Representation Theory of Diagram Algebras: Subalgebras and Generalisations of the Partition Algebra

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Submitted in accordance with the requirements for the degree of Doctor of Philosophy

The University of Leeds
School of Mathematics

August 2016

The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

Work from the following jointly-authored preprint is included in this thesis:

C. Ahmed, P. Martin and V. Mazorchuk, On the number of principal ideals in d-tonal partition monoids. arXiv preprint arXiv:1503.06718

Chapter 2 of this thesis is based on the above pre-print. The above work came after a visit of Prof. Mazorchuk (Uppsala University) to the Department of Mathematics in the University of Leeds in October 2014. The Mathematics in this paper comes from some discussions between the three authors after this visit. One third of the Mathematics of this work is contributed to me, and the online version of this preprint is agreed on by all the authors. However, Chapter 2 of this thesis is written entirely by me.

Acknowledgements

I would like to show my deepest gratitude for my main supervisor Prof. Paul Martin for his patience and guidance through writing this thesis over the last four years. Without his constant support and valuable feedback it would not have been possible for me to produce this document. I would also like to thank my co-supervisor Dr. Alison Parker for her encouragement, support, and reading parts of this thesis.

In addition, I would like to thank all the people in the School of Mathematics at the University of Leeds, especially the algebra group members and all the PhD students for making my time in Leeds more enjoyable, especially Raphael Bennett-Tennenhaus, Daniel Wolf, Ricardo Bello Aguirre and Daoud Siniora.

I would like to thank Prof. Mazorchuk for the discussions that we had during writing the preprint version of Chapter 2 of this thesis.

I gratefully acknowledge the comments on this thesis by both of my thesis examiners Prof. R. Marsh and Dr. R Gray.

I also wish to thank the Kurdistan Regional Government and the University of Sulaimani for their financial support and for offering me an HCDP scholarship.

Writing this document would not have been possible without my family's support. I wish to express my love and many thanks to my parents, my sister, my brothers and their families.

Abstract

This thesis concerns the representation theory of diagram algebras and related problems. In particular, we consider subalgebras and generalisations of the partition algebra. We study the d-tonal partition algebra and the planar d-tonal partition algebra. Regarding the d-tonal partition algebra, a complete description of the $\mathcal{J}$-classes of the underlying monoid of this algebra is obtained. Furthermore, the structure of the poset of $\mathcal{J}$-classes of the d-tonal partition monoid is also studied and numerous combinatorial results are presented. We observe a connection between canonical elements of the d-tonal partition monoids and some combinatorial objects which describe certain types of hydrocarbons, by using the alcove system of some reflection groups.

We show that the planar d-tonal partition algebra is quasi-hereditary and generically semisimple. The standard modules of the planar d-tonal partition algebra are explicitly constructed, and the restriction rules for the standard modules are also given. The planar 2-tonal partition algebra is closely related to the two coloured Fuss-Catalan algebra. We use this relation to transfer information from one side to the other. For example, we obtain a presentation of the 2-tonal partition algebra by generators and relations. Furthermore, we present a necessary and sufficient condition for semisimplicity of the two colour Fuss-Catalan algebra, under certain known restrictions.

To my family

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Chapter 1

Background

We devote this chapter to set up the notation of the thesis and recall some necessary background for the later chapters. While we are developing the notation, we state the aims of this thesis, and record some of the history of the development of this area of research.

In section 1 we define the partition algebras. Roughly speaking the partition algebras are a tower of finite dimensional algebras that possess a basis which can be described by diagrams. Multiplication of the basis can be described by concatenating of diagrams. Many well known algebras are embedded in the partition algebra, for example the symmetric group algebra, Temperley-Lieb algebra, full transformation semigroup algebra, and the Brauer algebra, see Figure 1.4. In the first section we define all the mentioned algebras and describe their relation with each other. Further, we define two other new classes of algebras which are going to be the main focus of the thesis, namely the d-tonal partition algebra and the planar d-tonal partition algebra.

The Schur-Weyl duality phenomenon relates the representation theory of some subgroups of the general linear group with some subalgebras of the partition algebra.

Describing this phenomenon is going to be the main concern of the subsection 1.1.4. In Theorem 1.1.9 we see the d-tonal partition algebra occurs naturally as a centraliser algebra of the unitary reflection groups.

In section 2, we formulate the questions that we would like to answer regarding the representation theory of the d-tonal partition algebra and the planar d-tonal partition algebra in the later chapters. In short, we would like to know the answer of the following questions, whether these algebras are simisimple or not? What are precisely the simple modules? Can we describe the blocks? Possibly in a geometrical way by using the action of some reflection groups, and what are their Cartan matrices? The answer to these questions is known, at least over the field of complex numbers, for the partition algebras. To demonstrate the aims of this thesis, in section 2, we recall the known answers for the above questions in the case of the partition algebra.

Beside recalling the definition of Green's relations about monoids one of the purposes of section 3 is to state the so called the *Clifford-Munn-Ponizovskiĭ* theorem. This theorem describes the relation between the $\mathcal{J}$-classes of a finite monoid and the irreducible representations of its monoid algebra. One can look at Clifford-Munn-Ponizovskiĭ theorem as one of the motivations for Chapter 2, since in Chapter 2 of this thesis we give a complete description of the $\mathcal{J}$-classes of the underlying monoid of the d-tonal partition algebra.

The final section of this chapter concerns reminding the reader of the definition of two closely related classes of finite dimensional algebras, namely quasi-hereditary algebras and cellular algebras.

1.1 The partition algebra $\mathfrak{P}_n(\delta)$.

The partition algebra is defined by Martin in [52] as a generalisation of two dimensional statistical mechanics into higher dimensions. In this section we recall the definition of the partition algebra and some of its subalgebras.

1.1.1 The basic partition category

Fix $n, m, l \in \mathbb{N}$, define $\underline{n} := \{1, 2, \ldots, n\}$, $\underline{n}' := \{1', 2', \ldots, n'\}$, $\underline{n}'' := \{1'', 2'', \ldots, n''\}$. Consider the following set maps:

$$\iota^+ : \underline{n} \cup \underline{m}' \to \underline{n}' \cup \underline{m}''$$
$$i \mapsto i'$$

and

$$\iota^- : \underline{n} \cup \underline{m}'' \to \underline{n} \cup \underline{m}'$$
$$i \mapsto i \text{ for } i \in \underline{n}$$
$$i'' \mapsto i' \text{ for } i \in \underline{m}$$

Denote by $\mathfrak{P}_S$ the set of all set partitions of a given set S. Let $\mathfrak{e} \in \mathfrak{P}_{\underline{n}\cup\underline{m}'}$ with parts $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_k$ then we define $\iota^+(\mathfrak{e}) := \{\iota^+(\mathfrak{e}_1), \iota^+(\mathfrak{e}_2), \ldots, \iota^+(\mathfrak{e}_k)\}$, similarly we define $\iota^-(\mathfrak{f})$ where $\mathfrak{f} \in \mathfrak{P}_{\underline{n}\cup\underline{m}''}$.

There is a bijection between the set of all equivalence relations on S and $\mathfrak{P}_S$. For a partition $\mathfrak{d} \in \mathfrak{P}_S$ we denote by $\hat{\mathfrak{d}}$ the equivalence relation obtained from $\mathfrak{d}$.

For $\mathfrak{a} \in \mathfrak{P}_{\underline{n}\cup\underline{m}'}$ and $\mathfrak{b} \in \mathfrak{P}_{\underline{m}\cup\underline{l}'}$ define their multiplication $\mathfrak{a}\circ\mathfrak{b}$ to be the corresponding partition to the equivalence relation $\iota^-((\hat{\mathfrak{a}} \cup_{cl} \widehat{\iota^+(\mathfrak{b})})\restriction_{\underline{n}\cup\underline{l}''})$ in $\mathfrak{P}_{\underline{n}\cup\underline{l}'}$, here $\hat{\mathfrak{a}} \cup_{cl} \widehat{\iota^+(\mathfrak{b})}$ denotes the smallest equivalence relation containing $\hat{\mathfrak{a}}$ and $\widehat{\iota^+(\mathfrak{b})}$, and $\restriction_{\underline{n}\cup\underline{l}''}$ denote the restriction to the set $\underline{n} \cup \underline{l}''$. We will give an example when we define the notion of diagram.

For a positive integer n, let $1_n = \{\{1, 1'\}, \{2, 2'\}, \ldots, \{n, n'\}\}$.

Proposition 1.1.1 (Proposition 1 [52]). *The multiplication* $\circ$ *defined above is associative. Furthermore, for any* $\mathfrak{a} \in \mathfrak{P}_{\underline{m}\cup\underline{l}'}$ *we have* $1_m \circ \mathfrak{a} = \mathfrak{a} = \mathfrak{a} \circ 1_l$.

The *basic partition category* , denoted by $(\mathbb{N}, \mathfrak{P}_{-\cup-}, \circ)$, is constructed in the following way: the objects are given by natural numbers. The morphisms are given by the set $\mathfrak{P}_{\underline{m}\cup\underline{l}'}$, for any $m, l \in \mathbb{N}$. The composition of morphisms is given by $\circ$.

1.1.2 Partition diagrams

In this subsection we give an alternative useful construction to the objects of the basic partition category.

An (n, m)-*partition diagram* (or just a diagram when it is clear from the context) is a graph drawn inside a rectangle with n vertices positioned on the top edge labelled by $1, 2, \ldots, n$ from left to right, and m vertices on the bottom edge labelled by $1', 2', \ldots, m'$ from left to right, for example see diagram $\mathfrak{a}$ in Figure 1.1.

The connected components of an (n, m)-partition diagram determines a partition of $\underline{n} \cup \underline{m}'$. Two such diagrams are defined to be equivalent if they determine the same partition. On the other hand any partition of $\underline{n} \cup \underline{m}'$ gives an equivalence class of (n, m)-diagrams, by letting two vertices be in the same connected component if and only if they are in the same part. Using the above correspondence we use the same symbol for a partition and its equivalence class of diagrams.

Example 1.1.2. *The set partition* $\{\{1, 4\}, \{2, 5'\}, \{3, 1'\}, \{5\}, \{2', 3', 4'\}\}$ *of* $\underline{5} \cup \underline{5}'$ *is represented by the diagram* $\mathfrak{a}$ *in Figure 1.1.*

Given an (n, m)-partition digram $\mathfrak{a}$ and an (m, l)-partition diagram $\mathfrak{b}$ we define their multiplication to be an (n, l)-diagram $\mathfrak{a} \star \mathfrak{b}$, in the following way. Place the digram $\mathfrak{a}$ above the digram $\iota^+(\mathfrak{b})$ and stack them in the matching vertices, denote the resulting diagram by $\mathfrak{a} \star'' \iota^+(\mathfrak{b})$. Then remove all the connected components that lie entirely in the middle. Thereafter remove the vertices that come from $\underline{m}'$ without causing

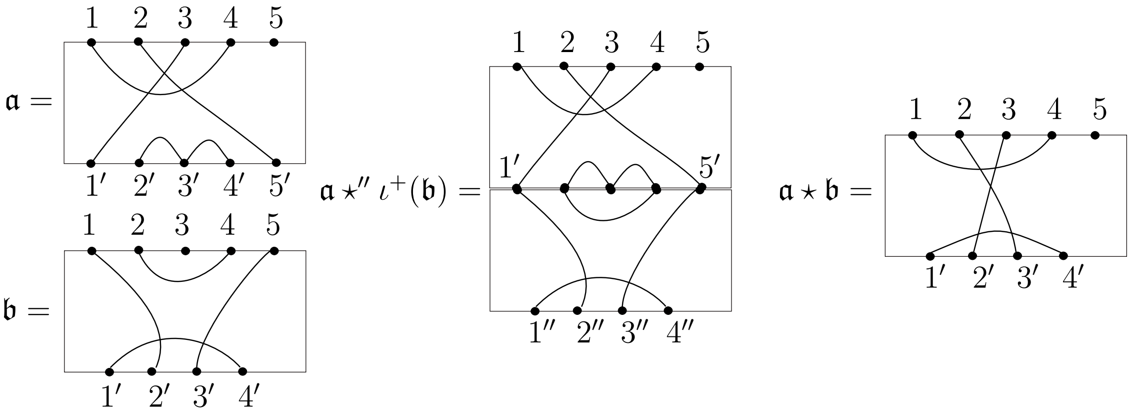


Figure 1.1: Example of the product of partition diagrams

any cut in the strings that they are lying on. This gives a partition of $\underline{n} \cup \underline{l}''$, denote it by $\mathfrak{a} \star' \iota^+(\mathfrak{b})$. Finally set $\mathfrak{a} \star \mathfrak{b} := \iota^-(\mathfrak{a} \star' \iota^+(\mathfrak{b}))$. See Figure 1.1. The operation $\star$ is associative, see for example Proposition 4.7 of [55] .

Lemma 1.1.3 (Section 4 [55])**.** *For $\mathfrak{a} \in \mathfrak{P}_{\underline{n} \cup \underline{m}'}$ and $\mathfrak{b} \in \mathfrak{P}_{\underline{m} \cup \underline{l}'}$, we have $\mathfrak{a} \star \mathfrak{b} = \mathfrak{a} \circ \mathfrak{b}$.*

We recall from Section 6 of [55], the horizontal product $\otimes : \mathfrak{P}_{\underline{n} \cup \underline{m}'} \times \mathfrak{P}_{\underline{k} \cup \underline{l}'} \to \mathfrak{P}_{\underline{n+k} \cup \underline{(m+l)}'}$ is given by placing diagrams side by side in a new diagram, see Figure 1.2 below.

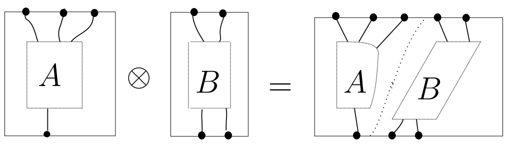


Figure 1.2: Horizontal product of diagrams

Example 1.1.4. *Let $n = 3, m = 1, k = 2$ and $l = 4$. If $\mathfrak{a} = \{\{1, 1'\}, \{2, 3\}\}$ and $\mathfrak{b} = \{\{1, 4'\}, \{2, 1', 2', 3'\}\}$ then $\mathfrak{a} \otimes \mathfrak{b} = \{\{1, 1'\}, \{2, 3\}, \{4, 5'\}, \{5, 2', 3', 4,'\}\}$.*

Note that the horizontal product gives the basic partition category a *monoidal* structure, see [55].

In Figure 1.3 we defined some special elements in the partition category. As a set

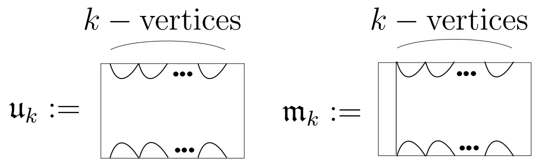


Figure 1.3: The elements $\mathfrak{u}_k$ and $\mathfrak{m}_k$

$\mathfrak{m}_i = \{\{1,,2,\dots,i,1',2,',\dots,i'\}\}$ and $\mathfrak{u}_d = \{\{1,2,\dots,d\},\{1',2',\dots,d'\}\}$. We set $\mathfrak{m}_i^{\otimes k} := \underbrace{\mathfrak{m}_{\mathrm{i}} \otimes \mathfrak{m}_{\mathrm{i}} \otimes \cdots \otimes \mathfrak{m}_{\mathrm{i}}}_{k-\text{ times}}$ and $\mathfrak{u}_d^{\otimes k} := \underbrace{\mathfrak{u}_{\mathrm{d}} \otimes \mathfrak{u}_{\mathrm{d}} \otimes \cdots \otimes \mathfrak{u}_d}_{k-\text{ times}}$, where $k+1, d, i \in \mathbb{Z}_{>0}$. Note that the empty set is the identity with respect to $\otimes$ in $(\mathbb{N}, \mathfrak{P}_{-\cup-}, \circ)$.

1.1.3 The partition algebra $\mathfrak{P}_n(\delta)$ and some of its subalgebras.

The pair $(\mathfrak{P}_{\underline{n}\cup\underline{n}'}, \circ)$ is a monoid with identity $1 := 1_n$, denote it by

$$\mathfrak{P}_n := (\mathfrak{P}_{\underline{n}\cup\underline{n}'}, \circ) \tag{1.1}$$

and call it the *partition monoid.*

Definition 1.1.5 (Partition algebra, see Definition 10 in [52])**.** *Let R be a commutative ring, $\delta \in R$ and $n \in \mathbb{N}$. The partition algebra, denoted by $\mathfrak{P}_n(\delta)$, is an associative R-algebra with identity* 1 *and defined as follows. The $\mathfrak{P}_n(\delta)$ is a free R-module has basis given by the set $\mathfrak{P}_n$, that is $\mathfrak{P}_n(\delta) = R$-span $(\mathfrak{P}_n)$, and the multiplication of basis elements $\mathfrak{e}, \mathfrak{d} \in \mathfrak{P}_n$ is given by*

$$\mathfrak{d}.\mathfrak{e} := \delta^{k_{\mathfrak{d},\mathfrak{e}}} \mathfrak{d} \circ \mathfrak{e} \tag{1.2}$$

where $k_{\mathfrak{d},\mathfrak{e}}$ is the number of connected components removed when we multiply $\mathfrak{d}$ and $\mathfrak{e}$.

From now on we do not write the $\circ$ and . while multiplying its elements whenever we believe it is clear from the context.

Let $\tau = \{\{1, 2'\}, \{2, 1'\}\}$, we define the following elements:

$$\begin{aligned} \boldsymbol{s}_k &:= \mathfrak{m}_1^{\otimes k-1} \otimes \tau \otimes \mathfrak{m}_1^{\otimes(n-k-1)} \\ \boldsymbol{\mu}_i &:= \mathfrak{m}_1^{\otimes i} \otimes \mathfrak{m}_2 \otimes \mathfrak{m}_1^{\otimes(n-i-2)} \\ \boldsymbol{\nu}_j &:= \mathfrak{m}_1^{\otimes j} \otimes \mathfrak{u}_1 \otimes \mathfrak{m}_1^{\otimes(n-j-1)} \end{aligned} \tag{1.3}$$

Where $0 \leq i, k-1 \leq n-2$ and $0 \leq j \leq n-1$, and $\mathfrak{m}_1^0 = \phi$.

In the next theorem we present a characterisation of the partition algebra as an abstract algebra defined by generators and relations.

Theorem 1.1.6 (see [25]Lemma 35,[35]Theorem 1.11)**.** *For $\delta \in \mathbb{F}$ the partition algebra $\mathfrak{P}_n(\delta)$ is presented by generators $\boldsymbol{\mu}_0 \ldots, \boldsymbol{\mu}_{n-2}, \boldsymbol{\nu}_0, \ldots, \boldsymbol{\nu}_{n-1}, \boldsymbol{s}_1 \ldots, \boldsymbol{s}_{n-1}$ subject to the following relations:*

$$\begin{aligned} \boldsymbol{s}_i^2 &= 1 && 1 \leq i \leq n-1, \\ \boldsymbol{s}_{i+1}\boldsymbol{s}_i\boldsymbol{s}_{i+1} &= \boldsymbol{s}_i\boldsymbol{s}_{i+1}\boldsymbol{s}_i && 1 \leq i \leq n-2, \\ \boldsymbol{s}_i\boldsymbol{s}_j &= \boldsymbol{s}_j\boldsymbol{s}_i && \mid i-j \mid > 1, \\ \boldsymbol{\nu}_i^2 = \delta\boldsymbol{\nu}_i, &\quad \boldsymbol{\mu}_j^2 = \boldsymbol{\mu}_j && 0 \leq i \leq n-1, 0 \leq j \leq n-2 \\ \boldsymbol{s}_i\boldsymbol{\mu}_{i-1} &= \boldsymbol{\mu}_{i-1}\boldsymbol{s}_i = \boldsymbol{\mu}_{i-1} && 1 \leq i \leq n-1 \\ \boldsymbol{s}_i\boldsymbol{\nu}_{i-1}\boldsymbol{\nu}_i &= \boldsymbol{\nu}_{i-1}\boldsymbol{\nu}_i\boldsymbol{s}_i = \boldsymbol{\nu}_{i-1}\boldsymbol{\nu}_i && 0 \leq i, j \leq n-1 \\ \boldsymbol{\nu}_i\boldsymbol{\nu}_j &= \boldsymbol{\nu}_j\boldsymbol{\nu}_i && 0 \leq i, j \leq n-2 \\ \boldsymbol{\mu}_i\boldsymbol{\mu}_j &= \boldsymbol{\mu}_j\boldsymbol{\mu}_i && j \neq i, i+1 \\ \boldsymbol{\nu}_i\boldsymbol{\mu}_j = \boldsymbol{\mu}_j\boldsymbol{\nu}_i, &\quad s_i\boldsymbol{\nu}_j = \boldsymbol{\nu}_j s_i && i \neq j-1, j \\ \boldsymbol{\nu}_i &= \boldsymbol{s}_i\boldsymbol{\nu}_{i-1}\boldsymbol{s}_i && 1 \leq i \leq n-1 \\ \boldsymbol{\mu}_i &= \boldsymbol{s}_i\boldsymbol{s}_{i+1}\boldsymbol{\mu}_{i-1}\boldsymbol{s}_{i+1}\boldsymbol{s}_i && 1 \leq i \leq n-2 \\ \boldsymbol{\mu}_i &= \boldsymbol{\mu}_i\boldsymbol{\nu}_j\boldsymbol{\mu}_i && j = i, i+1 \end{aligned} \tag{1.4}$$

□

Lemma 1.1.7. *Let R be a commutative ring and $\delta \in R$. Let $\mathcal{M}$ be a submonoid of $\mathfrak{P}_n$. Denote by $R\mathcal{M}$ the free R-module with a basis given by the set $\mathcal{M}$. Then $(R\mathcal{M}, .)$ is a subalgebra of the partition algebra $(\mathfrak{P}_n(\delta), .)$.*

Proof. To prove the Lemma it is enough to show that $R\mathcal{M}$ is closed under . , but this follows from the fact that $\mathcal{M}$ is a submonoid of the partition monoid. □

We call $\mathcal{M}$ the *underlying monoid* of the algebra $R\mathcal{M}$.

For $\mathfrak{d}, \mathfrak{e} \in \mathfrak{P}_n$ and $\mathbf{p} \in \mathfrak{d}$, we say $\mathbf{p}$ is a *propagating part* if $\mathbf{p} \cap \underline{n} \neq \phi$ and $\mathbf{p} \cap \underline{n}' \neq \phi$ otherwise call it a *non-propagating part* . We denote by $\#(\mathfrak{d})$ the number of propagating parts of $\mathfrak{d}$. one can easily show that

$$\#(\mathfrak{d}\mathfrak{e}) \leq min(\{\#(\mathfrak{d}), \#(\mathfrak{e})\}). \tag{1.5}$$

A non-propagating part $\mathbf{p} \in \mathfrak{d}$ is called *northern* if $\mathbf{p} \subseteq \underline{n}$, and it is called *southern* if $\mathbf{p} \subseteq \underline{n}'$. It is straightforward to see that a non-propagating part is either northern or southern.

For example, the part $\{1', 2', 3', 4'\} \in \mathfrak{u}_4 \otimes \mathfrak{m}_2 \in \mathfrak{P}_6$ is a southern non-propagating part but $\{5, 6, 5', 6'\} \in \mathfrak{u}_4 \otimes \mathfrak{m}_2$ is a propagating part, and $\#(\mathfrak{u}_4 \otimes \mathfrak{m}_2) = 1$.

A partition of $\mathfrak{d} \in \mathfrak{P}_{\underline{n} \cup \underline{m}'}$ is called *planar* if the equivalence class of diagrams of $\mathfrak{d}$ contains a diagram with the property that no two edges, or strings, cross each other. Denote the set of all planar partitions of $\underline{n} \cup \underline{n}'$ by $\mathcal{T}_n^1$, we shortly explain the reason behind this notation.

For example, $\mathfrak{u}_2 \otimes \mathfrak{m}_1 \in \mathcal{T}_3^1$ is planar, but the partition $\{\{1, 2'\}, \{2, 1'\}, \{3, 3'\}\} \in \mathfrak{P}_3$ is not planar.

In the rest of this section we use Lemma 1.1.7 to define some subalgebras of the partition algebra.

Let $\mathcal{TL}_n := \{\mathfrak{d} \in \mathcal{T}_n^1 \mid |\mathbf{p}| = 2 \text{ for any } \mathbf{p} \in \mathfrak{d}\}$. The set $\mathcal{TL}_n$ with the operation $\circ$ is a monoid and it is called the *Temperley-Lieb* monoid, see [78] and [51].

Lemma 1.1.8 (cf. Equation 1.5 [35], [51]). *The set $\mathcal{T}_n^1$ with the operation $\circ$ is a submonoid of $\mathfrak{P}_n$. Moreover, we have $\mathcal{T}_n^1 \simeq \mathcal{TL}_{2n}$ as a monoid.* □

The *Temperley-Lieb* algebra, see [78], denoted by $\mathcal{TL}_n(\delta)$, is a subalgebra of $\mathfrak{P}_n(\delta)$ with the basis given by the set $\mathcal{TL}_n$. The structure of Temperley-Lieb algebra is well understood and we refer the reader to [50, 51, 68, 79] for the representation theory of $\mathcal{TL}_n(\delta)$. Figure 1.4 describes where the Temperley-Lieb algebra is located in the poset of some other subalgebras of the partition algebra.

The *Brauer* monoid [9], denoted by $\mathcal{B}_n$, defined by $\mathcal{B}_n = \{\mathfrak{d} \in \mathfrak{P}_{\underline{n}\cup\underline{n}'} \mid |\mathbf{p}| = 2 \text{ for each } \mathbf{p} \in \mathfrak{d}\}$ is a submonoid of the partition monoid $\mathfrak{P}_n$. Using Lemma 1.1.7 one can define the *Brauer algebra* as a subalgebra of the partition algebra, denoted by $\mathcal{B}_n(\delta)$. See Figure 1.4 for the location of $\mathcal{B}_n(\delta)$ in the poset of subalgebras of the partition algebra that we are interested in here. The representation theory of the Brauer algebra over the field of complex numbers (that is $R = \mathbb{C}$) has been studied extensively. The closest reference to the kind of questions that we are interested in and would like to answer for our algebras (the planar d-tonal partition algebras and the d-tonal partition algebras) in this thesis are answered for the Brauer algebras in [13, 56, 70].

The *symmetric* group, denoted by $\mathcal{S}_n$, defined by $\mathcal{S}_n = \{\mathfrak{d} \in \mathfrak{P}_{\underline{n}\cup\underline{n}'} \mid |\mathbf{p} \cap \underline{n}| = |\mathbf{p} \cap \underline{n}'| = 1 \text{ for each } \mathbf{p} \in \mathfrak{d}\}$ is a subgroup of the partition monoid $\mathfrak{P}_n$. Using Proposition 1.1.7 one can define the *symmetric group algebra* as a subalgebra of the partition algebra, denoted by $R\mathcal{S}_n$. See Figure 1.4 for the location of $R\mathcal{S}_n(\delta)$ in the poset of subalgebras of the partition algebra. Note that the symmetric group algebra does not depend on the parameter δ, since $\#(\mathfrak{d}) = n$ for each $\mathfrak{d} \in \mathcal{S}_n$. References for the representation theory of the symmetric group are [42, 71].

The *full transformation* monoid, cf. [28, 39], $\mathcal{T}_n = \langle \mathcal{S}_n \cup \{\tau_n\}\rangle$, where $\tau_n =$

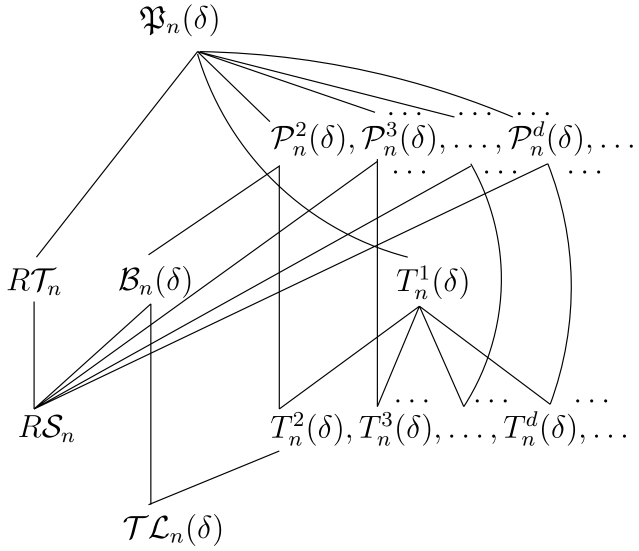


Figure 1.4: A poset describes the embedding of some subalgebras of the partition algebra.

$\{\{1,2,1'\},\{2'\},\{3,3'\},\ldots,\{n,n'\}\}$, is also a submonoid of the partition monoid. Similar to the symmetric group one can define the monoid algebra of the *full transformation monoid* , denote it by $R\mathcal{T}_n$. Hewitt and Zuckerman in [37] initiated the representation theory $\mathbb{C}\mathcal{T}_n$, see Theorem 3.2 [37], and they showed that the irreducible representations of $\mathbb{C}\mathcal{T}_n$ are indexed by the union of the sets of partitions of of k, where $1 \leq k \leq n$. Putcha in [66] Theorem 2.1 constructed the simple modules of $\mathbb{C}\mathcal{T}_n$, and in [67] Theorem 4.3 he showed that $\mathbb{C}\mathcal{T}_n$ has two blocks. Further, Steinberg recently in [76] computed the global dimension of $\mathbb{C}\mathcal{T}_n$.

We describe the reminder parts of the Figure 1.4 in the next section.

1.1.4 Schur-Weyl duality for some subalgebras of the partition algebra.

Let N be a positive integer fix a basis $\{e_1, \ldots, e_N\}$ for an N-dimensional $\mathbb{C}$-vector space V, then $\mathrm{End}_{\mathbb{C}}(V^{\otimes n}) \simeq M_{N^n \times N^n}(\mathbb{C})$. The following map which is defined on the generators of $\mathfrak{P}_n(N)$ is a morphism of $\mathbb{C}$-algebras:

$$\begin{aligned} \rho_N : \mathfrak{P}_n(N) &\to \mathrm{End}_{\mathbb{C}}(V^{\otimes n}) \\ \boldsymbol{\mu}_i &\mapsto \hat{\mu}_i \\ \boldsymbol{\nu}_i &\mapsto \hat{\nu}_i \\ \boldsymbol{s}_i &\mapsto \hat{s}_i \end{aligned} \tag{1.6}$$

Where

$$\begin{aligned} \hat{\mu}_r(e_{i_1} \otimes \cdots \otimes e_{i_n}) &= \delta_{i_{r+1}, i_{r+2}} e_{i_1} \otimes \cdots \otimes e_{i_n}, \\ \hat{\nu}_r(e_{i_1} \otimes \cdots \otimes e_{i_n}) &= \sum_{k=1}^{N} e_{i_1} \otimes \cdots \otimes e_{i_{r-1}} \otimes e_{i_k} \otimes e_{i_{r+1}} \otimes \cdots \otimes e_{i_n} \\ \hat{s}_r(e_{i_1} \otimes \cdots \otimes e_{i_n}) &= e_{i_1} \otimes \cdots \otimes e_{i_{r-1}} \otimes e_{i_{r+1}} \otimes e_{i_r} \otimes e_{i_{r+2}} \otimes \cdots \otimes e_{i_n}, \end{aligned}$$

for a basis element $e_{i_1} \otimes \cdots \otimes e_{i_n} \in V^{\otimes n}$. See, for example, Section 6 of [55] and Section 4 of [54] for details.

Let d be a positive integer, and $\mathcal{P}_n^d := \{\mathfrak{d} \in \mathfrak{P}_n \mid d \text{ divides } |\mathbf{p} \cap \underline{n}| - |\mathbf{p} \cap \underline{n}'| \text{ for each } \mathbf{p} \in \mathfrak{d}\}$. Then $\mathcal{P}_n^d$ is a submonoid of the partition monoid, see Theorem 2.1.1. The *d-tonal partition algebra*, denoted by $\mathcal{P}_n^d(\delta)$, is a subalgebra of the partition algebra and defined in a similar way to the Brauer algebra. We denote by $T_n^d(\delta)$ the subalgebra of $\mathcal{P}_n^d(\delta)$ which is spanned by all the planar partitions in $\mathcal{P}_n^d$, see Proposition 3.1.2 for the formal definition of $T_n^d(\delta)$. We call $T_n^d(\delta)$ the *planar d-tonal partition algebra*. The representation theory of $T_n^d(\delta)$ is going to be one of the main concerns of this thesis.

Let $\mathrm{GL}_N(\mathbb{C})$ be the group of all N by N invertible matrices over $\mathbb{C}$. The vector space V defined above is the natural module for $\mathrm{GL}_N(\mathbb{C})$. This natural action can be extended diagonally to make $V^{\otimes n}$ a $\mathrm{GL}_N(\mathbb{C})$-module. On the other hand, the group $\mathcal{S}_n$ acts from right on $V^{\otimes n}$ by permuting the tensor factors. In general, these two actions on $V^{\otimes n}$ commute, and gives $V^{\otimes n}$ the structure of $(\mathbb{C}\,\mathrm{GL}_N(\mathbb{C}), \mathbb{C}\mathcal{S}_n)$-bimodule. The "*classical Schur-Weyl duality*" [72] states that if $\rho \in \mathrm{End}_{\mathbb{C}}(V^{\otimes n})$ and ρ commutes with the action of $\mathrm{GL}_N(\mathbb{C})$ on $V^{\otimes n}$ then $\rho \in \mathbb{C}\mathcal{S}_n$, and vice versa. In other words, the action of a $\mathbb{C}\,\mathrm{GL}_N(\mathbb{C})$ and $\mathbb{C}\mathcal{S}_n$ have the *double-centraliser property.* We will see in Theorem 1.1.9 that the d-tonal partition algebra has the double centraliser property.

The classical Schur-Weyl duality has been generalised in many different ways. Brauer in [9] replaced $\mathrm{GL}_N(\mathbb{C})$ by the orthogonal group $O_N(\mathbb{C})$ and obtained the so called Brauer algebra $\mathcal{B}_n(N)$.

The d-tonal partition algebra is defined by Tanabe in [77] in the context of Schur-Weyl duality, and as a generalisation of partition algebras in this context. In the remainder of this section we describe Tanabe's duality.

Let $\zeta_d = e^{\frac{2\pi i}{d}}$, and let $G_d = \langle \zeta_d \rangle$ be the cyclic group of order d. The wreath product of G_d with the symmetric group $\mathcal{S}_N$, denoted by $\mathcal{S}_{d,N} = G_d \wr \mathcal{S}_N$, is a *unitary reflection group*, cf. [49]. In matrix notation, $\mathcal{S}_{d,N}$ can be identified with N by N permutation matrices with entries from G_d. If $d = 1$ then $\mathcal{S}_{1,N}$ is the symmetric group on N letters, and if $d = 2$ then $\mathcal{S}_{2,N}$ is known as the *Weyl group* of type B_N, see Example 2.11 of [49].

The group $\mathcal{S}_{d,N}$ is a subgroup of $\mathrm{GL}_N(\mathbb{C})$ and acts on $V^{\otimes n}$ by restricting the action of $\mathrm{GL}_N(\mathbb{C})$. On the other hand, $\mathcal{P}_n^d(N)$ also acts on $V^{\otimes n}$ by restricting the action of the partition algebra $\mathfrak{P}_n(N)$, see Equation 1.6.

The following theorem is a generalisation of the classical Schur-Weyl duality. When

$d = 1$ it was originally proved by Jones in [43]. Then Tanabe in [77] proved the result for any d.

Theorem 1.1.9. *Let $R = \mathbb{C}$. Then the action of the group algebra $\mathbb{C}\mathcal{S}_{d,N}$ and the d-tonal partition algebra $\mathcal{P}_n^d(N)$ have double centraliser property on $V^{\otimes n}$. That is, if $N \geq 2n$ then:*

1. *We have $\mathcal{P}_n^d(N) \simeq \mathrm{End}_{\mathbb{C}\mathcal{S}_{d,N}}(V^{\otimes n})$.*

2. *The group $\mathcal{S}_{d,N}$ generates $\mathrm{End}_{\mathcal{P}_n^d(N)}(V^{\otimes n})$.* □

1.2 Representation theory of $\mathfrak{P}_n(\delta)$ over the complex field.

In this section we state the questions that we are interested in regarding the representation theory of finite dimensional algebras. Some of these questions are motivated by physics in the case of the Temperley-Lieb algebra [51] and the partition algebra [52, 54, 58]. But in general non-semisimple algebras can be understood by the answer to these questions. All of the questions that we would like to ask in this section are answered in the case of the partition algebra over the complex field, mainly by Martin. Therefore, we recall his results to give the flavour of the results that we wish to obtain.

Suppose we are given a finite dimensional A over an algebraically closed field $\mathbb{F}$, for us A later is going to be either the d-tonal partition algebra or the planar d-tonal partition algebra. We are interested in the answer to the following questions:

1. Whether A is semisimple or not?

The answer of this question about the partition algebras over the field of complex numbers is due to Martin and Saleur.

Theorem 1.2.1 ([58]). *Let $R = \mathbb{C}$. The partition algebra $\mathfrak{P}_n(\delta)$ is semisimple if and only if $\delta \notin \{0, 1, 2, 3, \ldots, 2n-1\}$.* $\square$

We obtain a similar result for $T_n^2(\delta)$. Furthermore, we show that the algebra $T_n^d(\delta)$ is generically semisimple, for each n and d.

2. Describing the simple modules of A. This includes,
i) giving an indexing set for the simple modules of A,
ii) and constructing the simple modules of A explicitly.

Martin has given an indexing set to the simple modules of the partition algebra in terms of partitions of some positive integers. Therefore, to state his result we need to recall the following piece of standard notation, which also going to be used intensively in Chapter 2. A k-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is called a *partition of* n if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$ and $\sum_{i=1}^{k} \lambda_i = n$, write $\lambda \vdash n$. By definition we may assume 0 to be the only partition of 0.

Let $\Lambda(A)$ denote an indexing set of the simple modules of A. Then the following result is a partial answer to the question 2.
Theorem 1.2.2 (Corollary 6.1 [52]). *Let $R = \mathbb{C}$ and $\delta \neq 0$. Then we have* $\Lambda(\mathfrak{P}_n(\delta)) = \{\lambda \mid \lambda \vdash n, n-1, \ldots, 1, 0\}$. $\square$

Let $l \leq n$, and $L_{n,l}$ be the $\mathcal{L}$-class of $\mathfrak{P}_n$(see Section 1.3 for the definition of $\mathcal{L}$-classes) containing $\mathfrak{e}_l := \{\{1, 1'\}, \ldots, \{l, l'\}, \{l+1\}, \{(l+1)'\}, \ldots, \{n\}, \{n'\}\}$. Then $\mathbb{C}L_{n,l} = \mathbb{C}\text{-span}(L_{n,l})$ is a left $\mathfrak{P}_n(\delta)$-module with the following action (defined on basis of $\mathfrak{P}_n(\delta)$ and $\mathbb{C}L_{n,l}$):

$$\begin{aligned} \rho : \mathfrak{P}_n(\delta) \times \mathbb{C}L_{n,l} &\to \mathbb{C}L_{n,l} \\ (\mathfrak{d}, \mathfrak{t}) &\mapsto \begin{cases} \mathfrak{d}.\mathfrak{t} & \text{if} \quad \mathfrak{d} \circ \mathfrak{t} \in L_{n,l} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{1.7}$$

For simplicity we write $\mathfrak{d}.\mathfrak{t}$ to denote $\rho(\mathfrak{d}, \mathfrak{t})$.

Furthermore, $\mathbb{C}L_{n,l}$ is a right $\mathbb{C}\mathcal{S}_l$-module. The action of $\mathcal{S}_l$ on $L_{n,l}$ is given by permuting the first (from the left hand side) l vertices of the bottom part of a diagram $\mathfrak{p} \in L_{n,l}$.

Recall from [42, Theorem 2.1.11], we have $\Lambda(\mathbb{C}\mathcal{S}_l) = \{\lambda \mid \lambda \vdash l\}$. Let $\{\mathcal{S}(\lambda) \mid \lambda \vdash l\}$ be a representative of the isoclasses of simple $\mathbb{C}\mathcal{S}_l$-modules. The modules $\mathcal{S}(\lambda)$ are known as *Specht* modules.

Let $\lambda \in \Lambda(\mathfrak{P}_n(\delta))$, and $\Delta_n(\lambda) := \mathbb{C}L_{n,l} \otimes_{\mathbb{C}\mathcal{S}_l} \mathcal{S}(\lambda)$. Then $\Delta_n(\lambda)$ is a left $\mathfrak{P}_n(\delta)$-module, with the following action. Let $\mathfrak{d} \in \mathfrak{P}_n$ and $\mathfrak{p} \otimes_{\mathbb{C}\mathcal{S}_l} \mathfrak{s}$ be a basis element of $\Delta_n(\lambda)$. Then the action of $\mathfrak{P}_n(\delta)$ on $\Delta_n(\lambda)$ is defined by

$$\mathfrak{d}.(\mathfrak{p} \otimes_{\mathbb{C}\mathcal{S}_l} \mathfrak{s}) = (\mathfrak{d}.\mathfrak{p}) \otimes_{\mathbb{C}\mathcal{S}_l} \mathfrak{s}$$

Before stating the next theorem we recall some necessary definitions, from [4, Section 1.2]. For a left A-module M, denote by $\operatorname{rad}(M)$ the intersection of all maximal submodules of M. The *socle* of M, written as $\operatorname{soc}(M)$, is the maximal semisimple submodule of M. Finally, we write $\operatorname{head}(M)$ to denote $M/\operatorname{rad}(M)$.

Theorem 1.2.3 (Proposition 4, Proposition 6[53])**.** *Let $R = \mathbb{C}$ and $\delta \neq 0$. Then the partition algebra $\mathfrak{P}_n(\delta)$ is quasi-hereditary(see Section 1.4) with the follwoing hereditary chain:*

$$\mathfrak{P}_n(\delta)\mathfrak{e}_0\mathfrak{P}_n(\delta) \subset \mathfrak{P}_n(\delta)\mathfrak{e}_1\mathfrak{P}_n(\delta) \subset \cdots \subset \mathfrak{P}_n(\delta)\mathfrak{e}_n\mathfrak{P}_n(\delta)$$

Furthermore, the set $\{\Delta_n(\lambda) \mid \lambda \vdash n, n-1, \dots, 1, 0\}$ is a complete set of pairwise non-isomorphic standard modules of $\mathfrak{P}_n(\delta)$. Each $\operatorname{head}(\Delta_n(\lambda))$ *is a simple module, and every simple module of $\mathfrak{P}_n(\delta)$ arises in this way. In addition, if $\mathfrak{P}_n(\delta)$ semisimple then each standard module is simple.* □

Remark 1.2.4. One can define the modules $\Delta_n(\lambda)$ over $\mathbb{Z}$. This is because the Specht modules are defined over $\mathbb{Z}$, and it is possible to consider the $\mathbb{Z}$-span of the set $L_{n,p}$ instead of $\mathbb{C}L_{n,l}$, see [44] for such approach. However, we believe defining them over $\mathbb{C}$ is enough for the purpose of showing the flavour of the results that we desire at the moment.

Let $\{P(\lambda)\}_{\lambda\in\Lambda(A)}$ and $\{S(\lambda)\}_{\lambda\in\Lambda(A)}$ be complete lists of pairwise non-isomorphic indecomposable projective and simple modules of A respectively. Then the next question that we are interested in is:

3. What are the blocks of A? That is, describing the equivalence classes of the equivalence closure of a relation on $\Lambda(A)$ given by $\lambda \sim \mu$ if the simple A-modules $S(\lambda)$ and $S(\mu)$ are composition factors of the same indecomposable projective A-module. See Section 1.8 of [4] for general definition of block.

A *Young diagram* corresponding to a partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ is, denoted by $[\lambda]$, defined by

$$[\lambda] := \{(i,j) \mid (i,j) \in \mathbb{N}^2 \text{ with } 1 \le i \le k \text{ and } 1 \le j \le \lambda_i\}.$$

Let δ be a positive integer, and λ, μ be two partitions such that $[\lambda] \subset [\mu]$. Then the pair (λ, μ) is called *δ-pair* if there exist non-negative integers i, j_1 and j_2 such that $\mu \setminus \lambda = \{(i, j_1), (i, j_1+1), \ldots, (i, j_1+j_2)\}$, and $j_1 + j_2 - i = \delta - |\lambda|$.

