Representation Theory Of Algebras Related To The Bubble Algebra

Mufida Mohamed A. Hmaida

A thesis submitted in accordance with the requirements for the degree of Doctor of Philosophy

The University of Leeds
School of Mathematics

September 2016

The candidate confirms that the work submitted is her own and that appropriate credit has been given where reference has been made to the work of others.

This copy has been supplied on the understanding that it is copyright material and that no quotation from the thesis may be published without proper acknowledgement.

Acknowledgements

My greatest appreciation and gratitude go to both my supervisor, Dr. Alison Parker, and my co-supervisor, Prof. Paul Martin, for their invaluable guidance and advice throughout my PhD research.

In addition, I wish to thank all my friends and colleagues in the school of Mathematics at the University of Leeds for their wonderful collaboration, friendship and numerous interesting discussions.

Finally, thanks goes to my dear parents, sisters, brothers and darling aunty, Halima, for their unconditional love, support and cheering me up throughout my entire life.

Thank you very much, everyone!

Mufida Mohamed Hmaida
The University of Leeds,
Sept. 2016

Abstract

In this thesis we study several algebras which are related to the bubble algebra, including the bubble algebra itself. We introduce a new class of multi-parameter algebras, called the multi-colour partition algebra $\mathbb{P}_{n,m}(\breve{\delta})$, which is a generalization of both the partition algebra and the bubble algebra. We also define the bubble algebra and the multi-colour symmetric groupoid algebra as sub-algebras of the algebra $\mathbb{P}_{n,m}(\breve{\delta})$.

We investigate the representation theory of the multi-colour symmetric groupoid algebra $\mathbb{F}\mathfrak{S}_{n,m}$. We show that $\mathbb{F}\mathfrak{S}_{n,m}$ is a cellular algebra and it is isomorphic to the generalized symmetric group algebra $\mathbb{F}\mathbb{Z}_m \wr \mathfrak{S}_n$ when m is invertible and $\mathbb{F}$ is an algebraically closed field. We then prove that the algebra $\mathbb{P}_{n,m}(\breve{\delta})$ is also a cellular algebra and define its cell modules. We are therefore able to classify all the irreducible modules of the algebra $\mathbb{P}_{n,m}(\breve{\delta})$. We also study the semi-simplicity of the algebra $\mathbb{P}_{n,m}(\breve{\delta})$ and the restriction rules of the cell modules to lower rank n over the complex field.

The main objective of this thesis is to solve some open problems in the representation theory of the bubble algebra $\mathbb{T}_{n,m}(\breve{\delta})$. The algebra $\mathbb{T}_{n,m}(\breve{\delta})$ is known to be cellular. We use many results on the representation theory of the Temperley-Lieb algebra to compute bases of the radicals of cell modules of the algebra $\mathbb{T}_{n,m}(\breve{\delta})$ over an arbitrary field. We then restrict our attention to study representations of $\mathbb{T}_{n,m}(\breve{\delta})$ over the complex field, and we determine the entire Loewy structure of cell modules of the algebra $\mathbb{T}_{n,m}(\breve{\delta})$. In particular, the main theorem is Theorem 5.41.

Contents

List of Figures

List of Tables

Index of Notation

Introduction

In 2003, Grimm and Martin [23] introduced a new algebra, called the bubble algebra $\mathbb{T}_{n,m}(\delta_0, \ldots, \delta_{m-1})$, this algebra defined entirely diagrammatically. They found the generic representations of the bubble algebra and proved that it is semi-simple when none of parameters δ_i is a root of unity. Later, Jegan [28] showed that the bubble algebra is a cellular algebra over any field, and it is a tower of recollement when all of the δ_i are non-zero. Also Jegan [28] showed how certain idempotent sub-algebras of the bubble algebra correspond to tensor products of Temperley-Lieb algebras and investigated the homomorphisms between the cell modules of the algebra $\mathbb{T}_{n,m}$. The problem of computing the Cartan matrix of the algebra is still open when some of the parameters are roots of unity. This has been the starting point of the work we present here.

In this thesis we deal with many algebras, all of them contained in the multi-colour partition algebra, which is defined in Chapter 2. Although this algebra is much bigger than the partition algebra, many techniques that are used to study the partition algebra still work on the multi-colour partition algebra $\mathbb{P}_{n,m}(\delta_0, \ldots, \delta_{m-1})$. The partition algebra was defined by Martin [37, 38, 39] and independently by Jones [29], and its representation theory has been investigated by many people, for example Doran and Wales [14], Halverson [24], Halverson and Ram [25], Martin [39, 40], Martin and Saleur [42], Martin and Woodcock [44] and Xi [55].

A key technique used in this thesis consists of reducing problems in the bubble algebra to problems in the Temperley-Lieb algebra. The Temperley-Lieb algebra was first introduced in [52] and its representation theory is well known, see Martin [37], Ridout and Saint [48] and Westbury [54]. Not surprisingly we have found a number of features in common with of the Temperley-Lieb algebra, as both of them are cellular algebras. The notion of a cellular algebra was first introduced by Graham and Lehrer [20]. Many properties of the representation theory of a cellular algebra can be determined from the cellular structure alone, see [32], [33], [53] and [56].

Chapter overview:

In the first chapter, we shall recall the preliminary results required to proceed with the following chapters. This will mainly be a review of the Temperley-Lieb algebra and the partition algebra and some results regarding their representation theories.

In chapter two we define our main algebras. We begin by defining the multi-colour partition algebra $\mathbb{P}_{n,m}(\delta_0, \ldots, \delta_{m-1})$, which is a generalization of the partition algebra and the bubble algebra, and giving some of its properties such as its dimension. Also we redefine it by using generators and relations. In Section 2.5, we define the bubble algebra as a sub-algebra of the multi-colour partition algebra and determine its dimension and a generating set for it. In the end of this chapter we discuss certain special idempotent sub-algebras of the multi-colour algebra and show that they are isomorphic to products of partition algebras.

In Chapter 3, we study the multi-colour symmetric groupoid $\mathfrak{S}_{n,m}$. It is the same as the groupoid $\mathscr{G}(m,n)$ in Section 2 in [46]. In this chapter we show that for n and m positive integers, we have

$$\mathbb{F}\mathfrak{S}_{n,m} \cong \bigoplus_{\lambda \in \Gamma_{(n,m)}} \left(\mathbb{F}\Big(\prod_{i=0}^{m-1} \mathfrak{S}_{\lambda_i} \Big) \otimes_{\mathbb{F}} M_{n_\lambda}(\mathbb{F}) \right),$$

where $n_\lambda = \begin{pmatrix} n \\ \lambda_0, \ldots, \lambda_{m-1} \end{pmatrix}$. We use this to determine the complete set of non-isomorphic simple $\mathbb{F}\mathfrak{S}_{n,m}$-modules. In Section 3.3 we show that the generalized symmetric group algebra is isomorphic to the algebra $\mathbb{F}\mathfrak{S}_{n,m}$ when m is invertible in $\mathbb{F}$ and $\mathbb{F}$ is algebraically closed.

The main objective of the fourth chapter is to study the representation theory of the multi-colour partition algebra. We will analyse the irreducible representations of the multi-colour partition algebra. We do this by showing that the algebra $\mathbb{P}_{n,m}$ is a cellular algebra and then study the cell modules of it and find some of its properties such as the restriction rule. Xi [55] has proved that the partition algebra $\mathbb{P}_n(\delta)$ is a cellular algebra, by using the fact that the symmetric group algebra is a cellular algebra. We will do the same, showing that $\mathbb{P}_{n,m}(\breve{\delta})$ is a cellular by using the fact that the tensor product of finitely many symmetric group algebras is cellular. The main result of this chapter is that the algebra $\mathbb{P}_{n,m}(\breve{\delta})$ is non-semisimple over the complex field if and only if δ_j is a non-negative integer less than $2n-1$ for some $j \in \mathbb{Z}_m$.

In the final chapter we study the representation theory of the bubble algebra $\mathbb{T}_{n,m}(\delta_0, \ldots, \delta_{m-1})$. We have shown that we can use the cell modules of the algebra $\mathsf{TL}_n(\delta)$ to construct the cell modules of the bubble algebra. We begin by defining its cell modules and then study their properties such as the dimensions and their radicals. The last part of this thesis deals with the Cartan matrix of the bubble algebra over the complex field.

Chapter 1

Background

As mentioned in the introduction, we review some structures, known results and technical details that we will be using through the thesis. We start in Section 1.1 with fundamental facts about algebras. Next we define the groupoid and discuss some of its properties. Our aim in this thesis is studying the bubble algebra and the multi-colour partition algebras, relying on the results of the representation theory of both the Temperley-Lieb algebra and the partition algebra. Thus it will be convenient to recall the main results of the representation theory of the Temperley-Lieb algebra, the partition algebra and the symmetric group. Furthermore, as all these algebras are cellular algebras, we briefly summarise the basic facts about cellular algebra.

We denote the set of non-negative integers by $\mathbb{N}$, all integers are denoted by $\mathbb{Z}$ and the complex numbers by $\mathbb{C}$.

1.1 Basics

Throughout the thesis, we assume that $\mathbb{F}$ is an arbitrary field of a charcteristic $\mathsf{p} \geq 0$, $\mathbb{A}$ is a unital associative $\mathbb{F}$-algebra of finite dimension. We take $n, m \in \mathbb{N}$ and fix parameters δ, δ_0, $\delta_1, \ldots, \delta_{m-1}$ in the field $\mathbb{F}$. The symbol $\breve{\delta}$ is used to refer a tuple $(\delta_0, \ldots, \delta_{m-1})$. All modules in this thesis will be left modules of finite dimension unless explicitly stated otherwise.

