Involutive Algebras and Locally Compact Quantum Groups

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Abstract

In this thesis we will be concerned with some questions regarding involutions on dual and predual spaces of certain algebras arising from locally compact quantum groups. In particular we have the $\mathrm{L}^1(\mathbb{G})$ predual of a von Neumann algebraic quantum group $(\mathrm{L}^\infty(\mathbb{G}), \Delta)$. This is a Banach algebra (where the product is given by the pre-adjoint of the coproduct Δ), however in general we cannot make this into a Banach $*$-algebra in such a way that the regular representation is a $*$-homomorphism. We can however find a dense $*$-subalgebra $\mathrm{L}^1_\sharp(\mathbb{G})$ that satisfies this property and is a Banach algebra under a new norm. This was originally considered in Kustermans (2001) when defining the universal C*-algebraic quantum group, however little else has been studied regarding this algebra in general.

In this thesis we study the $\mathrm{L}^1_\sharp$-algebra of a locally compact quantum group in this thesis. In particular we show how this has a (not necessarily unique) operator space structure such that this forms a completely contractive Banach algebra, we study some properties for compact quantum groups, we study the object for the compact quantum group $\mathrm{SU}_q(2)$ and we study the operator biprojectivity of the $\mathrm{L}^1_\sharp$-algebra.

In addition we also briefly study some related properties of $\mathrm{C}_0(\mathbb{G})^*$ and its $*$-subalgebra $\mathrm{C}_0(\mathbb{G})^*{}_\sharp$.

Contents

Introduction

Locally compact quantum groups are a generalisation of locally compact groups that first appeared in a definitive form in the reduced C*-algebraic setting and afterwards in the von Neumann algebraic setting as detailed in the papers Kustermans & Vaes (2000) and Kustermans & Vaes (2003). In the von Neumann algebraic setting of a locally compact quantum group we obtain a Banach algebra from the predual of the von Neumann algebra and from this Kustermans introduced a dense $*$-subalgebra called the $\mathrm{L}^1_\sharp$-algebra (see Kustermans (2001)). In this thesis we offer a detailed comprehensive study of this $\mathrm{L}^1_\sharp$-algebra. In addition to this work for arbitrary locally compact quantum groups we will study this object for the compact quantum group $\mathrm{SU}_q(2)$ and obtain new results on this quantum group as a result.

We now give an outline of this thesis. In Chapter 1 we review some advanced topics in operator theory. In particular we study operator spaces, basic homological algebra of operator spaces, one-parameter groups on Banach and operator algebras and lastly weight theory (which is important for defining a locally compact quantum group). Chapter 1 will give us a good stable footing in order to develop the rest of the thesis.

In Chapter 2 we define a locally compact quantum group $\mathbb{G}$ in the von Neumann algebraic setting $(\mathrm{L}^\infty(\mathbb{G}), \Delta)$ and the reduced C*-algebraic setting $(\mathrm{C}_0(\mathbb{G}), \Delta)$ and then give the common properties we will use. In addition we will give details of duality, the L^1-algebra and of the product of locally compact quantum groups (a subject that to the author's knowledge is not currently recorded explicitly in the literature though we do use many results from Vaes & Vainerman (2003) to show this).

After this we study the special cases of compact and discrete quantum groups, Kac algebras and coamenable quantum groups in Chapter 3. For us the compact case is the most important as we shall see in the remainder of the thesis. We spend some time discussing compact quantum groups in the locally compact setting where, for example, we calculate the multiplicative unitary and scaling group of a compact quantum group (objects that feature predominantly in the locally compact case).

We then come onto the research chapters of this thesis. In Chapter 4 we begin our study of the $\mathrm{L}^1_\sharp$-algebra for a locally compact quantum group. We start by detailing known results regarding this object, many of which were known to Kustermans in his original paper on universal locally compact quantum groups. We then add an operator space structure on the $\mathrm{L}^1_\sharp$-algebra making it a completely contractive Banach algebra and we prove various properties of this as an operator space. We prove finally that a locally compact quantum group is compact if and only if its $\mathrm{L}^1_\sharp$-algebra is an ideal in its own double dual with respect to the Arens product.

In Chapter 5 we make a detailed study of the $\mathrm{SU}_q(2)$ compact quantum group for $q \in (0,1)$ that was originally discovered by Woronowicz. We begin by giving the basic definition, the Haar state and the corepresentation theory of $\mathrm{SU}_q(2)$ before moving on to discuss new results as obtained by the author. We then show that we have a compact space K such that the commutative C*-algebra generated by the normal element c of $\mathrm{SU}_q(2)$ is $*$-isomorphic to $\mathrm{C}(K)$. We then use this new object to study the $\mathrm{L}^1_\sharp$-algebra of $\mathrm{SU}_q(2)$ and the quantum group product of $\mathrm{SU}_q(2)$ with $\mathrm{SU}_{q'}(2)$ (for $q' \in (0,1)$ also). We then finish this chapter with a brief study of $\mathrm{C}(\mathrm{SU}_q(2))^*{}_\sharp$ and we show that the set $\overline{\mathrm{lin}\ \{(\nu \otimes \mathrm{id})(W) \mid \nu \in \mathrm{C}_0(\mathrm{SU}_q(2))^*\}}^{\|\cdot\|}$ is not closed under adjoint and therefore not a C*-algebra.

In the final chapter we study some homological algebra for the $\mathrm{L}^1_\sharp$-algebra. We study projective modules over the $\mathrm{L}^1_\sharp$-algebra and the adjoint map of the multiplication map of the $\mathrm{L}^1_\sharp$-algebra as a completely contractive Banach algebra. Finally we study its relationship to compact quantum groups where we prove that if the $\mathrm{L}^1_\sharp$-algebra is operator biprojective then the quantum group must be compact and we give a structure theorem for the $L^1_\sharp$-algebra to be operator biprojective.

We give an appendix for some further results used in functional analysis, measure theory and operator theory. We do assume however that the reader is familiar with the basics of these subjects including Banach space, Banach algebra, C*-algebra and von Neumann algebra theory.

Most of the notation we use is standard in functional analysis, operator theory and quantum group theory. The notation for operator spaces and quantum groups is described in the first two chapters. We will also have use for some algebraic notions: for example we denote by $\odot$ the algebraic tensor product, $\operatorname{lin} X$ will denote the linear span of a subset X of a linear space and $\operatorname{alg} A$ will denote the algebra generated by a subset A of an algebra. We will only ever use the norm, weak and, when applicable, the weak* topologies on Banach spaces with the exception of von Neumann algebras where we will also use the weak operator topologies (we assume the reader is comfortable with these notions). We will refer interchangeably to the equivalent σ-weak topology and weak*-topology on a von Neumann algebra. We denote the norm closure of a set X by $\overline{X}^{\|\cdot\|}$, the weak closure by $\overline{X}^{w}$ and the weak*-closure by $\overline{X}^{w^*}$. We denote a weak limit x of a net (x_α) by $x_\alpha \xrightarrow{w} x$ and similarly a weak*-limit by $x_\alpha \xrightarrow{w^*} x$. We will often denote a norm with a subscript when it is not clear where it comes from.

We also assume the reader is familiar with the Banach space projective tensor product denoted $\hat{\otimes}$, the minimal tensor product denoted by $\otimes_{min}$ on C*-algebras and the von Neumann algebraic tensor product $\overline{\otimes}$ on von Neumann algebras. See Ryan (2002) and Takesaki (2003a) for further details on tensor products in Banach spaces and operator theory.

Chapter 1

Operator Theory

In this chapter we discuss preliminaries required for reading this thesis that may not necessarily follow from standard courses in functional analysis and operator theory. In particular we describe operator spaces and homological algebra, one-parameter groups and weights on von Neumann algebras.

1.1 Operator Spaces

We begin with a discussion of operator space theory. It has been seen that operator space theory is useful for studying locally compact groups (see for example Effros & Ruan (2003) and Junge *et al.* (2009)) and locally compact quantum groups (see Hu *et al.* (2011), Aristov (2004) and Daws (2010)). We will make use of operator space structures in this thesis with our study of the $\mathrm{L}^1_\sharp(\mathbb{G})$ algebra for a locally compact quantum group $\mathbb{G}$.

Standard references in this section are Effros & Ruan (2000), Pisier (2003) and Blecher & Le Merdy (2004) which will be quoted regularly throughout.

1.1.1 Operator Spaces and Ruan's Theorem

Given any Banach space we can show there is some compact space Ω such that X can be isometrically embedded inside $\mathrm{C}(\Omega)$, that is any Banach space is a subspace of a function space. This can in turn be represented as bounded operators on a Hilbert space. We ask what kind of extra structure embedding a Banach space as a subspace of $\mathcal{B}(\mathcal{H})$ will give us? We characterise this according to Ruan's theorem below (Theorem 1.1.3).

We now make a small digression about matrices with entries in a linear space X. The following is just to standardise notation.

Notation 1.1.1 *Let X be a linear space and let $\mathbb{M}_{n,m}(X)$ denote the space of $n \times m$-matrices with entries in X. We make this a linear space with operations*

$$(x_{ij})_{i,j=1}^{n,m} + (y_{ij})_{i,j=1}^{n,m} = (x_{ij} + y_{ij})_{i,j=1}^{n,m} \qquad \text{and} \qquad \lambda\,(x_{i,j})_{i,j=1}^{n,m} = (\lambda x_{i,j})_{i,j=1}^{n,m}$$

where $x_{ij}, y_{ij} \in X$ for $1 \leqslant i, j \leqslant n$ and $\lambda \in \mathbb{C}$. If $n = m$ then the $n \times n$ matrices with entries in X are denoted $\mathbb{M}_n(X)$.

Given a linear map $T : X \to Y$ between linear spaces we let $T_{n,m} : \mathbb{M}_{n,m}(X) \to \mathbb{M}_{n,m}(Y)$ be the map given by $(x_{i,j})_{i,j=1}^{n,m} \mapsto (T(x_{i,j}))_{i,j=1}^{n,m}$ and if $n = m$ then we denote by $T_n : \mathbb{M}_n(X) \to \mathbb{M}_n(Y)$ the obvious map between square matrices.

Say X is a $*$-algebra, then we make $\mathbb{M}_n(X)$ into a $*$-algebra with multiplication and adjoint given by

$$(x_{i,j})_{i,j=1}^{n} \cdot (y_{i,j})_{i,j=1}^{n} = \left(\sum_{k=1}^{n} x_{i,k} y_{k,j}\right)_{i,j=1}^{n} \qquad \text{and} \qquad \left((x_{ij})_{i,j=1}^{n}\right)^{*} = \left(x_{ji}^{*}\right)_{i,j=1}^{n}$$

where $x_{i,j}, y_{i,j} \in X$ for $1 \leqslant i, j \leqslant n$.

It is reasonably straightforward to see that for a linear space X we have an isomorphism $\mathbb{M}_{n,m}(X) = \mathbb{M}_{n,m} \odot X$ given by $(x_{ij})_{i,j=1}^{n} \mapsto \sum_{i,j=1}^{n} e_{ij} \otimes x_{ij}$ where e_{ij} is the $n \times m$-matrix with entry 1 in the i, j-th place and 0 elsewhere where $\odot$ denotes the algebraic tensor product.

Now let $X \subset \mathcal{B}(\mathcal{H})$ be a linear subspace for some Hilbert space $\mathcal{H}$. We can consider

$\mathcal{H}^{(n)} = \mathcal{H} \oplus_2 \cdots \oplus_2 \mathcal{H}$ (n times) as a Hilbert space with inner product

$$((\xi_i)_{i=1}^n | (\eta_i)_{i=1}^n)_{\mathcal{H}^{(n)}} = \sum_{i=1}^n (\xi_i|\eta_i)_{\mathcal{H}} .$$

Let $\Phi : \mathbb{M}_n(X) \to \mathcal{B}(\mathcal{H}^{(n)})$ by

$$\left[\Phi\left((x_{ij})_{i,j=1}^n\right)\right](\xi_i)_{i=1}^n = \left(\sum_{j=1}^n x_{ij}\xi_j\right)_{i=1}^n$$

for $(\xi_i)_{i=1}^n \in \mathcal{H}^{(n)}$. We can show that this preserves the adjoint and multiplication operations if X is a $*$-algebra. As the space $\mathcal{B}(\mathcal{H}^{(n)})$ is normed we can use this to define a norm on $\mathbb{M}_n(X)$ by letting

$$\|(x_{ij})_{i,j=1}^n\|_{\mathbb{M}_n(X)} = \sup\left\{ \left\| \left(\sum_{j=1}^n x_{ij}\xi_j\right)_{i=1}^n \right\|_{\mathcal{H}^n} \;\middle|\; (\xi_i)_{i=1}^n \in H^n,\ \|(\xi)_{i=1}^n\| \leqslant 1 \right\}. \quad (1.1)$$

So for all $n \in \mathbb{N}$ we have defined a norm on $\mathbb{M}_n(X)$ with an embedding inside $\mathbb{M}_n(\mathcal{B}(\mathcal{H}))$ such that $\mathbb{M}_n(\mathcal{B}(\mathcal{H})) \cong_i \mathcal{B}(\mathcal{H}^{(n)})$, i.e. we have a sequence of matrix norms $\|\cdot\|_n$. We can show that the following is true.

Proposition 1.1.2 *Let $X \subset \mathcal{B}(\mathcal{H})$ be a linear subspace with the matrix norms $\|\cdot\|_n$ given above. Then we have*

(R1) $\|axb\|_n \leqslant \|a\|\|x\|_m\|b\|$ *for all* $x \in \mathbb{M}_m(X)$, $a \in \mathbb{M}_{n,m}$ *and* $b \in \mathbb{M}_{m,n}$;

(R2) $\|x \oplus y\|_{m+n} = \max\{\|x\|_m, \|y\|_n\}$ *for* $x \in \mathcal{M}_m(X)$ *and* $y \in \mathbb{M}_n(X)$;

where $x \oplus y \in \mathcal{B}(\mathcal{H}^{(n)} \oplus_2 \mathcal{H}^{(m)})$ *denotes the operator given by* $(\xi, \eta)^t \mapsto (x\xi, y\eta)^t$ *for* $\xi \in \mathcal{H}^{(n)}$ *and* $\eta \in \mathcal{H}^{(m)}$.

The more difficult and interesting point is that the converse of the proposition holds which forms Ruan's theorem as follows. See Effros & Ruan (2000) or Pisier (2003) for a proof of this.

Theorem 1.1.3 *Let X be a linear space such that for all $n \in \mathbb{N}$ we have a norm on $\mathbb{M}_n(X)$ satisfying* (R1) *and* (R2) *in Proposition 1.1.2. Then there exists a Hilbert space $\mathcal{H}$ and an embedding $J : X \to \mathcal{B}(\mathcal{H})$ such that the map $J_n : \mathbb{M}_n(X) \to \mathcal{B}(\mathcal{H}^{(n)})$ from Notation 1.1.1 is an isometry for all $n \in \mathbb{N}$.*

We state the following that is not difficult to prove for convenience here. For our purposes we will define operator spaces to be complete so as to be analogous to Banach spaces.

Proposition 1.1.4 *Let $X \subset \mathcal{B}(\mathcal{H})$, and let $(x_{ij})_{i,j=1}^{n} \in \mathbb{M}_n(X)$, then X is complete if and only if $\mathbb{M}_n(X)$ is complete for all $n \in \mathbb{N}$.*

We now make the following definition of an operator space.

Definition 1.1.5 *An operator space X is a linear space with a collection of norms $\| \cdot \|_n$ on $\mathbb{M}_n(X)$ for all $n \in \mathbb{N}$ that satisfy Ruan's axioms in Proposition 1.1.2 and such that X is complete with respect to the $\| \cdot \|_1$ norm.*

So we can either define an operator space through a sequence of norms or through a closed embedding into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. We will define operator spaces using both of these methods in this thesis.

Remark 1.1.6 *In the case where we have a non-square matrix then we can embed this in a larger square matrix by adding either columns or rows of zeros until it is square. As such, if we have an operator space structure X, we define norms on rectangular matrices by simply embedding the rectangular matrices into square matrices this way and calculating the norm. The norms of square matrices do not change as we add rows or columns of zeros so this is well defined.*

We now give some basic examples of operator spaces with more to follow in the next section.

Example 1.1.7 *We know from the start of this section that any Banach space X is an operator space. It also follows that any C*-algebra can be made into an operator space by the GNS construction. Similarly any von Neumann algebra is automatically an operator space.*

Example 1.1.8 *Let X be an operator space, then $\mathbb{M}_n(X)$ is an operator space by identifying $\mathbb{M}_n(\mathbb{M}_m(X)) = \mathbb{M}_{mn}(X)$ with the map $\left((x_{k,l}^{i,j})_{k,l=1}^m\right)_{i,j=1}^n \mapsto \left(x_{k,l}^{i,j}\right)_{(i,k),(j,l)=(1,1)}^{(n,m)}$.*

Let Y be a closed subspace of X and we have that Y is an operator space with the obvious embedding.

1.1.2 Completely Bounded Maps, Duals and Quotients

We have defined an operator space in the previous section and now in this section we consider the appropriate morphisms between operator spaces. As we have norms on matrices with entries in an operator space it will be useful to have a morphism that respects this structure in a similar way as to how bounded maps do for Banach spaces. We define now the completely bounded maps (along with their various counterparts of complete contractions, complete isometries, etc) and give some of their properties.

We will show in this section how we can make dual spaces and quotient spaces of operator spaces into operator spaces themselves and we will give various theorems regarding completely bounded adjoint maps. We note for the following definition that a quotient map between Banach spaces is given in Definition A.1.5.

Definition 1.1.9 *Let X and Y be operator spaces, let $T : X \to Y$ be a linear map and let $T_n : \mathbb{M}_n(X) \to \mathbb{M}_n(Y)$ be the map given in Notation 1.1.1. As $\mathbb{M}_n(X)$ and $\mathbb{M}_n(Y)$ are both normed then we can define the following norm on T_n*

$$\|T_n\| = \sup\left\{\|T_n\,(x_{ij})_{i,j=1}^n\| \;\middle|\; (x_{ij})_{i,j=1}^n \in \mathbb{M}_n(X),\ \left\|(x_{ij})_{i,j=1}^n\right\| \leqslant 1\right\}.$$

*We say T is **completely bounded** if there is some $M > 0$ such that $\|T_n\| \leqslant M$ for all*

$n \in \mathbb{N}$ in which case we define the following norm

$$\|T\|_{cb} = \sup_{n \in \mathbb{N}} \|T_n\|.$$

We denote by $\mathcal{CB}(X, Y)$ the completely bounded maps from X to Y and we have the following important definitions:

*(i) T is **completely contractive** if $\|T\|_{cb} \leqslant 1$;*

*(ii) T is a **complete isometry** if T_n is an isometry for all $n \in \mathbb{N}$;*

*(iii) T is a **complete quotient map** if T_n is a quotient map for all $n \in \mathbb{N}$ (see Example 1.1.17 below for the operator space structure on quotient spaces);*

*(iv) T is a **complete isomorphism** if T is a linear isomorphism and T and T^{-1} are completely bounded.*

*We denote $X \cong_{ci} Y$ where X and Y are completely isometric and completely isomorphic and say they are **completely isometrically isomorphic**.*

Clearly for any completely bounded map $T : X \to Y$ between operator spaces we have $\|T_n\|_n \leqslant \|T\|_{cb}$ for all $n \in \mathbb{N}$. In particular we have $\|T\| \leqslant \|T\|_{cb}$.

Throughout the rest of this section we give some more examples of operator spaces and morphisms between them.

Proposition 1.1.10 *Let A and B be C*-algebras and $T : A \to B$ a $*$-homomorphism, then T is a complete contraction. If T is a $*$-isomorphism than it is a complete isometry.*

Proof (Sketch)

We know that a $*$-homomorphism between C*-algebras is a contraction from C*-algebra theory. By representing A on a Hilbert space $\mathcal{H}$ and using that $\mathbb{M}_n(A)$ is a closed subspace of $\mathcal{B}(\mathcal{H}^{(n)})$ then $\mathbb{M}_n(A)$ is a C*-subalgebra. We can show that $\Phi_n : \mathbb{M}_n(A) \to \mathbb{M}_n(A)$ is a $*$-homomorphism and so Φ_n is a contraction for all $n \in \mathbb{N}$.

If Φ is a $*$-isomorphism then we show that Φ_n is injective and surjective for all $n \in \mathbb{N}$ and thus each Φ_n is a $*$-isomorphism as required. □

We now show that we can make $\mathcal{CB}(X, Y)$ into an operator space itself. See section 1.2.19 in Blecher & Le Merdy (2004) for further details.

Example 1.1.11 *For all $n \in \mathbb{N}$ we can define a bijective linear map $\Phi : \mathbb{M}_n(\mathcal{CB}(X, Y)) \to \mathcal{CB}(X, \mathbb{M}_n(Y))$ by $\Phi\left((T_{ij})_{i,j=1}^n\right)(x) = (T_{ij}(x))_{i,j=1}^n$ and we can define the norm on $\mathbb{M}_n(\mathcal{CB}(X, Y))$ such that Φ is an isometry.*

As a result of this we see that choosing $Y = \mathbb{C}$ in the above we turn X^* into an operator space.

We record the following two results from Proposition 2.2.2 and Corollary 2.2.5 from Effros & Ruan (2000).

Proposition 1.1.12 *Let X be an operator space, $n \in \mathbb{N}$ and $T : X \to \mathbb{M}_n$ a linear map, then $\|T\|_{cb} = \|T_n\|$.*

Corollary 1.1.13 *Let X and Y be n-dimensional operator spaces, then there is an isomorphism $T : X \to Y$ such that $\|T\|_{cb}\|T^{-1}\|_{cb} \leqslant n^2$.*

Example 1.1.14 *Let X be an operator space and X^* the Banach dual of X. We have from Proposition 1.1.12 that $X^* = \mathcal{B}(X, \mathbb{C}) = \mathcal{CB}(X, \mathbb{C})$.*

For $n \in \mathbb{N}$ we define a map $\Phi : \mathbb{M}_n(X^*) \to \mathcal{CB}(X, \mathbb{M}_n)$ given by $\Phi\left((\omega_{ij})_{i,j=1}^n\right)(x) = (\langle x, \omega_{ij}\rangle)_{i,j=1}^n$ and we show this defines a bijective linear map. Clearly we have a linear injective map. Say $f \in \mathcal{CB}(X, \mathbb{M}_n)$, then we define $\omega_{ij} : X \to \mathbb{C}$ by $\omega_{ij}(x) = (f(x))_{i,j}$; that is we define it as the i, j-th entry of $f(x) \in \mathbb{M}_n$. Then we have $|\langle x, \omega_{ij}\rangle| = |(f(x))_{ij}| \leqslant \|f(x)\| \leqslant \|f\|\|x\|$ and so $\omega_{ij} \in X^*$ for all $1 \leqslant i, j \leqslant n$. Also we have

$$\Phi((\omega_{ij})_{i,j=1}^n)(x) = (\langle x, \omega_{ij}\rangle)_{i,j=1}^n = ((f(x))_{i,j})_{i,j=1}^n = f(x)$$

and so Φ *is surjective. It follows that we can define an operator space structure on* X^* *making* Φ *a complete isometry using the operator space structure on* $\mathcal{CB}(X, \mathbb{M}_n)$ *for* $n \in \mathbb{N}$.

If X *has a predual* X_* *then* $X_* \subset X^*$ *has an operator space structure by restriction of the operator space structure on* X^* *by Example 1.1.8. In particular we have an operator space on the predual* M_* *of a von Neumann algebra* M *and furthermore, we have that the operator space structure on* $(M_*)^*$ *as the dual space is completely isometric to that of the operator space structure on* M.

We have the following property for the canonical embedding of a space into its double dual. See Proposition 3.2.1 in Effros & Ruan (2000) and Section 2.1 of Pisier (2003) for proofs.

Proposition 1.1.15 *Let* X *be an operator space, then the canonical embedding* $\iota : X \to X^{**}$ *of* X *into its double dual is a complete isometry.*

Next we consider adjoint maps. Let $T : X \to Y$ be a bounded map between Banach spaces X and Y, then there exists a unique bounded map $T^* : Y^* \to X^*$ called the **adjoint** map such that

$$\langle T^*\omega, x\rangle = \langle \omega, Tx\rangle \tag{1.2}$$

for $\omega \in Y^*$ and $x \in X$ where $\|T\| = \|T^*\|$. We have the following in the case of operator spaces.

Proposition 1.1.16 *Let* $T : X \to Y$ *be a linear map between operator spaces* X *and* Y, *then the adjoint map* $T^* : Y^* \to X^*$ *in Equation (1.2) is completely bounded if and only if* T *is completely bounded. Furthermore if these conditions hold we have* $\|T^*\|_{cb} = \|T\|_{cb}$.

The reader is referred to Section 2.4 in Pisier (2003) for further details on adjoint maps. We now establish some properties of quotient spaces of operator spaces.

Example 1.1.17 *Let X be an operator space, Y a closed subspace of X and fix $n \in \mathbb{N}$. Then $\mathbb{M}_n(Y)$ is closed in $\mathbb{M}_n(X)$ and so we may identify $\mathbb{M}_n(X/Y)$ with $\mathbb{M}_n(X)/\mathbb{M}_n(Y)$ by defining an isomorphism $(x_{ij} + Y)_{i,j=1}^n \mapsto (x_{ij})_{i,j=1}^n + \mathbb{M}_n(Y)$. In particular for $(x_{ij})_{i,j=1}^n \in \mathbb{M}_n(X/Y)$ we have*

$$\|(x_{ij})_{i,j=1}^n\| = \inf\left\{\|(y_{ij})_{i,j=1}^n\| \;\middle|\; (y_{ij})_{i,j=1}^n \in \mathbb{M}_n(X),\ (y_{ij}+Y)_{i,j=1}^n = (x_{ij})_{i,j=1}^n\right\}.$$

It can be shown this forms an operator space, see Example 3.1.1 in Effros & Ruan (2000).

Proposition 1.1.18 *Let X, Y and Z be operator spaces, let $q : X \to X/Y$ denote the canonical surjection map and let $u : X/Y \to Z$ be a linear map. We let X/Y have the operator space structure in Example 1.1.17 and then we have $u \in \mathcal{CB}(X/Y, Z)$ if and only if $uq \in \mathcal{CB}(X, Z)$ with $\|u\|_{cb} = \|uq\|_{cb}$.*

For a proof of the following see Section 2.4 in Pisier (2003) and 1.4.3 in Blecher & Le Merdy (2004). We have a similar well known result for Banach spaces.

Proposition 1.1.19 *Let $T : X \to Y$ be a completely bounded map between operator spaces. Then we have the following:*

(i) T is a complete isometry if and only if T^* is a complete quotient map;

(ii) T is a complete quotient map if and only if T^* is a complete isometry.

Lemma 1.1.20 *Let $T : X \to Y$ and $S : Y \to X$ be complete contractions between operator spaces with $ST = \mathrm{id}_X$. Then T is a complete isometry and S a complete quotient map.*

Proof

Let $x \in X$ and $n \in \mathbb{N}$ and we have

$$\begin{aligned}\|(x_{ij})_{i,j=1}^n\|_n &= \|S_nT_n(x_{ij})_{i,j=1}^n\|_n \leqslant \|S\|_{cb}\|T_n(x_{ij})_{i,j=1}^n\|_n \\ &\leqslant \|T_n(x_{ij})_{i,j=1}^n\|_n \leqslant \|T\|_{cb}\|(x_{ij})_{i,j=1}^n\|_n \leqslant \|(x_{ij})_{i,j=1}^n\|_n\end{aligned}$$

and so we have equality throughout and T_n is an isometry for all $n \in \mathbb{N}$.

Now taking adjoints we find completely contractive maps S^* and T^* such that $T^*S^* = \mathrm{id}_{X^*}$. Then we see similarly that S^* is a complete isometry and thus by Proposition 1.1.19 we have that S is a complete quotient map. □

Corollary 1.1.21 *Let $T : X \to Y$ and $S : Y \to X$ be complete contractions between operator spaces with $ST = \mathrm{id}_X$ and $TS = \mathrm{id}_Y$, then S and T are completely isometric isomorphisms and $X \cong_{ci} Y$.*

Example 1.1.22 *Let X be a linear space and let $\overline{X} = \{\overline{x} \mid x \in X\}$ be a linear space with addition and scalar multiplication given by*

$$\overline{x} + \overline{y} = \overline{x + y}, \qquad \lambda\overline{x} = \overline{\overline{\lambda}x}$$

for $x, y \in X$ and $\lambda \in \mathbb{C}$.

Given a finite dimensional linear space X with a basis $\{e_i \mid 1 \leqslant i \leqslant n\}$ we can write $x = \sum_{i=1}^n x_i e_i$ for $x_i \in \mathbb{C}$ $(1 \leqslant i \leqslant n)$. Now let $\overline{x} \in \overline{X}$ and we have $x \in X$ and can write this as above. Then we have $\overline{x} = \sum_{i=1}^n \overline{x_i e_i} = \sum_{i=1}^n \overline{x_i}\,\overline{e_i}$ and so we have a basis $\{\overline{e_i} \mid 1 \leqslant i \leqslant n\}$ for $\overline{X}$.

Let $T : X \to Y$ be a linear map between linear spaces and define a map $\overline{T} : \overline{X} \to \overline{Y}$ by $\overline{T}(\overline{x}) = \overline{Tx}$. In particular we have linear functionals on $\overline{X}$ say. If X is finite dimensional and $T : X \to X$ is given by a matrix $(T_{ij})_{i,j=1}^n$ then we can write

$$\overline{T}(\overline{e_i}) = \overline{T(e_i)} = \sum_{j=1}^n \overline{T_{ij}}\,\overline{e_j}$$

and so we define $\overline{(T_{ij})_{i,j=1}^n} = \left(\overline{T_{ij}}\right)_{i,j=1}^n$.

If we have a Hilbert space $\mathcal{H}$ we can make $\overline{\mathcal{H}}$ a Hilbert space with inner product $\left(\overline{\xi}|\overline{\eta}\right) = \overline{(\xi|\eta)} = (\eta|\xi)$ for all $\xi, \eta \in \mathcal{H}$. We have for $T \in \mathcal{B}(\mathcal{H})$ that $\left\|\overline{T}(\overline{\xi})\right\| = \|T\xi\|$ for all $\xi \in \mathcal{H}$ and so $\left\|\overline{T}\right\| = \|T\|$ giving $\overline{T} \in \mathcal{B}\left(\overline{\mathcal{H}}\right)$.

Given an operator space $X \subset \mathcal{B}(\mathcal{H})$ we have an embedding $\overline{\pi} : \overline{X} \to \mathcal{B}(\overline{\mathcal{H}})$ and we can show that

$$\left\| (\overline{x_{ij}})_{i,j=1}^n \right\|_{\mathbb{M}_n(\overline{X})} = \left\| \overline{(x_{ij})_{i,j=1}^n} \right\|_{\overline{\mathbb{M}_n(X)}} = \left\| (x_{ij})_{i,j=1}^n \right\|_{\mathbb{M}_n(X)}.$$

It also follows that for a completely bounded map between operator spaces $T \in \mathcal{CB}(X,Y)$ that $\overline{T} \in \mathcal{CB}(\overline{X},\overline{Y})$ and $\|T\|_{cb} = \left\|\overline{T}\right\|_{cb}$. Furthermore the map $\mathcal{CB}(X,Y) \to \mathcal{CB}(\overline{X},\overline{Y})$ given by $T \mapsto \overline{T}$ is an anti-linear completely isometric isomorphism.

Finally consider a completely bounded linear map $T : \overline{X} \to \overline{Y}$, then we have a map $\overline{T} : X \to Y$ such that for all $x, x' \in X$ and $\lambda \in \mathbb{C}$ we have

$$\overline{T}(x + \lambda y) = \overline{T}\left(\overline{\overline{x} + \overline{\lambda}\overline{y}}\right) = \overline{T(\overline{x} + \overline{\lambda}\overline{y})} = \overline{T\overline{x}} + \lambda\overline{T\overline{y}} = \overline{T}x + \lambda\overline{T}y$$

and so $\overline{T} : X \to Y$ is linear. We know that $\|T\|_{cb} = \left\|\overline{T}\right\|_{cb}$ and so we have a completely isometric isomorphism $\mathcal{CB}(\overline{X},\overline{Y}) \cong_{ci} \overline{\mathcal{CB}(X,Y)}$.

There are potentially many different operator space structures on a Banach space X; however we always have a minimal and maximal operator space structure. See Section 3.3 in Effros & Ruan (2000) and Chapter 3 in Pisier (2003) for further details.

Definition 1.1.23 *We define two operator space structures* $\mathrm{MIN}(X)$ *and* $\mathrm{MAX}(X)$ *as follows. For all $n \in \mathbb{N}$ and $x = (x_{ij})_{i,j=1}^n \in \mathbb{M}_n(X)$ let*

$$\|x\|_{\mathbb{M}_n(\mathrm{MIN}(X))} := \sup\left\{\|f_n(x)\| \;\middle|\; f \in X^*,\ \|f\| \leqslant 1\right\}$$

and let

$$\|x\|_{\mathbb{M}_n(\mathrm{MAX}(X))} := \sup\left\{ \|\theta_n(x)\| \;\middle|\; \begin{array}{c} \mathcal{H} \text{ is a Hilbert space and } \theta : X \to \mathcal{B}(\mathcal{H}) \\ \text{an embedding such that } \|\theta\| \leqslant 1 \end{array} \right\}.$$

We call $\mathrm{MIN}(X)$ *the* ***minimal operator space structure on*** X *and* $\mathrm{MAX}(X)$ *the* ***maximal***

operator space structure on X.

Let $\|\cdot\|_n$ a be collection of norms defining an operator space structure, then for all $x \in \mathbb{M}_n(X)$ we have $\|x\|_{\mathbb{M}_n(\mathrm{MIN}(X))} \leqslant \|x\|_n \leqslant \|x\|_{\mathbb{M}_n(\mathrm{MAX}(X))}$. We have the following important proposition.

Proposition 1.1.24 *For a Banach space X we have*

$$(\mathrm{MAX}(X))^* \cong_{ci} \mathrm{MIN}(X^*), \qquad (\mathrm{MIN}(X))^* \cong_{ci} \mathrm{MAX}(X^*).$$

1.1.3 Direct Sums and Tensor Products of Operator Spaces

We now move on to discussing direct sum and tensor products of operator spaces. See Section 2.6 and Chapters 4 in Pisier (2003) and Chapters 7 and 8 in Effros & Ruan (2000) for further details. We offer more proofs in this section where the author could not find proofs in the standard references given.

Definition 1.1.25 *Let $X_i \subset \mathcal{B}(\mathcal{H}_i)$ be operator spaces for $i = 1, 2$, then we define the operator space $X_1 \oplus_\infty X_2$ with the obvious embedding into $\mathcal{B}(\mathcal{H}_1 \oplus_2 \mathcal{H}_2)$.*

Note that given Banach spaces X_1 and X_2 we have a Banach space direct sum $X_1 \oplus_\infty X_2$ with norm

$$\|(x^1, x^2)\|_{X_1 \oplus_\infty X_2} = \sup\{\|x^1\|_{X_1}, \|x^2\|_{X_2}\}.$$

The following shows that our operator space embedding in Definition 1.1.25 gives us the Banach space $X_1 \oplus_\infty X_2$ when X_1 and X_2 are considered as Banach spaces.

Proposition 1.1.26 *Let $X_i \subset \mathcal{B}(\mathcal{H}_i)$ be operator spaces for $i = 1, 2$ and let $((x^1_{ij}, x^2_{ij}))^n_{i,j=1} \in \mathbb{M}_n(X_1 \oplus_\infty X_2)$, then we have*

$$\left\| \left((x^1_{ij}, x^2_{ij})\right)^n_{i,j=1} \right\|_{\mathbb{M}_n(X_1 \oplus_\infty X_2)} = \sup\left\{\|(x^1_{ij})^n_{i,j=1}\|_{\mathbb{M}_n(X_1)}, \|(x^2_{ij})^n_{i,j=1}\|_{\mathbb{M}_n(X_2)}\right\}.$$

Proposition 1.1.27 *Let $X_i \subset \mathcal{B}(\mathcal{H}_i)$ be operator spaces, $\pi_i : X_1 \oplus_\infty X_2 \to X_i$ the canonical projection maps and $\iota_i : X_i \to X_1 \oplus_\infty X_2$ the canonical injection maps. Then for $i = 1, 2$ we have that π_i and ι_i are complete contractions and furthermore π_i are complete quotient maps.*

Proposition 1.1.28 *(i) Let X, Y and Z be operator spaces, $T \in \mathcal{CB}(X, Y)$ and $S \in \mathcal{CB}(X, Z)$, then we have a map $T \oplus S \in \mathcal{CB}(X, Y \oplus_\infty Z)$ given by $x \mapsto (Tx, Sx)$ with $\|T \oplus S\|_{cb} = \max\{\|T\|_{cb}, \|S\|_{cb}\}$;*

(ii) Let X_1, X_2, Y_1 and Y_2 be operator spaces, $T_1 \in \mathcal{CB}(X_1, Y_1)$ and $T_2 \in \mathcal{CB}(X_2, Y_2)$. We have a map $T_1 \oplus_\infty T_2 \in \mathcal{CB}(X_1 \oplus_\infty X_2, Y_1 \oplus_\infty Y_2)$ given by $(x_1, x_2) \mapsto (T_1 x_1, T_2 x_2)$. If T_1 and T_2 are complete contractions (isometries) then $T_1 \oplus_\infty T_2$ is also a complete contraction (isometry).

Proof

(i) Let $(x_{ij}) \in \mathbb{M}_n(X)$ and then $(T \oplus S)_n((x_{ij})_{i,j=1}^n) = \big((T(x_{ij}))_{i,j=1}^n, (S(x_{ij}))_{i,j=1}^n\big)$ and so

$$\begin{aligned}\|(T \oplus S)_n((x_{ij})_{i,j=1}^n)\|_{\mathbb{M}_n(Y \oplus_\infty Z)} &= \max\left\{\|T_n((x_{ij})_{i,j=1}^n)\|, \|S_n((x_{ij})_{i,j=1}^n)\|\right\} \\ &\leqslant \max\{\|T_n\|, \|S_n\|\}\|(x_{ij})_{i,j=1}^n\|.\end{aligned}$$

Then it follows that $\|(T \oplus S)_n\| \leqslant \max\{\|T_n\|, \|S_n\|\} \leqslant \max\{\|T\|_{cb}, \|S\|_{cb}\}$ and so taking the limit $n \to \infty$ we get $\|T \oplus S\|_{cb} \leqslant \max\{\|T\|_{cb}, \|S\|_{cb}\}$. On the other hand we have $\|(T \oplus S)_n(x_{ij})\| \geqslant \|T_n((x_{ij})_{i,j=1}^n\|$ and similarly for S_n. So

$$\max\{\|T_n\|, \|S_n\|\} \leqslant \|(T \oplus S)_n\| \leqslant \|T \oplus S\|_{cb}$$

for all $n \in \mathbb{N}$ and then taking the limit of $n \to \infty$ we get $\max\{\|T\|_{cb}, \|S\|_{cb}\} \leqslant \|T \oplus S\|_{cb}$ as required.

(ii) Let $(x_{ij})_{i,j=1}^n \in \mathbb{M}_n(X_1)$ and $(y_{ij}) \in \mathbb{M}_n(X_2)$ and we have

$$\begin{aligned}
&\left\|(T_1 \oplus_\infty T_2)_n \left((x_{ij})_{i,j=1}^n, (y_{ij})_{i,j=1}^n\right)\right\|_{\mathbb{M}_n(Y_1 \oplus_\infty Y_2)} \\
&\qquad = \max\left\{\left\|(T_1(x_{ij}))_{i,j=1}^n\right\|_{\mathbb{M}_n(Y_1)}, \left\|(T_2(y_{ij}))_{i,j=1}^n\right\|_{\mathbb{M}_n(Y_2)}\right\} \\
&\qquad \leqslant \max\left\{\|T_1\|_{cb}\left\|(x_{ij})_{i,j=1}^n\right\|_{\mathbb{M}_n(X_1)}, \|T_2\|_{cb}\left\|(y_{ij})_{i,j=1}^n\right\|_{\mathbb{M}_n(X_2)}\right\} \\
&\qquad \leqslant \max\{\|T_1\|_{cb}, \|T_2\|_{cb}\} \max\left\{\left\|(x_{ij})_{i,j=1}^n\right\|_{\mathbb{M}_n(X_1)}, \left\|(y_{ij})_{i,j=1}^n\right\|_{\mathbb{M}_n(X_2)}\right\}
\end{aligned}$$

and so $\|T_1 \oplus_\infty T_2\|_{cb} \leqslant \|T_1\|_{cb}\|T_2\|_{cb}$ and $T_1 \oplus_\infty T_2$ is a completely bounded map. Clearly if T_1 and T_2 are contractions this shows that so is $T_1 \oplus_\infty T_2$. If T_1 and T_2 are complete isometries then

$$\begin{aligned}
&\left\|(T_1 \oplus_\infty T_2)_n \left((x_{ij})_{i,j=1}^n, (y_{ij})_{i,j=1}^n\right)\right\|_{\mathbb{M}_n(Y_1 \oplus_\infty Y_2)} \\
&\qquad = \max\left\{\left\|(T_1(x_{ij}))_{i,j=1}^n\right\|_{\mathbb{M}_n(Y_1)}, \left\|(T_2(y_{ij}))_{i,j=1}^n\right\|_{\mathbb{M}_n(Y_2)}\right\} \\
&\qquad = \max\left\{\left\|(x_{ij})_{i,j=1}^n\right\|_{\mathbb{M}_n(Y_1)}, \left\|(y_{ij})_{i,j=1}^n\right\|_{\mathbb{M}_n(Y_2)}\right\} \\
&\qquad = \left\|\left((x_{ij})_{i,j=1}^n, (y_{ij})_{i,j=1}^n\right)\right\|_{\mathbb{M}_n(X_1 \oplus_\infty X_2)}
\end{aligned}$$

and so $T_1 \oplus_\infty T_2$ is also a complete isometry. □

Corollary 1.1.29 *Let X, Y and Z denote operator spaces, then we have $\mathcal{CB}(X, Y \oplus_\infty Z) \cong_{ci} \mathcal{CB}(X,Y) \oplus_\infty \mathcal{CB}(X,Z)$.*

Proof

We have an isometry $\Psi : \mathcal{CB}(X,Y) \oplus_\infty \mathcal{CB}(X,Z) \to \mathcal{CB}(X, Y \oplus_\infty Z)$ by the previous proposition given by $(T,S) \mapsto T \oplus S$ and we show this is onto. Let $T \in \mathcal{CB}(X, Y \oplus_\infty Z)$ and we consider the maps $\pi_1 \circ T \in \mathcal{CB}(X,Y)$ and $\pi_2 \circ T \in \mathcal{CB}(X,Z)$. Then for all $x \in X$ we have

$$\psi((\pi_1 \circ T), (\pi_2 \circ T))(x) = ((\pi_1 \circ T)(x), (\pi_2 \circ T)(x)) = Tx$$

and so $\psi\left(((\pi_1 \circ T), (\pi_2 \circ T))\right) = T$. So Ψ is an isometric isomorphism.

Now fix $n \in \mathbb{N}$ and using this isometric isomorphism and Proposition 1.1.26 we have

$$\begin{aligned}\mathbb{M}_n(\mathcal{CB}(X,Y) \oplus_\infty \mathcal{CB}(X,Z)) &\cong_i \mathbb{M}_n(\mathcal{CB}(X,Y)) \oplus_\infty \mathbb{M}_n(\mathcal{CB}(X,Z)) \\ &\cong_i \mathcal{CB}(X, \mathbb{M}_n(Y)) \oplus_\infty \mathcal{CB}(X, \mathbb{M}_n(Z)) \cong_i \mathcal{CB}(X, \mathbb{M}_n(Y) \oplus_\infty \mathbb{M}_n(Z)) \\ &\cong_i \mathcal{CB}(X, \mathbb{M}_n(Y \oplus_\infty Z)) \cong_i \mathbb{M}_n(\mathcal{CB}(X, Y \oplus_\infty Z))\end{aligned}$$

and so Ψ is a complete isometry as required. □

We consider another example of a direct sum of operator spaces now.

Example 1.1.30 *Let $X_i \subset \mathcal{B}(\mathcal{H}_i)$ denote operator spaces for $i = 1, 2$, let $\iota_i : X_i \to \mathcal{B}(\mathcal{H}_1 \oplus_2 \mathcal{H}_2)$ denote the embeddings $x \mapsto (x, 0)$ and $y \mapsto (0, y)$ and let P denote the set of all pairs (u_1, u_2) of completely contractive maps $u_i : X_i \to \mathcal{B}(\mathcal{H}_u)$ for some Hilbert space $\mathcal{H}_u$ (dependent on (u_1, u_2)). Note that P is non-empty as we have the completely isometric embeddings $(\iota_1, \iota_2) \in P$. We let $\mathcal{H} = \bigoplus^2{}_{u \in P} \mathcal{H}_u$ for convenience.*

Let $J : X_1 \oplus X_2 \to \mathcal{B}(\mathcal{H})$ be given by $(x, y) \mapsto (u_1(x) + u_2(y))_{u \in P}$. We define an operator space $X_1 \oplus_1 X_2$ such that J is a complete isometry here.

This satisfies the following universal property: for any operator space Y and complete contractions $u_1 : X_1 \to Y$ and $u_2 : X_2 \to Y$ then the map $X_1 \oplus_1 X_2 \to Y$ given by $(x_1, x_2) \mapsto u_1(x_1) + u_2(x_2)$ for $x_1 \in X_1$ and $x_2 \in X_2$ is a complete contraction.

Proposition 1.1.31 *Let X_1, X_2 and Y denote operator spaces, let $T \in \mathcal{CB}(X_1, Y)$ and $S \in \mathcal{CB}(X_2, Y)$. Let $T \oplus_1 S : X_1 \oplus_1 X_2 \to Y$ be the map given by $(x, x') \mapsto Tx + Sx'$. If T and S are complete contractions then so is $T \oplus_1 S$. Furthermore we have $\|T \oplus_1 S\|_{cb} = \max\{\|T\|_{cb}, \|S\|_{cb}\}$.*

Proof

Let T and S be complete contractions and $\mathcal{H}$ the Hilbert space such that $Y \subset \mathcal{B}(\mathcal{H})$ as an operator space, then we have that $(T, S) \in P$ for P given in Example 1.1.30 and it follows that $\|(T \oplus_1 S)(x, y)\| \leqslant \|J(x, y)\|$. So $T \oplus_1 S$ is completely contractive.

Let $T \in \mathcal{CB}(X,Z)$ and $S \in \mathcal{CB}(Y,Z)$ and let $\alpha = \max\{\|T\|_{cb}, \|S\|_{cb}\}$. Then $T' = \frac{T}{\alpha}$ and $S' = \frac{S}{\alpha}$ are both complete contractions and thus $T' \oplus_1 S'$ is also a complete contraction. Then we have $\|T \oplus_1 S\| \leqslant \alpha = \max\{\|T\|_{cb}, \|S\|_{cb}\}$. Also we have for $n \in \mathbb{N}$ that

$$\|T_n((x_{ij})_{i,j=1}^n)\|_{\mathbb{M}_n(Z)} = \|(T \oplus_1 S)_n((x_{ij})_{i,j=1}^n, 0)\| \leqslant \|(T \oplus_1 S)_n\| \|(x_{ij})_{i,j=1}^n\|_{\mathbb{M}_n(X)}$$

and so in particular we have $\|T_n\| \leqslant \|(T \oplus_1 S)_n\| \leqslant \|T \oplus_1 S\|_{cb}$. Then we have $\|T\|_{cb} \leqslant \|T \oplus_1 S\|_{cb}$ and similarly $\|S\|_{cb} \leqslant \|T \oplus_1 S\|_{cb}$ so it follows that $\max\{\|T\|_{cb}, \|S\|_{cb}\} \leqslant \|T \oplus_1 S\|_{cb}$. □

Proposition 1.1.32 *For operator spaces X, Y and Z we have completely isometric isomorphisms $\mathcal{CB}(X \oplus_1 Y, Z) \cong_{ci} \mathcal{CB}(X,Z) \oplus_\infty \mathcal{CB}(Y,Z)$. In particular it then follows that $(X \oplus_1 Y)^* \cong_{ci} X^* \oplus_\infty Y^*$.*

Proof

It follows similar to that of the proof to Corollary 1.1.29 that we have an isometric isomorphism $\mathcal{CB}(X \oplus_1 Y, Z) \cong_i \mathcal{CB}(X,Z) \oplus_\infty \mathcal{CB}(Y,Z)$. Then for all $n \in \mathbb{N}$ we have the following isometric isomorphisms as required

$$\mathbb{M}_n(\mathcal{CB}(X \oplus_1 Y, Z)) \cong_i \mathcal{CB}(X \oplus_1 Y, \mathbb{M}_n(Z)) \cong_i \mathcal{CB}(X, \mathbb{M}_n(Z)) \oplus_\infty \mathcal{CB}(Y, \mathbb{M}_n(Z))$$
$$\mathbb{M}_n(\mathcal{CB}(X,Z)) \oplus_\infty \mathbb{M}_n(\mathcal{CB}(Y,Z)) \cong_i \mathbb{M}_n(\mathcal{CB}(X,Z) \oplus_\infty \mathcal{CB}(Y,Z)). \qquad \square$$

We now move on to the topic of tensor products of operator spaces. We have already introduced the minimal tensor product of two operator spaces; we now introduce the projective and injective tensor products and give further properties of each of these.

Notation 1.1.33 *Let X and Y be operator spaces. Let $(x_{ij})_{i,j=1}^n \in \mathbb{M}_n(X)$ and $(y_{ij})_{i,j=1}^m \in \mathbb{M}_m(Y)$, then we have $(x_{ij} \otimes y_{kl})_{(i,k),(j,l)=(1,1)}^{(n,m)} \in \mathbb{M}_{mn}(X \odot Y)$. We will always identify $\mathbb{M}_n(X) \odot \mathbb{M}_m(Y)$ with $\mathbb{M}_{mn}(X \odot Y)$ in this way.*

Definition 1.1.34 *Let X and Y be operator spaces, then we can form the algebraic tensor product $X \odot Y$ and we can consider norms on $\mathbb{M}_n(X \odot Y)$. We say that a collection of norms $\left\{\|\cdot\|_{\mathbb{M}_n(X\odot Y)} \;\middle|\; n \in \mathbb{N}\right\}$ on $X \odot Y$ is a **subcross matrix norm** if for all $(x_{ij})_{i,j=1}^n \in \mathbb{M}_n(X)$ and $(y_{ij})_{i,j=1}^m \in \mathbb{M}_m(Y)$ we have*

$$\left\|(x_{ij} \otimes y_{kl})_{(i,k),(j,l)=(1,1)}^{(n,m)}\right\|_{\mathbb{M}_{mn}(X\odot Y)} \leqslant \left\|(x_{ij})_{i,j=1}^n\right\|_{\mathbb{M}_n(X)} \left\|(y_{kl})_{k,l=1}^m\right\|_{\mathbb{M}_m(Y)}.$$

*We say it is a **cross matrix norm** if this is an equality for all $(x_{ij})_{i,j=1}^n \in \mathbb{M}_n(X)$ and $(y_{ij})_{i,j=1}^m \in \mathbb{M}_m(Y)$. We call the resulting completion of $X \odot Y$ the operator space tensor product.*

For a proof of the following see Theorem 7.1.1 in Effros & Ruan (2000).

Definition-Theorem 1.1.35 *Let X and Y be operator spaces, then there is a subcross matrix norm satisfying Ruan's axioms on $X \odot Y$ such that for all $(u_{ij})_{i,j=1}^n \in \mathbb{M}_n(X \odot Y)$ we have*

$$\left\|(u_{ij})_{i,j=1}^n\right\|_{\mathbb{M}_n(X\odot Y)} = \inf\left\{\|\alpha\|\|x\|\|y\|\|\beta\| \;\middle|\; u_{ij} = \sum_{p,r=1}^{m}\sum_{q,s=1}^{m'} \alpha_{i,pq}(x_{p,r} \otimes y_{q,s})\beta_{rs,j}\right\}$$

where on the right hand side we range over all $m, m' \in \mathbb{N}$, $\alpha \in \mathbb{M}_{n,mm'}$, $x \in \mathbb{M}_m(X)$, $y \in \mathbb{M}_{m'(Y)}$ and $\beta \in \mathbb{M}_{mm',n}$ and the norms are calculated in the appropriate space.

*This is the largest cross matrix norm on $X \odot Y$ and we denote by $X \widehat{\otimes} Y$ the operator space given by the completion of $X \odot Y$ with respect to the norm above. We call this the **operator space projective tensor product**.*

Most of following propositions are proved in Chapter 7 of Effros & Ruan (2000) or Chapter 4 of Pisier (2003). We quote the results we will use here and refer the reader to these references for further details.

Proposition 1.1.36 *Let X, X', Y, Y' and Z be operator spaces. Then we have the following:*

(i) The operator space projective tensor product $X \widehat{\otimes} Y$ *of* X *and* Y *is the unique operator space such that* $(X \widehat{\otimes} Y)^* \cong_{ci} \mathcal{CB}(X, Y^*)$ *or more generally we have*

$$\mathcal{CB}(X \widehat{\otimes} Y, Z) \cong_{ci} \mathcal{CB}(X, \mathcal{CB}(Y, Z));$$

(ii) If $T \in \mathcal{CB}(X, Y)$ *and* $T' \in \mathcal{CB}(X', Y')$ *are completely bounded maps, then we have a completely bounded map* $T \otimes T' : X \widehat{\otimes} X' \to Y \widehat{\otimes} Y'$ *that extends the map* $T \odot T' : X \odot X' \to Y \odot Y'$ *such that* $\|T \otimes T'\|_{cb} \leqslant \|T\|_{cb} \|T'\|_{cb}$*;*

(iii) If $T \in \mathcal{CB}(X, Y)$ *and* $T' \in \mathcal{CB}(X', Y')$ *are complete quotient maps then there is a complete quotient map* $T \otimes T' : X \widehat{\otimes} X' \to Y \widehat{\otimes} Y'$ *extending* $T \otimes T' : X \odot X' \to Y \odot Y'$ *and furthermore we have*

$$\mathrm{Ker}\,(T \otimes T') = \overline{(\mathrm{Ker}\, T) \odot X' + X \odot (\mathrm{Ker}\, T')}^{\|\cdot\|};$$

(iv) The flip map $\Sigma : X \widehat{\otimes} Y \to Y \widehat{\otimes} X$ *is a completely isometric isomorphism;*

(v) We have that $\overline{X} \widehat{\otimes} \overline{Y} \cong_{ci} \overline{X \widehat{\otimes} Y}$*.*

Proof

We prove only the last property here which follows as for any operator space Z we have

$$\begin{aligned}\mathcal{CB}(\overline{X} \widehat{\otimes} \overline{Y}, \overline{Z}) &\cong_{ci} \mathcal{CB}(\overline{X}, \mathcal{CB}(\overline{Y}, \overline{Z})) \cong_{ci} \mathcal{CB}(\overline{X}, \overline{\mathcal{CB}(Y, Z)}) \\ &\cong_{ci} \overline{\mathcal{CB}(X, \mathcal{CB}(Y, Z))} \cong_{ci} \overline{\mathcal{CB}(X \widehat{\otimes} Y, Z)} \cong_{ci} \mathcal{CB}(\overline{X \widehat{\otimes} Y}, \overline{Z}). \quad \square\end{aligned}$$

Remarks 1.1.37 *(i) We have a notion of a "completely bounded bilinear map" which gives us a space* $\mathcal{CB}(X \times Y, Z)$ *that is completely isometrically isomorphic to* $\mathcal{CB}(X \widehat{\otimes} Y, Z)$*. We do not pursue this here but this is a motivation for the definition of a completely contractive Banach algebra below;*

(ii) *It does* ***not*** *follow in general that if we have two complete isometries that the projective tensor product of these maps is a complete isometry. In fact given a complete isometry* $T : X \to Y$ *we don't necessarily have a complete isometry* $T \otimes \mathrm{id}_Z : X \widehat{\otimes} Z \to Y \widehat{\otimes} Z$ *for an operator space* Z.

Proposition 1.1.38 *Let* X, X', Y *and* Y' *be operator spaces, then we have completely isometric isomorphisms* $(X \oplus_1 X') \widehat{\otimes} Y \cong_{ci} X \widehat{\otimes} Y \oplus_1 X' \widehat{\otimes} Y$ *and* $X \widehat{\otimes} (Y \oplus_1 Y') \cong_{ci} X \widehat{\otimes} Y \oplus_1 X \widehat{\otimes} Y'$.

Proof

This follows as for all operator spaces Z we have

$$\begin{aligned}
\mathcal{CB}((X \oplus_1 X') \widehat{\otimes} Y, Z) &\cong_{ci} \mathcal{CB}((X \oplus_1 X'), \mathcal{CB}(Y, Z)) \\
&\cong_{ci} \mathcal{CB}(X, \mathcal{CB}(Y, Z)) \oplus_\infty \mathcal{CB}(X', \mathcal{CB}(Y, Z)) \\
&\cong_{ci} \mathcal{CB}(X \widehat{\otimes} Y, Z) \oplus_\infty \mathcal{CB}(X' \widehat{\otimes} Y, Z) \cong_{ci} \mathcal{CB}(X \widehat{\otimes} Y \oplus_1 X' \widehat{\otimes} Y, Z)
\end{aligned}$$

and the second follows from a similar calculation or from Proposition 1.1.36 (iv). □

Definition-Theorem 1.1.39 *Let* X *and* Y *be operator spaces, then there is a cross matrix norm satisfying Ruan's axioms on* $X \odot Y$ *such that for all* $u = (u_{ij})_{i,j=1}^n \in \mathbb{M}_n(X \odot Y)$ *we have*

$$\|u\|_\vee = \sup\left\{\|(f \otimes g)(u)\|_{\mathbb{M}_{pqn}} \;\middle|\; f \in \mathbb{M}_p(X^*),\ g \in \mathbb{M}_q(Y^*),\ \|f\| \leqslant 1,\ \|g\| \leqslant 1\right\}$$

We let $X \check{\otimes} Y$ *be the operator space given by the completion of* $X \odot Y$ *with respect to this norm and we call this the* ***operator space injective tensor product****.*

Notation 1.1.40 *Consider the identity map* $X \odot Y \to X \odot Y$ *which can be coextended to a map* $\psi : X \odot Y \to X \check{\otimes} Y$*. For any* $x \in \mathbb{M}_n(X)$ *and* $y \in \mathbb{M}_m(Y)$ *we have*

$$\|\psi(x \otimes y)\| = \|x \otimes y\| \leqslant \|x\|\|y\|$$

*and so ψ is a complete contraction. It follows that we can extend this to a linear completely contractive map $\Psi : X \hat{\otimes} Y \to X \check{\otimes} Y$ called the **canonical complete contraction** from $X \hat{\otimes} Y$ to $X \check{\otimes} Y$.*

We won't explore many details about this tensor product in this thesis; however we mention the following remark and proposition.

Remark 1.1.41 *Let $X \subset \mathcal{B}(\mathcal{H})$ and $Y \subset \mathcal{B}(\mathcal{K})$ be operator spaces, then we have a completely isometric embedding $X \check{\otimes} Y \hookrightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. It the follows that for C*-algebras A and B, then operator space injective tensor product $A \check{\otimes} B$ coincides with that of the minimal tensor product $A \otimes_{min} B$ as C*-algebras with respect to their canonical operator space structure.*

Proposition 1.1.42 *Say $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$ are completely contractive maps, then there is a completely contractive map $T_1 \otimes T_2 : X_1 \check{\otimes} X_2 \to Y_1 \check{\otimes} Y_2$ such that $(T_1 \otimes T_2)(x \otimes y) = T_1(x) \otimes T_2(y)$.*

We will use the following from Lemma 7.2.2 in Effros & Ruan (2000) shortly to define some additional operator space tensor products.

Lemma 1.1.43 *Let $\mathcal{H}$ and $\mathcal{K}$ denote Hilbert spaces and $\omega \in \mathcal{B}(\mathcal{H})_*$, then there is a unique weak*-continuous linear extension $\omega \otimes \mathrm{id} : \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{B}(\mathcal{K})$ of the map $x \otimes y \mapsto \omega(x)y$ for $x \in \mathcal{B}(\mathcal{H})$ and $y \in \mathcal{B}(\mathcal{K})$ such that $\|\omega \otimes \mathrm{id}\|_{cb} \leqslant \|\omega\|$. Similarly for $\kappa \in \mathcal{B}(\mathcal{K})_*$ we have a unique weak*-continuous extension $\mathrm{id} \otimes \kappa : \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{B}(\mathcal{H})$ of the map $x \otimes y \mapsto \kappa(y)x$ such that $\|\mathrm{id} \otimes \kappa\|_{cb} \leqslant \|\kappa\|$.*

Consider the duals X^* and Y^* for operator spaces X and Y. As X^* and Y^* are operator spaces there are Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and embeddings $\pi_1 : X^* \to \mathcal{B}(\mathcal{H})$ and $\pi_2 : Y^* \to \mathcal{B}(\mathcal{K})$. Furthermore we can assume this is a weak*-homeomorphic completely isometric injection from Proposition 3.2.4 in Effros & Ruan (2000). From this we can define the following tensor products on dual operator space structures.

Example 1.1.44 *Let X and Y be operator spaces with $X^* \subset \mathcal{B}(\mathcal{H})$ and $Y^* \subset \mathcal{B}(\mathcal{K})$ for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. We define the* ***normal tensor product*** *of X^* and Y^*, denoted by $X^* \overline{\otimes} Y^*$, as the weak*-closure of $X^* \odot Y^*$ in $\mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{K}) \cong_{ci} \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. This coincides with the von Neumann tensor product when X^* and Y^* are von Neumann algebras.*

We define the ***Fubini tensor product*** *of X^* and Y^*, denoted by $X^* \overline{\otimes}_{\mathcal{F}} Y^*$, as follows*

$$X^* \overline{\otimes}_{\mathcal{F}} Y^* = \left\{ x \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \;\middle|\; \begin{array}{r} (\omega \otimes \mathrm{id})(x) \in Y^*, \ (\mathrm{id} \otimes \kappa) \in X^*, \\ \forall\, \omega \in \mathcal{B}(\mathcal{H})_*, \ \kappa \in \mathcal{B}(\mathcal{K})_* \end{array} \right\}$$

where we've used the previous lemma.

We let $X \overset{nuc}{\otimes} Y = (X \widehat{\otimes} Y)/\operatorname{Ker} \psi$ for ψ the complete contraction in Notation 1.1.40.

The following is proved in Sections 7.2 and 8.1 in Effros & Ruan (2000). See Effros & Ruan (2003) for the notion of the nuclear tensor product.

Theorem 1.1.45 *We have*

(i) $X^* \overline{\otimes} Y^* \subset X^* \overline{\otimes}_{\mathcal{F}} Y^*$;

(ii) $(X \widehat{\otimes} Y)^*$ *is weak*-homeomorphic completely isometric to* $X^* \overline{\otimes}_{\mathcal{F}} Y^*$;

(iii) $(X \overset{nuc}{\otimes} Y)^* \cong_{ci} X^* \overline{\otimes} Y^*$.

Proposition 1.1.46 *For any two weak*-closed operator spaces $X^* \subset \mathcal{B}(\mathcal{H})$ and $Y^* \subset \mathcal{B}(\mathcal{K})$ we have $X^* \overline{\otimes} Y^* = X^* \overline{\otimes}_{\mathcal{F}} Y^*$ if and only if the canonical complete contraction $\psi : X \widehat{\otimes} Y \to X \check{\otimes} Y$ from Notation 1.1.40 satisfies* $\operatorname{Ker} \psi = 0$.

We have the following properties of these morphisms between operator spaces. See Proposition 3.2.1 in Effros & Ruan (2000) and Section 2.1 of Pisier (2003) for proofs.

Proposition 1.1.47 *Let M and N be von Neumann algebras, then there is a completely isometric isomorphism $(M \overline{\otimes} N)_* \cong_{ci} M_* \widehat{\otimes} N_*$. In particular it follows that the canonical complete contraction ψ given by Notation 1.1.40 has* $\operatorname{Ker} \psi = 0$.

The following can be found in Section 8.2 in Effros & Ruan (2000).

Proposition 1.1.48 *Let X and Y be Banach spaces, then we have*

$$\mathrm{MIN}(X) \check{\otimes} \mathrm{MIN}(Y) \cong_{ci} \mathrm{MIN}(X \check{\otimes} Y)$$

and

$$\mathrm{MAX}(X) \widehat{\otimes} \mathrm{MAX}(Y) \cong_{ci} \mathrm{MAX}(X \hat{\otimes} Y)$$

where $\check{\otimes}$ and $\hat{\otimes}$ denote the injective and projective tensor products as Banach spaces respectively.

1.1.4 Completely Contractive Banach Algebras

In this section we give the definition of a completely contractive Banach algebra. For a completely contractive Banach algebra A we have that the multiplication map $A \widehat{\otimes} A \to A$ given by $x \otimes y \mapsto xy$ for all $x, y \in A$ is completely contractive. In general there is no guarantee that this map is onto, however we show that we can extend any completely contractive Banach algebra to a unital completely contractive Banach algebra and in this case this map is clearly always onto.

Definition 1.1.49 *A **completely contractive Banach Algebra** is an algebra A that is an operator space and such that the multiplication map $A \times A \to A$ gives rise to a completely contractive map $m : A \widehat{\otimes} A \to A$ where $m(x \otimes y) = xy$ for all $x, y \in A$.*

Example 1.1.50 *Let A denote a completely contractive Banach algebra A and let $A^\flat$ denote the operator space $A \oplus_1 \mathbb{C}$ given by Example 1.1.30. We want to make this into a completely contractive Banach algebra such that we have product*

$$(a, \lambda) \cdot (b, \lambda') = (ab + \lambda b + \lambda' a, \lambda\lambda') \tag{1.3}$$

for all $a, b \in A$ and $\lambda, \lambda' \in \mathbb{C}$. Note that we adjoin an identity $e^\flat = (0,1)$ (even if there already is one), meaning we have a new identity. We show now that there is a completely contractive map $m^\flat : A^\flat \widehat{\otimes} A^\flat \to A^\flat$ satisfying this equation.

We let $m_1 : A \widehat{\otimes} A \to A^\flat$ be the map $x \mapsto (m(x), 0)$, $m_2 : A \widehat{\otimes} \mathbb{C} \to A^\flat$ be the map $a \otimes \lambda \mapsto (\lambda a, 0)$, $m_3 : \mathbb{C} \widehat{\otimes} A \to A^\flat$ be the map $\lambda \otimes a \mapsto (\lambda a, 0)$ and finally let $m_4 : \mathbb{C} \widehat{\otimes} \mathbb{C} \to A^\flat$ be the map $\lambda \otimes \lambda' \mapsto (0, \lambda\lambda')$. Then each of these maps is completely contractive. Also we have by Proposition 1.1.38 that $A^\flat \widehat{\otimes} A^\flat \cong_{ci} A \widehat{\otimes} A \oplus_1 A \widehat{\otimes} \mathbb{C} \oplus_1 \mathbb{C} \widehat{\otimes} A \oplus_1 \mathbb{C} \widehat{\otimes} \mathbb{C}$. Furthermore we have $m^\flat = m_1 \oplus_1 m_2 \oplus_1 m_3 \oplus_1 m_4$ satisfies Equation (1.3) above and using Proposition 1.1.31 it follows that $m^\flat$ is completely contractive.

Definition 1.1.51 *Let A be a completely contractive Banach algebra, then a **left ideal** I of A is a closed subspace of A such that $m(a \otimes i) \in I$ for all $a \in A$ and $i \in I$.*

1.2 Homological Algebra in Operator Spaces

We now move on to consider modules over completely contractive Banach algebras and operator biprojectivity. In Chapter 6 we will discuss the subject of operator biprojectivity for the $\mathrm{L}^1_\sharp$-algebra of a locally compact quantum group.

1.2.1 Basic Definitions

We first discuss some preliminaries concerning operator A-bimodules over completely contractive Banach algebras.

Definition 1.2.1 *Let A and B denote completely contractive Banach algebras and let X be a Banach A-B-bimodule, that is we have two operations $A \widehat{\otimes} X \to X$ and $X \widehat{\otimes} B \to X$ given by $a \otimes x \mapsto a \cdot x$ and $x \otimes b \mapsto x \cdot b$ such that for all $a, a' \in A$ and $x, x' \in X$ we have*

$$(a + a') \cdot x = a \cdot x + a' \cdot x, \qquad a \cdot (x + x') = a \cdot x + a \cdot x'$$

$$(aa') \cdot x = a \cdot (a' \cdot x), \qquad \|a \cdot x\| \leqslant \|a\|\|x\|$$

and similarly for the other operation and such that $a \cdot (x \cdot b) = (a \cdot x) \cdot b$ *for all* $a \in A$, $x \in X$ *and* $b \in B$. *We say* X *is a* ***completely bounded*** A***-***B ***bimodule*** *if the operation* $A \widehat{\otimes} X \widehat{\otimes} B \to X$ *given by* $a \otimes x \otimes b \mapsto a \cdot x \cdot b$ *is completely bounded and a* ***completely contractive*** A***-***B ***bimodule*** *if this operation is completely contractive. We also have completely bounded (contractive) left* A*-modules that are operator* A*-*$\mathbb{C}$*-bimodules and similarly for completely bounded (contractive) right* A*-modules.*

Definition 1.2.2 *Let* $T : X \to Y$ *denote a completely bounded (contractive) linear map between operator* A*-*B*-bimodules* X *and* Y. *We say* T *is a completely bounded (contractive)* A*-*B*-bimodule homomorphism if* $T(a \cdot x \cdot b) = a \cdot T(x) \cdot b$ *for all* $a \in A$, $x \in X$ *and* $b \in B$ *and denote such maps by* ${}_A\mathcal{CB}_B(X, Y)$. *If* $A = B$ *we will simply refer to a completely bounded* A*-bimodule homomorphism. Similarly we have left and right completely bounded (contractive)* A*-module homomorphisms.*

We now introduce some basic definitions necessary for introducing the notion of operator projectivity and biprojectivity. The following can be applied to various objects and morphisms, in particular we will apply this to operator spaces and completely bounded maps.

Definition 1.2.3 *A* ***short exact sequence*** *is a collection* $\{X, Y, Z\}$ *of objects and morphisms* $f : X \to Y$ *and* $g : Y \to Z$ *such that* f *is injective,* g *is surjective and* $\operatorname{Image} f = \operatorname{Ker} g$. *We will often denote this by the following*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0.$$

Definition 1.2.4 *A map* $f : X \to Y$ *is* ***admissible*** *if* $\operatorname{Ker} f$ *and* $\operatorname{Image} f$ *are both closed and complemented subspaces of* X *and* Y *respectively.*

Definition 1.2.5 *Consider a short exact sequence with X, Y and Z operator spaces and f and g completely bounded maps as follows*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0.$$

We say this is ***admissible*** *if g has a completely bounded right inverse $G \in \mathcal{CB}(Z, Y)$ and that it* ***splits*** *if G can be chosen to be a left completely bounded A-module homomorphism.*

We will also use the following definition occasionally.

Definition 1.2.6 *Let X be a completely bounded right A-module and Y a completely bounded left A-module. Then the* ***Balanced tensor product*** *of X and Y, denoted by $X \widehat{\otimes}_A Y$, is the quotient of $X \widehat{\otimes} Y$ by the set*

$$\overline{\text{lin }\{x \cdot a \otimes y - x \otimes a \cdot y \mid x \in X,\ y \in Y,\ a \in A\}}^{\|\cdot\|}.$$

1.2.2 Operator Biprojectivity

We now move on to discuss projectivity of completely bounded left A-modules. We use Wood (2002) as our basis for operator biprojectivity in this thesis. The standard reference for biprojectivity of Banach algebras Helemskiĭ (1989) is still useful to us here and we offer Aristov (2002) as an additional reference for the biprojectivity of completely contractive Banach algebras.

Throughout the rest of this section let A denote a completely contractive Banach algebra and let X, Y and P denote completely bounded left A-modules. We only work with completely bounded left A-modules in this section but we have the obvious counterparts for right completely bounded A-modules and completely bounded A-B-bimodules (for a completely contractive Banach algebra B).

Let P be a completely bounded left A-module with map $\pi : A \widehat{\otimes} P \to P$. Using similar methods to that of Example 1.1.50 we can extend π to a map $\pi^\flat \in \mathcal{CB}(A^\flat \widehat{\otimes} P, P)$

such that $\pi^\flat(e^\flat \otimes x) = x$. We let $\theta : P \to A^\flat \widehat{\otimes} P$ denote the map $x \mapsto e^\flat \otimes x$. Then as the operator projective tensor product is a subcross matrix norm we have for all $n \in \mathbb{N}$ and $(x_{ij})_{i,j=1}^n \in \mathbb{M}_n(P)$ that

$$\|\theta_n((x_{ij})_{i,j=1}^n)\| = \|(e^\flat \otimes x_{ij})_{i,j=1}^n\| \leqslant \|e^\flat\| \|(x_{ij})_{i,j=1}^n\|$$

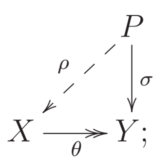
and so θ is completely bounded. Clearly we have $\pi(\theta(x)) = x$ and so we have a right inverse to π. In the special case that θ is a completely bounded left A-module homomorphism we make the following definition.

Definition 1.2.7 *A completely bounded left A-module P is* ***projective*** *if the multiplication map $\pi : A^\flat \widehat{\otimes} P \to P$ has a right inverse in ${}_A\mathcal{CB}(P, A^\flat \widehat{\otimes} P)$. We have similar definitions for right completely bounded A modules and completely bounded A-B-bimodules.*

Theorem 1.2.8 *Let P be a completely bounded left A-module. The following are equivalent:*

(i) P is projective;

(ii) For any completely bounded left A-modules X and Y, $\theta \in {}_A\mathcal{CB}(X, Y)$ a surjective, admissible homomorphism and $\sigma \in {}_A\mathcal{CB}(P, Y)$, then there exists some $\rho \in {}_A\mathcal{CB}(P, X)$ such that the following diagram commutes



(iii) Any admissible, short exact sequence of completely bounded left A-modules as follows

$$0 \longrightarrow Y \longrightarrow X \longrightarrow P \longrightarrow 0$$

splits.

We reiterate that all definitions and results in this section have obvious corresponding definitions and results for right completely bounded A-modules and completely bounded A-B-bimodules (for B a completely contractive Banach algebras).

Definition 1.2.9 *A completely contractive Banach algebra A is **operator biprojective** if it is projective as a completely bounded A-bimodule.*

Proposition 1.2.10 *A completely contractive Banach algebra A is operator biprojective if and only if the multiplication map $\Delta : A \widehat{\otimes} A \to A$ has a right inverse $\rho \in {}_A\mathcal{CB}_A(A)$.*

1.2.3 Additional Results

We prove some useful lemmas in this section that originally were recorded in Aristov (2004). We give our own full proofs of these results expanding on those given by Aristov.

Lemma 1.2.11 *Let A denote a completely contractive Banach algebra and I a left ideal of A, then there is a completely contractive map $A \widehat{\otimes} A^\flat/I \to A^\flat/I$ making $A^\flat/I$ a completely contractive left A-module such that $a \cdot ((b, \lambda) + I) = (ab + \lambda a, 0) + I$ for $a, b \in A$ and $\lambda \in \mathbb{C}$ (where we've used the identification $A^\flat = A \oplus_1 \mathbb{C}$).*

Proof

As A is a completely contractive Banach algebra then so is $A^\flat$ by Example 1.1.50. Clearly I is a subspace of $A^\flat$ as well with embedding $i \mapsto (i, 0)$. Let $q : A^\flat \to A^\flat/I$ be the complete quotient map $x \mapsto x + I$ for $x \in A^\flat$ and let $m^\flat : A \widehat{\otimes} A^\flat \to A$ be the extension of the completely contractive map m making A into a completely contractive Banach algebra. Consider the following commutative diagram

$$\begin{array}{ccc} A \widehat{\otimes} A^\flat & \xrightarrow{m^\flat} & A^\flat \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes q} & & \Big\downarrow{\scriptstyle q} \\ A \widehat{\otimes} A^\flat/I & \underset{S}{\dashrightarrow} & A^\flat/I. \end{array}$$

We want to show that a complete contraction S exists making this diagram commute. Clearly we need $\operatorname{Ker}(\mathrm{id} \otimes q) \subset \operatorname{Ker}(q \circ m^\flat)$ and so we show this first. By Proposition 1.1.36 (iii) we have $\operatorname{Ker}(\mathrm{id} \otimes q) = \overline{A \odot \operatorname{Ker} q} = \overline{A \odot I}$. Furthermore for $a \in A$ and $i \in I$ we have $q(m^\flat(a \otimes i)) = q(ai) = ai + I = I$ and then by linearity it follows that $q(m^\flat(A \odot I)) = 0$. It then follows from continuity that $\operatorname{Ker}(\mathrm{id} \otimes q) \subset \operatorname{Ker}(q \circ m^\flat)$.

As $\mathrm{id} \otimes q$ is onto then for any $y \in A \widehat{\otimes} A^\flat / I$ we have some $x \in A \widehat{\otimes} A^\flat$ such that $(\mathrm{id} \otimes q)(x) = y$. Now we define $S(y) = q(m^\flat(x))$ and we show this is well defined: that is given any $x, x' \in A \widehat{\otimes} A$ with $(\mathrm{id} \otimes q)(x) = (\mathrm{id} \otimes q)(x')$ we have $q(m^\flat(x)) = q(m^\flat(x'))$. For any $x, x' \in A \widehat{\otimes} A$ with $(\mathrm{id} \otimes q)(x) = (\mathrm{id} \otimes q')(x)$ then there is some $x_0 \in \operatorname{Ker}(\mathrm{id} \otimes q)$ such that $x = x' + x_0$. We have shown that $\operatorname{Ker}(\mathrm{id} \otimes q) = \operatorname{Ker}(q \circ m^\flat)$ and so

$$q(m^\flat(x)) = q(m^\flat(x' + x_0)) = q(m^\flat(x'))$$

as required.

We have shown there is a map S making this diagram commutative. By Proposition 1.1.36 (iii) the map $\mathrm{id} \otimes q$ is a complete quotient map and so by Proposition 1.1.18 we have

$$\|S\|_{cb} = \|S \circ (\mathrm{id} \otimes q_1)\|_{cb} = \|q_2 \circ m^\flat\|_{cb} \leqslant \|q_2\|_{cb} \|m^\flat\|_{cb} \leqslant 1$$

and therefore S is a complete contraction.

Finally we have for $a, b \in A$ and $\lambda \in \mathbb{C}$ that

$$\begin{aligned} S(a \otimes ((b, \lambda) + I)) &= (S \circ (\mathrm{id} \otimes q))(a \otimes (b, \lambda)) = q(m^\flat(a \otimes (b, \lambda))) \\ &= q((ab + \lambda a, 0)) = (ab + \lambda a, 0) + I \end{aligned}$$

as required. □

Given a right completely bounded A-module X and a completely bounded left A-module we can form the following tensor product of these two.

Notation 1.2.12 *Let N denote the operator subspace of $X \widehat{\otimes} Y$ given by*

$$N = \overline{\text{lin}\ \{xa \otimes y - x \otimes ay \mid x \in X,\ a \in A,\ y \in Y\}}$$

and let $X \widehat{\otimes}_A Y = X \widehat{\otimes} Y / N$.

We record the following proposition that is a more general version of the result in Section 6.2.2. The result was originally proved in the quantum groups work as part of Section 6.2.2 and then rewritten as the more general result given here.

Proposition 1.2.13 *Let A be a completely contractive Banach algebra with a left contractive approximate identity. Then the multiplication map $m : A \widehat{\otimes} A \to A$ is a complete quotient map.*

Proof

Fix $n \in \mathbb{N}$ throughout this proof. As m is a completely contractive map it follows that it maps the open unit ball of $\mathbb{M}_n(A \widehat{\otimes} A)$ into the open unit ball of $\mathbb{M}_n(A)$ and so by Proposition A.1.6 we need only show that it maps the open unit ball of $\mathbb{M}_n(A \widehat{\otimes} A)$ onto that of $\mathbb{M}_n(A)$.

We first show that $\mathbb{M}_n(A)$ is an essential left Banach A-module with the operation map $A \widehat{\otimes} \mathbb{M}_n(A) \to \mathbb{M}_n(A)$ given by $x \otimes (y_{ij})_{i,j=1}^n \mapsto (xy_{ij})_{i,j=1}^n$ for $x \in A$ and $(y_{ij})_{i,j=1}^n \in \mathbb{M}_n(A)$. The algebraic relations follow easily and as m is completely contractive and the operator space projective tensor product is a subcross norm (see Definition 1.1.34 and Definition-Theorem 1.1.35) we have

$$\begin{aligned}\left\|(xy_{ij})_{i,j=1}^n\right\|_{\mathbb{M}_n(A)} &= \left\|m_n((x \otimes y_{ij})_{i,j=1}^n)\right\|_{\mathbb{M}_n(A)} \\ &\leqslant \left\|(x \otimes y_{ij})_{i,j=1}^n\right\|_{\mathbb{M}_n(A)} \leqslant \|x\|_A \left\|(y_{ij})_{i,j=1}^n\right\|_{\mathbb{M}_n(A)}\end{aligned}$$

as required for a left Banach A-module. Clearly this is essential as A has a left contractive approximate identity.

Now fix $(y_{ij})_{i,j=1}^n \in \mathbb{M}_n(A)$ and let $\varepsilon > 0$ such that $((1+\varepsilon)y_{ij})_{i,j=1}^n$ is also in the open unit ball of $\mathbb{M}_n(A)$. Let $z_{ij} = (1+\varepsilon)y_{ij}$ for all $1 \leqslant i,j \leqslant n$ and we show there is some element of $\mathbb{M}_n(A \widehat{\otimes} A)$ such that m_n maps this to $(z_{ij})_{i,j=1}^n$. As A has a left contractive approximate identity $(e_\alpha) \in A$, from Cohen's factorisation theorem (see Theorem A.3.3), for any $\delta > 0$ there is some $x \in A$ and $(x_{ij})_{i,j=1}^n \in \mathbb{M}_n(A)$ such that

$$(z_{ij})_{i,j=1}^n = x \cdot (x_{ij})_{i,j=1}^n = m_n((x \otimes x_{ij})_{i,j=1}^n)$$

$$\|x\|_A \leqslant 1, \qquad \|(z_{ij})_{i,j=1}^n - (x_{ij})_{i,j=1}^n\|_{\mathbb{M}_n(A)} < \delta.$$

Now as $\|(z_{ij})_{i,j=1}^n\| < 1$ we have

$$\|(x_{ij})_{i,j=1}^n\|_{\mathbb{M}_n(A)} \leqslant \|(x_{ij})_{i,j=1}^n - (z_{ij})_{i,j=1}^n\|_{\mathbb{M}_n(A)} + \|(z_{ij})_{i,j=1}^n\|_{\mathbb{M}_n(A)} < 1 + \delta$$

and so, as the operator space projective tensor product is a subcross norm, we have

$$\|(x \otimes x_{ij})_{i,j=1}^n\|_{\mathbb{M}_n(A \widehat{\otimes} A)} \leqslant \|x\|_A \|(x_{ij})_{i,j=1}^n\|_{\mathbb{M}_n(A)} < 1 + \delta.$$

As $\delta > 0$ was arbitrary we may assume that $\delta < \varepsilon$ and thus $\dfrac{1+\delta}{1+\varepsilon} < 1$ and so it follows that $\dfrac{1}{1+\varepsilon}(x \otimes x_{ij})_{i,j=1}^n$ is in the open unit ball of $\mathbb{M}_n(A \widehat{\otimes} A)$. Then we have shown that m_n maps this to $(y_{ij})_{i,j=1}^n$ as required. $\square$

The following two lemmas are recorded in Aristov (2004) without proof and in Aristov (2002) with sketch proofs given there. We expand on these here and prove them in full now.

Lemma 1.2.14 *Let I be a closed left ideal in a completely contractive Banach algebra A. Then we have*

$$A/\overline{AI} \cong_{ci} A \widehat{\otimes}_A (A^\flat/I).$$

Proof Let $m^\flat : A \widehat{\otimes} A^\flat \to A$ denote the completely contractive map that extends the multiplication map of A. Let $q_1 : A^\flat \to A^\flat/I$ and let $q_2 : A \to A/\overline{AI}$ be the quotient map. We can show similarly to that of the proof of Lemma 1.2.11 that there exists a completely contractive map $S : A \widehat{\otimes} (A^\flat/I) \to A/\overline{AI}$ such that

$$\begin{array}{ccc} A \widehat{\otimes} A^\flat & \xrightarrow{m^\flat} & A \\ \downarrow{\scriptstyle \mathrm{id}\otimes q_1} & & \downarrow{\scriptstyle q_2} \\ A \widehat{\otimes} (A^\flat/I) & \xrightarrow[S]{} & A/\overline{AI} \end{array}$$

is commutative.

For $a, b \in A$ and $c \in A^\flat$ we have

$$S(a \otimes (b + I)) = S(\mathrm{id} \otimes q_1)(a \otimes b) = q_2(m(a \otimes b)) = q_2(ab) = ab + \overline{AI}$$

and

$$S(ab \otimes (c + I) - a \otimes (bc + I)) = (abc + \overline{AI}) - (abc + \overline{AI}) = 0.$$

So for all $u \in \overline{\mathrm{lin}\ \{ab \otimes (c + I) - a \otimes (bc + I) \mid a, b \in A, c \in A^\flat\}}$ we have shown that $S(u) = 0$. Thus there exists a completely contractive $T : A \widehat{\otimes}_A (A^\flat/I) \to A/\overline{AI}$ with $\|T\|_{cb} = \|S\|_{cb}$ (again using Proposition 1.1.18) such that we have a commutative diagram

$$\begin{array}{ccc} A \widehat{\otimes} (A^\flat/I) & \xrightarrow{S} & A/\overline{AI} \\ \downarrow{\scriptstyle Q} & \nearrow{\scriptstyle T} & \\ A \widehat{\otimes}_A (A^\flat/I) & & \end{array}$$

where $Q : A \widehat{\otimes} (A^\flat/I) \to A \widehat{\otimes}_A (A^\flat/I)$ is the quotient map to the balanced projective tensor product. As S is a completely contractive left A-module homomorphism and Q is a completely contractive left A-module homomorphism it follows that T is also a completely contractive left A-module homomorphism.

Clearly T is surjective as $S(a \otimes e + I) = a + \overline{AI}$ for all $a \in A$.

We now consider the dual map $T^* : (A/\overline{AI})^* \to (A\hat{\otimes}_A(A^\flat/I))^*$ and we show this is surjective. We have for the domain of T^* that

$$\begin{aligned}(A/\overline{AI})^* = (\overline{AI})^\perp &= \left\{\mu \in A^* \;\middle|\; \langle \mu, z\rangle = 0 \;\; \forall\, z \in \overline{AI}\right\}\\ &= \{\mu \in A^* \mid \langle \mu, z\rangle = 0 \;\; \forall\, z \in AI\}\\ &= \{\mu \in A^* \mid \langle \mu, ax\rangle = 0 \;\; \forall\, a \in A,\; x \in I\}\\ &= \{\mu \in A^* \mid \langle \mu \cdot a, x\rangle = 0 \;\; \forall\, a \in A,\; x \in I\}\\ &= \left\{\mu \in A^* \;\middle|\; \mu \cdot a \in I^\perp \;\; \forall\, a \in A\right\}\end{aligned}$$

and for the codomain of T^* let

$$Z = \overline{\operatorname{lin}\; \left\{ab \otimes (c + I) - a \otimes (bc + I) \mid a, b \in A,\; c \in A^\flat\right\}} \subset A \widehat{\otimes} (A^\flat/I).$$

Then we have

$$(A \widehat{\otimes}_A (A^\flat/I))^* = ((A \widehat{\otimes} (A^\flat/I))/Z)^* = Z^\perp$$

where

$$\begin{aligned}Z^\perp &= \left\{\mu \in (A \widehat{\otimes} (A^\flat/I))^* \;\middle|\; \mu(ab \otimes c + I) = \mu(a \otimes b(c + I)) \;\; \forall\, a, b \in A, c \in A^\flat\right\}\\ &= \left\{\mu : A \to (A^\flat/I)^* \;\middle|\; \langle \mu(ab), c + I\rangle = \langle \mu(a), bc + I\rangle \;\; \forall\, a, b \in A, c \in A^\flat\right\}\\ &= \left\{\mu : A \to (A^\flat/I)^* \;\middle|\; \langle \mu(ab), c + I\rangle = \langle \mu(a) \cdot b, c + I\rangle \;\; \forall\, a, b \in A, c \in A^\flat\right\}\\ &= \left\{\mu : A \to (A^\flat/I)^* \;\middle|\; \mu(ab) = \mu(a) \cdot b \;\; \forall\, a, b \in A\right\} = \mathcal{CB}_A(A, (A^\flat/I)^*).\end{aligned}$$

Let $\mu \in (A/\overline{AI})^*$, then $T^*(\mu) \in (A \widehat{\otimes}_A (A^\flat/I))^* \cong_{ci} \mathcal{CB}_A(A, (A^\flat/I)^*)$ and thus for all

$a \in A$ and $b \in A^\flat$ we have

$$\begin{aligned}\langle T^*(\mu)(a), b+I\rangle &= \langle T^*(\mu), a\otimes(b+I)\rangle = \langle \mu, T(a\otimes(b+I))\rangle \\ &= \langle \mu, ab+\overline{AI}\rangle = \langle \mu, ab\rangle = \langle \mu\cdot a, b\rangle = \langle \mu\cdot a, b+I\rangle\end{aligned}$$

giving $T^*(\mu)(a) = \mu\cdot a \in (A^\flat/I)^*$ for all $a \in A$.

Now let $\alpha \in \mathcal{CB}_A(A, (A^\flat/I)^*)$ and we show that there is some $\mu \in (A/\overline{AI})^*$ such that $T^*(\mu) = \alpha$. We can define $\mu : A^2 \to \mathbb{C}$, where we denote $A^2 = \operatorname{lin}\{ab \mid a, b \in A\}$, by $\mu(ab) = \langle \alpha(a), b\rangle = \langle \alpha(a)\cdot b, e\rangle = \langle \alpha(ab), e\rangle$. So for any $c \in A^2$ we have $\mu(c) = \langle \alpha(c), e\rangle$ and μ is well defined as α is well defined. Also

$$|\mu(c)| = |\langle \alpha(c), e\rangle| \leqslant \|\alpha(c)\|\|e\| \leqslant \|\alpha\|\|c\|$$

and so $\|\mu\| \leqslant \|\alpha\|$. We can then use Hahn-Banach to extend this to an element $\mu \in A^*$ with $\|\mu\|_{cb} \leqslant \|\alpha\|_{cb}$. As $\mathcal{B}(A,\mathbb{C}) = \mathcal{CB}(A,\mathbb{C})$ then μ is completely bounded.

Let $a \in A$ and $b \in I$, then we have $\langle ab, \mu\rangle = \langle \alpha(a), b\rangle = 0$ as $\alpha(a) \in I^\perp$ as a subset of $(A^\flat)^*$ and thus $\mu \in (AI)^\perp \cong_{ci} (A/\overline{AI})^*$. We have

$$\langle T^*(\mu)(a), b\rangle = \langle \mu, ab\rangle = \langle \alpha(a), b\rangle$$

and thus $T^*(\mu)(a) = \alpha(a)$ for all $a \in A$ and therefore $T^*(\mu) = \alpha$. So we have shown that T^* is surjective and therefore T must be injective as required.

We have that T is completely contractive and bijective and so there exists a bounded inverse $T^{-1} : A/\overline{AI} \to A \widehat{\otimes}_A (A^\flat/I)$. We can easily see that $T^{-1}(a+\overline{AI}) = a\otimes(e+I)$

and so we have a commutative diagram

$$
\begin{array}{ccc}
A \widehat{\otimes} A^\flat & \xrightarrow{m^\flat} & A \\
\downarrow {\scriptstyle \mathrm{id} \otimes q_1} & & \\
A \widehat{\otimes} (A^\flat / I) & & \downarrow {\scriptstyle q_2} \\
\downarrow {\scriptstyle Q} & & \\
A \widehat{\otimes}_A (A^\flat / I) & \underset{T^{-1}}{\overset{T}{\rightleftarrows}} & A/\overline{AI}.
\end{array}
$$

We show that T is a complete quotient map and then as $\operatorname{Ker} T = 0$ we have that T is a completely isometric isomorphism by Definition 1.1.9 (see also Definition A.1.5). We have that $m^\flat$ is a complete quotient map by following a similar proof to that of Proposition 1.2.13. As q_2 is a complete quotient map it follows that we must have that T is a complete quotient map as required. $\square$

Lemma 1.2.15 *Let A be a biprojective completely contractive algebra and Y a left completely bounded A-module, then $A \widehat{\otimes}_A Y$ is a left projective completely bounded A-module with the operation $a \cdot (b \otimes y) = ab \otimes y$ for all $a, b \in A$ and $y \in Y$.*

Proof

Let $m : A \widehat{\otimes} A \to A$ denote the multiplication map and ${}^\flat m$ the extension of m to $A^\flat \widehat{\otimes} A$. We have that $X = A \widehat{\otimes}_A Y$ is a completely bounded left A-module with operation $a \cdot (b \otimes y) = ab \otimes y = a \otimes b \cdot y$ for $a, b \in A$ and $y \in Y$, that is the module operation on X is given by the map $m \otimes \mathrm{id}_Y : A \widehat{\otimes} A \widehat{\otimes}_A Y \to A \widehat{\otimes}_A Y$. We want to show that $A \widehat{\otimes}_A Y$ is projective as a completely bounded left A-module, that is there is a completely bounded left A-module homomorphism $A \widehat{\otimes}_A Y \to A^\flat \widehat{\otimes} A \widehat{\otimes}_A Y$ that is a right inverse to ${}^\flat m \otimes \mathrm{id}_Y : A^\flat \widehat{\otimes} A \widehat{\otimes}_A Y \to A \widehat{\otimes}_A Y$.

As A is biprojective there is a completely bounded A-bimodule homomorphism from A to $A \widehat{\otimes} A$ that is a right inverse to m. Let $\rho : A \to A^\flat \otimes A$ denote its coextension and define $\tau : A \widehat{\otimes}_A Y \to A^\flat \widehat{\otimes} A \widehat{\otimes}_A Y$ by $a \otimes y \mapsto \rho(a) \otimes y$. We want to show this is well defined. Let $Q : A \widehat{\otimes} Y \to A \widehat{\otimes}_A Y$ be the projection map onto the balanced tensor product, then we have a diagram

$$\begin{array}{ccc} A \widehat{\otimes} Y & \xrightarrow{\rho\otimes \mathrm{id}} & A^\flat \widehat{\otimes} A \widehat{\otimes} Y \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes Q} \\ A \widehat{\otimes}_A Y & \xrightarrow[\tau]{} & A^\flat \widehat{\otimes} A \widehat{\otimes}_A Y. \end{array}$$

Let $u, v \in A \widehat{\otimes} Y$ with $Q(u) = Q(v)$ and we show that $\tau(Q(u)) = \tau(Q(v))$. This is equivalent to showing that if $u \in \operatorname{Ker} Q$ then $(\mathrm{id} \otimes Q)((\rho \otimes \mathrm{id})(u)) = 0$. We have $\operatorname{Ker} Q = \overline{\mathrm{lin}}\, \{ab \otimes y - a \otimes b \cdot y \mid a, b \in A,\ y \in Y\}$ and for $a, b \in A$ and $y \in Y$ we have

$$(\rho\otimes \mathrm{id})(ab\otimes y - a\otimes b\cdot y) = \rho(ab)\otimes y - \rho(a)\otimes b\cdot y = \rho(a)\cdot b\otimes y - \rho(a)\otimes b\cdot y \in \operatorname{Ker}(\mathrm{id}\otimes Q)$$

and thus $(\mathrm{id} \otimes Q)((\rho \otimes \mathrm{id})(ab \otimes y - a \otimes b \cdot y)) = 0$. Then it follows by linearity and continuity that $(\mathrm{id} \otimes Q)((\rho \otimes \mathrm{id})(u)) = 0$ for all $u \in \operatorname{Ker} Q$ as required.

We have that

$$\tau(a \cdot (b \otimes y)) = \tau(ab \otimes y) = \rho(ab) \otimes y = (a \cdot \rho(b)) \otimes y = a \cdot (\rho(b) \otimes y) = a \cdot \tau(b \otimes y)$$

so τ is a completely contractive left A-module homomorphism.

Finally we have

$$(m \otimes \mathrm{id}_Y) \circ \tau \circ Q = (m \otimes \mathrm{id}_Y) \circ (\mathrm{id} \otimes Q) \circ (\rho \otimes \mathrm{id}_Y) = (\mathrm{id} \otimes Q)(m \circ \rho \otimes \mathrm{id}_Y) = \mathrm{id} \otimes Q$$

and thus $(m \otimes \mathrm{id}_Y) \circ \tau = \mathrm{id}$ as required. □

1.3 One-parameter Groups and Smearing

In this section we introduce one-parameter groups and smearing techniques on Banach spaces, C*-algebras and von Neumann algebras. These techniques will be used throughout this thesis. They will be used when we discuss the modular automorphism groups for weights, in Chapter 2 when we introduce the scaling group and in Chapter 4 onwards when we investigate the $\mathrm{L}^1_\sharp$-algebra of a locally compact quantum group.

1.3.1 One-parameter Groups

We introduce one-parameter groups, analytic functions with values in Banach spaces, analytic extensions of one-parameter groups and further properties of these objects in this section. The standard reference for this section is Ciorănescu *et al.* (1976) and Kustermans (1997b). See also Kustermans' notes in Applebaum *et al.* (2005).

Definition 1.3.1 *Let X denote a Banach space, then a* ***one-parameter group on*** *X is a map $\sigma : \mathbb{R} \to \mathcal{B}(X)$, where we denote $\sigma_t = \sigma(t)$ for convenience, such that $\sigma_{t+s} = \sigma_t \circ \sigma_s$ for all $t, s \in \mathbb{R}$, $\sigma_0 = \mathrm{id}$ and $\|\sigma_t\| \leqslant 1$ for all $t \in \mathbb{R}$. If X is a Banach algebra and $\sigma : \mathbb{R} \to \mathrm{Aut}(X)$ we say it is a* ***one-parameter group of automorphisms on*** *X and similarly if X is a Banach $*$-algebra and $\sigma : \mathbb{R} \to \mathrm{Aut}^*(X)$ we say it is a* ***one-parameter group of $*$-automorphisms on*** *X.*

We say σ is ***norm continuous*** *(****weak continuous****,* ***weak*-continuous****) if for all $x \in X$ the map $\mathbb{R} \to X$ given by $t \mapsto \sigma_t(x)$ is continuous with respect to the norm topology (weak topology, weak*-topology) on X. If X is a von Neumann algebra then we can define similar properties with respect to any of the weak operator topologies.*

Note it follows that σ_t is an isometry and invertible for all $t \in \mathbb{R}$ with $(\sigma_t)^{-1} = \sigma_{-t}$. This follows as for all $t \in \mathbb{R}$ we have $\sigma_t \circ \sigma_{-t} = \sigma_0 = \mathrm{id} = \sigma_{-t} \circ \sigma_t$ and

$$\|\sigma_t(x)\| \leqslant \|x\| = \|\sigma_{-t}(\sigma_t(x))\| \leqslant \|\sigma_t(x)\|$$

and thus we have equality throughout.

Remark 1.3.2 *We have mentioned several possibilities of continuity for one-parameter groups in the above definition. We can have norm and weak continuity for a Banach space, for a Banach space with a predual we can have weak* continuity and for a von Neumann algebra we have σ-strong*, σ-strong, σ-weak, strong*, strong and weak operator continuity. In this thesis we will mostly use σ-weak (or equivalently weak*) continuity for von Neumann algebras and norm continuity otherwise, however we mention that the Kustermans and Vaes' defined one-parameter groups on von Neumann algebraic quantum groups with respect to the σ-strong* topology. We will refer to a one-parameter group in this section when the results are not dependent on the choice of topology, however we will always assume some continuity property on all one-parameter groups.*

We wish to discuss the analytic continuation of one-parameter groups, in order to do so we now give details of the analyticity of functions from a complex domain D into a Banach space X. We see from the following lemma that in fact analyticity is the same whether we are working with the norm, weak or weak* topologies on X.

