker ϕ. For $x \in X$,

$$x\tau\rho^\natural\psi = ((x\tau)\rho)\psi = x\tau\phi = xg$$

so that diagram $(*)$ commutes as required. So therefore $(\tau\rho^\natural, F_X/\rho)$ is an initial object in $\mathbf{PLR}(X, f, S)$.

□

We now have the desired result of this section:

Theorem 9.4.8. *The proper left restriction monoid F_X/ρ is isomorphic to $\mathcal{M}(X, f, S)$. If θ is the $(2,1,0)$-morphism making the diagram*

$$\begin{tikzcd} & X \arrow[dl, "\tau"'] \arrow[dr, "\tau_{\mathcal{M}}"] & \\ F_X \arrow[rr, "\theta"'] & & \mathcal{M}(X, f, S) \end{tikzcd}$$

commute, then Ker $\theta = \rho$.

Proof. By Theorem 9.2.7 and Lemma 9.4.7, both $(\tau_{\mathcal{M}}, \mathcal{M}(X,f,S))$ and $(\tau\rho^{\natural}, F_X/\rho)$ are initial objects in $\mathbf{PLR}(X,f,S)$. Since initial objects in $\mathbf{PLR}(X,f,S)$ are unique up to isomorphism,

$$\mathcal{M}(X,f,S) \cong F_X/\rho.$$

As $(\tau_{\mathcal{M}}, \mathcal{M}(X,f,S))$ is an initial object, there exists $\varphi : \mathcal{M}(X,f,S) \to F_X/\rho$ such that

$$\begin{tikzcd} & X \arrow[dl, "\tau_{\mathcal{M}}"'] \arrow[dr, "\tau\rho^{\natural}"] & \\ \mathcal{M}(X,f,S) \arrow[rr, "\varphi"'] & & F_X/\rho \end{tikzcd}$$

commutes. It follows from the definition of initial objects that φ is an isomorphism.

Considering $\varphi^{-1} = \psi : F_X/\rho \to \mathcal{M}(X,f,S)$, the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "\tau\rho^{\natural}"'] \arrow[dr, "\tau_{\mathcal{M}}"] & \\ F_X/\rho \arrow[rr, "\psi"'] & & \mathcal{M}(X,f,S) \end{tikzcd}$$

As $\tau\rho^{\natural}\psi = \tau_{\mathcal{M}}$, we have $\theta = \rho^{\natural}\psi$ by uniqueness within the definition of free objects. We also have

$$a\theta = b\theta \Leftrightarrow a\rho^{\natural}\psi = b\rho^{\natural}\psi \Leftrightarrow a\rho^{\natural} = b\rho^{\natural} \Leftrightarrow a \,\rho\, b.$$

Therefore Ker $\theta = \rho$. $\square$

9.5 Proper covers and varieties

An inverse semigroup S has a *proper cover over* $\mathcal{V}$, where $\mathcal{V}$ is a variety of groups, if S has a proper cover $\hat{S}$ such that $\hat{S}/\sigma \in \mathcal{V}$. Groups and inverse monoids form varieties and it is proved in [47] that the class of inverse monoids having a proper cover over $\mathcal{V}$ is a variety of inverse monoids. This variety is determined by

$$\Sigma = \{\bar{u}^2 \equiv \bar{u} : \bar{u} \equiv 1 \text{ is a law in } \mathcal{V}\}.$$

When trying to prove an analogous result for left ample monoids, technical issues are encountered due to left ample monoids forming a *quasivariety* rather than a variety. It is proved in [23] that the class of left ample monoids having a proper cover over $\mathcal{V}$ forms a quasivariety, where $\mathcal{V}$ is a subquasivariety of the quasivariety $\mathcal{RC}$ of right cancellative monoids defined (within $\mathcal{RC}$) by equations.

Weakly (left) ample semigroups also form quasivarieties, so similar difficulties would be encountered when trying to produce such a theorem. However, (left) restriction semigroups form a variety, so it is possible to prove such a theorem for (left) restriction semigroups.

By Theorem 5.1.7, any left restriction monoid has a proper cover. Let $\mathcal{V}$ be a variety of monoids. Denoting the class of left restriction monoids by $\mathcal{LR}$, we have the following definition:

Definition 9.5.1. A left restriction monoid has a *proper cover over* $\mathcal{V}$ if it has a proper cover M such that $M/\sigma_M \in \mathcal{V}$. We put

$$\hat{\mathcal{V}} = \{N \in \mathcal{LR} : N \text{ has a proper cover over } \mathcal{V}\}.$$

We shall show that the class of left restriction monoids having a proper cover over $\mathcal{V}$, where $\mathcal{V}$ is a variety of monoids, is a variety of left restriction monoids, showing that this variety is determined by

$$\Sigma = \{\bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u} : \bar{u} \equiv \bar{v} \text{ is a law in } \mathcal{V}\}.$$

Using the techniques based around graph expansions in [23], we shall deduce this result for left restriction monoids. In Chapter 10, we shall use the method in [47] to deduce the result for restriction semigroups.

9.6 A class of left restriction semigroups having a cover over a variety of monoids

We shall generalise the results in Section 5 of [23] with the alteration that we are considering varieties instead of quasivarieties. In particular, the following result uses ideas from the proof of Proposition 5.2 in [23]. Throughout we shall continue to let $\mathcal{V}$ denote a variety of monoids.

Proposition 9.6.1. *Suppose M is a left restriction monoid that has a proper cover over $\mathcal{V}$ and let $\theta : M \to N$ be an onto $(2,1,0)$-morphism, where N is also a left restriction monoid. Then N has a proper cover over $\mathcal{V}$.*

Proof. As M has a proper cover over $\mathcal{V}$, there is a proper left restriction monoid P and an onto $(2,1,0)$-morphism $\varphi : P \to M$ such that φ is E_P-separating and $S = P/\sigma_P \in \mathcal{V}$. Putting $\psi = \varphi\theta : P \to N$, we have that ψ is an onto $(2,1,0)$-morphism. We also note that $\sigma_P^\natural : P \to S$ is an onto $(2,1,0)$-morphism.

As N is a left restriction monoid and S is a monoid, which can be regarded as a left restriction monoid, $N \times S$ is also a left restriction monoid as $\mathcal{LR}$ is a variety. We have for $(n,s),(m,t) \in N \times S$,

$$(n,s) \;\widetilde{\mathscr{R}}_{E_{N\times S}}\; (m,t) \text{ if and only if } n \;\widetilde{\mathscr{R}}_{E_N}\; m$$

as

$$\begin{aligned}(n,s)\ \widetilde{\mathscr{R}}_{E_{N\times S}}\ (m,t) &\Leftrightarrow (n,s)^+ = (m,t)^+\\ &\Leftrightarrow (n^+,s^+) = (m^+,t^+)\\ &\Leftrightarrow (n^+,1) = (m^+,1)\\ &\Leftrightarrow n^+ = m^+\\ &\Leftrightarrow n\ \widetilde{\mathscr{R}}_{E_N}\ m.\end{aligned}$$

We note that $E_{N\times S} = E_N \times \{1\}$.

We wish to show that

$$K = \{(n,s) \in N \times S : \text{ there exists } p \in P \text{ with } p\psi = n \text{ and } p\sigma_P^\natural = s\}$$

is our required cover for N. We have that K is a subalgebra of $N \times S$ as we shall show. Take $(n,s),(m,t) \in K$. Then there exist $p,q \in P$ with $p\psi = n, p\sigma_P^\natural = s, q\psi = m$ and $q\sigma_P^\natural = t$. As ψ and $\sigma_P^\natural$ are $(2,1,0)$-morphisms,

$$nm = (p\psi)(q\psi) = (pq)\psi$$

and

$$st = (p\sigma_P^\natural)(q\sigma_P^\natural) = (pq)\sigma_P^\natural.$$

So $(nm,st) \in K$ since $pq \in P$. Considering $(n,s) \in K$ where $p\psi = n$ and $p\sigma_P^\natural = s$ for some $p \in P$, we wish to show $(n,s)^+ \in K$, i.e. $(n^+,1) \in K$. We have

$$n^+ = (p\psi)^+ = p^+\psi$$

as ψ is a $(2,1,0)$-morphism. We also have $p^+\sigma_P^\natural = 1$ as $p^+ \in E_P$. Hence $(n,s)^+ \in K$. Now, as P is a monoid, $1_P \in P$ and since ψ and $\sigma_P^\natural$ are $(2,1,0)$-morphisms, $1_P\psi = 1_N$ and $1_P\sigma_P^\natural = 1_S$. So $(1_N,1_S) \in K$. Clearly $E_K \subseteq E_1 \times \{1\}$. Considering $(n^+,1) \in E_N \times \{1\}$, we have $p\psi = n^+$ as ψ is onto. As ψ is a $(2,1,0)$-morphism it follows that $p^+\psi = n^+$. As $p^+\sigma_P^\natural = 1$, $(n^+,1) \in E_K$. Therefore $E_K = E_N \times \{1\}$ and K is a subalgebra of $N \times S$ and so K is a left restriction monoid with respect to the distinguished semilattice $E_N \times \{1\}$. So for $(n,s),(m,t) \in K$,

$$(n,s)\ \widetilde{\mathscr{R}}_{E_K}\ (m,t) \text{ in } K \text{ if and only if } n\ \widetilde{\mathscr{R}}_{E_K}\ m \text{ in } N.$$

Suppose that $(n,s),(m,t) \in K$ and

$$(n,s)\ (\widetilde{\mathscr{R}}_{E_K} \cap \sigma_K)\ (m,t).$$

So $(n,s)\ \widetilde{\mathscr{R}}_{E_K}\ (m,t)$, $(n,s)\ \sigma_K\ (m,t)$ and $(e,1)(n,s) = (e,1)(m,t)$ for some $e \in E_N$. Therefore $n^+ = m^+$, $n\,\sigma_M\,m$ and $s = t$. Take $p,q \in P$ such that $p\psi = n, p\sigma_P^\natural = s, q\psi = m$ and $q\sigma_P^\natural = t$. As $s = t$ we have that $p\ \sigma_P\ q$ and so $p^+q = q^+p$ by Lemma 2.8.4. We have

$$(p^+q)\psi = (q^+p)\psi,$$

i.e.
$$(p\psi)^+q\psi = (q\psi)^+p\psi$$
as ψ is a $(2,1,0)$-morphism. So $n^+m = m^+n$. As $n^+ = m^+$, we have
$$m = m^+m = n^+n = n.$$
So $(n,s) = (m,t)$ and hence K is proper.

Since ψ is onto it follows that the morphism $\gamma : K \to N$, where $(n,s)\gamma = n$ for $(n,s) \in K$, is also onto as if $n \in N$, there exists $p \in P$ such that $p\psi = n$ since ψ is onto. Considering $s = p\sigma_P$, $p\sigma_P^\natural = p\sigma_P$ and so $(n, p\sigma_P) \in K$ such that $(n, p\sigma_P)\gamma = n$. Finally, γ is E_K-separating as
$$(n^+,1)\gamma = (m^+,1)\gamma \Rightarrow n^+ = m^+ \Rightarrow (n^+,1) = (m^+,1)$$
for $(n^+,1),(m^+,1) \in E_K$. So K is a proper cover of N.

It remains to show that $K/\sigma_K \in \mathcal{V}$. As $\mathcal{V}$ is a variety of monoids, we shall show $K/\sigma_K \in \mathcal{V}$ by showing it is a homomorphic image of $P/\sigma_P \in \mathcal{V}$. Note that this method could not be applied in [23].

Let us define $\mu : P \to K$ by
$$p\mu = (p\psi, p\sigma_P^\natural)$$
for $p \in P$ and $\delta : P/\sigma_P \to K/\sigma_K$ by
$$(p\sigma_P)\delta = (p\psi, p\sigma_P^\natural)\sigma_K$$
for $p\sigma_P \in P/\sigma_P$. First we shall show that δ is well-defined. Suppose that $p\,\sigma_P q$ for $p,q \in P$. We note $p\sigma_P^\natural = q\sigma_P^\natural$ and $p^+q = q^+p$ since P is proper. We wish to show that $(p\psi, p\sigma_P^\natural)\,\sigma_K\,(q\psi, q\sigma_P^\natural)$, i.e.
$$(e,1)(p\psi, p\sigma_P^\natural) = (e,1)(q\psi, q\sigma_P^\natural)$$
for some $e \in E_N$. As $p\sigma_P^\natural = q\sigma_P^\natural$, this is equivalent to showing that
$$e(p\psi) = e(q\psi), \text{ i.e. } p\psi\,\sigma_N\,q\psi$$
for some $e \in E_N$. We have
$$\begin{aligned} p\,\sigma_P\,q &\Rightarrow fp = fq \text{ for some } f \in E_P \\ &\Rightarrow (fp)\psi = (fq)\psi \\ &\Rightarrow (f\psi)(p\psi) = (f\psi)(q\psi) \\ &\Rightarrow p\psi\,\sigma_N\,q\psi. \end{aligned}$$
So $(p\psi, p\sigma_P^\natural)\,\sigma_K\,(q\psi, q\sigma_P^\natural)$ and hence δ is well-defined.

Now, using the fact that ψ and $\sigma_P^\natural$ are $(2,1,0)$-morphisms, we have for

$(p\sigma_P), (q\sigma_P) \in P/\sigma_P$,

$$\begin{aligned} ((p\sigma_P)(q\sigma_P))\delta &= (pq\sigma_P)\delta \\ &= ((pq)\psi, (pq)\sigma_P^\natural)\sigma_K \\ &= ((p\psi, p\sigma_P^\natural)(q\psi, q\sigma_P^\natural))\sigma_K \\ &= (p\psi, p\sigma_P^\natural)\sigma_K(q\psi, q\sigma_P^\natural)\sigma_K \\ &= (p\sigma_P)\delta(q\sigma_P)\delta \end{aligned}$$

and

$$\begin{aligned} (p\sigma_P)^+\delta &= (p^+\sigma_P)\delta \\ &= (p^+\psi, p^+\sigma_P^\natural)\sigma_K \\ &= ((p\psi)^+, 1)\sigma_K \\ &= (p\psi, p\sigma_P^\natural)^+\sigma_K \\ &= ((p\psi, p\sigma_P^\natural)\sigma_K)^+ \\ &= ((p\sigma_P)\delta)^+. \end{aligned}$$

Considering $1_{P/\sigma_P} = 1_P\sigma_P$, we have

$$\begin{aligned} (1_P\sigma_P)\delta &= (1_P\psi, 1_P\sigma_P^\natural)\sigma_K \\ &= (1_N, 1_S)\sigma_K \\ &= 1_K\sigma_K \\ &= 1_{K/\sigma_K} \end{aligned}$$

so δ is a $(2,1,0)$-morphism.