For any family of $\mathbb{F}$-algebras $\mathbb{A}_n$ with an inclusion $\mathbb{A}_{n-1} \hookrightarrow \mathbb{A}_n$, we will use $M \downarrow_{\mathbb{A}_{n-1}}$ to denote the restriction of an left $\mathbb{A}_n$-module M to the algebra $\mathbb{A}_{n-1}$.

Let N be an $\mathbb{A}_{n-1}$-module, then $N \uparrow^{\mathbb{A}_n} := \mathbb{A}_n \otimes_{\mathbb{A}_{n-1}} N$ is an $\mathbb{A}_n$-module, the *induced module*, with action defined by $a(b \otimes n) = (ab) \otimes n$ for all $a, b \in \mathbb{A}_n$ and $n \in N$.

Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be finite dimensional algebras over $\mathbb{F}$ and $\mathbb{A}_1 \hookrightarrow \mathbb{A}_2$, and M and N are $\mathbb{A}_1$ and $\mathbb{A}_2$ modules respectively. Then we have

$$\mathrm{Hom}_{\mathbb{A}_2}(M \uparrow^{\mathbb{A}_2}, N) \cong \mathrm{Hom}_{\mathbb{A}_1}(M, N \downarrow_{\mathbb{A}_1}), \tag{1.1}$$

which is known as Frobenius reciprocity, see for example Proposition 3.3.1 in [3].

Proposition 1.1. *[e.g. 6, Section 6.2]. Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be algebras over $\mathbb{F}$. Suppose that $\mathbb{A}_1$ is given by generators and relations: $\mathbb{A}_1 = \mathbb{F}\langle X\rangle/(r_i)$. Then algebra homomorphisms are in bijection with maps $f : X \to \mathbb{A}_2$ such that $f(r_i) = 0$ for all i.*

Two idempotents $\mathsf{e}, \mathsf{e}' \in \mathbb{A}$ are conjugate if there exists an invertible element $u \in \mathbb{A}$ such that $u\mathsf{e}u^{-1} = \mathsf{e}'$.

Lemma 1.2. *[e.g. 1, Corollary 5.11]. The idempotent e is primitive if and only if the $\mathsf{e}\mathbb{A}\mathsf{e}$ is a local ring.*

Let $\mathbb{A}$ be an algebra over a field $\mathbb{F}$ and e be an idempotent in $\mathbb{A}$, then $\mathsf{e}\mathbb{A}\mathsf{e}$ is also an algebra and it is called an idempotent sub-algebra of $\mathbb{A}$. There are additive $\mathbb{F}$-linear covariant functors between $\mathbb{A}$-mod(the category of left $\mathbb{A}$-modules) and $\mathsf{e}\mathbb{A}\mathsf{e}$-mod

$$\mathsf{e}\mathbb{A}\mathsf{e}\text{-mod} \xrightarrow{G} \mathbb{A}\text{-mod} \xrightarrow{F} \mathsf{e}\mathbb{A}\mathsf{e}\text{-mod}$$

where $F(N) = \mathsf{e}N$ and $G(M) = \mathbb{A}\mathsf{e} \otimes_{\mathsf{e}\mathbb{A}\mathsf{e}} M$. The functors F and G are called localisation and globalisation with respect to e, respectively. Note that $FG(M) = M$ and G is a full embedding. For more details see for example Section 5.3 in [37]. Note that F takes simples to simples or zero.

Theorem 1.3. *[21, Section 6.2]. Let e be an idempotent in $\mathbb{A}$ and $\{S_\lambda \mid \lambda \in \Lambda\}$ be a complete set of non-isomorphic simple left modules of $\mathbb{A}$, and set $\Lambda^{\mathsf{e}} = \{\lambda \in \Lambda \mid$*

$\mathsf{e}S_\lambda \neq 0\}$. *Then* $\{\mathsf{e}S_\lambda \mid \lambda \in \Lambda^{\mathsf{e}}\}$ *is a complete set of inequivalent simple left modules of* $\mathsf{e}\mathbb{A}\mathsf{e}$, *and the remaining simple modules* S_λ *where* $\lambda \in \Lambda \setminus \Lambda^{\mathsf{e}}$ *are a complete set of simple modules of* $\mathbb{A}/\mathbb{A}\mathsf{e}\mathbb{A}$.

We will use the symbol $\bigoplus^{l} \mathbb{A}$ to denote the direct sum of l copies of a $\mathbb{F}$-algebra $\mathbb{A}$, and $M_l(\mathbb{F})$ to be the $l \times l$ matrix algebra over $\mathbb{F}$.

1.2 Groupoids

In order to study the multi-colour symmetric groupoid, it will be useful to recall some facts about groupoids and groupoid algebras. We follow Khalkhali [31].

Definition 1.4. [e.g. 31, Definition 2.1.1]**.** A groupoid $\mathcal{G}$ is a small category in which every morphism is an isomorphism.

A small category is a category where its objects form a set. The set of objects of $\mathcal{G}$ is denoted by $\mathcal{G}^{(0)}$ and the set of morphisms of $\mathcal{G}$ by $\mathcal{G}^{(1)}$. Every morphism has a source, a target and an inverse. They define maps, denoted by s, t, and i, respectively ($s : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(0)}$, $t : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(0)}$, ${}^{-1} : \mathcal{G}^{(1)} \longrightarrow \mathcal{G}^{(1)}$), there is also a canonical map $id : \mathcal{G}^{(0)} \longrightarrow \mathcal{G}^{(1)}$, which sends an object x to the unit morphism id_x from the object to itself. The composition $\rho_1 \bullet \rho_2$ of morphisms ρ_1 and ρ_2 is only defined if $s(\rho_1) = t(\rho_2)$.

The groupoid algebra $\mathbb{F}\mathcal{G}$ is the $\mathbb{F}$-algebra that is generated by the set $\mathcal{G}^{(1)}$ where the multiplication $\rho_1\rho_2$ is defined to be $\rho_1 \bullet \rho_2$ if the composition $\rho_1 \bullet \rho_2$ is defined, otherwise it will be zero, where $\rho_1, \rho_2 \in \mathcal{G}^{(1)}$. Note that $\mathbb{F}\mathcal{G}$ is unital algebra if and only if the set $\mathcal{G}^{(0)}$ is finite. The identity in this case is given by $1 = \sum_{x \in \mathcal{G}^{(0)}} id_x$.

Any groupoid $\mathcal{G}$ can be canonically decomposed as a disjoint union of connected groupoids (there is a morphism between any two objects), see Section 2.2 in [31]. From that we obtain

$$\mathbb{F}\mathcal{G} = \bigoplus_i \mathbb{F}\mathcal{G}_i,$$

where $\mathcal{G}_i$ is a connected groupoid. Choose a point $x_0 \in \mathcal{G}_i^{(0)}$ and let

$$G_i := \mathrm{Hom}_{\mathcal{G}_i}(x_0, x_0). \tag{1.2}$$

The set G_i forms a group and it is called the *isotropy group* of x_0. The isomorphism class of G_i is independent of the choice of the base point x_0.

Theorem 1.5. *[13, Proposition 3.1]. Let $\mathcal{G}$ be connected groupoid and $l = |\mathcal{G}^{(0)}|$ finite. Let $x_0 \in \mathcal{G}^{(0)}$ and G be the isotropy group of x_0. Then $\mathbb{F}\mathcal{G} \cong \mathbb{F}G \otimes_{\mathbb{F}} M_l(\mathbb{F})$, where $M_l(\mathbb{F})$ is the $l \times l$ matrix algebra over $\mathbb{F}$.*

In particular, if a groupoid $\mathcal{G}$ has all connected components of $\mathcal{G}$ finite, then from the previous theorem we obtain

$$\mathbb{F}\mathcal{G} \cong \bigoplus_i \mathbb{F}G_i \otimes_{\mathbb{F}} M_{n_i}(\mathbb{F}), \tag{1.3}$$

where n_i is the cardinality of the connected component and G_i is the corresponding isotropy group.

1.3 Generalized symmetric group

In this section we give a brief summary on results about the generalized symmetric group and the symmetric group itself that will be useful later.

We will use the wreath product to define the generalized symmetric group. Let $\mathfrak{S}_n$ denotes the symmetric group on the set $\{1, \ldots, n\}$ and $\mathbb{Z}_m$ be the cyclic group of order m generated by 1 under addition modulo m.

Definition 1.6. [e.g. 27, Chapter 4]**.** Let G be a group and H a subgroup of $\mathfrak{S}_n$. The wreath product of G by H, $G \wr H$, is the set $G^n \times H$ with the composition defined by

$$(x;\pi)(y;\sigma) = (xy^{\pi};\pi\sigma)\,,$$

where

$$y^{\pi} = (y_{\pi^{-1}(1)}, y_{\pi^{-1}(2)}, \ldots, y_{\pi^{-1}(n)}), \tag{1.4}$$

and $x, y \in G^n$, $\pi, \sigma \in H$.

The set $G \wr H$ forms a group with this composition. The identity of $G \wr H$ is the element $((e, \ldots, e); id)$, where e is the identity of G and id is the identity of the group $\mathfrak{S}_n$. Also, $(x; \pi)^{-1} = (x^{-1^{\pi^{-1}}}; \pi^{-1})$. If the group G is finite then the order of $G \wr H$ is $|G|^n |H|$. For proofs and more details see chapter 4 in [27].