Notation 1.3.3 *Let $z \in \mathbb{C}\backslash\mathbb{R}$, then we denote*

$$S(z) = \begin{cases} \{w \in \mathbb{C} \mid \mathfrak{Im}\ w \in [0, \mathfrak{Im}\ z]\} & \text{if } \mathfrak{Im}\ z > 0 \\ \{w \in \mathbb{C} \mid \mathfrak{Im}\ w \in [\mathfrak{Im}\ z, 0]\} & \text{if } \mathfrak{Im}\ z < 0 \end{cases}$$

and $S(z)^o$ is the interior of $S(z)$.

The reader is referred to A.1 in Takesaki (2003b) for a proof of the following lemma.

Lemma 1.3.4 *Let X be a Banach space, let $D \subset \mathbb{C}$ be a complex domain (i.e. an open connected subset of $\mathbb{C}$) and let $f : D \to X$. Then the following are equivalent:*

(i) For all $w_0 \in D$ and $\delta > 0$ such that $B_\delta(w_0) \subset S(z)^o$ (where $B_\delta(w_0)$ is the open ball of radius δ around w_0) there is a sequence $(x_n)_{n=0}^{\infty} \subset X$ such that for all w with

$|w - w_0| < \delta$ the following is norm convergent

$$f(w) = \sum_{n=0}^{\infty} (w - w_0)^n x_n;$$

(ii) For all $\omega \in X^*$ we have a holomorphic function $D \to \mathbb{C}$ given by $w \mapsto \langle f(w), \omega \rangle$;

(iii) Let $Y \subset X^*$ be a norm closed subspace such that for all $x \in X$ we have

$$\|x\| = \sup \{ |\langle x, \omega \rangle| \mid \omega \in Y, \ \|\omega\| \leqslant 1 \} .$$

For each $\omega \in Y$ we have a holomorphic function $D \to \mathbb{C}$ given by $w \mapsto \langle f(w), \omega \rangle$.

Definition 1.3.5 *Let X be a Banach space and $D \subset \mathbb{C}$ be a complex domain, then a function $f : D \to X$ is an* ***analytic function*** *if any of the equivalent conditions in Lemma 1.3.4 hold.*

The following lemma belongs to complex analysis, we prove it here as it will be useful in this section.

Lemma 1.3.6 *Let $F : S(z) \to \mathbb{C}$ be a function that is continuous, analytic on $S(z)^o$ and $F(t) = 0$ for all $t \in \mathbb{R}$, then $F = 0$ everywhere.*

Proof

We may assume without loss of generality that $\mathfrak{Im}\ z > 0$ and we define a map $G : S(z) \cup S(-z) \to \mathbb{C}$ by

$$G(w) = \begin{cases} \overline{F(\overline{w})} & \text{if } \mathfrak{Im}\ w < 0 \\ F(w) & \text{if } \mathfrak{Im}\ w \geqslant 0. \end{cases}$$

Clearly G is continuous on $S(z)$ and $S(-z)$ and thus everywhere. We also have $G(t) = F(0) = 0$ for all $t \in \mathbb{R}$ and so by the Schwarz reflection principle (see Theorem 11.14 in

Rudin (1987)) this is analytic on $\mathrm{Dom}(G)$. From Theorem 10.18 in Rudin (1987) we have that the set $Z(G) = \{w \in \mathrm{Dom}(G) \mid G(w) = 0\}$ is either $\mathrm{Dom}(G)$ or there is no limit point of $Z(G)$ in $Z(G)$. However any $t \in \mathbb{R}$ is a limit point of $Z(G)$ with $t \in Z(G)$ and so we must have $Z(G) = \mathrm{Dom}(G)$. So we have shown that $G(w) = 0$ for all $w \in \mathrm{Dom}(G)$ and thus $F = 0$ as required. □

Note that we don't specify a continuity property for the one-parameter group in the following lemma, however for there to exist such an F in the lemma then σ would need to satisfy such a continuity property also.

Lemma 1.3.7 *Let X be a Banach space, $x \in X$, $\sigma : \mathbb{R} \to \mathcal{B}(X)$ a one-parameter group on X and $z \in \mathbb{C}\backslash\mathbb{R}$. Say there exists a function $F : S(z) \to X$ such that (i) F is continuous with respect to either the norm topology, weak topology or (if X has a predual) a weak*-topology on X, (ii) F is analytic on $S(z)$ and (iii) $F(t) = \sigma_t(x)$ for all $t \in \mathbb{R}$. Then F is necessarily unique.*

Proof

Fix $z \in \mathbb{C}$ and let $F_1, F_2 : S(z) \to X$ be two functions satisfying these conditions. Let $F : S(z) \to X$ be the map $F = F_1 - F_2$, then clearly F is continuous, analytic and we have $F(t) = 0$ for all $t \in \mathbb{R}$. Let $\omega \in X^*$ and we consider the map $G_\omega : S(z) \to \mathbb{C}$ given by $G_\omega(w) = \langle F(w), \omega\rangle$ for all $w \in S(z)$. Clearly $G_\omega(t) = 0$ for all $t \in \mathbb{R}$. As F is analytic it follows from Lemma 1.3.4 that G_ω is analytic.

Say F_1 and F_2 are norm or weak continuous, then G_ω is easily seen to be continuous for all $\omega \in X^*$ and so it follows from Lemma 1.3.6 that $G_\omega = 0$ for all $\omega \in X^*$ and so $F_1 = F_2$ as required. If there exists a predual X_* of X such that $X \cong_i (X_*)^*$ then for all $\omega \in X_*$ we have G_ω is continuous and as this separates X we can similarly conclude that $F_1 = F_2$. □

Finally we can introduce the analytic continuation of a one-parameter group.

Definition 1.3.8 *Let X be a Banach space and σ a one-parameter group on X. Then we define the following:*

(i) For any $z \in \mathbb{C}$ we define $\mathrm{Dom}(\sigma_z)$ as the set of $x \in X$ such that there exists a function (necessarily unique by the previous lemma) $F : S(z) \to X$ such that (i) F is continuous with respect to the appropriate topology on X, (ii) F is analytic on $S(z)^o$ and (iii) $F(t) = \sigma_t(x)$ for all $t \in \mathbb{R}$.

(ii) For $x \in \mathrm{Dom}(\sigma_z)$ we define $\sigma_z(x) = F(z)$ for $F : S(z) \to X$ the function in (i).

We say σ_z is the ***analytic extension*** *of σ at $z \in \mathbb{C}$.*

Proposition 1.3.9 *Let X be a Banach space, σ a one-parameter group on X, $z \in \mathbb{C}$ and $x \in \mathrm{Dom}(\sigma_z)$. Then we have:*

(i) For $w \in S(z)$ we have $\mathrm{Dom}(\sigma_w) \supset \mathrm{Dom}(\sigma_z)$ and if $\mathfrak{Im}\ w = \mathfrak{Im}\ z$ we have $\mathrm{Dom}(\sigma_z) = \mathrm{Dom}(\sigma_w)$;

(ii) For all $t \in \mathbb{R}$ we have $\sigma_t(\sigma_z(x)) = \sigma_{z+t}(x) = \sigma_z(\sigma_t(x))$;

(iii) σ_z is injective.

Proof (Sketch)

Part (i) follows by considering the restriction of the function from Definition 1.3.8 from $S(z)$ to $S(w)$. Part (ii) follows from considering the function $G : S(z) \to X$ given by $w \mapsto \sigma_t(\sigma_w(x)) - \sigma_{t+w}(x)$. For part (iii) let $x, y \in \mathrm{Dom}(\sigma_z)$ such that $\sigma_z(x) = \sigma_z(y)$ and let $F, G : S(z) \to X$ be the functions that are are continuous, analytic on $S(z)^o$, $F(t) = \sigma_t(x)$ for all $t \in \mathbb{R}$ and $G(t) = \sigma_t(y)$ for all $t \in \mathbb{R}$. Then from (ii) we have $\sigma_{t+z}(x) = \sigma_t(\sigma_z(x)) = \sigma_t(\sigma_z(y)) = \sigma_{t+z}(y)$ for all $t \in \mathbb{R}$. By considering the function $H : S(z) \to X$ given by $H(w) = F(z-w) - G(z-w)$ we find that we must have $F = G$, and so $x = F(0) = G(0) = y$ as required. □

Proposition 1.3.10 *Let X be a Banach algebra, σ a one-parameter group of automorphisms on X and $z \in \mathbb{C}$, then $\mathrm{Dom}(\sigma_z)$ is a subalgebra of X and σ_z acts as a homomorphism on $\mathrm{Dom}(\sigma_z)$. If in addition X is a Banach $*$-algebra and σ is a one-parameter group of $*$-automorphisms then $\mathrm{Dom}(\sigma_z)^* := \{x \in X \mid x^* \in \mathrm{Dom}(\sigma_z)\} = \mathrm{Dom}(\sigma_{\overline{z}})$ and $\sigma_z(x)^* = \sigma_{\overline{z}}(x^*)$ for all $x \in \mathrm{Dom}(\sigma_z)$.*

Proof

Let $x, y \in \mathrm{Dom}(\sigma_z)$ and let $F, G : S(z) \to X$ be the functions that are continuous, analytic on $S(z)^o$, $F(t) = \sigma_t(x)$ for all $t \in \mathbb{R}$ and $G(t) = \sigma_t(y)$ for all $t \in \mathbb{R}$ and thus $\sigma_z(x) = F(z)$ and $\sigma_z(y) = G(z)$. Let $H : S(z) \to X$ be given by $H(w) = F(w)G(w)$ for all $w \in S(z)$, then H is a product of continuous functions and a product of analytic functions on $S(z)^o$ and so is continuous and analytic on $S(z)^o$. Also we have $H(t) = F(t)G(t) = \sigma_t(x)\sigma_t(y) = \sigma_t(xy)$ for all $t \in \mathbb{R}$. So $xy \in \mathrm{Dom}(\sigma_z)$ with $\sigma_z(xy) = H(z) = F(z)G(z) = \sigma_z(x)\sigma_z(y)$.

Now let X be a Banach $*$-algebra with σ a one-parameter group of $*$-automorphisms. Let $x \in \mathrm{Dom}(\sigma_z)$, then there is some $F : S(z) \to X$ that is continuous, analytic on $S(z)^o$ and $F(t) = \sigma_t(x)$ for all $t \in \mathbb{R}$. Then consider the map $G : S(\overline{z}) \to X$ given by $w \mapsto F(\overline{w})^*$. It is easy to show that G is continuous and analytic on $S(\overline{z})^o$ and $G(t) = F(t)^* = \sigma_t(x)^* = \sigma_t(x^*)$ for all $t \in \mathbb{R}$. So we have $x^* \in \mathrm{Dom}(\sigma_{\overline{z}})$ with $\sigma_{\overline{z}}(x^*) = G(\overline{z}) = F(z)^* = \sigma_z(x)^*$. □

Proposition 1.3.11 *Let σ, σ' denote two one-parameter groups of $*$-automorphisms on a $*$-algebra M. Then $\sigma = \sigma'$ if and only if $\sigma_z = \sigma'_z$ for any $z = ti \in \mathbb{C}$ with $t \neq 0$.*

We now examine the tensor product of one-parameter groups to finish this subsection. Fix norm continuous one-parameters groups σ and τ on Banach spaces X and Y respectively throughout this section and fix a subcross norm $\|\cdot\|_\mu$ on $X \odot Y$ and let $X \otimes_\mu Y$ denote the completion of $X \odot Y$ with respect to this norm. Then we define a one-parameter group $(\sigma \otimes \tau) : \mathbb{R} \to \mathcal{B}(X \otimes_\mu Y)$ in the rest of section. This work is largely influenced by that of Section 4 in Kustermans (1997b).

For fixed $t \in \mathbb{R}$ we have $\sigma_t \in \mathcal{B}(X)$ and $\tau_t \in \mathcal{B}(Y)$ and so we can consider $\sigma_t \odot \tau_t : X \odot Y \to X \odot Y$ for any $t \in \mathbb{R}$. We will assume that σ and τ are such that the map $\sigma_t \odot \tau_t$ is continuous with respect to the $\|\cdot\|_\mu$ and has norm less than 1.

Definition 1.3.12 *For all $t \in \mathbb{R}$ let $(\sigma \otimes \tau)_t : X \otimes Y \to X \otimes Y$ be the unique continuous linear extension of $\sigma_t \odot \tau_t$.*

For any $z \in \mathbb{C}$ we have (unbounded) linear maps $\sigma_z : X \to X$ and $\tau_z : Y \to Y$ defined in Definition 1.3.8. We consider the map $\sigma_z \odot \tau_z : \mathrm{Dom}(\sigma_z) \odot \mathrm{Dom}(\tau_z) \to X \odot Y$ such that $x \otimes y \mapsto \sigma_z(x) \otimes \tau_z(y)$ and we have the following.

Proposition 1.3.13 *For all $z \in \mathbb{C}$ the map $\sigma_z \odot \tau_z$ given above is closable with closure equal to the analytic extension $(\sigma \otimes \tau)_z$ of $\sigma \otimes \tau$ at z (see Definition 1.3.8).*

1.3.2 Smearing of a One-parameter Group on a Banach space

We now consider smearing on Banach space in this section, this will be very important in this thesis and is one of the key techniques that we will use frequently. Throughout this section let X be a Banach space and $\sigma : \mathbb{R} \to \mathrm{Aut}(X)$ be a one-parameter group of automorphisms on X that is norm continuous.

Fix $x \in X$ and consider the function $f : \mathbb{R} \to X$ given by

$$t \mapsto \frac{n}{\sqrt{\pi}} e^{-n^2t^2} \sigma_t(x).$$

Then this is continuous with respect to the norm topology on X. Also we have

$$\int_{\mathbb{R}} \|f(t)\| \, dt = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \|\sigma_t(x)\| \, dt = \|x\|$$

where we've used the Gaussian integral formula

$$\int_{\mathbb{R}} e^{-a(x-b)^2} \, dx = \sqrt{\frac{\pi}{a}} \tag{1.4}$$

and that σ_t is an isometry for all $t \in \mathbb{R}$. It follows by Proposition A.6.3 that there is a unique $x(n) \in X$ that is the weak integral of this function f so we have the following definition.

Definition 1.3.14 *Let $n \in \mathbb{N}$ and we let $x(n)$ denote the element in X that is the weak integral of the function $\mathbb{R} \to X$ given by $t \mapsto \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \sigma_t(x)\, dt$. Then we can define a map $X \to X$ given by $x \mapsto x(n)$ such that for all $x \in X$ and $\omega \in X^*$ we have*

$$\langle x(n), \omega \rangle = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \langle \sigma_t(x), \omega \rangle\, dt.$$

*We call $x(n)$ the **smear** of x with respect to $n \in \mathbb{N}$.*

Proposition 1.3.15 *Let $x \in X$, then the sequence $(x(n))_{n=1}^{\infty}$ has norm limit $x \in X$.*

Proof

Using the Gaussian integral formula (1.4) we have

$$\|x(n) - x\| = \left\| \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} (\sigma_t(x) - x)\, dt \right\| \leqslant \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \|\sigma_t(x) - x\|\, dt.$$

Now we define a map $f : \mathbb{R} \to \mathbb{R}^+$ given by $t \mapsto \|\sigma_t(x) - x\|$. Then f is continuous as σ is norm continuous, $f(0) = 0$ and $f(t) \leqslant 2\|x\|$ for all $t \in \mathbb{R}$.

Fix $\varepsilon > 0$. As $f(0) = 0$ there is some $\delta > 0$ such that $f(t) < \frac{\varepsilon}{2}$ for all $t \in \mathbb{R}$ with $|t| < \delta$. Furthermore there exists some $N \in \mathbb{N}$ such that $\frac{n}{\sqrt{\pi}} \int_{\mathbb{R}\setminus[-\delta,\delta]} e^{-n^2t^2}\, dt < \frac{\varepsilon}{4\|x\|}$ for all $n \geqslant N$. Then putting all this together and using Equation (1.5) we have

$$\|x(n) - x\| \leqslant \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}\setminus[-\delta,\delta]} e^{-n^2t^2} 2\|x\|\, dt + \frac{n}{\sqrt{\pi}} \int_{[-\delta,\delta]} e^{-n^2t^2} f(t)\, dt < \varepsilon$$

for all $n \geqslant N$ where we've used that $\int_{[-\delta,\delta]} e^{-n^2t^2}\, dt < \frac{\sqrt{\pi}}{n}$ for all $\delta > 0$. □

1.3.3 Smearing of a One-parameter Group on a von Neumann algebra

We now consider smearing on von Neumann algebras in this section. Throughout this section let M be a von Neumann algebra, M_* its unique predual and $\sigma : \mathbb{R} \to \mathrm{Aut}(M)$ be a one-parameter group of $*$-automorphisms on M that is σ-weakly continuous.

Similarly to the previous section for $x \in M$ we can consider the function $f : \mathbb{R} \to M$ given by

$$t \mapsto \frac{n}{\sqrt{\pi}} e^{-n^2t^2} \sigma_t(x)$$

that is continuous with respect to the σ-weak topology on M. Also from the Gaussian integral formula we have $\int_{\mathbb{R}} \|f(t)\|\, dt = \|x\|$ and from Proposition A.6.4 we can define the smear $x(n)$ similarly to Definition 1.3.14.

Proposition 1.3.16 *For $x \in M$ the sequence $x(n)$ has weak*-limit x.*

Proof (Sketch)

We sketch the proof here as it is very similar to the proof of Proposition 1.3.15. For $\omega \in M_*$ we have

$$|\langle x(n), \omega\rangle - \langle x, \omega\rangle| \leqslant \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \, |\langle \sigma_t(x) - x, \omega\rangle| \, dt.$$

We define a map $f : \mathbb{R} \to \mathbb{R}^+$ by $t \mapsto |\langle \sigma_t(x) - x, \omega\rangle|$, then f is continuous as σ is σ-weakly continuous and for all $t \in \mathbb{R}$ we have $f(t) \leqslant 2\|\omega\|\|x\|$. Finally fix $\varepsilon > 0$. Then the rest of the proof follows verbatim from that of Proposition 1.3.15 but we pick $N \in \mathbb{N}$ such that $\frac{n}{\sqrt{\pi}} \int_{\mathbb{R}\setminus[-\delta,\delta]} e^{-n^2t^2} f(t)\, dt < \frac{\varepsilon}{4\|\omega\|\|x\|}$. $\square$

Theorem 1.3.17 *Let $x \in M$ then for all $z \in \mathbb{C}$ we have that $x(n) \in \mathrm{Dom}(\sigma_z)$ and furthermore*

$$\sigma_z(x(n)) = \frac{n}{\sqrt{\pi}} \int e^{-n^2(t-z)^2} \sigma_t(x)\, dt;$$

and $\|\sigma_z(x(n))\| \leqslant e^{n^2(\mathfrak{Im}\, z)^2} \|x\|$.

Proof

Fix $z \in \mathbb{C}$ and consider the map $f : \mathbb{R} \to M$ given by

$$t \mapsto \frac{n}{\sqrt{\pi}} e^{-n^2(t-z)^2} \sigma_t(x). \tag{1.5}$$

For $a, b \in \mathbb{R}$ we have $\left| e^{-n^2(a+ib)^2} \right| = \left| e^{-n^2(a^2+2iab-b^2)} \right| = e^{-n^2a^2+n^2b^2}$ and so setting $a = t - \mathfrak{Re}\ z$ and $b = -\mathfrak{Im}\ z$ we get

$$\int_{\mathbb{R}} \|f(t)\|\, dt = e^{n^2(\mathfrak{Im}\ z)^2} \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \|x\|\, dt = e^{n^2(\mathfrak{Im}\ z)^2} \|x\|.$$

Then we have from Proposition A.6.4 that f is weak* integrable and so we have

$$\frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2(t-z)^2} \sigma_t(x)\, dt \in M$$

for all $z \in \mathbb{C}$.

So we can define a map $F : \mathbb{C} \to M$ given by

$$z \mapsto \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2(t-z)^2} \sigma_t(x)\, dt$$

Fix $s \in \mathbb{R}$, then for all $\omega \in M_*$ we have

$$\begin{aligned} \langle F(s), \omega \rangle &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2(t-s)^2} \langle \sigma_t(x), \omega \rangle\, dt \\ &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \langle \sigma_{s+t}(x), \omega \rangle\, dt = \langle \sigma_s(x(n)), \omega \rangle \end{aligned}$$

and thus $F(s) = \sigma_s(x(n))$ for all $s \in \mathbb{R}$.

We show that F is analytic on $\mathbb{C}$. In particular we show that for all $\omega \in M_*$ that the map $\mathbb{C} \to \mathbb{C}$ given by $z \mapsto \langle F(z), \omega \rangle$ is analytic and then by Lemma 1.3.4 we have that F is analytic on $\mathbb{C}$ and thus also continuous on $\mathbb{C}$. Fix $\omega \in M_*$ and define $g : \mathbb{C} \to \mathbb{C}$ as the map

$$z \mapsto \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} e^{2n^2tz} \langle \sigma_t(x), \omega \rangle \, dt$$

and we have $\langle F(z), \omega \rangle = e^{-n^2z^2} g(z)$ and so we need only show that g is analytic on $\mathbb{C}$. Expanding e^{2n^2tz} we have

$$g(z) = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \left(\sum_{k=0}^{\infty} \frac{(2n^2tz)^k}{k!} \right) \langle \sigma_t(x), \omega \rangle \, dt$$

and so letting $N \in \mathbb{N}_0$ be arbitrary and using that we can interchange infinite sums and integrals for positive numbers (see Theorem 1.27 in Rudin (1987)) we have

$$\begin{aligned}
&\left| \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} e^{2n^2tz} \langle \sigma_t(x), \omega \rangle \, dt - \sum_{k=0}^{N} \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \frac{(2n^2tz)^k}{k!} \langle \sigma_t(x), \omega \rangle \, dt \right| \\
&\qquad = \left| \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \left(\sum_{k=N+1}^{\infty} \frac{(2n^2tz)^k}{k!} \right) \langle \sigma_t(x), \omega \rangle \, dt \right| \\
&\qquad \leqslant \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \left(\sum_{k=N+1}^{\infty} \frac{(2n^2)^k \, |tz|^k}{k!} \right) |\langle \sigma_t(x), \omega \rangle| \, dt \\
&\qquad \leqslant \frac{n\|x\|\|\omega\|}{\sqrt{\pi}} \sum_{k=N+1}^{\infty} \frac{(2n^2)^k}{k!} |z|^k \int_{\mathbb{R}} e^{-n^2t^2} |t|^k \, dt \\
&\qquad = \frac{2n\|x\|\|\omega\|}{\sqrt{\pi}} \sum_{k=N+1}^{\infty} \frac{(2n^2)^k}{k!} |z|^k \int_0^{\infty} e^{-n^2t^2} t^k \, dt.
\end{aligned}$$

We define $a_k = \frac{2n\|x\|\|\omega\|}{\sqrt{\pi}} \frac{(2n^2)^k}{k!} \int_0^{\infty} e^{-n^2t^2} t^k \, dt$ and we find

$$\begin{aligned}
\sum_{k=0}^{\infty} a_k |z|^k &= \frac{2n\|x\|\|\omega\|}{\sqrt{\pi}} \int_0^{\infty} e^{-n^2t^2} e^{2n^2|z|t} \, dt \leqslant \frac{2n\|x\|\|\omega\|}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} e^{2n^2|z|t} \, dt \\
&= \frac{2n\|x\|\|\omega\|}{\sqrt{\pi}} e^{n^2|z|^2} \int_{\mathbb{R}} e^{-n^2(t-|z|)^2} \, dt = 2\|x\|\|\omega\| e^{n^2|z|^2}
\end{aligned}$$

where we've used the Gaussian integral formula (1.4). As $e^{n^2|z|^2}$ is finite for all fixed $z \in \mathbb{C}$ it follows that for all $\varepsilon > 0$ and fixed $z \in \mathbb{C}$ there is some $N \in \mathbb{N}$ such that

$\sum_{k=N}^{\infty} a_k |z|^k < \varepsilon$. It then follows from the derivation above that for all $\varepsilon > 0$ and $z \in \mathbb{C}$ there is some $N \in \mathbb{N}$ such that

$$\left| g(z) - \sum_{k=0}^{N} \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2 t^2} \frac{(2n^2 tz)^k}{k!} \langle \sigma_t(x), \omega \rangle \, dt \right| < \varepsilon$$

and so g is analytic.

It follows that for any $z \in \mathbb{C}$ and $n \in \mathbb{N}$ we have $x(n) \in \mathrm{Dom}(\sigma_z)$ with

$$\sigma_z(x) = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2(t-z)^2} \sigma_t(x) \, dt$$

and from the first paragraph we have $\|\sigma_z(x(n))\| \leqslant e^{n^2(\mathfrak{Im}\, z)^2} \|x\|$. □

We omit the proofs of the following two results and refer the reader to Kustermans (1997b) and Ciorănescu *et al.* (1976).

Proposition 1.3.18 *For all $z \in \mathbb{C}$ the map σ_z is densely defined, has dense range and is closed in the σ-weak topology (see Definition A.2.2).*

Proposition 1.3.19 *Let X be a Banach space, let Y denote a dense subspace of X, let σ denote a one-parameter group on X and let $z \in \mathbb{C}$. Then the set $\{x(n) \mid x \in Y,\ n \in \mathbb{N}\}$ is a core for σ_z.*

Proposition 1.3.20 *Let M be a von Neumann algebra, $z \in \mathbb{C}$ and $x \in \mathrm{Dom}(\sigma_z) \subset M$. Then we have the following:*

(i) $\sigma_z(x(n)) = \sigma_z(x)(n)$;

(ii) Any normal linear or antilinear map $T : M \to M$ that commutes with σ_t for all $t \in \mathbb{R}$ satisfies $T(x(n)) = T(x)(n)$ for all $n \in \mathbb{N}$.

Proof

Let $\omega \in M_*$ and consider the map $F : S(z) \to M$ given by

$$z \mapsto \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2(t-z)^2} \sigma_t(x)\, dt.$$

Using similar methods from the proof of Theorem 1.3.17 we can show that this is continuous and analytic on $S(z)^o$. Also for $s \in \mathbb{R}$ we have

$$\langle \sigma_s(x)(n), \omega \rangle = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \langle \sigma_{t+s}(x), \omega \rangle\, dt = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2(t-s)^2} \langle \sigma_t(x), \omega \rangle\, dt = F(s)$$

from which (i) follows.

Let $T : M \to M$ be a normal map such that $T \circ \sigma_t = \sigma_t \circ T$ for all $t \in \mathbb{R}$. Let $\omega \in M_*$ and we have

$$\begin{aligned} \langle (Tx)(n), \omega \rangle &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \langle \tau_t(Tx), \omega \rangle\, dt = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \langle \tau_t(x), T_*(\omega) \rangle\, dt \\ &= \langle x(n), T_*(\omega) \rangle = \langle T(x(n)), \omega \rangle \end{aligned}$$

and we have (ii). □

The following is highly important for us as it gives us a notion of smearing in the predual M_*.

Theorem 1.3.21 *Let M be a von Neumann algebra $x \in M$, $n \in \mathbb{N}$ and define a map $\Phi(n) : M \to M$ by $x \mapsto x(n)$. Then $\Phi(n)$ is contractive and normal. Furthermore if we consider M with its natural operator space structure as a von Neumann algebra, then $\Phi(n)$ is completely contractive.*

Proof

Using that σ_t is an isometry we can show that for all $\omega \in M_*$ we have $|\langle x(n), \omega \rangle| \leqslant \|x\|\|\omega\|$ and so $\|x(n)\| \leqslant \|x\|$, i.e. $\Phi(n)$ is a contraction.

Let $(x_\alpha) \subset M$ with σ-weak limit $x \in M$. As σ_t is normal for all $t \in \mathbb{R}$ then for all $\omega \in M_*$ and $t \in \mathbb{R}$ we have $\langle x_\alpha, \omega \circ \sigma_t \rangle \to \langle x, \omega \circ \sigma_t \rangle$. It then follows that

$$|\langle x_\alpha(n), \omega \rangle - \langle x_\alpha(n), \omega \rangle| \leqslant \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2 t^2} \, |\langle x_\alpha - x, \omega \circ \sigma_t \rangle| \, dt \to 0$$

and so $\Phi(n)$ is normal.

Let M have the usual operator space structure as a von Neumann algebra from Example 1.1.7. Let $m \in \mathbb{N}$ and consider the map $\Phi(n)_m : \mathbb{M}_m(M) \to \mathbb{M}_m(M)$ given by Definition 1.1.9. We have that σ_t is a $*$-automorphism for all $t \in \mathbb{R}$ and thus a complete isometry by Proposition 1.1.10. Let $(x_{ij})_{i,j=1}^m \in \mathbb{M}_m(M)$. Then using that $\mathbb{M}_m(M) = \mathbb{M}_m((M_*)^*) \cong_{ci} \mathcal{CB}(M_*, \mathbb{M}_m)$ and the operator space structure on the dual (and thus on the predual) we have

$$\begin{aligned}\left(\Phi(n)_m \left((x_{ij})_{i,j=1}^m\right)\right)(\omega) &= \left(\langle x_{ij}(n), \omega \rangle\right)_{i,j=1}^m = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2 t^2} \left(\langle \sigma_t(x_{ij}), \omega \rangle\right)_{i,j=1}^m \, dt \\ &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2 t^2} \left((\sigma_t(x_{ij}))_{i,j=1}^m\right)(\omega) \, dt = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2 t^2} \left((\sigma_t)_m \left((x_{ij})_{i,j=1}^m\right)\right)(\omega) \, dt\end{aligned}$$

where we used that $\left(\int_{\mathbb{R}} f_{ij}(t) \, dt\right)_{i,j=1}^n = \int_{\mathbb{R}} (f_{ij}(t))_{i,j=1}^n \, dt$. As this holds for any $\omega \in M_*$ we read off that

$$\Phi(n)_m \left((x_{ij})_{i,j=1}^m\right) = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2 t^2} (\sigma_t)_m \left((x_{ij})_{i,j=1}^m\right) \, dt.$$

Now we calculate the norm to get

$$\left\|\Phi(n)_m \left((x_{ij})_{i,j=1}^m\right)\right\| \leqslant \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2 t^2} \left\|(\sigma_t)_m \left((x_{ij})_{i,j=1}^m\right)\right\| dt \leqslant \left\|(x_{ij})_{i,j=1}^m\right\|$$

and so $\Phi(n)_m$ is contractive for all $m \in \mathbb{N}$. □

1.4 Weight Theory

Let μ be a measure on the σ-algebra generated by the open sets on a locally compact space Ω. Then we can consider a map $\phi : C_0(\Omega)^+ \to [0, \infty]$ on the continuous positive functions on Ω given by $f \mapsto \int_\Omega f \, d\mu$. On the other hand, given any positive linear functional $\phi \in C_0(\mathbb{G})^*_+$ we know there is some measure μ such that $\phi(f) = \int_\Omega f \, d\mu$ so for the "unbounded" functionals we might want to consider $\phi : C_0(\mathbb{G})^+ \to [0, \infty]$.

In this section we will consider weights which are the non-commutative analogue of such "unbounded" functionals on arbitrary C*-algebras and von Neumann algebras. These are important for the development of C*-algebras and von Neumann algebras in general and will be needed in order to define locally compact quantum groups in the next chapter.

Proofs are scarce in this section as it would lead us too far astray to include them here. We refer the reader to Takesaki (2003b), Strătilă *et al.* (1979), Combes (1968), Kustermans & Vaes (1999) and Kustermans (1997a).

Notation 1.4.1 *We denote by* $[0, \infty]$ *the set* $[0, \infty) \cup \{\infty\}$*. We have a totally ordered set with the usual order on* $[0, \infty)$ *and by letting* $a < \infty$ *for all* $a \in [0, \infty)$*. We also define addition and multiplication on* $[0, \infty]$ *with the usual operations on* $[0, \infty)$ *and by letting* $a + \infty = \infty + a = \infty$ *for all* $a \in [0, \infty]$, $0 \cdot \infty = \infty \cdot 0 = 0$ *and* $a \cdot \infty = \infty \cdot a = \infty$ *for* $a \in (0, \infty]$.

1.4.1 Weights on C*-algebras and von Neumann algebras

Throughout this section fix a C*-algebra A and a von Neumann algebra M.

Definition 1.4.2 *A map* $\phi : A^+ \to [0, \infty]$ *is called a* ***weight*** *if for all* $x, y \in A^+$ *and* $\lambda \in \mathbb{R}^+$ *we have* $\phi(x + y) = \phi(x) + \phi(y)$ *and* $\phi(\lambda x) = \lambda \phi(x)$.

Furthermore we define the following sets

$$\mathcal{M}_\phi^+ = \left\{a \in A^+ \;\middle|\; \phi(a) < \infty\right\}, \qquad \mathcal{N}_\phi = \{a \in A \mid \phi(a^*a) < \infty\}$$

$$\mathcal{M}_\phi = \text{lin } \mathcal{M}_\phi^+.$$

*We say a weight ϕ on A is **faithful** if for $a \in A^+$ we have $\phi(a) = 0$ implies $a = 0$.*

Example 1.4.3 *In the case of a commutative von Neumann algebra $\mathrm{L}^\infty(\Omega, \mu)$ where μ is a positive measure on a locally compact space Ω we have a weight $\phi : \mathrm{L}^\infty(\Omega, \mu)^+ \to [0, \infty]$ defined by*

$$\phi(f) = \int_X f \, d\mu$$

for all $f \in \mathrm{L}^\infty(\Omega, \mu)^+$ and we have

$$\mathcal{M}_\phi^+ = \mathrm{L}^1(\Omega, \mu) \cap \mathrm{L}^\infty(\Omega, \mu)^+, \qquad \mathcal{N}_\phi^+ = \mathrm{L}^2(\Omega, \mu) \cap \mathrm{L}^\infty(\Omega, \mu)$$

$$\mathcal{M}_\phi = \mathrm{L}^1(\Omega, \mu) \cap \mathrm{L}^\infty(\Omega, \mu).$$

In particular, given a locally compact group G with left (or right) Haar measure μ we have a weight defined on $\mathrm{L}^\infty(G, \mu)$.

Of course a linear functional restricted to the positive elements of a von Neumann algebra are trivial examples of a weight. Non-trivial examples of weights (i.e. including ∞ in the range) in the non-commutative case are often not so easy to construct as they are in the commutative case. We can however construct weights on C*-algebras from GNS-constructions using similar functionals to those in 1.4.7, see Section 3 in Kustermans (1997a) for further details.

Part (vi) in the following is from Theorem 3.20 in Strătilă *et al.* (1979). The remainder can be found in Chapter VII of Takesaki (2003b).

Proposition 1.4.4 *Let ϕ be a weight on a C*-algebra A then we have the following properties of ϕ:*

(i) $\mathcal{M}_\phi^+$ *is a hereditary cone in* A^+*, that is if* $x, y \in \mathcal{M}_\phi^+$ *then* $x + y \in \mathcal{M}_\phi^+$ *and if* $x \in \mathcal{M}_\phi^+$*,* $y \in A^+$ *and* $0 \leqslant y \leqslant x$ *then* $y \in \mathcal{M}_\phi^+$*;*

(ii) If $0 \leqslant x \leqslant y$ *for* $x, y \in \mathcal{M}_\phi^+$ *then* $\phi(x) \leqslant \phi(y)$*;*

(iii) $\mathcal{N}_\phi$ *is a left ideal in* A*;*

(iv) $\mathcal{M}_\phi = \text{lin}\,\{y^*x \mid x, y \in \mathcal{N}_\phi\}$ *is a* $*$*-subalgebra of* A *with* $\mathcal{M}_\phi \cap A^+ = \mathcal{M}_\phi^+$ *and* $\mathcal{M}_\phi \subset \mathcal{N}_\phi$*. Furthermore any element of* $\mathcal{M}_\phi$ *can be written uniquely as a linear combination* $\sum_{k=0}^{3} i^k z_k$ *of* 4 *elements* $z_k \in \mathcal{M}_\phi^+$ *for* $0 \leqslant k \leqslant 3$*;*

(v) There exists a net (e_α) *in* $\mathcal{M}_\phi^+$ *such that* $0 \leqslant e_\alpha \leqslant 1$ *for all* α*, for* $\alpha, \beta \in I$ *with* $\alpha \leqslant \beta$ *we have* $e_\alpha \leqslant e_\beta$ *and* $\|x - xe_\alpha\| \to 0$ *for all* $x \in \mathcal{N}_\phi$*;*

(vi) There exists a unique linear map $\phi : \mathcal{M}_\phi \to \mathbb{C}$ *(also denoted by* ϕ*) that is equivalent to* ϕ *on* $\mathcal{M}_\phi^+$ *and such that* $\phi(x^*) = \overline{\phi(x)}$ *for all* $x \in \mathcal{M}_\phi$*.*

We also have the following Cauchy-Schwarz type equation. The proof is similar to that of the standard Cauchy-Schwarz equation.

Proposition 1.4.5 *Let ϕ denote a weight on a C*-algebra A, then we have*

$$|\phi(y^*x)|^2 \leqslant \phi(x^*x)\phi(y^*y)$$

for all $x, y \in \mathcal{N}_\phi$*.*

There is a GNS construction for a weight on a C*-algebra to that given by a positive linear functional that follows similarly to that of the usual GNS construction.

Theorem 1.4.6 *Let ϕ denote a weight on a C*-algebra A, then there exists a triple $(\mathcal{H}_\phi, \pi_\phi, \Lambda_\phi)$ where $\mathcal{H}_\phi$ is a Hilbert space, $\Lambda_\phi : \mathcal{N}_\phi \to \mathcal{H}_\phi$ is a map into $\mathcal{H}_\phi$ with dense range and $\pi_\phi : A \to \mathcal{B}(\mathcal{H}_\phi)$ is a $*$-homomorphism such that*

$$(\Lambda_\phi(x)|\Lambda_\phi(y)) = \phi(y^* x)$$

for all $x, y \in \mathcal{N}_\phi$ and $\pi_\phi(x)\Lambda_\phi(y) = \Lambda_\phi(xy)$ for all $x \in A$ and $y \in \mathcal{N}_\phi$. Furthermore this triple is unique up to unitary isomorphism of Hilbert spaces. We call the triple $(\mathcal{H}_\phi, \pi_\phi, \Lambda_\phi)$ the ***GNS construction*** *of ϕ.*

In order to handle the unboundedness of Λ_ϕ we give approximations for a weight ϕ by positive linear functionals. In particular we define two sets $\mathcal{F}_\phi$ and $\mathcal{G}_\phi$ that both approximate ϕ. We will see below that the advantage of $\mathcal{G}_\phi$ over $\mathcal{F}_\phi$ is that $\mathcal{G}_\phi$ is directed upwards enabling us to take limits over $\mathcal{G}_\phi$. We will prove this in Proposition 1.4.9 but first we need some preliminary lemmas. We also define similar sets for von Neumann algebras but taken as subsets of M_*^+ and show similar properties hold.

Notation 1.4.7 *Let A be a C*-algebra and ϕ a weight on A. Then we define the sets*

$$\mathcal{F}_\phi = \left\{\omega \in A_+^* \;\middle|\; \omega(x) \leqslant \phi(x) \quad \forall x \in A^+\right\}$$
$$\mathcal{G}_\phi = \{\lambda\omega \mid \omega \in \mathcal{F}_\phi,\ \lambda \in (0,1)\} \subset \mathcal{F}_\phi$$

where we let $\mathcal{F}_\phi$ have the order inherited from A_*^+.

Now let M be a von Neumann algebra and ϕ a weight on M. In this case we define the sets to be

$$\mathcal{F}_\phi = \left\{\omega \in M_*^+ \;\middle|\; \omega(x) \leqslant \phi(x) \quad \forall x \in M^+\right\},$$
$$\mathcal{G}_\phi = \{\lambda\omega \mid \omega \in \mathcal{F}_\phi,\ \lambda \in (0,1)\} \subset \mathcal{F}_\phi.$$

where we let $\mathcal{F}_\phi$ have the order inherited from M_*^+. We note in this case of von Neumann *algebras that we only consider functionals from the predual in accordance with weights on a von Neumann algebra.*

The proof of this can be found in Combes (1968) Lemma 2.3 and Proposition 2.4. See also Chapter VII in Takesaki (2003b).

Proposition 1.4.8 *Let ϕ be a weight on a C*-algebra A, let $(\mathcal{H}_\phi, \pi_\phi, \Lambda_\phi)$ be the GNS construction of ϕ and let $\omega \in \mathcal{F}_\phi$ with GNS representation $(\mathcal{H}_\omega, \pi_\omega, \eta_\omega)$. Then we have:*

(i) There exists a unique contraction $T \in \pi_\phi(A)'$ where $0 \leqslant T \leqslant 1$ such that $(T\Lambda_\phi(x)|\Lambda_\phi(y)) = \omega(y^* x)$ for all $x, y \in \mathcal{N}_\phi$;

(ii) There is a unique element $\xi_\omega \in \mathcal{H}_\phi$ such that $\langle x, \omega \rangle = (\pi_\phi(x)\xi_\omega|\xi_\omega)$ and $T^{1/2}\Lambda_\phi(x) = \pi_\phi(x)\xi_\omega$ for all $x \in \mathcal{N}_\phi$.

For the proof of the following two Propositions see Chapter 3 in Kustermans (1997b). See also Quaegebeur & Verding (1999).

Proposition 1.4.9 *The sets $\mathcal{G}_\phi$ for a weight ϕ on a C*-algebra and $\tilde{\mathcal{G}}_\phi$ for a weight ϕ on a von Neumann algebra are both upwards directed.*

1.4.2 Normal Semi-finite Faithful Weights on von Neumann algebras

Similar to the development of measure theory on locally compact spaces, we want to impose some further conditions on weights such that we obtain a more complete theory as a result. We describe further conditions for weights on von Neumann algebras that generalise some of the conditions of measure theory. See Kustermans & Vaes (1999) for the case of weights on a C*-algebra.

Most theorems in this section have complex proofs and we refer the reader to the standard references Takesaki (2003b) and Strătilă (1981) for further details.

Definition 1.4.10 *A weight ϕ on a von Neumann algebra M is:*

(i) ***normal*** *if for all $\lambda \in \mathbb{R}^+$ the set $\{a \in M^+ \mid \phi(a) \leqslant \lambda\}$ is σ-weakly closed;*

(ii) ***semi-finite*** *if $\mathcal{M}_\phi^+$ is σ-weakly dense in M^+;*

(iii) ***n.s.f.*** *if it is normal, semi-finite and faithful.*

Proposition 1.4.11 *Let ϕ be a weight on a von Neumann algebra M, then ϕ is semi-finite if and only any of the following conditions are satisfied:*

(i) $\mathcal{M}_\phi^+$ is dense in M^+ in any of the weak topologies;

(ii) $\mathcal{M}_\phi$ is dense in M in any of the weak topologies;

(iii) $\mathcal{N}_\phi$ is dense in M in any of the weak topologies.

Proposition 1.4.12 *Let ϕ be an n.s.f. weight on a von Neumann algebra M and let $(\mathcal{H}_\phi, \pi_\phi, \Lambda_\phi)$ denote the GNS construction. Then $\pi_\phi : M \to \mathcal{B}(\mathcal{H}_\phi)$ is a normal $*$-isomorphism of M onto $\pi_\phi(M) \subset \mathcal{B}(\mathcal{H}_\phi)$.*

The following theorem is from Haagerup (1975). See also Chapter VII of Takesaki (2003b).

Theorem 1.4.13 *Let ϕ be a weight on a von Neumann algebra M then the following conditions are equivalent:*

(i) ϕ is normal;

(ii) for any bounded increasing net $(x_\alpha) \subset M^+$ we have $\phi(\sup_\alpha x_\alpha) = \sup \phi(x_\alpha)$;

(iii) $\phi(x) = \sup\{\omega(x) \mid \omega \in \mathcal{F}_\phi\} = \lim\{\omega(x) \mid \omega \in \mathcal{G}_\phi\}$ for all $x \in M^+$.

See Theorem VII.2.7 in Takesaki (2003b) for a proof of the following.

Theorem 1.4.14 *Let M be a von Neumann algebra, then there exists an n.s.f. weight on M.*

The following is from Lemma IX.1.5 in Takesaki (2003b).

Proposition 1.4.15 *Let ϕ and ψ denote two n.s.f. weights on a von Neumann algebra M with GNS constructions $(\mathcal{H}_\phi, \pi_\phi, \Lambda_\phi)$ and $(\mathcal{H}_\psi, \pi_\psi, \Lambda_\psi)$ respectively. Then there exists a unitary operator $U : \mathcal{H}_\phi \to \mathcal{H}_\psi$ such that $U\pi_\phi(x)U^* = \pi_\psi(x)$ for all $x \in M$.*

We now give the KMS properties of n.s.f. weights on von Neumann algebras. The notation $S(i)$ and $S(i)^o$ in the next theorem is introduced in Notation 1.3.3.

Definition-Theorem 1.4.16 *Let ϕ be a weight on a von Neumann algebra M then there exists a unique strongly continuous one-parameter automorphism group σ^ϕ (see Definition 1.3.1) on M such that*

(i) $\phi \circ \sigma_t^\phi = \phi$ for all $t \in \mathbb{R}$ and

(ii) for all $x, y \in \mathcal{N}_\phi \cap \mathcal{N}_\phi^*$ there exists a function $F_{x,y} : S(i) \to \mathbb{C}$ which is analytic on $S(i)^o$ and such that

$$F_{x,y}(t) = \phi(\sigma_t^\phi(x)y) \qquad \text{and} \qquad F_{x,y}(t+i) = \phi(x\sigma_t^\phi(y))$$

for all $t \in \mathbb{R}$.

*We call σ^ϕ the **modular automorphism group** of ϕ.*

Proposition 1.4.17 *Let M be a von Neumann algebra, ϕ a weight on M and σ the modular automorphism group of ϕ. Then we have:*

(i) $\sigma_t^\phi(1) = 1$ for all $t \in \mathbb{R}$;

(ii) Let $x \in \mathrm{Dom}(\sigma_{-i})$ and $y \in \mathcal{M}_\phi$, then xy and $y\sigma_{-i}(x)$ are in $\mathcal{M}_\phi$ and $\phi(xy) = \phi(y\sigma_{-i}(x))$.

Definition-Theorem 1.4.18 *For an n.s.f. weight ϕ on a von Neumann algebra M with GNS construction $(\mathcal{H}, \pi, \Lambda)$ and with modular automorphism group σ there exists:*

(i) *a unique injective positive operator* ∇ *on* $\mathcal{H}$ *such that* $\nabla^{it}\Lambda(x) = \Lambda(\sigma_t(x))$ *for all* $t \in \mathbb{R}$ *and* $x \in \mathcal{N}_\phi$*;*

(ii) *an anti-unitary operator* J *on* $\mathcal{H}$ *such that* $J\Lambda(x) = \Lambda(\sigma_{i/2}(x)^*)$ *for all* $x \in \mathcal{N}_\phi \cap \mathrm{Dom}(\sigma_{i/2})$ *such that* $\sigma_{i/2}(x)^* \in \mathcal{N}_\phi$*. We call* J *the* ***modular conjugation*** *and* ∇ *the* ***modular operator*** *of* ϕ *for any n.s.f. weight* ϕ*.*

Proposition 1.4.19 *Let* M *be a von Neumann algebra,* ϕ *a weight on* M*,* J *the modular conjugation and* ∇ *the modular operator of* ϕ*. Then we have:*

(i) $J^2 = 1$*;*

(ii) *For all* $x \in \mathcal{N}_\phi$ *and* $y \in \mathrm{Dom}(\sigma_{i/2})$ *we have* $xa \in \mathcal{N}_\phi$ *and*

$$\Lambda(xa) = J\pi(\sigma_{i/2}(y))^* J\Lambda(x);$$

(iii) *The set* $\Lambda(\mathcal{N}_\phi \cap \mathcal{N}_\phi^*)$ *is a core for* $\nabla^{1/2}$ *and* $\Lambda(x^*) = J\nabla^{1/2}\Lambda(x)$ *for all* $x \in \mathcal{N}_\phi \cap \mathcal{N}_\phi^*$*.*

1.4.3 Slicing and Tensor Products of Weights on von Neumann algebras

In this section we consider the tensor product of weights and slice maps on weights, that is given von Neumann algebras M and N and n.s.f. weights ϕ on M and ψ on N we want to make sense of the maps $\phi \otimes \mathrm{id}$ on $(M \overline{\otimes} N)^+$ and $\mathrm{id} \otimes \psi$ on $(M \overline{\otimes} N)^+$. In order to do this we define the extended positive part M_{ext}^+ of any von Neumann algebra M and we will define an operator valued weight from $(M \overline{\otimes} N)^+$ into N_{ext}^+ (or M_{ext}^+ for the other weight ψ).

We follow Strătilă (1981) chapters 8, 9 and 11 in this section. See also Takesaki (2003b) and the papers Haagerup (1979a) and Haagerup (1979b). For the C*-algebra case see Kustermans & Vaes (1999).

Definition-Theorem 1.4.20 *Let M and N be von Neumann algebras and ϕ and ψ be normal semi-finite weights on M and N respectively. Then there exists a unique weight $\phi \otimes \psi$ on $M \overline{\otimes} N$ such that for $x \in \mathcal{M}_\phi$ and $y \in \mathcal{M}_\psi$ we have $x \otimes y \in \mathcal{M}_{\phi\otimes\psi}$ and*

$$(\phi \otimes \psi)(x \otimes y) = \phi(x)\psi(y).$$

Proposition 1.4.21 *Let M and N be von Neumann algebras with weights ϕ and ψ respectively. Then for all $x \in (M \overline{\otimes} N)^+$ we have*

$$\begin{aligned}(\phi \otimes \psi)(x) &= \sup \{(\omega \otimes \kappa)(x) \mid \omega \in \mathcal{F}_\phi,\ \kappa \in \mathcal{F}_\psi\} \\ &= \lim \{\langle x, \omega \otimes \kappa\rangle \mid \omega \in \mathcal{G}_\phi,\ \kappa \in \mathcal{G}_\psi\}\end{aligned}$$

where we take the supremum or limit of an unbounded set to be infinity and $\omega \otimes \kappa$ is the unique positive linear functional in $(M \overline{\otimes} N)^+_*$ such that $(\omega \otimes \kappa)(x \otimes y) = \omega(x)\kappa(y)$ for all $x \in M$ and $y \in N$ (see Proposition IV.5.13 *in Takesaki (2003a)).*

We have the following important theorem.

Theorem 1.4.22 *Let ϕ and ψ denote normal semi-finite weights on von Neumann algebras M and N respectively. Then $\phi \otimes \psi$ is a normal semi-finite weight on the von Neumann algebra $M \overline{\otimes} N$ and*

$$\mathcal{N}_\phi \odot \mathcal{N}_\psi \subset \mathcal{N}_{\phi\otimes\psi}.$$

If ϕ and ψ are faithful then so is $\phi \otimes \psi$.

Let σ^ϕ and σ^ψ denote the modular automorphism groups of ϕ and ψ respectively (see Theorem 1.4.16), then the modular automorphism group of the weight $\phi \otimes \psi$ satisfies the property $\sigma_t^{\phi\otimes\psi} = \sigma_t^\phi \otimes \sigma_t^\psi$ for all $t \in \mathbb{R}$.

Proposition 1.4.23 *Let M and N denote von Neumann algebras and ϕ and ψ denote n.s.f. weights on M and N respectively. Let J^M and ∇^M denote the modular conjugation and modular operator of ϕ on M and similarly for ψ on N. Then $J^M \otimes J^N$ is the modular* *conjugation of $\phi \otimes \psi$ and the one-parameter group $\nabla^M \otimes \nabla^N$ (as per Definition 1.3.12) is the modular operator of $\phi \otimes \psi$.*

We now move on to discuss operator valued weights in order to discuss the slice maps we mentioned at the start of this section.

Definition 1.4.24 *Let M be a von Neumann algebra and let M^+_{ext} denote the set of maps $m : M^+_* \to [0, \infty]$ such that*

(i) $m(\omega + \kappa) = m(\omega) + m(\kappa)$ for all $\omega, \kappa \in M^+_*$;

(ii) $m(\lambda\omega) = \lambda m(\omega)$ for all $\lambda \geqslant 0$ and $\omega \in M^+_*$;

(iii) m is lower semi-continuous on M^+_*.

*We call M^+_{ext} the **extended positive part** of a von Neumann algebra M. For $m, n \in M^+_{ext}$, $\lambda \geqslant 0$ and $a \in M$ we define $m + n$, λn and $a^*ma \in M^+_{ext}$ by*

$$(m + n)(\omega) = m(\omega) + n(\omega)$$
$$(\lambda m)(\omega) = \lambda m(\omega)$$
$$(a^*ma)(\omega) = m(a\omega a^*)$$

for $\omega \in M^+_*$ where $a\omega a^* \in M^+_*$ is given by $x \mapsto \langle a^*xa, \omega\rangle$. We can order M^+_{ext} by $m_1 \leqslant m_2$ if $m_1(\omega) \leqslant m_2(\omega)$ for all $\omega \in M^+_*$.

We now move on to the main definition of the section.

Definition 1.4.25 *Let M be a von Neumann algebra and N a von Neumann subalgebra of M. An **operator valued weight** is a map $T : M^+ \to N^+_{ext}$ such that:*

(i) $T(x + y) = T(x) + T(y)$ for $x, y \in M^+$;

(ii) $T(\lambda x) = \lambda T(x)$ for $x \in M^+$ and $\lambda \geqslant 0$;

(iii) $T(y^*xy) = y^*T(x)y$ *for all* $x \in M^+$ *and* $y \in N$.

Similarly to the case of a weight on M *we define*

$$\mathcal{M}_T^+ = \left\{x \in M^+ \mid T(x) \in N^+\right\}, \qquad \mathcal{N}_T = \left\{x \in M \mid T(x^*x) \in N^+\right\}$$

$$\mathcal{M}_T = \text{lin } \mathcal{M}_T^+.$$

By the previous proposition we have that $x \in \mathcal{M}_T^+$ if and only if we have $(T(x))(\omega)$ is finite for all $\omega \in N_*^+$ with similar condition for $\mathcal{N}_T$ also.

Proposition 1.4.26 *Let* $T : M \to N_{ext}^+$ *be an operator valued weight. Then we have the following properties:*

(i) $\mathcal{M}_T^+$ *is a hereditary cone in* M^+ *and if* $0 \leqslant x \leqslant y$ *for* $x, y \in \mathcal{M}_T^+$ *then* $T(x) \leqslant T(y)$*;*

(ii) $\mathcal{N}_T$ *is a left ideal in* M*;*

(iii) $\mathcal{M}_T = \text{lin }\{y^*x \mid x, y \in \mathcal{N}_T\}$ *is a* $*$*-subalgebra of* M *with* $\mathcal{M}_T \cap M^+ = \mathcal{M}_T^+$ *and* $\mathcal{M}_T \subset \mathcal{N}_T$. *Furthermore any element of* $\mathcal{M}_T$ *can be written uniquely as a linear combination* $\sum_{k=0}^{3} i^k z_k$ *of* 4 *elements* $z_k \in \mathcal{M}_T^+$ *for* $0 \leqslant k \leqslant 3$*;*

(iv) There is a unique linear map $T : \mathcal{M}_T \to N$ *(also denoted by* T*) such that* $T(axb) = aT(x)b$ *for all* $x \in \mathcal{M}_T$ *and* $a, b \in N$ *that is equivalent to the given* T *on* $\mathcal{M}_T^+$*;*

(v) $\mathcal{M}_T$ *and* $\mathcal{N}_T$ *are* N*-bimodules.*

We again want to define additional requirements on operator valued weights in order to define normal, semi-finite and faithful (n.s.f.) operator valued weights. We define each of these concepts now.

Definition 1.4.27 *An operator valued weight* $T : M \to N_{ext}^+$ *from a von Neumann algebra* M *to a von Neumann subalgebra* $N \subset M$ *is* *(i)* ***normal*** *if for all bounded increasing nets* $(x_\alpha) \subset M^+$ *with limit* $x \in M^+$ *we have* $\lim T(x_\alpha) = T(x)$*;*

(ii) ***semi-finite*** *if* n_T *is* σ*-weakly dense in* M*;*

(iii) ***faithful*** *if* $T(x^*x) = 0$ *implies* $x = 0$*;*

(iv) ***n.s.f.*** *if it is normal, semi-finite and faithful.*

Remark 1.4.28 *We have that an n.s.f. operator valued weight* $T : M^+ \to N^+_{ext}$ *can be uniquely extended to a map* $T : M^+_{ext} \to N^+_{ext}$*. It follows that we can also extend a weight* $\phi : M^+ \to [0, \infty]$ *to a map* $\phi : M^+_{ext} \to [0, \infty]$ *and so the following theorem makes sense.*

We have the following important theorems proved by Haagerup in Haagerup (1979b). See also Sections 12.1-12.5 and 12.8 in Strătilă (1981).

Theorem 1.4.29 *For* $i = 1, 2$ *let* M_i *be von Neumann algebras with* $N_i \subset M_i$ *a von Neumann subalgebra and let* $T_i : M_i^+ \to (N_i)^+_{ext}$ *be n.s.f. operator valued weights. Then there exists a unique n.s.f. operator valued weight* $T_1 \otimes T_2 : (M_1 \otimes M_2)^+ \to (N_1 \overline{\otimes} N_2)^+_{ext}$ *such that for all n.s.f. weights* ψ_i *on* N_i *(*$i = 1, 2$*) we have*

$$(\psi_1 \otimes \psi_2) \circ (T_1 \otimes T_2) = (\psi_1 \circ T_1) \otimes (\psi_2 \otimes T_2).$$

Furthermore for $x_1 \in \mathcal{M}_{T_1}$ *and* $x_2 \in \mathcal{M}_{T_2}$ *we have* $x_1 \otimes x_2 \in \mathcal{M}_{T_1 \otimes T_2}$ *and* $(T_1 \otimes T_2)(x_1 \otimes x_2) = T_1(x) \otimes T_2(x)$.

Now we consider slice maps with weights.

Corollary 1.4.30 *Let* ϕ *and* ψ *be n.s.f. weights on von Neumann algebras* ϕ *and* ψ *respectively. Then there exist n.s.f. operator valued weights* $\phi \otimes \mathrm{id} : (M \overline{\otimes} N)^+ \to N^+_{ext}$ *and* $\mathrm{id} \otimes \psi : (M \overline{\otimes} N)^+ \to M^+_{ext}$ *such that* $\psi \circ (\phi \otimes \mathrm{id}) = \phi \otimes \psi = \phi \circ (\mathrm{id} \otimes \psi)$*. Furthermore for* $x \in \mathcal{M}_\phi$ *and* $y \in N$ *we have* $x \otimes y \in \mathcal{M}_{\phi \otimes \mathrm{id}}$ *with* $(\phi \otimes \mathrm{id})(x \otimes y) = \phi(x)y$ *and similarly for* $x \in M$ *and* $y \in \mathcal{M}_\psi$ *we have* $x \otimes y \in \mathcal{M}_{\mathrm{id} \otimes \psi}$ *with* $(\mathrm{id} \otimes \psi)(x \otimes y) = \psi(y)x$.

Chapter 2

Locally Compact Quantum Groups

There are three types of locally compact quantum groups that we will use in this thesis, namely the von Neumann algebraic setting, the reduced C*-algebraic setting and the universal C*-algebraic setting. Locally compact quantum groups first appear in the literature in the reduced C*-algebraic setting in Kustermans & Vaes (2000), then in the universal C*-algebraic setting in Kustermans (2001) and finally in the von Neumann algebraic setting in Kustermans & Vaes (2003). We will reference these regularly in this chapter. See also Johan Kustermans' lecture notes in Applebaum *et al.* (2005), Stefan Vaes' PhD Thesis Vaes (2001), Thomas Timmermann's book Timmermann (2008) and van Daele's alternative approach for the von Neumann algebraic setting Van Daele (2014).

2.1 Introduction

In this section we briefly introduce locally compact groups in the language of locally compact quantum groups and we discuss quantum semigroups for the C*-algebraic and von Neumann algebraic settings. We will also briefly discuss Hopf algebras and the difficulties in using them for locally compact quantum groups.

2.1.1 Locally Compact Groups as Quantum Groups

It is recommended that the reader use this as a motivation for what follows and as a reference for the properties of quantum groups in the special case where we have a group. The results are easier to prove here because of the commutativity of $\mathrm{C}_0(G)$, however we feel that it may give the reader some motivation as to why we are interested in the results in the more general case of quantum groups.

Example 2.1.1 *Let G be a locally compact group, then we can consider the commutative C*-algebra of continuous functions vanishing at infinity* $\mathrm{C}_0(G)$*. Given the structure on the group G we define the map* $\Delta : \mathrm{C}_0(G) \to \mathrm{C}_b(G \times G)$ *given by* $(\Delta(f))(x, y) \mapsto f(xy)$ *for* $f \in \mathrm{C}_0(G)$*. We have that* $\Delta(f)$ *is bounded as we have*

$$\|\Delta(f)\| = \sup_{x,y\in G} |(\Delta(f))(x, y)| = \sup_{x,y\in G} |f(xy)| = \sup_{x\in G} |f(x)| = \|f\|.$$

We note that $\mathrm{C}_b(G \times G) \cong_i \mathrm{M}(\mathrm{C}_0(G \times G)) \cong_i \mathrm{M}(\mathrm{C}_0(G) \otimes_{min} \mathrm{C}_0(G))$ *where* M *denotes the multiplier algebra (see Section A.5) and so* $\Delta : \mathrm{C}_0(G) \to \mathrm{M}(\mathrm{C}_0(G) \otimes_{min} \mathrm{C}_0(G))$*.*

We have for $x, y, z \in G$ *and* $f \in \mathrm{C}_0(G)$ *that*

$$f((xy)z) = [\Delta(f)](xy, z) = [(\Delta \otimes \mathrm{id})(\Delta(f))](x, y, z)$$

and

$$f(x(yz)) = [\Delta(f)](x, yz) = [(\mathrm{id} \otimes \Delta)(\Delta(f))](x, y, z)$$

and so by associativity of G we have $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$*. We call this property coassociativity.*

We also have a left Haar weight $\phi : \mathrm{C}_0(G)^+ \to [0, \infty]$ *given by* $f \mapsto \int_X f \, d\mu$*. Similarly we have a right Haar weight* ψ *given by the right Haar measure. We can consider* $\mathrm{L}^2(G, \mu)$ *as the space of square integrable (up to almost everywhere) functions on G, that is* $f \in \mathrm{L}^2(G, \mu)$ *if and only if* $\int_G f^* f \, d\mu < \infty$*. Then* $\mathrm{L}^2(G, \mu)$ *is a Hilbert space with inner* *product* $(f|g) = \phi(fg^*)$.

For $\omega \in \mathrm{C}_0(G)^*_+$ *we have a positive measure* $\nu \in \mathrm{M}(G)$ *such that* $\langle f, \omega \rangle = \int_X f\, d\nu$. *It then follows from Fubini's theorem that for all* $f \in \mathrm{C}_0(G)^+$ *we have*

$$\begin{aligned}\phi((\omega \otimes \mathrm{id})(\Delta(f))) &= \int_G [(\omega \otimes \mathrm{id})(\Delta(f))](y)\, d\mu(y) \\ &= \int_G \left(\int_G f(xy)\, d\nu(x) \right) d\mu(y) = \int_G \left(\int_G f(y)\, d\mu(y) \right) d\nu(x) = \omega(1)\phi(f)\end{aligned}$$

It follows that left invariance of the Haar measure μ *is equivalent to having* $\phi((\omega \otimes \mathrm{id})(\Delta(f))) = \phi(f)\omega(1)$ *for all* $\omega \in \mathrm{C}_0(G)^*_+$ *and* $f \in \mathrm{C}_0(\mathbb{G})^+$. *It also follows that*

$$\overline{\mathrm{lin}\ \{(\omega \otimes \mathrm{id})\Delta(f) \mid f \in \mathrm{C}_0(G),\ \omega \in \mathrm{C}_0(G)^*\}} = \mathrm{C}_0(G)$$

and similarly for ω *acting on the right.*

We can define an isometry $W : \mathrm{L}^2(G,\mu) \otimes \mathrm{L}^2(G,\mu) \to \mathrm{L}^2(G,\mu) \otimes \mathrm{L}^2(G,\mu)$ *by* $W(F)(x,y) = F(x, x^{-1}y)$ *for* $F \in \mathrm{L}^2(G \times G, \mu \times \mu) = \mathrm{L}^2(G,\mu) \otimes \mathrm{L}^2(G,\mu)$ *and it follows that* $W^*(f \otimes g) = \Delta(g)(f \otimes 1)$. *We denote* $W_{12} = W \otimes 1$, $W_{23} = 1 \otimes W$ *and* $W_{13} = \sigma_{23} W_{12}$ *where* σ *denotes the flip map on* $\mathrm{L}^2(G,\mu) \otimes \mathrm{L}^2(G,\mu)$. *Then we have for* $F \in \mathrm{L}^2(G \times G \times G, \mu \times \mu \times \mu)$

$$\begin{aligned}((W_{12}W_{13}W_{23})(F))(x,y,z) &= (W_{13}W_{23}(F))(x, x^{-1}y, z) \\ &= (W_{23}(F))(x, x^{-1}y, x^{-1}z) = F(x, x^{-1}y, y^{-1}z)\end{aligned}$$

and

$$\begin{aligned}((W_{23}W_{12})(F))(x,y,z) &= (W_{12}(F))(x,y,y^{-1}z) \\ &= F(x, x^{-1}y, y^{-1}z) = .((W_{12}W_{13}W_{23})(F))(x,y,z).\end{aligned}$$

As this then holds for all such F *we have* $W_{12}W_{13}W_{23} = W_{23}W_{12}$.

Example 2.1.2 *Let G be a locally compact group and μ the left Haar measure on G. Then we have the von Neumann algebra of measurable essentially bounded functions $\mathrm{L}^\infty(G,\mu)$. We also have a map $\Delta : \mathrm{L}^\infty(G,\mu) \to \mathrm{L}^\infty(G,\mu) \overline{\otimes} \mathrm{L}^\infty(G,\mu)$ given by $\Delta(f)(x,y) = f(xy)$ where we've identified $\mathrm{L}^\infty(G,\mu) \overline{\otimes} \mathrm{L}^\infty(G,\mu) \cong_i \mathrm{L}^\infty(G \times G, \mu \times \mu)$.*

We define $\phi : \mathrm{L}^\infty(G,\mu)^+ \to [0,\infty]$ by $f \mapsto \int_X f d\mu$ and we have a n.s.f. weight on $\mathrm{L}^\infty(G,\mu)$. As μ is a Haar measure then we have

$$\int_X f(yx)\, d\mu(x) = \int_X f(x)\, d\mu(x)$$

for all $y \in G$.

2.1.2 Quantum Semigroups

We now define the coproduct on an algebra to give us a notion of a "quantum semigroup" or a bialgebra. On the way to defining a locally compact quantum group we need to define a C*-algebraic quantum semigroup or von Neumann algebraic quantum semigroup in order to capture the "topology" at the quantum level.

Definition 2.1.3 *Let A be an algebra and $\Delta : A \to \mathrm{M}(A \otimes A)$ a non-degenerate map, then we say Δ is **coassociative** if we have $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$. A non-degenerate coassociative homomorphism $\Delta : A \to \mathrm{M}(A \otimes A)$ is called a **coproduct** on A where if A is unital we require Δ to be unital and if A is a $*$-algebra we require Δ to be a $*$-homomorphism.*

*A **C*-algebraic quantum semigroup** is a pair (A,Δ) where A is a C*-algebra and $\Delta : A \to \mathrm{M}(A \otimes_{min} A)$ is a coproduct on A. A **von Neumann algebraic quantum semigroup** is a pair (M,Δ) where M is a von Neumann algebra and $\Delta : M \to M \overline{\otimes} M$ is a normal coproduct on M.*

In Example 2.1.1 we gave a coproduct $\Delta : \mathrm{C}_0(G) \to \mathrm{M}(\mathrm{C}_0(G) \otimes_{min} \mathrm{C}_0(G)) \cong_i \mathrm{C}_b(G \times G)$ on $\mathrm{C}_0(G)$ for a locally compact group G. Similarly from Example 2.1.2 we have a coproduct $\Delta : \mathrm{L}^\infty(G,\mu) \to \mathrm{L}^\infty(G,\mu)\overline{\otimes}\mathrm{L}^\infty(G,\mu)$ on $\mathrm{L}^\infty(G,\mu)$.

2.1.3 Hopf Algebras

For a quantum group we need operations that are equivalent to the identity and inversion operations at a "quantum level". We will discuss the "inversion" operation with Hopf algebras now. First we give the definition of a Hopf algebra, then we discuss this definition in terms of the algebra of polynomials over a finite group and finally we discuss some of the problems for defining locally compact quantum groups using Hopf algebras in this way.

Definition 2.1.4 *A **Hopf algebra** is a unital algebra A with multiplication map $m : A \otimes A \to A$ and unit given by $\eta : \mathbb{C} \to A$ such that m is associative (i.e. $m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m)$ and $m \circ (\eta \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes \eta)$). Furthermore we have the following:*

*(i) We have a unital, coassociative homomorphism $\Delta : A \to A \otimes A$ and a homomorphism $\varepsilon : A \to \mathbb{C}$ called the **counit** such that $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$;*

(ii) There is an antihomomorphism $S : A \to A$ such that

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta$$

*called the **antipode**.*

*A **Hopf $*$-algebra** is a Hopf algebra A with involution such that A is a $*$-algebra and Δ and ε are $*$-homomorphisms.*

Now consider the algebra of polynomials A over a finite group G with multiplication $m : A \odot A \to A$ given by $m(f,g) \mapsto fg$ for all $f,g \in A$ (where $(fg)(x) = f(x)g(x)$ for all $x \in G$) and identity map $\eta : \mathbb{C} \to A$ given by $\lambda \mapsto \lambda 1$ for $\lambda \in \mathbb{C}$ and 1 the unit of A.

We let Δ be the usual coproduct $(\Delta(f))(x, y) = f(xy)$ for all $f \in A$ and $x, y \in G$. We can define the counit $\varepsilon : A \to \mathbb{C}$ given by $f \mapsto f(e)$ for all $f \in A$ (where e is the unit element of G) and the antipode $S : A \to A$ given by $(S(f))(x) = f(x^{-1})$ for all $f \in A$ and $x \in G$. We can easily show that this gives us a Hopf algebra. Note how we have used the group product to define Δ, the group identity to define ε and the group inverse to define S.

Unfortunately for locally compact quantum groups it does not necessarily follow that the maps ε and S are bounded. In particular we will see that S is unbounded for the example of $\mathrm{SU}_q(2)$ that we will give in Chapter 5. As such it is difficult to define locally compact quantum groups using Hopf algebras, however the antipode in particular does still play an important role. We show in the next section an appropriate generalisation to what we now consider a locally compact quantum group.

2.2 Locally Compact Quantum Groups

In this section we give details of locally compact quantum groups in the von Neumann algebraic and reduced C*-algebraic settings. We will also discuss the relationship from one to another though in essence they describe the same "object" as we will see.

Proofs are scarcely given in this section and the reader is referred to the literature for further details.

2.2.1 Von Neumann Algebraic Quantum Groups

In this section we give details of von Neumann algebraic quantum groups. We give details of the Haar weights, the main definition, the multiplicative unitary, the antipode and its related objects and then we give further properties of each of these objects as needed.

The main reference for this is Kustermans & Vaes (2003) (see also Vaes (2001)). We remind the reader that we defined $\mathcal{M}_\phi^+ = \{a \in A^+ \mid \phi(a) < \infty\}$ in Definition 1.4.2.

Definition 2.2.1 *Let (M, Δ) be a von Neumann algebraic quantum semigroup. Then a weight ϕ on M is:*

(i) ***left invariant*** *if $\phi((\omega \otimes \mathrm{id})\Delta(x)) = \omega(1)\phi(x)$ for all $x \in \mathcal{M}_\phi^+$ and $\omega \in M_*^+$;*

(ii) ***right invariant*** *if $\phi((\mathrm{id} \otimes \omega)\Delta(x)) = \omega(1)\phi(x)$ for all $x \in \mathcal{M}_\phi^+$ and $\omega \in M_*^+$.*

Let (M, Δ) be a von Neumann algebraic quantum semigroup where M acts on a Hilbert space $\mathcal{H}$ and let ϕ be a left or right invariant weight on M. We may assume that M acts on $\mathcal{H}$ in standard form and we have a GNS construction $(\mathcal{H}, \pi, \Lambda)$ of ϕ and $(\mathcal{H}, \pi, \Gamma)$ of ψ.

We now turn to the main definition of this section.

Definition 2.2.2 *A* ***von Neumann algebraic quantum group*** *(M, Δ, ϕ, ψ) consists of a von Neumann algebraic quantum semigroup (M, Δ) with a left invariant n.s.f. weight ϕ and a right invariant n.s.f. weight ψ. We call ϕ the* ***left Haar weight*** *and ψ the* ***right Haar weight****.*

We will often denote a von Neumann algebraic quantum group (M, Δ, ϕ, ψ) simply by M or (M, Δ) with ϕ and ψ understood. Throughout the rest of this section we let M denote a von Neumann algebraic quantum group. We also fix a Hilbert space $\mathcal{H}$ on which M acts in standard form and which is also the GNS Hilbert space of ϕ and ψ.

The following is proved in Proposition 3.17 in Kustermans & Vaes (2000).

Proposition 2.2.3 *The coproduct $\Delta : M \to M \overline{\otimes} M$ of a locally compact quantum group M is injective.*

We require the following frequently in quantum groups so we introduce this now.

Notation 2.2.4 *Let $\mathcal{H}$ denote a Hilbert spaces and let $X \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, then we let $X_{12} := X \otimes 1 \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$ where 1 is the identity in $\mathcal{B}(\mathcal{H})$ and similarly we let*

$X_{23} := 1 \otimes X \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$. *Let* $\Sigma \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ *denote the flip map given by* $\xi \otimes \eta \mapsto \eta \otimes \xi$ *and we define* $X_{13} = \Sigma_{23} X_{12} \Sigma_{23}$.

We call the notation X_{12}, X_{13} *and* X_{23} *the* ***leg numbering notation***. *It can easily be extended to higher tensor products of* $\mathcal{H}$ *and to different Hilbert spaces when necessary. This should be clear from the context.*

We now introduce the multiplicative unitary.

Definition-Theorem 2.2.5 *There is a unique unitary operator* $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ *such that*

$$W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)(\Delta(y)(x \otimes 1))$$

for all $x, y \in \mathcal{N}_\phi$. *Furthermore we have the following properties:*

(i) *For all* $x \in M$ *we have*

$$\Delta(x) = W^*(1 \otimes x)W; \tag{2.1}$$

(ii) $W_{12} W_{13} W_{23} = W_{23} W_{12}$;

(iii) $(\Delta \otimes \mathrm{id})(W) = W_{13} W_{23}$.

We call W *the* ***multiplicative unitary*** *of* (M, Δ).

We have the following density conditions as a theorem for a von Neumann algebraic quantum group.

Theorem 2.2.6 *Let* M *denote a von Neumann algebraic quantum group, then we have*

$$\begin{aligned} M &= \{(\omega \otimes \mathrm{id})\Delta(x) \mid x \in M,\ \omega \in M_*\} \\ &= \{(\mathrm{id} \otimes \omega)\Delta(x) \mid x \in M,\ \omega \in M_*\} \\ &= \{(\omega \otimes \mathrm{id})W \mid \omega \in \mathcal{B}(\mathcal{H})_*\}. \end{aligned}$$

We now give details of the antipode as mentioned earlier.

Definition-Theorem 2.2.7 *There exists a unique σ-strong* closed operator $S : M \to M$ with a σ-strong* core*

$$\operatorname{lin}\{(\mathrm{id} \otimes \omega)(W) \mid \omega \in M_*\}$$

such that

$$S((\mathrm{id} \otimes \omega)(W)) = (\mathrm{id} \otimes \omega)(W^*)$$

for all $\omega \in M_*$. Furthermore we have a polar decomposition $S = R \circ \tau_{-i/2}$ where τ is a σ-strong continuous one-parameter group of $*$-automorphisms on M and R is a $*$-anti-automorphism on A with $R^2 = \mathrm{id}$ such that*

(i) *S is densely defined and has dense range;*

(ii) *S is injective with $S^{-1} = R \circ \tau_{i/2}$;*

(iii) *S is an anti-homomorphism on its domain, i.e. for $x, y \in \mathrm{Dom}(S)$ we have $S(xy) = S(y)S(x)$;*

(iv) *For all $x \in \mathrm{Dom}(S)$ we have $S(x)^* \in \mathrm{Dom}(S)$ with $S(S(x)^*)^* = x$;*

(v) *We have $1 \in \mathrm{Dom}(S)$ with $S(1) = 1$. In particular $\tau_t(1) = 1$ for all $t \in \mathbb{R}$ and $R(1) = 1$.*

We call S the ***antipode****, τ the* ***scaling group*** *and R the* ***unitary antipode*** *of M.*

Proposition 2.2.8 *We have the following relations for R and τ:*

(i) *$\tau_t \circ R = R \circ \tau_t$ for all $t \in \mathbb{R}$;*

(ii) *$\Delta \circ \tau_t = (\tau_t \otimes \tau_t) \circ \Delta$ for all $t \in \mathbb{R}$ and $\Delta \circ R = \sigma \circ (R \otimes R) \circ \Delta$ where σ is the flip map on $M \overline{\otimes} M$;*

(iii) *$S \circ \tau_t = \tau_t \circ S$ for all $t \in \mathbb{R}$ and $S \circ R = R \circ S$;*

(iv) *For all $t \in \mathbb{R}$ we have τ_t and R are isometries and normal.*

It follows easily that $S = \tau_{-i/2} \circ R$. We now turn to look at modular automorphism groups and the uniqueness of Haar weights. It follows from the properties given that $(\phi \circ R)((\mathrm{id} \otimes \omega)\Delta(x)) = \omega(1)(\phi \circ R)(x)$ for all $x \in \mathcal{M}_\phi^+$ and $\omega \in M_*^+$ and so from the following we may always assume that $\psi = \phi R$.

Notation 2.2.9 *We let σ and σ' denote the modular automorphism groups of ϕ and ψ respectively as given by Theorem 1.4.16.*

Proposition 2.2.10 *There exists a number $\nu > 0$ such that $\phi\tau_t = \nu^{-t}\phi$ for all $t \in \mathbb{R}$ and a unique injective positive operator P on $\mathcal{H}$ such that $P^{it}\Lambda(x) = \nu^{t/2}\Lambda(\tau_t(x))$ for all $t \in \mathbb{R}$ and $x \in \mathcal{N}_\phi$. We call ν the* ***scaling constant*** *of (M, Δ).*

Proposition 2.2.11 *The one parameter groups τ, σ and σ' all commute with each other and we have for all $t \in \mathbb{R}$ that*

$$\Delta \circ \sigma_t = (\tau_t \otimes \sigma_t) \circ \Delta, \qquad \Delta \circ \sigma'_t = (\sigma'_t \otimes \tau_{-t}) \circ \Delta,$$
$$\Delta \circ \tau_t = (\tau_t \otimes \tau_t) \circ \Delta, \qquad \Delta \circ \tau_t = (\sigma_t \otimes \sigma'_{-t}) \circ \Delta.$$

Theorem 2.2.12 *Let ϕ' denote a left invariant normal semi-finite weight on (M, Δ), then there exists some $r > 0$ such that $\phi' = r\phi$. There is a similar result for right invariant normal semi-finite weights.*

We record the following useful results in von Neumann algebraic quantum groups for later use. A proof is given in Lemma 4.6 in Aristov (2004). See also Proposition 5.13 in Kustermans & Vaes (2000).

Lemma 2.2.13 *Let $x, y \in M$ such that $\Delta(x) = y \otimes 1$, then $x, y \in \mathbb{C} \cdot 1$.*

2.2.2 C*-algebraic Quantum Groups

In this section we give an overview of locally compact quantum groups in the C*-algebraic setting, in particular in the reduced C*-algebraic setting. This section is so similar to section 2.2.1 we will often only point out the differences in definitions and theorems and not restate them. The main reference C*-algebraic quantum groups is the work Kustermans & Vaes (2000) and we refer there for all such proofs or justification (see also Vaes (2001)). We don't give the definitions of KMS weights and approximate KMS weights for the following definition as it is not required for this thesis, see Kustermans & Vaes (1999) for further details.

We note how in the following definition we have some density conditions that are absent in the von Neumann algebraic setting. These conditions follow from the axioms for a von Neumann algebraic quantum group but we must assume them in the C*-algebraic setting.

Definition 2.2.14 *A **C*-algebraic quantum group** is a C*-algebraic quantum semigroup* (A, Δ) *such that*

$$\overline{\text{lin}\ \{\Delta(x)(1 \otimes y) \mid x, y \in A\}}^{\|\cdot\|} = A \otimes_{min} A = \overline{\text{lin}\ \{\Delta(x)(y \otimes 1) \mid x, y \in A\}}^{\|\cdot\|} \quad (2.2)$$

and there exists a left and right invariant approximate KMS weights ϕ *and* ψ *on* (A, Δ) *respectively.*

We now give the main definition of the section. Note that it follows from Proposition 5.1 in Woronowicz (1996) that if the density conditions in the following definition are satisfied then the density conditions of the previous definition are automatically satisfied.

Definition 2.2.15 *A **reduced C*-algebraic quantum group** is a C*-algebraic quantum group* (A, Δ, ϕ, ψ) *such that the left invariant approximate KMS weight* ϕ *is faithful and that satisfy the following density conditions*

$$\begin{aligned} A &= \overline{\text{lin}\ \{(\omega \otimes \text{id})\Delta(x) \mid \omega \in A^*,\ x \in A\}}^{\|\cdot\|} \\ &= \overline{\text{lin}\ \{(\text{id} \otimes \omega)\Delta(x) \mid \omega \in A^*,\ x \in A\}}^{\|\cdot\|}. \end{aligned}$$

Throughout the rest of this section let $\mathcal{H}$ denote the GNS Hilbert space given by ϕ and we let $\Lambda : \mathcal{N}_\phi \to \mathcal{H}$ be the GNS embedding of A into $\mathcal{H}$. In this thesis we will assume that our reduced C*-algebraic quantum groups act on $\mathcal{H}$ and we will often drop the GNS map $\pi : A \to \mathcal{B}(\mathcal{H})$.

Theorem 2.2.16 *There exists a unique unitary operator* $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ *such that* $W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)(\Delta(y)(x \otimes 1))$ *for all* $x, y \in \mathcal{N}_\phi$. *Furthermore we have:*

(i) For all $x \in A$ *we have* $\Delta(x) = W^*(1 \otimes x)W$*;*

(ii) $W_{12}W_{13}W_{23} = W_{23}W_{12}$*;*

(iii) $A = \overline{\text{lin } \{(\text{id} \otimes \omega)(W) \mid \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\cdot\|}$.

*We call this operator the **multiplicative unitary** of* (A, Δ). *It is also sometimes referred to as the **left regular corepresentation** of the locally compact quantum group.*

We note that in the setting of Kustermans & Vaes (2000) we have an operator $V \in \mathrm{M}(A \otimes_{min} \mathcal{B}_0(\mathcal{H}))$ such that $(\pi \otimes \text{id})(V) = W$ and we call V the left regular corepresentation. As we are assuming A acts on $\mathcal{H}$ these two objects are the same for us so we use both names to refer to W.

The definition of the antipode for a C*-algebraic quantum group is identical to that of a von Neumann algebraic quantum group in Definition-Theorem 2.2.7 but with the σ-strong* topology replaced with the norm topology. So it is a closed map $S : A \to A$ with a core given by lin $\{(\text{id} \otimes \omega)(W) \mid \omega \in \mathcal{B}(\mathcal{H})_*\}$ where we have $S((\text{id} \otimes \omega)(W)) = (\text{id} \otimes \omega)(W^*)$ for all $\omega \in \mathcal{B}(\mathcal{H})_*$. Also τ is a norm continuous one-parameter group on A in the decomposition $S = R \circ \tau_{-i/2}$. The commutation relations are similar to the previous section.

2.2.3 Reduced C*-algebraic and von Neumann algebraic Quantum Groups

We show now that the two settings we have given above are essentially equivalent. In particular given a von Neumann algebraic quantum group we can associate a reduced C*-algebraic quantum group and similarly vice versa. We give details in this section, see Kustermans & Vaes (2000) and Kustermans & Vaes (2003) for further details.

First fix a von Neumann algebraic quantum group (M, Δ, ϕ, ψ). Then we have the following (see Proposition 1.6 in Kustermans & Vaes (2003) for the proof).

Theorem 2.2.17 *Let* $A = \overline{\text{lin}\ \{(\text{id} \otimes \omega)(W) \mid \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\|\cdot\|}$*, then* A *is a C*-algebra and there exists a restriction and corestriction of* $\Delta : M \to M \overline{\otimes} M$ *to a contractive map* $A \to \mathrm{M}(A \otimes_{min} A)$*. Similarly we can restriction* ϕ *and* ψ *such that* $(A, \Delta|_A, \phi|_A, \psi|_A)$ *is a reduced C*-algebraic quantum group.*

Proposition 2.2.18 *Let* $(A, \Delta^A, \phi^A, \psi^A)$ *denote the reduced C*-algebraic quantum group associated with a von Neumann algebraic quantum group* (M, Δ, ϕ, ψ) *from Theorem 2.2.17. Let* S*,* R *and* τ *denote the antipode, unitary antipode and scaling group of* M *respectively. Let* $R^A = R|_A$*,* $\tau_t^A = \tau_t|_A$ *for all* $t \in \mathbb{R}$ *and let* $\text{Dom}(S^A) = \text{Dom}(S) \cap A$ *with* $S^A(x) = S(x)$ *for all* $x \in \text{Dom}(S^A)$*. Then* S^A*,* R^A *and* τ^A *are the antipode, unitary antipode and scaling group of* A *respectively.*

Now let (A, Δ, ϕ, ψ) denote a reduced C*-algebraic quantum group, let $(\mathcal{H}, \pi, \Lambda)$ denote the GNS construction of ϕ and let $M = \pi(A)''$. Then there is an extension $\tilde{\Delta} : M \to M \overline{\otimes} M$ and extensions $\tilde{\phi}$ and $\tilde{\psi}$ of ϕ and ψ such that $(M, \tilde{\Delta}, \tilde{\phi}, \tilde{\psi})$ is a von Neumann algebraic quantum group. So in fact we have the following.

Proposition 2.2.19 *For every reduced C*-algebraic quantum group there is a von Neumann algebraic quantum group and similarly for every von Neumann algebraic quantum group there is a reduced C*-algebraic quantum group.*

2.2.4 The Locally Compact Quantum Group $\mathbb{G}$

We now have two different settings for locally compact quantum groups, namely the von Neumann algebraic setting and the reduced C*-algebraic setting. We have seen in the previous section that given a von Neumann algebraic quantum group we can form a reduced C*-algebraic quantum group and vice versa.

Consider the case where we have a locally compact group G, then we have a reduced C*-algebraic quantum group $(\mathrm{C}_0(G), \Delta, \phi, \psi)$ with $\mathrm{C}_0(G)$ a commutative C*-algebra (see Example 2.1.1). We also have a von Neumann algebraic quantum group $(\mathrm{L}^\infty(G), \Delta, \phi, \psi)$ such that $\mathrm{L}^\infty(G)$ is a commutative von Neumann algebra where we've used the same notation for the coproduct and Haar weights. So in this case the von Neumann algebraic and reduced C*-algebraic quantum groups are two different operator algebras generated by functions over the same group G. We mimic this for the case of quantum groups and we say there is some underlying locally compact quantum group $\mathbb{G}$ such that we have a reduced C*-algebraic version $(\mathrm{C}_0(\mathbb{G}), \Delta, \phi, \psi)$ and a von Neumann algebraic version $(\mathrm{L}^\infty(\mathbb{G}), \Delta, \phi, \psi)$. We have no way of computing such an object; however notationally we will still speak of locally compact quantum group $\mathbb{G}$. So we have the following.

Notation 2.2.20 *In the remainder of this thesis we will denote a locally compact quantum group by $\mathbb{G}$ (or $\mathbb{H}$) and then we denote the reduced C*-algebraic quantum group by $\mathrm{C}_0(\mathbb{G})$ and the von Neumann algebraic quantum group by $\mathrm{L}^\infty(\mathbb{G})$ where Δ, ϕ and ψ are implied. We will denote the GNS Hilbert space by $\mathrm{L}^2(\mathbb{G})$ and the predual of $\mathrm{L}^\infty(\mathbb{G})$ by $\mathrm{L}^1(\mathbb{G})$.*

We will discuss the $\mathrm{L}^1(\mathbb{G})$ object further in section 2.4. Given a locally compact quantum group we can form its opposite locally compact quantum group, we give the details of this in the next example that will be referred back to later.

Example 2.2.21 *Let $\mathbb{G}$ be a locally compact quantum group and let $\Delta^{op} = \sigma \circ \Delta : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{G})$ where $\sigma \in \mathcal{B}(\mathrm{L}^2(\mathbb{G}) \otimes \mathrm{L}^2(\mathbb{G}))$ is the flip map. Then we can show that $(\mathrm{L}^\infty(\mathbb{G}), \Delta^{op}, \psi, \phi)$ is a von Neumann algebraic quantum group called the opposite quantum group.*

2.2.5 The Universal C*-algebraic Quantum Group

We introduce briefly the universal C*-algebraic quantum group. We give only the existence theorem and properties that we require here. We refer the reader to Kustermans (2001) for further details.

Definition-Theorem 2.2.22 *There exists a C*-algebraic quantum group called the* ***universal C*-algebraic quantum group*** *and denoted* $(\mathrm{C}_0^u(\mathbb{G}), \Delta^u, \phi^u, \psi^u)$ *with a unique embedding* $\iota_0 : \mathrm{C}_0(\mathbb{G}) \to \mathrm{C}_0^u(\mathbb{G})$ *and a surjective* $*$*-homomorphism* $\pi : \mathrm{C}_0^u(\mathbb{G}) \to \mathrm{C}_0(\mathbb{G})$.

Proposition 2.2.23 *There exists a unique non-zero* $*$*-homomorphism* $\varepsilon^u : \mathrm{C}_0^u(\mathbb{G}) \to \mathbb{C}$ *such that* $(\varepsilon^u \otimes \mathrm{id}) \circ \Delta^u = \mathrm{id} = (\mathrm{id} \otimes \varepsilon^u) \circ \Delta^u$.

It can be shown that there exist an antipode, a scaling group, a unitary antipode and modular automorphism groups for the universal C*-algebraic quantum group, all denoted with a superscript u when needed and detailed further in Kustermans (2001).

2.3 Duality in Locally Compact Quantum Groups

Let $\mathbb{G}$ denote a locally compact quantum group, then in this section we will define a locally compact quantum group $\hat{\mathbb{G}}$ through its von Neumann algebraic quantum group $(\mathrm{L}^\infty(\hat{\mathbb{G}}), \hat{\Delta}^{\mathbb{G}}, \hat{\phi}^{\mathbb{G}}, \hat{\psi}^{\mathbb{G}})$. We define the von Neumann algebra and the coproduct first. Note the flip map on the coproduct below is a matter of convention where we have chosen to follow Kustermans & Vaes (2000).

We will also discuss the self-duality of locally compact quantum groups as a generalisation of the Pontryagin duality of locally compact Abelian groups.

We again refer to Kustermans & Vaes (2000) and Kustermans & Vaes (2003) as references of this section.

Notation 2.3.1 *We define the following*

$$\mathrm{L}^\infty(\hat{\mathbb{G}}) = \overline{\mathrm{lin}\ \{(\omega \otimes \mathrm{id})(W) \mid \omega \in \mathcal{B}(\mathrm{L}^2(\mathbb{G}))_*\}}^{\sigma-\mathrm{strong}*}$$

and the map $\hat{\Delta} : \mathrm{L}^\infty(\hat{\mathbb{G}}) \to \mathcal{B}(\mathrm{L}^2(\mathbb{G}) \otimes \mathrm{L}^2(\mathbb{G}))$ *is defined by*

$$\hat{\Delta}(x) = \Sigma W(x \otimes 1)W^*\Sigma$$

for all $x \in \mathrm{L}^\infty(\hat{\mathbb{G}})$ *where* Σ *is the flip map on* $\mathrm{L}^2(\mathbb{G}) \otimes \mathrm{L}^2(\mathbb{G})$.

Theorem 2.3.2 *We have that* $\mathrm{L}^\infty(\hat{\mathbb{G}})$ *is a von Neumann algebra and* $\hat{\Delta} : \mathrm{L}^\infty(\hat{\mathbb{G}}) \to \mathrm{L}^\infty(\hat{\mathbb{G}}) \overline{\otimes} \mathrm{L}^\infty(\hat{\mathbb{G}})$ *is the unique normal unital coassociative* $*$*-homomorphism such that* $(\mathrm{L}^\infty(\hat{\mathbb{G}}), \hat{\Delta})$ *is a von Neumann algebraic quantum group.*

Remark 2.3.3 *Equivalently we can define this as a reduced C*-algebraic quantum group with C*-algebra* $\mathrm{C}_0(\hat{\mathbb{G}}) = \overline{\mathrm{lin}\ \{(\omega \otimes \mathrm{id})(W) \mid \omega \in \mathcal{B}(L^2(\mathbb{G}))_*\}}^{\|\cdot\|}$ *and coproduct* $\hat{\Delta} : \mathrm{C}_0(\hat{\mathbb{G}}) \to \mathrm{M}(\mathrm{C}_0(\hat{\mathbb{G}}) \otimes_{min} \mathrm{C}_0(\hat{\mathbb{G}}))$ *given by* $\hat{\Delta}(x) = \Sigma W(x \otimes 1)W^*\Sigma$ *for all* $x \in \mathrm{C}_0(\hat{\mathbb{G}})$.

We have that $W \in \mathrm{L}^\infty(\mathbb{G}) \overline{\otimes} \mathrm{L}^\infty(\hat{\mathbb{G}})$ and from the definition of the coproducts and Definition-Theorem 2.2.5 we have

$$(\Delta \otimes \mathrm{id})(W) = W_{12}^* W_{23} W_{12} = W_{13} W_{23}$$

and

$$(\mathrm{id} \otimes \hat{\Delta})(W) = (1 \otimes \Sigma) W_{23} W_{12} W_{23}^* (1 \otimes \Sigma) = (1 \otimes \Sigma) W_{12} W_{13} (1 \otimes \Sigma) = W_{13} W_{12}.$$

As $\hat{\mathbb{G}}$ is a locally compact quantum group then we have an antipode, a unitary antipode and a scaling group denoted by $\hat{S}$, $\hat{R}$ and $\hat{\tau}$ respectively. It can be shown that the scaling constant of $\hat{\mathbb{G}}$ is given by ν^{-1} for ν the scaling constant for $\mathbb{G}$.

Given a locally compact quantum group $\mathbb{G}$ we can form its double dual $\hat{\hat{\mathbb{G}}}$ as the dual of $\hat{\mathbb{G}}$. We have the following theorem showing that locally compact quantum groups are closed under this duality and generalises the Pontryagin duality theorem for locally compact Abelian groups (see Chapter 4 in Folland (1994)).

Theorem 2.3.4 *Let $\mathbb{G}$ denote a locally compact quantum group, then there is a map θ : $L^\infty(\mathbb{G}) \to L^\infty(\hat{\hat{\mathbb{G}}})$ that is a $*$-isomorphism and satisfies $(\theta \otimes \theta) \circ \Delta = \hat{\hat{\Delta}} \circ \theta$. We can also identify $\hat{\hat{\phi}}$ with ϕ, $\hat{\hat{\psi}}$ with ψ and $\hat{\hat{\Lambda}}$ with Λ.*

For a locally compact quantum group $\mathbb{G}$, we have its dual locally compact quantum group $\hat{\mathbb{G}}$ and so we have a multiplicative unitary $\hat{W}$ of $\hat{\mathbb{G}}$ satisfying the appropriate relations. We also have the following relation of $\hat{W}$ with W.

Proposition 2.3.5 *For W the multiplicative unitary of a locally compact quantum group $\mathbb{G}$ and $\hat{W}$ the multiplicative unitary of its dual $\hat{\mathbb{G}}$ we have*

$$\hat{W} = \Sigma W^* \Sigma$$

where $\Sigma \in \mathcal{B}(L^2(\mathbb{G}) \otimes L^2(\mathbb{G}))$ is the flip map.

We will make use of the following formulas for the scaling group.

Proposition 2.3.6 *Let $\mathbb{G}$ be a locally compact quantum group with ϕ the left invariant Haar weight for $(L^\infty(\mathbb{G}), \Delta)$ and $\hat{\phi}$ the left invariant Haar weight for $(L^\infty(\hat{\mathbb{G}}), \hat{\Delta})$. Let J and ∇ denote the modular conjugation and modular operator of ϕ respectively, let $\hat{J}$ and $\hat{\nabla}$ denote the modular conjugation and modular operator of $\hat{\phi}$ respectively and let P denote the operator given in proposition 2.2.10. Then for all $x \in L^\infty(\mathbb{G})$, $y \in L^\infty(\hat{\mathbb{G}})$ and $t \in \mathbb{R}$ we have*

$$\tau_t(x) = \hat{\nabla}^{it} x \hat{\nabla}^{-it} = P^{it} x P^{-it}, \qquad R(x) = \hat{J} x^* \hat{J},$$
$$\hat{\tau}_t(y) = \nabla^{it} y \nabla^{-it} = P^{it} y P^{-it}, \qquad \hat{R}(y) = J y^* J.$$

Corollary 2.3.7 *As τ_t is normal for all $t \in \mathbb{R}$ we can consider the pre-adjoint $(\tau_t)_*$: $L^1(\mathbb{G}) \to L^1(\mathbb{G})$. Then we have a norm continuous one-parameter group τ_* on the Banach space $L^1(\mathbb{G})$ where $(\tau_*)_t(\omega) = (\tau_t)_*(\omega)$ for all $t \in \mathbb{R}$ and $\omega \in L^1(\mathbb{G})$.*

Proof

We show norm continuity as the rest is a straight forward application of the definition. It is enough to show that for all $\omega \in \mathrm{L}^1(\mathbb{G})$ that we have $\lim_{t\to 0}(\tau_*)_t(\omega) = \omega$ as $(\tau_*)_{t+s} = (\tau_*)_s \circ (\tau_*)_t$ for all $s, t \in \mathbb{R}$.

As $\mathrm{L}^\infty(\mathbb{G})$ is in standard position with respect to $\mathrm{L}^2(\mathbb{G})$ then a typical element of $\mathrm{L}^1(\mathbb{G})$ is given by $\omega_{\xi,\eta}$ for some $\xi, \eta \in \mathrm{L}^2(\mathbb{G})$. Therefore it is sufficient to show that $\lim_{t\to 0}(\tau_*)_t(\omega_{\xi,\eta}) = \omega_{\xi,\eta}$. First we show that the map $\mathbb{R} \to \mathrm{L}^2(\mathbb{G})$ given by $t \mapsto e^{itP}\xi$ is continuous. As P is positive and injective we can define $\ln P$ and we consider the map $\mathbb{R} \to \mathrm{L}^2(\mathbb{G})$ given by $t \mapsto e^{it\ln P}\xi = P^{it}\xi$. Then by Stone's theorem (see for example Section 10.5 in Conway (1990)) we have that this map is continuous.