We shall now show that δ is onto. Considering $(a,b)\sigma_K \in K/\sigma_K$, we know $a \in N$ and $b \in S$ and there exists $p \in P$ such that $p\psi = a$ and $p\sigma_P^\natural = b$. So

$$(a,b)\sigma_K = (p\psi, p\sigma_P^\natural)\sigma_K.$$

As $p \in P$, $p\sigma_P \in P/\sigma_P$ and we know

$$(p\sigma_P)\delta = (a,b)\sigma_K.$$

Hence δ is onto. So K/σ_K is a homomorphic image of P/σ_P. Since $P/\sigma_P \in \mathcal{V}$, we have $K/\sigma_K \in \mathcal{V}$ as $\mathcal{V}$ is a variety of monoids. Hence K is a proper cover for N over $\mathcal{V}$. $\square$

The following result shows that the class of left restriction monoids having a cover over a variety of monoids is a variety of left restriction monoids.

Theorem 9.6.2. *Let $\mathcal{V}$ be a variety of monoids. Then*

$$\widehat{\mathcal{V}} = \{N \in \mathcal{LR} : N \models \Sigma\}$$

where

$$\Sigma = \{\bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u} : \bar{u} \equiv \bar{v} \text{ is a law in } \mathcal{V}\}$$

for $(2,0)$*-terms* $\bar{u}$ *and* $\bar{v}$*.*

Proof. Suppose first that $N \in \widehat{\mathcal{V}}$. So N is a left restriction monoid and N has a proper cover over $\mathcal{V}$, i.e. there is a proper left restriction monoid M such that $M/\sigma_M \in \mathcal{V}$ and an onto $(2,1,0)$-morphism $\zeta : M \to N$ which is E_M-separating. If $\bar{u} \equiv \bar{v}$ is a law in $\mathcal{V}$ then since M is proper and $M/\sigma_M \in \mathcal{V}$, $M \models \bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u}$ by Lemma 2.8.4. As ζ is onto, $N \models \bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u}$ and so $N \models \Sigma$.

Conversely, suppose that N is a left restriction monoid and $N \models \Sigma$. Let X be a (not necessarily finite) set of generators for N, that is, there is a map $g : X \to N$ such that $\langle Xg \rangle = N$. Let S be the free object in $\mathcal{V}$ on X and let $f : X \to S$ denote the canonical embedding of X into S, so $\langle Xf \rangle = S$ and (X, f, S) is a monoid presentation of S.

Since F_X is the free (proper) left restriction monoid on $X\tau$ with canonical embedding τ, there is a morphism $\theta : F_X \to N$ such that

$$\begin{tikzcd} & X \arrow[dl, "\tau"'] \arrow[dr, "g"] & \\ F_X \arrow[rr, "\theta"'] & & N \end{tikzcd}$$

commutes. We shall show that $\rho \subseteq \operatorname{Ker}\theta$. Let $\bar{u}$ and $\bar{v}$ be $(2,0)$-terms in the free term algebra on a countably infinite set Y. Say

$$\bar{u} = s(y_1, \ldots, y_n) \text{ and } \bar{v} = t(y_1, \ldots, y_n).$$

Suppose that

$$s(x_1, \ldots, x_n)\bar{f} = t(y_1, \ldots, y_n)\bar{f}.$$

For any $T \in \mathcal{V}$ and $a_1, \ldots a_n \in T$, define $g : X \to T$ by $x_i g = a_i$. Then there exists $\theta : S \to T$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & S \arrow[dl, "\theta"] \\ T & \end{tikzcd}$$

commutes. As

$$s(x_1, \ldots, x_n)\bar{f} = t(x_1, \ldots, x_n)\bar{f},$$

it follows that

$$s(x_1 f\theta, \ldots, x_n f\theta) = t(x_1 f\theta, \ldots, x_n f\theta),$$

i.e.

$$s(a_1, \ldots, a_n) = t(a_1, \ldots, a_n).$$

It follows that $\bar{u} \equiv \bar{v}$ is a law in $\mathcal{V}$.

Let $\bar{g}$ denote the lifting of g to a monoid morphism from X^* to N. Since $N \models \Sigma$ we have

$$s(x_1g, \ldots, x_ng)^+t(x_1g, \ldots, x_ng) = t(x_1g, \ldots, x_ng)^+s(x_1g, \ldots, x_ng).$$

Since $\bar{g} = \bar{\tau}\theta$ and θ is a $(2,1,0)$-morphism, we have

$$((\bar{u}\bar{\tau})^+\bar{v}\bar{\tau})\theta = ((\bar{v}\bar{\tau})^+\bar{u}\bar{\tau})\theta$$

so that $H \subseteq \text{Ker } \theta$, where

$$H = H_{(X,f,S)} = \{((\bar{u}\bar{\tau})^+\bar{v}\bar{\tau}, (\bar{v}\bar{\tau})^+\bar{u}\bar{\tau}) \in F_X \times F_X : \bar{u}, \bar{v} \in X^* \text{ and } \bar{u}\bar{f} = \bar{v}\bar{f}\}.$$

As H generates the congruence ρ we have $\rho \subseteq \text{Ker } \theta$.

By Corollary 1.2.11 it follows that there is an onto $(2,1,0)$-morphism $\phi : F_X/\rho \to N$, defined by

$$(a\rho)\phi = a\theta,$$

so that the following diagram commutes:

$$\begin{tikzcd}
F_X \arrow[d, "\rho^\natural"'] \arrow[dr, "\theta"] & \\
F_X/\rho \arrow[r, "\phi"] & N
\end{tikzcd}$$

We wish to show ϕ is onto by considering $n \in N$. As $N = \langle Xg \rangle$, $\tau\theta = g$ and $\theta = \rho^\natural\phi$, we have

$$\begin{aligned}
n &= t(x_1g, \ldots x_ng) \\
&= t(x_1\tau\theta, \ldots x_n\tau\theta) \\
&= t(x_1\tau, \ldots x_n\tau)\theta \\
&= t(x_1\tau, \ldots x_n\tau)\rho^\natural\phi.
\end{aligned}$$

So ϕ is onto and hence is an onto $(2,1,0)$-morphism.

From Proposition 9.4.4, F_X/ρ is a proper left restriction monoid. As F_X/ρ is proper, it has itself as a proper cover. It remains to show that $F_X/\rho \in \widehat{\mathcal{V}}$. From the proof of Lemma 9.4.5, $\eta : F_X/\rho \to S$ defined by $(a\rho)\eta = a\sigma^\sharp_{F_X}\bar{f}$, for $a\rho \in F_X/\rho$, is a $(2,1,0)$-morphism and $\tau\rho^\natural\eta = f$. By Proposition 9.2.2, η is onto since (f, S) and $(\tau\rho^\natural, F_X/\rho)$ are objects in $\mathbf{PLR}(X, f, S)$. Hence η is an onto $(2,1,0)$-morphism and $\text{Im } \eta = S$.

Also by the proof of Lemma 9.4.5, $\text{Ker } \eta = \sigma_F$, where $F = F_X/\rho$. By the fundamental theorem of semigroup morphisms,

$$F/\sigma_F \cong S.$$

Since $S \in \mathcal{V}$, $F/\sigma_F \in \mathcal{V}$. Hence $F = F_X/\rho \in \widehat{\mathcal{V}}$. As $F_X/\rho \in \widehat{\mathcal{V}}$, $\phi : F_X/\rho \to N$ is an onto $(2,1,0)$-morphism and N is a left restriction monoid, N has a proper cover over $\mathcal{V}$ by Proposition 9.6.1. □

Theorem 9.6.3. *Let $\mathcal{V}$ be a variety of monoids. Let X be a set and (X, f, S) the canonical monoid presentation of the free object in $\mathcal{V}$ on X. Then $\mathcal{M}(X, f, S)$ is the free object in $\widehat{\mathcal{V}}$ on X with canonical embedding $\tau_{\mathcal{M}}$.*

Proof. Take $N \in \widehat{\mathcal{V}}$ and let $g : X \to N$ be a map. We wish to show that $\mathcal{M}(X, f, S) \in \widehat{\mathcal{V}}$, that $\tau_{\mathcal{M}} : X \to \mathcal{M}(X, f, S)$ is a one-to-one map and that there is a unique morphism $\xi : \mathcal{M}(X, f, S) \to N$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[d, "\tau_{\mathcal{M}}"'] \arrow[dr, "g"] & \\ \mathcal{M}(X,f,S) \arrow[r, "\xi"] & N \end{tikzcd}$$

Let F_X be $\mathcal{M}(X, \iota, X^*)$ so that F_X is the free left restriction monoid $X\tau$ where $\tau : X \to F_X$ is as in Theorem 9.2.13. So there is a morphism $\theta : F_X \to N$ such that

$$\begin{tikzcd} & X \arrow[dl, "\tau"'] \arrow[dr, "g"] & \\ F_X \arrow[rr, "\theta"'] & & N \end{tikzcd}$$

commutes. As in the proof of Theorem 9.6.2 we have $\rho \subseteq \operatorname{Ker}\theta$ so that there is a unique onto morphism $\psi : F_X/\rho \to N$ such that $\rho^\natural\psi = \theta$. Also from the proof of Theorem 9.6.2, we have $F_X/\rho \in \widehat{\mathcal{V}}$. As in the proof of Theorem 9.4.8, there is an isomorphism $\phi : \mathcal{M}(X, f, S) \to F_X/\rho$ such that $\tau_{\mathcal{M}}\phi = \tau\rho^\natural$. Let $\xi = \phi\psi$ so that $\xi : \mathcal{M}(X, f, S) \to N$ is a morphism. For any $x \in X$ we have

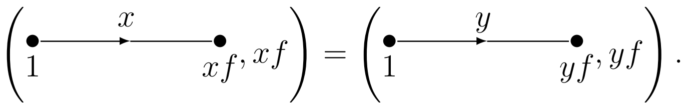
$$x\tau_{\mathcal{M}}\xi = x\tau_{\mathcal{M}}\phi\psi = x\tau\rho^\natural\psi = x\tau\theta = xg$$

so that the following diagram commutes:

$$\begin{tikzcd} & X \arrow[dl, "\tau_{\mathcal{M}}"'] \arrow[dr, "g"] & \\ \mathcal{M}(X,f,S) \arrow[rr, "\xi"'] & & N \end{tikzcd}$$

By Proposition 1.2.12, ξ is unique. As $\mathcal{M}(X,f,S) \cong F_X/\rho$, we have $\mathcal{M}(X,f,S) \in \widehat{\mathcal{V}}$ and by Proposition 9.2.6, $X\tau_{\mathcal{M}}$ generates $\mathcal{M}(X,f,S)$.

Let us suppose that $x\tau_{\mathcal{M}} = y\tau_{\mathcal{M}}$, so



As the edge $(1, x, xf)$ equals $(1, y, yf)$, then $x = y$. Therefore $\tau_{\mathcal{M}}$ is a one-to-one map. Thus $\mathcal{M}(X,f,S)$ is the free object on X in $\widehat{\mathcal{V}}$ with canonical embedding $\tau_{\mathcal{M}}$. $\square$

Chapter 10

Varieties of restriction semigroups

10.1 Proper covers and varieties

We shall consider a variety of restriction semigroups and provide alternative conditions for when such a variety has proper covers by proving an analogue of a result by Petrich and Reilly [47].

First of all, we shall look at a relation, $\rho_{\min}$, the dual of which is defined for a right ample semigroup in [15]. We shall define $\rho_{\min}$ on a left restriction semigroup S.

Definition 10.1.1. Let ρ be a $(2,1)$-congruence on a left restriction semigroup S. Then we define $\rho_{\min}$ on S by

$$a\,\rho_{\min}\,b \text{ if and only if } ea = eb \text{ and } e\,\rho\,a^+\,\rho\,b^+ \text{ for some } e \in E_S$$

for $a, b \in S$.

Proposition 10.1.2. *(cf. [15]) Let ρ be a $(2,1)$-congruence on a left restriction semigroup S. Then $\rho_{\min}$ is a $(2,1)$-congruence on S and $\rho_{\min} \subseteq \rho$.*

Proof. Let S be a left restriction semigroup. Suppose $a\,\rho_{\min}\,b$. Then there exists $e \in E_S$ such that $ea = eb$ and $e\,\rho\,a^+\,\rho\,b^+$. We have

$$e\,\rho\,a^+ \Rightarrow ea\,\rho\,a^+a \Rightarrow ea\,\rho\,a$$

and dually $e\,\rho\,b^+$ implies $eb\,\rho\,b$, so $a\,\rho\,b$. Hence $\rho_{\min} \subseteq \rho$.

We wish to show, for $a, b, c \in S$, that the following hold:

(i) $a\,\rho_{\min}\,a$;

(ii) $a\,\rho_{\min}\,b \Rightarrow b\,\rho_{\min}\,a$;

(iii) $a\,\rho_{\min}\,b,\ b\,\rho_{\min}\,c \Rightarrow a\,\rho_{\min}\,c$;

(iv) $a\,\rho_{\min}\,b \Rightarrow ca\,\rho_{\min}\,cb$;

(v) $a \,\rho_{\min}\, b \Rightarrow ac \,\rho_{\min}\, bc$;

(vi) $a \,\rho_{\min}\, b \Rightarrow a^+ \,\rho_{\min}\, b^+$.

(i) Let $a \in S$. Then $a^+ \in E_S$; also $a^+a = a^+a$ and $a^+ \,\rho\, a^+ \,\rho\, a^+$ since ρ is a congruence. So $a \,\rho_{\min}\, a$ for all $a \in S$.

Part (ii) is clear.

(iii) Suppose $a \,\rho_{\min}\, b$ and $b \,\rho_{\min}\, c$. Then there exist $e, f \in E_S$ such that $ea = eb$, $e \,\rho\, a^+ \,\rho\, b^+$, $fb = fc$ and $f \,\rho\, b^+ \,\rho\, c^+$. Considering $fe \in E_S$,

$$fea = feb = efb = efc = fec.$$

As $e \,\rho\, b^+$ and $f \,\rho\, b^+$, we have $e \,\rho\, f$. Also,

$$e \,\rho\, f \Rightarrow ee \,\rho\, fe \Rightarrow e \,\rho\, fe,$$

so we also have $fe \,\rho\, a^+ \,\rho\, c^+$ and hence $a \,\rho_{\min}\, c$.

So $\rho_{\min}$ is an equivalence relation.