The group $\mathbb{Z}_m \wr \mathfrak{S}_n$ is called the *generalized symmetric group* and it has been investigated for some time, see for example [7], [47] and [51]. Since the group $\mathbb{Z}_m$ has m elements, so $|\mathbb{Z}_m \wr \mathfrak{S}_n| = m^n n!$. In the group $\mathbb{Z}_m^n$, define $e_i := (0, \ldots, 1, \ldots, 0)$ with 1 at the i^{th} position, and

$$je_i = \underbrace{e_i + \cdots + e_i}_{j \ \ times} = (0, \ldots, j, \ldots, 0) \ ,$$

where $j \in \mathbb{N}$. In the group $\mathbb{Z}_m \wr \mathfrak{S}_n$, we will let $\overline{e}_i = (e_i; id)$, and $\overline{\mathsf{s}}_j = ((0, \ldots, 0); \mathsf{s}_j)$ where s_j is the transposition $(j \ j+1)$ in the group $\mathfrak{S}_n$, $1 \leq i \leq n$ and $1 \leq j \leq n-1$.

Proposition 1.7. *[12, Lemma 1]. The group $\mathbb{Z}_m \wr \mathfrak{S}_n$ is generated by the elements $\overline{e}_1, \ldots, \overline{e}_n$, $\overline{\mathsf{s}}_1, \ldots, \overline{\mathsf{s}}_{n-1}$ satisfying the following relations:*

1. $\overline{e}_i^m = 1$ *for all* $1 \leq i \leq n$.
2. $\overline{e}_i \overline{e}_j = \overline{e}_j \overline{e}_i$ *for all* $1 \leq i, j \leq n$.
3. $\overline{\mathsf{s}}_i \overline{e}_i \overline{\mathsf{s}}_i = \overline{e}_{i+1}$ *for all* $1 \leq i \leq n-1$.
4. $\overline{\mathsf{s}}_j \overline{e}_i \overline{\mathsf{s}}_j = \overline{e}_i$ *for all* $i \neq j, j+1$.
5. $\overline{\mathsf{s}}_i^2 = 1$ *for all* $1 \leq i \leq n-1$.
6. $\overline{\mathsf{s}}_i \overline{\mathsf{s}}_j = \overline{\mathsf{s}}_j \overline{\mathsf{s}}_i$ *for all* $|i - j| > 1$.
7. $\overline{\mathsf{s}}_i \overline{\mathsf{s}}_{i+1} \overline{\mathsf{s}}_i = \overline{\mathsf{s}}_{i+1} \overline{\mathsf{s}}_i \overline{\mathsf{s}}_{i+1}$ *for all* $1 \leq i \leq n-2$.

1.3.1 Partitions and multi-partitions

Recall that a *composition* λ of a positive integer n, denoted by $\lambda \vDash n$, is a sequence of non-negative integers $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_t)$ such that $n = \sum_{i=0}^{t} \lambda_i$. Let $t = m-1$ for some m, we called the composition λ an m-composition of n, denoted by $\lambda \vDash_m n$. Let $\Gamma_{(n,m)}$ denote the set of all m-compositions of n.

A *partition* is a composition $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_t)$ such that $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_t > 0$. The sum of all parts of λ, denoted by $|\lambda|$, is the weight of λ. If $|\lambda| = n$ we say λ is a partition of n. We write $\lambda \vdash n$ to denote that λ is a partition of n and $\Lambda_{(n)}$ is the set of all possible partitions of n. For example, $\Lambda_{(3)} = \{(3), (2,1), (1,1,1)\}$. For more details see for example Section 1.8 in [4] or Chapter 1 in [35].

Definition 1.8. [7, Definition 1.2]**.** Let $\lambda = (\lambda_0, \dots, \lambda_{m-1}) \vDash n$. A *m-multi-partition* of n of type λ, $\boldsymbol{\mu} = (\mu^{(0)}, \dots, \mu^{(m-1)})$, consists of m partitions $\mu^{(0)}, \dots, \mu^{(m-1)}$ such that $\mu^{(i)} = (\mu_0^{(i)}, \dots, \mu_{t_i}^{(i)}) \vdash \lambda_i$. We denote this by $\boldsymbol{\mu} \vdash \lambda$ or $\boldsymbol{\mu} \vdash_m n$. Note, for any i, if $\lambda_i = 0$, we still need to write $\mu^{(i)} = 0$.

We define $\Lambda_{(n,m)}$ to be the set of all possible m-multi-partitions of the non-negative integer n.

1.3.2 Representation theory of the group $\mathfrak{S}_n$

Let $\mathscr{S}_\mu$ be the *Specht module* of the symmetric group $\mathfrak{S}_n$ associated to a partition μ, as defined in [27] and [17]. Over fields of characteristic 0 or greater than n, the Specht modules are simple, and form a complete set of non-isomorphic simple modules of the symmetric group. Also in this case, the algebra $\mathbb{F}\mathfrak{S}_n$ is semi-simple by Maschke's theorem, see for example Theorem 3.5 in [2] or Theorem 4.1.1 in [16].

A partition is called p-*regular* if it does not have p parts of the same (positive) size. For p-regular partitions, Specht modules have a unique irreducible head, and these irreducible quotient modules form a complete set of irreducible modules of the group algebra of the symmetric group.

Theorem 1.9. *[26, Theorem 11.5]. Let $\mathbb{F}$ be a field of characteristic $\mathsf{p} \geq 0$, then the non-isomorphic simple modules of the algebra $\mathbb{F}\mathfrak{S}_n$ are parametrized by $\{\lambda \vdash n \mid \lambda$ is a p-regular partition $\}$.*

1.4 Properties of some matrix operations

For any arbitrary matrices A_i where $i = 1, \ldots, t$, the *direct sum* of these matrices, is denoted by $A_1 \oplus \cdots \oplus A_t$ and is defined to be the matrix

$$\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_t \end{bmatrix}.$$

If $A_i = A$ for each i, we set $\bigoplus^t A = \underbrace{A \oplus \cdots \oplus A}_{t \text{ copies}}$. Also, it is known that

$$\operatorname{rank}(A_1 \oplus A_2) = \operatorname{rank}(A_1) + \operatorname{rank}(A_2), \tag{1.5}$$

see for example Proposition 2.11.13 in [5].

Let $B = A_1 \oplus \cdots \oplus A_t$, where each matrix A_i is a square matrix, then

$$\det B = \prod_{i=1}^{t} \det(A_i). \tag{1.6}$$

For the proof see Proposition 2.8.1 in [5].

Let A be a $s \times s$ matrix and B a $r \times r$ matrix. The determinant of the *Kronecker* (or *tensor*) product of A and B, denoted $A \otimes B$, satisfies $\det(A \otimes B) = \det(A)^r \times \det(B)^s$, see for example Proposition 7.1.11 in [5]. Also,

$$\operatorname{rank}(A \otimes B) = \operatorname{rank}(A) \operatorname{rank}(B), \tag{1.7}$$

see Proposition 7.4.23 in [5].

Lemma 1.10. *[28, Lemma 3.2.8]. Let $B = A_1 \otimes \cdots \otimes A_t$ where A_i is $n_i \times n_i$ matrix, then*

$$\det B = \Big(\prod_{i=1}^{t} (\det A_i)^{n_i^{-1}}\Big)^{\prod_{i=1}^{t} n_i}. \tag{1.8}$$

1.4.1 Vandermonde matrix

Let $\mathbb{F}$ be an algebraically closed field and m a positive integer, then $\mathbb{F}$ contains all m^{th} roots of unity. Assume that ω is a primitive m^{th} root of unity, so we can define a special case of the *Vandermonde* matrix $\mathbf{F}_m$ to be:

$$\mathbf{F}_m := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{m-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(m-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{m-1} & \omega^{2(m-1)} & \cdots & \omega^{(m-1)^2} \end{bmatrix} = \Big(w^{(i-1)(j-1)}\Big)_{\substack{1\le i\le m \\ 1\le j\le m}}.$$

From the definition, it is evident that $\mathbf{F}_m$ is symmetric. Consider the matrix $\mathbf{F}_m^* := \Big(w^{-(i-1)(j-1)}\Big)_{\substack{1\le i\le m \\ 1\le j\le m}}$. Note that $\mathbf{F}_m\mathbf{F}_m^* = mI_m$, so $\mathbf{F}_m$ is invertible as long as m is a unit in the field $\mathbb{F}$. In other words, the matrix $\mathbf{F}_m$ is invertible if and only if $\gcd(m, \mathrm{Char}(\mathbb{F})) = 1$. For more details see Chapter 4 in [17] or [36] .

Now, we define the matrix $\mathbf{F}_m^{(n)} := \bigotimes^{n} \mathbf{F}_m = \underbrace{\mathbf{F}_m \otimes \cdots \otimes \mathbf{F}_m}_{n \text{ times}}$. From the definition of $\mathbf{F}_m^{(n)}$, it is clear that

$$\mathbf{F}_m^{(n)} = \Big(w^{ij}\mathbf{F}_m^{(n-1)}\Big)_{\substack{0\le i\le m-1 \\ 0\le j\le m-1}}. \tag{1.9}$$

By relation (1.8), if the matrix $\mathbf{F}_m$ is invertible, the matrix $\mathbf{F}_m^{(n)}$ will be also invertible for all n.

1.5 Cellular algebra

In this section, we recall the definition of a *cellular algebra*, which was introduced by Graham and Lehrer [20], and some results about the representation theory of cellular algebras. The original definition of the cellular algebra is over a ring, but we use a field since all our work is over a field.