Now let $\varepsilon > 0$ and as the previous map is continuous we can find $\delta > 0$ such that $\|(P^{-it} - \mathrm{id})\xi\| < \dfrac{\varepsilon}{2\|\eta\|}$ and $\|(P^{-it} - \mathrm{id})\eta\| < \dfrac{\varepsilon}{2\|\xi\|}$ for all $|t| < \delta$. For all $x \in \mathrm{L}^\infty(\mathbb{G})$ and $t \in \mathbb{R}$ and using that $\tau_t(x) = P^{it}xP^{-it}$ from the previous proposition we have

$$\begin{aligned}
|\langle x, (\tau_*)_t(\omega_{\xi,\eta})\rangle - \langle x, \omega_{\xi,\eta}\rangle| &= \left|\left(xP^{-it}\xi\middle|P^{-it}\eta\right) - (x\xi|\eta)\right| \\
&\leqslant \left|\left(xP^{-it}\xi\middle|P^{-it}\eta\right) - \left(xP^{-it}\xi\middle|\eta\right)\right| + \left|\left(xP^{-it}\xi\middle|\eta\right) - (x\xi|\eta)\right| \\
&\leqslant \left|\left(xP^{-it}\xi\middle|(P^{-it} - \mathrm{id})\eta\right)\right| + \left|\left(x(P^{-it} - \mathrm{id})\xi\middle|\eta\right)\right|.
\end{aligned}$$

Then for all $x \in \mathrm{L}^\infty(\mathbb{G})$ with $\|x\| \leqslant 1$ we have

$$\begin{aligned}
|\langle x, (\tau_*)_t(\omega_{\xi,\eta})\rangle - \langle x, \omega_{\xi,\eta}\rangle| &\leqslant \|x\|\|P^{-it}\xi\|\|(P^{-it} - \mathrm{id})\eta\| + \|x\|\|(P^{-it} - \mathrm{id})\xi\|\|\eta\| \\
&\leqslant \|\xi\|\|(P^{-it} - \mathrm{id})\eta\| + \|(P^{-it} - \mathrm{id})\xi\|\|\eta\| < \varepsilon.
\end{aligned}$$

Then we can take the supremum over all such x in the left hand side to get

$$\|(\tau_*)_t(\omega_{\xi,\eta}) - \omega_{\xi,\eta}\| < \varepsilon$$

and thus $\lim_{t\to 0}(\tau_*)_t(\omega_{\xi,\eta}) = \omega_{\xi,\eta}$. □

2.4 Duals and Preduals of Operator Algebraic Quantum Groups

Again let $\mathbb{G}$ denote a locally compact quantum group throughout this section. We discuss the dual space of the C*-algebra $\mathrm{C}_0(\mathbb{G})$ and the predual of the von Neumann algebra $\mathrm{L}^\infty(\mathbb{G})$. Given the coproduct Δ on $\mathrm{C}_0(\mathbb{G})$ we can define a bilinear map $m : \mathrm{C}_0(\mathbb{G})^* \hat{\otimes} \mathrm{C}_0(\mathbb{G})^* \to \mathrm{C}_0(\mathbb{G})^*$ that gives $\mathrm{C}_0(\mathbb{G})^*$ the structure of a Banach algebra as follows: let $\omega, \kappa \in \mathrm{C}_0(\mathbb{G})^*$ and denote by $\omega * \kappa \in \mathrm{C}_0(\mathbb{G})^*$ the map

$$\langle x, \omega * \kappa \rangle = \langle \Delta(x), \omega \otimes \kappa \rangle$$

for all $x \in \mathrm{C}_0(\mathbb{G})$ where $\omega \otimes \kappa$ denotes one of the maps $\omega \circ (\mathrm{id} \otimes \kappa) = \kappa \circ (\mathrm{id} \otimes \omega)$. It follows that $\mathrm{C}_0(\mathbb{G})^*$ is a Banach algebra by using that Δ is a $*$-homomorphism and thus contractive and so

$$|\langle x, \omega * \kappa \rangle| \leqslant \|\Delta(x)\| \|\omega \otimes \kappa\| \leqslant \|x\| \|\omega\| \|\kappa\|.$$

So we have $\|\omega * \kappa\| \leqslant \|\omega\| \|\kappa\|$ for all $\omega, \kappa \in \mathrm{C}_0(\mathbb{G})^*$.

Definition 2.4.1 *Given a locally compact quantum group $\mathbb{G}$ we consider $\mathrm{C}_0(\mathbb{G})^*$ as a Banach algebra with multiplication given as above.*

We can also consider $\mathrm{L}^1(\mathbb{G}) \subset \mathrm{C}_0(\mathbb{G})^*$ as a Banach subalgebra where we remind that $\mathrm{L}^1(\mathbb{G})$ is defined as the predual of the von Neumann algebra $\mathrm{L}^\infty(\mathbb{G})$.

Proposition 2.4.2 *We have that*

$$\mathrm{L}^1(\mathbb{G}) = \overline{\mathrm{lin}\ \{x \cdot \phi \cdot y^* \mid x, y \in \mathcal{N}_\phi\}}^{\|\cdot\|} = \overline{\mathrm{lin}\ \{\omega \circ \pi \mid \omega \in \mathcal{B}(\mathrm{L}^2(\mathbb{G}))_*\}}^{\|\cdot\|}$$

where we let $\pi : \mathrm{C}_0(\mathbb{G}) \to \mathcal{B}(\mathrm{L}^2(\mathbb{G}))$ denote the isometric GNS map $\pi(a)\Lambda(b) = \Lambda(ab)$

for all $a, b \in \mathrm{C}_0(\mathbb{G})$ *from the GNS construction of* ϕ*. Furthermore* $\mathrm{L}^1(\mathbb{G})$ *is a two sided ideal of* $\mathrm{C}_0(\mathbb{G})^*$*.*

Definition 2.4.3 *Let* $\lambda : \mathrm{L}^1(\mathbb{G}) \to \mathcal{B}(\mathcal{H})$ *be the map* $\omega \mapsto (\omega \otimes \mathrm{id})(W)$*. Then* λ *is the* ***left regular representation*** *of* $\mathbb{G}$*.*

We have that λ is a homomorphism as

$$\begin{aligned}\lambda(\omega * \kappa) &= ((\omega * \kappa) \otimes \mathrm{id})(W) = (\omega \otimes \kappa \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(W) \\ &= (\omega \otimes \kappa \otimes \mathrm{id})(W_{13}W_{23}) = (\omega \otimes \mathrm{id})(W)(\kappa \otimes \mathrm{id})(W) = \lambda(\omega)\lambda(\kappa)\end{aligned}$$

where we've used Definition-Theorem 2.2.5 property (iii). We also have the following.

Proposition 2.4.4 *We can extend* λ *to a contractive injective linear homomorphism* $\lambda : \mathrm{C}_0(\mathbb{G})^* \to \mathrm{M}(\mathrm{C}_0(\hat{\mathbb{G}}))$ *given by* $\omega \mapsto (\omega \otimes \mathrm{id})(W)$ *for all* $\omega \in \mathrm{C}_0(\mathbb{G})^*$*. Furthermore we have* $\lambda(\mathrm{L}^1(\mathbb{G}))$ *is a dense subalgebra of* $\mathrm{C}_0(\hat{\mathbb{G}})$*.*

2.5 Products of Locally Compact Quantum Groups

Let $\mathbb{G}$ and $\mathbb{H}$ denote locally compact quantum groups as per Notation 2.2.20. We will now define a locally compact quantum group $\mathbb{G} \times \mathbb{H}$ in a similar fashion to the product of two groups. Whilst this section is not necessarily new work the author is unaware of a suitable reference in this section so we give proofs (though we will heavily make use of the work in Vaes & Vainerman (2003)).

Throughout this section, to distinguish maps associated to a particular quantum group, we will add a superscript of the quantum group to the map. For example $\Delta^{\mathbb{G}}$, $\tau^{\mathbb{G}}$ and $R^{\mathbb{G}}$ refer to the coproduct, scaling group and unitary antipode of the locally compact quantum group $\mathbb{G}$.

We give some motivation first and discuss products of groups now.

Example 2.5.1 *Let G and H be locally compact groups, then we can form a locally compact group $G \times H$ given by the Cartesian product of G and H and given product $(x, y) \cdot (x', y') = (xx', yy')$.*

We would like to define a map $\Delta^{G\times H}$ on $\mathrm{C}_0(G \times H) \cong_i \mathrm{C}_0(\mathbb{G}) \otimes_{min} \mathrm{C}_0(\mathbb{H})$ that gives this group product in terms of Δ^G and Δ^H. So for $F \in \mathrm{C}_0(G \times H)$, $x, x' \in G$ and $y, y' \in H$ we wish to have

$$\Delta^{G\times H}(F)((x, y), (x', y')) = F(xx', yy')$$

but we have

$$\begin{aligned} F(xx', yy') &= [(\Delta^G \otimes \Delta^H)(F)]((x, x'), (y, y')) \\ &= [(\sigma_{23} \circ (\Delta^G \otimes \Delta^H))(F)]((x, y), (x', y')) \end{aligned}$$

where σ is the flip map and thus $\sigma_{23} : \mathrm{C}_0(G \times G \times H \times H) \to \mathrm{C}_0(G \times H \times G \times H)$ flips the middle two legs. So we can define $\Delta^{G\times H} := \sigma_{23} \circ (\Delta^G \otimes \Delta^H)$.

We now state the main theorem of this section that we will take as a definition of the product $\mathbb{G} \times \mathbb{H}$ of two locally compact quantum groups $\mathbb{G}$ and $\mathbb{H}$. Whilst this is not necessarily a new result to the author's knowledge it is not previously recorded in the literature in this form.

Definition-Theorem 2.5.2 *Let $\mathbb{G}$ and $\mathbb{H}$ denote locally compact quantum groups. Then there exists a locally compact quantum group $\mathbb{G} \times \mathbb{H}$ such that we have:*

(i) $\mathrm{L}^\infty(\mathbb{G} \times \mathbb{H}) = \mathrm{L}^\infty(\mathbb{G}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{H})$;

(ii) coproduct $\Delta^{\mathbb{G}\times\mathbb{H}} = \sigma_{23} \circ (\Delta^{\mathbb{G}} \otimes \Delta^{\mathbb{H}})$ *where σ is the flip map on* $\mathrm{L}^\infty(\mathbb{G}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{H})$;

(iii) multiplicative unitary $W^{\mathbb{G}\times\mathbb{H}} = \sigma_{23}(W^{\mathbb{G}} \otimes W^{\mathbb{H}})$;

(iv) $\mathrm{C}_0(\mathbb{G} \times \mathbb{H}) = \mathrm{C}_0(\mathbb{G}) \otimes_{min} \mathrm{C}_0(\mathbb{H})$.

For the following proof we will reference many results in Vaes & Vainerman (2003). The reader may find it helpful while reading this proof to have a copy of this at hand.

Proof

Let $\hat{\mathbb{G}}$ denote the dual of $\mathbb{G}$ and let $\left(\mathrm{L}^\infty(\hat{\mathbb{G}}), \hat{\Delta}^{\mathbb{G}}, \hat{\phi}^{\mathbb{G}}, \hat{\psi}^{\mathbb{G}}\right)$ denote its von Neumann algebraic quantum group. We let $\tau : \mathrm{L}^\infty(\hat{\mathbb{G}}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{H}) \to \mathrm{L}^\infty(\hat{\mathbb{G}}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{H})$ be the identity map, $\mathcal{U} = 1_{\hat{\mathbb{G}}} \otimes 1_{\hat{\mathbb{G}}} \otimes 1_{\mathbb{H}}$ and $\mathcal{V} = 1_{\hat{\mathbb{G}}} \otimes 1_{\mathbb{H}} \otimes 1_{\mathbb{H}}$ to give the triple $(\tau, \mathcal{U}, \mathcal{V})$. Then by Definition 2.1 in Vaes & Vainerman (2003) we have an action $\alpha : \mathrm{L}^\infty(\mathbb{H}) \to \mathrm{L}^\infty(\hat{\mathbb{G}}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{H})$ given by $y \mapsto 1 \otimes y$ and an action $\beta : \mathrm{L}^\infty(\hat{\mathbb{G}}) \to \mathrm{L}^\infty(\hat{\mathbb{G}}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{H})$ given by $x \mapsto x \otimes 1$ satisfying the properties of the same definition to give a cocycle matching $(\tau, \mathcal{U}, \mathcal{V})$.

We let $\phi^{\mathbb{G}}$ denote the left invariant n.s.f. Haar weight of $\mathbb{G}$, we let $(\mathrm{L}^2(\mathbb{G}), \pi^{\mathbb{G}}, \Lambda^{\mathbb{G}})$ denote the GNS construction of $\phi^{\mathbb{G}}$ and $W^{\mathbb{G}}$ the multiplicative unitary of $\mathbb{G}$ and similarly for $\mathbb{H}$. Also we let $\hat{\phi}^{\mathbb{G}}$ denote the Haar weight of $\hat{\mathbb{G}}$ which has GNS construction $(\mathrm{L}^2(\mathbb{G}), \pi^{\mathbb{G}}, \hat{\Lambda}^{\mathbb{G}})$. We have the multiplicative unitary $\hat{W}^{\mathbb{G}}$ of $\hat{\mathbb{G}}$ given by Proposition 2.3.5 and so from Definition 2.2.2 in Vaes & Vainerman (2003) we have a von Neumann algebra generated in $\mathcal{B}(\mathrm{L}^2(\mathbb{G})) \overline{\otimes} \mathrm{L}^\infty(\mathbb{H})$ by

$$\left\{(\omega \otimes \mathrm{id} \otimes \mathrm{id}) \left(\hat{W}^{\mathbb{G}} \otimes 1\right) \;\middle|\; \omega \in \mathrm{L}^1(\hat{\mathbb{G}})\right\} \text{ and } \alpha(\mathrm{L}^\infty(\mathbb{H})) = \{1 \otimes y \mid y \in \mathrm{L}^\infty(\mathbb{G})\}.$$

We have

$$\begin{aligned}\mathrm{L}^\infty(\mathbb{G}) = \mathrm{L}^\infty(\hat{\hat{\mathbb{G}}}) &= \left\{(\omega \otimes \mathrm{id})(\hat{W}^{\mathbb{G}}) \;\middle|\; \omega \in \mathcal{B}(\mathrm{L}^2(\mathbb{G}))_*\right\} \\ &= \left\{(\omega \otimes \mathrm{id})(\hat{W}^{\mathbb{G}}) \;\middle|\; \omega \in \mathrm{L}^1(\hat{\mathbb{G}})\right\}\end{aligned}$$

and so the von Neumann algebra generated is $\mathrm{L}^\infty(\mathbb{G}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{H})$. We let $\mathrm{L}^\infty(\mathbb{G} \times \mathbb{H})$ denote this von Neumann algebra.

We need several flip maps that we denote as follows

(i) Σ be the flip map from $\mathrm{L}^2(\mathbb{G}) \otimes \mathrm{L}^2(\mathbb{H})$ to $\mathrm{L}^2(\mathbb{H}) \otimes \mathrm{L}^2(\mathbb{G})$ and Σ^* the reverse flip;

(ii) $\Sigma^{\mathbb{G} \times \mathbb{H}}$ denotes the flip map on $\mathrm{L}^2(\mathbb{G} \times \mathbb{H}) \otimes \mathrm{L}^2(\mathbb{G} \times \mathbb{H})$;

(iii) $\Sigma^{\mathbb{G}}$ the flip map on $\mathrm{L}^2(\mathbb{G})\otimes\mathrm{L}^2(\mathbb{G})$ and similarly for $\mathbb{H}$;

(iv) σ will denote the flip on C*-algebra or von Neumann algebra tensors, e.g. for C*-algebras A and B we let $\sigma : A\otimes_{min} B \to B\otimes_{min} A$ be the usual flip map.

We will also use the leg notation on many of these maps.

It also follows from Definition 2.2 in Vaes & Vainerman (2003) that we have

$$\hat{W}^{\mathbb{G}\times\mathbb{H}} = \sigma_{23}(\hat{W}^{\mathbb{G}}\otimes\hat{W}^{\mathbb{H}}), \qquad W^{\mathbb{G}\times\mathbb{H}} = \Sigma\left(\hat{W}^{\mathbb{G}\times\mathbb{H}}\right)^{*}\Sigma^{*}$$

and we calculate for $\xi_1,\eta_1,\xi_3,\eta_3\in\mathcal{H}_\phi$ and $\xi_2,\eta_2,\xi_4,\eta_4\in\mathcal{H}_\psi$ that

$$\begin{aligned}
&\big(W^{\mathbb{G}\times\mathbb{H}}(\xi_1\otimes\xi_2\otimes\xi_3\otimes\xi_4)\big|\eta_1\otimes\eta_2\otimes\eta_3\otimes\eta_4\big)\\
&\quad=\left(\hat{W}^{\mathbb{G}\times\mathbb{H}}(\xi_3\otimes\xi_4\otimes\xi_1\otimes\xi_2)\Big|\eta_3\otimes\eta_4\otimes\eta_1\otimes\eta_2\right)\\
&\quad=\left(\xi_3\otimes\xi_1\otimes\xi_4\otimes\xi_2\Big|(\hat{W}^{\mathbb{G}}\otimes\hat{W}^{\mathbb{H}})(\eta_3\otimes\eta_1\otimes\eta_4\otimes\eta_2)\right)\\
&\quad=\big(\xi_1\otimes\xi_3\otimes\xi_2\otimes\xi_4\big|((W^{\mathbb{G}})^*\otimes(W^{\mathbb{H}})^*)(\eta_1\otimes\eta_3\otimes\eta_2\otimes\eta_4)\big)\\
&\quad=\big((\Sigma_{23}(W^{\mathbb{G}}\otimes W^{\mathbb{H}})\Sigma_{23}^*)(\xi_1\otimes\xi_2\otimes\xi_3\otimes\xi_4)\big|\eta_1\otimes\eta_2\otimes\eta_3\otimes\eta_4\big)
\end{aligned}$$

and so we have $W^{\mathbb{G}\times\mathbb{H}} = \sigma_{23}(W^{\mathbb{G}}\otimes W^{\mathbb{H}})$.

We claim that $\Delta^{\mathbb{G}\times\mathbb{H}} = \sigma_{23}\circ(\Delta^{\mathbb{G}}\otimes\Delta^{\mathbb{H}})$. We know from Definition 2.2.1 in Vaes & Vainerman (2003) that $\Delta^{\mathbb{G}\times\mathbb{H}}(x) = (W^{\mathbb{G}\times\mathbb{H}})^*(1\otimes x)W^{\mathbb{G}\times\mathbb{H}}$ for all $x\in\mathrm{L}^\infty(\mathbb{G}\times\mathbb{H})$ and we show that $\big((\sigma_{23}\circ(\Delta^{\mathbb{G}}\otimes\Delta^{\mathbb{H}}))(x)\big)\,(W^{\mathbb{G}\times\mathbb{H}})^* = (W^{\mathbb{G}\times\mathbb{H}})^*(1\otimes x)$ for all $x\in\mathrm{L}^\infty(\mathbb{G}\times\mathbb{H})$. For all $y\in\mathrm{L}^\infty(\mathbb{G})$, $z\in\mathrm{L}^\infty(\mathbb{H})$, $a,a'\in\mathrm{C}_0(\mathbb{G})$ and $b,b'\in\mathrm{C}_0(\mathbb{H})$ we have

$$\begin{aligned}
&\big((\sigma_{23} \circ (\Delta^{\mathbb{G}} \otimes \Delta^{\mathbb{H}}))(y \otimes z)\big)\, (W^{\mathbb{G}\times\mathbb{H}})^* (\Lambda^{\mathbb{G}}(a) \otimes \Lambda^{\mathbb{H}}(b) \otimes \Lambda^{\mathbb{G}}(a') \otimes \Lambda^{\mathbb{H}}(b')) \\
&\qquad = (\Sigma_{23}(\Delta(y) \otimes \Delta(z))\Sigma_{23}{}^*)\, \Sigma_{23}((W^{\mathbb{G}})^* \otimes (W^{\mathbb{H}})^*)(\Lambda^{\mathbb{G}}(a) \otimes \Lambda^{\mathbb{G}}(a') \otimes \Lambda^{\mathbb{H}}(b) \otimes \Lambda^{\mathbb{H}}(b')) \\
&\qquad = (\Sigma_{23}(\Delta(y) \otimes \Delta(z)))\, \big((\Lambda^{\mathbb{G}} \otimes \Lambda^{\mathbb{G}})(\Delta(a')(a \otimes 1)) \otimes (\Lambda^{\mathbb{H}} \otimes \Lambda^{\mathbb{H}})(\Delta(b')(b \otimes 1))\big) \\
&\qquad = \Sigma_{23}\, \big((\Lambda^{\mathbb{G}} \otimes \Lambda^{\mathbb{G}})(\Delta(ya')(a \otimes 1)) \otimes (\Lambda^{\mathbb{H}} \otimes \Lambda^{\mathbb{H}})(\Delta(zb')(b \otimes 1)\big) \\
&\qquad = \Sigma_{23}\, \big((W^{\mathbb{G}})^*(\Lambda^{\mathbb{G}}(a) \otimes \Lambda^{\mathbb{G}}(ya')) \otimes (W^{\mathbb{H}})^*(\Lambda^{\mathbb{H}}(b) \otimes \Lambda^{\mathbb{H}}(zb'))\big) \\
&\qquad = \Sigma_{23}((W^{\mathbb{G}})^* \otimes (W^{\mathbb{H}})^*)\Sigma^*{}_{23}(\Lambda^{\mathbb{G}}(a) \otimes \Lambda^{\mathbb{H}}(b) \otimes \Lambda^{\mathbb{G}}(ya') \otimes \Lambda^{\mathbb{H}}(zb')) \\
&\qquad = (W^{\mathbb{G}\times\mathbb{H}})^*(1 \otimes 1 \otimes y \otimes z)(\Lambda^{\mathbb{G}}(a) \otimes \Lambda^{\mathbb{H}}(b) \otimes \Lambda^{\mathbb{G}}(a') \otimes \Lambda^{\mathbb{H}}(b')).
\end{aligned}$$

As this holds for all $a, a' \in \mathrm{C}_0(\mathbb{G})$ and $b, b' \in \mathrm{C}_0(\mathbb{H})$ we have

$$\big((\sigma_{23} \circ (\Delta^{\mathbb{G}} \otimes \Delta^{\mathbb{H}}))(y \otimes z)\big)\, (W^{\mathbb{G}\times\mathbb{H}})^* = (W^{\mathbb{G}\times\mathbb{H}})^*(1 \otimes 1 \otimes y \otimes z)$$

for all $y \in \mathrm{L}^\infty(\mathbb{G})$ and $z \in \mathrm{L}^\infty(\mathbb{H})$ and thus by linearity and continuity we have

$$\big((\sigma_{23} \circ (\Delta^{\mathbb{G}} \otimes \Delta^{\mathbb{H}}))(x)\big)\, (W^{\mathbb{G}\times\mathbb{H}})^* = (W^{\mathbb{G}\times\mathbb{H}})^*(1 \otimes x)$$

for all $x \in \mathrm{L}^\infty(\mathbb{G} \times \mathbb{H})$ as was to be shown.

So we have shown that $\mathbb{G} \times \mathbb{H}$ is a locally compact quantum group with $(\mathrm{L}^\infty(\mathbb{G} \times \mathbb{H}), \Delta^{\mathbb{G}\times\mathbb{H}})$ the von Neumann algebraic quantum group and we now consider the reduced C*-algebraic quantum group. We have

$$\mathrm{C}_0(\mathbb{G} \times \mathbb{H}) = \overline{\mathrm{lin}\ \big\{(\mathrm{id}^{\mathbb{G}\times\mathbb{H}} \otimes \omega)(W^{\mathbb{G}\times\mathbb{H}})\ \big|\ \omega \in \mathrm{L}^1(\mathbb{G} \times \mathbb{H})\big\}}^{\|\cdot\|}$$

forming the reduced C*-algebraic quantum group and

$$\mathrm{C}_0(\mathbb{G}) \otimes_{min} \mathrm{C}_0(\mathbb{H}) = \overline{\mathrm{lin}\ \{x \otimes y\ |\ x \in \mathrm{C}_0(\mathbb{G}), y \in \mathrm{C}_0(\mathbb{H})\}}^{\|\cdot\|}$$

with the closures taken inside $\mathcal{B}(\mathrm{L}^2(\mathbb{G}) \otimes \mathrm{L}^2(\mathbb{H}))$. We want to show these are equal.

Let $\omega \in \mathrm{L}^1(\mathbb{G})$ and $\kappa \in \mathrm{L}^1(\mathbb{H})$, then we have

$$(\mathrm{id} \otimes \omega)(W^{\mathbb{G}}) \otimes (\mathrm{id} \otimes \kappa)(W^{\mathbb{H}}) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega \otimes \kappa)(W^{\mathbb{G}\times\mathbb{H}}) \in \mathrm{C}_0(\mathbb{G} \times \mathbb{H})$$

and so $\{(\mathrm{id} \otimes \omega)(W^{\mathbb{G}}) \mid \omega \in \mathrm{L}^1(\mathbb{G})\} \odot \{(\mathrm{id} \otimes \kappa)(W^{\mathbb{H}}) \mid \kappa \in \mathrm{L}^1(\mathbb{H})\} \subset \mathrm{C}_0(\mathbb{G} \times \mathbb{H})$. Taking the closure of the left hand side we get

$$\mathrm{C}_0(\mathbb{G}) \otimes_{min} \mathrm{C}_0(\mathbb{H}) \subset \mathrm{C}_0(\mathbb{G} \times \mathbb{H}).$$

Now let $\Omega \in \mathrm{L}^1(\mathbb{G} \times \mathbb{H})$ and so $(\mathrm{id} \otimes \Omega)(W^{\mathbb{G}\times\mathbb{H}}) \in \mathrm{C}_0(\mathbb{G} \times \mathbb{H})$. As $\mathrm{L}^1(\mathbb{G} \times \mathbb{H}) \cong_i \mathrm{L}^1(\mathbb{G}) \hat{\otimes} \mathrm{L}^1(\mathbb{H})$ we have a net $(\Omega_\alpha) \subset \mathrm{L}^1(\mathbb{G}) \odot \mathrm{L}^1(\mathbb{H})$ such that $\lim_\alpha \Omega_\alpha = \Omega$ and so $\lim_\alpha \langle z, \Omega_\alpha \rangle = \langle z, \Omega \rangle$ for all $z \in \mathrm{L}^\infty(\mathbb{G} \times \mathbb{H})$. For all α we have

$$(\mathrm{id} \otimes \Omega_\alpha)(W^{\mathbb{G}\times\mathbb{H}}) = (\Omega_\alpha)_{24}(W^{\mathbb{G}} \otimes W^{\mathbb{H}}) \in \mathrm{C}_0(\mathbb{G}) \odot \mathrm{C}_0(\mathbb{H})$$

and so $(\mathrm{id} \otimes \Omega)(W^{\mathbb{G}\times\mathbb{H}}) = \lim_\alpha (\mathrm{id} \otimes \Omega_\alpha)(W^{\mathbb{G}\times\mathbb{H}}) \in \mathrm{C}_0(\mathbb{G}) \otimes_{min} \mathrm{C}_0(\mathbb{H})$. It then follows that $\mathrm{C}_0(\mathbb{G} \times \mathbb{H}) \subset \mathrm{C}_0(\mathbb{G}) \otimes_{min} \mathrm{C}_0(\mathbb{H})$. □

We now calculate explicit formulas for the left Haar weight, the unitary antipode and the scaling group of the locally compact quantum group $\mathbb{G} \times \mathbb{H}$.

Lemma 2.5.3 *For $x \in \mathrm{L}^\infty(\mathbb{G} \times \mathbb{H})^+$ we have*

$$(\mathrm{id} \otimes \mathrm{id} \otimes \phi^{\mathbb{H}})(\mathrm{id} \otimes \Delta^{\mathbb{H}})(x) = (\mathrm{id} \otimes \phi^{\mathbb{H}})(x) \otimes 1^{\mathbb{H}} \tag{2.3}$$

and

$$(\phi^{\mathbb{G}} \otimes \mathrm{id} \otimes \mathrm{id})((\Delta^{\mathbb{G}})^{op} \otimes \mathrm{id})(x) = 1^{\mathbb{G}} \otimes (\phi^{\mathbb{G}} \otimes \mathrm{id})(x) \tag{2.4}$$

where each side of both equations is in $\mathrm{L}^\infty(\mathbb{G} \times \mathbb{H})^+_{ext}$.

Proof

Equation (2.3) is given by Proposition 3.1 in Kustermans & Vaes (2003) and we prove Equation (2.4) as a consequence of this. We fix $x \in \mathrm{L}^\infty(\mathbb{G} \times \mathbb{H})^+$ throughout this proof. We have an operator valued weight $\phi^{\mathbb{G}} \otimes \mathrm{id} \otimes \mathrm{id} : \mathrm{L}^\infty(\mathbb{G} \times \mathbb{G} \times \mathbb{H})^+ \to \mathrm{L}^\infty(\mathbb{G} \times \mathbb{H})^+_{ext}$ and as $((\Delta^{\mathbb{G}})^{op} \otimes \mathrm{id})(x) \in \mathrm{L}^\infty(\mathbb{G} \times \mathbb{G} \times \mathbb{H})^+$ we have a map

$$(\phi^{\mathbb{G}} \otimes \mathrm{id} \otimes \mathrm{id})((\Delta^{\mathbb{G}})^{op} \otimes \mathrm{id})(x)) : \mathrm{L}^1(\mathbb{G} \times \mathbb{H})^+ \to [0, \infty] \tag{2.5}$$

given by $\omega \mapsto \phi^{\mathbb{G}}((\mathrm{id} \otimes \omega)((\Delta^{\mathbb{G}})^{op} \otimes \mathrm{id})(x))$. Also let $\sigma : \mathrm{L}^\infty(\mathbb{G}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{H}) \to \mathrm{L}^\infty(\mathbb{H}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{G})$ and $\sigma' : \mathrm{L}^\infty(\mathbb{H}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{H})$ be the flip maps with pre-adjoints $\sigma_* : \mathrm{L}^1(\mathbb{H} \times \mathbb{G}) \to \mathrm{L}^1(\mathbb{G} \times \mathbb{H})$ and $\sigma'_* : \mathrm{L}^1(\mathbb{G} \times \mathbb{H}) \to \mathrm{L}^1(\mathbb{H} \times \mathbb{G})$. Then we have $(\mathrm{id} \otimes \Delta^{\mathbb{G}})(\sigma(x)) \in \mathrm{L}^\infty(\mathbb{H} \times \mathbb{G} \times \mathbb{G})^+$ and so we have a map

$$(\mathrm{id} \otimes \mathrm{id} \otimes \phi^{\mathbb{G}})(\mathrm{id} \otimes \Delta^{\mathbb{G}})(x) \circ \sigma'_* : \mathrm{L}^1(\mathbb{G} \times \mathbb{H})^+ \to [0, \infty]. \tag{2.6}$$

We claim the maps in Equations (2.5) and (2.6) are equal, that is for any $\omega \in \mathrm{L}^1(\mathbb{G} \times \mathbb{H})^+$ we have either both acting on ω are finite and equal or both are infinite.

Let $\omega \in \mathrm{L}^1(\mathbb{G} \times \mathbb{H})^+$ and $y = z \otimes z' \in \mathrm{L}^\infty(\mathbb{G} \times \mathbb{H})^+$ for $z \in \mathrm{L}^\infty(\mathbb{G})^+$ and $z' \in \mathrm{L}^\infty(\mathbb{H})^+$, then we have

$$(\sigma'_*(\omega) \otimes \mathrm{id}) \big((\mathrm{id} \otimes \Delta^{\mathbb{G}})(\sigma(y))\big) = (\sigma'_*(\omega) \otimes \mathrm{id})(z' \otimes \Delta^{\mathbb{G}}(z)) = (\mathrm{id} \otimes \omega) \big(((\Delta^{\mathbb{G}})^{op} \otimes \mathrm{id})(y)\big)$$

and so by linearity and continuity we have

$$(\sigma'_*(\omega) \otimes \mathrm{id}) \big((\mathrm{id} \otimes \Delta^{\mathbb{G}})(\sigma(x))\big) = (\mathrm{id} \otimes \omega) \big(((\Delta^{\mathbb{G}})^{op} \otimes \mathrm{id})(x)\big)$$

for all $x \in \mathrm{L}^\infty(\mathbb{G} \times \mathbb{H})^+$. So if ω is such that $[(\mathrm{id} \otimes \mathrm{id} \otimes \phi^{\mathbb{G}})(\mathrm{id} \otimes \Delta^{\mathbb{G}})(\sigma(x))](\sigma'_*(\omega))$ is finite, then we have

$$\begin{aligned}[(\mathrm{id}\otimes\mathrm{id}\otimes\phi^{\mathbb{G}})(\mathrm{id}\otimes\Delta^{\mathbb{G}})(\sigma(x))](\sigma'_*(\omega)) &= \phi^{\mathbb{G}}[(\sigma'_*(\omega)\otimes\mathrm{id})(\mathrm{id}\otimes\Delta^{\mathbb{G}})(\sigma(x))] \\ &= \phi^{\mathbb{G}}[(\mathrm{id}\otimes\omega)((\Delta^{\mathbb{G}})^{op}\otimes\mathrm{id})(x)] = [(\phi^{\mathbb{G}}\otimes\mathrm{id}\otimes\mathrm{id})((\Delta^{\mathbb{G}})^{op}\otimes\mathrm{id})(x)](\omega)\end{aligned}$$

and so $[(\phi^{\mathbb{G}}\otimes\mathrm{id}\otimes\mathrm{id})((\Delta^{\mathbb{G}})^{op}\otimes\mathrm{id})(x)](\omega)$ is finite and they are equal. It follows similarly that if the latter is finite then so is the former and they are equal. Also if either of these is infinite then so is the other and the maps (2.5) and (2.6) are equal.

From Equation (2.3) (with $\mathbb{G}$ and $\mathbb{H}$ reversed) we have

$$\begin{aligned}(\phi^{\mathbb{G}}\otimes\mathrm{id}\otimes\mathrm{id})((\Delta^{\mathbb{G}})^{op}\otimes\mathrm{id})(x) &= (\mathrm{id}\otimes\mathrm{id}\otimes\phi^{\mathbb{G}})(\mathrm{id}\otimes\Delta^{\mathbb{G}})(\sigma(x))\circ\sigma'_* \\ &= ((\mathrm{id}\otimes\phi^{\mathbb{G}})(\sigma(x))\otimes 1^{\mathbb{G}})\circ\sigma'_* = 1^{\mathbb{G}}\otimes(\phi^{\mathbb{G}}\otimes\mathrm{id})(x)\end{aligned}$$

as required. □

Proposition 2.5.4 *For locally compact quantum groups $\mathbb{G}$ and $\mathbb{H}$ we have left Haar weight $\phi^{\mathbb{G}\times\mathbb{H}} = \phi^{\mathbb{G}}\otimes\phi^{\mathbb{H}}$ on $\mathbb{G}\times\mathbb{H}$.*

Proof

This follows largely from the work in Sections 1 and 2 of Vaes & Vainerman (2003). From Definition 1.13 in Vaes & Vainerman (2003) we have a left invariant n.s.f. weight $\phi^{\mathbb{G}\times\mathbb{H}}$ on $(\mathrm{L}^\infty(\mathbb{G}\times\mathbb{H}),\Delta^{\mathbb{G}\times\mathbb{H}})$ given by

$$\phi^{\mathbb{G}\times\mathbb{H}} = \phi^{\mathbb{H}}\circ\alpha^{-1}\circ(\phi\otimes\mathrm{id}\otimes\mathrm{id})\circ\hat{\alpha}$$

where $\alpha : \mathrm{L}^\infty(\mathbb{H}) \to \mathrm{L}^\infty(\hat{\mathbb{G}})\overline{\otimes}\mathrm{L}^\infty(\mathbb{H})$ is given by $y\mapsto 1\otimes y$ and $\hat{\alpha}$ is the map from Propositions 1.4 and 1.5 in Vaes & Vainerman (2003) such that $\hat{\alpha}(z) = (\hat{W}^{\mathbb{G}}\otimes 1)(\mathrm{id}\otimes\alpha)(z)((\hat{W}^{\mathbb{G}})^*\otimes 1)$ for all $z\in\mathrm{L}^\infty(\mathbb{G})\overline{\otimes}\mathrm{L}^\infty(\mathbb{H})$.

It follows that for $y \in \mathrm{L}^\infty(\mathbb{G})$ and $z \in \mathrm{L}^\infty(\mathbb{H})$ that

$$\begin{aligned}\hat{\alpha}(y \otimes z) &= (\hat{W}^{\mathbb{G}} \otimes 1)(y \otimes \alpha(z))((\hat{W}^{\mathbb{G}})^* \otimes 1) = \hat{W}^{\mathbb{G}}(y \otimes 1)(\hat{W}^{\mathbb{G}})^* \otimes z \\ &= \Sigma(W^{\mathbb{G}})^*(1 \otimes y)W^{\mathbb{G}}\Sigma \otimes z = \Sigma\Delta(y)\Sigma \otimes z = \Delta^{op}(y) \otimes z\end{aligned}$$

and so it follows by linearity and continuity that $\hat{\alpha}(x) = (\Delta^{op} \otimes \mathrm{id})(x)$ for all $x \in \mathrm{L}^\infty(\mathbb{G}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{H})$. Now let $x \in \mathrm{L}^\infty(\mathbb{G} \times \mathbb{H})^+$, clearly $\hat{\alpha}(x) \in \mathrm{L}^\infty(\mathbb{G} \times \mathbb{G} \times \mathbb{H})^+$ then applying Lemma 2.5.3 we have

$$((\phi \otimes \mathrm{id} \otimes \mathrm{id}) \circ \hat{\alpha})(x) = 1_{\mathbb{G}} \otimes (\phi \otimes \mathrm{id})(x) \in \mathrm{L}^\infty(\mathbb{G} \times \mathbb{H})^+_{ext}.$$

Clearly we have a restriction and corestriction $\alpha' : \mathrm{L}^\infty(\mathbb{H})^+ \to (\mathrm{L}^\infty(\mathbb{G} \times \mathbb{H})^+$ and we can extend this to a map $\alpha'_{ext} : \mathrm{L}^\infty(\mathbb{H})^+_{ext} \to \mathrm{L}^\infty(\mathbb{G} \times \mathbb{H})^+_{ext}$ such that $[\alpha'_{ext}(y)](\omega) = \omega(1 \otimes y)$ for all $x \in \mathrm{L}^\infty(\mathbb{H})^+_{ext}$ and $\omega \in \mathrm{L}^1(\mathbb{G} \times \mathbb{H})^+$. Then ${\alpha'_{ext}}^{-1}(1_{\mathbb{G}} \otimes (\phi \otimes \mathrm{id})(x)) = (\phi \otimes \mathrm{id})(x)$ and thus we have

$$(\alpha^{-1} \circ (\phi \otimes \mathrm{id} \otimes \mathrm{id}) \circ \hat{\alpha})(x) = (\phi \otimes \mathrm{id})(x) \in \mathrm{L}^\infty(\mathbb{H})^+_{ext}.$$

Finally by 1.4.30 we have

$$(\psi \circ \alpha^{-1} \circ (\phi \otimes \mathrm{id} \otimes \mathrm{id}) \circ \hat{\alpha})(x) = \psi((\phi \otimes \mathrm{id})(x)) = (\phi \otimes \psi)(x)$$

for all $x \in \mathrm{L}^\infty(\mathbb{G} \times \mathbb{H})^+$ as required. □

Proposition 2.5.5 *As $\mathbb{G} \times \mathbb{H}$ is a locally compact quantum group by Definition-Theorem 2.2.7 have a scaling group $\tau^{\mathbb{G}\times\mathbb{H}}$ and a unitary antipode $R^{\mathbb{G}\times\mathbb{H}}$. We have the following:*

$$\tau_t^{\mathbb{G}\times\mathbb{H}} = \tau_t^{\mathbb{G}} \otimes \tau_t^{\mathbb{H}} \qquad \textit{and} \qquad R^{\mathbb{G}\times\mathbb{H}} = R^{\mathbb{G}} \otimes R^{\mathbb{H}}.$$

Proof

We have from Proposition 2.3.6 that the scaling group $\tau^{\mathbb{G}\times\mathbb{H}}$ is given by

$$\tau_t^{\mathbb{G}\times\mathbb{H}}(x) = (\hat{\Delta}^{\mathbb{G}\times\mathbb{H}})^{it}\, x\, (\hat{\Delta}^{\mathbb{G}\times\mathbb{H}})^{-it}$$

for all $x \in \mathrm{L}^\infty(\mathbb{G} \times \mathbb{H})$ and $t \in \mathbb{R}$ where $\hat{\Delta}^{\mathbb{G}\times\mathbb{H}}$ is the modular operator of the left Haar weight of $(\mathrm{L}^\infty(\widehat{\mathbb{G} \times \mathbb{H}}), \hat{\Delta}^{\mathbb{G}\times\mathbb{H}})$. From Proposition 1.4.23 we have that $\hat{\Delta}^{\mathbb{G}\times\mathbb{H}} = \hat{\Delta}^{\mathbb{G}} \otimes \hat{\Delta}^{\mathbb{H}}$ where $\hat{\Delta}^{\mathbb{G}}$ and $\hat{\Delta}^{\mathbb{H}}$ are the modular operators of the left Haar weight of $(\mathrm{L}^\infty(\hat{\mathbb{G}}), \hat{\Delta}^{\mathbb{G}})$ and $(\mathrm{L}^\infty(\hat{\mathbb{H}}), \hat{\Delta}^{\mathbb{H}})$. It follows that $\tau^{\mathbb{G}\times\mathbb{H}}(y \otimes z) = \tau_t^{\mathbb{G}}(x) \otimes \tau_t^{\mathbb{H}}(y) = (\tau_t^{\mathbb{G}} \otimes \tau_t^{\mathbb{H}})(y \otimes z)$ for all $y \in \mathrm{L}^\infty(\mathbb{G})$ and $z \in \mathrm{L}^\infty(\mathbb{H})$ and by linearity and continuity we have $\tau_t^{\mathbb{G}\times\mathbb{H}}(x) = (\tau_t^{\mathbb{G}} \otimes \tau_t^{\mathbb{H}})(x)$.

Similarly from Proposition 2.3.6 we have $R^{\mathbb{G}\times\mathbb{H}}(x) = \hat{J}^{\mathbb{G}\times\mathbb{H}}\, x^*\, \hat{J}^{\mathbb{G}\times\mathbb{H}}$ for all $x \in \mathrm{L}^\infty(\mathbb{G} \times \mathbb{H})$ and from Proposition 1.4.23 we have $\hat{J}^{\mathbb{G}\times\mathbb{H}} = \hat{J}^{\mathbb{G}} \otimes \hat{J}^{\mathbb{H}}$ and the rest follows as above. □

Chapter 3

Special Quantum Groups

In the previous chapter we have worked at a high level of generality by describing locally compact quantum groups. In this chapter we now study the special cases of Kac algebras, compact and discrete quantum groups and coamenable quantum groups.

3.1 Kac Algebras

As we have introduced locally compact quantum groups (which are a generalisation of Kac algebras) we will give our definition based on quantum groups for Kac algebras. We only give the definition of Kac algebras here and refer the reader to Enock & Schwartz (2013) for further details.

We can see from Definitions 1.2.1 and 2.2.1 in Enock & Schwartz (2013) that our definition and the usual definition coincide.

Definition 3.1.1 *A locally compact quantum group* $\mathbb{G}$ *is of* ***Kac type*** *if we have* $\tau_t = \mathrm{id}$ *for all* $t \in \mathbb{R}$ *and* $\sigma' = \sigma$. *We say* $(\mathrm{L}^\infty(\mathbb{G}), \Delta, \phi, \psi)$ *is a* ***Kac algebra*** *in this case.*

3.2 Compact Quantum Groups

We now introduce the special case of compact quantum groups. Historically compact quantum groups were defined before the locally compact setting by Woronowicz after discovering $\mathrm{SU}_q(2)$ that didn't fit in the Kac algebra framework. Originally he generalised this to what are now called the compact matrix quantum groups Woronowicz (1987a) and then later he made a generalisation of this to what are now called compact quantum groups Woronowicz (1998). We give details of this in this section now. See also Maes & Van Daele (1998) and Neshveyev & Tuset (2013).

We will introduce some conditions for verifying when a locally compact quantum group is compact and then come back to discussing how to construct examples of compact quantum groups. We then discuss corepresentation theory, a cornerstone for the subject and compact matrix quantum groups. Lastly we discuss the multiplicative unitary and products of compact quantum groups.

3.2.1 Compact and Locally Compact Quantum Groups

We start with the following as a definition based on our work with locally compact quantum groups and in the following sections make contact with Woronowicz' original definition.

Definition 3.2.1 *A locally compact quantum group $\mathbb{G}$ is a **compact quantum group** if the reduced C*-algebra $\mathrm{C}_0(\mathbb{G})$ is unital in which case we denote the reduced C*-algebra by $\mathrm{C}(\mathbb{G})$ (in accordance with the convention for continuous functions on a compact group).*

Proposition 3.2.2 *The following are equivalent for a locally compact quantum group $\mathbb{G}$:*

(i) $\mathbb{G}$ is compact;

(ii) The left Haar weight of $\mathrm{L}^\infty(\mathbb{G})$ is finite;

(iii) There exists a normal left invariant state.

Proof

(i) $\implies$ (ii): It follows from the theory of compact quantum groups that we have a unique state ϕ on the reduced C*-algebra quantum group $C(\mathbb{G})$ such that for all $x \in C(\mathbb{G})$ we have $(\mathrm{id} \otimes \phi)\Delta(x) = \phi(x)1 = (\phi \otimes \mathrm{id})\Delta(x)$. We refer the reader to Section 4 in Maes & Van Daele (1998) and Section 5.1 in Timmermann (2008) for a proof of this. Then from Result 2.3 in Kustermans & Vaes (2000) we have that $\phi(x)1 = (\phi \otimes \mathrm{id})\Delta(x) \in \mathcal{N}_\phi$ and thus $1 \in \mathcal{N}_\phi$. As $\mathcal{N}_\phi$ is a left ideal it follows that $x = x1 \in \mathcal{N}_\phi$ for all $x \in C_0(\mathbb{G})$, or indeed $\mathcal{N}_\phi = C_0(\mathbb{G})$, and therefore ϕ is finite.

(ii) $\implies$ (iii): As the left Haar weight ϕ is finite we have $\mathcal{N}_\phi = \mathcal{M}_\phi = L^\infty(\mathbb{G})$. We can show using the definition of normal functionals that $\phi \in L^1(\mathbb{G})^+$ and so $\psi := \phi/\|\phi\| \in L^1(\mathbb{G})^+$ is a normal state that is easily seen to be left invariant.

(iii) $\implies$ (i): We have that $L^1(\mathbb{G})$ is a left $C_0(\mathbb{G})$-module by the map $a \otimes \omega \mapsto a \cdot \omega$ for $a \in C_0(\mathbb{G})$ and $\omega \in L^1(\mathbb{G})$ where we denote by $a \cdot \omega \in L^1(\mathbb{G})$ the functional $\langle b, a \cdot \omega \rangle = \langle ba, \omega \rangle$ for all $b \in C_0(\mathbb{G})$. We show that this is essential, see Definition A.3.2. Let $\omega = \omega_{\Lambda(a),\eta}$ for $a \in C_0(\mathbb{G})$ and $\eta \in L^2(\mathbb{G})$ first. Then as $C_0(\mathbb{G})$ has a bounded approximate identity it follows that this is essential as a left Banach module over itself. Then by the Cohen Factorisation Theorem (see Theorem A.3.3) we have that $a = bc$ for some $b, c \in C_0(\mathbb{G})$. Then for all $x \in C_0(\mathbb{G})$ it follows that

$$\omega_{\Lambda(a),\eta}(x) = (x\Lambda(bc)|\eta) = (xb\Lambda(c)|\eta) = \omega_{\Lambda(c),\eta}(xb) = (b \cdot \omega_{\Lambda(c),\eta})(x)$$

where we've used the $C_0(\mathbb{G})$-bimodule structure on $L^1(\mathbb{G})$ from Example A.3.1. Then it follows that $\omega_{\Lambda(a),\eta} = b \cdot \omega_{\Lambda(c),\eta}$ and thus as Λ has dense range we have $\omega_{\xi,\eta} \in \overline{\mathrm{lin}\ \{a \cdot \omega \mid a \in C_0(\mathbb{G}),\ \omega \in L^1(\mathbb{G})\}}^{\|\cdot\|}$ for all $\xi, \eta \in L^2(\mathbb{G})$. It the follows from linearity and continuity that $L^1(\mathbb{G})$ is essential as a left $C_0(\mathbb{G})$-module.

Now we can use Cohen's Factorisation Theorem again to show that there exists $a \in C_0(\mathbb{G})$ and $\omega \in L^1(\mathbb{G})$ such that $\phi = a \cdot \omega$. As ϕ is left invariant for all $\kappa \in L^1(\mathbb{G})$ and

$x \in C_0(\mathbb{G})$ we have

$$\langle \Delta(x), \kappa \otimes \phi \rangle = \langle 1, \kappa \rangle \langle x, \phi \rangle$$

and then substituting $\phi = a \cdot \omega$ we get

$$\langle \Delta(x)(1 \otimes a), \kappa \otimes \omega \rangle = \langle 1, \kappa \rangle \langle x, \phi \rangle.$$

As this holds for all $\kappa \in L^1(\mathbb{G})$ we have for all $x \in C_0(\mathbb{G})$ that

$$(\mathrm{id} \otimes \omega)\,(\Delta(x)(1 \otimes a)) = \langle x, \phi \rangle 1.$$

By the density conditions in Definition 2.2.14 we have $\Delta(x)(1 \otimes a) \in C_0(\mathbb{G}) \otimes_{min} C_0(\mathbb{G})$ and so $\langle x, \phi \rangle 1 = (\mathrm{id} \otimes \omega)\,(\Delta(x)(1 \otimes a)) \in C_0(\mathbb{G})$. As $\langle e_\alpha, \phi \rangle \to 1$ there is some $x \in C_0(\mathbb{G})$ such that $\langle x, \phi \rangle \neq 0$ and so $1 \in C_0(\mathbb{G})$ and $\mathbb{G}$ is compact. □

3.2.2 C*-algebraic Compact Quantum Groups

So we have seen that we can check whether a locally compact quantum group is compact from the reduced C*-algebraic or von Neumann algebraic version. It is more common, however, when constructing compact quantum groups to find a unital C*-algebraic quantum semigroup that satisfies some density conditions. We can then form the reduced C*-algebraic or von Neumann algebraic version from this when required.

The following important proposition was proved in Woronowicz (1998) in the separable case and in Van Daele (1995) in the general case.

Definition-Theorem 3.2.3 *Let A be a unital C*-algebra with $\Delta : A \to A \otimes_{min} A$ a unital $*$-homomorphism making (A, Δ) a C*-algebraic quantum semigroup such that that the following equations hold:*

$$A \otimes_{min} A = \overline{\mathrm{lin}\ \{\Delta(x)(y \otimes 1) \mid x, y \in A\}}^{\|\cdot\|} = \overline{\mathrm{lin}\ \{\Delta(x)(1 \otimes y) \mid x, y \in A\}}^{\|\cdot\|}. \quad (3.1)$$

Then we have a KMS state ϕ such that for all $x \in A$ we have $(\mathrm{id} \otimes \phi)\Delta(x) = \phi(x)1 = (\phi \otimes \mathrm{id})\Delta(x)$. In particular (A, Δ) is a C-algebraic quantum group as per Definition 2.2.14 and a compact quantum group. We call ϕ the* ***Haar state*** *of (A, Δ).*

Proposition 3.2.4 *Let (A, Δ) denote an arbitrary C*-algebraic quantum group satisfying the properties of Definition-Theorem 3.2.3, that is A is a unital C*-algebra and ϕ the left and right invariant Haar state. Let $A_r = A/\mathrm{Ker}\,\phi$ and we can restrict Δ to a map $\Delta_r : A_r \to A_r \otimes_{min} A_r$ such that (A_r, Δ_r) is a reduced C*-algebraic quantum group with A_r unital, Δ_r a unital map and there is a left and right Haar state ϕ_r such that $\phi_r \circ \pi = \phi$ where $\pi : A \to \mathcal{B}(\mathrm{L}^2(\mathbb{G}))$ is the GNS map.*

So given a unital C*-algebra A and a unital coproduct $\Delta : A \to A \otimes_{min} A$ satisfying Equation (3.1) we consider this a C*-algebraic quantum group. Then there is a compact quantum group $\mathbb{G}$ such that we can form the reduced C*-algebraic quantum group $(\mathrm{C}(\mathbb{G}), \Delta_r)$. We treat both (A, Δ) and $(\mathrm{C}(\mathbb{G}), \Delta)$ as C*-algebraic quantum groups for the same compact quantum group $\mathbb{G}$.

3.2.3 Corepresentation Theory

For compact quantum groups we have a good understanding of corepresentation theory. This is fundamental to a lot of work in compact quantum groups and there are many presentations of this in the literature, see Maes & Van Daele (1998) and Woronowicz (1998) for example. However, many conventions used in the case of locally compact quantum groups are different to those used by Woronowicz and van Daele for compact quantum groups, for example in the compact case we tend use the right regular representation where in the locally compact case we tend to use the left regular corepresentation. Additionally it is difficult to find some of the results as stated here in the literature and as such we present a thorough treatment of this subject here.

Throughout this section let (A, Δ) be a C*-algebraic quantum group satisfying the conditions of Definition-Theorem 3.2.3 and $\mathbb{G}$ the underlying compact quantum group, that is the reduced C*-algebra $\mathrm{C}(\mathbb{G})$ is constructed as in Proposition 3.2.4.

Definition 3.2.5 *A **corepresentation** of (A, Δ) on a Hilbert space $\mathcal{H}$ is an element $U \in \mathrm{M}(A \otimes \mathcal{B}_0(\mathcal{H}))$ such that $(\Delta \otimes \mathrm{id})U = U_{13}U_{23}$. If U is unitary then we say this is a **unitary corepresentation**.*

If $\mathcal{H}$ is n-dimensional with orthonormal basis $\{e_k \mid 1 \leqslant k \leqslant n\}$ then for $1 \leqslant i, j \leqslant n$ we define $u_{ij} \in \mathrm{M}(A) = A$ by $u_{ij} = (\mathrm{id} \otimes \omega_{e_j,e_i})(U)$ and we have

$$\Delta(u_{i,j}) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega_{e_j,e_i})(\Delta \otimes \mathrm{id})(U) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega_{e_j,e_i})(U_{13}U_{23}).$$

As $\omega_{e_j,e_i}(xy) = (xye_j|e_i) = \sum_{k=1}^n (ye_j|e_k)(e_k|x^*e_i) = \sum_{k=1}^n \omega_{e_k,e_i}(x)\omega_{e_j,e_k}(y)$ for $x, y \in \mathcal{B}(\mathcal{H})$ then we have

$$\begin{aligned}\Delta(u_{i,j}) &= \sum_{k=1}^n (\mathrm{id} \otimes \mathrm{id} \otimes \omega_{e_k,e_i})(U_{13})(\mathrm{id} \otimes \mathrm{id} \otimes \omega_{e_j,e_k})(U_{23}) \\ &= \sum_{k=1}^n (\mathrm{id} \otimes \omega_{e_k,e_i})(U) \otimes (\mathrm{id} \otimes \omega_{e_j,e_k})(U) = \sum_{k=1}^n u_{i,k} \otimes u_{k,j}.\end{aligned}$$

We can also similarly show that if this equation holds then so does Definition 3.2.5. So equivalently, for a finite dimensional corepresentation, we can consider a matrix $(u_{ij})_{i,j=1}^n \in \mathbb{M}_n(A)$ such that $\Delta u_{ij} = \sum_{i=1}^n u_{ik} \otimes u_{kj}$.

We now move on to consider irreducible corepresentations.

Definition 3.2.6 *Let $U \in \mathrm{M}(A \otimes \mathcal{B}_0(\mathcal{H}))$ be a corepresentation of $\mathbb{G}$. Then a closed subspace $\mathcal{K}$ of $\mathcal{H}$ is **invariant** if for e the orthogonal projection from $\mathcal{H}$ to $\mathcal{K}$ we have $(1 \otimes e)U(1 \otimes e) = U(1 \otimes e)$. We say U is **irreducible** if the only invariant subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$.*

Definition 3.2.7 *Let U and V be corepresentations of (A, Δ) on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. Then we say a linear map $T : \mathcal{H} \to \mathcal{K}$ **intertwines** U and V if $(1 \otimes T)U = V(1 \otimes T)$. We say corepresentations U and V are **equivalent** if there is an invertible* *intertwiner from U to V and* ***unitarily equivalently*** *if there is a unitary intertwiner from U to V.*

We now offer some theorems on corepresentations and refer the reader to Maes & Van Daele (1998) for proofs.

Proposition 3.2.8 *Any invertible, finite-dimensional corepresentation of (A, Δ) is equivalent to a unitary corepresentation.*

The following is one of the most important theorems on corepresentations of compact quantum groups.

Theorem 3.2.9 *There exists a maximal family denoted*

$$\{U^\alpha \in A \otimes \mathcal{B}(\mathcal{H}_\alpha) \mid \alpha \in \mathbb{A}\}$$

of mutually inequivalent, finite-dimensional, irreducible, unitary corepresentations of (A, Δ) such that U^α is contained in the left regular corepresentation for all $\alpha \in \mathbb{A}$. Furthermore any unitary irreducible corepresentation is equivalent to some U^α.

Let $U \in \mathrm{M}(A \otimes \mathcal{B}_0(\mathcal{H}))$ be a corepresentation where $\mathcal{H}$ is finite dimensional with orthonormal basis $\{e_i \mid 1 \leqslant i \leqslant n\}$. Then we have $U = \sum_{i,j=1}^{n} u_{ij} \otimes e_{ij}$ for some $u_{ij} \in A$ such that $\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$ for all $1 \leqslant i, j \leqslant n$. In particular for the maximal family given in the previous theorem we can write this as

$$\left\{u = (u_{ij}^\alpha)_{i,j=1}^{n_\alpha} \in \mathbb{M}_{n_\alpha}(A) \;\middle|\; \alpha \in \mathbb{A},\ 1 \leqslant i, j \leqslant n_\alpha\right\}.$$

We will use this notation throughout the rest of this section for this maximal family.

Notation 3.2.10 *We denote by* $\mathrm{Hopf}(\mathbb{G})$ *the linear span of* $\left\{u_{ij}^\alpha \;\middle|\; \alpha \in \mathbb{A},\ 1 \leqslant i, j \leqslant n_\alpha\right\}$ *in A.*

Proposition 3.2.11 *We have that* $\mathrm{Hopf}(\mathbb{G})$ *is a unital Hopf* $*$*-algebra that is dense in* A *such that the Haar state* ϕ *is faithful on* $\mathrm{Hopf}(\mathbb{G})$*. Furthermore, for all* $\alpha \in \mathbb{A}$ *and* $1 \leqslant i, j \leqslant n_\alpha$*, we have the following relations*

$$\Delta(u_{ij}^\alpha) = \sum_{k=1}^{n_\alpha} u_{ik}^\alpha \otimes u_{kj}^\alpha, \qquad S(u_{ij}^\alpha) = (u_{ji}^\alpha)^*, \qquad \epsilon(u_{ij}^\alpha) = \delta_{ij}, \qquad \phi(u_{ij}^\alpha) = \delta_{\alpha,\alpha_0}$$

where $\alpha_0 \in \mathbb{A}$ *is the unique* 1*-dimensional corepresentation consisting of the identity* $1 \in A$*.*

Example 3.2.12 *We now give examples for constructing corepresentation matrices from given corepresentation matrices. Say* $(u_{ij})_{i,j=1}^n \in \mathbb{M}_n(A)$ *is a corepresentation matrix, then we have*

$$\Delta(u_{ij}^*) = \Delta(u_{ij})^* = \sum_{k=1}^n u_{ik}^* \otimes u_{kj}^*$$

and so we have another corepresentation $\bar{U} = \sum_{i,j=1}^n u_{ij}^* \otimes e_{ij}$*. We can show that if* U *is irreducible then so is* $\bar{U}$*. Clearly the definition of* $\bar{U}$ *depends on the choice of orthonormal basis however given any other orthonormal basis we can show that the two corepresentations obtained are equivalent.*

Say $u \in \mathbb{M}_n(A)$ *is a corepresentation. Similarly to Example 2.2.21 we have a* **C****-algebraic quantum group* (A, Δ^{op}) *where* $\Delta^{op} = \sigma \circ \Delta$ *for* σ *the flip map on* $A \otimes_{min} A$*. Then we have*

$$\Delta^{op}((u^t)_{ij}) = \Delta^{op}(u_{ji}) = \sum_{k=1}^n u_{ki} \otimes u_{jk} = \sum_{k=1}^n (u^t)_{ik} \otimes (u^t)_{kj}$$

and so $u^t \in \mathbb{M}_n(A)$ *is a corepresentation matrix of* (A, Δ^{op})*. Similarly* $u^* = (\bar{u})^t \in \mathbb{M}_n(A)$ *is a corepresentation matrix of* (A, Δ^{op})*.*

We can show that for an n*-dimensional unitary irreducible corepresentation* U *that* $\bar{U}$ *is equivalent to a unitary corepresentation. Then there exists an invertible matrix* $T \in \mathbb{M}_n$ *such that* $V = (1 \otimes T)\bar{U}(1 \otimes T^{-1})$ *is unitary. Then we have* $\bar{U}$ *is invertible with inverse*

$\bar{U}^{-1} = (1 \otimes T^{-1})V^*(1 \otimes T)$. *Also* $U^t = (\bar{U})^*$ *is invertible with*

$$(U^t)^{-1} = ((\bar{U})^*)^{-1} = (1 \otimes T^*)V(1 \otimes (T^{-1})^*) = (1 \otimes T^*T)\bar{U}(1 \otimes (T^*T)^{-1}).$$

It then follows that $(U^t)^{-1}$ *is a corepresentation as*

$$\begin{aligned}(\Delta \otimes \mathrm{id})(U^t)^{-1} &= (1 \otimes 1 \otimes T^*T)(\Delta \otimes \mathrm{id})(\bar{U})(1 \otimes 1 \otimes (T^*T)^{-1}) \\ &= (1 \otimes 1 \otimes T^*T)(\bar{U})_{13}(1 \otimes 1 \otimes (T^*T)^{-1}) \\ &\qquad \times (1 \otimes 1 \otimes T^*T)(\bar{U})_{23}(1 \otimes 1 \otimes (T^*T)^{-1}) \\ &= ((U^t)^{-1})_{13}((U^t)^{-1})_{23}\end{aligned}$$

and thus $(1 \otimes T^*T)\bar{U} = (U^t)^{-1}(1 \otimes T^*T)$.

We offer a proof of the next few results as they differ slightly from that found in the literature. We first record the following which comes from Lemma 6.3 in Maes & Van Daele (1998).

Lemma 3.2.13 *Let* U *and* V *be corepresentations of* (A, Δ) *on Hilbert spaces* $\mathcal{H}$ *and* $\mathcal{K}$ *respectively, and let* $x \in \mathcal{B}_0(\mathcal{H}, \mathcal{K})$ *and let*

$$y = (\phi \otimes \mathrm{id})(V^*(1 \otimes x)U).$$

Then $y \in \mathcal{B}_0(\mathcal{H}, \mathcal{K})$ *and* $V^*(1 \otimes y)U = 1 \otimes y$.

Proof

Because $x \in \mathcal{B}_0(\mathcal{H}, \mathcal{K})$ we have $(1 \otimes x)U \in \mathcal{B}_0(\mathcal{H}, \mathcal{K}) \otimes A$ and thus $V^*(1 \otimes x)U \in \mathcal{B}_0(\mathcal{H}, \mathcal{K}) \otimes A$ also. In particular $y \in \mathcal{B}_0(\mathcal{H}, \mathcal{K})$.

Also we have

$$(\Delta \otimes \mathrm{id})(V^*(1 \otimes x)U) = V_{23}^* V_{13}^*(1 \otimes 1 \otimes x)U_{13}U_{23}$$

and then applying $\phi \otimes \mathrm{id} \otimes \mathrm{id}$ we get the result. □

Lemma 3.2.14 (Schur's Lemma) *Let U and V be finite-dimensional corepresentations of (A, Δ) on $\mathcal{H}$ and $\mathcal{K}$ respectively and let T an intertwiner from U to V. Then*

(i) $\mathrm{Ker}\, T$ *is invariant for U and* $\mathrm{Image}\, T$ *is invariant for V;*

(ii) If either U or V is irreducible then either U and V are inequivalent and there is only the 0 intertwiner from U to V or U is equivalent to V and there is some invertible $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that the set $\{\lambda T \mid \lambda \in \mathbb{C}\}$ gives all the intertwiners from U to V.

In particular we have that the intertwiners from U to itself are given by $\{\lambda\, \mathrm{id} \mid \lambda \in \mathbb{C}\}$.

Proof

Let e denote the orthogonal projection of $\mathcal{H}$ onto $\mathrm{Ker}\, T$ and f the orthogonal projection of $\mathcal{K}$ onto $\overline{\mathrm{Image}\, T}$. Then $0 = (1 \otimes Te)$ and thus $0 = V(1 \otimes Te) = (1 \otimes T)U(1 \otimes e)$ and so we have $(1 \otimes e)U(1 \otimes e) = U(1 \otimes e)$. Similarly we have

$$(1 \otimes f)V(1 \otimes T) = (1 \otimes fT)U = (1 \otimes T)U = V(1 \otimes T)$$

and as $f(\mathcal{K}) = \overline{T\mathcal{H}}$ then we have $(1 \otimes f)V(1 \otimes f) = V(1 \otimes f)$.

It follows that if U is irreducible then either $\mathrm{Ker}\, T = \{0\}$ or $\mathrm{Ker}\, T = U$, that is T is either injective (and thus an isomorphism as it is finite dimensional) or 0. Similarly if V is irreducible then $\mathrm{Image}\, T = \{0\}$ or $\mathrm{Image}\, T = \mathcal{K}$, that is T is 0 or surjective (and thus an isomorphism as it is finite dimensional). In either case if U and V are inequivalent then the only intertwiner is 0. On the other hand if T is a non-zero intertwiner from U to V then T is bijective and U and V are equivalent.

Now say S is a non-zero intertwiner from U to V, then for all $\lambda \in \mathbb{C}$ we have $\lambda T - S$ is an intertwiner from U to V and so is either bijective (and thus an isomorphism) or 0. Choose λ such that $\det(\lambda T - S) = 0$ and then we have $S = \lambda T$ for some $\lambda \in \mathbb{C}$ as required. □

The diagonalisation argument in the following theorem was originally given in Daws (2010).

Theorem 3.2.15 *For all $\alpha \in \mathbb{A}$ there exists a unique invertible positive definite matrix $F^\alpha \in \mathbb{M}_{n_\alpha}$ with $\mathrm{Tr}\ F^\alpha = \mathrm{Tr}\ (F^\alpha)^{-1}$ such that for all $\alpha, \beta \in \mathbb{A}$, $1 \leqslant k, l \leqslant n_\alpha$ and $1 \leqslant i, j \leqslant n_\beta$ we have*

$$\phi(u_{ij}^\alpha (u_{kl}^\beta)^*) = \delta_{\alpha\beta}\delta_{ik}\frac{F_{lj}^\alpha}{\Lambda^\alpha}, \qquad \phi((u_{ij}^\alpha)^* u_{kl}^\beta) = \delta_{\alpha\beta}\delta_{jl}\frac{((F^\alpha)^{-1})_{ki}}{\Lambda^\alpha} \tag{3.2}$$

*where $\Lambda^\alpha = \mathrm{Tr}\ F^\alpha$. We can assume that the maximal family of corepresentations is chosen such that for all $\alpha \in \mathbb{A}$ we have $F^\alpha = \mathrm{diag}(\lambda_1^\alpha, \ldots, \lambda_{n_\alpha}^\alpha)$ with $\lambda_i^\alpha > 0$ for all i and $\Lambda^\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha = \sum_{i=1}^{n_\alpha} (\lambda_i^\alpha)^{-1} > 0$. Furthermore we have that F^α intertwines $\bar{U}^\alpha$ and $((U^\alpha)^t)^{-1}$ and $(F^\alpha)^{-1}$ intertwines U^α and $\kappa_{n_\alpha}^2(U^\alpha)$. We call these matrices the F**-matrices** of (A, Δ).*

Proof

Let $\alpha \in \mathbb{A}$. We have from Example 3.2.12 that there is some $T \in \mathbb{M}_{n_\alpha}$ such that $V = (1 \otimes T)\bar{U}^\alpha(1 \otimes T^{-1})$ is unitary, $\bar{U}^\alpha$ is invertible and $1 \otimes T^*T$ intertwines $\bar{U}^\alpha$ with $((U^\alpha)^t)^{-1}$ as corepresentations of (A, Δ). We have T^*T is positive definite as T is invertible. We define $F^\alpha = \lambda(T^*T)^t$ where we choose λ such that $\mathrm{Tr}\ F^\alpha = \mathrm{Tr}\ (F^\alpha)^{-1} > 0$ and we have that $(F^\alpha)^t$ intertwines $\bar{U}^\alpha$ and $((U^\alpha)^t)^{-1}$. As $\bar{U}^\alpha$ is irreducible it follows from Schur's Lemma (3.2.14) that F^α is the unique operator such that $(F^\alpha)^t$ intertwines $\bar{U}^\alpha$ and $((U^\alpha)^t)^{-1}$ and such that $\mathrm{Tr}\ F^\alpha = \mathrm{Tr}\ (F^\alpha)^{-1}$.

Let $\alpha, \beta \in \mathbb{A}$, $i, j \in \mathbb{N}_0$ such that $1 \leqslant i \leqslant n_\alpha$, $1 \leqslant j \leqslant n_\beta$ and let $x = e_{i,j}^{\beta,\alpha}$ be the $n_\beta \times n_\alpha$-matrix in $\mathcal{B}(\mathcal{H}_\alpha, \mathcal{H}_\beta)$ with 1 in the i, j-th position and 0 elsewhere. Then by

Lemma 3.2.13 we have an intertwiner given by

$$\begin{aligned} y &= (\phi \otimes \mathrm{id}) \left((U^\beta)^* (1 \otimes e_{ij}^{\beta,\alpha}) U^\alpha \right) \\ &= \sum_{p,q=1}^{n_\beta} \sum_{s,t=1}^{n_\alpha} (\phi \otimes \mathrm{id}) \left((u_{pq}^\beta)^* \otimes e_{qp}^\beta \right) \left(1 \otimes e_{ij}^{\beta,\alpha} \right) \left(u_{st}^\alpha \otimes e_{st}^\alpha \right) \\ &= \sum_{p,q=1}^{n_\alpha} \sum_{s,t=1}^{n_\beta} \phi \left((u_{pq}^\beta)^* u_{st}^\alpha \right) \delta_{pi} \delta_{js} e_{qt}^{\beta,\alpha} = \sum_{q=1}^{n_\alpha} \sum_{t=1}^{n_\beta} \phi \left((u_{iq}^\beta)^* u_{jt}^\alpha \right) e_{qt}^{\beta,\alpha} \end{aligned}$$

such that $(1 \otimes y) U^\alpha = U^\beta (1 \otimes y)$, i.e. y intertwines U^α and U^β. As U^α and U^β are irreducible then, by Schur's Lemma 3.2.14, we have that $y = 0$ if $\alpha \neq \beta$.

Let $\alpha \in \mathbb{A}$, $1 \leqslant i, j, k, l \leqslant n_\alpha$ and we let

$$y_{ik} = (\phi \otimes \mathrm{id}) \left((U^\alpha)^* (1 \otimes e_{ik}^\alpha) U^\alpha \right) = \sum_{p,q=1}^{n_\alpha} \phi \left((u_{ip}^\alpha)^* u_{kq}^\alpha \right) e_{pq}^\alpha$$

and in (A, Δ^{op}) (as $(\bar{U}^\alpha)^*$ is a corepresentation of (A, Δ^{op})) we let

$$\begin{aligned} z_{jl} &= (\phi \otimes \mathrm{id}) \left(\bar{U}^\alpha (1 \otimes e_{jl}^\alpha) (\bar{U}^\alpha)^* \right) = (\phi \otimes \mathrm{id}) \left(\bar{U}^\alpha (1 \otimes e_{jl}^\alpha) (U^\alpha)^t \right) \\ &= \sum_{p,q,s,t=1}^{n_\alpha} (\phi \otimes \mathrm{id}) \left(((u_{ps}^\alpha)^* \otimes e_{ps}^\alpha)(1 \otimes e_{jl}^\alpha)(u_{qt}^\alpha \otimes e_{tq}^\alpha) \right) = \sum_{p,q=1}^{n_\alpha} \phi((u_{pj}^\alpha)^* u_{ql}^\alpha) e_{pq}^\alpha. \end{aligned}$$

We have from Lemma 3.2.13 that $(1 \otimes y_{ik}) U^\alpha = U^\alpha (1 \otimes y_{ik})$ and $(1 \otimes z_{jl})((U^\alpha)^t)^{-1} = \bar{U}^\alpha (1 \otimes z_{jl})$. So by Schur's Lemma 3.2.14 we must have $y_{ik} = \mu_{ik} \mathrm{id}$ and $z_{jl} = \nu_{jl}((F^\alpha)^t)^{-1}$ for some collections $(\mu_{ik}), (\nu_{jl}) \subset \mathbb{C}$. Then we have

$$\nu_{jl}((F^\alpha)^t)^{-1} = \sum_{p,q=1}^{n_\alpha} \phi((u_{pj}^\alpha)^* u_{ql}^\alpha) e_{pq}^\alpha.$$

We now show that $\nu_{jl} = \delta_{jl} \dfrac{1}{\mathrm{Tr}\ F^\alpha}$. We have

$$\nu_{jl}((F^\alpha)^{-1})_{ki} = \nu_{jl}(((F^\alpha)^t)^{-1})_{ik} = \phi((u_{ij}^\alpha)^* u_{kl}^\alpha) = \mu_{ik} \delta_{jl} \tag{3.3}$$

and so for $j \neq l$ we have $\nu_{jl} = 0$ and if $j = l$ then ν_{jj} is independent of j so we let $\nu_{jj} = \nu$ for some $\nu \in \mathbb{C}$. Using that U^α is unitary we have

$$\sum_{i=1}^{n_\alpha} y_{ii} = (\phi\otimes\mathrm{id})\left((U^\alpha)^*\left(1\otimes\sum_{i=1}^{n_\alpha} e_{ii}^\alpha\right)U^\alpha\right) = (\phi\otimes\mathrm{id})((U^\alpha)^*U^\alpha) = (\phi\otimes\mathrm{id})(1\otimes 1) = 1$$

and so $\sum_{i=1}^{n_\alpha} \mu_{ii} = 1$. Then it follows from Equation (3.3) that

$$\nu\,\mathrm{Tr}\,((F^\alpha)^t)^{-1} = \sum_{i=1}^{n_\alpha} \mu_{ii} = 1$$

and so $\nu = \dfrac{1}{\mathrm{Tr}\,((F^\alpha)^t)^{-1}} = \dfrac{1}{\mathrm{Tr}\,F^\alpha}$ and we have the second equation in (3.2).

The first equation is proved similarly by considering $(\phi\otimes\mathrm{id})\left(U^\alpha(1\otimes e_{jl}^\alpha)(U^\alpha)^*\right)$ and $(\phi\otimes\mathrm{id})\left((U^\alpha)^t(1\otimes e_{ik}^\alpha)\bar{U}^\alpha\right)$.

For $\alpha \in \mathbb{A}$ we have that F^α is positive and so there is some unitary $Q^\alpha \in \mathbb{M}_{n_\alpha}$ such that $(Q^\alpha)^* F^\alpha Q^\alpha$ is diagonal with matrix $\mathrm{diag}(\lambda_1^\alpha, \ldots, \lambda_{n_\alpha}^\alpha)$. It follows that $\sum_{i=1}^{n_\alpha} \lambda_i^\alpha = \mathrm{Tr}\,F^\alpha = \mathrm{Tr}\,(F^\alpha)^{-1} = \sum_{i=1}^{n_\alpha}(\lambda_i^\alpha)^{-1}$ which are all greater than 0. For $1 \leqslant i, j \leqslant n_\alpha$ let

$$v_{ij}^\alpha = ((Q^\alpha)^* u^\alpha Q^\alpha)_{ij} = \sum_{k,l=1}^{n_\alpha} \overline{Q_{ki}^\alpha} u_{kl}^\alpha Q_{lj}^\alpha.$$

We then have that $V^\alpha = (v_{ij}^\alpha)_{i,j=1}^{n_\alpha}$ is a corepresentation of (A, Δ) as

$$\begin{aligned}\sum_{p=1}^{n_\alpha} v_{ip}^\alpha \otimes v_{pj}^\alpha &= \sum_{k,l,p,s,t=1}^{n_\alpha} \overline{Q_{ki}^\alpha} u_{kl}^\alpha Q_{lp}^\alpha \otimes \overline{Q_{s,p}^\alpha} u_{st}^\alpha Q_{tj}^\alpha \\ &= \sum_{k,l,t=1}^{n_\alpha} \overline{Q_{ki}^\alpha}(u_{kl}^\alpha \otimes u_{lt}^\alpha) Q_{tj}^\alpha = \sum_{k,t=1}^{n_\alpha} \overline{Q_{ki}^\alpha}\Delta(u_{kt}^\alpha) Q_{tj}^\alpha = \Delta(v_{ij}^\alpha).\end{aligned}$$

Using similar techniques we can show that V^α is unitary and satisfies Equation (3.2) with F_{ij} the diagonal matrix $\mathrm{diag}(\lambda_1^\alpha, \ldots, \lambda_{n_\alpha}^\alpha)$.

Finally we show that F^α intertwines U^α and $S_{n_\alpha}^2(U^\alpha)$. Above we have a $T \in \mathbb{M}_{n_\alpha}$ such that we have a unitary matrix $V = (1\otimes T)\bar{U}^\alpha(1\otimes T^{-1})$. Then using that $S_{n_\alpha}(W)^t =$

$(W^*)^t = \bar{W}$ for any corepresentation $W \in \mathbb{M}_{n_\alpha}(A)$ of (A, Δ) (where we've used Notation 1.1.1 for the antipode S) and so $S_{n_\alpha}(\bar{W})^t = S_{n_\alpha}^2(W)$. Using these equations it follows that

$$(1 \otimes \bar{T})U^\alpha(1 \otimes \bar{T}^{-1}) = \bar{V} = S_{n_\alpha}(V)^t = ((1 \otimes T)S_{n_\alpha}(\bar{U}^\alpha)(1 \otimes T^{-1}))^t$$
$$= (1 \otimes (T^{-1})^t)S_{n_\alpha}(\bar{U}^\alpha)^t(1 \otimes T^t) = (1 \otimes (T^{-1})^t)S_{n_\alpha}^2(U^\alpha)(1 \otimes T^t)$$

and so rearranging we get

$$(1 \otimes T^t\bar{T})U^\alpha = S_{n_\alpha}^2(U^\alpha)(1 \otimes T^t\bar{T})$$

and using that $T^t\bar{T} = (T^*T)^t$ we are done. □

Definition 3.2.16 *Let (A, Δ) be a compact quantum group with $\{U^\alpha \mid \alpha \in \mathbb{A}\}$ the maximal family of mutually inequivalent, finite-dimensional, irreducible, unitary corepresentations. We define $f_z : \mathrm{Hopf}(\mathbb{G}) \to \mathbb{C}$ for all $z \in \mathbb{C}$ as the map $u_{ij}^\alpha \mapsto ((F^\alpha)^z)_{ij}$, that is the i, j-th entry of the matrix $(F^\alpha)^z$.*

We can calculate the modular automorphism group from Definition-Theorem 1.4.16 and the scaling group from Definition-Theorem 2.2.7 for a compact quantum group in terms of the collection $(f_z)_{z\in\mathbb{C}}$. We have the following properties for the collection $(f_z)_{z\in\mathbb{C}}$. See Theorem 3.2.19 in Timmermann (2008) for a proof.

Proposition 3.2.17 *For all $z \in \mathbb{C}$ we have that f_z is a character on $\mathrm{Hopf}(\mathbb{G})$, that is it is a non-zero homomorphism $f_z : \mathrm{Hopf}(\mathbb{G}) \to \mathbb{C}$. Furthermore we have the following*

(i) For any $x \in \mathrm{Hopf}(\mathbb{G})$ we have that the map $w \mapsto f_w(x)$ is entire and there exists $C > 0$ and $d \in \mathbb{R}$ such that $|f(z)| \leqslant Ce^{d\mathfrak{Re}\, z}$ for all $z \in \mathbb{C}$ with $\mathfrak{Re}\, z > 0$;

(ii) $f_0 = \varepsilon$ (the counit) and $f_z * f_w = f_{z+w}$ where $f_z * f_w = (f_z \otimes f_w) \circ \Delta$ for all $z, w \in \mathbb{C}$;

(iii) $f_z(1) = 1$, $f_z(S(x)) = f_{-z}(x)$ *and* $f_{it}(x^*) = \overline{f_{it}(x)}$ *for all* $x \in \mathrm{Hopf}(\mathbb{G})$, $z \in \mathbb{C}$ *and* $t \in \mathbb{R}$.

Proposition 3.2.18 *Let σ denote the modular automorphism group and let τ denote the scaling group of the reduced C*-algebra* $\mathrm{C}(\mathbb{G})$ *of a compact quantum group* $\mathbb{G}$*. Then for any* $z \in \mathbb{C}$ *we have* $\mathrm{Hopf}(\mathbb{G}) \subset \mathrm{Dom}(\sigma_z)$ *and* $\mathrm{Hopf}(\mathbb{G}) \subset \mathrm{Dom}(\tau_z)$ *and furthermore for all* $\alpha \in \mathbb{A}$ *and* $1 \leqslant i, j \leqslant n_\alpha$ *we have*

$$\sigma_z(u^\alpha_{ij}) = \sum_{k,l=1}^{n_\alpha} f_{iz}(u^\alpha_{ik}) f_{iz}(u^\alpha_{lj}) u^\alpha_{kl} \qquad \textit{and} \qquad \tau_z(u^\alpha_{ij}) = \sum_{k,l=1}^{n_\alpha} f_{iz}(u^\alpha_{ik}) f_{-iz}(u^\alpha_{lj}) u^\alpha_{kl}.$$

If we assume that $F^\alpha = \mathrm{diag}(\lambda^\alpha_1, \ldots, \lambda^\alpha_{n_\alpha})$ *as per Theorem 3.2.15 we have*

$$\sigma_z(u^\alpha_{ij}) = (\lambda^\alpha_i)^{iz} (\lambda^\alpha_j)^{iz} u^\alpha_{ij}, \qquad \textit{and} \qquad \tau_z(u^\alpha_{ij}) = (\lambda^\alpha_i)^{iz} (\lambda^\alpha_j)^{-iz} u^\alpha_{ij}.$$

Proof

For all $t \in \mathbb{R}$ we define $\sigma^0_t : \mathrm{Hopf}(\mathbb{G}) \to \mathrm{Hopf}(\mathbb{G})$ by

$$\sigma^0_t = (f_{it} \otimes \mathrm{id} \otimes f_{it}) \circ (\Delta \otimes \mathrm{id}) \circ \Delta \tag{3.4}$$

and we show that this can be extended to a one-parameter group of $*$-automorphisms on the reduced C*-algebra $\mathrm{C}(\mathbb{G})$. Using Proposition 3.2.11, for $\alpha \in \mathbb{A}$ and $1 \leqslant i, j \leqslant n_\alpha$, we have

$$\sigma^0_t(u^\alpha_{ij}) = (f_{it} \otimes \mathrm{id} \otimes f_{it}) \sum_{k,l=1}^{n_\alpha} u^\alpha_{ik} \otimes u^\alpha_{kl} \otimes u^\alpha_{lj} = \sum_{k,l=1}^{n_\alpha} f_{it}(u^\alpha_{ik}) f_{it}(u^\alpha_{lj}) u^\alpha_{kl}. \tag{3.5}$$

For all $\alpha \in \mathbb{A}$ we have that $(F^\alpha)^{it}$ is unitary for all $t \in \mathbb{R}$ and thus for all $\alpha \in \mathbb{A}$ and $1 \leqslant i, j \leqslant n_\alpha$ we have

$$\left| f_{it}(u^\alpha_{ij}) \right| = \left| ((F^\alpha)^{it})_{ij} \right| \leqslant \| (F^\alpha)^{it} \| = 1.$$

We may assume F^{it} is diagonal without loss of generality and so from Equation (3.5) we have

$$\|\sigma_t^0(u_{ij}^\alpha)\| = \left\| \sum_{k,l=1}^{n_\alpha} f_{it}(u_{ik}^\alpha) f_{it}(u_{lj}^\alpha) u_{kl}^\alpha \right\| \leqslant \|u_{ij}^\alpha\|$$

and so σ_t^0 is contractive.

It follows easily that $\sigma_0^0 = \mathrm{id}$ and for $s, t \in \mathbb{R}$ we have $\sigma_{t+s}^0 = \sigma_t^0 \circ \sigma_s^0$. We now show that for $t \in \mathbb{R}$ that σ_t^0 is a $*$-automorphism on $\mathrm{Hopf}(\mathbb{G})$. Let $\alpha, \beta \in \mathbb{A}$, $1 \leqslant i, j \leqslant n_\alpha$ and $1 \leqslant k, l \leqslant n_\beta$, then using Equation (3.4), that Δ is a $*$-homomorphism and that f_{it} is a character we have

$$\begin{aligned}
\sigma_t^0(u_{ij}^\alpha u_{kl}^\beta) &= \sum_{p,q=1}^{n_\alpha} \sum_{r,s=1}^{n_\alpha} (f_{it} \otimes \mathrm{id} \otimes f_{it}) \left(u_{ip}^\alpha u_{kr}^\beta \otimes u_{pq}^\alpha u_{rs}^\beta \otimes u_{qj}^\alpha u_{sl}^\beta \right) \\
&= \sum_{p,q=1}^{n_\alpha} \sum_{r,s=1}^{n_\beta} f_{it}(u_{ip}^\alpha u_{kr}^\beta) f_{it}(u_{qj}^\alpha u_{sl}^\beta) u_{pq}^\alpha u_{rs}^\beta \\
&= \left(\sum_{p,q=1}^{n_\alpha} f_{it}(u_{ip}^\alpha) f_{it}(u_{qj}^\alpha) u_{pq}^\alpha \right) \left(\sum_{r,s=1}^{n_\beta} f_{it}(u_{kr}^\beta) f_{it}(u_{sl}^\beta) u_{rs}^\beta \right) = \sigma_t^0(u_{ij}^\alpha) \sigma_t^0(u_{kl}^\beta)
\end{aligned}$$

and similarly using Proposition 3.2.17 we have

$$\begin{aligned}
\sigma_t^0((u_{ij}^\alpha)^*) &= \sum_{p,q=1}^{n_\alpha} (f_{it} \otimes \mathrm{id} \otimes f_{it})((u_{ip}^\alpha)^* \otimes (u_{pq}^\alpha)^* \otimes u_{qj}^\alpha)^*) \\
&= \sum_{p,q=1}^{n_\alpha} \overline{f_{it}(u_{ip}^\alpha) f_{it}(u_{qj}^\alpha)} (u_{pq}^\alpha)^* = \sigma_t^0(u_{ij}^\alpha)^*.
\end{aligned}$$

Let $z \in \mathbb{C}$, $\alpha \in \mathbb{A}$ and $1 \leqslant i, j \leqslant n_\alpha$, then we show that $u_{ij}^\alpha \in \mathrm{Dom}(\sigma_z^0)$. Let $G : \overline{S(z)} \to \mathbb{C}$ be the map

$$w \mapsto \sum_{k,l=1}^{n_\alpha} f_{iw}(u_{ik}^\alpha) f_{iw}(u_{lj}^\alpha) u_{kl}^\alpha.$$

From Proposition 3.2.17 we have that $w \mapsto f_w(x)$ is entire for any $x \in \mathrm{Hopf}(\mathbb{G})$ and thus G is analytic on $S(z)$ and continuous. We have from Equation (3.5) that $G(t) = \sigma_t^0(u_{ij}^\alpha)$

and so $u_{ij}^{\alpha} \in \mathrm{Dom}(\sigma_z)$ with

$$\sigma_z(u_{ij}^{\alpha}) = \sum_{k,l=1}^{n_\alpha} f_{iz}(u_{ik}^{\alpha}) f_{iz}(u_{lj}^{\alpha}) u_{kl}^{\alpha}.$$

We calculate for $\alpha, \beta \in \mathbb{A}$, $1 \leqslant i, j \leqslant n_\alpha$ and $1 \leqslant k, l \leqslant n_\beta$ that

$$\begin{aligned}\phi((u_{ij}^{\alpha})^* \sigma_{-i}^0(u_{kl}^{\beta})) &= \sum_{p,q=1}^{n_\beta} f_1(u_{kp}^{\beta}) f_1(u_{ql}^{\beta}) \phi((u_{ij}^{\alpha})^* u_{pq}^{\beta}) \\ &= \delta_{\alpha\beta} \sum_{p=1}^{n_\alpha} F_{kp}^{\alpha} F_{jl}^{\alpha} \frac{((F^\alpha)^{-1})_{pi}}{\mathrm{Tr}(F^\alpha)} = \delta_{\alpha\beta}\delta_{ki} \frac{F_{jl}^{\alpha}}{\mathrm{Tr}(F^\alpha)} = \phi(u_{kl}^{\beta}(u_{ij}^{\alpha})^*).\end{aligned}$$

It then follows by linearity that for all $x, y \in \mathrm{Hopf}(\mathbb{G})$ we have $\phi(x\sigma_{-i}^0(y)) = \phi(yx)$. Then for all $x, y \in \mathrm{Hopf}(\mathbb{G})$ we have $x\sigma_{-i}^0(y) \in \mathrm{Hopf}(\mathbb{G})$ and also

$$\phi(x\sigma_{-i}^0(y)) = \phi(yx) = \phi(x\sigma_{-i}(y)).$$

Now using that ϕ is faithful on $\mathrm{Hopf}(\mathbb{G})$ it follows that $x\sigma_{-i}^0(y) = x\sigma_{-i}(y)$ for all $x, y \in \mathrm{Hopf}(\mathbb{G})$ and then letting $x = 1$ we have $\sigma_{-i}^0(y) = \sigma_{-i}(y)$ for all $y \in \mathrm{Hopf}(\mathbb{G})$.

So $\sigma_{-i}^0(y) = \sigma_{-i}(y)$ for all $y \in \mathrm{Hopf}(\mathbb{G})$ and thus for all $y \in \mathrm{Dom}(\sigma_{-i})$. It then follows from Proposition 1.3.11 that we have $\sigma = \sigma^0$.

Define $\tau_z^0 : \mathrm{Hopf}(\mathbb{G}) \to \mathrm{Hopf}(\mathbb{G})$ be the map $u_{ij}^{\alpha} \mapsto \sum_{k,l=1}^{n_\alpha} f_{-iz}(u_{ik}^{\alpha}) f_{iz}(u_{lj}^{\alpha}) u_{kl}^{\alpha}$. As we have F^α intertwines U^α and $S_{n_\alpha}^2(U^\alpha)$ we have $S_{n_\alpha}^2(U^\alpha) = F^\alpha U^\alpha (F^\alpha)^{-1}$ and so it follows that

$$S^2(u_{ij}^{\alpha}) = S_{n_\alpha}^2(U^\alpha)_{ij} = \sum_{k,l=1}^{n_\alpha} F_{ik}^{\alpha} u_{kl}^{\alpha} ((F^\alpha)^{-1})_{lj} = \sum_{k,l=1}^{n_\alpha} f_1(u_{ik}^{\alpha}) f_{-1}(u_{lj}^{\alpha}) u_{kl}^{\alpha}$$

and so $\tau_{-i}(u_{ij}^{\alpha}) = \tau_{-i}^0(u_{ij}^{\alpha})$. It follows that τ^0 is the scaling group restricted to $\mathrm{Hopf}(\mathbb{G})$.

We have from 3.2.16 that $f_z(u_{ij}^{\alpha}) = ((F^\alpha)^z)_{ij} = \delta_{ij}(\lambda_i^\alpha)^z$ where $t^z = \exp(z \ln t)$ for all $t \geqslant 0$. Then it follows that

$$\sigma_z(u_{ij}^\alpha) = \sum_{k,l=1}^{n_\alpha} f_{iz}(u_{ik}^\alpha) f_{iz}(u_{lj}^\alpha) u_{kl}^\alpha = \sum_{k,l=1}^{n_\alpha} \delta_{ki}\delta_{lj}(\lambda_i^\alpha)^{iz}(\lambda_j^\alpha)^{iz} u_{kl}^\alpha = (\lambda_i^\alpha)^{iz}(\lambda_j^\alpha)^{iz} u_{ij}^\alpha$$

and similarly for τ_z. □

Corollary 3.2.19 *For any $z \in \mathbb{C}$ and compact quantum group $\mathbb{G}$ we have that* $\mathrm{Hopf}(\mathbb{G})$ *is a core for τ_z. Similarly* $\mathrm{Hopf}(\mathbb{G})$ *is a σ-weak core for τ_z on* $\mathrm{L}^\infty(\mathbb{G})$.

Proof

From Proposition 1.3.19 we have that $\{x(n) \mid x \in \mathrm{Hopf}(\mathbb{G}),\ n \in \mathbb{N}\}$ is a core for $\mathrm{Dom}(\tau_z)$ as $\mathrm{Hopf}(\mathbb{G})$ is dense in A. Fix $n \in \mathbb{N}$ and let $\{(u_{ij}^\alpha) \mid \alpha \in \mathbb{A}, 1 \leqslant i,j \leqslant n_\alpha\}$ be the family from Theorem 3.2.9 that is a basis for $\mathrm{Hopf}(\mathbb{G})$. Then we need only show that for any $\alpha \in \mathbb{A}$ and $1 \leqslant i,j \leqslant n_\alpha$ we have $u_{ij}^\alpha(n) \in \mathrm{Hopf}(\mathbb{G})$.

Fix $n \in \mathbb{N}$, $\alpha \in \mathbb{A}$ and $1 \leqslant i,j \leqslant n_\alpha$ and assume the F^α matrices are diagonal with $F^\alpha = \mathrm{diag}(\lambda_1^\alpha, \ldots, \lambda_{n_\alpha}^\alpha)$ for convenience. Then using Proposition 3.2.18 we have

$$\begin{aligned} u_{ij}^\alpha(n) &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \tau_t(u_{ij}^\alpha)\, dt = \left(\frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} (\lambda_i^\alpha)^{it}(\lambda_j^\alpha)^{-it}\, dt\right) u_{ij}^\alpha \\ &= \exp\left(\frac{(\ln(\lambda_i^\alpha) - \ln(\lambda_j^\alpha))^2}{4n^2}\right) u_{ij}^\alpha. \end{aligned}$$

Then we have $u_{ij}^\alpha(n) \in \mathrm{Hopf}(\mathbb{G})$ as required. □

Corollary 3.2.20 *For any $z \in \mathbb{C}$ we have that σ_z and τ_z are automorphisms on* $\mathrm{Hopf}(\mathbb{G})$.

3.2.4 Compact Matrix Quantum Groups

We end this section with the following proposition. This will be our method for constructing the compact quantum group $\mathrm{SU}_q(2)$ in Chapter 5.

Proposition 3.2.21 *Let A denote a unital C*-algebra, $(u_{ij})_{i,j=1}^n \in \mathbb{M}_n(A)$ and $\Delta : A \to A \otimes_{min} A$ a unital $*$-homomorphism such that:*

(i) $u = (u_{ij})_{i,j=1}^n$ *is unitary and* $\bar{u}$ *is invertible in* $\mathbb{M}_n(A)$*;*

(ii) *the set* $\{u_{ij} \mid 1 \leqslant i, j \leqslant n\}$ *generates* A*;*

(iii) $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$ *for all* $1 \leqslant i, j \leqslant n$*.*

Then there is a compact quantum group $\mathbb{G}$ *such that the reduced C*-algebra* $A_r = \mathrm{C}(\mathbb{G})$ *and* $(\mathrm{C}(\mathbb{G}), \Delta)$ *is a C*-algebraic quantum group.*

Definition 3.2.22 *We say a C*-algebra quantum group* (A, Δ) *generated as in Proposition 3.2.21 by a corepresentation* u *is a **compact matrix quantum group*** (A, Δ, u)*.*

We have immediately that u and $\bar{u}$ are corepresentation matrices of (A, Δ).

3.2.5 The Multiplicative Unitary on Compact Quantum Groups

Throughout this section let $\mathbb{G}$ be a compact quantum group and we consider the reduced C*-algebraic quantum group $(\mathrm{C}(\mathbb{G}), \Delta)$. Let $\left\{u_{ij}^\alpha \;\middle|\; \alpha \in \mathbb{A},\ 1 \leqslant i, j \leqslant n_\alpha\right\}$ be the usual basis for $\mathrm{Hopf}(\mathbb{G})$.

We now show that we can give a formula for the multiplicative unitary in the compact case.

Notation 3.2.23 *Let* $\alpha \in \mathbb{A}$ *and consider* $\mathcal{H}_\alpha = \mathrm{lin}\left\{\Lambda((u_{ij}^\alpha)^*) \;\middle|\; 1 \leqslant i, j \leqslant n_\alpha\right\}$ *as a subspace of* $\mathrm{L}^2(\mathbb{G})$*.*

As $\mathrm{Hopf}(\mathbb{G})$ is a dense $*$-subalgebra of $\mathrm{C}(\mathbb{G})$ we have $\mathrm{L}^2(\mathbb{G}) = \overline{\bigoplus_{\alpha\in\mathbb{A}}^2 \mathcal{H}_\alpha}^{\|\cdot\|_2}$.

Proposition 3.2.24 *For* W *the multiplicative unitary of* $(\mathrm{C}(\mathbb{G}), \Delta)$ *we have*

$$W\left(\xi \otimes \Lambda((u_{ij}^\alpha)^*)\right) = \sum_{k=1}^{n_\alpha} u_{ki}^\alpha \xi \otimes \Lambda((u_{kj}^\alpha)^*)$$

for all $\alpha \in \mathbb{A}$*,* $1 \leqslant i, j \leqslant n_\alpha$ *and* $\xi \in \mathrm{L}^2(\mathbb{G})$*. In particular it follows that for all* $\omega \in \mathrm{L}^1(\mathbb{G})$ *we have*

$$(\omega \otimes \mathrm{id})(W)\Lambda((u_{ij}^\alpha)^*) = \sum_{k=1}^{n_\alpha} \omega(u_{ki}^\alpha)\Lambda((u_{kj}^\alpha)^*).$$

Proof

We have from Theorem 2.2.16 that $W \in \mathcal{B}(\mathrm{L}^2(\mathbb{G}) \otimes \mathrm{L}^2(\mathbb{G}))$ is given by $W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)(\Delta(y)(x \otimes 1))$ for all $x, y \in \mathrm{C}(\mathbb{G})$. Then for $\eta \in \mathrm{L}^2(\mathbb{G})$, $\beta \in \mathbb{A}$ and $1 \leqslant p, q \leqslant n_\beta$ we have

$$W^*(\eta \otimes \Lambda((u^\beta_{pq})^*)) = \left(\sum_{r=1}^{n_\beta} (u^\beta_{pr})^* \otimes (u^\beta_{rq})^*\right)(\eta \otimes \Lambda(1)) = \sum_{r=1}^{n_\beta} (u^\beta_{pr})^* \eta \otimes \Lambda((u^\beta_{rq})^*)$$

and so for $\alpha \in \mathbb{A}$, $1 \leqslant i, j \leqslant n_\alpha$ and $\xi \in \mathrm{L}^2(\mathbb{G})$ we calculate

$$\begin{aligned}
\big(W(\xi \otimes \Lambda((u^\alpha_{ij})^*))\big|\eta \otimes \Lambda((u^\beta_{pq})^*)\big) &= \sum_{r=1}^{n_\beta} \big(\xi \otimes \Lambda((u^\alpha_{ij})^*)\big|(u^\beta_{pr})^*\eta \otimes \Lambda((u^\beta_{rq})^*)\big) \\
= \sum_{r=1}^{n_\beta} \big(u^\beta_{pr}\xi\big|\eta\big)\, \phi(u^\beta_{rq}(u^\alpha_{ij})^*) &= \sum_{r=1}^{n_\beta} \big(u^\beta_{pr}\xi\big|\eta\big)\, \delta_{\alpha,\beta}\delta_{ri}\frac{F^\alpha_{jq}}{\Lambda^\alpha} = \sum_{r=1}^{n_\alpha} (u^\alpha_{ri}\xi|\eta)\, \delta_{\alpha,\beta}\delta_{rp}\frac{F^\alpha_{jq}}{\Lambda^\alpha} \\
= \sum_{r=1}^{n_\alpha} (u^\alpha_{ri}\xi|\eta)\, \phi(u^\beta_{pq}(u^\alpha_{rj})^*) &= \sum_{r=1}^{n_\alpha} \big(u^\alpha_{ri}\xi \otimes \Lambda((u^\alpha_{rj})^*)\big|\eta \otimes \Lambda((u^\beta_{pq})^*)\big)
\end{aligned}$$

from which the result follows. □

Proposition 3.2.25 *The Hilbert space $\mathcal{H} = \bigoplus^2_{\alpha \in \mathbb{A}} \mathcal{H}_\alpha$ is unitarily equivalent to $\mathrm{L}^2(\mathbb{G})$ and for $\alpha \in \mathbb{A}$ and $1 \leqslant i, j \leqslant n_\alpha$ we have that W acts on the restriction $\mathrm{L}^2(\mathbb{G}) \otimes \mathcal{H}_\alpha$ of $\mathrm{L}^2(\mathbb{G}) \otimes \mathcal{H}$ by*

$$W(\xi \otimes e^{n_\alpha}_{ij}) = \sum_{k=1}^{n_\alpha} u^\alpha_{ik}\xi \otimes e^{n_\alpha}_{kj}.$$

Proof We define a map $T^\alpha : \mathcal{H}_\alpha \to \mathbb{M}_{n_\alpha}$ by $\Lambda((u^\alpha_{ij})^*) \mapsto \sqrt{\frac{\lambda^\alpha_j}{\Lambda^\alpha}} e^{n_\alpha}_{ij}$ where $e^{n_\alpha}_{ij} \in \mathbb{M}_{n_\alpha}$ is the unit matrix with 1 in the i, j-th entry and 0 elsewhere and we treat $\mathbb{M}_{n_\alpha} \cong_i \ell^2(n_\alpha) \otimes \ell^2(n_\alpha)$ as a Hilbert space with this isometric identification. Clearly T^α is onto and we have

$$\big(T^\alpha(\Lambda((u^\alpha_{ij})^*))\big|T^\alpha(\Lambda((u^\alpha_{pq})^*))\big) = \delta_{jq}\delta_{ip}\frac{\lambda^\alpha_j}{\Lambda^\alpha} = \phi(u^\alpha_{pq}(u^\alpha_{ij})^*) = \big(\Lambda((u^\alpha_{ij})^*)\big|\Lambda((u^\alpha_{pq})^*)\big)$$

for λ^α_j the diagonal elements of the F-matrices as per Theorem 3.2.15. So we have shown that T^α is unitary.