(iv) Suppose $a \,\rho_{\min}\, b$, with $ea = eb$ and $e \,\rho\, a^+ \,\rho\, b^+$. We have

$$ea = eb \Rightarrow cea = ceb \Rightarrow (ce)^+ca = (ce)^+cb$$

using the ample condition. We also have

$$\begin{aligned} e \,\rho\, a^+ \,\rho\, b^+ &\Rightarrow ce \,\rho\, ca^+ \,\rho\, cb^+ \\ &\Rightarrow (ce)^+ \,\rho\, (ca^+)^+ \,\rho\, (cb^+)^+ \\ &\Rightarrow (ce)^+ \,\rho\, (ca)^+ \,\rho\, (cb)^+ \end{aligned}$$

using Proposition 2.6.2. Therefore, $ca \,\rho_{\min}\, cb$.

(v) Suppose $a \,\rho_{\min}\, b$, with $ea = eb$ and $e \,\rho\, a^+ \,\rho\, b^+$. First we note

$$ea = eb \Rightarrow eac = ebc \Rightarrow e(ac)^+ac = e(bc)^+bc.$$

As $\rho_{\min} \subseteq \rho$, $a \,\rho\, b$ and we have

$$a \,\rho\, b \Rightarrow ac \,\rho\, bc \Rightarrow (ac)^+ \,\rho\, (bc)^+ \Rightarrow e \,\rho\, (ac)^+ \,\rho\, (bc)^+.$$

It remains to show that $e(ac)^+ \,\rho\, (ac)^+$. We have

$$e \,\rho\, a^+ \Rightarrow e(ac)^+ \,\rho\, a^+(ac)^+ = (ac)^+$$

using Lemma 2.6.2. So $e(ac)^+ \,\rho\, (ac)^+$ and therefore $ac \,\rho_{\min}\, bc$.

(vi) Suppose $a \,\rho_{\min}\, b$, with $ea = eb$ and $e \,\rho\, a^+ \,\rho\, b^+$. We have

$$ea^+ \,\widetilde{\mathscr{R}}_{E_S}\, ea = eb \,\widetilde{\mathscr{R}}_{E_S}\, eb^+,$$

so $ea^+ = eb^+$. Since $e\,\rho\,(a^+)^+\,\rho\,(a^+)^+$, $a^+\,\rho_{\min} b^+$ and so $\rho_{\min}$ is a $(2,1)$-congruence.

□

The following proposition tells us that $\rho_{\min}$ is the least congruence on a left restriction semigroup S with the same trace as ρ.

Proposition 10.1.3. *(cf. [15]) For $e, f \in E_S$,*

$$e\,\rho_{min}\,f \text{ if and only if } e\,\rho\,f.$$

If μ is any congruence on the left restriction semigroup S with the property that for $e, f \in E_S$,

$$e\,\mu\,f \text{ if and only if } e\,\rho\,f,$$

then $\rho_{min} \subseteq \mu$.

Proof. As in the proof above, $\rho_{\min} \subseteq \rho$. Now suppose $e, f \in E_S$ with $e\,\rho\,f$. We have $ef \in E_S$ and $efe = eff$. We also have

$$\begin{aligned} e\,\rho\,f &\Rightarrow ef\,\rho\,ff \\ &\Rightarrow ef\,\rho\,f \\ &\Rightarrow ef\,\rho\,e^+\,\rho\,f^+. \end{aligned}$$

So $e\,\rho_{\min}\,f$ and hence we have the first part of the required result.

Now suppose that μ is a congruence on S with the property that for $e, f \in E_S$,

$$e\,\mu\,f \text{ if and only if } e\,\rho\,f$$

and that $a\,\rho_{\min}\,b$ for $a, b \in S$. Then there exists $g \in E_S$ such that $ga = gb$ and $g\,\rho\,a^+\,\rho\,b^+$. Therefore $g\,\mu\,a^+\,\mu\,b^+$ as $g, a^+, b^+ \in E_S$. As μ is a congruence,

$$g\,\mu\,a^+ \Rightarrow ga\,\mu\,a^+a \Rightarrow ga\,\mu\,a$$

and dually $g\,\mu\,b^+$ implies $gb\,\mu\,b$. Since $ga = gb$, $a\,\mu\,b$ and therefore $\rho_{\min} \subseteq \mu$. □

As in [15] for right ample semigroups, we define the $(2,1)$-congruence $\rho'_{\min}$ on a right restriction semigroup:

Definition 10.1.4. Let ρ be a $(2,1)$-congruence on a right restriction semigroup S. Then we define $\rho'_{\min}$ on S by

$$a\,\rho'_{\min}\,b \text{ if and only if } af = bf \text{ and } f\,\rho\,a^*\,\rho\,b^* \text{ for some } f \in E_S$$

for $a, b \in S$.

As discussed in [18] for ample semigroups, note that if ρ is a $(2,1,1)$-congruence on a restriction semigroup S, then $\rho_{\min}$ and $\rho'_{\min}$ are both defined on S. So, by Proposition 10.1.3 and its dual, we have the following result:

Corollary 10.1.5. *If S is a restriction semigroup and ρ is a $(2,1,1)$-congruence on S, then*

$$\rho_{min} = \rho'_{min}.$$

Proof. Let $a, b \in S$. Then

$$\begin{aligned} a\,\rho_{\min}\,b &\Rightarrow ea = eb \text{ and } e\,\rho\,a^+\,\rho\,b^+ \text{ for some } e \in E_S \\ &\Rightarrow a(ea)^* = b(eb)^* \text{ and } ea\,\rho\,a^+a \\ &\Rightarrow a(ea)^* = b(eb)^* \text{ and } ea\,\rho\,a \\ &\Rightarrow a(ea)^* = b(eb)^* \text{ and } (ea)^*\,\rho\,a^*. \end{aligned}$$

We have $(ea)^* = (eb)^*$ as $ea = eb$. Similarly we can deduce $(eb)^*\,\rho\,a^*$ and hence $a\,\rho'_{\min}\,b$. Dually $a\,\rho'_{\min}\,b$ implies $a\,\rho_{\min}\,b$. So $\rho_{\min}$ and $\rho'_{\min}$ are $(2,1,1)$-congruences.

We have

$$e\,\rho_{\min}\,f \Leftrightarrow e\,\rho\,f \Leftrightarrow e\,\rho'_{\min}\,f$$

for $e, f \in E$, by the first part of Proposition 10.1.3 and its dual. By the second part of Proposition 10.1.3 and its dual,

$$\rho_{\min} \subseteq \rho'_{\min} \text{ and } \rho'_{\min} \subseteq \rho_{\min}.$$

Therefore $\rho_{\min} = \rho_{\min}$.

□

Here we begin to look at analogues of results by Petrich and Reilly in [47] for restriction semigroups. The following result, a generalisation of Theorem 3.3, gives us alternative conditions for a variety of restriction semigroups to have proper covers.

Theorem 10.1.6. *Let $\mathscr{V}$ be a variety of restriction semigroups. Then the following are equivalent:*

(i) $\mathscr{V}$ has proper covers;

(ii) the free objects in $\mathscr{V}$ are proper;

(iii) $\mathscr{V}$ is generated by its proper members.

Proof. (i) $\Rightarrow$ (ii) Suppose that $\mathscr{V}$ is a variety of restriction semigroups and that $\mathscr{V}$ has proper covers. So for every $S \in \mathscr{V}$ there is a proper cover of S in $\mathscr{V}$.

Let F, with map $\iota : X \to F$, be the free restriction semigroup on X in $\mathscr{V}$. As F is in the variety $\mathscr{V}$, there exists a proper restriction semigroup $S \in \mathscr{V}$ and there is an onto $(2,1,1)$-morphism $\varphi : S \to F$ which is E_S-separating.

We note that for $u, v \in S$,we have $u\,\mathrm{Ker}\varphi\,v$ if and only if $u\varphi = v\varphi$. Let T be a cross section of $\mathrm{Ker}\varphi$, i.e. let T consist of a representative from each $\mathrm{Ker}\,\varphi$ class.

Let us define a map $\psi : X \to T$ by $x\psi = t$ if $t \in T$ and $t\varphi = x\iota$. Since F is free, there is a unique extension of ψ, namely $\overline{\psi} : F \to S$, such that

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[d, "\psi"'] & F \arrow[dl, "\overline{\psi}"] \\ S & \end{tikzcd}$$

commutes. We note that for $x \in X$,

$$(x\iota)\overline{\psi}\varphi = x\psi\varphi = x\iota.$$

We have morphisms $\overline{\psi} : F \to S$ and $\varphi : S \to F$, so $\overline{\psi}\varphi : F \to F$ is a morphism which restricts to the identity map on $X\iota$.

However, as $X\iota$ is the set of generators of F , so it follows that $\overline{\psi}\varphi$ is the identity map on F.

Recall that if $\overline{\psi}\varphi$ is one-to-one, then $\overline{\psi}$ is one-to-one. Since $\overline{\psi}\varphi$ is the identity map on F, it is one-to-one and so $\overline{\psi} : F \to S$ is a one-to-one morphism.

We have $U = F\overline{\psi}$ is a subalgebra of S. By Proposition 2.8.6, U is proper since S is proper. As $F \cong F\overline{\psi}$, F is also proper.

(ii) $\Rightarrow$ (iii) By Proposition 1.4.4, a variety is generated by its free objects, so $\mathscr{V}$ is generated by its proper members.

(iii) $\Rightarrow$ (i) Suppose that $\mathscr{V}$ is a variety of restriction semigroups and that $\mathscr{V}$ is generated by its proper members. We wish to show $S \in \mathscr{V}$ has a proper cover in $\mathscr{V}$.

By Theorem 1.3.5 in Section 1.3, there exist proper restriction semigroups $T_\alpha \in \mathscr{V}$, a restriction semigroup $T \in \mathcal{V}$ which is a subalgebra of $\prod_{\alpha\in\Lambda} T_\alpha$ (where Λ is an indexing set), and an onto $(2, 1, 1)$-morphism $\varphi : T \to S$.

As each T_α is proper, $\prod_{\alpha\in\Lambda} T_\alpha$ is proper by Proposition 2.8.7. Since T is a subalgebra of $\prod_{\alpha\in\Lambda} T_\alpha$, T is also proper by Proposition 2.8.6.

Let ρ be the congruence on T induced by φ, so $\rho = \text{Ker } \varphi$ and

$$a\,\rho\,b \Leftrightarrow a\varphi = b\varphi \Leftrightarrow (a, b) \in \text{Ker } \varphi$$

for $a, b \in T$. Let $\rho_{\min}$ be defined as in Definition 10.1.1. So $\rho_{\min}$ is the least congruence on T with the same trace as ρ, by Proposition 10.1.3, and is also equal to $\rho'_{\min}$, by Corollary 10.1.5.

As $\varphi : T \to S$ is an onto $(2,1,1)$-morphism,

$$T/\text{ Ker }\varphi \cong S$$

and so there is an isomorphism $\overline{\varphi} : T/\rho \to S$ such that $(a\rho)\overline{\varphi} = a\varphi$ for $a \in T$.

Let $\tau : T/\rho_{\min} \to T/\rho$ be defined by

$$(t\rho_{\min})\tau = t\rho$$

for $t \in T$, which exists due to Proposition 1.2.10 as $\rho_{\min} \subseteq \rho$. Clearly τ is an onto $(2,1)$-morphism.

So we have the following commutative diagram of onto $(2,1)$-morphisms:

$$\begin{tikzcd}
T \arrow[r, "\varphi"] \arrow[d, "{\rho^{\natural}}_{\min}"'] & S \\
T/\rho_{\min} \arrow[r, "\tau"] & T/\rho \arrow[u, "\overline{\varphi}"']
\end{tikzcd}$$

We wish to show that $T/\rho_{\min}$ is a proper cover of S. We know that $T/\rho_{\min}$ is a restriction semigroup as T is a restriction semigroup, and that $T/\rho_{\min} \in \mathscr{V}$ as we have closure under homomorphic images.

Let E denote the subsemilattice associated with $T/\rho_{\min}$ which is equal to

$$\{e\rho_{\min} : e \in E_T\}.$$

Let us consider the onto $(2,1)$-morphism $\tau\overline{\varphi} : T/\rho_{\min} \to S$. For $e, f \in E_T$, we have

$$(e\rho_{\min})\tau = (f\rho_{\min})\tau \Rightarrow e\rho = f\rho \Rightarrow e\rho_{\min} = f\rho_{\min}$$

by Proposition 10.1.3, so τ is E-separating. As τ is E-separating and $\overline{\varphi}$ is one-to-one, $\tau\overline{\varphi}$ is E-separating.

We shall show that $T/\rho_{\min}$ is proper. We will prove $\widetilde{\mathscr{R}}_E \cap \sigma_{T/\rho_{\min}} = \iota$, where $\sigma_{T/\rho_{\min}}$ is defined on $T/\rho_{\min}$. Suppose that for $a, b \in T$, $a\,\rho_{\min}\,(\widetilde{\mathscr{R}}_E \cap \sigma_{T/\rho_{\min}})\,b\,\rho_{\min}$. We wish to show that $a\,\rho_{\min} = b\,\rho_{\min}$. Let us denote $a\,\rho_{\min}$ by $[a]$ for $a \in T$.

We have $[a]\,\widetilde{\mathscr{R}}_E\,[b]$ and $[a]\,\sigma_{T/\rho_{\min}}\,[b]$, so

$$[a^+] = [b^+]$$

and

$$[e][a] = [e][b]$$

for some $e \in E_T$. We have $ea\,\rho_{\min}\,eb$, so there exists $f \in E_T$ such that $fea = feb$ and $f\,\rho\,(ea)^+\,\rho\,(eb)^+$. We note that

$$f\,\rho\,(ea)^+\,\rho\,(eb)^+ \Rightarrow fe\,\rho\,e(ea)^+\,\rho\,e(eb)^+.$$

As $a^+\,\rho_{\min}\,b^+$, there exists $g \in E_T$ such that $ga^+ = gb^+$ and $g\,\rho\,a^+\,\rho\,b^+$. We have

$$\begin{aligned} fea = feb &\Rightarrow a\,\sigma_T\,b \\ &\Rightarrow ga\,\sigma_T\,gb. \end{aligned}$$

As $ga^+ = gb^+$, $(ga)^+ = (gb)^+$ and so

$$ga\,\widetilde{\mathscr{R}}_{E_T}\,gb.$$

As T is proper, we have $ga = gb$. As $g\,\rho\,a^+\,\rho\,b^+$, we have $a\,\rho_{\min} = b\,\rho_{\min}$. Therefore $\widetilde{\mathscr{R}}_E \cap \sigma_{T/\rho_{\min}} = \iota$. As $\rho_{\min} = \rho'_{\min}$, the dual argument gives $\widetilde{\mathscr{L}}_E \cap \sigma_{T/\rho_{\min}} = \iota$.