For example let $\mu = (5,4,1)$ and $\lambda = (5,2,1)$. Then (λ, μ) is 10-pair, here $j_1 = 3$, $j_2 = 1$ and $i = 2$. Hence, (λ, μ) are differing in the 2nd row.

Theorem 1.2.5 (Proposition 9 [53])**.** *Each block of $\mathfrak{P}_n(\delta)$ is given by a chain partition*

$$\lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)}$$

where for each i, $(\lambda^{(i-1)}, \lambda^{(i)})$ is a δ-pair. Furthermore, there is an exact sequence of $\mathfrak{P}_n(\delta)$-modules associated to the above chain:

$$0 \to \Delta_n(\lambda^{(r)}) \to \Delta_n(\lambda^{(r-1)}) \to \cdots \to \Delta_n(\lambda^{(0)}) \to \mathrm{head}(\Delta_n(\lambda^{(0)})) \to 0$$

such that the image of each morphism in the above sequence is simple. In particular, for each i we have $\mathrm{head}(\Delta_n(\lambda^{(i)}))$ *and* $\mathrm{rad}(\Delta_n(\lambda^{(i)}))$ *are both simple, and* $\mathrm{head}(\Delta_n(\lambda^{(r)})) = \Delta_n(\lambda^{(r)})$. □

Remark 1.2.6. The notation used in Theorem 1.2.5 is slightly different from Martin's original notation. In fact we follow [44] in our notation. Bowman and others in [44] have given a reformulation of Theorem 1.2.5 in terms of the action of reflection group of type A_n. Furthermore, a description of the blocks of the partition algebra over a field of positive characteristic is also given in [44].

4. Describing the composition multiplicities of the indecomposable projective modules of A. Let $\lambda \in \Lambda(A)$ and M be a left A-module. Then we write $[M : S(\lambda)]$ for the number of occurrences of the simple module $S(\lambda)$ in the Jordan-Holder series of M. Let $c_{\lambda,\mu} = [P(\lambda) : S(\mu)]$ then the integral matrix $(c_{\lambda,\mu})$ is called a *Cartan matrix* of A, see for e.g. section 1.9 of [4]. A Cartan matrix of A is unique up to conjugation by permutation matrices. Therefore, one might say $(c_{\lambda,\mu})$ is *the* Cartan matrix of A.

Note that if the Cartan matrix of A is known one can obtain the blocks of A in the following way. Let $\sim'_c$ be a relation defined by $\lambda \sim'_c \mu$ if $c_{\lambda,\mu} \neq 0$, for each $\lambda, \mu \in \Lambda(A)$. Let $\sim_c$ be the equivalence closure of $\sim'_c$. Then $S(\lambda)$ and $S(\mu)$ are in the same block if and only if $\lambda \sim_c \mu$.

Suppose A possess an involutory anti-automorphism of algebras, denote it by $a \mapsto a^t$. Then a right A-module N can be regarded as a left A-module with the following action

$$a.x := xa^t. \tag{1.8}$$

The right A-module $\hat{M} := \mathrm{Hom}_{\mathbb{F}}({}_AM, \mathbb{F})$ is called the dual of ${}_AM$. By using the involutory anti-automorphism t we may regard $\hat{M}$ as a left A-module, denote it by M^o to stress the left action of A via t. The map $M \mapsto M^o$ is called the *contravariant duality*, M^o is the *contravariant dual* of M. In addition, ${}_AM$ is called *self dual* if ${}_AM \simeq {}_AM^o$.

Let A be a quasi-hereditary algebra such that each simple A-module is self dual, in which case some times A is called a BGG algebra in the literature. Let $d_{\lambda,\mu} = [\Delta(\lambda) : S(\mu)]$, then the matrix $(d_{\lambda,\mu})$ is called the *decomposition matrix* of A. We have the following identity, see Corollary 3.4 of [41],

$$(c_{\lambda,\mu}) = (d_{\lambda,\mu})^{tr}(d_{\lambda,\mu}). \tag{1.9}$$

In the case of the partition algebra the Cartan matrix can be computed from Equation 1.9 and Theorem 1.2.5.

Unfortunately, we have not been able to compute the Cartan matrix of the planar d-tonal partition algebra yet. However, we have shown that $T_n^d(\delta)$ is a BGG algebra, and hence satisfies the Equation 1.9.

1.3 Green's Relations.

Recall from [32] for a finite monoid $\mathcal{M}$ two elements $\alpha, \beta \in \mathcal{M}$ we define the following three relations,

1. The elements α and β are $\mathcal{J}$- *related* if and only if $\mathcal{M}\alpha\mathcal{M} = \mathcal{M}\beta\mathcal{M}$.
2. The elements α and β are $\mathcal{L}$- *related* if and only if $\mathcal{M}\alpha = \mathcal{M}\beta$.
3. The elements α and β are $\mathcal{R}$- *related* if and only if $\alpha\mathcal{M} = \beta\mathcal{M}$.

These relations are equivalence relations and they are known as *Green's relations*. We write $\bar{\alpha}$ for the $\mathcal{J}$-equivalence class containing α.

Note that, there is a partial order relation on $\mathcal{M}/\mathcal{J}$ induced from the inclusion of $\mathcal{J}$-classes of $\mathcal{M}$, in the following way; for $\bar{\alpha}, \bar{\beta} \in \mathcal{M}/\mathcal{J}$ we say $\bar{\beta} \preceq \bar{\alpha}$ if and only if $\beta \in \mathcal{M}\alpha\mathcal{M}$.

An element α of monoid $\mathcal{M}$ is called *regular* if $\alpha \in \alpha\mathcal{M}\alpha$, cf. [28, 39]. A monoid $\mathcal{M}$ is called *regular* monoid if every element in $\mathcal{M}$ is regular. In fact, it is not hard to see that a finite $\mathcal{M}$ is regular if and only if every $\mathcal{J}$-class of $\mathcal{M}$ contains an idempotent.

Let $e \in \mathcal{M}$ be an idempotent then we denote by G_e the group of units of $e\mathcal{M}e$.

Definition 1.3.1. *Let R be a commutative ring and $(\mathcal{M}, \bullet)$ be a finite monoid. Then the R-monoid algebra $(R\mathcal{M}, +, .)$ is the free R-module with basis given by $\mathcal{M}$, and the operation . is given by extending $\bullet$ R-linearly.*

The following result might answer partially the question, for a given monoid $\mathcal{M}$ why should we care about its $\mathcal{J}$-classes when we are interested in the representation theory of $\mathcal{M}$?

Theorem 1.3.2 (Clifford, Munn, Ponizovskiĭ, Theorem 7 [29])**.** *Let $\mathbb{F}$ be an algebraically closed field, and $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ be a complete list of idempotent representatives of the $\mathcal{J}$-classes of of a finite monoid $\mathcal{M}$ which contains an idempotent. Then there is a bijection between $\Lambda(\mathbb{F}\mathcal{M})$ and the disjoint union $\bigsqcup_{i=1}^{k} \Lambda(\mathbb{F}G_{\epsilon_i})$.* □

1.4 Quasi-hereditary and Cellular algebras.

We devote this section to recall in the definition of two classes of finite dimensional algebras. Later in this thesis we show that the algebras we interested in belong to both of the classes.

The quasi-heredity algebras were defined by Ringel, Cline, Parshall and Scott , see p.280 of [73], in the study of algebraic groups to describe the so called highest weight categories.

Through this thesis $\mathbb{F}$ will denote an algebraically closed field.

Fix a finite dimensional $\mathbb{F}$- algebra A. A two sided ideal J in A is called *hereditary* ideal if;

I. There is an idempotent $e \in A$ with $J = AeA$.

II. ${}_AJ$ is a projective module.

III. $\mathrm{End}_A(Ae) \simeq eAe$ is a semisimple $\mathbb{F}$-algebra.

The algebra A is *quasi-hereditary* algebra if there is a chain of of two sided ideals

$$0 =: J_r \subset J_{r-1} \subset \cdots \subset J_1 \subset J_0 := A$$

such that J_i/J_{i+1} is a hereditary ideal in A/J_{i+1} for $0 \leq i \leq r-1$. The above chain is called a *hereditary chain.*

Let $\mathfrak{C}$ be a class of left A-modules and M be a left A-module. We say M is filtered by members of $\mathfrak{C}$ if there is a sequence of submodules of M:

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{k-1} \subset M_k = M$$

such that for each $1 \leq i \leq k$ the quotient M_i/M_{i-1} is in $\mathfrak{C}$ up to isomorphism of left A-modules.

In [12] another useful characterisation of the quasi-hereditary algebras is given, which we recall it in the following theorem.

Theorem 1.4.1 (see Theorem 3.6 of [12])**.** *Let A be a finite dimensional $\mathbb{F}$-algebra. Fix a complete set of pairwise non-isomorphic simple A- modules $\{S(\lambda) \mid \lambda \in \Lambda(A)\}$, where $\Lambda(A)$ is an indexing set of the simple modules of A. Then A is a quasi-hereditary algebra if and only if there is a partial order relation $\leq$ on the set $\Lambda(A)$, such that for any $\lambda \in \Lambda(A)$ there exists a left A-module $\Delta(\lambda)$ satisfying the following two conditions;*

I. There exists a surjective morphism $\theta_\lambda : \Delta(\lambda) \to S(\lambda)$, such that if $S(\mu)$ is a composition factor of the $ker(\theta_\lambda)$ then $\mu < \lambda$.

II. Let $P(\lambda)$ be the projective cover of $S(\lambda)$. There exists a surjective morphism

$\vartheta_\lambda : P(\lambda) \to \Delta(\lambda)$ *such that the kernel of* ϑ_λ *is filtered by modules* $\Delta(\mu)$ *with* $\mu > \lambda$.

□

We recall from p 137 of [22] the modules $\Delta(\lambda)$, $\lambda \in \Lambda(A)$ defined in Theorem 1.4.1 are called *standard* modules and they are unique up to isomorphism.

Let A be a quasi-hereditary algebra, and $\mathfrak{C}$ be a set of representatives of isoclasses of standard modules of A. If M is a finite dimensional A-module filtered by members of $\mathfrak{C}$, then we say M has a *standard filtration*. Further, we denote by $(M : \Delta(\lambda))$ the number of times when $\Delta(\lambda)$ appear in a standard filtration of M. See $A1 - 7$ of [22] for the well definedness of the multiplicity of standard modules in a standard filtration of finite dimensional modules of quasi-hereditary algebras.

For a finite dimensional $\mathbb{F}$- algebra A we denote the category of finite dimensional left A- modules by A-*mod*. Given any idempotent $e \in A$ there are functors (see §6.2 of [33] for more details, and [14] for further development and applications of these functors to diagram algebras)

$$\begin{aligned} \mathcal{F}_e : A - mod &\to eAe - mod \\ M &\mapsto eM \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} \mathcal{G}_e : eAe - mod &\to A - mod \\ N &\mapsto Ae \otimes_{eAe} N \end{aligned} \tag{1.11}$$

We call $\mathcal{F}_e$ the localisation, and $\mathcal{G}_e$ the globalisation functor associated to e, we shall write $\mathcal{F}$ and $\mathcal{G}$ when it is clear from the context. This pair of functors have interesting properties; $\mathcal{G}$ is a right inverse to $\mathcal{F}$, $\mathcal{F}$ is exact, and $\mathcal{G}$ takes simple eAe-modules to indecomposable A-modules . We denote by $N_{(e)}$ the largest submodule of an A-module N contained in $(1 - e)N$. Let S be a simple eAe- module, then $\mathcal{G}(S)/\mathcal{G}(S)_{(e)}$ is a simple A- module (the proof of all of these claims can be found in [33] page 55-57).

Theorem 1.4.2 ([33]). *For any finite dimensional* $\mathbb{F}$*-algebra* A *and an idempotent* $e \in A$ *we have*

$$\Lambda(A) \simeq \Lambda(eAe) \sqcup \Lambda(A/AeA).$$

□

The globalisation and localisation functors behave very nicely with quasi-hereditary algebras. Indeed the following two Propositions explain how these functors preserve standard modules, which we will make use of this property in Chapter 3.

Proposition 1.4.3 (A1-4 of [22], Proposition 3 of [57]). *We keep the notation of Theorem 1.4.1. Let* A *be a quasi-hereditary algebra and* $e \in A$ *be an idempotent such that* eAe *is also quasi-hereditary. For* $\lambda \in \Lambda(eAe)$ *we have:*

1. $\mathcal{F}_e(\Delta(\lambda)) = \Delta_e(\lambda)$, *where* $\Delta_e(\lambda)$ *is the standard module of* eAe *labelled by* λ.

2. $\mathcal{F}_e(P(\lambda)) = P_e(\lambda)$, *where* $P_e(\lambda)$ *is the indecomposable projective module of* eAe *labelled by* λ. □

Proposition 1.4.4 (Proposition 4 [57]). *Let* A *be a quasi-hereditary algebra and* e *be part of a hereditary chain of* A. *Let* M *be a finite dimensional left* eAe*-module having a standard filtration. Then* $\mathcal{G}_e(M)$ *also has a standard filtration (in the sense of Section 1.4). Further, the standard multiplicity is*

$$(\mathcal{G}_e(M) : \Delta(\lambda)) = \begin{cases} (M : \Delta_e(\lambda)) & \text{if } \lambda \in \Lambda(eAe) \\ 0 & \text{otherwise} \end{cases} \tag{1.12}$$

Where $\Delta_e(\lambda)$ *is the standard module of* eAe *labelled by* λ. □

Finally, we recall the definition of Cellular algebras.

Definition 1.4.5 (Definition 1.1 of [30]). *A Cellular algebra is an associative* R*-algebra* A *with identity possessing a tuple (known as a cell datum)* $(\Lambda, M, C, \star)$ *satisfying the following conditions:*

C1. Λ *is a poset with the property that for each* $\lambda \in \Lambda$, $M(\lambda)$ *is a finite set such that* $C : \bigsqcup_{\lambda\in\Lambda} M(\lambda) \times M(\lambda) \to A$ *is an injective map and* A *is a free* R*-module with* *basis* $C(\bigsqcup_{\lambda\in\Lambda} M(\lambda)\times M(\lambda))$.

***C**2. For* $\lambda \in \Lambda$ *and* $S, T \in M(\lambda)$ *write* $C^{\lambda}_{S,T} = C(S,T)$. *Then* $\star$ *is an anti-automorphism of* R*-algebras such that* $\star(C^{\lambda}_{S,T}) = C^{\lambda}_{T,S}$.

***C**3. For* $\lambda \in \Lambda$, $S, T \in M(\lambda)$ *and each* $\alpha \in A$ *we have*

$$\alpha.C^{\lambda}_{S,T} = \sum_{S'\in M(\lambda)} r_{\alpha}(S',S)C^{\lambda}_{S',T} \pmod{A} < \lambda$$

where $r_{\alpha}(S',S)$ *is independent of* T, *and* $A(<\lambda)$ *is the* R*-module generated by* $\{C^{\mu}_{S'',T''} \mid \mu < \lambda$ *and* $S'', T'' \in M(\mu)\}$.

Lemma 1.4.6. *Let* A *be a cellular algebra with a cell datum* $(\Lambda, M, C, \star)$, *and let* $\circ$ *be the contravariant duality defined on page 17 with* $t = \star$. *Then we have* ${}_AS \simeq {}_A(S)^{\circ}$ *for each simple* A*-module.* □

Chapter 2

On the number of principal ideals in d-tonal partition monoids

Tanabe in [77] defined a family of subalgebras of the partition algebras for each $d \in \mathbb{Z}_{>0}$. We call them *d-tonal partition algebras*, see also Theorem 1.1.9. Kosuda studied these algebras under the name of the *modular party algebra* in [46, 47, 48]. In particular, Kosuda characterised the d-tonal partition algebras by generators and relations, and showed that these algebras are cellular, in the sense of Graham and Lehrer [30]. In Kosuda's proof the cell modules are not constructed because the method of Changchang Xi [81] is used. Orellana independently constructed the Bratteli diagram for the d-tonal partition algebras in [63, 64]. Moreover, Orellana studied the representation theory of these algebras over the field of complex numbers by using the double centraliser theorem, and described the simple modules in the semisimple case.

In this chapter, as a step toward the study of the representation theory of the d-tonal partition algebras over an algebraically closed field we study the ideal structure of the d-tonal partition monoids. For each d, the d-tonal partition monoid is the underlying monoid of the d-tonal partition algebra. We will denote this monoid by $\mathcal{P}_n^d$ where n is a non-negative integer. We start by giving the definition of the $\mathcal{P}_n^d$ in the first section. In the second section, we define a special type of elements in $\mathcal{P}_n^d$ called *canonical elements*. Canonical elements turn out to be crucial for representation theory as they are in a bijection with the $\mathcal{J}$-classes of $\mathcal{P}_n^d$. Moreover, each canonical element is idempotent, this proves the regularity of $\mathcal{P}_n^d$. In fact, the set of all canonical elements of $\mathcal{P}_n^d$ form a partially ordered set with the order induced from the inclusion of principal two sided ideals of $\mathcal{P}_n^d$. We denoted the induced order by $\preceq_d$. In section 3, we define a family of graded posets combinatorially which will be denoted by $(\mathscr{I}(n\mathbf{e}_1), \leq_{X_d})$. In the fourth section, we show in Theorem 2.4.5 that these two posets are isomorphic. This will provide a useful combinatorial characterisation for the set of all canonical elements of $\mathcal{P}_n^d$.

In section 5, a closed formula is obtained for the number of canonical elements of $\mathcal{P}_n^d$ which is denoted by $I_n^{(d)}$. The sequence $I_n^{(d)}$ is a two parameter sequence. When $d \leq 3$ these sequences are known but the other cases seems to be new. In particular, they have have no match on Slone [1] at the moment. Furthermore, we show that $(\mathscr{I}(n\mathbf{e}_1), \leq_{X_d})$ is isomorphic to the quotient of the poset of partitions of a positive integer, partially ordered by refinement, by some equivalence relation.

In the partition monoid, which is equal to $\mathcal{P}_n^1$, two elements are $\mathcal{J}$- equivalent if and only if they have the same number of propagating parts. However, this statement is not true in general when $d > 1$ in $\mathcal{P}_n^d$. This observation lead us to study the set of canonical elements with h propagating parts, denote it by $I_{n,h}^{(d)}$. In general we still do not know a closed formula for this three parameter sequence. In Section 6, for small vales of d the two parameter sequence $I_{n,h}^{(d)}$ is studied and some explicit formulas are achieved.

The polycyclic hydrocarbons are aromatic chemical compounds whose molecular shape may be represented by the so called hollow hexagons, see [16, 18, 17]. Furthermore, the “Cyvin sequence”, which matches the sequence $A028289$ in [1], counts the number of equivalence classes of hollow hexagons. We devote section 7 to defining explicitly these hollow hexagons and construct a representative element for each equivalence class. In Theorem 2.7.5 we construct an explicit isomorphism between the set of rank n hollow hexagons and the poset $\mathscr{I}(n\mathbf{e}_1)$, where $d = 3$. This isomorphism leads to a new way of constructing the hollow hexagons and puts a partial order on them.

2.1 Definition and order of the d-tonal monoid $\mathcal{P}_n^d$.

For a fixed $d, n \in \mathbb{Z}^+$, define

$$\mathcal{P}_n^d := \{\mathfrak{d} \in \mathfrak{P}_n \mid d \text{ divides } |\alpha \cap \underline{n}| - |\alpha \cap \underline{n}'| \text{ for any } \alpha \in \mathfrak{d}\}$$

For example, the left hand side diagram in 2.1 is an example of an element in $\mathcal{P}_6^2$, but the right hand side diagram in 2.1 does not belong to $\mathcal{P}_6^2$.

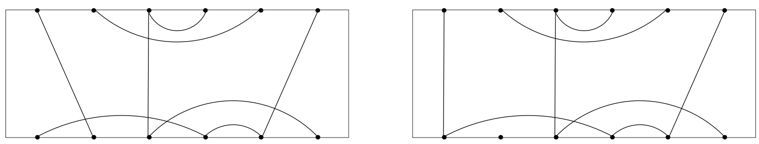


Figure 2.1: An example and a non-example of an elements in $\mathcal{P}_6^2$

Theorem 2.1.1 ([77]). *The set $\mathcal{P}_n^d$ is a submonoid of $\mathfrak{P}_n$.*

Proof. By definition of $\mathcal{P}_n^d$ we have $1 \in \mathcal{P}_n^d$, for all d and n. It only remains to show that $\mathcal{P}_n^d$ is closed under $\circ$. Let $\mathfrak{d}, \mathfrak{e} \in \mathcal{P}_n^d$, and $\mathbf{p} \in \mathfrak{d} \circ \mathfrak{e}$. Then from the definition of $\circ$ there exist a unique $\mathbf{r} \in \mathfrak{d} \star'' \iota^+(\mathfrak{e})$ such that after removing all the vertices that come from $\underline{n}'$, denote it by $\mathbf{r}'$, we have $\iota^-(\mathbf{r}') = \mathbf{p}$. Definition of $\star''$ implies that $\mathbf{r}$ consists of k parts of $\mathfrak{d}$, say $\mathbf{p}_1, \ldots, \mathbf{p}_k$, and l parts of $\iota^+(\mathfrak{e})$, say $\mathbf{q}_1, \ldots, \mathbf{q}_l$. Furthermore we have

$$\sum_{i=1}^{k} |\mathbf{p}_i \cap \underline{n}'| = \sum_{i=1}^{l} |\mathbf{q}_i \cap \underline{n}'|.$$

Then

$$\begin{aligned} |\mathbf{r} \cap \underline{n}| - |\mathbf{r} \cap \underline{n}''| &= \sum_{i=1}^{k} |\mathbf{p}_i \cap \underline{n}| - \sum_{i=1}^{l} |\mathbf{q}_i \cap \underline{n}''| \\ &= \left(\sum_{i=1}^{k} |\mathbf{p}_i \cap \underline{n}| - \sum_{i=1}^{k} |\mathbf{p}_i \cap \underline{n}'| \right) + \left(\sum_{i=1}^{l} |\mathbf{q}_i \cap \underline{n}'| - \sum_{i=1}^{l} |\mathbf{q}_i \cap \underline{n}''| \right) \end{aligned} \tag{2.1}$$

Since $\mathfrak{d}, \mathfrak{e} \in \mathcal{P}_n^d$, the right hand side of the second equality of Equation 2.1 is divisible by d. Therefore, after removing the vertices come from $\underline{n}'$ from $\mathbf{r}'$ and applying ι^- we obtain that the expression $|\mathbf{p} \cap \underline{n}| - |\mathbf{p} \cap \underline{n}'|$ is divisible by d. Hence $\mathfrak{d} \circ \mathfrak{e} \in \mathcal{P}_n^d$, and the claim follows. □

We will call $\mathcal{P}_n^d$ the *d-tonal partition monoid*. Note that $\mathcal{P}_n^1 = \mathfrak{P}_n$.

Some small values of the cardinality of $\mathcal{P}_n^d$, denoted by $|\mathcal{P}_n^d|$, is given in Table 2.1, the first two rows occur on [1]. The first row matches the even *Bill numbers* $A000110(2n)$ and the second row matches $A005046$. The following recursion formula can be found, for example, on p611 in [64],

$$|\mathcal{P}_n^2| = \sum_{k=1}^{n} \binom{2n-1}{2k-1} |\mathcal{P}_{n-k}^2|$$

The final row of Table 2.1 describes the cardinality of $\mathcal{P}_n^d$ as $d \to \infty$, denote it by $\mathcal{P}_n^\infty$, that is:

$$\mathcal{P}_n^\infty := \{\mathfrak{d} \in \mathfrak{P}_n \mid |\alpha \cap \underline{n}| = |\alpha \cap \underline{n}'| \text{ for any } \alpha \in \mathfrak{d}\}$$

Remark 2.1.2. The monoid algebra of $\mathcal{P}_n^\infty$ is studied by Kosuda in [45] and it is called *party algebra*.

d \ n	0	1	2	3	4	5	6	7	...
1	1	2	15	203	4140	115975	4213597	190899322	...
2	1	1	4	31	379	6556	150349	4373461	...
3	1	1	3	17	155	2041	.	.	...
4	1	1	3	16	132	1531	.	.	...
$\vdots$									
$d \to \infty$	1	1	3	16	131	1496	22482	426833	...

Table 2.1: Cardinality of $\mathcal{P}_n^d$, for some small values of d and n.

However, for $d \geq 3$ the sequences $|\mathcal{P}_n^d|$ seems to be new.

Problem 2.1.3. *Find a closed formula for $|\mathcal{P}_n^d|$ for all n and $d \geq 3$.*

In the following remark we present some ideas that might help to solve the Problem 2.1.3

Remark 2.1.4. We note that the d-tonal partition algebras are generically semisimple, see Theorem 5.4 in [64]. They form a chain of embedding of algebras with multiplicity free restriction. The Bratteli diagram for the semisimple cases is constructed in [64], see subsection 3.8.2 for the definition of Bratteli diagram. Therefore, the dimension of each simple module is given by the number of different walks from the top of the Bratteli diagram to the vertex representing it. Hence, by Artin-Wedderburn theorem the Problem 2.1.3 admits a combinatorial description in terms of number of walks on a certain Bratteli diagram.

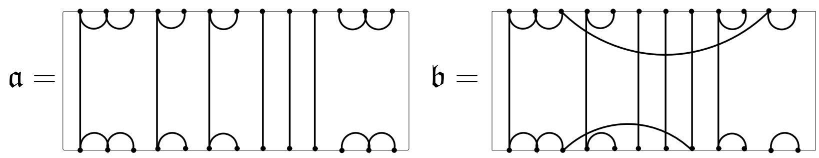


Figure 2.2: The diagram $\mathfrak{a}$ is a canonical element in $\mathcal{P}^3_{13}$ while the digram $\mathfrak{b}$ is not a canonical element in $\mathcal{P}^2_{12}$.

2.2 Canonical representatives of the $\mathscr{J}$-classes of $\mathcal{P}^d_n$.

In this section we give the definition of special kind of elements of $\mathcal{P}^d_n$, called canonical elements. Then, we show each element of $\mathcal{P}^d_n$ is $\mathscr{J}$-equivalent to a canonical element. In Lemma 2.2.2 we show that the set of all canonical elements of $\mathcal{P}^d_n$ is minimal with this property.

We start by introducing a linear order on the set $\underline{n} \cup \underline{n}'$. Let $n' < \cdots < 2' < 1' < 1 < 2 < \cdots < n$, and let $\leqslant$ be the reflexive transitive closure of the relation $<$ on the set $\underline{n} \cup \underline{n}'$, then $(\underline{n} \cup \underline{n}', \leqslant)$ is linearly ordered set.

Definition 2.2.1. *Let* $\mathfrak{d} \in \mathcal{P}^d_n$ *we say* $\mathfrak{d}$ canonical *whenever it is of the form* $(\bigotimes_{l=1}^{r} \mathfrak{m}_{i_l}) \otimes \mathfrak{u}_d^{\otimes k}$ *such that* $1 \leq i_r \leq i_{r-1} \leq \cdots \leq i_1 \leq d$ *and* $kd + \sum_{l=1}^{r} i_l = n$.

For example of a canonical element and a non-canonical element see Figure 2.2. Note also that each canonical element in $\mathcal{P}^d_n$ is planar.

Before stating the next result we would like to define the following elements in $\mathcal{P}^d_n$;

$$\begin{aligned} \boldsymbol{\mu}_i &:= \mathfrak{m}_1^{\otimes i} \otimes \mathfrak{m}_2 \otimes \mathfrak{m}_1^{\otimes(n-i-2)} \\ \boldsymbol{\nu}_j &:= \mathfrak{m}_1^{\otimes j} \otimes \mathfrak{u}_d \otimes \mathfrak{m}_1^{\otimes(n-j-d)} \end{aligned} \tag{2.2}$$

Where $0 \leq i \leq n-2$ and $0 \leq j \leq n-d$, and $\mathfrak{m}_1^0 = \phi$.

We would like to remark that in $\mathcal{P}^1_n$ the $\boldsymbol{\mu}_i$ defined in Equation 2.2 is the same $\boldsymbol{\mu}_i$

defined in Equation 1.3, for all $0 \leq i \leq n-2$. However, the $\boldsymbol{\nu}_j$ in Equation 2.2 can be considered as a generalisation of $\boldsymbol{\nu}_j$ in Equation 1.3. In fact if $d = 1$ then the elements defined in Equation 2.2 are the same as the elements in Equation 1.3.

Let $\mathfrak{d} \in \mathcal{P}_n^d$ we say a propagating part $\mathbf{p} \in \mathfrak{d}$ is of *type* $k \in \{1, 2, \ldots, d\}$ if $|\mathbf{p} \cap \underline{n}| = k \pmod{d}$.

Lemma 2.2.2. *Let* $\mathfrak{d} \in \mathcal{P}_n^d$ *then there exist a unique canonical element* $\mathfrak{p} \in \mathcal{P}_n^d$ *such that* $\mathfrak{d} \mathcal{J} \mathfrak{p}$.

Proof. First note that $\mathcal{S}_n$ is a subgroup of $\mathcal{P}_n^d$. Hence by using the action of $\mathcal{S}_n$ we can permute any two vertices in the northern (respectively southern) part of any element in $\mathcal{P}_n^d$. Therefore, there exist $\mathfrak{s}, \mathfrak{t} \in \mathcal{S}_n$ such that $\mathfrak{s}\mathfrak{d}\mathfrak{t}$ is planar and has the following property; If $\mathbf{p} \in \mathfrak{s}\mathfrak{d}\mathfrak{t}$ is a northern non-propagating part then for any two positive integers k_1 and k_2 we have $\min(\mathbf{p}) - k_1$ and $\max(\mathbf{p}) + k_2$ are not in the same part, similarly for the southern non-propagating part. Let $\mathbf{q}$ be any part of $\mathfrak{s}\mathfrak{d}\mathfrak{t}$ with $|\mathbf{q} \cap \underline{n}| > d$ and let $\max(\mathbf{q}) = q$, then $\boldsymbol{\nu}_{q-d}\mathfrak{s}\mathfrak{d}\mathfrak{t}$ is the same as $\mathfrak{s}\mathfrak{d}\mathfrak{t}$ except the part $\mathbf{q}$ is split to two parts; one of the parts contains q and it is a subset of $\underline{n}$ with cardinality d. Note that $\mathfrak{s}\mathfrak{d}\mathfrak{t} = \boldsymbol{\mu}_{q-d-1}\boldsymbol{\nu}_{q-d}\mathfrak{s}\mathfrak{d}\mathfrak{t}$, this means the above type of cutting process can be reversed. Similar steps can be taken if there is a part $\mathbf{q}$ of $\mathfrak{s}\mathfrak{d}\mathfrak{t}$ with $|\mathbf{q} \cap \underline{n}'| > d$. Proceeding inductively, we may assume that there is an element $\mathfrak{r} \in \mathcal{P}_n^d$ such that $\mathfrak{d} \mathcal{J} \mathfrak{r}$, and for each part $\mathbf{r} \in \mathfrak{r}$ we have $|\mathbf{r} \cap \underline{n}| < d$ and $|\mathbf{r} \cap \underline{n}'| < d$. Finally there are permutations $\mathfrak{s}', \mathfrak{t}' \in \mathcal{S}_n$ such that $\mathfrak{s}'\mathfrak{r}\mathfrak{t}'$ is canonical. The existence of a canonical element follows.

To prove the uniqueness, from Equation 1.5 we have any two elements in the same $\mathcal{J}$-class have the same number of propagating parts. To change the type of a propagating part we must connect it to another propagating part, and this process reduces the number of the propagating parts. Moreover, if $n = kd$ then the only canonical element with no non-propagating parts is $\mathfrak{u}_d^{\otimes k}$. Hence, there exist a unique canonical element in each $\mathcal{J}$-class. □

Corollary 2.2.3. *The monoid $\mathcal{P}_n^d$ is regular.*

Proof. By Proposition 2.2.2 each $\mathcal{J}$-class contains a canonical element, and each canonical element is idempotent. Hence $\mathcal{P}_n^d$ is a regular monoid. □

Corollary 2.2.4. *The monoid $\mathcal{P}_n^d$ is generated by $\boldsymbol{\mu}_0$, $\boldsymbol{\nu}_0$ and the symmetric group $\mathcal{S}_n$.*

Proof. It is not hard to see that each canonical element is generated by $\boldsymbol{\mu}_0$, $\boldsymbol{\nu}_0$ and the symmetric group $\mathcal{S}_n$, for example for the canonical element in Figure 2.2 we have $\mathfrak{a} = \boldsymbol{\mu}_0\boldsymbol{\mu}_1\boldsymbol{\mu}_3\boldsymbol{\nu}_{10}$. By Lemma 2.2.2 each element of $\mathcal{P}_n^d$ is $\mathcal{J}$-equivalent to a canonical element, and in the proof of Lemma 2.2.2 we only used canonical elements and permutations to convert arbitrary element to a canonical element. □

2.3 Graded posets related to the set of all canonical elements of $\mathcal{P}_n^d$.

In this section we define combinatorially a family of posets. The motivation for defining and studying these posts manifests itself in Theorem 2.4.5.

Consider the free $\mathbb{Z}$-module $\mathbb{Z}^d$, we define the following two maps

$$\begin{aligned}\mathsf{ht} &: \mathbb{Z}^d \to \mathbb{Z}\\ (v_1, v_2, \ldots, v_d) &\mapsto (1, 1, \ldots, 1)_{1\times d}(v_1, v_2, \ldots, v_d)^{tr}\end{aligned} \tag{2.3}$$

$$\begin{aligned}\mathsf{wt} &: \mathbb{Z}^d \to \mathbb{Z}\\ (v_1, v_2, \ldots, v_d) &\mapsto (1, 2, \ldots, d)_{1\times d}(v_1, v_2, \ldots, v_d)^{tr}\end{aligned} \tag{2.4}$$

call **ht** height function and **wt** weight function. Both of these maps are surjective $\mathbb{Z}$-module homomorphisms.

For a finite poset $(P, \leq)$ and $x, y \in P$, we say x is *covered* by y (or y *covers* x) and denote it by $x \lessdot y$ if $x < y$ and there is no $z \in P$ with $x < z < y$. The relation $\lessdot$ is called the *covering relation* of $\leq$, cf. [20] Section 1.14. In this case the relation $\leq$ is the transitive reflective closure of $\lessdot$.

Let $\mathbf{e}_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^d$ be a vector having all components zero except the i^{th} component which is one. Then the set $\{\mathbf{e}_i | \, i = 1, 2, \ldots, d\}$ forms a $\mathbb{Z}$-basis of $\mathbb{Z}^d$, they are called standard basis vector. Note that $\textbf{ht}(\mathbf{e}_i) = 1$ for each i.

Let

$$\Omega_d = \{\mathbf{v} \in \mathbb{Z}^d \,|\, v_i \geq 0 \text{ for all } i \in \underline{d}\},$$

and for a fixed integer h let

$$\Omega_d^{(h)} = \{\mathbf{v} \in \mathbb{Z}^d \,|\, \textbf{ht}(\mathbf{v}) = h\}.$$

For any subset $X \subseteq \Omega_d^{(h)}$ we define the following relation, denote it by $\lessdot_X$, on Ω_d;

for $\mathbf{v}, \mathbf{w} \in \Omega_d$ we have $\mathbf{v} \lessdot_X \mathbf{w}$ if and only if there is $\mathbf{x} \in X$ such that $\mathbf{v} = \mathbf{w} + \mathbf{x}$

Note that from the definition of **ht** map if $\mathbf{w} \lessdot_X \mathbf{v}$ we cannot have $\mathbf{v} \lessdot_X \mathbf{w}$, therefore $\lessdot_X$ is antisymmetric.

Let $\leq_X$ be the transitive reflexive closure of $\lessdot_X$. Then $(\Omega_d, \leq_X)$ is a poset with the covering relation $\lessdot_X$.

Lemma 2.3.1. *Let $X, X' \subseteq \Omega_d^{(h)}$. If $X \neq X'$ then $\leq_X \neq \leq_{X'}$.*

Proof. Without loss of generality we may assume $X \not\subseteq X'$. Suppose we have $\leq_X = \leq_{X'}$. Let $x \in X$ and $x \notin X'$. Then for any $v \in \Omega_d$ we have $v + x \lessdot_X v$. This implies that $v + x \lessdot_{X'} v$ and we obtain $x \in X'$, which is a contradiction. The result follows. □

If $X \subseteq \Omega_d^{(-1)}$ then for any $\mathbf{v}, \mathbf{w} \in \Omega_d$ and $\mathbf{x} \in X$ with $\mathbf{v} \leq_X \mathbf{w}$ we have $\mathsf{ht}(\mathbf{w}) = 1 + \mathsf{ht}(\mathbf{w} + \mathbf{x})$ and $\mathsf{ht}(\mathbf{v}) \leq \mathsf{ht}(\mathbf{w})$. Therefore, $(\Omega_d, \leq_X)$ is a graded poset with the rank function $\mathsf{ht} : \Omega_d \to \mathbb{N}$.

We define an infinite sequence of sets $\{X_d\}_{d=1}^{\infty}$ as follows:

$$X_d := \{\mathbf{e}_k - \mathbf{e}_i - \mathbf{e}_j \mid (i, j, k) \in \underline{d}^3 \text{ such that } d \text{ is a divisor of } k - i - j\}$$

We record the first few values of the above sequence:

$$\begin{aligned} X_1 &= \{(-1)\}; \\ X_2 &= \{(-2, 1), (0, -1)\}; \\ X_3 &= \{(-2, 1, 0), (0, 0, -1), (1, -2, 0), (-1, -1, 1)\}; \\ X_4 &= \{(-2, 1, 0, 0), (0, 0, 0, -1), (0, 1, -2, 0), (-1, -1, 1, 0), (-1, 0, -1, 1), \\ &\quad (1, -1, -1, 0), (0, -2, 0, 1)\}; \\ &\ \vdots \end{aligned}$$

One can use the following lemma as a verification tool while writing the elements of X_d.

Lemma 2.3.2. *We have* $|X_d| = \frac{d(d-1)}{2} + 1$.

Proof. We keep the notation of the definition of X_d. There are $\binom{d}{2}$ ways of choosing an not necessarily different pair i and j from $\underline{d-1}$. If the set $\{i, j\}$ is given then the value of k is forced as follows: If $i + j \leq d$ then since we have $k \in \underline{d}$ we must have $k = i + j$, and if $i + j > d$ then $k = i + j - d$.

We claim different choices of $\{i, j\}$ lead to different elements of X_d. To this end, let $\{i, j\} \neq \{i_1, j_1\}$ then without loss of generality we have the following case;

Case1 If $i + j \leq d$ and $i_1 + j_1 \leq d$ then $\mathbf{e}_{i+j} - \mathbf{e}_i - \mathbf{e}_j \neq \mathbf{e}_{i_1+j_1} - \mathbf{e}_{i_1} - \mathbf{e}_{j_1}$.

Case2 If $i + j > d$ and $i_1 + j_1 \leq d$ then $\mathbf{e}_{i+j-d} - \mathbf{e}_i - \mathbf{e}_j \neq \mathbf{e}_{i_1+j_1} - \mathbf{e}_{i_1} - \mathbf{e}_{j_1}$.

Case3 If $i+j>d$ and $i_1+j_1>d$ then $\mathbf{e}_{i+j-d}-\mathbf{e}_i-\mathbf{e}_j \neq \mathbf{e}_{i_1+j_1-d}-\mathbf{e}_{i_1}-\mathbf{e}_{j_1}$.

Finally, we have $\mathbf{e}_k-\mathbf{e}_i-\mathbf{e}_j=-\mathbf{e}_d$ if $\{i,j\}\cap\underline{d}\neq\phi$. Therefore, $|X_d|=\frac{d(d-1)}{2}+1$. □

We denote by $\mathscr{P}_d$ the poset $(\Omega_d, \leq_{X_d})$, and for any vector $\mathbf{v}\in\Omega_d$ we define $\mathscr{I}(\mathbf{v})$ as follows:

$$\mathscr{I}(\mathbf{v}) := \{\mathbf{w}\in\mathscr{P}_d | \mathbf{w}\leq_{X_d}\mathbf{v}\} \tag{2.5}$$

That is, $\mathscr{I}(\mathbf{v})$ is the principal ideal of $\mathscr{P}_d$ generated by $\mathbf{v}$, for example see Figure 2.3.

For any non-negative integer n we denote by $I_n^{(d)}$ the cardinality of $\mathscr{I}(n\mathbf{e}_1)$, and define $I_{n,h}^{(d)} := |\mathscr{I}(n\mathbf{e}_1)\cap\Omega_d^{(h)}|$, note that $\textsf{ht}(n\mathbf{e}_1)=n$. Therefore, we have

$$I_n^{(d)} = I_{n,0}^{(d)} + I_{n,1}^{(d)} + \cdots + I_{n,n}^{(d)} \tag{2.6}$$

We would like to observe the following structural property of $\mathscr{P}_d$, for $k\in\{1,2,\ldots,d\}$ consider the set

$$\Omega_{d,k} := \{\mathbf{v}\in\Omega_d \mid d \text{ divides } \textsf{wt}(\mathbf{v})-k\}$$

Denote by $\mathscr{P}_{d,k}$ the poset with the underlying set $\Omega_{d,k}$ obtained by restricting the relation $\leq_{X_d}$ to the set $\Omega_{d,k}$. For a non-negative integer h, set $\Omega_{d,k}^{(h)} := \Omega_{d,k}\cap\Omega_d^{(h)}$. Note that if $k\neq k'$ then $\Omega_{d,k}$ and $\Omega_{d,k'}$ are disjoint.

Proposition 2.3.3. *i. We have $\mathscr{P}_d=\mathscr{P}_{d,1}\cup\mathscr{P}_{d,2}\cup\cdots\cup\mathscr{P}_{d,d}$.*

ii. For $k=1,2,\ldots,d$ the poset $\mathscr{P}_{d,k}$ is indecomposable(can not be written as a union of two disjoint subposets).

Proof. i. The claim follows from the Euclid's division Lemma for integers.

ii. We have $\mathbf{e}_j\in\mathscr{P}_{d,k}$ if and only if $j=k$, since $\textsf{wt}(\mathbf{e}_k)=k$. Let $\mathbf{v}\in\mathscr{P}_{d,k}$ such that $\textsf{ht}(\mathbf{v})\geq 2$ we claim $\mathbf{e}_k\leq_{X_d}\mathbf{v}$. To prove the claim, assume $\textsf{ht}(\mathbf{v})\geq 2$. Then either $\mathbf{v}$ has a component greater than or equal to 2, or it has at least two

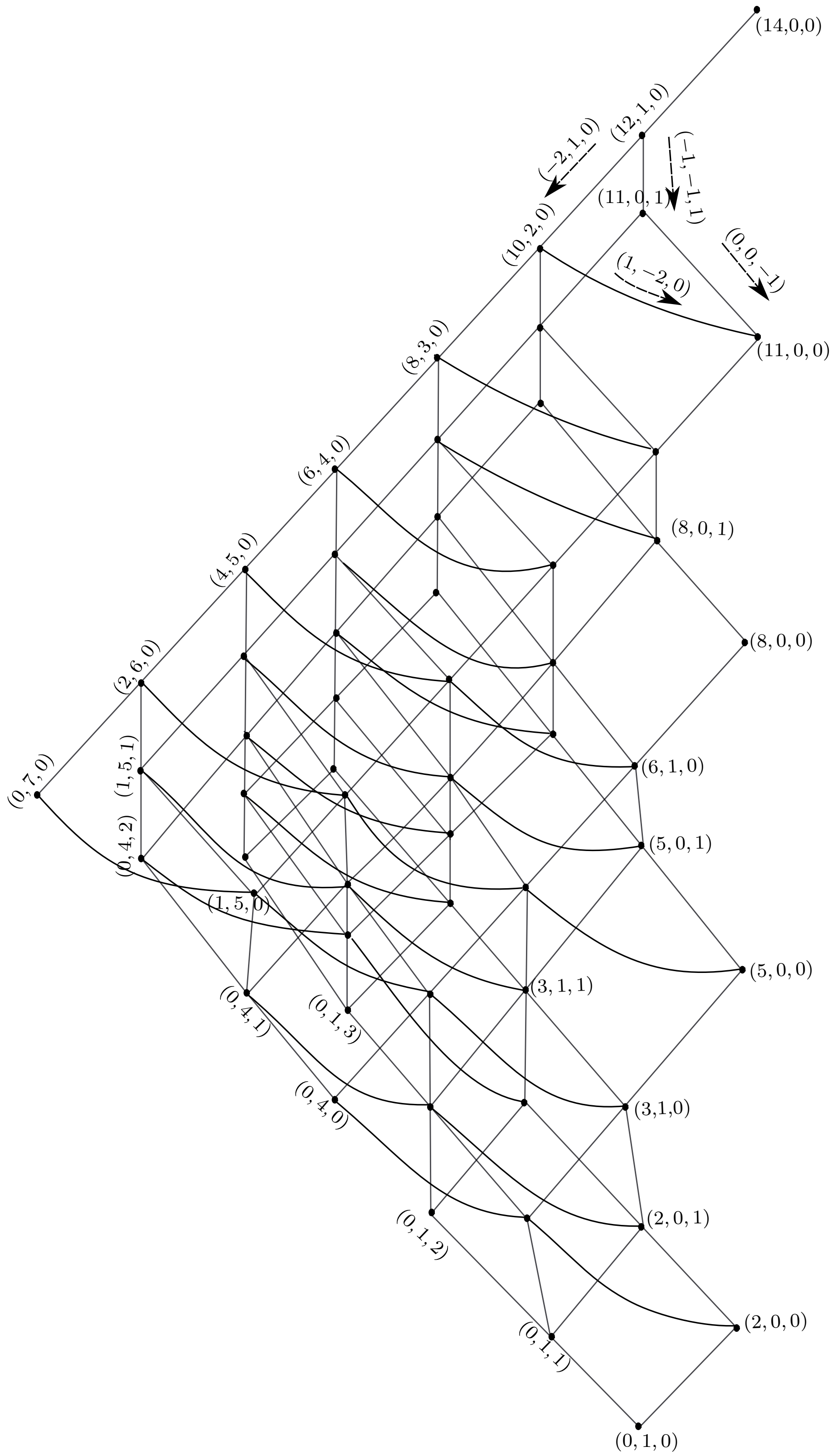


Figure 2.3: Case $d = 3$, $\mathscr{I}(14\boldsymbol{e}_1)$ poset

non- zero components. In either cases there is $\mathbf{x} \in X_d$ such that $\mathbf{v}+\mathbf{x} \in \Omega_d$ and hence $\mathbf{v}+\mathbf{x} \lessdot_{X_d} \mathbf{v}$. Further, for any $\mathbf{x} \in X_d$ we have $\mathbf{wt}(\mathbf{x}) \in \{-d, 0\}$, which implies that $\mathbf{v}+\mathbf{x} \in \mathscr{P}_{d,k}$. Therefore, using induction on the hight of $\mathbf{v}$ implies that $\mathbf{e}_k \leq_{X_d} \mathbf{v}$. Which completes the proof.