Let $\mathcal{H} := \overline{\bigoplus^2_{\alpha\in\mathbb{A}} \mathbb{M}_{n_\alpha}}^{\|\cdot\|_2}$ as a Hilbert space direct sum, where again we treat $\mathbb{M}_{n_\alpha} \cong_i \ell^2(n_\alpha) \otimes \ell^2(n_\alpha)$ as a Hilbert space. Then we can then define a map $T : \mathrm{L}^2(\mathbb{G}) \to \mathcal{H}$ as a direct sum of the maps $(T_\alpha)_{\alpha\in\mathbb{A}}$ as given in Proposition A.5.2. Let $\xi = (\xi_\alpha), \eta = (\eta_\alpha) \in \mathcal{H}$ and we have

$$(T\xi|T\eta) = \sum_\alpha (T^\alpha\xi_\alpha|T^\alpha\eta_\alpha) = \sum_\alpha (\xi_\alpha|\eta_\alpha)$$

and as T has dense range by construction then T is unitary.

Now using the formula for W given by Proposition 3.2.24 we can consider $(1 \otimes T)W(1 \otimes T^*) \in \mathcal{B}(\mathrm{L}^2(\mathbb{G}) \otimes \mathcal{H})$ for $\mathcal{H}$ given above. As T^α is unitary we have $(T^\alpha)^* = (T^\alpha)^{-1}$ and thus

$$(T^\alpha)^*(e^{n_\alpha}_{ij}) = \sqrt{\frac{\Lambda^\alpha}{\lambda^\alpha_j}}\Lambda((u^\alpha_{ij})^*).$$

The result follows as for all $\alpha \in \mathbb{A}$, $1 \leqslant i, j \leqslant n_\alpha$ and $\xi \in \mathrm{L}^2(\mathbb{G})$ we have

$$\begin{aligned}((1 \otimes T)W(1 \otimes T^*))\,(\xi \otimes e^{n_\alpha}_{ij}) &= \sqrt{\frac{\Lambda^\alpha}{\lambda^\alpha_j}}\,((1 \otimes T)W)\,(\xi \otimes \Lambda((u^\alpha_{ij})^*)) \\ &= \sqrt{\frac{\Lambda^\alpha}{\lambda^\alpha_j}}\sum_{k=1}^{n_\alpha}(1 \otimes T)\left(u^\alpha_{ki}\xi \otimes \Lambda((u^\alpha_{kj})^*)\right) = \sum_{k=1}^{n_\alpha} u^\alpha_{ki}\xi \otimes e^{n_\alpha}_{kj}. \quad \square\end{aligned}$$

3.2.6 Products of Compact Quantum Groups

In Section 2.5 we defined the product of two locally compact quantum groups. We now consider products of compact quantum groups.

Proposition 3.2.26 *If $\mathbb{G}$ and $\mathbb{H}$ are compact quantum groups then the product $\mathbb{G} \times \mathbb{H}$ of Definition-Theorem 2.5.2 is compact.*

Theorem 3.2.27 *Let $\mathbb{G}$ and $\mathbb{H}$ be compact matrix quantum groups with matrices $u^{\mathbb{G}} \in \mathbb{M}_N(\mathbb{G})$ and $u^{\mathbb{H}} \in \mathbb{M}_M(\mathbb{H})$ that generate $\mathrm{C}(\mathbb{G})$ and $\mathrm{C}(\mathbb{H})$ respectively. Let $\iota^{\mathbb{G}} : \mathrm{C}(\mathbb{G}) \to$*

$\mathrm{C}(\mathbb{G}) \otimes_{min} \mathrm{C}(\mathbb{H})$ *and* $\iota^{\mathbb{H}} : \mathrm{C}(\mathbb{H}) \to \mathrm{C}(\mathbb{G}) \otimes_{min} \mathrm{C}(\mathbb{H})$ *denote the maps given by* $x \mapsto x \otimes 1$ *and* $y \mapsto 1 \otimes y$ *respectively for* $x \in \mathrm{C}(\mathbb{G})$ *and* $y \in \mathrm{C}(\mathbb{H})$. *Then the product* $\mathbb{G} \times \mathbb{H}$ *as locally compact quantum groups given by Definition-Theorem* 2.5.2 *is a compact matrix quantum group with reduced C*-algebraic quantum group* $(\mathrm{C}(\mathbb{G} \times \mathbb{H}), \Delta^{\mathbb{G}\times\mathbb{H}}, \iota_N^{\mathbb{G}}(u^{\mathbb{G}}) \oplus \iota_M^{\mathbb{H}}(u^{\mathbb{H}}))$ *where*

$$\iota_N^{\mathbb{G}}(u^{\mathbb{G}}) \oplus \iota_M^{\mathbb{H}}(u^{\mathbb{H}}) = \begin{pmatrix} \iota_N^{\mathbb{G}}(u^{\mathbb{G}}) & 0 \\ 0 & \iota_M^{\mathbb{H}}(u^{\mathbb{H}}) \end{pmatrix} \in \mathbb{M}_{N+M}(\mathrm{C}(\mathbb{G}) \otimes_{min} \mathrm{C}(\mathbb{H}))$$

and with dense Hopf algebra $\mathrm{Hopf}(\mathbb{G}) \odot \mathrm{Hopf}(\mathbb{H})$.

Proof

It is easy to show that $u^{\mathbb{G}\times\mathbb{H}} := \iota_N^{\mathbb{G}}(u^{\mathbb{G}}) \oplus \iota_M^{\mathbb{H}}(u^{\mathbb{H}})$ is unitary and invertible and we show it is a corepresentation. For $1 \leqslant i, j \leqslant N$ we have

$$\begin{aligned} \Delta^{\mathbb{G}\times\mathbb{H}}(u_{ij}^{\mathbb{G}\times\mathbb{H}}) &= \Delta^{\mathbb{G}\times\mathbb{H}}(u_{ij}^{\mathbb{G}} \otimes 1) = \sigma_{23}(\Delta^{\mathbb{G}}(u_{ij}^{\mathbb{G}}) \otimes \Delta^{\mathbb{G}}(1)) \\ &= \sum_{k=1}^{N} u_{ik}^{\mathbb{G}} \otimes 1 \otimes u_{kj}^{\mathbb{G}} \otimes 1 = \sum_{k=1}^{N+M} u_{ik}^{\mathbb{G}\times\mathbb{H}} \otimes u_{kj}^{\mathbb{G}\times\mathbb{H}} \end{aligned}$$

where we can take the sum to $N + M$ as the off diagonal entries are 0. Similarly for $N + 1 \leqslant i, j \leqslant N + M$ we have

$$\begin{aligned} \Delta^{\mathbb{G}\times\mathbb{H}}(u_{ij}^{\mathbb{G}\times\mathbb{H}}) &= \Delta^{\mathbb{G}\times\mathbb{H}}(1 \otimes u_{i-N,j-N}^{\mathbb{H}}) = \sigma_{23}(\Delta^{\mathbb{G}}(1) \otimes \Delta^{\mathbb{H}}(u_{i-N,j-N}^{\mathbb{H}})) \\ &= \sum_{k=N+1}^{N+M} 1 \otimes u_{i-N,k-N}^{\mathbb{H}} \otimes 1 \otimes U_{k-N,j-N}^{\mathbb{H}} = \sum_{k=1}^{N+M} u_{ik}^{\mathbb{G}\times\mathbb{H}} \otimes u_{kj}^{\mathbb{G}\times\mathbb{H}}. \end{aligned}$$

The off diagonal entries are all zero so for all $1 \leqslant i, j \leqslant N + M$ we have shown that

$$\Delta^{\mathbb{G}\times\mathbb{H}}(u_{ij}^{\mathbb{G}\times\mathbb{H}}) = \sum_{k=1}^{N+M} u_{ik}^{\mathbb{G}\times\mathbb{H}} \otimes u_{kj}^{\mathbb{H}}$$

and so $u^{\mathbb{G}\times\mathbb{H}}$ is a unitary, invertible corepresentation.

Let $\mathrm{Hopf}(\mathbb{G} \times \mathbb{H})$ be the Hopf $*$-algebra generated by the corepresentation $u^{\mathbb{G}\times\mathbb{H}}$. Let $x \in \mathrm{C}(\mathbb{G})$ and $y \in \mathrm{C}(\mathbb{H})$, then we have nets $(x_\alpha) \subset \mathrm{Hopf}(\mathbb{G})$ and $(y_\beta) \subset \mathrm{Hopf}(\mathbb{H})$ with limits x and y respectively. Clearly $(x_\alpha \otimes 1), (1 \otimes y_\beta) \subset \mathrm{Hopf}(\mathbb{G} \times \mathbb{H})$ and $(x_\alpha \otimes y_\beta) \subset \mathrm{Hopf}(\mathbb{G} \times \mathbb{H})$ with limit $x \otimes y \in \mathrm{C}(\mathbb{G}) \odot \mathrm{C}(\mathbb{H})$. Using that $\mathrm{C}(\mathbb{G}) \odot \mathrm{C}(\mathbb{H})$ is dense in $\mathrm{C}(\mathbb{G} \times \mathbb{H})$ we have that $\mathrm{Hopf}(\mathbb{G} \times \mathbb{H})$ is dense in $\mathrm{C}(\mathbb{G} \times \mathbb{H})$. Then by Proposition 3.2.21 we have a compact matrix quantum group. It follows that $\mathrm{Hopf}(\mathbb{G} \times \mathbb{H}) = \mathrm{Hopf}(\mathbb{G}) \odot \mathrm{Hopf}(\mathbb{H})$ from the matrix generating $\mathrm{Hopf}(\mathbb{G} \times \mathbb{H})$. $\square$

3.3 Discrete Quantum Groups

We will only need a few minor facts about discrete quantum groups for this thesis. We give only basic properties here and refer the reader to Sołtan (2006) for further details.

Definition 3.3.1 *A* ***discrete quantum group*** *is a locally compact quantum group that is the dual of a compact quantum group.*

For $\mathbb{G}$ a discrete quantum group we denote by $c_0(\mathbb{G})$ the reduced C*-algebraic quantum group to mimic $c_0(G)$ for a discrete group G.

Theorem 3.3.2 *Let $\mathbb{G}$ be a discrete quantum group, then $\widehat{\mathbb{G}}$ is a compact quantum group and we let $\{U^\alpha \mid \alpha \in \mathbb{A}\}$ denote the set of all mutually inequivalent, finite-dimensional, irreducible, unitary corepresentations. Then we have*

$$c_0(\mathbb{G}) = \bigoplus^{\infty}{}_{\alpha \in \mathbb{A}} \mathbb{M}_{n_\alpha}$$

that is $c_0(\mathbb{G})$ is the set of all families $\{(m_\alpha)_{\alpha\in\mathbb{A}} \mid m_\alpha \in \mathbb{M}_{n_\alpha}\}$ such that for any $\varepsilon > 0$ there is a finite subset $F \subset \mathbb{A}$ such that $\|m_\beta\| < \varepsilon$ for all $\beta \in \mathbb{A}\backslash F$. Also we have the multiplier C-algebra $\mathrm{M}(c_0(\mathbb{G}))$ given by the family $\{(m_\alpha)_{\alpha\in\mathbb{A}} \mid m_\alpha \in \mathbb{M}_{n_\alpha}\}$ such that $\sup_{i\in I} \|m_\alpha\|$ is finite.*

3.4 Coamenable Quantum Groups

We now define coamenable quantum groups. Coamenability has the interesting property that the universal and reduced C*-algebras are equal and therefore the dual of $\mathrm{C}_0(\mathbb{G})$ is a unital Banach algebra. We refer to the paper Bédos & Tuset (2003) for further details and proofs. In particular we have that $\mathrm{SU}_q(2)$ is coamenable.

Theorem 3.4.1 *Let $\mathbb{G}$ be a locally compact quantum group. Then the following are equivalent:*

(i) *There exists a state ε on $\mathrm{C}_0(\mathbb{G})$ such that $(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$;*

(ii) *There exists a state ε on $\mathrm{C}_0(\mathbb{G})$ such that $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$;*

(iii) *There exists a bounded approximate identity in $\mathrm{L}^1(\mathbb{G})$;*

(iv) $\mathrm{C}_0(\mathbb{G})^*$ *is unital;*

(v) $\mathrm{C}_0(\mathbb{G}) = \mathrm{C}_0^u(\mathbb{G})$.

Definition 3.4.2 *A locally compact quantum group $\mathbb{G}$ is **coamenable** if any of the equivalent conditions of Theorem 3.4.1 hold.*

Chapter 4

The $L^1_\sharp(\mathbb{G})$ Algebra

We now move on to the research topics of the thesis. In this chapter we will define a Banach $*$-subalgebra of $L^1(\mathbb{G})$ that we will denote $L^1_\sharp(\mathbb{G})$ and we will give a comprehensive study of the properties of this object for a locally compact quantum group $\mathbb{G}$.

We begin in Section 4.1 with an overview of $L^1_\sharp(\mathbb{G})$ and properties that were already known before this work. We show it is a Banach $*$-algebra, investigate properties of smearing elements of $L^1_\sharp(\mathbb{G})$ as a Banach space, investigate further properties of $L^1_\sharp(\mathbb{G})$ and its dual and we consider the related space of $C_0(\mathbb{G})^*{}_\sharp$. In Section 4.2 we will place an operator space structure on $L^1_\sharp(\mathbb{G})$ and show that we have a completely contractive Banach algebra. Then we review smearing properties on $L^1_\sharp(\mathbb{G})$ as a completely contractive Banach algebra and for the $L^1_\sharp$ algebra of a product of locally compact quantum groups. Finally in Section 4.3 we investigate properties of the $L^1_\sharp(\mathbb{G})$ algebra for compact quantum groups and we show that a locally compact quantum group $\mathbb{G}$ is compact if and only if $L^1_\sharp(\mathbb{G})$ is an ideal in its double dual with respect to either Arens products.

With the exception of Section 4.1 all the work in this chapter is original research by the author.

4.1 Basic Properties of $\mathrm{L}^1(\mathbb{G})$ and $\mathrm{L}^1_\sharp(\mathbb{G})$

In this section we define $\mathrm{L}^1_\sharp(\mathbb{G})$ in Definition 4.1.1 and we investigate the elementary properties of this object. We show first it is a Banach $*$-algebra and then we investigate the smearing to show that it is dense in $\mathrm{L}^1(\mathbb{G})$ under the $\|\cdot\|_{\mathrm{L}^1(\mathbb{G})}$ norm.

Note that whilst all the results in this section are known, the proofs, whilst not necessarily difficult, are not always recorded in the original literature on the subject and we offer them here for completeness.

4.1.1 $\mathrm{L}^1_\sharp(\mathbb{G})$ as a Banach $*$-algebra

We showed in Section 2.4 that $\mathrm{L}^1(\mathbb{G})$ is a Banach algebra. We can make $\mathrm{L}^1(\mathbb{G})$ a Banach $*$-algebra with involution given by $\omega \mapsto \omega^\natural$ where we define $\langle x, \omega^\natural\rangle = \overline{\langle R(x)^*, \omega\rangle}$ for $x \in \mathrm{C}_0(\mathbb{G})$ or indeed $\omega^\natural = \omega^* \circ R$; however in general λ is not a $*$-homomorphism with this involution. In fact there is no involution on $\mathrm{L}^1(\mathbb{G})$ such that λ is a $*$-homomorphism but we can define a $*$-subalgebra such that the restriction of λ to this $*$-subalgebra is a $*$-homomorphism. We define this object now and then show that there is a norm such that this is a Banach $*$-algebra.

Definition 4.1.1 *We define the space*

$$\mathrm{L}^1_\sharp(\mathbb{G}) = \left\{\omega \in \mathrm{L}^1(\mathbb{G}) \;\middle|\; \exists\, \kappa \in \mathrm{L}^1(\mathbb{G}) \text{ such that } \langle x, \kappa\rangle = \overline{\langle S(x)^*, \omega\rangle} \quad \forall\, x \in \mathrm{Dom}(S)\right\}. \tag{4.1}$$

Let $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$, then we have a unique $\kappa \in \mathrm{L}^1(\mathbb{G})$ such that $\langle x, \kappa\rangle = \overline{\langle S(x)^*, \omega\rangle}$ for all $x \in \mathrm{Dom}(S)$. This follows by considering two such $\kappa_1, \kappa_2 \in \mathrm{L}^1(\mathbb{G})$ that satisfy this equation and we have then $\langle x, \kappa_1\rangle = \overline{\langle S(x)^*, \omega\rangle} = \langle x, \kappa_2\rangle$ for all $x \in \mathrm{Dom}(S)$. As $\mathrm{Dom}(S)$ is dense in $\mathrm{C}_0(\mathbb{G})$ we have $\kappa_1 = \kappa_2$.

Note that we haven't mentioned whether we are considering the antipode in the C*-algebraic or von Neumann algebraic setting. It seems more natural to consider the von

Neumann algebraic antipode as $\mathrm{L}^1(\mathbb{G})$ is the predual of $\mathrm{L}^\infty(\mathbb{G})$ and clearly if we assume this then the equation holds for all the x in the domain of the C*-algebraic antipode. On the other hand let $x \in \mathrm{Dom}(S)$ for S the von Neumann algebraic antipode, then from Definition-Theorem 2.2.7 we have a σ-strong* core of $\mathrm{Dom}(S)$ given by $\mathrm{Dom}(S) \cap \mathrm{C}_0(\mathbb{G})$ and thus also a σ-weak core as this is weaker than the σ-strong* topology. Then there is some $(x_\alpha) \subset \mathrm{Dom}(S) \cap \mathrm{C}_0(\mathbb{G})$ such that $x_\alpha \xrightarrow{w^*} x$ and $S(x_\alpha) \xrightarrow{w^*} S(x)$ and so for all $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ we have

$$\begin{aligned}\left|\overline{\langle S(x)^*, \omega\rangle} - \langle x, \omega^\sharp\rangle\right| &\leqslant \left|\overline{\langle S(x)^*, \omega\rangle} - \overline{\langle S(x_\alpha)^*, \omega\rangle}\right| + \left|\langle x_\alpha, \omega^\sharp\rangle - \langle x, \omega^\sharp\rangle\right| \\ &= |\langle S(x) - S(x_\alpha), \omega^*\rangle| + \left|\langle x_\alpha - x, \omega^\sharp\rangle\right| \to 0.\end{aligned}$$

So it is sufficient to consider the C*-algebraic antipode.

The following proposition was proved in Proposition 3.1 in Kustermans (2001) but we reproduce the proof here for convenience of the reader.

Proposition 4.1.2 *Let* $\lambda : \mathrm{L}^1(\mathbb{G}) \to \mathrm{C}_0(\widehat{\mathbb{G}})$ *be the left regular representation from Definition 2.4.3. We have the following identity*

$$\mathrm{L}^1_\sharp(\mathbb{G}) = \left\{\omega \in \mathrm{L}^1(\mathbb{G}) \;\middle|\; \exists\, \kappa \in \mathrm{L}^1(\mathbb{G}),\;\; \lambda(\omega)^* = \lambda(\kappa)\right\}$$

and $\lambda(\omega^\sharp) = \lambda(\omega)^*$ *for all* $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$.

Proof

Let $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ and we show that $\lambda(\omega^\sharp) = \lambda(\omega)^*$. By Definition-Theorem 2.2.7 for all $\rho \in \mathrm{L}^1(\mathbb{G})$ we have $(\mathrm{id} \otimes \rho)(W) \in \mathrm{Dom}(S)$ and using the formula $S((\mathrm{id} \otimes \rho)(W)) = (\mathrm{id} \otimes \rho)(W^*)$ we have

$$\begin{aligned}\langle \lambda(\omega^\sharp), \rho\rangle &= \langle (\omega^\sharp \otimes \mathrm{id})(W), \rho\rangle = \langle (\mathrm{id} \otimes \rho)(W), \omega^\sharp\rangle = \langle S((\mathrm{id} \otimes \rho)(W)), \omega^*\rangle \\ &= \langle (\mathrm{id} \otimes \rho)(W^*), \omega^*\rangle = \overline{\langle W, \omega \otimes \rho^*\rangle} = \langle (\omega \otimes \mathrm{id})(W)^*, \rho\rangle = \langle \lambda(\omega)^*, \rho\rangle.\end{aligned}$$

As this holds for all $\rho \in L^1(\mathbb{G})$ we have $\lambda(\omega^\sharp) = \lambda(\omega)^*$.

Using that $\lambda(\omega^\sharp) = \lambda(\omega)^*$ for all $\omega \in L^1_\sharp(\mathbb{G})$ it follows immediately that we have $L^1_\sharp(\mathbb{G}) \subset \{\omega \in L^1(\mathbb{G}) \mid \exists\, \kappa \in L^1(\mathbb{G}),\quad \lambda(\omega)^* = \lambda(\kappa)\}$. Conversely, let $\omega \in L^1(\mathbb{G})$ such that there exists $\kappa \in L^1(\mathbb{G})$ with $\lambda(\omega)^* = \lambda(\kappa)$. Then for all $\rho \in L^1(\mathbb{G})$ we have

$$\begin{aligned}\overline{\langle S((\mathrm{id}\otimes\rho)(W))^*, \omega\rangle} &= \langle(\mathrm{id}\otimes\rho)(W^*), \omega^*\rangle = \overline{\langle W, \omega\otimes\rho^*\rangle} = \overline{\langle\lambda(\omega), \rho^*\rangle}\\ &= \langle\lambda(\omega)^*, \rho\rangle = \langle\lambda(\kappa), \rho\rangle = \langle(\mathrm{id}\otimes\rho)(W), \kappa\rangle.\end{aligned}$$

We have from Definition-Theorem 2.2.7 that $\{(\mathrm{id}\otimes\rho)(W) \mid \rho \in L^1(\mathbb{G})\}$ is a core for S and so we have shown that for all $x \in \mathrm{Dom}(S)$ we have $\overline{\langle S(x)^*, \omega\rangle} = \langle x, \kappa\rangle$, that is $\omega \in L^1_\sharp(\mathbb{G})$ with $\omega^\sharp = \kappa$. □

In particular we have shown in the previous proposition that the left regular representation λ restricts to a $*$-homomorphism on $L^1_\sharp(\mathbb{G})$.

Proposition 4.1.3 *We have that $L^1_\sharp(\mathbb{G})$ is a Banach $*$-algebra under the norm $\|\cdot\|_\sharp$ given by*

$$\|\omega\|_\sharp = \max\{\|\omega\|_{L^1(\mathbb{G})}, \|\omega^\sharp\|_{L^1(\mathbb{G})}\}$$

for $\omega \in L^1_\sharp(\mathbb{G})$.

Proof

Let $\omega_1, \omega_2 \in L^1_\sharp(\mathbb{G})$, then as λ is a $*$-homomorphism on $L^1_\sharp(\mathbb{G})$ we have

$$\lambda(\omega_1 * \omega_2)^* = (\lambda(\omega_1)\lambda(\omega_2))^* = \lambda(\omega_2)^*\lambda(\omega_1)^* = \lambda(\omega_2^\sharp)\lambda(\omega_1^\sharp) = \lambda(\omega_2^\sharp * \omega_1^\sharp)$$

which implies that $\omega_1 * \omega_2 \in L^1_\sharp(\mathbb{G})$ with $(\omega_1 * \omega_2)^\sharp = \omega_2^\sharp * \omega_1^\sharp$. It is easy to see that $\omega \mapsto \omega^\sharp$ is an involution on $L^1_\sharp(\mathbb{G})$ and thus $L^1_\sharp(\mathbb{G})$ is a $*$-algebra.

We now show that $\|\cdot\|_\sharp$ is indeed a norm. It is clear that $\|\omega\|_\sharp = 0$ if and only if $\omega = 0$

and $\|t\omega\|_\sharp = |t|\,\|\omega\|_\sharp$ for all $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ and $t \in \mathbb{C}$. Finally we have

$$\|\omega_1 + \omega_2\|_{\mathrm{L}^1(\mathbb{G})} \leqslant \|\omega_1\|_{\mathrm{L}^1(\mathbb{G})} + \|\omega_1\|_{\mathrm{L}^1(\mathbb{G})} \leqslant \|\omega_1\|_\sharp + \|\omega_2\|_\sharp$$

and similarly $\|\omega_1^\sharp + \omega_2^\sharp\|_{\mathrm{L}^1(\mathbb{G})} \leqslant \|\omega_1\|_\sharp + \|\omega_2\|_\sharp$ and so it follow that $\|\omega_1 + \omega_2\|_\sharp \leqslant \|\omega_1\|_\sharp + \|\omega_2\|_\sharp$ as required.

We want to show that $\mathrm{L}^1_\sharp(\mathbb{G})$ is complete under the $\|\cdot\|_\sharp$ norm. Let $(\omega_n) \subset \mathrm{L}^1_\sharp(\mathbb{G})$ be a Cauchy sequence under the $\|\cdot\|_\sharp$ norm. Then we have from the definition of the $\|\cdot\|_\sharp$ norm that both $\|\omega_n - \omega_m\|_{\mathrm{L}^1(\mathbb{G})} \to 0$ and $\|\omega_n^\sharp - \omega_m^\sharp\|_{\mathrm{L}^1(\mathbb{G})} \to 0$ as $n, m \to \infty$ and so as $\mathrm{L}^1(\mathbb{G})$ is a Banach algebra there exists $\omega, \kappa \in \mathrm{L}^1(\mathbb{G})$ with $\|\omega - \omega_n\|_{\mathrm{L}^1(\mathbb{G})} \to 0$ and $\|\kappa - \omega_n^\sharp\|_{\mathrm{L}^1(\mathbb{G})} \to 0$. We now show that $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$. For all $x \in \mathrm{Dom}(S)$ we have

$$\left|\overline{\langle S(x)^*, \omega\rangle} - \langle x, \kappa\rangle\right| \leqslant \left|\overline{\langle S(x)^*, \omega\rangle} - \overline{\langle S(x)^*, \omega_n\rangle}\right| + \left|\langle x, \omega_n^\sharp\rangle - \langle x, \kappa\rangle\right| \to 0$$

and so for all $x \in \mathrm{Dom}(S)$ we have $\overline{\langle S(x)^*, \omega\rangle} = \langle x, \kappa\rangle$ as required.

Finally we show that it satisfies the additional properties for it to be a Banach $*$-algebra. For $\omega_1, \omega_2 \in \mathrm{L}^1_\sharp(\mathbb{G})$ we have $\|\omega_1 * \omega_2\|_{\mathrm{L}^1(\mathbb{G})} \leqslant \|\omega_1\|_{\mathrm{L}^1(\mathbb{G})}\|\omega_2\|_{\mathrm{L}^1(\mathbb{G})} \leqslant \|\omega_1\|_\sharp\|\omega_2\|_\sharp$ and similarly $\|(\omega_1 * \omega_2)^\sharp\|_{\mathrm{L}^1(\mathbb{G})} \leqslant \|\omega_1\|_\sharp\|\|\omega_2\|_\sharp$ giving $\|\omega_1 * \omega_2\|_\sharp \leqslant \|\omega_1\|_\sharp\|\omega_2\|_\sharp$. Also clearly $\|\omega\|_\sharp = \|\omega^\sharp\|_\sharp$ for all $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ and we are done. □

We quote the following theorem now which will be proved in the next section after a discussion of smearing in $\mathrm{L}^1_\sharp(\mathbb{G})$.

Theorem 4.1.4 *The Banach algebra $\mathrm{L}^1_\sharp(\mathbb{G})$ is dense in $\mathrm{L}^1(\mathbb{G})$ with respect to the norm on $L^1(\mathbb{G})$.*

We now show a few elementary known properties regarding $\mathrm{L}^1(\mathbb{G})$ and $\mathrm{L}^1_\sharp(\mathbb{G})$ and in particular the involution. The next Proposition follows immediately as τ_t and R are normal.

Proposition 4.1.5 *Let $\omega \in \mathrm{L}^1(\mathbb{G})$ and $t \in \mathbb{R}$, then $\omega \circ \tau_t \in \mathrm{L}^1(\mathbb{G})$ and $\omega \circ R \in \mathrm{L}^1(\mathbb{G})$.*

Proposition 4.1.6 *Let* $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ *and* $t \in \mathbb{R}$, *then* $\omega \circ \tau_t \in \mathrm{L}^1_\sharp(\mathbb{G})$ *and we have* $(\omega \circ \tau_t)^\sharp = \omega^\sharp \circ \tau_t$.

Proof

Let $x \in \mathrm{Dom}(S)$, then by Proposition 1.3.9 (ii) we have $\tau_t(x) \in \mathrm{Dom}(S)$ and furthermore $\tau_t(\tau_{-i/2}(x)) = \tau_{t-i/2}(x) = \tau_{-i/2}(\tau_t(x))$. So using Proposition 2.2.8 we have $\tau_t(S(x)) = S(\tau_t(x))$ and for all $x \in \mathrm{Dom}(S)$ we have

$$\overline{\langle S(x)^*, \omega \circ \tau_t \rangle} = \overline{\langle S(\tau_t(x))^*, \omega \rangle} = \langle \tau_t(x), \omega^\sharp \rangle = \langle x, \omega^\sharp \circ \tau_t \rangle$$

As $\omega^\sharp \in \mathrm{L}^1(\mathbb{G})$ it follows that $\omega^\sharp \circ \tau_t \in \mathrm{L}^1(\mathbb{G})$ and $(\omega \circ \tau_t)^\sharp = \omega^\sharp \circ \tau_t$. □

4.1.2 Smearing for Locally Compact Quantum Groups

We now move on to describe some further properties of $\mathrm{L}^1_\sharp(\mathbb{G})$ as a Banach $*$-algebra using smearing techniques. Some of the properties of smearing here were known previously as evidenced in Kustermans (2001) however we give more details here. This section also contains a proof of Theorem 4.1.4.

By Definition-Theorem 2.2.7 we have that the scaling group τ is a σ-strong* continuous one-parameter group of $*$-automorphisms on $\mathrm{L}^\infty(\mathbb{G})$. So by Definition 1.3.1, for fixed $x \in \mathrm{L}^\infty(\mathbb{G})$, for all sequences $(t_n) \subset \mathbb{R}$ with limit $t \in \mathbb{R}$ we have that $\tau_{t_n}(x)$ converges to $\tau_t(x)$ in the σ-strong* topology. As the σ-weak topology is weaker than the σ-strong* topology we have that $\tau_{t_n}(x) \to \tau_t(x)$ in the σ-weak topology and so the map $\mathbb{R} \to \mathbb{C}$ given by $t \mapsto \langle \tau_t(x), \omega \rangle$ is continuous for any fixed $x \in \mathrm{L}^\infty(\mathbb{G})$ and $\omega \in \mathrm{L}^1(\mathbb{G})$.

It follows from Section 1.3.3 that we can consider the smear $x(n) \in \mathrm{L}^\infty(\mathbb{G})$ for $n \in \mathbb{N}$ and we have for all $\omega \in \mathrm{L}^1(\mathbb{G})$ that

$$\langle x(n), \omega \rangle = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2 t^2} \langle \tau_t(x), \omega \rangle \, dt.$$

We have that τ_t is normal for all $t \in \mathbb{R}$ and so we have a map $(\tau_t)_* : \mathrm{L}^1(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G})$. In particular for all $t \in \mathbb{R}$ and $x \in \mathrm{L}^\infty(\mathbb{G})$ we have

$$\langle (\tau_t)_*(\omega), x\rangle = \langle \omega, \tau_t(x)\rangle = \langle \omega \circ \tau_t, x\rangle$$

and so $(\tau_t)_*(\omega) = \omega \circ \tau_t$ for all $t \in \mathbb{R}$. We make the following definition (which is easily seen to be a one-parameter group).

Definition 4.1.7 *Let τ_* be the one-parameter group on $\mathrm{L}^1(\mathbb{G})$ given by $(\tau_*)_t = (\tau_t)_*$ for all $t \in \mathbb{R}$. In particular we have $(\tau_*)_t(\omega) = \omega \circ \tau_t$ for all $\omega \in \mathrm{L}^1(\mathbb{G})$ and $t \in \mathbb{R}$.*

We know that the map $\mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G})$ given by $x \mapsto x(n)$ is normal and contractive by Theorem 1.3.21 and thus there is a contractive map which we denote $\Phi(n) : \mathrm{L}^1(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G})$ given by

$$\langle x, (\Phi(n))(\omega)\rangle = \langle x(n), \omega\rangle = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \langle \tau_t(x), \omega\rangle \, dt$$

and so we see that

$$(\Phi(n))(\omega) = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \omega \circ \tau_t \; dt \in \mathrm{L}^1(\mathbb{G})$$

where the integral is considered in the weak topology. We show now that in fact $\Phi(n)$ has image inside $\mathrm{L}^1_\sharp(\mathbb{G})$ after showing how smearing interacts with the antipode and unitary antipode. The following is immediately from Proposition 1.3.20 and Proposition 1.3.10.

Proposition 4.1.8 *Let $x \in \mathrm{L}^\infty(\mathbb{G})$ and $n \in \mathbb{N}$, then we have $R(x(n)) = R(x)(n)$ and $(x^*)(n) = x(n)^*$. Furthermore let $x \in \mathrm{Dom}(S) \subset \mathrm{L}^\infty(\mathbb{G})$, then we have $(S(x))(n) = S(x(n))$ and $(S(x)^*)(n) = S(x(n))^*$.*

The proof of the following theorem is original work by the author (though it is likely that a similar theorem was already known by Kustermans in Kustermans (2001) but no proof is offered.)

Theorem 4.1.9 *For all $\omega \in \mathrm{L}^1(\mathbb{G})$ we have $(\Phi(n))(\omega) \in \mathrm{L}^1_\sharp(\mathbb{G})$. In particular we have a corestriction $\Phi'(n) : \mathrm{L}^1(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ of the map $\Phi(n)$ and we have $\|\Phi'(n)(\omega)\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \leqslant e^{n^2/4}\|\omega\|_{\mathrm{L}^1(\mathbb{G})}$.*

Proof We note that $\left|e^{-n^2(t+i/2)^2}\right| = \left|e^{-n^2t^2-in^2t+n^2/4}\right| = e^{-n^2t^2}e^{n^2/4}$ and so we have

$$\begin{aligned}\left|\frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2(t+i/2)^2}\langle x, \omega^* \circ R \circ \tau_t\rangle dt\right| &\leqslant e^{n^2/4}\frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2t^2}\left|\langle R(\tau_t(x))^*, \omega\rangle\right| dt \\ &\leqslant e^{n^2/4}\frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2t^2}\|x\|\|\omega\|\, dt = e^{n^2/4}\|x\|\|\omega\|\end{aligned}$$

where we've used that τ_t and R are isometries from Proposition 2.2.8. So we can define a map $\kappa \in \mathrm{L}^\infty(\mathbb{G})^*$ by

$$\kappa(x) = \frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2(t+i/2)^2}\langle x, \omega^* \circ R \circ \tau_t\rangle\, dt.$$

with $\|\kappa\| \leqslant e^{n^2/4}\|\omega\|$. As ω, R are normal and τ_t is normal for all $t \in \mathbb{R}$ we have $\omega^* \circ R \circ \tau_t \in \mathrm{L}^1(\mathbb{G})$ for all $t \in \mathbb{R}$ and we have a continuous function $\mathbb{R} \to \mathrm{L}^1(\mathbb{G})$ given by $t \mapsto \frac{n}{\sqrt{\pi}}e^{-n^2(t+i/2)^2}\omega^* \circ R \circ \tau_t$. Furthermore we have that

$$\int_{\mathbb{R}}\left\|\frac{n}{\sqrt{\pi}}e^{-n^2(t+i/2)^2}\omega^* \circ R \circ \tau_t\right\| dt = e^{n^2/4}\frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2t^2}\|\omega^* \circ R \circ \tau_t\| dt \leqslant e^{n^2/4}\|\omega\| < \infty$$

and so we have that $\kappa \in \mathrm{L}^1(\mathbb{G})$ from Proposition A.6.3.

From Theorem 1.3.17 and Propositions 1.3.20 and 4.1.8, for all $x \in \mathrm{Dom}(S)$ we have $x^* \in \mathrm{Dom}(\tau_{i/2})$ and

$$\begin{aligned}\langle x, \kappa\rangle &= \frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2(t+i/2)^2}\overline{\langle R(\tau_t(x))^*, \omega\rangle} dt = \overline{\frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2(t-i/2)^2}\langle \tau_t(x^*), \omega \circ R\rangle dt} \\ &= \overline{\langle (\tau_{i/2}(x^*))(n), \omega \circ R\rangle} = \overline{\langle \big(R(\tau_{-i/2}(x))^*\big)(n), \omega\rangle} = \overline{\langle S(x)^*, \omega(n)\rangle}\end{aligned}$$

and so $\omega(n) \in \mathrm{L}^1_\sharp(\mathbb{G})$ with $(\omega(n))^\sharp = \kappa$.

Finally we have

$$\begin{aligned}\|\omega(n)\|_{\mathrm{L}^1_\sharp(\mathbb{G})} &= \max\{\|\omega(n)\|_{\mathrm{L}^1(\mathbb{G})}, \|\omega(n)^\sharp\|_{\mathrm{L}^1(\mathbb{G})}\} \\ &\leqslant \max\{\|\omega\|_{\mathrm{L}^1(\mathbb{G})}, e^{n^2/4}\|\omega\|_{\mathrm{L}^1(\mathbb{G})}\} = e^{n^2/4}\|\omega\|_{\mathrm{L}^1(\mathbb{G})}. \quad \square\end{aligned}$$

We now show the following as a corollary of this theorem as promised earlier.

Proof of Theorem 4.1.4

Let $\omega \in \mathrm{L}^1(\mathbb{G})$ and consider the sequence $(\omega(n))$ in $\mathrm{L}^1_\sharp(\mathbb{G})$ from the previous theorem. Then as τ_* is norm continuous by Corollary 2.3.7 it follows from Proposition 1.3.15 that $\lim_{n\to\infty}\omega(n) = \omega$ in the norm topology. $\square$

We also have the following straightforward proposition.

Proposition 4.1.10 *For $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ it follows that $\omega^\sharp(n) = \omega(n)^\sharp$.*

Proof

Using Proposition 4.1.8 for $x \in \mathrm{Dom}(S)$ we have

$$\langle x, \omega(n)^\sharp\rangle = \overline{\langle S(x)^*, \omega(n)\rangle} = \overline{\langle S(x(n))^*, \omega\rangle} = \langle x(n), \omega^\sharp\rangle = \langle x, \omega^\sharp(n)\rangle$$

and so it follows from the density of $\mathrm{Dom}(S)$ in $\mathrm{C}_0(\mathbb{G})$. $\square$

4.1.3 Further Properties of $\mathrm{L}^1_\sharp(\mathbb{G})$

We have the following useful characterisation of $\mathrm{L}^1_\sharp(\mathbb{G})$ in terms of the scaling group. See Definition A.2.10 in the appendix for the definition of the pre-adjoint of $\tau_{i/2}$.

Proposition 4.1.11 *We have that $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ if and only if $\omega \circ \tau_{i/2} \in \mathrm{L}^1(\mathbb{G})$, in which case $\omega^\sharp = (\omega \circ \tau_{i/2} \circ R)^*$. It then follows that $\mathrm{L}^1_\sharp(\mathbb{G}) = \mathrm{Dom}((\tau_{i/2})_*)$.*

Proof

First say $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ and let $x \in \mathrm{Dom}(\tau_{i/2})$. Then by Proposition 1.3.20 (ii) we have

$x^* \in \mathrm{Dom}(\tau_{-i/2})$ and $\tau_{-i/2}(x^*) = \tau_{i/2}(x)^*$ and so we have the following

$$(\omega \circ \tau_{i/2})(x) = \langle \tau_{-i/2}(x^*)^*, \omega \rangle = \langle S(R(x^*))^*, \omega \rangle = \overline{\langle R(x^*), \omega^\sharp \rangle} = \langle x, (\omega^\sharp \circ R)^* \rangle$$

where we've used that R, S and $\tau_{-i/2}$ commute from Proposition 2.2.8 and $R^2 = \mathrm{id}$ from Definition-Theorem 2.2.7. It follows from this that for $x \in \mathrm{Dom}(\tau_{i/2})$ we have

$$\left|(\omega \circ \tau_{i/2})(x)\right| = \left|\langle x^*, \omega^\sharp \circ R \rangle\right| \leqslant \|R(x^*)\| \|\omega^\sharp\|_{L^1(\mathbb{G})} = \|x\| \|\omega^\sharp\|_{L^1(\mathbb{G})}$$

and as $\mathrm{Dom}(\tau_{i/2})$ is dense in $L^\infty(\mathbb{G})$ we then have that $\|\omega \circ \tau_{i/2}\| \leqslant \|\omega^\sharp\|_{L^1(\mathbb{G})}$. Also we as $\omega \circ \tau_{i/2} = (\omega^\sharp \circ R)^*$ and R is normal it follows that $\omega \circ \tau_{i/2}$ is normal and so we have shown that $\omega \circ \tau_{i/2} \in L^1(\mathbb{G})$ as required.

Conversely, say $\omega \circ \tau_{i/2} \in L^1(\mathbb{G})$. Then for all $x \in \mathrm{Dom}(S)$ we have

$$\overline{\langle S(x)^*, \omega \rangle} = \overline{\langle \tau_{i/2}(R(x^*)), \omega \rangle} = \overline{\langle x^*, \omega \circ \tau_{i/2} \circ R \rangle} = \langle x, (\omega \circ \tau_{i/2} \circ R)^* \rangle$$

where we've used that $x^* \in \mathrm{Dom}(\tau_{i/2})$ again and that R is a $*$-map. As $\omega \circ \tau_{i/2} \in L^1(\mathbb{G})$ it follows that $(\omega \circ \tau_{i/2} \circ R)^* \in L^1(\mathbb{G})$ and so $\omega \in L^1_\sharp(\mathbb{G})$ with $\omega^\sharp = (\omega \circ \tau_{i/2} \circ R)^*$.

It follows immediately from this and Definition A.2.10 that $L^1_\sharp(\mathbb{G}) = \mathrm{Dom}((\tau_{i/2})_*)$. □

We create the following notation, note however that this does not give us an involution as generally $(\omega^\flat)^\flat = \omega \circ \tau_i \neq \omega$ for $\omega \in L^1_\sharp(\mathbb{G})$.

Notation 4.1.12 *Given $\omega \in L^1_\sharp(\mathbb{G})$ we let $\omega^\flat := \omega \circ \tau_{i/2} \in L^1(\mathbb{G})$.*

It follows from Proposition 4.1.11 that if $\omega \in L^1_\sharp(\mathbb{G})$ then the map $\omega \circ \tau_{i/2} : \mathrm{Dom}(\tau_{i/2}) \to \mathbb{C}$ given by $x \mapsto \langle \tau_{i/2}(x), \omega \rangle$ extends uniquely to a map in $L^1(\mathbb{G})$. We might ask the following question: given $\omega \in L^1(\mathbb{G})$ such that the map $\omega \circ \tau_{i/2} : \mathrm{Dom}(\tau_{i/2}) \to \mathbb{C}$ is bounded, then by the Hahn-Banach theorem there is a map $\kappa \in C_0(\mathbb{G})^*$ such that $\langle x, \kappa \rangle = \langle \tau_{i/2}(x), \omega \rangle$ for all $x \in \mathrm{Dom}(\tau_{i/2})$ and such that $\|\kappa\| = \|\omega \circ \tau_{i/2}\|$, then does it follows that $\kappa \in \mathrm{L}^1(\mathbb{G})$? If so we can give a much more palatable definition of the $\mathrm{L}^1_\sharp(\mathbb{G})$ algebra. The general answer to this question is still open to the best of the author's knowledge, however we will investigate this further in the case of $\mathrm{SU}_q(2)$ in Section 5.3 below.

We refer the reader to Daws & Salmi (2013) for a proof of the following useful theorem.

Theorem 4.1.13 *For $\mathbb{G}$ a coamenable locally compact quantum group there is a contractive approximate identity (e_α) of $\mathrm{L}^1_\sharp(\mathbb{G})$.*

Finally we mention that as $\mathrm{L}^1_\sharp(\mathbb{G})$ is a Banach $*$-algebra we have a multiplication map $m : \mathrm{L}^1_\sharp(\mathbb{G}) \hat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ linearising the multiplication map where $\hat{\otimes}$ denotes the Banach space projective tensor product. We show in the next section that we can extend this to a completely contractive Banach algebra as per Definition 1.1.49.

4.1.4 The Dual of $\mathrm{L}^1_\sharp(\mathbb{G})$

Consider the map $\theta : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G})}$ given by

$$\omega \mapsto (\omega, \overline{\omega^\sharp}). \tag{4.2}$$

It follows easily that this is a linear map and for $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ we have that $\|\theta(\omega)\| = \max\left\{\|\omega\|_{\mathrm{L}^1(\mathbb{G})}, \|\omega^\sharp\|_{\mathrm{L}^1(\mathbb{G})}\right\} = \|\omega\|_\sharp$ and so θ is an isometric embedding. Because this is an isometry the adjoint is a quotient map and we show below that we can form an explicit representation of $\mathrm{L}^1_\sharp(\mathbb{G})^*$. We begin with a result that comes from Proposition A.1 from Brannan *et al.* (2013). Note however we use smearing techniques for the proof here.

Lemma 4.1.14 *Let $\mathbb{G}$ be a locally compact quantum group and $x, y \in \mathrm{L}^\infty(\mathbb{G})$ such that $\langle x^*, \omega^* \rangle = \langle y, \omega^\sharp \rangle$ for all $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$. Then $y \in \mathrm{Dom}(S)$ and $S(y) = x^*$.*

Proof

We consider the sequence $(y(n))_{n=1}^\infty \subset \mathrm{Dom}(S)$ where $y(n)$ is the smear of y with respect to τ for $n \in \mathbb{N}$ given by Definition 1.3.14. It follows from Theorem 1.3.17 that for all $n \in \mathbb{N}$ we have $y(n) \in \mathrm{Dom}(S)$ and from Proposition 1.3.16 we have $y(n) \xrightarrow{w^*} y$. We show that $S(y(n)) \xrightarrow{w^*} x^*$.

Let $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$. From Proposition 4.1.6 we have that $\omega \circ \tau_t \in \mathrm{L}^1_\sharp(\mathbb{G})$ with $(\omega \circ \tau_t)^\sharp = \omega^\sharp \circ \tau_t$ and using Proposition 4.1.8 we have

$$\begin{aligned}\langle S(y(n)), \omega^* \rangle = \langle y(n), \omega^\sharp \rangle = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \langle \tau_t(y), \omega^\sharp \rangle \, dt = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \overline{\langle x, \omega \circ \tau_t \rangle} \, dt \\ = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \langle \tau_t(x)^*, \omega^* \rangle \, dt = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \langle \tau_t(x^*), \omega^* \rangle \, dt = \langle x(n)^*, \omega^* \rangle.\end{aligned}$$

As $\left\{\omega^* \;\middle|\; \omega \in \mathrm{L}^1_\sharp(\mathbb{G})\right\}$ is dense in $\mathrm{L}^1(\mathbb{G})$ it follows that $S(y(n)) = x(n)^*$ for all $n \in \mathbb{N}$. But as $x(n)^* \xrightarrow{w^*} x^*$ then $S(y(n)) \xrightarrow{w^*} x^*$. Then using Proposition A.2.3 we have $y \in \mathrm{Dom}(S)$ and $S(y) = x^*$. $\square$

Theorem 4.1.15 *We have an isometric isomorphism*

$$\mathrm{L}^1_\sharp(\mathbb{G})^* \cong_{ci} (\mathrm{L}^\infty(\mathbb{G}) \oplus_1 \overline{\mathrm{L}^\infty(\mathbb{G})})/K_\sharp$$

where $K_\sharp = \left\{(x, -\overline{S(x)^*}) \;\middle|\; x \in \mathrm{Dom}(S)\right\}$. *In particular we can represent any element of* $\mathrm{L}^1_\sharp(\mathbb{G})^*$ *by* $(x, \overline{y}) + K_\sharp$ *for some (non-unique)* $x, y \in \mathrm{L}^\infty(\mathbb{G})$.

Proof

As per Equation (1.2) we can form the adjoint $\theta^* : \mathrm{L}^1(\mathbb{G})^* \oplus_1 \overline{\mathrm{L}^1(\mathbb{G})^*} \to \mathrm{L}^1_\sharp(\mathbb{G})^*$ or indeed $\theta^* : \mathrm{L}^\infty(\mathbb{G}) \oplus_1 \overline{\mathrm{L}^\infty(\mathbb{G})} \to \mathrm{L}^1_\sharp(\mathbb{G})^*$ of the map $\theta : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G})}$ given in Equation 4.2 as follows. Let $x, y \in \mathrm{L}^\infty(\mathbb{G})$ and $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ and we have

$$\langle \theta^*(x, \overline{y}), \omega \rangle = \langle (x, \overline{y}), \theta(\omega) \rangle = \langle x, \omega \rangle + \langle \overline{y}, \overline{\omega^\sharp} \rangle = \langle x, \omega \rangle + \overline{\langle y, \omega^\sharp \rangle}.$$

As θ is an isometry then θ^* is a quotient map and we have an induced isometric isomorphism $\widetilde{\theta^*} : (\mathrm{L}^\infty(\mathbb{G}) \oplus_1 \overline{\mathrm{L}^\infty(\mathbb{G})})/\mathrm{Ker}\,\theta^* \to \mathrm{L}^1_\sharp(\mathbb{G})^*$. We calculate $\mathrm{Ker}\,\theta^*$. Let $x, y \in \mathrm{L}^\infty(\mathbb{G})$ such that $\theta^*((x, \overline{y})) = 0$, then for all $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ we have

$$0 = \langle \theta^*((x, \overline{y})), \omega \rangle = \langle (x, \overline{y}), (\omega, \overline{\omega^\sharp}) \rangle = \langle x, \omega \rangle + \overline{\langle y, \omega^\sharp \rangle}$$

and so $\theta^*((x, \overline{y})) = 0$ if and only if for all $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ we have $\langle y, \omega^\sharp \rangle = \langle -x^*, \omega^* \rangle$. It follows from the above lemma that this is true if and only if $y \in \mathrm{Dom}(S)$ with $S(y) = -x^*$. Then we have $x = -S(y)^* \in \mathrm{Dom}(S)$ and $S(x)^* = -S(S(y)^*)^* = -y$. □

It can easily be shown that the adjoint $\iota^* : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})^*$ of the inclusion $\iota : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G})$ is given by $\iota^*(x) = (x, 0) + K_\sharp$ for $x \in \mathrm{L}^\infty(\mathbb{G})$. See the proof of Lemma 6.2.7 for example.

4.1.5 $\mathrm{C}_0(\mathbb{G})^*{}_\sharp$

We now offer brief details of a related space $\mathrm{C}_0(\mathbb{G})^*{}_\sharp$ in preparation for Section 5.5. We define $\mathrm{C}_0(\mathbb{G})^*{}_\sharp$ as a subspace of $\mathrm{C}_0(\mathbb{G})^*$ in a similar way to how we defined $\mathrm{L}^1_\sharp(\mathbb{G})$ as a subspace of $\mathrm{L}^1(\mathbb{G})$.

Definition 4.1.16 *We define*

$$\mathrm{C}_0(\mathbb{G})^*{}_\sharp = \left\{ \omega \in \mathrm{C}_0(\mathbb{G})^* \;\middle|\; \exists\, \kappa \in \mathrm{C}_0(\mathbb{G})^* \text{ such that } \langle x, \kappa \rangle = \overline{\langle S(x)^*, \omega \rangle} \quad \forall\, x \in \mathrm{Dom}(S) \right\}$$

and for $\mu \in \mathrm{C}_0(\mathbb{G})^*{}_\sharp$ we let $\mu^\sharp$ denote its (necessarily unique) involution.

We know from Proposition 2.4.4 that we have a contractive injective linear map $\lambda : \mathrm{C}_0(\mathbb{G})^* \to \mathrm{M}(\mathrm{C}_0(\widehat{\mathbb{G}}))$ given by $\mu \mapsto (\mu \otimes \mathrm{id})(W)$ for all $\mu \in \mathrm{C}_0(\mathbb{G})^*$. A similar proof to that of Proposition 4.1.2 shows that when restricted to $\mathrm{C}_0(\mathbb{G})^*{}_\sharp$ we have a $*$-homomorphism.

Proposition 4.1.17 *We have the following identity*

$$C_0(\mathbb{G})^*{}_\sharp = \{\mu \in C_0(\mathbb{G})^* \mid \exists\, \nu \in C_0(\mathbb{G})^*, \;\; \lambda(\mu)^* = \lambda(\nu)\}$$

and λ is a $*$-homomorphism on $C_0(\mathbb{G})^*{}_\sharp$.

We define the smear of $\mu \in C_0(\mathbb{G})^*$ by

$$\langle x, \mu(n)\rangle = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \langle \tau_t(x), \mu\rangle \, dt = \langle x(n), \mu\rangle.$$

Clearly we have $\|\mu(n)\| \leqslant \|\mu\|$ by similar proofs to that for the case of $L^1(\mathbb{G})$ and so $\mu(n) \in C_0(\mathbb{G})^*$. It follows from a similar proof to that of Theorem 4.1.9 that $\mu \in C_0(\mathbb{G})^*{}_\sharp$ and so we have the following proposition and corollary.

Proposition 4.1.18 *Let $\mu \in C_0(\mathbb{G})^*$, then $\mu(n) \in C_0(\mathbb{G})^*{}_\sharp$.*

Corollary 4.1.19 *We have that $C_0(\mathbb{G})^*{}_\sharp$ is weak*-dense in $C_0(\mathbb{G})^*$.*

4.2 Operator Space Structures on $L^1_\sharp(\mathbb{G})$

In this section we show that we can place an operator space structure on $L^1_\sharp(\mathbb{G})$ to make this into a completely contractive Banach algebra and we discuss further properties of this object with this operator space structure.

Let $\mathbb{G}$ denote a locally compact quantum group. Then we have a σ-weakly continuous embedding $L^\infty(\mathbb{G}) \hookrightarrow \mathcal{B}(L^2(\mathbb{G}))$ and in particular, by Proposition 1.1.7 we have that $L^\infty(\mathbb{G})$ has a natural operator space structure. We also have a natural operator space structure on $L^1(\mathbb{G})$ given by Example 1.1.14. We will assume these operator space structures on $L^\infty(\mathbb{G})$ and $L^1(\mathbb{G})$ throughout this section.

We have that $L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$ has an operator space structure by Example 1.1.44 and we have a unital normal injective $*$-homomorphism $\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$ and so this is a completely isometric map. It follows from Definition-Theorem 1.1.16 that we can take the pre-adjoint $\Delta_* : \mathrm{L}^1(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G})$ to make $\mathrm{L}^1(\mathbb{G})$ an algebra where associativity of Δ_* follows from coassociativity of Δ. In fact we have that Δ_* is a complete contraction by Proposition 1.1.16 and furthermore we have that

$$\langle x, \Delta_*(\omega \otimes \kappa)\rangle = \langle \Delta(x), \omega \otimes \kappa\rangle = \langle x, \omega * \kappa\rangle$$

for all $x \in \mathrm{L}^\infty(\mathbb{G})$ and so $\Delta_*(\omega \otimes \kappa) = \omega * \kappa$. Then from Definition 1.1.49 we have that $(\mathrm{L}^1(\mathbb{G}), \Delta_*)$ is a completely contractive Banach algebra. We show next that there is an operator space structure on $\mathrm{L}^1_\sharp(\mathbb{G})$ such that this is a completely contractive Banach algebra.

4.2.1 $\mathrm{L}^1_\sharp(\mathbb{G})$ as a Completely Contractive Banach Algebra

We remind the reader that we have the map $\theta : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G})}$ from Equation (4.2) given by $\omega \mapsto (\omega, \overline{\omega^\sharp})$ that is an isometric embedding. We show now that using this map we can make $\mathrm{L}^1_\sharp(\mathbb{G})$ into a completely contractive Banach algebra.

Theorem 4.2.1 *Let $\mathrm{L}^\infty(\mathbb{G})$ and $\mathrm{L}^1(\mathbb{G})$ have the operator space structures given in the introduction to this section, let $\overline{\mathrm{L}^1(\mathbb{G})}$ have the operator space structure given from Example 1.1.22 and let $\mathrm{L}^1(\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G})}$ have the operator space structure given in Definition 1.1.25 and Proposition 1.1.26. Then there is a unique operator space structure on $\mathrm{L}^1_\sharp(\mathbb{G})$ making the map θ in Equation (4.2) a complete isometry and there is a completely contractive map $m_\sharp : \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ such that $m_\sharp(\omega \otimes \kappa) = \omega * \kappa$ for all $\omega, \kappa \in \mathrm{L}^1_\sharp(\mathbb{G})$ making $(\mathrm{L}^1_\sharp(\mathbb{G}), m_\sharp)$ a completely contractive Banach algebra with this operator space structure.*

Proof

For all $n \in \mathbb{N}$ we define a norm on $\mathbb{M}_n(\mathrm{L}^1_\sharp(\mathbb{G}))$ by

$$\|(\omega_{ij})_{i,j=1}^n\|_{\mathbb{M}_n(\mathrm{L}^1_\sharp(\mathbb{G}))} = \|\theta_n((\omega_{ij})_{i,j=1}^n)\|_{\mathbb{M}_n(\mathrm{L}^1(\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G})})}.$$

We know that $\mathrm{L}^1(\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G})}$ has an operator space structure as stated in the theorem and so there is a unique operator space structure on $\mathrm{L}^1_\sharp(\mathbb{G})$ making θ a complete isometry.

We can define a complete contraction $T_1 : \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G})$ given by

$$T_1 := \Delta_* \circ (\pi_1 \otimes \pi_1) \circ (\theta \otimes \theta) : \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G})$$

where θ is the embedding above, π_1 is the projection onto the first coordinate of $\mathrm{L}^1(\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G})}$ and Δ_* is the multiplication map on the completely contractive Banach algebra $\mathrm{L}^1(\mathbb{G})$. We have that θ and π_1 are complete contractions and so by Proposition 1.1.36 we have that $\theta \otimes \theta$ and $\pi_1 \otimes \pi_1$ are complete contractions. So as $\theta \otimes \theta$, $\pi_1 \otimes \pi_1$ and Δ_* are all complete contractions it follows that T_1 is a complete contraction. Furthermore we have for $\omega, \kappa \in \mathrm{L}^1_\sharp(\mathbb{G})$ that $T_1(\omega \otimes \kappa) = \omega * \kappa \in \mathrm{L}^1_\sharp(\mathbb{G})$.

Similarly we can define a complete contraction $T_2 : \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) \to \overline{\mathrm{L}^1(\mathbb{G})}$ given by $T_2 := \overline{\Delta_*} \circ \overline{\Sigma_*} \circ (\pi_2 \otimes \pi_2) \circ (\theta \otimes \theta)$ where Σ_* is the completely isometric flip map on $\mathrm{L}^1(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1(\mathbb{G})$ and where we've used Proposition 1.1.36 (v). We have $T_2(\omega \otimes \kappa) = \overline{\kappa^\sharp * \omega^\sharp}$ and as $T_1(\omega \otimes \kappa) \in \mathrm{L}^1_\sharp(\mathbb{G})$ we have $T_1(\omega \otimes \kappa)^\sharp = \overline{T_2(\omega \otimes \kappa)}$ which extends by linearity to an equality on $\mathrm{L}^1_\sharp(\mathbb{G}) \odot \mathrm{L}^1_\sharp(\mathbb{G})$. Given any $\Omega \in \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G})$ we can approximate this in $\mathrm{L}^1_\sharp(\mathbb{G}) \odot \mathrm{L}^1_\sharp(\mathbb{G})$ and so it follows that for all $x \in \mathrm{Dom}(S)$ we have $\overline{\langle S(x)^*, T_1(\Omega)\rangle} = \langle x, \overline{T_2(\Omega)}\rangle$ and thus $T_1(\Omega) \in \mathrm{L}^1_\sharp(\mathbb{G})$ with $T_1(\Omega)^\sharp = \overline{T_2(\Omega)}$.

So we have shown that T_1 has image in $\mathrm{L}^1_\sharp(\mathbb{G})$ and finally we show that the map $m_\sharp : \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ defined as the corestriction of T_1 is a complete contraction. We clearly have that $T_1 \oplus T_2 = \theta \circ m_\sharp$ where $T_1 \oplus T_2$ is given in Proposition 1.1.28 (i). Also from Proposition 1.1.28 (i) we have that $\|T_1 \oplus T_2\|_{cb} = \max\{\|T_1\|_{cb}, \|T_2\|_{cb}\} \leqslant 1$ and so $T_1 \oplus T_2$ is a complete contraction. As θ is a complete isometry we must have that $m_\sharp$ is also a complete contraction. □

We have the following corollary using the same proof as that of Theorem 4.1.15 but using that θ is now a complete isometry and thus its adjoint is a complete quotient map.

Corollary 4.2.2 *We have a completely isometric isomorphism*

$$L^1_\sharp(\mathbb{G})^* \cong_{ci} L^\infty(\mathbb{G}) \oplus_1 \overline{L^\infty(\mathbb{G})}/K_\sharp.$$

The following proposition will be used a few times in this thesis. We give it here as we feel it is an interesting characteristic of the operator space structure we have given on $L^1_\sharp(\mathbb{G})$.

Proposition 4.2.3 *The linear map $Q : L^1(\mathbb{G}) \to \overline{L^1(\mathbb{G})}$ given by $\omega \mapsto \overline{\omega^* \circ R}$ is a completely isometric isomorphism.*

Proof

We have the adjoint $Q^* : \overline{L^\infty(\mathbb{G})} \to L^\infty(\mathbb{G})$ of Q and for $x \in L^\infty(\mathbb{G})$ and $\omega \in L^1(\mathbb{G})$ we have

$$\langle Q^*(\overline{x}), \omega\rangle = \langle \overline{x}, Q(\omega)\rangle = \overline{\langle x, \omega^* \circ R\rangle} = \langle R(x^*), \omega\rangle.$$

Thus by Proposition 2.3.6 we have $Q^*(\overline{x}) = R(x^*) = \hat{J}x\hat{J}$ for all $x \in L^\infty(\mathbb{G})$ where $\hat{J}$ is the modular conjugation of the left invariant weight $\hat{\phi}$ of $(L^\infty(\hat{\mathbb{G}}), \hat{\Delta})$.

We show that Q^* is a complete contraction now. Let $(\xi_i)_{i=1}^n \in L^2(\mathbb{G})^{(n)}$, then using that $\mathbb{M}_n(L^\infty(\mathbb{G})) \subset \mathcal{B}(L^2(\mathbb{G})^{(n)})$ and $\hat{J}$ is an anti-unitary operator and thus an isometry we have

$$\begin{aligned}
\left\|(Q^*)_n\left((\overline{x_{ij}})_{i,j=1}^n\right)\cdot(\xi_i)_{i=1}^n\right\|^2_{L^2(\mathbb{G})^{(n)}} &= \left\|\left(\sum_{j=1}^n \hat{J}x_{ij}\hat{J}\xi_j\right)_{i=1}^n\right\|^2_{L^2(\mathbb{G})^{(n)}} \\
= \sum_{i=1}^n \left\|\sum_{j=1}^n \hat{J}x_{ij}\hat{J}\xi_j\right\|^2_{L^2(\mathbb{G})} &= \sum_{i=1}^n \left\|\sum_{j=1}^n x_{ij}\hat{J}\xi_j\right\|^2_{L^2(\mathbb{G})} = \left\|\left(\sum_{j=1}^n x_{ij}\hat{J}\xi_j\right)_{i=1}^n\right\|^2_{L^2(\mathbb{G})^{(n)}} \\
= \left\|(x_{ij})_{i,j=1}^n \cdot \left(\hat{J}\xi_j\right)_{i=1}^n\right\|^2_{L^2(\mathbb{G})^{(n)}} &\leqslant \left\|(x_{ij})_{i,j=1}^n\right\|^2_{\mathbb{M}_n(L^\infty(\mathbb{G}))}\left\|\left(\hat{J}\xi_i\right)_{i=1}^n\right\|^2_{L^2(\mathbb{G})^{(n)}} \\
= \left\|(x_{ij})_{i,j=1}^n\right\|^2_{\mathbb{M}_n(L^\infty(\mathbb{G}))}\left\|(\xi_i)_{i=1}^n\right\|^2_{L^2(\mathbb{G})^{(n)}}.
\end{aligned}$$

Then taking the supremum over $(\xi_i)_{i=1}^n \in L^2(\mathbb{G})^{(n)}$ with norm less than 1 we have

$$\left\|(Q^*)_n\left((\overline{x_{ij}})_{i,j=1}^n\right)\right\|_{\mathbb{M}_n(L^\infty(\mathbb{G}))} \leqslant \left\|(x_{ij})_{i,j=1}^n\right\|_{\mathbb{M}_n(L^\infty(\mathbb{G}))} = \left\|(\overline{x_{ij}})_{i,j=1}^n\right\|_{\mathbb{M}_n(L^\infty(\mathbb{G}))}$$

and thus Q^* is a complete contraction.

As Q^* is a complete contraction then Q and $\overline{Q}$ are complete contractions. Also for $\kappa \in L^1(\mathbb{G})$ and $x \in L^\infty(\mathbb{G})$ we have

$$\langle \overline{Q}(\overline{\kappa}), x\rangle = \langle \kappa^*, R(x)\rangle = \overline{\langle \kappa, R(x^*)\rangle}$$

and then letting $\overline{\kappa} = Q(\omega)$ for $\omega \in L^1(\mathbb{G})$ we have

$$\langle (\overline{Q}Q)(\omega), x\rangle = \overline{\langle \overline{Q(\omega)}, R(x^*)\rangle} = \langle Q(\omega), \overline{R(x^*)}\rangle = \overline{\langle \omega^*, x^*\rangle} = \langle \omega, x\rangle.$$

As this holds for all $\omega \in L^1(\mathbb{G})$ and $x \in L^\infty(\mathbb{G})$ and so we have $\overline{Q}Q = \mathrm{id}$. In particular, Q is completely contractive with a completely contractive inverse and so Q is a complete isometric isomorphism. □

The next corollary follows from Propositions 4.2.3, 4.1.11 and 1.1.28 (ii).

Corollary 4.2.4 *Let θ be the map from Equation (4.2), then we have a complete isometry* $(\mathrm{id} \oplus_\infty \overline{Q}) \circ \theta : L^1_\sharp(\mathbb{G}) \to L^1(\mathbb{G}) \oplus_\infty L^1(\mathbb{G})$ *such that*

$$\omega \mapsto (\omega, \omega \circ \tau_{i/2}) = (\omega, \omega^\flat)$$

and so we can calculate the norm on $L^1_\sharp(\mathbb{G})$ with the formula

$$\|\omega\|_{L^1_\sharp(\mathbb{G})} = \max\{\|\omega\|_{L^1(\mathbb{G})}, \|\omega^\flat\|_{L^1(\mathbb{G})}\}.$$

4.2.2 Smearing $\mathrm{L}^1_\sharp(\mathbb{G})$ as a Completely Contractive Banach Algebra

We know from Theorem 1.3.21 that smearing in $\mathrm{L}^\infty(\mathbb{G})$ is completely contractive and normal. In this section we will prove some propositions regarding smearing in $\mathrm{L}^1(\mathbb{G})$ and $\mathrm{L}^1_\sharp(\mathbb{G})$ as completely contractive Banach algebras that will be useful later.

Theorem 4.2.5 *Let $n \in \mathbb{N}$ and $\Phi(n) : \mathrm{L}^1(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G})$ be the map $\omega \mapsto \omega(n)$ from Proposition 4.1.9. Then there is a completely bounded corestriction $\Phi'(n) : \mathrm{L}^1(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$.*

Proof

By Proposition 4.1.9 we have that $\Phi(n)(\omega) \in \mathrm{L}^1_\sharp(\mathbb{G})$ and so the corestriction $\Phi' : \mathrm{L}^1(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ exists and we show it is completely bounded. We let $\rho := \theta \circ \Phi'(n) : \mathrm{L}^1(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G})}$ where θ is the usual embedding of $\mathrm{L}^1_\sharp(\mathbb{G})$ into $\mathrm{L}^1(\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G})}$. Then as θ is a complete isometry we need only show that ρ is completely bounded.

Let $m \in \mathbb{N}$ and $(\omega_{ij})_{i,j=1}^m \in \mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))$, then we have

$$\rho_m\left((\omega_{ij})_{i,j=1}^m\right) = \left((\omega_{ij}(n), \overline{\omega_{ij}(n)^\sharp})\right)_{i,j=1}^m$$

and so

$$\begin{aligned}
&\left\|\rho_m\left((\omega_{ij})_{i,j=1}^m\right)\right\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G})})} \\
&\qquad = \max\left\{\left\|(\omega_{ij}(n))_{i,j=1}^m\right\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))}, \left\|(\overline{\omega_{ij}(n)^\sharp})_{i,j=1}^m\right\|_{\mathbb{M}_m(\overline{\mathrm{L}^1(\mathbb{G})})}\right\} \\
&\qquad = \max\left\{\left\|(\omega_{ij}(n))_{i,j=1}^m\right\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))}, \left\|(\omega_{ij}(n)^\sharp)_{i,j=1}^m\right\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))}\right\}.
\end{aligned}$$

We consider each of these norms in $\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))$ in turn now. Firstly, as the map $\mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G})$ given by $x \mapsto x(n)$ is completely contractive then so is its pre-adjoint as a map $\mathrm{L}^1(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G})$ and we have

$$\left\|(\omega_{ij}(n))_{i,j=1}^m\right\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))} \leqslant \left\|(\omega_{ij})_{i,j=1}^m\right\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))}.$$

Also from Proposition 4.2.3 we have

$$\begin{aligned}
\left\|(\omega_{ij}(n)^\sharp)_{i,j=1}^m\right\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))} &= \left\|\left(\frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2(t+i/2)^2}\omega_{ij}^* \circ R \circ \tau_t \, dt\right)_{i,j=1}^m\right\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))} \\
&\leqslant \frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} \left\|e^{-n^2(t+i/2)^2}\left(\omega_{ij}^* \circ R \circ \tau_t\right)_{i,j=1}^m\right\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))} dt \\
&\leqslant e^{n^2/4}\frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2t^2}\left\|\left(\omega_{ij}^* \circ R\right)_{i,j=1}^m\right\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))} \|\tau_t\| dt \\
&= e^{n^2/4}\frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2t^2}\left\|\left(\omega_{ij}\right)_{i,j=1}^m\right\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))} dt \\
&= e^{n^2/4}\left\|\left(\omega_{ij}\right)_{i,j=1}^m\right\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))}.
\end{aligned}$$

Then it follows from the preceding equations that $\|\rho_m\| \leqslant e^{n^2/4}$ for all $m \in \mathbb{N}$ and so ρ is completely bounded as required. □

Proposition 4.2.6 *Let $\Phi'(n) : \mathrm{L}^1(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ be the map $\omega \mapsto \omega(n)$ given by Theorem 4.2.5, then for all $x, y \in \mathrm{L}^\infty(\mathbb{G})$ (and thus $(x, \overline{y}) + K_\sharp \in \mathrm{L}^1_\sharp(\mathbb{G})^*$) we have*

$$\Phi'(n)^*((x,\overline{y}) + K_\sharp) = x(n) + S(y(n))^*.$$

Proof

We have the adjoint $\Phi'(n)^* : \mathrm{L}^1_\sharp(\mathbb{G})^* \to \mathrm{L}^1(\mathbb{G})$ and using $\mathrm{L}^1_\sharp(\mathbb{G})^* \cong_{ci} (\mathrm{L}^\infty(\mathbb{G}) \oplus_1 \overline{\mathrm{L}^1(\mathbb{G})})/K_\sharp$ for all $\omega \in \mathrm{L}^1(\mathbb{G})$ we have

$$\langle \Phi'(n)^*((x,\overline{y}) + K_\sharp), \omega\rangle = \langle (x,\overline{y}) + K_\sharp, \Phi'(n)(\omega)\rangle = \langle x, \Phi'(n)(\omega)\rangle + \overline{\langle y, \Phi'(n)(\omega)^\sharp\rangle}.$$

For $y \in \mathrm{Dom}(S)$ we have by Corollary 4.1.8 that

$$\overline{\langle y, \Phi'(n)(\omega)^\sharp\rangle} = \langle S(y)^*, \Phi'(n)(\omega)\rangle = \langle S(y(n))^*, \omega\rangle$$

and as $\mathrm{Dom}(S)$ is σ-weakly dense in $\mathrm{L}^\infty(\mathbb{G})$ and $y(n) \in \mathrm{Dom}(S)$ for all $y \in \mathrm{L}^\infty(\mathbb{G})$ (Theorem 1.3.17) we have $\overline{\langle y, \Phi'(n)^\sharp\rangle} = \langle S(y(n))^*, \omega\rangle$ for all $y \in \mathrm{L}^\infty(\mathbb{G})$. Then it follows that

$$\langle \Phi'(n)^*((x,\overline{y}) + K_\sharp), \omega\rangle = \langle x(n) + S(y(n))^*, \omega\rangle$$

for all $\omega \in \mathrm{L}^1(\mathbb{G})$ and the theorem follows. □

We now define the following notation for use later on.

Notation 4.2.7 *For $n \in \mathbb{N}$ we let $\Upsilon(n) : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ be the map $\Phi'(n) \circ \iota$ where $\iota : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G})$ is the inclusion map.*

Proposition 4.2.8 *We have that $\Upsilon(n)$ is a completely contractive map and for $x, y \in \mathrm{L}^\infty(\mathbb{G})$ (so $(x,y) + K_\sharp \in \mathrm{L}^1_\sharp(\mathbb{G})^*$) we have $\Upsilon(n)^*((x,y) + K_\sharp) = (x(n), y(n)) + K_\sharp$.*

Proof Fix $m \in \mathbb{N}$ and let $(\omega_{ij})_{i,j=1}^m \in \mathbb{M}_m(\mathrm{L}^1_\sharp(\mathbb{G}))$. As the map $x \mapsto x(n)$ is completely contractive it follows that its pre-adjoint map is also completely contractive and so

$$\|(\omega_{ij}(n))_{i,j=1}^n\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))} \leqslant \|(\omega_{ij})_{i,j=1}^n\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))}$$

and similarly $\|(\omega_{ij}^\sharp(n))_{i,j=1}^m\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))} \leqslant \|(\omega_{ij}^\sharp)_{i,j=1}^m\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))}$. Then as the map $\theta : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G})}$ given by Equation (4.2) is a complete isometry we have

$$\begin{aligned}
\|\Upsilon(n)_m((\omega_{ij})_{i,j=1}^m)\|_{\mathbb{M}_m(\mathrm{L}^1_\sharp(\mathbb{G}))} &= \|(\theta_m \circ \Upsilon(n)_m)((\omega_{ij})_{i,j=1}^m)\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G})\oplus_\infty\overline{\mathrm{L}^1(\mathbb{G})})} \\
&= \max\{\|(\omega_{ij}(n))_{i,j=1}^m\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))}, \|(\omega_{ij}(n)^\sharp)_{i,j=1}^n\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))}\} \\
&\leqslant \max\{\|(\omega_{ij})_{i,j=1}^m\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))}, \|(\omega_{ij}^\sharp)_{i,j=1}^n\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G}))}\} \\
&= \|\theta_m((\omega_{ij})_{i,j=1}^m)\|_{\mathbb{M}_m(\mathrm{L}^1(\mathbb{G})\oplus_\infty\overline{\mathrm{L}^1(\mathbb{G})})} = \|(\omega_{ij})_{i,j=1}^m\|_{\mathbb{M}_m(\mathrm{L}^1_\sharp(\mathbb{G}))}
\end{aligned}$$

where we've used Proposition 4.1.10. So we have that $\Upsilon(n)$ is a complete contraction.

We have that $\Upsilon(n)^* = \iota^* \circ \Phi'(n)^*$ and so using Proposition 4.2.6 we have

$$\Upsilon(n)^*\left((x,\overline{y}) + K_\sharp\right) = (x(n) + S(y(n))^*, 0) + K_\sharp.$$

As $y(n) \in \mathrm{Dom}(S)$ we have $S(y(n))^* \in \mathrm{Dom}(S)$ and so $(S(y(n))^*, -S(S(y(n))^*)^*) = (S(y(n))^*, -y(n)) \in K_\sharp$ and thus we also have

$$\Upsilon(n)^* \left((x, \overline{y}) + K_\sharp\right) = (x(n), y(n)) + K_\sharp. \quad \square$$

We now prove some useful results about the convergence of $(\Upsilon(n))$ and $(\Upsilon(n) \otimes \Upsilon(n))$ for use later.

Proposition 4.2.9 *We have* $\Upsilon(n) \to \mathrm{id}$ *weakly as* $n \to \infty$*, that is for all* $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ *we have* $\|\Upsilon(n)(\omega) - \omega\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \to 0$.

Proof

Let $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ and $x \in \mathrm{L}^\infty(\mathbb{G})$, then by Proposition 1.3.15 we have

$$\|\Upsilon(n)(\omega) - \omega\|_{\mathrm{L}^1(\mathbb{G})} = \|\omega(n) - \omega\|_{\mathrm{L}^1(\mathbb{G})} \to 0$$

and so $\lim \Upsilon(n)(\omega) = \omega$ in the $\mathrm{L}^1(\mathbb{G})$ norm.

Let $x \in \mathrm{Dom}(S)$ and $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$, then from Proposition 4.1.8 we have

$$\langle x, \Upsilon(n)(\omega)^\sharp\rangle = \overline{\langle S(x)^*, \Upsilon(n)(\omega)\rangle} = \overline{\langle S(x(n))^*, \omega\rangle} = \langle x(n), \iota(\omega^\sharp)\rangle = \langle x, \Upsilon(n)(\omega^\sharp)\rangle$$

and as $\mathrm{Dom}(S)$ is weak*-dense in $\mathrm{L}^\infty(\mathbb{G})$ we have $\Upsilon(n)(\omega)^\sharp = \Upsilon(n)(\omega^\sharp)$. Then similarly to above we have $\|(\Upsilon(n)(\omega) - \omega)^\sharp\|_{\mathrm{L}^1(\mathbb{G})} = \|\Upsilon(n)(\omega^\sharp) - \omega^\sharp\|_{\mathrm{L}^1(\mathbb{G})} \to 0$ and so $\|\Upsilon(n)(\omega) - \omega\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \to 0$ as required. $\square$

Proposition 4.2.10 *For all* $\Omega \in \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G})$ *we have*

$$\lim_{n\to\infty} (\Upsilon(n) \otimes \Upsilon(n))(\Omega) = \Omega.$$

Proof

Let $\omega, \omega' \in \mathrm{L}^1_\sharp(\mathbb{G})$, then using Proposition 4.2.8 we have

$$\begin{aligned}
&\|(\Upsilon(n) \otimes \Upsilon(n))(\omega \otimes \omega') - \omega \otimes \omega'\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \\
&\quad \leqslant \|\Upsilon(n)(\omega) \otimes \Upsilon(n)(\omega') - \omega \otimes \Upsilon(n)(\omega')\|_{\mathrm{L}^1_\sharp(\mathbb{G})} + \|\omega \otimes \Upsilon(n)(\omega') - \omega \otimes \omega'\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \\
&\quad \leqslant \|\Upsilon(n)(\omega) - \omega\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \|\Upsilon(n)(\omega')\|_{\mathrm{L}^1_\sharp(\mathbb{G})} + \|\omega\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \|\Upsilon(n)(\omega') - \omega'\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \\
&\quad \leqslant \|\Upsilon(n)(\omega) - \omega\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \|\omega'\|_{\mathrm{L}^1_\sharp(\mathbb{G})} + \|\omega\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \|\Upsilon(n)(\omega') - \omega'\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \to 0
\end{aligned}$$

and so by linearity this holds on all $\mathrm{L}^1_\sharp(\mathbb{G}) \odot \mathrm{L}^1_\sharp(\mathbb{G})$.

Now let $\Omega \in \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G})$, then for all $\varepsilon > 0$ there is some $\Omega' \in \mathrm{L}^1_\sharp(\mathbb{G}) \odot \mathrm{L}^1_\sharp(\mathbb{G})$ such that $\|\Omega - \Omega'\| < \varepsilon$ and there is some $n \in \mathbb{N}$ such that $\|(\Upsilon(n) \otimes \Upsilon(n))(\Omega) - \Omega'\| < \varepsilon$. Then as $\Upsilon(n)$ is (completely) contractive by Proposition 4.2.8 we have

$$\begin{aligned}
&\|(\Upsilon(n) \otimes \Upsilon(n))(\Omega) - \Omega'\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \\
&\quad \leqslant \|(\Upsilon(n) \otimes \Upsilon(n))(\Omega) - (\Upsilon(n) \otimes \Upsilon(n))(\Omega')\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \\
&\qquad\qquad + \|(\Upsilon(n) \otimes \Upsilon(n))(\Omega') - \Omega'\|_{\mathrm{L}^1_\sharp(\mathbb{G})} + \|\Omega' - \Omega\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \\
&\quad \leqslant \|\Upsilon(n) \otimes \Upsilon(n)\| \|\Omega - \Omega'\|_{\mathrm{L}^1_\sharp(\mathbb{G})} + \|(\Upsilon(n) \otimes \Upsilon(n))(\Omega') - \Omega'\|_{\mathrm{L}^1_\sharp(\mathbb{G})} + \|\Omega' - \Omega\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \\
&\quad < \varepsilon(\|\Upsilon(n) \otimes \Upsilon(n)\| + 2) \leqslant 3\varepsilon
\end{aligned}$$

as required. □

4.2.3 Smearing for Products of Quantum Groups

Fix two locally compact quantum groups $\mathbb{G}$ and $\mathbb{H}$. In Section 2.5 we gave a definition of a locally compact quantum group $\mathbb{G} \times \mathbb{H}$. We defined the von Neumann algebra by $\mathrm{L}^\infty(\mathbb{G} \times \mathbb{H}) = \mathrm{L}^\infty(\mathbb{G}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{H})$ and so it follows that $\mathrm{L}^1(\mathbb{G} \times \mathbb{H}) = (\mathrm{L}^\infty(\mathbb{G} \times \mathbb{H}))_* = (\mathrm{L}^\infty(\mathbb{G}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{H}))_* = \mathrm{L}^1(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1(\mathbb{H})$. In this section we ask if we can find a similar relation between $\mathrm{L}^1_\sharp(\mathbb{G} \times \mathbb{H})$ and $\mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{H})$?

We begin this section by showing that we have a weak*-core for $S^{\mathbb{G}\times\mathbb{H}}$ and that we have an embedding of $\mathrm{L}^1_\sharp(\mathbb{G}) \odot \mathrm{L}^1_\sharp(\mathbb{H})$ inside $\mathrm{L}^1_\sharp(\mathbb{G}\times\mathbb{H})$. We then prove Theorem 4.2.14 showing that in the case that $\mathbb{G} = \mathbb{H}$ we have a complete contraction from $\mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G})$ to $\mathrm{L}^1_\sharp(\mathbb{G}\times\mathbb{G})$. It follows immediately by density of $\mathrm{L}^1_\sharp$-algebras in L^1-algebras that if $\mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{H})$ is completely isometrically isomorphic to $\mathrm{L}^1_\sharp(\mathbb{G}\times\mathbb{H})$ that the map T in Theorem 4.2.14 must be a completely isometric isomorphism. We will return to this theorem in Section 5.4 where we prove that this is not a completely isometric isomorphism for the case $\mathbb{G} = \mathbb{H} = \mathrm{SU}_q(2)$.

Proposition 4.2.11 *We have* $\mathrm{Dom}(S^{\mathbb{G}}) \odot \mathrm{Dom}(S^{\mathbb{H}})$ *is a weak*-core for* $S^{\mathbb{G}\times\mathbb{H}}$ *in the von Neumann algebraic setting (where* $\mathrm{Dom}(S^{\mathbb{G}}) \subset \mathrm{L}^\infty(\mathbb{G})$ *and similarly for* $\mathbb{H}$*).*

Proof

We have that $\left\{(\mathrm{id}\otimes\omega)(W^{\mathbb{G}\times\mathbb{H}}) \;\middle|\; \omega \in \mathrm{L}^1(\mathbb{G}\times\mathbb{H})\right\}$ is a weak*-core for $S^{\mathbb{G}\times\mathbb{H}}$. Fix $\omega \in \mathrm{L}^1(\mathbb{G}\times\mathbb{H})$ and $\varepsilon > 0$, then there exists some $\omega' \in \mathrm{L}^1(\mathbb{G}) \odot \mathrm{L}^1(\mathbb{H})$ such that $\|\omega - \omega'\|_{\mathrm{L}^1(\mathbb{G}\times\mathbb{H})} < \varepsilon$. Then using that $W^{\mathbb{G}\times\mathbb{H}}$ is unitary we have

$$\|(\mathrm{id}\otimes\omega)(W^{\mathbb{G}\times\mathbb{H}}) - (\mathrm{id}\otimes\omega')(W^{\mathbb{G}\times\mathbb{H}})\| = \|(\mathrm{id}\otimes(\omega-\omega'))(W^{\mathbb{G}\times\mathbb{H}})\| \leqslant \|\omega-\omega'\| < \varepsilon$$

and similarly

$$\|S((\mathrm{id}\otimes\omega)(W^{\mathbb{G}\times\mathbb{H}})) - S((\mathrm{id}\otimes\omega')(W^{\mathbb{G}\times\mathbb{H}}))\| = \|(\mathrm{id}\otimes(\omega-\omega'))((W^{\mathbb{G}\times\mathbb{H}})^*)\| < \varepsilon$$

and so as $S^{\mathbb{G}\times\mathbb{H}}$ is weak*-closed then $\left\{(\mathrm{id}\otimes\omega)(W^{\mathbb{G}\times\mathbb{H}}) \;\middle|\; \omega \in \mathrm{L}^1(\mathbb{G}) \odot \mathrm{L}^1(\mathbb{H})\right\}$ is also a weak*-core for $S^{\mathbb{G}\times\mathbb{H}}$.

We have

$$\begin{aligned}
&\left\{(\mathrm{id}^{\mathbb{G}\times\mathbb{H}} \otimes \omega)(W^{\mathbb{G}\times\mathbb{H}}) \;\middle|\; \omega \in \mathrm{L}^1(\mathbb{G}) \odot \mathrm{L}^1(\mathbb{H})\right\} \\
&\qquad = \mathrm{lin}\left\{(\mathrm{id}^{\mathbb{G}\times\mathbb{H}} \otimes \omega_1 \otimes \omega_2)(W^{\mathbb{G}\times\mathbb{H}}) \;\middle|\; \omega_1 \in \mathrm{L}^1(\mathbb{G}), \omega_2 \in \mathrm{L}^1(\mathbb{H})\right\} \\
&\qquad = \mathrm{lin}\left\{(\mathrm{id}^{\mathbb{G}} \otimes \omega_1)(W^{\mathbb{G}}) \otimes (\mathrm{id}^{\mathbb{H}} \otimes \omega_2)(W^{\mathbb{H}}) \;\middle|\; \omega_1 \in \mathrm{L}^1(\mathbb{G}), \omega_2 \in \mathrm{L}^1(\mathbb{H})\right\} \\
&\qquad \subset \mathrm{Dom}(S^{\mathbb{G}}) \odot \mathrm{Dom}(S^{\mathbb{H}}) \subset \mathrm{Dom}(S^{\mathbb{G}\times\mathbb{H}})
\end{aligned}$$

and so as $\mathrm{Dom}(S^{\mathbb{G}}) \odot \mathrm{Dom}(S^{\mathbb{H}})$ is a superset of a weak*-core for $\mathrm{Dom}(S^{\mathbb{G}\times\mathbb{H}})$ and a subset of $\mathrm{Dom}(S^{\mathbb{G}\times\mathbb{H}})$ it must be a weak*-core for $\mathrm{Dom}(S^{\mathbb{G}\times\mathbb{H}})$. □

Lemma 4.2.12 *We have $\mathrm{L}^1_\sharp(\mathbb{G}) \odot \mathrm{L}^1_\sharp(\mathbb{H}) \subset \mathrm{L}^1_\sharp(\mathbb{G} \times \mathbb{H})$, in particular for all $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ and $\kappa \in \mathrm{L}^1_\sharp(\mathbb{H})$ we have $\omega \otimes \kappa \in \mathrm{L}^1_\sharp(\mathbb{G} \times \mathbb{H})$ with $(\omega \otimes \kappa)^\sharp = \omega^\sharp \otimes \kappa^\sharp$.*

Proof

Let $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ and $\kappa \in \mathrm{L}^1_\sharp(\mathbb{H})$. We have $\omega \otimes \kappa \in \mathrm{L}^1(\mathbb{G} \times \mathbb{H})$ and for $y \in \mathrm{Dom}(S^{\mathbb{G}})$ and $z \in \mathrm{Dom}(S^{\mathbb{H}})$ we have

$$\overline{\langle ((S^{\mathbb{G}} \otimes S^{\mathbb{H}})(y \otimes z))^*, \omega \otimes \kappa \rangle} = \langle y \otimes z, \omega^\sharp \otimes \kappa^\sharp \rangle$$

and so by linearity we have for all $x \in \mathrm{Dom}(S^{\mathbb{G}}) \odot \mathrm{Dom}(S^{\mathbb{H}})$ we have

$$\overline{\langle S^{\mathbb{G}\times\mathbb{H}}(x), \omega \otimes \kappa \rangle} = \langle x, \omega^\sharp \otimes \kappa^\sharp \rangle.$$

It follows from Proposition 4.2.11 that we have a weak*-core $\mathrm{Dom}(S^{\mathbb{G}}) \odot \mathrm{Dom}(S^{\mathbb{H}})$ of $S^{\mathbb{G}\times\mathbb{H}}$ and so for all $x \in \mathrm{Dom}(S^{\mathbb{G}\times\mathbb{H}})$ we have a net $(x_\alpha) \in \mathrm{Dom}(S^{\mathbb{G}})\odot\mathrm{Dom}(S^{\mathbb{H}})$ such that $|\langle x, \Omega \rangle - \langle x_\alpha, \Omega \rangle| \to 0$ and $\left|\langle S^{\mathbb{G}\times\mathbb{H}}(x), \Omega \rangle - \langle S^{\mathbb{G}\times\mathbb{H}}(x_\alpha), \Omega \rangle\right| \to 0$ for all $\Omega \in \mathrm{L}^1(\mathbb{G}\times\mathbb{H})$. Then from above we have

$$\begin{aligned}
&\left|\overline{\langle S^{\mathbb{G}\times\mathbb{H}}(x), \omega \otimes \kappa \rangle} - \langle x, \omega^\sharp \otimes \kappa^\sharp \rangle\right| \\
&\quad \leqslant \left|\overline{\langle S^{\mathbb{G}\times\mathbb{H}}(x), \omega \otimes \kappa \rangle} - \overline{\langle S^{\mathbb{G}\times\mathbb{H}}(x_\alpha), \omega \otimes \kappa \rangle}\right| + \left|\langle x_\alpha, \omega^\sharp \otimes \kappa^\sharp \rangle - \langle x, \omega^\sharp \otimes \kappa^\sharp \rangle\right| \to 0.
\end{aligned}$$

It follows then that $\omega \otimes \kappa \in L^1_\sharp(\mathbb{G})$ with $(\omega \otimes \kappa)^\sharp = \omega^\sharp \otimes \kappa^\sharp$. $\square$

Let $x \in L^\infty(\mathbb{G} \times \mathbb{H})$, then as this is a von Neumann algebra then we have the smear $x(n)$ for $n \in \mathbb{N}$ given by Section 1.3.3. We define the smear of an element in $L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{H})$ as the smear in $L^\infty(\mathbb{G} \times \mathbb{H})$. We consider such smears now.

We need one more lemma to prove the last theorem of this section.

Lemma 4.2.13 *Fix $n \in \mathbb{N}$. Then for $x \in L^\infty(\mathbb{G})$ we have*

$$\Delta^{\mathbb{G}}(x(n)) = (\Delta^{\mathbb{G}}(x))(n) \tag{4.3}$$

where in the right hand side we take the smear in $L^\infty(\mathbb{G} \times \mathbb{G})$ and

$$(\sigma \circ \Delta^{\mathbb{G}})(x(n)) = ((\sigma \circ \Delta^{\mathbb{G}})(x))(n) \tag{4.4}$$

where $\sigma : L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$ is the flip map. Furthermore we have that $\Delta^{\mathbb{G}}(x(n)) \in \mathrm{Dom}(S^{\mathbb{G} \times \mathbb{G}})$ and

$$S^{\mathbb{G} \times \mathbb{G}}\left(\Delta^{\mathbb{G}}(x(n))\right) = (\sigma \circ \Delta^{\mathbb{G}} \circ S^{\mathbb{G}})(x(n)). \tag{4.5}$$

Proof

Fix $n \in \mathbb{N}$. It follows from Proposition 2.2.8 (iii) and Proposition 2.5.5 that $\Delta^{\mathbb{G}} \circ \tau_t^{\mathbb{G}} = (\tau_t^{\mathbb{G}} \otimes \tau_t^{\mathbb{G}}) \circ \Delta^{\mathbb{G}} = \tau_t^{\mathbb{G} \times \mathbb{G}} \circ \Delta^{\mathbb{G}}$ for all $t \in \mathbb{R}$ and so for all $x \in L^\infty(\mathbb{G})$ and $\kappa \in L^1(\mathbb{G} \times \mathbb{G})$ we have

$$\begin{aligned}\langle \Delta^{\mathbb{G}}(x(n)), \kappa \rangle = \langle x(n), \Delta^{\mathbb{G}}_*(\kappa) \rangle &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2 t^2} \langle \tau_t^{\mathbb{G}}(x), \Delta^{\mathbb{G}}_*(\kappa) \rangle \, dt \\ &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2 t^2} \langle \tau_t^{\mathbb{G} \times \mathbb{G}}(\Delta^{\mathbb{G}}(x)), \kappa \rangle \, dt = \langle (\Delta^{\mathbb{G}}(x))(n), \kappa \rangle\end{aligned}$$

and so Equation (4.3) follows. The proof of Equation (4.4) is almost identical.

Let $x \in L^\infty(\mathbb{G})$, then by Equation (4.3) and Propositions 4.1.8, 2.2.8 and 2.5.5 we have

$$(R^{\mathbb{G}\times\mathbb{G}} \circ \Delta^{\mathbb{G}})(x(n)) = ((R^{\mathbb{G}\times\mathbb{G}} \circ \Delta^{\mathbb{G}})(x))(n) = ((\sigma \circ \Delta^{\mathbb{G}} \circ R^{\mathbb{G}})(x))(n) \in \mathrm{Dom}(\tau^{\mathbb{G}\times\mathbb{G}}_{-i/2}).$$

Then for all $\kappa \in \mathrm{L}^1(\mathbb{G})$ we have

$$\begin{aligned}
\langle (S^{\mathbb{G}\times\mathbb{G}} \circ \Delta^{\mathbb{G}})(x(n)), \kappa \rangle &= \left\langle \tau^{\mathbb{G}\times\mathbb{G}}_{-i/2} \left(\left((\sigma \circ \Delta^{\mathbb{G}} \circ R^{\mathbb{G}})(x) \right)(n) \right), \kappa \right\rangle \\
&= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2(t+i/2)^2} \langle \tau^{\mathbb{G}\times\mathbb{G}}_t ((\sigma \circ \Delta^{\mathbb{G}} \circ R^{\mathbb{G}})(x)), \kappa \rangle \, dt \\
&= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2(t+i/2)^2} \langle (\tau^{\mathbb{G}}_t \circ R^{\mathbb{G}})(x), (\Delta^{\mathbb{G}}_* \circ \sigma_*)(\kappa) \rangle \, dt \\
&= \langle (\tau^{\mathbb{G}}_{-i/2} \circ R^{\mathbb{G}})(x(n)), (\Delta^{\mathbb{G}}_* \circ \sigma_*)(\kappa) \rangle = \langle \sigma \circ \Delta^{\mathbb{G}} \circ S^{\mathbb{G}})(x(n)), \kappa \rangle
\end{aligned}$$

as required. □

Theorem 4.2.14 *There is a unique completely contractive map* $T : \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G} \times \mathbb{G})$ *such that we have a commutative diagram*

$$\begin{array}{ccc}
\mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) & \xrightarrow{T} & \mathrm{L}^1_\sharp(\mathbb{G} \times \mathbb{G}) \\
\Big\downarrow{\scriptstyle \iota^{\mathbb{G}} \otimes \iota^{\mathbb{G}}} & & \Big\downarrow{\scriptstyle \iota^{\mathbb{G}\times\mathbb{G}}} \\
\mathrm{L}^1(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1(\mathbb{G}) & = & \mathrm{L}^1(\mathbb{G} \times \mathbb{G})
\end{array}$$

and there exists a map $m'_\sharp : \mathrm{L}^1_\sharp(\mathbb{G} \times \mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ *such that we have* $m'_\sharp \circ T = m_\sharp$ *for* $m_\sharp$ *the map in Theorem 4.2.1.*

Proof

We have a complete isometry $\theta^{\mathbb{G}} : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G})}$ given by $\omega \mapsto (\omega, \overline{\omega^\sharp})$ and complete contractions $\pi_1 : \mathrm{L}^1(\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G})} \to \mathrm{L}^1(\mathbb{G})$ and $\pi_2 : \mathrm{L}^1(\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G})} \to \overline{\mathrm{L}^1(\mathbb{G})}$ as coordinate projections. Then similarly to the proof of Theorem 4.2.1, using the identification $\mathrm{L}^1(\mathbb{G} \times \mathbb{G}) = \mathrm{L}^1(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1(\mathbb{G})$, we can define complete contractions $T_1 : \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G} \times \mathbb{G})$ given by $(\pi_1 \otimes \pi_1) \circ (\theta^{\mathbb{G}} \otimes \theta^{\mathbb{G}})$ and $T_2 : \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) \to$

$\overline{\mathrm{L}^1(\mathbb{G}\times\mathbb{G})}$ given by $(\pi_2 \otimes \pi_2) \circ (\theta^{\mathbb{G}} \otimes \theta^{\mathbb{G}})$ where we've used that $\overline{\mathrm{L}^1(\mathbb{G})}\,\widehat{\otimes}\,\overline{\mathrm{L}^1(\mathbb{G})} \cong_{ci} \overline{\mathrm{L}^1(\mathbb{G})\,\widehat{\otimes}\,\mathrm{L}^1(\mathbb{G})} \cong_{ci} \overline{\mathrm{L}^1(\mathbb{G}\times\mathbb{G})}$.