□

10.2 Subhomomorphisms

We shall introduce the definition of a subhomomorphism for restriction semigroups, which was previously defined for inverse semigroups, and generalise results by Petrich and Reilly [47]. The definition of a subhomomorphism is as follows in the inverse case, which we shall define as an inverse subhomomorphism to distinguish them.

Definition 10.2.1. [47] Let S and T be inverse semigroups. Then a mapping $\varphi : S \to 2^T$ is an *inverse subhomomorphism* of S into T, if for all $s, t \in S$,

(i) $s\varphi \neq \emptyset$;

(ii) $(s\varphi)(t\varphi) \subseteq (st)\varphi$;

(iii) $s'\varphi = (s\varphi)'$,

where for any subset A of T, $A' = \{a' : a \in A\}$.

Adapting this definition for left/right restriction semigroups, we have to take the following definition of subhomomorphisms. These are also known as relational morphisms as in the left ample case [50]:

Definition 10.2.2. Let S and T be left restriction semigroups. Then a mapping $\varphi : S \to 2^T$ is a *left subhomomorphism* of S into T, if for all $s, t \in S$,

(i) $s\varphi \neq \emptyset$;

(ii) $(s\varphi)(t\varphi) \subseteq (st)\varphi$;

(iii) $(s\varphi)^+ \subseteq s^+\varphi$,

where for any subset A of T, $A^+ = \{a^+ : a \in A\}$. Dually, we define a *right subhomomorphism* to be a map φ such that Conditions (i), (ii) and $(s\varphi)^* \subseteq s^*\varphi$ hold. If S and T are restriction semigroups, $\varphi : S \to 2^T$ is a *subhomomorphism* if it is both a left and right subhomomorphism.

A left or right subhomomorphism is said to be *surjective* if $S\varphi = T$, where $S\varphi = \cup\{s\varphi : s \in S\}$.

As an inverse semigroups is a restriction semigroups with distinguished semilattice $E(S)$, we have the following connection between the definitions of inverse subhomomorphism and subhomomorphism:

Proposition 10.2.3. *Let S and T be inverse semigroups. If φ is an inverse subhomomorphism of S into T, then φ is a subhomomorphism of S into T.*

Proof. Suppose φ is an inverse subhomomorphism of S into T and $s \in S$. Then as $s'\varphi = (s\varphi)'$, we have

$$\begin{aligned}(s\varphi)^+ &= \{uu' : u \in s\varphi\}\\ &\subseteq (s\varphi)(s\varphi)'\\ &= (s\varphi)(s'\varphi)\\ &\subseteq (ss')\varphi\\ &= s^+\varphi.\end{aligned}$$

Dually, $(s\varphi)^* \subseteq s^*\varphi$.

□

Note that we shall state and prove results for left restriction semigroups and left subhomomorphisms. The dual, and hence the two-sided, corresponding results hold true as well.

Proposition 10.2.4. *Let φ be a left subhomomorphism of S into T, where S and T are left restriction semigroups. Then $S\varphi$ is a left restriction semigroup with respect to the distinguished semilattice $E_{S\varphi} = \cup\{(s\varphi)^+ : s \in S\}$.*

Proof. As $S\varphi \subseteq T$ and $S\varphi \neq \emptyset$, we just need to check closure under the binary and unary operations.

Considering $u, v \in S\varphi$, we have $u \in s\varphi$ and $v \in t\varphi$ for some $s, t \in S$. So

$$uv \in (s\varphi)(t\varphi) \subseteq (st)\varphi,$$

where $st \in S$. Hence $uv \in S\varphi$.

Let $a \in S\varphi$. So $a \in s\varphi$ for some $s \in S$. Then $a^+ \in (s\varphi)^+ \subseteq s^+\varphi$. As $s^+ \in S$, $a^+ \in S\varphi$.

□

We shall generalise Proposition 2.2 in [41].

Proposition 10.2.5. *Let S and T be left restriction semigroups and let φ be a (surjective) left subhomomorphism of S into T. Then*

$$\Pi(S,T,\varphi) = \{(s,t) \in S \times T : t \in s\varphi\}$$

is a left restriction semigroup (which is a subdirect product of S and T).

Conversely, suppose that V is a left restriction semigroup which is a subdirect product of S and T. Then φ, defined by

$$s\varphi = \{t \in T : (s,t) \in V\},$$

is a surjective left subhomomorphism of S into T. Furthermore, $V = \Pi(S,T,\varphi)$.

Proof. To show that $\Pi(S,T,\varphi)$ is a left restriction semigroup, we shall show that $\Pi(S,T,\varphi)$ is a subalgebra of $S \times T$. As the class of left restriction semigroups forms a variety, if S and T are left restriction semigroups, then $S \times T$ is a left restriction semigroup with respect to distinguished semilattice $E_{S\times T} = E_S \times E_T$. If $\Pi(S,T,\varphi)$ is a subalgebra of $S \times T$, then this would imply that $\Pi(S,T,\varphi)$ is a left restriction semigroup.

First note that $\Pi(S,T,\varphi)$ is non-empty as for $s \in S$, $s\varphi \neq \emptyset$. To show that $\Pi(S,T,\varphi)$ is a subalgebra of $S \times T$, we need to show closure under the binary and unary operations.

Take $(s,t),(u,v) \in \Pi(S,T,\varphi)$. So $s,u \in S$, $t,v \in T$, $t \in s\varphi$ and $v \in u\varphi$. We have $tv \in (s\varphi)(u\varphi) \subseteq (su)\varphi$, and so $tv \in (su)\varphi$. Therefore

$$(s,u)(t,v) = (su,tv) \in \Pi(S,T,\varphi).$$

Hence the binary operation is closed.

Let us consider the semilattice

$$E_\Pi = \{(s,t) \in E_S \times E_T : (s,t) \in \Pi(S,T,\varphi)\} \subseteq E_S \times E_T.$$

Take $(s,t) \in \Pi(S,T,\varphi)$. Within $S \times T$, $(s,t)^+ = (s^+,t^+)$. We require $(s^+,t^+) \in E_\Pi$. We have

$$\begin{aligned}
(s,t) \in \Pi(S,T,\varphi) &\Rightarrow t \in s\varphi \\
&\Rightarrow t^+ \in (s\varphi)^+ \\
&\Rightarrow t^+ \in s^+\varphi \\
&\Rightarrow (s^+,t^+) \in \Pi(S,T,\varphi) \\
&\Rightarrow (s^+,t^+) \in E_\Pi.
\end{aligned}$$

So $\Pi(S,T,\varphi)$ is a left restriction semigroup with distinguished semilattice E_Π.

Suppose now that φ is surjective. We wish to show that $\Pi(S,T,\varphi)$ is a subdirect product of S and T (i.e. the coordinate maps are onto) if φ is surjective. Let $p_1 : \Pi(S,T,\varphi) \to S$ and $p_2 : \Pi(S,T,\varphi) \to T$ be defined by $(s,t)p_1 = s$ and $(s,t)p_2 = t$ for $(s,t) \in \Pi(S,T,\varphi)$. Take $s \in S$. Then as $s\varphi \neq \emptyset$, there exists $u \in s\varphi$ and so $(s,u) \in \Pi(S,T,\varphi)$. Take $t \in T$. As φ is surjective, $t \in S\varphi$ and so $t \in v\varphi$ for some $v \in S$. Hence $(v,t) \in \Pi(S,T,\varphi)$. Therefore p_1 and p_2 are onto and so $\Pi(S,T,\varphi)$ is a subdirect product of S and T.

Let V be a left restriction semigroup which is a subdirect product of S and T, so $V \subseteq S \times T$. We shall show that $s\varphi = \{t \in T : (s,t) \in V\}$ is a surjective left subhomomorphism of S into T.

(i) Take $s \in S$. As p_1 is onto, there exists $(s,v) \in V$ such that $(s,v)p_1 = s$. So $v \in s\varphi$ and hence $s\varphi \neq \emptyset$.

(ii) Consider $u \in s\varphi$ and $v \in t\varphi$. Then $(s,u),(t,v) \in V$. Then $(st,uv) \in V$ and hence $uv \in (st)\varphi$, so that $(s\varphi)(t\varphi) \subseteq (st)\varphi$.

(iii) Let $u \in s\varphi$ and consider $u^+ \in (s\varphi)^+ = \{u^+ : u \in s\varphi\}$. As $u \in s\varphi$, $(s,u) \in V\psi$. We have

$$\begin{aligned}(s,u) \in V &\Rightarrow (s,u)^+ \in V\\ &\Rightarrow (s^+,u^+) \in V\\ &\Rightarrow u^+ \in s^+\varphi.\end{aligned}$$

Hence $(s\varphi)^+ \subseteq s^+\varphi$.

Take $t \in T$. We wish to show that $t \in s\varphi$ for some $s \in S$, i.e. there exists $s \in S$ such that $(s,t) \in V$. As p_2 is onto, $t \in T$ implies there exists $(u,t) \in V$ so that $t \in u\varphi$. Hence $T \subseteq S\varphi$. Since $S\varphi \subseteq T$, $S\varphi = T$.

Considering $(s,t) \in S \times T$,

$$\begin{aligned}(s,t) \in \Pi(S,T,\varphi) &\Leftrightarrow t \in s\varphi\\ &\Leftrightarrow (s,t) \in V.\end{aligned}$$

Hence $V = \Pi(S,T,\varphi)$.

□

We shall now generalise Theorem 4.3 from [47].

Theorem 10.2.6. *Let R, S and T be left restriction semigroups. Let $\alpha : R \to S$ be an onto morphism and $\beta : R \to T$ a morphism. Then $\varphi = \alpha^{-1}\beta$ is a left subhomomorphism of S into T and every such left subhomomorphism is obtained in this way.*

Proof. If R, S and T are left restriction semigroups, $\alpha : R \to S$ an onto morphism and $\beta : R \to T$ a morphism such that $\varphi = \alpha^{-1}\beta$ is a left subhomomorphism of S into T, then this will be represented as

$$\begin{tikzcd} R \arrow[r, "\beta"] \arrow[d, "\alpha"'] & T \\ S \arrow[ur, dotted, "\varphi = \alpha^{-1}\beta"'] & \end{tikzcd}$$

Considering left restriction semigroups R, S and T with onto morphism $\alpha : R \to S$ and morphism $\beta : R \to T$, we wish to show that $\varphi = \alpha^{-1}\beta$ is a left subhomomorphism of S into T. We need to show for $s, t \in S$,

(i) $s(\alpha^{-1}\beta) \neq \emptyset$;

(ii) $(s(\alpha^{-1}\beta))(t(\alpha^{-1}\beta)) \subseteq (st)(\alpha^{-1}\beta)$;

(iii) $(s(\alpha^{-1}\beta))^+ \subseteq s^+(\alpha^{-1}\beta)$.

(i) Take $s \in S$. As α is onto, there exists $r \in R$ such that $r\alpha = s$. So $r \in s\alpha^{-1}$. Put $r\beta = t$. So

$$t = r\beta \in s(\alpha^{-1}\beta)$$

and hence $s(\alpha^{-1}\beta) \neq \emptyset$ for all $s \in S$.

(ii) Let $t_1 \in (s_1(\alpha^{-1}\beta))$ and $t_2 \in (s_2(\alpha^{-1}\beta))$. So

$$t_1t_2 \in (s_1(\alpha^{-1}\beta))(s_2(\alpha^{-1}\beta)).$$

There exist $r_1, r_2 \in R$ such that $r_1\beta = t_1$, $r_2\beta = t_2$, $r_1 \in s_1\alpha^{-1}$ and $r_2 \in s_2\alpha^{-1}$. We have

$$\begin{aligned} r_1 \in s_1\alpha^{-1}, r_2 \in s_2\alpha^{-1} &\Rightarrow r_1\alpha = s_1, r_2\alpha = s_2 \\ &\Rightarrow (r_1\alpha)(r_2\alpha) = s_1s_2 \\ &\Rightarrow (r_1r_2)\alpha = s_1s_2 \\ &\Rightarrow r_1r_2 \in (s_1s_2)\alpha^{-1}. \end{aligned}$$

We therefore have $t_1t_2 = (r_1\beta)(r_2\beta) = (r_1r_2)\beta$ and as $r_1r_2 \in (s_1s_2)\alpha^{-1}$, $t_1t_2 \in (s_1s_2)(\alpha^{-1}\beta)$.

(iii) Consider $t \in s(\alpha^{-1}\beta)$ and $t^+ \in (s(\alpha^{-1}\beta))^+ = \{t^+ : t \in s(\alpha^{-1}\beta)\}$. We wish to show that $t^+ \in s^+(\alpha^{-1}\beta)$, i.e. that there exists $p \in R$ such that $p \in s^+\alpha^{-1}$ and $p\beta = t^+$. We have

$$\begin{aligned} t \in s(\alpha^{-1}\beta) &\Rightarrow r \in s\alpha^{-1} \text{ and } r\beta = t \text{ for some } r \in R \\ &\Rightarrow s = r\alpha \\ &\Rightarrow s^+ = (r\alpha)^+ = r^+\alpha \\ &\Rightarrow r^+ \in s^+\alpha^{-1}. \end{aligned}$$

As $t^+ = (r\beta)^+ = r^+\beta$, we have $t^+ \in s^+(\alpha^{-1}\beta)$. Hence $(s(\alpha^{-1}\beta))^+ \subseteq s^+(\alpha^{-1}\beta)$ and so $\varphi = \alpha^{-1}\beta$ is a left subhomomorphism of S into T.

Conversely, let S and T be left restriction semigroups and let φ be a left subhomomorphism from S into T. Let

$$R = \Pi(S, T, \varphi).$$

Then R is a left restriction semigroup due to Proposition 10.2.5. Let $\alpha : R \to S$ and $\beta : R \to T$ be defined by $(s,t)\alpha = s$ and $(s,t)\beta = t$. We see that α and β are clearly morphisms. If $s \in S$, then as $s\varphi \neq \emptyset$, there exists $t \in T$ such that $t \in s\varphi$. Hence there exists $(s,t) \in R$ such that $(s,t)\alpha = s$ and so α is onto.