□

Note that in the proof of the above result we showed that if k does not divide d then $\mathbf{e}_k$ is the minimum element of the poset $\mathscr{P}_{d,k}$, and $(0, \ldots, 0)$ is the minimum of $\mathscr{P}_{d,d}$.

Proposition 2.3.4. *For any $d \in \mathbb{Z}_{>2}$ and $1 \leq r < d$ the posets $\mathscr{P}_{d,d-r}$ and $\mathscr{P}_{d,r}$ are isomorphic.*

Proof. The symmetric group $\mathcal{S}_d$ acts on $\mathbb{Z}^d$ in the following way: For $\mathbf{v} = (v_1, \ldots, v_{d-1}, v_d) \in \mathbb{Z}^d$ and $\pi \in \mathcal{S}_d$ set $\pi.\mathbf{v} := (v_{\pi(1)}, \ldots, v_{\pi(2)}, \ldots, v_{\pi(d)})$. Let $\pi := (1, d-1)(2, d-2)\ldots(\lceil \frac{d}{2} \rceil - 1, \lceil \frac{d+1}{2} \rceil) \in \mathcal{S}_d$, then for $\mathbf{e}_k - \mathbf{e}_i - \mathbf{e}_j \in X_d$ we have $\pi.(\mathbf{e}_k - \mathbf{e}_i - \mathbf{e}_j) = \mathbf{e}_{d-k} - \mathbf{e}_{d-i} - \mathbf{e}_{d-j} \in X_d$.

Restricting the action of π to the set Ω_d and considering the fact that the set X_d is invariant under this action we obtain a poset automorphism of $\mathscr{P}_d$. Under this automorphism and for $1 \leq r < d$ the vectors $\mathbf{e}_r$ and $\mathbf{e}_{d-r}$ are mapped to each other. The claim follows. □

We record the following lemma which will be used later to obtain a characterisation of the elements of $\mathscr{I}(n\mathbf{e}_1)$ in terms of the weight function.

Lemma 2.3.5. *If $\mathbf{v} \in \mathscr{I}(n\mathbf{e}_1)$ then $\mathbf{wt}(\mathbf{v}) \in \{n, n-d, n-2d, \ldots, n-d\lfloor \frac{n}{d} \rfloor\}$.*

Proof. Keep the notations of the definition of X_d and let $\mathbf{v} \in X_d$ then $\mathbf{wt}(\mathbf{v}) = k - i - j$, which is either 0 or $-d$. We also have $\mathbf{wt}(n\mathbf{e}_1) = n$, now let $\mathbf{v} \in \mathscr{I}(n\mathbf{e}_1)$ hence $\mathbf{w} <_{X_d} n\mathbf{e}_1$, thus there exist vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r \in X_d$ such that $\mathbf{v} = \sum_{l=1}^{r} \mathbf{x}_l + n\mathbf{e}_1$. Therefore, $\mathbf{wt}(\mathbf{v}) \in \{n, n-d, n-2d, \ldots, n-d\lfloor \frac{n}{d} \rfloor\}$. □

2.4 Relation between the set of canonical elements of $\mathcal{P}_n^d$ and $\mathscr{I}(n\mathbf{e}_1)$.

Define a set map

$$\mathbf{\Psi} : \mathcal{P}_n^d \to \Omega_d \tag{2.7}$$

as follows: For an element $\mathfrak{p}$ in $\mathcal{P}_n^d$ with the propagating parts $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$, we set $\mathbf{\Psi}(\mathfrak{p}) = (v_1, v_2, \ldots, v_d)$ where v_r is equal to the cardinality of the set $\{\mathbf{p}_l : |\mathbf{p}_l \cap \underline{n}| = r \mod d\}$ for each $r \in \underline{d}$.

It is evident from the definition of the maps ht and $\mathbf{\Psi}$ that for any element $\mathfrak{p}$ in $\mathcal{P}_n^d$ we have the following equality:

$$\#(\mathfrak{p}) = \mathsf{ht}(\mathbf{\Psi}(\mathfrak{p})) \tag{2.8}$$

We recall that, by$\#(\mathfrak{p})$ we mean the number of propagating parts of $\mathfrak{p}$.

Example 2.4.1. Let $\mathfrak{a}$ and $\mathfrak{b}$ be as given in Figure 2.2. Then $\mathbf{\Psi}(\mathfrak{a}) = (3, 2, 1)$ and $\mathbf{\Psi}(\mathfrak{b}) = (2, 3)$.

Proposition 2.4.2. *We have $\mathbf{\Psi}(\mathcal{P}_n^d) = \mathscr{I}(n\boldsymbol{e}_1)$ and $\mathbf{\Psi}\restriction_{\boldsymbol{\mathcal{C}}}$ is bijection, where $\boldsymbol{\mathcal{C}}$ is the set of all canonical elements of $\mathcal{P}_n^d$.*

Proof. Let $\mathfrak{d} \in \mathcal{P}_n^d$ and $\mathfrak{c}$ be the canonical element $\mathcal{J}$-equivalent to $\mathfrak{d}$. Then by the proof of Lemma 2.2.2 the element $\mathfrak{d}$ has the same number of propagating parts of each type as $\mathfrak{c}$. Therefore $\mathbf{\Psi}(\mathfrak{d}) = \mathbf{\Psi}(\mathfrak{c})$ and hence $\mathbf{\Psi}(J_c) = \mathbf{\Psi}(\mathfrak{c})$ where $J_\mathfrak{c}$ is the $\mathcal{J}$-class containing $\mathfrak{c}$. Hence to prove $\mathbf{\Psi}(\mathcal{P}_n^d) = \mathscr{I}(n\mathbf{e}_1)$ it is enough to show that $\mathbf{\Psi}(\boldsymbol{\mathcal{C}}) = \mathscr{I}(n\mathbf{e}_1)$.

We prove $\mathbf{\Psi}(\boldsymbol{\mathcal{C}}) \subseteq \mathscr{I}(n\mathbf{e}_1)$ by downward induction on the number of propagating parts of canonical elements. The only canonical element with n propagating parts is the identity and $\mathbf{\Psi}(1) = n\mathbf{e}_1$, hence we have the base case. Let $\mathfrak{p} \in \mathcal{P}_n^d$ be canonical and $\#(\mathfrak{p}) = k$, for some $k \leq n-1$. Then $\mathbf{v} = (v_1, v_2, \ldots, v_d) = \mathbf{\Psi}(\mathfrak{p})$ where $\mathbf{v}$ is described in the definition of $\mathbf{\Psi}$. We have the following two cases:

i) If $\mathbf{wt}(\mathbf{v}) \neq n$ then $\mathfrak{p}$ has at least two non-propagating parts one contains $\mathbf{wt}(\mathbf{v})+1$ and the other contains $(\mathbf{wt}(\mathbf{v})+1)'$, call them $\mathbf{p}$ and $\mathbf{p}'$ respectively. Let $\mathfrak{p}' = \mathfrak{p} \setminus \{\mathbf{p}, \mathbf{p}'\} \cup \{\mathbf{p} \cup \mathbf{p}'\}$ then $\mathfrak{p}'$ is canonical element with $k+1$ propagating parts and $\mathbf{\Psi}(\mathfrak{p}') = \mathbf{v} + \mathbf{e}_d$. By induction we have $\mathbf{\Psi}(\mathfrak{p}') \in \mathscr{I}(n\mathbf{e}_1)$, and from the definition of $<_{X_d}$ we obtain $\mathbf{v} \lessdot_{X_d} \mathbf{v} + \mathbf{e}_d$ hence $\mathbf{\Psi}(\mathfrak{p}) = \mathbf{v} \in \mathscr{I}(n\mathbf{e}_1)$.

ii) If $n = \mathbf{wt}(\mathbf{v})$, that is $\mathfrak{p}$ has no non- propagating parts, we let i be minimum with the property $i > 1$ and $v_i \neq 0$. Let $\mathbf{q}_1 = \{v_1+1, (v_1+1)'\}$ and $\mathbf{q}_2 = \mathbf{p} \setminus \mathbf{q}_1$ where $\mathbf{p}$ is the part of $\mathfrak{p}$ containing $n - v_1$, then $\mathfrak{p}' := \mathfrak{p} \setminus \mathbf{p} \cup \{\mathbf{q}_1, \mathbf{q}_2\}$ is canonical element with $k+1$ propagating parts and $\mathbf{\Psi}(\mathfrak{p}') = \mathbf{v} + (\mathbf{e}_1 + \mathbf{e}_{i-1} - \mathbf{e}_i)$. By induction $\mathbf{\Psi}(\mathfrak{p}') \in \mathscr{I}(n\mathbf{e}_1)$, and definition of $\leq_{X_d}$ implies that $\mathbf{\Psi}(\mathfrak{p}) = \mathbf{v} \in \mathscr{I}(n\mathbf{e}_1)$.

To show $\mathscr{I}(n\mathbf{e}_1) \subseteq \mathbf{\Psi}(\boldsymbol{\mathcal{C}})$, let $\mathbf{v} = (v_1, \ldots, v_d) \in \mathscr{I}(n\mathbf{e}_1)$ then by Lemma 2.3.5 we have $\mathbf{wt}(\mathbf{v}) \in \{n, n-d, n-2d, \ldots, n - d\lfloor \frac{n}{d} \rfloor\}$. Therefore, there is a canonical element $\mathfrak{p}$ with v_l propagating parts of size $2l$, for $l \in \underline{d}$, and hence $\mathbf{\Psi}(\mathfrak{p}) = \mathbf{v}$. Finally, the injectivity of $\mathbf{\Psi}\restriction_{\boldsymbol{\mathcal{C}}}$ follows from the definition of canonical elements and the map $\mathbf{\Psi}$. □

Corollary 2.4.3. *Let* $\mathbf{v} \in \Omega_d$ *then* $\mathbf{v} \in \mathscr{I}(n\mathbf{e}_1)$ *if and only if* $\mathbf{wt}(\mathbf{v}) \in \{n, n-d, n-2d, \ldots, n - d\lfloor \frac{n}{d} \rfloor\}$.

Proof. The if part is Lemma 2.3.5, for the only if part let $\mathbf{v} \in \Omega_d$ and $\mathbf{wt}(\mathbf{v}) \in \{n, n-d, n-2d, \ldots, n - d\lfloor \frac{n}{d} \rfloor\}$. Then by the first part of Proposition 2.4.2 there is a canonical element $\mathfrak{p} \in \mathcal{P}_n^d$ such that $\mathbf{v} = \mathbf{\Psi}(\mathfrak{p}) \subseteq \mathscr{I}(n\mathbf{e}_1)$. □

Lemma 2.4.4. *Let* $\mathfrak{p} \in \mathcal{P}_n^d$ *then*

i. If $\boldsymbol{\tau} \in \mathcal{S}_n$ *then* $\mathbf{\Psi}(\boldsymbol{\tau}\mathfrak{p}) = \mathbf{\Psi}(\mathfrak{p}\boldsymbol{\tau}) = \mathbf{\Psi}(\mathfrak{p})$.

ii. For $0 \leq l \leq n-2$, *we have* $\mathbf{\Psi}(\boldsymbol{\mu}_l\mathfrak{p}) \leq_{X_d} \mathbf{\Psi}(\mathfrak{p})$ *and* $\mathbf{\Psi}(\mathfrak{p}\boldsymbol{\mu}_l) \leq_{X_d} \mathbf{\Psi}(\mathfrak{p})$.

iii. For $0 \leq l \leq n-d$, *we have* $\mathbf{\Psi}(\boldsymbol{\nu}_l\mathfrak{p}) \leq_{X_d} \mathbf{\Psi}(\mathfrak{p})$ *and* $\mathbf{\Psi}(\mathfrak{p}\boldsymbol{\nu}_l) \leq_{X_d} \mathbf{\Psi}(\mathfrak{p})$.

Proof. i. Follows from the fact that $\mathfrak{p}\boldsymbol{\tau}\,\mathcal{J}\,\mathfrak{p}$ and $\mathfrak{p}\,\mathcal{J}\,\boldsymbol{\tau}\mathfrak{p}$.

ii. We only prove $\boldsymbol{\Psi}(\boldsymbol{\mu}_l\mathfrak{p}) \leq_{X_d} \boldsymbol{\Psi}(\mathfrak{p})$, one can prove the other case similarly. There are two cases to be considered;

(a) If l and $l+1$ are in the same part of $\mathfrak{p}$, or at least on of them is in a non-propagating part then $\mathfrak{p}\,\mathcal{J}\,\boldsymbol{\mu}_l\mathfrak{p}$, hence $\boldsymbol{\Psi}(\boldsymbol{\mu}_l\mathfrak{p}) = \boldsymbol{\Psi}(\mathfrak{p})$.

(b) If l and $l+1$ are in different propagating parts of $\mathfrak{p}$, namely $\mathbf{p}_l$ and $\mathbf{p}_{l+1}$ respectively. Then $\boldsymbol{\Psi}(\boldsymbol{\mu}_l\mathfrak{p}) = \boldsymbol{\Psi}(\mathfrak{p}) + \mathbf{e}_k - \mathbf{e}_i - \mathbf{e}_j$ where $i = |\mathbf{p}_l \cap \underline{n}|$ (mod d), $j = |\mathbf{p}_{l+1} \cap \underline{n}|$ (mod d), and $\mathbf{e}_k - \mathbf{e}_i - \mathbf{e}_j \in X_d$.

iii. We only prove the first part, one can prove the second part by a similar argument. Let $\mathbf{l} = \{l+1, l+2, \ldots, l+d\}$ we assume $\mathbf{l} \subseteq \mathbf{p}$ for some part $\mathbf{p}$ of $\mathfrak{p}$. If not then $\boldsymbol{\nu}_l\mathfrak{p} = \boldsymbol{\nu}_l\boldsymbol{\mu}_l\boldsymbol{\mu}_{l+1}\ldots\boldsymbol{\mu}_{l+d-2}\mathfrak{p}$ and by *ii* of this Lemma $\boldsymbol{\Psi}(\boldsymbol{\mu}_l\boldsymbol{\mu}_{l+1}\ldots\boldsymbol{\mu}_{l+d-2}\mathfrak{p}) \leq_{X_d} \boldsymbol{\Psi}(\mathfrak{p})$, moreover $\boldsymbol{\mu}_l\boldsymbol{\mu}_{l+1}\ldots\boldsymbol{\mu}_{l+d-2}\mathfrak{p}$ has the desired property. Therefore we may only need to consider the following two cases:

(a) If $\mathbf{p}$ is a propagating part, we then have two more cases;

i. If $|\mathbf{p} \cap \underline{n}| > d$ then $\mathfrak{p}\,\mathcal{J}\,\boldsymbol{\nu}_l\mathfrak{p}$, and hence $\boldsymbol{\Psi}(\boldsymbol{\nu}_l\mathfrak{p}) = \boldsymbol{\Psi}(\mathfrak{p})$.

ii. If $|\mathbf{p} \cap \underline{n}| = d$ then $\boldsymbol{\Psi}(\boldsymbol{\nu}_l\mathfrak{p}) = \boldsymbol{\Psi}(\mathfrak{p}) - \mathbf{e}_d$.

(b) If $\mathbf{p}$ is a non-propagating part then $\mathfrak{p}\,\mathcal{J}\,\boldsymbol{\nu}_l\mathfrak{p}$.

□

We recall from Section 1.3, there is a natural partial order relation on the set $\mathcal{P}_n^d/\mathcal{J}$ induced from the inclusion of two sided ideals of $\mathcal{P}_n^d$, and here we denote this partial order relation by $\preceq_d$.

Theorem 2.4.5. *The induced map* $\boldsymbol{\Psi} : (\mathcal{P}_n^d/\mathcal{J}, \preceq_d) \to (\mathscr{I}(n\boldsymbol{e}_1), \leq_{X_d})$ *is an isomorphism of posets.*

Proof. By Proposition 2.4.2 and Lemma 2.2.2 the map $\boldsymbol{\Psi}$ is a set bijection.

Let $\mathfrak{p}, \mathfrak{q} \in \mathcal{P}_n^d$ such that $\bar{\mathfrak{p}} \preceq_d \bar{\mathfrak{q}}$ then there exist $\mathfrak{r}, \mathfrak{s} \in \mathcal{P}_n^d$ such that $\mathfrak{p} = \mathfrak{r}\mathfrak{q}\mathfrak{s}$. By Corollary 2.2.4 and Lemma 2.4.4 we have $\mathbf{\Psi}(\mathfrak{p}) \leq_{X_d} \mathbf{\Psi}(\mathfrak{q})$.

Let $\mathbf{v}, \mathbf{w} \in \mathscr{I}(n\mathbf{e}_1)$ such that $\mathbf{v} \lessdot_{X_d} \mathbf{w}$ and $\mathfrak{p} \in \mathcal{P}_n^d$ such that $\mathbf{\Psi}(\mathfrak{p}) = \mathbf{w}$. From the definition of $\leq_{X_d}$ there exist $\mathbf{e}_k - \mathbf{e}_i - \mathbf{e}_j \in X_d$ such that $\mathbf{w} + \mathbf{e}_k - \mathbf{e}_i - \mathbf{e}_j = \mathbf{v}$. Consider the element $\mathfrak{q} = \mathfrak{p} \setminus \{\mathbf{p}_i, \mathbf{p}_j\} \cup \{\mathbf{p}_k\}$ where $\mathbf{p}_i$ and $\mathbf{p}_j$ are both propagating parts of $\mathfrak{p}$, and they have the following property: $|\mathbf{p}_i \cap \underline{n}| = i$ and $|\mathbf{p}_j \cap \underline{n}| = j$. Then $\mathfrak{p}\mathfrak{q} = \mathfrak{q}$ hence $\bar{\mathfrak{q}} \preceq_d \bar{\mathfrak{p}}$ and $\mathbf{\Psi}(\mathfrak{q}) = \mathbf{v}$.

Moreover, it is not hard to see that the covering relation of $(\mathcal{P}_n^d/\mathcal{J}, \preceq_d)$ and $(\mathscr{I}(n\mathbf{e}_1), \leq_{X_d})$ match via the map $\mathbf{\Psi}$. □

2.5 Relation between $(\mathscr{I}(n\mathbf{e}_1), \leq_{X_d})$ and integer partition combinatorics.

Let $\Pi_n^{\leq d}$ denote the set of all partitions of n with at most d parts, then define $P_n^{(d)} := |\Pi_n^{\leq d}|$. The generating function of the sequence $P_n^{(d)}$ is given by the following formula, cf. Equation 1.76 of [75],

$$\sum_{n \geq 1} P_n^{(d)} t^n = \prod_{i=1}^{d} \frac{1}{1 - t^i} \tag{2.9}$$

Proposition 2.5.1. *We have* $I_n^{(d)} = P_n^{(d)} + P_{n-d}^{(d)} + P_{n-2d}^{(d)} + \cdots + P_{n-d\lfloor n/d \rfloor}^{(d)}$.

Proof. Let $(\underbrace{d, d, \ldots, d}_{v_d-\text{times}}, \underbrace{d-1, d-1, \ldots, d-1}_{v_{d-1}-\text{times}}, \ldots, \underbrace{1, 1, \ldots, 1}_{v_1-\text{times}})$ be a partition of $n - kd$, for $k = 0, 1, \ldots, \lfloor n/d \rfloor$, then $(v_1, v_2, \ldots, v_d)$ is a an element in $\mathscr{I}(n\mathbf{e}_1)$ with weight $n - kd$. The converse is also true. Therefore, the result follows from the Corollary 2.4.3. □

$d \backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16 ...
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17 ...
2	1	1	3	3	6	6	10	10	15	15	21	21	28	28	36	36	45 ...
3	1	1	2	4	5	7	11	13	17	23	27	33	42	48	57	69	78 ...
4	1	1	2	3	6	7	11	14	21	25	34	41	56	67	83	99	120 ...
⋮																	

Table 2.2: Some values of $I_n^{(d)}, n \leq 16$ and $d \leq 4$

Note that the second row of Table 2.2 is the sequence $A008805(n)$[1]. However, we do not know a general formula for the other rows, $n \geq 4$, for example they do not occur on [1] at the moment.

Corollary 2.5.2. *For $d \geq 1$, we have*

$$\sum_{n \geq 1} I_n^{(d)} t^n = \frac{1}{(1-t^d)(1-t)(1-t^2)(1-t^3)\dots(1-t^d)}.$$

Proof. The proof follows from combining equation 2.9 and Proposition 2.5.1. □

Let Π_n be the set of all partitions of n, and $P_n = |\Pi_n|$. For any two elements $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_l)$ of Π_n we say λ *refines* μ, denote it by $\mu \leq_{\Pi_n} \lambda$, if $k \geq l$ and there exist a set partition $J_1 \cup J_2 \cup \dots \cup J_l$ of $\underline{k}$ such that for each $i \in \underline{l}$ we have $\mu_i = \sum_{j \in J_i} \lambda_j$. This makes the pair $(\Pi_n, \leq_{\Pi_n})$ a poset. Moreover, it is a graded with the rank function $(\lambda_1, \lambda_2, \dots, \lambda_k) \mapsto k$. For example see the Figure 2.4.

The poset $(\Pi_n, \leq_{\Pi_n})$ is originally defined by Birkhoff (see [5]) then later discussed in [7] and studied in detail by G.Ziegler in [83].

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_l)$ be two partitions of n, then we write $\lambda \sim_d \mu$ if $k = l$ and there exist a permutation $\sigma \in \mathcal{S}_k$ such that for each $i \in \underline{k}$ d divides $\lambda_i - \mu_{\sigma(i)}$. The relation $\sim_d$ is an equivalence relation and we denote the $\sim_d$-class of λ by $\bar{\lambda}^{(d)}$. Then the set $\Pi_{n,d} := \Pi_n / \sim_d$ has a partial order relation on it, denote by $\leq_{\sim_d}$, induced from $\leq_{\Pi_n}$ as follows. For $\lambda, \mu \in \Pi_n$ we have $\bar{\lambda}^{(d)} \leq_{\sim_d} \bar{\mu}^{(d)}$ if for all $\lambda' \in \bar{\lambda}^{(d)}$ there exist $\mu' \in \bar{\mu}^{(d)}$ such that $\lambda' \leq_{\Pi_n} \mu'$. This is well-defined and it is easy to check the requirements. Moreover, $(\Pi_{n,d}, \leq_{\sim_d})$ is also a graded poset with the induced rank function on Π_n.

Define the poset $\Pi^*_{n,d}$ as follows: If d does not divide n, set $\Pi^*_{n,d} := \Pi_{n,d}$ with the order $\leq_{\sim_d}$, if d divides n, define $\Pi^*_{n,d}$ as a poset obtained from $(\Pi_{n,d}, \leq_{\sim_d})$ by adding a minimum element, denoted $\varnothing$ (for simplicity, we will keep the notation $\leq_{\sim_d}$ for the partial order on $\Pi^*_{n,d}$). The structure of a graded poset on Π_n induced the structure of a graded poset on $\Pi^*_{n,d}$ by defining the degree of $\varnothing$ to be zero. The class $\overline{(1, 1, \dots, 1)}^{(d)}$ of the partition $(1, 1, \dots, 1)$ is the maximum element in $(\Pi^*_{n,d}, \leq_{\sim_d})$. For example see the Figure 2.4.

Theorem 2.5.3. *The graded posets $(\Pi^*_{n,d}, \leq_{\sim_d})$ and $(\mathscr{I}(n\mathbf{e}_1), \leq_{X_d})$ are isomorphic.*

Proof. For $\lambda \vdash n$ define a vector $(v_1^\lambda, v_2^\lambda, \dots, v_d^\lambda) \in \Omega_d$ such that the component v_i^λ, for $1 \leq i \leq d$, is obtained as follows:

$$v_i^\lambda := |\{j \mid \lambda_j = i \pmod{d}\}|$$

It is evident that $\mathbf{wt}((v_1^\lambda, v_2^\lambda, \dots, v_d^\lambda)) \in \{n, n-d, n-2d, \dots, n-d\lfloor \frac{n}{d} \rfloor\}$ then by Corollary 2.4.3 we have $(v_1^\lambda, v_2^\lambda, \dots, v_d^\lambda) \in \mathscr{I}(n\mathbf{e}_1)$.

Let $\lambda \sim_d \mu$ then for each $1 \leq i \leq d$ we have $v_i^\lambda = |\{j \mid \lambda_j = i \pmod{d}\}| = |\{j \mid \mu_{\sigma(j)} = i \pmod{d} \text{ for some } \sigma \in \mathcal{S}_d\}| = v_i^\mu$. Thus, the assignment is a well-

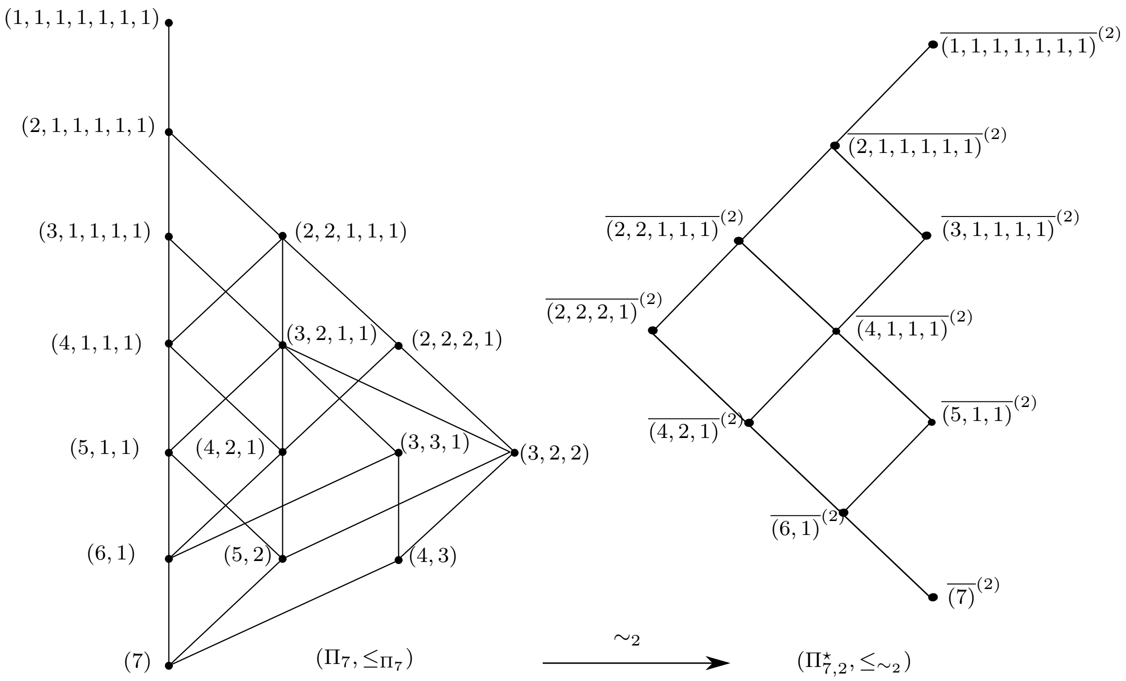


Figure 2.4: Quotient of the poset Π_7 by the equivalence relation $\sim_2$.

defined map from $\Pi_{n,d}$ to $\mathscr{I}(n\mathbf{e}_1)$. If d divides n then extend this map to $\Pi^\star_{n,d}$ by sending $\varnothing$ to the zero vector in $\mathbb{Z}^d$. Call the defined map Φ.

If $\lambda \vdash n$ has k parts then $\mathbf{ht}(\Phi(\lambda)) = k$ and hence Φ preserves the degree.

We claim that Φ is an isomorphism of graded posets. To this end, we start by showing that Φ is a morphism of posets. Any refinement of $\mu \vdash n$ can be obtained as a composition of a sequence of *elementary* refinements, by elementary refinement we mean a partition $\lambda \vdash n$ obtained from μ by refining one part into two smaller parts, in this case $\mu \leq_{\sim_d} \lambda$. Further, if μ has k parts then λ has $k+1$ parts, hence these elementary refinements determine the covering relation of $\leq_{\sim_d}$, denote it by $\lessdot_{\sim_d}$.

Let $\mu \lessdot_{\sim_d} \lambda$ be obtained by refining μ_i to λ_s and λ_t, that is $\mu_i = \lambda_s + \lambda_t$. Then we have

$$\mu_i \pmod d = \lambda_s \pmod d + \lambda_t \pmod d$$

Let $a, b, c \in \{1, 2, \ldots, d\}$ such that $\mu_i = a \pmod d$, $\lambda_s = b \pmod d$ and $\lambda_t = c \pmod d$, then $\mathbf{e}_a - \mathbf{e}_b - \mathbf{e}_c \in X_d$. Therefore, we have $\Phi(\mu) \lessdot_{X_d} \Phi(\lambda)$. It follows that Φ is a morphism of posets.

The injectivity of Φ follows from the definition. We prove the surjectivety of Φ by downward induction on the height of the elements of $\mathscr{I}(n\mathbf{e}_1)$. One can see directly from the definition that $\Phi(\overline{(1, 1, \ldots, 1)}^{(d)}) = n\mathbf{e}_1$ hence the base case is hold.

For the inductive step $h \to h - 1$, let $\mathbf{v} \in \mathscr{I}(n\mathbf{e}_1)$ such that $\textsf{ht}(\mathbf{v}) = h - 1$ then there is $\mathbf{w} \in \mathscr{I}(n\mathbf{e}_1)$ with $\textsf{ht}(\mathbf{w}) = h$ and $\mathbf{v} \lessdot_{X_d} \mathbf{w}$. This implies that there are $i, j, k \in \{1, 2, \ldots, d\}$ such that $\mathbf{w} + \mathbf{e}_k - \mathbf{e}_i - \mathbf{e}_j = \mathbf{v}$. By using the inductive step there is a partition $\lambda \vdash n$ such that $\Phi(\lambda) = \mathbf{w}$. Let λ_s and λ_t be two different parts of λ with $\lambda_s = i \pmod d$ and $\lambda_t = j \pmod d$. Define μ as the partition obtained from λ by uniting λ_s and λ_t. Then $\Phi(\mu) = \mathbf{v}$. Therefore, Φ is a bijection.

The above argument shows that if $\mathbf{v}, \mathbf{w} \in \mathscr{I}(n\mathbf{e}_1)$ with $\mathbf{v} \lessdot_{X_d} \mathbf{w}$ then $\Phi^{-1}(\mathbf{v}) \lessdot_{\sim_d} \Phi^{-1}(\mathbf{w})$. This implies that the covering relations of $(\Pi^*_{n,d}, \leq_{\sim_d})$ and $(\mathscr{I}(n\mathbf{e}_1), \leq_{X_d})$ match under the morphism Φ. This completes the proof. ☐

Corollary 2.5.4. *For $n \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{>0}$, we have $|\Pi^*_{n,d}| = P^{(d)}_n + P^{(d)}_{n-d} + \cdots + P^{(d)}_{n-d\lfloor n/d \rfloor}$.*

Proof. Follows from Theorem 2.5.3 and Proposition 2.5.1. ☐

Proposition 2.5.5. *If $n - h < d$ and $n < 2h$, then $I^{(d)}_{n,h} = P_{n-h}$.*

Proof. We prove the result by establishing a set bijection between Π_{n-h} and $\mathscr{I}(n\mathbf{e}_1) \cap \Omega^{(h)}_d$. For any $\mathbf{v} \in \Omega_d$, set $\alpha(\mathbf{v}) = \textsf{wt}(\mathbf{v}) - \textsf{ht}(\mathbf{v})$. For a given $e_{i+j} - \mathbf{e}_i - \mathbf{e}_j \in X_d$ with $i + j < d$ we have $\alpha(\mathbf{v} + \mathbf{e}_{i+j} - \mathbf{e}_i - \mathbf{e}_j) = \alpha(\mathbf{v}) + 1$.

Fix $n-h<d$ and let $\mathbf{v}\in\mathscr{I}(n\mathbf{e}_1)$ with height h. Then $\mathbf{v}$ can be obtained from $n\mathbf{e}_1$ by adding $n-h$ vectors from X_d with the above property. Therefore, $v_2+2v_3+\cdots+(d-1)v_d=\alpha(\mathbf{v})=n-h$ and hence $(\underbrace{d-1,d-1,\ldots,d-1}_{v_d-\text{times}},\ldots,\underbrace{1,1,\ldots,1}_{v_2-\text{times}})$ is a partition of $n-h$. The injectivity of α is evident.

To show that the map α is surjective, let $v_2+2v_3+\cdots+(d-1)v_d+\cdots=n-h$ for some non- negative integers $v_2,v_3,\ldots$. Then from assumption condition $n<2h$ and Corollary 2.4.3 we have $\mathbf{v}:=(h-\sum_{i=2}^{d}v_i,v_2,\ldots,v_d)\in\mathscr{I}(n\mathbf{e}_1)$, and it is clear that $\mathsf{ht}(\mathbf{v})=h$. Moreover, from the definition of α we have $\alpha(\mathbf{v})=v_2+2v_3+\cdots+(d-1)v_d$. □

Problem 2.5.6. *Find a closed formula for* $I_{n,h}^{(d)}$ *for all* $n,h,$ *and* d.

2.6 Combinatorial results about $\mathscr{P}_d$, for small values of d.

In this section we try to answer the Problem 2.5.6 partially and for small values of d. On the other hand, one might see the results in this section as a demonstration of the level of difficulty of the problem.

2.6.1 The case $d=1$.

This is the easiest case to describe, in fact we have the following isomorphism of posets;

$$\begin{aligned}\mathscr{P}_1&\to(\mathbb{N},\le)\\(i)&\mapsto i\end{aligned}\tag{2.10}$$

Moreover, for $n\in\mathbb{N}$ we have $\mathscr{I}(n\mathbf{e}_1)=\{(0),(1),\ldots,(n)\}$. Therefore, the poset of principal two sided ideals of the partition monoid $\mathcal{P}_n^1$ is isomorphic to the poset

$(\underline{n+1}, \leq)$.

2.6.2 The case $d = 2$.

It follows from Proposition 2.3.3 that the poset $\mathscr{P}_2$ decomposes in to the disjoint union of the indecomposable posets $\mathscr{P}_{2,1}$ and $\mathscr{P}_{2,2}$. Moreover, the following map is isomorphism of posets:

$$\begin{aligned} \mathscr{P}_{2,1} &\to \mathscr{P}_{2,2} \\ \mathbf{v} &\mapsto \mathbf{v} - (1,0) \end{aligned} \tag{2.11}$$

In particular, we have $\mathscr{I}((2n+1)\mathbf{e}_1)$ is isomorphic to $\mathscr{I}((2n)\mathbf{e}_1)$ as a poset. Therefore, combining this isomorphism with Corollary 2.5.2 we obtain

$$I^{(2)}_{2n+1} = I^{(2)}_{2n} = \frac{(n+1)(n+2)}{2}. \tag{2.12}$$

for each $n \in \mathbb{N}$. Moreover, we have $I^{(1)}_{n,h} = 1$ for all $0 \leq h \leq n$.

We also note that the poset $\mathscr{I}(2n\mathbf{e}_1)$ is an ideal in $\mathscr{I}(2(n+1)\mathbf{e}_1)$ for all $n \in \mathbb{N}$, and

$$\bigcup_{n\in\mathbb{N}} \mathscr{I}(2n\mathbf{e}_1) = \mathscr{P}_{2,2}.$$

Proposition 2.6.1. *For* $0 \leq h \leq n$ *we have* $I^{(2)}_{2n,2n-h} = I^{(2)}_{2n,h} = \lceil \frac{h+1}{2} \rceil$ *and* $I^{(2)}_{2n+1,h+1} = I^{(2)}_{2n,h}$.

Proof. The first equation $I^{(2)}_{2n,2n-h} = I^{(2)}_{2n,h}$ is a direct consequence of the following set bijection:

$$\begin{aligned} \mathscr{I}(2n\mathbf{e}_1) \cap \Omega_2^{(h)} &\to \mathscr{I}(2n\mathbf{e}_1) \cap \Omega_2^{(2n-h)} \\ (v_1, v_2) &\mapsto (2n - 2h + v_1, v_2). \end{aligned} \tag{2.13}$$

Moreover, for $0 \leq 2k < n$ the following set map is also a bijection:

$$\begin{aligned} \mathscr{I}(2n\mathbf{e}_1) \cap \Omega_2^{(2k)} &\to \mathscr{I}(2n\mathbf{e}_1) \cap \Omega_2^{(2k+1)} \\ (v_1, v_2) &\mapsto (v_1, v_2 + 1). \end{aligned} \tag{2.14}$$

This implies that $I^{(2)}_{2n,2k} = I^{(2)}_{2n,2k+1}$.

If $(v_1, v_2) \in \mathscr{I}(2n\mathbf{e}_1) \cap \Omega_2^{(2k)}$ then both v_1 and v_2 have to be even non-negative integers. Therefore, $(v_1, v_2) \in \mathscr{I}(2ne_1) \cap \Omega_2^{(2k)}$ if and only if $(v_1, v_2) \in \{(2k, 0), (2k-2, 2), \ldots, (0, 2k)\}$. The first part of the claim follows.

The equality $I^{(2)}_{2n+1,h+1} = I^{(2)}_{2n,h}$ is an immediate consequence of isomorphism between $\mathscr{I}((2n+1)\mathbf{e}_1)$ and $\mathscr{I}((2n)\mathbf{e}_1)$. □

2.6.3 The case $d = 3$.

The case $d = 3$ seems to have a very rich combinatorial structure. We are going to focus on it in more detail and study its relation with other integral sequences.

We try to construct the poset $\mathscr{P}_3$ from $\mathbb{Z}^3$ by restriction instead of constructing it directly from Ω_3. Consider the relation $\lessdot_3'$ on the set $\mathbb{Z}^3$ defined as follows:

$$\text{for } \mathbf{v}, \mathbf{w} \in \mathbb{Z}^3 \text{ we have } \mathbf{v} \lessdot_3' \mathbf{w} \text{ if and only if there is } \mathbf{x} \in X_3 \text{ such that } \mathbf{v} = \mathbf{w} + \mathbf{x}$$

Let $\leq_3'$ denote the partial order on $\mathbb{Z}^3$ induced by this covering relation $\lessdot_3'$. Our main observation here is the following:

Proposition 2.6.2. *The relation $\leq_{X_3}$ coincides with the restriction of the relation $\leq_3'$ to Ω_3.*

Proof. Denote by $\leq_3' \restriction_{\Omega_3}$ the restriction of the relation $\leq_3'$ to Ω_3. It follows from the definition of both of the relations that $\leq_{X_3} \subseteq \leq_3' \restriction_{\Omega_3}$. It remains to show that $\leq_3' \restriction_{\Omega_3} \subseteq \leq_{X_3}$.

Let $\mathbf{v}, \mathbf{w} \in \Omega_3$ be such that $\mathbf{v} \leq_3' \mathbf{w}$. We would like to show that $\mathbf{v} \leq_{X_3} \mathbf{w}$. Suppose the claim is not true. Choose a pair $(\mathbf{v}, \mathbf{w})$ with $\mathbf{v} \leq_3' \mathbf{w}$ and $\mathbf{v} \not\leq_{X_3} \mathbf{w}$ such that $\mathsf{ht}(\mathbf{w} - \mathbf{v}) = k \in \mathbb{Z}_{>0}$ is minimum. As $\mathbf{v} \leq_3' \mathbf{w}$, there is a sequence of elements

$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in X_3$ such that

$$\mathbf{v} = \mathbf{w} + \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k.$$

Let $\mathbf{v}_i := \mathbf{v} - \mathbf{x}_i$ for $i = 1, 2, \ldots, k$. We claim that $\mathbf{v}_i \notin \Omega_3$ for all i. Assume this is not the case and $\mathbf{v}_i \in \Omega_3$ for some i, then we have $\mathbf{v} \leq_{X_3} \mathbf{v}_i$ and $\mathbf{v}_i \leq_3' \mathbf{w}$. This would imply $\mathbf{v}_i \not\leq_{X_3} \mathbf{w}$ which would contradict the assumption that k is minimum. Particularly, for $i = 1, 2, \ldots, k$ we must have $\mathbf{x}_i \neq (0, 0, -1)$ since $\mathbf{y} - (0, 0, -1) \in \Omega_3$ for any $\mathbf{y} \in \Omega_3$.

Next we show that $\mathbf{x}_i \neq (-1, -1, 1)$ for $i = 1, 2, \ldots, k$. Otherwise, without loss of generality we may assume that $\mathbf{x}_k = (-1, -1, 1)$. Since $\mathbf{v}_k \notin \Omega_3$ we obtain $\mathbf{v} = (*, *, 0)$ and $\mathbf{v}_k = (*, *, -1)$. Furthermore, we have

$$\mathbf{v}_k = \mathbf{w} + \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_{k-1}$$

Then $w_3 \geq 0$ since $\mathbf{w} \in \Omega_3$, and this implies that $\mathbf{x}_i = (0, 0, -1)$ for some i. However, we have already proven that $\mathbf{x}_i \neq (0, 0, -1)$ for all i, which is a contradiction.

Therefore each $\mathbf{x}_i$ is equal to either $(-2, 1, 0)$ or $(1, -2, 0)$. Firstly, we may assume that $\mathbf{x}_i = (-2, 1, 0)$ for all i, the case when $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_k = (1, -2, 0)$ can be treated similarly. Then $\mathbf{v} = \mathbf{w} + k(-2, 1, 0)$. For $i = 1, 2, \ldots, k$ we have $\mathbf{w} + i(-2, 1, 0) \in \Omega_3$ since $\mathbf{v}, \mathbf{w} \in \Omega_3$. Hence $\mathbf{v} \leq_{X_3} \mathbf{w}$, a contradiction.

Therefore, each of the equation $\mathbf{x}_i = (1, -2, 0)$ and $\mathbf{x}_i = (-2, 1, 0)$ has at least one solution. This implies $\mathbf{v} - (-2, 1, 0) - (1, -2, 0) = \mathbf{v} + (1, 1, 0) \leq_3' \mathbf{w}$. At the same time, we have

$$\mathbf{v} + (0, 0, 1), \mathbf{v} + (1, 1, 0) \in \Omega_3$$

as $\mathbf{v} \in \Omega_3$ and

$$\mathbf{v} \leq_{X_3} \mathbf{v} + (0, 0, 1) \leq_{X_3} \mathbf{v} + (0, 0, 1) + (1, 1, -1) = \mathbf{v} + (1, 1, 0).$$

This implies $\mathbf{v} + (1, 1, 0) \not\leq_{X_3} \mathbf{w}$ which again contradicts our minimal choice of k. The claim follows. □

We recall that from [1] we have $P_n^{(3)} = A001399(n)$. The following result tells us the values of the sequence $I_{n,h}^{(3)}$ for sufficiently large h.