Definition 1.11. [20, Definition 1.1]**.** A cellular algebra over $\mathbb{F}$ is an associative unital algebra $\mathbb{A}$, together with a tuple $(\Lambda, T, \mathrm{C}, *)$ such that

1. The set Λ is finite and partially ordered.
2. For every $\lambda \in \Lambda$, there is a non-empty finite set $T(\lambda)$ such that the map $\mathrm{C} : \bigcup_{\lambda\in\Lambda} T(\lambda) \times T(\lambda) \to \mathbb{A}$ is injective, and its image forms a $\mathbb{F}$ -basis of $\mathbb{A}$. The images under this map are notated with an upper index $\lambda \in \Lambda$ and two lower indices $s, t \in T(\lambda)$ so that the image is written as $\mathrm{C}^{\lambda}_{st}$.
3. The map $* : \mathbb{A} \to \mathbb{A}$ is $\mathbb{F}$-linear involution (This means that $*$ is an anti-automorphism with $*^2 = id_{\mathbb{A}}$ and $*(\mathrm{C}^{\lambda}_{st}) = \mathrm{C}^{\lambda}_{ts}$ for all $\lambda \in \Lambda$, $s, t \in T(\lambda)$).
4. For $\lambda \in \Lambda, s, t \in T(\lambda)$ and any $a \in \mathbb{A}$ we have
$$a\mathrm{C}^{\lambda}_{st} \equiv \sum_{u\in T(\lambda)} r_a(u, s)\mathrm{C}^{\lambda}_{ut} \mod \mathbb{A}^{<\lambda}, \tag{1.10}$$
where $r_a(u, s) \in \mathbb{F}$ depends only on a, u and s. Here $\mathbb{A}^{<\lambda}$ denotes the $\mathbb{F}$-span of all basis elements with upper index strictly less than λ.

Definition 1.12. [20, Definition 2.1]**.** The cell module $\Delta(\lambda)$, $\lambda \in \Lambda$, is an $\mathbb{A}$-module with $\mathbb{F}$-basis $\{\mathrm{C}_s \mid s \in T(\lambda)\}$ and an action given by $a\mathrm{C}_s = \sum_{u\in T(\lambda)} r_a(s, u)\mathrm{C}_u$ for any $a \in \mathbb{A}$, $s \in T(\lambda)$ where $r_a(s, u)$ are the same coefficients as in equation (1.10) .

Definition 1.13. [e.g. 26, Section 1.5]**.** Let V be a finite-dimensional vector space with an inner product $\langle\ ,\ \rangle$. The *Gram matrix* of V, G, is defined with respect to a basis $v_1, \ldots, v_k$ of V by letting the $(i, j)^{\text{th}}$ entry of G be $\langle v_i, v_j\rangle$. The radical of the form $\langle\ ,\ \rangle$ is the set $\{v \in V \mid \langle v, w\rangle = 0 \text{ for all } w \in V\}$.

On each cell module $\Delta(\lambda)$, there is an contravariant, symmetric bilinear form $\langle\ ,\ \rangle_\lambda : \Delta(\lambda) \times \Delta(\lambda) \to \mathbb{F}$ defined by the relation (1.10). For a proof see Proposition 2.9 in [45] or Proposition 2.4 [20]. Let G_λ be the Gram matrix for $\Delta(\lambda)$ of this bilinear form with respect to a basis $\{\mathrm{C}_s \mid s \in T(\lambda)\}$. All Gram matrices of any cell module that will be mentioned later are with respect to the cellular basis with the bilinear form defined by (1.10).

If $\mathbb{A}$ is an algebra over a field, the module $\Delta(\lambda)$ is a simple module if and only if $\det \mathsf{G}_\lambda \neq 0$ as long as $\langle\ ,\ \rangle_\lambda \neq 0$, see for example Proposition 3.1 in [20] or Proposition 2.7e [21]. Let Λ^0 be the subset $\{\lambda \in \Lambda \mid \langle\ ,\ \rangle_\lambda \neq 0\}$. The radical $\mathrm{Rad}_{\langle\ ,\ \rangle_\lambda}(\Delta(\lambda))$ of the form $\langle\ ,\ \rangle_\lambda$ is an $\mathbb{A}$-sub-module.

Theorem 1.14. *[45, Chapter 2]. Let $\mathbb{A}$ be a cellular algebra over a field $\mathbb{F}$. Then*

1. *$\mathbb{A}$ is semi-simple if and only if $\det \mathsf{G}_\lambda \neq 0$ for each $\lambda \in \Lambda$.*

2. *The quotient module $\Delta(\lambda)/\mathrm{Rad}_{\langle\ ,\ \rangle_\lambda}(\Delta(\lambda))$ is either irreducible or zero. That means that $\mathrm{Rad}_{\langle\ ,\ \rangle_\lambda}(\Delta(\lambda))$ is the radical of the module $\Delta(\lambda)$ if $\langle\ ,\ \rangle_\lambda \neq 0$.*

3. *The set $\{L(\lambda) := \Delta(\lambda)/\mathrm{Rad}_{\langle\ ,\ \rangle_\lambda}(\Delta(\lambda)) \mid \lambda \in \Lambda^0\}$ consists of all non-isomorphic irreducible $\mathbb{A}$-modules.*

4. *Let $L(\mu) \neq 0$ and M be a sub-module of $\Delta(\lambda)$, and suppose that $\theta : \Delta(\mu) \to \Delta(\lambda)/M$ is a non-zero $\mathbb{A}$-module homomorphism, then $\lambda \geq \mu$.*

5. *Each cell module $\Delta(\lambda)$ of $\mathbb{A}$ has a composition series with sub-quotients isomorphic to $L(\mu)$, where $\mu \in \Lambda^0$. The multiplicity of $L(\mu)$ is the same in any composition series of $\Delta(\lambda)$ and we write $d_{\lambda\mu} = [\Delta(\lambda) : L(\mu)]$ for this multiplicity.*

6. *The* decomposition matrix $\mathsf{D} = \left(d_{\lambda\mu}\right)_{\lambda\in\Lambda,\mu\in\Lambda^0}$ *is upper uni-triangular, i.e. $d_{\lambda\mu} = 0$ unless $\lambda \leq \mu$ and $d_{\lambda\lambda} = 1$ for $\lambda \in \Lambda^0$.*

7. *If Λ is a finite set and $\mathscr{C}$ is the Cartan matrix of $\mathbb{A}$, then $\mathscr{C} = \mathsf{D}^t\mathsf{D}$.*

Now since $\mathbb{A}$ is a cellular algebra over a field $\mathbb{F}$, then the rank of the matrix G_λ equals $\dim L(\lambda)$ and the nullity of G_λ is equal to $\dim \mathrm{Rad}_{\langle\, ,\, \rangle_\lambda}(\Delta(\lambda))$ as long as $\lambda \in \Lambda^0$, see Chapter 2, exercise 6 in [45].

Theorem 1.15. *[55, Theorem 3.3]. Let $\mathbb{A}$ be an algebra with an involution $*$. Suppose there is a decomposition of $\mathbb{A}$:*

$$\mathbb{A} = \bigoplus_{j=1}^{m} V_j \otimes_{\mathbb{F}} V_j \otimes_{\mathbb{F}} B_j$$

as direct sum of vector spaces, where V_j is a vector space and B_j is a cellular algebra with respect to an involution σ_j and we have a cell chain $J_1^{(j)} \subset \cdots \subset J_{s_j}^{(j)} = B_j$ for each j. Define $J_t = \bigoplus_{j=1}^{t} V_j \otimes_{\mathbb{F}} V_j \otimes_{\mathbb{F}} B_j$. Assume that the restriction of $*$ on $V_j \otimes_{\mathbb{F}} V_j \otimes_{\mathbb{F}} B_j$ is given by $w \otimes v \otimes b \mapsto v \otimes w \otimes \sigma_j(b)$. If for each j there is a bilinear form $\phi_j : V_j \otimes_{\mathbb{F}} V_j \to B_j$ such that $\sigma_j(\phi_j(w, v)) = \phi_j(v, w)$ for all $w, v \in V_j$ and that the multiplication of two elements in $V_j \otimes_{\mathbb{F}} V_j \otimes_{\mathbb{F}} B_j$ is governed by ϕ_j modulo J_{j-1}, that is, for $x, y, u, v \in V_j$ and $b, c \in B_j$, we have

$$(x \otimes y \otimes b)(u \otimes v \otimes c) = x \otimes v \otimes b\phi_j(y, u)c \mod J_{j-1},$$

and if $V_j \otimes V_j \otimes J_l^{(j)} + J_{j-1}$ is an ideal in $\mathbb{A}$ for all l and j, then $\mathbb{A}$ is a cellular algebra. Furthermore, $V_j \otimes v_j \otimes \Delta_t^{(j)}$ is a cell module of $\mathbb{A}$ for each j where $\Delta_t^{(j)}$ is a cell module of B_j and v_j is any non-zero vector in V_j.

Proposition 1.16. *[22, Section 3]. The tensor product and direct sum of finitely many cellular algebras is a cellular algebra.*

Proposition 1.17. *[56, Proposition 3.4]. Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two cellular algebras over a perfect field. Then $\mathbb{A}_1 \otimes \mathbb{A}_2$ is semi-simple if and only if $\mathbb{A}_1$, $\mathbb{A}_2$ are semi-simple.*

1.6 Quasi-hereditary algebra

There is another class of algebras, called *quasi-hereditary*, related to a cellular algebra. It was introduced for the first time in [8].

Definition 1.18. [8, Section 3]**.** A two-sided ideal J of $\mathbb{A}$ is a heredity ideal if

1. $J^2 = J$,
2. $J\,\mathrm{Rad}(\mathbb{A})J = 0$,
3. $J_{\mathbb{A}}$ and ${}_{\mathbb{A}}J$ are projective $\mathbb{A}$-modules.

Definition 1.19. [8, Section 3]**.** An algebra $\mathbb{A}$ is quasi-hereditary if there is a finite chain of two-sided ideals

$$0 = J_0 \subset J_1 \subset \cdots \subset J_l = A$$

such that for all $1 \leq i \leq l$, J_i/J_{i-1} is a heredity ideal of $\mathbb{A}/J_{i-1}$.