We have $(s,t) \in R$ if and only if $t \in s\varphi$, by the definition of R, so we have

$$\begin{aligned} t \in s(\alpha^{-1}\beta) &\Leftrightarrow (u,v)\beta = t \text{ and } (u,v) \in s\alpha^{-1} \text{ for some } (u,v) \in R \\ &\Leftrightarrow v = t \text{ and } (u,v)\alpha = s \text{ for some } (u,v) \in R \\ &\Leftrightarrow v = t \text{ and } u = s \text{ for some } (u,v) \in R \\ &\Leftrightarrow (s,t) \in R \\ &\Leftrightarrow t \in s\varphi. \end{aligned}$$

Hence $\varphi = \alpha^{-1}\beta$.

$\square$

Proposition 10.2.7. *Let φ be a left subhomomorphism of S into M, where S is a left restriction semigroup and M a monoid. Then $\Pi(S, M, \varphi)$ is proper if and only if φ satisfies*

$$a\varphi \cap b\varphi \neq \emptyset, a\,(\widetilde{\mathcal{R}}_{E_S} \cap \sigma_S)\,b \Rightarrow a = b, \tag{S1}$$

for $a, b \in S$. If φ is a right subhomomorphism of S into M, where S is a right restriction semigroup and M a monoid, then $\Pi(S, M, \varphi)$ is proper if and only if φ satisfies

$$a\varphi \cap b\varphi \neq \emptyset, a\,(\widetilde{\mathcal{L}}_{E_S} \cap \sigma_S)\,b \Rightarrow a = b, \tag{S2}$$

for $a, b \in S$.

Proof. Let φ be a left subhomomorphism of S into M, where S is a left restriction semigroup and M a monoid. Suppose that $\Pi(S, M, \varphi)$ is proper, so

$$(s,t)\,(\widetilde{\mathcal{R}}_{E_\Pi} \cap \sigma_\Pi)\,(u,v) \Rightarrow (s,t) = (u,v).$$

Suppose that $s\varphi \cap u\varphi \neq \emptyset$ and $s\,(\widetilde{\mathcal{R}}_{E_S} \cap \sigma_S)\,u$ for $s, u \in S$. We wish to show $s = u$.

We have

$$\begin{aligned} s\varphi \cap u\varphi \neq \emptyset &\Rightarrow m \in s\varphi, u\varphi \text{ for some } m \in M \\ &\Rightarrow (s,m),(u,m) \in \Pi(S,M,\varphi) \end{aligned}$$

and

$$\begin{aligned} s\, \widetilde{\mathcal{R}}_{E_S}\, u &\Rightarrow s^+ = u^+ \\ &\Rightarrow (s^+,1) = (u^+,1) \\ &\Rightarrow (s,m)^+ = (u,m)^+ \\ &\Rightarrow (s,m)\, \widetilde{\mathcal{R}}_{E_\Pi}\, (u,m). \end{aligned}$$

If we consider $e \in E_S$, then $e \in S$ and so $e\varphi \neq \emptyset$. So there exists $m \in M$ such that $m \in e\varphi$. Hence $(e,m) \in \Pi(S,M,\varphi)$ and so $(e,m)^+ = (e,1) \in E_\Pi$. So we have

$$\begin{aligned} s\, \sigma_S\, u &\Rightarrow es = eu \text{ for some } e \in E_S \\ &\Rightarrow (e,1)(s,m) = (e,1)(u,m) \text{ for } (e,1) \in E_\Pi \\ &\Rightarrow (s,m)\, \sigma_\Pi\, (u,m). \end{aligned}$$

Since $\Pi(S,M,\varphi)$ is proper,

$$\begin{aligned} (s,m)\, (\widetilde{\mathcal{R}}_{E_\Pi} \cap \sigma_\Pi)\, (u,m) &\Rightarrow (s,m) = (u,m) \\ &\Rightarrow s = u. \end{aligned}$$

Therefore Condition (S1) holds.

Conversely, suppose that φ satisfies Condition (S1), i.e.

$$a\varphi \cap b\varphi \neq \emptyset, a\, (\widetilde{\mathcal{R}}_{E_S} \cap \sigma_S)\, b \Rightarrow a = b,$$

for $a, b \in S$. We wish to show that

$$(s,t)\, (\widetilde{\mathcal{R}}_{E_\Pi} \cap \sigma_\Pi)\, (u,v) \Leftrightarrow (s,t) = (u,v)$$

for $(s,t),(u,v) \in \Pi(S,M,\varphi)$. Suppose $(s,t)\, (\widetilde{\mathcal{R}}_{E_\Pi} \cap \sigma_\Pi)\, (u,v)$. We have

$$\begin{aligned} (s,t)\, \widetilde{\mathcal{R}}_{E_\Pi}\, (u,v) &\Leftrightarrow (s,t)^+ = (u,v)^+ \\ &\Leftrightarrow (s^+,1) = (u^+,1) \\ &\Leftrightarrow s^+ = u^+ \\ &\Leftrightarrow s\, \widetilde{\mathcal{R}}_{E_S}\, u \end{aligned}$$

and

$$\begin{aligned} (s,t)\, \sigma_\Pi\, (u,v) &\Leftrightarrow t = v \text{ and } es = eu \text{ for some } e \in E_S \\ &\Leftrightarrow t = v \text{ and } s\, \sigma_S\, u. \end{aligned}$$

It remains to show $s = u$.

As $(s,t),(u,v) \in \Pi(S,M,\varphi)$, $t \in s\varphi$ and $v \in u\varphi$. As $t = v$, $t \in s\varphi \cap u\varphi$ and so $s\varphi \cap u\varphi \neq \emptyset$. So, by Condition (S1), $s = u$. Hence $(s,t) = (u,v)$ and $\Pi(S,M,\varphi)$ is proper.

□

Proposition 10.2.8. *Let φ be a left subhomomorphism of S into M, where S is a left restriction semigroup and M a monoid. Then $\Pi(S,M,\varphi)$ is E_Π-unitary if and only if φ satisfies*

$$1 \in s\varphi, es \in E_S \Rightarrow s \in E_S, \tag{S3}$$

for $s \in S$ and $e \in E_S$.

Proof. Suppose $\Pi(S,M,\varphi)$ is E-unitary, i.e. $(e,1),(e,1)(s,t) \in E_\Pi$ implies $(s,t) \in E_\Pi$. Suppose $1 \in s\varphi$ and $es \in E_S$ for $s \in S$ and $e \in E_S$. Then $(s,1) \in \Pi(S,M,\varphi)$ as $1 \in s\varphi$. We have $(es,1) \in E_\Pi$ as $es \in E_S$. So

$$\begin{aligned} (e,1),(e,1)(s,1) = (es,1) \in E_\Pi &\Rightarrow (s,1) \in E_\Pi \\ &\Rightarrow s \in E_S. \end{aligned}$$

Conversely, suppose that Condition (S3) holds. Taking $(e,1),(e,1)(s,t) \in E_\Pi$, we wish to show $(s,t) \in E_\Pi$. We have $es \in E_S$ and $t = 1$. As $(s,t) \in \Pi(S,M,\varphi)$, $t \in s\varphi$, i.e. $1 \in s\varphi$. By Condition (S3), $s \in E_S$ and so $(s,t) = (s,1) \in E_\Pi$. □

As a proper left restriction semigroup S is E_S-unitary, we have the following:

Proposition 10.2.9. *Suppose φ is a left subhomomorphism of S into M, where S is a left restriction semigroup and M a monoid. Then*

(S1) $a\varphi \cap b\varphi \neq \emptyset, a\,(\widetilde{\mathcal{R}}_{E_S} \cap \sigma_S)\,b \Rightarrow a = b$ for $a,b \in S$

implies

(S3) $1 \in s\varphi, es \in E_S \Rightarrow s \in E_S$ for $s \in S$ and $e \in E_S$.

Dually, Condition (S2) implies Condition (S3).

We shall consider conditions on subhomomorphisms that will allow us to generalise results in [47]. Conditions (S1) and (S2) give us conditions for $\Pi(S,M,\varphi)$ to be proper. We shall introduce Conditions (S4) and (S5) which will allow us to generalise Proposition 5.5 in [47] in the proper cases. We note that Condition (S4) is the left proper condition from [50].

$$a\varphi \cap b\varphi \neq \emptyset \Rightarrow a^+b = b^+a, \tag{S4}$$

for $a,b \in S$;

$$a\varphi \cap b\varphi \neq \emptyset \Rightarrow ab^* = ba^*, \tag{S5}$$

for $a, b \in S$.

We shall also introduce Conditions (S6) and (S7), which we will show to be equivalent to (S4) and (S5) respectively:

$$a\varphi \cap b\varphi \neq \emptyset, a^+ = b^+ \Rightarrow a = b, \tag{S6}$$

for $a, b \in S$;

$$a\varphi \cap b\varphi \neq \emptyset, a^* = b^* \Rightarrow a = b, \tag{S7}$$

for $a, b \in S$.

Proposition 10.2.10. *Let φ be a left subhomomorphism of S into M, where S is a left restriction semigroup and M is a monoid. Then the following are equivalent for $a, b \in S$:*

(S4) $a\varphi \cap b\varphi \neq \emptyset \Rightarrow a^+b = b^+a$;

(S6) $a\varphi \cap b\varphi \neq \emptyset, a^+ = b^+ \Rightarrow a = b$.

Dually, if φ is a right subhomomorphism of S into M, where S is a right restriction semigroup and M is a monoid, then the following are equivalent for $a, b \in S$:

(S5) $a\varphi \cap b\varphi \neq \emptyset \Rightarrow ab^* = ba^*$;

(S7) $a\varphi \cap b\varphi \neq \emptyset, a^* = b^* \Rightarrow a = b$.

Proof. Suppose φ is a left subhomomorphism of S into M, where S is a left restriction semigroup and M is a monoid. Let Condition (S4) hold, $a\varphi \cap b\varphi \neq \emptyset$ and $a^+ = b^+$. As $a\varphi \cap b\varphi \neq \emptyset$, $a^+b = b^+a$ by Condition (S4). So

$$a = a^+a = b^+a = a^+b = b^+b = b.$$

Conversely, suppose that Condition (S6) holds and $a\varphi \cap b\varphi \neq \emptyset$ for $a, b \in S$. We note

$$a^+b^+ = b^+a^+ \Rightarrow (a^+b^+)^+ = (b^+a^+)^+ \Rightarrow (a^+b)^+ = (b^+a)^+$$

by Proposition 2.6.2. Suppose $m \in a\varphi \cap b\varphi$. Then $m \in M$ such that $m \in a\varphi$ and $m \in b\varphi$. Hence $m^+ \in (a\varphi)^+$ and $m^+ \in (b\varphi)^+$. So $1 \in a^+\varphi$ and $1 \in b^+\varphi$. As $m \in b\varphi$,

$$m = 1m \in (a^+\varphi)(b\varphi) \subseteq (a^+b)\varphi.$$

Similarly $m \in (b^+a)\varphi$. Therefore

$$m \in (a^+b)\varphi \cap (b^+a)\varphi.$$

As $(a^+b)\varphi \cap (b^+a)\varphi \neq \emptyset$ and $(a^+b)^+ = (b^+a)^+$, $a^+b = b^+a$ by Condition (S6).

Dually, Conditions (S5) and (S7) are equivalent. $\square$

Condition (S3) is the condition required for $\Pi(S, M, \varphi)$ to be E-unitary and will be used to prove Proposition 10.3.3 in Section 10.3. We shall introduce Condition (S8), which is the *unitary* condition in the inverse case in [47]:

$$1 \in s\varphi \Rightarrow s \in E_S, \tag{S8}$$

for $s \in S$.

Clearly, Condition (S8) implies Condition (S3):

$$1 \in s\varphi, es \in E_S \Rightarrow s \in E_S, \tag{S3}$$

for $s \in S$ and $e \in E_S$.

We shall consider Condition (S9), which is required when we consider covers in both the proper and E-unitary cases:

$$a\varphi \cap b\varphi \neq \emptyset \Rightarrow a \, \sigma_S \, b, \tag{S9}$$

for $a, b \in S$.

We see that in the inverse case Condition (S8) implies (S9):

Proposition 10.2.11. *Let φ be an inverse subhomomorphism of S into G, where S is an inverse semigroup and G a group. Then*

(S8) $1 \in s\varphi \Rightarrow s \in E(S)$, *for* $s \in S$,

implies

(S9) $s\varphi \cap t\varphi \neq \emptyset \Rightarrow s \, \sigma_S \, t$, *for* $s, t \in S$.

Proof. Suppose φ is a subhomomorphism of S into G, where S is an inverse semigroup and G a group. Let Condition (S8) hold and $m \in s\varphi \cap t\varphi$. Then $m \in s\varphi$ and

$$m^{-1} \in (s\varphi)^{-1} = s'\varphi.$$

Therefore

$$1 = m^{-1}m \in (s'\varphi)(t\varphi) \subseteq (s't)\varphi.$$

By Condition (S8), $s't \in E(S)$ as $1 \in (s't)\varphi$. We have $(ss')t = s(s't)$, where $ss', s't \in E(S)$. By Proposition 2.7.6, $s \, \sigma_S \, t$. □

In the restriction cases, we shall see that Conditions (S4) and (S5) both imply Condition (S9), but Condition (S3) does not necessarily imply Condition (S9) as we see from the following example:

Example 10.2.12. Let S be a reduced left restriction semigroup with at least three distinct elements, so along with the identity there are at least two other elements. Plenty of such examples exist as we can consider any monoid with three or more elements. Let $T = \{0, 1\}$ where $01 = 10 = 00 = 0$ and $11 = 1$. It can easily be seen that T is also a reduced left restriction semigroup.

Let us define $\varphi : S \to 2^T$ by

$$a\varphi = \begin{cases} \{1\} & \text{if } a = 1 \\ \{0\} & \text{if } a \neq 1. \end{cases}$$

Clearly $a\varphi \neq \emptyset$ for $a \in S$. We have

$$(a\varphi)(b\varphi) = \begin{cases} \{1\} & \text{if } a = b = 1 \\ \{0\} & \text{otherwise.} \end{cases}$$

and

$$(ab)\varphi = \begin{cases} \{1\} & \text{if } a = b = 1 \\ \{0\} & \text{otherwise,} \end{cases}$$

so $(a\varphi)(b\varphi) = (ab)\varphi$ for $a, b \in S$. As

$$(a\varphi)^+ = \{1\} = 1\varphi = a^+\varphi$$

for any $a \in S$, φ is a left subhomomorphism of S into T. Condition (S8) holds as

$$1 \in s\varphi \Rightarrow s = 1 \Rightarrow s \in E_S.$$

We note that in S,

$$\begin{aligned} a \,\sigma_S\, b &\Leftrightarrow ea = eb \text{ for some } e \in E_S \\ &\Leftrightarrow a = b \text{ as } E_S = \{1\}. \end{aligned}$$

Let $a, b \in S$ such that $a \neq b$ and neither are equal to 1. Then

$$a\varphi \cap b\varphi = \{0\} \cap \{0\} = \{0\} \neq \emptyset,$$

but $a \neq b$, so Condition (S9) does not hold.