Proposition 2.6.3. *For $h \geq \lceil \frac{n}{2} \rceil$ we have $I_{n,h}^{(3)} = P_{n-h}^{(3)}$.*

Proof. We prove the result by showing that the following assignment is a set bijection:

$$\begin{aligned}\boldsymbol{\pi} :\ & \Pi_{n-h}^{\leq 3} \to \mathscr{I}(n\mathbf{e}_1) \cap \Omega_3^{(h)} \\ & (a,b,c) \mapsto (n,0,0) + a(-2,1,0) + b(-1,-1,1) + c(0,0,-1)\end{aligned}$$

The condition $h \geq \lceil \frac{n}{2} \rceil$ implies that $2a + b \leq n$. We have $a \geq b \geq c \geq 0$, hence Proposition 2.6.2 implies that $\boldsymbol{\pi}(a,b,c) \in \mathscr{I}(n\mathbf{e}_1) \cap \Omega_3^{(h)}$. Therefore the relation $\boldsymbol{\pi}$ is a set map.

The set $\{(-2,1,0), (-1,-1,1), (0,0,-1)\}$ is $\mathbb{Q}$-linearly independent . Therefore, the map $\boldsymbol{\pi}$ is injective.

Let a, b, c be three non-negative integers such that $a + b + c = n - h$ and

$$\mathbf{v}_{(a,b,c)} := (n,0,0) + a(-2,1,0) + b(-1,-1,1) + c(0,0,-1) \leq_{X_3} (n,0,0) \tag{2.15}$$

Then $\mathbf{v}_{(a,b,c)} \in \Omega_3$ implies that (a,b,c) is a partition of $n - h$.

To prove that $\boldsymbol{\pi}$ is onto we only need to show that any $\mathbf{v} \in \mathscr{I}(n\mathbf{e}_1) \cap \Omega_3^{(h)}$ is of the form of Equation 2.15 for some $(a,b,c) \in \Omega_3$ with $a+b+c = n-h$. Suppose this is not the case, and let $\mathbf{v} \in \mathscr{I}(n\mathbf{e}_1) \cap \Omega_3^{(h)}$ be not of the desired form for the maximum possible h.

By definition of $\leq_{X_3}$ there exist $a, b, c, f \in \mathbb{N}$ such that $a + b + c + f = n - h$ and

$$\mathbf{v} = (n,0,0) + a(-2,1,0) + b(-1,-1,1) + c(0,0,-1) + f(1,-2,0)$$

If $f > 1$, by using the condition $h \geq \lceil \frac{n}{2} \rceil$ and Proposition 2.6.2 we have $\mathbf{v} - (1,-2,0) \leq_{X_3} (n,0,0)$. But $\mathsf{ht}(\mathbf{v} - (1,-2,0)) = h+1$ hence by the maximality of h the vector $\mathbf{v} - (1,-2,0)$ is of the form Equation 2.15 with the desired conditions.

Therefore, we may assume $f = 1$. Then one might apply the above argument and see $\mathbf{v} - (1, -2, 0)$ has the form of Equation 2.15. Since $\mathbf{v} - (1, -2, 0) \in \mathscr{I}(n\mathbf{e}_1)$, we have $a > 0$. Using the equality $(-2, 1, 0) + (1, -2, 0) = (-1, -1, 1) + (0, 0, -1)$ enables us to write $\mathbf{v}$ in the form of Equation 2.15 with the desired conditions. Which is a contradiction. The surjectivity follows. □

The next observation is the first step toward determining the value of the sequence $I^{(3)}_{n,h}$ for relatively small vales of h.

Proposition 2.6.4. *Let $d > 1$, for $h \leq \lceil \frac{n}{d} \rceil$ we have $\Omega^{(h)}_d \cap \mathscr{I}(n\mathbf{e}_1) = \Omega^{(h)}_{d,k}$, where $k \in \{1, \dots, d\}$ such that $n \equiv k \pmod d$. In particular, $I^{(d)}_{n,h} = |\Omega^{(h)}_{d,k}|$.*

Proof. As $n = d(\lceil \frac{n}{d} \rceil - 1) + k$ for some $k \in \{1, \dots, d\}$, we have $\mathscr{I}(n\mathbf{e}_1) \subset \Omega_{d,k}$. Hence, to prove the statement of the proposition we need to show that $\Omega^{(h)}_d \cap \Omega_{d,k} \subseteq \Omega^{(h)}_d \cap \mathscr{I}(n\mathbf{e}_1)$.

Let $\mathbf{v} = (v_1, \dots, v_d) \in \Omega_{d,k}$ with $\mathbf{ht}(\mathbf{v}) = h \leq \lceil \frac{n}{d} \rceil < n$. Then, $\mathbf{wt}(\mathbf{v}) = d.r + k$ for some non negative integer r. This implies $\mathbf{wt}(\mathbf{v}) \in \{k, k+d, k+2d, \dots, n-d, n, n+d, \dots\}$. But we have $\mathbf{wt}(\mathbf{v}) < d\,\mathbf{ht}(\mathbf{v}) \leq d.\lceil \frac{n}{d} \rceil \leq d(\lceil \frac{n}{d} \rceil - 1) + k + d - k = n + d - k$ and hence $\mathbf{wt}(\mathbf{v}) < n + d - k$. Thus, $\mathbf{wt}(\mathbf{v}) = d.r + k \in \{n, n-d, \dots, k\}$. Corollary 2.4.3 implies that $\mathbf{v} \in \mathscr{I}(n\mathbf{e}_1)$. □

It is necessary to mention that the above result is not true for $h > \lceil \frac{n}{d} \rceil$. To see this, let $h = \lceil \frac{n}{d} \rceil + 1$ and $\mathbf{v} = \mathbf{e}_k + (h-1)\mathbf{e}_d$, then $\mathbf{v} \in \Omega^{(h)}_d \cap \Omega_{d,k}$. However, $\mathbf{v} \notin \mathscr{I}(n\mathbf{e}_1)$ as $\mathbf{wt}(\mathbf{v}) = k + (h-1)d = k + d\lceil \frac{n}{d} \rceil > n$.

From Figure 2.5 one can see that for all $h \in \mathbb{N}$ we have

$$|\Omega^{(h)}_3| = \frac{(h+1)(h+2)}{2} \tag{2.16}$$

Proposition 2.6.5.

(*i*) *If 3 does not divide h, then $|\Omega^{(h)}_{3,1}| = |\Omega^{(h)}_{3,2}| = |\Omega^{(h)}_{3,3}|$.*

$$
\begin{array}{cccccccc}
(0,0,h) & (1,0,h-1) & (2,0,h-2) & (3,0,h-3) & (4,0,h-4) & (5,0,h-5) & (6,0,h-6) & \ldots \\
 & (0,1,h-1) & (1,1,h-2) & (2,1,h-3) & (3,1,h-4) & (4,1,h-5) & (5,1,h-6) & \ldots \\
 & & (0,2,h-2) & (1,2,h-3) & (2,2,h-4) & (3,2,h-5) & (4,2,h-6) & \ldots \\
 & & & (0,3,h-3) & (1,3,h-4) & (2,3,h-5) & (3,3,h-6) & \ldots \\
 & & & & (0,4,h-4) & (1,4,h-5) & (2,4,h-6) & \ldots \\
 & & & & & (0,5,h-5) & (1,5,h-6) & \ldots \\
 & & & & & & (0,6,h-6) & \ldots \\
 & & & & & & & \ddots
\end{array}
$$

Figure 2.5: Triangular arrangement of $\Omega_d^{(h)}$

(ii) If 3 *divides* h*, then* $|\Omega_{3,1}^{(h)}| = |\Omega_{3,2}^{(h)}| = |\Omega_{3,3}^{(h)}| - 1$.

Proof. We prove both statements simultaneously by induction on h. We start by arranging the elements of $\Omega_3^{(h)}$ in a triangular array as shown on Figure 2.5.

Writing down the residue modulo 3 of the $\mathbf{wt}(\mathbf{v})$ for each element $\mathbf{v} \in \Omega_3^{(h)}$ in

Figure 2.5 we get

$$
\begin{array}{cccccccc}
0 & 1 & 2 & 0 & 1 & 2 & 0 & \dots \\
 & 2 & 0 & 1 & 2 & 0 & 1 & \dots \\
 & & 1 & 2 & 0 & 1 & 2 & \dots \\
 & & & 0 & 1 & 2 & 0 & \dots \\
 & & & & 2 & 0 & 1 & \dots \\
 & & & & & 1 & 2 & \dots \\
 & & & & & & 0 & \dots \\
 & & & & & & & \ddots
\end{array}
$$

For a fixed h the set $\Omega_3^{(h)}$ corresponds the the first $h+1$ columns from left. The induction step $h \to h+1$ corresponds to adding the next column.

We claim that the residues in each column and each row follow a cyclic order on $0, 1, 2$. To this end, let $(a, b, c) \in \Omega_3^{(h)}$ then one step moving down along the column which (a, b, c) belongs to means going to $(a-1, b+1, c)$, but $\mathbf{wt}\left((a-1, b+1, c)\right) = \mathbf{wt}\left((a, b, c)\right) + 1$. In addition, moving one step directly to right means going to $(a+1, b, c-1)$, in this case we also have $\mathbf{wt}\left((a+1, b, c-1)\right) = \mathbf{wt}\left((a, b, c)\right) + 1$.

Moreover, if we analyse the number of 0's, 1's and 2's in each column we have the following three cases;

Case 1 If a column starts with 0 then it ends with 0, this means it has one 0 more than 1's and 2's.

Case 2 If a column starts with 1 then it ends with 2, this means it has one 1 more than 0's and the number of 1's and 2's is the same.

Case 3 If a column starts with 2 then it ends with 1 , this means it has the same number of of 0's, 1's and 2's.

The claim follows by using induction on h and having the above information in hand. □

For a set X, we denote by δ_X the indicator function of X, that is

$$\delta_X(x) = \begin{cases} 1, & x \in X; \\ 0, & x \notin X. \end{cases} \tag{2.17}$$

Combining Proposition 2.6.5 and Formula 2.16, we obtain the following formula, for $k \in \{1, 2, 3\}$

$$|\Omega_{3,k}^{(h)}| = \frac{(h+1)(h+2) + (6\delta_{3\mathbb{Z}}(k) - 2)\delta_{3\mathbb{Z}}(h)}{6}. \tag{2.18}$$

Corollary 2.6.6. *For $h \leq \lceil \frac{n}{3} \rceil$ we have*

$$I_{n,h}^{(3)} = \frac{(h+1)(h+2) + (6\delta_{3\mathbb{Z}}(n) - 2)\delta_{3\mathbb{Z}}(h)}{6}.$$

Proof. Follows from Equation 2.18 and Proposition 2.6.4. □

If 3 does not divide n then

$$I_{n,h}^{(3)} = \frac{(h+1)(h+2) - 2\delta_{3\mathbb{Z}}(h)}{6}.$$

Which coincides with the sequence $A001840$ from [1].

If 3 divides n then

$$I_{n,h}^{(3)} = \frac{(h+1)(h+2) + 4\delta_{3\mathbb{Z}}(h)}{6}.$$

Which coincides with the sequence $A007997(h+2)$ from [1].

However, it seems that our interpretation of both these sequences does not appear on [1] at the moment.

Finally, we would like to record the following problem to summarise the remaining cases, where $d = 3$.

Problem 2.6.7. *Find a closed formula for $I_{n,h}^{(3)}$ where $\lceil \frac{n}{3} \rceil \leq h \leq \lceil \frac{n}{2} \rceil$.*

2.7 Relation with Hollow Hexagons by using reflection groups.

We begin this section by recalling the definition of affine Weyl groups which will be needed for triangular tiling of the Euclidean space $\mathbb{R}^2$ by equilateral triangles. Tiling $\mathbb{R}^2$ by regular hexagons, under certain conditions, can be considered as a dual graph of triangulation of $\mathbb{R}^2$ by equilateral triangles. This duality enables us formally to define the so called t -hexagons and their hexagonal envelope, and consequently define hollow hexagons.

We follow Chapters 1 and 2 of [40] in our exposition. Let V be a finite dimensional real vector space with the inner product $\langle .,. \rangle$. For a non-zero vector $\alpha \in V$ and $k \in \mathbb{Z}$ the set

$$H_{\alpha,k} := \{\lambda \in V \mid \langle \lambda, \alpha \rangle = k\}$$

is an *affine hyperplane* in V, and if $k = 0$ then $H_{\alpha,k}$ is a *hyperplane*. Moreover, there is an affine transformation, call it *affine reflection* throughout $H_{\alpha,k}$, associated to α and k as follows:

$$\begin{aligned} s_{\alpha,k} &: V \to V \\ \beta &\mapsto \beta - 2\frac{\langle \alpha, \beta \rangle - k}{\langle \alpha, \alpha \rangle}\alpha \end{aligned} \tag{2.19}$$

It is evident that for any $\beta \in V$, $s_{\alpha,k}(\beta) = s_{\alpha,0}(\beta) + k\frac{2\alpha}{\langle \alpha,\alpha \rangle}$.

Geometrically, $s_{\alpha,0}$ fixes the hyperplane $H_{\alpha,0}$ pointwise and sends any vector orthogonal to $H_{\alpha,0}$ to its negative.

A finite subset $\mathbf{R}$ of V is a *root system* if it satisfies the following conditions:

i $\mathbf{R}$ spans V and it does not contain zero.

ii If $\alpha \in \mathbf{R}$ then $\mathbb{R}\alpha \cap \mathbf{R} = \{\alpha, -\alpha\}$.

iii If $\alpha \in \mathbf{R}$ then $s_{\alpha,0}(\mathbf{R}) = \mathbf{R}$.

iv If $\alpha, \beta \in \mathbf{R}$ then $2\frac{\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle} \in \mathbb{Z}$.

The *Weyl group* of $\mathbf{R}$ is the subgroup of $GL(V)$ generated by reflections $s_{\alpha,0}$ for $\alpha \in \mathbf{R}$, denote it by $\mathcal{W}_{\mathbf{R}}$.

A subset $\mathbf{B}$ of $\mathbf{R}$ is called *simple* root system (or basis) if

1. The set $\mathbf{B}$ spans V.

2. Every root can be written as a linear combination of elements of $\mathbf{B}$ such that either all the coefficients are non-negative integers or all the coefficients are non-positive integers.

The Weyl group $\mathcal{W}_{\mathbf{R}}$ is generated $s_{\alpha,0}$ with $\alpha \in \mathbf{B}$.

It is not hard to see that for $\alpha \in \mathbf{B}$ and $k \in \mathbb{Z}$ we have $\lambda \in H_{\alpha,0}$ if and only if $\lambda + \frac{k\alpha}{(\alpha,\alpha)} \in H_{\alpha,k}$.

The *affine Weyl group* of $\mathbf{R}$ is the subgroup of the affine linear group *Aff*(V) generated by the affine reflections $s_{\alpha,k}$ with $\alpha \in \mathbf{R}$ and $k \in \mathbb{Z}$. Denote it by $\widetilde{\mathcal{W}_{\mathbf{R}}}$. Note that $\mathcal{W}_{\mathbf{R}}$ is a subgroup of $\widetilde{\mathcal{W}_{\mathbf{R}}}$ and normalizes the translation group corresponding to the *coroot lattice* $L_{\mathbf{R}}$, defined by;

$$L_{\mathbf{R}} = \sum_{\alpha \in \mathbf{R}} \mathbb{Z} \frac{2\alpha}{\langle \alpha, \alpha \rangle}$$

Therefore, $\widetilde{\mathcal{W}_{\mathbf{R}}}$ is the semidirect product of $\mathcal{W}_{\mathbf{R}}$ acting on the translation group associated to $L_{\mathbf{R}}$ (see p. 88 [40]).

Let $\mathcal{H}$ be the collection of all hyperplanes $H_{\alpha,k}$, for $\alpha \in \mathbf{R}$ and $k \in \mathbb{Z}$. The open connected components of the open set $V^{\triangle} := V \backslash \bigcup_{H \in \mathcal{H}} H$ are called *alcoves*. Further, the group $\widetilde{\mathcal{W}_{\mathbf{R}}}$ acts regularly (simply transitively) on the set $V^{\triangle}$ and permutes the alcoves. For example see 2.6.

Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ be the standard basis for $\mathbb{R}^3$ and let $V = \{(v_1, v_2, v_3) \in \mathbb{R}^3 \mid v_1 + v_2 + v_3 = 0\}$. Let $\mathbf{R} = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ with $\alpha_1 = \boldsymbol{e}_1 - \boldsymbol{e}_2$ and $\alpha_2 = \boldsymbol{e}_2 - \boldsymbol{e}_3$,

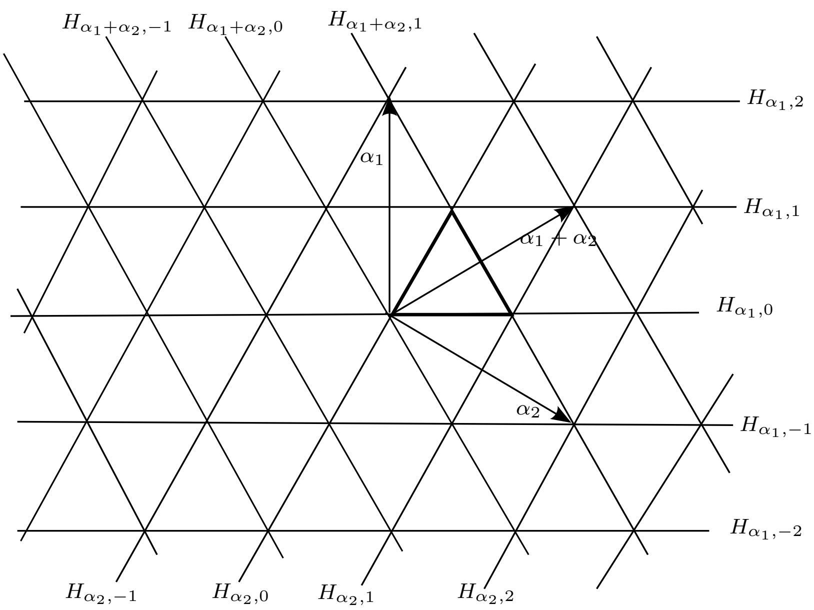


Figure 2.6: The alcove system of $\widetilde{A}_2$.

then $\mathbf{R}$ is a root system in V with a basis $\{\alpha_1, \alpha_2\}$. Therefore, $\mathcal{W}_{\mathbf{R}}$ is generated by the reflections $s_{\alpha_1,0}$ and $s_{\alpha_2,0}$, and $\mathcal{W}_{\mathbf{R}} \simeq \mathcal{S}_3$. Furthermore, $L_{\mathbf{R}} = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ and $\widetilde{\mathcal{W}_{\mathbf{R}}} = L_{\mathbf{R}} \rtimes \mathcal{S}_3$. This group is commonly denoted by $\widetilde{A}_2$ and it has the following presentation by generators and relations;

$$\widetilde{A}_2 = \{s_0, s_1, s_2 \mid s_0^2 = s_1^2 = s_2^2 = (s_0 s_1)^3 = (s_0 s_2)^3 = (s_1 s_2)^3 = 1\}$$

The group $\widetilde{A}_2$ is also known as a triangle group cf. p246 of [62].

We associate to the $\widetilde{A}_2$ alcove system graph the dual graph and called *hexagonal tiling* in the following way; for any alcove we assign a vertex and an edge between two different vertices whenever the closure of their alcoves intersect non-trivially. Finally we remove all the affine hyperplanes see Figure 2.7.

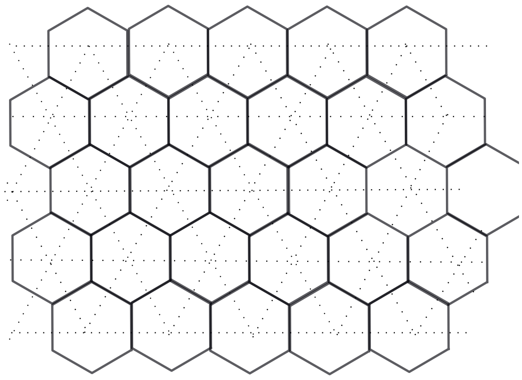


Figure 2.7: The dual graph of the the alcove system of $\widetilde{A}_2$.

2.7.1 T-hexagons and their H-envelope.

We follow [34] Chapter 8 and [17] in the introduction of this subsection. Assume $p \geq 6$, a cycle C on the hexagonal tiling in Figure 2.7 consists of a list $e_1, e_2, \ldots, e_{p-1}$ of different edges such that for each $1 \leq i \leq p-1$ the edges e_i and e_{i+1} have a common vertex, in addition, e_1 and e_p have a common vertex. Let C_1 and C_2 be two cycles such that C_2 is in the interior of the region formed by C_1. Then a *coronoid system* consists of the vertices and edges on C_1 and C_2, and in the interior of C_1, but exterior of C_2. The cycle C_1 is called outer perimeter and C_2 inner perimeter, see Figure 2.8. Furthermore, coronoid systems are mathematical representation of chemical objects called coronoid hydrocarbons.

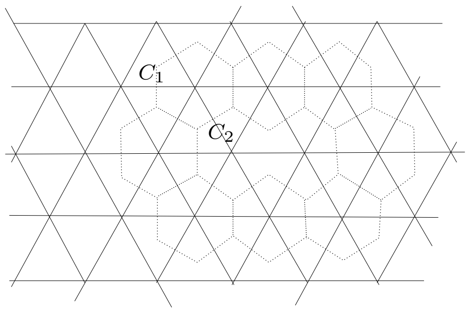


Figure 2.8: An example of a coronoid system.

A *hollow hexagon* is a coronoid system consisting only a chain of hexagons such that its interior cycle is a hexagonal envelope of some t-hexagon. We devote the rest of this chapter to define formally t-hexagons and their hexagonal envelope. Moreover, we establish a precise bijection between the set of iso-classes of t-hexagons and the set $\mathscr{I}(n\mathbf{e}_1)$. This bijection allows one to construct efficiently the set of all hollow hexagons for a given perimeter up to isomorphism.

We call the hyperplanes *tiling lines*, and for $k \in \mathbb{Z}$ the hyperplane $H_{\alpha_1,k}$ (resp. $H_{\alpha_1+\alpha_2,k}$, $H_{\alpha_2,k}$) is called tiling line of *type 1* (resp. *type 2*, *type 3*).

For $i = 1, 2, 3$ a *tiling stripe* of type i is the area between two (not necessarily different) tiling lines of type i, and if these two tiling lines coincide then the corresponding tiling stripe coincides with each one of these two lines. See Figure 2.9 for an example of tiling stripe of type 1.

A t-*hexagon* is defined to be the intersection of three tiling line stripes, one of each type. For example, the polygon shape in figure 2.10 (with the bold boundary lines) is a t-hexagon obtained from the intersection of a tiling stripe of type 1 given by the area between $H_{\alpha_1,0}$ and $H_{\alpha_1,2}$, the tiling stripe of type 2 given by the area between the lines $H_{\alpha_1+\alpha_2,0}$ and $H_{\alpha_1+\alpha_2,4}$ and tiling stripe of type 3 given by the area between the lines $H_{\alpha_2,0}$ and $H_{\alpha_2,3}$.

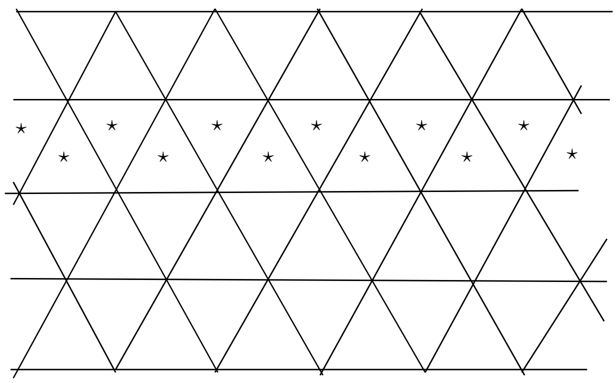


Figure 2.9: Triangles marked with $\star$ form a tiling strip of type 1.

Lemma 2.7.1. *A t-hexagon can only be one of the following*

i. Empty,

ii. A vertex of an alcove,

iii. A bounded line segment of the tiling line,

iv. A convex polygon with three, four, five or six vertices.

Proof. It is not difficult to construct an example for each of the t-hexagons described in $i - iv$. Hence we only need to show that a t-hexagon is convex and it cannot have more than 6 vertices. The intersection of a tiling stripe of type 1 and 2 has 1 vertex if both of the tiling stripes are actually a tiling line, has two vertices and it is a bounded line segment if one of the tiling stripes is obtained from a tiling line, or otherwise it is a parallelogram. The tiling lines of a tiling stripe of type 3 is not parallel to the tiling lines of type 1 and 2 hence it intersects a point in at most one point, a bounded line segment in at most two points and a parallelogram in at most four points. Therefore, the obtained polygon has at most six vertices. It remains to show that a t-hexagon is convex, but this follows from the fact that if two tiling

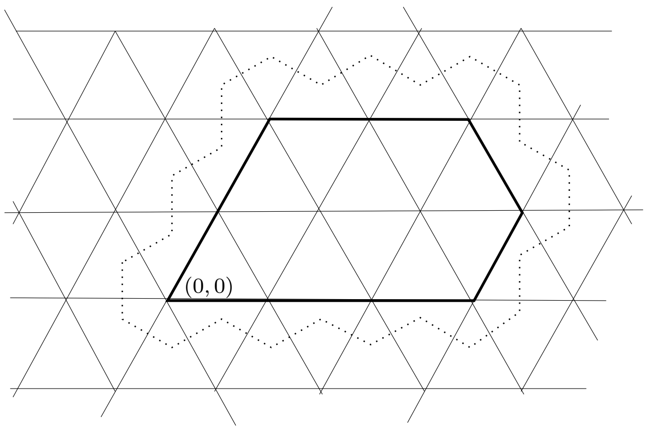


Figure 2.10: The bold convex polygon is a t-hexagon and its hexagonal envelope is the doted line shape.

lines are not parallel the angle of their intersection is either $\pi/3$ or $2\pi/3$. Hence an interior angle of a t-hexagon with more than two points is either $\pi/3$ or $2\pi/3$. $\square$

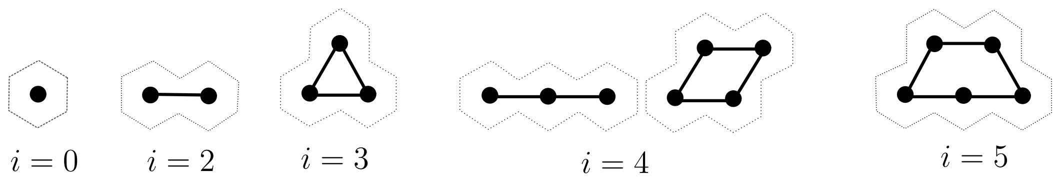
From now on we set the length of the sides of an alcove in the hyperplane arrangement of $\widetilde{A}_2$ to be 1. The *perimeter* of a t-hexagon is defined to be equal to its perimeter when it considered as a polygon. In particular, the perimeter of a vertex is zero and the perimeter of a bounded line segment with length l is equal to $2l$. For example, the perimeter of the t-hexagon in Figure 2.10 is 9.

Let H a t-hexagon and $e(H)$ be the set of all hexagons, in the hexagonal tiling, that intersect H non-trivially. The *hexagonal envelope* of H is the boundary of the region obtained from the closure of the set $e(H)$. We write $E(H)$ to refer to the hexagonal envelope of H and $\#(E(H))$ to refer to the number of vertices of $E(H)$, an example is given in Figure 2.10.

Two t-hexagons are said to be *isomorphic* if they can be obtained from each other by applying the elements of $\widetilde{A}_2$. For $n \in \mathbb{N}$ we denote by T_{2n} the number isomorphism classes of t-hexagons with perimeter $2n$.

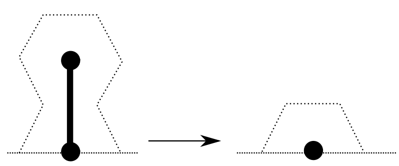
Lemma 2.7.2. *Let T be a t-hexagon of perimeter i for some $i \in \mathbb{N}$. Then the hexagonal envelope of T has $6 + 2i$ vertices.*

Proof. We start by listing all the t-hexagons of perimeter $i = 0, 1, 2, 3, 4, 5$.



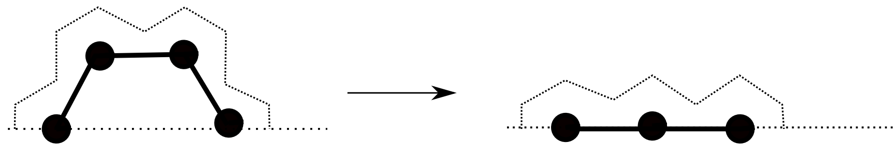
Let T be a t-hexagon with perimeter $i \geq 6$, by definition T is the intersection of three tiling strips, one of each type. At least one of the tiling strips is obtained from two tiling lines $H_{\alpha,k'}$ and $H_{\alpha,k}$ with $k' < k$ and $\alpha \in \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. Let T' be a t-hexagon obtained from T by moving the tiling line $H_{\alpha,k}$ to $H_{\alpha,k-1}$ or $H_{\alpha,k'}$ to $H_{\alpha,k'+1}$. Then, T' is a smaller t-hexagon than T, the perimeter is reduced. Therefore we can use induction on i. In fact we only have the following possible ceases in the process of obtaining T' from T.

Case 1. The t-hexagon T' is obtained from T by projecting a line segment to a vertex as it has be depicted in the following figure:



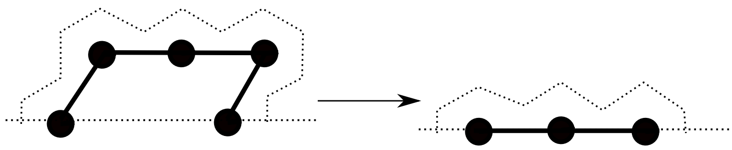
In this case the perimeter of T' is $i - 2$ and $\#(E(T')) = \#(E(T)) - 4$.

Case 2. The t-hexagon T' is obtained from T by projecting a trapezoid segment onto its basis as it has been depicted in the following figure (note that the length of the segment in the figure is 3 but in general it can be arbitrary):



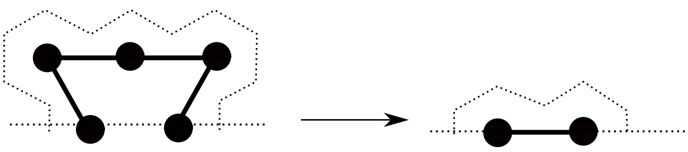
In this case the perimeter of T' is $i-1$ and $\#(E(T')) = \#(E(T)) - 2$.

Case 3. The t-hexagon T' is obtained from T by projecting a trapezoid segment onto its basis as it has been depicted the following figure (note that the length of the segment in the figure is 4 but in general it can be arbitrary):



In this case the perimeter of T' is $i-2$ and $\#(E(T')) = \#(E(T)) - 4$.

Case 4. The t-hexagon T' is obtained from T by projecting a trapezoid segment onto its basis as it has been depicted in the following figure (note that the length of the segment in the figure is 4 but in general it can be arbitrary):



In this case the perimeter of T' is $i-3$ and $\#(E(T')) = \#(E(T)) - 6$.

All the above cases confirm the desired formula, and hence the claim of the lemma follows by induction. □

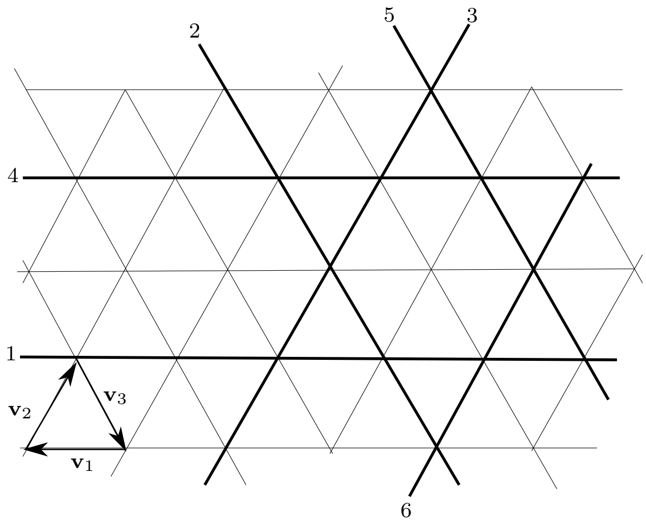


Figure 2.11: Basic vectors and lines .

2.7.2 Characters of t-hexagons

In this subsection we associate a vector with non-negative coordinates to each t-hexagon in a unique way, this allows us to to construct a bijection between a special class of t-hexagons with perimeter $2n$ and the poset $\mathscr{I}(n\boldsymbol{e}_1)$.

Let $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ be the three vectors defined in Figure 2.11, each of them has length 1 and satisfy the condition $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = 0$. Given a t-hexagon H we number the tiling lines of its tiling strips as it has been explained in Figure 2.11. In particular, if two tingling lines coincide we still number them differently according to our numbering scheme. This numbering corresponds to walking along the boundary of H, starting with tiling line numbered by 1 and walking along the path (with the same initial and terminal point)

$$\mathbf{v}_1 \to -\mathbf{v}_3 \to \mathbf{v}_2 \to -\mathbf{v}_1 \to \mathbf{v}_3 \to -\mathbf{v}_2. \tag{2.20}$$

From the definition of t-hexagons, the intersection of a tiling line of a tiling stripe of H with the boundary of H is either a vertex or a side of H. For $i = 1, 2, 3, 4, 5, 6$ let a_i be the length of the intersection of a tiling line i with the boundary of H, we denote by $\chi(H)$ the vector $(a_1, a_2, a_3, a_4, a_5, a_6)$, and it will be called the *character* of H. For example, for t-hexagon obtained from the tiling lines $1, 2, 3, 4, 5, 6$ in Figure 2.11 we have $\chi(H) = (1, 1, 1, 1, 1, 1)$, and for the t-hexagon in Figure 2.10 we have $\chi(H) = (3, 0, 2, 2, 1, 1)$.

A t-hexagon H with character $\chi(H) = (a_1, a_2, a_3, a_4, a_5, a_6)$ is said to be *distinguished* if it satisfies the following conditions;

$$a_1 + a_3 + a_5 \leq a_2 + a_4 + a_6 \quad \text{and} \quad a_1 \geq a_3 \geq a_5 \tag{2.21}$$

A *distinguished character* is the character of a distinguished t-hexagon.

The action of $\widetilde{A}_2$ induced on the set of characters of t-hexagons is a finite group generated by the following three permutations corresponding to the reflection through the three different types of tiling lines :

Reflecting H through a tiling line of type 1 $\chi(H) \mapsto (a_4, a_3, a_2, a_1, a_6, a_5)$

Reflecting H through a tiling line of type 2 $\chi(H) \mapsto (a_6, a_5, a_4, a_3, a_2, a_1)$

Reflecting H through a tiling line of type 3 $\chi(H) \mapsto (a_2, a_1, a_6, a_5, a_4, a_3)$

Using the action of $\widetilde{A}_2$ we can change some t-hexagons to a distinguished t-hexagon. Moreover, each t-hexagon is isomorphic to at most one distinguished t-hexagon up to isomorphism of t-hexagons. For example the t-hexagon in Figure 2.10 is not distinguished and it is not isomorphic to any distinguished t-hexagon, while the t-hexagon in Figure 2.11 is distinguished.

Lemma 2.7.3. *An element* $(a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{N}^6$ *is a character of a t-hexagon if and only if*

$$(a_1 - a_4)\mathbf{v}_1 + (a_3 - a_6)\mathbf{v}_2 + (a_5 - a_2)\mathbf{v}_3 = 0 \tag{2.22}$$

Proof. First note that from $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = 0$, and linear independence of $\mathbf{v}_1$ and $\mathbf{v}_2$ the equation 2.22 is equivalent to

$$a_1 + a_2 = a_4 + a_5 \quad \text{and} \quad a_2 + a_3 = a_5 + a_6 \tag{2.23}$$

The *only if* part follows from the definition of the character of a t-hexagon. To prove the *if* part, assume $(a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{N}^6$ satisfies the equation 2.22. Consider the a t-hexagon H with the tiling stripe of type 1 obtained from the lines $H_{\alpha_1,0}$ and $H_{\alpha_1,a_2+a_3} = H_{\alpha_1,a_5+a_6}$, the tiling stripe of type 2 obtained from the lines $H_{\alpha_1+\alpha_2,0}$ and $H_{\alpha_1+\alpha_2,a_1+a_6}$, and the tiling stripe type 3 obtained from the lines $H_{\alpha_2,-a_2}$ and H_{α_2,a_1}. Then by construction $\chi(H) = (a_1, a_2, a_3, a_4, a_5, a_6)$. □

Lemma 2.7.4. *Let H be a distinguished t-hexagon with $\chi(H) = (a_1, a_2, a_3, a_4, a_5, a_6)$ then*

$$a_1 \leq a_4, \qquad a_5 \leq a_2 \quad \textit{and} \quad a_3 \leq a_6 \tag{2.24}$$

Proof. From equation 2.23 we have $a_1 - a_4 = a_5 - a_2 = a_3 - a_6$, the statement of the lemma follows by substituting this equality in the first part of the equation 2.21 in the three different ways. □

2.7.3 Elementary operations on distinguished t-hexagons.

In this subsection we define some operations on the distinguished t-hexagons which allows us to change a distinguished t-hexagon to another distinguished t-hexagon.

Let H be a distinguished t-hexagon with $\chi(H) = (a_1, a_2, a_3, a_4, a_5, a_6)$, we define four operations on $\chi(H)$ and call them *elementary operations* as follows;

Operation Φ. If $a_1 - a_3 \geq 2$ then by using Lemma 2.7.4 the following vector

$$(a_1 - 1, a_2, a_3 + 1, a_4 - 1, a_5, a_6 + 1)$$

is in $\mathbb{N}^6$ and satisfies the equations 2.21 and 2.23. Therefore, it is the character of a unique distinguished t-hexagon, and it well be denoted by $\Phi(H)$.

Operation Ψ. If $a_1 > a_3 > a_5$ then by using Lemma 2.7.4 the following vector

$$(a_1 - 1, a_2 + 1, a_3, a_4 - 1, a_5 + 1, a_6)$$

is in $\mathbb{N}^6$ and satisfies the equations 2.21 and 2.23. Therefore, it is the character of a unique distinguished t-hexagon, and it well be denoted by $\Psi(H)$.

Operation Θ. If $a_5 > 0$ then by using Lemma 2.7.4 the following vector

$$(a_1 - 1, a_2 + 1, a_3 - 1, a_4 + 1, a_5 - 1, a_6 + 1)$$

is in $\mathbb{N}^6$ and satisfies the equations 2.21 and 2.23. Therefore, it is the character of a unique distinguished t-hexagon, and it well be denoted by $\Theta(H)$.

Operation Λ. If $a_3 - a_5 \geq 2$ then by using Lemma 2.7.4 the following vector

$$(a_1 - 1, a_2 + 2, a_3 - 2, a_4 + 1, a_5, a_6)$$

is in $\mathbb{N}^6$ and satisfies the equations 2.21 and 2.23. Therefore, it is the character of a unique distinguished t-hexagon, and it well be denoted by $\Lambda(H)$.

It is not hard to see that all the assignments Φ, Ψ, Θ, and Λ are set maps when they are restricted to their domains. Moreover, they do not change the perimeter of the distinguished t-hexagons.

For a distinguished t-hexagon H with $\chi(H) = (a_1, a_2, a_3, a_4, a_5, a_6)$ we define $\mathbf{sign}(H) := (a_1 - a_3, a_3 - a_5, a_5) \in \mathbb{N}^3$ and call it *signature* of H. For example, the t-hexagon in Figure 2.11 has a signature $(0, 0, 1)$. Moreover, it is not hard to see that *sign* is a map from the set of all distinguished t-hexagons to $\mathbb{N}^3$.

For any distinguished t-hexagon H, directly from the definition of signature and the maps $\Phi, \Psi, \Theta, \Lambda$ we have;

$$\begin{aligned}
\mathbf{sign}(\Phi(H)) &= \mathbf{sign}(H) + (-2,1,0) \text{ whenever } \Phi(H) \text{ is defined,}\\
\mathbf{sign}(\Psi(H)) &= \mathbf{sign}(H) + (-1,-1,1) \text{ whenever } \Psi(H) \text{ is defined,}\\
\mathbf{sign}(\Theta(H)) &= \mathbf{sign}(H) + (0,0,-1) \text{ whenever } \Theta(H \text{ is defined,}\\
\mathbf{sign}(\Lambda(H)) &= \mathbf{sign}(H) + (1,-2,0) \text{ whenever } \Lambda(H) \text{ is defined.}
\end{aligned} \tag{2.25}$$

We define the *defect* of H to be;

$$\mathbf{def}(H) := a_2 + a_4 + a_6 - a_1 - a_3 - a_5$$

Theorem 2.7.5. *For $n \in \mathbb{N}$, the map $H \mapsto \mathbf{sign}(H)$ induces a set bijection between the set of isomorphism classes of distinguished t-hexagons of perimeter $2n$ and the underlying set of the poset $(\mathscr{I}(n\boldsymbol{e}_1), \leq_{X_3})$.*

Proof. Let H be a distinguished t-hexagon with character $(n,0,0,n,0,0)$, that is a horizontal line segment with perimeter $2n$. Then the signature of H is $n\boldsymbol{e}_1$. Let A_{2n} be the set of all t-hexagons obtained by applying a sequence of operations Φ, Ψ, Θ and Λ to H whenever possible. Then each element of A_{2n} has perimeter $2n$. Moreover, from the definition of $\mathscr{I}(n\boldsymbol{e}_1)$ and 2.25 we have $\mathbf{sign}(A_{2n}) = \mathscr{I}(n\boldsymbol{e}_1)$.

We claim that A_{2n} is the set of all distinguished t-hexagons of perimeter $2n$. If the claim is true, it implies that the image of **sign** is $\mathscr{I}(n\boldsymbol{e}_1)$ and hence **sign** is surjective. To prove the claim, let K be a distinguished t- hexagon with character $(a_1, a_2, a_3, a_4, a_5, a_6)$. Assume $a_5 > 0$, then by Lemma 2.7.4 we have $a_2 > 0$ and hence the vector

$$(a_1 + 1, a_2 - 1, a_3, a_4 + 1, a_5 - 1, a_6)$$

is in $\mathbb{N}^6$; moreover, it satisfies the equations 2.21 and 2.23. Thus it is the character of a unique distinguished t-hexagon, say K' up to isomorphism. We have $\Psi(K') = K$ and the fifth coordinate of K' is smaller than the fifth coordinate of K. Therefore by repeating the above sequence of Ψ's, the t- hexagon K can obtained from another distinguished t-hexagon K'' which has zero in its fifth coordinate.

Let K be a distinguished t-hexagon with character $(a_1, a_2, a_3, a_4, a_5, a_6)$, this time we assume $a_3 > 0$. Then by an argument similar to the case when $a_5 > 0$ there is a distinguished t-hexagon K'' which has zero in its third coordinate, and $\Phi^k(K'') = K$ for some positive integer k.

Now let K be a distinguished t-hexagon with character $(a_1, a_2, 0, a_4, 0, a_6)$. Then by using Lemma 2.24 we have $a_4 \geq a_1$. If $a_1 = a_4$ then by substituting it in equation 2.23 we obtain $a_2 = a_6 = 0$ and hence $K = H$. If $a_4 > a_1$ then the vector

$$(a_1 + 1, a_2 - 1, a_3 + 1, a_4 - 1, a_5 + 1, a_6 - 1)$$

is in $\mathbb{N}^6$; moreover, it satisfies the equations 2.21 and 2.23. Thus it is the character of a unique distinguished t-hexagon, say K'. We have $\Theta(K') = K$ and $\mathbf{def}(K') < \mathbf{def}(K)$.

Therefore, by using induction on the defect of K and the previous steps we obtain that any distinguished t-hexagon of perimeter $2n$ is obtained from a distinguished t-hexagon having character of the form $(a_1, 0, 0, a_4, 0, 0)$. But equation 2.23 implies that $a_1 = a_4 = n$ and consequently the claim follows.