Remark 1.20. Let $\mathbb{A}$ be cellular algebra over a field, then $\mathbb{A}$ is quasi-hereditary if and only if $\Lambda = \Lambda^0$, see for example Corollary 2.23 in [45] and Corollary 4.2 in [32].

1.7 The partition algebra

Let $\mathcal{P}_X$ to be the set of all partitions of a finite set X:

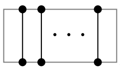
$$\mathcal{P}_X = \{\, \{X_1, X_2, \dots\} \mid \emptyset \neq X_i \subset X,\ \cup_i X_i = X, X_i \cap X_j = \emptyset \text{ if } i \neq j\}.$$

The subsets $X_1, X_2, \dots$ are called parts (or blocks).

The set $\mathcal{P}_X$ is a lattice with a partial order: if α, β are two partitions in $\mathcal{P}_X$, we say that α is smaller than or equal to β if and only if each part of α is a subset of a part of β.

For $n \in \mathbb{N}$, the symbol $\mathcal{P}_n$ denotes the set of all partitions of the set $\underline{n} \cup \underline{n'}$, where $\underline{n} = \{1, 2, \dots, n\}$ and $\underline{n'} = \{1', 2', \dots, n'\}$.

Each individual set partition can be represented by a graph: the graph is drawn in a rectangle with n nodes on the top row representing the elements in the set $\underline{n}$ and with n nodes on the bottom row of the rectangle representing the elements in the set $\underline{n}'$, and the vertices that are in the same part at the partition are represented as lines drawn inside the rectangle connecting these vertices. The diagram representing a partition is not unique, since there are different ways to drawing the edges. Two such diagramss are equivalent if they have the same connected components. A *partition diagram* (or sometimes (n, n)-partition diagram) is the equivalence class of a graph.

Now the composition $\beta \circ \alpha$ in $\mathcal{P}_n$, where $\alpha, \beta \in \mathcal{P}_n$, is the partition obtained by placing α above β, identifying the bottom vertices of α with the top vertices of β, and ignoring any connected components that are isolated from boundaries. The product on $\mathcal{P}_n$ is associative and well-defined up to equivalence, so $\mathcal{P}_n$ forms a monoid with the identity . The proof can be found in [38].

A (n, m)-partition diagram for any $n, m \in \mathbb{N}^+$ is a diagram representing a set partition of the set $\underline{n} \cup \underline{m'}$ in the obvious way.

We can generalize the product on $\mathcal{P}_n$ to define a product of (n, m)-partition diagrams when it is defined: Let α be (n_1, n_2)-diagram and β be (m_1, m_2)-diagram, $\beta \circ \alpha$ is defined if and only if $n_2 = m_1$ and it is (n_1, m_2)-diagram. For example, see figure 1.1.

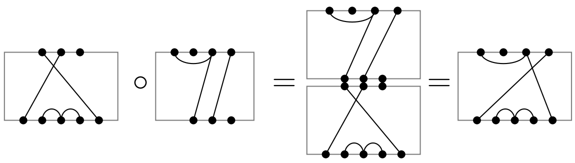


FIGURE 1.1: The composition of two partition diagrams.

Definition 1.21. [39, Definition 5]**.** The propagating number of a diagram, $\#(d)$, is the number of parts which include elements from both the top and the bottom rows.

A string in a diagram which is connecting a point in the top row and a point in the bottom row is called a *propagating line.* Martin in [38] has proved that the propagating number satisfies the property

$$\#(d_1 \circ d_2) \leq \min(\#(d_1), \#(d_2)), \tag{1.11}$$

where $d_1, d_2 \in \mathcal{P}_n$.

Definition 1.22. [39, Definition 4]**.** Let $\mathbb{P}_n(\delta) = \mathbb{F}\mathcal{P}_n$ be the $\mathbb{F}$-vector space with basis $\mathcal{P}_n$, and product on $\mathbb{P}_n(\delta)$ defined by $\alpha\beta = \delta^l(\beta \circ \alpha)$, where $\beta \circ \alpha$ is the monoid multiplication, the parameter $\delta \in \mathbb{F}$ and l is the number of connected components removed from the middle row when constructing the product $\beta \circ \alpha$.

The space $\mathbb{P}_n(\delta)$, or simply $\mathbb{P}_n$, is an associative $\mathbb{F}$-algebra with identity and it is known as the *partition algebra*. For more details, see [38] or [39].

A *planar diagram* in $\mathcal{P}_n$ is a partition diagram where there are no edges crossing in the diagram.

Define the following subsets of the partition monoid $\mathcal{P}_n$:

$$\left.\begin{aligned} \mathfrak{S}_n &= \{d \in \mathcal{P}_n \mid \#(d) = n\}, \\ \mathcal{A}_n &= \{d \in \mathcal{P}_n \mid d \text{ is planar}\}, \\ \mathcal{Q}_n &= \{d \in \mathcal{P}_n \mid \#(d) < n\}, \\ \mathcal{B}_n &= \{d \in \mathcal{P}_n \mid \text{all blocks of } d \text{ have size } 2\}, \\ \mathcal{T}_n &= \mathcal{A}_n \cap \mathcal{B}_n. \end{aligned}\right\} \tag{1.12}$$

All of them are sub-monoids except the subset $\mathcal{Q}_n$. Therefore the following algebra can be defined: the *Temperley-Lieb algebra*, $\mathsf{TL}_n(\delta)$, is the sub-algebra of $\mathbb{P}_n(\delta)$ which is generated by the set $\mathcal{T}_n$. The *Brauer algebra*, $\mathbb{B}_n(\delta)$, is the monoid algebra generated by the set $\mathcal{B}_n$.

Note that we used the same symbol in the last equation of the symmetric group, since this set and the symmetric group are isomorphic. Hence the symmetric group algebra $\mathbb{F}\mathfrak{S}_n$ is embedded in $\mathbb{P}_n(\delta)$. For more details see [24] and [29].

Each partition $d \in \mathcal{P}_n$ can be written as $\sigma_1 t \sigma_2$, where $\sigma_1, \sigma_2 \in \mathfrak{S}_n$ and $t \in \mathcal{A}_n$, see Relation 1.6 [25] or [15]. So

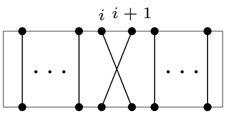
$$\mathcal{P}_n = \mathfrak{S}_n \mathcal{A}_n \mathfrak{S}_n. \tag{1.13}$$

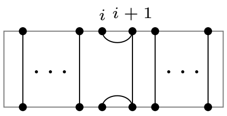
Let $t, b \in \mathcal{A}_n$ and $\pi \in \mathfrak{S}_n$, so there is $d \in \mathcal{A}_n$ and $\sigma \in \mathfrak{S}_n$ such that

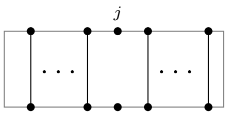
$$t\pi b = td\sigma, \tag{1.14}$$

for more details see the proof of Theorem 1.11 in [25].

Next, we define generators of $\mathbb{P}_n(\delta)$ that are represented by diagrams as following:

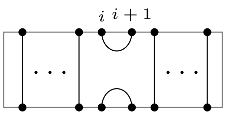
$\mathsf{s}_i =$ 

, $\mathsf{q}_i =$ 

, $\mathsf{p}_j =$ 

,

where $i \in \underline{n-1}$ and $j \in \underline{n}$. The monoid $\mathcal{A}_n$ is generated by the elements $\mathsf{q}_1, \ldots, \mathsf{q}_{n-1}$ and $\mathsf{p}_1, \ldots, \mathsf{p}_n$, and the monoid $\mathcal{B}_n$ is generated by the elements $\mathsf{u}_1, \ldots, \mathsf{u}_{n-1}$ and $\mathsf{s}_1, \ldots, \mathsf{s}_{n-1}$, where $\mathsf{u}_i := \mathsf{q}_i\mathsf{p}_i\mathsf{p}_{i+1}\mathsf{q}_i$, for more details see [25] and [24]. The element u_i is represented by the diagram



.

Martin [38] has proved that the previous elements generate the algebra $\mathbb{P}_n(\delta)$, and Halverson-Ram [25] have found a presentation for $\mathbb{P}_n(\delta)$ using these elements.

Theorem 1.23. *[25, Theorem 1.11]. The algebra* $\mathbb{P}_n(\delta)$ *is generated by* $1, \mathsf{s}_1, \ldots, \mathsf{s}_{n-1}$, $\mathsf{q}_1, \ldots, \mathsf{q}_{n-1}, \mathsf{p}_1, \ldots, \mathsf{p}_n$ *and relations*

$$\mathsf{s}_i^2 = 1, \text{ for } i = 1, \ldots, n-1. \qquad \mathsf{s}_i\mathsf{s}_j = \mathsf{s}_j\mathsf{s}_i, \text{ if } j \neq i \pm 1.$$

$$\mathsf{s}_i\mathsf{s}_{i+1}\mathsf{s}_i = \mathsf{s}_{i+1}\mathsf{s}_i\mathsf{s}_{i+1}, \text{ for } i = 1, \ldots, n-2.$$

$$\mathsf{p}_i^2 = \delta\mathsf{p}_i, \text{ for } i = 1, \ldots, n. \qquad \mathsf{q}_i^2 = \mathsf{q}_i, \text{ for } i = 1, \ldots, n-1.$$

$$\mathsf{s}_i\mathsf{q}_i = \mathsf{q}_i\mathsf{s}_i = \mathsf{q}_i, \qquad \mathsf{s}_i\mathsf{p}_i\mathsf{p}_{i+1} = \mathsf{p}_i\mathsf{p}_{i+1}\mathsf{s}_i = \mathsf{p}_i\mathsf{p}_{i+1}, \text{ for } i = 1, \ldots, n-1.$$