Proposition 10.2.13. *Let φ be a left subhomomorphism of S into T, where S and T are left restriction semigroups. Then*

(S4) $a\varphi \cap b\varphi \neq \emptyset \Rightarrow a^+b = b^+a$, for $a, b \in S$,

implies

(S9) $a\varphi \cap b\varphi \neq \emptyset \Rightarrow a \,\sigma_S\, b$, for $a, b \in S$.

Dually, if φ is a right subhomomorphism of S into T, where S and T are right restriction semigroups, then

(S5) $a\varphi \cap b\varphi \neq \emptyset \Rightarrow ab^* = ba^*$, for $a, b \in S$,

implies

(S9) $a\varphi \cap b\varphi \neq \emptyset \Rightarrow a\,\sigma_S\,b$, *for* $a, b \in S$.

Proof. Suppose φ is a left subhomomorphism of S into T, where S and T are left restriction semigroups and Condition (S4) holds. Suppose $a\varphi \cap b\varphi \neq \emptyset$. Then by Condition (S4), $a^+b = b^+a$. By Proposition 2.7.3, $a\,\sigma_S\,b$. Dually, Condition (S5) implies Condition (S9). $\square$

In fact, we have the following result:

Proposition 10.2.14. *Let* φ *be a left subhomomorphism of* S *into* T, *where* S *and* T *are left restriction semigroups. Then the following are equivalent for* $a, b \in S$:

(i) (S4) $a\varphi \cap b\varphi \neq \emptyset \Rightarrow a^+b = b^+a$;

(ii) (S6) $a\varphi \cap b\varphi \neq \emptyset, a^+ = b^+ \Rightarrow a = b$;

(iii) Conditions (S1) and (S9),

where

(S1) $a\varphi \cap b\varphi \neq \emptyset, a\,(\widetilde{\mathcal{R}}_{E_S} \cap \sigma_S)\,b \Rightarrow a = b$ *and*

(S9) $a\varphi \cap b\varphi \neq \emptyset \Rightarrow a\,\sigma_S\,b$.

Dually, if φ *be a right subhomomorphism of* S *into* T, *where* S *and* T *are right restriction semigroups, then the following are equivalent for* $a, b \in S$:

(i) (S5) $a\varphi \cap b\varphi \neq \emptyset \Rightarrow ab^* = ba^*$;

(ii) (S7) $a\varphi \cap b\varphi \neq \emptyset, a^* = b^* \Rightarrow a = b$;

(iii) Conditions (S2) and (S9),

where (S2) $a\varphi \cap b\varphi \neq \emptyset, a\,(\widetilde{\mathcal{L}}_{E_S} \cap \sigma_S)\,b \Rightarrow a = b$.

Proof. Suppose φ is a left subhomomorphism of S into T, where S and T are left restriction semigroups. We have already shown Conditions (S4) and (S6) are equivalent in Proposition 10.2.10 and that Condition (S4) implies Condition (S9) in Proposition 10.2.13. In the presence of Condition (S9), Condition (S1) becomes

$$a\varphi \cap b\varphi \neq \emptyset, a\,\widetilde{\mathcal{R}}_{E_S}\,b \Rightarrow a = b,$$

for $a, b \in S$, i.e.

$$a\varphi \cap b\varphi \neq \emptyset, a^+ = b^+ \Rightarrow a = b,$$

for $a, b \in S$, which is Condition (S6). It is clear that Condition (S6) implies Condition (S1). Therefore, Condition (S4) is equivalent to both Condition (S6) and Conditions (S1) and (S9) combined. Dually, Condition (S5) is equivalent to both Condition (S7) and Conditions (S2) and (S9) combined. $\square$

The conditions we require to prove results in Section 10.3 are Conditions (S4), (S5) and (S8). We note the following relationship between them.

Proposition 10.2.15. *Let φ be a left subhomomorphism of S into M, where S is a left restriction semigroup and M a monoid. Then*

(S4) $a\varphi \cap b\varphi \neq \emptyset \Rightarrow a^+b = b^+a$, *for* $a, b \in S$,

implies

(S8) $1 \in s\varphi \Rightarrow s \in E_S$, *for* $s \in S$.

If φ is a right subhomomorphism of S into M, where S is a right restriction semigroup and M a monoid, then

(S5) $a\varphi \cap b\varphi \neq \emptyset \Rightarrow ab^* = ba^*$, *for* $a, b \in S$,

implies

(S8) $1 \in s\varphi \Rightarrow s \in E_S$, *for* $s \in S$.

Proof. Suppose φ is a left subhomomorphism of S into M, where S is a left restriction semigroup and M a monoid, and that Condition (S4) holds. Suppose $1 \in s\varphi$. Then

$$1^+ \in (s\varphi)^+ \subseteq s^+\varphi.$$

So

$$1 \in s\varphi \cap s^+\varphi.$$

As $s\varphi \cap s^+\varphi \neq \emptyset$, by Condition (S4), $s^+s^+ = s^+s$, i.e. $s^+ = s$. So $s \in E_S$. Dually, Condition (S5) implies Condition (S8). $\square$

Before looking at results similar to the last part of Theorem 4.3 in [47], we need to distinguish the two definitions of kernel and the two interpretations of

$$\text{Ker } \beta \subseteq \text{Ker } \alpha,$$

where $\alpha : R \to S$ and $\beta : R \to T$ are morphisms between restriction semigroups R, S and T. We shall let

$$\text{Ker } \alpha = \{(a, b) \in R \times R : a\alpha = b\alpha\}$$

and take the corresponding definition for Ker β. We shall let

$$\text{ker } \alpha = \{a \in R : a\alpha \in E_S\}$$

and take the corresponding definition for ker β. We note the following proposition which shows the connection between the two definitions when considering restriction semigroups:

Proposition 10.2.16. *Let R, S and T be left restriction semigroups. Let $\alpha : R \to S$ and $\beta : R \to T$ be morphisms. Then*

$$\textit{Ker } \beta \subseteq \textit{Ker } \alpha \textit{ implies ker } \beta \subseteq \textit{ker } \alpha.$$

Proof. Suppose Ker $\beta \subseteq$ Ker α and $a \in \ker \beta$. Then $a\beta \in E_T$, i.e. $a\beta = e$ for some $e \in E_T$. So

$$(a\beta)^+ = e^+, \text{ i.e. } a^+\beta = e.$$

Hence $a\beta = a^+\beta$. So $(a, a^+) \in$ Ker β. Hence $(a, a^+) \in$ Ker α and so

$$a\alpha = a^+\alpha = (a\alpha)^+.$$

Hence $a\alpha \in \ker \alpha$ and so $\ker \beta \subseteq \ker \alpha$.

$\square$

Note that by Theorem 10.2.6, if R, S and T are left restriction semigroups, then $\varphi = \alpha^{-1}\beta$ is a left subhomomorphism, where $\alpha : R \to S$ is an onto morphism and $\beta : R \to T$ a morphism, and every left subhomomorphism is in this form. In Section 10.3, we will need to consider Conditions (S4) and (S5). We have the following result that is similar to the last part of Theorem 4.3 in [47], which provides an alternative condition for a left subhomomorphism to satisfy Condition (S4). However, we need a slightly weaker condition than Ker $\beta \subseteq$ Ker α.

Proposition 10.2.17. *Let R, S and T be left restriction semigroups. Let $\alpha : R \to S$ be an onto morphism and $\beta : R \to T$ a morphism. Then the left subhomomorphism $\varphi = \alpha^{-1}\beta$ satisfies*

$$a\varphi \cap b\varphi \neq \emptyset \Rightarrow a^+b = b^+a, \tag{S4}$$

for $a, b \in S$, if and only if

$$s\beta = t\beta \Rightarrow (s^+t)\alpha = (t^+s)\alpha, \tag{*}$$

for $s, t \in R$.

Proof. Suppose Condition (S4) holds and $s\beta = t\beta$ for $s, t \in R$. Let

$$s\beta = t\beta = m \in T,$$

$$s\alpha = p \text{ and } t\alpha = q$$

for some $p, q \in S$. Then $m \in p\varphi$ and $m \in q\varphi$. So $p\varphi \cap q\varphi \neq \emptyset$ and by Condition (S4), $p^+q = q^+p$, i.e.

$$(s\alpha)^+t\alpha = (t\alpha)^+s\alpha$$

and so

$$(s^+t)\alpha = (t^+s)\alpha.$$

Conversely suppose that Condition (*) holds and $a\varphi \cap b\varphi \neq \emptyset$ for $a, b \in S$. Then there exist $m \in T$ such that $m \in a\alpha^{-1}\beta$ and $m \in b\alpha^{-1}\beta$. So there exists $u, v \in R$ such that

$$u\alpha = a \text{ and } u\beta = m,$$

$$v\alpha = b \text{ and } v\beta = m.$$

Hence $u\beta = v\beta$. By (*), $(u^+v)\alpha = (v^+u)\alpha$, i.e.

$$(u\alpha)^+v\alpha = (v\alpha)^+u\alpha$$

and so

$$a^+b = b^+a.$$

□

We shall also state the dual of this proposition:

Proposition 10.2.18. *Let R, S and T be right restriction semigroups. Let $\alpha : R \to S$ be an onto morphism and $\beta : R \to T$ a morphism. Then the right subhomomorphism $\varphi = \alpha^{-1}\beta$ satisfies*

$$a\varphi \cap b\varphi \neq \emptyset \Rightarrow ab^* = ba^*, \tag{S5}$$

for $a, b \in S$, if and only if

$$s\beta = t\beta \Rightarrow (st^*)\alpha = (ts^*)\alpha, \tag{**}$$

for $s, t \in R$.

We can make the following connections with the condition Ker $\beta \subseteq$ Ker α. Note that we could consider either left or right restriction semigroups and obtain the same result.

Proposition 10.2.19. *Let R and S be left restriction semigroups and let T be a monoid. Let $\alpha : R \to S$ be an onto morphism and $\beta : R \to T$ a morphism. Then the left subhomomorphism $\varphi = \alpha^{-1}\beta$ satisfies*

$$a\varphi \cap b\varphi \neq \emptyset \Rightarrow a = b,$$

for $a, b \in S$, if and only if

$$Ker\ \beta \subseteq Ker\ \alpha.$$

for $s, t \in R$.

Proof. First suppose that if $a\varphi \cap b\varphi \neq \emptyset$, then $a = b$. Let $(a, b) \in \text{Ker}\beta$, i.e. $a\beta = b\beta$. We wish to show $a\alpha = b\alpha$. As $a, b \in R$, $a\beta, b\beta \in T$ and $a\alpha, b\alpha \in S$. Let

$$a\beta = c \text{ and } a\alpha = m,$$

$$b\beta = c \text{ and } b\alpha = n.$$

So $c \in m\alpha^{-1}\beta$ and $c \in n\alpha^{-1}\beta$, i.e. $c \in m\varphi$ and $c \in n\varphi$. Hence $c \in m\varphi \cap n\varphi$. Therefore $m = n$, i.e. $a\alpha = b\alpha$.

Conversely, suppose Ker $\beta \subseteq$ Ker α and that $a\varphi \cap b\varphi \neq \emptyset$ for $a, b \in S$. So there exists $c \in T$ such that $c \in a\alpha^{-1}\beta$ and $c \in b\alpha^{-1}\beta$. There also exist $r, s \in R$ such that

$$r\beta = c \text{ and } r\alpha = a,$$

$$s\beta = c \text{ and } s\alpha = b.$$

Since $r\beta = s\beta$, we have $(r, s) \in \text{Ker } \beta$. Hence $(r, s) \in \text{Ker } \alpha$ and so $r\alpha = s\alpha$, i.e. $a = b$. $\square$

The corresponding result when considering Condition (S8) is more like the result in the inverse case. We state and prove the result for either left or right restriction semigroups. The proof is identical in both cases.

Proposition 10.2.20. *Let R and S be left/right restriction semigroups and T a monoid. Let $\alpha : R \to S$ be an onto morphism and $\beta : R \to T$ a morphism. Then the left/right subhomomorphism $\varphi = \alpha^{-1}\beta$ satisfies*

$$1 \in s\varphi \Rightarrow s \in E_S, \tag{S8}$$

for $s \in S$, if and only if

$$\textit{ker } \beta \subseteq \textit{ker } \alpha.$$

Proof. Suppose Condition (S8) holds and $a \in \ker \beta$. As T is a monoid,

$$\ker \beta = \{a \in R : a\beta \in E_T\} = \{a \in R : a\beta = 1\}$$

and so $a\beta = 1$. Let $a\alpha = b$ for $b \in S$. So $1 \in b\varphi$ and by Condition (S8), $b \in E_S$. Hence $a\alpha \in E_S$.

Conversely, suppose that $\ker \beta \subseteq \ker \alpha$, i.e.

$$a\beta = 1 \Rightarrow a\alpha \in E_S \text{ for } a \in R.$$

Suppose $1 \in b\varphi$. Then $a\beta = 1$ and $a\alpha = b$ for some $a \in R$. As $a\beta = 1$, we have $a\alpha \in E_S$, i.e. $b \in E_S$. $\square$

We end this section by generalising Proposition 4.4 from [47]:

Proposition 10.2.21. *Let θ be a subhomomorphism of a restriction semigroup S into a restriction semigroup T. Then there exist a free restriction semigroup F, an onto morphism $\alpha : F \to S$, and a morphism $\beta : F \to T$ such that $\theta = \alpha^{-1}\beta$.*

Proof. Suppose θ is a subhomomorphism of a restriction semigroup S into a restriction semigroup T. By Theorem 10.2.6 and its dual, there exist a restriction semigroup R, an onto morphism $\gamma : R \to S$ and a morphism $\delta : R \to T$ such that $\theta = \gamma^{-1}\delta$.