It remains to show that **sign** is injective. To this end, let K be a distinguished t-hexagon with $\mathbf{sign}(K) = (x, y, z) \in \mathscr{I}(n\boldsymbol{e}_1)$. Then $\chi(K) = (x + y + z, a_2, y + z, a_4, z, a_6)$ for some non-negative integers a_2, a_4 and a_6. By using 2.23 and the fact that perimeter of K is $2n$ we have the following system of equations

$$x + y + z - a_4 = z - a_2 = y + z - a_6$$

$$x + y + z + a_2 + y + z + a_4 + z + a_6 = 2n$$

Which has a unique solution in terms of x, y, z and n, it is given by $a_2 = z + 2k, a_4 = n - k - 2z - y$ and $a_6 = y + z + 2k$ where $3k = n - x - 2y - 3z$. Thus the character of K is uniquely determined and hence **sign** is injective. This completes the proof. $\square$

Corollary 2.7.6. *For a non-negative integer n we have $T_{2n} = I_n^{(3)}$.*

We recall the following equations from page 85 [16].

$$
\begin{aligned}
T_{6i} &= \frac{1}{8}\left((i+1)(2i^2+i+1) - \frac{1}{2}(1+(-1)^i)\right)\\
T_{6i+2} &= \frac{1}{8}\left((i+1)(2i^2+3i-1) + \frac{1}{2}(1+(-1)^i)\right)\\
T_{6i+4} &= \frac{1}{8}\left((i+1)(2i^2+5i+1) - \frac{1}{2}(1+(-1)^i)\right)
\end{aligned}
\tag{2.26}
$$

Corollary 2.7.7. *For $i \in \mathbb{Z}_{>0}$ we have:*

$$
\begin{aligned}
I_{3i}^{(3)} &= \frac{1}{8}\left((i+1)(2i^2+i+1) - \frac{1}{2}(1+(-1)^i)\right)\\
I_{3i+1}^{(3)} &= \frac{1}{8}\left((i+1)(2i^2+3i-1) + \frac{1}{2}(1+(-1)^i)\right)\\
I_{3i+2}^{(3)} &= \frac{1}{8}\left((i+1)(2i^2+5i+1) - \frac{1}{2}(1+(-1)^i)\right)
\end{aligned}
$$

Proof. This corollary is a direct consequence of the Corollary 2.7.6 and Equation 2.26. □

Note that the above Corollary can also be obtained from Proposition 2.5.1 and Equation 2.26 .

Chapter 3

Representation theory of the planar d-tonal partition algebra $T_n^d(\delta)$.

Let R be a commutative ring and $\delta \in R$. For each $n, d \in \mathbb{N}$, in this chapter we introduce and study the representation theory of a family of finite dimensional algebras. We call them the planar d-tonal partition algebra, denote it by $T_n^d(\delta)$. The bases of $T_n^d(\delta)$ consists of all planar diagrams of $\mathcal{P}_n^d$.

In particular, if $d = 1$ then the planar 1-tonal partition algebra is isomorphic to the Temperley-Lieb algebra $\mathcal{TL}_{2n}(\delta)$. If $d = 2$ then the Temperley-Lieb algebra is a subalgebra of $T_n^2(\delta)$. Therefore, we may consider the planar d-tonal partition algebras as a generalisation of the Temperley-Lieb algebra.

In the first section we present the definition of $T_n^d(\delta)$ and its underlying monoid $\mathcal{T}_n^d$. In section two we make some observations about the order of the monoid $\mathcal{T}_n^d$. In particular we show that the order of $\mathcal{T}_n^2$ is given by the 2-Fuss-Catalan numbers, which is the same as $A001764$ in [1].

In Section 3 we study the monoid $\mathcal{T}_n^d$. A complete description of the of the $\mathcal{J}$-classes of $\mathcal{T}_n^d$ is obtained. We define a planar version of the canonical elements of $\mathcal{P}_n^d$, and call them planer canonical elements of $\mathcal{T}_n^d$. Precisely, in Theorem 3.3.4 we show that each element of $\mathcal{T}_n^d$ is $\mathcal{J}$-equivalent to a unique canonical element of $\mathcal{T}_n^d$. Consequently, we show that $\mathcal{T}_n^d$ is a regular monoid. Furthermore, a set of generators for $\mathcal{T}_n^d$ is obtained. The obtained set of generators of $\mathcal{T}_n^d$ are all idempotent, therefore, $\mathcal{T}_n^d$ is an example of an idempotent generated monoid.

Regarding the Representation theory of $T_n^d(\delta)$, we answer some fundamental questions. In particular, in Theorem 3.5.7 we show that the simple modules of $T_n^d(\delta)$ are indexed by the set of all compositions of the integers $n, n-d, \ldots, \lfloor n/d \rfloor$ with each part less than or equal to d. Further, in Proposition 3.7.4 the simple modules of $T_n^d(\delta)$ are constructed explicitly. We also show that $T_n^d(\delta)$ is quasi-hereditary and give the restriction rules for the standard modules. Analogues to the other types of diagram algebras we show that $T_n^d(\delta)$ is generically semisimple.

When $d = 2$ in Theorem 3.4.2 it is shown that $T_n^2(\delta)$ is isomorphic to the two colour Fuss-Catalan algebra, defined by Bisch and Jones [6], under some restrictions. We use this isomorphism to pass our knowledge form one side to the other. For example the presentation of $T_n^2(\delta)$ by generators and relations is obtained from the Fuss-Catalan algebras. By using the axiomatic frame work developed in [14] in Theorem 3.9.16 we present a necessary and sufficient condition on $T_n^2(\delta)$ to be semisimple. Consequently, we improve a result of Bisch and Jones regarding the semisimplicity of the two colour Fuss-Catalan algebras.

3.1 Definition of the planar d-tonal partition monoid $\mathcal{T}_n^d$ and the planar d-tonal algebra $T_n^d(\delta)$.

Proposition 3.1.1. *Let $\mathcal{T}_n^d := \mathcal{T}_n^1 \cap \mathcal{P}_n^d$. Then $(\mathcal{T}_n^d, \circ)$ is a submonoid of $\mathfrak{P}_n$.*

Proof. The intersection of two submonoids is again submonoid. □

We call the monoid $\mathcal{T}_n^d$ the *planar d-tonal partition monoid.* It is not hard to show that $\mathcal{TL}_n$ is a submonoid of $\mathcal{T}_n^2$. Therefore one might consider the d-tonal partition monoids as a generalisation of Tepereley-Lieb monoids (later, algebras).

Proposition 3.1.2. *Let R be commutative ring and $\delta \in R$. Let $T_n^d(\delta)$ be a free R-submodule of $\mathfrak{P}_n(\delta)$ with basis given by the set $\mathcal{T}_n^d$. Then $T_n^d(\delta)$ is subalgebra of $\mathfrak{P}_n(\delta)$.*

Proof. It is enough to show that $T_n^d(\delta)$ is closed under multiplication. By equation 1.2 the multiplication of two bases elements of $T_n^d(\delta)$ is scalar multiplication of another basis element, since $(\mathcal{T}_n^d, \circ)$ is a monoid. □

We call the algebra $T_n^d(\delta)$ a *planar d-tonal partition algebra* and it is going to be the main object of our study in this chapter.

3.2 On the order of $\mathcal{T}_n^d$.

Let $(S, \leqq)$ be a finite linear ordered set, a partition $\mathfrak{d} = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_r\}$ of S is called a *non-crossing* partition if for any $w, x, y, z \in S$ with $w < x < y < z$ there does not exist a distinct pair of i and j in $\underline{r}$ such that $w, y \in \mathbf{p}_i$ and $x, z \in \mathbf{p}_j$.

A non-crossing partition $\mathfrak{d}$ is said to be *d-divisible* partition if d divides $|\mathbf{p}_i|$ for all $i \in \underline{r}$.

Example 3.2.1. Let $A = \{1, 2, 3, 4\}$ and $\leq$ be the relation defined by, for $a, b \in A$ we have $a \leq b$ if and only if $b - a$ is non-negative integer. Then, $(A, \leq)$ is a linearly ordered set. The partition $\{\{1, 4\}, \{2, 3\}\}$ is a 2-divisible non-crossing partition of A, whereas $\{\{1, 3\}\ \{2, 4\}\}$ is a crossing partition of A

We denote by $NC^d(n)$ the number different d-divisible non-crossing partitions of $\underline{dn}$, here we consider $\underline{dn}$ as a linear ordered set with the usual order of natural numbers. In [27] it has been shown that

$$CN^d(n) = \frac{\binom{(d+1)n}{n}}{dn + 1} \tag{3.1}$$

The numbers $CN^d(n)$ are known as *d-Fuss-Catalan* numbers, see [31] p.347.

The sequence $CN^2(n)$ is the same as $A001764$ in [1] and the first few values of this is given in the Table 3.1, where $d = 2$.

The sequence $CN^2(n)$ is also known as ternary numbers and it is shown to have numerous interesting properties, see [2, 11, 21, 65].

Proposition 3.2.2. *Let E_n be the set of all 2-divisible non-crossing partitions of $(\underline{n} \cup \underline{n}', \leqslant)$ (the order $\leqslant$ is defined in Section 2.2).Then $\mathcal{T}_n^2 = E_n$ as a set. In particular, $|\mathcal{T}_n^2| = CN^2(n)$.*

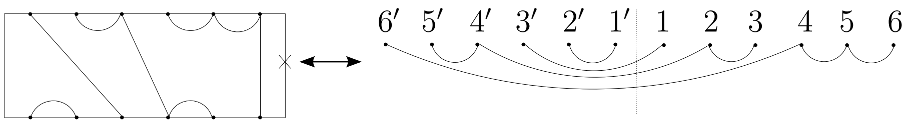


Figure 3.1: An example describing the bijection in the Proposition 3.2.2, with $n = 6$.

Proof. Let $\mathfrak{d} = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_r\} \in \mathcal{T}_n^2$, then for any $i \in \underline{r}$ there exist an $l \in \mathbb{Z}$ such that $|\mathbf{p}_i \cap \underline{n}| - |\mathbf{p}_i \cap \underline{n}'| = 2l$. Therefore, $|\mathbf{p}_i \cap \underline{n}|$ and $|\mathbf{p}_i \cap \underline{n}'|$ are both even or both odd, and hence $|\mathbf{p}_i|$ is also even. Let $w, x, y, z \in \underline{n} \cup \underline{n}'$ such that $w < x < y < z$

and w, y are in the same part, say $\mathbf{p}_j$ where $j \in \underline{r}$. If x and z are in a part $\mathbf{p}_k$, for the unique $k \in \underline{r}$, then we must have $k = j$ otherwise it is not possible, by Jordan curve theorem, in any diagram representing $\mathfrak{d}$ to connect x and z by an edge without crossing the edge connecting w and y. Hence $\mathfrak{d} \in E_n$. For example see Figure 3.1.

Conversely, given $\mathfrak{d} = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_r\} \in E_n$. Let $\mathbf{p}_i = \{x_1, x_2, \ldots, x_k\}, \mathbf{p}_j = \{y_1, y_2, \ldots, y_l\} \in \mathfrak{d}$ be two different parts such that $x_p < x_{p+1}$ and $y_q < y_{q+1}$, for every $p \in \underline{k-1}$ and $q \in \underline{l-1}$. Without loss of generality we have following two cases

i. We have $x_k < y_1$.

ii. There is $p \in \underline{k-1}$ such that $x_p < y_1 < \cdots < y_l < x_{p+1}$.

In both cases in there is at least a diagrams representing $\mathfrak{d}$ in which the parts $\mathbf{p}_i$ and $\mathbf{p}_j$ do not intersect for all $i \neq j$, hence $\mathfrak{d}$ is planar. It remains to show that for any $i \in \underline{r}$, $|\mathbf{p}_i \cap \underline{n}| - |\mathbf{p}_i \cap \underline{n}'|$ is even, but this follows from the fact that $|\mathbf{p}_i| = |\mathbf{p}_i \cap \underline{n}| + |\mathbf{p}_i \cap \underline{n}'|$ is even. For example see Figure 3.1. □

In the Table 3.1 some vales of the two parameter sequence $|\mathcal{T}_n^d|$ is given. Note that the numbers in the first row are the even Catalan numbers, that is $A000108(2n) = CN^1(2n)$ in [1]. However, when $d > 2$ it seems that the sequence $|\mathcal{T}_n^d|$ is a new sequences, for example they have no match on [1] at the moment.

3.3 The $\mathcal{J}$-classes of $\mathcal{T}_n^d$.

In this section we study the monoid $\mathcal{T}_n^d$. We show that is $\mathcal{T}_n^d$ regular and give a unique *canonical* representative for each $\mathcal{J}$-class. Furthermore, a generating set to $\mathcal{T}_n^d$ is obtained.

Definition 3.3.1. *An element in* $\mathcal{T}_n^d$ *will be called planar canonical if it is of the form* $(\bigotimes_{l=1}^r \mathfrak{m}_{i_l}) \otimes \mathfrak{u}_d^{\otimes k}$ *such that* $1 \leq i_l \leq d$ *and* $kd + \sum_{l=1}^r i_l = n$.

d \ n	0	1	2	3	4	5	6	7	...
1	1	2	14	132	1430	16796	208012	2674440	...
2	1	1	3	12	55	273	1428	7752	...
3	1	1	2	5	16	54	186	689	...
4	1	1	2	4	9	24	70	202	...
$\vdots$									
$d \to \infty$	1	1	2	4	8	16	32	64	...

Table 3.1: Cardinality of $\mathcal{T}_n^d$, for some small values of d and n.

Every canonical element in $\mathcal{P}_n^d$ is planar canonical, but the converse is not true. For example, $\{\{1,1'\},\{2,3,2',3'\}\}$ is a planar canonical element in $\mathcal{T}_3^2 \subset \mathcal{P}_3^2$ but it is not a canonical element in $\mathcal{P}_3^2$.

Lemma 3.3.2. *For any element $\mathfrak{d} \in \mathcal{T}_n^d$ there is a planar canonical element $\mathfrak{e} \in \mathcal{T}_n^d$ such that $\mathfrak{d}\mathcal{J}\mathfrak{e}$.*

Proof. First we claim $\mathfrak{d}$ can be factorised $\mathfrak{d} = \mathfrak{t}\mathfrak{d}'\mathfrak{t}'$, for some $\mathfrak{t}, \mathfrak{d}', \mathfrak{t}' \in \mathcal{T}_n^d$ with $\mathfrak{t}, \mathfrak{t}' \neq \mathfrak{d}$ and $\mathfrak{d}'$ is constrained as follows. If $\mathbf{p} \in \mathfrak{d}'$ is a northern non- propagating part then given any positive integer $1 \leq k \leq \min(\mathbf{p}) - 1$ we have $\min(\mathbf{p}) - k$ and $\max(\mathbf{p}) + 1$ are not in the same part. Similarly for the southern non-propagating part. To prove this claim, let $\mathbf{p} \in \mathfrak{d}$ be a northern non-propagating part such that $\min(\mathbf{p}) - k$ and $\max(\mathbf{p}) + 1$ are in the same part, for some positive integer k and there is no other part $\mathbf{q} \in \mathfrak{d}$ with $\max(\mathbf{q}) < \max(\mathbf{p})$ and $\min(\mathbf{p}) < \min(\mathbf{q})$.

We set $\mathfrak{d}''$ to be the same as $\mathfrak{d}$ except the non-propagating part $\mathbf{p}$ is shifted to right one vertex and the vertex $\max(\mathbf{p})+1$ is shifted to left $|\mathbf{p}|$ vertices, this means $\mathfrak{d} = (\mathfrak{m}_1^{\otimes(\min(\mathbf{p})-1)} \otimes \mathfrak{u}_{|\mathbf{p}|} \otimes \mathfrak{m}_1^{\otimes(n-\max(\mathbf{p}))})\mathfrak{d}''$. In this way we can move $\mathbf{p}$ to right as many vertices as we would like. Hence, by using the elements defined in Equation 2.2 we can move any northern non-propagating part as many as vertices to right as we need to obtain the desired $\mathfrak{d}'$, similarly for the southern non-propagating parts. Applying this process to all the non-propagating parts we achieve the desired $\mathfrak{d}'$. By a similar argument, but this time moving a non- propagating part to left if it is necessary, we can find $\mathfrak{k}, \mathfrak{k}' \in \mathcal{T}_n^d$ such that $\mathfrak{d}' = \mathfrak{k}\mathfrak{d}\mathfrak{k}'$. For example see Figure 3.2.

Now let $\mathfrak{d}'$ be given as above, suppose there is a part $\mathbf{p} \in \mathfrak{d}'$ with $|\mathbf{p} \cap \underline{n}| > d$. Let $k := \max(\mathbf{p} \cap \underline{n})$ then $\mathfrak{d}' = (\mathfrak{m}_1^{\otimes(k-3)} \otimes \mathfrak{m}_2 \otimes \mathfrak{m}_1^{\otimes(n-k+1)})(\mathfrak{m}_1^{\otimes(k-d)} \otimes \mathfrak{u}_{k-d} \otimes \mathfrak{m}_1^{\otimes(n-k)})\mathfrak{d}'$, and $(\mathfrak{m}_1^{\otimes(k-d)} \otimes \mathfrak{u}_{k-d} \otimes \mathfrak{m}_1^{\otimes(n-k)})\mathfrak{d}'$ is exactly the same as $\mathfrak{d}'$ except the part $\mathbf{p}$ is replaced by two other parts namely $\mathbf{p}^{(1)} := \mathbf{p} \setminus \{k, k-1, \ldots, k-d\}$ and $\{k, k-1, \ldots, k-d\}$, see the right equality of Figure 3.2. Now if $|\mathbf{p}^{(1)} \cap \underline{n}| > d$ we apply the previous step's mechanism, after repeating this process as many times as required, say r times, we may assume $|\mathbf{p}^{(r)} \cap \underline{n}| \leq d$, where $\mathbf{p}^{(r)}$ is obtained from $\mathbf{p}^{(r-1)}$ in the same way as $\mathbf{p}^{(1)}$ obtained from $\mathbf{p}$. We apply a similar technique if $|\mathbf{p} \cap \underline{n}'| > d$. Using this cutting procedure we may assume that $\mathfrak{d} = \mathfrak{r}\mathfrak{f}\mathfrak{r}'$ where $\mathfrak{r}, \mathfrak{r}', \mathfrak{f} \in \mathcal{T}_n^d$ and $\mathfrak{f}$ has the following property; $|\mathbf{f} \cap \underline{n}| \leq d$ and $|\mathbf{f} \cap \underline{n}'| \leq d$ for any $\mathbf{f} \in \mathfrak{f}$. Finally, move all non-propagating parts to right by the moving technique given in the first paragraph. Therefore, there exist $\mathfrak{t}, \mathfrak{t}' \in \mathcal{T}_n^d$ such that $\mathfrak{d} = \mathfrak{t}\mathfrak{e}\mathfrak{t}'$, where $\mathfrak{e}$ is a planar canonical. All the above process can be reversed to obtain $\mathfrak{g}, \mathfrak{g}' \in \mathcal{T}_n^d$ such that $\mathfrak{g}\mathfrak{d}'\mathfrak{g}' = \mathfrak{e}$, hence $\mathfrak{g}\mathfrak{k}\mathfrak{d}\mathfrak{k}'\mathfrak{g}' = \mathfrak{e}$. □

Proposition 3.3.3. *The monoid $\mathcal{T}_n^d$ is generated by identity and the following elements $\{1, \boldsymbol{\mu}_0, \boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_{n-2}, \boldsymbol{\nu}_0, \boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_{n-d}\}$–see Equation 2.2 for the definition of $\boldsymbol{\mu}_i$ and $\boldsymbol{\nu}_j$*

Proof. First note that $\boldsymbol{\mu}_i, \boldsymbol{\nu}_j \in \mathcal{T}_n^d$, for $0 \leq i \leq n-2$ and $0 \leq j \leq n-d$. .

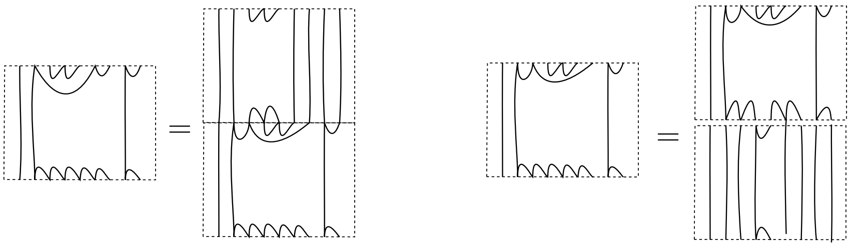


Figure 3.2: The first equality illustrates moving a northern non-propagating part to left, and the second equality explains gluing a non-propagating part to an other part.

Furthermore, any planar canonical element in $\mathcal{T}_n^d$ is a product of some $\boldsymbol{\mu}_i$ and $\boldsymbol{\nu}_j$. In the proof of Lemma 3.3.2 we have seen to change a non-canonical element to a planar canonical, and conversely, it is enough to only use elements which are products of $\boldsymbol{\mu}_i$ and $\boldsymbol{\nu}_j$. $\square$

Theorem 3.3.4. *Each $\mathcal{J}$-class of $\mathcal{T}_n^d$ contains a unique planar canonical element.*

Proof. First note that from Lemma 3.3.2 we have each $\mathcal{J}$-class in $\mathcal{T}_n^d$ contains at least one planar canonical element. Let $\mathfrak{a} := (\bigotimes_{l=1}^{r} \mathfrak{m}_{i_l}) \otimes \mathfrak{u}_d^{\otimes k}$ and $\mathfrak{b} := (\bigotimes_{l=1}^{s} \mathfrak{m}_{j_l}) \otimes \mathfrak{u}_d^{\otimes m}$ be two planar canonical elements in $\mathcal{T}_n^d$ such that $\mathfrak{a}\mathcal{J}\mathfrak{b}$. Then $\mathfrak{a}\mathcal{J}\mathfrak{b}$ in $\mathcal{P}_d(n)$ and by Lemma 2.2.2 we have $k = m$, $r = s$ and $\mathfrak{b} = (\bigotimes_{l=1}^{r} \mathfrak{m}_{i_{\sigma(l)}}) \otimes \mathfrak{u}_d^{\otimes k}$ for some $\sigma \in \mathcal{S}_r$, where $\mathcal{S}_r$ is the permutation group on r letters. But $\mathfrak{a}\mathcal{J}\mathfrak{b}$ also implies that there exist $\mathfrak{x}, \mathfrak{y} \in \mathcal{T}_n^d$ such that $\mathfrak{a} = \mathfrak{x}\mathfrak{b}\mathfrak{y}$, and hence $\mathfrak{a} = (\mathfrak{a}\mathfrak{x}\mathfrak{b})\mathfrak{b}(\mathfrak{b}\mathfrak{y}\mathfrak{a})$. This means there exist $\mathfrak{x}', \mathfrak{y}' \in \mathcal{T}_{n-kd}^d$ with $\bigotimes_{l=1}^{r} \mathfrak{m}_{i_l} = \mathfrak{x}'(\bigotimes_{l=1}^{r} \mathfrak{m}_{i_{\sigma(l)}})\mathfrak{y}'$. We claim $\mathfrak{x}'\mathfrak{b}' = \mathfrak{b}'$ and $\mathfrak{b}'\mathfrak{y}' = \mathfrak{b}'$ where $\mathfrak{b}' = \bigotimes_{l=1}^{r} \mathfrak{m}_{i_{\sigma(l)}}$, and thus $\mathfrak{a} = \mathfrak{b}$ as desired.

To prove $\mathfrak{x}'\mathfrak{b}' = \mathfrak{b}'$ we first show that each part of $\mathfrak{x}'$ is a propagating part. Let $\mathfrak{a}' = \bigotimes_{l=1}^{r} \mathfrak{m}_{i_l}$ and suppose $\mathfrak{x}'$ has a southern non-propagating part. Then this southern non-propagating part either glues two propagating parts of $\mathfrak{b}'$ or changes at least one propagating part to non-propagating part, in either cases $\#(\mathfrak{x}'\mathfrak{b}') < \#(\mathfrak{b}')$.

Hence $\#(\mathfrak{a}') = \#(\mathfrak{x}'\mathfrak{b}'\mathfrak{y}') \leq \#(\mathfrak{x}'\mathfrak{b}') < \#(\mathfrak{b}') = \#(\mathfrak{a}')$ which is absurd. If $\mathfrak{x}'$ has at least one northern non-propagating part, then from the previous step we only need to consider the possibility that there exist a propagating part $\mathbf{p} \in \mathfrak{x}'$ such that $d < |\mathbf{p} \cap \underline{n}'| \leq n - kd$. This implies that the part $\mathbf{p}$ glues at least two parts of $\mathfrak{b}'$. Thus $\#(\mathfrak{x}'\mathfrak{b}') < \#(\mathfrak{b}')$ which is a contradiction again. Moreover, for any part $\mathbf{p} \in \mathfrak{x}'$ we have $|\mathbf{p} \cap \underline{n}| = |\mathbf{p} \cap \underline{n}'| \leq d$, since otherwise similar to the previous cases we get $\#(\mathfrak{x}'\mathfrak{b}') < \#(\mathfrak{b}')$. Therefore, $\mathfrak{x}'$ is planar canonical and hence $\mathfrak{x}' = \bigotimes_{l=1}^{t} \mathfrak{m}_{p_l}$. Again we use the fact $\#(\mathfrak{x}'\mathfrak{b}') \not< \#(\mathfrak{b}')$ to assume no part of $\mathfrak{x}'$ can connect two parts of $\mathfrak{b}'$. This means for each $1 \leq l \leq r$ there exist $0 \leq t_l \leq r$ such that $\sigma(l) = \sum_{u=t_{l-1}+1}^{t_l} p_u$, consequently, we have $\mathfrak{x}'\mathfrak{b}' = \mathfrak{b}'$. Similar steps can be taken to prove $\mathfrak{b}'\mathfrak{y}' = \mathfrak{b}'$. □

Corollary 3.3.5. *For each n and d the monoid $\mathcal{T}_n^d$ is regular.*

Proof. Each planar canonical element is idempotent, now use Theorem 3.3.4. □

In Theorem 3.3.4 we showed that each $\mathcal{J}$- class contains a unique planar canonical element, we let this planar canonical element to be representative of its class.

We define another partial order relation on $\mathcal{T}_n^d/\mathcal{J}$ and then show that it is equal to $\preceq$. However, this new way of considering the relation $\preceq$ enables us to determine its covering relation which is crucial in the proof that the algebra is quasi-hereditary.

Let $\mathfrak{d}, \mathfrak{d}' \in \mathcal{T}_n^d$ with $\mathfrak{d} = (\bigotimes_{j=1}^{l} \mathfrak{m}_{i_j}) \otimes \mathfrak{u}_d^{\otimes r}$ and $1 \leq i_j \leq d$, that is $\mathfrak{d}$ is planar canonical. We say $\bar{\mathfrak{d}}' \lessdot' \bar{\mathfrak{d}}$ if and only if one of the following holds;

- If $\mathfrak{d}' \mathcal{J}\, (\mathfrak{d} \backslash \{\sigma_{i_j}, \sigma_{i_{j+1}}\}) \cup \{\sigma_{i_j} \cup \sigma_{i_{j+1}}\}$, where σ_{i_j} is the part containing $\sum_{k=1}^{j-1} i_k + 1$, $\sigma_{i_{j+1}}$ is the part containing $\sum_{k=1}^{j} i_k + 1$ and $j \leq l$. This means $\mathfrak{d}'$ obtained from $\mathfrak{d}$ by gluing two successive (neighbouring) propagating parts, and this can be done by multiplying by a suitable element. For example, $\mathfrak{d}' = \mathfrak{d}(\mathfrak{m}_1^{\otimes s} \otimes \mathfrak{m}_{i_j + i_{j+1}} \otimes \mathfrak{m}_1^{\otimes t})$, where $s = \sum_{k=1}^{j-1} i_k$ and $t = \sum_{k=j+2}^{l} i_k + rd$.

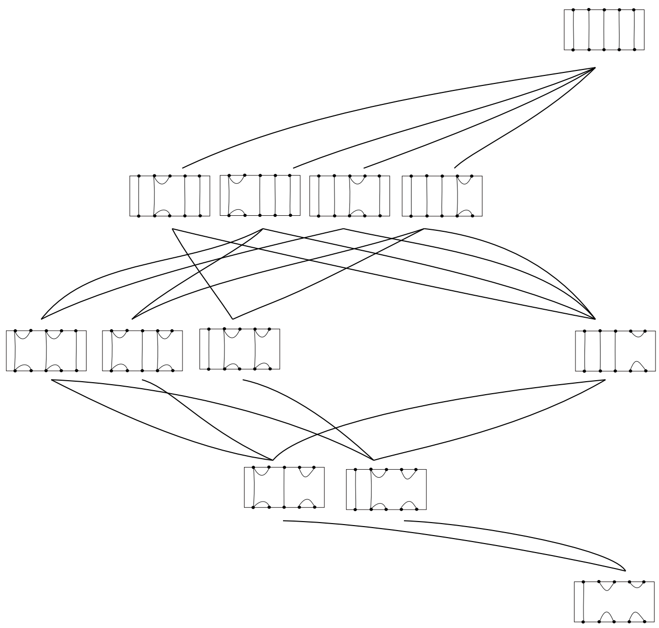


Figure 3.3: The covering relation $\lessdot'$, when $d = 2$ and $n = 5$.

- If $i_j = d$ and $\mathfrak{d}'\mathcal{J}\,(\mathfrak{d} \setminus \sigma_{i_j}) \cup \{\sigma_{i_j} \cap \underline{n}, \sigma_{i_j} \cap \underline{n}'\}$. This can be done by multiplying by $\boldsymbol{\nu}_{i_j}$, that is $\mathfrak{d}' = \mathfrak{d}\boldsymbol{\nu}_{i_j}$.

Let $\preceq'$ be the reflexive transitive closure of the relation $\lessdot'$, see Figure 3.3. We then have the following important lemma.

Lemma 3.3.6. *The two relations $\preceq$ and $\preceq'$ on $\mathcal{T}_n^d/\mathcal{J}$ are the same.*

Proof. Follows from the definitions of $\preceq$, $\preceq'$ and the argument in the proof of Proposition 3.3.3. □

Lemma 3.3.7. *The relation $\lessdot'$ is the covering relation of $\preceq'$.*

Proof. Directly from the definition of $\lessdot'$ we see that a necessary condition for $\mathfrak{d}' \lessdot' \mathfrak{d}$ is $\#(\mathfrak{d}') = \#(\mathfrak{d}) - 1$. Hence, the relation $\lessdot'$ is a transitive reduction of $\preceq'$. □

Lemma 3.3.8. *Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{T}_n^d$ and $\mathfrak{a} := (\bigotimes_{l=1}^{r} \mathfrak{m}_{i_l}) \otimes \mathfrak{u}_d^{\otimes k}$ be planar canonical. If $\mathfrak{a}\mathcal{J}\mathfrak{b}$ then $\#(\mathfrak{b}) = r$, and if we let $\beta_1, \beta_2, \ldots, \beta_r$ be the complete list of propagating parts of*

$\mathfrak{b}$ *such that* $\max(\beta_j \cap \underline{n}) < \min(\beta_{j+1} \cap \underline{n})$ *for* $1 \leq j < r$*, then* $|\beta_t \cap \underline{n}| \equiv |\beta_t \cap \underline{n}'| \equiv i_t$ mod d *for* $1 \leq t \leq r$.

Proof. The first part of the Lemma is a straightforward application of Equation 1.5. For the second part of the Lemma, note that in the proof of Lemma 3.3.2 when we changed a non-canonical element to a planar canonical element we did not permute the propagating parts, and for $1 \leq t \leq r$ we reduced $|\beta_t \cap \underline{n}|$ or $|\beta_t \cap \underline{n}'|$ by d at each step whenever it was necessary, the rest of the Lemma follows from uniqueness of the planar canonical elements. □

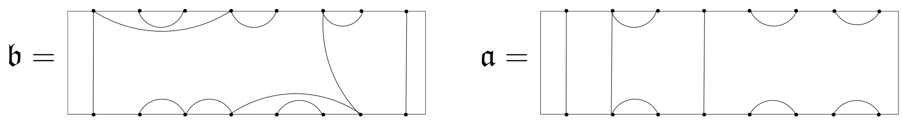


Figure 3.4: An example of two different $\mathcal{J}$-equivalent elements in $\mathcal{T}_8^2$, describing Lemma 3.3.8.

3.4 Relation between $T_n^2(\delta^2)$ and the Fuss-Catalan algebra.

In this subsection we establish an isomorphism between the Fuss-Catalan algebra of two colours and the 2-tonal partition algebra. Fuss-Catalan algebras have been introduced in [6] as a coloured generalisation of Temperley-Lieb algebra and used in the study of intermediate subfactor. They also appear in other contexts such as integrable lattice models.

We start by recalling the definition of the Fuss-Catalan algebras. It is necessary to mention that in [6] Bisch and Jones are working over the field of complex numbers

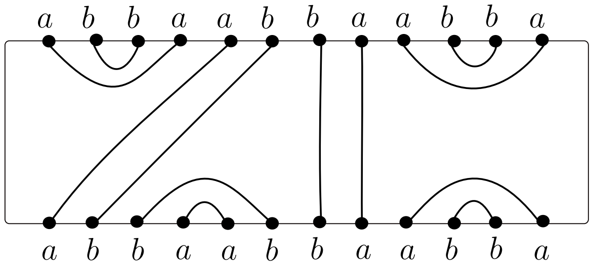


Figure 3.5: An element of $\mathbf{F}(6,1)$

but all their results which we refer to here are still true over a commutative ring, unless we explicitly specify the base field.

Let $a_0, a_1, \ldots, a_k$ be $k+1$ colours and $\mathfrak{d} \in \mathcal{TL}_{n(k+1)}$, $k, n \in \mathbb{N}$, as defined in Section 1.1.3. Label the top vertices of $\mathfrak{d}$ by the colours $a_0, a_1, \ldots, a_{k-1}, a_k, a_k, a_{k-1}, \ldots, a_1, a_0, a_0, \ldots, a_k, a_{k-1}, \ldots$ in order from left to right, similarly for the bottom vertices. This labelling will replace the original labelling of the vertices where the labels come from the set $\underline{n(k+1)} \cup \underline{(n(k+1))}'$. Note that according to this colouring if n is even then the vertex $n(k+1)$ is going to be labelled by the colour a_0, otherwise by a_k. We say $\mathfrak{d}$ is a Fuss-Catalan diagram if only vertices with the same colour are connected. Denote by $\mathbf{F}(n,k)$ the set of all Fuss-Catalan $(n(k+1), n(k+1))$ diagrams. An example of an element of $\mathbf{F}(6,1)$ is given in Figure 3.5.

Fix a commutative ring R and $\delta_{a_0}, \ldots, \delta_{a_k} \in R$, let $FC_{k,n}(\delta_{a_0}, \delta_{a_1}, \ldots, \delta_{a_k})$ be the free R-module with basis $\mathbf{F}(n,k)$. Define the multiplication of basis of $FC_{k,n}(a_0, a_1, \ldots, a_k)$ in the following way; for $\mathfrak{d}, \mathfrak{t} \in FC_{k,n}(a_0, a_1, \ldots, a_k)$ set

$$\mathfrak{d}.\mathfrak{t} := \delta_{a_0}^{n_0} \ldots \delta_{a_k}^{n_k} \mathfrak{d} \circ \mathfrak{t} \tag{3.2}$$

Where $\mathfrak{d} \circ \mathfrak{t}$ is the multiplication of diagrams $\mathfrak{d}$ and $\mathfrak{t}$ in $\mathcal{TL}_{n(k+1)}$, and n_i is the number of removed connected components of type a_i. Extending the multiplication in 3.2 linearly to all elements of $FC_{k,n}(a_0, a_1, \ldots, a_k)$ makes it a R-algebra.

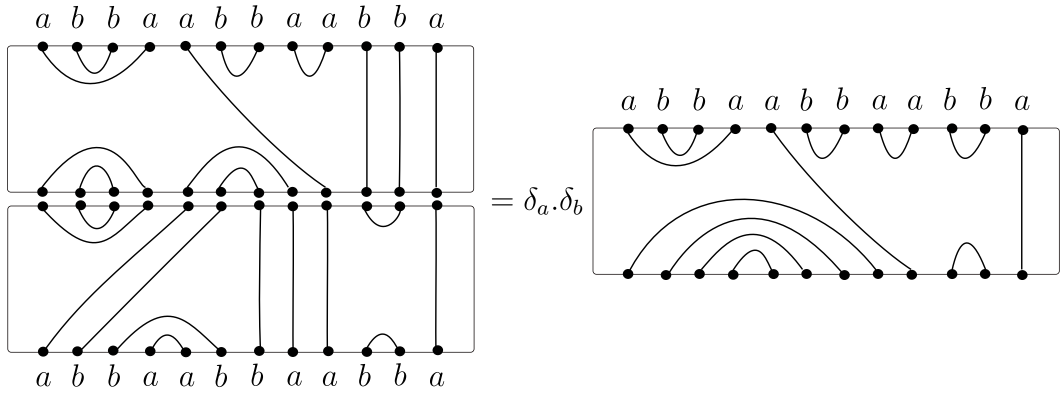


Figure 3.6: An element of $\mathbf{F}(6,1)$

We are mainly interested in the case when we have $k = 1$ and the following theorem will give the presentation of $FC_{1,n}(\delta_a, \delta_b)$ by generators and relations. See Figure 3.6 for example.

Theorem 3.4.1 (Proposition 4.1.4 [6], Theorem 4.2.14 [6])**.** *Let δ_a, δ_b be two non-zero elements in $\mathbb{F}$. Then the algebra $FC_{1,n}(\delta_a, \delta_b)$ is presented by generators*

$$G := \{1, P_1, \ldots, P_{n-1}, E_1, \ldots, E_{n-1}\}$$

and the following relations:

$$\begin{aligned} E_i^2 &= \delta_a \delta_b E_i && 1 \le i \le n-1, \\ E_i E_j &= E_j E_i && 1 \le i, j \le n-1 \text{ and } |i-j| \ge 2, \\ E_i E_{i\pm 1} E_i &= E_i && 1 \le i, i \pm 1 \le n-1. \end{aligned} \tag{3.3}$$

$$\begin{aligned} P_{2i}^2 &= \delta_a P_{2i} && 1 \le 2i \le n-1, \\ P_{2i+1}^2 &= \delta_b P_{2i+1} && 1 \le 2i+1 \le n-1, \\ P_i P_j &= P_j P_i && 1 \le i, j \le n-1. \end{aligned} \tag{3.4}$$

$$\begin{aligned} E_{2i} P_{2i} = P_{2i} E_{2i} &= \delta_a E_{2i} && 1 \le 2i \le n-1, \\ E_{2i+1} P_{2i+1} = P_{2i+1} E_{2i+1} &= \delta_b E_{2i+1} && 1 \le 2i+1 \le n-1, \\ P_i E_j &= E_i P_j && 1 \le i, j \le n-1 \text{ and } |i-j| \ge 2. \end{aligned} \tag{3.5}$$

$$\begin{aligned} E_{2i\pm1}P_{2i}E_{2i\pm1} &= \delta_b E_{2i\pm1} && 1 \leq 2i, 2i \pm 1 \leq n-1, \\ E_{2i}P_{2i\pm1}E_{2i} &= \delta_a E_{2_i} && 1 \leq 2i, 2i \pm 1 \leq n-1. \end{aligned} \tag{3.6}$$

$$\begin{aligned} P_{2i}E_{2i\pm1}P_{2i} &= P_{2i\pm1}P_{2i} && 1 \leq 2i, 2i \pm 1 \leq n-1, \\ P_{2i\pm1}E_{2i}P_{2i\pm1} &= P_{2i}P_{2i\pm1} && 1 \leq 2i, 2i \pm 1 \leq n-1. \end{aligned} \tag{3.7}$$

□

We also recall from [6] Corollary 2.1.7,

$$dim_{\mathbb{F}}(FC_{k,n}(\delta_{a_0}, \delta_{a_1}, \ldots, \delta_{a_k})) = CN^{k+1}(n). \tag{3.8}$$

Theorem 3.4.2. *Let $0 \neq \delta_a = \delta_b = \delta \in \mathbb{F}$. There is an isomorphism of $\mathbb{F}$-algebras*

$$\Phi : FC_{1,n}(\delta, \delta) \to T_n^2(\delta^2) \tag{3.9}$$

such that $\Phi(E_i) = \boldsymbol{\nu}_{i-1}$ and $\Phi(P_i) = \delta\boldsymbol{\mu}_{i-1}$, for all $1 \leq i \leq n-1$.

Proof. Define a set map $f : G \to T_n^2(\delta^2)$ by $f(1) = 1, f(E_i) = \boldsymbol{\nu}_{i-1}$ and $f(P_i) = \delta\boldsymbol{\mu}_{i-1}$, for all $1 \leq i \leq n-1$. By the universal property of free associative algebras there is a unique $\mathbb{F}$-algebra homomorphism $\hat{\Phi} : \mathbb{F}\langle G\rangle \to T_n^2(\delta^2)$ such that $\hat{\Phi}\restriction_G = f$, where $\mathbb{F}\langle G\rangle$ is the free associative $\mathbb{F}$-algebra generated by the set G. It is elementary diagram calculation to verify that the elements $\delta\boldsymbol{\mu}_i$ and $\boldsymbol{\nu}_i$, for all $0 \leq i \leq n-2$, satisfy the relations 3.3 to 3.7 in Theorem 3.4.1, hence the ideal generated by these relations is contained in the kernel of $\hat{\Phi}$. Therefore, by using the factor lemma and Theorem 3.4.1 there is a unique $\mathbb{F}$-algebra homomorphism $\Phi : FC_{1,n}(\delta, \delta) \to T_n^2(\delta^2)$ such that $\hat{\Phi} = \Phi\pi$.

The morphism Φ is onto since by Proposition 3.8 the elements $\boldsymbol{\mu}_i$ and $\boldsymbol{\nu}_i$, for $0 \leq i \leq n-2$ generate the algebra $T_n^2(\delta^2)$. On the other hand, by Proposition 3.2.2 and Equation 3.8 both algebras have the same dimension, which is $CN^2(n)$. Thus by the rank-nullity theorem Φ is an isomorphism of $\mathbb{F}$-algebras. □

It is necessary to mention that the isomorphism in Theorem 3.4.2 does not work for $d > 2$, since the dimensions are different.

Corollary 3.4.3. *For $\delta \in \mathbb{F} \setminus \{0\}$ the $\mathbb{F}$-algebra $T_n^2(\delta^2)$ is presented by generators $G = \{1, \boldsymbol{\mu}_0, \boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_{n-2}, \boldsymbol{\nu}_0, \boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_{n-2}\}$ and the following relations:*

$$\begin{aligned} \boldsymbol{\nu}_i^2 &= \delta^2 \boldsymbol{\nu}_i^2 && 0 \le i \le n-2, \\ \boldsymbol{\nu}_i \boldsymbol{\nu}_j &= \boldsymbol{\nu}_j \boldsymbol{\nu}_i && 0 \le i, j \le n-2 \text{ and } |i-j| \ge 2, \\ \boldsymbol{\nu}_i \boldsymbol{\nu}_{i\pm1} \boldsymbol{\nu}_i &= \boldsymbol{\nu}_i && 0 \le i, i \pm 1 \le n-2. \end{aligned} \tag{3.10}$$

$$\begin{aligned} \boldsymbol{\mu}_i^2 &= \boldsymbol{\mu}_i && 0 \le i \le n-2, \\ \boldsymbol{\mu}_i \boldsymbol{\mu}_j &= \boldsymbol{\mu}_j \boldsymbol{\mu}_i && 0 \le i, j \le n-2 \text{ and } |i-j| \ge 1, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \boldsymbol{\mu}_i \boldsymbol{\nu}_i = \boldsymbol{\nu}_i \boldsymbol{\mu}_i &= \boldsymbol{\nu}_i && 0 \le i \le n-2, \\ \boldsymbol{\mu}_i \boldsymbol{\nu}_j &= \boldsymbol{\nu}_j \boldsymbol{\mu}_i && 0 \le i, j \le n-2 \text{ and } |i-j| \ge 2, \\ \boldsymbol{\nu}_{i\pm1} \boldsymbol{\mu}_i \boldsymbol{\nu}_{i\pm1} &= \boldsymbol{\nu}_{i\pm1} && 0 \le i, i \pm 1 \le n-2, \\ \boldsymbol{\mu}_{i\pm1} \boldsymbol{\nu}_i \boldsymbol{\mu}_{i\pm1} &= \boldsymbol{\mu}_i \boldsymbol{\mu}_{i+1} && 0 \le i, i \pm 1 \le n-2. \end{aligned} \tag{3.12}$$

Proof. Follows straightforwardly from Theorem 3.4.2. □

3.5 An indexing set for the simple modules of $T_n^d(\delta)$.

In this subsection we obtain an explicit indexing set for the isomorphism classes of simple modules of $T_n^d(\delta)$ over an algebraically closed field. Therefore we fulfil one of the aims of the representation theory of $T_n^d(\delta)$.