Using Lemma 4.2.12, for $\omega_1, \omega_2 \in \mathrm{L}^1_\sharp(\mathbb{G})$ we have $T_2(\omega_1 \otimes \omega_2) = \overline{\omega_1^\sharp \otimes \omega_2^\sharp}$ and so $T_1(\omega_1 \otimes \omega_2) = \omega_1 \otimes \omega_2 \in \mathrm{L}^1_\sharp(\mathbb{G}\times\mathbb{G})$ with

$$T_1(\omega_1 \otimes \omega_2)^\sharp = \omega_1^\sharp \otimes \omega_2^\sharp = \overline{T_2(\omega_1 \otimes \omega_2)}.$$

Then by linearity and continuity we have $T_1(\Omega) \in \mathrm{L}^1_\sharp(\mathbb{G}\times\mathbb{G})$ for all $\Omega \in \mathrm{L}^1_\sharp(\mathbb{G})\,\widehat{\otimes}\,\mathrm{L}^1_\sharp(\mathbb{G})$ with $T_1(\Omega)^\sharp = \overline{T_2(\Omega)}$. Then similar to the proof of Theorem 4.2.1, we have a corestriction $T : \mathrm{L}^1_\sharp(\mathbb{G})\,\widehat{\otimes}\,\mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G}\times\mathbb{G})$ and we can show that the map $\theta^{\mathbb{G}\times\mathbb{G}} \circ T = T_1 \oplus T_2$ is completely contractive for $\theta^{\mathbb{G}\times\mathbb{G}}$ the usual isometric embedding of $\mathrm{L}^1_\sharp(\mathbb{G}\times\mathbb{G})$ into $\mathrm{L}^1(\mathbb{G}\times\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G}\times\mathbb{G})}$. It follows that T must be completely contractive. By construction this T makes the diagram in the theorem commute. Also T is unique as if there is a map T' making this diagram commute then we have $\iota^{\mathbb{G}\times\mathbb{G}} \circ T = \iota^{\mathbb{G}\times\mathbb{G}} \circ T'$ and we use that $\iota^{\mathbb{G}\times\mathbb{G}}$ is injective.

Now we define a completely contractive map $m_1 : \mathrm{L}^1_\sharp(\mathbb{G}\times\mathbb{G}) \to \mathrm{L}^1(\mathbb{G})$ by $\Delta_* \circ \pi_1 \circ \theta^{\mathbb{G}\times\mathbb{G}}$ where π_1 is now the projection of $\mathrm{L}^1(\mathbb{G}\times\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G}\times\mathbb{G})}$ onto the first component. Similarly we define $m_2 : \mathrm{L}^1_\sharp(\mathbb{G}\times\mathbb{G}) \to \overline{\mathrm{L}^1(\mathbb{G})}$ by $\overline{\Delta_*} \circ \overline{\sigma_*} \circ \pi_2 \circ \theta^{\mathbb{G}\times\mathbb{G}}$. Fix $n \in \mathbb{N}$ and $\kappa \in \mathrm{L}^1_\sharp(\mathbb{G}\times\mathbb{G})$, then for all $x \in \mathrm{Dom}(S^{\mathbb{G}})$ we have

$$\begin{aligned}\overline{\langle S^{\mathbb{G}}(x)^*, m_1(\kappa(n))\rangle} &= \overline{\langle S^{\mathbb{G}}(x(n))^*, m_1(\kappa)\rangle} = \overline{\langle (\Delta^{\mathbb{G}} \circ S^{\mathbb{G}})(x(n))^*, \iota^{\mathbb{G}}(\kappa)\rangle} \\ &= \overline{\langle (S^{\mathbb{G}\times\mathbb{G}} \circ \sigma \circ \Delta^{\mathbb{G}})(x(n))^*, \kappa\rangle} = \langle (\sigma \circ \Delta^{\mathbb{G}})(x(n)), \kappa^\sharp\rangle \\ &= \left\langle x(n), (\Delta_*^{\mathbb{G}} \circ \sigma_*)\left(\overline{(\pi_2 \circ \theta^{\mathbb{G}\times\mathbb{G}})(\kappa)}\right)\right\rangle = \left\langle x, \overline{m_2(\kappa(n))}\right\rangle\end{aligned}$$

where we've used Lemma 4.2.13 and Proposition 4.1.8. So we have shown that for all $\kappa \in \mathrm{L}^1_\sharp(\mathbb{G}\times\mathbb{G})$ and $n \in \mathbb{N}$ we have $m_1(\kappa(n)) \in \mathrm{L}^1_\sharp(\mathbb{G})$ and $m_1(\kappa(n))^\sharp = \overline{m_2(\kappa(n))}$.

Again for $\kappa \in \mathrm{L}^1_\sharp(\mathbb{G} \times \mathbb{G})$ and for all $x \in \mathrm{Dom}(S^{\mathbb{G}\times\mathbb{G}})$ we have

$$\begin{aligned}\overline{\langle S^{\mathbb{G}\times\mathbb{G}}(x)^*, m_1(\kappa)\rangle} &= \overline{\lim\langle S^{\mathbb{G}\times\mathbb{G}}(x)^*, m_1(\kappa(n))\rangle} = \lim\langle x, m_1(\kappa(n))^\sharp\rangle \\ &= \lim\langle x, \overline{m_2(\kappa(n))}\rangle = \langle x, \overline{m_2(\kappa)}\rangle\end{aligned}$$

and so $m_1(\kappa) \in \mathrm{L}^1_\sharp(\mathbb{G})$ with $m_1(\kappa)^\sharp = \overline{m_2(\kappa)}$. We can show the restriction to a map $m'_\sharp : \mathrm{L}^1_\sharp(\mathbb{G} \times \mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ is a completely contractive map in a similar way to that of the proof of 4.2.1. By construction we have $m'_\sharp \circ T = m_\sharp$ as required. □

So we have proved that we have a unique complete contraction T from $\mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G})$ to $\mathrm{L}^1_\sharp(\mathbb{G} \times \mathbb{G})$ such that $\iota^{\mathbb{G}\times\mathbb{G}} \circ T = \iota^{\mathbb{G}} \otimes \iota^{\mathbb{G}}$. Furthermore it follows easily that if there were a completely isometric isomorphism from $\mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G})$ to $\mathrm{L}^1_\sharp(\mathbb{G} \times \mathbb{G})$ then it would be equal to this map T. At this point we might hope that next we can prove that this is a completely isometric isomorphism, however as alluded to earlier we will show that we have a counterexample to this in Section 5.4.

4.3 Compact Quantum Groups

We now investigate $\mathrm{L}^1_\sharp(\mathbb{G})$ in the case that $\mathbb{G}$ is a compact quantum group. We show in the first section that for $\mathbb{G}$ compact we have a dense subset of $\mathrm{L}^1(\mathbb{G})$ and $\mathrm{L}^1_\sharp(\mathbb{G})$ that can be built using $\mathrm{Hopf}(\mathbb{G})$. In the next section we give a new criterion for compactness in terms of the $\mathrm{L}^1_\sharp(\mathbb{G})$ algebra similar to a criterion of Runde for $\mathrm{L}^1(\mathbb{G})$ given in Runde (2008).

4.3.1 $\mathrm{L}^1_\sharp$-algebra for a Compact Quantum Group

We assume throughout this section that $\mathbb{G}$ is a compact quantum group as per Definition 3.2.1 and that we have a maximal set of irreducible corepresentations $\{U^\alpha \mid \alpha \in \mathbb{A}\}$ as per Theorem 3.2.9 indexed by a set $\mathbb{A}$.

We define the following subset of $\mathrm{L}^1(\mathbb{G})$ and we show that this subset is dense in both

$\mathrm{L}^1(\mathbb{G})$ and $\mathrm{L}^1_\sharp(\mathbb{G})$ with respect to the appropriate norms. In particular we see we have a nice way of calculating the $\mathrm{L}^1_\sharp(\mathbb{G})$ algebra of a compact quantum group $\mathbb{G}$ using its Hopf algebra structure $\mathrm{Hopf}(\mathbb{G})$.

Notation 4.3.1 *Let $\alpha \in \mathbb{A}$, $1 \leqslant i, j \leqslant n_\alpha$ and let Λ^α and λ_j^α be as in Theorem 3.2.15. We denote $\omega_{ij}^\alpha := \dfrac{\Lambda^\alpha}{\lambda_j^\alpha}(u_{ij}^\alpha)^* \cdot \phi \in \mathrm{L}^1(\mathbb{G})$ and we let*

$$D = \mathrm{lin}\left\{\omega_{ij}^\alpha \in \mathrm{L}^1(\mathbb{G}) \;\middle|\; \alpha \in \mathbb{A},\ 1 \leqslant i, j \leqslant n_\alpha\right\}.$$

Note that it follows immediately that $D = \{x \cdot \phi \in \mathrm{L}^1(\mathbb{G}) \mid x \in \mathrm{Hopf}(\mathbb{G})\}$. We show that $D \subset \mathrm{L}^1_\sharp(\mathbb{G})$ first.

Proposition 4.3.2 *For $\alpha \in \mathbb{A}$ and $1 \leqslant i, j \leqslant n_\alpha$ we have $\omega_{ij}^\alpha \in \mathrm{L}^1_\sharp(\mathbb{G})$ with $(\omega_{ij}^\alpha)^\sharp = \omega_{ji}^\alpha$.*

Proof

Let $\beta \in \mathbb{A}$ and $1 \leqslant k, l \leqslant n_\beta$ then, as $\lambda_j^\alpha > 0$ for all $1 \leqslant j \leqslant n_\alpha$ and $\Lambda^\alpha > 0$ we have

$$\begin{aligned}\overline{\langle S(u_{kl}^\beta)^*, \omega_{ij}^\alpha\rangle} = \overline{\langle u_{lk}^\beta, \omega_{ij}^\alpha\rangle} &= \overline{\frac{\Lambda^\alpha}{\lambda_j^\alpha}\phi(u_{lk}^\beta(u_{ij}^\alpha)^*)} \\ = \delta_{\alpha\beta}\delta_{il}\delta_{jk} &= \frac{\Lambda^\alpha}{\lambda_i^\alpha}\phi(u_{kl}^\beta(u_{ji}^\alpha)^*) = \langle u_{kl}^\beta, \omega_{ji}^\alpha\rangle\end{aligned}$$

where we've used Proposition 3.2.11 and Theorem 3.2.15. We have from Theorem 3.2.9 that $\left\{u_{kl}^\beta \;\middle|\; \beta \in \mathbb{A},\ 1 \leqslant k, l \leqslant n_\beta\right\}$ is a basis for $\mathrm{Hopf}(\mathbb{G})$ and so $\overline{\langle S(x)^*, \omega_{ij}^\alpha\rangle} = \langle x, \omega_{ji}^\alpha\rangle$ for all $x \in \mathrm{Hopf}(\mathbb{G})$. By Proposition 3.2.19 we have that $\mathrm{Hopf}(\mathbb{G})$ is a core for $\mathrm{Dom}(S)$, so for any $x \in \mathrm{Dom}(S)$ we have a net $(x_\alpha) \subset \mathrm{Hopf}(\mathbb{G})$ such that $x_\alpha \xrightarrow{w^*} x$ and $S(x_\alpha) \xrightarrow{w^*} S(x)$. So we have

$$\overline{\langle S(x)^*, \omega_{ij}^\alpha\rangle} = \lim \overline{\langle S(x_\alpha)^*, \omega_{ij}^\alpha\rangle} = \lim\langle x_\alpha, \omega_{ji}^\alpha\rangle = \langle x, \omega_{ji}^\alpha\rangle$$

as required. □

Now we show that the set D in Notation 4.3.1 gives us a dense subset of $\mathrm{L}^1(\mathbb{G})$ first and then $\mathrm{L}^1_\sharp(\mathbb{G})$ with respect to each of their norms.

Theorem 4.3.3 *The set D in Notation 4.3.1 is dense in $\mathrm{L}^1(\mathbb{G})$.*

Proof

Clearly $D \subset \mathrm{L}^1(\mathbb{G})$. Consider the set $D' = \mathrm{lin}\,\{x \cdot \phi \cdot y^* \mid x, y \in \mathrm{Hopf}(\mathbb{G})\}$, then clearly $D \subset D'$. We show first that $D' = D$ and then we show that D' is dense in $\mathrm{L}^1(\mathbb{G})$.

Let $n \in \mathbb{N}$ and $x, y \in \mathrm{Hopf}(\mathbb{G})$, then $y^* \in \mathrm{Hopf}(\mathbb{G}) \subset \mathrm{Dom}(\sigma_{-i})$ and by Proposition 3.2.18 we have $\sigma_{-i}(\mathrm{Hopf}(\mathbb{G})) \subset \mathrm{Hopf}(\mathbb{G})$. Using this and Proposition 1.4.17 we have

$$\langle z, x \cdot \phi \cdot y^* \rangle = \phi(y^* z x) = \phi(z x \sigma_{-i}(y^*)) = \langle z, (x\sigma_{-i}(y^*)) \cdot \phi \rangle$$

and so as $x\sigma_{-i}(y^*) \in \mathrm{Hopf}(\mathbb{G})$ we have $x \cdot \phi \cdot y^* = (x\sigma_{-i}(y^*)) \cdot \phi \in D$ and so $D' \subset D$.

Now we show that D' is dense in $\mathrm{L}^1(\mathbb{G})$. Fix $\varepsilon > 0$ and let $x, y \in \mathrm{C}(\mathbb{G})$. There exists $x' \in \mathrm{Hopf}(\mathbb{G})$ such that $\|x - x'\| < \dfrac{\varepsilon}{2\|y\|}$ then there exists $y' \in \mathrm{Hopf}(\mathbb{G})$ such that $\|y - y'\| < \dfrac{\varepsilon}{2\|x'\|}$. Then for any $z \in \mathrm{C}(\mathbb{G})$ we calculate

$$\begin{aligned}\left|\langle z, x \cdot \phi \cdot y^* \rangle - \langle z, x' \cdot \phi \cdot y'^* \rangle\right| &= \left|\phi(y^* z x - y'^* z x')\right| \leqslant \|y^* z x - y'^* z x'\| \\ &\leqslant \|y^* z x - y^* z x'\| + \|y^* z x' - y'^* z x'\| \\ &\leqslant \|y\|\|z\|\|x - x'\| + \|y - y'\|\|z\|\|x'\| < \varepsilon\|z\|\end{aligned}$$

and so taking the supremum over $z \in \mathrm{C}(\mathbb{G})$ with $\|z\| \leqslant 1$ we get

$$\|x \cdot \phi \cdot y^* - x' \cdot \phi \cdot y'^*\| < \varepsilon. \tag{4.6}$$

Now let $\omega \in \mathrm{L}^1(\mathbb{G})$ and fix $\varepsilon > 0$, then it follows from Proposition 2.4.2 that we have some $n \in \mathbb{N}$ and $\{x_i\}_{i=1}^n, \{y_i\}_{i=1}^n \subset \mathrm{C}(\mathbb{G})$ such that $\|\omega - \sum_{k=1}^n x_k \cdot \phi \cdot y_k^*\| < \dfrac{\varepsilon}{2}$. Let $1 \leqslant k \leqslant n$, then from Equation (4.6), we can find $x'_k, y'_k \in \mathrm{Hopf}(\mathbb{G})$ such that

$\|x_k \cdot \phi \cdot y_k^* - x_k' \cdot \phi \cdot {y_k'}^*\| < \dfrac{\varepsilon}{2n}$. Then we have

$$\begin{aligned}\left\|\omega - \sum_{k=1}^n x_k' \cdot \phi \cdot {y_k'}^*\right\| &\leqslant \left\|\omega - \sum_{k=1}^n x_k \cdot \phi \cdot y_k^*\right\| + \left\|\sum_{k=1}^n x_k \cdot \phi \cdot y_k^* - \sum_{k=1}^n x_k' \cdot \phi \cdot {y_k'}^*\right\| \\ &\leqslant \left\|\omega - \sum_{k=1}^n x_k \cdot \phi \cdot y_k^*\right\| + \sum_{k=1}^n \|x_k \cdot \phi \cdot y_k^* - x_k' \cdot \phi \cdot {y_k'}^*\| \\ &< \frac{\varepsilon}{2} + \sum_{k=1}^n \frac{\varepsilon}{2n} = \varepsilon.\end{aligned}$$

As $\sum_{k=1}^n x_k' \cdot \phi \cdot {y_k'}^* \in D'$ then it follows that D' is dense in $\mathrm{L}^1(\mathbb{G})$ as required. □

We can use this proposition to show that D is also dense in $\mathrm{L}^1_\sharp(\mathbb{G})$. The following proof uses similar techniques to that of the proof of Lemma 3 in Daws & Salmi (2013).

Theorem 4.3.4 *The set D in Notation 4.3.1 is dense in $\mathrm{L}^1_\sharp(\mathbb{G})$.*

Proof

Let $\alpha \in \mathbb{A}$, $1 \leqslant i, j \leqslant n_\alpha$ and let $\omega := \omega_{ij}^\alpha \in D$ for ease of notation and we show that $\omega(r) \in D$. Let $\beta \in \mathbb{A}$, $1 \leqslant k, l \leqslant n_\beta$ and $\mu_{kl}^\beta = (\ln(\lambda_k^\beta) - \ln(\lambda_l^\beta)) \in \mathbb{R}$. From Proposition 3.2.18 we calculate

$$\begin{aligned}\langle u_{kl}^\beta, \omega(r)\rangle &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \langle \tau_t(u_{kl}^\beta), \omega\rangle \, dt = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} (\lambda_k^\beta)^{it} (\lambda_l^\beta)^{-it} \langle u_{kl}^\beta, \omega\rangle \, dt \\ &= \left(\frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} e^{it\mu_{kl}^\beta} \, dt\right) \langle u_{kl}^\beta, \omega\rangle = \exp\left(-\frac{(\mu_{kl}^\beta)^2}{4n^2}\right) \langle u_{kl}^\beta, \omega\rangle.\end{aligned}$$

As $\left\{u_{kl}^\beta \;\middle|\; \beta \in \mathbb{A},\ 1 \leqslant k, l \leqslant n_\beta\right\}$ is dense in $\mathrm{C}(\mathbb{G})$ we have $\omega(r) = \exp\left(-\frac{(\mu_{kl}^\beta)^2}{4n^2}\right)\omega \in D$.

Let $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$, then $\omega \in \mathrm{L}^1(\mathbb{G})$ and so by Theorem 4.3.3 there is a net $(\omega_\alpha) \subset D$ such that $\lim \omega_\alpha = \omega$ in the $\mathrm{L}^1(\mathbb{G})$ norm. By Theorem 4.1.9, for fixed $r \in \mathbb{N}$, we have $\omega(r), \omega_\alpha(r) \in \mathrm{L}^1_\sharp(\mathbb{G})$ where $\omega_\alpha(r) \in D$ for all α and so $\|\omega(r) - \omega_\alpha(r)\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \leqslant e^{r^2/4}\|\omega - \omega_\alpha\|_{\mathrm{L}^1(\mathbb{G})}$. So taking the limit of α whilst holding r fixed we see that $\omega(r)$ is in the closure of D with respect to the $\mathrm{L}^1_\sharp(\mathbb{G})$ norm. As $r \to \infty$ we have $\omega(r) \to \omega$ and by Proposition 4.1.10 we have $\omega(r)^\sharp = \omega^\sharp(r) \to \omega^\sharp$ and so $\|\omega - \omega(r)\|_{\mathrm{L}^1_\sharp(\mathbb{G})} = \max\{\|\omega - \omega(r)\|_{\mathrm{L}^1(\mathbb{G})}, \|\omega^\sharp - \omega^\sharp(r)\|_{\mathrm{L}^1(\mathbb{G})}\} \to 0$ and thus ω is in the $\mathrm{L}^1_\sharp(\mathbb{G})$ closure of D as required. □

4.3.2 Criterion for Compactness in terms of $\mathrm{L}^1_\sharp(\mathbb{G})$ and $\mathrm{L}^1_\sharp(\mathbb{G})^{**}$

It was shown in Runde (2008) that a locally compact quantum group $\mathbb{G}$ is compact if and only if $\mathrm{L}^1(\mathbb{G})$ is an ideal in $\mathrm{L}^1(\mathbb{G})^{**}$ with respect to either Arens product in $\mathrm{L}^1(\mathbb{G})^{**}$. We show in this section that a similar result holds in the case of $\mathrm{L}^1_\sharp(\mathbb{G})$. We quote this theorem now to begin with and will spend the rest of this section proving this result.

Theorem 4.3.5 *Let $\mathbb{G}$ be a locally compact quantum group, then $\mathrm{L}^1_\sharp(\mathbb{G})$ is an ideal in $\mathrm{L}^1_\sharp(\mathbb{G})^{**}$ if and only if $\mathbb{G}$ is compact.*

We begin with some preparatory lemmas for the case that we have a locally compact quantum group $\mathbb{G}$ that is not compact.

Lemma 4.3.6 *Let $\mathbb{G}$ be a locally compact quantum group, $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ and $y \in \mathrm{Dom}(S)$. Then $\omega \cdot y \in \mathrm{L}^1_\sharp(\mathbb{G})$ with $(\omega \cdot y)^\sharp = \omega^\sharp \cdot S(y)^*$.*

Proof

Let $x \in \mathrm{Dom}(S)$, then using that $S(S(y)^*)^* = y$ and that S is an anti-homomorphism we have

$$\overline{\langle S(x)^*, \omega \cdot y\rangle} = \overline{\langle yS(x)^*, \omega\rangle} = \overline{\langle S(S(y)^* x)^*, \omega\rangle} = \langle S(y)^* x, \omega^\sharp\rangle = \langle x, \omega^\sharp \cdot S(y)^*\rangle$$

and as $y \in \mathrm{Dom}(S)$ it follows that $\omega^\sharp \cdot S(y)^* \in \mathrm{L}^1(\mathbb{G})$ and the result follows. □

Lemma 4.3.7 *Let $\mathbb{G}$ be a non-compact, locally compact quantum group, then there exists a non-zero net of states $(\kappa_\alpha) \subset \mathrm{L}^1(\mathbb{G})^+$ such that for all $x \in \mathrm{C}_0(\mathbb{G})$ we have $\lim_\alpha \langle x, \kappa_\alpha\rangle = 0$.*

We remind the reader that for the next proof we have $\mathcal{N}_\phi = \{x \in \mathrm{L}^\infty(\mathbb{G}) \mid \phi(x^*x) < \infty\}$ from Definition 1.4.2, $\mathcal{F}_\phi = \{\omega \in \mathrm{L}^1(\mathbb{G})^+_* \mid \omega(x) \leqslant \phi(x) \quad \forall\, x \in \mathrm{L}^\infty(\mathbb{G})^+\}$ and $\mathcal{G}_\phi = \{\lambda\omega \mid \omega \in \mathcal{F}_\phi,\ \lambda \in (0,1)\}$ from Notation 1.4.7.

Proof

Let $\omega \in \mathcal{F}_\phi$ be non-zero, let $\mu = \dfrac{\omega}{\|\omega\|}$ so that μ is a state in $\mathrm{L}^1(\mathbb{G})$ and let $(\mathcal{H}_\mu, \pi_\mu, \xi_\mu)$ be the GNS representation of μ. Let $x \in \mathcal{N}_\phi \cap \operatorname{Ker}\pi_\mu$, then we have $\mu(x^*x) = \|\pi_\mu(x)\xi_\mu\|^2 \leqslant \|\pi_\mu(x)\|^2\|\xi_\mu\|^2 = 0$ and thus $\omega(x^*x) = 0$. Also by Proposition 1.4.8 there is a $T \in \mathrm{C}_0(\mathbb{G})' \subset \mathcal{B}(\mathrm{L}^2(\mathbb{G}))$ such that $0 \leqslant T \leqslant 1$ and for all $y \in \mathcal{N}_\phi$ we have

$$|(T\Lambda_\phi(x)|\Lambda_\phi(y))| = |\omega(y^*x)| \leqslant \omega(y^*y)\omega(x^*x) = 0$$

and thus $T\Lambda_\phi(x) = 0$. Then we have $\|T^{1/2}\Lambda_\phi(x)\|^2 = (T\Lambda_\phi(x)|\Lambda_\phi(x)) = 0$ and so $T^{1/2}\Lambda_\phi(x) = 0$ for all $x \in \mathcal{N}_\phi \cap \operatorname{Ker}\pi_\mu$.

By Proposition 1.4.8 we have a unique element $\xi_\omega \in \mathrm{L}^2(\mathbb{G})$ such that $\langle x, \omega\rangle = (\pi_\omega(x)\xi_\omega|\xi_\omega)$ and $T^{1/2}\Lambda_\phi(x) = x\xi_\omega$ (where we've suppressed the π_ϕ map). We now show that $\|\xi_\omega\| = \|\omega\|^{1/2}$. Define $U : \mathcal{H}_\mu \to \mathrm{L}^2(\mathbb{G})$ as the map

$$\pi_\mu(x)\xi_\mu \mapsto \|\omega\|^{-1/2}T^{1/2}\Lambda_\phi(x) = \|\omega\|^{-1/2}x\xi_\omega$$

for all $x \in \mathcal{N}_\phi$. It follows from the previous paragraph that this is well-defined. Then for all $x, y \in \mathcal{N}_\phi$ we have

$$(U\pi_\mu(x)\xi_\mu|U\pi_\mu(y)\xi_\mu) = \frac{(T\Lambda_\phi(x)|\Lambda_\phi(y))}{\|\omega\|} = \frac{\omega(y^*x)}{\|\omega\|} = \mu(y^*x) = (\pi_\mu(x)\xi_\mu|\pi_\mu(y)\xi_\mu)$$

and so as $\mathcal{N}_\phi$ is dense in $\mathrm{C}_0(\mathbb{G})$ then U is an isometry. It follows that $1 = \|\xi_\mu\| = \|U\xi_\mu\| = \|\omega\|^{-1/2}\|\xi_\omega\|$ and thus $\|\xi_\omega\| = \|\omega\|^{1/2}$.

By Proposition 1.4.13 we have a net $(\omega_\alpha) \subset \mathcal{G}_\phi \subset \mathcal{F}_\phi \subset \mathrm{C}_0(\mathbb{G})^*_+$ such that for all $x \in \mathrm{L}^\infty(\mathbb{G})^+$ we have $\lim_\alpha\langle x, \omega_\alpha\rangle = \phi(x)$ and thus this also holds for all $x \in \mathcal{M}_\phi$. As

$\omega_\alpha \in \mathcal{F}_\phi$ for all α then, by Proposition 1.4.8, there is a $T_\alpha \in C_0(\mathbb{G})' \subset \mathcal{B}(L^2(\mathbb{G}))$ with $0 \leqslant T_\alpha \leqslant 1$ such that for all $x, y \in \mathcal{N}_\phi$ we have $(T_\alpha \Lambda_\phi(x)|\Lambda_\phi(y)) = \langle y^*x, \omega_\alpha\rangle$ and there exists a $\xi_\alpha \in L^2(\mathbb{G})$ such that for all $x \in \mathcal{N}_\phi$ we have $T_\alpha^{1/2}\Lambda_\phi(x) = x\xi_\alpha$.

Now let $\eta_\alpha = \dfrac{\xi_\alpha}{\|\xi_\alpha\|}$ for all α (we may assume without loss of generality $\omega_\alpha \neq 0$ for all α) and define $\kappa_\alpha := \omega_{\eta_\alpha,\eta_\alpha} \in L^1(\mathbb{G})^+$. As $\|\eta_\alpha\| = 1$ we have a net of states (κ_α). We calculate for fixed α and $x, y \in \mathcal{N}_\phi$ that

$$|\langle y^*x, \kappa_\alpha\rangle| = |(x\eta_\alpha|y\eta_\alpha)| = \frac{|(x\xi_\alpha|y\xi_\alpha)|}{\|\xi_\alpha\|^2} = \frac{|(T_\alpha\Lambda_\phi(x)|\Lambda_\phi(y))|}{\|\xi_\alpha\|^2} \leqslant \frac{\|\Lambda_\phi(x)\|\|\Lambda_\phi(y)\|}{\|\xi_\alpha\|^2}.$$

However as $\mathbb{G}$ is non-compact it follows from Proposition 3.2.2 that ϕ is not finite and as $\phi(x) = \lim\langle x, \omega_\alpha\rangle$ it follows that $\|\omega_\alpha\| \to \infty$. Then from the first two paragraphs of this proof we have that $\|\xi_\alpha\| = \|\omega_\alpha\|^{1/2} \to \infty$ and thus we have $\langle y^*x, \kappa_\alpha\rangle \to 0$ for all $x, y \in \mathcal{N}_\phi$. By Proposition 1.4.4 (iv) we have $\langle x, \kappa_\alpha\rangle \to 0$ for all $x \in \mathcal{M}_\phi$ and by density of $\mathcal{M}_\phi$ in $C_0(\mathbb{G})$ we thus have this for all $C_0(\mathbb{G})$ as required. □

Lemma 4.3.8 *Let $\mathbb{G}$ denote a non-compact locally compact quantum group, then there exists a net $(\omega_\alpha) \subset L^1_\sharp(\mathbb{G}) \cap L^1(\mathbb{G})^+$ that is bounded in $L^1_\sharp(\mathbb{G})$ and such that ω_α is a state in $L^1(\mathbb{G})^+$ for all α and for all $x \in C_0(\mathbb{G})$ we have $\lim_\alpha\langle x, \omega_\alpha\rangle = 0$ but $\langle 1, \omega_\alpha\rangle = 1$ for all α.*

Proof

Let $(\kappa_\alpha) \subset L^1(\mathbb{G})^+$ be a non-zero net of states given by Lemma 4.3.7 such that for all $x \in C_0(\mathbb{G})$ we have $\lim_\alpha \langle x, \kappa_\alpha\rangle = 0$. As τ_t is a $*$-homomorphism and κ_α is positive for all α we have $\langle \tau_t(x), \kappa_\alpha\rangle \geqslant 0$ for all $t \in \mathbb{R}$ and α. Fix $n \in \mathbb{N}$, then for all $x \in C_0(\mathbb{G})^+$ and α we have

$$\langle x, \kappa_\alpha(n)\rangle = \frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2t^2}\langle \tau_t(x), \kappa_\alpha\rangle\, dt \geqslant 0$$

and so $\kappa_\alpha(n) \in L^1(\mathbb{G})^+$. It also follows that

$$\langle x, \kappa_\alpha(n)\rangle = \frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2t^2}\langle \tau_t(x), \kappa_\alpha\rangle\, dt = \left\langle \frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2t^2}\tau_t(x)\, dt, \kappa_\alpha \right\rangle \to 0$$

where we note that we have n fixed and the limit is in α. However we have

$$\langle 1, \kappa_\alpha(n)\rangle = \frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2t^2}\langle \tau_t(1), \kappa_\alpha\rangle\, dt = \langle 1, \kappa_\alpha\rangle = 1$$

and also

$$\|\kappa_\alpha(n)\|_{L^1_\sharp(\mathbb{G})} \leqslant e^{n^2/4}\|\kappa_\alpha\|_{L^1(\mathbb{G})} = e^{n^2/4}$$

and so $(\kappa_\alpha(n))$ is bounded in $L^1_\sharp(\mathbb{G})$ for any fixed $n \in \mathbb{N}$. So for fixed $n \in \mathbb{N}$ the net $(\kappa_\alpha(n))$ satisfies the required properties. □

Now we consider some preparatory lemmas in the case that we have a compact quantum group $\mathbb{G}$. For any compact quantum group $\mathbb{G}$ we will denote by $\{U^\alpha \mid \alpha \in \mathbb{A}\}$ the maximal family of irreducible corepresentations by Theorem 3.2.9 for $\mathbb{A}$ the index set and D the set from Notation 4.3.1.

Lemma 4.3.9 *Let $\mathbb{G}$ denote a compact quantum group, $\omega \in D$ and let $T_\omega : L^1_\sharp(\mathbb{G}) \to L^1_\sharp(\mathbb{G})$ be the map $\kappa \mapsto \kappa * \omega$. Then T_ω has finite rank.*

Proof

Let $\alpha \in \mathbb{A}$, $1 \leqslant i, j \leqslant n_\alpha$ and $\omega^\alpha_{ij} \in D$ the set from Notation 4.3.1. For fixed $\kappa \in L^1_\sharp(\mathbb{G})$ we calculate

$$\begin{aligned}\sum_{l=1}^{n_\alpha}\langle u^\alpha_{li}, \kappa\rangle\langle u^\beta_{pq}, \omega^\alpha_{lj}\rangle &= \sum_{l=1}^{n_\alpha}\delta_{\alpha\beta}\delta_{pl}\delta_{qj}\langle u^\alpha_{li}, \kappa\rangle = \delta_{\alpha\beta}\delta_{qj}\langle u^\alpha_{pi}, \kappa\rangle = \sum_{r=1}^{n_\beta}\langle u^\beta_{pr}, \kappa\rangle\delta_{\alpha\beta}\delta_{ri}\delta_{qj}\\ &= \sum_{r=1}^{n_\beta}\langle u^\beta_{pr}, \kappa\rangle\langle u^\beta_{rq}, \omega^\alpha_{ij}\rangle = \langle \Delta(u^\beta_{pq}), \kappa \otimes \omega^\alpha_{ij}\rangle = \langle u^\beta_{pq}, T^\alpha_{ij}(\kappa)\rangle.\end{aligned}$$

So by linearity for all $x \in \mathrm{Hopf}(\mathbb{G})$ we have $\langle x, T^\alpha_{ij}(\kappa)\rangle = \sum_{l=1}^{n_\alpha}\langle u^\alpha_{li}, \kappa\rangle\langle x, \omega^\alpha_{lj}\rangle$ and as $\mathrm{Hopf}(\mathbb{G})$ is dense in $C(\mathbb{G})$ we have $T^\alpha_{ij}(\kappa) = \sum_{l=1}^{n_\alpha}\langle u^\alpha_{li}, \kappa\rangle\omega^\alpha_{lj}$. Thus we have shown that the image of T^α_{ij} is a finite linear combination of ω^α_{lj} for $1 \leqslant l \leqslant n_\alpha$ and thus T^α_{ij} has finite rank.

Now for $\omega \in D$ we consider the map $T_\omega : L^1_\sharp(\mathbb{G}) \to L^1_\sharp(\mathbb{G})$ given by $\kappa \mapsto \kappa * \omega$, then as ω is a finite linear combination of elements ω_{ij}^{α} it follows that T_ω is a finite linear combination of elements T_{ij}^{α} and so T_ω is of finite rank. $\square$

Lemma 4.3.10 *Let $\mathbb{G}$ denote a compact quantum group and $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$, then the maps $\kappa \mapsto \omega * \kappa$ and $\kappa \mapsto \kappa * \omega$ from $\mathrm{L}^1_\sharp(\mathbb{G})$ to $\mathrm{L}^1_\sharp(\mathbb{G})$ are weakly compact.*

Proof

Let $T_\omega : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ denote the map $\kappa \mapsto \kappa * \omega$. It follows from Notation 4.3.1 and Theorem 4.3.4 that the set D (the linear span of all ω_{ij}^{α}) is dense in $\mathrm{L}^1_\sharp(\mathbb{G})$ and so there is a net $(\omega_\alpha) \subset D$ with $\|\omega - \omega_\alpha\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \to 0$. Then for $\kappa \in \mathrm{L}^1_\sharp(\mathbb{G})$ we have

$$\|T_\omega(\kappa) - T_{\omega_\alpha}(\kappa)\|_{\mathrm{L}^1_\sharp(\mathbb{G})} = \|\kappa * \omega - \kappa * \omega_\alpha\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \leqslant \|\kappa\|_{\mathrm{L}^1_\sharp(\mathbb{G})}\|\omega - \omega_\alpha\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \to 0$$

and so $\|T_\omega - T_{\omega_\alpha}\| \to 0$. From Lemma 4.3.9 we have T_{ω_α} is of finite rank so T_ω is compact and thus weakly compact by Proposition A.4.4.

Let $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ and we consider the map $\kappa \mapsto \omega * \kappa$ now. We have by Proposition A.4.3 that the composition of arbitrary maps with weakly compact maps is weakly compact. Then as $\omega * \kappa = (\kappa^\sharp * \omega^\sharp)^\sharp$ and as the map $\kappa \mapsto \kappa * \omega$ is weakly compact it follows that $\kappa \mapsto \omega * \kappa$ is weakly compact. $\square$

Proof of Theorem 4.3.5

Let $\mathbb{G}$ be a compact quantum group, then by Lemma 4.3.10 we have that multiplication from the left and right in $\mathrm{L}^1_\sharp(\mathbb{G})$ is weakly compact and thus it follows from Proposition A.4.6 that $\mathrm{L}^1_\sharp(\mathbb{G})$ is an ideal in $\mathrm{L}^1_\sharp(\mathbb{G})^{**}$.

Now assume $\mathbb{G}$ is not compact and we show that we have some $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ such that the map $\kappa \mapsto \omega * \kappa$ is not weakly compact and thus it follows by Proposition A.4.6 that $\mathrm{L}^1_\sharp(\mathbb{G})$ is not an ideal in $\mathrm{L}^1_\sharp(\mathbb{G})^{**}$. As $\mathbb{G}$ is not compact then $1 \notin \mathrm{C}_0(\mathbb{G})$. Fix $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ and $y \in \mathrm{Dom}(S) \cap \mathrm{C}_0(\mathbb{G})$ such that $\langle y, \omega \rangle \neq 0$, then by Lemma 4.3.6 we have $\omega \cdot y \in \mathrm{L}^1_\sharp(\mathbb{G})$.

Assume that the map $T : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ given by $\kappa \mapsto (\omega \cdot y) * \kappa$ is weakly compact and it suffices to show we have a contradiction. By Lemma 4.3.8 we have a net $(\kappa_\alpha) \subset L^1_\sharp(\mathbb{G}) \cap L^1(\mathbb{G})^+$ such that for all $x \in C_0(\mathbb{G})$ we have $\lim_\alpha \langle x, \kappa_\alpha \rangle = 0$ and $\langle 1, \kappa_\alpha \rangle = 1$ for all α. As left multiplication by $\omega \cdot y$ is weakly compact there is a subnet (κ_β) of (κ_α) with a weak*-limit of $(\omega \cdot y) * \kappa_\beta$. For all $x \in C_0(\mathbb{G})$ we have

$$\langle x, (\omega \cdot y) * \kappa_\beta \rangle = \langle (y \otimes 1)\Delta(x), \omega \otimes \kappa_\beta \rangle$$

and as $(y \otimes 1)\Delta(x) \in C_0(\mathbb{G}) \otimes_{min} C_0(\mathbb{G})$ we have $(\omega \otimes \mathrm{id})((y \otimes 1)\Delta(x)) \in C_0(\mathbb{G})$ and taking the limit we see that $\langle x, (\omega \cdot y) * \kappa \rangle \to 0$ for all $x \in C_0(\mathbb{G})$. Now let $x \in L^\infty(\mathbb{G})$ and we have a net (x_γ) with weak*-limit x. Then we have

$$\langle x, (\omega \cdot y) * \kappa_\beta \rangle = \lim_\gamma \langle x_\gamma, (\omega \cdot y) * \kappa_\beta \rangle = 0$$

and so $(\omega \cdot y) * \kappa_\beta \xrightarrow{w} 0$. But we also have

$$\lim_\beta \langle 1, (\omega \cdot y) * \kappa_\beta \rangle = \lim_\beta \langle y, \omega \rangle \langle 1, \kappa_\beta \rangle = \langle y, \omega \rangle \neq 0$$

and so we have a contradiction as required. □

Chapter 5

The Compact Quantum Group $\mathrm{SU}_q(2)$

By considering the C*-algebra of continuous functions on the compact Lie group $\mathrm{SU}(2)$ Woronowicz found a non-trivial example of a compact quantum group in the C*-algebraic setting in Woronowicz (1987b). He showed that for $q \in [-1,1]\backslash\{0\}$ there is a C*-algebra $\mathrm{C}(\mathrm{SU}_q(2))$ such that as q tends to 1 we obtain a commutative C*-algebra isomorphic to the continuous functions on $\mathrm{SU}(2)$. He then showed there is a map $\Delta : \mathrm{C}(\mathrm{SU}_q(2)) \to \mathrm{C}(\mathrm{SU}_q(2)) \otimes_{min} \mathrm{C}(\mathrm{SU}_q(2))$ that implements the group product when $q = 1$ which subsequently led to the definition of a compact matrix quantum group. Furthermore he detailed the corepresentation theory in the same paper and he showed there was a non-trivial Haar state in Woronowicz (1987a) analogous to that of the Haar integral of $\mathrm{SU}(2)$.

We will give an overview of $\mathrm{SU}_q(2)$ in Section 5.1 where we define the C*-algebra $\mathrm{C}(\mathrm{SU}_q(2))$, the coproduct Δ and show that we get a compact quantum matrix group. In addition we give details of the corepresentation theory for $\mathrm{SU}_q(2)$ and we give formulas for the antipode on $\mathrm{Hopf}(\mathrm{SU}_q(2))$ and the Haar state on $\mathrm{C}(\mathrm{SU}_q(2))$. In Section 5.2 we move on to prove some new results about the C*-algebraic and von Neumann algebraic quantum group structure of $\mathrm{SU}_q(2)$. Most importantly we will show that the commutative unital C*-subalgebra $\mathrm{C}^*(c, 1)$ of $\mathrm{C}(\mathrm{SU}_q(2))$ can be realised as the continuous functions $\mathrm{C}(K)$ on a compact subset $K \subset \mathbb{C}$ and we use this to study $\mathrm{C}^*(c, 1)$ further. In Section 5.3 we discuss the $\mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ algebra and prove some new results about this. In particular we show that we have a subalgebra $\mathrm{L}^1_\sharp(K,\nu)$ as a subspace of $\mathrm{L}^1(K,\nu)$ and we prove some structure theorems regarding this. Then in Section 5.4 we consider the product $\mathrm{SU}_q(2)\times\mathrm{SU}_{q'}(2)$ and we use this as a counterexample to a question asked in Chapter 4 regarding tensoring $\mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$. Finally in Section 5.5 we discuss adjoints of elements $(\mu\otimes\mathrm{id})(W^{\mathrm{SU}_q(2)})\in\mathcal{B}(\mathrm{L}^2(\mathrm{SU}_q(2)))$ where $\mu\in\mathrm{C}(\mathrm{SU}_q(2))^*$ and show that there is some $\nu\in\mathrm{C}(\mathrm{SU}_q(2))^*$ where $(\nu\otimes\mathrm{id})(W)^*$ is not in the closure of $\mathrm{lin}\,\{(\mu\otimes\mathrm{id})(W)\mid\mu\in\mathrm{C}(\mathrm{SU}_q(2))^*\}$ answering a question resulting from work in Das & Daws (2014).

Throughout this chapter we let $q\in(0,1)$ with the exception of Section 5.1 where the results hold for $q\in[-1,0)\cup(0,1]$. We let $K\subset\mathbb{C}$ denote the set $K=\{0\}\cup\left\{q^re^{2\pi i\theta}\;\middle|\;r\in\mathbb{N}_0,\ \theta\in[0,1)\right\}$. For all $n\in\mathbb{N}$ we let $\underline{z}^n:K\to\mathbb{C}$ be the map $z\mapsto z^n$ and $\underline{z}^{*n}:K\to\mathbb{C}$ the map $z\mapsto\overline{z^n}$. Finally we let $\alpha_s=(1-q^{2s})^{1/2}$ for $s\in\mathbb{N}$.

We note that all results from Section 5.2 would also hold for $q\in(-1,0)$ excepting the notational inconvenience of having to consider $|q|$ instead of q.

5.1 Basics of $\mathrm{SU}_q(2)$

In this section we give details of $\mathrm{SU}_q(2)$. Nothing in this section is new, we simply give the background required for the new results in the following section. We give details of the C*-algebraic quantum group, the corepresentation theory, the antipode and the Haar state of $\mathrm{SU}_q(2)$.

5.1.1 C*-algebraic Quantum Group

Let $\mathrm{Hopf}(\mathrm{SU}_q(2))$ denote the free unital $*$-algebra generated by two elements a and c satisfying the following commutation relations

$$\begin{gathered}a^*a+c^*c=1=aa^*+q^2c^*c,\\ cc^*=c^*c,\qquad ac=qca,\qquad ac^*=qc^*a.\end{gathered}\tag{5.1}$$

We notice that for $q = 1$ we have a commutative $*$-algebra and otherwise we have a non-commutative $*$-algebra.

Consider the Hilbert space $\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{Z}) \cong_i \ell^2(\mathbb{N}_0 \times \mathbb{Z})$ with canonical orthonormal basis $\{e_{k,l} \mid k \in \mathbb{N}_0, l \in \mathbb{Z}\}$ and let $\pi_0 : \mathrm{Hopf}(\mathrm{SU}_q(2)) \to \mathcal{B}(\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{Z}))$ be defined as follows

$$\pi_0(a)e_{k,l} = \begin{cases} \alpha_k e_{k-1,l} & \text{if } k > 0 \\ 0 & \text{if } k = 0 \end{cases} \quad \text{and} \quad \pi_0(c)e_{k,l} = q^k e_{k,l+1}. \tag{5.2}$$

With a little bit of work we can show that this satisfies the relations given by 5.1 with adjoints given by

$$\pi_0(a^*)e_{k,l} = \alpha_{k+1} e_{k+1,l} \quad \text{and} \quad \pi_0(c^*)e_{k,l} = q^k e_{k,l-1} \tag{5.3}$$

and so this forms a $*$-representation of $\mathrm{Hopf}(\mathrm{SU}_q(2))$.

Notation 5.1.1 *Throughout this chapter for $k < 0$ we let $a^k = (a^*)^{-k}$ and $(a^*)^k = a^{-k}$. For all $k \in \mathbb{Z}$ and $m, n \in \mathbb{N}_0$ we denote $a_{kmn} := a^k (c^*)^m c^n$.*

The following theorem is due to Woronowicz. For a proof see Theorem 1.2 in Woronowicz' paper Woronowicz (1987b) or Proposition 6.2.5 in Timmermann's book Timmermann (2008).

Theorem 5.1.2 *The set $\{a_{kmn} \mid k \in \mathbb{Z},\ m, n \in \mathbb{N}_0\}$ forms a basis for* $\mathrm{Hopf}(\mathrm{SU}_q(2))$.

We now want to build a C*-algebra A with $\mathrm{Hopf}(\mathrm{SU}_q(2))$ as a dense $*$-subalgebra. First we need to give an appropriate norm on $\mathrm{Hopf}(\mathrm{SU}_q(2))$.

Lemma 5.1.3 *Let $\pi : \mathrm{Hopf}(\mathrm{SU}_q(2)) \to \mathcal{B}(\mathcal{H})$ be a $*$-representation for any Hilbert space $\mathcal{H}$, then we have $\|\pi(a_{kmn})\| \leqslant 1$ for all $k \in \mathbb{Z}$ and $m, n \in \mathbb{N}_0$.*

Proof

For $\xi \in \mathcal{H}$ we have

$$\begin{aligned}\|\pi(a)\xi\|^2 + \|\pi(c)\xi\|^2 &= (\pi(a)\xi|\pi(a)\xi) + (\pi(c)\xi|\pi(c)\xi)\\ &= (\pi(a)^*\pi(a)\xi + \pi(c)^*\pi(c)\xi|\xi) = (\pi(a^*a + c^*c)\xi|\xi) = (\xi|\xi) = \|\xi\|^2.\end{aligned}$$

Then it follows by varying $\xi \in \mathrm{L}^2(\mathrm{SU}_q(2))$ with $\|\xi\| \leqslant 1$ that $\|\pi(a)\| \leqslant 1$ and $\|\pi(c)\| \leqslant 1$ and thus $\|\pi(a_{kmn})\| \leqslant 1$ for all $k \in \mathbb{Z}$ and $m, n \in \mathbb{N}_0$ as π is a $*$-homomorphism. □

Using that we have a $*$-representation π_0 of $\mathrm{Hopf}(\mathrm{SU}_q(2))$ given by Equation (5.2) we can define a map $\|\cdot\| : \mathrm{Hopf}(\mathrm{SU}_q(2)) \to \mathbb{R}^+$ given by

$$\|x\| = \sup\left\{ \|\pi(x)\| \;\middle|\; \begin{array}{l}\pi : \mathrm{Hopf}(\mathrm{SU}_q(2)) \to \mathcal{B}(\mathcal{H}) \text{ for } \mathcal{H} \text{ a}\\ \qquad\qquad \text{Hilbert space and } \pi \text{ a } *\text{-representation}\end{array}\right\} \tag{5.4}$$

for $x \in \mathrm{Hopf}(\mathrm{SU}_q(2))$. We note this is finite by Lemma 5.1.3. We show in Proposition 5.1.5 that it is non-zero and thus a norm satisfying $\|x^*x\| = \|x\|^2$ for all $x \in \mathrm{Hopf}(\mathrm{SU}_q(2))$. We have a simple lemma first.

Lemma 5.1.4 *We have for $k, t \in \mathbb{Z}$ and $m, n, s \in \mathbb{N}_0$ that*

$$\pi_0(a_{kmn})e_{s,t} = \begin{cases} q^{s(n+m)}\alpha_s \dots \alpha_{s-(k-1)}e_{s-k,t+n-m} & \text{if } 0 \leqslant k \leqslant s\\ q^{s(n+m)}\alpha_{s+1} \dots \alpha_{s-k}e_{s-k,t+n-m} & \text{if } k < 0\\ 0 & \text{if } k > s\end{cases} \tag{5.5}$$

where if $k = 0$ this reduces to $\pi_0(a_{0mn})e_{s,t} = q^{s(n+m)}e_{s,t+n-m}$.

Proof

For $k > s$ we have $\pi_0(a^k)e_{s,t} = 0$. For $0 \leqslant k \leqslant s$ we have

$$\begin{aligned}\pi_0(a^k)e_{s,t} &= \alpha_s a^{k-1} e_{s-1,t} = \alpha_s \alpha_{s-1} (a^*)^{k-2} e_{s-2,t} \\ &= \alpha_s \alpha_{s-1} \alpha_{s-2} (a^*)^{k-3} e_{s-3,t} = \cdots = \alpha_s \alpha_{s-1} \cdots \alpha_{s-(k-1)} e_{s-k,t}\end{aligned}$$

and the result follows as $\pi_0((c^*)^m c^n)e_{s,t} = q^{s(n+m)} e_{s,t+n-m}$. The case of $k < 0$ is similar. □

Proposition 5.1.5 *We have* $\|a_{kmn}\| \neq 0$ *and* $\|a_{kmn}\| \leqslant 1$ *for all* $k \in \mathbb{Z}$ *and* $m, n \in \mathbb{N}_0$. *In particular the completion of* $\mathrm{Hopf}(\mathrm{SU}_q(2))$ *with respect to the norm given is a C*-algebra.*

Proof

If $k \geqslant 0$ let $s \geqslant k$ and then

$$\|\pi_0(a_{kmn})e_{s,t}\| = q^{s(n+m)} \alpha_s \cdots \alpha_{s-(k-1)} \|e_{s-k,t+n-m}\| = q^{s(n+m)} \alpha_s \cdots \alpha_{s-(k-1)} > 0$$

and similarly if $k < 0$ then $\|\pi_0(a_{kmn})e_{s,t}\| > 0$ and so

$$\|a_{kmn}\| \geqslant \|\pi_0(a_{kmn})\| = \sup\left\{\|\pi_0(a_{kmn})\xi\| \;\middle|\; \xi \in \ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{Z}),\ \|\xi\| \leqslant 1\right\} > 0.$$

It is almost immediate from the definition that $\|x^*x\| = \|x\|^2$ for all $x \in \mathrm{Hopf}(\mathrm{SU}_q(2))$ and then it follows easily that the completion is a C*-algebra. □

We are now in a position to define the following C*-algebraic completion of $\mathrm{Hopf}(\mathrm{SU}_q(2))$. In fact as we will see shortly, this gives us the C*-algebra from the reduced C*-algebraic quantum group $(\mathrm{C}(\mathrm{SU}_q(2)), \Delta)$.

Definition 5.1.6 *We define* A *to be the completion of* $\mathrm{Hopf}(\mathrm{SU}_q(2))$ *with respect to the norm given by Equation* (5.4).

We can make A into a compact matrix quantum group as follows.

Theorem 5.1.7 *Consider* $u = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \in \mathbb{M}_2(A)$. *Then we have a map* $\Delta : A \to A \otimes_{min} A$ *given by*

$$\Delta(a) = a \otimes a - qc^* \otimes c, \qquad \Delta(c) = c \otimes a + a^* \otimes c \tag{5.6}$$

making this a compact matrix quantum group (A, Δ, u) *as per definition 3.2.22. The antipode* S *as defined in Definition-Theorem 2.2.7 is given by*

$$S(a) = a^*, \qquad S(a^*) = a, \qquad S(c) = -qc, \qquad S(c^*) = -\frac{1}{q}c^*. \tag{5.7}$$

Proof

We show that the conditions of Proposition 3.2.21 are satisfied. It is easy to see that u is unitary. We can also easily see that the inverse of $\bar{u}$ is given by $\begin{pmatrix} a & q^2 c \\ -\frac{1}{q}c^* & a^* \end{pmatrix}$.

Clearly $A = \overline{\mathrm{alg}\ \{u_{ij} \mid 1 \leqslant i, j \leqslant 2\}}^{\|\cdot\|}$ by construction where alg denotes the algebra generated. It is easy to show that Δ satisfies condition (iv) in Proposition 3.2.21 and so we have a compact matrix quantum group. The antipode S follows from the equation $S(u_{ij}) = u_{ji}^*$ for $1 \leqslant i, j \leqslant 2$. $\square$

We now calculate the $(f_z)_{z \in \mathbb{C}}$ characters from 3.2.16.

Proposition 5.1.8 *For* $z \in \mathbb{C}$ *we have* $f_z(a) = q^{-z}$, $f_z(a^*) = q^z$ *and* $f_z(c) = f_z(c^*) = 0$. *Furthermore we have*

$$(\mathrm{id} \otimes f_z)\Delta(a) = q^{-z}a = (f_z \otimes \mathrm{id})\Delta(a),$$

$$(\mathrm{id} \otimes f_z)\Delta(a^*) = q^z a^* = (f_z \otimes \mathrm{id})\Delta(a^*),$$

$$(\mathrm{id} \otimes f_z)\Delta(c) = q^{-z}c, \qquad (f_z \otimes \mathrm{id})\Delta(c) = q^z c,$$

$$(\mathrm{id}\otimes f_z)\Delta(c^*) = q^z c^* \qquad \textit{and} \qquad (f_z\otimes\mathrm{id})\Delta(c^*) = q^{-z}c^*.$$

Proof

We calculate the F-matrix for the corepresentation u. We have by Theorem 3.2.15 that F intertwines u and $S_2^2(u)$ and we have $S_2^2(u) = \begin{pmatrix} a & -\frac{1}{q}c^* \\ q^2 c & a^* \end{pmatrix}$. Setting $F = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}$ we can easily show that $Fu = S_2^2(u)F$ and furthermore this satisfies $\mathrm{Tr}(F) = \mathrm{Tr}(F^{-1})$ giving the F-matrix for u_2 of Theorem 3.2.15.

We can then calculate the f_z values on the generators a and c and their adjoints easily (for example $f_z(a) = (F_{11})^z = q^{-z}$). We can then use these formulas and Equation (5.5) to calculate the remainder of the equations. □

The formulas in the following corollary can be extended to all $\mathrm{Hopf}(\mathrm{SU}_q(2))$ as R is a $*$-anti-homomorphism and τ_z is a homomorphism on $\mathrm{Hopf}(\mathrm{SU}_q(2))$.

Corollary 5.1.9 *Let $z\in\mathbb{C}$. We have the following formulas for the scaling group on* $\mathrm{Hopf}(\mathrm{SU}_q(2))$*:*

$$\tau_z(a) = a, \qquad \tau_z(c) = q^{2iz}c, \qquad \tau_z(c^*) = q^{-2iz}c^*, \qquad \tau_z(a^*) = a^*$$

and for the unitary antipode R on $\mathrm{Hopf}(\mathrm{SU}_q(2))$ *we have*

$$R(a) = a^*, \qquad R(c) = \begin{cases} -c & \textit{if } 0 < q \leqslant 1 \\ c & \textit{if } -1 \leqslant q < 0. \end{cases}$$

Proof

The first set of formulas for τ_z follow easily from Proposition 3.2.18 and the formulas in Proposition 5.1.8. It is easy to calculate R knowing S and $\tau_{-i/2}$ on the Hopf algebra elements. It follows that R is a $*$-anti-homomorphism from Definition-Theorem 2.2.7 and that τ_z is a homomorphism from Proposition 1.3.10. □

Finally we have the following important theorem. We refer the reader to Bédos *et al.* (2001) for a proof.

Theorem 5.1.10 *The compact quantum group* $\mathrm{SU}_q(2)$ *is co-amenable.*

As $\mathrm{SU}_q(2)$ is coamenable we have that $\mathrm{C}(\mathrm{SU}_q(2)) \cong_i A$ from Theorem 3.4.1 and so we can always work with the reduced C*-algebraic quantum group $\mathrm{C}(\mathrm{SU}_q(2)) \subset \mathcal{B}(\mathrm{L}^2(\mathrm{SU}_q(2)))$.

5.1.2 Corepresentation Theory for $\mathrm{SU}_q(2)$

In this section we discuss corepresentations of $\mathrm{SU}_q(2)$. This was investigated first in Woronowicz (1987b) and the explicit formulas for the irreducible unitary corepresentations were found in Koornwinder (1989). The following is a summary of the results we need from these papers.

We first give some definitions and basic propositions on q-hypergeometric polynomials and we give the main theorem regarding the irreducible finite-dimensional unitary corepresentations of $\mathrm{SU}_q(2)$ (note that by Theorem 3.2.9 we need only consider finite-dimensional corepresentations for the unitary corepresentations).

Definition 5.1.11 *Let* $t \in \mathbb{C}$ *and* $k \in \mathbb{N}$, *then the* q***-shifted factorial*** *is defined inductively by* $(t;q)_0 = 1$ *and*

$$(t;q)_k = \prod_{j=0}^{k-1}(1-tq^j) = (1-t)(1-tq)\cdots(1-tq^{k-1}).$$

For $n,k \in \mathbb{N}$ *the* q***-combinatorial coefficient*** *is defined by*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n;q^{-1})_k}{(q;q)_k} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

It is straightforward to show that we have the following relations.

Proposition 5.1.12 *For $n \in \mathbb{N}_0$ and $0 \leqslant k \leqslant n$ we have the relations*

$$\begin{bmatrix} n \\ n-k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q, \qquad \begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1 = \begin{bmatrix} n \\ n \end{bmatrix}_q$$

and

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^{n-(k-1)} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

Proposition 5.1.13 *For x and y indeterminates such that $xy = qyx$ we have*

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k x^{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} x^k y^{n-k}.$$

We will need the following definition in Section 5.5.

Definition 5.1.14 *For $n \in \mathbb{N}$ we define the* ***little q-Jacobi polynomial*** *by*

$$p_n(x; a, b|q) := \sum_{k=0}^{\infty} \frac{(q^{-n}; q)_k (abq^{n+1}; q)_k}{(aq; q)_k} (qx)^k.$$

Finally we have the following important theorems. The proof of the following is given in Theorem 5.4 in Woronowicz (1987b) and Section 4 and Proposition 5.2 of Koornwinder (1989).

Theorem 5.1.15 *Let U be a unitary finite-dimensional corepresentation, then there is a half-integer $l \in \frac{1}{2}\mathbb{N}_0 = \{0, \frac{1}{2}, 1, 1\frac{1}{2}, \dots\}$ such that $\dim U = 2l+1$ and the matrix can be indexed by $n, m \in \{-l, -l+1, \dots, l-1, l\}$ such that the entries are given by*

$$u^l_{n,m} = \begin{bmatrix} 2l \\ l-n \end{bmatrix}_{q^{-2}}^{1/2} \begin{bmatrix} 2l \\ l-m \end{bmatrix}_{q^{-2}}^{-1/2} \sum_{i=\max\{0,(m-n)\}}^{\min\{(l-n),(l+m)\}} q^{(l-n-i)(n-m+2i)} q^{-i(n-m+i)}$$
$$\times \begin{bmatrix} l-n \\ i \end{bmatrix}_{q^{-2}} \begin{bmatrix} l+n \\ l+m-i \end{bmatrix}_{q^{-2}} (-qc^*)^i c^{n-m+i} a^{l-n-i} (a^*)^{l+m-i}.$$

In particular we have $U^l = \sum_{n,m=-l}^{l} u^l_{n,m} e^{2l+1}_{l+n+1,l+m+1}$.

The following is Theorem 5.3 from Koornwinder (1989).

Theorem 5.1.16 *Let* $l \in \frac{1}{2}\mathbb{N}_0$ *and* $-l \leqslant n, m \leqslant l$. *If* $n \geqslant m \geqslant -n$ *we have*

$$u^l_{n,m} = \begin{bmatrix} l-m \\ n-m \end{bmatrix}^{1/2}_{q^2} \begin{bmatrix} l+n \\ n-m \end{bmatrix}^{1/2}_{q^2} q^{-(n-m)(l-n)} (a^*)^{n+m} \\ p_{l-n}(c^*c; q^{2(n-m)}, q^{2(n+m)} | q^2) c^{n-m}$$

and if $m \geqslant n \geqslant -m$ *we have*

$$u^l_{n,m} = \begin{bmatrix} l-n \\ m-n \end{bmatrix}^{1/2}_{q^2} \begin{bmatrix} l+m \\ m-n \end{bmatrix}^{1/2}_{q^2} q^{-(m-n)(l-m)} (a^*)^{m+n} \\ p_{l-m}(c^*c; q^{2(m-n)}, q^{2(m+n)} | q^2)(-qc^*)^{m-n}$$

for p_n *the* q*-Jacobi Polynomial of Definition 5.1.14.*

Example 5.1.17 *We can show that* $U^0 = (1)$ *and* $U^{1/2} = u$ *for* u *the corepresentation in Theorem 5.1.7.*

5.1.3 The Haar State

In this section we calculate the Haar state from Definition-Theorem 3.2.3 on $\mathrm{Hopf}(\mathrm{SU}_q(2))$. Later we will show that we can extend this to $\mathrm{C}(\mathrm{SU}_q(2))$. Throughout this section we let π_0 be the representation given by equation (5.2).

Consider the Hilbert space $\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}_0)$ with an orthonormal basis given by $\{e_{r,s,t} \mid r, t \in \mathbb{N}_0, s \in \mathbb{Z}\}$ and $x \in \mathrm{C}(\mathrm{SU}_q(2))$ acting on this Hilbert space as

$$\xi \mapsto (\pi_0(x) \otimes 1)\xi$$

for $\xi \in \ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}_0)$. We define

$$\xi_0 = (1-q^2)^{1/2} \sum_{p=0}^{\infty} q^p e_{p,0,p} \tag{5.8}$$

as per Equation (1.8) in Lance (1994) which is easily seen to be a unit vector in the Hilbert space $\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}_0)$. We can easily prove the following formula.

Lemma 5.1.18 *We have*

$$(\pi_0(a_{kmn}) \otimes 1)\xi_0 = \begin{cases} (1-q^2)^{1/2} \sum_{s=k}^{\infty} q^{s(1+n+m)} \alpha_s \ldots \alpha_{s-(k-1)} e_{s-k,n-m,s} & \text{if } k \geqslant 0 \\ (1-q^2)^{1/2} \sum_{s=0}^{\infty} q^{s(1+n+m)} \alpha_{s+1} \ldots \alpha_{s-k} e_{s-k,n-m,s} & \text{if } k < 0 \end{cases} \tag{5.9}$$

where we remind that $\alpha_k = (1-q^{2k})^{1/2}$. *If* $k = 0$ *this collapses to* $(\pi_0(a_{0mn}) \otimes 1)\xi_0 = (1-q^2)^{1/2} \sum_{s=0}^{\infty} q^{s(1+n+m)} e_{s,n-m,s}$.

Let ϕ denote the Haar state on $\mathrm{Hopf}(\mathrm{SU}_q(2))$ (see Definition-Theorem 3.2.3). We now given an explicit description of ϕ in terms of π_0 and ξ_0. We offer a proof of the following as this is in a different form than sometimes found in the literature (though the proof is very similar to that of Theorem $6.2.17$ in Timmermann (2008)).

Proposition 5.1.19 *For all* $x \in \mathrm{C}(\mathrm{SU}_q(2))$ *we have*

$$\phi(x) = ((\pi_0(x) \otimes 1)\xi_0 | \xi_0) \tag{5.10}$$

and in particular for $k \in \mathbb{Z}$ *and* $m, n \in \mathbb{N}_0$ *we have*

$$\phi(a_{kmn}) = \delta_{k,0}\delta_{m,n} \frac{1-q^2}{1-q^{2(1+n)}}.$$

Proof

Let $k \in \mathbb{Z}$ and $m, n \in \mathbb{N}_0$ throughout this proof, then from Lemma 5.1.18 we have that $((\pi_0(a_{kmn}) \otimes 1)\xi_0 | \xi_0) = \delta_{k,0}\delta_{m,n} \dfrac{1-q^2}{1-q^{2(1+n)}}$ immediately. We show that Equation

(5.10) holds for $a_{kmn} \in \mathrm{Hopf}(\mathrm{SU}_q(2))$ and then extending by linearity and continuity the result follows. Let $z \in \mathbb{C}$, then using the multiplicativity of f_z by Proposition 3.2.17 and Proposition 5.1.8 we have

$$\begin{aligned}(\mathrm{id} \otimes f_z)\Delta(a_{kmn}) &= ((\mathrm{id} \otimes f_z)\Delta(a))^k((\mathrm{id} \otimes f_z)\Delta(c^*))^m((\mathrm{id} \otimes f_z)\Delta(c))^n \\ &= (q^{-z}a)^k(q^z c^*)^m(q^{-z}c)^n = q^{z(-k+m-n)}a_{kmn}.\end{aligned}$$

Now acting on this with ϕ we get

$$q^{z(-k+m-n)}\phi(a_{kmn}) = (\phi \otimes f_z)\Delta(a_{kmn}) = f_z(1)\phi(a_{kmn}) = \phi(a_{kmn}) \tag{5.11}$$

for all $z \in \mathbb{C}$. Similarly by considering $(f_z \otimes \mathrm{id})\Delta(a_{kmn})$ we can show that

$$\phi(a_{kmn}) = q^{z(-k-m+n)}\phi(a_{kmn}) \tag{5.12}$$

for all $z \in \mathbb{C}$. Then, from Equations (5.11) and (5.12), if $\phi(a_{kmn}) \neq 0$ we have

$$q^{z(-k+m-n)} = 1 = q^{z(-k-m+n)}$$

for all $z \in \mathbb{C}$ and so we must have $k-m+n = 0 = k+m-n$ or indeed $k = m-n = n-m$. This is only possible if $k = 0$ and $m = n$.

If $k \neq 0$ or $m \neq n$ then from Lemma 5.1.18 we have

$$\begin{aligned}&((\pi_0(a_{kmn}) \otimes 1)\xi_0|\xi_0) \\ &= \begin{cases} (1-q^2)\sum_{r=k}^{\infty}\sum_{s=0}^{\infty} q^{r+s}q^{r(n+m)}\alpha_r \dots \alpha_{r-(k-1)}\,(e_{r-k,n-m,r}|e_{s,0,s}) & \text{if } k \geqslant 0 \\ (1-q^2)\sum_{r,s=0}^{\infty} q^{r+s}q^{r(n+m)}\alpha_{r+1} \dots \alpha_{r-k}\,(e_{r-k,n-m,r}|e_{s,0,s}) & \text{if } k < 0 \end{cases} \\ &= 0 = \phi(a_{kmn})\end{aligned}$$

and so we need to verify this equation for $k = 0$ and $m = n$.

We have

$$((a_{0nn} \otimes 1)\xi_0|\xi_0) = (1-q^2)\sum_{r=0}^{\infty} q^{2r(1+n)} = \frac{1-q^2}{1-q^{2(1+n)}} \tag{5.13}$$

and we show that $\phi(a_{0nn})$ satisfies this equation also. From Proposition 3.2.18 we have

$$\sigma_{-i}(a) = \sum_{k,l=-1/2}^{1/2} f_1(u^{1/2}_{-1/2,k}) f_1(u^{1/2}_{l,-1/2}) u^{1/2}_{kl} = f_1(a) f_1(a) a = q^{-2} a$$

and then from Proposition 1.4.17 (ii) we have

$$q^{2n+2}\phi(aa^* a_{0nn}) = q^{2n+2}\phi(a^* a_{0nn}\sigma_{-i}(a)) = q^{2n}\phi(a^* a_{0nn} a). \tag{5.14}$$

We have

$$q^{2n+2} aa^* a_{0nn} = q^{2n+2}(1-q^2 c^* c) a_{0nn} = q^{2n+2} a_{0nn} - q^{2n+4} a_{0,n+1,n+1} \tag{5.15}$$

and as $a_{0nn} a = q^{-2n} a a_{0nn}$ from relations (5.1) we have

$$q^{2n} a^* a_{0nn} a = a^* a a_{0nn} = (1-c^* c) a_{0nn} = a_{0nn} - a_{0,n+1,n+1}. \tag{5.16}$$

Applying ϕ to Equations (5.15) and (5.16), subtracting and using (5.14) we get

$$\phi(a_{0nn}) - \phi(a_{0,n+1,n+1}) = q^{2n+2}\phi(a_{0nn}) - q^{2n+4}\phi(a_{0,n+1,n+1})$$

and then rearranging we have

$$\phi(a_{0,n+1,n+1}) = \phi(a_{0nn}) \frac{(1-q^{2(n+1)})}{(1-q^{2(n+2)})}.$$

It then follows easily that $\phi(a_{0nn})$ is equal to the final Equation in (5.13) as required. □

5.2 New Results on $\mathrm{SU}_q(2)$

In the previous section we have described the basic results largely due to Woronowicz. For the rest of this chapter we will move on to discuss new results as obtained by the author.

We know by the Gelfand-Naimark theorem (see Theorem 4.4.3 in Kadison & Ringrose (1997)) that as c is normal there exists a compact space $K \subset \mathbb{C}$ such that the commutative unital C*-algebra $\mathrm{C}^*(c, 1)$ is $*$-isomorphic to $\mathrm{C}(K)$ (the continuous functions on K). Our main result in this section is Proposition 5.2.4 where we calculate this space K. We then find a measure ν that implements the Haar state $\phi \in \mathrm{C}(\mathrm{SU}_q(2))^*$ on $\mathrm{C}(K)$, that is for any $f \in \mathrm{C}(K)$ with corresponding $x \in \mathrm{C}^*(c, 1)$ we have $\phi(x) = \int_K f \, d\nu$. Then we will study the von Neumann algebra $\mathrm{L}^\infty(K, \nu)$ and its predual $\mathrm{L}^1(K, \nu)$ in relation to $\mathrm{L}^\infty(\mathrm{SU}_q(2))$ and $\mathrm{L}^1(\mathrm{SU}_q(2))$ respectively. In particular we show that we have an isometric normal $*$-homomorphism that embeds $\mathrm{L}^\infty(K, \nu)$ in $\mathrm{L}^\infty(\mathrm{SU}_q(2))$ that has a left inverse that is a normal quotient map. Lastly we will calculate the P operator for $\mathrm{SU}_q(2)$ given by Theorem 2.2.10.

In the remaining sections of this chapter we will use the space K to enable a deeper study of $\mathrm{SU}_q(2)$.

5.2.1 The GNS Space $\mathrm{L}^2(\mathrm{SU}_q(2))$

The following Theorem is quoted in Lance (1994) however to the author's knowledge this is not proved anywhere in the literature. We feel this is a non-trivial result and so we offer a proof here.

We remind the reader that we have a GNS representation $(\mathrm{L}^2(\mathrm{SU}_q(2)), \pi_\phi, \xi_\phi)$ where $\phi(x) = (x\xi_\phi | \xi_\phi)$ for all $x \in \mathrm{C}(\mathrm{SU}_q(2))$ (where we omit the π_ϕ map as we will do in this section).

Theorem 5.2.1 *We have an isometric isomorphism of $\mathrm{L}^2(\mathrm{SU}_q(2))$ onto the Hilbert space $\ell^2(\mathbb{N}_0)\otimes\ell^2(\mathbb{Z})\otimes\ell^2(\mathbb{N}_0)$ such that $x\xi_\phi \mapsto (x\otimes 1)\xi_0$ for all $x\in \mathrm{C}(\mathrm{SU}_q(2))$ and where ξ_0 is defined by Equation (5.8).*

Proof

Throughout this proof let $\mathcal{H} = \ell^2(\mathbb{N}_0)\otimes\ell^2(\mathbb{Z})\otimes\ell^2(\mathbb{N}_0)$ and $\psi : \mathrm{L}^2(\mathrm{SU}_q(2)) \to \mathcal{H}$ be the map given by $x\xi_\phi \mapsto (x\otimes 1)\xi_0$ for convenience.

We have already observed that ϕ is realised as a vector state, given by the vector ξ_0 and the representation $\pi_0\otimes 1$. To identify this representation and ξ_0 with the GNS construction for ϕ, we must show that ξ_0 is actually a cyclic vector for $(\pi_0\otimes 1)(\mathrm{C}(\mathrm{SU}_q(2)))$.

We now show that ψ issurjective. We denote by $\mathcal{K}$ the image of $\mathrm{L}^2(\mathrm{SU}_q(2))$ under ψ inside $\mathcal{H}$, that is let

$$\mathcal{K} = \overline{\mathrm{lin}\ \{(a_{kmn}\otimes 1)\xi_0 \mid k\in\mathbb{Z},\ m,n\in\mathbb{N}_0\}}^{\|\cdot\|_2}.$$

Let $\eta\in\mathcal{K}^\perp$ and write $\eta = \sum_{s,u=0}^{\infty}\sum_{t=-\infty}^{\infty}\eta_{s,t,u}e_{s,t,u}$ in terms of the orthonormal basis $\{e_{s,t,u} \mid s,u\in\mathbb{N}_0, t\in\mathbb{Z}\}$ for $\mathcal{H}$. We show that $\eta_{s,t,u} = 0$ for all $s,u\in\mathbb{N}_0$ and $t\in\mathbb{Z}$ meaning $\mathcal{K}^\perp = \{0\}$ and thus $\mathcal{K} = \mathcal{H}$ as required.

Fix $n,m\in\mathbb{N}_0$. As $\eta\in\mathcal{K}^\perp$, from Equation (5.9) in Lemma 5.1.18, for $k\geqslant 0$ we have

$$\begin{aligned} 0 = ((a_{kmn}\otimes 1)\xi_0|\eta) &= \sum_{r=k}^{\infty}\sum_{t=-\infty}^{\infty}\sum_{s,u=0}^{\infty} q^{r(1+n+m)}\alpha_r\ldots\alpha_{r-(k-1)}\eta_{s,t,u}\left(e_{s,t,u}|e_{r-k,n-m,r}\right)\\ &= \sum_{r=k}^{\infty} q^{r(1+n+m)}\alpha_r\ldots\alpha_{r-(k-1)}\eta_{r-k,n-m,r} = \sum_{r=0}^{\infty} q^{(r+k)(1+n+m)}\alpha_{r+k}\ldots\alpha_{r+1}\eta_{r,n-m,r+k}\end{aligned}$$

and so dividing by through $q^{k(1+n+m)} \neq 0$ we have equivalently that

$$0 = \sum_{r=0}^{\infty} q^{r(1+n+m)} \alpha_{r+k} \dots \alpha_{r+1} \eta_{r,n-m,r+k}.$$

We can also use Equation (5.9) for the case $k < 0$ and we have in general that

$$0 = \begin{cases} \sum_{r=0}^{\infty} q^{r(1+n+m)} \alpha_{r+1} \cdots \alpha_{r+k} \eta_{r,n-m,r+k} & \text{if } k \geqslant 0 \\ \sum_{r=0}^{\infty} q^{r(1+n+m)} \alpha_{r+1} \cdots \alpha_{r-k} \eta_{r-k,n-m,r} & \text{if } k < 0. \end{cases} \tag{5.17}$$

Let $p = 1 + n + m$ and $p' = n - m$, that is

$$\begin{pmatrix} p-1 \\ p' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix}$$

or inverting

$$\begin{pmatrix} n \\ m \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} p-1 \\ p' \end{pmatrix}.$$

As $n, m \geqslant 0$ it follows that $p \geqslant 1$ and from the previous equation that $p - 1 + p' \geqslant 0$ and $p - 1 - p' \geqslant 0$ or indeed $1 - p \leqslant p' \leqslant p - 1$. Also as $n, m \in \mathbb{N}_0$ we must have $p - 1 + p' \in 2\mathbb{N}_0$ and $p - 1 - p' \in 2\mathbb{N}_0$. So if p is odd then p' must be even and if p is even then p' must be odd. So we must have

$$\left\{ (p, p') \in \mathbb{N} \times \mathbb{Z} \;\middle|\; \begin{array}{l} 1 - p \leqslant p' \leqslant p - 1 \text{ and} \\ \qquad ((p \text{ is even and } p' \text{ is odd}) \text{ or } (p \text{ is odd and } p' \text{ is even})) \end{array} \right\}.$$

Solving for p in terms of p' we get $1 - p' \leqslant p$ and $p' + 1 \leqslant p$ or indeed $p \geqslant \max\{1 - p', p' + 1\}$.

For $k \geqslant 0$ it follows from Equation (5.17) that

$$\sum_{r=0}^{\infty} q^{rp} \alpha_{r+1} \cdots \alpha_{r+k} \eta_{r,p',r+k} = 0$$

for all $p \in \mathbb{N}$ and $p' \in \mathbb{Z}$ such that $p \geqslant 1$ and $1 - p \leqslant p' \leqslant p - 1$. Say there is some $p' \in \mathbb{Z}$ such that $\eta_{0,p',k} \neq 0$, then for all $p \geqslant \max\{1 - p', p' - 1\}$ we have

$$\eta_{0,p',k} = -\frac{1}{\alpha_1 \cdots \alpha_k} \sum_{r=1}^{\infty} q^{rp} \alpha_{r+1} \cdots \alpha_{r+k} \eta_{r,p',r+k}. \tag{5.18}$$

We have

$$\begin{aligned} \left| \sum_{r=1}^{\infty} q^{rp} \alpha_{r+1} \cdots \alpha_{r+k} \eta_{r,p',r+k} \right| &\leqslant \sum_{r=1}^{\infty} \left| q^{rp} \alpha_{r+1} \cdots \alpha_{r+k} \eta_{r,p',r+k} \right| \leqslant \sum_{r=1}^{\infty} q^{rp} \left| \eta_{r,p',r+k} \right| \\ &\leqslant \left(\sum_{r=1}^{\infty} q^{2rp} \right)^{1/2} \left(\sum_{r=1}^{\infty} \left| \eta_{0,p',k} \right|^2 \right)^{1/2} = \left(\frac{q^{2p}}{1 - q^{2p}} \right)^{1/2} \left(\sum_{r=1}^{\infty} \left| \eta_{0,p',k} \right|^2 \right)^{1/2} \end{aligned}$$

where we've used the Cauchy-Schwarz inequality. We have $\sum_{r=1}^{\infty} \left| \eta_{0,p',k} \right|^2 \leqslant \|\eta\|^2 < \infty$ and so letting $p \to \infty$ (which satisfies $p \geqslant \max\{1 - p', p' - 1\}$) we see that $\dfrac{q^{2p}}{1 - q^{2p}} \to 0$ and thus $\left| \sum_{r=1}^{\infty} q^{rp} \alpha_{r+1} \cdots \alpha_{r+k} \eta_{r,p',r+k} \right| \to 0$ as $p \to \infty$. However from Equation (5.18) we see that $\eta_{0,p',k} = 0$ contradicting the existence of such a p' making this non-zero. So as $k \geqslant 0$ was fixed we have shown that for all $p' \in \mathbb{Z}$ and $k \geqslant 0$ we have $\eta_{0,p',k} = 0$.

Now say for $N \in \mathbb{N}$ we have $\eta_{r,p',r+k} = 0$ for all $0 \leqslant r \leqslant N - 1$ and $p' \in \mathbb{Z}$. Then we have

$$\begin{aligned} 0 &= \sum_{r=N}^{\infty} q^{r(1+p'+2n)} \alpha_{r+1} \cdots \alpha_{r+k} \eta_{r,p',r+k} \\ &= q^{N(1+p'+2n)} \sum_{r=0}^{\infty} q^{r(1+p'+2n)} \alpha_{r+N+1} \cdots \alpha_{r+N+k} \eta_{r+N,p',r+N+k} \end{aligned}$$

and a similar proof from above shows that $\eta_{N,r',N+k}$ and thus we have $\eta_{r,p',r+k} = 0$ for all $r, k \geqslant 0$ and $p' \in \mathbb{Z}$.

A similar proof follows to show that $\eta_{r-k,p',r} = 0$ for all $k < 0$, $r \geqslant 0$ and $p' \in \mathbb{Z}$ or indeed $\eta_{r+k,p',r} = 0$ for all $r, k \geqslant 0$ and $p' \in \mathbb{Z}$. Thus it follows that $\eta_{s,t,u} = 0$ for all $s, u \in \mathbb{N}_0$ and $t \in \mathbb{Z}$ as required. □

The following shows that we can in fact consider $\mathrm{C}(\mathrm{SU}_q(2))$ as acting on $\ell^2(\mathbb{N}_0)\otimes\ell^2(\mathbb{Z})$ given by the representation π_0 in Equation (5.2).

Corollary 5.2.2 *Let $A \subset \mathcal{B}(\ell^2(\mathbb{N}_0)\otimes\ell^2(\mathbb{Z}))$ be the C*-algebra generated by 1, $\pi_0(a)$ and $\pi_0(c)$ where $a, c \in \mathrm{Hopf}(\mathrm{SU}_q(2))$ and π_0 is the representation from Equation (5.2). Then A is isometrically isomorphic to the reduced C*-algebra $\mathrm{C}(\mathrm{SU}_q(2))$.*

Proof
We have by the previous theorem that the GNS space $\mathrm{L}^2(\mathrm{SU}_q(2))$ is unitary equivalent to $\ell^2(\mathbb{N}_0)\otimes\ell^2(\mathbb{Z})\otimes\ell^2(\mathbb{N}_0)$. The map $\mathcal{B}(\ell^2(\mathbb{N}_0)\otimes\ell^2(\mathbb{Z})) \to \mathcal{B}(\ell^2(\mathbb{N}_0)\otimes\ell^2(\mathbb{Z})\otimes\ell^2(\mathbb{N}_0))$ given by $x \mapsto x\otimes 1$ for $x \in \mathcal{B}(\ell^2(\mathbb{N}_0)\otimes\ell^2(\mathbb{Z}))$ is an injective $*$-homomorphism onto its image and thus is an isometry. Then as $\mathrm{C}(\mathrm{SU}_q(2))$ acts on $\ell^2(\mathbb{N}_0)\otimes\ell^2(\mathbb{Z})\otimes\ell^2(\mathbb{N}_0)$ as $\pi_0(x)\otimes 1$ for all $x \in \mathrm{C}(\mathrm{SU}_q(2))$ the restriction of this to the map A is also a complete isometry onto $\mathrm{C}(\mathrm{SU}_q(2))$. □

5.2.2 The C*-algebra $\mathrm{C}(K)$ and the Hilbert space $\mathrm{L}^2(K,\nu)$

We can consider the C*-subalgebra $\mathrm{C}^*(c,1)$ of $\mathrm{C}(\mathrm{SU}_q(2))$ generated by c and 1. As c is normal we have that this forms a unital commutative C*-subalgebra and so it follows from the Gelfand-Naimark theorem that we have some compact space $K \subset \mathbb{C}$ (given by the spectrum of c) such that $\mathrm{C}^*(c,1)$ is $*$-isomorphic to $\mathrm{C}(K)$ where c maps to $\underline{z}$ $(z \mapsto z)$. In this section we show that we have a conditional expectation from $\mathrm{C}(\mathrm{SU}_q(2))$ onto $\mathrm{C}^*(c,1)$ and we explicitly find the compact space $K \subset \mathbb{C}$. We will then move on to consider a measure ν on $\mathrm{C}(K)$ that implements the Haar state equivalent on $\mathrm{C}(K)$ and we study $\mathrm{L}^2(K,\nu)$ further including showing how this embeds in $\mathrm{L}^2(\mathrm{SU}_q(2))$.

Theorem 5.2.3 *There exists a unique conditional expectation (see Definition A.5.4) $P : \mathrm{C}(\mathrm{SU}_q(2)) \to \mathrm{C}^*(c,1)$ such that $P(a_{kmn}) = \delta_{k0}a_{0mn}$ for all $k \in \mathbb{Z}$ and $m, n \in \mathbb{N}_0$ where $a_{kmn} \in \mathrm{Hopf}(\mathbb{G})$ is from Notation 5.1.1. Furthermore we have $\phi \circ P = \phi$ for the Haar state ϕ on $\mathrm{C}(\mathrm{SU}_q(2))$.*

Proof

We define $P_0 : \mathrm{Hopf}(\mathbb{G}) \to \mathrm{C}^*(c, 1)$ as the unique linear map such that $a_{kmn} \mapsto \delta_{k,0} a_{0mn}$ where $\mathrm{Hopf}(\mathbb{G})$ is considered as a normed subspace of $\mathrm{C}(\mathrm{SU}_q(2))$. We show that P_0 is contractive and then P_0 extends uniquely to the conditional expectation P by Theorem A.5.5. As $(\mathrm{C}(\mathrm{SU}_q(2)), \Delta)$ is coamenable we need only consider the reduced C*-algebraic setting and by Corollary 5.2.2 we can consider this as a acting on the Hilbert space $\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{Z})$ by the representation π_0 from Equation (5.2).

Consider the Hilbert space

$$\mathcal{H} = \left\{ (\xi_i)_{i=0}^{\infty} \;\middle|\; \xi_i \in \ell^2(\mathbb{Z}) \;\; \forall\, i \in \mathbb{N}_0, \;\; \sum_{i=0}^{\infty} \|\xi_i\|^2 < \infty \right\} \tag{5.19}$$

with inner product $((\xi_i)_{i=0}^{\infty} | (\eta_i)_{i=0}^{\infty})_{\mathcal{H}} := \sum_{i=0}^{\infty} (\xi_i | \eta_i)_{\ell^2(\mathbb{Z})}$. We have a unique unitary isomorphism $\psi : \ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{Z}) \to \mathcal{H}$ such that

$$\Psi(e_{s,t}) = (\delta_{i,s} e_t)_{i=0}^{\infty} \tag{5.20}$$

for all $s \in \mathbb{N}_0$ and $t \in \mathbb{Z}$, that is Ψ maps $e_{s,t}$ into the vector with 0 in all but the s-th entry where it has entry $e_t \in \ell^2(\mathbb{Z})$. Then we have a $*$-isomorphism $\Psi : \mathcal{B}(\ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{Z})) \to \mathcal{B}(\mathcal{H})$ given by $\Psi(x) = \psi x \psi^{-1}$. We have that $\mathcal{B}(\mathcal{H})$ consists of infinite matrices with entries in $\mathcal{B}(\ell^2(\mathbb{Z}))$. Consider $c \in \mathrm{C}(\mathrm{SU}_q(2))$ first. Let $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the map $e_t \mapsto e_{t+1}$ for all $t \in \mathbb{Z}$, then from Equation (5.2) we have

$$\Psi(c) = \begin{pmatrix} T & 0 & 0 & \cdots \\ 0 & qT & 0 & \cdots \\ 0 & 0 & q^2T & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5.21}$$

with usual adjoint. Similarly for $a \in \mathrm{C}(\mathrm{SU}_q(2))$ we have

$$\Psi(a) = \begin{pmatrix} 0 & \alpha_1 \mathrm{id} & 0 & 0 & \cdots \\ 0 & 0 & \alpha_2 \mathrm{id} & 0 & \cdots \\ 0 & 0 & 0 & \alpha_3 \mathrm{id} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with the usual adjoint. By construction, P_0 acts on the matrices in $\mathcal{B}(\mathcal{H})$ by sending the non-diagonal entries to 0. That is for $x = (x_{ij})_{i,j=0}^{\infty} \in \mathcal{B}(\mathcal{H})$ we have $P_0 x = \mathrm{diag}(x_{ij})$.

From Proposition A.5.3 we have that $\|x_{ii}\| \leqslant \|x\|$ for all $i \in \mathbb{N}_0$ and so $\sup \|x_{ii}\| \leqslant \|x\|$. In Proposition A.5.2 we let $\mathcal{H}_i = \mathcal{K}_i = \ell^2(\mathbb{Z})$ for all $i \in \mathbb{N}_0$ and we have $\|\mathrm{diag}(x_{ij})\| = \sup_{i \in \mathbb{N}_0} \|x_{ii}\| \leqslant \|x\|$. So for $x \in \mathrm{C}(\mathrm{SU}_q(2))$ it follows from Proposition that

$$\|P_0 x\| = \|\mathrm{diag}(x_{ij})\| \leqslant \|x\|$$

and P_0 is contractive.

Finally for $k \in \mathbb{Z}$ and $m, n \in \mathbb{N}_0$ we have

$$(\phi \circ P)(a_{kmn}) = \delta_{k,0} \phi(a_{0mn}) = \phi(a_{kmn})$$

which extends by linearity to all of $\mathrm{Hopf}(\mathrm{SU}_q(2))$. Then as $\mathrm{Hopf}(\mathrm{SU}_q(2))$ is dense in $\mathrm{C}(\mathrm{SU}_q(2))$ we have $\phi \circ P = \phi$ as required. □

We remind that in this chapter we denote $K = \{0\} \cup \left\{ q^r e^{2\pi i \theta} \;\middle|\; r \in \mathbb{N}_0,\ \theta \in [0,1) \right\}$. That is K is a compact subset of $\mathbb{C}$ (as it is closed and bounded) consisting of 0 and every circle in $\mathbb{C}$ of radius q^r for all $r \in \mathbb{N}_0$. We now show as a consequence of this previous theorem that K is the spectrum of $c \in \mathrm{C}(\mathrm{SU}_q(2))$.

Proposition 5.2.4 *We have $\sigma(c) = K$ and consequently we have a $*$-isomorphism Ψ : $\mathrm{C}^*(c, 1) \to \mathrm{C}(K)$ such that $(c^*)^m c^n \mapsto \underline{z}^{*m} \underline{z}^n$.*

Proof

Consider the Hilbert space $\mathcal{H}$ from Equation (5.19) and the $*$-isomorphism Ψ from equation (5.20). We have $\Psi(c)$ is the operator in Equation (5.21) from the proof of the previous theorem and we know from Proposition A.7.4 that we have $\sigma(T) = \mathbb{T}$ (the unit circle in $\mathbb{C}$). So it follows immediately that the spectrum of $\Psi(c)$, and thus the spectrum of c, consists of the closure $\{q^r e^{2\pi i\theta} \mid r \in \mathbb{N}_0,\ \theta \in [0,1)\}$. The rest follows from the Gelfand-Naimark theorem (see Theorem 2.1.13 in Murphy (1990) for example). □

We define a measure ν on K where for all measurable $A \subset K$ we let

$$\nu(A) = (1-q^2)\sum_{r=0}^{\infty} q^{2r}\int_0^1 \chi_A(q^r e^{2\pi i\theta})\,d\theta \tag{5.22}$$

and then for all $f \in \mathrm{C}(K)$ it follows that

$$\int_K f\,d\nu = (1-q^2)\sum_{r=0}^{\infty} q^{2r}\int_0^1 f(q^r e^{2\pi i\theta})\,d\theta.$$

Let $m, n \in \mathbb{N}_0$, then by Proposition 5.1.19 we have

$$\begin{aligned}\int_K \underline{z}^{*m}\,\underline{z}^n\,d\nu &= (1-q^2)\sum_{r=0}^{\infty} q^{2r} q^{r(n+m)}\int_0^1 e^{2\pi i(n-m)\theta}\,d\theta\\ &= (1-q^2)\sum_{r=0}^{\infty} q^{2r(1+n)}\delta_{n,m} = \delta_{n,m}\frac{1-q^2}{1-q^{2(1+n)}} = \phi((c^*)^m c^n)\end{aligned}$$

where we remind that $\underline{z}^n, \underline{z}^{*m} : K \to \mathbb{C}$ are the maps $z \mapsto z^n$ and $z \mapsto \overline{z^m}$ respectively. We also have $\int_K 1\,d\nu = 1$ and so we have a probability measure ν on K. We have from Proposition 5.2.4 that $\mathrm{C}(K) \cong_i \mathrm{C}^*(c, 1)$ with a_{0mn} corresponding to the function $\underline{z}^{*m}\,\underline{z}^n$ and so ν is the measure on K corresponding to the restriction of the Haar state ϕ. We will use $\mathrm{C}(K)$ with measure ν to study $\mathrm{C}^*(c, 1)$ and thus $\mathrm{C}(\mathrm{SU}_q(2))$ further.

We now study $\mathrm{L}^2(K, \nu)$ briefly.

Proposition 5.2.5 *Let* $f : K \to \mathbb{C}$*. Then* $f \in \mathrm{L}^2(K,\nu)$ *if and only if there exists a sequence of functions* $(f_r)_{r=0}^{\infty} \subset \mathrm{L}^2(\mathbb{T})$ *such that* $\sum_{r=0}^{\infty} q^{2r}\|f_r\|^2_{\mathrm{L}^2(\mathbb{T})} < \infty$ *and* $f(q^r e^{2\pi i\theta}) = f_r(e^{2\pi i\theta})$ *for all* $r \in \mathbb{N}_0$ *and* $\theta \in [0,1)$.

Proof

Let $f \in \mathrm{L}^2(K,\nu)$, then we have

$$\|f\|^2_{\mathrm{L}^2(K,\nu)} = \int_0^1 |f|^2 \, d\nu = (1-q^2)\sum_{r=0}^{\infty} q^{2r} \int_0^1 \left|f(q^r e^{2\pi i\theta})\right|^2 \, d\theta < \infty.$$

For $s \in \mathbb{N}_0$ let $f_s : K \to \mathbb{C}$ be the map $f_s(e^{2\pi i\theta}) = f(q^s e^{2\pi i\theta})$ for all $\theta \in [0,1)$ and we have

$$q^{2s}\|f_s\|^2_{\mathrm{L}^2(\mathbb{T})} = q^{2s}\int_0^1 \left|f_s(e^{2\pi i\theta})\right|^2 d\theta \leqslant \sum_{r=0}^{\infty} q^{2r}\int_0^1 \left|f_r(e^{2\pi i\theta})\right|^2 d\theta < \infty$$

and so $f_s \in \mathrm{L}^2(\mathbb{T})$. Then we have $\sum_{r=0}^{\infty} q^{2r}\|f_r\|^2_{\mathrm{L}^2(\mathbb{T})} = \dfrac{1}{1-q^2}\|f\|^2_{\mathrm{L}^2(K,\nu)} < \infty$ as required.

The converse is immediate from considering $\|f\|^2_{\mathrm{L}^2(K,\nu)}$. □

Notation 5.2.6 *Let* $s \in \mathbb{N}_0$ *and* $t \in \mathbb{Z}$*, then we let* $\phi_{s,t} : K \to \mathbb{C}$ *be the function* $q^r e^{2\pi i\theta} \mapsto \dfrac{1}{\sqrt{1-q^2}} q^{-s}\delta_{r,s}e^{2\pi i t\theta}$ *for all* $r \in \mathbb{N}_0$ *and* $\theta \in [0,1)$.

Proposition 5.2.7 *We have an orthonormal basis for* $\mathrm{L}^2(K,\nu)$ *given by the functions*

$$\{\phi_{s,t} \mid s \in \mathbb{N}_0, t \in \mathbb{Z}\}$$

where $\phi_{s,t}$ *is given by Notation 5.2.6 above.*

Proof

We have that $\phi_{s,t}$ is orthonormal as for $s, s' \in \mathbb{N}_0$ and $t, t' \in \mathbb{Z}$ we have

$$(\phi_{s,t}|\phi_{s',t'}) = (1-q^2)\sum_{r=0}^{\infty} q^{2r}\int_0^1 \frac{1}{1-q^2} q^{-s-s'}\delta_{r,s}\delta_{r,s'}e^{2\pi i(t-t')\theta} d\theta = \delta_{s,s'}\delta_{t,t'}.$$

It is sufficient to show that this set spans $\mathrm{L}^2(K,\nu)$. Let $f \in \mathrm{L}^2(K,\nu)$, then by Proposition 5.2.5 we have a sequence of functions $(f_r)_{r=0}^{\infty} \subset \mathrm{L}^2(\mathbb{T})$ such that $f(q^r e^{2\pi i\theta}) = f_r(e^{2\pi i\theta})$ and $\sum_{r=0}^{\infty} q^{2r}\|f\|^2_{\mathrm{L}^2(\mathbb{T})} < \infty$. We have

$$\left\| \sum_{s=0}^{\infty}\sum_{t=0}^{\infty} q^s \widehat{f}_s(t)\phi_{s,t} \right\|^2 = \sum_{s,s'}^{\infty}\sum_{t,t'=0}^{\infty} q^{s+s'}\widehat{f}_s(t)\overline{\widehat{f_{s'}}(t)}\,(\phi_{s,t}|\phi_{s',t'})$$
$$= \sum_{s=0}^{\infty}\sum_{t=-\infty}^{\infty} q^{2s}\left|\widehat{f}_s(t)\right|^2 = \sum_{s=0}^{\infty} q^{2s}\|f_s\|^2_{\mathrm{L}^2(\mathbb{T})} < \infty$$

using Equation (A.1) for the last equality and so we have $\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} q^s \widehat{f}_s(t)\phi_{s,t} \in \mathrm{L}^2(K,\nu)$. We can then calculate that

$$\left\| f - \sqrt{1-q^2}\sum_{s=0}^{\infty}\sum_{t=-\infty}^{\infty} q^s \widehat{f}_s(t)\phi_{s,t} \right\|_{\mathrm{L}^2(K,\nu)} = 0$$

using Hilbert space techniques and therefore the set $\{\phi_{s,t} \mid s \in \mathbb{N}_0, t \in \mathbb{Z}\}$ is an orthonormal spanning set, i.e. a basis. □

We prove in the following that we have an isometric embedding of $\mathrm{L}^2(K,\nu)$ inside $\mathrm{L}^2(\mathrm{SU}_q(2))$ with some further useful properties.

Proposition 5.2.8 *We have an isometric embedding $U : \mathrm{L}^2(K,\nu) \to \mathrm{L}^2(\mathrm{SU}_q(2))$ such that for all $m,n \in \mathbb{N}_0$ we have $U(\underline{z}^{*m}\underline{z}^n) = (c^*)^m c^n \xi_\phi$ (where $\xi_\phi \in \mathrm{L}^2(\mathrm{SU}_q(2))$ is the cyclic vector from the GNS space of the Haar state ϕ). Furthermore the adjoint is a contractive map $U^* : \mathrm{L}^2(\mathrm{SU}_q(2)) \to \mathrm{L}^2(K,\nu)$ such that for Ψ the $*$-isomorphism from Proposition 5.2.4 we have:*

(i) for $x \in \mathrm{C}(\mathrm{SU}_q(2))$ we have $U^*(x\xi_\phi) = (\Psi \circ P)(x) \in \mathrm{C}(K) \subset \mathrm{L}^2(K,\nu)$;

(ii) for $x \in \mathrm{C}^*(c,1)$ and $y \in \mathrm{C}(\mathrm{SU}_q(2))$ we have $U^*(xy\xi_\phi) = \Psi(x)U^*(y\xi_\phi)$;

(iii) for $x \in \mathrm{C}(\mathrm{SU}_q(2))$ and $y \in \mathrm{C}^*(c,1)$ we have $U^*(xy\xi_\phi) = (\Psi \circ P)(x)U^*(y\xi_\phi)$.

Proof

We can define a unique linear map from the polynomials in $\underline{z}$ and $\underline{z^*}$ to $\mathrm{L}^2(\mathrm{SU}_q(2))$ such that $\underline{z}^n\underline{z^*}^m \mapsto (c^*)^m c^n \xi_\phi$ for $m, n \in \mathbb{N}_0$ and ξ_ϕ the cyclic vector for the Haar state ϕ of $\mathrm{C}(\mathrm{SU}_q(2))$. Then for $m, n, m', n' \in \mathbb{N}_0$ and we have

$$\begin{aligned}\left(U(\underline{z^*}^m\underline{z}^n)\middle|U(\underline{z}^{n'}\underline{z^*}^{m'})\right) &= \left((c^*)^m c^n\xi_\phi\middle|(c^*)^{m'}c^{n'}\xi_\phi\right) = \phi(c^{m'}(c^*)^{n'}(c^*)^m c^n)\\ &= \int_K \underline{z}^{m'}\underline{z^*}^{n'}\underline{z^*}^m\underline{z}^n\, d\nu = \left(\underline{z^*}^m\underline{z}^n\middle|\underline{z^*}^{m'}\underline{z}^{n'}\right)\end{aligned}$$

and so we can extend this to an isometric embedding $U : \mathrm{L}^2(K,\nu) \to \mathrm{L}^2(\mathrm{SU}_q(2))$.