Let F_R, along with the map $\mu : R \to F_R$, be the free restriction semigroup on the set R. Let $\pi : F_R \to R$ be the unique morphism making the following diagram commute:

$$\begin{tikzcd} R \arrow[r, "\mu"] \arrow[d, "I_R"'] & F_R \arrow[dl, "\pi"] \\ R & \end{tikzcd}$$

So we have $\mu\pi = I_R$. Therefore π is onto. As $\pi : F_R \to R$, $\gamma : R \to S$ and $\delta : R \to T$, let us define morphisms $\alpha : F_R \to S$ and $\beta : F_R \to T$ by

$$\alpha = \pi\gamma \text{ and } \beta = \pi\delta$$

respectively. As γ and π are onto morphisms, α is also onto. We have the following diagram:

$$\begin{tikzcd}
F_R \arrow[dr, "\pi"] \arrow[drr, "\beta = \pi\delta"] \arrow[ddr, "\alpha = \pi\gamma"'] & & \\
& R \arrow[r, "\delta"'] \arrow[d, "\gamma"] & T \\
& S \arrow[ur, dotted, "\theta = \gamma^{-1}\delta"'] &
\end{tikzcd}$$

It remains to show that $\theta = \alpha^{-1}\beta$. Let $x \in S$. If $y \in x\theta$, then $x = z\gamma$ and $y = z\delta$ for some $z \in R$. We have

$$x = (z\mu\pi)\gamma = (z\mu)\pi\gamma = (z\mu)\alpha$$

and

$$y = (z\mu\pi)\delta = (z\mu)\pi\delta = (z\mu)\beta,$$

where $z\mu \in F_R$. Hence $y \in x(\alpha^{-1}\beta)$ and so $\theta \subseteq \alpha^{-1}\beta$.

Conversely, let $y \in x(\alpha^{-1}\beta)$. Then $x = z\alpha$ and $y = z\beta$ for some $z \in F_R$. So

$$x = (z\pi)\gamma \text{ and } y = (z\pi)\delta,$$

where $z\pi \in R$. Therefore $y \in x(\gamma^{-1}\delta)$, i.e. $y \in x\theta$. Hence $\theta = \alpha^{-1}\beta$. $\square$

We note that we also have the result for left restriction semigroups and dually for right restriction:

Proposition 10.2.22. *Let θ be a left subhomomorphism of S into T, where S and T are left restriction semigroups. Then there exist a free left restriction semigroup F, an onto morphism $\alpha : F \to S$, and a morphism $\beta : F \to T$ such that $\theta = \alpha^{-1}\beta$.*

10.3 Proper covers and varieties

We shall show that the class of restriction semigroups having a proper cover over a variety of monoids $\mathcal{U}$ is a variety of restriction semigroups. A left restriction semigroup has a *proper cover over $\mathcal{U}$* if it has a proper cover P such that $P/\sigma_P \in \mathcal{U}$. If S is a left restriction semigroup, then also we say that P is a proper cover of S over M if P is a proper cover such that $M \cong P/\sigma_P$. We begin by looking at analogous results to Proposition 3.2 from [41], which is proved in the left ample case in [50].

Proposition 10.3.1. *Let R be a left restriction semigroup and M a monoid. Let ϕ be a surjective left subhomomorphism of R into M such that*

$$a\phi \cap b\phi \neq \emptyset \Rightarrow a^+b = b^+a, \tag{S4}$$

for $a, b \in R$. Then

$$\Pi(R, M, \phi) = \{(r, m) \in R \times M : m \in r\phi\}$$

is a proper cover of R over M.

Conversely, let P be a proper cover of R over M along with $(2,1)$-morphism $\alpha : P \to R$. Let the induced morphism $\psi : P \to R \times M$ be defined by

$$p\psi = (p\alpha, p\beta),$$

where $p\beta = p\sigma_P$ for $p \in P$. Then ϕ, defined by

$$s\phi = \{g \in M : (s, g) \in P\psi\},$$

for $s \in R$, is a surjective left subhomomorphism of R into M such that Condition (S4) holds and

$$P \cong \Pi(R, M, \phi).$$

Proof. Suppose R is a left restriction semigroup, M is a monoid and ϕ is a surjective left subhomomorphism of R into M such that Condition (S4) holds. We wish to show the following:

(i) $\Pi(R, M, \phi)$ is a proper left restriction semigroup;

(ii) there is an onto $(2,1)$-morphism $\psi : \Pi(R, M, \phi) \to R$ which is E_Π-separating;

(iii) $\Pi(R, M, \phi)/\sigma_\Pi \cong M$.

(i) A monoid can be regarded as a left restriction semigroup with distinguished semilattice $\{1\}$. By Proposition 10.2.5, $\Pi(R, M, \phi)$ is a left restriction semigroup. By Proposition 10.2.7, $\Pi(R, M, \phi)$ is proper if and only if Condition (S1) is satisfied. However, by Proposition 10.2.14, Condition (S1) is implied by Condition (S4) and so $\Pi(R, M, \phi)$ is proper.

(ii) Let us consider $p_1 : \Pi(R, M, \phi) \to R$ where

$$(s, t)p_1 = s$$

for $(s, t) \in \Pi(R, M, \phi)$. As in the proof of Proposition 10.2.5, p_1 is an onto $(2,1)$-morphism. Now, considering $(s, 1), (u, 1) \in E_\Pi$, we clearly have

$$(s, 1)p_1 = (u, 1)p_1 \Rightarrow (s, 1) = (u, 1)$$

and hence p_1 is E_Π-separating.

(iii) Let us consider $p_2 : \Pi(R, M, \phi) \to M$ defined by

$$(s,t)p_2 = t.$$

As in the proof of Proposition 10.2.5, p_2 is also an onto $(2,1)$-morphism.

Let $(s,t),(u,v) \in \Pi(R, M, \phi)$. We wish to show that $(s,t)\,\sigma_\Pi\,(u,t)$ if and only if $t = v$. If $t = v$, we have

$$\begin{aligned}(s,t),(u,t) \in \Pi(R,M,\phi) &\Rightarrow t \in s\phi \text{ and } t \in u\phi\\ &\Rightarrow t \in s\phi \cap u\phi\\ &\Rightarrow s^+u = u^+s \text{ by Condition (S4)}\\ &\Rightarrow (u^+,1)(s,t) = (s^+,1)(u,t)\\ &\Rightarrow (s,t)\,\sigma_\Pi\,(u,t) \text{ by Proposition 2.7.3.}\end{aligned}$$

Conversely if $(s,t)\,\sigma_\Pi\,(u,v)$, then $(e,1)(s,t) = (e,1)(u,v)$ for some $e \in E$. Hence $(es,t) = (eu,v)$ and so $t = v$. We have

$$\begin{aligned}(s,t),(u,v) \in \text{ Ker } p_2 &\Leftrightarrow (s,t)p_2 = (u,v)p_2\\ &\Leftrightarrow t = v\\ &\Leftrightarrow (s,t)\,\sigma_\Pi\,(u,v).\end{aligned}$$

Therefore $\Pi(R,M,\phi)/\sigma_\Pi$ is isomorphic to M and so $\Pi(R,M,\phi)$ is a proper cover of R over M.

Conversely, let P be a proper cover of R over M. So P is a proper left restriction semigroup, $\alpha : P \to R$ is an onto $(2,1)$-morphism which is E_P-separating and $P/\sigma_P \cong M$. Let $\beta : P \to M$ be given by $p\beta = p\sigma_P$ for $p \in P$. Let $\psi : P \to R \times M$ be the induced morphism given by

$$p\psi = (p\alpha, p\beta) = (p\alpha, p\sigma_P)$$

for $p \in P$.

We wish to show that ϕ, defined by

$$s\phi = \{m \in M : (s,m) \in P\psi\},$$

for $s \in R$, is a surjective left subhomomorphism of R into M such that Condition (S4) holds. As α is an onto morphism and β is a morphism, then by Theorem 10.2.6, $\alpha^{-1}\beta$ is a left subhomomorphism. As

$$\begin{aligned}m \in s\phi &\Leftrightarrow (s,m) = (p\alpha, p\beta)\\ &\Leftrightarrow m \in s\alpha^{-1}\beta,\end{aligned}$$

$\phi = \alpha^{-1}\beta$ and hence ϕ is a left subhomomorphism. It remains to show $R\phi = M$ and $s\phi \cap t\phi \neq \emptyset \Rightarrow s^+t = t^+s$ for $s,t \in R$.

We wish to show that $R\phi = M$, where $R\phi = \{r\phi : r \in R\}$. Take $u \in R\phi$. Then $u = r\phi$ for some $r \in R$ and hence $u \in M$. So $R\phi \subseteq M$. Conversely, consider $m \in M$. Then as $P/\sigma_P \cong M$, $m = p\sigma_P$ for some $p \in P$. We also have $p\alpha \in R$. So

$$(p\alpha, p\sigma_P) \in P\psi.$$

Therefore $m \in (p\alpha)\phi$, where $p\alpha \in R$. So $m \in R\phi$. Hence $M \subseteq R\phi$ and so $M = R\phi$.

Suppose $s\phi \cap t\phi \neq \emptyset$. We wish to show $s^+t = t^+s$. Let $m \in M$ be such that $m \in s\phi \cap t\phi$, i.e. $m \in s\phi$ and $m \in t\phi$. We have

$$(s, m) = (p\alpha, p\beta) \text{ and } (t, m) = (q\alpha, q\beta)$$

for some $p, q \in P$. So

$$s = p\alpha, t = q\alpha \text{ and } m = p\sigma_P = q\sigma_P.$$

As $p\,\sigma_P\,q$, we have

$$(q^+p)\,\sigma_P\,(p^+q).$$

As in the proof of Proposition 10.2.10, $p^+q^+ = q^+p^+$ implies $(q^+p)^+ = (p^+q)^+$. Hence $q^+p = p^+q$ as P is proper. So

$$\begin{aligned} s^+t &= (p\alpha)^+q\alpha \\ &= (p^+q)\alpha \\ &= (q^+p)\alpha \\ &= (q\alpha)^+p\alpha \\ &= t^+s. \end{aligned}$$

Therefore ϕ is a surjective left subhomomorphism of R into M such that Condition (S4) holds.

It remains to show that $P \cong \Pi(R, M, \phi)$. Consider $\psi : P \to R \times M$. We have

$$(r, m) \in P\psi \Leftrightarrow m \in r\phi \Leftrightarrow (r, m) \in \Pi(R, M, \phi).$$

So $\psi : P \to \Pi(R, M, \phi)$ is an onto morphism. Consider $p, q \in P$ such that $p\psi = q\psi$. We have

$$\begin{aligned} p\psi = q\psi &\Rightarrow (p\alpha, p\sigma_P) = (q\alpha, q\sigma_P) \\ &\Rightarrow p\alpha = q\alpha \text{ and } p\sigma_P q \end{aligned}$$

and

$$\begin{aligned} p\alpha = q\alpha &\Rightarrow (p\alpha)^+ = (q\alpha)^+ \\ &\Rightarrow p^+\alpha = q^+\alpha \\ &\Rightarrow p^+ = q^+ \text{ as } \alpha \text{ is } E_P\text{-separating} \\ &\Rightarrow p\,\widetilde{\mathcal{R}}_{E_P}\,q. \end{aligned}$$

As P is proper and $p\,(\widetilde{\mathcal{R}}_{E_P} \cap \sigma_P)\,q$, we have $p = q$. So $\psi : P \to \Pi(R, M, \phi)$ is an isomorphism and hence $P \cong \Pi(R, M, \phi)$. $\square$

Combining Proposition 10.3.1 with its dual, we get:

Proposition 10.3.2. *Let R be a restriction semigroup and M a monoid. Let ϕ be a surjective subhomomorphism of R into M such that*

$$a\phi \cap b\phi \neq \emptyset \Rightarrow a^+b = b^+a, \tag{S4}$$

and

$$a\phi \cap b\phi \neq \emptyset \Rightarrow ab^* = ba^*, \tag{S5}$$

for $a, b \in S$. Then

$$\Pi(R, M, \phi) = \{(r, m) \in R \times M : m \in r\phi\}$$

is a proper cover of R over M.

Conversely, let P be a proper cover of R over M along with $(2, 1, 1)$-morphism $\alpha : P \to R$. Let $\psi : P \to R \times M$ be the induced morphism defined by

$$p\psi = (p\alpha, p\beta),$$

where $p\beta = p\sigma_P$ for $p \in P$. Then ϕ, defined by

$$s\phi = \{g \in M : (s, g) \in P\psi\},$$

for $s \in R$, is a surjective subhomomorphism of R into M such that Conditions (S4) and (S5) hold and

$$P \cong \Pi(R, M, \phi).$$

Proposition 10.3.3. *Let R be a left/right restriction semigroup and M a monoid. Let ϕ be a surjective left/right subhomomorphism of R into M such that*

$$1 \in s\phi \Rightarrow s \in E_R \tag{S8}$$

and

$$s\phi \cap t\phi \neq \emptyset \Rightarrow s\,\sigma_R\,t, \tag{S9}$$

for $s, t \in R$. Then

$$\Pi(R, M, \phi) = \{(r, m) \in R \times M : m \in r\phi\}$$

is an E-unitary cover of R over M.

Conversely, let P be an E-unitary cover of R over M along with $(2,1)$-morphism $\alpha : P \to R$. Let the induced morphism $\psi : P \to R \times M$ be defined by

$$p\psi = (p\alpha, p\beta),$$

where $p\beta = p\sigma_P$ for $p \in P$. Then ϕ defined by

$$s\phi = \{g \in M : (s,g) \in P\psi\},$$

for $s \in R$, is a surjective left/right subhomomorphism of R into M such that Conditions (S8) and (S9) hold.

Proof. Suppose R is a left restriction semigroup, M is a monoid and ϕ is a surjective left subhomomorphism of R into M such that Conditions (S8) and (S9) hold. We wish to show the following:

(i) $\Pi(R,M,\phi)$ is an E-unitary left restriction semigroup;

(ii) there is an onto $(2,1)$-morphism $\psi : \Pi(R,M,\phi) \to R$ which is E_Π-separating;

(iii) $\Pi(R,M,\phi)/\sigma_\Pi \cong M$.

(i) As in Proposition 10.3.1, $\Pi(R,M,\phi)$ is a left restriction semigroup. By Proposition 10.2.8, $\Pi(R,M,\phi)$ is E-unitary if and only if Condition (S3) is satisfied. However, Condition (S3) is implied by Condition (S8) and so $\Pi(R,M,\phi)$ is E-unitary.

(ii) As in Proposition 10.3.1, $p_1 : \Pi(R,M,\phi) \to R$ where

$$(s,t)p_1 = s$$

for $(s,t) \in \Pi(R,M,\phi)$, is an onto morphism that is E_Π-separating.