From now on we assume $R = \mathbb{F}$ and $\mathbb{F}$ is algebraically closed field.

Proposition 3.5.1. *For $n \ge d$ we have the following isomorphism of $\mathbb{F}$-algebras*

$$\begin{aligned} \phi : T_{n-d}^d(\delta) &\to \mathfrak{e}_{1,d} T_n^d(\delta) \mathfrak{e}_{1,d} \\ \mathfrak{d} &\mapsto \mathfrak{e}_{1,d}(\mathfrak{d} \otimes \mathfrak{u}_d)\mathfrak{e}_{1,d} \end{aligned} \tag{3.13}$$

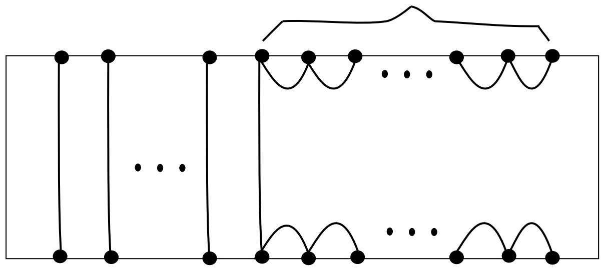


Figure 3.7: The idempotent $\mathfrak{e}_{i,d}$, where $0 \leq i < \lfloor n/d \rfloor$

where $\mathfrak{e}_{1,d}$ is as given in Figure 3.7

Proof. To show ϕ is an onto morphism, let $\mathfrak{t} \in \mathfrak{e}_{1,d}T_n^d(\delta)\mathfrak{e}_{1,d}$ be a diagram basis element. We have $\mathfrak{t} = \mathfrak{e}_{1,d}\mathfrak{t}\mathfrak{e}_{1,d} = (\mathfrak{e}_{1,d}\boldsymbol{\nu}_{n-d}\mathfrak{e}_{1,d})\mathfrak{t}(\mathfrak{e}_{1,d}\boldsymbol{\nu}_{n-d}\mathfrak{e}_{1,d}) = \mathfrak{e}_{1,d}(\boldsymbol{\nu}_{n-d}\mathfrak{e}_{1,d}\mathfrak{t}\mathfrak{e}_{1,d}\boldsymbol{\nu}_{n-d})\mathfrak{e}_{1,d}$. Let $\mathfrak{d} = \boldsymbol{\nu}_{n-d}\mathfrak{e}_{1,d}\mathfrak{t}\mathfrak{e}_{1,d}\boldsymbol{\nu}_{n-d} \backslash \{S, S'\}$, where $S = \{n, n-1, \ldots, n-d-1\}$, then $\mathfrak{d} \in T_{n-d}^d(\delta)$ and $\mathfrak{t} = \mathfrak{e}_{1,d}(\mathfrak{d} \otimes \mathfrak{u}_d)\mathfrak{e}_{1,d}$, thus ϕ is onto. For one to one, let $\phi(\mathfrak{t}) = \phi(\mathfrak{d})$ then $\mathfrak{t} \otimes \mathfrak{u}_d = \boldsymbol{\nu}_{n-d}\phi(\mathfrak{t})\boldsymbol{\nu}_{n-d} = \boldsymbol{\nu}_{n-d}\phi(\mathfrak{d})\boldsymbol{\nu}_{n-d} = \mathfrak{d} \otimes \mathfrak{u}_d$, hence $\mathfrak{t} = \mathfrak{d}$. □

We also denote the quotient algebra by;

$$T_n^{d,1} := T_n^d(\delta)/J_1 \tag{3.14}$$

Where $J_1 = T_n^d(\delta)\boldsymbol{\nu}_{n-d}T_n^d(\delta)$.

Our strategy is to pass our knowledge of $T_{n-d}^d(\delta)$ to $T_n^d(\delta)$, and to do this we use the machinery of functors. We shall use globalisation and the localisation functors, defined in Equations 1.10,1.11

In particular, using Theorem 1.4.2 we obtain

$$\Lambda(T_n^d(\delta)) \simeq \Lambda(T_{n-d}^d(\delta)) \sqcup \Lambda(T_n^{d,1}) \tag{3.15}$$

Lemma 3.5.2. *The set*
$\hat{\Lambda}(T_n^{d,1}) := \{\mathfrak{p} + J_1 \mid \mathfrak{p}$ *is planar canonical with no non-propagating parts*$\}$ *forms a basis for* $T_n^{d,1}$.

Proof. Let $\mathfrak{t}$ be a planar canonical element. To prove the Lemma it is enough to show

I. If $\mathfrak{t}$ has no non-propagating part then $\mathfrak{t} \npreceq \boldsymbol{\nu}_{n-d}$.

II. If $\mathfrak{t}$ has at least two non- propagating parts then $\mathfrak{t} \preceq \boldsymbol{\nu}_{n-d}$.

Both cases are direct consequence of the definition of $\lessdot'$, since we cannot decrease the number of $\mathfrak{u}_d$ in any element. Moreover, by using the two given conditions, in the definition of $\lessdot'$, we can reach any other planar canonical element with at least one $\mathfrak{u}_d$. □

Let $\mathfrak{p} := (\bigotimes_{j=1}^{l} \mathfrak{m}_{i_j}) \otimes \mathfrak{u}_d^{\otimes r}$ be a planar canonical diagram, by $S_\mathfrak{p}$ we mean the set of all elements covered by $\mathfrak{p}$ with respect to the relation $\lessdot'$.

Proposition 3.5.3. *The $\mathbb{F}$- algebra $T_n^{d,1}$ is semisimple and commutative.*

Proof. For $\mathfrak{w} + J_1 \in \hat{\Lambda}(T_n^{d,1})$, $\widehat{T_n^{d,1}}\mathfrak{w} := T_n^{d,1}(\mathfrak{w} + J_1)/\sum_{\mathfrak{e}\in S_\mathfrak{w}} T_n^{d,1}(\mathfrak{e} + J_1)$ is a one dimensional left $T_n^{d,1}$ module, since by Lemma 3.3.6 and Lemma 3.3.7 $\mathfrak{t}\mathfrak{m} \preceq \mathfrak{m}$ for any $\mathfrak{t} + J_1 \in T_n^{d,1}$, and $\mathfrak{s}\mathfrak{m} + J_1 = \mathfrak{m} + J_1$ if $\mathfrak{m} \preceq \mathfrak{s}$, where $\mathfrak{s} \in T_n^{d,1}$. □

Theorem 3.5.4. *The $\mathbb{F}$-algebra $T_n^d(\delta)$ obeys*

$$\Lambda(T_n^d(\delta)) = \bigsqcup_{i=0}^{\lfloor n/d \rfloor} \Lambda(T_{n-di}^{d,1})$$

Proof. The proof follows by using induction on n and Equation 3.15. □

A *composition* of a positive integer r is defined to be, see for e.g. [36] and [75], a k-tuple of positive integers $(a_1, \ldots, a_k)$ such that $\sum_{i=1}^{k} a_i = r$.

Denote by $\mathcal{C}(r)$ the set of all compositions of r. By definition we set 0 to be the only composition of 0.

Let $C(r) := |\mathcal{C}(r)|$ then $C(r) = 2^{r-1}$ (see Theorem 3.3 of [36]), for convenience we set $C(-r) := 0$.

We write $\mathcal{C}^d(r)$ for the set of all compositions of r with $a_i \leq d$ for all $1 \leq i \leq k$.

Set $C^d(r) := |\mathcal{C}^d(r)|$. Note that if $d \geq r$ then $C^d(r) = C(r)$, and $C^1(r) = 1$.

Example 3.5.5. *Let $d = 3$ and $n = 5$. Then we have:*

$$\mathcal{C}^3(5) = \{(1,1,1,1,1),(2,1,1,1),(1,2,1,1),(1,1,2,1),(1,1,1,2),(2,2,1),(2,1,2),$$
$$(1,2,2),(3,1,1),(1,3,1),(1,1,3),(3,2),(2,3)\}.$$

Lemma 3.5.6. *For $d \geq 2$ and a positive integer r, $C^d(r)$ is given by the following recurrence relation*

$$C^d(r) = \sum_{i=1}^{d} C^d(r-i) \tag{3.16}$$

Proof. Let $(a_1, \ldots, a_k)$ be a composition of r, then $(a_1, \ldots, a_{k-1})$ is a composition of m for some $r - d \leq m \leq r - 1$. Therefore, $C^d(r) = \sum_{i=1}^{d} C^d(r-i)$. □

We recall from [61] a generalised Fibonacci sequence. Here we call it d-Fibonacci sequence and denoted by $F^d(r)$. The sequence $F^d(r)$ is recursively given by

$$F^d(r) = \sum_{i=1}^{d} F^d(r-i) \tag{3.17}$$

where $F^d(k) = 0$ for $0 \leq k \leq d-2$ and $F^d(d-1) = 1$. The 2-Fibonacci sequence is the usual Fibonacci sequence.

Theorem 2.2 in [38] states that $F^d(r) = 2^{r-d}$ for $d \leq r \leq 2d-1$. Now, if $1 \leq r \leq d$ we obtain $C^d(r) = 2^{r-1} = F^d(r+d-1)$. Thus, from equations 3.16 and 3.17 we have the following equation, for all $r \in \mathbb{N}$;

$$C^d(r) = F^d(r+d-1) \tag{3.18}$$

It has been proven in [23] Theorem 1 that $C^d(r)$ can be computed by the following formula;

$$C^d(r) = \sum_{i=1}^{d} \frac{\alpha_i - 1}{2 + (d+1)(\alpha_i - 1)} \alpha_i^{r-1} \tag{3.19}$$

where $\alpha_1 \ldots, \alpha_d$ are the roots of the polynomial $x^d - x^{d-1} - \cdots - 1$.

It is time to return to our main story and connect the notion of composition of a positive integer to an indexing set of simple modules of $T_n^d(\delta)$.

Theorem 3.5.7. *There exist a set bijection*

$$\Lambda(T_n^d(\delta)) \simeq \bigsqcup_{i=0}^{\lfloor n/d \rfloor} \mathcal{C}^d(n-id) \tag{3.20}$$

In particular,

$$|\Lambda(T_n^d(\delta))| = \sum_{i=0}^{\lfloor n/d \rfloor} C^d(n-id)$$

Proof. From Theorem 3.5.4 we note that to prove the result it is enough to show that $\mathcal{C}^d(m) \simeq \hat{\Lambda}(T_m^{d,1})$ for any $m \in \mathbb{N}$. To prove this, let $(a_1, \ldots, a_k)$ be any composition of m such that for each i we have $a_i \leq d$ then $\bigotimes_{i=1}^k \mathfrak{m}_{a_i}$ is a planar canonical element with no non-propagating parts in $\mathcal{T}_m^d$, and hence by Lemma 3.5.2 the element $\bigotimes_{i=1}^k \mathfrak{m}_{a_i} + J_1$ is a basis element of $T_m^{d,1}$. It is not hard to see that the above assignment is bijective. The claim follows. □

We would also like to record the following Lemma which is going to be useful later, and its proof is quite similar to the above theorem.

Lemma 3.5.8. *Let* $\mathbf{P}$ *be the set of all planar canonical elements of* $\mathcal{T}_n^d$ *then we have the following set bijection;*

$$\Xi : \bigsqcup_{i=0}^{\lfloor n/d \rfloor} \mathcal{C}^d(n-id) \to \mathbf{P}$$

$$(a_1, \ldots, a_k) \mapsto (\otimes_{i=1}^k \mathfrak{m}_{a_i}) \otimes \mathfrak{u}_d^{\otimes(n-\sum_{i=1}^k a_i)/d}$$

□

3.6 On the quasi-heredity and Cellularity of $T_n^d(\delta)$.

In this section we show that for $\delta \neq 0$ or $d \nmid n$ the $\mathbb{F}$-algebra $T_n^d(\delta)$ is both quasi-heredity and cellular.

Let $J_i := T_n^d(\delta)\mathfrak{e}_{i,d}T_n^d(\delta)$ where $\mathfrak{e}_{i,d}$ be as given in Figure3.7 for $0 \leq i \leq \lfloor n/d \rfloor$ and $d \nmid n$, we let $\mathfrak{e}_{r,d} := \frac{1}{\delta}\{\underline{n}, \underline{n}'\}$ when $n = rd$ and $\delta \neq 0$. We have the following chain of two sided ideals

$$0 \subset J_{\lfloor n/d \rfloor} \subset J_{\lfloor n/d \rfloor -1} \subset \cdots \subset J_1 \subset J_0 := T_n^d(\delta)$$

For $0 \leq i \leq \lfloor n/d \rfloor$ we define

$$T_n^{d,i}(\delta) := T_n^d(\delta)/J_i$$

Clearly we have $T_n^{d,0}(\delta) = 0$ and $T_n^{d,1}(\delta) = T_n^{d,1}$

Lemma 3.6.1. *For $1 \leq i \leq \lfloor n/d \rfloor$, $\delta \neq 0$ or $d \nmid n$ the module ${}_{T_n^{d,i}(\delta)}J_{i-1}/J_i$ is projective.*

Proof. The module ${}_{T_n^{d,i}(\delta)}T_n^{d,i}(\delta)\mathfrak{e}_{i-1,d}$ is projective, since if $\delta \neq 0$ or $d \nmid n$ the element $\mathfrak{e}_{i-1,d} + J_i$ is idempotent in $T_n^{d,i}(\delta)$. Furthermore, ${}_{T_n^{d,i}(\delta)}T_n^{d,i}(\delta)\mathfrak{e}_{i-1,d} \simeq {}_{T_n^{d,i}(\delta)}T_n^{d,i}(\delta)\mathfrak{e}_{i-1,d}\mathfrak{d}$ for any $\mathfrak{d} \in \mathcal{T}_n^d$ such that $\mathfrak{e}_{i-1,d}\mathfrak{d} \notin J_i$. But we have

$${}_{T_n^{d,i}(\delta)}J_{i-1}/J_i = \bigoplus_{\mathfrak{d} \in \mathcal{T}_n^d \text{ and } \mathfrak{e}_{i-1}\mathfrak{d} \notin J_i} {}_{T_n^{d,i}(\delta)}T_n^{d,i}(\delta)\mathfrak{e}_{i-1,d}\mathfrak{d}$$

The claim follows from above and the fact that direct sum of projective modules is again projective. □

Lemma 3.6.2. *Let $1 \leq i \leq \lfloor n/d \rfloor$, $\delta \neq 0$ or $d \nmid n$ we have the following isomorphism of algebras*

$$\psi : T_{n-(i-1)d}^{d,1} \to \mathfrak{e}_{i-1,d}T_n^{d,i}(\delta)\mathfrak{e}_{i-1,d} \tag{3.21}$$

which is defined on basis of $T^{d,1}_{n-(i-1)d}$ *and extended linearly to all* $T^{d,1}_{n-(i-1)d}$ *as follows. Let* $\mathfrak{d} + J_1 \in T^{d,1}_{n-(i-1)d}$ *be a basis element then define* $\psi(\mathfrak{d} + J_1) = \mathfrak{e}_{i-1,d}(\mathfrak{d} \otimes \mathfrak{u}_d^{\otimes(i-1)})\mathfrak{e}_{i-1,d} + J_i$.

Proof. We have $\psi(1 + j_1) = \mathfrak{e}_{i-1,d}(\mathfrak{m}_1^{\otimes n-(i-1)d} \otimes \mathfrak{u}_d^{\otimes(i-1)})\mathfrak{e}_{i-1,d} + J_i = \mathfrak{e}_{i-1,d} + J_i$, which means ψ preserves identity. It is straightforward to see that for any two basis elements $\mathfrak{a}+J_1, \mathfrak{b}+J_1 \in T^{d,1}_{n-(i-1)d}$ we have $\psi((\mathfrak{a}+J_1)(\mathfrak{b}+J_1)) = \psi(\mathfrak{a}+J_1)\psi(\mathfrak{b}+J_1)$. Hence, by extending linearly ψ is an $\mathbb{F}$-algebra homomorphism.

By an argument similar to Proposition 3.5.1 one can show that ψ is one to one.

For onto, let $\mathfrak{d} + J_i$ be a non-zero basis element in $\mathfrak{e}_{i-1,d}T_n^{d,i}(\delta)\mathfrak{e}_{i-1,d}$, we show that $\mathfrak{d} = (\bigotimes_{k=1}^{l-1} \mathfrak{m}_{r_k}) \otimes \mathfrak{m}_{i_l+(i-1)d}$ such that $1 \leq r_k \leq d$ for $1 \leq k \leq l$. Let α be the part containing n then we have $(i-1)d <\mid \alpha \cap n \mid\leq id$, since if $\mid \alpha \cap n \mid> id$ we obtain $\mathfrak{d} \in J_i$, and the ineqality $(i-1)d <\mid \alpha \cap n \mid$ follows from the fact that $\mathfrak{e}_{i-1,d}$ is the identity element of $\mathfrak{e}_{i-1,d}T_n^{d,i}(\delta)\mathfrak{e}_{i-1,d}$. Similarly one can get $(i-1)d <\mid \alpha \cap n' \mid\leq id$. Moreover, n and n' are in propagating parts, otherwise we would get a non propagating part of size id. Let $p := \max(\alpha_n)$ and $q := \max(\alpha_{n'})$, where α_n and $\alpha_{n'}$ are the parts containing n and n' respectively. Thus $q < n < n' < p$, which implies that n and n' has to be in the same part. Consequently we have $\mid \alpha \cap n \mid=\mid \alpha \cap n' \mid$. Let β is any part does not containing n then $1 \leq\mid \beta \cap n \mid=\mid \beta \cap n' \mid\leq d$ and it is a propagating part, since otherwise we would have $\mathfrak{d}\mathcal{J}(\mathfrak{d}' \otimes \mathfrak{u}_d^{\otimes i})$ for some $\mathfrak{d}' \in \mathcal{T}^d_{n-id}$. Therefore $\mathfrak{d} = (\bigotimes_{k=1}^{l-1} \mathfrak{m}_{r_k}) \otimes \mathfrak{m}_{i_l+(i-1)d}$, and hence $\mathfrak{d} + J_i = \psi(\bigotimes_{k=1}^{l} \mathfrak{m}_{r_k} + J_1)$. □

Theorem 3.6.3. *The* $\mathbb{F}$*-algebra* $T_n^d(\delta)$ *is quasi-hereditary with the heredity chain*

$$0 \subset J_{\lfloor n/d \rfloor} \subset J_{\lfloor n/d \rfloor -1} \subset \cdots \subset J_1 \subset J_0 := T_n^d(\delta) \tag{3.22}$$

if and only if $\delta \neq 0$ *or* $d \nmid n$.

Proof. By the Lemmas 3.6.1 and 3.6.2 with Theorem 3.5.3 the chain 3.22 is a heredity chain of $T_n^d(\delta)$, when $\delta \neq 0$ or $d \nmid n$.

Conversely, if $\delta = 0$ and $n = d$, then ${}_{T_d^d(0)}T_d^d(0)(\mathfrak{m}_d)$ is an indecomposable module and $\mathrm{rad}(T_d^d(0)(\mathfrak{m}_d)) \simeq \mathrm{head}(T_d^d(0)(\mathfrak{m}_d))$. Therefore, it is not possible to put a partial order relation on the labelling set of simple modules of $T_d^d(0)$ for which it becomes a quasi-hereditary algebra. If $n = 2d$ then $[P_{(d)} : \mathrm{head}(P_{(d)})] > 1$ where $P_{(d)}$ is the projective cover of the simple $T_{2d}^d(0)(\mathfrak{u}_d^{\otimes 2})$, hence $T_{2d}^d(0)$ is not quasi-hereditary algebra. If $n = kd$ with $k \in \mathbb{Z}_{>1}$, the functor $\mathcal{G}$ associated to the idempotent $\mathfrak{e}_{1,d} \in T_{n+d}^d(\delta)$ is left exact, hence it preserves isomorphisms. Therefore, by induction on n, $T_n^d(\delta)$ is not quasi-hereditary for all $n = kd$. $\square$

Let R be a commutative ring and $\delta \in R$. In this subsection we show that if $\delta \neq 0$ or $d \nmid n$ then the R-algebra $T_n^d(\delta)$ is a cellular algebra.

Let ι be a set map defined as follows:

$$\iota : \underline{n} \cup \underline{n}' \to \underline{n} \cup \underline{n}'$$
$$i \mapsto i'$$
$$j' \mapsto j$$

For $\mathfrak{e} \in \mathcal{T}_n^d$ with parts $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_k$ we define

$$\mathfrak{e}^\star := \{\iota(\mathfrak{e}_1), \iota(\mathfrak{e}_2), \ldots, \iota(\mathfrak{e}_k)\}.$$

Pictorially, $\mathfrak{e}^\star$ is obtained from $\mathfrak{e}$ by reflecting the diagram of $\mathfrak{e}$ around the x-axes. Note that $(\mathfrak{e}^\star)^\star = \mathfrak{e}$ and $(\mathfrak{e}\mathfrak{f})^\star = \mathfrak{f}^\star\mathfrak{e}^\star$, for each $\mathfrak{e}, \mathfrak{f} \in \mathcal{T}_n^d$.

Lemma 3.6.4. *The flipping map*

$$\star : T_n^d(\delta) \to T_n^d(\delta)$$
$$\mathfrak{d} \mapsto \mathfrak{d}^\star$$

is an anti-automorphism of R-algebras.

Proof. Follows from the definition of $\star$. $\square$

Remark 3.6.5. The map $\star$ makes $\mathcal{T}_n^d$ a regular $\star$-semigroup, and consequently the canonical elements would become some special kind of projections (projections which are fixed under $\star$)—see [26] for more details in this direction. We would like to mention that the above remark is also hold for $\mathcal{P}_n^d$.

Proposition 3.6.6. *If $\delta \neq 0$ or $d \nmid n$ then the R-algebra $T_n^d(\delta)$ is cellular.*

Proof. Let I be the poset $(\mathcal{T}_n^d/\mathcal{J}, \preceq)$ and let $\mathfrak{p}$ be a canonical element with r northen non-propagating parts. Denote by $\mathcal{L}_{\mathfrak{p}\mathfrak{e}_{r,d}}$ the $\mathcal{L}$-class of $\mathcal{T}_n^d$ containing $\mathfrak{p}\mathfrak{e}_{r,d}$. Then the $\mathcal{R}$-class containing $\mathfrak{e}_{r,d}\mathfrak{p}$ is equal to $(\mathcal{L}_{\mathfrak{p}\mathfrak{e}_{r,d}})^\star$ and the $\mathcal{J}$-class containing $\mathfrak{p}\mathfrak{e}_{r,d}$ is equal to $\mathcal{L}_{\mathfrak{p}\mathfrak{e}_{r,d}} \circ (\mathcal{L}_{\mathfrak{p}\mathfrak{e}_{r,d}})^\star$. Moreover, the following set map is injective,

$$\begin{aligned} C^{\mathfrak{p}} : \mathcal{L}_{\mathfrak{p}\mathfrak{e}_{r,d}} \times \mathcal{L}_{\mathfrak{p}\mathfrak{e}_{r,d}} &\to \mathcal{J}_{\mathfrak{p}} \\ (\mathfrak{d}, \mathfrak{s}) &\mapsto \mathfrak{d} \circ \mathfrak{s}^\star. \end{aligned} \tag{3.23}$$

By using Equation 3.23 and the fact that the $\mathcal{L}$-classes of $\mathcal{T}_n^d$ are pairwise disjoint the following map also is injective when $\delta \neq 0$ or $d \nmid n$,

$$C : \bigcup_{\mathfrak{p} \in I} \mathcal{L}_{\mathfrak{p}\mathfrak{e}_{r,d}} \times \mathcal{L}_{\mathfrak{p}\mathfrak{e}_{r,d}} \to T_n^d(\delta)$$

where $C = \bigcup C^{\mathfrak{p}}$. The image of C is the union of all $\mathcal{J}$-classes of $\mathcal{T}_n^d$. Therefore it forms an R-basis for $T_n^d(\delta)$. Hence, axiom C_1 of the definition 1.4.5 is satisfied. The axiom C_2 follows from Lemma 3.9 and the fact that the $\mathcal{J}$-classes are fixed under $\star$. Finally, the axiom C_3 is a consequence of the definition of the partial order $\preceq$ on the $\mathcal{J}$-classes of $\mathcal{T}_n^d$. □

Remark 3.6.7. One might be able to show that the algebra $T_n^d(\delta)$ is Cellular by showing that it is a twisted semigroup algebra—see [24, 60, 80]. The above references tell us that the Proposition 3.6.6 is expected; however, we did not uses these methods here because we believe that the direct prove of Cellularity is easier.

3.7 On the standard modules of $T_n^d(\delta)$.

We construct and study the quasi-hereditary standard modules of $T_n^d(\delta)$, in the context of Theorem 1.4.1. The standard modules of a quasi-hereditary algebra have simple head. We use this fact about the standard modules of $T_n^d(\delta)$ to construct the simple modules of $T_n^d(\delta)$. Therefore, we achieve another aim of the representation theory of $T_n^d(\delta)$.

3.7.1 Construction of the standard modules of $T_n^d(\delta)$

For $\mathfrak{q} \in \mathcal{T}_n^d$ let $\mathcal{L}_\mathfrak{q}$ be the $\mathcal{L}$-class containing $\mathfrak{q}$ and $L_n(\mathfrak{q})$ be an $\mathbb{F}$-vector space spanned by the set $\mathcal{L}_\mathfrak{q}$. Then $L_n(\mathfrak{q})$ is a left $T_n^d(\delta)$-module with the following action defined on diagram basis of $T_n^d(\delta)$ and $L_n(\mathfrak{q})$:

$$\rho : T_n^d(\delta) \times L_n(\mathfrak{q}) \to L_n(\mathfrak{q})$$
$$(\mathfrak{d}, \mathfrak{t}) \mapsto \begin{cases} \delta^{\kappa_{\mathfrak{d},\mathfrak{t}}}\mathfrak{d} \circ \mathfrak{t} & \text{if} \quad \mathfrak{d} \circ \mathfrak{t} \,\mathcal{L}\, \mathfrak{q} \\ 0 & \text{otherwise} \end{cases} \tag{3.24}$$

and extended linearly to both $T_n^d(\delta)$, $L_n(\mathfrak{q})$. Note that the definition of $\kappa_{\mathfrak{d},\mathfrak{t}}$ is given in Proposition 1.1.5.

We shall write $\mathfrak{d} \diamond \mathfrak{t}$ for $\rho(\mathfrak{d}, \mathfrak{t})$, or just simply $\mathfrak{d}\mathfrak{t}$ when it is clear from the context.

Fix $\mathfrak{p} = \left(\bigotimes_{j=1}^{l} \mathfrak{m}_{i_j}\right) \otimes \mathfrak{u}_d^{\otimes r}$ a planar canonical element and let $\lambda = \Xi^{-1}(\mathfrak{p})$, where Ξ is defined in Lemma 3.5.8. Denote by $L_n(\lambda)$ the left $T_n^d(\delta)$-module $L_n(\mathfrak{p})$.

Example 3.7.1. Let $d = 2$ and $\mathfrak{p} = \mathfrak{m}_1 \otimes \mathfrak{m}_2 \otimes \mathfrak{u}_2$, that is $\lambda = (1, 2)$. Then $\mathcal{L}_\mathfrak{p} = \{\mathfrak{p}, \boldsymbol{\nu}_2\mathfrak{p},\ \boldsymbol{\nu}_1\boldsymbol{\nu}_2\mathfrak{p},\ \boldsymbol{\nu}_0\boldsymbol{\nu}_1\boldsymbol{\nu}_2\mathfrak{p},\ \boldsymbol{\mu}_0\boldsymbol{\nu}_1\boldsymbol{\nu}_2\mathfrak{p},\ \boldsymbol{\mu}_2\mathfrak{p}\}$ is a basis for $L_5((1, 2))$. We have $\boldsymbol{\mu}_1 \diamond \mathfrak{p} = 0$, but $\boldsymbol{\nu}_2 \diamond \mathfrak{p} = \boldsymbol{\nu}_2\mathfrak{p} \in L_5((1, 2))$, and $\mathfrak{p} \diamond \mathfrak{p} = \delta\mathfrak{p} \in L_5((1, 2))$.

Lemma 3.7.2. *If $\delta \neq 0$ or $d \nmid n$ then $L_n(\lambda)$ is a left $T_n^{d,r+1}(\delta)$-module. Furthermore,*

1. *We have $T_n^{d+r+1}(\delta)\mathfrak{e}_{r,d}L_n(\lambda) = L_n(\lambda)$,*

2. *and* $\operatorname{rad}(L_n(\lambda)) = \{\mathfrak{t} \in L_n(\lambda) \mid \mathfrak{e}_{r,d} T_n^{d,r+1}(\delta)\mathfrak{t} = 0\}$.

Proof. For any $\mathfrak{t} \in \mathcal{T}_n^d$ we have $\#(\mathfrak{e}_{r+1,d}\mathfrak{t}\mathfrak{p}) \leq \#(\mathfrak{e}_{r+1,d}\mathfrak{p}) < \#(\mathfrak{p})$, then $J_{r+1}L_n(\lambda) = 0$, hence $L_n(\lambda)$ is a left $T_n^{d,r+1}(\delta)$-module.

1. From the definition of $\mathcal{L}$-equivalence relation we have ${}_{T_n^d(\delta)}L_n(\lambda)$ is a cyclic module generated by any element of $\mathcal{L}_\mathfrak{p}$. Since $\delta \neq 0$ or $d \nmid n$ we have $\mathfrak{p} \notin J_{r+1}$, moreover $\mathfrak{p} = (\mathfrak{p}\mathfrak{e}_{r,d} + J_{r+1}) \diamond \mathfrak{p} \in T_n^{d,r+1}(\delta)\mathfrak{e}_{r,d}L_n(\lambda)$ hence $T_n^{d,r+1}(\delta)\mathfrak{e}_{r,d}L_n(\lambda) = L_n(\lambda)$.

2. Let $N = \{\mathfrak{t} \in L_n(\lambda) \mid \mathfrak{e}_{r,d}T_n^{d,r+1}(\delta)\mathfrak{t} = 0\}$ then $N \subseteq \operatorname{rad}(L_n(\lambda))$, otherwise there exist a maximal submodule M of $L_n(\lambda)$ such that $N \nsubseteq M$, and hence $M + N = L_n(\lambda)$. By part 1 we have $L_n(\lambda) = T_n^{d,r+1}(\delta)\mathfrak{e}_{r,d}L_n(\lambda) = T_n^{d,r+1}(\delta)\mathfrak{e}_{r,d}(N + M) = T_n^{d,r+1}(\delta)\mathfrak{e}_{r,d}M \subseteq M$ which is a contradiction. On the other hand, ${}_{T_{n-dr}^{d,1}}\,\mathfrak{e}_{r,d}L_n(\lambda)$ is one dimensional $\mathbb{F}$-space, since $\mathfrak{e}_{r,d}\mathfrak{p}\,\mathcal{L}\,\mathfrak{p}$ and $\#(\mathfrak{e}_{r,d}\mathfrak{t}) < \#(\mathfrak{p})$ if $\mathfrak{e}_{r,d}\mathfrak{t} \neq \mathfrak{e}_{r,d}\mathfrak{p}$, hence it is simple. Therefore, by part 1 $\mathfrak{e}_{r,d}\operatorname{rad}(L_n(\lambda)) = \mathfrak{e}_{r,d}\operatorname{rad}(T_n^{d,r+1}(\delta))L_n(\lambda) = \mathfrak{e}_{r,d}\operatorname{rad}(T_n^{d,r+1}(\delta))T_n^{d,r+1}(\delta)\mathfrak{e}_{r,d}L_n(\lambda) = \operatorname{rad}(T_{n-dr}^{d,1})\mathfrak{e}_{r,d}L_n(\lambda) = 0$, and hence $N = \operatorname{rad}(L_n(\lambda))$. □

Proposition 3.7.3. *Let $\delta \neq 0$ or $d \nmid n$ then the left $T_n^d(\delta)$-module $L_n(\lambda)$ is indecomposable with simple head and* $\operatorname{rad}(L_n(\lambda))$ *is maximal.*

Proof. Let L_1 and L_2 be two submodules of $L_n(\lambda)$ such that $L_n(\lambda) = L_1 \oplus L_2$ then $\mathfrak{e}_{r,d}L_1 = 0$ or $\mathfrak{e}_{r,d}L_2 = 0$, since $\mathfrak{e}_{r,d}L_n(\lambda)$ is a simple $T_{n-dr}^{d,1}$ module. Without loss of generality we assume $\mathfrak{e}_{r,d}L_2 = 0$ then by Lemma 3.7.2 part 1 we have $L_n(\lambda) = T_n^{d,r+1}(\delta)\mathfrak{e}_{r,d}L_n(\lambda) = T_n^{d,r+1}(\delta)\mathfrak{e}_{r,d}(L_1 \oplus L_2) \subseteq L_1$, hence $L_n(\lambda) = L_1$. Therefore, $L_n(\lambda)$ is indecomposable $T_n^{d,r+1}(\delta)$ module.

We prove $L_n(\lambda)$ has a simple head by showing $\operatorname{rad}(L_n(\lambda))$ is the unique maximal submodule of $L_n(\lambda)$. To this end, let M be a submodule of $L_n(\lambda)$ such that $\operatorname{rad}(L_n(\lambda)) \subseteq M$ and $\operatorname{rad}(L_n(\lambda)) \neq M$. If there is an element $\mathfrak{a} \in M \setminus \operatorname{rad}(_n(\lambda))$ then $\mathfrak{e}_{r,d}T_n^{d,r+1}(\delta)\mathfrak{a} \neq 0$, and hence $\mathfrak{e}_{r,d}L_n(\lambda) = \mathfrak{e}_{r,d}T_n^{d,r+1}(\delta)\mathfrak{a} \subseteq M$. Therefore, $\mathfrak{e}_{r,d}\mathfrak{p} \in M$, hence $\mathfrak{p} = \mathfrak{p} \diamond \mathfrak{e}_{r,d}\mathfrak{p} \in M$ which implies $M = L_n(\lambda)$.

Now by restriction of the action of $T_n^{d,r+1}(\delta)$ to $T_n^d(\delta)$ we obtain the result. $\square$

Proposition 3.7.4. *If $\delta \neq 0$ or $d \nmid n$ then $L_n(\lambda)$ is the projective cover of* $\text{head}(L_n(\lambda))$ *as a left $T_n^{d,r+1}(\delta)$-module. Moreover, the set $\{\text{head}(L_n(\lambda)) \mid \lambda \in \Lambda(T_n^d(\delta))\}$ is a complete list of isomorphism classes of simple $T_n^d(\delta)$-modules.*

Proof. The module ${}_{T_{n-dr}^{d,1}}\mathfrak{e}_{r,d}L_n(\lambda)$ is projective simple and $\mathcal{G}_{\mathfrak{e}_{r,d}}$ preserves the indecomposable projectivity hence it is enough to show ${}_{T_n^{d,r+1}(\delta)}\mathcal{G}_{\mathfrak{e}_{r,d}}(\mathfrak{e}_{r,d}L_n(\lambda)) \simeq {}_{T_n^{d,r+1}(\delta)}L_n(\lambda)$. The pair $(\mathcal{G}_{\mathfrak{e}_{r,d}}, \mathcal{F}_{\mathfrak{e}_{r,d}})$ is adjunction hence $\text{Hom}_{T_n^{d,r+1}(\delta)}(\mathcal{G}_{\mathfrak{e}_{r,d}}(\mathfrak{e}_{r,d}L_n(\lambda)), L_n(\lambda)) \simeq \text{Hom}_{T_{n-dr}^{d,1}}(\mathfrak{e}_{r,d}L_n(\lambda), \mathcal{F}_{\mathfrak{e}_{r,d}}(L_n(\lambda))) = \text{End}_{T_{n-dr}^{d,1}}(\mathfrak{e}_{r,d}L_n(\lambda))$. This means there is a non-zero module morphism

$$\theta : {}_{T_n^{d,r+1}(\delta)}\mathcal{G}_{\mathfrak{e}_{r,d}}(\mathfrak{e}_{r,d}L_n(\lambda)) \to {}_{T_n^{d,r+1}(\delta)}L_n(\lambda)$$

such that θ restricted to $\mathfrak{e}_{r,d}L_n(\lambda)$ is module isomorphism. The map θ is surjective, for if not by Proposition 3.7.3 we have $\theta(\mathcal{G}_{\mathfrak{e}_{r,d}}(\mathfrak{e}_{r,d}L_n(\lambda)) \subseteq \text{rad}(L_n(\lambda))$ and hence $0 = \mathfrak{e}_{r,d}T_n^{d,r+1}(\delta)\theta(\mathcal{G}_{\mathfrak{e}_{r,d}}(\mathfrak{e}_{r,d}L_n(\lambda)) = \theta(\mathfrak{e}_{r,d}L_n(\lambda))$, which is a contradiction.

We have $\mathcal{G}_{\mathfrak{e}_{r,d}}(\mathfrak{e}_{r,d}L_n(\lambda)) = T_n^{d,r+1}(\delta)\mathfrak{e}_{r,d} \otimes_{T_{n-dr}^{d,1}} ((\mathfrak{e}_{r,d} + J_{r+1}) \diamond \mathfrak{p})$ and it is spanned by the set $B = \{\mathfrak{t}\mathfrak{e}_{r,d}\mathfrak{p}\mathfrak{e}_{r,d} + J_{r+1} \otimes_{T_{n-dr}^{d,1}} ((\mathfrak{e}_{r,d} + J_{r+1}) \diamond \mathfrak{p}) \mid \mathfrak{t} \in \mathcal{T}_n^d\}$. Write $f(\mathfrak{t})$ for $\mathfrak{t}\mathfrak{e}_{r,d}\mathfrak{p}\mathfrak{e}_{r,d} + J_{r+1} \otimes_{T_{n-dr}^{d,1}} ((\mathfrak{e}_{r,d} + J_{r+1}) \diamond \mathfrak{p})$, we show that if $f(\mathfrak{t}) \neq 0$ then there exist $\mathfrak{t}' \in \mathcal{L}_{\mathfrak{p}}$ such that $f(\mathfrak{t}) = f(\mathfrak{t}')$. Hence we have $\dim_{\mathbb{F}}(\mathcal{G}_{\mathfrak{e}_{r,d}}(\mathfrak{e}_{r,d}L_n(\lambda)) \leq \dim_{\mathbb{F}}(L_n(\lambda))$. Therefore, by the rank-nullity theorem θ is an isomorphism. To this end, let $f(\mathfrak{t}) \neq 0$ and $\mathfrak{t}\mathfrak{e}_{r,d}\mathfrak{p}$ be $\mathscr{J}$- equivalent to a canonical element $\mathfrak{p}_1$. Then it is evident from the definition of $\preceq$ that $\bar{\mathfrak{p}}_1 \preceq \bar{\mathfrak{p}}$. Moreover, since $J_{r+1}\mathcal{G}_{\mathfrak{e}_{r,d}}(\mathfrak{e}_{r,d}L_n(\lambda)) = 0$ we must have $\overline{\mathfrak{p}_1} \npreceq \overline{\mathfrak{e}_{r+1,d}}$. Let $\mathfrak{t}\mathfrak{e}_{r,d}\mathfrak{p} = \mathfrak{t}_1\mathfrak{p}_1\mathfrak{t}_2$ for some $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathcal{T}_n^d$. If $\bar{\mathfrak{p}}_1 \prec \bar{\mathfrak{p}}$ then $\mathfrak{p}_1 = \mathfrak{p}_1\mathfrak{e}_{r,d}\mathfrak{p}_1$. Hence $f(\mathfrak{t}) = \alpha f(\mathfrak{t}\mathfrak{e}_{r,d}\mathfrak{p}) = \alpha f(\mathfrak{t}_1\mathfrak{p}_1\mathfrak{t}_2) = \alpha f(\mathfrak{t}_1\mathfrak{p}_1\mathfrak{e}_{r,d}\mathfrak{p}_1\mathfrak{t}_2) = 0$ for some $\alpha \in \mathbb{F}$, which is a contradiction. Therefore, we have $\mathfrak{p}_1 = \mathfrak{p}$, by Lemma 3.3.8 and the fact that $\{\mathbf{p} \cap \underline{n}' \mid \mathbf{p} \in \mathfrak{p}\} = \{\mathbf{p} \cap \underline{n}' \mid \mathbf{p} \in \mathfrak{t}\mathfrak{e}_{r,d}\mathfrak{p}\}$ we obtain $\mathfrak{t}\mathfrak{e}_{r,d}\mathfrak{p} \,\mathcal{L}\, \mathfrak{p}$ and $f(\mathfrak{t}) = \alpha f(\mathfrak{t}\mathfrak{e}_{r,d}\mathfrak{p})$ for some $0 \neq \alpha \in \mathbb{F}$.

By restriction of the action of $T_n^{d,r+1}(\delta)$ to $T_n^d(\delta)$ we obtain that $L_n(\lambda)$ is an indecomposable $T_n^d(\delta)$-module with simple head. The completeness of the list follows from Lemma 3.5.8. □

We introduce a partial order on the labelling set $\bigsqcup_{i=0}^{\lfloor n/d \rfloor} \mathcal{C}^d(n-id)$ of simple modules of $T_n^d(\delta)$ in the following way; for $\lambda, \mu \in \bigsqcup_{i=0}^{\lfloor n/d \rfloor} \mathcal{C}^d(n-id)$ we say $\lambda \leq_\Delta \mu$ if and only if $\overline{\Xi(\mu)} \preceq \overline{\Xi(\lambda)}$. Recall that, Ξ is defined Lemma 3.5.8, $\overline{\Xi(\lambda)}$ and $\preceq$ are defined in Section 1.3.

Proposition 3.7.5. *Let $\delta \neq 0$ or $d \nmid n$ then the set $\{L_n(\lambda) \mid \lambda \in \bigsqcup_{i=0}^{\lfloor n/d \rfloor} \mathcal{C}^d(n-id)\}$ is a complete set of pairwise non-isomorphic standard modules of $T_n^d(\delta)$ with the hereditary order $\leq_\Delta$.*

Proof. We first show that the given set of modules with the order $\leq_\Delta$ satisfies I of Theorem 1.4.1. That is we wish to show if $[L_n(\lambda) : \text{head}(L_n(\mu))] \neq 0$ then $\mu \leq_\Delta \lambda$. Fix $\lambda \in \bigsqcup_{i=0}^{\lfloor n/d \rfloor} \mathcal{C}^d(n-id)$ and suppose $[L_n(\lambda) : \text{head}(L_n(\mu))] \neq 0$ for some $\mu \in \bigsqcup_{i=0}^{\lfloor n/d \rfloor} \mathcal{C}^d(n-id)$ with $\mu \not\leq_\Delta \lambda$. From the definition of $\leq_\Delta$ we have $\mu \not\leq_\Delta \lambda$ implies that $\overline{\Xi(\lambda)} \not\preceq \overline{\Xi(\mu)}$. Further, from 3.24 we have $\Xi(\mu)L_n(\lambda) = 0$ and $\Xi(\mu)\,\text{head}(L_n(\mu)) \neq 0$. Therefore, $\text{head}(L_n(\mu))$ is not a factor of $L_n(\lambda)$ which is a contradiction.

Second we show that the given set of modules are standard modules then by the uniqueness property of the standard modules we will be done. To prove that we fix d and use induction on n. Assume the result is true for $T_n^d(\delta)$. By using induction step, Proposition 1.4.4 and the fact that $L_{n+d}(\lambda) = \mathcal{G}_{\mathfrak{e}_{1,d}}(L_n(\lambda))$ we have $L_{n+d}(\lambda) = \mathcal{G}_{\mathfrak{e}_{1,d}}(L_n(\lambda))$ is a standard module for each $\lambda \in \Lambda(T_n^d(\delta))$. If $\lambda \in \Lambda(T_{n+d}^{d,1})$ then by Proposition 3.7.4 we have $L_{n+d}(\lambda) = \text{head}(L_{n+d}(\lambda))$. Hence, $L_{n+d}(\lambda)$ is also standard. □

Remark 3.7.6. For each $\lambda \in \bigsqcup_{i=0}^{\lfloor n/d \rfloor} \mathcal{C}^d(n-id)$, from now on we write $\Delta_n(\lambda)$ for the standard module $L_n(\lambda)$ to match the standard notation of quasi-hereditary algebras.