$$\mathsf{p}_i\mathsf{p}_j = \mathsf{p}_j\mathsf{p}_i, \text{ for all } 1 \leq i, j \leq n. \qquad \mathsf{q}_i\mathsf{q}_j = \mathsf{q}_j\mathsf{q}_i, \text{ for all } 1 \leq i, j \leq n-1.$$

$$\mathsf{p}_i\mathsf{q}_j = \mathsf{q}_j\mathsf{p}_i, \text{if } j \neq i, i+1. \qquad \mathsf{s}_i\mathsf{p}_j = \mathsf{p}_j\mathsf{s}_i, \text{ if } j \neq i, i+1.$$

$$\mathsf{s}_i\mathsf{q}_j = \mathsf{q}_j\mathsf{s}_i, \text{ if } j \neq i-1, i+1. \qquad \mathsf{s}_i\mathsf{p}_i\mathsf{s}_i = \mathsf{p}_{i+1}, \text{ for } i = 1, \dots, n-1.$$

$$\mathsf{s}_i\mathsf{s}_{i+1}\mathsf{q}_i\mathsf{s}_{i+1}\mathsf{s}_i = \mathsf{q}_{i+1}, \qquad \mathsf{p}_i\mathsf{q}_i\mathsf{p}_i = \mathsf{p}_i = \mathsf{p}_i\mathsf{q}_{i-1}\mathsf{p}_i, \text{ for } i = 1, \dots, n-2,$$

$$\mathsf{q}_i\mathsf{p}_i\mathsf{q}_i = \mathsf{q}_i = \mathsf{q}_i\mathsf{p}_{i+1}\mathsf{q}_i,, \text{ for } i = 1, \dots, n-1.$$

Later we will need a presentation of the Brauer algebra, so we give one here.

Theorem 1.24. *[e.g. 50, Definition 2.1]. The Brauer algebra $\mathbb{B}_n(\delta)$ is generated by the elements $1, \mathsf{s}_1, \dots, \mathsf{s}_{n-1}, \mathsf{u}_1, \dots, \mathsf{u}_{n-1}$ subject to the relations*

$$\mathsf{s}_i^2 = 1, \qquad \mathsf{s}_i\mathsf{s}_j = \mathsf{s}_j\mathsf{s}_i, \qquad \mathsf{s}_k\mathsf{s}_{k+1}\mathsf{s}_k = \mathsf{s}_{k+1}\mathsf{s}_k\mathsf{s}_{k+1},$$

$$\mathsf{u}_i^2 = \delta\mathsf{u}_i, \qquad \mathsf{u}_i\mathsf{u}_j = \mathsf{u}_j\mathsf{u}_i, \qquad \mathsf{u}_k\mathsf{u}_{k+1}\mathsf{u}_k = \mathsf{u}_k, \qquad \mathsf{u}_{k+1}\mathsf{u}_k\mathsf{u}_{k+1} = \mathsf{u}_{k+1},$$

$$\mathsf{s}_i\mathsf{u}_i = \mathsf{u}_i = \mathsf{u}_i\mathsf{s}_i, \qquad \mathsf{s}_i\mathsf{u}_j = \mathsf{u}_j\mathsf{s}_i, \qquad \mathsf{s}_k\mathsf{u}_{k+1}\mathsf{u}_k = \mathsf{s}_{k+1}\mathsf{u}_k, \qquad \mathsf{s}_{k+1}\mathsf{u}_k\mathsf{u}_{k+1} = \mathsf{s}_k\mathsf{u}_{k+1},$$

where $1 \leq i, j \leq n-1$, with $j \neq i \pm 1$, and $1 \leq k \leq n-2$.

1.7.1 Representation theory of the algebra $\mathbb{P}_n(\delta)$

As it was shown by Xi [55], the partition algebra is cellular. We will mention some theorems that discuss when the partition algebra is semi-simple, an index set for its simple modules and the generic restriction rule for its cell modules.

Theorem 1.25. *[42, Corollary 10.3]. For each integer $n \geq 0$, the algebra $\mathbb{P}_n(\delta)$ is semi-simple over $\mathbb{C}$ whenever δ is not an integer in the range $[0, 2n-1]$.*

Let $\delta \neq 0$ and $E_i = \prod_{j=i+1}^{n} \frac{1}{\delta^{n-i}}\mathsf{p}_j$, $\lambda \vdash i$ and e_λ is the primitive idempotent corresponding to the Specht module $\mathscr{S}_\lambda$ of the group $\mathfrak{S}_{\sum \lambda_i}$. As it is shown in Corollary 10.1 in [38], the element $E_i\mathsf{e}_\lambda$ is a primitive idempotent modulo $\mathbb{P}_nE_{i-1}\mathbb{P}_n$, by $\mathbb{P}_n$ we mean the algebra $\mathbb{P}_n(\delta)$. The cell modules of the algebra $\mathbb{P}_n$, as they are defined in [40] and [38], are

$$\mathfrak{V}_n(\lambda) := \mathbb{P}_nE_i\mathsf{e}_\lambda \pmod{\mathbb{P}_nE_{i-1}\mathbb{P}_n}. \tag{1.15}$$

Let λ be a partition. We write $\mu \rhd \lambda$ to denote that μ is a partition obtained from the partition λ by adding a box to λ after regarding them as Young diagrams. Also $\mu \lhd \lambda$ means that μ is a partition obtained from λ by removing a box, for more details see Chapter 1 in [35]. Additionally, $\mu \lhd \rhd \lambda$ means that μ is a partition obtained from λ by removing a box after adding a box.

We say something is generic if it holds on an (Zariski) open subset of parameter space, as it is described in [9].

Proposition 1.26. *[38, Proposition 13].* *Let λ be a partition of a non-negative integer is less than or equal to n. The generic restriction from the algebra $\mathbb{P}_n(\delta)$ to $\mathbb{P}_{n-1}(\delta)$ is*

$$\mathfrak{V}_n(\lambda) \downarrow_{\mathbb{P}_{n-1}} \cong \Big(\bigoplus_{\mu \rhd \lambda} \mathfrak{V}_{n-1}(\mu)\Big) \oplus \Big(\bigoplus_{\mu \lhd \rhd \lambda} \mathfrak{V}_{n-1}(\mu)\Big) \oplus \Big(\bigoplus_{\mu \lhd \lambda} \mathfrak{V}_{n-1}(\mu)\Big).$$

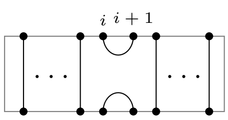
1.8 A review of the Temperley-Lieb algebra

A key technique used in this thesis is to reduce problems in the bubble algebra to problems in the Temperley-Lieb algebra. Therefore it will be helpful to give a brief description of the Temperley-Lieb algebra and its representation theory.

Let $n_1, n_2 \in \mathbb{N}$, with $n_1 + n_2$ is an even number. A (n_1, n_2)-*Kauffman diagram* is a planar (n_1, n_2)-partition diagram such that all its blocks are of size two. Then the set $\mathcal{T}_n$ is the set of all (n, n)-Kauffman diagrams. More details can be found in [30] and [37].

There are many ways to prove that the dimension of the algebra $\mathsf{TL}_n(\delta)$ is the n^{th} *Catalan number* $\mathsf{C}_n = \frac{(2n)!}{(n+1)!n!}$, for example see Theorem 2.3 in [48].

The algebra $\mathsf{TL}_n(\delta)$ is generated by the set $\{1, \mathsf{u}_1, \dots, \mathsf{u}_{n-1}\}$ where

$\mathsf{u}_i =$  .

These diagrams satisfy the relations:

$$\mathsf{u}_i^2 = \delta \mathsf{u}_i \ , \ \text{for all } i = 1, \ldots, n-1,$$
$$\mathsf{u}_i \mathsf{u}_j = \mathsf{u}_j \mathsf{u}_i \ , \ \text{ for } |i-j| \geq 2, \qquad \mathsf{u}_i \mathsf{u}_j \mathsf{u}_i = \mathsf{u}_i \ , \ \text{for } |i-j| = 1.$$

More details can be found in [38] and [48].

1.8.1 The dimension of the algebra $\mathsf{TL}_n(\delta)$

In this subsection, we are going to show that there are $\dim \mathsf{TL}_n(\delta)$ ways to connect $2n$ nodes in pairs without crossing, but these nodes need not be divided equally on the top and the bottom of the diagram, as this fact will be used to compute the dimension of bubble algebra. To prove this, we need to define the set of all *non-crossing perfect matchings.*

A *p-matching*, or simply matching, of the set $\underline{n}$ is an unordered collection of p-pairs of vertices and $n-2p$ single vertices all contained in $\underline{n}$ without repeating. A matching is called *crossing* if it contains a pair $\{i,j\}$ and a vertex k such that $i<k<j$ or if it contains pairs $\{i,j\}$ and $\{k,l\}$ such that $i<k<j<l$.

Definition 1.27. [19, Section 2.1]**.** A non-crossing perfect matching of $\underline{2n}$ is a non-crossing p-matching, where $p=n$. Denote by $\mathcal{F}_{2n}$ the set of all non-crossing perfect matchings of the set $\underline{2n}$.

The elements of $\mathcal{F}_{2n}$ are represented as *cups.* $(2n,n)$-cups are diagrams with one row of $2n$ dots and where edges connect pairs of dots with the restriction that edges can not cross. For example,

There is a bijection between the set $\mathcal{F}_{2n}$ and the diagrams of TL_n, which means $|\mathcal{F}_{2n}| = \dim \mathsf{TL}_n$. Consider the top and the bottom rows of $\mathcal{T}_n$-diagrams as bars, now take the lower bar and move it by rotating the bar up and putting it next to the other bar.