We now consider the Hilbert space adjoint $U^* : \mathrm{L}^2(\mathrm{SU}_q(2)) \to \mathrm{L}^2(K,\nu)$. Clearly U^* is contractive as U is an isometry. Furthermore for $k \in \mathbb{Z}$ and $m, n, s, t \in \mathbb{N}_0$ we have

$$\left(U^*(a_{kmn}\xi_\phi)\middle|\underline{z^*}^s\underline{z}^t\right) = (a_{kmn}\xi_\phi|a_{0st}\xi_\phi) = \left(\delta_{k,0}\underline{z^*}^m\underline{z}^n\middle|\underline{z^*}^s\underline{z}^t\right)$$

and so $U^*(a_{kmn}\xi_\phi) = \delta_{k,0}\underline{z^*}^m\underline{z}^n$ from which property (i) follows. Let $x \in \mathrm{C}^*(c,1)$ and $y \in \mathrm{C}(\mathrm{SU}_q(2))$, then as P is a conditional expectation we have from Definition A.5.4 that $P(xy) = xP(y)$ and so

$$U^*(xy\xi_\phi) = (\Psi\circ P)(xy) = \Psi(xP(y)) = \Psi(x)U^*(y\xi_\phi)$$

and we have property (ii). Finally for $x \in \mathrm{C}(\mathrm{SU}_q(2))$ and $y \in \mathrm{C}^*(c,1)$ we have $P(xy) = P(x)y$ and so $U^*(xy\xi_\phi) = (\Psi\circ P)(xy) = \Psi(P(x)y) = (\Psi\circ P)(x)\Psi(y) = (\Psi\circ P)(x)U^*(y\xi_\phi)$ giving (iii). □

5.2.3 The von Neumann algebras $\mathrm{L}^\infty(\mathrm{SU}_q(2))$ and $\mathrm{L}^\infty(K,\nu)$

In the previous section we studied the C*-subalgebra $\mathrm{C}^*(c,1)$ of $\mathrm{C}(\mathrm{SU}_q(2))$ and we showed that $\mathrm{C}^*(c,1) \cong_i \mathrm{C}(K)$. We now consider what this means for the von Neumann algebraic setting, that is we consider relations between the von Neumann algebra $\mathrm{L}^\infty(\mathrm{SU}_q(2))$ (which can be given as the double commutant of $\mathrm{C}(\mathrm{SU}_q(2))$ inside $\mathcal{B}(\mathrm{L}^2(\mathrm{SU}_q(2))))$ and the commutative von Neumann algebra $\mathrm{L}^\infty(K,\nu)$. We also consider $\mathrm{L}^1(\mathrm{SU}_q(2)) = \mathrm{L}^\infty(\mathrm{SU}_q(2))_*$ and $\mathrm{L}^1(K,\nu)$ the integrable functions with respect to the measure ν.

We show in this section that there is a normal, completely isometric embedding of $\mathrm{L}^\infty(K,\nu)$ in $\mathrm{L}^\infty(\mathrm{SU}_q(2))$ with a left inverse given by a normal $*$-map that is also a complete quotient map (where we let $\mathrm{L}^\infty(K,\nu)$ have the operator space structure given by its embedding inside $\mathcal{B}(\mathrm{L}^2(K,\nu))$ and let $\mathrm{L}^1(K,\nu)$ have the predual operator space structure).

For convenience in this section we let $\Theta : \mathrm{C}(K) \to \mathrm{C}(\mathrm{SU}_q(2))$ be the isometric $*$-homomorphism given as the composition of Ψ^{-1} in Proposition 5.2.4 and the isometric embedding of $\mathrm{C}^*(c,1)$ into $\mathrm{C}(\mathrm{SU}_q(2))$.

We note that the following theorem is similar to that of a variety of theorems on Borel functional calculus for normal operators on a Hilbert space. We could not find a theorem in the literature that states this theorem in the form we give here however and so we offer a proof in full that gives details. We note however that the applying $\mathbb{F}$ in the following theorem to bounded Borel functions on K gives the same result as that of the usual Borel functional calculus homomorphism for c.

Theorem 5.2.9 *There exists an isometric normal $*$-homomorphism $\mathbb{F} : \mathrm{L}^\infty(K,\nu) \to \mathrm{L}^\infty(\mathrm{SU}_q(2))$ that is the normal extension of the isometric $*$-homomorphism $\Theta : \mathrm{C}(K) \to \mathrm{C}(\mathrm{SU}_q(2))$ with image $\overline{\mathrm{C}^*(c,1)}^{w^*} \subset \mathrm{L}^\infty(\mathrm{SU}_q(2))$.*

Proof

We define a map $\alpha : \mathrm{L}^1(\mathrm{SU}_q(2)) \to \mathrm{C}(K)^*$ by $\omega \mapsto \omega \circ \Theta$. Clearly as α is a composition of contractions then α is a contraction and so $\alpha(\omega) \in \mathrm{C}(K)^*$. We show that α has image $\mathrm{L}^1(K,\nu)$. Let $y \in \mathrm{C}(\mathrm{SU}_q(2))$ and $z \in \mathrm{Dom}(\sigma_{-i}) \cap \mathrm{C}(\mathrm{SU}_q(2))$, then $\omega_{y\xi_\phi, z\xi_\phi} \in \mathrm{L}^1(\mathrm{SU}_q(2))$. Let $f \in \mathrm{C}(K)$ and $x = \Theta(f) \in \mathrm{C}^*(c,1)$. Using that $\phi \circ P = \phi$ and that P is a conditional expectation from Theorem 5.2.3 and Proposition 1.4.17 (ii) we have

$$\begin{aligned}\left\langle f, \alpha\left(\omega_{y\xi_\phi, z\xi_\phi}\right)\right\rangle &= \langle x, \omega_{y\xi_\phi, z\xi_\phi}\rangle = (xy\xi_\phi | z\xi_\phi) = \phi(z^*xy) = \phi(xy\sigma_{-i}(z^*)) \\ &= \phi(P(xy\sigma_{-i}(z^*))) = \phi(xP(y\sigma_{-i}(z^*))) = \phi(\Theta(f)\Theta(g)) = \int_K fg\, d\nu\end{aligned}$$

where we've set $g = \Psi(P(y\sigma_{-i}(z^*))) \in \mathrm{C}(K) \subset \mathrm{L}^1(K,\nu)$ for Ψ the $*$-isomorphism in Proposition 5.2.4. Then we have shown that

$$\left\langle f, \alpha\left(\omega_{y\xi_\phi, z\xi_\phi}\right)\right\rangle = \langle f, g\rangle$$

for all $f \in \mathrm{C}(K)$ and thus $\alpha\left(\omega_{y\xi_\phi, z\xi_\phi}\right) = g \in \mathrm{L}^1(K,\nu)$.

Now let $y, z \in \mathrm{C}(\mathrm{SU}_q(2))$. By Proposition 1.3.18 we have that $\mathrm{Dom}(\sigma_{-i})$ is dense in $\mathrm{C}(\mathrm{SU}_q(2))$, so we have a net $(z_\alpha) \subset \mathrm{Dom}(\sigma_{-i}) \cap \mathrm{C}(\mathrm{SU}_q(2))$ such that $\lim z_\alpha = z$ and by the above we have $\omega_{y\xi_\phi, z_\alpha\xi_\phi} \in \mathrm{L}^1(\mathrm{SU}_q(2))$ for all α. So it follows that

$$\|\alpha(\omega_{y\xi_\phi, z\xi_\phi}) - \alpha(\omega_{y\xi_\phi, z_\alpha\xi_\phi})\| \leqslant \|\omega_{y\xi_\phi,(z-z_\alpha)\xi_\phi}\| = \|y\xi_\phi\|\|(z-z_\alpha)\xi_\phi\| \to 0$$

and thus $\alpha(\omega_{y\xi_\phi, z\xi_\phi}) \in \mathrm{L}^1(\mathrm{SU}_q(2))$. As $\mathrm{lin}\,\{\omega_{y\xi,z\xi} \mid y, z \in \mathrm{C}(\mathrm{SU}_q(2))\}$ is dense in $\mathrm{L}^1(\mathrm{SU}_q(2))$ and α is contractive we have $\alpha(\omega) \in \mathrm{L}^1(K,\nu)$ for all $\omega \in \mathrm{L}^1(\mathrm{SU}_q(2))$.

We show α is surjective. Let $f \in \mathrm{L}^1(K,\nu)$, then there exists $g, h \in \mathrm{L}^2(K,\nu)$ such that $f = g\overline{h}$ and let $\xi = U(f)$ and $\eta = U(g)$ for $U : \mathrm{L}^2(K,\nu) \to \mathrm{L}^2(\mathrm{SU}_q(2))$ the isometric embedding given in Proposition 5.2.8. Let $F \in \mathrm{C}(K)$ with $x = \Theta(F) \in \mathrm{C}^*(c,1)$ and we have

$$\langle F, \alpha(\omega_{\xi,\eta})\rangle = \langle x, \omega_{\xi,\eta}\rangle = (xUg|Uh) = \langle F, g\overline{h}\rangle = \langle F, f\rangle.$$

As this holds for all $F \in \mathrm{C}(K)$ we have $\alpha(\omega_{\xi,\eta}) = f$.

We let $\mathbb{F}_* : \mathrm{L}^1(\mathrm{SU}_q(2)) \to \mathrm{L}^1(K,\nu)$ denote the corestriction of α to $\mathrm{L}^1(K,\nu)$ and we consider the adjoint $\mathbb{F} : \mathrm{L}^\infty(K,\nu) \to \mathrm{L}^\infty(\mathrm{SU}_q(2))$. We have that $\mathbb{F}$ is contractive as $\mathbb{F}_*$ is contractive. Let $F \in \mathrm{C}(K) \subset \mathrm{L}^\infty(K,\nu)$ then for all $\omega \in \mathrm{L}^1(\mathrm{SU}_q(2))$ we have

$\langle \mathbb{F}(F), \omega\rangle = \langle \Theta(F), \omega\rangle$ and so $\mathbb{F}(F) = \Theta(F)$ and $\mathbb{F}$ is the normal extension of Θ. Then $\mathbb{F}|_{\mathrm{C}(K)}$ is the map Θ and thus is a $*$-homomorphism. In particular from Proposition 5.2.4 we have that $\mathbb{F}(\underline{z}^{*m}\underline{z}^n) = (c^*)^m c^n$.

Clearly we have that Image $\mathbb{F} \subset \overline{\mathrm{C}^*(c,1)}^{w*}$ and so we show that $\mathbb{F}$ is onto $\overline{\mathrm{C}^*(c,1)}^{w*}$. Let $x \in \overline{\mathrm{C}^*(c,1)}^{w*}$, then by the Kaplansky density theorem there exists a bounded net $(x_\alpha) \subset \mathrm{C}^*(c,1)$ such that $x_\alpha \xrightarrow{w*} x$. For all α we let $F_\alpha \in \mathrm{C}(K)$ such that $\mathbb{F}(F_\alpha) = x_\alpha$ which exists as $\mathbb{F}$ restricted to $\mathrm{C}(K)$ and corestricted to $\mathrm{C}^*(c,1)$ is a $*$-isomorphism. Also it follows that $\mathbb{F}$ is an isometry on $\mathrm{C}(K)$ again as $\mathbb{F}$ restricted to $\mathrm{C}(K)$ and corestricted to $\mathrm{C}^*(c,1)$ is a $*$-isomorphism, therefore we have a bounded net $(F_\alpha) \subset \mathrm{C}(K)$. Also as $\mathbb{F}_*$ is surjective for all $f \in \mathrm{L}^1(K,\nu)$ we have some $\omega \in \mathrm{L}^1(\mathrm{SU}_q(2))$ such that $\mathbb{F}_*(\omega) = f$ and so

$$|\langle F_\alpha - F_\beta, f\rangle| = |\langle \mathbb{F}(F_\alpha - F_\beta), \omega\rangle| = |\langle x_\alpha - x_\beta, \omega\rangle| \to 0.$$

So (F_α) is a weak* Cauchy net and we have a unique $F \in \mathrm{L}^\infty(K,\nu)$ such that $F_\alpha \xrightarrow{w*} F$. Using the triangle inequality and that $\mathbb{F}$ is a $*$-map on $\mathrm{C}(K)$, for all $\omega \in \mathrm{L}^1(\mathrm{SU}_q(2))$ we have

$$|\langle x, \omega\rangle - \langle \mathbb{F}(F), \omega\rangle| \leqslant |\langle x - x_\alpha, \omega\rangle| + |\langle F_\alpha - F, \mathbb{F}_*(\omega)\rangle| \to 0.$$

and so $\mathbb{F}(F) = x$.

Finally we show that $\mathbb{F}$ is a $*$-homomorphism. Let $F \in \mathrm{L}^\infty(K,\nu)$ and let $(F_\alpha) \subset \mathrm{C}(K)$ such that $F_\alpha \xrightarrow{w*} F$. Then for all $\omega \in \mathrm{L}^1(\mathrm{SU}_q(2))$ we have

$$\begin{aligned}|\langle \mathbb{F}(F^*), \omega\rangle - \langle \mathbb{F}(F)^*, \omega\rangle| &\leqslant |\langle \mathbb{F}(F^* - F_\alpha^*), \omega\rangle| + |\langle \mathbb{F}(F_\alpha - F)^*, \omega\rangle| \\ &= |\langle F - F_\alpha, \mathbb{F}_*(\omega)^*\rangle| + |\langle F_\alpha - F, \mathbb{F}_*(\omega^*)\rangle| \to 0\end{aligned}$$

and so $\mathbb{F}$ is a $*$-map.

Now let $F \in \mathrm{L}^\infty(K,\nu)$ and $G \in \mathrm{C}(K)$ and using that $\mathbb{F}$ is a $*$-homomorphism on

$\mathrm{C}(K)$ we have

$$\begin{aligned}|\langle \mathbb{F}(FG),\omega\rangle - \langle \mathbb{F}(F)\mathbb{F}(G),\omega\rangle| \\ &\leqslant |\langle \mathbb{F}(FG) - \mathbb{F}(F_\alpha G),\omega\rangle| + |\langle \mathbb{F}(F_\alpha)\mathbb{F}(G),\omega\rangle - \langle \mathbb{F}(F)\mathbb{F}(G),\omega\rangle| \\ &= |\langle F - F_\alpha, G\cdot \mathbb{F}_*(\omega)\rangle| + |\langle F_\alpha - F, \mathbb{F}_*(\mathbb{F}(G)\omega)\rangle| \to 0\end{aligned}$$

and so $\mathbb{F}(FG) = \mathbb{F}(F)\mathbb{F}(G)$. Similarly we have $\mathbb{F}(GF) = \mathbb{F}(G)\mathbb{F}(F)$. Now say $G \in \mathrm{L}^\infty(K,\nu)$ and let $(G_\alpha) \subset \mathrm{C}(K)$ such that $G_\alpha \xrightarrow{w^*} G$, then using the results above we have

$$\begin{aligned}|\langle \mathbb{F}(FG) - \mathbb{F}(F)\mathbb{F}(G),\omega\rangle| \\ &\leqslant |\langle \mathbb{F}(FG) - \mathbb{F}(FG_\alpha),\omega\rangle| + |\langle \mathbb{F}(F)\mathbb{F}(G_\alpha) - \mathbb{F}(F)\mathbb{F}(G),\omega\rangle| \\ &= |\langle G - G_\alpha, \mathbb{F}_*(\omega)\cdot F\rangle| + |\langle G_\alpha - G, \mathbb{F}_*(\omega\cdot \mathbb{F}(F))\rangle| \to 0\end{aligned}$$

and so $\mathbb{F}$ is a homomorphism as required. □

Theorem 5.2.10 *There exists a normal quotient map $\mathbb{E} : \mathrm{L}^\infty(\mathrm{SU}_q(2)) \to \mathrm{L}^\infty(K,\nu)$ that is a $*$-map where for all $f, g \in \mathrm{L}^2(K,\nu)$ and $x \in \mathrm{L}^\infty(\mathrm{SU}_q(2))$ we have*

$$(\mathbb{E}(x)f|g) = (xU(f)|U(g))$$

(where $U : \mathrm{L}^2(K,\nu) \to \mathrm{L}^2(\mathrm{SU}_q(2))$ is the embedding from Proposition 5.2.8) and where $\mathbb{E}$ is a left inverse to $\mathbb{F}$. Additionally we have

(i) a normal extension $\mathbb{P} := \mathbb{F}\circ\mathbb{E}$ of the conditional expectation $P : \mathrm{C}(\mathrm{SU}_q(2)) \to \mathrm{C}^*(c,1)$ given in Theorem 5.2.3 where $\mathbb{P}(x) = x$ if and only if $x \in \overline{\mathrm{C}^*(c,1)}^{w^*}$;

(ii) $\mathbb{E}(x) \in \mathrm{C}(K)$ for all $x \in \mathrm{C}(\mathrm{SU}_q(2))$;

(iii) $\mathbb{E}$ is a $*$-homomorphism when restricted to $\overline{\mathrm{C}^*(c,1)}^{w^*}$.

Proof

We define $\mathbb{E} : \mathrm{L}^\infty(\mathrm{SU}_q(2)) \to \mathcal{B}(\mathrm{L}^2(K,\nu))$ by the $\mathbb{E}(x) = U^* x U$. Then clearly $\mathbb{E}$ is contractive and satisfies the formula given in the theorem. Let $(x_\alpha) \subset \mathrm{L}^\infty(\mathrm{SU}_q(2))$ with a σ-weak limit $x \in \mathrm{L}^\infty(\mathrm{SU}_q(2))$. Then in particular for any $(\xi_i)_{i=1}^\infty, (\eta_i)_{i=1}^\infty \subset \mathrm{L}^2(\mathrm{SU}_q(2))$ such that $\sum_{i=1}^\infty \|\xi_i\|^2, \sum_{i=1}^\infty \|\eta_i\|^2 < \infty$ we have $\left|\sum_{i=1}^\infty (([x - x_\alpha]\xi_i|\eta_i)\right| \to 0$. Let $(f_i)_{i=1}^\infty, (g_i)_{i=1}^\infty \subset \mathrm{L}^2(K,\nu)$ such that $\sum_{i=1}^\infty \|f_i\|^2 < \infty$ and $\sum_{i=1}^\infty \|g_i\|^2 < \infty$, then we have

$$\left|\sum_{i=1}^\infty ([\mathbb{E}(x) - \mathbb{E}(x_\alpha)] f_i|g_i)\right| = \left|\sum_{i=1}^\infty ([x - x_\alpha]U(f_i)|U(g_i))\right|.$$

As U is an isometry then we have $\sum_{i=1}^\infty \|U(f_i)\|^2 = \sum_{i=1}^\infty \|f_i\|^2 < \infty$ and similarly $\sum_{i=1}^\infty \|U(g_i)\|^2 < \infty$ and so this equation tends to 0 and $\mathbb{E}$ is a normal map. For $g, h \in \mathrm{L}^2(K,\nu)$ and $x \in \mathrm{L}^\infty(\mathrm{SU}_q(2))$ we have

$$(\mathbb{E}(x^*)g|h) = (x^*U(g)|U(h)) = \overline{(xU(h)|U(g))} = \overline{(\mathbb{E}(x)h|g)} = (\mathbb{E}(x)^*g|h)$$

and so we have shown that we have a normal contractive $*$-map $\mathbb{E} : \mathrm{L}^\infty(\mathrm{SU}_q(2)) \to \mathcal{B}(\mathrm{L}^2(K,\nu))$.

Let $x \in \mathrm{C}(\mathrm{SU}_q(2)) \subset \mathrm{L}^\infty(\mathrm{SU}_q(2))$ and $g, h \in \mathrm{L}^2(\mathrm{SU}_q(2))$, then letting $y = U(g)$ in Proposition 5.2.8 we have

$$\begin{aligned}(\mathbb{E}(x)g|h) &= (xU(g)|U(h)) = (U^*(xU(g))|h) \\ &= ((\Psi \circ P)(x)U^*U(g)|h) = ((\Psi \circ P)(x)g|h)\end{aligned}$$

where we've used the $*$-isomorphism Ψ from Proposition 5.2.4 and that U is unitary and so $U^*U = \mathrm{id}$. So $\mathbb{E}(x) = (\Psi \circ P)(x)$ for all $x \in \mathrm{C}(\mathrm{SU}_q(2))$.

Now we show that $\mathbb{E}$ has image inside $\mathrm{L}^\infty(K,\nu)$. We have that $\mathrm{C}(\mathrm{SU}_q(2))$ is σ-weakly dense inside $\mathrm{L}^\infty(\mathrm{SU}_q(2))$ and similarly $\mathrm{C}(K)$ is σ-weakly dense inside $\mathrm{L}^\infty(K,\nu)$, so we have a commutative diagram as follows

$$\begin{array}{ccccc} \mathrm{L}^\infty(\mathrm{SU}_q(2)) & & & & \mathrm{L}^\infty(K,\nu) \overset{\mathbb{E}}{\hookrightarrow} \mathcal{B}(\mathrm{L}^2(K,\nu)) \\ \big\uparrow & & & & \big\uparrow \\ \mathrm{C}(\mathrm{SU}_q(2)) & \xrightarrow[P]{} & \mathrm{C}^*(c,1) & \xrightarrow[\Psi]{} & \mathrm{C}(K). \end{array}$$

As $\mathbb{E}$ is normal and restricts to $\Psi \circ P$ on $\mathrm{C}(\mathrm{SU}_q(2))$ it follows that $\mathbb{E}$ has image inside $\mathrm{L}^\infty(K,\nu)$. We redefine $\mathbb{E}$ to be the map $\mathbb{E} : \mathrm{L}^\infty(\mathrm{SU}_q(2)) \to \mathrm{L}^\infty(K,\nu)$ and it follows from above that $\mathbb{E}$ is a normal contractive $*$-map with $\mathbb{E}(x) = (\Psi \circ P)(x)$ for all $x \in \mathrm{C}(\mathrm{SU}_q(2))$.

As $\mathbb{F}$ and $\mathbb{E}$ are normal we can define a normal map $\mathbb{P} = \mathbb{F} \circ \mathbb{E}$. We have shown above that $\mathbb{P}$ is the normal extension of P, that is we have $\mathbb{P}(x) = P(x)$ for all $x \in \mathrm{C}(\mathrm{SU}_q(2))$. Now say $x \in \mathrm{C}^*(c,1)$ then we have $(\mathbb{F} \circ \Psi)(x) = x$ and so $\mathbb{P}(x) = (\mathbb{F} \circ \Psi \circ P)(x) = x$.

We show that $\mathbb{E}$ is a left inverse of $\mathbb{F}$. For $g, h \in \mathrm{L}^2(K,\nu)$ and $m, n \in \mathbb{N}_0$ we have

$$\begin{aligned} \langle (\mathbb{F}_* \circ \mathbb{E}_*)(\omega_{g,h}), \underline{z}^{*m}\underline{z}^n \rangle &= \langle \omega_{g,h}, \mathbb{E}((c^*)^m c^n) \rangle = ((c^*)^m c^n U(g) | U(h)) \\ &= (U^*((c^*)^m c^n U(g)) | h) = (\underline{z}^{*m}\underline{z}^n U^*Ug | h) = \langle \omega_{g,h}, \underline{z}^{*m}\underline{z}^n \rangle \end{aligned}$$

where we've used Proposition 5.2.8 (iii) and that $U^*U = \mathrm{id}$ again. It follows that for all $F \in \mathrm{C}(K)$ we have $\langle (\mathbb{F}_* \circ \mathbb{E}_*)(\omega_{g,h}), F \rangle = \langle \omega_{g,h}, F \rangle$ and, as any element of $\mathrm{L}^1(K,\nu)$ can be written as a product of two elements of $\mathrm{L}^2(K,\nu)$, we have $\mathbb{F}_* \circ \mathbb{E}_* = \mathrm{id}_{\mathrm{L}^1(K,\nu)}$. Then taking the adjoint of this we see that $\mathbb{E} \circ \mathbb{F} = \mathrm{id}_{\mathrm{L}^\infty(K,\nu)}$. From this and using that $\mathbb{F}$ and $\mathbb{E}$ are contractions and the Banach space version of Lemma 1.1.20 we have that $\mathbb{E}$ is a quotient map.

We show that for $x \in \mathrm{L}^\infty(\mathrm{SU}_q(2))$ we have $\mathbb{P}(x) = x$ if and only if $x \in \overline{\mathrm{C}^*(c,1)}^{w^*}$. Say $\mathbb{P}(x) = x$, then as $\mathbb{F}$ has image $\overline{\mathrm{C}^*(c,1)}^{w^*}$ we must have $x \in \overline{\mathrm{C}^*(c,1)}^{w^*}$. Conversely let $x \in \overline{\mathrm{C}^*(c,1)}^{w^*}$, then there is a net $(x_\alpha) \subset \mathrm{C}^*(c,1)$ such that $x_\alpha \xrightarrow{w^*} x$. Then for all $\omega \in \mathrm{L}^1(\mathrm{SU}_q(2))$ we have

$$\langle \mathbb{P}(x) - x, \omega \rangle = \langle \mathbb{P}(x) - \mathbb{P}(x_\alpha), \omega \rangle + \langle x_\alpha - x, \omega \rangle = \langle x - x_\alpha, \mathbb{P}_*(\omega) + \omega) \rangle \to 0$$

and so $\mathbb{P}(x) = x$.

Finally it is clear that $\mathbb{E}$ is a $*$-homomorphism on $\overline{\mathrm{C}^*(c,1)}^{w*}$ as $\mathbb{P} = \mathbb{F} \circ \mathbb{E}$ is clearly a $*$-homomorphism on $\overline{\mathrm{C}^*(c,1)}^{w*}$ and $\mathbb{F}$ is a $*$-homomorphism. □

We have natural operator space structures on $\mathrm{C}(K)$ and $\mathrm{L}^\infty(K,\nu)$ as subspaces of $\mathcal{B}(\mathrm{L}^2(K,\nu))$ and thus we have an operator space on the predual $\mathrm{L}^1(K,\nu)$ of $\mathrm{L}^\infty(K,\nu)$. Then with these operator space structures we have the following. We note that this gives us the minimal operator space structure on $\mathrm{C}(K)$ and $\mathrm{L}^\infty(K,\nu)$ and the maximal operator space structure on $\mathrm{L}^1(K,\nu)$.

Proposition 5.2.11 *The map $\mathbb{E}$ is a complete quotient map and $\mathbb{F}$ is a complete isometry.*

Proof

Using Stinespring's theorem we have $\|\mathbb{E}\|_{cb} = \|U\|\|U^*\| \leqslant 1$ and so $\mathbb{E}$ is a complete contraction. We also have that $\mathbb{F}$ is a $*$-homomorphism and so this is a complete contraction. It follows that $\mathbb{E}_*$ and $\mathbb{F}_*$ are also complete contractions. By Lemma 1.1.20 and using that $\mathbb{F}_* \circ \mathbb{E}_* = \mathrm{id}_{\mathrm{L}^1(K,\nu)}$ it follows that $\mathbb{E}_*$ is a complete isometry with $\mathbb{F}_*$ a complete quotient map. The result then follows from Proposition 1.1.19. □

We finish this section by proving a decomposition theorem for $\mathrm{L}^1(K,\nu)$ that will be useful later.

Proposition 5.2.12 *Let $f : K \to \mathbb{C}$, then we have $f \in \mathrm{L}^1(K,\nu)$ if and only if there exists a sequence of functions $(f_r)_{r=0}^\infty \subset \mathrm{L}^1(\mathbb{T})$ such that $\sum_{r=0}^\infty q^{2r}\|f_r\|_1$ is finite with $f(q^r e^{2\pi i\theta}) = f_r(e^{2\pi i\theta})$.*

Proof

Say we have a sequence $(f_r)_{r=0}^\infty \subset \mathrm{L}^1(\mathbb{T})$ such that $\sum_{r=0}^\infty q^{2r}\|f_r\|_1$ is finite and let f be the function $f(q^r e^{2\pi i\theta}) = f_r(e^{2\pi i\theta})$ for $r \in \mathbb{N}_0$ and $\theta \in [0,1)$ and $f(0)$ arbitrary. Then

$$\|f\|_{\mathrm{L}^1(K,\nu)} = (1-q^2)\sum_{r=0}^\infty q^{2r}\int_0^1 \left|f(q^r e^{2\pi i\theta})\right| d\theta = (1-q^2)\sum_{r=0}^\infty q^{2r}\|f_r\|_1 < \infty$$

and so $f \in \mathrm{L}^1(K, \nu)$.

Conversely, say $f \in \mathrm{L}^1(K, \nu)$ and for all $r \in \mathbb{N}_0$ let $f_r : \mathbb{T} \to \mathbb{C}$ be the map $e^{2\pi i\theta} \mapsto f(q^r e^{2\pi i\theta})$ for $\theta \in [0, 1)$. Then

$$(1-q^2)\sum_{r=0}^{\infty} q^{2r}\|f_r\|_1 = (1-q^2)\sum_{r=0}^{\infty} q^{2r}\int_0^1 \left|f(q^r e^{2\pi i\theta})\right| d\theta = \|f\|_{\mathrm{L}^1(K,\nu)}$$

which is finite as $f \in \mathrm{L}^1(K, \nu)$ and so we must have $\|f_r\|_1$ finite for all $r \in \mathbb{N}_0$ and therefore $(f_r)_{r=0}^{\infty} \subset \mathrm{L}^1(\mathbb{T})$. □

5.2.4 The P Operator for $\mathrm{SU}_q(2)$

In this section we show that we have an explicit formula for the unbounded operator $P : \mathrm{L}^2(\mathrm{SU}_q(2)) \to \mathrm{L}^2(\mathrm{SU}_q(2))$ from Proposition 2.2.10 for the case of $\mathrm{SU}_q(2)$. Note that this is not used for the remainder of the thesis and is given for interest only.

As $\mathrm{SU}_q(2)$ is compact it follows immediately that $\nu = 1$ for ν the scaling constant. We show that we have the following formula for P.

Proposition 5.2.13 *We have $\Lambda(\mathrm{Hopf}(\mathrm{SU}_q(2))) \subset \mathrm{Dom}(P)$ and $P(e_{k,l,m}) = q^{2l}e_{k,l,m}$ for $k, m \in \mathbb{N}_0$ and $l \in \mathbb{Z}$.*

Proof

Let $Q : \mathrm{L}^2(\mathrm{SU}_q(2)) \to \mathrm{L}^2(\mathrm{SU}_q(2))$ be the map $e_{k,l,m} \mapsto q^{2l}e_{k,l,m}$ for $k, m \in \mathbb{N}_0$ and $l \in \mathbb{Z}$, i.e. let Q be the map given in the proposition. It follows easily that Q is positive, injective, self-adjoint and unbounded. It follows from the theory of unbounded operators (see Strătilă *et al.* (1979) Chapter 9 and Conway (1990) Chapter 10) that Q^{it} is a unitary operator and $Q^{it}e_{k,l,m} = q^{2ilt}e_{k,l,m}$ for all $k, m \in \mathbb{N}_0$ and $l \in \mathbb{Z}$.

Let $k \in \mathbb{Z}$, $m, n \in \mathbb{N}_0$ and $t \in \mathbb{R}$, then we show that $P^{it}\Lambda(a_{kmn}) = \tau_t(\Lambda(a_{kmn}))$. It follows from Corollary 5.1.9 that $\Lambda(\tau_t(a_{kmn})) = q^{2it(n-m)}\Lambda(a_{kmn})$. Also from Equation (5.8) and Theorem 5.2.1 we have GNS space $\mathrm{L}^2(\mathrm{SU}_q(2)) = \ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}_0)$. We calculate from Lemma 5.1.18 for $k \geqslant 0$ that

$$\Lambda(a_{kmn}) = (a_{kmn} \otimes 1)\xi_0 = (1-q^2)^{1/2} \sum_{p=0}^{\infty} q^p (a_{kmn} \otimes 1) e_{p,0,p}$$
$$= \begin{cases} (1-q^2) \sum_{p=k}^{\infty} q^{p(1+n+m)} \alpha_p \dots \alpha_{p-(k-1)} e_{p-k,n-m,p} & \text{if } k \geqslant 0 \\ (1-q^2) \sum_{p=0}^{\infty} q^{p(1+n+m)} \alpha_{p+1} \dots \alpha_{p-k} e_{p-k,n-m,p} & \text{if } k < 0 \end{cases}$$

and so

$$P^{it}\Lambda(a_{kmn}) = q^{2it(n-m)}\Lambda(a_{kmn}) = \Lambda(\tau_t(a_{kmn})).$$

Clearly we can extend this linearly to all of $\mathrm{Hopf}(\mathrm{SU}_q(2))$ and the result then follows from Proposition 2.2.10. □

5.3 $\mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$

In this section we study the $\mathrm{L}^1_\sharp$-algebra from Chapter 4 of the quantum group $\mathrm{SU}_q(2)$. Consider the antipode S on $\mathrm{C}(\mathrm{SU}_q(2))$ with domain $\mathrm{Dom}(S)$ where $\mathrm{Hopf}(\mathrm{SU}_q(2)) \subset \mathrm{Dom}(S) \subset \mathrm{C}(\mathrm{SU}_q(2)) \subset \mathrm{L}^\infty(\mathrm{SU}_q(2))$ with $S(a) = a^*$, $S(a^*) = a$, $S(c) = -qc$ and $S(-qc^*) = c^*$ for the generators of $\mathrm{C}(\mathrm{SU}_q(2))$. As S is an anti-homomorphism we have that $S((c^*)^m) = \left(-\frac{1}{q}\right)^m (c^*)^m$ and so we see that $\|S((c^*)^m)\| \to \infty$ as $m \to \infty$ and S is unbounded on $\mathrm{C}(\mathrm{SU}_q(2))$. As a result of this unboundedness we will see that $\mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ is a proper subalgebra of $\mathrm{L}^1(\mathrm{SU}_q(2))$. Also as the unboundedness of S seems to be largely dependent on the c generator then it is worth to study the effects of this on $\mathrm{C}^*(c,1) \cong_{ci} \mathrm{C}(K)$. We now turn to the study of this.

5.3.1 The Antipode of $\mathrm{C}^*(c,1)$ and $\mathrm{C}(K)$

First we consider what happens with S acting on $\mathrm{Dom}(S) \cap \mathrm{C}^*(c,1)$. We show in this section that for $x \in \mathrm{Dom}(S) \cap \mathrm{C}^*(c,1)$ that we have $S(x) \in \mathrm{C}^*(c,1)$.

Lemma 5.3.1 *We have $\tau_t \circ \mathbb{P} = \mathbb{P} \circ \tau_t$ for all $t \in \mathbb{R}$. Then it follows that there is a σ-weakly continuous one-parameter group τ^c on $\overline{\mathrm{C}^*(c,1)}^{w*}$ such that for all $t \in \mathbb{R}$ we have that τ_t^c is the restriction of τ_t. Furthermore we can restrict this again to a norm continuous one-parameter group on $\mathrm{C}^*(c,1)$.*

Proof

By Corollary 5.1.9 we have $\tau_t(a) = a$ and $\tau_t(c) = q^{2it}c$ for all $t \in \mathbb{R}$, so using that τ_t is a $*$-homomorphism for all $t \in \mathbb{R}$ we have

$$(P \circ \tau_t)(a_{kmn}) = q^{2it(n-m)} P(a_{kmn}) = q^{2it(n-m)} \delta_{k,0} (c^*)^m c^n = (\tau_t \circ P)(a_{kmn}).$$

By linearity and continuity it follows that $\tau_t \circ P = P \circ \tau_t$ on $\mathrm{C}(\mathrm{SU}_q(2))$ for all $t \in \mathbb{R}$.

Now let $x \in \mathrm{L}^\infty(\mathrm{SU}_q(2))$, then we have a net $(x_\alpha) \subset \mathrm{C}(\mathrm{SU}_q(2))$ that has σ-weak limit x. Then using that τ_t is normal for all $t \in \mathbb{R}$ and $\mathbb{P}$ is normal from Theorems 5.2.9 and 5.2.10 it follows that for all $\omega \in \mathrm{L}^1(\mathrm{SU}_q(2))$ we have

$$\begin{aligned}
&|\langle (\mathbb{P} \circ \tau_t)(x), \omega\rangle - \langle (\tau_t \circ \mathbb{P})(x), \omega\rangle| \\
&\qquad \leqslant |\langle (\mathbb{P} \circ \tau_t)(x) - (\mathbb{P} \circ \tau_t)(x_\alpha), \omega\rangle| + |\langle (\tau_t \circ \mathbb{P})(x_\alpha) - (\tau_t \circ \mathbb{P})(x), \omega\rangle| \\
&\qquad = |\langle x - x_\alpha, (\mathbb{P} \circ \tau_t)_*(\omega)\rangle| + |\langle x - x_\alpha, (\tau_t \circ \mathbb{P})_*(\omega)\rangle| \to 0
\end{aligned}$$

where we've used that $\mathbb{P}(y) = P(y)$ for all $y \in \mathrm{C}(\mathrm{SU}_q(2))$. So we have $\tau_t \circ \mathbb{P} = \mathbb{P} \circ \tau_t$ for all $t \in \mathbb{R}$.

By Theorem 5.2.10, for $x \in \overline{\mathrm{C}^*(c,1)}^{w*}$ we have $\mathbb{P}(x) = x$, then from above we have $\tau_t(x) = \mathbb{P}(\tau_t(x))$ and so $\tau_t(x) \in \overline{\mathrm{C}^*(c,1)}^{w*}$. Then we can define a restriction and corestriction $\tau_t^c : \overline{\mathrm{C}^*(c,1)}^{w*} \to \overline{\mathrm{C}^*(c,1)}^{w*}$. As for all $t \in \mathbb{R}$ we have τ_t^c is the restriction of τ_t then for all $x \in \overline{\mathrm{C}^*(c,1)}^{w*}$ the map $t \mapsto \tau_t^c(x)$ from $\mathbb{R}$ to is $\overline{\mathrm{C}^*(c,1)}^{w*}$ is continuous with respect to the weak*-topology on its codomain, that is τ^c is a weak*-continuous one-parameter group. The proof for the norm-continuous one-parameter group on $\mathrm{C}^*(c,1)$ follows similarly. □

Proposition 5.3.2 *Fix $z \in \mathbb{C}$ and let τ be the σ-weakly continuous scaling group on $\mathrm{L}^\infty(\mathrm{SU}_q(2))$ given by Definition-Theorem 2.2.7. Then for $x \in \mathrm{Dom}(\tau_z)$ we have $\mathbb{P}(x) \in \mathrm{Dom}(\tau_z)$ with $\tau_z(\mathbb{P}(x)) = \mathbb{P}(\tau_z(x))$ and in particular if $x \in \overline{\mathrm{C}^*(c,1)}^{w*} \cap \mathrm{Dom}(\tau_z)$ then $\tau_z(x) \in \overline{\mathrm{C}^*(c,1)}^{w*}$.*

Similarly let τ be the norm continuous scaling group on $\mathrm{C}(\mathrm{SU}_q(2))$. Then for $x \in \mathrm{C}(\mathrm{SU}_q(2)) \cap \mathrm{Dom}(\tau_z)$ we have $P(x) \in \mathrm{C}^*(c,1) \cap \mathrm{Dom}(\tau_z)$ with $\tau_z(P(x)) = P(\tau_z(x))$ and in particular if $x \in \mathrm{C}^*(c,1) \cap \mathrm{Dom}(\tau_z)$ then $\tau_z(x) \in \mathrm{C}^*(c,1)$.

Proof

First let τ be the σ-weakly continuous scaling group on $\mathrm{L}^\infty(\mathrm{SU}_q(2))$ and fix $x \in \mathrm{Dom}(\tau_z) \subset \mathrm{L}^\infty(\mathrm{SU}_q(2))$. Then there is a unique $F : S(z) \to \mathrm{L}^\infty(\mathrm{SU}_q(2))$ such that F is continuous with respect to the σ-weak topology on $\mathrm{L}^\infty(\mathrm{SU}_q(2))$, analytic on $S(z)^o$ and $F(t) = \tau_t(x)$ for all $t \in \mathbb{R}$. Define a map $G : S(z) \to \overline{\mathrm{C}^*(c,1)}^{w*}$ by $G(w) = \mathbb{P}(F(w))$ for all $w \in S(z)$. We show that G is continuous with respect to the σ-weak topology on $\overline{\mathrm{C}^*(c,1)}^{w*}$, analytic on $S(z)^o$ and $G(t) = \tau_t(\mathbb{P}(x))$.

Clearly G is continuous as it is the composition of continuous functions. Fix $w_0 \in S(z)^o$. As F is analytic we have some sufficiently small $\delta > 0$ that is not bigger than the radius of convergence and a sequence $(x_n)_{n=0}^\infty \subset \mathrm{L}^\infty(\mathrm{SU}_q(2))$ such that we have a norm convergent sum

$$F(w) = \sum_{n=0}^{\infty}(w-w_0)^n x_n$$

for all $w \in S(z)^o$ such that $|w - w_0| < \delta$. As δ is less than or equal to the radius of convergence of F we have $\sum_{n=0}^\infty |w-w_0|^n \|x_n\| < \infty$. Consider the sequence $(y_n)_{n=0}^\infty$ where $y_n = \sum_{i=0}^n (w-w_0)^i \mathbb{P}(x_i)$ for all $n \in \mathbb{N}_0$. Then for all $n \in \mathbb{N}$ we have

$$\begin{aligned}\left\| G(w) - \sum_{i=0}^{n}(w-w_0)^i \mathbb{P}(x_i) \right\| &= \left\| \mathbb{P}\left(F(w) - \sum_{i=0}^{n}(w-w_0)^i x_i \right) \right\| \\ &\leqslant \left\| F(w) - \sum_{i=0}^{n}(w-w_0)^i x_i \right\|\end{aligned}$$

which converges to 0 as $n \to \infty$. So it follows that $G(z) = \sum_{n=0}^{\infty}(w - w_0)^n \mathbb{P}(x_n)$ converges in the norm for $w \in S(z)^o$ with $|w - w_0| < \delta$ where δ is less than the radius of convergence of F and thus G is analytic on $S(z)^o$. Lastly, from Lemma 5.3.1 we have $G(t) = \mathbb{P}(F(t)) = \mathbb{P}(\tau_t(x)) = \tau_t(\mathbb{P}(x))$. It then follows that $\mathbb{P}(x) \in \mathrm{Dom}(\tau_z)$ and $\tau_z(\mathbb{P}(x)) = G(z) = \mathbb{P}(F(z)) = \mathbb{P}(\tau_z(x))$.

The C*-algebra case follows similarly but by considering the function $G : S(z) \to \mathrm{C}^*(c, 1)$ given by $G(w) = P(F(w))$ for all $w \in S(z)$ and using the norm topology on $\mathrm{C}(\mathrm{SU}_q(2))$ and $\mathrm{C}^*(c, 1)$. □

Proposition 5.3.3 *We have* $\mathbb{P} \circ R = R \circ \mathbb{P}$.

Proof

We show that $P \circ R = R \circ P$ on $\mathrm{C}(\mathrm{SU}_q(2))$ and then $\mathbb{P} \circ R = R \circ \mathbb{P}$ on $\mathrm{L}^\infty(\mathrm{SU}_q(2))$. Assume for now that $0 < q \leqslant 1$, then from Corollary 5.1.9 we have $R(c) = -c$ and $R(a) = a^*$. So as R is a $*$-anti-homomorphism, for $k \geqslant 0$ and $m, n \in \mathbb{N}_0$ we have

$$\begin{aligned} R(a_{kmn}) &= R(c)^n R(c^*)^m R(a)^k = (-1)^{n+m} c^n (c^*)^m (a^*)^k \\ &= (-1)^{n+m} q^{k(n+m)} (a^*)^k (c^*)^m c^n = (-1)^{n+m} q^{k(n+m)} a_{-k,m,n} \end{aligned}$$

and similarly for $k < 0$ we have $R(a_{kmn}) = (-1)^{n+m} q^{-k(n+m)} a_{-k,m,n}$. Thus we have

$$(P \circ R)(a_{kmn}) = (-1)^{n+m} \delta_{k,0} (c^*)^m c^n = R(\delta_{k,0} (c^*)^m c^n) = (R \circ P)(a_{kmn}).$$

Similarly if $-1 \leqslant q < 0$ we have $R(c) = c$ and $R(a) = a^*$ and so again $(P \circ R)(a_{kmn}) = (R \circ P)(a_{kmn})$. It then follows by continuity that $(P \circ R)(x) = (R \circ P)(x)$ for all $x \in \mathrm{C}(\mathrm{SU}_q(2))$.

Now let $x \in \mathrm{L}^\infty(\mathrm{SU}_q(2))$ and let $(x_\alpha) \subset \mathrm{C}(\mathrm{SU}_q(2))$ be a net such that $x_\alpha \xrightarrow{w^*} x$. Then using that R and $\mathbb{P}$ are normal, for all $\omega \in \mathrm{L}^1(\mathrm{SU}_q(2))$ we have

$$
\begin{aligned}
&|\langle(\mathbb{P}\circ R)(x),\omega\rangle - \langle(R\circ\mathbb{P})(x),\omega\rangle| \\
&\qquad \leqslant |\langle(\mathbb{P}\circ R)(x) - (\mathbb{P}\circ R)(x_\alpha),\omega\rangle| + |\langle(R\circ\mathbb{P})(x_\alpha) - (R\circ\mathbb{P})(x),\omega\rangle| \\
&\qquad = |\langle x - x_\alpha, (\mathbb{P}\circ R)_*(\omega)\rangle| + |\langle x - x_\alpha, (R\circ\mathbb{P})_*(\omega)\rangle| \to 0
\end{aligned}
$$

and so $\mathbb{P}\circ R = R\circ\mathbb{P}$ as required. □

The next theorem follows immediately from the preceding three lemmas and propositions and the decomposition $S = R\circ\tau_{-i/2}$.

Theorem 5.3.4 *Let S denote the von Neumann algebraic antipode and $x \in \mathrm{Dom}(S) \subset \mathrm{L}^\infty(\mathrm{SU}_q(2))$, then $\mathbb{P}(x) \in \mathrm{Dom}(S)$ and $S(\mathbb{P}(x)) = \mathbb{P}(S(x))$. In particular for $x \in \overline{\mathrm{C}^*(c,1)}^{w^*}$ we have $S(x) \in \overline{\mathrm{C}^*(c,1)}^{w^*}$.*

*Similarly let S denote the reduced **C***-algebraic antipode and $x \in \mathrm{Dom}(S) \subset \mathrm{C}(\mathrm{SU}_q(2))$, then $P(x) \in \mathrm{Dom}(S)$ and $S(P(x)) = P(S(x))$. In particular for $x \in \mathrm{C}^*(c,1)$ we have $S(x) \in \mathrm{C}^*(c,1)$.*

We want to define an antipode, unitary antipode and a scaling group on $\mathrm{C}(K)$ and $\mathrm{L}^\infty(K,\nu)$. The preceding theorems and propositions of this section ensure the following definitions make sense.

Definition 5.3.5 *(i) Let $\mathrm{Dom}(S^K) = \{f \in \mathrm{L}^\infty(K,\nu) \mid \mathbb{F}(f) \in \mathrm{Dom}(S)\}$ and let $S^K : \mathrm{L}^\infty(K,\nu) \to \mathrm{L}^\infty(K,\nu)$ be given by $S^K(f) = \mathbb{E}(S(\mathbb{F}(f)))$ for $f \in \mathrm{Dom}(S^K)$;*

(ii) For $z \in \mathbb{C}$, let $\mathrm{Dom}(\tau^K_z) = \{f \in \mathrm{L}^\infty(K,\nu) \mid \mathbb{F}(f) \in \mathrm{Dom}(\tau_z)\}$ and let $\tau^K_z : \mathrm{L}^\infty(K,\nu) \to \mathrm{L}^\infty(K,\nu)$ be given by $\tau^K_z(f) = \mathbb{E}(\tau_z(\mathbb{F}(f)))$ for $f \in \mathrm{Dom}(\tau^K_z)$;

(iii) Let $R^K : \mathrm{L}^\infty(K,\nu) \to \mathrm{L}^\infty(K,\nu)$ be the map $\mathbb{E}\circ R\circ\mathbb{F}$.

In the following straightforward proposition we show that τ^K_z is equivalent to the one-parameter group τ^c_z from Lemma 5.3.1 as would be expected.

Proposition 5.3.6 *Let $f \in \mathrm{Dom}(\tau^K_z)$ and let $x = \mathbb{F}(f)$, then $x \in \mathrm{Dom}(\tau_z) \cap \overline{\mathrm{C}^*(c,1)}^{w^*}$ and $\mathbb{F}(\tau^K_z(f)) = \tau_z(x)$.*

Proof

It follows by definition of $\mathrm{Dom}(\tau_z^K)$ that $x = \mathbb{F}(f) \in \mathrm{Dom}(\tau_z)$ and as $\mathbb{F}(f) \in \overline{\mathrm{C}^*(c,1)}^{w*}$ from Theorem 5.2.9 we have $x \in \mathrm{Dom}(\tau_z^K) \cap \overline{\mathrm{C}^*(c,1)}^{w*}$. It follows from Definition 5.3.5 (ii) that $\mathbb{F}(\tau_z^K(f)) = \mathbb{F}(\mathbb{E}(\tau_z(x)) = \mathbb{P}(\tau_z(x)) = x$ where the final equality follows as $x \in \overline{\mathrm{C}^*(c,1)}^{w*}$ and Theorem 5.2.10 (i). □

Proposition 5.3.7 *For all $t \in \mathbb{R}$ we have that τ_t^K is a $*$-automorphism on $\mathrm{L}^\infty(K,\nu)$ and the restriction of τ_t^K to $\mathrm{C}(K)$ is also a $*$-automorphism.*

Proof

This follows on $\mathrm{L}^\infty(K,\nu)$ as $\mathbb{F}$ is a $*$-homomorphism from Theorem 5.2.9, $\mathbb{E}$ is a $*$-homomorphism on $\overline{\mathrm{C}^*(c,1)}^{w*}$ from Theorem 5.2.10 (iii), τ_t is a $*$-homomorphism for all $t \in \mathbb{R}$ and $\tau_t^K = \mathbb{E} \circ \tau_t \circ \mathbb{F}$ for all $t \in \mathbb{R}$.

Let $x \in \mathrm{C}^*(c,1)$. When restricted to $\mathrm{C}(K)$ we have that $\mathbb{F}$ has image $\mathrm{C}^*(c,1)$, for all $t \in \mathbb{R}$ we have that $\tau_t(x) \in \mathrm{C}^*(c,1)$ and $\mathbb{E}(x) \in \mathrm{C}(K)$ by Theorem 5.2.10 (ii) and thus τ_t^K is a $*$-automorphism on $\mathrm{C}(K)$. □

Proposition 5.3.8 *We have the following relations*

$$S^K \circ \mathbb{E} = \mathbb{E} \circ S, \qquad S \circ \mathbb{F} = \mathbb{F} \circ S^K, \qquad \tau_z^K \circ \mathbb{E} = \mathbb{E} \circ \tau_z, \qquad \tau_z \circ \mathbb{F} = \mathbb{F} \circ \tau_z^K$$
$$R^K \circ \mathbb{E} = \mathbb{E} \circ R, \qquad R \circ \mathbb{F} = \mathbb{F} \circ R^K, \qquad S^K = R^K \circ \tau_{-i/2}^K = \tau_{-i/2}^K \circ R^K$$
$$(R^K)^2 = \mathrm{id}, \qquad R^K \circ \tau_t^K = \tau_t^K \circ R^K$$

and R^K and τ_t^K are normal operators where $z \in \mathbb{C}$ and $t \in \mathbb{R}$.

Proof

If $x \in \mathrm{Dom}(S)$ then from Theorem 5.3.4 we have $\mathbb{F}(\mathbb{E}(x)) = \mathbb{P}(x) \in \mathrm{Dom}(S)$ and so by definition $\mathbb{E}(x) \in \mathrm{Dom}(S^K)$ and $S^K(\mathbb{E}(x)) = (\mathbb{E} \circ S \circ \mathbb{P})(x) = (\mathbb{E} \circ \mathbb{P} \circ S)(x) = \mathbb{E}(S(x))$, that is $S^K \circ \mathbb{E} = \mathbb{E} \circ S$. For $F \in \mathrm{Dom}(S^K)$ by definition we have $\mathbb{F}(F) \in \mathrm{Dom}(S)$ and

$$(S \circ \mathbb{F})(F) = (S \circ \mathbb{P} \circ \mathbb{F})(F) = (\mathbb{P} \circ S \circ \mathbb{F})(F) = (\mathbb{F} \circ S^K \circ \mathbb{E} \circ \mathbb{F})(F) = (\mathbb{F} \circ S^K)(F)$$

and so $S \circ \mathbb{F} = \mathbb{F} \circ S^K$. We can prove the results on τ_z^K for $z \in \mathbb{C}$ and R^K similarly.

Let $f \in \mathrm{Dom}(S^K)$ and we have

$$S^K(f) = (\mathbb{E} \circ S \circ \mathbb{F})(f) = (\mathbb{E} \circ R \circ \tau_{-i/2} \circ \mathbb{F})(f).$$

We have $\mathbb{F}(f) \in \mathrm{Dom}(S) = \mathrm{Dom}(\tau_{-i/2})$ and so by Proposition 5.3.2 it follows that $(\mathbb{P} \circ \tau_{-i/2} \circ \mathbb{F})(f) = (\tau_{-i/2} \circ \mathbb{P} \circ \mathbb{F})(f) = (\tau_{-i/2} \circ \mathbb{F})(f)$. Then

$$S^K(f) = (\mathbb{E} \circ R \circ \mathbb{P} \circ \tau_{-i/2} \circ \mathbb{F})(f) = (R^K \circ \tau_{-i/2}^K)(f)$$

and we have $S^K = R^K \circ \tau_{-i/2}^K$. The others follow similarly.

For all $t \in \mathbb{R}$ we have R^K and τ_t^K are normal operators as $\mathbb{E}$, $\mathbb{F}$, R and τ_t are normal for all $t \in \mathbb{R}$. $\square$

We now show that for any $z \in \mathbb{C}$ that $\mathrm{Poly}(K)$ is a core for τ_z^K and thus also for S^K.

Proposition 5.3.9 *Fix $z \in \mathbb{C}$. We have that $\mathrm{Poly}(K)$ is a core for τ_z^K on $\mathrm{C}(K)$ and is a weak*-core for τ_z^K on $\mathrm{L}^\infty(K, \nu)$.*

Proof

We show the case where τ^K is a one-parameter group on $\mathrm{L}^\infty(K, \nu)$, the C*-algebra case is similar. Let $F \in \mathrm{Dom}(\tau_z^K)$ so that $\mathbb{F}(F) \in \mathrm{Dom}(\tau_z))$. Then using Corollary 3.2.19 we have $\mathrm{Hopf}(\mathrm{SU}_q(2))$ is a σ-weak core in $\mathrm{L}^\infty(\mathrm{SU}_q(2))$ and so we have a net $(x_\alpha) \subset \mathrm{Hopf}(\mathrm{SU}_q(2))$ such that $x_\alpha \xrightarrow{w*} \mathbb{F}(F)$ and $\tau_z(x_\alpha) \xrightarrow{w*} \tau_z(\mathbb{F}(F))$. From Theorems 5.2.9 and 5.2.10 we have $\mathbb{E}(x_\alpha) \in \mathrm{Poly}(K)$ and as $\mathbb{F}(\mathbb{E}(x_\alpha)) = \mathbb{P}(x_\alpha) = P(x_\alpha)$ then $\mathbb{E}(x_\alpha) \in \mathrm{Dom}(\tau_z^K)$. Then for all $G \in \mathrm{L}^1(K, \nu)$ and using that $\mathbb{E} \circ \mathbb{F} = \mathrm{id}$ we have

$$|\langle \mathbb{E}(x_\alpha) - F, G \rangle| = |\langle \mathbb{E}(x_\alpha) - (\mathbb{E} \circ \mathbb{F})(F), G \rangle| = |\langle x_\alpha - \mathbb{F}(F), \mathbb{E}_*(G) \rangle| \to 0$$

and using that $\tau_z^K \circ \mathbb{E} = \mathbb{E} \circ \tau_z^K$ from Proposition 5.3.8 and $\tau_z \circ \mathbb{P} = \mathbb{P} \circ \tau_z$ from Proposition

5.3.2 we have

$$\begin{aligned}\left|\langle\tau_z^K(\mathbb{E}(x_\alpha)) - \tau_z^K(F), G\rangle\right| &= |\langle\mathbb{E}(\tau_z(\mathbb{P}(x_\alpha))) - \mathbb{E}(\tau_z(\mathbb{F}(F))), G\rangle| \\ &= |\langle\tau_z(x_\alpha) - \tau_z(\mathbb{F}(F)), \mathbb{E}_*(G)\rangle| \to 0.\end{aligned}$$

Both of these equations tend to 0 because $x_\alpha \xrightarrow{w^*} \mathbb{F}(F)$ and $\tau_z(x_\alpha) \xrightarrow{w^*} \tau_z^K(\mathbb{F}(F))$ and so $\mathrm{Poly}(K)$ is a core for $\mathrm{Dom}(\tau_z^K)$ as required. □

For $t \in \mathbb{R}$ we have that τ_t^K acts by rotating the domain. We make this precise in the next proposition.

Proposition 5.3.10 *Let $F \in \mathrm{C}(K)$, then we have $\tau_t^K(F)(q^r e^{2\pi i\theta}) = F(q^r e^{2\pi i\theta + 2it\ln q})$ for all $r \in \mathbb{N}_0$ and $\theta \in [0,1)$ and $\tau_t^K(F)(0) = F(0)$.*

Proof

For all $t \in \mathbb{R}$ we have that τ_t^K is a $*$-automorphism on $\mathrm{C}(K)$ from Proposition 5.3.7 and from Lemma 1.33 in Williams (2007) we have a homeomorphism $h_t : K \to K$ such that $\tau_t^K(F)(z) = F(h_t(z))$ for all $F \in \mathrm{C}(K)$ and $z \in K$. Now consider the function $\underline{z}$, we know that $\tau_t^K(\underline{z}) = \mathbb{E}(\tau_t(c)) = q^{2it}\underline{z}$ and thus for all $r \in \mathbb{N}_0$ and $\theta \in [0,1)$ we have

$$h_t(q^r e^{2\pi i\theta}) = \underline{z}(h_t(q^r e^{2\pi i\theta})) = \tau_t^K(\underline{z})(q^r e^{2\pi i\theta}) = e^{2it\ln q}\underline{z}(q^r e^{2\pi i\theta}) = q^r e^{2\pi i\theta + 2it\ln q}$$

and similarly $h_t(0) = \underline{z}(h_t(0)) = \tau_t^K(\underline{z})(0) = 0$ as required. □

To finish this section we show that we have the following complete isometry similar to Proposition 4.2.3.

Proposition 5.3.11 *The map $Q^K : \mathrm{L}^1(K,\nu) \to \overline{\mathrm{L}^1(K,\nu)}$ given by $f \mapsto \overline{R_*^K(f^*)}$ for $f \in \mathrm{L}^1(K,\nu)$ is a complete isometry.*

Proof

We define a map $Q^K : \mathrm{L}^1(K,\nu) \to \overline{\mathrm{L}^1(K,\nu)}$ by $Q^K = \overline{\mathbb{F}_*} \circ Q \circ \mathbb{E}_*$ where Q :

$\mathrm{L}^1(\mathrm{SU}_q(2)) \to \overline{\mathrm{L}^1(\mathrm{SU}_q(2))}$ is given by Proposition 4.2.3. Then using that $\mathbb{E}$ and $\mathbb{F}$ are $*$-maps, for all $f \in \mathrm{L}^1(K,\nu)$ and $F \in \mathrm{L}^\infty(K,\nu)$ we have

$$\begin{aligned}\langle Q^K(f), \overline{F}\rangle &= \langle(\overline{\mathbb{F}_*} \circ Q)(\mathbb{E}_*(f)), \overline{F}\rangle = \langle\overline{\mathbb{F}_*((\mathbb{E}_*(f))^* \circ R)}, \overline{F}\rangle \\ &= \langle\mathbb{E}_*(f), R(\mathbb{F}(F))^*\rangle = \langle f, R^K(F)^*\rangle = \langle\overline{R^K_*(f^*)}, \overline{F}\rangle\end{aligned}$$

and so $Q^K(f) = \overline{R^K_*(f^*)}$.

We have from Proposition 4.2.3 that Q is a completely isometric isomorphism and so Q^* is a completely isometric isomorphism given by $\overline{x} \mapsto R(x^*)$. Taking the adjoint of Q^K we get $(Q^K)^* = \mathbb{E} \circ Q^* \circ \overline{\mathbb{F}}$ and then for $F \in \mathrm{L}^\infty(K,\nu)$, and using Proposition 5.3.8 we have

$$(Q^K)^*(\overline{F}) = (\mathbb{E} \circ R)(\mathbb{F}(F)^*) = R^K(F^*).$$

Then we have $((Q^K)^* \circ \overline{(Q^K)^*})(F) = (Q^K)^*(\overline{R^K(F^*)}) = R^K(R^K(F^*)^*) = F$ and so $(Q^K)^* \circ \overline{(Q^K)^*} = \mathrm{id}$. Also we have $\mathbb{F}(F) \in \overline{\mathrm{C}^*(c,1)}^{w^*}$ and $\overline{Q^*}\left(\overline{\mathbb{F}(F)}\right) \in \overline{\mathrm{C}^*(c,1)}^{w^*}$ meaning $\mathbb{P}\left(\overline{Q^*(\mathbb{F}(F))}\right) = \overline{Q^*}(\overline{\mathbb{F}(F)})$ and as $\mathbb{F}$ and Q^* are complete isometries we have

$$\begin{aligned}\left\|\overline{(Q^K)^*}\left(\overline{F}\right)\right\|_{cb} &= \left\|(\mathbb{E} \circ \overline{Q^*} \circ \overline{\mathbb{F}})(\overline{F})\right\|_{cb} = \left\|\mathbb{P}\left(\overline{Q^*(\mathbb{F}(F))}\right)\right\|_{cb} \\ &= \left\|\overline{Q^*(\mathbb{F}(F))}\right\|_{cb} = \|\mathbb{F}(F)\|_{cb} = \|F\|_{cb} = \left\|\overline{F}\right\|_{cb}.\end{aligned}$$

As $(Q^K)^* \circ \overline{(Q^K)^*} = \mathrm{id}$ and $(Q^K)^*$ is a complete isometry then $(Q^K)^*$ is a completely isometric isomorphism and so is Q^K. □

5.3.2 $\mathrm{L}^1_\sharp(K,\nu)$

In this section we introduce the $\mathrm{L}^1_\sharp(K,\nu)$ with a natural inclusion into $\mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ and then investigate this space further. As $\mathrm{L}^1(K,\nu)$ are the integrable functions with respect to the measure ν we can use some measure theoretic techniques in this section to gain a better understanding of $\mathrm{L}^1_\sharp(K,\nu)$.

We begin by introducing $\mathrm{L}^1_\sharp(K,\nu)$.

Proposition 5.3.12 *Let $f \in \mathrm{L}^1(K,\nu)$. Then $\mathbb{E}_*(f) \in \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ if and only if there exists some $g \in \mathrm{L}^1(K,\nu)$ such that $\langle F, g\rangle = \langle S^K(F), f^*\rangle$ for all $F \in \mathrm{C}(K) \cap \mathrm{Dom}(S^K)$.*

Proof

Let $\omega := \mathbb{E}_*(f)$ and assume $\omega \in \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$. Let $F \in \mathrm{C}(K) \cap \mathrm{Dom}(S^K)$ and then $\mathbb{F}(F) \in \mathrm{C}^*(c,1) \cap \mathrm{Dom}(S)$. Then using that $\mathbb{E}$ is a $*$-map from Theorem 5.2.10 and Definition 5.3.5 we have

$$\langle S^K(F), f^*\rangle = \overline{\langle (\mathbb{E}\circ S\circ \mathbb{F})(F)^*, f\rangle} = \overline{\langle S(\mathbb{F}(F))^*, \omega\rangle} = \langle \mathbb{F}(F), \omega^\sharp\rangle = \langle F, \mathbb{F}_*(\omega^\sharp)\rangle.$$

And so letting $g = \mathbb{F}_*(\omega^\sharp)$ we have $\langle F, g\rangle = \langle S^K(F), f\rangle$ for all $F \in \mathrm{C}(K) \cap \mathrm{Dom}(S)$.

Conversely say there exists some $g \in \mathrm{L}^1(K,\nu)$ such that $\langle F, g\rangle = \langle S^K(F), f^*\rangle$ for all $F \in \mathrm{C}(K) \cap \mathrm{Dom}(S^K)$. Let $x \in \mathrm{Dom}(S) \cap \mathrm{C}(\mathrm{SU}_q(2))$, then by Theorem 5.2.10 we have $\mathbb{E}(x) \in \mathrm{C}(K) \cap \mathrm{Dom}(S^K)$ and using that $\mathbb{E}$ is a $*$-map and Proposition 5.3.8 we have

$$\overline{\langle S(x)^*, \mathbb{E}_*(f)\rangle} = \overline{\langle \mathbb{E}(S(x)^*), f\rangle} = \langle S^K(\mathbb{E}(x)), f^*\rangle = \langle \mathbb{E}(x), g\rangle = \langle x, \mathbb{E}_*(g)\rangle.$$

Now let $x \in \mathrm{Dom}(S) \subset \mathrm{L}^\infty(\mathrm{SU}_q(2))$, then there exists a bounded net $(x_\alpha) \subset \mathrm{Dom}(S) \cap \mathrm{C}(\mathrm{SU}_q(2))$ such that $x_\alpha \xrightarrow{w^*} x$ and $S(x_\alpha) \xrightarrow{w^*} S(x)$. Then we have

$$\begin{aligned}&\left|\overline{\langle S(x)^*, \mathbb{E}_*(f)\rangle} - \langle x, \mathbb{E}_*(g)\rangle\right| \\ &\qquad \leqslant \left|\overline{\langle S(x)^*, \mathbb{E}_*(f)\rangle} - \overline{\langle S(x_\alpha)^*, \mathbb{E}_*(f)\rangle}\right| + |\langle x_\alpha, \mathbb{E}_*(g)\rangle - \langle x, \mathbb{E}_*(g)\rangle| \to 0.\end{aligned}$$

So for all $x \in \mathrm{Dom}(S)$ we have $\overline{\langle S(x)^*, \mathbb{E}_*(f)\rangle} = \langle x, \mathbb{E}_*(g)\rangle$ where $\mathbb{E}_*(g) \in \mathrm{L}^1(\mathrm{SU}_q(2))$ as required. □

So we make the following obvious definition for $\mathrm{L}^1_\sharp(K,\nu)$.

Definition 5.3.13 *We let*

$$\mathrm{L}^1_\sharp(K,\nu) = \left\{ f \in \mathrm{L}^1(K,\nu) \;\middle|\; \begin{array}{l} \exists\, g \in \mathrm{L}^1(K,\nu) \text{ such that} \\ \qquad \langle S^K(F), f^* \rangle = \langle F, g \rangle \quad \forall\, F \in \mathrm{C}(K) \cap \mathrm{Dom}(S^K) \end{array} \right\}.$$

Similarly to the case of $\mathrm{L}^1_\sharp(\mathbb{G})$ for a general locally compact quantum group $\mathbb{G}$ we have a unique $f^\sharp$ satisfying the condition of $\mathrm{L}^1_\sharp(K,\nu)$ and we have a norm

$$\|f\|_{\mathrm{L}^1_\sharp(K,\nu)} = \max\{\|f\|_{\mathrm{L}^1(K,\nu)}, \|f^\sharp\|_{\mathrm{L}^1(K,\nu)}\}$$

for $f \in \mathrm{L}^1_\sharp(K,\nu)$ under which $\mathrm{L}^1_\sharp(K,\nu)$ is a Banach space. We show we have an an isometric embedding of $\mathrm{L}^1_\sharp(K,\nu)$ into $\mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ next that we can use to define an operator space structure on $\mathrm{L}^1_\sharp(K,\nu)$.

Proposition 5.3.14 *There is an isometric embedding* $\mathbb{E}^\sharp_* : \mathrm{L}^1_\sharp(K,\nu) \to \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ *given as the restriction and corestriction of the map* $\mathbb{E}_* : \mathrm{L}^1(K,\nu) \to \mathrm{L}^1(\mathrm{SU}_q(2))$ *from Theorem 5.2.10.*

Proof

We have an isometry $\theta^K : \mathrm{L}^1_\sharp(K,\nu) \to \mathrm{L}^1(K,\nu) \oplus_\infty \overline{\mathrm{L}^1(K,\nu)}$ given by $f \mapsto (f, \overline{f^\sharp})$ for all $f \in \mathrm{L}^1_\sharp(K,\nu)$. We also have an isometric embedding $\theta^{\mathrm{SU}_q(2)} : \mathrm{L}^1_\sharp(\mathrm{SU}_q(2)) \to \mathrm{L}^1(\mathrm{SU}_q(2)) \oplus_\infty \overline{\mathrm{L}^1(\mathrm{SU}_q(2))}$ given by Equation (4.2). Lastly from Theorem 5.2.10 and Proposition 1.1.28 (ii) we have an isometric embedding $\mathbb{E}_* \oplus_\infty \overline{\mathbb{E}_*} : \mathrm{L}^1(K,\nu) \oplus_\infty \overline{\mathrm{L}^1(K,\nu)} \to \mathrm{L}^1(\mathrm{SU}_q(2)) \oplus_\infty \overline{\mathrm{L}^1(\mathrm{SU}_q(2))}$. Then by Theorem 5.3.12 there exists a bounded map $\mathbb{E}^\sharp_* : \mathrm{L}^1_\sharp(K,\nu) \to \mathrm{L}^1_\sharp(\mathbb{G})$ that is the restriction and corestriction of $\mathbb{E}_*$ such that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{L}^1_\sharp(K,\nu) & \xrightarrow{\theta^K} & \mathrm{L}^1(K,\nu) \oplus_\infty \overline{\mathrm{L}^1(K,\nu)} \\ \Big\downarrow{\scriptstyle \mathbb{E}^\sharp_*} & & \Big\downarrow{\scriptstyle \mathbb{E}_* \oplus_\infty \overline{\mathbb{E}_*}} \\ \mathrm{L}^1_\sharp(\mathrm{SU}_q(2)) & \xrightarrow[\theta^{\mathrm{SU}_q(2)}]{} & \mathrm{L}^1(\mathrm{SU}_q(2)) \oplus_\infty \overline{\mathrm{L}^1(\mathrm{SU}_q(2))}. \end{array}$$

As θ^K, $\theta^{\mathrm{SU}_q(2)}$ and $\mathbb{E}_* \oplus_\infty \overline{\mathbb{E}_*}$ are all isometries then $\mathbb{E}_*^\sharp$ must also be an isometry. □

Definition 5.3.15 *We let $\mathrm{L}^1_\sharp(K,\nu)$ have the operator space structure such that the embedding $\mathbb{E}_*^\sharp : \mathrm{L}^1_\sharp(K,\nu) \to \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ from the preceding proposition is a complete isometry.*

Proposition 5.3.16 *(i) The map $\theta^K : \mathrm{L}^1_\sharp(K,\nu) \to \mathrm{L}^1(K,\nu) \oplus_\infty \overline{\mathrm{L}^1(K,\nu)}$ given by $f \mapsto (f, \overline{f^\sharp})$ is a completely isometric embedding.*

(ii) For $\omega \in \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ we have $\mathbb{F}_*(\omega) \in \mathrm{L}^1_\sharp(K,\nu)$ and we have a map $\mathbb{F}_*^\sharp : \mathrm{L}^1_\sharp(\mathrm{SU}_q(2)) \to \mathrm{L}^1_\sharp(K,\nu)$ that is the restriction and corestriction of the map $\mathbb{F}_*$ from Theorem 5.2.9. Furthermore this map is a complete quotient map that is a left inverse of $\mathbb{E}_*^\sharp$.

Proof

(i) From the preceding proof we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{L}^1_\sharp(K,\nu) & \xrightarrow{\quad\theta^K\quad} & \mathrm{L}^1(K,\nu) \oplus_\infty \overline{\mathrm{L}^1(K,\nu)} \\
\downarrow{\scriptstyle \mathbb{E}_*^\sharp} & & \downarrow{\scriptstyle \mathbb{E}_*\oplus_\infty\overline{\mathbb{E}_*}} \\
\mathrm{L}^1_\sharp(\mathrm{SU}_q(2)) & \xrightarrow[\theta^{\mathrm{SU}_q(2)}]{} & \mathrm{L}^1(\mathrm{SU}_q(2)) \oplus_\infty \overline{\mathrm{L}^1(\mathrm{SU}_q(2))}.
\end{array}$$

We know that $\mathbb{E}_*^\sharp$ and $\theta^{\mathrm{SU}_q(2)}$ are complete isometries. Also as $\mathbb{E}_* : \mathrm{L}^1(K,\nu) \to \mathrm{L}^1(\mathrm{SU}_q(2))$ is a complete isometry then by Proposition 1.1.28 (ii) it follows that $\mathbb{E}_* \oplus_\infty \overline{\mathbb{E}_*}$ is a complete isometry and so θ^K must be a complete isometry.

(ii) Let $\omega \in \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$, then for all $F \in \mathrm{Dom}(S^K)$ we have

$$\overline{\langle S^K(F)^*, \mathbb{F}_*(\omega)\rangle} = \overline{\langle S(\mathbb{F}(F))^*, \omega\rangle} = \langle \mathbb{F}(F), \omega^\sharp\rangle = \langle F, \mathbb{F}_*(\omega^\sharp)\rangle$$

and so $\mathbb{F}_*(\omega) \in \mathrm{L}^1_\sharp(K,\nu)$ with $\mathbb{F}_*(\omega)^\sharp = \mathbb{F}_*(\omega^\sharp)$.

Then we have shown that there is a completely bounded map $\mathbb{F}_*^\sharp : \mathrm{L}^1_\sharp(\mathrm{SU}_q(2)) \to \mathrm{L}^1_\sharp(K,\nu)$ given by the restriction and corestriction of $\mathbb{F}_*$ such that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{L}^1_\sharp(\mathrm{SU}_q(2)) & \xrightarrow{\theta^{\mathrm{SU}_q(2)}} & \mathrm{L}^1(\mathrm{SU}_q(2)) \oplus_\infty \overline{\mathrm{L}^1(\mathrm{SU}_q(2))} \\ \downarrow \mathbb{F}^\sharp_* & & \downarrow \mathbb{F}_* \oplus_\infty \overline{\mathbb{F}_*} \\ \mathrm{L}^1_\sharp(K,\nu) & \xrightarrow[\theta^K]{} & \mathrm{L}^1(K,\nu) \oplus_\infty \overline{\mathrm{L}^1(K,\nu)}. \end{array}$$

As $\mathbb{F}_*$ is a complete contraction it follows from Proposition 1.1.28 that $\mathbb{F}_* \oplus_\infty \overline{\mathbb{F}_*}$ is a complete contraction and then using that $\theta^{\mathrm{SU}_q(2)}$ and θ^K are complete isometries it follows from the commutative diagram that $\mathbb{F}^\sharp_*$ is a complete contraction.

As $\mathbb{E}^\sharp_*$ and $\mathbb{F}^\sharp_*$ are restrictions of the maps $\mathbb{E}_*$ and $\mathbb{F}_*$ respectively it follows that $\mathbb{F}^\sharp_*$ is the left inverse for $\mathbb{E}^\sharp_*$. Then from Lemma 1.1.20 it follows that $\mathbb{F}^\sharp_*$ is a complete quotient map. □

We have the following which is similar to Proposition 4.1.11.

Proposition 5.3.17 *Let $f \in \mathrm{L}^1(K,\nu)$. We have that the following are equivalent:*

(i) $f \in \mathrm{L}^1_\sharp(K,\nu)$*;*

(ii) For all $F \in \mathrm{Dom}(\tau^K_{i/2})$ we have some $g \in \mathrm{L}^1(K,\nu)$ such that $\langle \tau^K_{i/2}(F), f\rangle = \langle F, g\rangle$;

(iii) $f \in \mathrm{Dom}((\tau^K_{i/2})_*)$.

In the case that these conditions hold we have $(\tau^K_{i/2})_*(f) = R^K_*(f^\sharp)^*$.

Proof

Say (i) holds, that is $f \in \mathrm{L}^1_\sharp(K,\nu)$. By Proposition 1.3.20 (ii) for all $F \in \mathrm{Dom}(\tau^K_{i/2})$ we have $F^* \in \mathrm{Dom}(\tau^K_{-i/2})$ and $\tau^K_{-i/2}(F^*) = \tau^K_{i/2}(F)^*$ and so

$$\langle \tau^K_{i/2}(F), f\rangle = \langle \tau^K_{-i/2}(F^*)^*, f\rangle = \langle S^K(R^K(F^*))^*, f\rangle = \langle R^K(F^*), f^\sharp\rangle = \langle F, R^K_*(f^\sharp)^*\rangle$$

where we've used that $S^K = \tau^K_{-i/2} \circ R^K$ and $(R^K)^2 = \mathrm{id}$. Then as $R^K_*(f^\sharp)^* \in \mathrm{L}^1(K,\nu)$ we have condition (ii).

We have that condition (ii) implies condition (iii) straight from Definition A.2.10. It also follows that $(\tau^K_{i/2})_*(f) = R^K_*(f^\sharp)^*$ from the same definition.

Now assume $f \in \mathrm{Dom}((\tau^K_{i/2})_*)$, then again by Definition A.2.10 there exists $g \in \mathrm{L}^1(K,\nu)$ such that for all $F \in \mathrm{Dom}(\tau^K_{i/2})$ we have $\langle \tau^K_{i/2}(F), f\rangle = \langle F, g\rangle$. Let $G \in \mathrm{Dom}(\tau^K_{-i/2})$ and using that $G^* \in \mathrm{Dom}(\tau^K_{i/2})$ with $\tau^K_{i/2}(G^*) = \tau^K_{-i/2}(G)^*$ we have

$$\langle S^K(G), f^*\rangle = \langle \tau^K_{i/2}(R^K(G)^*)^*, f^*\rangle = \overline{\langle R^K(G)^*, g\rangle} = \langle G, R^K_*(g^*)\rangle$$

for all $G \in \mathrm{Dom}(\tau^K_{-i/2})$ and so $f \in \mathrm{L}^1_\sharp(K,\nu)$. □

By this proposition the following notation is well defined.

Notation 5.3.18 *Given $f \in \mathrm{L}^1_\sharp(K,\nu)$ we let $f^\flat \in \mathrm{L}^1(K,\nu)$ denote the function $(\tau^K_{i/2})_*(f)$ where $(\tau^K_{i/2})_*$ is defined by Definition A.2.10.*

We give another method for calculating the norm on $\mathrm{L}^1_\sharp(K,\nu)$ before moving on to further investigation. This follows directly from Propositions 5.3.17, 5.3.16, 5.3.11 and 1.1.28.

Corollary 5.3.19 *For $\theta^K : \mathrm{L}^1_\sharp(K,\nu) \to \mathrm{L}^1(K,\nu) \oplus_\infty \overline{\mathrm{L}^1(K,\nu)}$ the map in Proposition 5.3.16 and $Q^K : \overline{\mathrm{L}^1(K,\nu)} \to \mathrm{L}^1(K,\nu)$ the map from Proposition 5.3.11, we have a complete isometry $\Psi := (\mathrm{id} \oplus_\infty \overline{Q^K}) \circ \theta^K : \mathrm{L}^1_\sharp(K,\nu) \to \mathrm{L}^1(K,\nu) \oplus_\infty \mathrm{L}^1(K,\nu)$ such that*

$$f \mapsto (f, (\tau^K_{i/2})_*(f)) = (f, f^\flat)$$

and furthermore it follows that

$$\|f\|_{\mathrm{L}^1_\sharp(K,\nu)} = \max\{\|f\|_{\mathrm{L}^1(K,\nu)}, \|f^\flat\|_{\mathrm{L}^1(K,\nu)}\}.$$

5.3.3 Structure of $\mathrm{L}^1_\sharp(K,\nu)$

In this section we now prove Theorem 5.3.26 that states informally that a function $f \in \mathrm{L}^1(K,\nu)$ is in $\mathrm{L}^1_\sharp(K,\nu)$ if the function $f \circ \tau^K_{i/2}$ is bounded. This goes part way towards answering the question after Proposition 4.1.11 for $\mathrm{SU}_q(2)$, however we will see in the next section that this is not easy to extend to the case of $\mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$.

We remind the reader that given $f \in \mathrm{L}^1(\mathbb{T})$ we have the Fourier transform $\hat{f} \in c_0(\mathbb{Z})$ where $\hat{f}(n) = \int_0^1 f(e^{2\pi i\theta})e^{2\pi in\theta}\, d\theta$ for $n \in \mathbb{N}_0$ (see Appendix A.7). We now give a couple of straightforward lemmas before proving our first proposition of this section.

Lemma 5.3.20 *Fix $z_0 \in \mathbb{C}$ and $s \in \mathbb{N}_0$. For all $l \in \mathbb{Z}$ the function F on K given by $q^r e^{2\pi i\theta} \mapsto \delta_{r,s}e^{2\pi il\theta}$ is in $\mathrm{C}(K) \cap \mathrm{Dom}(\tau^K_{z_0})$ and $\tau^K_{z_0}(F)(q^r e^{2\pi i\theta}) = \delta_{r,s}e^{2\pi il\theta+2ilz_0 \ln q}$.*

Proof

Clearly for any given $l \in \mathbb{Z}$ the function F given in the lemma is in $\mathrm{C}(K)$. We define $\alpha : S(z_0) \to \mathrm{C}(K)$ by $z \mapsto \delta_{r,s}e^{2\pi il\theta+2ilz \ln q}$. This is clearly continuous and analytic on $S(z_0)^o$ by properties of the exponential function and by Proposition 5.3.10 we have $\alpha(t) = \delta_{r,s}e^{2\pi il\theta+2ilt \ln q} = F(q^r e^{2\pi i\theta+2it \ln q}) = \tau^K_t(F)(q^r e^{2\pi i\theta})$. So it follows that $F \in \mathrm{Dom}(\tau^K_{z_0})$ with $(\tau^K_{z_0}(F))(q^r e^{2\pi i\theta}) = F(z_0) = \delta_{r,s}e^{2\pi il\theta+2ilz_0 \ln q}$. $\square$

Lemma 5.3.21 *Let $z_0 \in \mathbb{C}$, $r \in \mathbb{N}_0$, $m, n \in \mathbb{N}_0$ be fixed, $F = \underline{z}^{*m}\underline{z}^n \in \mathrm{C}(K)$ and $\phi \in \mathrm{L}^1(\mathbb{T})$. Then we have*

$$\int_0^1 \tau^K_{z_0}(F)(q^r e^{2\pi i\theta})\phi(e^{2\pi i\theta})\, d\theta = q^{r(n+m)}e^{2i(n-m)z_0 \ln q}\hat{\phi}(m-n).$$

Proof

We let $\alpha : S(z_0) \to \mathbb{C}$ denote the map $z \mapsto q^{r(n+m)}e^{2i(n-m)z \ln q}\hat{\phi}(m-n)$. Clearly this is continuous and analytic on $S(z_0)^o$ and for $t \in \mathbb{R}$ we have

$$\begin{aligned}\alpha(t) &= q^{r(n+m)}e^{2i(n-m)t \ln q}\int_0^1 \phi(e^{2\pi i\theta})e^{-2\pi i(m-n)\theta}\, d\theta \\ &= \int_0^1 F(q^r e^{2\pi i\theta+2it \ln q})\phi(e^{2\pi i\theta})\, d\theta = \int_0^1 \tau^K_t(F)(q^r e^{2\pi i\theta})\phi(e^{2\pi i\theta})\, d\theta\end{aligned}$$

from which the result follows. $\square$

Proposition 5.3.22 *Let $f \in \mathrm{L}^1(K,\nu)$ and let $(f_r)_{r=0}^{\infty} \subset \mathrm{L}^1(\mathbb{T})$ denote the decomposition of $f \in \mathrm{L}^1(K,\nu)$ as per Proposition 5.2.12. We have $f \in \mathrm{L}^1_\sharp(K,\nu)$ if and only if there exists a sequence of functions $(g_r)_{r=0}^{\infty} \subset \mathrm{L}^1(\mathbb{T})$ such that $\sum_{r=0}^{\infty} q^{2r}\|g_r\|_1$ is finite and $\widehat{g_s}(l) = q^l\widehat{f_s}(l)$ for all $l \in \mathbb{Z}$ and $s \in \mathbb{N}_0$. If these conditions are true we can define $g \in \mathrm{L}^1(K,\nu)$ given by $q^r e^{2\pi i\theta} \mapsto g_r(e^{2\pi i\theta})$ (and $g(0)$ arbitrary) and we have $f^\flat = g$.*

Proof

Let $f \in \mathrm{L}^1_\sharp(K,\nu)$, then by Proposition 5.3.17 we have some $f^\flat \in \mathrm{L}^1(K,\nu)$ such that for all $F \in \mathrm{Dom}(\tau^K_{i/2})$ we have $\langle \tau^K_{i/2}(F), f\rangle = \langle F, f^\flat\rangle$ (where $f^\flat$ is from Notation 5.3.18). By Proposition 5.2.12 we have a decomposition of $f^\flat$ into a sequence of functions $({f^\flat}_r)_{r=0}^{\infty} \subset \mathrm{L}^1(\mathbb{T})$ such that $f^\flat_r(e^{2\pi i\theta}) = f^\flat(q^r e^{2\pi i\theta})$ and $\sum_{r=0}^{\infty} q^{2r}\|f^\flat_r\|_1 < \infty$.

Fix $s \in \mathbb{N}_0$ and $l \in \mathbb{Z}$ and let F denote the function $q^r e^{2\pi i\theta} \mapsto \delta_{r,s}e^{-2\pi i l\theta}$, then clearly $F \in \mathrm{C}(K)$ and by Lemma 5.3.20 we have $F \in \mathrm{Dom}(\tau^K_{i/2})$ and

$$(\tau^K_{i/2}(F))(q^r e^{2\pi i\theta}) = \delta_{r,s}e^{-2\pi i l\theta - 2il(i/2)\ln q} = \delta_{r,s}q^l e^{-2\pi i l\theta}$$

and so it follows that we have

$$\langle \tau^K_{i/2}(F), f\rangle = (1-q^2)\sum_{r=0}^{\infty} q^{2r}\delta_{r,s}q^l \int_0^1 f_r(e^{2\pi i\theta})e^{-2\pi i l\theta}\,d\theta = (1-q^2)q^{2s}q^l\widehat{f_s}(l).$$

Also we have

$$\langle F, f^\flat\rangle = (1-q^2)\sum_{r=0}^{\infty} q^{2r}\delta_{r,s}\int_0^1 f^\flat_r(e^{2\pi i\theta})e^{-2\pi i l\theta}\,d\theta = (1-q^2)q^{2s}\widehat{f^\flat_s}(l)$$

and as $\langle \tau^K_{i/2}(F), f\rangle = \langle F, f^\flat\rangle$ we can equate these to get

$$\widehat{f^\flat_s}(l) = q^l\widehat{f_s}(l)$$

for all $l \in \mathbb{Z}$ and $s \in \mathbb{N}_0$.

Conversely, say there exists a sequence of functions $(g_r)_{r=0}^{\infty} \subset \mathrm{L}^1(\mathbb{T})$ such that

$\sum_{r=0}^{\infty} q^{2r}\|g_r\|_1 < \infty$ and $\widehat{g}_s(l) = q^l \widehat{f}_s(l)$ for all $s \in \mathbb{N}_0$ and $l \in \mathbb{Z}$. Let $g : K \to \mathbb{C}$ be the map $g(q^r e^{2\pi i\theta}) = g_r(e^{2\pi i\theta})$ for $r \in \mathbb{N}_0$ and $\theta \in [0, 1)$ with $g(0)$ arbitrary. By Proposition 5.2.12 we have $g \in \mathrm{L}^1(K, \nu)$. We show that $\langle F, g\rangle = \langle \tau^K_{i/2}(F), f\rangle$ for all $F \in \mathrm{Dom}(\tau^K_{i/2})$ and then by Proposition 5.3.17 we have $f \in \mathrm{L}^1_\sharp(K, \nu)$.

By Proposition 5.3.9 we have that $\mathrm{lin}\,\{\underline{z}^{*m}\underline{z}^n \mid n, m \in \mathbb{N}_0\}$ is a core for $\mathrm{Dom}(\tau^K_{i/2}) \subset \mathrm{C}(K)$. For fixed $m, n \in \mathbb{N}_0$ we consider $F = \underline{z}^{*m}\underline{z}^n$ and we show that $\langle \tau^K_{i/2}(F), f\rangle = \langle F, g\rangle$. We have by Lemma 5.3.21 that

$$\begin{aligned}
\langle \tau^K_{i/2}(F), f\rangle &= (1 - q^2)\sum_{r=0}^{\infty} q^{2r} \int_0^1 \tau^K_{i/2}(F)(q^r e^{2\pi i\theta}) f_r(e^{2\pi i\theta})\, d\theta \\
&= (1 - q^2)\sum_{r=0}^{\infty} q^{r(2+n+m)} e^{-(n-m)\ln q}\, \widehat{f}_r(m - n) \\
&= (1 - q^2)\sum_{r=0}^{\infty} q^{r(2+n+m)} \widehat{g}_r(m - n) \\
&= (1 - q^2)\sum_{r=0}^{\infty} q^{2r} \int_0^1 q^{r(n+m)} e^{2\pi i(n-m)\theta} g_r(e^{2\pi i\theta})\, d\theta \\
&= (1 - q^2)\sum_{r=0}^{\infty} q^{2r} \int_0^1 F(q^r e^{2\pi i\theta}) g(q^r e^{2\pi i\theta})\, d\theta = \langle F, g\rangle
\end{aligned}$$

where we've used that $\widehat{g}_r(l) = q^l \widehat{f}_r(l)$ for all $r \in \mathbb{Z}$.

So by linearity we have $\langle F, g\rangle = \langle \tau^K_{i/2}(F), f\rangle$ for all $F \in \mathrm{Poly}(K)$ and as this is a core for $\mathrm{Dom}(\tau^K_{i/2})$ this also holds for all $F \in \mathrm{Dom}(\tau^K_{i/2})$. Then by Proposition 5.3.17 we have $f \in \mathrm{L}^1_\sharp(K, \nu)$ and by construction $f^\flat = g$. □

We now move on to proving the main theorem of this section. We still need a bit more preparation however. We begin with the following which is essentially notation.

Proposition 5.3.23 *Let $\phi \in M(K)$ be a measure on K. Then for all $r \in \mathbb{N}_0$ there is a measure $\phi_r \in M(\mathbb{T})$ and $\phi_\infty \in M(\{0\})$ such that for any $F \in \mathrm{L}^1(K, \phi)$ we have*

$$\int_K F\, d\phi = \left(\sum_{r=0}^{\infty} \int_{\mathbb{T}} F_r\, d\phi_r\right) + F(0)\phi_\infty(\{0\})$$

where we've decomposed F as per Proposition 5.2.12.

Proof

For all $r \in \mathbb{N}_0$ we let $\phi_r \in M(\mathbb{T})$ be given by $\phi_r(A) = \phi(q^r A)$ for all measurable $A \subset \mathbb{T}$ and we let $\phi_\infty \in M(\{0\})$ the measure on 0 given by $\phi_\infty(\{0\}) = \phi(\{0\})$. Then for any measurable $A \subset K$ we have a sequence $(A_r)_{r=0}^\infty \subset \mathbb{T}$ such that $A = \left(\bigcup_{r=0}^\infty q^r A_r\right) \cup A_\infty$ where $A_\infty = \{0\}$ if $0 \in A$ and is empty otherwise. Then we have

$$\phi(A) = \phi\left(\bigcup_{r=0}^\infty q^r A_r\right) + \phi(A_\infty) = \left(\sum_{r=0}^\infty \phi_r(A_r)\right) + \phi_\infty(A_\infty)$$

from which the result follows. □

The following lemma is the main ingredient in the proof of Theorem 5.3.26 below.

Lemma 5.3.24 *Let $f \in \mathrm{L}^1(K,\nu)$ and $\phi \in M(K)$ such that for all $F \in \mathrm{Dom}(\tau_{i/2}^K) \cap \mathrm{C}(K)$ we have $\int F\, d\phi = \langle \tau_{i/2}^K(F), f\rangle$. Then for all $r \in \mathbb{N}_0$ there exists a function $g_r \in \mathrm{L}^1(\mathbb{T})$ such that $g_r d\theta = d\phi_r$ (where $\phi_r \in M(\mathbb{T})$ is the measure in the decomposition of ϕ given by Proposition 5.3.23).*

Proof

Fix $s \in \mathbb{N}_0$ and $l \in \mathbb{Z}$ and let $F : K \to \mathbb{C}$ be the function $q^r e^{2\pi i\theta} \mapsto \delta_{r,s} e^{-2\pi i l\theta}$. From Lemma 5.3.20 we have $F \in \mathrm{Dom}(\tau_{i/2}^K)$ and $(\tau_{i/2}^K(F))(q^r e^{2\pi i\theta}) = \delta_{r,s} q^l e^{-2\pi i l\theta}$ and so

$$\langle \tau_{i/2}^K(F), f\rangle = (1-q^2)q^{2s}q^l \int_0^1 f_s(e^{2\pi i\theta})e^{-2\pi i l\theta}\, d\theta = (1-q^2)q^{2s}q^l \widehat{f}_s(l).$$

We also have

$$\int_K F\, d\phi = \int_{\mathbb{T}} \underline{z}^{-l} d\phi_s = \widehat{\phi}_s(l)$$

where $\widehat{\phi}_s \in \ell^\infty(\mathbb{Z})$ is the Fourier transform of ϕ_s (see Definition A.7.3). Then as $\int_K F\, d\phi = \langle \tau_{i/2}^K(F), f\rangle$ and because s was arbitrary we can equate to get

$$\widehat{\phi}_r(l) = (1-q^2)q^{2r}q^l \widehat{f}_r(l) \tag{5.23}$$

for all $r \in \mathbb{N}_0$ and $l \in \mathbb{Z}$.

We know that if $A \subset \mathbb{T}$ is open then $\overline{A} = \{x \in \mathbb{T} \mid \overline{x} \in A\}$ (not to be confused with closure here) is also open and so if A is measurable then $\overline{A}$ is measurable. So we can define a measure $\psi_r \in M(\mathbb{T})$ by $\psi_r(A) = \phi_r(\overline{A})$ for all Borel subsets A of $\mathbb{T}$ making the map $\phi_r \mapsto \psi_r$ an isometry on $M(\mathbb{T})$. We have $\chi_{\overline{A}}(z) = \chi_A(\overline{z})$ and by linearity and density of all χ_A in $C(\mathbb{T})$ we have

$$\int_0^1 h(e^{2\pi i\theta})\, d\psi_r(\theta) = \int_0^1 h(e^{-2\pi i\theta}) d\phi_r(\theta)$$

for all $h \in C(\mathbb{T})$. So in particular for all $l \in \mathbb{Z}$ we have $\widehat{\psi_r}(l) = \widehat{\phi}_r(-l)$ and then by Equation (5.23) we have

$$\widehat{\psi_r}(l) = (1-q^2)q^{2r}q^{-l}\widehat{f}_r(-l) \tag{5.24}$$

for all $l \in \mathbb{Z}$ and $r \in \mathbb{N}_0$.

We define a function $\lambda_r^- : \mathbb{Z} \to \mathbb{C}$ by

$$\lambda_r^-(l) = \begin{cases} \widehat{\psi_r}(l) & \text{if } l < 0 \\ 0 & \text{if } l \geqslant 0 \end{cases} = \begin{cases} (1-q^2)q^{2r}q^{-l}\widehat{f}_r(-l) & \text{if } l < 0 \\ 0 & \text{if } l \geqslant 0, \end{cases}$$

that is λ_r^- is the negative coefficients in Equation (5.24). We know that $f_r \in L^1(K,\nu)$ and so $\widehat{f}_r \in c_0(\mathbb{Z})$ and so

$$\sum_{l\in\mathbb{Z}} \left|\lambda_r^-(l)\right| = (1-q^2)q^{2r}\sum_{l=1}^{\infty} q^l \left|\widehat{f}_r(l)\right| \leqslant (1-q^2)q^{2r}\left\|\widehat{f}_r\right\|_\infty \sum_{l=1}^{\infty} q^l = q(1+q)q^{2r}\left\|\widehat{f}_r\right\|_\infty$$

where we've used that $\sum_{l=1}^{\infty} q^l = \frac{q}{1-q}$. As this is finite we have $\lambda_r^- \in \ell^1(\mathbb{Z})$ and as $\ell^1(\mathbb{Z}) \subset A(\mathbb{Z})$, where $A(\mathbb{Z})$ is the Fourier algebra, there exists some $h_r^- \in L^1(\mathbb{T})$ such that $\widehat{h_r^-}(l) = \lambda_r^-(l)$ for all $l \in \mathbb{Z}$.

We define a measure ψ_r^- such that $d\psi_r^- = h_r^-\, d\theta$ and we let $\psi_r^+ = \psi_r - \psi_r^-$. Then we have

$$\widehat{\psi_r^+}(l) = \begin{cases} \widehat{\psi_r}(l) & \text{if } l \geqslant 0 \\ 0 & \text{if } l < 0 \end{cases}$$

and so for all $l < 0$ we have $\int_0^1 e^{-2\pi i l\theta}\, d\psi_r^+(\theta) = 0$. Then by the Theorem of F. and M. Riesz (Theorem 17.13 in Rudin (1987)) we have that this is absolutely continuous and so there is some $h_r^+ \in \mathrm{L}^1(\mathbb{T})$ such that $\psi_r^+(A) = \int_0^1 h_r^+(e^{2\pi i\theta})\chi_A(e^{2\pi i\theta})\, d\theta$. We then define $h_r = h_r^+ + h_r^-$ and we have that

$$h_r d\theta = (h_r^+ + h_r^-)d\theta = \psi_r^+ + \psi_r^- = \psi_r.$$

Finally let $g_r \in \mathrm{L}^1(\mathbb{T})$ be the function $g_r(e^{2\pi i\theta}) = h_r(e^{-2\pi i\theta})$ for all $\theta \in [0,1)$. We then have $g_r d\theta = \overline{h_r} d\theta = \overline{\psi_r} = \phi_r$ as required. □

Lemma 5.3.25 *Fix $f \in \mathrm{L}^1(K,\nu)$ and let $\phi \in M(K)$ such that for all $F \in \mathrm{Dom}(\tau_{i/2}^K) \cap \mathrm{C}(K)$ we have $\int F\, d\phi = \langle \tau_{i/2}^K(F), f\rangle$. Then $\phi_\infty(\{0\}) = 0$ for $\phi_\infty \in M(\{0\})$ the measure in 5.3.23*

Proof

Let $s \in \mathbb{N}_0$ be fixed and let $F_s : K \to \mathbb{C}$ be the function $q^r e^{2\pi i\theta} \mapsto \delta_{r,s}$ and $F_s(0) = 0$. Clearly $F_s \in \mathrm{C}(K)$ and from Proposition 5.3.10 we have $(\tau_t^K(F))(q^r e^{2\pi i\theta}) = \delta_{r,s}$ and so clearly $F \in \mathrm{Dom}(\tau_{i/2}^K)$ with $\tau_{i/2}^K(F) = F$. Then it follows from $\int F\, d\phi = \langle \tau_{i/2}^K(F), f\rangle$ that for all $s \in \mathbb{N}_0$ we have

$$\int_{\mathbb{T}} 1\, d\phi_s = (1-q^2)q^{2s} \int_0^1 f(q^s e^{2\pi i\theta})\, d\theta$$

Now let $F \in \mathrm{C}(K)$ be the function $z \mapsto 1$. Then clearly $F \in \mathrm{Dom}(\tau_{i/2}^K)$ with $\tau_{i/2}^K(F) = F$ and thus from $\int F\, d\phi = \langle \tau_{i/2}^K(F), f\rangle$ we have

$$\phi_\infty(0) + \sum_{r=0}^{\infty} \int_{\mathbb{T}} 1\, d\phi_r = (1-q^2)\sum_{r=0}^{\infty} q^{2r} \int_0^1 f(q^r e^{2\pi i\theta}) d\theta.$$

So we have $\phi_\infty(0) = 0$ as required. □

Theorem 5.3.26 *Fix $f \in \mathrm{L}^1(K,\nu)$ such that the map $\mathrm{Dom}(\tau^K_{i/2}) \cap \mathrm{C}(K) \to \mathbb{C}$ given by $F \mapsto \langle \tau^K_{i/2}(F), f\rangle$ is bounded. Then there exists some $g \in \mathrm{L}^1(K,\nu)$ such that $F \mapsto \langle F, g\rangle$ extends this given map to a map $\mathrm{C}(K) \to \mathbb{C}$. In particular it follows that $f \in \mathrm{L}^1_\sharp(K,\nu)$.*

Proof

As the map $\mathrm{Dom}(\tau^K_{i/2}) \to \mathbb{C}$ given by $F \mapsto \langle \tau^K_{i/2}(F), f\rangle$ is bounded and as $\mathrm{Dom}(\tau^K_{i/2})$ is dense in $\mathrm{C}(K)$ we can extend this to a bounded map $\mathrm{C}(K) \to \mathbb{C}$. Then identifying $\mathrm{C}(K)^*$ with $M(K)$ it follows that there is some $\phi \in M(K)$ such that

$$\int_K F\, d\phi = \langle \tau^K_{i/2}(F), f\rangle$$

for all $F \in \mathrm{Dom}(\tau^K_{i/2})$. We show that there is some $g \in \mathrm{L}^1(K,\nu)$ such that $\int_K F\, d\phi = \langle F, g\rangle$.