3.7.2 Restriction rules for the standard modules of $T_n^d(\delta)$.

For each $n \in \mathbb{N}$ the algebra $T_n^d(\delta)$ can be embedded in $T_{n+1}^d(\delta)$. A natural question would be, how the modules of $T_{n+1}^d(\delta)$ behave under the action of $T_n^d(\delta)$? In this subsection we answer this question for the standard modules, which arguably are the most important types of $T_n^d(\delta)$-modules. One may also ask, why the answer of the above question is important? One of the main advantages of understanding the action of $T_n^d(\delta)$ on standard $T_{n+1}^d(\delta)$-modules is, it enables us to use the so called "Frobenius reciprocity". For example, if $d = 2$ then as we shall see in section 3.9 globalising a standard modules using the functor $\mathcal{G}$ and then restricting the globalised module to the action of $T_{n+1}^2(\delta)$ is the same as inducing the initial standard module. This compatibility of local behaviour reduces considerably the amount of work needed to understand the structure of $T_n^2(\delta)$, see Theorem 3.9.2. Furthermore, we shall also see that in the subsection 3.8.2 understanding the restriction of the standard models can be very useful combinatorially.

Theorem 3.7.7. *Let* $\mathbf{res}_{n+1} : T_{n+1}^d(\delta) \to T_n^d(\delta)$ *be the right restriction corresponding to the embedding (defined on the diagram basis of* $T_n^d(\delta)$ *and extended linearly)*

$$\begin{aligned} \mathbf{inc}_n : T_n^d(\delta) &\hookrightarrow T_{n+1}^d(\delta) \\ \mathfrak{d} &\mapsto \mathfrak{d} \otimes \mathfrak{m}_1 \end{aligned} \tag{3.25}$$

Let $\lambda = (i_1, i_2 \ldots, i_l) \in \bigsqcup_{i=0}^{\lfloor n/d \rfloor} \mathcal{C}^d(n-id)$, $\lambda_1 = (i_1 \ldots, i_l, d-1)$, $\lambda_2 = (i_1, \ldots, i_{l-1}, i_l - 1)$, $\lambda_3 = (i_1, \ldots, i_{l-1})$ *and* $\lambda_4 = (i_1, \ldots, i_{l-1}, d)$. *Let* $r := \frac{n-\sum_{i=1}^{l} i_j}{d}$ *then we have the following short exact sequences of* $T_n^d(\delta)$*-modules:*

when $r \neq 0$ *and* $1 < i_l \leq d$

$$0 \to \Delta_n(\lambda_2) \to \mathbf{res}_{n+1}(\Delta_{n+1}(\lambda)) \to \Delta_n(\lambda_1) \to 0 \tag{3.26}$$

when $r \neq 0$ *and* $i_l = 1$

$$0 \to \Delta_n(\lambda_3) \oplus \Delta_n(\lambda_4) \to \mathbf{res}_{n+1}(\Delta_{n+1}(\lambda)) \to \Delta_n(\lambda_1) \to 0 \tag{3.27}$$

if $r \neq 0$ *and* $n = rd$ *we have*

$$\mathbf{res}_{n+1}(\Delta_{n+1}((0))) \simeq \Delta_n((d-1)) \tag{3.28}$$

and if $r = 0$ *we have*

$$\mathbf{res}_{n+1}(\Delta_{n+1}(\lambda)) \simeq \Delta_n(\lambda_2) \tag{3.29}$$

Proof. First we would like to remark that the arguments of this proof is a generalisation of the case when $d = 1$, which can be found, for example, in [68] Proposition 4.1.

We begin our proof by the case when $r \neq 0$ and $1 < i_l \leq d$. Let $\mathcal{H}$ be the set of all basis elements $\mathfrak{d} \in \Delta_{n+1}(\lambda)$ with the property that $n+1$ is in a propagating part. The only possible case here is that $n+1$ is connected to the right most propagating part, since otherwise we would have a non-planar diagram. We also let $\mathbb{F}\mathcal{H}$ be the $\mathbb{F}$-subspace of $\Delta_{n+1}(\lambda)$ spanned by the set $\mathcal{H}$. In fact $\mathbb{F}\mathcal{H}$ is a $T_n^d(\delta) \simeq \mathbf{inc}_n(T_n^d(\delta))$ submodule of $\Delta_{n+1}(\lambda)$. To prove that, let $\mathfrak{s} \in \mathcal{H}$ and $\mathfrak{d} \in \mathcal{T}_n^d(\delta)$. If $\mathbf{inc}_n(\mathfrak{d}) \diamond \mathfrak{s} \neq 0$ then the vertex $n+1$ in $\mathbf{inc}_n(\mathfrak{d})\mathfrak{s}$ remains connected to the first propagating part from right, since $\mathbf{inc}_n(T_n^d(\delta))$ fixes $n+1$ and does not permute the propagating parts. Moreover, $\mathbb{F}\mathcal{H}$ is equal to the left $\mathbf{inc}_n(T_n^d(\delta))$- module generated by $\mathfrak{q}$ where $\mathfrak{q} := (\mathfrak{m}_1^{\otimes(\sum_{t=1}^{l} i_t - 1)} \otimes \mathfrak{u}_d^{\otimes r} \otimes \mathfrak{m}_1)\mathfrak{p}$, since by multiplying by suitable $\boldsymbol{\nu}_j$ as many times as necessary, where $0 \leq j \leq n-d$, we can move northern non-propagating parts of $\mathfrak{q}$ to left, in the same way as it has been done in the Lemma 3.3.2. For $0 \leq i \leq n-2$, by using an appropriate $\boldsymbol{\mu}_i$ we can connect these northern non-propagating parts with a desired propagating part.

Let Ξ be as defined in 3.5.8 and $\mathfrak{d}$ be a diagram basis element of $\Delta(\lambda_2)$. Let $\mathbf{p}$ be the part of $\mathfrak{d}$ containing $(\sum_{j=1}^{l} i_j - 1)'$, define $\mathfrak{d}_{\lambda_2} = (\mathfrak{d} \setminus \mathbf{p}) \cup \{\mathbf{p}'\}$ such that

$\mathbf{p}' = \mathbf{p} \cup \{n+1, (n+1)'\}$ then $\mathfrak{d}_{\lambda_2}\mathfrak{p} \in \mathcal{H}$, for example $\Xi(\lambda_2)_{\lambda_2}\mathfrak{p} = \mathfrak{q}$. The following map which is defined on the diagram basis is an isomorphism of $T_n^d(\delta)$-modules;

$$\begin{aligned} \phi : \Delta_n(\lambda_2) &\to \mathbb{F}\mathcal{H} \\ \mathfrak{d} &\mapsto \mathfrak{d}_{\lambda_2}\mathfrak{p} \end{aligned} \tag{3.30}$$

It is not hard to see ϕ is invertible map on basis, one way is by reversing the operation of defining ϕ. To show ϕ is a morphism of $T_n^d(\delta)$-modules, let $\mathfrak{d}$ be a basis element of $\Delta(\lambda_2)$ then $(\boldsymbol{\nu}_i\mathfrak{d})_{\lambda_2} = \mathbf{inc}_n(\boldsymbol{\nu}_i)\mathfrak{d}_{\lambda_2}$ and $(\boldsymbol{\mu}_j\mathfrak{d})_{\lambda_2} = \mathbf{inc}_n(\boldsymbol{\mu}_j)\mathfrak{d}_{\lambda_2}$, for $0 \leq i \leq n-d$ and $0 \leq j \leq n-2$. Therefore, ϕ is a module morphism.

Let $\mathfrak{q}' = (\mathfrak{m}_1^{\otimes(\sum_{t=1}^{l} i_t + d - 1)} \otimes \mathfrak{u}_d^{\otimes r-1} \otimes \mathfrak{m}_1)\mathfrak{p}$ then $L_{n+1}(\mathfrak{q}') = \Delta_{n+1}(\lambda)$ as a left $T_{n+1}^d(\delta)$-module. Let $\mathfrak{d}$ be a diagram basis of $\Delta_n(\lambda_1)$ and let $\mathbf{p}$ be the part of $\mathfrak{d}$ containing $(\sum_{j=1}^{l} i_j + 1)'$, define $\mathfrak{d}_{\lambda_1} = (\mathfrak{d} \setminus \mathbf{p}) \cup \{\mathbf{p}'\}$ such that $\mathbf{p}' = \mathbf{p} \cup \{n+1, (n+1)'\}$ then $\mathfrak{d}_{\lambda_1}\mathfrak{p} + \mathbb{F}\mathcal{H} \in \Delta_{n+1}(\lambda)/\mathbb{F}\mathcal{H}$, for example $\Xi(\lambda_1)_{\lambda_1}\mathfrak{p} = \mathfrak{q}'$.

Similar to the morphism ϕ we have the following isomorphism of left $T_n^d(\delta)$-modules

$$\begin{aligned} \psi : \Delta_n(\lambda_1) &\to \Delta_{n+1}(\lambda)/\mathbb{F}\mathcal{H} \\ \mathfrak{d} &\mapsto \mathfrak{d}_{\lambda_1}\mathfrak{p} + \mathbb{F}\mathcal{H} \end{aligned} \tag{3.31}$$

Hence we have we the short exact sequence 3.26.

Consider the case when $r \neq 0$ and $i_l = 1$, let $\mathcal{H}_1$ to be the set of all elements $\mathfrak{d} \in \Delta_{n+1}(\lambda)$ such that $n+1$ is in a propagating part $\mathbf{p}$ with the property $|\mathbf{p} \cap \underline{n}| = 1$. Then $\mathbb{F}\mathcal{H}_1$ is a left $T_n^d(\delta)$-module. Moreover, one can argue as above and show that $\mathbb{F}\mathcal{H}_1 \simeq \Delta_n(\lambda_3)$ as a left a left $T_n^d(\delta)$-module. Similarly, we have ${}_{T_n^d(\delta)}\mathbb{F}\mathcal{H}_2 \simeq {}_{T_n^d(\delta)}\Delta_n(\lambda_4)$ where $\mathcal{H}_2$ is the set of all elements $\mathfrak{d} \in \Delta_{n+1}(\lambda)$ such that $n+1$ is in a propagating part $\mathbf{p}$ such that $|\mathbf{p} \cap \underline{n+1}| > 1$. Further, since $\mathcal{H}_1 \cap \mathcal{H}_2 = \phi$ we have $\mathbb{F}\mathcal{H}_1 \oplus \mathbb{F}\mathcal{H}_2 = \mathbb{F}\mathcal{H}_1 + \mathbb{F}\mathcal{H}_2$.

Arguing as 3.31 we can obtain an isomorphism of $T_n^d(\delta)$ modules $\Delta_{n+1}(\lambda)/(\mathbb{F}\mathcal{H}_1 \oplus \mathbb{F}\mathcal{H}_2) \simeq \Delta_n(\lambda_1)$.

If $r \neq 0$ and $n = rd$, this case also similar to the first case except here we have $\mathcal{H} = \phi$, and hence $\mathbb{F}\mathcal{H} = 0$.
Finally, if $r = 0$ this case can also be proven by the similar way as the first case and using the fact that both $\mathbf{res}_n(\Delta_{n+1}(\lambda))$ and $\Delta_n(\lambda_2)$ are one dimensional modules. □

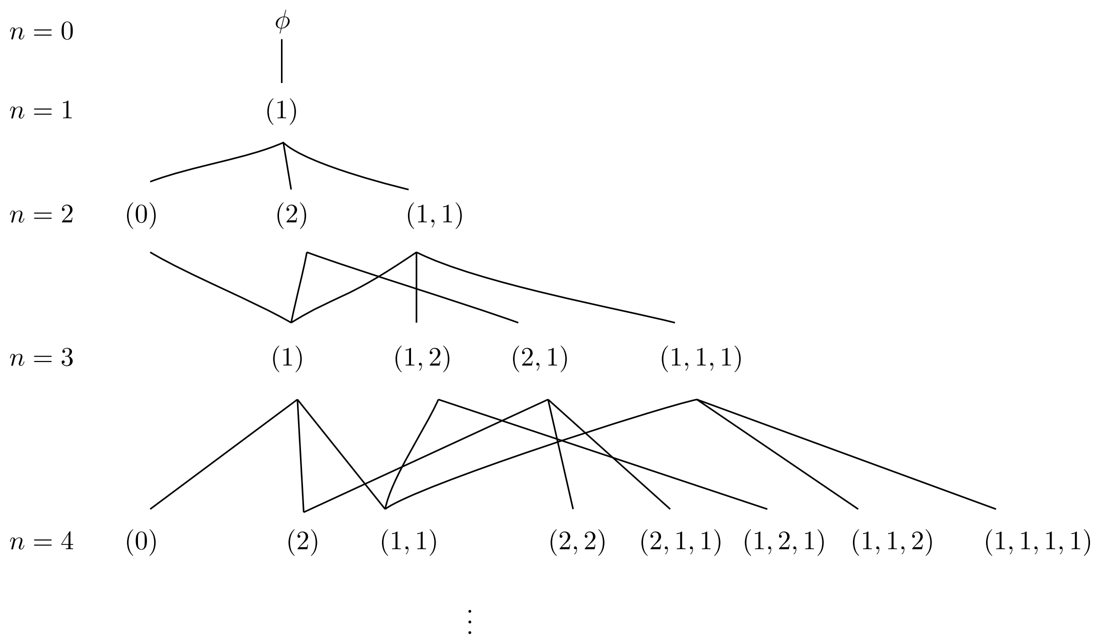


Figure 3.8: Right restriction rule, for $n \leq 4$ and $d = 2$

Remark 3.7.8. In Figure 3.8 we explain the first few terms of restriction of the standard modules $T_n^2(\delta)$. If, for each n, the algebra $T_n^2(\delta)$ is semisimple (see later) then the diagram in Figure 3.8 is the Bratteli diagram for $\{T_n^d(\delta)\}_{n=1}^{\infty}$, see Section 3.8.2.

The diagram 3.8 appears also in [6, Figure 21]. There the authors describe the restriction rules when the algebra $FC_{1,n}(\delta_a, \delta_b)$ is semisimple [6, Theorem 3.2.1]. Theorem 3.7.7 generalises the part of Theorem 3.2.1 of [6] regarding $FC_{1,n}(\delta_a, \delta_a)$, indeed the algebra $FC_{1,n}(\delta_a, \delta_b)$ by generic semisimplicity, in two ways. Firstly, to any algebraically closed field, secondly to include the non-semisimple cases of

$FC_{1,n}(\delta_a, \delta_a)$, except when $\delta_a = 0$.

3.8 On the semisimple cases of $T_n^d(\delta)$.

In this section we prove that for all but finitely many $\delta \in \mathbb{C}$ the algebra $T_n^d(\delta)$ is semisimple. We then present explicitly the simple modules of $T_n^d(\delta)$ in the semisimple case. Furthermore, we construct the Bratelli diagram for $T_n^d(\delta)$ which describes the embedding $T_n^d(\delta)$ in to $T_{n+1}^d(\delta)$. Finally by using the action of $T_n^d(\delta)$ on $V^{\otimes k}$ we show that $T_n^d(1)$ is not semisimple for sufficiently large n.

3.8.1 Generic Semisimplicity of $T_n^d(\delta)$.

Let M be a finite dimensional left A-module, and t be as defined on page 17. An $\mathbb{F}$-bilinear form $\langle -, - \rangle$ on ${}_AM$ is called *contravariant* if

$$\langle \alpha.a, b \rangle = \langle a, \alpha^t.b \rangle \qquad \text{for all } \alpha \in A \text{ and } a, b \in M$$

We denote by $R_{\langle -,- \rangle}$ the set of all elements of $a \in M$ such that $\langle a, b \rangle = 0$ for all $b \in M$. In fact $R_{\langle -,- \rangle}$ is a submodule M and it will be called the *radical* of $\langle -, - \rangle$.

Let $\Delta_n(\lambda)$ be as given in Remark 3.7.6. We define a contravariant form on the basis of $\Delta_n(\lambda)$, denote it by $\langle -, - \rangle_\lambda$, in the following way: for diagram basis elements $\mathfrak{a}, \mathfrak{b} \in \Delta_n(\lambda)$, that is $\mathfrak{a}, \mathfrak{b} \in \mathcal{L}_{\Xi_{(\lambda)}}$ where Ξ is defined in Lemma 3.5.8 and $\mathcal{L}_{\Xi_{(\lambda)}}$ is the left $\mathcal{L}$-class contains $\Xi(\lambda)$, we define the following bilinear form on the basis of $\Delta_n(\lambda)$ and extend bilinearly to $\Delta_n(\lambda)$,

$$\langle \mathfrak{a}, \mathfrak{b} \rangle_\lambda := \begin{cases} \delta^{\kappa_{\mathfrak{a}^\star, \mathfrak{b}}} & \text{if } \mathfrak{a}^\star \circ \mathfrak{b} \ \mathcal{L} \ \Xi(\lambda) \\ 0 & \text{otherwise} \end{cases} \tag{3.32}$$

Where $\star$ is defined in Lemma 3.6.4. and $\kappa_{-,-}$ is defined in Proposition 1.1.5.

Example 3.8.1. Continue from Example 3.7.1, that is $\lambda = (1, 2)$ and $d = 2$. To illustrate how the bilinear form $\langle -, - \rangle_\lambda$ works we calculate $\langle -, - \rangle_\lambda$ for some cases as follows:

$$\begin{aligned}
\langle \mathfrak{p}, \mathfrak{p} \rangle_\lambda &= \delta, \\
\langle \mathfrak{p}, \boldsymbol{\nu}_0\boldsymbol{\nu}_1\boldsymbol{\nu}_2\mathfrak{p} \rangle_\lambda = \langle \boldsymbol{\nu}_0\boldsymbol{\nu}_1\boldsymbol{\nu}_2\mathfrak{p}, \mathfrak{p} \rangle_\lambda &= 0, \\
\langle \boldsymbol{\nu}_0\boldsymbol{\nu}_1\boldsymbol{\nu}_2\mathfrak{p}, \boldsymbol{\nu}_0\boldsymbol{\nu}_1\boldsymbol{\nu}_2\mathfrak{p} \rangle_\lambda = \langle \boldsymbol{\nu}_1\boldsymbol{\nu}_2\mathfrak{p}, \boldsymbol{\nu}_1\boldsymbol{\nu}_2\mathfrak{p} \rangle_\lambda = \langle \boldsymbol{\nu}_2\mathfrak{p}, \boldsymbol{\nu}_2\mathfrak{p} \rangle_\lambda &= \delta, \\
\langle \boldsymbol{\mu}_2\mathfrak{p}, \mathfrak{p} \rangle_\lambda = \langle \boldsymbol{\mu}_2\mathfrak{p}, \boldsymbol{\nu}_2\mathfrak{p} \rangle_\lambda = \langle \boldsymbol{\mu}_2\mathfrak{p}, \boldsymbol{\mu}_2\mathfrak{p} \rangle_\lambda &= 1.
\end{aligned}$$

We record some properties the above function in the following Lemma, which has a straightforward proof.

Lemma 3.8.2. *For each λ the function given in equation 3.32 is symmetric contravariant $\mathbb{F}$-bilinear form on $\Delta_n(\lambda)$.* □

Note this form defined in Equation 3.32 is analogue of the usual contravariant form on the standard modules of the Temperley-Lieb algebras, see for example Equation 3.5 of [68].

Definition 3.8.3. *Let M be a finite dimensional $\mathbb{F}$-vector space with ordered basis $\{m_1, m_2, \dots, m_n\}$. Suppose $\langle -, - \rangle$ is an $\mathbb{F}$-bilinear form on M. Then the matrix $G(M) = (\langle m_i, m_j \rangle)_{n \times n}$ is called the gram matrix of M with respect to the given ordered basis and the form $\langle -, - \rangle$.*

In general there might be more than one contravariant bilinear form on a given module ${}_AM$. The following propositions give a necessary and sufficient condition for the uniqueness of a bilinear form of ${}_AM$.

Proposition 3.8.4 (Section 2.7 of [33])**.** *Let A be a finite dimensional $\mathbb{F}$-algebra with an involutory anti-automorphism t, as defined on page 17. For each finite dimensional module ${}_AM$, there is a one to one correspondence between contravariant forms $\langle -, - \rangle : M \times M \to \mathbb{F}$ and morphisms $f \in \mathrm{Hom}_A({}_AM, {}_AM^\circ)$ given by*

$$f(a)(b) = \langle a, b \rangle \qquad \text{for all } a, b \in M.$$

□

Proposition 3.8.5 (see for e.g. Proposition B10,Corollary B 11[10])**.** *Let A and M be given as in Proposition 3.8.4. Suppose M has a simple* head *which is self dual and* $\text{rad}(M)$ *is maximal submodule of M, and $[M : \text{head}(M)] = 1$. Then there exist only one form on M up to multiplication by scalars.*

The form is non-singular if and only if ${}_AM$ is simple if and only if ${}_AM$ self dual.

Lemma 3.8.6. *Let $\delta \neq 0$ or $d \nmid n$. Then for each $\lambda \in \Lambda(T_n^d(\delta))$ we have $R_{\langle -,- \rangle_\lambda} =$* $\text{rad}(\Delta_n(\lambda))$.

Proof. From Proposition 3.7.3 we have $R_{\langle -,- \rangle_\lambda} \subseteq \text{rad}(\Delta_n(\lambda))$. Let $\mathfrak{t} \in \Delta_n(\lambda) \setminus R_{\langle -,- \rangle_\lambda}$ then there exist $\mathfrak{s} \in \Delta_n(\lambda)$ and $\alpha \in \mathbb{C}$ (in fact $\alpha \in \mathbb{Z}[\delta]$) such that $0 \neq \mathfrak{s}^\star \diamond \mathfrak{t} = \alpha \Xi(\lambda)$. From the definition of $\Delta_n(\lambda)$ we may be able to consider $\mathfrak{s}$ as an element of $T_n^d(\delta)$. Let r be the number of northern non-propagating parts of $\Xi(\lambda)$, the definition of $\preceq$ implies that $\mathfrak{s}\mathfrak{e}_{r,d} \notin J_{r+1}$. Therefore, $\mathfrak{e}_{r,d}\mathfrak{s}^\star + J_{r+1} \in \mathfrak{e}_{r,d}T^{d,r+1}(\delta)$ and $(\mathfrak{e}_{r,d}\mathfrak{s}^\star + J_{r+1}) \diamond \mathfrak{t} \neq 0$. By Lemma 3.7.2 we have $\mathfrak{t} \notin \text{rad}(\Delta_n(\lambda))$ and hence $R_{\langle -,- \rangle_\lambda} = \text{rad}(\Delta_n(\lambda))$. □

Lemma 3.8.7. *Let $\mathbb{F} = \mathbb{C}$. The determinant of the gram matrix of $\Delta_n(\lambda)$ with respect to the form 3.32 is non-zero for all but finitely many $\delta \in \mathbb{C}$.*

Proof. For any two basis elements for $\mathfrak{a}, \mathfrak{b} \in \Delta_n(\lambda)$ we have $\langle \mathfrak{a}, \mathfrak{b} \rangle$ is a monomial in δ. Let $G(\lambda)$ denote the gram matrix of $\Delta_n(\lambda)$ with respect to the form 3.32. Hence the entries of the $G(\lambda)$ are monomials in δ. Let $1 \leq k \leq dim_{\mathbb{C}}(\Delta_n(\lambda))$, organise the basis of $\Delta_n(\lambda)$ by the number of non-propagating parts in order to $G(\lambda)$ has the following form, if the rows k and $k+1$ are the rows of $\mathfrak{a}$ and $\mathfrak{b}$ respectively then the number of northern non-propagating parts of $\mathfrak{a}$ is greater than or equal to the number of northern non-propagating parts of $\mathfrak{b}$. In this case $degree(\langle \mathfrak{a}, \mathfrak{b} \rangle) < degree(\langle \mathfrak{a}, \mathfrak{a} \rangle)$, and hence the rows of $G(\lambda)$ are linearly independent for all but finitely many $\delta \in \mathbb{C}$. □

For us the involutory anti-automorphism t defined on page 17 is the flipping map $\star$ in Lemma 3.6.4.

Corollary 3.8.8. *Let $\delta \neq 0$ or $d \nmid n$. Then for each $\lambda \in \Lambda(T_n^d(\delta))$ we have* $\text{head}(\Delta_n(\lambda)) \simeq \text{head}(\Delta_n(\lambda))^\circ$, *where $\circ$ is defined in Lemma 1.4.6.*

Proof. Follows from Proposition 3.6.6, Lemma 1.4.6 and Proposition 3.7.3. □

Corollary 3.8.9. *Let $\delta \neq 0$ or $d \nmid n$. Then for each $\lambda \in \Lambda(T_n^d(\delta))$ the bilinear form $\langle -, - \rangle_\lambda$ defined on $\Delta_n(\lambda)$ is unique up to scalar multiplication. Furthermore, $\Delta_n(\lambda)$ is simple if and only if the determinant of the gram matrix of $\Delta_n(\lambda)$ with respect to the form $\langle -, - \rangle_\lambda$ is non-zero.*

Proof. For each λ, by Theorem 3.6.3 and Proposition 3.7.3 we have $\Delta_n(\lambda)$ satisfies the conditions of Proposition 3.8.5. □

Proposition 3.8.10. *Let A be a quasi-hereditary algebra with a contavariant duality, as given on page 17. If each standard module is isomorphic to its contavariant dual as an A-module then each standard module is simple.*

Proof. We denote by $\circ$ the given contravariant duality. Keep the notation of Theorem 1.4.1. Let $\lambda \in \Lambda$, in general we have ${}_A(\text{soc}((\Delta(\lambda)))^\circ \simeq {}_A\left((\Delta(\lambda))^\circ / \text{rad}((\Delta(\lambda))^\circ)\right)$(see for e.g. p162 [3]). Hence the socle of ${}_A\Delta(\lambda)$ is isomorphic to the dual of its head. We claim ${}_A\,\text{head}(\Delta(\lambda)) \simeq {}_A(\text{head}(\Delta(\lambda)))^o$, hence $\text{head}(\Delta(\lambda)) \simeq \text{soc}(\Delta_n(\lambda))$ and consequently $\Delta(\lambda)$ is simple. To this end, from the quasi-heredity of A we have $(\text{head}(\Delta(\lambda)))^o = \text{head}(\Delta(\lambda'))$ for an other standard module $\Delta(\lambda')$. Thus ${}_A\,\text{head}(\Delta(\lambda)) \simeq {}_A\,\text{soc}(\Delta(\lambda'))$, but by the the unitriangular property of quasi-hereditary algebras this impossible unless ${}_A\Delta(\lambda') \simeq {}_A\Delta(\lambda)$. □

Corollary 3.8.11. *For all but finitely many $\delta \in \mathbb{C}$, each standard $T_n^d(\delta)$-module $\Delta_n(\lambda)$ is simple.*

Proof. By Lemma 3.8.7 each standard module is isomorphic to its contravariant dual for all but finitely many $\delta \in \mathbb{C}$. Now use Proposition 3.8.10. □

Theorem 3.8.12. *The algebra $T_n^d(\delta)$ is semisimple for all but finitely many $\delta \in \mathbb{C}$, with the complete set of simple modules $\{\Delta_n(\lambda) \mid \lambda \in \bigsqcup_{i=0}^{\lfloor n/d \rfloor} \mathcal{C}^d(n-id)\}$.*

Proof. By Theorem 3.5.4 and Corollary 3.8.11 the modules $\Delta_n(\lambda)$ for $\lambda \in \bigsqcup_{i=0}^{\lfloor n/d \rfloor} \mathcal{C}^d(n-id)$ form a complete list of pairwise non-isomorphic simple modules of $T_n^d(\delta)$. Let $\mathbf{P}$ be the set of all planar canonical elements, Lemma 3.5.8, and $\mathcal{J}_{\mathfrak{p}}$ denote the $\mathcal{J}$-class containing $\mathfrak{p}$. Denote by $J(T_n^d(\delta))$ the Jacobson radical of $T_n^d(\delta)$.Then

$$\begin{aligned}\dim_{\mathbb{C}}(T_n^d(\delta)) &= \sum_{\mathfrak{p}\in\mathbf{P}} |\mathcal{J}_{\mathfrak{p}}| = \sum_{\mathfrak{p}\in\mathbf{P}} |\mathcal{L}_{\mathfrak{p}}||\star(\mathcal{L}_{\mathfrak{p}})| = \sum_{\mathfrak{p}\in\mathbf{P}} |\mathcal{L}_{\mathfrak{p}}|^2 \\ &= \sum_{\mathfrak{p}\in\mathbf{P}} (\dim(\Delta_n(\Xi^{-1}(\mathfrak{p}))))^2 = \dim_{\mathbb{C}}(T_n^d(\delta)/J(T_n^d(\delta)))\end{aligned}$$

such that $\star$ is the map in 3.9. Therefore, $\dim_{\mathbb{C}}(J(T_n^d(\delta))) = 0$ and hence $T_n^d(\delta)$ is semisimple. □

3.8.2 The Bratteli diagram for $\{T_n^d(\delta)\}_n$ in the semisimple cases.

Let

$$A_0 \to A_1 \to \cdots \to A_i \xrightarrow{\iota_i} A_{i+1} \to \dots \tag{3.33}$$

be a chain of embeddings of semisimple $\mathbb{F}$-algebras. Let $\mathbf{res}_i$ be the restriction corresponding to the embedding ι_i. For each $i \in \mathbb{N}$ choose a complete set of pairwise non-isomorphic simple A_i-modules, say $\{\pi_s^i \mid s \in \{1, 2, \dots, |\Lambda(A_i)\}|\}$. By definition if $A_0 = 0$ then we set ϕ to be the only simple module of A_0. The *Bratteli diagram* (see [8], and page 13 of [69] for more examples of Bratteli daigrams) corresponding to the chain 3.33 is a graph constructed pictorially in the following way:

1. If $i = 0$ draw $|\Lambda(A_i)|$ vertices and label them bijectively by $\pi_s^0, i \in \{1, \ldots, |\Lambda(A_0)|\}$. For $i \geq 1$ draw $|\Lambda(A_i)|$ vertices on a horizontal row bellow the row $i-1$. By below we mean the y-coordinate of each vertex of row $i-1$ is greater than the y-coordinate of the vertices of row i. Label the vertices of row i by the labelling set of simple modules of A_i bijectively.

2. For any $i > 1$, for each $s \in \{1, 2 \ldots, |\Lambda(A_i)|\}$ and $r \in \{1, 2 \ldots, |\Lambda(A_{i-1})|\}$ draw $d_{\pi_r^{i-1}}^{\pi_s^i}$ edges between the vertex labelled by π_s^i and the vertex labelled by π_r^{i-1}, where $d_{\pi_r^{i-1}}^{\pi_s^i}$ is the number of times when ${}_{A_{i-1}}\pi_r^{i-1}$ appears up to isomorphism as a composition factor of ${}_{A_{i-1}}\mathbf{res}_i(\pi_s^i)$. If $A_0 = 0$ then draw an edge between each vertex on level 1 and the vertex labelled by ϕ.

A Bratteli diagram is called *multiplicity free* if the number of edges between any two vertices is at most one.

The Bratteli diagram carries both algebraic and combinatorial information about the sequence of algebras (3.33). Algebraically, it encodes the embedding of A_i in to A_{i+1}. Combinatorially, the Bratteli diagram gives information about the dimension of each irreducible representation. In fact, we have

$$dim_{\mathbb{F}}(\pi_s^i) = \sum_{r=1}^{|\Lambda(A_{i-1})|} d_{\pi_r^{i-1}}^{\pi_s^i} dim_{\mathbb{F}}(\pi_r^{i-1})$$

In particular, assume the Bratelli diagram is multiplicity free and $dim_{\mathbb{F}} A_1 = 1$. Then the dimension of π_s^i is equal to the number of different paths between π_1^1 and π_s^i.

In Theorem 3.8.12 we showed that $\{T_n^d(\delta)\}_n$ is semisimple for infinitely many values of $\delta \in \mathbb{C}$, indeed when δ is an element of some Zariski open set in $\mathbb{C}$. Moreover, we showed that $\{\Delta_n(\lambda) \mid \lambda \in \bigsqcup_{i=0}^{\lfloor n/d \rfloor} \mathcal{C}^d(n - id)\}$ is a complete set of pair wise non-isomorphic simple modules of $T_n^d(\delta)$. When $T_n^d(\delta)$ is semisimple the short exact sequences in Theorem 3.7.7 are split. Therefore, for each n the value of $d_{\Delta_{n-1}(\mu)}^{\Delta_n(\lambda)}$ is obtained from the short exact sequences in Theorem 3.7.7. Hence, for each d

we have constructed the Bratteli diagram $\{T_n^d(\delta)\}_n$ with respect to the inclusion given Theorem 3.7.7. Furthermore, Theorem 3.7.7 implies that the Bratteli diagram of $\{T_n^d(\delta)\}_n$ is multiplicity free. Consequently, we obtain the following recurrence relation regarding the dimension of each simple of $T_n^d(\delta)$ when it is semisimple. Hence the dimension of each standard modules when $T_n^d(\delta)$ is not semisimple,

Proposition 3.8.13. *For each* $\lambda \in \Lambda(T_n^d(\delta))$ *the dimension of the standard module* $\Delta_n(\lambda)$ *obeys the following recurrence relation*

$$dim_{\mathbb{F}}(\Delta_n(\lambda)) = \sum_{\mu \in \Lambda(T_{n-1}^d(\delta))} d_{\Delta_{n-1}(\mu)}^{\Delta_n(\lambda)} dim_{\mathbb{F}}(\Delta_{n-1}(\mu)) \tag{3.34}$$

Proof. Follows from Theorem 3.7.7 and the Bratelli diagram argument. □

Example 3.8.14. When $d = 2$ the graph in Figure 3.8 is the Bratteli diagram of $\{T_n^2(\delta)\}_n$ with respect to the inclusion given Theorem 3.7.7.

3.8.3 Action of $T_n^d(\delta)$ on $V^{\otimes n}$.

In this subsection we use the action $T_n^d(\delta)$ on $V^{\otimes n}$ to show that $T_n^d(\delta)$ is not semisimple for some values of δ.

For a finitely generated representation (M, ρ) of A, there is an associated function $\chi_M : A \to \mathbb{F}$ defined by $\chi_M(\alpha) = tr(\rho(\alpha))$, where $tr(\rho(\alpha))$ is the trace of $\rho(\alpha)$ for $\alpha \in A$. The function χ_M is called the *character* of M afforded by the representation ρ.

Theorem 3.8.15 (Theorem 54.16 [15])**.** *Let* A *be a finite dimensional algebra over an algebraically closed field* $\mathbb{F}$*, and let* f *be a primitive idempotent in* A*. Then for a finite dimensional left* A*-module* M *we have* $[M : \text{head}(Af)] = dim_{\mathbb{F}}(fM)]$. □

Lemma 3.8.16. *Let* $\mathbb{F}$ *be an algebraically closed field with* $char(\mathbb{F}) = 0$ *and* $\{\alpha_i, \dots, \alpha_m\}$ *be an* $\mathbb{F}$*-basis for* A*. Let* $f = \sum_{i=1}^m a_i \alpha_i$ *be a primitive idempotent* *in A and $S = \text{head}(Af)$. Then for a finite dimensional left A-module M we have $[M : S] = \sum_{i=1}^{m} a_i \chi_M(\alpha_i)$.*

Proof. Let ρ be the representation afforded by M, then by Theorem 3.8.15 we have $[M : S] = dim_{\mathbb{F}}(fM)$. Let $rk(\rho(f))$ denote the rank of the matrix $\rho(f)$ then we have $dim_{\mathbb{F}}(fM) = rk(\rho(f)) = tr(\rho(f)) = tr(\rho(\sum_{i=1}^{m} a_i\alpha_i)) = \sum_{i=1}^{m} a_i tr(\rho(\alpha_i)) = \sum_{i=1}^{m} a_i\chi_M(\alpha_i)$. □

Proposition 3.8.17. *1. If $n > d$ the $\mathbb{C}$-algebra $T_n^d(1)$ is not semisimple.*

2. If $n > d = 2$ and n is sufficiently large the $\mathbb{C}$-algebra $\mathcal{T}_n^2(2)$ is not semisimple.

Proof. If $\delta \neq 0$ or $d \nmid n$ then $\Delta_n((n - \lfloor \frac{n}{d} \rfloor d))$ is indecomposable projective $T_n^d(\delta)$-module. We shall use the notation of the subsection 1.1.4. The vector space $V^{\otimes n}$ is a left $T_n^d(N)$-module by restricting the action of the partition algebra $\mathfrak{P}_n(N)$ on $V^{\otimes n}$.

1. Since $\delta = N = 1$ we have $dim_{\mathbb{C}}(V^{\otimes n}) = 1$ and by Lemma 3.8.16 $[V^{\otimes n} : \text{head}(\Delta_n((n - \lfloor \frac{n}{d} \rfloor d)))] = 1$. But $dim_{\mathbb{C}}(\Delta_n((n - \lfloor \frac{n}{d} \rfloor d))) > 1$, therefore there is a $T_n^d(1)$-morphism from $\Delta_n((n - \lfloor \frac{n}{d} \rfloor d))$ to $V^{\otimes n}$ with a non-zero kernel. Hence, $\Delta_n((n - \lfloor \frac{n}{d} \rfloor d))$ is not simple, and consequently $T_n^d(1)$ is not semisimple.

2. If $N = 2$ then $dim_{\mathbb{C}}(V^{\otimes n}) = 2^n$, and there are two cases;

i. If $n = 2k$ then $\rho_2(\mathfrak{e}_{k,d})(e_{i_1} \otimes \cdots \otimes e_{i_n}) = \frac{\delta_{i_1,i_2,\ldots,i_n}}{2} \sum_{i=1}^{2} e_i \otimes \cdots \otimes e_i$, where $\delta_{i_1,i_2,\ldots,i_n} = 1$ if $i_1 = i_2 = \cdots = i_n$ and zero otherwise. Therefore by Lemma 3.8.16 we have $[V^{\otimes n} : \text{head}(\Delta_n((0)))] = 1$. On the other hand, $dim_{\mathbb{C}}(\Delta_n((0))) = CN^2(k) > 4^k$ for all $k \geq 10$, and by the same argument as in the first part we have $T_{2k}^2(2)$ is not semisimple for $k \geq 10$.

ii. If $n = 2k + 1$ then $\rho_2(\mathfrak{e}_{k,d})(e_{i_1} \otimes \cdots \otimes e_{i_n}) = \delta_{i_1,i_2,\ldots,i_n} e_{i_1} \otimes \cdots \otimes e_{i_1}$ and by Lemma 3.8.16 $[V^{\otimes n} : \text{head}(\Delta_n((1)))] = 2$. Hence, a necessary condition for $T_{2k+1}^2(2)$

to be semisimple is $2^{2k+1} \geq 2CN^2(k+1)$ but by induction on k this only happens if $k < 5$. □

3.9 A necessary and sufficient condition for the semisimplicity of $T_n^2(\delta)$.

In [14] Cox, Martin, Parker and Xi developed an axiomatic framework to study the representation theory of a tower of finite dimensional algebras satisfying some specific axioms. It has been shown in [14] that if a tower of algebra satisfies their axiomatic framework then in order to understand the morphisms between the modules and determine whether each algebra is semisimple or not we only need to study the morphisms between a relatively very small set of modules, see Theorem 3.9.2. We begin this section by recalling Cox and others axiomatic frame work and then show that the tower of algebras $\{T_n^2(\delta)\}_n$ satisfy their axioms. We then use this powerful technique to determine explicitly when each $T_n^2(\delta)$ is semisimple.

Let $\mathbb{F}$ be algebraically closed, recall from [14] we call a family of $\mathbb{F}$-algebras $\{A_n \mid n \in \mathbb{N}\}$ with an idempotent $e_n \in A_n$ for all n, *towers of recollement* if it satisfies the following axioms;

A$_1$) For $n \geq 2$ there is an isomorphism of $\mathbb{F}$- algebras $\Phi_n : A_{n-2} \to e_n A_n e_n$.

Therefore, if we let Λ_n and Λ^n be the indexing set for the simple modules of A_n and $A_n/A_n e_n A_n$ respectively, then $\Lambda_n = \Lambda_{n-2} \sqcup \Lambda^n$.

A$_2$) Each A_n is quasi-hereditary with a hereditary chain $0 \subset A_n e_1 A_n \subset A_n e_2 A_n \subset \cdots \subset A_n e_{n-1} A_n \subset A_n e_n A_n = A_n$. For each $\lambda \in \Lambda_n$ we denote by $\Delta_n(\lambda)$ the standard module of A_n labelled by λ.

$\mathbf{A}_3$) For $n \geq 0$ we have the embedding of $\mathbb{F}$-algebras $A_n \hookrightarrow A_{n+1}$. This means there is the restriction functor $\mathbf{res}_n : A_n\text{-}mod \to A_{n-1}\text{-}mod$ and the induction functor $\mathbf{ind}_n : A_n\text{-}mod \to A_{n+1}\text{-}mod$ given by $\mathbf{ind}_n(M) = A_{n+1} \otimes_{A_n} M$ for a left A_n module M.

$\mathbf{A}_4$) For $n \geq 1$ there is an isomorphism $A_n e_n \simeq A_{n-1}$ of (A_{n-1}, A_{n-2})-bimodules.

For a left A_n-module M let $supp(M)$ denote the set of all labels $\mu \in \Lambda_{n-1}$ such that $\Delta_{n-1}(\mu)$ occurs in the standard filtration of M, in the sense of Section 1.4.

Let $m, l \in \mathbb{N}$ such that $m - l$ is even. If $m \geq l$ we set $\Lambda_m^l := \Lambda^l$ considered as a subset of Λ_m, otherwise we set $\Lambda_m^l := \phi$.

$\mathbf{A}_5$) For each $\lambda \in \Lambda_n^m$, we have $\mathbf{res}_n(\Delta_n(\lambda))$ is filtered by standard modules of A_{n-1}, and we must have

$$supp(\mathbf{res}_n(\Delta_n(\lambda))) \subseteq \Lambda_{n-1}^{m-1} \sqcup \Lambda_{n-1}^{m+1}.$$

$\mathbf{A}_6$) For each $\lambda \in \Lambda_n^n$ there exist $\mu \in \Lambda_{n+1}^{n-1}$ such that

$$\lambda \in supp(\mathbf{res}_{n+1}(\Delta_{n+1}(\mu))).$$

In order to avoid confusion we write $\mathfrak{e}_n$ for the idempotent $\mathfrak{e}_{1,2} \in T_n^2(\delta)$.

Lemma 3.9.1. *Let $\delta \neq 0$ then the tower of $\mathbb{F}$-algebras $\{T_n^2(\delta) \mid n \in \mathbb{N}\}$ and the set of idempotents $\{\mathfrak{e}_n \mid n \in \mathbb{N}\}$ is a tower of recollement.*

Proof. Axiom 1 follows from Proposition 3.5.1. By Theorem 3.6.3 the algebra $T_n^2(\delta)$ is quasi-hereditary when $\delta \neq 0$ or n is odd, therefore $\mathbf{A}_2$ is satisfied. The axiom $\mathbf{A}_3$ follow from the map defined in Equation 3.25.

To check $\mathbf{A}_4$, let $\mathfrak{d} \in \mathcal{T}_n^2 \mathfrak{e}_n$ and $\mathfrak{d}_{n-1} = (\mathfrak{d} \setminus \{\mathbf{p}_n, \mathbf{p}_{n'}\}) \cup \{\mathbf{p}'_n, \mathbf{p}'_{n'}\}$ such that $n \in \mathbf{p}_n$, $n' \in \mathbf{p}_{n'}$ and $\mathbf{p}'_n = (\mathbf{p}_n \setminus \{n\}) \cup \{(n-1)'\}$, $\mathbf{p}'_{n'} = \mathbf{p}_{n'} \setminus \{(n-1)', n'\}$, then $\mathfrak{d}_{n-1} \in \mathcal{T}_{n-1}^2$, for example $(\boldsymbol{\nu}_{n-2}\mathfrak{e}_n)_{n-1} = 1 \in \mathcal{T}^2_{n-1}$. By argument similar to 3.30 one can show that the following map which is defined on the diagram basis of $T_n^2(\delta)\mathfrak{e}_n$

$$\vartheta : T_n^2(\delta)\mathfrak{e}_n \to T_{n-1}^2(\delta)$$

$$\mathfrak{d} \mapsto \mathfrak{d}_{n-1}$$

is a $(T_{n-1}^2(\delta), T_{n-2}^2(\delta))$-bimodule isomorphism.