(iii) Let us consider $p_2 : \Pi(R,M,\phi) \to M$, where $(s,t)p_2 = t$. Suppose $(s,t),(u,v) \in \Pi(R,M,\phi)$. We wish to show $(s,t)\,\sigma_\Pi\,(u,v)$ if and only if $t = v$. If $t = v$, we have

$$\begin{aligned}
(s,t),(u,t) \in \Pi(R,M,\phi) &\Rightarrow t \in s\phi \text{ and } t \in u\phi \\
&\Rightarrow t \in s\phi \cap u\phi \\
&\Rightarrow s\,\sigma_R\,u \text{ by Condition (S9)} \\
&\Rightarrow es = eu \text{ for some } e \in E_R \\
&\Rightarrow (e,1)(s,t) = (e,1)(u,t) \text{ for some } (e,1) \in E_\Pi \\
&\Rightarrow (s,t)\,\sigma_\Pi\,(u,t).
\end{aligned}$$

Conversely if $(s,t)\,\sigma_\Pi\,(u,v)$, then $t = v$ as in Proposition 10.3.1. Also, as in Proposition 10.3.1,

$$(s,t),(u,v) \in \text{ Ker } p_2 \Leftrightarrow (s,t)\,\sigma_\Pi\,(u,v).$$

Therefore, $\theta : \Pi(R, M, \phi)/\sigma_\Pi \to M$ and hence $\Pi(R, M, \phi)$ is an E-unitary cover of R over M.

Conversely, let P be an E-unitary cover of R over M. So P is an E-unitary left restriction semigroup, $\alpha : P \to R$ is an onto $(2,1)$-morphism which is E_P-separating and $P/\sigma_P \cong M$. Let $\beta : P \to M$ by given by $p\beta = p\sigma_P$ for $p \in P$. Let $\psi : P \to R \times M$ be the induced morphism given by

$$p\psi = (p\alpha, p\beta) = (p\alpha, p\sigma_P)$$

for $p \in P$.

As in Proposition 10.3.1, ϕ, defined by

$$s\phi = \{m \in M : (s, m) \in P\psi\},$$

for $s \in R$, is a surjective left subhomomorphism of R into M. It remains to show that Conditions (S8) and (S9) hold for $s, t \in R$:

$$1 \in s\phi \Rightarrow s \in E_R; \tag{S8}$$

$$s\phi \cap t\phi \neq \emptyset \Rightarrow s\,\sigma_R\,t. \tag{S9}$$

(S8) Suppose that $1 \in s\phi$. We wish to show that $s \in E_R$. As $1 \in s\phi$, we have $(s, 1) \in P\psi$. So there exists $p \in P$ such that $(p\alpha, p\sigma_P) = (s, 1)$. As $p\sigma_P 1$, we have $p \in E_P$ by Proposition 2.7.7 and 2.8.5. Hence $p = p^+$. So

$$p\alpha = s \Rightarrow p^+\alpha = s \Rightarrow (p\alpha)^+ = s \Rightarrow s \in E_P.$$

Therefore Condition (S8) holds.

(S9) Let $s, t \in R$ and let $u \in M$ be such that $u \in s\phi \cap t\phi$. We wish to show $s\,\sigma_R\,t$. We have

$$\begin{aligned}
u \in s\phi \cap t\phi &\Rightarrow u \in s\phi \text{ and } u \in t\phi \\
&\Rightarrow (s, u), (t, u) \in P\psi \\
&\Rightarrow (s, u) = (p\alpha, p\sigma_P) \text{ and } (t, u) = (q\alpha, q\sigma_P) \text{ for some } p, q \in P \\
&\Rightarrow s = p\alpha, t = q\alpha \text{ and } u = p\sigma_P = q\sigma_P \\
&\Rightarrow s = p\alpha, t = q\alpha \text{ and } ep = eq \text{ for some } e \in E_P \\
&\Rightarrow s = p\alpha, t = q\alpha \text{ and } (ep)\alpha = (eq)\alpha \\
&\Rightarrow s = p\alpha, t = q\alpha \text{ and } (e\alpha)(p\alpha) = (e\alpha)(q\alpha) \\
&\Rightarrow (e\alpha)s = (e\alpha)t \\
&\Rightarrow s\,\sigma_R\,t \text{ as } e\alpha \in E_R.
\end{aligned}$$

Therefore Condition (S9) holds. Thus ϕ is a surjective left subhomomorphism of R into M such that Conditions (S8) and (S9) hold.

□

Using Proposition 10.3.1, we have our desired result. We note that the first part of the result is the same as for Theorem 9.6.2, but we shall state it for completeness.

Theorem 10.3.4. *Let S be a left restriction semigroup and $\mathcal{U}$ a variety of monoids. Then the following are equivalent:*

(1) S has proper covers over $\mathcal{U}$;

(2) if $\bar{u} \equiv \bar{v}$ is a law in $\mathcal{U}$, then $\bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u}$ is a law in S, where $\bar{u}$ and $\bar{v}$ are $(2,1)$-terms.

Proof. Suppose S has a proper cover over $\mathcal{U}$. Then there is a proper left restriction monoid T such that $T/\sigma_T \in \mathcal{U}$ and an onto $(2,1)$-morphism $\psi : T \to S$ which is E_T-separating. If $\bar{u} \equiv \bar{v}$ is a law in $\mathcal{U}$ then since T is proper and $T/\sigma_T \in \mathcal{U}$, $T \models \bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u}$ by Lemma 2.8.4. As ψ is onto, $S \models \bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u}$ and so $\bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u}$ is a law in S.

Conversely, suppose that (2) holds. Let F_S, along with $\mu : S \to F_S$, be the free left restriction semigroup on S. Suppose $\alpha : F_S \to S$ is the unique morphism making the following diagram commute:

$$\begin{tikzcd} S \arrow[r, "\mu"] \arrow[d, "I_S"'] & F_S \arrow[dl, "\alpha"] \\ S & \end{tikzcd}$$

So α is defined on the set of generators of F_S sending $s\mu$ to s. Clearly, α is onto. Let M, along with the map $\delta : S\mu \to M$, be the free monoid in $\mathcal{U}$ on the set of generators $S\mu$. As $\mu : S \to F_S$ and $\delta : S\mu \to M$,

$$\mu\delta : S \to M.$$

As M is reduced left restriction, there exists a unique morphism

$$\beta : F_S \to M$$

such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\mu"] \arrow[d, "\mu\delta"'] & F_S \arrow[dl, "\beta"] \\ M & \end{tikzcd}$$

Set $\theta = \alpha^{-1}\beta$. This is a left subhomomorphism of S into M by Theorem 10.2.6. Let $m \in M$. As $(S\mu)\delta$ generates M,

$$m = (s_1\mu\delta)...(s_n\mu\delta)$$

for some $s_1...s_n \in S$. So we have

$$m = (s_1\mu\beta)...(s_n\mu\beta) = ((s_1\mu)...(s_n\mu))\beta,$$

where $s_1\mu...s_n\mu \in F_S$. So β is onto. As β is onto, θ is surjective. We wish to show that θ satisfies Condition (S4). By Proposition 10.2.17, this is equivalent to showing that

$$s\beta = t\beta \Rightarrow (s^+t)\alpha = (t^+s)\alpha$$

for $s, t \in F_S$. We have

$$s = h(s_1\mu, ..., s_n\mu)^+k(s_1\mu, ..., s_n\mu)$$

and

$$t = p(s_1\mu, ..., s_n\mu)^+q(s_1\mu, ..., s_n\mu)$$

by Lemma 2.2.15 where h, k, p and q are n-ary functions and h and p are products of terms of the form $(x_1...x_n)^+$. Suppose $s\beta = t\beta$. Then

$$h(s_1\mu\beta, ..., s_n\mu\beta)^+k(s_1\mu\beta, ..., s_n\mu\beta) = p(s_1\mu\beta, ..., s_n\mu\beta)^+q(s_1\mu\beta, ..., s_n\mu\beta).$$

Therefore

$$h(s_1\mu\delta, ..., s_n\mu\delta)^+k(s_1\mu\delta, ..., s_n\mu\delta) = p(s_1\mu\delta, ..., s_n\mu\delta)^+q(s_1\mu\delta, ..., s_n\mu\delta)$$

and so

$$k(s_1\mu\delta, ..., s_n\mu\delta) = q(s_1\mu\delta, ..., s_n\mu\delta).$$

For $a_1, \ldots a_n \in M$, define $\upsilon : S \to M$ by

$$s\upsilon = a_i.$$

As M is the free monoid on $S\mu$, there exists a morphism $\theta : M \to M$ making the following diagram commute:

$$\begin{tikzcd}
S \arrow[r, "\mu\delta"] \arrow[d, "\upsilon"'] & M \arrow[dl, "\theta"] \\
M &
\end{tikzcd}$$

As

$$k(s_1\mu\delta, \ldots, s_n\mu\delta) = q(s_1\mu\delta, \ldots, s_n\mu\delta),$$

we have

$$k(s_1\mu\delta\theta, \ldots, s_n\mu\delta\theta) = q(s_1\mu\delta\theta, \ldots, s_n\mu\delta\theta),$$

i.e.

$$k(a_1, \ldots, a_n) = q(a_1, \ldots, a_n).$$

It follows that $q^+k \equiv k^+q$ is a law in S by (2) and so

$$\begin{aligned} s^+t &\equiv (h^+k)^+p^+q \\ &\equiv h^+k^+p^+q \\ &\equiv h^+p^+k^+q \\ &\equiv h^+p^+q^+k \\ &\equiv (p^+q)^+h^+k \\ &\equiv t^+s. \end{aligned}$$

Therefore $s^+t \equiv t^+s$ is a law in S. Hence $(s^+t)\alpha = (t^+s)\alpha$.

As θ is a surjective left subhomomorphism of S into M such that Condition (S4) holds, $\Pi(S, M, \theta)$ is a proper cover of S over M by Proposition 10.3.1. Therefore S has proper covers over $\mathcal{U}$. $\square$

Combining the previous result and its proof with the dual, we have the following result:

Theorem 10.3.5. *Let S be a restriction semigroup and $\mathcal{U}$ a variety of monoids. Then the following are equivalent:*

(1) S has proper covers over $\mathcal{U}$;

(2) if $\bar{u} \equiv \bar{v}$ is a law in $\mathcal{U}$, then $\bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u}$ and $\bar{u}\bar{v}^* \equiv \bar{v}\bar{u}^*$ are laws in S, where $\bar{u}$ and $\bar{v}$ are $(2,1)$-terms.

We would ideally like to have a similar theorem for E-unitary semigroups. However, in Proposition 10.3.3 we need both Conditions (S8) and (S9) for

$$\Pi(R, M, \phi) = \{(r, m) \in R \times M : m \in r\phi\}$$

to be an E-unitary cover of R over M. In Proposition 2.2 in [41], which is used to prove the covering result for inverse semigroups in [47], only Condition (S8) is needed. This is due to Condition (S9) being a consequence of Condition (S8) in the inverse case as proved in Proposition 10.2.11. The requirement of the extra condition poses problems when trying to deduce such a theorem in the restriction case.

We have two ways to show that the class of left restriction monoids having a proper cover over a variety of monoids $\mathcal{U}$ is itself a variety of left restriction monoids, where this variety is determined by

$$\Sigma = \{\bar{u}^+\bar{v} \equiv \bar{v}^+\bar{u} : \bar{u} \equiv \bar{v} \text{ is a law in } \mathcal{U}\}.$$

Firstly, in Chapter 9 we used graph expansions to obtain our desired result using techniques that were used when considering the class of left ample monoids which form a quasivariety. Unlike the left ample case we were also able to apply the techniques of Petrich and Reilly to obtain the result using left subhomomorphisms. In both methods the proof that if a left restriction monoid S has a proper cover over $\mathcal{U}$ then S satisfies Σ is the same.

In our method involving graph expansions in Chapter 9 we considered the free left restriction monoid on $X\tau_{\mathcal{M}}$, namely $F_X = \mathcal{M}(X, \iota, X^*)$, and ρ which was the $(2,1,0)$-congruence on F_X generated by

$$H = \{((\bar{u}\tau)^+\bar{v}\tau, (\bar{v}\tau)^+\bar{u}\tau) \in F_X \times F_X : \bar{u}, \bar{v} \in X^* \text{ and } \bar{u}\bar{f} = \bar{v}\bar{f}\}.$$

Assuming the left restriction monoid S satisfied Σ we showed that S had a proper cover over $M = F/\sigma_F$, where $F = F_X/\rho$. The proper cover was given by

$$K = \{(s, m) \in S \times M : \exists a\rho \in F_X/\rho \text{ with } (a\rho)\phi = s \text{ and } (a\rho)\sigma_F^\natural = m\},$$

along with morphism $\gamma : K \to S$ defined by $(s, m)\gamma = s$ in Proposition 9.6.1 and Theorem 9.6.2. We have $\phi : F_X/\rho \to S$ defined as $(a\rho)\phi = a\theta$, where $\theta : F_X \to S$ is the morphism which exists due to F_X being the free left restriction monoid on $X\tau_{\mathcal{M}}$.

Assuming the left restriction monoid S satisfied Σ we can show S has a proper cover over $\mathcal{U}$ using the subhomomorphisms method. Instead of taking F_S to be the free left restriction semigroup on S, we take the F_S to be free left restriction monoid on S in the proof of Theorem 10.3.4. Considering $\alpha : F_S \to S$ and $\beta : F_S \to M$, which both exist due to F_S being the free left restriction monoid on S, we consider the left subhomomorphism $\theta = \alpha^{-1}\beta$ and show that S has a proper cover over $\mathcal{U}$. The proper cover is given by

$$\Pi(S, M, \theta) = \{(s, m) \in S \times M : m \in s\theta\}.$$

By the monoid version of Proposition 10.3.1, the proper cover K of S over $M = F/\sigma_F$ from Chapter 9, is also of this form. Let $\psi : K \to S \times M$ be the induced morphism given by

$$p\psi = (p\gamma, p\beta),$$

where $\gamma : K \to S$ is given by $(s, m)\gamma = s$ and $\beta : K \to M$ is given by $(s, m)\beta = (s, m)\sigma_K$ for $(s, m) \in K$. So we have

$$(s, m)\psi = (s, (s, m)\sigma_K).$$

Then ϕ, defined by

$$s\phi = \{m \in M : (s, m) \in K\psi\},$$

for $s \in S$, is a surjective left subhomomorphism of S into M, represented as:

$$\begin{tikzcd}
P \arrow[r, "\sigma_P^\natural"] \arrow[d, "\gamma"'] & M \\
S \arrow[ur, dotted, "\phi"'] &
\end{tikzcd}$$

Furthermore, Condition (S4) holds and

$$K \cong \Pi(S, M, \phi).$$

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