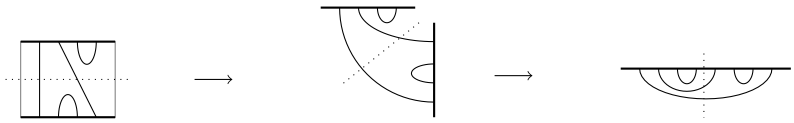
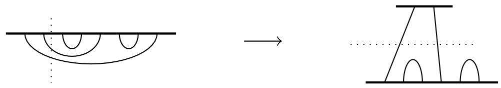


FIGURE 1.2: Illustration of the bijection between the sets $\mathcal{T}_n$ and $\mathcal{F}_{2n}$.

This establishes a bijection between TL_n-diagrams and the set $\mathcal{F}_{2n}$, more details can be found in [19].

Now if we cut a $(2n, n)$-cup in a different position and do the same but in reverse order, we obtain a Kauffman diagram. For example, see the following figure.



Let the cutting be after n_1 points. We will have a bijection between $(2n, n)$-cup and the $(n_1, 2n - n_1)$-Kauffman diagrams. Then

$$\dim \mathsf{TL}_n = \text{The number of } (n_1, 2n - n_1)\text{-Kauffman diagrams.} \tag{1.16}$$

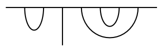
This proves that $\dim \mathsf{TL}_n$ equals to the number of ways of connecting $2n$ vertices without crossing whatever the distribution of these vertices on the frame was.

1.8.2 The cell modules of Temperley-Lieb algebra

We will briefly describe the cell modules of the algebra $\mathsf{TL}_n(\delta)$, which will be of use to us in later chapters.

The *link module* M_n, as defined in Section 3 in [48], of $\mathsf{TL}_n(\delta)$ is the left module that is spanned by all the *half-diagrams* that are obtained from all diagrams in $\mathsf{TL}_n(\delta)$ by cutting horizontally in the middle only cutting propogating lines. The action of $\mathsf{TL}_n(\delta)$ on M_n is defined by the concatenation of diagram with half-diagram then proceeding as we would with two diagrams in the algebra $\mathsf{TL}_n(\delta)$.

A string in a half-diagram that connects two points is called an *arc*. If a half-diagram has p arcs, then there will be $n - 2p$ points which are not connected to anything, we will refer them as *defects* and a half-diagram with n points and p arcs will be called an (n,p)-*link state*. For more details see [37] or [48]. For example, the next half-diagram is a $(7,3)$-link state



.

As the number of propagating lines can not be increased by the multiplication, we could define left TL_n -submodules $\mathsf{M}_{n,p} \subseteq \mathsf{M}_n$ which are spanned by the (n,p')-link states with $p' \geq p$. Note that

$$0 \subset \mathsf{M}_{n,[^n/_2]} \subset \ldots \subset \mathsf{M}_{n,1} \subset \mathsf{M}_{n,0} = \mathsf{M}_n.$$

The quotient modules will be denoted by

$$\mathsf{V}_{n,p} := \frac{\mathsf{M}_{n,p}}{\mathsf{M}_{n,p+1}}. \tag{1.17}$$

The quotient sends any half-diagram with more than p arcs to zero. The Temperley-Lieb algebra $\mathsf{TL}_n(\delta)$ is a cellular algebra, with the involution sending each diagram d to its reflection d^* in the horizontal plane and $\Lambda = \{0, 1, \ldots, [^n/_2]\}$. The modules $\mathsf{V}_{n,p}$ where $p \in \Lambda$ are the cell modules of the algebra $\mathsf{TL}_n(\delta)$. The proof can be found in Theorem 3.8 in [20].

The dimension of $\mathsf{V}_{n,p}$ is given by the formula

$$\dim \mathsf{V}_{n,p} = \binom{n}{p} - \binom{n}{p-1} := \mathsf{d}_{n,p}. \tag{1.18}$$

The proof of this can be found in Section 2 [48] or [37]. Note that $\binom{n}{-1} = 0$.

On each module $\mathsf{V}_{n,p}$, we define a bilinear form $\langle\ ,\ \rangle \equiv \langle\ ,\ \rangle_{n,p,\delta}$ as follows. If x and y are two (n,p)-link states, the scalar $\langle x, y\rangle$ is computed by reflecting x in a horizontal axis and identifying its vertical border with that of y. The value $\langle x, y\rangle$ is then non-zero only if every defect of x ends up being connected to one of y, and in this case $\langle x, y\rangle = \delta^l$ where l is the number of closed loops which is obtained from connecting x and y. For more details see Section 9.5.2 in [37].

The matrix $\mathsf{G}_{n,p,\delta}$ is defined to be the Gram matrix for the module $\mathsf{V}_{n,p}$ that represents the form $\langle\ ,\ \rangle_{n,p,\delta}$ with respect to a basis that contains all (n,p)-link states. For example,

$$\mathsf{G}_{n,0,\delta} = (1), \quad \mathsf{G}_{n,1,\delta} = \begin{pmatrix} \delta & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \delta & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \delta & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & \delta \end{pmatrix}.$$

Let $\mathsf{R}_{n,p,\delta}$ be the *radical* of the previous bilinear form on the module $\mathsf{V}_{n,p}$. Recall that the radical $\mathsf{R}_{n,p}$ is a submodule of $\mathsf{V}_{n,p}$.

Lemma 1.28. *[e.g. 48, Section 3]. If $\mathsf{TL}_n(\delta)$ is an $\mathbb{F}$-algebra, and $\langle\ ,\ \rangle$ is not identically zero on $\mathsf{V}_{n,p}$, then $\mathsf{V}_{n,p}$ is cyclic and indecomposable. Moreover, $\mathsf{L}_{n,p} := \mathsf{V}_{n,p}/R_{n,p}$ is irreducible.*

1.8.3 Irreducibility of the cell modules

The cell modules $\mathsf{V}_{n,p}$ of the algebra $\mathsf{TL}_n(\delta)$ are irreducible except for particular values of the scalar δ. Throughout this section, let $\delta = q + q^{-1}$ with $q \in \mathbb{F}$.

Proposition 1.29. *[37, Section 6.4, Theorem 1]. If q is not a root of unity, then the algebra $\mathsf{TL}_n(\delta)$ is semi-simple, and the modules $\mathsf{V}_{n,p}$, where $0 \leq p \leq [{}^{n}/_{2}]$, form a complete set of non-isomorphic irreducible modules of $\mathsf{TL}_n(\delta)$.*

For the values δ where $\mathsf{TL}_n(\delta)$ is not semi-simple, *non-generic cases*, many different studies of this have been made. Assume that q is a primitive l^{th} root of unity. If $n < l$, in this case the algebra $\mathsf{TL}_n(\delta)$ will be semi-simple.

The module $\mathsf{V}_{n,p}$ (or the pair (n,p)) is called *critical* for a given q if $q^{2(n-2p+1)} = 1$.

Lemma 1.30. *[e.g. 54, Section 9]. Let* $\operatorname{Char}\mathbb{F} = 0$*. For all* n *and* $l > p > 0$*, we have*

$$\dim \operatorname{Hom}(\mathsf{V}_{n,0}, \mathsf{V}_{n,p}) = \begin{cases} 1 & \text{if } n - p + 1 = 0 \pmod{l}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.31. *[37, Section 7.3, Theorem 2]. If* $\operatorname{Char}\mathbb{F} = 0$*,* $0 \le p_1 - p_2 < l$ *and* $n - p_1 - p_2 + 1 = 0 \pmod{l}$*, then there is a non-trivial homomorphism* $\theta_{n,p_1} : \mathsf{V}_{n,p_2} \to \mathsf{V}_{n,p_1}$*. Otherwise, there is no non-trivial homomorphism from* V_{n,p_2} *to* V_{n,p_1}*.*

Theorem 1.32. *[10, Theorem 5.3]. Let* $\operatorname{Char}\mathbb{F} = \mathsf{p}$*. There is a non-trivial homomorphism* $\theta_{n,p_1} : \mathsf{V}_{n,p_2} \to \mathsf{V}_{n,p_1}$ *if and only if* $n - p_1 - p_2 + 1 = 0 \pmod{l\mathsf{p}^j}$ *with* $0 \le p_1 - p_2 < l\mathsf{p}^j$ *for some non-negative integer* j*.*

Theorem 1.33. *[e.g. 54, Section 9]. The kernels and co-kernels of the homomorphism* θ_{n,p_1} *are irreducible.*

1.8.4 Further properties over the complex field

In this subsection, we give relations that are helpful to compute the dimension of the head of each cell module and the dimension of the radical. Let q be a root of unity and let l be the minimal positive integer satisfying $q^{2l} = 1$.

Proposition 1.34. *[48, Proposition 5.1]. The dimensions of the radical of* $\mathsf{V}_{n,p}$ *satisfy the recursion relation*

$$\dim \mathsf{R}_{n,p,\delta} = \begin{cases} 0 & \text{if } n - 2p + 1 = 0 \pmod{l}, \\ \dim \mathsf{R}_{n-1,p,\delta} + \dim \mathsf{V}_{n-1,p-1} & \text{if } n - 2p + 1 = l - 1 \pmod{l}, \\ \dim \mathsf{R}_{n-1,p,\delta} + \dim \mathsf{R}_{n-1,p-1,\delta} & \text{otherwise.} \end{cases}$$

Subject to initial conditions $\dim \mathsf{R}_{n,0,\delta} = 0$ *and* $\dim \mathsf{R}_{2p-1,p,\delta} = 0$*.*

Corollary 1.35. *[48, Corollary 5.2]. The dimensions of the simple quotients* $\mathsf{L}_{n,p,\delta} = \mathsf{V}_{n,p}/\mathsf{R}_{n,p,\delta}$ *satisfy the recursion relation*

$$\dim \mathsf{L}_{n,p,\delta} = \begin{cases} \dim \mathsf{V}_{n,p} & \text{if } n - 2p + 1 = 0 \pmod{l}, \\ \dim \mathsf{L}_{n-1,p,\delta} & \text{if } n - 2p + 1 = l - 1 \pmod{l}, \\ \dim \mathsf{L}_{n-1,p,\delta} + \dim \mathsf{L}_{n-1,p-1,\delta} & \text{otherwise.} \end{cases}$$

Subject to initial conditions $\dim \mathsf{L}_{n,0,\delta} = 1$ *and* $\dim \mathsf{L}_{2p-1,p,\delta} = 0$.