Next we show that $\{T_n^2(\delta) \mid n \in \mathbb{N}\}$ satisfies $\mathbf{A}_5$ and $\mathbf{A}_6$. Note that here we have $\Lambda^n = \mathcal{C}^2(n)$, $\Lambda_n = \bigsqcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} \mathcal{C}^2(n-2i)$. To show that the given tower satisfies $\mathbf{A}_5$, let $\lambda \in \Lambda_n^m = \mathcal{C}^2(m)$ then we have the following cases:

i If $\lambda = (0) \in \Lambda_n^0$ then n must be even and by 3.28 we have $\mathbf{res}_n(\Delta_n((0))) = \Delta_{n-1}((1))$ and $(1) \in \Lambda_{n-1}^1$.

ii If $\lambda = (i_1, \dots, i_l) \in \Lambda_n^n = \mathcal{C}^2(n)$ then by 3.29 $\mathbf{res}_n(\Delta_n(\lambda)) = \Delta_{n-1}((i_1, \dots, i_l - 1))$ and $(i_1, \dots, i_l - 1) \in \Lambda_{n-1}^{n-1}$.

iii If $\lambda = (i_1, \dots, i_l, 1) \in \Lambda_n^m$ with $m < n$ then by 3.27 we have $supp(\mathbf{res}_n(\Delta_n(\lambda))) = \{(i_1, \dots, i_l), (i_1, \dots, i_l, 1, 1), (i_1, \dots, i_l, 2)\} \subset \Lambda_{n-1}^{m-1} \cup \Lambda_{n-1}^{m+1}$.

iv If $\lambda = (i_1, \dots, i_l, 2) \in \Lambda_n^m$ with $m < n$ then by 3.26 we have $supp(\mathbf{res}_n(\Delta_n(\lambda))) = \{(i_1, \dots, i_l, 1), (i_1, \dots, i_l, 2, 1)\} \subset \Lambda_{n-1}^{m-1} \cup \Lambda_{n-1}^{m+1}$.

Hence the given tower of algebras satisfies $\mathbf{A}_5$. It remains to check the axiom $\mathbf{A}_6$. To this end, let $\lambda \in \Lambda_n^n = \mathcal{C}^2(n)$. Then we have the following two cases:

i If $\lambda = (i_1, \dots, i_l, 1)$ then by 3.27 we have $\lambda \in supp(\mathbf{res}_{n+1}(\Delta_{n+1}(i_1, \dots, i_l))$, and $(i_1, \dots, i_l) \in \Lambda_{n+1}^{n-1}$.

ii If $\lambda = (i_1, \dots, i_l, 2)$ then by 3.26 we have $\lambda \in supp(\mathbf{res}_{n+1}(\Delta_{n+1}(i_1, \dots, i_l, 1))$, and $(i_1, \dots, i_l, 1) \in \Lambda_{n+1}^{n-1}$.

Thus, the given tower of algebras satisfies the axioms of tower of recollement. □

The following result from [14] shows that to find all the maps between the standard modules in the non-semisimple case it is enough to study the maps between certain standard modules of $T_n^2(\delta)$. Moreover, the result helps to find the precise values where $T_n^2(\delta)$ is non-semisimple.

Theorem 3.9.2. *Let $\{A_n \mid n \in \mathbb{N}\}$ be a tower of algebras satisfy the axioms A_1 to A_6.*

i)For all pairs of weights $\lambda \in \Lambda_n^m$ and $\mu \in \Lambda_n^l$ we have

$$\mathrm{Hom}(\Delta_n(\lambda), \Delta_n(\mu)) = \begin{cases} \mathrm{Hom}(\Delta_m(\lambda), \Delta_m(\mu)) & \text{if } l \leq m \\ 0 & \text{otherwise} \end{cases} \tag{3.35}$$

ii) Let m be a positive integer. Suppose for all $0 \leq n \leq m$ and all pairs $\lambda \in \Lambda_n^n$ and $\mu \in \Lambda_n^{n-2}$ we have

$$\mathrm{Hom}_{A_n}(\Delta_n(\lambda), \Delta_n(\mu)) = 0$$

then the algebra A_m is semisimple.

Proof. For the proof of part i see part i of Theorem 1.1 of [14]. Part ii is a mild generalisation of the statement of part ii of Theorem 1.1 [14], and the proof is identical to the proof of part ii of Theorem 1.1 [14]. □

In our case, if $\lambda \in \Lambda_n^n$ then $\Delta_n(\lambda)$ is always simple $T_n^2(\delta)$-module, therefore by the above theorem and using the fact that $\dim_{\mathbb{F}}(\Delta_n(\mu)) > 1$ for $\mu \in \Lambda_n^{n-2}$ to show that $T_n^2(\delta)$ is semisimple it is enough to prove for $\mu \in \Lambda_n^{n-2}$ the module $\Delta_n(\mu)$ is simple, for each μ.

It is necessary to mention that if $A_n = T_n^2(\delta)$ for all n, then we have

$$\Lambda_n^{n-2} = \mathcal{C}^2(n-2) \tag{3.36}$$

In the remainder of this section, we obtain a general formula to the determinant of the gram matrix of $\Delta_n(\lambda)$ when $d = 2$, which be considered as a function depending

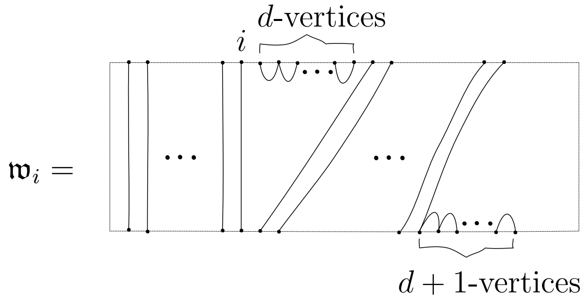


Figure 3.9: The element $\mathfrak{w}_i \in \mathcal{T}_n^d$, where $0 \leq i < n - d$

on the Chebyshev polynomial of the second kind. We start by fixing a planar canonical element $\mathfrak{p} = (\bigotimes_{j=1}^{l} \mathfrak{m}_{i_j}) \otimes \mathfrak{u}_2$, and $\lambda = \Xi^{-1}(\mathfrak{p})$.

Lemma 3.9.3. *Let $d = 2$ and $\mathfrak{w}_i$ be as given in Figure 3.9 and $\mathfrak{n}_i = \boldsymbol{\mu}_{i-1}\mathfrak{w}_i$. Then $B(\lambda) = \{\mathfrak{p}, \mathfrak{w}_{n-3}\mathfrak{p}, \mathfrak{w}_{n-4}\mathfrak{p}, \ldots, \mathfrak{w}_0\mathfrak{p}, \mathfrak{n}_{\sum_{j=1}^{l} i_j}\mathfrak{p}, \mathfrak{n}_{\sum_{j=1}^{l-1} i_j}\mathfrak{p}, \ldots, \mathfrak{n}_{i_1}\mathfrak{p}\}$ is the $\mathcal{L}$-class contains $\mathfrak{p}$. Hence $B(\lambda)$ is a basis for the $T_n^2(\delta)$-module $\Delta_n(\lambda)$.*

Proof. It is not hard to see that each element of $B(\lambda)$ is $\mathcal{L}$-related to $\mathfrak{p}$. Let $\mathfrak{a} \in \mathcal{T}_n^d$ and $\mathfrak{a} \mathcal{L} \mathfrak{p}$. Then we have $\mathfrak{a} = \mathfrak{a}\mathfrak{p}$. By Lemma 3.3.8 the element $\mathfrak{a}$ can not join two propagating parts of $\mathfrak{p}$. Hence, $\mathfrak{a}$ either moves the only northern non-propagating part of $\mathfrak{p}$ to left or join it to a propagating part, in either cases $\mathfrak{a} = \mathfrak{a}\mathfrak{p} \in B(\lambda)$. $\square$

To make the calculation of the gram matrix of $\Delta_n(\lambda)$ easier for each $\mu = (i_1, i_2, \ldots, i_l) \in \mathbb{Z}_{>0}^l$, we define the matrix B_μ where its transpose denoted by B_μ^t, is given by:

$$B_\mu^t := \begin{array}{c} \begin{matrix} 1 & 2 & 3 & \dots & & i_l+1 & & \dots & & i_l+i_{l-1}+1 & & & & & \sum_j^l i_j+1 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ \\ l \end{matrix} \end{array}$$

In general, for any matrix M we denote its transpose by M^t.

Lemma 3.9.4. *For each $\lambda \in \mathcal{C}^2(n-2)$ the Gram matrix of $\Delta_n(\lambda)$ with respect to the form* (3.32) *is given by the following matrix*

$$G(\lambda) = \left(\begin{array}{c|c} D_{n-1} & B_\lambda \\ \hline B_\lambda^t & I_{l\times l} \end{array}\right) \tag{3.37}$$

Where

$$D_{n-1} := \begin{pmatrix} \delta & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & \delta & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \delta & 1 & 0 & \dots & 0 & 0 & 0 \\ & & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & \delta & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & \delta \end{pmatrix}_{n-1\times n-1}$$

Proof. Keep the notation from Lemma 3.9.3 and let the row r of the gram matrix $G(\lambda)$ of $\Delta_n(\lambda)$ with respect to the form (3.32) to be the row of $\mathfrak{g}_r := \mathfrak{w}_{n-1-r}\mathfrak{p}$ where $1 \leq r \leq n-1$, and the row $n-1+s$ to be the row of $\mathfrak{f}_s := \mathfrak{n}_{\sum_{j=1}^{l-s+1} i_j}\mathfrak{p}$ where $1 \leq s \leq l$, and $\mathfrak{n}_i$ be as given in Lemma 3.9.3. The rest of the lemma follows from the following equations;

$$\langle \mathfrak{g}_p, \mathfrak{g}_q \rangle = \begin{cases} 1 & \text{if } |p-q| = 1 \\ \delta & \text{if } p = q \\ 0 & \text{otherwise} \end{cases} \tag{3.38}$$

$$\langle \mathfrak{g}_p, \mathfrak{f}_s \rangle = b_{p,s} \tag{3.39}$$

$$\langle \mathfrak{f}_s, \mathfrak{f}_{s'} \rangle = \begin{cases} 1 & \text{if } s = s' \\ 0 & \text{otherwise} \end{cases} \tag{3.40}$$

Where $b_{p,s}$ is the (p,s)-th entry of the matrix B_λ. □

Example 3.9.5. Let $\lambda = (2,1)$ then $\mathfrak{p} = \mathfrak{m}_2 \otimes \mathfrak{m}_1 \otimes \mathfrak{u}_2$. The gram matrix of ${}_{T_5^2(\delta)}\Delta_5(\lambda)$ with respect to the ordered basis $\{\mathfrak{g}_1, \ldots, \mathfrak{g}_4, \mathfrak{f}_1, \mathfrak{f}_2\}$ and the form $\langle -,- \rangle_\lambda$ is

$$G((2,1)) = \left(\begin{array}{cccc|cc} \delta & 1 & 0 & 0 & 1 & 0 \\ 1 & \delta & 1 & 0 & 1 & 1 \\ 0 & 1 & \delta & 1 & 0 & 1 \\ 0 & 0 & 1 & \delta & 0 & 1 \\ \hline 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right).$$

Corollary 3.9.6. *Let $\lambda \in \mathcal{C}^2(n-2)$ and $G_\lambda(\delta) = \det(G(\lambda))$ then $G_\lambda(\delta) = \det(D_{n-1} - B_\lambda B_\lambda^t)$.*

Proof. From Lemma 3.9.4 we have

$$G_\lambda(\delta) = \det \begin{pmatrix} D_{n-1} & B_\lambda \\ B_\lambda^t & I_{l\times l} \end{pmatrix} = \det \left(\begin{pmatrix} I_{n-1\times n-1} & B_\lambda \\ 0 & I_{l\times l} \end{pmatrix} \begin{pmatrix} D_{n-1} - B_\lambda B_\lambda^t & 0 \\ B_\lambda^t & I_{l\times l} \end{pmatrix} \right).$$

Which implies that $G_\lambda(\delta) = \det(D_{n-1} - B_\lambda B_\lambda^t)$. □

Before stating our main result about the Gram matrices of $T_n^2(\delta)$, we recall the definition and some facts about the Chebyshev polynomial of the second kind. The determinant of the Gram matrix of $\Delta_n(\lambda)$ is expressible in terms of Chebyshev polynomials. We use this information to factorise the $G_\lambda(\delta)$ into linear factors.

We follow [59] in our exposition.

Definition 3.9.7. *The Chebyshev polynomial of the second kind, denoted by* $\mathfrak{U}_n(x)$, *is recursively given by the following formula*

$$\mathfrak{U}_{n+2}(x) = 2x\mathfrak{U}_{n+1}(x) - \mathfrak{U}_n(x) \tag{3.41}$$

with $\mathfrak{U}_0(x) = 1$ *and* $\mathfrak{U}_1(x) = 2x$.

The Chebyshev polynomials of the second kind are pairwise orthogonal in the interval $[-1, 1]$. Furthermore $\mathfrak{U}_n(x)$ is a polynomial of degree n, having n different real roots in $[-1, 1]$.

For each n, the polynomial $\mathfrak{U}_n(x)$ satisfies the following relation,

$$\begin{aligned} \mathfrak{U}_n(x) &= \frac{r_+^{n+1}(x) - r_-^{n+1}(x)}{r_+(x) - r_-(x)} \\ &= \sum_{k=0}^{n} (-2)^k \binom{n+k+1}{2k+1} (1-x)^k \end{aligned} \tag{3.42}$$

where $r_+(x) = x + \sqrt{x^2 - 1}$ and $r_-(x) = x - \sqrt{x^2 - 1}$. See, for example, Equation 9 of [19] for the second equality.

The Chebyshev polynomial of the second kind is also expressed as a determinant of some tridiagonal matrix as follows:

$$\mathfrak{U}_m(x) = \det \begin{pmatrix} 2x & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 2x & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2x & 1 & 0 & \dots & 0 & 0 & 0 \\ & & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 2x & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2x \end{pmatrix}_{m \times m}.$$

For each $m \in \mathbb{N}$ and an indeterminate x we define a matrix $M_{m+1}(x)$ as follows:

$$M_{m+1}(x) = \begin{pmatrix} x-2 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & x-2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & x-2 & -1 & 0 & \dots & 0 & 0 & 0 \\ & & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & x-2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & x-1 \end{pmatrix}_{m+1 \times m+1}$$

Let

$$\mathcal{M}_{m+1}(x) := \det(M_{m+1}(x)) \tag{3.43}$$

then the proof of the following lemma is straightforward.

Lemma 3.9.8. *Let m be a non-negative integer, then we have*

$$\mathcal{M}_{m+1}(x) = (-1)^m \left((x-1)\mathfrak{U}_m \left(\frac{2-x}{2} \right) + \mathfrak{U}_{m-1} \left(\frac{2-x}{2} \right) \right). \quad \square \tag{3.44}$$

Lemma 3.9.9. *For each non-negative integer l we have*

$$(x-1)^l((x-1)\mathcal{M}_l(x) - \mathcal{M}_{l-1}(x)) = (-1)^l x (x-1)^l \, \mathfrak{U}_l \left(\frac{2-x}{2} \right).$$

Proof. We have

$$\begin{aligned} (x-1)\mathcal{M}_l(x) - \mathcal{M}_{l-1}(x) &= (-1)^{l-1} x \left((x-2)\mathfrak{U}_{l-1} \left(\frac{2-x}{2} \right) + \mathfrak{U}_{l-2} \left(\frac{2-x}{2} \right) \right) \\ &= (-1)^{l-1} x \left(-2 \left(\frac{2-x}{2} \right) \mathfrak{U}_{l-1} \left(\frac{2-x}{2} \right) + \mathfrak{U}_{l-2} \left(\frac{2-x}{2} \right) \right) \\ &= (-1)^l x \, \mathfrak{U}_l \left(\frac{2-x}{2} \right). \end{aligned}$$

□

The first six values of $\mathcal{M}_n(\delta)$ are

$$\begin{aligned}
\mathcal{M}_0(x) &= 1,\\
\mathcal{M}_1(x) &= x-1,\\
\mathcal{M}_2(x) &= x^2-3x+1,\\
\mathcal{M}_3(x) &= x^3-5x^2+6x-1,\\
\mathcal{M}_4(x) &= x^4-7x^3+15x^2-10x+1,\\
\mathcal{M}_5(x) &= x^5-9x^4+28x^3-35x^2+15x-1,\\
&\vdots
\end{aligned}$$

Lemma 3.9.10. *Let* $x = \left(\frac{2-\delta}{2}\right)$*, the polynomial* $\mathcal{M}_{m+1}(\delta)$ *obeys*

$$\mathcal{M}_{m+1}(\delta) = (-1)^m \frac{r_-(x)-1}{r_+(x)-r_-(x)}\left(r_+^{m+2}(x)+r_-^{m+1}(x)\right)$$

Where $r_+(x) = x+\sqrt{x^2-1}$, $r_-(x) = x-\sqrt{x^2-1}$.

Proof. From Equation 3.42 and definition of $\mathcal{M}_{m+1}(\delta)$ we have,

$$\begin{aligned}
\mathcal{M}_{m+1}(\delta) &= (-1)^m \frac{1}{r_+(x)-r_-(x)}\left((1-2x)(r_+^{m+1}(x)-r_-^{m+1}(x))+(r_+^m(x)-r_-^m(x))\right)\\
&= (-1)^m \frac{r_-(x)-1}{r_+(x)-r_-(x)}\left(r_+^{m+2}(x)+r_-^{m+1}(x)\right)
\end{aligned}$$

such that $x = \frac{2-\delta}{2}$. □

Part i of the the following theorem can be found, for example, on p 59 of [74], and part ii is Theorem 1 of [82].

Lemma 3.9.11 ([74], [82])**.** *Let*

$$Q_m = \begin{pmatrix} b & c & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ a & b & c & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & a & b & c & 0 & \dots & 0 & 0 & 0 \\ & & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & a & b & c \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & a & b-\beta \end{pmatrix}_{m\times m}$$

with $a, b, c, \beta \in \mathbb{C}$ then we have the following:

i If $\beta = 0$ then the eigenvalues of Q_m are

$$\lambda_k = b + 2\sqrt{ac}\cos\left(\frac{k\pi}{m+1}\right), \qquad k = 1, 2, \dots, m.$$

ii If $\beta = \sqrt{ac} \neq 0$ then the eigenvalues of Q_m are

$$\lambda_k = b + 2\sqrt{ac}\cos\left(\frac{2k\pi}{2m+1}\right), \qquad k = 1, 2, \dots, m.$$

□

Lemma 3.9.12. *For each $m \in \mathbb{Z}_{\geq 1}$ we have the following equations:*

$$\mathfrak{U}_m\left(\frac{2-x}{2}\right) = (-1)^m \prod_{k=1}^{m}\left(x - 4\cos^2\left(\frac{k\pi}{2(m+1)}\right)\right). \tag{3.45}$$

$$\mathcal{M}_m(x) = \prod_{k=1}^{m}\left(x - 4\cos^2\left(\frac{k\pi}{2m+1}\right)\right). \tag{3.46}$$

Proof. We first proof Equation 3.45. By substituting $\frac{2-x}{2}$ in 3.42 we obtain

$$\mathfrak{U}_m\left(\frac{2-x}{2}\right) = \sum_{k=0}^{m}(-1)^k\binom{m+k+1}{2k+1}x^k. \tag{3.47}$$

Hence, the leading coefficient of $\mathfrak{U}_m\left(\frac{2-x}{2}\right)$ is $(-1)^m$. On the other hand, we have $\mathfrak{U}_m\left(\frac{2-x}{2}\right) = \det(xI_m - Q_m)$, where Q_m is given in Lemma 3.9.11 with $a = c = 1$,

$b = 2$ and $\beta = 0$. Now we may apply Lemma 3.9.11 part i to obtain the eigenvalues of Q_m, which are given by,

$$\lambda_k = 2 + 2\cos\left(\frac{k\pi}{m+1}\right), \quad k = 1, 2, \ldots, m.$$

Furthermore, the eigenvalues of Q_m are the roots of $\mathfrak{U}_m\left(\frac{2-x}{2}\right)$. Combining the above information about the roots and the coefficient of the leading term of $\mathfrak{U}_m\left(\frac{2-x}{2}\right)$ implies 3.45.

Next we proof Equation 3.46. By substituting 3.47 in 3.44, and then examining the cases when m is either even or odd we conclude that the leading coefficient of $\mathcal{M}_m(x)$ is 1. Furthermore, we have $\mathcal{M}_m(x) = \det(xI_m - Q_m)$, where Q_m is given in Lemma 3.9.11 with $a = c = 1$, $b = 2$ and $\beta = 1$. By using Lemma 3.9.11 part ii and a similar argument as in the proof of 3.45 we obtain 3.46. □

Let $\lambda = (\underbrace{i'_1, i'_1, \ldots, i'_1}_{k_1-\text{times}}, \ldots, \underbrace{i'_v, i'_v, \ldots, i'_v}_{k_v-\text{times}}) \in \Lambda(T_n^2(\delta))$, we shall say $({i'_1}^{k_1}, {i'_2}^{k_2}, \ldots, {i'_v}^{k_v})$ is *frequency representation* of λ, and write $\lambda = ({i'_1}^{k_1}, {i'_2}^{k_2}, \ldots, {i'_v}^{k_v})$, if $i'_s \neq i'_{s+1}$ for $1 \leq s \leq v-1$.

We say λ is of the form $(2^{k_1}, 1^{k_2}, 2^{k_3}, \ldots, 1^{k_{2v}}, 2^{k_{2v+1}})$ if it satisfies one of the following conditions

1. We have $k_1 = 0$ and $\lambda = (1^{k_2}, 2^{k_3}, \ldots, 1^{k_{2v}}, 2^{k_{2v+1}})$ is frequency representation of an element of $\Lambda(T_n^2(\delta))$.

2. We have $k_{2v+1} = 0$ and $\lambda = (2^{k_1}, 1^{k_2}, 2^{k_3}, \ldots, 1^{k_{2v}})$ is frequency representation of an element of $\Lambda(T_n^2(\delta))$.

3. We have $k_1 = k_{2v+1} = 0$ and $\lambda = (1^{k_2}, 2^{k_3}, \ldots, 1^{k_{2v}})$ is frequency representation of an element of $\Lambda(T_n^2(\delta))$.

For example, $(2^3) \in \Lambda(T_8^2(\delta))$ is of the form (2^3), $(1^6) \in \Lambda(T_8^2(\delta))$ is of the form $(2^0, 1^6, 2^0)$ and $(2, 1^4) \in \Lambda(T_8^2(\delta))$ is of the form $(2, 1^4, 2^0)$.

Proposition 3.9.13. *For each* $\lambda = (i_1, i_2, \ldots, i_l) \in \mathcal{C}^2(n-2)$ *the polynomial* $G_\lambda(\delta) = \det(G(\lambda))$ *has one of the following forms:*

1. *If* λ *is of the form* (2^l) *then* $G_\lambda(\delta) = (-1)^l\delta(\delta-1)^l\,\mathfrak{U}_l\left(\frac{2-\delta}{2}\right)$.

2. *If* λ *is of the form* $(2^{k_1}, 1^{k_2}, 2^{k_3}, \ldots, 1^{k_{2v}}, 2^{k_{2v+1}})$ *with* $v \neq 0$ *then* $G_\lambda(\delta) = \alpha(\delta-1)^{\sum_{t=0}^{v} k_{2t+1}}(\delta-2)^{\sum_{t=1}^{v} k_{2t}-v}\prod_{t=1}^{v-1}\mathfrak{U}_{k_{2t+1}+1}\left(\frac{2-\delta}{2}\right)\mathcal{M}_{k_1+1}(\delta)\mathcal{M}_{k_{2v+1}+1}(\delta)$, *where* $\alpha \in \{-1, 1\}$.

Proof. First we prove part 1 of the proposition. If $\lambda = (2^l)$ then $n-1 = 2l+1$ and

$$B^t_{(2^l)} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & & & \cdots & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & & & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & & & \cdots & 0 \\ \vdots & & & & & & & \ddots & & & & \\ 0 & 0 & 0 & \cdots & & & & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & & & & 0 & 1 & 1 & 1 \end{pmatrix}_{l\times n-1}.$$

We have

$$D_{n-1} - B_{(2^l)}B^t_{(2^l)} = \begin{pmatrix} \delta-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & 0 \\ 0 & \delta-1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & & 0 \\ -1 & 0 & \delta-2 & 0 & -1 & 0 & 0 & 0 & \cdots & & & & 0 \\ 0 & 0 & 0 & \delta-1 & 0 & 0 & 0 & 0 & \cdots & & & & 0 \\ 0 & 0 & -1 & 0 & \delta-2 & 0 & -1 & 0 & \cdots & & & & 0 \\ & \vdots & & & & & & \ddots & & & & & \\ 0 & 0 & 0 & & \cdots & & & 0 & -1 & 0 & \delta-2 & 0 & -1 \\ 0 & 0 & 0 & & \cdots & & & 0 & 0 & 0 & 0 & \delta-1 & 0 \\ 0 & 0 & 0 & & \cdots & & & 0 & 0 & 0 & -1 & 0 & \delta-1 \end{pmatrix}.$$

Hence by the above step, Laplace expansion, Corollary 3.9.6, Equation 3.43 and Lemma 3.9.9 we obtain

$$\begin{aligned} G_{(2^l)}(\delta) = \det(D_{n-1} - B_{(2^l)}B^t_{(2^l)}) &= (\delta-1)^l((\delta-1)\mathcal{M}_l(\delta) - \mathcal{M}_{l-1}(\delta)) \\ &= (-1)^l\delta(\delta-1)^l\,\mathfrak{U}_l\left(\frac{2-\delta}{2}\right). \end{aligned}$$

Regarding part 2 of the proposition we only need to consider the following special cases, since the general case can be obtained by combining these special cases,

Case 1. If $\lambda = (2^p, 1^{n-2(p+1)})$ with $p > 0$ then λ is of the form $(2^p, 1^{n-2(p+1)}, 2^0)$

$$B^t_{(2^p,1^{n-2(p+1)})} = \begin{pmatrix} 1 & 1 & 0 & & & & \cdots & & & 0 \\ 0 & 1 & 1 & 0 & & & & & & \\ & & & \ddots & & & & & & \\ & & & 0 & 1 & 1 & 0 & & & \\ & & & 0 & 0 & 1 & 1 & 1 & & \\ \vdots & & & & & \ddots & & & & \vdots \\ & & \cdots & & & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & & & \cdots & & & 0 & 1 & 1 & 1 \end{pmatrix}_{l \times n-1}.$$

such that the row of $B^t_{(2^p,1^{n-2(p+1)})}$ in which three 1's occur in the same row for the first time is the $(n - 2(p + 1) + 1)$-th row.

Hence

$$D_{n-1} - B_\lambda B^t_\lambda = \begin{pmatrix} \delta-1 & 0 & & & & & & & & & & 0 \\ 0 & \delta-2 & 0 & & & & & & & & & \\ \vdots & & \ddots & & & & & & & & & \vdots \\ & & 0 & \delta-2 & 0 & & & & & & & \\ 0 & \cdots & & 0 & \delta-2 & 0 & -1 & 0 & & & & \\ & & & 0 & 0 & \delta-1 & 0 & 0 & 0 & & & \\ & & & 0 & -1 & 0 & \delta-2 & 0 & -1 & 0 & & \\ & & & & & & 0 & \delta-1 & 0 & 0 & & \\ & & & & & & -1 & 0 & \delta-2 & 0 & -1 & \\ & & & & & & 0 & 0 & 0 & \delta-1 & 0 & \\ \vdots & & & \vdots & & & & & & & \ddots & \\ & & & & & & & -1 & 0 & \delta-2 & 0 & -1 \\ & & & & & & & 0 & 0 & 0 & \delta-1 & 0 \\ 0 & & & & & \cdots & & 0 & 0 & -1 & 0 & \delta-1 \end{pmatrix}$$

such that the row of $D_{n-1} - B_\lambda B^t_\lambda$ in which -1 occurs for the first time is the $(n - 2(p+1) + 1)$-th row. By using Equation 3.43 and Laplace expansion we obtain

$$\det(D_{n-1} - B_{(2^p,1^{n-2(p-1)})} B^t_{(2^p,1^{n-2(p-1)})}) = (\delta - 1)^p (\delta - 2)^{n-2(p+1)-1} \mathcal{M}_1(\delta) \mathcal{M}_{p+1}(\delta).$$

Case 2. If $\lambda = (1^l)$ then λ is of the form $(2^0, 1^l, 2^0)$ and it is not hard to see

$$D_{n-1} - B_\lambda B^t_\lambda = \begin{pmatrix} \delta-1 & 0 & & & \cdots & 0 \\ 0 & \delta-2 & 0 & & & \\ & 0 & \delta-2 & 0 & & \vdots \\ & & & \ddots & & \\ \vdots & & & & \delta-2 & 0 \\ 0 & \cdots & & & 0 & \delta-1 \end{pmatrix}_{n-1 \times n-1}$$

Laplace expansion implies that

$$\det(D_{n-1} - B_{(1^l)} B^t_{(1^l)}) = (\delta - 2)^{l-1} \mathcal{M}_1 \mathcal{M}_1 = (\delta - 1)^2 (\delta - 2)^{n-3}.$$

Case 3. If $\lambda = (1^{s_1}, 2^{s_2}, 1^{s_3})$ such that s_1, s_2 and s_3 are positive integers then λ is of the form $(2^0, 1^{s_1}, 2^{s_2}, 1^{s_3}, 2^0)$ and

$$D_{n-1} - B_\lambda B_\lambda^t = \left(\begin{array}{cccccccccccccccccccc}
\delta-1 & 0 & & & & & & & & & & & & & & & & & \cdots & 0 \\
0 & \delta-2 & 0 & \cdots & & & & & & & & & & & & & & & & \\
\vdots & & \ddots & & & & & & & & & & & & & & & & & \vdots \\
 & \cdots & 0 & \delta-2 & 0 & 0 & 0 & 0 & & & & & & & & & & & & \\
0 & \cdots & & 0 & \delta-2 & 0 & -1 & 0 & & & & & & & & & & & & \\
 & & & 0 & 0 & \delta-1 & 0 & 0 & 0 & & & & & & & & & & & \\
 & & & 0 & -1 & 0 & \delta-2 & 0 & -1 & 0 & & & & & & & & & & \\
 & & & 0 & 0 & 0 & 0 & \delta-1 & 0 & 0 & & & & & & & & & & \\
 & & & 0 & 0 & 0 & -1 & 0 & \delta-2 & 0 & -1 & & & & & & & & & \\
 & & & & & & 0 & 0 & 0 & \delta-1 & 0 & & & & & & & & & \\
\vdots & & & & \vdots & & & & & \ddots & & & & & & & & & & \\
 & & & & & & & & & & & \delta-2 & 0 & -1 & & & & & & 0 \\
 & & & & & & & & & & & 0 & \delta-1 & 0 & & & & & & \\
 & & & & & & & & & & & -1 & 0 & \delta-2 & 0 & & & & & \\
 & & & & & & & & & & & & & 0 & \delta-2 & 0 & & & & \\
 & & & & & & & & & & & & & & 0 & \delta-2 & 0 & & & \\
 & & & & & & & & & & & & & & & & \ddots & & & \vdots \\
 & & & & & & & & & & & & & & & & & 0 & \delta-2 & 0 \\
0 & & & & & & \cdots & & & & & 0 & 0 & 0 & 0 & & & \cdots & 0 & \delta-1
\end{array}\right)$$

such that the row of $D_{n-1} - B_\lambda B_\lambda^t$ in which -1 occurs for the first time is the (s_3+1)-th row, and the row of $D_{n-1} - B_\lambda B_\lambda^t$ in which -1 occurs for final time is $(n - s_1 - 1)$-th row. Since $n - 1 - s_1 - s_3 = 2s_2 + 1$ we have

$$G_\lambda(\delta) = (\delta - 2)^{s_3+s_1-2} \mathcal{M}_1(\delta) \mathcal{M}_1(\delta) X_{2s_2+1}(\delta)$$

such that

$$X_{2s_2+1}(\delta) = \det \left(\begin{array}{ccccccccccc}
\delta-2 & 0 & -1 & 0 & & & & & & \cdots & 0 \\
0 & \delta-1 & 0 & 0 & 0 & & & & & & \\
-1 & 0 & \delta-2 & 0 & -1 & 0 & & & & & \\
 & 0 & 0 & \delta-1 & 0 & 0 & & & & & \\
 & 0 & -1 & 0 & \delta-2 & 0 & -1 & & & & \\
 & & 0 & 0 & 0 & \delta-1 & 0 & & & & \\
\vdots & & & & & & \ddots & & & \vdots & \\
 & & & & & & & & \delta-2 & 0 & -1 \\
 & & & & & & & 0 & 0 & \delta-1 & 0 \\
0 & \cdots & & & & & & 0 & -1 & 0 & \delta-2
\end{array}\right)_{2s_2+1 \times 2s_2+1}.$$

By applying Laplace expansion to $X_{2s_2+1}(\delta)$ and using the determinant form of the Chebyshev polynomial of the second kind we obtain $X_{2s_2+1}(\delta) = (\delta-1)^{s_2} \mathfrak{U}_{s_2+1}\left(\frac{2-\delta}{2}\right)$, and hence

$$G_\lambda(\delta) = (\delta-1)^{s_2}(\delta-2)^{s_3+s_1-2}\mathcal{M}_1(\delta)\mathcal{M}_1(\delta)\mathfrak{U}_{s_2+1}\left(\frac{2-\delta}{2}\right)$$

Now let λ be of the form $(2^{k_1}, 1^{k_2}, 2^{k_3}, \ldots, 1^{k_{2v}}, 2^{k_{2v+1}})$ with $v \neq 0$. By an argument similar to the Case 1, we have $G_\lambda(\delta) = (\delta-1)^{k_1+k_{2v+1}}\mathcal{M}_{k_1+1}(\delta)\mathcal{M}_{k_{2v+1}+1}(\delta)G'_\lambda(\delta)$, for some polynomial $G'_\lambda(\delta)$.

By the argument in Case 2 the existence of $k_2, k_4, \ldots, k_{2v}$ implies that $G_\lambda(\delta) = (\delta-1)^{k_1+k_{2v+1}}(\delta-1)^{k_2+k_4+\cdots+k_{2v}-v}\mathcal{M}_{k_1+1}(\delta)\mathcal{M}_{k_{2v+1}+1}(\delta)G''_\lambda(\delta)$ for some polynomial $G''_\lambda(\delta)$.

By the argument in Case 3 the existence of $k_3, k_5, \ldots, k_{2v-1}$ implies that $G_\lambda(\delta) = (\delta-1)^{\sum_{t=0}^{v} k_{2t+1}}(\delta-2)^{\sum_{t=1}^{v} k_{2t}-v}\prod_{t=1}^{v-1}\mathfrak{U}_{k_{2t+1}+1}\left(\frac{2-\delta}{2}\right)\mathcal{M}_{k_1+1}(\delta)\mathcal{M}_{k_{2v+1}+1}(\delta)G'''_\lambda(\delta)$ for some polynomial $G'''_\lambda(\delta)$.

It remains to show that $G'''_\lambda(\delta) \in \{-1, 1\}$. To this end, let $Y_n(\delta) = (\delta-1)^{\sum_{t=0}^{v} k_{2t+1}}(\delta-2)^{\sum_{t=1}^{v} k_{2t}-v}\prod_{t=1}^{v-1}\mathfrak{U}_{k_{2t+1}+1}\left(\frac{2-\delta}{2}\right)\mathcal{M}_{k_1+1}(\delta)\mathcal{M}_{k_{2v+1}+1}(\delta)$. First note that $\deg(G_\lambda(\delta)) = n-1 = \deg(Y_n(\delta))$. Hence, $G'''_\lambda(\delta)$ is a constant polynomial. From Equation 3.46 we have $\mathcal{M}_m(\delta)$ is monic polynomial, for each m. By Equation 3.45 the leading coefficient of $\mathfrak{U}_m\left(\frac{2-\delta}{2}\right)$ is equal to $(-1)^m$. Therefore, the leading coefficient of $Y_n(\delta)$ is in $\{-1, 1\}$. By Corollary 3.9.6 and Laplace expansion of determinant of a matrix, one can see that the leading coefficient of $G_\lambda(\delta)$ is 1. The claim follows. □

Corollary 3.9.14. *For each $\lambda \in \mathcal{C}^2(n-2)$ all the roots of the polynomials $G_\lambda(\delta)$ are real and lie in the interval $[0, 4)$.*

Proof. Follows directly from the Lemma 3.9.12. □

Example 3.9.15. We continue from Example 3.9.5.Then λ is of the form $(2, 1, 2^0)$ and by Laplace expansion we have $G_{(2,1)}(\delta) = (\delta-1)^2(\delta^2-3\delta+1)$.

If we compare this result with the Proposition 3.9.13 we see, according to the Proposition 3.9.13 we have $G_{(2,1)}(\delta) = (\delta-1)^1(\delta-2)^0\mathcal{M}_2(\delta)\mathcal{M}_1(\delta)$ which agrees with our direct calculation.

We define the following sequence of polynomials and then use their zeros to obtain a necessary and sufficient condition for the semisimplicity of $T_n^2(\delta)$.

$$K_{2r}(\delta) = \prod_{i=1}^{r-1} \mathcal{M}_i(\delta)\mathfrak{U}_i\left(\frac{2-\delta}{2}\right) \tag{3.48}$$

$$K_{2r+1}(\delta) = \prod_{i=1}^{r} \mathcal{M}_i(\delta) \prod_{i=1}^{r-1} \mathfrak{U}_i\left(\frac{2-\delta}{2}\right) \tag{3.49}$$

Let $K_n = \{\alpha \in \mathbb{C} \mid K_n(\alpha) = 0\}$. Then we have

$$K_3 \subset K_4 \subset \cdots \subset K_n \subset K_{n+1} \subset \cdots \tag{3.50}$$

Theorem 3.9.16. *If $\delta \neq 0$ then the $\mathbb{C}$-algebra $T_n^2(\delta)$ is semisimple if and only if $\delta \notin K_n$. In particular, if $\delta \notin [0,4)$ then $T_n^2(\delta)$ is semisimple. Furthermore, $T_{2n}^2(0)$ is not semisimple.*

Proof. By using Theorem 3.6.3 the algebra $T_{2n}^2(0)$ is not quasi-hereditary, hence it is not semisimple.
For the rest of the proof we assume $\delta \neq 0$. Let $\delta \notin K_n$ to show that $T_n^2(\delta)$ is semisimple. By Lemma 3.9.1, the sequence of $\mathbb{C}$-algebras $\{T_n^2(\delta) \mid n \in \mathbb{N}\}$ and the set of idempotents $\{\mathfrak{e}_n \mid n \in \mathbb{N}\}$ satisfies the axioms towers of recollement. Therefore, by Theorem 3.9.2 for $n \in \mathbb{Z}_{>0}$ to show $T_n^2(\delta)$ is semisimple it is enough to prove ${}_{T_m^2(\delta)}\Delta_m(\lambda)$ is simple, for each $m \leq n$ and $\lambda \in \Lambda_m^{m-2} = \mathcal{C}^2(m-2)$. By Corollary 3.8.9, for any $\lambda \in \Lambda(T_m^2(\delta))$ the module ${}_{T_m^2(\delta)}\Delta_m(\lambda)$ is simple if and only if $G_\lambda(\delta) \neq 0$. By Proposition 3.9.13 for each $\lambda \in \Lambda_m^{m-2}$ the polynomial $G_\lambda(\delta)$ can be factorised into products of the polynomials $\delta, \mathcal{M}_i(\delta)$ and $\mathfrak{U}_j(\frac{2-\delta}{\delta})$, for some positive integers $i, j \in \mathbb{Z}_{>0}$. Therefore, to prove the if part of the result it is enough to check which degrees of the polynomials $\delta, \mathcal{M}_i(\delta)$ and $\mathfrak{U}_j(\frac{2-\delta}{\delta})$ occur as a factor of

$$\prod_{\substack{\lambda \in \Lambda_m^{m-2} \\ m \leq n}} G_\lambda(\delta).$$

In fact we have the following two cases:

Case 1. If $n = 2r$, then by Proposition 3.9.13 we have

$$\prod_{\substack{\lambda\in\Lambda_m^{m-2}\\ m\leq n}} G_\lambda(\delta) = \delta^k \prod_{i=1}^{r-1} (\mathcal{M}_i(\delta))^{k_i} \left(\mathfrak{U}_i\left(\frac{2-\delta}{2}\right)\right)^{s_i}$$

for some $k, k_i, s_i \in \mathbb{Z}_{>0}$, where $i \in \underline{r-1}$.

Case 2. If $n = 2r+1$, then by Proposition 3.9.13 we have

$$\prod_{\substack{\lambda\in\Lambda_m^{m-2}\\ m\leq n}} G_\lambda(\delta) = \prod_{i=1}^{r} (\mathcal{M}_i(\delta))^{k_i} \prod_{j=1}^{r-1} \left(\mathfrak{U}_j\left(\frac{2-\delta}{2}\right)\right)^{s_j}$$

for some $k_i, s_j \in \mathbb{Z}_{>0}$, where $i \in \underline{r}, j \in \underline{r-1}$. If $\delta \neq 0$ then from the definition of K_n and Proposition 3.9.13 we have

$$K_n = \left\{\alpha \in \mathbb{C} \mid \prod_{\substack{\lambda\in\Lambda_m^{m-2}\\ m\leq n}} G_\lambda(\alpha) = 0\right\} = \left\{\alpha \in \mathbb{C} \mid \prod_{\lambda\in\mathcal{C}^2(n-2)} G_\lambda(\alpha) = 0\right\} \qquad (3.51)$$

Hence if $\delta \notin K_n$ we have for each $m \leq n$ and $\lambda \in \Lambda_m^{m-2} = \mathcal{C}^2(m-2)$ the module ${}_{T_m^2(\delta)}\Delta_m(\lambda)$ is simple, as desired.

Let $T_n^2(\delta)$ be semisimple to show that $\delta \notin K_n$. If $\delta \in K_n$ then by Equation 3.51, there exist $\lambda \in \mathcal{C}^2(n-2)$ such that $G_\lambda(\delta) = 0$. Hence, ${}_{T_n^2(\delta)}\Delta_n(\lambda)$ is not simple, which is a contradiction.

Corollary 3.9.14 implies that for each $\lambda \in \mathcal{C}^2(n-2)$ if $\delta \notin [0, 4)$ then $\delta \notin K_n$, hence ${}_{T_n^2(\delta)}\Delta_n(\lambda)$ is simple. Consequently, $T_n^2(\delta)$ is semisimple. □

In Proposition 2.2.3 [6] Bisch and Jones have shown that if $\delta \geq 2$ then the $\mathbb{C}$-algebra $FC_{1,n}(\delta, \delta)$ is semisimple. In the next Corollary we improve their result by presenting a necessary and sufficient condition for $FC_{1,n}(\delta, \delta)$ to be semisimple.

Corollary 3.9.17. *Let $\delta \neq 0$. Then the $\mathbb{C}$-algebra $FC_{1,n}(\delta, \delta)$ is semisimple if and only if $\delta \notin K_n$. In particular, if $\delta \notin (-2, 2)$ then the the Fuss-Catalan algebra $FC_{1,n}(\delta, \delta)$ is semisimple.*

Proof. Follows from Theorems 3.4.2 and 3.9.16. $\square$

We list the first few terms of the sequence $\{K_n\}_{n\in\mathbb{N}}$, which describes the values where $T_n^2(\delta)$ is not semisimple,

$$
\begin{aligned}
K_2 &= \phi\\
K_3 &= \{1\},\\
K_4 &= \{1, 2\},\\
K_5 &= \left\{1, 2, \frac{3}{2} \pm \frac{\sqrt{5}}{2}\right\},\\
K_6 &= \left\{1, 2, 3, \frac{3}{2} \pm \frac{\sqrt{5}}{2}\right\},\\
K_7 &= \left\{1, 2, 3, \frac{3}{2} \pm \frac{\sqrt{5}}{2}, 4\cos^2\left(\frac{\pi}{7}\right), 4\cos^2\left(\frac{2\pi}{7}\right), 4\cos^2\left(\frac{3\pi}{7}\right)\right\},\\
&\vdots
\end{aligned}
$$

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