From Proposition 5.3.23 we have a decomposition of $\int_K F\, d\phi$. From Lemma 5.3.24 we have for each $\phi_r \in M(\mathbb{T})$ that there is some $g_r \in \mathrm{L}^1(\mathbb{T})$ such that $g_r d\theta = d\phi_r$ and from Lemma 5.3.25 we have $\phi_\infty(0) = 0$. From this we have

$$\phi(A) = \int_K \chi_A\, d\phi = \sum_{r=0}^{\infty} \int_0^1 \chi_A(q^r e^{2\pi i\theta})\, d\phi_r(\theta) = \sum_{r=0}^{\infty} \int_0^1 \chi_A(q^r e^{2\pi i\theta}) g_r(e^{2\pi i\theta})\, d\theta$$

for all measurable A on K.

Let A be a measurable subset of K such that $\nu(A) = 0$, then we have

$$\sum_{r=0}^{\infty} q^{2r} \int_0^1 \chi_A(q^r e^{2\pi i\theta})\, d\theta = 0.$$

As χ_A is positive it follows that $\int_0^1 \chi_A(q^r e^{2\pi i\theta})\, d\theta = 0$ for all $r \in \mathbb{N}_0$, i.e. A is negligible on each circle. Define $h_r : \mathbb{T} \to \mathbb{C}$ by $h_r(e^{2\pi i\theta}) = \chi_A(e^{2\pi i\theta}) g_r(e^{2\pi i\theta})$ and we have

$$\int_0^1 \left|h_r(e^{2\pi i\theta})\right| d\theta = \int_0^1 \chi_A(e^{2\pi i\theta}) \left|g_r(e^{2\pi i\theta})\right| d\theta \leqslant \int_0^1 \left|g_r(e^{2\pi i\theta})\right| d\theta < \infty$$

and so $h_r \in \mathrm{L}^1(\mathbb{T})$ and $h_r = \chi_A$ almost everywhere so $\int_0^1 h_r(e^{2\pi i\theta})\,d\theta = 0$. It follows that ϕ is absolutely continuous and so from the Radon-Nikodym theorem there exists some $g \in \mathrm{L}^1(K,\nu)$ such that $d\phi = g\,d\nu$ and furthermore we have $\langle F, g\rangle = \langle \tau^K_{i/2}(F), f\rangle$ for all $F \in \mathrm{C}(K) \cap \mathrm{Dom}(\tau^K_{i/2})$. □

5.3.4 $\mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ and $\mathrm{L}^1_\sharp(K,\nu)$

We might try to extend Theorem 5.3.26 to $\mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ after proving this for $\mathrm{L}^1_\sharp(K,\nu)$. We show however in this section that this is a non-trivial question to answer. This is still an open problem for $\mathrm{SU}_q(2)$.

It will be difficult to transcribe the proof of Theorem 5.3.26 to the $\mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ case as this relies heavily on measure theoretic techniques. However, given any $f \in \mathrm{L}^1(K,\nu)$ we have $\omega := \mathbb{E}_*(f) \in \mathrm{L}^1(\mathrm{SU}_q(2))$ where $\mathbb{E}$ is given by Theorem 5.2.10 such that for all $x \in \mathrm{L}^\infty(\mathrm{SU}_q(2))$ we have $\langle x, \omega\rangle = \langle \mathbb{E}(x), f\rangle$. We can consider "shifting" ω with the a^k and $(a^*)^k$ elements of $\mathrm{Hopf}(\mathrm{SU}_q(2))$ for $k \in \mathbb{Z}$. That is for all $k \in \mathbb{Z}$ we can consider $\omega \cdot (a^*)^k$ as the map $\langle x, \omega \cdot (a^*)^k\rangle = \langle (a^*)^k x, \omega\rangle$ for $x \in \mathrm{L}^\infty(\mathrm{SU}_q(2))$ (where we remind that for $k < 0$ we let $(a^*)^k = a^{-k}$ and $a^k = (a^*)^{-k}$). We let $g, h \in \mathrm{L}^2(K,\nu)$ such that $f = g\overline{h}$ and $\|f\|_1 = \|g\|_2\|h\|_2$. Then for all $x \in \mathrm{L}^\infty(\mathrm{SU}_q(2))$ we have

$$\langle x, \omega \cdot (a^*)^k\rangle = \langle \mathbb{E}((a^*)^k x), f\rangle = \big(\mathbb{E}((a^*)^k x)g \big| h\big) = \big((a^*)^k x Ug \big| Uh\big) = \langle x, \omega_{Ug, a^k Uh}\rangle$$

where $U : \mathrm{L}^2(K,\nu) \to \mathrm{L}^2(\mathrm{SU}_q(2))$ is the map from Proposition 5.2.8 and so we have $\omega \cdot (a^*)^k = \omega_{Ug, a^k Uh}$. We have the following easy proposition.

Proposition 5.3.27 *Let $f, g \in \mathrm{L}^2(K,\nu)$ and for $k \in \mathbb{Z}$ let $\omega_k := \omega_{U(f), a^k U(g)} \in \mathrm{L}^1(\mathrm{SU}_q(2))$. We have $\omega_k \in \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ for all $k \in \mathbb{Z}$ if and only if $\omega_{f,g} := f\overline{g} \in \mathrm{L}^1_\sharp(K,\nu)$ in which case we have $\langle x, \omega_k^\sharp\rangle = \langle \mathbb{E}((a^*)^k x), \omega_{f,g}^\sharp\rangle$.*

Proof

Say $\omega_k \in \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ for all $k \in \mathbb{Z}$, then for $F \in \mathrm{Dom}(S^K)$ we have

$$\begin{aligned}\overline{\langle S^K(F)^*, \omega_{f,g}\rangle} &= \overline{(S^K(F)^* f|g)} = \overline{(\mathbb{E}(S(\mathbb{F}(F))^*)f|g)} \\ &= \overline{(S(\mathbb{F}(F))^* U(f)|U(g))} = \overline{\langle S(\mathbb{F}(F))^*, \omega_0\rangle} = \langle F, \mathbb{F}_*(\omega_0^\sharp)\rangle\end{aligned}$$

(where ω_0 denotes ω_k with $k = 0$) and so $\omega_{f,g} \in \mathrm{L}^1_\sharp(K,\nu)$ with $\omega^\sharp_{f,g} = \mathbb{F}_*(\omega_0^\sharp)$.

Conversely, say $\omega_{f,g} \in \mathrm{L}^1_\sharp(K,\nu)$. Let $k \in \mathbb{Z}$ and $x \in \mathrm{Dom}(S)$, then using that S is an anti-homomorphism, $S(a^*) = a$ and that $\mathbb{E}$ is a $*$-map we have

$$\begin{aligned}\overline{\langle S(x)^*, \omega_k\rangle} &= \overline{(S(x)^* U(f)|a^k U(g))} = \overline{(S((a^*)^k x)^* U(f)|U(g))} = \overline{(\mathbb{E}(S((a^*)^k x))^* f|g)} \\ &= \langle S^K(\mathbb{E}((a^*)^k x)), \omega^*_{f,g}\rangle = \langle \mathbb{E}((a^*)^k x), \omega^\sharp_{f,g}\rangle = \langle x, \mathbb{E}_*(\omega^\sharp_{f,g}) \cdot (a^*)^k\rangle.\end{aligned}$$

So $\omega_k \in \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ for all $k \in \mathbb{Z}$.

We have shown that for all $x \in \mathrm{Dom}(S)$ we have $\langle x, \omega_k^\sharp\rangle = \langle \mathbb{E}((a^*)^k x), \omega^\sharp_{f,g}\rangle$ and as $\mathrm{Dom}(S)$ is weak*-dense in $\mathrm{L}^\infty(\mathrm{SU}_q(2))$ this holds for all $x \in \mathrm{L}^\infty(\mathrm{SU}_q(2))$. □

We now ask is this sufficient to capture all of $\mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ in some way? One might hope that we can build $\mathrm{L}^1_\sharp(K,\nu)$ by taking all sequences $(f_k)_{k\in\mathbb{Z}} \subset \mathrm{L}^1_\sharp(K,\nu)$ such that $\omega := \sum_{k=-\infty}^{\infty} \mathbb{E}_*(f_k) \cdot (a^*)^k$ is convergent in $\mathrm{L}^1(\mathrm{SU}_q(2))$. However we show in the next example that we can find such a sequence of functions such that $\omega \in \mathrm{L}^1(\mathrm{SU}_q(2))$ but $\omega \notin \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$. We have not yet found a satisfactory way of "reconstructing" $\mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ from $\mathrm{L}^1_\sharp(K,\nu)$ at present.

Lemma 5.3.28 *Fix $s \geqslant 0$ and let $f \in \mathrm{L}^1(K,\nu)$ be the map $q^r e^{2\pi i\theta} \mapsto \delta_{r,s} e^{2\pi i\alpha\theta}$ for $\alpha \in \mathbb{R}$. Then $f_s \in \mathrm{L}^1_\sharp(K,\nu)$ with $f^\flat(q^r e^{2\pi i\theta}) = \delta_{r,s} q^\alpha e^{2\pi i\alpha\theta}$.*

Proof

For all $r \geqslant 0$ let $\phi_r \in \mathrm{L}^1(\mathbb{T})$ be given by $\phi_r(e^{2\pi i\theta}) = f(q^r e^{2\pi i\theta})$ and let $\psi_r : \mathbb{T} \to \mathbb{C}$ be given by

$$\psi_r(e^{2\pi i\theta}) = \delta_{r,s} q^{\alpha} e^{2\pi i\alpha\theta}$$

Clearly $\psi_r \in \mathrm{L}^1(\mathbb{T})$ for all $r \geqslant 0$ and $\sum_{r=0}^{\infty} q^{2r}\|\psi_r\|_1$ is finite. We show that $\widehat{\psi_r}(l) = q^l\widehat{\phi}_r(l)$ for all $r \in \mathbb{N}_0$ and $l \in \mathbb{Z}$ then by Proposition 5.3.22 we have $f \in \mathrm{L}^1_\sharp(K,\nu)$ with $f^\flat$ given by $q^r e^{2\pi i\theta} \mapsto \psi_r(e^{2\pi i\theta})$. Fix $r \in \mathbb{N}_0$ and $l \in \mathbb{Z}$, then

$$\widehat{\phi}_r(l) = \int_0^1 f(q^r e^{2\pi i\theta}) e^{-2\pi i l\theta}\, d\theta = \delta_{r,s}\int_0^1 e^{2\pi i(\alpha-l)\theta}\, d\theta = \delta_{r,s}\delta_{l,\alpha}$$

and so using this we have

$$\widehat{\psi_r}(l) = \int_0^1 \psi_r(e^{2\pi i\theta}) e^{-2\pi i l\theta}\, d\theta = \delta_{r,s} q^{\alpha}\delta_{l,\alpha} = q^l\widehat{\phi}_r(l)$$

and thus $f \in \mathrm{L}^1_\sharp(K,\nu)$. $\square$

Lemma 5.3.29 *Fix $s \geqslant 0$ and let $f \in \mathrm{L}^1_\sharp(K,\nu)$ be the map $q^r e^{2\pi i\theta} \mapsto \delta_{r,s} e^{2\pi i\alpha\theta}$ for $\alpha \in \mathbb{R}$. Then for $k \geqslant 0$ we have $\mathbb{E}_*(f)\cdot(a^*)^k \in \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ and for $p \geqslant 0$ we have*

$$\begin{aligned}\langle a^p x, (\mathbb{E}_*(f)\cdot(a^*)^k)^\flat\rangle = \delta_{p,k}(1-q^2)q^{2k}(1-q^2)\ldots(1-q^{2k})q^{\alpha}\\ \times\int_0^1 (\mathbb{E}(x))(q^r e^{2\pi i\theta}) e^{2\pi i\alpha\theta}\, d\theta.\end{aligned}$$

Proof

We have f is non-zero everywhere and so we can define $g = \dfrac{f}{|f|^{1/2}}$ and $h = |f|^{1/2}$ where $|f|^{1/2}(z) = |f(z)|^{1/2}$ for $z \in K$. Then $g, h \in \mathrm{L}^2(K,\nu)$, $f = \omega_{g,h}$ and $\|f\|_{\mathrm{L}^1(K,\nu)} = \|g\|_{\mathrm{L}^2(K,\nu)}\|h\|_{\mathrm{L}^2(K,\nu)}$. Then we can identify $\mathbb{E}_*(f)\cdot(a^*)^k = \omega_{Ug,a^kUh}$ and it follows from Proposition 5.3.27 that $\mathbb{E}_*(f)\cdot(a^*)^k \in \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$.

Let $x \in \mathrm{C}^*(c,1) \cap \mathrm{Dom}(\tau_{i/2})$, then using that $\tau_{i/2}$ is a homomorphism by Proposition 3.2.20 and that $a^* \in \mathrm{Dom}(\tau_{i/2})$ with $\tau_{i/2}(a^*) = a^*$ by Corollary 5.1.9 we have

$$\begin{aligned}\langle a^p x, (\mathbb{E}_*(f)\cdot(a^*)^k)^\flat\rangle &= \langle (a^*)^k \tau_{i/2}(a^p x), \mathbb{E}_*(f)\rangle \\ &= \langle \tau^K_{i/2}(\mathbb{E}((a^*)^k a^p x)), f_k\rangle = \langle \mathbb{E}((a^*)^k a^p x), f_k^\flat\rangle\end{aligned}$$

where we've also used Proposition 5.3.8. We can use Relations 5.1 to calculate $(a^*)^k a^p$ dependent on p. If $p \geqslant k$ we have

$$\begin{aligned}(a^*)^k a^p &= (a^*)^{k-1}(1-c^*c)a^{p-1} = (a^*)^{k-1}a^{p-1}(1-q^{-2(p-1)}c^*c) \\ &= (a^*)^{k-2}a^{p-2}(1-q^{-2(p-2)}c^*c)(1-q^{-2(p-1)}c^*c) \\ &= \cdots = a^{p-k}(1-q^{-2(p-k)}c^*c)\ldots(1-q^{-2(p-1)}c^*c)\end{aligned}$$

and similarly if $p < k$ we have

$$(a^*)^k a^p = (a^*)^{k-p}(1-c^*c)(1-q^{-2}c^*c)\ldots(1-q^{-2(p-1)}c^*c).$$

So it follows that $\mathbb{E}((a^*)^k a^p x) = \delta_{p,k}(1-\underline{z^*z})\ldots(1-q^{-2(k-1)}\underline{z^*z})\mathbb{E}(x)$ and using Lemma 5.3.28 for $f^\flat$ we have

$$\begin{aligned}\langle a^p x, (\mathbb{E}_*(f)\cdot(a^*)^k)^\flat\rangle &= \delta_{k,p}\langle(1-\underline{z^*z})\ldots(1-q^{-2(k-1)}\underline{z^*z})\mathbb{E}(x), f_k^\flat\rangle \\ &= \delta_{k,p}(1-q^2)\sum_{r=0}^{\infty} q^{2r}(1-q^{2r})(1-q^{2r-2})\ldots(1-q^{2r-2(k-1)})\delta_{r,k}q^{\alpha} \\ &\qquad\qquad \times \int_0^1 (\mathbb{E}(x))(q^r e^{2\pi i\theta})e^{2\pi i\alpha\theta}\,d\theta \\ &= \delta_{k,p}(1-q^2)q^{2k}(1-q^2)\ldots(1-q^{2k})q^{\alpha}\int_0^1 (\mathbb{E}(x))(q^r e^{2\pi i\theta})e^{2\pi i\alpha\theta}\,d\theta. \quad \square\end{aligned}$$

Example 5.3.30 *For $k < 0$ let $f_k = 0$ and for $k \geqslant 0$ let $f_k \in \mathrm{L}^1(K,\nu)$ be given by*

$$f_k(q^r e^{2\pi i\theta}) = \delta_{r,k}e^{2\pi i(-4k-\alpha(k))\theta}$$

where for each k we choose $\alpha(k) \in \mathbb{N}$ such that $\alpha(k) \geqslant \dfrac{\ln(1-q^2) + \sum_{i=1}^{k} \ln(1-q^2)}{\ln q}$.

Clearly $f_k \in \mathrm{L}^1_\sharp(K,\nu)$ for all $k \in \mathbb{Z}$ and from Lemma 5.3.28 we have $f_k \in \mathrm{L}^1_\sharp(K,\nu)$ for all $k \geqslant 0$ with $f_k^\flat(q^r e^{2\pi i\theta}) = \delta_{r,k} e^{2\pi i(-4k-\alpha(k))\theta}$ for $k \geqslant 0$. From the proof of Lemma 5.3.29 we have $g_k, h_k \in \mathrm{L}^2(K,\nu)$ such that $f_k = \omega_{g_k,h_k}$ and $\|f_k\|_{\mathrm{L}^1(K,\nu)} = \|g_k\|_{\mathrm{L}^2(K,\nu)}\|h_k\|_{\mathrm{L}^2(K,\nu)}$. Now define $\omega := \sum_{k=0}^{\infty} \omega_{U(g_k),a^kU(h_k)}$ and we will show that $\omega \in \mathrm{L}^1(\mathrm{SU}_q(2))$ but $\omega \notin \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$.

We have $\|f_k\| = (1-q^2)q^{2k}$ for all $k \geqslant 0$ and so

$$\|\omega\| \leqslant \sum_{k=0}^{\infty} \|\omega_{U(g_k),a^kU(h_k)}\| = \sum_{k=0}^{\infty} \|U(g_k)\|\|a^kU(h_k)\| \leqslant \sum_{k=0}^{\infty} \|g_k\|\|h_k\| = \sum_{k=0}^{\infty} \|f_k\| = 1$$

and thus $\omega \in \mathrm{L}^1(\mathrm{SU}_q(2))$.

Let $x \in \mathrm{Dom}(\tau_{i/2})$, then from Proposition 5.3.27 we have $\omega_{U(g_k),a^kU(h_k)} \in \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ for all $k \in \mathbb{Z}$ and so from Proposition 4.1.11 and Notation 4.1.12 we have

$$\langle \tau_{i/2}(x), \omega \rangle = \sum_{k=0}^{\infty} \langle \tau_{i/2}(x), \omega_{U(g_k),a^kU(h_k)} \rangle = \sum_{k=0}^{\infty} \langle x, \omega^\flat_{U(g_k),a^kU(h_k)} \rangle.$$

So it follows that $\omega \in \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ if and only if $\sum_{k=0}^{\infty} \omega^\flat_{U(g_k),a^kU(h_k)} \in \mathrm{L}^1(\mathrm{SU}_q(2))$.

As $\mathrm{C}(K) \cong_i \mathrm{C}^*(c,1)$ we consider $F \in \mathrm{C}(K)$ given by $q^r e^{2\pi i\theta} \mapsto \delta_{r,k} e^{2\pi i(4k+\alpha(k))\theta}$ (and $F(0) = 0$), then we have

$$\|F\|_{\mathrm{C}(K)} = \sup_{z \in K} |F(z)| = \sup_{\theta \in [0,1)} \left|F(q^k e^{2\pi i\theta})\right| = 1$$

and

$$\left|\int_0^1 F(q^k e^{2\pi i\theta}) e^{2\pi i(-4k-\alpha(k))\theta}\, d\theta\right| = \left|\int_0^1 1\, d\theta\right| = 1.$$

So letting $x \in \mathrm{C}^*(c,1)$ with $\mathbb{E}(x) = F$ we have from Lemma 5.3.29 that

$$\left|\langle a^k x, \omega^\flat_{U(g_k),a^kU(h_k)} \rangle\right| = (1-q^2)q^{-2k-\alpha(k)} \prod_{l=1}^{k} (1-q^{2l}).$$

We have chosen $\alpha(k) \in \mathbb{N}$ *such that* $\alpha(k) \geqslant \dfrac{\ln(1-q^2) + \sum_{l=1}^{k} \ln(1-q^{2l})}{\ln q}$. *As* $0 < q < 1$ *then* $\ln q < 0$ *and so*

$$\alpha(k) \ln q \leqslant \ln(1-q^2) + \sum_{l=1}^{k} \ln(1-q^{2l})$$

and taking exponentials we have

$$q^{\alpha(k)} \leqslant (1-q^2) \prod_{l=1}^{k} (1-q^{2l})$$

meaning

$$q^{-\alpha(k)} \geqslant \frac{1}{(1-q^2) \prod_{l=1}^{k} (1-q^{2l})}.$$

Then substituting this above we have

$$\left| \left\langle a^k x, \omega^{\flat}_{U(g_k), a^k U(h_k)} \right\rangle \right| \geqslant q^{-2k}$$

and so

$$\left| \left\langle a^k x, \sum_{k=0}^{\infty} \omega^{\flat}_{U(g_k), a^k U(h_k)} \right\rangle \right| = \left| \left\langle a^k x, \omega^{\flat}_{U(g_k), a^k U(h_k)} \right\rangle \right| \geqslant q^{-2k}$$

and $\left\| \sum_{k=0}^{\infty} \omega^{\flat}_{U(g_k), a^k U(h_k)} \right\| \geqslant q^{-2k}$ *for all* $k \geqslant 0$. *In particular we have shown that* $\sum_{k=0}^{\infty} \omega^{\flat}_{U(g_k), a^k U(h_k)}$ *does not converge and is not in* $\mathrm{L}^1(\mathrm{SU}_q(2))$ *and therefore* $\omega \notin \mathrm{L}^1_{\sharp}(\mathrm{SU}_q(2))$.

5.4 $\mathrm{SU}_q(2) \times \mathrm{SU}_{q'}(2)$

In this section we investigate the quantum group product $\mathrm{SU}_q(2) \times \mathrm{SU}_{q'}(2)$ for $q, q' \in (-1,1) \backslash \{0\}$ as per Definition-Theorem 2.5.2. In investigating this we will show that $\mathrm{L}^1_{\sharp}(\mathrm{SU}_q(2) \times \mathrm{SU}_q(2))$ is not isometrically isomorphic to $\mathrm{L}^1_{\sharp}(\mathrm{SU}_q(2)) \widehat{\otimes} \mathrm{L}^1_{\sharp}(\mathrm{SU}_q(2))$ answering a question posed at the beginning of Section 4.2.3.

Take two compact matrix quantum groups $(\mathrm{C}(\mathrm{SU}_q(2)), \Delta, u)$ and $(\mathrm{C}(\mathrm{SU}_{q'}(2)), \nabla, v)$.

Let a, c and a', c' denote the generators of $\mathrm{C}(\mathrm{SU}_q(2))$ and $\mathrm{C}(\mathrm{SU}_{q'}(2))$ respectively and then we have $u = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}$ and $v = \begin{pmatrix} a' & -q'c'^* \\ c' & a'^* \end{pmatrix}$. We consider the quantum group $\mathrm{SU}_q(2) \times \mathrm{SU}_{q'}(2)$. It follows from Proposition 3.2.26 and Theorem 3.2.27 that $\mathrm{SU}_q(2) \times \mathrm{SU}_{q'}(2)$ is a compact matrix quantum group with generators $a \otimes 1, c \otimes 1, 1 \otimes a', 1 \otimes c'$ for $\mathrm{Hopf}(\mathrm{SU}_q(2) \times \mathrm{SU}_{q'}(2))$.

Let $K = \{q^r e^{2\pi i\theta} \mid r \in \mathbb{N}_0, \theta \in [0,1]\} \cup \{0\}$ with measures

$$\nu(A) = (1-q^2)\sum_{r=0}^{\infty} q^{2r} \int_0^1 \chi_A(q^r e^{2\pi i\theta})\, d\theta$$

and similarly for K'. We will now investigate properties of $\mathrm{SU}_q(2) \times \mathrm{SU}_{q'}(2)$ and $K \times K'$ similar to those given previously for $\mathrm{SU}_q(2)$ and K.

Proposition 5.4.1 *We have completely isometric isomorphisms*

$$\mathrm{C}(K \times K') \cong_{ci} \mathrm{C}(K) \otimes_{min} \mathrm{C}(K') \cong_{ci} \mathrm{C}^*(c,1) \otimes_{min} \mathrm{C}^*(c',1) \cong_{ci} \mathrm{C}^*(c\otimes 1, 1\otimes c', 1\otimes 1).$$

Proof

We know from C*-algebra theory that $\mathrm{C}(K \times K') \cong_i \mathrm{C}(K) \otimes_{min} \mathrm{C}(K')$. Also from Proposition 5.2.4 we have that $\mathrm{C}(K) \cong_i \mathrm{C}^*(c,1)$ and $\mathrm{C}(K') \cong_i \mathrm{C}^*(c',1)$ and therefore $\mathrm{C}(K) \otimes_{min} \mathrm{C}(K') \cong_i \mathrm{C}^*(c,1) \otimes_{min} \mathrm{C}^*(c',1)$. We have a unital commutative C*-algebra $\mathrm{C}^*(c\otimes 1, 1\otimes c', 1\otimes 1)$ as $c\otimes 1$ and $1\otimes c'$ are normal and commute and we have isometric $*$-homomorphisms $\mathrm{C}^*(c,1) \hookrightarrow \mathrm{C}(\mathrm{SU}_q(2))$ and $\mathrm{C}^*(c',1) \hookrightarrow \mathrm{C}(\mathrm{SU}_{q'}(2))$. Then by Proposition IV.4.22 in Takesaki (2003a) it follows easily that we have an isometric $*$-homomorphism

$$\mathrm{C}(K) \otimes_{min} \mathrm{C}(K') \hookrightarrow \mathrm{C}(\mathrm{SU}_q(2)) \otimes_{min} \mathrm{C}(\mathrm{SU}_{q'}(2)) = \mathrm{C}(\mathrm{SU}_q(2) \otimes_{min} \mathrm{SU}_{q'}(2)).$$

Clearly the image of $\mathrm{C}(K \times K') \cong_i \mathrm{C}(K) \otimes_{min} \mathrm{C}(K')$ under this map is $\mathrm{C}^*(c \otimes 1, 1 \otimes c', 1 \otimes 1)$ and so we have $\mathrm{C}(K \times K') \cong_i \mathrm{C}^*(c \otimes 1, 1 \otimes c', 1 \otimes 1)$ as required. These are then all completely isometric isomorphisms as they are C*-algebraic $*$-isomorphisms. □

For convenience we will often identify $\mathrm{L}^1(K \times K', \nu \times \nu')$ with $\mathrm{L}^1(K, \nu) \widehat{\otimes} \mathrm{L}^1(K', \nu')$ as per the following proposition.

Proposition 5.4.2 *We have a* $\mathrm{L}^1(K, \nu) \widehat{\otimes} \mathrm{L}^1(K', \nu') \cong_{ci} \mathrm{L}^1(K \times K', \nu \times \nu')$ *where* $(f \otimes g)(q^r e^{2\pi i\theta}, (q')^{r'} e^{2\pi i\theta'}) = f(q^r e^{2\pi i\theta}) g((q')^{r'} e^{2\pi i\theta'})$ *for all* $f \in \mathrm{L}^1(K, \nu)$ *and* $g \in \mathrm{L}^1(K', \nu')$.

Proof

We have that $\mathrm{L}^\infty(K, \nu)$ is a commutative von Neumann algebra with the minimal operator space structure and so, by Proposition, 1.1.24 its predual $\mathrm{L}^1(K, \nu)$ has the maximal operator space structure. By Proposition 1.1.48 we have $\mathrm{L}^1(K, \nu) \widehat{\otimes} \mathrm{L}^1(K', \nu') \cong_{ci} \mathrm{MAX}(\mathrm{L}^1(K, \nu) \hat{\otimes} \mathrm{L}^1(K', \nu'))$. As ν and ν' are clearly σ-finite it follows from a result in Banach space and measure theory (see Chapter 2 in Ryan (2002)) that as Banach spaces we have $\mathrm{L}^1(K, \nu) \hat{\otimes} \mathrm{L}^1(K', \nu') \cong_i \mathrm{L}^1(K \times K', \nu \times \nu')$ with the equation given in the theorem. The result follows as $\mathrm{L}^1(K \times K', \nu \times \nu')$ has the maximal operator space structure as the predual of the space $\mathrm{L}^\infty(K \times K', \nu \times \nu')$ with minimal operator space structure. □

For ease of notation, in the remainder of this section we use a superscript q for an operator on $\mathrm{SU}_q(2)$, q' for an operator on $\mathrm{SU}_{q'}(2)$ and $q \times q'$ for an operator on $\mathrm{SU}_q(2) \times \mathrm{SU}_{q'}(2)$. We also consider the K spaces and we will similarly use superscripts K, K' and $K \times K'$.

Proposition 5.4.3 *Using that* $\mathrm{L}^\infty(\mathrm{SU}_q(2) \times \mathrm{SU}_{q'}(2)) = \mathrm{L}^\infty(\mathrm{SU}_q(2)) \overline{\otimes} \mathrm{L}^\infty(\mathrm{SU}_{q'}(2))$ *and* $\mathrm{L}^1(K \times K', \nu \times \nu') \cong_{ci} \mathrm{L}^1(K, \nu) \widehat{\otimes} \mathrm{L}^1(K', \nu')$ *we have a normal $*$-map and complete quotient map given by*

$$\mathbb{E}^{q \times q'} := \mathbb{E}^q \otimes \mathbb{E}^{q'} : \mathrm{L}^\infty(\mathrm{SU}_q(2) \times \mathrm{SU}_{q'}(2)) \to \mathrm{L}^\infty(K \times K', \nu \times \nu')$$

and a normal $*$-homomorphism and completely isometric embedding given by

$$\mathbb{F}^{q \times q'} := \mathbb{F}^q \otimes \mathbb{F}^{q'} : \mathrm{L}^\infty(K \times K', \nu \times \nu') \to \mathrm{L}^\infty(\mathrm{SU}_q(2) \times \mathrm{SU}_{q'}(2)).$$

We have that $\mathbb{F}^{q\times q'}$ *is a right inverse to* $\mathbb{E}^{q\times q'}$ *and for all* $x \in \overline{\mathrm{C}^*(c\otimes 1, 1\otimes c', 1\otimes 1)}^{w*}$ *we have* $(\mathbb{F}^{q\times q'} \circ \mathbb{E}^{q\times q'})(x) = x$ *and* $\mathrm{Image}\,\mathbb{F}^{q\times q'} = \overline{\mathrm{C}^*(c\otimes 1, 1\otimes c', 1\otimes 1)}^{w*}$.

Proof

From the maps $\mathbb{E}^q : \mathrm{L}^\infty(\mathrm{SU}_q(2)) \to \mathrm{L}^\infty(K,\nu)$ and $\mathbb{F}^q : \mathrm{L}^\infty(K,\nu) \to \mathrm{L}^\infty(\mathrm{SU}_q(2))$ given by Theorems 5.2.9 and 5.2.10 we have complete contractions $\mathbb{E}^q \otimes \mathbb{E}^{q'}$ and $\mathbb{F}^q \otimes \mathbb{F}^{q'}$. Clearly we have $\mathbb{E}^{q\times q'} \circ \mathbb{F}^{q\times q'} = \mathrm{id}_{\mathrm{L}^\infty(K\times K',\nu\times\nu')}$ and so it follows from Lemma 1.1.20 that $\mathbb{E}^{q\times q'}$ is a complete quotient map and $\mathbb{F}^{q\times q'}$ is a completely isometric embedding.

The map $\mathbb{E}^{q\times q'}$ is normal as it has a pre-adjoint map given by $\mathbb{E}^q{}_* \otimes \mathbb{E}^{q'}{}_*$ and it is a $*$-map as $\mathbb{E}^q$ and $\mathbb{E}^{q'}$ are $*$-maps. Similarly $\mathbb{F}^{q\times q'}$ is a normal $*$-homomorphism. For $x \in \overline{\mathrm{C}^*(c\otimes 1, 1\otimes c', 1\otimes 1)}^{w*}$ we have

$$(\mathbb{F}^{q\times q'} \circ \mathbb{E}^{q\times q'})(x) = ((\mathbb{F}^q \circ \mathbb{E}^q) \otimes (\mathbb{F}^{q'} \circ \mathbb{E}^{q'}))(x) = x.$$

It then follows that $\mathrm{Image}\,\mathbb{F}^{q\times q'} \supset \overline{\mathrm{C}^*(c\otimes 1, 1\otimes c', 1\otimes 1)}^{w*}$. We know that $\mathbb{F}^q|_{\mathrm{C}(K)}$ has image $\mathrm{C}^*(c,1)$ and similarly for $\mathbb{F}^{q'}$ and so we have $\mathrm{Image}\ \mathbb{F}^q \otimes \mathbb{F}^{q'}|_{\mathrm{C}(K)\otimes_{min}\mathrm{C}(K')} = \mathrm{C}^*(c,1) \otimes \mathrm{C}^*(c',1) = \mathrm{C}^*(c\otimes 1, 1\otimes c', 1\otimes 1)$. Then taking the weak*-closure we have that $\mathrm{Image}\,\mathbb{F}^{q\times q'} \subset \overline{\mathrm{C}^*(c\otimes 1, 1\otimes c', 1\otimes 1)}^{w*}$. □

Similarly to the case of $\mathrm{SU}_q(2)$ we define the following notation.

Notation 5.4.4 *Let* $\mathbb{P}^{q\times q'} : \mathrm{L}^\infty(\mathrm{SU}_q(2) \times \mathrm{SU}_{q'}(2)) \to \overline{\mathrm{C}^*(c,1) \otimes \mathrm{C}^*(c',1)}^{w*}$ *denote the map* $\mathbb{F}^{q\times q'} \circ \mathbb{E}^{q\times q'}$.

We define the scaling group on $L^\infty(K\times K',\nu\times\nu')$ and then prove some properties related to this. We could also define the antipode and unitary antipode in the obvious way but these will not be needed.

Definition 5.4.5 *For* $z \in \mathbb{C}$ *let*

$$\mathrm{Dom}(\tau_z^{K\times K'}) = \left\{ f \in \mathrm{L}^\infty(K\times K', \nu\times\nu') \;\middle|\; \mathbb{F}^{q\times q'}(f) \in \mathrm{Dom}(\tau_z^{q\times q'}) \right\}$$

and let $\tau_z^{K\times K'} : \mathrm{L}^\infty(K \times K', \nu \times \nu') \to \mathrm{L}^\infty(K \times K', \nu \times \nu')$ *be given by* $\tau_z^{K\times K'}(f) = \mathbb{E}^{q\times q'}(\tau_z^{q\times q'}(\mathbb{F}^{q\times q'}(f)))$ *for* $f \in \mathrm{Dom}(\tau_z^{K\times K'})$.

The proof of the following proposition is almost identical to that of Proposition 5.3.2 so we omit the details.

Proposition 5.4.6 *Let* $z \in \mathbb{C}$ *and* $x \in \mathrm{Dom}(\tau_z^{q\times q'})$*, then* $\mathbb{P}^{q\times q'}(x) \in \mathrm{Dom}(\tau_z^{q\times q'})$ *with*

$$\tau_z^{q\times q'}(\mathbb{P}^{q\times q'}(x)) = \mathbb{P}^{q\times q'}(\tau_z^{q\times q'}(x)).$$

Corollary 5.4.7 *For* $z \in \mathbb{C}$ *and* $x \in \mathrm{Dom}(\tau_z^{q\times q'})$ *we have* $\mathbb{E}^{q\times q'}(x) \in \mathrm{Dom}(\tau_z^{K\times K'})$ *and* $(\tau_z^{K\times K'} \circ \mathbb{E}^{q\times q'})(x) = (\mathbb{E}^{q\times q'} \circ \tau_z^{q\times q'})(x)$. *Similarly for* $F \in \mathrm{Dom}(\tau_z^{K\times K'})$ *then* $\mathbb{F}^{q\times q'}(F) \in \mathrm{Dom}(\tau_z^{q\times q'})$ *and* $(\tau_z^{q\times q'} \circ \mathbb{F}^{q\times q'})(F) = (\mathbb{F}^{q\times q'} \circ \tau_z^{K\times K'})(F)$.

The proof of the following proposition is similar to that of Proposition 5.3.9.

Proposition 5.4.8 *For* $z \in \mathbb{C}$ *we have* $\mathrm{Poly}(K) \odot \mathrm{Poly}(K')$ *is a core for* $\tau_z^{K\times K'}$.

Proof

We have that $\mathrm{Hopf}(\mathrm{SU}_q(2) \times \mathrm{SU}_{q'}(2)) = \mathrm{Hopf}(\mathrm{SU}_q(2)) \odot \mathrm{Hopf}(\mathrm{SU}_{q'}(2))$ is a core for $\tau_z^{q\times q'}$ by Proposition 3.2.19, then for $F \in \mathrm{Dom}(\tau_z^{K\times K'})$ we have $\mathbb{F}^{q\times q'}(F) \in \mathrm{Dom}(\tau_z^{q\times q'})$ and so there is a net $(x_\alpha) \subset \mathrm{Hopf}(\mathrm{SU}_q(2) \times \mathrm{SU}_{q'}(2))$ such that

$$\|\mathbb{F}^{q\times q'}(F) - x_\alpha\| \to 0 \qquad \text{and} \qquad \|\tau_z^{q\times q'}(\mathbb{F}^{q\times q'}(F)) - \tau_z^{q\times q'}(x_\alpha)\| \to 0.$$

We have $\mathbb{E}^{q\times q'}(x_\alpha) \in \mathrm{Poly}(K) \odot \mathrm{Poly}(K')$ as $\mathbb{E}^{q\times q'} = \mathbb{E}^q \otimes \mathbb{E}^{q'}$ and $\mathbb{F}^{q\times q'} = \mathbb{F}^q \otimes \mathbb{F}^{q'}$ and so

$$\begin{aligned}\mathbb{F}^{q\times q'}(\mathbb{E}^{q\times q'}(x_\alpha)) &= \Big((\mathbb{F}^q \circ \mathbb{E}^q) \otimes (\mathbb{F}^{q'} \circ \mathbb{E}^{q'})\Big)(x_\alpha) = (P^q \otimes P^{q'})(x_\alpha) \\ &\in \big(\mathrm{Hopf}(\mathrm{SU}_q(2)) \cap \mathrm{C}^*(c,1)\big) \odot \big(\mathrm{Hopf}(\mathrm{SU}_{q'}(2)) \cap \mathrm{C}^*(c',1)\big).\end{aligned}$$

We have

$$\|\mathbb{E}^{q\times q'}(x_\alpha) - F\| = \|\mathbb{E}^{q\times q'}(x_\alpha) - \mathbb{E}^{q\times q'}(\mathbb{F}^{q\times q'}(F))\| \leqslant \|x_\alpha - \mathbb{F}^{q\times q'}(F)\| \to 0$$

and from Proposition 5.4.6 we have

$$\begin{aligned}\|\tau_z^{K\times K'}(\mathbb{E}^{q\times q'}(x_\alpha)) - \tau_z^{K\times K'}(F)\| &= \|\mathbb{E}^{q\times q'}(\tau_z^{q\times q'}(\mathbb{P}^{q\times q'}(x_\alpha))) - \mathbb{E}^{q\times q'}(\tau_z^{q\times q'}(\mathbb{F}^{q\times q'}(F)))\| \\ &\leqslant \|\tau_z^{q\times q'}(x_\alpha) - \tau_z^{q\times q'}(\mathbb{F}^{q\times q'}(F))\| \to 0\end{aligned}$$

as required. □

The proof of the following is very similar to that of Theorem 5.3.12.

Theorem 5.4.9 *Let $f \in \mathrm{L}^1(K\times K', \nu\times\nu')$. Then $\omega := \mathbb{E}_*^{q\times q'}(f) \in \mathrm{L}^1_\sharp(\mathrm{SU}_q(2)\times \mathrm{SU}_{q'}(2))$ if and only if there exists a $g \in \mathrm{L}^1(K\times K', \nu\times\nu')$ such that for all $F \in \mathrm{Dom}(\tau_{i/2}^{K\times K'})$ we have*

$$\langle F, g\rangle = \langle \tau_{i/2}^{K\times K'}(F), f\rangle.$$

We now make the following definition of $\mathrm{L}^1_\sharp(K\times K', \nu\times\nu')$. This is slightly different to the definition of $\mathrm{L}^1_\sharp(K,\nu)$. We could have started with a definition similar to that of Definition 5.3.13 and we would have derived our definition as a consequence. The two definitions can be shown to be equivalent using similar techniques as we have used in Proposition 5.3.17 for example, and we choose this definition here as this is all we require for our main result in this section.

Definition 5.4.10 *We define*

$$\mathrm{L}^1_\sharp(K\times K', \nu\times\nu') := \left\{ f \in \mathrm{L}^1(K\times K', \nu\times\nu') \;\middle|\; \begin{array}{l} \exists\, g \in \mathrm{L}^1(K\times K', \nu\times\nu') \text{ such that} \\ \quad \langle \tau_{i/2}^{K\times K'}(F), f\rangle = \langle F, g\rangle \quad \forall\, F \in \mathrm{Dom}(\tau_{i/2}^{K\times K'}) \end{array} \right\}.$$

We denote the map g in the definition of $\mathrm{L}^1_\sharp(K \times K, \nu \times \nu)$ by $f^\flat$ for $f \in \mathrm{L}^1_\sharp(K \times K, \nu \times \nu)$. Using a very similar proof to that of Proposition 4.1.3 we can show that this is a Banach space with norm

$$\|f\|_{\mathrm{L}^1_\sharp(K \times K', \nu \times \nu')} = \max\{\|f\|_{\mathrm{L}^1(K \times K', \nu \times \nu')}, \|f^\flat\|_{\mathrm{L}^1(K \times K', \nu \times \nu')}\}$$

for $f \in \mathrm{L}^1_\sharp(K \times K', \nu \times \nu')$. We wish to introduce the operator space structure on $\mathrm{L}^1_\sharp(K \times K', \nu \times \nu')$ next. The next two propositions have proofs that follow similarly to that of Propositions 5.3.14 and 5.3.16.

Proposition 5.4.11 *The map* ${\mathbb{E}^{q\times q'}}^\sharp_* : \mathrm{L}^1_\sharp(K \times K', \nu \times \nu') \to \mathrm{L}^1_\sharp(\mathrm{SU}_q(2) \times \mathrm{SU}_{q'}(2))$ *as the restriction and corestriction of the map* $\mathbb{E}^{q\times q'}_* : \mathrm{L}^1(K \times K', \nu \times \nu') \to \mathrm{L}^1(\mathrm{SU}_q(2) \times \mathrm{SU}_q(2))$ *from Proposition 5.4.3 is an isometric embedding.*

Definition 5.4.12 *We let* $\mathrm{L}^1_\sharp(K \times K', \nu \times \nu')$ *have the operator space structure such that the map* ${\mathbb{E}^{q\times q'}}^\sharp_*$ *from the preceding proposition is a completely isometry.*

Proposition 5.4.13 *(i) The map* $\Psi^{K \times K'} : \mathrm{L}^1_\sharp(K \times K', \nu \times \nu') \to \mathrm{L}^1(K \times K', \nu \times \nu') \oplus_\infty \mathrm{L}^1(K \times K', \nu \times \nu')$ *given by* $f \mapsto (f, f^\flat)$ *is a complete isometry.*

(ii) For $\omega \in \mathrm{L}^1_\sharp(\mathrm{SU}_q(2) \times \mathrm{SU}_{q'}(2))$ *we have* $\mathbb{F}_*(\omega) \in \mathrm{L}^1_\sharp(K \times K', \nu \times \nu')$ *and we have a map* ${\mathbb{F}^{q\times q'}}^\sharp_* : \mathrm{L}^1_\sharp(\mathrm{SU}_q(2)) \to \mathrm{L}^1_\sharp(K, \nu)$ *that is the restriction and corestriction of* $\mathbb{F}^{q\times q}_*$*. Furthermore this map is a complete quotient map that is a left inverse of* ${\mathbb{E}^{q\times q'}}^\sharp_*$*.*

We now concentrate on $\mathrm{SU}_q(2) \times \mathrm{SU}_q(2)$ and we answer the question from Section 4.2.3 by showing that $\mathrm{L}^1_\sharp(\mathrm{SU}_q(2)) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ is not completely isometrically isomorphic to $\mathrm{L}^1_\sharp(\mathrm{SU}_q(2) \times \mathrm{SU}_q(2))$. Assume there exists a completely isometric isomorphism $T : \mathrm{L}^1_\sharp(\mathrm{SU}_q(2)) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathrm{SU}_q(2)) \to \mathrm{L}^1_\sharp(\mathrm{SU}_q(2) \times \mathrm{SU}_q(2))$. Then as $\mathrm{L}^1_\sharp$-algebras are dense in

L^1-algebras the following diagram must commute

$$\begin{array}{ccc} \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))\,\widehat{\otimes}\,\mathrm{L}^1_\sharp(\mathrm{SU}_q(2)) & \xrightarrow{T} & \mathrm{L}^1_\sharp(\mathrm{SU}_q(2)\times\mathrm{SU}_q(2)) \\ {\scriptstyle \iota^q\otimes\iota^q}\downarrow & & \downarrow{\scriptstyle \iota^{q\times q}} \\ \mathrm{L}^1(\mathrm{SU}_q(2))\,\widehat{\otimes}\,\mathrm{L}^1(\mathrm{SU}_q(2)) & = & \mathrm{L}^1(\mathrm{SU}_q(2)\times\mathrm{SU}_q(2)) \end{array}$$

and so T is the completely contractive map from Theorem 4.2.14. We show in the following counterexample that this map cannot give us a completely isometric isomorphism after some preparatory lemmas.

Lemma 5.4.14 *Let $\mathbb{P}^{q\sharp}_* := \mathbb{E}^{q\sharp}_* \circ \mathbb{F}^{q\sharp}_*$ and $\mathbb{P}^{q\times q\sharp}_* := \mathbb{F}^{q\times q\sharp}_* \circ \mathbb{E}^{q\times q\sharp}_*$. Then $T\circ(\mathbb{P}^{q\sharp}_*\otimes\mathbb{P}^{q\sharp}_*) = \mathbb{P}^{q\times q\sharp}_* \circ T$ where T is the completely contractive map from Theorem 4.2.14.*

Proof

We show that $\iota^q \circ \mathbb{P}^{q\sharp}_* = \mathbb{P}^q_* \circ \iota^q$ first. Let $\omega \in \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ and $x \in \mathrm{L}^\infty(\mathrm{SU}_q(2))$, then we have $(\iota^q \circ \mathbb{P}^{q\sharp}_*)(\omega) \in \mathrm{L}^1(\mathrm{SU}_q(2))$ and so have that following

$$\langle x, (\iota^q \circ \mathbb{P}^{q\sharp}_*)(\omega)\rangle = \langle x, \mathbb{P}^q_*(\omega)\rangle = \langle \mathbb{P}^q(x), \iota^q(\omega)\rangle = \langle x, (\mathbb{P}^q_* \circ \iota^q)(\omega)\rangle$$

where we've used that $\mathbb{P}^{q\sharp}_*$ is the restriction of $\mathbb{P}^q_*$. Then as this holds for all $x \in \mathrm{L}^\infty(\mathrm{SU}_q(2))$ and $\omega \in \mathrm{L}^1_\sharp(\mathrm{SU}_q(2))$ we have the formula stated. We can show similarly that $\iota^{q\times q}\circ\mathbb{P}^{q\times q\sharp}_* = \mathbb{P}^{q\times q}_* \circ \iota^{q\times q}$.

Let $\Omega \in \mathrm{L}^1_\sharp(\mathrm{SU}_q(2)\times\mathrm{SU}_q(2))$, then by Theorem 4.2.14 we have $\iota^{q\times q}\circ T = \iota^q \otimes \iota^q$ and so

$$\begin{aligned}(T\circ(\mathbb{P}^{q\sharp}_*\otimes\mathbb{P}^{q\sharp}_*))(\Omega) &= (\iota^{q\times q}\circ T\circ(\mathbb{P}^{q\sharp}_*\otimes\mathbb{P}^{q\sharp}_*))(\Omega) = ((\iota^q\otimes\iota^q)\circ(\mathbb{P}^{q\sharp}_*\otimes\mathbb{P}^{q\sharp}_*))(\Omega) \\ &= ((\mathbb{P}^q_*\otimes\mathbb{P}^q_*)\circ(\iota^q\otimes\iota^q))(\Omega) = (\mathbb{P}^{q\times q}_*\circ\iota^{q\times q}\circ T)(\Omega) = (\mathbb{P}^{q\times q\sharp}_*\circ T)(\Omega)\end{aligned}$$

where we've used that $\mathbb{P}^{q\times q}_* = \mathbb{P}^q_*\otimes\mathbb{P}^q_*$ by Proposition 5.4.3. □

Lemma 5.4.15 *Fix $s, l \geqslant 0$ and let $G \in \mathrm{L}^1(K \times K, \nu \times \nu)$ be the map*

$$G(q^r e^{2\pi i\theta}, q^{r'} e^{2\pi i\theta'}) = \delta_{r,s}\delta_{r',s}\frac{1}{(1-q^2)^2}e^{-2\pi il\theta+2\pi il\theta'}.$$

Then $G \in \mathrm{L}^1_\sharp(K \times K, \nu \times \nu)$ with $G = G^\flat$.

Proof

For $F \in \mathrm{C}(K \times K)$ it follows from Definition 5.4.5, Propositions 5.4.3 and 2.5.5 that

$$\tau_t^{K\times K} = \mathbb{E}^{q\times q} \circ \tau_t^{q\times q} \circ \mathbb{F}^{q\times q} = (\mathbb{E}^q \otimes \mathbb{E}^q) \circ (\tau_t^q \otimes \tau_t^q) \circ (\mathbb{F}^q \otimes \mathbb{F}^q) = \tau_t^K \otimes \tau_t^K$$

and so

$$(\tau_t^{K\times K}(F))(q^r e^{2\pi i\theta}, q^{r'} e^{2\pi i\theta'}) = F(q^r e^{2\pi i\theta+2it\ln q}, q^{r'} e^{2\pi i\theta'+2it\ln q}).$$

Let $m, n, m', n' \in \mathbb{N}_0$ and consider $F = \underline{z}^{*m}\underline{z}^n \otimes \underline{z}^{*m'}\underline{z}^{n'} \in \mathrm{Poly}(K) \odot \mathrm{Poly}(K)$. By considering maps $S(i/2) \to \mathrm{C}(K \times K)$ given by

$$z \mapsto q^{r(m+n)}q^{r'(m'+n')}e^{2\pi i(n-m)\theta+2\pi i(n'-m')\theta'+2i(n+n'-m-m')z\ln q}$$

it is straightforward to show that

$$(\tau_{i/2}^{K\times K}(F))(q^r e^{2\pi i\theta}, q^{r'} e^{2\pi i\theta'}) = q^{r(m+n)}q^{r'(m'+n')}q^{m+m'-n-n'}e^{2\pi i(n-m)\theta+2\pi i(n'-m')\theta'}.$$

Using the completely isometric isomorphism from Proposition 5.4.2 we can calculate

$$\begin{aligned}\langle\tau_{i/2}^{K\times K}(F), G\rangle &= \int_K \left(\int_{K'} (\tau_{i/2}^{K\times K}(F))G\, d\nu'\right) d\nu \\ &= \sum_{r=0}^\infty q^{2r}\int_0^1 \left(\sum_{r'=0}^\infty q^{2r'}\int_0^1 q^{r(m+n)}q^{r'(m'+n')}q^{m+m'-n-n'}\right. \\ &\qquad \left. e^{2\pi i(n-m)\theta+2\pi i(n'-m')\theta'}\delta_{r,s}\delta_{r',s}e^{-2\pi il\theta}e^{2\pi il\theta'}\,d\theta'\right) d\theta \\ &= q^{4s}q^{s(m+n+m'+n')}q^{m-n+m'-n'}\delta_{l,n-m}\delta_{l,m'-n'} = q^{4s}q^{s(m+n+m'+n')}\delta_{l,n-m}\delta_{l,m'-n'}\end{aligned}$$

and similarly

$$\langle F, G\rangle = q^{4s} q^{s(m+n+m'+n')}\delta_{l,n-m}\delta_{l,m'-n'}.$$

Thus extending this by linearity we have that $\langle \tau_{i/2}^{K\times K}(F), G\rangle = \langle F, G\rangle$ for all $F \in \mathrm{Poly}(K) \odot \mathrm{Poly}(K)$ and then by Proposition 5.4.8 we have $G \in \mathrm{L}^1_\sharp(K \times K, \nu \times \nu)$ with $G^\flat = G$. $\square$

Lemma 5.4.16 *Let* $T : \mathrm{L}^1_\sharp(\mathrm{SU}_q(2)) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathrm{SU}_q(2)) \to \mathrm{L}^1_\sharp(\mathrm{SU}_q(2) \times \mathrm{SU}_q(2))$ *be the completely contractive map from Theorem 4.2.14 and let* $T^K : \mathrm{L}^1_\sharp(K,\nu) \widehat{\otimes} \mathrm{L}^1_\sharp(K,\nu) \to \mathrm{L}^1_\sharp(K \times K, \nu \times \nu)$ *be the map given by* $(\mathbb{F}^{q\times q})^\sharp_* \circ T \circ (\mathbb{E}^{q\sharp}_* \otimes \mathbb{E}^{q\sharp}_*)$*. If* T *is a completely isometric isomorphism then* T^K *is a completely isometric isomorphism.*

Proof

We have from Theorems 5.2.9 and 5.2.10, Definition 5.3.15 and Proposition 5.3.16 a commutative diagram

$$\begin{array}{ccc} \mathrm{L}^1_\sharp(K,\nu) & \overset{\mathbb{E}^{q\sharp}_*}{\underset{\mathbb{F}^{q\sharp}_*}{\rightleftarrows}} & \mathrm{L}^1_\sharp(\mathrm{SU}_q(2)) \\ \downarrow \iota^K & & \downarrow \iota^q \\ \mathrm{L}^1(K,\nu) & \overset{\mathbb{E}^{q}_*}{\underset{\mathbb{F}^{q}_*}{\rightleftarrows}} & \mathrm{L}^1(\mathrm{SU}_q(2)) \end{array}$$

where the bottom horizontal arrows are left inverses to the top horizontal arrows and $\iota^K : \mathrm{L}^1_\sharp(K,\nu) \to \mathrm{L}^1(K,\nu)$ is the canonical embedding. By expanding this composing with the diagram in Theorem 4.2.14 we get

$$\begin{array}{ccc} \mathrm{L}^1_\sharp(K,\nu) \widehat{\otimes} \mathrm{L}^1_\sharp(K,\nu) & \overset{T\circ(\mathbb{E}^{q\sharp}_*\otimes\mathbb{E}^{q\sharp}_*)}{\underset{(\mathbb{F}^{q\sharp}_*\otimes\mathbb{F}^{q\sharp}_*)\circ T^{-1}}{\rightleftarrows}} & \mathrm{L}^1_\sharp(\mathrm{SU}_q(2) \times \mathrm{SU}_q(2)) \\ \downarrow \iota^K\otimes\iota^K & & \downarrow \iota^{q\times q} \\ \mathrm{L}^1(K,\nu) \widehat{\otimes} \mathrm{L}^1(K,\nu) & \overset{\mathbb{E}^{q}_*\otimes\mathbb{E}^{q}_*}{\underset{\mathbb{F}^{q}_*\otimes\mathbb{F}^{q}_*}{\rightleftarrows}} & \mathrm{L}^1(\mathrm{SU}_q(2) \times \mathrm{SU}_q(2)) \end{array}$$

where we've used that T is invertible as it is a completely isometric isomorphism. Also from Definition 5.4.12 and Propositions 5.4.3 and 5.3.15 we have a commutative diagram

$$\begin{array}{ccc} \mathrm{L}^1_\sharp(\mathrm{SU}_q(2)\times\mathrm{SU}_q(2)) & \underset{(\mathbb{F}^{q\times q})^\sharp_*}{\overset{(\mathbb{E}^{q\times q})^\sharp_*}{\rightleftarrows}} & \mathrm{L}^1_\sharp(K\times K,\nu\times\nu) \\ \downarrow \iota^{q\times q} & & \downarrow \iota^{K\times K} \\ \mathrm{L}^1(\mathrm{SU}_q(2)\times\mathrm{SU}_q(2)) & \underset{\mathbb{F}^{q\times q}_*}{\overset{\mathbb{E}^{q\times q}_*}{\rightleftarrows}} & \mathrm{L}^1(K\times K,\nu\times\nu) \end{array}$$

where again the bottom horizontal arrows are left inverses to the top horizontal arrows. We can collapse all this to a diagram

$$\begin{array}{ccc} \mathrm{L}^1_\sharp(K,\nu)\widehat{\otimes}\mathrm{L}^1_\sharp(K,\nu) & \xrightarrow{T^K} & \mathrm{L}^1_\sharp(K\times K,\nu\times\nu) \\ \downarrow \iota^K\otimes\iota^K & & \downarrow \iota^{K\times K} \\ \mathrm{L}^1(K,\nu)\widehat{\otimes}\mathrm{L}^1(K,\nu) & = & \mathrm{L}^1(K\times K,\nu\times\nu) \end{array}$$

where $T^K : \mathrm{L}^1_\sharp(K,\nu)\widehat{\otimes}\mathrm{L}^1_\sharp(K,\nu) \to \mathrm{L}^1_\sharp(K\times K,\nu\times\nu)$ is a complete contraction given in the theorem and we've used Proposition 5.4.2 on the bottom row.

We show next that T^K is a completely isometric isomorphism. We have that T^K is a complete contraction and we consider the complete contraction from $\mathrm{L}^1_\sharp(K\times K,\nu\times\nu)$ to $\mathrm{L}^1_\sharp(K,\nu)\widehat{\otimes}\mathrm{L}^1_\sharp(K,\nu)$ given by $(\mathbb{F}^{q\sharp}_*\otimes\mathbb{F}^{q\sharp}_*)\circ T^{-1}\circ\mathbb{E}^{q\times q\sharp}_*$. Using Lemma 5.4.14 we have

$$\begin{aligned} T^K\circ(\mathbb{F}^{q\sharp}_*\otimes\mathbb{F}^{q\sharp}_*)\circ T^{-1}\circ\mathbb{E}^{q\times q\sharp}_* &= \mathbb{F}^{q\times q\sharp}_*\circ T\circ(\mathbb{P}^{q\sharp}_*\otimes\mathbb{P}^{q\sharp}_*)\circ T^{-1}\circ\mathbb{E}^{q\times q\sharp}_* \\ &= \mathbb{F}^{q\times q\sharp}_*\circ\mathbb{P}^{q\times q\sharp}_*\circ T\circ T^{-1}\circ\mathbb{E}^{q\times q\sharp}_* = \mathrm{id} \end{aligned}$$

and similarly $(\mathbb{F}^{q\sharp}_*\otimes\mathbb{F}^{q\sharp}_*)\circ T^{-1}\circ\mathbb{E}^{q\times q\sharp}_*\circ T^K = \mathrm{id}$ as required. □

We now give our counterexample and show that the map T is not a completely isometric isomorphism and therefore $\mathrm{L}^1_\sharp(\mathbb{G}\times\mathbb{H})$ is in general not completely isometrically isomorphic to $\mathrm{L}^1_\sharp(\mathbb{G})\widehat{\otimes}\mathrm{L}^1_\sharp(\mathbb{H})$.

Counterexample 5.4.17 *We can "twist" an element on $\mathrm{L}^1_\sharp(K,\nu)\widehat{\otimes}\mathrm{L}^1(K,\nu)$ on one side of this tensor product, however in $\mathrm{L}^1_\sharp(K\times K,\nu\times\nu)$ it is not clear that we can do this in* *the same way. We try and exploit this in this counterexample and we will show that if we apply the $\flat$ operation from Notation 4.1.12 that we cannot perform the same operation in* $\mathrm{L}^1_\sharp(K \times K, \nu \times \nu)$.

Assume the map T is a completely isometric isomorphism and we aim towards a contradiction. It follows from Lemma 5.4.16 that we have a completely isometric isomorphism $T^K : \mathrm{L}^1_\sharp(K,\nu) \widehat{\otimes} \mathrm{L}^1_\sharp(K,\nu) \to \mathrm{L}^1_\sharp(K \times K, \nu \times \nu)$. Let $\alpha^K : \mathrm{L}^1_\sharp(K,\nu) \to \mathrm{L}^1(K,\nu)$ be the map $\pi_2 \circ (\mathrm{id} \otimes \overline{Q^K}) \circ \theta^K$ where Q^K is from Proposition 5.3.11. Then by Corollary 5.3.19 α^K is a complete contraction with $\alpha^K(f) = f^\flat$ where $f^\flat = (\tau^K_{i/2})_*(f)$. As T^K is a completely isometric isomorphism there is a completely contractive map $S : \mathrm{L}^1_\sharp(K \times K, \nu \times \nu) \to \mathrm{L}^1(K \times K, \nu \times \nu)$ such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{L}^1_\sharp(K,\nu) \widehat{\otimes} \mathrm{L}^1_\sharp(K,\nu) & \xrightarrow{T^K} & \mathrm{L}^1_\sharp(K \times K, \nu \times \nu) \\ \downarrow{\scriptstyle \mathrm{id}\otimes\alpha^K} & & \downarrow{\scriptstyle S} \\ \mathrm{L}^1(K,\nu) \widehat{\otimes} \mathrm{L}^1(K,\nu) & = & \mathrm{L}^1(K \times K, \nu \times \nu). \end{array}$$

Fix $l, s \geqslant 0$ and let $f, g \in \mathrm{L}^1(K,\nu)$ be maps given by $f(q^r e^{2\pi i\theta}) = \delta_{r,s}\frac{1}{1-q^2}e^{-2\pi i l\theta}$ and $g(q^r e^{2\pi i\theta}) = \delta_{r,s}\frac{1}{1-q^2}e^{2\pi i l\theta}$ for $r \in \mathbb{N}_0$ and $\theta \in [0,1)$. Then from Proposition 5.3.22 we have $f, g \in \mathrm{L}^1_\sharp(K,\nu)$ with $f^\flat(q^r e^{2\pi i\theta}) = \delta_{r,s}\frac{1}{1-q^2}q^l e^{-2\pi i l\theta}$ and $g^\flat(q^r e^{2\pi i\theta}) = \delta_{r,s}q^{-l}\frac{1}{1-q^2}e^{2\pi i l\theta}$. So we can easily calculate

$$\|f\|_{\mathrm{L}^1_\sharp(K,\nu)} = \max\{\|f\|_{\mathrm{L}^1(K,\nu)}, \|f^\flat\|_{\mathrm{L}^1(K,\nu)}\} = \max\{q^{2s}, q^{2s}q^l\} = q^{2s}$$

and similarly

$$\|g\|_{\mathrm{L}^1_\sharp(K,\nu)} = \max\{q^{2s}, q^{2s}q^{-l}\} = q^{2s}q^{-l}.$$

As the operator space projective tensor product is a cross matrix norm it follows that

$$\|f \otimes g\|_{\mathrm{L}^1_\sharp(K,\nu)\widehat{\otimes}\mathrm{L}^1_\sharp(K,\nu)} = \|f\|_{\mathrm{L}^1_\sharp(K,\nu)}\|g\|_{\mathrm{L}^1_\sharp(K,\nu)} = q^{4s}q^{-l}.$$

From Lemma 5.4.15 we have $T^K(f \otimes g) \in \mathrm{L}^1_\sharp(K \times K, \nu \times \nu)$ *with* $(T^K(f \otimes g))^\flat = T^K(f \otimes g)$ *and so*

$$\|T^K(f \otimes g)\|_{\mathrm{L}^1_\sharp(K\times K,\nu\times\nu)} = \|T^K(f \otimes g)\|_{\mathrm{L}^1(K\times K,\nu\times\nu)} = \int_K \left(\int_{K'} \left|T^K(f \otimes g)\right| d\nu' \right) d\nu$$
$$= (1-q^2)^2 \sum_{r=0}^{\infty} q^{2r} \int_0^1 \left(\sum_{r'=0}^{\infty} q^{2r'} \int_0^1 \left| f(q^r e^{2\pi i\theta}) g(q^{r'} e^{2\pi i\theta'}) \right| d\theta \right) d\theta' = q^{4s}.$$

Lastly we calculate

$$\|(\mathrm{id} \otimes \alpha^K)(f \otimes g)\|_{\mathrm{L}^1(K,\nu)\hat{\otimes}\mathrm{L}^1(K,\nu)} = \|f\|_{\mathrm{L}^1(K,\nu)} \|g^\flat\|_{\mathrm{L}^1(K,\nu)} = q^{4s} q^{-l}$$

and as $\mathrm{id} \otimes \alpha^K = S \circ T^K$ *we have*

$$q^{4s}q^{-l} = \|(\mathrm{id} \otimes \alpha^K)(f \otimes g)\|_{\mathrm{L}^1(K,\nu)\hat{\otimes}\mathrm{L}^1(K,\nu)} = \|(S \circ T^K)(f \otimes g)\|_{\mathrm{L}^1(K\times K,\nu\times\nu)}$$
$$\leqslant \|T^K(f \otimes g)\|_{\mathrm{L}^1_\sharp(K\times K,\nu\times\nu)} = q^{4s} < q^{4s}q^{-l}$$

a contradiction.

5.5 Adjoint of $(\mu \otimes \mathrm{id})(W^{\mathrm{SU}_q(2)})$ for $\mu \in \mathrm{C}(\mathrm{SU}_q(2))^*$

Consider for a moment an arbitrary locally compact quantum group $\mathbb{G}$. Then for $W \in \mathrm{M}(\mathrm{C}_0(\mathbb{G}) \otimes_{min} \mathrm{C}_0(\widehat{\mathbb{G}}))$ we have that $\mathrm{C}_0(\widehat{\mathbb{G}}) = \overline{\mathrm{lin}\ \{(\omega \otimes \mathrm{id})(W) \mid \omega \in \mathrm{L}^1(\mathbb{G})\}}$ is a C*-algebra from Remark 2.3.3. We have $(\mu \otimes \mathrm{id})(W) \in \mathrm{M}(\mathrm{C}_0(\widehat{\mathbb{G}}))$ for $\mu \in \mathrm{C}_0(\mathbb{G})^*$ and so we can consider $\overline{\mathrm{lin}\ \{(\mu \otimes \mathrm{id})(W) \mid \mu \in \mathrm{C}_0(\mathbb{G})^*\}}$. Clearly this is a closed linear space and we can easily show it is an algebra. We show in this section that this is not a C*-algebra in general, in particular we show that for $W^{\mathrm{SU}_q(2)}$ the left regular corepresentation of $\mathrm{SU}_q(2)$ there is a $\mu \in \mathrm{C}_0(\mathrm{SU}_q(2))^*$ such that $(\mu \otimes \mathrm{id})(W)^* \notin \overline{\mathrm{lin}\ \{(\nu \otimes \mathrm{id})(W) \mid \nu \in \mathrm{C}_0(\mathbb{G})^*\}}$.

We consider the space K from Proposition 5.2.4 and throughout this section we fix $r \in \mathbb{N}_0$ and $\theta \in [0,1)$ and $z_0 = q^r e^{2\pi i\theta}$ and we consider the measure $\delta_{z_0} \in \mathrm{C}(K)^*$ given by

$f \mapsto f(z_0)$. We can extend this to a linear functional $\mu = \delta_{z_0} \circ P \in \mathrm{C}_0(\mathrm{SU}_q(2))^*$ where P is the conditional expectation from Theorem 5.2.3 and we have $\|\mu\| \leqslant \|\delta_{z_0}\| = 1$.

We begin by proving a couple of straightforward lemmas.

Lemma 5.5.1 *Let $\{U^l \mid l \in \frac{1}{2}\mathbb{N}_0\}$ be the irreducible corepresentations in Theorem 5.1.15, $l \in \frac{1}{2}\mathbb{N}_0$, $n, m \in \frac{1}{2}\mathbb{N}_0$ such that $-l \leqslant n, m \leqslant l$ and let $\mu = \delta_{z_0} \circ P \in \mathrm{C}(\mathrm{SU}_q(2))^*$. If $m \neq -n$ we have $\mu(u^l_{n,m}) = 0$, if $n \geqslant 0$ and $m = -n$ we have*

$$u^l_{n,-n} = \begin{bmatrix} l+n \\ 2n \end{bmatrix}_{q^2} q^{-2n(l-n)} p_{l-n}(c^*c; q^{4n}, 1|q^2) c^{2n} \tag{5.25}$$

and if $n < 0$ and $m = -n$ we have

$$u^l_{n,-n} = \begin{bmatrix} l-n \\ -2n \end{bmatrix}_{q^2} q^{2n(l+n)} p_{l+n}(c^*c; q^{-4n}, 1|q^2)(-qc^*)^{-2n}. \tag{5.26}$$

In particular for all $l \in \frac{1}{2}\mathbb{N}_0$ and $-l \leqslant n \leqslant l$ we have

$$u^l_{n,-n} = (-q)^{-2n}(u^l_{-n,n})^* \tag{5.27}$$

Proof

We have by construction of μ above that $\mu(a_{kmn}) = 0$ if $k \neq 0$. Then using the $\mathrm{SU}_q(2)$ relations (5.1) and Theorem 5.1.15 we see that $\mu(u^l_{n,m}) = 0$ if $n \neq -m$. If $n \geqslant 0$ we let $m = -n$ then $n \geqslant m \geqslant -n$ and using Theorem 5.1.16 we get Equation (5.25). If $n < 0$ we let $m = -n$ then $m \geqslant n \geqslant -m$ and again using Theorem 5.1.16 we get Equation (5.26). We also see that if $n \geqslant 0$ then $-n \leqslant 0$ and we have

$$\begin{aligned}
u^l_{n,-n} &= \begin{bmatrix} l-(-n) \\ -2(-n) \end{bmatrix}_{q^2} q^{2(-n)(l+(-n))} p_{l+(-n)}(c^*c; q^{-4(-n)}, 1|q^2) c^{-2(-n)} \\
&= \left(\begin{bmatrix} l-(-n) \\ -2(-n) \end{bmatrix}_{q^2} q^{2(-n)(l+(-n))} p_{l+(-n)}(c^*c; q^{-4(-n)}, 1|q^2)(c^*)^{-2(-n)} \right)^* \\
&= (-q)^{-2n}(u^l_{-n,n})^*
\end{aligned}$$

and similarly for $n < 0$ we have

$$\begin{aligned}
u^l_{n,-n} &= \left(\begin{bmatrix} l+(-n) \\ 2(-n) \end{bmatrix}_{q^2} q^{-2(-n)(l-(-n))} p_{l-(-n)}(c^*c; q^{4(-n)}, 1|q^2)(-qc)^{2(-n)} \right)^* \\
&= (-q)^{-2n}(u^l_{-n,n})^*. \quad \square
\end{aligned}$$

Lemma 5.5.2 *For $\mu = \delta_{z_0} \circ P$, $s \in \mathbb{N}_0$, $k \in \mathbb{Z}$ and $m, n \in \mathbb{N}_0$ we have*

$$(\mu(s))(a_{kmn}) = \mu(a_{kmn}) \exp\left(\frac{-(n-m)^2(\ln q)^2}{s^2} \right). \tag{5.28}$$

and thus for any $l \in \mathbb{N}_0$ and $-l \leqslant m', n' \leqslant l$ we have

$$(\mu(s))(u^l_{n',m'}) = \mu(u^l_{n',m'}) \exp\left(\frac{-4n'^2(\ln q)^2}{s^2} \right). \tag{5.29}$$

Proof

Using Corollary 5.1.9 we have

$$\begin{aligned}
(\mu(s))(a_{kmn}) &= \frac{s}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-s^2t^2} \langle \tau_t(a_{kmn}), \mu \rangle \, dt \\
&= \mu(a_{kmn}) \frac{s}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-s^2t^2} e^{2it(n-m)\ln q} \, dt \\
&= \mu(a_{kmn}) \exp\left(\frac{-(m-n)^2(\ln q)^2}{s^2} \right).
\end{aligned}$$

As $\mu(a_{kmn}) = 0$ if $k \neq 0$ then by Lemma 5.5.1 we need only show Equation (5.29) for $m' = -n'$. We notice from Lemma 5.5.1 that $u^l_{n',-n'}$ is a polynomial $\sum_{k'=0}^{N} \lambda_{k'} (c^*)^{k'} c^{k'+2n'}$ for $n' \geqslant 0$ and a polynomial $\sum_{k'=0}^{N} \lambda_{k'} c^{k'} (c^*)^{k'-2n'}$ for $n' < 0$ for some $(\lambda_{k'}) \subset \mathbb{C}$. However we have from above that for any $k' \in \mathbb{N}_0$ we have

$$(\mu(s))((c^*)^{k'} c^{k'+2n'}) = \mu((c^*)^{k'} c^{k'+2n'}) \exp\left(\frac{-4n'^2(\ln q)^2}{s^2}\right)$$

and similarly

$$(\mu(s))(c^{k'} (c^*)^{k'-2n'}) = \mu(c^{k'} (c^*)^{k'-2n'}) \exp\left(\frac{-4n'^2(\ln q)^2}{s^2}\right)$$

from which the result follows. □

Lemma 5.5.3 *For any $\mu \in \mathrm{C}(\mathrm{SU}_q(2))^*$ we have*

$$\|\lambda(\mu)\| = \sup_{l \in \frac{1}{2}\mathbb{N}_0} \|\mu_{2l+1}(U^l)\| = \sup_{l \in \frac{1}{2}\mathbb{N}_0} \left\|(\mu(U^l_{ij}))^{n_\alpha}_{i,j=1}\right\|.$$

Proof

We have from Proposition 3.2.25 that we can consider W acting on $\mathrm{L}^2(\mathrm{SU}_q(2)) \otimes \mathcal{H}$ where $\mathcal{H} = \bigoplus^2{}_{l \in \frac{1}{2}\mathbb{N}_0} \mathcal{H}_l$ by

$$W(\xi \otimes e^l_{ij}) = \sum_{k=-l}^{l} u^l_{ik} \xi \otimes e^l_{kj}$$

for $\xi \in \mathrm{L}^2(\mathrm{SU}_q(2))$, $l \in \frac{1}{2}\mathbb{N}_0$ and $-l \leqslant i, j \leqslant l$. So we have that W is a direct sum of matrices

$$W = \begin{pmatrix} u^0 & 0 & 0 & \dots \\ 0 & u^{1/2} & 0 & \dots \\ 0 & 0 & u^1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and so

$$(\mu \otimes \mathrm{id})(W) = \begin{pmatrix} \mu(u^0) & 0 & 0 & \cdots \\ 0 & \mu_2(u^{1/2}) & 0 & \cdots \\ 0 & 0 & \mu_3(u^1) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then we have a direct sum of matrices acting on Hilbert spaces $\mathcal{H}_l$ and from Proposition A.5.2 we have

$$\|(\mu \otimes \mathrm{id})(W)\| = \sup_{l \in \frac{1}{2}\mathbb{N}_0} \|\mu_{2l+1}(u^l)\|$$

as required. □

Theorem 5.5.4 *Let* $\mu = \delta_{z_0} \circ P \in \mathrm{C}(\mathrm{SU}_q(2))^*$. *Then the following are equivalent:*

(i) For all $\varepsilon > 0$ *there exists* $N \in \mathbb{N}_0$ *such that* $\left|\mu(u^l_{n,-n})\right| < \varepsilon$ *for all* $l \in \frac{1}{2}\mathbb{N}_0$ *and* $-l \leqslant n \leqslant l$ *where* $l \geqslant N$ *and* $|n| \geqslant N$*;*

(ii) $\lambda(\mu)^* = \lim_m \lambda(\mu(m)^\sharp)$ *where* $\mu(m)$ *is the smear of* μ *and* $\mu(m)^\sharp \in \mathrm{C}(\mathrm{SU}_q(2))^*$ *is the map such that* $\langle x, \mu(m)^\sharp \rangle = \overline{\langle S(x)^*, \mu(m) \rangle}$ *for all* $x \in \mathrm{Dom}(S) \cap \mathrm{C}(\mathrm{SU}_q(2))$.

(iii) For all $\varepsilon > 0$ *there exists* $\nu \in \mathrm{C}(\mathrm{SU}_q(2))^*$ *such that*

$$|(\nu \otimes \mathrm{id})(W) - (\mu \otimes \mathrm{id})(W)^*| < \varepsilon.$$

Proof

(i) $\Longrightarrow$ (ii): Fix $\varepsilon > 0$ and let $N \in \mathbb{N}_0$ such that for all $l \in \frac{1}{2}\mathbb{N}_0$ and n the integer steps from $-l$ to l with $l, |n| \geqslant N$ we have $\left|\mu(u^l_{n,-n})\right| < \varepsilon$. By Proposition 4.1.18 we have $\mu(m) \in \mathrm{C}_0(\mathrm{SU}_q(2))^*{}_\sharp$ and by Proposition 4.1.17 we have

$$\|\lambda(\mu(m)^\sharp) - \lambda(\mu)^*\| = \|\lambda(\mu(m))^* - \lambda(\mu)^*\| = \|\lambda(\mu(m)) - \lambda(\mu)\|.$$

We have from Lemma 5.5.3 that $\|\lambda(\mu)\| = \sup_{l\in\frac{1}{2}\mathbb{N}_0} \|\mu_{2l+1}(U^l)\|$ and so

$$\begin{aligned}\|\lambda(\mu(m)) - \lambda(\mu)\| &= \sup_{l\in\frac{1}{2}\mathbb{N}_0} \|\mu(s)_{2l+1}(U^l) - \mu_{2l+1}(U^l)\| \\ &= \sup_{l\in\frac{1}{2}\mathbb{N}_0} \left\| \sum_{n=-l}^{l} \left(\mu(s)(u^l_{n,-n}) - \mu(u^l_{n,-n})\right) e^{2l+1}_{n,-n} \right\|.\end{aligned}$$

By a Proposition A.5.2 (and rearranging rows by unitary matrices) we have

$$\left\| \sum_{n=-l}^{l} \left(\mu(s)(u^l_{n,-n}) - \mu(u^l_{n,-n})\right) e^{2l+1}_{n,-n} \right\| = \sup_{n\in\{-l,\dots,l\}} \left|\mu(s)(u^l_{n,-n}) - \mu(u^l_{n,-n})\right|$$

and then from Lemma 5.5.2 we have

$$\left|\mu(s)(u^l_{n,-n}) - \mu(u^l_{n,-n})\right| = \left|\mu(u^l_{n,-n})\right| \left| \exp\left(\frac{-4n^2(\ln q)^2}{m^2}\right) - 1 \right|.$$

We let $M > 0$ such that for all $n \in \frac{1}{2}\mathbb{Z}$ where $|n| < N$ and for all $m \geqslant M$ we have $\left|\exp\left(\frac{-4n^2(\ln q)^2}{m^2}\right) - 1\right| < \varepsilon$. For all $l \in \frac{1}{2}\mathbb{N}_0$ we have that U^l is unitary and μ is contractive and so using Proposition A.5.3 we have $\left|\mu(u^l_{n,-n})\right| \leqslant \|\mu\|\|u^l_{n,-n}\| \leqslant \|\mu\|\|U^l\| \leqslant 1$. So we have shown that for all $l \in \frac{1}{2}\mathbb{N}_0$ and $-l \leqslant n \leqslant l$ with $|n| < N$ and $m \geqslant M$ we have

$$\left|\mu(u^l_{n,-n})\right| \left| \exp\left(\frac{-4n^2(\ln q)^2}{m^2}\right) - 1 \right| < \varepsilon.$$

On the other hand if $l \in \frac{1}{2}\mathbb{N}_0$ and $-l \leqslant n \leqslant l$ with $l, |n| \geqslant N$ then we have by assumption that $\left|\mu(u^l_{n,-n})\right| < \varepsilon$ and $\left|\exp\left(\frac{-4n^2(\ln q)^2}{m^2}\right) - 1\right| \leqslant 1$. So we have shown that for all $l \in \mathbb{N}_0$ and $-l \leqslant n \leqslant l$ there is some $M > 0$ such that for all $m \geqslant M$ we have $\|\lambda(\mu(m)^\sharp) - \lambda(\mu)^*\| < \varepsilon$ and we have (ii).

Clearly (ii) $\implies$ (iii) and we show (iii) $\implies$ (i). Fix $\varepsilon > 0$, then by (iii) there is some $\nu \in C(\mathrm{SU}_q(2))^*$ such that

$$\|(\nu \otimes \mathrm{id})(W) - (\mu \otimes \mathrm{id})(W)^*\| < \varepsilon/2.$$

Equivalently by Lemma 5.5.3 we have

$$\sup_{l\in\frac{1}{2}\mathbb{N}_0}\left\|\sum_{n,m=-l}^{l}\nu(u^l_{n,m})e^{2l+1}_{l+n+1,l+m+1}-\sum_{n,m=-l}^{l}\overline{\mu(u^l_{n,m})}e^{2l+1}_{l+m+1,l+n+1}\right\|<\varepsilon/2$$

$$\Longleftrightarrow\quad \sup_{l\in\frac{1}{2}\mathbb{N}_0}\left\|\sum_{n,m=-l}^{l}\nu(u^l_{m,n})e^{2l+1}_{l+m+1,l+n+1}-\sum_{n=-l}^{l}\overline{\mu(u^l_{n,-n})}e^{2l+1}_{l-n+1,l+n+1}\right\|<\varepsilon/2$$

or equivalently for all $l \in \frac{1}{2}\mathbb{N}_0$ we have

$$\left\|\sum_{n=-l}^{l}\left(\sum_{m=-l}^{l}\nu(u^l_{m,n})e^{2l+1}_{l+m+1,l+n+1}-\overline{\mu(u^l_{n,-n})}e^{2l+1}_{l-n+1,l+n+1}\right)\right\|<\varepsilon/2.$$

Fix $l \in \frac{1}{2}\mathbb{N}_0$ and let $Q^l : \mathbb{M}_{2l+1} \to \mathbb{M}_{2l+1}$ be the map that $(a_{ij})^{2l+1}_{i,j=1} \mapsto (\delta_{i,-j}a_{ij})^{2l+1}_{i,j=1}$, that is Q^l maps an off anti-diagonal entry to 0 and an anti-diagonal entry to itself. We have that Q^l is a contractive projection by the similar result to that of Proposition A.5.2 referred to previously and Proposition A.5.3 we have

$$\|Q^lU^l\| = \sup_{n\in\{-l,\ldots,l\}}\|u^l_{n,-n}\| \leqslant \|U^l\|$$

and so it follows that

$$\left\|\sum_{n=-l}^{l}\left(\nu(u^l_{-n,n})-\overline{\mu(u^l_{n,-n})}\right)e^{2l+1}_{l-n+1,l+n+1}\right\|<\varepsilon/2.$$

From Lemma 5.5.2 we have that $u^l_{-n,n} = (-q)^{2n}(u^l_{n,-n})^*$ and so for all $-l \leqslant n \leqslant l$ we have

$$\left|(-q)^{2n}\nu((u^l_{n,-n})^*)-\overline{\mu(u^l_{n,-n})}\right|<\varepsilon/2$$

and so

$$\left|\mu(u^l_{n,-n})\right|<\varepsilon/2+\left|(-q)^{2n}\nu((u^l_{n,-n})^*)\right|\leqslant\varepsilon/2+q^{2n}\|\nu\|.$$

Also we have $u^l_{n,-n} = (-q)^{-2n}(u^l_{-n.n})^*$ and so

$$\left|((-q)^{-2n}\nu((u^l_{-n.n})^*) - \overline{\mu(u^l_{-n,n})}\right| < \varepsilon$$

or indeed

$$\left|\mu(u^l_{-n,n})\right| < \varepsilon + \left|((-q)^{-2n}\nu((u^l_{-n.n})^*)\right| \leqslant \varepsilon + q^{-2n}\|\nu\|.$$

Now we can choose $N \in \mathbb{N}_0$ such that for all $n \geqslant N$ we have $q^{2n}\|\nu\| < \varepsilon/2$ and for all $n \leqslant -N$ we have $q^{-2n}\|\nu\| < \varepsilon/2$. Then for all $l \in \frac{1}{2}\mathbb{N}_0$ and $-l \leqslant n \leqslant l$ with $|n| \geqslant N$ we have $\left|\mu(u^l_{n,-n})\right| < \varepsilon$ as required. □

Counterexample 5.5.5 *We consider the special case of* $r = 0$, $\theta \in [0, 1)$ *arbitrary and so* $z_0 = e^{2\pi i\theta}$ *and* $\mu = \delta_{z_0} \circ P$. *We now show that we cannot approximate* $(\mu \otimes \mathrm{id})(W)^*$ *by elements* $\{(\nu \otimes \mathrm{id})(W) \mid \nu \in \mathrm{C}_0(\mathrm{SU}_q(2))^*\}$. *Fix* $l \in \frac{1}{2}\mathbb{N}_0$ *and consider* $n = l$. *Then we have*

$$\mu(u^l_{l,-l}) = \mu(c^{2l}) = e^{2\pi i l\theta}. \tag{5.30}$$

According to Theorem 5.5.4 if $(\mu \otimes \mathrm{id})(W)^*$ *can be norm approximated by elements* $\{(\nu \otimes \mathrm{id})(W) \mid \nu \in \mathrm{C}_0(\mathrm{SU}_q(2))^*\}$ *then for all* $\varepsilon > 0$ *we must have some* $N > 0$ *such that for large enough* l *and* n *we have* $\mu(u^l_{n,-n}) < \varepsilon$, *however for* $\varepsilon < 1$ *it follows from Equation* (5.30) *that this is not possible.*

There is still an open problem, motivated by the work Das & Daws (2014), of finding the largest C*-subalgebra of $\overline{\mathrm{lin}\ \{(\mu \otimes \mathrm{id})(W) \mid \mu \in \mathrm{C}_0(\mathbb{G})^*\}}$. We have shown here however that it is not all of $\overline{\mathrm{lin}\ \{(\mu \otimes \mathrm{id})(W) \mid \mu \in \mathrm{C}_0(\mathbb{G})^*\}}$.

Chapter 6

Homological Algebra for $\mathrm{L}^1_\sharp(\mathbb{G})$

In this chapter we investigate the operator biprojectivity of $\mathrm{L}^1_\sharp(\mathbb{G})$ as a completely contractive Banach algebra where this structure is given in Theorem 4.2.1. The notion of biprojectivity for Banach algebras was introduced by Helemskiĭ when he investigated the biprojectivity of $\mathrm{L}^1(G)$ for a locally compact group G. He showed that $\mathrm{L}^1(G)$ is biprojective as a Banach algebra if and only if G is compact (see Helemskiĭ (1989) for further details). The notion of operator biprojectivity was introduced by Ruan in his study of the Fourier algebra of a group G and is described in Section 1.2.2.

The case of locally compact quantum groups was studied in Aristov (2004) where Aristov showed that if $\mathrm{L}^1(\mathbb{G})$ is operator biprojective then $\mathbb{G}$ is compact and on the other hand that if $\mathbb{G}$ is compact and a Kac algebra then $\mathrm{L}^1(\mathbb{G})$ is operator biprojective. It was then shown in Daws (2010) and Caspers *et al.* (2015) that if $\mathrm{L}^1(\mathbb{G})$ is operator biprojective then $\mathbb{G}$ is of Kac type (see Definition 3.1.1) so we have the following theorem:

$\mathbb{G}$ is compact and of Kac type if and only if $\mathrm{L}^1(\mathbb{G})$ is operator biprojective.

We investigate similar questions in this chapter.

In the first section we show that we have more completely bounded $\mathrm{L}^1_\sharp(\mathbb{G})$-modules than completely bounded $\mathrm{L}^1(\mathbb{G})$-modules. We then show that we have a dual completely contractive map $\Delta^\sharp : \mathrm{L}^1_\sharp(\mathbb{G})^* \to \mathrm{L}^1_\sharp(\mathbb{G})^* \overline{\otimes} \mathrm{L}^1_\sharp(\mathbb{G})^*$ (rather than into the Fubini tensor product) in Theorem 6.2.1 in preparation for an investigation of operator biprojectivity. For $\mathrm{L}^1_\sharp(\mathbb{G})$ to be operator biprojective it is clearly necessary that the multiplication map $m_\sharp : \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ in Theorem 4.2.1 is surjective; so we show in Section 6.2.2 that for all coamenable quantum groups that the map $m_\sharp$ is a complete quotient map and thus onto. Finally in Section 6.2.3 we show that if $\mathrm{L}^1_\sharp(\mathbb{G})$ is operator biprojective then $\mathbb{G}$ is compact and in Theorem 6.2.6 we give a give a structure theorem for compact quantum groups similar to that in Daws (2010).

It is still an open and possibly difficult question as to whether we can find similar conditions to that of $\mathrm{L}^1(\mathbb{G})$ that are necessary and sufficient to ensure operator biprojectivity of $\mathrm{L}^1_\sharp(\mathbb{G})$ in general.

6.1 Projective Modules over $\mathrm{L}^1_\sharp(\mathbb{G})$

In this section we show that all completely bounded $\mathrm{L}^1(\mathbb{G})$-modules are also completely bounded $\mathrm{L}^1_\sharp(\mathbb{G})$-modules and that there does exist a completely bounded $\mathrm{L}^1_\sharp(\mathbb{G})$-module that is not a completely bounded $\mathrm{L}^1(\mathbb{G})$-module. We work with completely bounded left modules in this section, however the same could be applied to completely bounded right and bimodules.

Proposition 6.1.1 *Let X be a completely bounded left $\mathrm{L}^1(\mathbb{G})$-module. Then X is a completely bounded left $\mathrm{L}^1_\sharp(\mathbb{G})$-module.*

Proof

We have a complete contraction $\iota := \pi_1 \circ \theta : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G})$ where θ is the map given by Equation (4.2) and π_1 is the projection onto the first coordinate.

Let X be a completely bounded left $\mathrm{L}^1(\mathbb{G})$-module: that is there is a completely bounded map $\Phi : \mathrm{L}^1(\mathbb{G}) \widehat{\otimes} X \to X$ such that $\omega \otimes x \mapsto \omega \cdot x$. We define a completely bounded map $\Phi_\sharp : \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} X \to X$ by $\Phi_\sharp := \Phi \circ (\iota \otimes \mathrm{id}_X)$. We know that ι has dense range and is a homomorphism and so for $\omega, \omega' \in \mathrm{L}^1_\sharp(\mathbb{G})$ we can easily show that

$\Phi_\sharp(\mathrm{id} \otimes \Phi_\sharp)(\omega \otimes \omega' \otimes x) = \Phi_\sharp(m_\sharp \otimes \mathrm{id})(\omega \otimes \omega' \otimes x)$. Then it follows that $\Phi_\sharp \circ (\mathrm{id} \otimes \Phi_\sharp) = \Phi_\sharp \circ (m_\sharp \otimes \mathrm{id})$ and thus E is also a left completely bounded Banach $\mathrm{L}^1_\sharp(\mathbb{G})$-module. □

We now spend the rest of this section showing there exists a completely bounded left $\mathrm{L}^1_\sharp(\mathbb{G})$-module that is not a completely bounded left $\mathrm{L}^1(\mathbb{G})$-module.

Example 6.1.2 *Assume $\mathbb{G}$ is coamenable giving us that $\mathrm{L}^1_\sharp(\mathbb{G})$ has a contractive approximate identity by Theorem 4.1.13. Clearly as $\mathrm{L}^1_\sharp(\mathbb{G})$ is a completely contractive Banach algebra then it is also a left completely contractive $\mathrm{L}^1_\sharp(\mathbb{G})$-module with the same operation $m_\sharp : \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$. Say this extends to a left completely bounded $\mathrm{L}^1(\mathbb{G})$-module; that is we have a map $\psi : \mathrm{L}^1(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ making the following diagram commute*

$$\begin{array}{ccc} \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) & \xrightarrow{m_\sharp} & \mathrm{L}^1_\sharp(\mathbb{G}) \\ \downarrow{\scriptstyle \iota \otimes \mathrm{id}} & \nearrow{\scriptstyle \psi} & \\ \mathrm{L}^1(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) & & \end{array}$$

where $\iota : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G})$ is the usual completely contractive embedding. We will show that these assumptions imply that the antipode S is bounded and thus we are in the Kac algebra case. In particular this is a contradiction for $\mathrm{SU}_q(2)$ for example.

Let $\omega \in \mathrm{L}^1(\mathbb{G})$, then by Theorem 4.1.4 there exists a net $(\omega_\alpha) \subset \mathrm{L}^1_\sharp(\mathbb{G})$ such that $\|\omega - \omega_\alpha\|_{\mathrm{L}^1(\mathbb{G})} \to 0$. Then for $\omega' \in \mathrm{L}^1_\sharp(\mathbb{G})$ we can consider $\Delta_*(\omega \otimes \iota(\omega')) \in \mathrm{L}^1(\mathbb{G})$ and we have

$$\begin{aligned} \|(\iota \circ \psi)(\omega \otimes \omega') - \omega * \iota(\omega')\|_{\mathrm{L}^1(\mathbb{G})} &\leqslant \|(\iota \circ \psi)(\omega \otimes \omega') - (\iota \circ \psi)(\omega_\alpha \otimes \omega')\|_{\mathrm{L}^1(\mathbb{G})} \\ &\quad + \|\iota(\omega_\alpha * \omega') - \omega * \iota(\omega')\|_{\mathrm{L}^1(\mathbb{G})} \to 0 \end{aligned}$$

and so $(\iota \circ \psi)(\omega \otimes \omega') = \omega * \iota(\omega')$ for all $\omega \in \mathrm{L}^1(\mathbb{G})$ and $\omega' \in \mathrm{L}^1_\sharp(\mathbb{G})$, or indeed we have *the following commutative diagram*

$$\begin{array}{ccccc}
\mathrm{L}^1_\sharp(\mathbb{G})\,\widehat{\otimes}\,\mathrm{L}^1_\sharp(\mathbb{G}) & \xrightarrow{m_\sharp} & \mathrm{L}^1_\sharp(\mathbb{G}) & \xrightarrow{\iota} & \mathrm{L}^1(\mathbb{G}) \\
\downarrow{\scriptstyle \iota\otimes\mathrm{id}} & \nearrow{\scriptstyle \psi} & & & \uparrow{\scriptstyle \Delta_*} \\
\mathrm{L}^1(\mathbb{G})\,\widehat{\otimes}\,\mathrm{L}^1_\sharp(\mathbb{G}) & & \xrightarrow{\mathrm{id}\otimes\iota} & & \mathrm{L}^1(\mathbb{G})\,\widehat{\otimes}\,\mathrm{L}^1(\mathbb{G}).
\end{array}$$

As $\mathbb{G}$ is coamenable it follows from Theorem 4.1.13 that we have a contractive approximate identity $(e_\alpha) \subset \mathrm{L}^1_\sharp(\mathbb{G})$. Let $\omega \in \mathrm{L}^1(\mathbb{G})$, then we have shown above that $\omega * e_\alpha = \psi(\omega \otimes e_\alpha) \in \mathrm{L}^1_\sharp(\mathbb{G})$ and so there exists $(\omega * e_\alpha)^\sharp$ such that for all $x \in \mathrm{Dom}(S)$ we have

$$\langle x, \psi(\omega \otimes e_\alpha)^\sharp \rangle = \overline{\langle S(x)^*, \omega * e_\alpha \rangle} \to \overline{\langle S(x)^*, \omega \rangle}.$$

Also for $x \in \mathrm{Dom}(S)$ we have

$$\begin{aligned}
\left|\langle x, \psi(\omega \otimes e_\alpha)^\sharp \rangle\right| &\leqslant \|x\| \|\psi(\omega \otimes e_\alpha)^\sharp\|_{\mathrm{L}^1(\mathbb{G})} \leqslant \|x\| \|\psi(\omega \otimes e_\alpha)\|_{\mathrm{L}^1_\sharp(\mathbb{G})} \\
&\leqslant \|x\| \|\psi\| \|\omega \otimes e_\alpha\|_{\mathrm{L}^1(\mathbb{G})\,\widehat{\otimes}\,\mathrm{L}^1_\sharp(\mathbb{G})} \leqslant \|x\| \|\psi\| \|\omega\|_{\mathrm{L}^1(\mathbb{G})}.
\end{aligned}$$

So we have

$$|\langle S(x)^*, \omega \rangle| = \lim \left|\langle x, \psi(\omega \otimes e_\alpha)^\sharp \rangle\right| \leqslant \|x\| \|\psi\| \|\omega\|$$

for all $x \in \mathrm{Dom}(S)$ and $\omega \in \mathrm{L}^1(\mathbb{G})$ and therefore

$$\|S(x)\| = \|S(x)^*\| \leqslant \|\psi\| \|x\|.$$

As $\mathrm{Dom}(S)$ is norm dense in $\mathrm{C}_0(\mathbb{G})$ we have for any $x \in \mathrm{C}_0(\mathbb{G})$ a net $(x_\alpha) \subset \mathrm{Dom}(S)$ with limit x. It follows that (x_α) is a Cauchy net and from above we have $\|S(x_\alpha) - S(x_{\alpha'})\| \leqslant \|\psi\| \|x_\alpha - x_{\alpha'}\|$ meaning $(S(x_\alpha))$ is also a Cauchy net. Then there is some $y \in \mathrm{C}_0(\mathbb{G})$ with $S(x_\alpha) \to y$ and so $x \in \mathrm{Dom}(S)$ and $y = S(x)$. Then $\mathrm{Dom}(S) = \mathrm{C}_0(\mathbb{G})$ and $\|S(x)\| \leqslant \|\psi\| \|x\|$ for all $x \in \mathrm{C}_0(\mathbb{G})$, i.e. S is bounded as was to be shown.

6.2 Operator Biprojectivity of $\mathrm{L}^1_\sharp(\mathbb{G})$

We investigate now the operator biprojectivity of $\mathrm{L}^1_\sharp(\mathbb{G})$. In order for $\mathrm{L}^1_\sharp(\mathbb{G})$ to be operator biprojective it is necessary that the multiplication map $m_\sharp : \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ must be onto. We begin an investigation into the circumstances under which this holds now. In particular we show that if $\mathbb{G}$ is coamenable then $m_\sharp$ is a complete quotient map (and thus onto) and we give a structure theorem for $\mathrm{L}^1_\sharp(\mathbb{G})$. First we will show that the dual of our map $m_\sharp$ has image $\mathrm{L}^1_\sharp(\mathbb{G})^* \overline{\otimes} \mathrm{L}^1_\sharp(\mathbb{G})^*$ where $\overline{\otimes}$ denotes the normal tensor product from Example 1.1.44.

6.2.1 The Adjoint of $m_\sharp$

We know that in the case of $\mathrm{L}^1(\mathbb{G})$ that $\Delta_* : \mathrm{L}^1(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G})$ is onto as the adjoint map $\Delta : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^\infty(\mathbb{G}) \overline{\otimes} \mathrm{L}^\infty(\mathbb{G})$ is an isometry. So we consider the map $\Delta^\sharp = (m_\sharp)^* : \mathrm{L}^1_\sharp(\mathbb{G})^* \to \mathrm{L}^1_\sharp(\mathbb{G})^* \overline{\otimes}_{\mathcal{F}} \mathrm{L}^1_\sharp(\mathbb{G})^*$ where we've used the Fubini tensor product from Example 1.1.44 and Theorem 1.1.45. To begin our investigation we show that the Fubini tensor product and normal tensor product of $\mathrm{L}^1_\sharp(\mathbb{G})^*$ with itself are equal (which follows in the case of $\mathrm{L}^\infty(\mathbb{G})$ as it is a von Neumann algebra).

Theorem 6.2.1 *Let $\psi : \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G}) \check{\otimes} \mathrm{L}^1_\sharp(\mathbb{G})$ denote the canonical complete contraction from Notation 1.1.40 that extends the identity map on $\mathrm{L}^1_\sharp(\mathbb{G}) \odot \mathrm{L}^1_\sharp(\mathbb{G})$. Then ψ is injective and thus by Proposition 1.1.46 we have that $\mathrm{L}^1_\sharp(\mathbb{G})^* \overline{\otimes}_{\mathcal{F}} \mathrm{L}^1_\sharp(\mathbb{G})^* = \mathrm{L}^1_\sharp(\mathbb{G})^* \overline{\otimes} \mathrm{L}^1_\sharp(\mathbb{G})^*$.*

Proof
Let $\Omega \in \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G})$ such that $\psi(\Omega) = 0$ and we show that $\Omega = 0$. Let $\phi : \mathrm{L}^1(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G}) \check{\otimes} \mathrm{L}^1(\mathbb{G})$ denote the canonical complete contraction from Notation 1.1.40 that extends the identity map on $\mathrm{L}^1(\mathbb{G}) \odot \mathrm{L}^1(\mathbb{G})$. We know that this is injective from Proposition 1.1.47. Then using Proposition 1.1.42 and that ϕ and ψ are the identity

maps on $\mathrm{L}^1_\sharp(\mathbb{G})\odot\mathrm{L}^1_\sharp(\mathbb{G})$ and $\mathrm{L}^1(\mathbb{G})\odot\mathrm{L}^1(\mathbb{G})$ respectively, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{L}^1_\sharp(\mathbb{G})\,\widehat{\otimes}\,\mathrm{L}^1_\sharp(\mathbb{G}) & \xrightarrow{\ \psi\ } & \mathrm{L}^1_\sharp(\mathbb{G})\,\check{\otimes}\,\mathrm{L}^1_\sharp(\mathbb{G}) \\ {\scriptstyle \iota\otimes\iota}\downarrow & & \downarrow{\scriptstyle \iota\otimes\iota} \\ \mathrm{L}^1(\mathbb{G})\,\widehat{\otimes}\,\mathrm{L}^1(\mathbb{G}) & \xrightarrow[\ \phi\ ]{} & \mathrm{L}^1(\mathbb{G})\,\check{\otimes}\,\mathrm{L}^1(\mathbb{G}). \end{array}$$

It follows then that we have

$$(\phi\circ(\iota\otimes\iota))(\Omega) = ((\iota\otimes\iota)\circ\psi)(\Omega) = 0$$

and as ϕ is injective we have

$$(\iota\otimes\iota)(\Omega) = 0. \tag{6.1}$$

Fix $n \in \mathbb{N}$. From Proposition 4.2.5 we have a completely bounded map $\Phi'(n) : \mathrm{L}^1(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ and so for any $T \in (\mathrm{L}^1_\sharp(\mathbb{G})\,\widehat{\otimes}\,\mathrm{L}^1_\sharp(\mathbb{G}))^* \cong_{ci} \mathcal{CB}(\mathrm{L}^1_\sharp(\mathbb{G}), \mathrm{L}^1_\sharp(\mathbb{G})^*)$ we have $\Phi'(n)^* \circ T \circ \Phi'(n) \in \mathcal{CB}(\mathrm{L}^1(\mathbb{G}), \mathrm{L}^\infty(\mathbb{G}))$. Then for all $\omega, \omega' \in \mathrm{L}^1(\mathbb{G})$ we have

$$\begin{aligned}\langle(\Phi'(n)^*\circ T\circ\Phi'(n))(\omega), \omega'\rangle &= \langle(T\circ\Phi'(n))(\omega), \Phi'(n)(\omega')\rangle \\ &= \langle T, (\Phi'(n)\otimes\Phi'(n))(\omega\otimes\omega')\rangle.\end{aligned}$$

We have $\Upsilon(n) : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ from Notation 4.2.7 and using that $(\iota\otimes\iota)(\Omega) = 0$ from Equation (6.1) we have

$$0 = \langle T, (\Upsilon(n)\otimes\Upsilon(n))(\Omega)\rangle$$

for all $T \in (\mathrm{L}^1_\sharp(\mathbb{G})\,\widehat{\otimes}\,\mathrm{L}^1_\sharp(\mathbb{G}))^*$. As we have $(\Upsilon(n)\otimes\Upsilon(n))(\Omega) \to \Omega$ as $n \to \infty$ by Proposition 4.2.10 then we have $\langle T, \Omega\rangle = 0$ for all $T \in (\mathrm{L}^1_\sharp(\mathbb{G})\,\widehat{\otimes}\,\mathrm{L}^1_\sharp(\mathbb{G}))^*$ and so $\Omega = 0$ as required. □

6.2.2 Coamenable Quantum Groups and Biprojectivity

Given a coamenable locally compact quantum group $\mathbb{G}$ we have a contractive approximate identity by Theorem 4.1.13. We record here the following result of interest that follows immediately as a special case of Proposition 1.2.13.

Proposition 6.2.2 *For $\mathbb{G}$ a coamenable locally compact quantum group the map $m_\sharp$: $\mathrm{L}^1_\sharp(\mathbb{G})\widehat{\otimes}\mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ is a complete quotient map.*

6.2.3 Compact Quantum Groups and Operator Biprojectivity of $\mathrm{L}^1_\sharp(\mathbb{G})$

It was shown in Aristov (2004) that if $\mathrm{L}^1(\mathbb{G})$ is operator biprojective then $\mathbb{G}$ is compact. We now show that we can generalise this result to the case of $\mathrm{L}^1_\sharp(\mathbb{G})$.

Throughout this section we let $\mathbb{G}$ denote a locally compact quantum group. We will prove the following result in this section.

Theorem 6.2.3 *If $\mathrm{L}^1_\sharp(\mathbb{G})$ is operator biprojective (that is the multiplication map $m_\sharp$: $\mathrm{L}^1_\sharp(\mathbb{G})\widehat{\otimes}\mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ has a right inverse that is a completely bounded $\mathrm{L}^1_\sharp(\mathbb{G})$-bimodule homomorphism, see Proposition 1.2.10) then $\mathbb{G}$ is compact.*

We remind the reader that we have a complete isometry $\theta : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G}) \oplus_\infty \overline{\mathrm{L}^1(\mathbb{G})}$ from Equation (4.2) given by $\omega \mapsto (\omega, \overline{\omega^\sharp})$ for $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$. We also know from Theorem 4.1.15 and Corollary 4.2.2 that the adjoint θ^* gives us $\mathrm{L}^1_\sharp(\mathbb{G})^* \cong_{ci} \mathrm{L}^\infty(\mathbb{G}) \oplus_1 \overline{\mathrm{L}^1(\mathbb{G})}/K_\sharp$ where $K_\sharp = \left\{(x, -\overline{S(x)^*}) \;\middle|\; x \in \mathrm{Dom}(S)\right\}$.

Following Aristov (2004) we can define a map $\tau : \mathrm{L}^1(\mathbb{G}) \to \mathbb{C}$ by $\omega \mapsto \langle 1, \omega\rangle$. Then we have immediately that $|\tau(\omega)| \leqslant \|\omega\|$ and so τ is contractive. It follows from Example 1.1.14 that τ is also completely contractive. Now let $\tau_\sharp : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathbb{C}$ be the map $\tau \circ \iota$ where $\iota : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G})$ is the usual embedding. We have that τ is a completely contractive homomorphism and so $\tau_\sharp$ is a completely contractive homomorphism. In particular we have $\tau_\sharp \in \mathrm{L}^1_\sharp(\mathbb{G})^*$. We will fix this map $\tau_\sharp$ throughout this section.

For all $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ we have

$$\langle \theta^*(1,0), \omega \rangle = \langle 1, \omega \rangle = \langle \tau, \omega \rangle = \langle \tau, \iota(\omega) \rangle = \langle \tau_\sharp, \omega \rangle$$

and so we can identify $\tau_\sharp = (1,0) + K_\sharp$. As $(1,0) + K_\sharp \neq 0$ we have some $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ such that $\langle \tau_\sharp, \omega \rangle \neq 0$ and thus $\tau_\sharp$ is surjective. Let $I_\sharp = \operatorname{Ker} \tau_\sharp$ and we have a commutative diagram

$$\begin{array}{ccc} \mathrm{L}^1_\sharp(\mathbb{G}) & \xrightarrow{\tau_\sharp} & \mathbb{C} \\ {\scriptstyle q}\downarrow & \nearrow_{\tilde{\tau}_\sharp} & \\ \mathrm{L}^1_\sharp(\mathbb{G})/I_\sharp. & & \end{array}$$

As τ and ι are contractive then so is $\tilde{\tau}_\sharp$ and by Proposition 1.1.12 we have that this is in fact completely contractive. Clearly it is surjective as it is non-zero and injective as we have quotiented by the kernel. Then by Corollary 1.1.13 we have that $\tilde{\tau}_\sharp^{-1}$ is completely contractive and thus $\tilde{\tau}_\sharp$ is a completely isometric isomorphism.

As $\tau_\sharp$ is a complete contraction we can make $\mathbb{C}$ a completely contractive left $\mathrm{L}^1_\sharp(\mathbb{G})$-module with module operation given by

$$\omega \cdot \lambda = \lambda \tau_\sharp(\omega) \tag{6.2}$$

for all $\lambda \in \mathbb{C}$ and $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$.

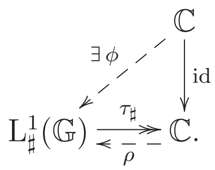
Lemma 6.2.4 *Suppose that $\mathbb{C}$ be projective as a left completely contractive $\mathrm{L}^1_\sharp(\mathbb{G})$-module with module operation given by Equation 6.2, then $\mathbb{G}$ is compact.*

Proof

We show that there is a normal left invariant state on $\mathrm{L}^\infty(\mathbb{G})$ and then by Proposition 3.2.2 it follows that $\mathbb{G}$ is compact.

As $\tau_\sharp$ is a homomorphism we have $\tau_\sharp(\omega_1 * \omega_2) = \tau_\sharp(\omega_1)\tau_\sharp(\omega_2) = \omega_1 \cdot \tau_\sharp(\omega_2)$ for $\omega_1, \omega_2 \in \mathrm{L}^1_\sharp(\mathbb{G})$ and so $\tau_\sharp$ is a left completely contractive $\mathrm{L}^1_\sharp(\mathbb{G})$-module homomorphism. Now fix $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ with $\tau_\sharp(\omega) = 1$ and let $\rho : \mathbb{C} \to \mathrm{L}^1_\sharp(\mathbb{G})$ be the map $\lambda \mapsto \lambda\omega$

and we have $\tau_\sharp(\rho(\lambda)) = \lambda\tau_\sharp(\omega) = \lambda$ and so $\tau_\sharp$ is admissible. As $\mathbb{C}$ is projective as a completely bounded left A-module then there exists a completely bounded left $\mathrm{L}^1_\sharp(\mathbb{G})$-module homomorphism making the following diagram commute



Let $\omega_0 = \phi(1)$ (for $1 \in \mathbb{C}$), then for all $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ we have

$$\omega * \omega_0 = \omega * \phi(1) = \phi(\omega \cdot 1) = \phi(\tau_\sharp(\omega)) = \tau_\sharp(\omega)\phi(1) = \langle 1, \omega\rangle\omega_0$$

where we've used that ϕ is a left $\mathrm{L}^1_\sharp(\mathbb{G})$-module homomorphism. Let $\omega \in \mathrm{L}^1(\mathbb{G})$, then as $\mathrm{L}^1_\sharp(\mathbb{G})$ is dense in $\mathrm{L}^1(\mathbb{G})$ by Theorem 4.1.4 we have a net in $(\omega_\alpha) \subset \mathrm{L}^1_\sharp(\mathbb{G})$ with $\lim \omega_\alpha = \omega$ and

$$\begin{aligned}\|\omega * \omega_0 - \langle 1, \omega\rangle\omega_0\| &\leqslant \|\omega * \omega_0 - \omega_\alpha * \omega_0\| + \|\langle 1, \omega_\alpha\rangle\omega_0 - \langle 1, \omega\rangle\omega_0\| \\ &\leqslant \|\omega - \omega_\alpha\|\|\omega_0\| + \|1\|\|\omega_\alpha - \omega\|\|\omega_0\| \to 0.\end{aligned}$$

So we have shown that for all $\omega \in \mathrm{L}^1(\mathbb{G})$ we have $\omega * \omega_0 = \langle 1, \omega\rangle\omega_0$ and thus ω_0 is a left invariant normal functional by Definition 2.2.1.

We have for $x \in \mathrm{L}^\infty(\mathbb{G})$ and $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ that

$$\begin{aligned}\langle x, \omega * \omega_0^*\rangle = \langle \Delta(x), \omega \otimes \omega_0^*\rangle &= \overline{\langle \Delta(x^*), \omega^* \otimes \omega_0\rangle} \\ &= \overline{\langle x^*, \omega^* * \omega_0\rangle} = \overline{\langle 1, \omega^*\rangle\langle x^*, \omega_0\rangle} = \langle 1, \omega\rangle\langle x, \omega_0^*\rangle\end{aligned}$$

and so ω_0^* is also a normal left invariant functional. So we have a self-adjoint normal left invariant functional $(\omega_0 + \omega_0^*)/2$ and thus we may assume without loss of generality that ω_0 is self-adjoint.

Now consider the Jordan decomposition $\omega_0 = \omega_0^+ - \omega_0^-$ with $\omega_0^+, \omega_0^- \in \mathrm{L}^1(\mathbb{G})$ such that $\|\omega_0\| = \|\omega_0^+\| + \|\omega_0^-\|$. Let $\omega \in \mathrm{L}^1(\mathbb{G})$ be any state, then we have

$$\omega_0 = \langle 1, \omega \rangle \omega_0 = \omega * \omega_0 = \omega * \omega_0^+ - \omega * \omega_0^-$$

where $\omega_0 * \omega_0^+$ and $\omega_0 * \omega_0^-$ are both positive as Δ is a $*$-homomorphism. Then

$$\|\omega_0\| = \|\omega * \omega_0\| = \|\omega * \omega_0^+ - \omega * \omega_0^-\| \leqslant \|\omega * \omega_0^+\| + \|\omega * \omega_0^-\| \leqslant \|\omega_0^+\| + \|\omega_0^-\| = \|\omega_0\|$$

and then we have equality throughout and so in particular $\|\omega_0\| = \|\omega * \omega_0^+\| + \|\omega * \omega_0^-\|$. By the uniqueness of the Jordan decomposition we have for all states $\omega \in \mathrm{L}^1(\mathbb{G})^+$ that

$$\omega * \omega_0^+ = \omega_0^+ = \langle 1, \omega \rangle \omega_0^+$$

and so for all $\omega \in \mathrm{L}^1(\mathbb{G})^+$ we have $\omega * \omega_0^+ = \langle 1, \omega \rangle \omega_0^+$.

Either ω_0^+ or ω_0^- must be non-zero, otherwise $\omega_0 = 0$ so we assume without loss of generality that $\omega_0^+ \neq 0$. Let $\omega_0' = \dfrac{\omega_0^+}{\langle 1, \omega_0^+ \rangle}$ and we have ω_0' is a normal left invariant state as required. □

We need one more technical lemma before proving Theorem 6.2.3.

Lemma 6.2.5 *We have* $\overline{\mathrm{L}^1_\sharp(\mathbb{G}) I_\sharp}^{\|\cdot\|} = I_\sharp$ *where* $I_\sharp = \operatorname{Ker} \tau_\sharp$.

Proof

We first show that $\overline{\mathrm{L}^1_\sharp(\mathbb{G}) I_\sharp}^{\|\cdot\|_\sharp} \subset I_\sharp$. As $\tau_\sharp$ is a homomorphism, then for $\omega_1 \in \mathrm{L}^1_\sharp(\mathbb{G})$ and $\omega_2 \in I_\sharp$ we have

$$\langle \omega_1 * \omega_2, \tau_\sharp \rangle = \langle \omega_1, \tau_\sharp \rangle \langle \omega_2, \tau_\sharp \rangle = 0$$

and so by linearity $\mathrm{L}^1_\sharp(\mathbb{G}) I_\sharp \subset I_\sharp$. Now let $(\omega_\alpha) \subset I_\sharp = \operatorname{Ker} \tau_\sharp$ be a net with limit $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$, then $\langle \omega, \tau_\sharp \rangle = \lim \langle \omega_\alpha, \tau_\sharp \rangle = 0$ and so $I_\sharp$ is a closed ideal. So as the right hand side is closed we can close the left hand side to get $\overline{\mathrm{L}^1_\sharp(\mathbb{G}) I_\sharp}^{\|\cdot\|_\sharp} \subset I_\sharp$.

We now need to show that $\overline{\mathrm{L}^1_\sharp(\mathbb{G})I_\sharp}^{\|\cdot\|} \supset I_\sharp$ or as both sides are closed we equivalently show that $(\mathrm{L}^1_\sharp(\mathbb{G})I_\sharp)^\perp \subset I_\sharp^\perp$. First we show that $I_\sharp^\perp = \{\lambda\tau_\sharp \mid \lambda \in \mathbb{C}\}$. As $\lambda\tau_\sharp \in I_\sharp^\perp$ for all $\lambda \in \mathbb{C}$ we have $I_\sharp^\perp \supset \{\lambda\tau_\sharp \mid \lambda \in \mathbb{C}\}$. Now let $T \in I_\sharp^\perp$ (that is $T(\omega) = 0$ for all $\omega \in I_\sharp$), then there is a unique $\tilde{T} : \mathrm{L}^1_\sharp(\mathbb{G})/I_\sharp \to \mathbb{C}$ such that for $q : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})/I_\sharp$ the quotient map $\omega \mapsto \omega + I_\sharp$ we have $T = \tilde{T} \circ q$. Then as $\mathrm{L}^1_\sharp(\mathbb{G})/I_\sharp \cong_i \mathbb{C}$ there is some $\lambda \in \mathbb{C}$ such that for all $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ we have $\tilde{T}(\omega + I_\sharp) = \lambda\tilde{\tau}_\sharp(\omega + I_\sharp)$. Then in particular we have $T(\omega) = \lambda\tau_\sharp(\omega)$ for all $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ and so $T = \lambda\tau_\sharp$ and $T \in \{\lambda\tau_\sharp \mid \lambda \in \mathbb{C}\}$.

Let $x, y \in \mathrm{L}^\infty(\mathbb{G})$ such that $(x, \overline{y}) + K_\sharp \in (\mathrm{L}^1_\sharp(\mathbb{G})I_\sharp)^\perp$, then we show this is in $I_\sharp^\perp$ or indeed that there is some λ such that $(x, \overline{y}) + K_\sharp = (\lambda 1, 0) + K_\sharp$. Let $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ and $\kappa \in I_\sharp$, then $\langle (x, \overline{y}) + K_\sharp, \omega * \kappa \rangle = 0$ and so

$$\begin{aligned} 0 &= \langle (x, \overline{y}) + K_\sharp, \omega * \kappa \rangle = \langle x, \omega * \kappa \rangle + \overline{\langle y, (\omega * \kappa)^\sharp \rangle} \\ &= \langle \Delta(x), \omega \otimes \kappa \rangle + \overline{\langle \Delta(y), \kappa^\sharp \otimes \omega^\sharp \rangle} = \langle (\omega \otimes \mathrm{id})\Delta(x), \kappa \rangle + \overline{\langle (\mathrm{id} \otimes \omega^\sharp)\Delta(y), \kappa^\sharp \rangle} \\ &= \left\langle \big((\omega \otimes \mathrm{id})\Delta(x), (\mathrm{id} \otimes \omega^\sharp)\Delta(y) \big) + K_\sharp, \kappa \right\rangle. \end{aligned} \tag{6.3}$$

As this holds for all $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ and $\kappa \in I_\sharp$ then $\big((\omega \otimes \mathrm{id})\Delta(x), (\mathrm{id} \otimes \omega^\sharp)\Delta(y) \big) + K_\sharp \in I_\sharp^\perp$ for all $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$. We showed above that $I_\sharp^\perp = \{\lambda\tau_\sharp \mid \lambda \in \mathbb{C}\}$ so for all $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ we have some $\alpha(\omega) \in \mathbb{C}$ such that

$$\big((\omega \otimes \mathrm{id})\Delta(x), (\mathrm{id} \otimes \omega^\sharp)\Delta(y) \big) + K_\sharp = (\alpha(\omega)1, 0) + K_\sharp. \tag{6.4}$$

It is easy to show that this defines a linear map $\alpha : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathbb{C}$ and we show that it is bounded and thus $\alpha \in \mathrm{L}^1_\sharp(\mathbb{G})^*$. By acting ω_2 on Equation (6.4) and using the calculation (6.3), for any $\omega_1, \omega_2 \in \mathrm{L}^1_\sharp(\mathbb{G})$ we have

$$\alpha(\omega_1)\langle 1, \omega_2 \rangle = \langle (x, \overline{y}) + K_\sharp, \omega_1 * \omega_2 \rangle$$

and so

$$|\alpha(\omega_1)|\,|\langle 1,\omega_2\rangle| \leqslant \|(x,\overline{y})+K_\sharp\|_{\mathrm{L}^1_\sharp(\mathbb{G})^*}\|\omega_1\|_{\mathrm{L}^1_\sharp(\mathbb{G})}\|\omega_2\|_{\mathrm{L}^1_\sharp(\mathbb{G})}.$$

Now taking the supremum over all $\omega_2 \in \mathrm{L}^1_\sharp(\mathbb{G})$ and then $\omega_1 \in \mathrm{L}^1_\sharp(\mathbb{G})$ we get

$$\|\alpha\| \leqslant \frac{\|(x,\overline{y})+K_\sharp\|_{\mathrm{L}^1_\sharp(\mathbb{G})^*}}{\|(1,0)+K_\sharp\|_{\mathrm{L}^1_\sharp(\mathbb{G})}}.$$

Finally using that $(1,0)+K_\sharp \neq 0$ we have $\|(1,0)+K_\sharp\| \neq 0$ and α is bounded.

As we have shown that $\alpha \in \mathrm{L}^1_\sharp(\mathbb{G})^*$ we can write $\alpha = (x',\overline{y'})+K_\sharp$ for some $x',y' \in \mathrm{L}^\infty(\mathbb{G})$. For all $\omega_1,\omega_2 \in \mathrm{L}^1_\sharp(\mathbb{G})$ we have from Equation (6.4) that

$$\langle\Delta(x),\omega_1\otimes\omega_2\rangle + \overline{\langle\Delta(y),\omega_2^\sharp\otimes\omega_1^\sharp\rangle} = \langle x'\otimes 1,\omega_1\otimes\omega_2\rangle + \overline{\langle y'\otimes 1,\omega_1^\sharp\otimes\omega_2^\sharp\rangle} \tag{6.5}$$

where we've used that $\langle 1,\omega_2^\sharp\rangle = \overline{\langle 1,\omega_2\rangle}$. We now smear and calculate as follows

$$\begin{aligned}
\langle (x'(n)+S(y'(n))^*)\otimes 1,\omega_1\otimes\omega_2\rangle &= \left(\langle x'(n),\omega_1\rangle + \overline{\langle y'(n),\omega_1^\sharp\rangle}\right)\langle 1,\omega_2\rangle \\
&= \left(\frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2t^2}\left(\langle\tau_t(x'),\omega_1\rangle + \overline{\langle\tau_t(y'),\omega_1^\sharp\rangle}\right)dt\right)\langle 1,\omega_2\rangle \\
&= \frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2t^2}\left(\langle x'\otimes 1,\omega_1\circ\tau_t\otimes\omega_2\circ\tau_t\rangle + \overline{\langle y'\otimes 1,\omega_1^\sharp\circ\tau_t\otimes\omega_2^\sharp\circ\tau_t\rangle}\right)dt \\
&= \frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2t^2}\left(\langle\Delta(x),\omega_1\circ\tau_t\otimes\omega_2\circ\tau_t\rangle + \overline{\langle\Delta(y),(\omega_2\circ\tau_t)^\sharp\otimes(\omega_1\circ\tau_t)^\sharp\rangle}\right)dt \\
&= \frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2t^2}\left(\langle\Delta(\tau_t(x)),\omega_1\otimes\omega_2\rangle + \overline{\langle\Delta(\tau_t(y)),\omega_2^\sharp\otimes\omega_1^\sharp\rangle}\right)dt \\
&= \frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2t^2}\left(\langle\tau_t(x),\omega_1*\omega_2\rangle + \overline{\langle\tau_t(y),(\omega_1*\omega_2)^\sharp\rangle}\right)dt \\
&= \langle x(n),\omega_1*\omega_2\rangle + \overline{\langle y(n),(\omega_1*\omega_2)^\sharp\rangle} \\
&= \langle x_1(n),\omega_1*\omega_2\rangle + \langle S(y(n))^*,\omega_1*\omega_2\rangle = \langle\Delta(x(n)+S(y(n))^*),\omega_1\otimes\omega_2\rangle
\end{aligned}$$

where we've used Equation (6.5), τ_t is normal, $\tau_t(1)=1$, $(\omega\circ\tau_t)^\sharp = \omega^\sharp\circ\tau_t \in \mathrm{L}^1_\sharp(\mathbb{G})$ for all $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ and $t \in \mathbb{R}$ and $(\tau_t\otimes\tau_t)\circ\Delta = \Delta\circ\tau_t$ for all $t \in \mathbb{R}$. See Definition-Theorem

2.2.7 and Propositions 2.2.8 and 4.1.5 for these results.

This holds for all $\omega_1, \omega_2 \in L^1_\sharp(\mathbb{G})$ and thus we have

$$\Delta(x(n) + S(y(n))^*) = (x'(n) + S(y'(n))^*) \otimes 1.$$

Then by Lemma 2.2.13 for all $n \in \mathbb{N}$ we have some $t_n \in \mathbb{C}$ such that $x(n) + S(y(n))^* = t_n 1$. Let $\omega \in L^1_\sharp(\mathbb{G})$ and we have

$$\begin{aligned} |(t_n - t_m)|\,|\langle 1, \omega\rangle| &= |\langle x(n) - x(m), \omega\rangle + \langle S(y(n))^* - S(y(m))^*, \omega\rangle| \\ &= \left|\langle x(n) - x(m), \omega\rangle + \overline{\langle y(n) - y(m), \omega^\sharp\rangle}\right| \to 0 \end{aligned}$$

and so as there exists $\omega \in L^1_\sharp(\mathbb{G})$ with $\langle 1, \omega\rangle \neq 0$ then $(t_n) \subset \mathbb{C}$ is a Cauchy sequence.

Let $t \in \mathbb{C}$ be the limit of (t_n) and for $\omega \in L^1_\sharp(\mathbb{G})$ we have

$$\begin{aligned} \langle (x, \overline{y}) + K_\sharp, \omega\rangle &= \langle x, \omega\rangle + \overline{\langle y, \omega^\sharp\rangle} = \lim\langle x(n), \omega\rangle + \overline{\lim\langle y(n), \omega^\sharp\rangle} \\ &= \lim\langle x(n) + S(y(n))^*, \omega\rangle = \lim t_n\langle 1, \omega\rangle = t\langle 1, \omega\rangle. \end{aligned}$$

As this holds for all $\omega \in L^1_\sharp(\mathbb{G})$ we have $(x, \overline{y}) + K_\sharp = t(1, 0) + K_\sharp$ as required. □

Proof of Theorem 6.2.3

As $L^1_\sharp(\mathbb{G})$ is a completely contractive Banach algebra it follows from Example 1.1.50 that the unitisation $L^1_\sharp(\mathbb{G})^\flat$ is a completely contractive Banach algebra. By Lemma 1.2.11 we then have that $L^1_\sharp(\mathbb{G})^\flat / I_\sharp$ is a left operator $L^1_\sharp(\mathbb{G})$-module with an operation such that

$$\omega \cdot ((\omega', \lambda) + I_\sharp) = \omega * \omega' + \lambda\omega + I_\sharp.$$

By Lemma 1.2.15 we have that $L^1_\sharp(\mathbb{G}) \widehat{\otimes}_{L^1_\sharp(\mathbb{G})} (L^1_\sharp(\mathbb{G})^\flat / I_\sharp)$ is a projective left completely bounded $L^1_\sharp(\mathbb{G})$-module. By Lemma 1.2.14 this is completely isometrically isomorphic to $L^1_\sharp(\mathbb{G}) / \overline{L^1_\sharp(\mathbb{G}) I_\sharp}$ which by Lemma 6.2.5 is equal to $L^1_\sharp(\mathbb{G}) / I_\sharp \cong \mathbb{C}$. So $\mathbb{C}$ is a left projective completely bounded $L^1_\sharp(\mathbb{G})$-module and then $\mathbb{G}$ is compact by Lemma 6.2.4. □

6.2.4 Structure Theorem for Operator Biprojectivity of $\mathrm{L}^1_\sharp(\mathbb{G})$ for Compact Quantum Group $\mathbb{G}$

We have seen in the previous section that if $\mathrm{L}^1_\sharp(\mathbb{G})$ is operator biprojective then $\mathbb{G}$ is compact. We assume in this section that $\mathbb{G}$ is compact and we prove a structure theorem for the operator biprojectivity of $\mathrm{L}^1_\sharp(\mathbb{G})$. In particular we prove some necessary and sufficient conditions for the $\mathrm{L}^1_\sharp(\mathbb{G})$ algebra of a compact quantum group $\mathbb{G}$ to be operator biprojective. The inspiration for this section comes from Section 3 in Daws (2010) where a similar theorem is proved for the $\mathrm{L}^1(\mathbb{G})$ algebra.

Let $\{U^\alpha \in \mathrm{C}(\mathbb{G}) \otimes \mathcal{B}(\mathcal{H}_\alpha) \mid \alpha \in \mathbb{A}\}$ denote the maximal family of corepresentations from Theorem 3.2.9 throughout this section.

We begin now by stating the main theorem and we spend the rest of this section proving this result.

Theorem 6.2.6 *Let $m_\sharp : \mathrm{L}^1_\sharp(\mathbb{G}) \widehat{\otimes} \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ be the multiplication map from Theorem 4.2.1 and $\Delta^\sharp$ its adjoint. Then the following are equivalent:*

(i) $\mathrm{L}^1_\sharp(\mathbb{G})$ *is operator biprojective;*

(ii) There exists a completely bounded normal map $\Psi : \mathrm{L}^1_\sharp(\mathbb{G})^* \overline{\otimes} \mathrm{L}^1_\sharp(\mathbb{G})^* \to \mathrm{L}^1_\sharp(\mathbb{G})^*$ such that

$$\Psi \circ \Delta^\sharp = \mathrm{id}_{\mathrm{L}^1_\sharp(\mathbb{G})^*}, \qquad \Delta^\sharp \circ \Psi = (\Psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta^\sharp) = (\mathrm{id} \otimes \Psi) \circ (\Delta^\sharp \otimes \mathrm{id}); \tag{6.6}$$

(iii) There exists a family of matrices $\{X^\alpha \in \mathbb{M}_{n_\alpha} \mid \alpha \in \mathbb{A}\}$ such that for $\alpha, \beta \in \mathbb{A}$, $1 \leqslant i, j \leqslant n_\alpha$ and $1 \leqslant k, l \leqslant n_\beta$ and a completely bounded normal map Ψ :

$\mathrm{L}^1_\sharp(\mathbb{G})^* \overline{\otimes} \mathrm{L}^1_\sharp(\mathbb{G})^* \to \mathrm{L}^1_\sharp(\mathbb{G})^*$ *such that*

$$\Psi\left(((u^\alpha_{ij},0)+K_\sharp)\otimes((u^\beta_{kl},0)+K_\sharp)\right)=\delta_{\alpha\beta}X^\alpha_{jk}((u^\alpha_{il},0)+K_\sharp),\qquad \sum_{r=1}^{n_\alpha}X^\alpha_{rr}=1. \tag{6.7}$$

We need two preparatory lemmas before proving Theorem 6.2.6.

Lemma 6.2.7 *Let $\iota : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1(\mathbb{G})$ be the completely contractive inclusion map, then the completely contractive adjoint $\iota^* : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})^*$ has weak*-dense range and we have*

$$(\iota^*\otimes\iota^*)\circ\Delta=\Delta^\sharp\circ\iota^*$$

where $\Delta^\sharp$ is the adjoint of the completely contractive map $m_\sharp : \mathrm{L}^1_\sharp(\mathbb{G})\widehat{\otimes}\mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ from Theorem 4.2.1.

Proof

We have for $x \in \mathrm{L}^\infty(\mathbb{G})$ and $\omega \in \mathrm{L}^1_\sharp(\mathbb{G})$ and using Theorems 4.1.15 and 4.2.6 that

$$\langle\iota^*(x),\omega\rangle=\langle x,\iota(\omega)\rangle=\langle(x,0)+K_\sharp,\omega\rangle$$

and so $\iota^*(x)=(x,0)+K_\sharp$.

By Theorem 4.1.15 for any element in $\mathrm{L}^1_\sharp(\mathbb{G})^*$ we have some $x, y \in \mathrm{L}^\infty(\mathbb{G})$ such that this element is given by $(x,\overline{y})+K_\sharp$. Let $n \in \mathbb{N}$ and consider $x(n)+S(y(n))^* \in \mathrm{L}^\infty(\mathbb{G})$, then as $K_\sharp=\left\{(x,-\overline{S(x)^*}) \;\middle|\; x\in\mathrm{Dom}(S)\right\}$ we have

$$\begin{aligned}\iota^*(x(n)+S(y(n))^*)&=(x(n)+S(y(n))^*,0)+K_\sharp\\&=(x(n)+S(y(n))^*,0)+(-S(y(n))^*,\overline{S(S(y(n))^*)^*})+K_\sharp\\&=(x(n),\overline{y(n)})+K_\sharp=\Upsilon(n)^*((x,\overline{y})+K_\sharp)\end{aligned}$$

where we've used $\Upsilon(n)$ from Notation 4.2.7. Using Proposition 4.2.9 we have for all

$T \in L^1_\sharp(\mathbb{G})^*$ and $\omega \in L^1_\sharp(\mathbb{G})$ that

$$|\langle \Upsilon(n)^*(T), \omega\rangle - \langle T, \omega\rangle| \leqslant \|T\|_{L^1_\sharp(\mathbb{G})^*} \|\Upsilon(n)(\omega) - \omega\|_{L^1_\sharp(\mathbb{G})} \to 0.$$

Therefore for all $\omega \in L^1_\sharp(\mathbb{G})$ we have

$$\lim \langle \iota^*(x(n) + S(y(n))^*), \omega\rangle = \lim \langle \Upsilon(n)^*((x, \overline{y}) + K_\sharp), \omega\rangle = \langle (x, \overline{y}) + K_\sharp, \omega\rangle$$

and so ι^* has weak*-dense range.

Using that ι is a homomorphism, for $x \in L^\infty(\mathbb{G})$ and $\omega_1, \omega_2 \in L^1_\sharp(\mathbb{G})$ we have

$$\begin{aligned}\langle (\Delta^\sharp \circ \iota^*)(x), \omega_1 \otimes \omega_2\rangle &= \langle \iota^*(x), m_\sharp(\omega_1 \otimes \omega_2)\rangle = \langle x, \iota(\omega_1 * \omega_2)\rangle \\ &= \langle \Delta(x), \iota(\omega_1) \otimes \iota(\omega_2)\rangle = \langle ((\iota^* \otimes \iota^*) \circ \Delta)(x)), \omega_1 \otimes \omega_2\rangle.\end{aligned}$$

Then for all $\Omega \in L^1(\mathbb{G}) \widehat{\otimes} L^1_\sharp(\mathbb{G})$ we have $\langle (\Delta^\sharp \circ \iota^*)(x), \Omega\rangle = \langle ((\iota^* \otimes \iota^*) \circ \Delta)(x)), \Omega\rangle$ and thus $\Delta^\sharp \circ \iota^* = (\iota^* \otimes \iota^*) \circ \Delta$ as required. □

Lemma 6.2.8 *For all $t \in \mathbb{R}$ there is an normal isometry $\tau^\sharp_t : L^1_\sharp(\mathbb{G})^* \to L^1_\sharp(\mathbb{G})^*$ such that $\tau^\sharp_t \circ \iota^* = \iota^* \circ \tau_t$ (where $\iota : L^1_\sharp(\mathbb{G}) \to L^1(\mathbb{G})$ is the inclusion) and where*

$$\tau^\sharp_t ((x, \overline{y}) + K_\sharp) = \Big((\tau_t(x), \overline{\tau_t(y)}) + K_\sharp \Big)$$

for all $x, y \in L^\infty(\mathbb{G})$. We have $\tau^\sharp : \mathbb{R} \to \mathcal{B}(L^1_\sharp(\mathbb{G})^*)$ is a weak-continuous one-parameter automorphism group on the Banach space $L^1_\sharp(\mathbb{G})^*$. In addition for $t \in \mathbb{R}$ we have*

$$(\tau^\sharp_t \otimes \tau^\sharp_t) \circ \Delta^\sharp = \Delta^\sharp \circ \tau^\sharp_t$$

where $\Delta^\sharp$ is the adjoint of the completely contractive map $m_\sharp : L^1_\sharp(\mathbb{G}) \widehat{\otimes} L^1_\sharp(\mathbb{G}) \to L^1_\sharp(\mathbb{G})$ from Theorem 4.2.1.

Proof

Fix $t \in \mathbb{R}$ and we consider the contraction $\tau_t^0 : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})^*$ given by $\iota^* \circ \tau_t$. Then as $\iota^* : \mathrm{L}^\infty(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})^*$ has weak*-dense range by Lemma 6.2.7 above we can uniquely define a contraction $\tau_t^\sharp : \mathrm{L}^1_\sharp(\mathbb{G})^* \to \mathrm{L}^1_\sharp(\mathbb{G})^*$ such that $\tau_t^\sharp \circ \iota^* = \iota^* \circ \tau_t$. As ι^* and τ_t are normal it follows that $\tau_t^\sharp$ is normal.

Let $x, y \in \mathrm{L}^\infty(\mathbb{G})$ and $t \in \mathbb{R}$, then by Lemma 6.2.7 we have

$$\begin{aligned}
\tau_t^\sharp\left((x,\overline{y}) + K_\sharp\right) &= \tau_t^\sharp\left(\lim((x(n) + S(y(n))^*, 0) + K_\sharp)\right) \\
&= \lim(\tau_t^\sharp \circ \iota^*)\left(x(n) + S(y(n))^*\right) = \lim(\iota^* \circ \tau_t)\left(x(n) + S(y(n))^*\right) \\
&= \lim \iota^*\left(\tau_t(x(n)) + S(\tau_t(y(n)))^*\right) = \lim\left((\tau_t(x)(n), \overline{\tau_t(y)(n)}) + K_\sharp\right) \\
&= \left(\left(\tau_t(x), \overline{\tau_t(y)}\right) + K_\sharp\right)
\end{aligned}$$

where we've used that $\iota^*(x) = (x, 0) + K_\sharp$ for all $x \in \mathrm{L}^\infty(\mathbb{G})$, $(x + S(y)^*, 0) + K_\sharp = (x, \overline{y}) + K_\sharp$ for all $x \in \mathrm{L}^\infty(\mathbb{G})$, $y \in \mathrm{Dom}(S)$ (see the proof of Proposition 4.2.8), $\tau_t \circ S = S \circ \tau_t$ from Proposition 2.2.8 and that $\tau_t(x)(n) = \tau_t(x(n))$ for all $x \in \mathrm{L}^\infty(\mathbb{G})$ and $n \in \mathbb{N}$.

It follows easily that $\tau_{t+s}^\sharp = \tau_t^\sharp \circ \tau_s^\sharp$ for all $t, s \in \mathbb{R}$ using the same property of τ_t. We have that $\tau_t^\sharp$ is an isometric automorphism as for all $t \in \mathbb{R}$ we have $\tau_{-t}^\sharp \circ \tau_t^\sharp = \mathrm{id} = \tau_t^\sharp \circ \tau_{-t}^\sharp$ and so $\tau_{-t}^\sharp$ is a contractive inverse for $\tau_t^\sharp$.

Let $x, y \in \mathrm{L}^\infty(\mathbb{G})$, then we have

$$\begin{aligned}
\left|\left\langle \tau_t^\sharp\left((x,\overline{y}) + K_\sharp\right), \omega\right\rangle - \left\langle \tau_{t_n}^\sharp\left((x,\overline{y}) + K_\sharp\right), \omega\right\rangle\right| &= \left|\left\langle \left(\tau_{t-t_n}(x), \overline{\tau_{t-t_n}(y)}\right) + K_\sharp, \omega\right\rangle\right| \\
= \left|\langle \tau_{t-t_n}(x), \omega\rangle - \overline{\langle \tau_{t-t_n}(y), \omega\rangle}\right| &\leqslant |\langle \tau_{t-t_n}(x), \omega\rangle| + |\langle \tau_{t-t_n}(y), \omega\rangle| \to 0
\end{aligned}$$

and so τ is a weak*-continuous one-parameter group of automorphisms.

The final statement follows as

$$\begin{aligned}
\Delta^\sharp \circ \tau_t^\sharp \circ \iota^* &= \Delta^\sharp \circ \iota^* \circ \tau_t = (\iota^* \otimes \iota^*) \circ \Delta \circ \tau_t = (\iota^* \otimes \iota^*) \circ (\tau_t \otimes \tau_t) \circ \Delta \\
&= (\tau_t^\sharp \otimes \tau_t^\sharp) \circ (\iota^* \otimes \iota^*) \circ \Delta = (\tau_t^\sharp \otimes \tau_t^\sharp) \circ \Delta^\sharp \circ \iota^*
\end{aligned}$$

where we've used Proposition 2.2.8 and the statements proved above $\Delta^\sharp \circ \iota^* = (\iota^* \otimes \iota^*) \circ \Delta$ and $\tau_t^\sharp \circ \iota^* = \iota^* \circ \tau_t$ for all $t \in \mathbb{R}$. As ι^* has dense range the result follows. □

The following lemma will help us prove (iii) $\implies$ (ii) in Theorem 6.2.6. Note that the proof gives us that β_k^α and γ_k^α both depend on j, however we will not use this in our proof of our main theorem.

Lemma 6.2.9 *Let $\Psi : \mathrm{L}^1_\sharp(\mathbb{G})^* \overline{\otimes} \mathrm{L}^1_\sharp(\mathbb{G})^* \to \mathrm{L}^1_\sharp(\mathbb{G})^*$ be a completely bounded normal map satisfying Equations (6.6), let $\alpha \in \mathbb{A}$, $1 \leqslant i, j \leqslant n_\alpha$ and $x \in \mathrm{L}^1_\sharp(\mathbb{G})^*$ be fixed and let $a_{ij} = \Psi\left(x \otimes ((u_{ij}^\alpha, 0) + K_\sharp)\right)$ and $b_{ij} = \Psi\left(((u_{ij}^\alpha, 0) + K_\sharp) \otimes x\right)$. Then there exist collections $\{\beta_k^\alpha \mid 1 \leqslant k \leqslant n_\alpha\}$ and $\{\gamma_k^\alpha \mid 1 \leqslant k \leqslant n_\alpha\}$ such that*

$$a_{ij} = \sum_{k=1}^{n_\alpha} \beta_k^\alpha ((u_{kj}^\alpha, 0) + K_\sharp) \qquad \textit{and} \qquad b_{ij} = \sum_{k=1}^{n_\alpha} \gamma_k^\alpha ((u_{ik}^\alpha, 0) + K_\sharp),$$

Proof

From the second Equation in (6.6) and using Lemma 6.2.7 we have

$$\begin{aligned} \Delta^\sharp(a_{ij}) &= (\Delta^\sharp \circ \Psi)\left(x \otimes \iota^*(u_{ij}^\alpha)\right) = (\Psi \otimes \mathrm{id})\left(x \otimes (\Delta^\sharp \circ \iota^*)(u_{ij}^\alpha)\right) \\ &= \sum_{r=1}^{n_\alpha} (\Psi \otimes \mathrm{id})\left(x \otimes \iota^*(u_{ir}^\alpha) \otimes \iota^*(u_{rj}^\alpha)\right) = \sum_{r=1}^{n_\alpha} a_{ir} \otimes ((u_{rj}^\alpha, 0) + K_\sharp). \end{aligned}$$

Then for all $\omega_1, \omega_2 \in \mathrm{L}^1_\sharp(\mathbb{G})$ we have

$$\langle a_{ij}, \omega_1 * \omega_2 \rangle = \sum_{r=1}^{n_\alpha} \langle a_{ir}, \omega_1 \rangle \langle u_{rj}^\alpha, \omega_2 \rangle.$$

As $\tau_t^\sharp$ is normal for all $t \in \mathbb{R}$ from Lemma 6.2.8 we have a pre-adjoint $(\tau_t^\sharp)_* : \mathrm{L}^1_\sharp(\mathbb{G}) \to \mathrm{L}^1_\sharp(\mathbb{G})$ and so replacing ω_1 with $(\tau_t^\sharp)_*(\omega_1)$ and similarly for ω_2 in the above equation we have

$$\left\langle a_{ij}, ((\tau_t^\sharp)_*(\omega_1)) * ((\tau_t^\sharp)_*(\omega_2)) \right\rangle = \sum_{r=1}^{n_\alpha} \langle a_{ir}, (\tau_t^\sharp)_*(\omega_1) \rangle \langle u_{rj}^\alpha, (\tau_t^\sharp)_*(\omega_2) \rangle.$$

Using that $(\tau_t^\sharp \otimes \tau_t^\sharp) \circ \Delta^\sharp = \Delta^\sharp \circ \tau_t^\sharp$ for all $t \in \mathbb{R}$ from Lemma 6.2.8 we have equivalently

$$\left\langle \tau_t^\sharp(a_{ij}), \omega_1 * \omega_2 \right\rangle = \sum_{r=1}^{n_\alpha} \langle \tau_t^\sharp(a_{ir}), \omega_1 \rangle \langle \tau_t(u^\alpha_{rj}), \omega_2 \rangle$$

and then weighting and integrating with respect to t and using Proposition 3.2.18 we get

$$\begin{aligned} &\frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \left\langle \tau_t^\sharp(a_{ij}), \omega_1 * \omega_2 \right\rangle dt \\ &\qquad = \sum_{r=1}^{n_\alpha} \left(\frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} (\lambda_r^\alpha)^{it} (\lambda_j^\alpha)^{-it} \langle \tau_t^\sharp(a_{ir}), \omega_1 \rangle \, dt \right) \langle u^\alpha_{rj}, \omega_2 \rangle. \end{aligned} \tag{6.8}$$

For all $1 \leqslant m, n \leqslant n_\alpha$ we let $a^1_{mn}, a^2_{mn} \in L^\infty(\mathbb{G})$ such that $a_{mn} = (a^1_{mn}, \overline{a^2_{mn}}) + K_\sharp$. We calculate the integrals in turn now, first we have

$$\begin{aligned} &\frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \left\langle \tau_t^\sharp(a_{ij}), \omega_1 * \omega_2 \right\rangle dt \\ &\qquad = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \left(\langle \tau_t(a^1_{ij}), \omega_1 * \omega_2 \rangle + \overline{\langle \tau_t(a^2_{ij}), (\omega_1 * \omega_2)^\sharp \rangle} \right) dt \\ &\qquad = \langle a^1_{ij}(n), \omega_1 * \omega_2 \rangle + \overline{\langle a^2_{ij}(n), (\omega_1 * \omega_2)^\sharp \rangle} \\ &\qquad = \langle a^1_{ij}(n), \omega_1 * \omega_2 \rangle + \langle S(a^2_{ij}(n))^*, \omega_1 * \omega_2 \rangle \\ &\qquad = \langle \Delta(a^1_{ij}(n) + S(a^2_{ij}(n))^*), \omega_1 \otimes \omega_2 \rangle \end{aligned}$$

where we've used that the smear of any element of $L^\infty(\mathbb{G})$ is in $\mathrm{Dom}(S)$. For all $1 \leqslant m, n \leqslant n_\alpha$ we let $\mu_{mn} = (\ln \lambda^\alpha_m - \ln \lambda^\alpha_n)/2$ and for the other integral in Equation (6.8)

we have

$$\begin{aligned}
&\frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2t^2}e^{2it\mu_{rj}}\langle\tau_t^\sharp(a_{ir}),\omega_1\rangle\,dt \\
&\qquad = \frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2(t-i\mu_{rj}/n^2)^2-\mu_{rj}^2/n^2}\left(\langle\tau_t(a^1_{ir}),\omega_1\rangle+\overline{\langle\tau_t(a^2_{ir}),\omega_1^\sharp\rangle}\right)dt \\
&\qquad = \exp\left(-\frac{\mu_{rj}^2}{n^2}\right)\left(\langle\tau_{i\mu_{rj}/n^2}(a^1_{ir}(n)),\omega_1\rangle+\overline{\langle\tau_{-i\mu_{rj}/n^2}(a^2_{ir}(n)),\omega_1^\sharp\rangle}\right) \\
&\qquad = \exp\left(-\frac{\mu_{rj}^2}{n^2}\right)\langle\tau_{i\mu_{rj}/n^2}(a^1_{ir}(n))+S(\tau_{-i\mu_{rj}/n^2}(a^2_{ir}(n)))^*,\omega_1\rangle
\end{aligned}$$

where we've used Theorem 1.3.17. Then substituting back into Equation (6.8) we have

$$\begin{aligned}
&\left\langle\Delta\left(a^1_{ij}(n)+S(a^2_{ij}(n))^*\right),\omega_1\otimes\omega_2\right\rangle \\
&\qquad = \sum_{r=1}^{n_\alpha}\exp\left(-\frac{\mu_{rj}^2}{n^2}\right)\langle\tau_{i\mu_{rj}/n^2}(a^1_{ir}(n))+S(\tau_{-i\mu_{rj}/n^2}(a^2_{ir}(n)))^*,\omega_1\rangle\langle u^\alpha_{rj},\omega_2\rangle
\end{aligned}$$

and as this is true for all $\omega_1,\omega_2\in\mathrm{L}^1_\sharp(\mathbb{G})$ we have

$$\begin{aligned}
&\Delta\left(a^1_{ij}(n)+S(a^2_{ij}(n))^*\right) \\
&\qquad = \sum_{r=1}^{n_\alpha}\exp\left(-\frac{\mu_{rj}^2}{n^2}\right)\left(\tau_{i\mu_{rj}/n^2}(a^1_{ir}(n))+S(\tau_{-i\mu_{rj}/n^2}(a^2_{ir}(n)))^*\right)\otimes u^\alpha_{rj}.
\end{aligned}$$

So we now have an equality in $\mathrm{L}^\infty(\mathbb{G})\overline{\otimes}\mathrm{L}^\infty(\mathbb{G})$. Fix $1\leqslant k,l\leqslant n_\alpha$ and using this equation and that Δ is a $*$-homomorphism we have

$$\begin{aligned}
\Delta\left((a^1_{ij}(n)+S(a^2_{ij}(n))^*)(u^\alpha_{kl})^*\right) &= \sum_{s=1}^{n_\alpha}\Delta((a^1_{ij}(n)+S(a^2_{ij}(n))^*))((u^\alpha_{ks})^*\otimes(u^\alpha_{sl})^*) \\
= \sum_{r,s=1}^{n_\alpha}\exp\left(-\frac{\mu_{rj}^2}{n^2}\right)&\left(\tau_{i\mu_{rj}/n^2}(a^1_{ir}(n))+S(\tau_{-i\mu_{rj}/n^2}(a^2_{ir}(n)))^*\right)(u^\alpha_{ks})^*\otimes u^\alpha_{rj}(u^\alpha_{sl})^*.
\end{aligned}$$

We can apply $(\mathrm{id}\otimes\phi)$ for ϕ the Haar state and using Definition-Theorem 3.2.3 and Theorem 3.2.15 we have

$$\begin{aligned}
&\phi\left((a^1_{ij}(n)+S(a^2_{ij}(n))^*)(u^\alpha_{kl})^*\right)1\\
&\quad=\sum_{r,s=1}^{n_\alpha}\exp\left(-\frac{\mu_{rj}^2}{n^2}\right)\left(\tau_{i\mu_{rj}/n^2}(a^1_{ir}(n))+S(\tau_{-i\mu_{rj}/n^2}(a^2_{ir}(n)))^*\right)(u^\alpha_{ks})^*\phi(u^\alpha_{rj}(u^\alpha_{sl})^*)\\
&\quad=\sum_{r=1}^{n_\alpha}\delta_{jl}\frac{\lambda^\alpha_l}{\Lambda^\alpha}\exp\left(-\frac{\mu_{rj}^2}{n^2}\right)\left(\tau_{i\mu_{rj}/n^2}(a^1_{ir}(n))+S(\tau_{-i\mu_{rj}/n^2}(a^2_{ir}(n)))^*\right)(u^\alpha_{kr})^*
\end{aligned}$$

where λ^α_l and Λ^α are as given in Theorem 3.2.15. This holds for all $1\leqslant k\leqslant n_\alpha$ and so we can multiply on the right by u^α_{ks} for $1\leqslant s\leqslant n_\alpha$ and sum over k. By doing this and using that U^α is unitary and so $\sum_{k=1}^{n_\alpha}(u^\alpha_{kr})^*u^\alpha_{ks}=\delta_{rs}1$ we have

$$\begin{aligned}
&\frac{\Lambda^\alpha}{\lambda^\alpha_j}\sum_{k=1}^{n_\alpha}\phi\left((a^1_{ij}(n)+S(a^2_{ij}(n))^*)(u^\alpha_{kj})^*\right)u^\alpha_{ks}\\
&\qquad\qquad=\exp\left(-\frac{\mu_{sj}^2}{n^2}\right)\left(\tau_{i\mu_{sj}/n^2}(a^1_{is}(n))+S(\tau_{-i\mu_{sj}/n^2}(a^2_{is}(n)))^*\right).
\end{aligned}$$

Then for any $\omega\in\mathrm{L}^1_\sharp(\mathbb{G})$ we have

$$\begin{aligned}
&\frac{\Lambda^\alpha}{\lambda^\alpha_j}\sum_{k=1}^{n_\alpha}\phi\left((a^1_{ij}(n)+S(a^2_{ij}(n))^*)(u^\alpha_{kj})^*\right)\langle u^\alpha_{ks},\omega\rangle\\
&\quad=\exp\left(-\frac{\mu_{sj}^2}{n^2}\right)\left\langle\tau_{i\mu_{sj}/n^2}(a^1_{is}(n))+S(\tau_{-i\mu_{sj}/n^2}(a^2_{is}(n)))^*,\omega\right\rangle\\
&\quad=\exp\left(-\frac{\mu_{sj}^2}{n^2}\right)\left(\left\langle\tau_{i\mu_{sj}/n^2}(a^1_{is}(n)),\omega\right\rangle+\overline{\left\langle\tau_{-i\mu_{sj}/n^2}(a^2_{is}(n)),\omega^\sharp\right\rangle}\right)\\
&\quad=\exp\left(-\frac{\mu_{sj}^2}{n^2}\right)\left(\frac{n}{\sqrt{\pi}}\int_{\mathbb{R}}e^{-n^2(t-i\mu_{sj}/n^2)^2}\left(\langle\tau_t(a^1_{is}),\omega\rangle+\overline{\langle\tau_t(a^2_{is}),\omega^\sharp\rangle}\right)\right)dt\\
&\quad=\exp\left(-\frac{\mu_{sj}^2}{n^2}\right)\frac{n}{\sqrt{\pi}}\int_{\mathbb{R}}e^{-n^2(t-i\mu_{sj}/n^2)^2}\langle\tau^\sharp_t(a_{is}),\omega\rangle\,dt \qquad (6.9)
\end{aligned}$$

where we've used Theorem 1.3.17 again. Using that $\tau^\sharp$ is a weak*-continuous one-parameter group we calculate

$$\left|\exp\left(-\frac{\mu_{sj}^2}{n^2}\right)\frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2(t-i\mu_{sj}/n^2)^2}\langle\tau_t^\sharp(a_{is}),\omega\rangle\,dt - e^{2i\mu_{sj}}\langle a_{is},\omega\rangle\right|$$
$$= \left|\frac{n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-n^2t^2}\left(\langle\tau_t^\sharp(a_{is}),\omega\rangle - \langle a_{is},\omega\rangle\right)dt\right| \to 0$$

and so using this, Proposition 4.3.2 and letting $n\to\infty$ Equation (6.9) becomes

$$\begin{aligned} e^{2i\mu_{sj}}\langle a_{is},\omega\rangle &= \frac{\Lambda^\alpha}{\lambda_j^\alpha}\sum_{k=1}^{n_\alpha}\lim_{n\to\infty}\phi((a_{ij}^1(n)+S(a_{ij}^2(n))^*)(u_{kj}^\alpha)^*)\langle u_{ks}^\alpha,\omega\rangle \\ &= \sum_{k=1}^{n_\alpha}\lim_{n\to\infty}\left(\langle a_{ij}^1(n),\omega_{kj}^\alpha\rangle + \langle S(a_{ij}^2(n))^*,\omega_{kj}^\alpha\rangle\right)\langle u_{ks}^\alpha,\omega\rangle \\ &= \sum_{k=1}^{n_\alpha}\left(\langle a_{ij}^1,\omega_{kj}^\alpha\rangle + \overline{\langle a_{ij}^2,(\omega_{kj}^\alpha)^\sharp\rangle}\right)\langle u_{ks}^\alpha,\omega\rangle = \sum_{k=1}^{n_\alpha}\langle a_{ij},\omega_{kj}^\alpha\rangle\langle u_{ks}^\alpha,\omega\rangle \end{aligned}$$

where we remind that $\omega_{kj}^\alpha = \dfrac{\Lambda^\alpha}{\lambda_j^\alpha}(u_{kj}^\alpha)^*\cdot\phi$ from Notation 4.3.1. As this holds for all $\omega\in\mathrm{L}^1_\sharp(\mathbb{G})$ we have

$$a_{is} = \sum_{k=1}^{n_\alpha}\lambda_k((u_{ks}^\alpha,0)+K_\sharp)$$

for all $\alpha\in\mathbb{A}$ and $1\leqslant i,j,s\leqslant n_\alpha$ where we've set $\lambda_k := e^{-2i\mu_{is}}\langle a_{ij},\omega_{kj}^\alpha\rangle\in\mathbb{C}$ for all $1\leqslant k\leqslant n_\alpha$.

Using similar techniques we can show that for $1\leqslant i,j,s\leqslant n_\alpha$ we have

$$b_{sj} = e^{-2i\mu_{is}}\Lambda^\alpha\lambda_j^\alpha\sum_{l=1}^{n_\alpha}\phi((u_{il}^\alpha)^*b_{ij})((u_{sl}^\alpha,0)+K_\sharp)$$

giving the other formula in the lemma. □

Proof of Theorem 6.2.6

(i) $\Longrightarrow$ (ii): Then there exists a completely bounded $\mathrm{L}^1_\sharp(\mathbb{G})$-bimodule homomorphism $\Psi_* : \mathrm{L}^1_\sharp(\mathbb{G})\to\mathrm{L}^1_\sharp(\mathbb{G})\widehat{\otimes}\mathrm{L}^1_\sharp(\mathbb{G})$ that is a right inverse to the multiplication map $m_\sharp$. By Theorem 6.2.1 we have the adjoint $\Psi : \mathrm{L}^1_\sharp(\mathbb{G})^*\overline{\otimes}\mathrm{L}^1_\sharp(\mathbb{G})^*\to\mathrm{L}^1_\sharp(\mathbb{G})^*$ and as Ψ_* is a right inverse to $m_\sharp$ we have $\Psi \circ \Delta^\sharp = \mathrm{id}_{L^1_\sharp(\mathbb{G})^*}$. By construction Ψ is normal. Also for $\omega_1, \omega_2 \in L^1_\sharp(\mathbb{G})$ and $T \in L^1_\sharp(\mathbb{G})^* \overline{\otimes} L^1_\sharp(\mathbb{G})^*$ we have

$$\begin{aligned}\langle (\Delta^\sharp \circ \Psi)(T), \omega_1 \otimes \omega_2 \rangle &= \langle T, \Psi_*(\omega_1 * \omega_2) \rangle = \langle T, \Psi_*(\omega_1) * \omega_2 \rangle \\ &= \langle T, (\mathrm{id} \otimes m_\sharp)(\Psi_*(\omega_1) \otimes \omega_2) \rangle = \langle (\Psi \otimes \mathrm{id})(\mathrm{id} \otimes \Delta^\sharp)(T), \omega_1 \otimes \omega_2 \rangle\end{aligned}$$

and similarly we can show $\langle (\Delta^\sharp \circ \Psi)(T), \omega_1 \otimes \omega_2 \rangle = \langle (\mathrm{id} \otimes \Psi)(\Delta^\sharp \otimes \mathrm{id})(T), \omega_1 \otimes \omega_2 \rangle$. As these hold for all $\omega_1, \omega_2 \in L^1_\sharp(\mathbb{G})$ and $T \in L^1_\sharp(\mathbb{G})^* \overline{\otimes} L^1_\sharp(\mathbb{G})^*$ we have Equation (6.6).

(ii) $\implies$ (i): As Ψ in (ii) is normal and using Theorem 6.2.1 again we have a predual map $\Psi_* : L^1_\sharp(\mathbb{G}) \to L^1_\sharp(\mathbb{G}) \widehat{\otimes} L^1_\sharp(\mathbb{G})$ that is completely bounded and such that $m_\sharp \circ \Psi_* = \mathrm{id}_{L^1_\sharp(\mathbb{G})}$, i.e. Ψ_* is a right inverse to $\Delta^\sharp$. It follows by similar calculations to above that Ψ_* is a completely bounded $L^1_\sharp(\mathbb{G})$-bimodule homomorphism.

(iii) $\implies$ (ii): Using Equation (6.7), Lemma 6.2.8 above and Proposition 3.2.11, for $\alpha \in \mathbb{A}$ and $1 \leqslant i, j \leqslant n_\alpha$ we have

$$\begin{aligned}(\Psi \circ \Delta^\sharp \circ \iota^*)(u^\alpha_{ij}) &= (\Psi \circ (\iota^* \otimes \iota^*) \circ \Delta)(u^\alpha_{ij}) = (\Psi \circ (\iota^* \otimes \iota^*)) \left(\sum_{k=1}^{n_\alpha} u^\alpha_{ik} \otimes u^\alpha_{kj} \right) \\ &= \sum_{k=1}^{n_\alpha} \Psi \left(((u^\alpha_{ik}, 0) + K_\sharp) \otimes ((u^\alpha_{kj}, 0) + K_\sharp) \right) = \sum_{k=1}^{n_\alpha} X^\alpha_{kk}((u^\alpha_{ij}, 0) + K_\sharp) = \iota^*(u^\alpha_{ij}).\end{aligned}$$

As $\mathrm{Hopf}(\mathbb{G})$ is weak*-dense in $L^\infty(\mathbb{G})$ we have $\Psi \circ \Delta^\sharp \circ \iota^* = \iota^*$ and then using that ι^* has weak*-dense range we have $\Psi \circ \Delta^\sharp = \mathrm{id}$. For $\alpha, \beta \in \mathbb{A}$, $1 \leqslant i, j \leqslant n_\alpha$ and $1 \leqslant k, l \leqslant n_\beta$

we have

$$
\begin{aligned}
&\left((\mathrm{id}\otimes\Psi)\circ(\Delta^\sharp\otimes\mathrm{id})\circ(\iota^*\otimes\iota^*)\right)\left(u^\alpha_{ij}\otimes u^\beta_{kl}\right)\\
&\quad=((\mathrm{id}\otimes\Psi))\left(((\iota^*\otimes\iota^*)\circ\Delta)(u^\alpha_{ij})\otimes((u^\beta_{kl},0)+K_\sharp)\right)\\
&\quad=\sum_{r=1}^{n_\alpha}(\mathrm{id}\otimes\Psi)\left(((u^\alpha_{ir},0)+K_\sharp)\otimes((u^\alpha_{rj},0)+K_\sharp)\otimes((u^\beta_{kl},0)+K_\sharp)\right)\\
&\quad=\delta_{\alpha\beta}X^\alpha_{jk}\sum_{r=1}^{n_\alpha}((u^\alpha_{ir},0)+K_\sharp)\otimes((u^\alpha_{rl},0)+K_\sharp)\\
&\quad=\delta_{\alpha\beta}X^\alpha_{jk}((\iota^*\otimes\iota^*)\circ\Delta)(u^\alpha_{il})=\delta_{\alpha\beta}X^\alpha_{jk}\Delta^\sharp\left((u^\alpha_{il},0)+K_\sharp\right)\\
&\quad=(\Delta^\sharp\circ\Psi\circ(\iota^*\otimes\iota^*))\left(u^\alpha_{ij}\otimes u^\beta_{kl}\right)
\end{aligned}
$$

and similarly we can show

$$
\left((\Psi\otimes\mathrm{id})\circ(\mathrm{id}\otimes\Delta^\sharp)\circ(\iota^*\otimes\iota^*)\right)\left(u^\alpha_{ij}\otimes u^\beta_{kl}\right)=(\Delta^\sharp\circ\Psi\circ(\iota^*\otimes\iota^*))\left(u^\alpha_{ij}\otimes u^\beta_{kl}\right).
$$

As before we can linearly extend both of these to $\mathrm{Hopf}(\mathbb{G})\odot\mathrm{Hopf}(\mathbb{G})$ which is weak*-dense in to $\mathrm{L}^\infty(\mathbb{G})\mathbin{\overline{\otimes}}\mathrm{L}^\infty(\mathbb{G})$ and which itself is weak*-dense in $\mathrm{L}^1_\sharp(\mathbb{G})^*\mathbin{\overline{\otimes}}\mathrm{L}^1_\sharp(\mathbb{G})^*$ and so we have Equations (6.6).

(ii) $\implies$ (iii): Let $\alpha,\beta\in\mathbb{A}$, $1\leqslant i,j\leqslant n_\alpha$ and $1\leqslant k,l\leqslant n_\beta$. Then by Lemma 6.2.9 we have $\Psi\left(((u^\alpha_{ij},0)+K_\sharp)\otimes((u^\beta_{kl},0)+K_\sharp)\right)$ is in both $\mathrm{lin}\left\{(u^\alpha_{is},0)+K_\sharp \mid 1\leqslant s\leqslant n_\alpha\right\}$ and $\mathrm{lin}\left\{(u^\beta_{rl},0)+K_\sharp \,\middle|\, 1\leqslant r\leqslant n_\beta\right\}$. Thus if $\beta\neq\alpha$ we have

$$
\Psi\left(((u^\alpha_{ij},0)+K_\sharp)\otimes((u^\beta_{kj},0)+K_\sharp)\right)=0.
$$

If $\alpha=\beta$ it follows by linear independence that there exists some $X^\alpha_{jk}\in\mathbb{C}$ such that

$$
\Psi\left(((u^\alpha_{ij},0)+K_\sharp)\otimes((u^\beta_{kl},0)+K_\sharp)\right)=X^\alpha_{jk}((u^\alpha_{il},0)+K_\sharp).
$$

Finally we have that Ψ is a left inverse to $\Delta^\sharp$ and so from Lemma 6.2.7 we have

$$\sum_{k=1}^{n_\alpha} X^\alpha_{kk}((u^\alpha_{ij},0)+K_\sharp) = \sum_{k=1}^{n_\alpha} \Psi\left(((u^\alpha_{ik},0)+K_\sharp)\otimes((u^\alpha_{kj},0)+K_\sharp)\right)$$
$$= (\Psi\circ(\iota^*\otimes\iota^*)\circ\Delta)(u^\alpha_{ij}) = (\Psi\circ\Delta^\sharp\circ\iota^*)(u^\alpha_{ij}) = (u^\alpha_{ij},0)+K_\sharp$$

and so $\sum_{k=1}^{n_\alpha} X^\alpha_{kk} = 1$ as required. □

Appendix A

Functional Analysis

In the appendix we record some results from functional analysis and measure theory that we will use in the text. We assume the reader is familiar with these subjects and will only discuss those results we feel add clarity to the thesis.

For a more comprehensive treatment of Banach spaces we recommend Helemskiĭ (2006), Reed & Simon (1980), Megginson (1998), Pedersen (1989) and Conway (1990) and for Banach algebras see Dales (2000), Palmer (1994) and Helemskiĭ (1993).

A.1 Banach Spaces

We now define some constructions on Banach spaces, in particular we discuss subspaces and quotients of Banach spaces.

Proposition A.1.1 *Let X be a Banach space and Y a subspace of X, then we let Y have the norm inherited from X and it follows that Y is a normed linear space. In particular for Y a closed subspace (with respect to the norm topology) we have that Y is a Banach space.*

Proposition A.1.2 *Let X be a normed space, M a closed subspace of X and X/M the* *quotient linear space. Then we have a norm given by*

$$\|x + M\| := \inf\{\|x - y\| \mid y \in M\} = \inf\{\|y\| \mid y \in x + M\}$$

*called the **quotient norm** and furthermore if X is a Banach space then so is X/M.*

Proposition A.1.3 *Let M be a closed subspace of a normed space X. Then the map $q : X \to X/M$ given by $x \mapsto x + M$ is a contractive linear operator that is an open mapping with kernel M and if $M \neq X$ then $\|q\| = 1$.*

Theorem A.1.4 *Let $T : X \to Y$ be a linear map between normed spaces and $q : X \to X/\mathrm{Ker}\,T$ the natural from X to $X/\mathrm{Ker}\,T$ given by $x \mapsto x + T$. Then there exists a unique injective linear map $\tilde{T} : X/\mathrm{Ker}\,T \to Y$ such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle q}\downarrow & \nearrow_{\tilde{T}} & \\ X/\mathrm{Ker}\,T. & & \end{array}$$

Furthermore T is bounded if and only if $\tilde{T}$ is bounded in which case we have $\|\tilde{T}\| = \|T\|$ and T is an open mapping if and only if $\tilde{T}$ is an open mapping. If T is surjective and bounded then $\tilde{T}$ is an isomorphism.

Definition A.1.5 *Let $T \in \mathcal{B}(X, Y)$ be a bounded linear map between normed spaces, then T is a **quotient map** if it is surjective and such that the map $\tilde{T} : X/\mathrm{Ker}\,T \to Y$ from Theorem A.1.4 is an isometry (and thus an isometric isomorphism).*

Proposition A.1.6 *Let $T \in \mathcal{B}(X, Y)$ be a bounded linear map between normed spaces, then T is a quotient map if and only if T maps the open unit ball of X onto the open unit ball of Y.*

A.2 Unbounded Maps in Banach Space Theory

We now give a brief overview of unbounded maps between Banach spaces.

Definition A.2.1 *In this thesis we define a linear map T between Banach spaces X and Y (either with the norm, weak or when applicable weak* topologies), denoted $T : X \to Y$ as a map from a subspace $\mathrm{Dom}(T)$ of X to Y that is linear on $\mathrm{Dom}(T)$. We say T is* ***densely defined*** *if $\mathrm{Dom}(T)$ is dense in X and* ***everywhere defined*** *if $\mathrm{Dom}(T) = X$. A linear map $T : X \to X$ is a* ***linear operator****.*

Definition A.2.2 *Let $T : X \to Y$ be a linear map, then the* ***graph*** *of T is the set*

$$\mathcal{G}(T) := \{(x, Tx) \in X \times Y \mid x \in \mathrm{Dom}(T)\}.$$

We say T is ***closed*** *if $\mathcal{G}(T)$ is closed with the product topology on X and Y.*

Proposition A.2.3 *Let $T : X \to Y$ be a linear map. Then $\mathcal{G}(T)$ is closed if and only if it satisfies the following condition: for all nets $(x_\alpha) \subset \mathrm{Dom}(T)$ with limit $x \in X$ such that there is some $y \in Y$ with $\lim_\alpha(Tx_\alpha) = y$ then $x \in \mathrm{Dom}(T)$ and $y = Tx$.*

It is well known in Banach space theory that if a densely defined map $T : X \to Y$ is bounded, then there is a unique map that is everywhere defined that extends T. This is not necessarily the case for unbounded maps however (by which we mean not necessarily bounded maps). In fact we have the following which shows that for closed unbounded maps we must allow for maps that are not everywhere defined.

Theorem A.2.4 (Closed Graph Theorem) *Consider X and Y with the norm topologies and let $T : X \to Y$ be an everywhere defined linear map such that the graph $\mathcal{G}(T)$ is closed in $X \oplus_\infty Y$, then T is bounded.*

We now give the notion of a core of an unbounded map which will make unbounded maps much easier to handle.

Definition A.2.5 *Let T be a closed operator and D_0 a subspace of $\mathrm{Dom}(T)$ such that T is the closure of $T_0 := T|_{D_0}$, that is $\mathcal{G}(T_0)$ is dense in $\mathcal{G}(T)$. Then we say that D_0 is a **core** for T.*

Proposition A.2.6 *A subspace $D_0 \subset \mathrm{Dom}(T)$ for a closed operator $T : X \to Y$ is a core for T if and only if for all $x \in \mathrm{Dom}(T)$ there exists a net $(x_\alpha) \subset D_0$ with $\lim_\alpha x_\alpha = x$ and $\lim_\alpha Tx_\alpha = Tx$.*

Proposition A.2.7 *Let $T, S : X \to Y$ be closed linear maps such that there is a subspace $V \subset \mathrm{Dom}(T) \cap \mathrm{Dom}(S)$ with V a core for S and T. If $S(x) = T(x)$ for all $x \in V$ then $S = T$.*

We now discuss unbounded linear maps that can be closed.

Definition A.2.8 *Let $T : X \to Y$, then T is **preclosed** if there is a closed map S such that $T \subset S$ and the closure of $\mathcal{G}(T)$ is $\mathcal{G}(S)$. We let $\bar{T}$ denote the smallest closed extension of T.*

It follows immediately from the definition of a core that for a preclosed operator $T : X \to Y$ we have that $\mathrm{Dom}(T)$ is a core for the closure $\mathrm{Dom}(\bar{T})$.

Proposition A.2.9 *Let $T : X \to Y$ be a linear map. Then the following conditions are equivalent on T:*

(i) T is preclosed;

(ii) For all $y, y' \in Y$ with $(x, y), (x, y') \in \overline{G(T)}$ then we have $y = y'$;

(iii) For all nets $(x_\alpha) \subset \mathrm{Dom}(T)$ converging to 0 such that (Tx_α) converges to some $y \in Y$, then $y = 0$.

Finally we consider the adjoints of unbounded maps. Consider Banach spaces X and Y and $T : X \to Y$ a densely defined map between the Banach spaces. Say for $\omega \in Y^*$ there exists $\kappa, \kappa' \in X^*$ such that for all $x \in \mathrm{Dom}(T)$ we have $\langle Tx, \omega \rangle = \langle x, \kappa \rangle$ and similarly for κ'. Clearly then $\langle x, \kappa \rangle = \langle x, \kappa' \rangle$ for all $x \in \mathrm{Dom}(T)$ and as T is densely defined we have that $\kappa = \kappa'$. Thus we have a well defined linear map $T^* : Y^* \to X^*$ given as follows (with similar reasoning for a pre-adjoint map).

Definition A.2.10 *Let $T : X \to Y$ be a densely defined map between Banach spaces X and Y. Then we define*

$$\mathrm{Dom}(T^*) = \{\omega \in Y^* \mid \exists\, \kappa \in X^* \textit{ such that } \langle Tx, \omega \rangle = \langle x, \kappa \rangle \;\; \forall\, x \in \mathrm{Dom}(T)\}$$

and $T^* : Y^* \to X^*$ by $T^*\omega = \kappa$ for all $\omega \in \mathrm{Dom}(T^*)$ for $\kappa \in X^*$ the unique element given in the definition of $\mathrm{Dom}(T^*)$.

If X and Y are dual Banach spaces with unique preduals X_* and Y_* we can similarly define

$$\mathrm{Dom}(T_*) = \{\omega \in Y_* \mid \exists\, \kappa \in X_* \textit{ such that } \langle Tx, \omega \rangle = \langle x, \kappa \rangle \;\; \forall\, x \in \mathrm{Dom}(T)\}$$

and then $T_* : Y_* \to X_*$ is given by $T_*\,\omega = \kappa$ for all $\omega \in \mathrm{Dom}(T_*)$ with $\kappa \in X_*$ the unique element given in the definition of $\mathrm{Dom}(T_*)$. *It follows that we have $T_*\,\omega = \omega \circ T \in X_*$.*

Theorem A.2.11 *If T is a closed operator then T^* is weak*-closed. On the other hand, if T is the dual of an operator T_* and T is weak*-closed then the pre-adjoint T_* is a closed operator.*

A.3 Banach Modules

Banach modules are very similar to that of operator space modules so we don't give the main definition. We do however give the example of how to make the dual of a Banach algebra into an A-bimodule and the Cohen Factorisation Theorem (or sometimes as the Cohen-Hewitt Factorisation theorem or the Doran-Wichman Factorisation theorem) in this appendix. The Cohen Factorisation Theorem is a highly non-trivial and important theorem in the study of Banach A-modules and we refer the reader to Doran & Wichmann (1979) for further details on this subject.

Example A.3.1 *Let A be a Banach algebra, then for all $\omega \in A^*$ and $a, b \in A$ we define $b \cdot \omega, \omega \cdot a : A \to \mathbb{C}$ by*

$$\langle a, b \cdot \omega \rangle = \langle ab, \omega \rangle = \langle b, \omega \cdot a \rangle$$

and we can easily see that $b \cdot \omega, a \cdot \omega \in A^*$ and that A^* is a Banach A-bimodule. In particular for $a, b \in A$ and $\omega \in A^*$ we have

$$\langle x, a \cdot \omega \cdot b \rangle = \langle bxa, \omega \rangle$$

for all $x \in A$. If A has a predual we can restrict the bimodule structure on A^* to this.

Definition A.3.2 *Let A be a Banach algebra and X a left Banach A-module. Then we say X is* ***essential*** *if $X = \overline{\mathrm{lin}\ \{ax \mid a \in A,\ x \in X\}}$.*

Theorem A.3.3 *Let A be a Banach algebra with a left approximate identity bounded by some $K \geqslant 1$ and X an essential left Banach A-module. Then for $x \in X$ and $\varepsilon > 0$ there exists $a \in A$ and $y \in X$ such that*

$$x = ay, \qquad \|a\| \leqslant K, \qquad y \in \overline{A \cdot x}^{\|\cdot\|}, \qquad \|y - x\| < \varepsilon.$$

A.4 Weakly Compact Operators and Arens Products

We begin this section by defining weakly compact maps between Banach spaces and then move onto a discussion of Arens products.

Proposition A.4.1 *Let $T : X \to Y$ be a bounded map between Banach spaces. Then the following are equivalent:*

(i) $T(\overline{B}_X)$ *is relatively weakly compact in* Y *for* $\overline{B}_X$ *the closed unit ball in* X*;*

(ii) $T(B)$ *is relatively weakly compact in* Y *for any bounded subset* B *of* X*;*

(iii) A bounded sequence $(x_n) \subset X$ *has a subsequence* (x_{n_k}) *such that* $(T(x_{n_k}))$ *converges weakly.*

Definition A.4.2 *A map* $T : X \to Y$ *between Banach spaces is* ***weakly compact*** *if any of the equivalent conditions in Proposition A.4.1 are satisfied and we let* $\mathcal{B}_0^w(X, Y)$ *denote the weakly compact operators from* X *to* Y*.*

Proposition A.4.3 *Let* $T : X \to Y$ *be a weakly compact map and* $S : Y \to Z$ *and* $R : Z \to X$ *be arbitrary bounded maps, then* $S \circ T$ *and* $T \circ R$ *are both weakly compact operators.*

Proposition A.4.4 *We have that all compact maps are weakly compact and all weakly compact maps are bounded, that is* $\mathcal{B}_0(X, Y) \subset \mathcal{B}_0^w(X, Y) \subset \mathcal{B}(X, Y)$ *for Banach spaces* X *and* Y*.*

Let A denote a normed algebra and $\iota : A \to A^{**}$ the canonical embedding of A as a normed space inside its double dual A^{**}. We have that there are two natural ways of making A^{**} into a Banach algebra with the left and right Arens' products. We define these products now. In general these two products will differ. For further details on this subject see Palmer (1994).

Definition A.4.5 *For* $m, n \in A^{**}$ *we define the* ***left Arens product*** $m \,\Box\, n \in A^{**}$ *and the* ***right Arens product*** $m \diamond n \in A^{**}$ *as follows. We remind that* A^* *is an* A*-bimodule with the structure given in Example A.3.1. For* $\omega \in A^*$ *and* $m, n \in A^{**}$ *we define* $n \,\Box\, \omega, \omega \diamond m \in A^*$ *by*

$$\langle n \,\Box\, \omega, a \rangle = \langle n, \omega \cdot a \rangle \qquad \langle a, \omega \diamond m \rangle = \langle a \cdot \omega, m \rangle$$

and then define $m \,\Box\, n, m \diamond n \in A^{**}$ *by*

$$\langle m \,\Box\, n, \omega \rangle = \langle m, n \,\Box\, \omega \rangle \qquad \langle \omega, m \diamond n \rangle = \langle \omega \diamond m, n \rangle.$$

It can be shown that A^{**} is a Banach algebra with either Arens product and that $\iota : A \to A^{**}$ is an injective homomorphism with respect to either Arens product. The most important theorem for us regarding Arens products is the following.

Proposition A.4.6 *Let* A *be a normed algebra, then* $\iota(A)$ *is a left (right) ideal in* A^{**} *if and only if for all* $x \in A$ *the multiplication map* $A \to A$ *given by* $y \mapsto yx$ *(*$y \mapsto xy$*) is weakly compact.*

A.5 Operator Theory

We give a brief discussion of the direct sum of Hilbert spaces now. For Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ we have that the linear space $\mathcal{H} \oplus \mathcal{K}$ is a Hilbert space when given the following inner product

$$\left((\xi_1, \eta_1)^t \middle| (\xi_2, \eta_2)^t\right)_{\mathcal{H} \oplus \mathcal{K}} = (\xi_1 | \xi_2)_{\mathcal{H}} \, (\eta_1 | \eta_2)_{\mathcal{K}}$$

where $\xi_1, \xi_2 \in \mathcal{H}$ and $\eta_1, \eta_2 \in \mathcal{K}$. We will also use the following notation for bounded linear operators on direct sums of Hilbert spaces.

Notation A.5.1 *Let* $x \in \mathcal{B}(\mathcal{H})$ *and* $y \in \mathcal{B}(\mathcal{K})$*. Then we let* $x \oplus y \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ *denote the operator given by* $(\xi, \eta)^t \mapsto (x\xi, y\eta)^t$ *for* $\xi \in \mathcal{H}$ *and* $\eta \in \mathcal{K}$*.*

We can also consider infinite direct sums of Hilbert spaces which have an additional convergence property. In particular we will need the following related results.

Proposition A.5.2 *Let* $(\mathcal{H}_i)$*,* $(\mathcal{K}_i)$ *be two collections of Hilbert spaces and let* $\mathcal{H} = \bigoplus_i \mathcal{H}_i$ *and* $\mathcal{K} = \bigoplus_i \mathcal{K}_i$*. Let* (x_i) *be a collection of maps that is bounded (i.e.* $\sup_i \|x_i\|$ *is finite) where* $x_i \in \mathcal{B}(\mathcal{H}_i, \mathcal{K}_i)$ *for all* i *and consider* $x := \bigoplus_i x_i$ *where* $(\bigoplus_i x_i) \cdot (\xi_i)_i = (x_i \xi_i)_i$ *for all* $(\xi_i)_i \in \mathcal{H}$*, then we have* $x \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ *and* $\|x\| = \sup_i \|x_i\|$*.*

Proof

As (x_i) is bounded it follows that $\sup_i \|x_i\| < \infty$. Let $\xi := (\xi_i)_i \in \mathcal{H}$ where $\xi_i \in \mathcal{H}_i$ for all i and we have

$$\|x\xi\|^2 = \|(x_i\xi_i)_i\|^2 = \sum_i \|x_i\xi_i\|^2 \leqslant \left(\sup_i \|x_i\|^2\right)\left(\sum_i \|\xi_i\|^2\right) = \left(\sup_i \|x_i\|\right)^2 \|\xi\|^2$$

and so taking the square root and supremum over $\xi \in \mathcal{H}$ with $\|\xi\| \leqslant 1$ we have $\|x\| \leqslant \sup_i \|x_i\|$.

Fix i, then for all $\xi_i \in \mathcal{H}_i$ such that $\|\xi_i\| \leqslant 1$ we have $\|x\| \geqslant \|x_i\xi_i\|$ and so taking the supremum over all such $\xi_i \in \mathcal{H}_i$ we have $\|x_i\| \leqslant \|x\|$ for all i. As this holds for all i we have $\sup_i \|x_i\| \leqslant \|x\|$ as required. □

Proposition A.5.3 *Let $A \subset \mathcal{B}(\mathcal{H})$ be a C*-algebra on a Hilbert space $\mathcal{H}$ with the usual operator space structure, that is for any $N \in \mathbb{N}_0$ (including $N = \infty$) and matrix $(x_{ij})_{i,j=1}^N \in \mathbb{M}_N(A)$ we have*

$$\left\|(x_{ij})_{i,j=1}^N\right\| = \sup\left\{\left\|\left(\sum_{j=1}^N x_{ij}\xi_j\right)_{i=1}^N\right\| \;\middle|\; \xi = (\xi_i)_{i=1}^N \in \mathcal{H}^{(N)},\ \ \|\xi\| \leqslant 1\right\}$$

where $\mathcal{H}^{(N)} = \bigoplus_{i=1}^N \mathcal{H}$. Let $(x_{ij})_{i,j=1}^N \in \mathbb{M}_N(A) \subset \mathcal{B}(\mathcal{H}^{(N)})$ (including $N = \infty$), then for all $1 \leqslant m, n \leqslant N$ we have $\|x_{mn}\| \leqslant \left\|(x_{ij})_{i,j=1}^N\right\|$.

Proof

Fix any $1 \leqslant n \leqslant N$, let $\xi_n \in \mathcal{H}$ and let $\xi = (\delta_{i,n}\xi_n)_{i=1}^N$, then we have

$$\|x\xi\|^2 = \left\|(x_{in}\xi_n)_{i=1}^N\right\|^2 = \sum_{i=1}^N \|x_{in}\xi_n\|^2 \geqslant \|x_{mn}\xi_n\|^2$$

for all $1 \leqslant m \leqslant N$ and so $\|x_{mn}\xi_n\| \leqslant \|x\xi\| \leqslant \|x\|\|\xi\| = \|x\|\|\xi_n\|$. Taking the supremum over $\xi_n \in \mathcal{H}$ with $\|\xi_n\| \leqslant 1$ we get the result. □

We discuss briefly conditional expectations on C*-algebras now.

Definition A.5.4 *Let A be a C*-algebra and B a C*-subalgebra of A. Then a* ***projection*** *$T : A \to B$ is a map such that $T(b) = b$ for all $b \in B$ and a* ***conditional expectation*** *is a contractive projection $\mathcal{E} : A \to B$ that is completely positive such that $\mathcal{E}(bab') = b\mathcal{E}(a)b'$ for all $b, b' \in B$ and $a \in A$.*

See Brown & Ozawa (2008) Theorem 1.5.10 for a proof of the following result.

Theorem A.5.5 (Tomiyama's Theorem) *Let A be a C*-algebra and B a C*-subalgebra of A. Then for a projection $\mathcal{E} : A \to B$ the following are equivalent:*

(i) $\mathcal{E}$ is a conditional expectation;

(ii) $\mathcal{E}$ is contractive and completely positive;

(iii) $\mathcal{E}$ is contractive.

Finally we give a brief overview of multiplier C*-algebras, see Chapter 2 in Murphy (1990) and Chapter 2 in Timmermann (2008) for further details. We can consider maps $L : A \to \mathcal{B}(A)$ and $R : A \to \mathcal{B}(A)$ given by $a \mapsto L_a$ and $a \mapsto R_a$ respectively where $L_a, R_a \in \mathcal{B}(A)$ are the maps $L_a(b) = ab$ and $R_a(b) = ba$ for all $b \in A$. It is easy to show that $L_a(bc) = L_a(b)c$, $R_a(bc) = bR_a(c)$ and $bL_a(c) = R_a(b)c$ for all $a, b, c \in A$. Also for $a \in A$ we have

$$\|a\| = \sup\{\|ab\| \mid b \in A, \ \ \|b\| \leqslant 1\} = \sup\{\|ba\| \mid b \in A, \ \ \|b\| \leqslant 1\}$$

and so it follows that $\|L_a\| = \|R_a\| = \|a\|$. This motivates the following definition.

Definition A.5.6 *Let A be a C*-algebra, then a pair (L, R) with $L, R \in \mathcal{B}(A)$ is a* ***double centraliser*** *for A if for all $a, b \in A$ we have*

$$L(ab) = L(a)b, \qquad R(ab) = aR(b), \qquad R(a)b = aL(b).$$

We denote the set of double centralisers of A by $\mathrm{M}(A)$.

In the following, for $L \in \mathcal{B}(A)$ we define $L^* : A \to A$ by $L^*(a) = L(a^*)^*$ for all $a \in A$. Then we have $\|L^*(a)\| = \|L(a^*)\| \leqslant \|L\|\|a\|$ and so $L^* \in \mathcal{B}(A)$.

Proposition A.5.7 *Let A be a C*-algebra, then $\mathrm{M}(A)$ is a unital C*-algebra with the following structure*

$$(L, R) + \lambda(L', R') = (L + \lambda L', R + \lambda R')$$

$$(L, R)(L', R') = (LL', R'R), \qquad (L, R)^* = (R^*, L^*)$$

$$\|(L, R)\| = \|L\| = \|R\|$$

for $(L, R), (L', R') \in \mathrm{M}(A)$ and $\lambda \in \mathbb{C}$.

We have that $\mathrm{M}(A)$ is in a sense a unitisation of A as shown by the following proposition.

Proposition A.5.8 *Let A be a C*-algebra, then there is an isometric $*$-homomorphism that embeds A as a C*-subalgebra of $\mathrm{M}(A)$. Furthermore A is an ideal in $\mathrm{M}(A)$. If A is unital then we have $A \cong_i \mathrm{M}(A)$.*

Definition A.5.9 *Let A and B be C*-algebras and let $\phi : A \to \mathrm{M}(B)$ be a homomorphism, then ϕ is* ***non-degenerate*** *if $\phi(A)B$ and $B\phi(A)$ are both linearly dense in B.*

A.6 Measure Theory and Banach Spaces

We will make use of various measure theoretic results throughout this thesis and the reader is assumed to have a background in this area. In particular we expect the reader to be familiar with integration, the Lebesgue measure, the L^p spaces for $1 \leqslant p \leqslant \infty$, the duality of $\mathrm{C}_0(\Omega)$ and $\mathrm{M}(\Omega)$ for a locally compact space Ω, complex Radon measures, the Radon-Nikodym theorem, products of measures, the Fubini theorem and the basics of the Fourier transform. We recommend Rudin (1987) Chapters 1–6, Folland (1984) and Cohn (1980) as references.

We cover now briefly some results in integration on Banach spaces. Let (Ω, μ) denote a measure space and X a Banach spaces and we consider a function $f : \Omega \to X$. We let $\|f\| : \Omega \to \mathbb{R}^+$ be the function given by $\|f\|(t) = \|f(t)\|$ for all $t \in \Omega$. We define now the notions of weak and weak* integrable and give some basic results on these concepts.

Definition A.6.1 *We say a function $f : \Omega \to X$ is **weakly integrable** (sometimes called Pettis integrable in the literature) if $\|f\|$ is integrable and there exists $x \in X$ such that for all $\omega \in X^*$ we have*

$$\langle x, \omega \rangle = \int_\Omega \langle f(t), \omega \rangle \, d\mu(t).$$

*We then say x is the **weak-integral** of f over X.*

Definition A.6.2 *Let Y be a closed separating subset of X^*. Then we say f is **integrable with respect to** Y if $\|f\|$ is integrable and there exists some $x \in X$ such that for all $\omega \in Y$ we have*

$$\langle x, \omega \rangle = \int_\Omega \langle f(t), \omega \rangle \, d\mu(t).$$

*In particular, if X is the dual of a unique Banach space X_* and $Y = X_*$, then we say x is the **weak* integral** of f over X.*

Say a function $f : \Omega \to X$ is integrable with respect to some closed separating set $Y \subset X^*$, then as Y is separating it is immediate that the $x \in X$ such that $\langle x, \omega \rangle = \int_\Omega \langle f(t), \omega \rangle \, d\mu(t)$ is unique. We now give the existence theorems we will use in this thesis.

Proposition A.6.3 *Let $f : \mathbb{R} \to X$ be function that is continuous with respect to the norm topology on X and such that the map $\|f\| : \mathbb{R} \to \mathbb{R}^+$ given by $\|f\|(t) = \|f(t)\|$ is integrable. Then f is weak-integrable with respect to the Lebesgue measure on $\mathbb{R}$.*

Proposition A.6.4 *Let X be a Banach space and let $f : \mathbb{R} \to X^*$ be a function that is continuous with respect to the weak*-topology on X^* with $\|f\|$ integrable. Then f is weak*-integrable with respect to the Lebesgue measure on $\mathbb{R}$.*

A.7 The Fourier Transform

Consider the circle $\mathbb{T} = \left\{ e^{2\pi i\theta} \;\middle|\; \theta \in [0,1) \right\} \subset \mathbb{C}$ and consider the Hilbert space $\mathrm{L}^2(\mathbb{T}) = \left\{ f : \mathbb{T} \to \mathbb{C} \;\middle|\; \int_0^1 \left| f(e^{2\pi i\theta}) \right|^2 d\theta < \infty \right\}$ with inner product $(f|g) = \int_0^1 f(e^{2\pi i\theta})\overline{g(e^{2\pi i\theta})}\, d\theta$.

We define a set of functions $\{\underline{z}^n \mid n \in \mathbb{Z}\}$ where $\underline{z}^n : \mathbb{T} \to \mathbb{C}$ is given by $e^{2\pi i\theta} \mapsto e^{2\pi i n\theta}$ for all $n \in \mathbb{Z}$. Then we have

$$(\underline{z}^n|\underline{z}^m) = \int_0^1 \underline{z}^n(e^{2\pi i\theta})\underline{z}^m(e^{2\pi i\theta})\, d\theta = \int_0^1 e^{2\pi i(n-m)\theta}\, d\theta = \delta_{n,m}$$

and so this is an orthonormal set. Furthermore we can show that this set is complete and so forms a basis for $\mathrm{L}^2(\mathbb{T})$ (see Rudin (1987) Sections 4.24 and 4.25).

We let $\hat{f} : \mathbb{Z} \to \mathbb{C}$ be the map $\hat{f}(n) = \int_0^1 f(e^{2\pi i\theta})e^{-2\pi in\theta}\, d\theta$ which is well defined as $\left| \int_0^1 f(e^{2\pi i\theta})e^{-2\pi in\theta}\, d\theta \right|^2 \leqslant \int_0^1 \left| f(e^{2\pi i\theta}) \right|^2 d\theta < \infty$. Given $f \in \mathrm{L}^2(\mathbb{T})$ we can write $f = \sum_{n\in\mathbb{Z}} (f|\underline{z}^n)\, \underline{z}^n = \sum_{n\in\mathbb{Z}} \hat{f}(n)\underline{z}^n$ and so $f(e^{2\pi i\theta}) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi in\theta}$. We also have

$$\|f\|_2^2 = (f|f) = \sum_{n,m\in\mathbb{Z}} \hat{f}(n)\overline{\hat{f}(m)}\, (\underline{z}^n|\underline{z}^m) = \sum_{n\in\mathbb{Z}} \left| \hat{f}(n) \right|^2 \tag{A.1}$$

and so we have the following.

Proposition A.7.1 *There exists a unitary isomorphism $\mathcal{F} : \mathrm{L}^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ given by $f \mapsto \hat{f}$.*

We have $\mathcal{F}(\underline{z}^n)(m) = \int_0^1 \underline{z}^n(e^{2\pi i\theta})e^{-2\pi im\theta}\, d\theta = \delta_{n,m}$ and so $\mathcal{F}(\underline{z}^n) = e_n$ (the entry with 1 in the n-th place and 0 elsewhere). It follows easily that we have $(f|\mathcal{F}^*(e_n)) = \left(\hat{f} \middle| e_n \right) = \hat{f}(n) = (f|\underline{z}^n)$ and so $\mathcal{F}^*(e_n) = \underline{z}^m$.

As we have a finite measure it follows that $\mathrm{L}^2(\mathbb{T}) \subset \mathrm{L}^1(\mathbb{T})$. We have the following that is proved in Rudin (1987) Section 5.14.

Lemma A.7.2 (Riemann-Lebesgue Lemma) *Let $f \in \mathrm{L}^1(\mathbb{T})$, then as $|n| \to \infty$ we have $\int_0^1 f(t)e^{-2\pi int}\, dt \to 0$.*

So by this lemma we can define an extension $\mathcal{F} : \mathrm{L}^1(\mathbb{T}) \to c_0(\mathbb{Z})$ by $f \mapsto \hat{f}$ where $\hat{f}(n) = \int_0^1 f(e^{2\pi i\theta})e^{-2\pi in\theta}d\theta$. Then we have for $f \in \mathrm{L}^1(\mathbb{T})$ that

$$\|\mathcal{F}(f)\| = \sup_{n\in\mathbb{Z}} \left|\hat{f}(n)\right| \leqslant \sup_{n\in\mathbb{Z}} \int_0^1 \left|f(e^{2\pi i\theta})e^{2\pi in\theta}\right| d\theta = \int_0^1 \left|f(e^{2\pi i\theta})\right| d\theta = \|f\|_1$$

and so $\mathcal{F}$ is a contraction. Let $f, g \in \mathrm{L}^1(\mathbb{T})$, then we define $f * g : \mathbb{T} \to \mathbb{C}$ by

$$(f * g)(e^{2\pi i\theta}) = \int_0^1 f(e^{2\pi i\phi})g(e^{2\pi i(\theta-\phi)})\, d\phi.$$

We have

$$\begin{aligned}\|f * g\|_{\mathrm{L}^1} &= \int_0^1 \left|(f * g)(e^{2\pi i\theta})\right| d\theta = \int_0^1 \left|\int_0^1 f(e^{2\pi i\phi})g(e^{2\pi i(\theta-\phi)})d\phi\right| d\theta \\ &\leqslant \int_0^1\int_0^1 \left|f(e^{2\pi i\theta})\right| \left|g(e^{2\pi i(\theta-\phi)})\right| d\phi d\theta = \|f\|_1\|g\|_1\end{aligned}$$

and so $f * g \in \mathrm{L}^1(\mathbb{T})$ and $\mathrm{L}^1(\mathbb{T})$ is a Banach algebra under this multiplication. We can also show that

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$$

in a similar way.

Also we can define the following Fourier transform on a measure.

Definition A.7.3 *Let $\phi \in M(\mathbb{T})$ be a measure on $\mathbb{T}$ and we define $\widehat{\phi} \in \ell^\infty(\mathbb{Z})$ by $\widehat{\phi}(n) = \int_{\mathbb{T}} \underline{z}^{-l}\, d\phi$ for all $n \in \mathbb{Z}$.*

We will make use of the following well known proposition in Chapter 5 that we briefly sketch a proof of now.

Proposition A.7.4 *Let $T \in \mathcal{B}(\ell^2(\mathbb{Z}))$ be the bilateral shift operator on $\ell^2(\mathbb{Z})$, that is the unique bounded linear operator such that $e_t \mapsto e_{t+1}$ for all $t \in \mathbb{Z}$, then the spectrum of T is given by $\sigma(T) = \mathbb{T}$.*

Proof (Sketch)

By mapping $\ell^2(\mathbb{Z})$ to $\mathrm{L}^2(\mathbb{T})$ with the unitary operator $\mathcal{F}$ we can easily see that T is unitarily equivalent to the operator $M_{\underline{z}} \in \mathcal{B}(\mathrm{L}^2(\mathbb{T}))$ where $M_{\underline{z}}$ is the multiplication operator for the identity function given by $(M_{\underline{z}} f)(e^{2\pi i\theta}) = e^{2\pi i\theta} f(e^{2\pi i\theta})$ for any $f \in \mathrm{L}^2(\mathbb{T})$. Furthermore it is easy to see that the spectrum of any multiplication operator M_f is given by the closure range of f, thus we have $\sigma(T) = \mathbb{T}$. □

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