Define $r_{(n,p)}$ to be the integer satisfying (see Section 5 in [48])

$$n - 2p + 1 = kl + r_{(n,p)},$$

where $k \in \mathbb{N}$ and $r_{(n,p)} \in \{1, \ldots, l\}$. The pair (n, p) is critical if $r_{(n,p)} = l$ when q is a root of unity.

Proposition 1.36. *[37, Section 7.3, Theorem 2]. Let q be a root of unity and (n, p) be non-critical. Then*

$$\dim \mathsf{R}_{n,p,\delta} = \begin{cases} \dim \mathsf{L}_{n,\, p+r_{(n,p)}-l,\, \delta} & \textit{if } p + r_{(n,p)} - l \geq 0, \\ 0 & \textit{otherwise.} \end{cases} \tag{1.19}$$

Chapter 2

The Multi-Colour Partition Algebra $\mathbb{P}_{n,m}(\breve{\delta})$

The purpose of this chapter is to introduce a new class of algebras, the multi-colour partition algebra, a generalization of both the bubble algebra and the partition algebra, and to define some of its sub-algebras. The algebra can be well studied by using similar techniques used in the investigation of the partition algebra. We also introduce some concepts required for the subsequent chapters. In Section 2.4, a generating form of the multi-colour partition algebra is given. In Section 2.5, we define the bubble algebra as a sub-algebra of the multi-colour partition algebra and determine its dimension and a generating set of it. In the end of this chapter we discuss certain special idempotent sub-algebras of the multi-colour algebra and show that they are isomorphic to products of partition algebras.

2.1 Definitions and structure

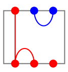
The aim of this section is to define the multi-colour partition algebra and give some of its properties such as its dimension. We begin by defining the two-colour partition algebra.

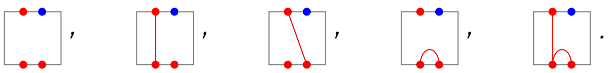
2.1.1 The two-colour partition algebra

In the definition of the partition algebra $\mathbb{P}_n(\delta)$, the set $\underline{n} \cup \underline{n'}$ is partitioned to define basis elements of $\mathbb{P}_n(\delta)$. For the two-colour partition algebra, we do that in two steps, first break up the set $\underline{n} \cup \underline{n'}$ into two different subsets and then partition each subset alone. The new partition can be represented by a diagram in the same way, since it is still a partition of the set $\underline{n} \cup \underline{n'}$.

Let $A \subseteq \underline{n} \cup \underline{n'}$ be fixed (note that A can be an empty set) and the sets $\mathcal{P}_A$ and $\mathcal{P}_{A^c}$ be the sets of all partitions of A and A^c respectively, where A^c is the set $\underline{n} \cup \underline{n'} \setminus A$. Define the set $\mathcal{P}_{A,A^c}$ to be the set $\mathcal{P}_A \times \mathcal{P}_{A^c}$.

Consider an element $(d_0, d_1) \in \mathcal{P}_{A,A^c}$, so the set $d_0 \cup d_1$ is a partition of $\underline{n} \cup \underline{n'}$, from that we can represent the element (d_0, d_1) by the same partition diagram of $d_0 \cup d_1$. In order to distinguish the partitions d_0 and d_1 in $d_0 \cup d_1$, we will colour them where red edges and red nodes are from the partition d_0 and blue edges and blue nodes are from the partition d_1. Thus we can think of (d_0, d_1) as a coloured image of the diagram $d_0 \cup d_1$. From this definition, it is clear that if any two nodes are connected then the nodes and the edge have the same colour.

Example 2.0.1. *The element* $(\{\{1, 1', 2'\}, \{3'\}\}, \{\{2, 3\}\})$ *can be represented by the digram* . *Also the elements in the set* $\mathcal{P}_{\{1,1',2'\},\{2\}}$ *are represented by the following diagrams:*



A diagram representing an element $(d_0, d_1) \in \mathcal{P}_{A,A^c}$ is not unique. We say two diagrams are equivalent if they represent the same partition in the set $\mathcal{P}_{A,A^c}$ for some subset A. The term *two-colour partition diagram* will be used to mean the equivalence class of diagrams that representing a two-colour partition. We are only interested in the equivalence classes of two-colour partition diagrams, and whenever two-colour diagram is mentioned we mean the equivalence class of it.

For simplicity we may say diagram instead of two-colour diagram when the type of the diagram is obvious.

Now we take all possibilities for the subset A and define the set:

$$\mathcal{P}_{n,2} := \bigcup_{A \subseteq \underline{n} \cup \underline{n}'} \mathcal{P}_{A,A^c}. \tag{2.1}$$

The product of partition diagrams on the monoid $\mathcal{P}_n$ can be extended to define a product on the set $\mathcal{P}_{n,2}$. First we need to define the subset $\text{top}(d_i) \subseteq \underline{n}$ to be the set of all nodes in the top row of d_i and similarly $\text{bot}(d_i) \subseteq \underline{n}'$ denotes the set of all nodes in the bottom row of d_i, where $(d_0, d_1) \in \mathcal{P}_{n,2}$. Note that $\text{top}(d_1) = \underline{n} \setminus \text{top}(d_0)$ and $\text{bot}(d_1) = \underline{n}' \setminus \text{bot}(d_0)$.

Let $\alpha = (\alpha_0, \alpha_1)$, $\beta = (\beta_0, \beta_1)$ be diagrams in $\mathcal{P}_{n,2}$, we say that $\text{bot}(\alpha_j) = \text{top}(\beta_j)$ when they satisfy $i' \in \text{bot}(\alpha_j)$ if and only if $i \in \text{top}(\beta_j)$ for any $i \in \underline{n}$, where $j = 0, 1$.

The composition $\beta \circ \alpha$ of elements $\alpha = (\alpha_0, \alpha_1)$, $\beta = (\beta_0, \beta_1)$ in the set $\mathcal{P}_{n,2}$ is defined as follows (which is the same multiplication that was described in [23]):

1. Place α above β and identify the vertices in the bottom row of α with the vertices in the top row of β regardless of the colour of dots.

2. If the colours match up, this means $\text{bot}(\alpha_0) = \text{top}(\beta_0)$, then the products $\beta_0 \circ \alpha_0$ and $\beta_1 \circ \alpha_1$ are well-defined as partition diagrams, and then define $\beta \circ \alpha$ to be $(\beta_0 \circ \alpha_0, \beta_1 \circ \alpha_1)$.

3. If the colours don't match up, this means $\text{bot}(\alpha_0) \neq \text{top}(\beta_0)$, then the product of α in β will be undefined.

Definition 2.1. Let $\mathbb{P}_{n,2}(\delta_0, \delta_1)$ be the $\mathbb{F}$-vector space with basis $\mathcal{P}_{n,2}$. We define a product on the algebra $\mathbb{P}_{n,2}(\delta_0, \delta_1)$ as follows.

$$\alpha\beta = \begin{cases} \delta_0^{c_0}\delta_1^{c_1}(\beta_0 \circ \alpha_0, \beta_1 \circ \alpha_1) & \text{if} \quad \text{bot}(\alpha_0) = \text{top}(\beta_0), \\ 0 & \text{otherwise}, \end{cases} \tag{2.2}$$

where $\alpha, \beta \in \mathcal{P}_{n,2}$, $\circ$ is the composition of partition diagrams and the scalars $\delta_0, \delta_1 \in \mathbb{F}$ and c_0 (similarly c_1) is the number of connected components removed from the middle row when constructing the product $\alpha_0 \circ \beta_0$ (from $\alpha_1 \circ \beta_1$).

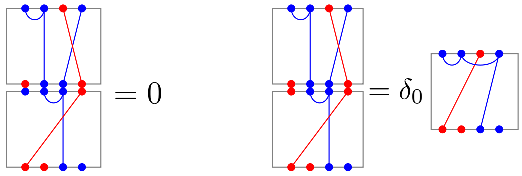


FIGURE 2.1: The composition of some diagrams in $\mathbb{P}_{4,2}(\delta_0, \delta_1)$.

Figure 2.1 is an example of the multiplication of some elements in the space $\mathbb{P}_{4,2}(\delta_0, \delta_1)$.

Theorem 2.2. *The product that is defined in* (2.2) *is associative.*

Proof. Let $\alpha = (\alpha_0, \alpha_1)$, $\beta = (\beta_0, \beta_1)$ and $\rho = (\rho_0, \rho_1)$ be partitions in the set $\mathcal{P}_{n,2}$. Note that $\text{top}(\alpha_0 \circ \beta_0) = \text{top}(\beta_0)$ and $\text{bot}(\alpha_0 \circ \beta_0) = \text{bot}(\alpha_0)$ as long as $\alpha \circ \beta$ is defined. From the multiplication on $\mathbb{P}_{n,2}$, the composition $\alpha \circ (\beta \circ \rho)$ is defined if and only if $\text{top}(\alpha_0) = \text{bot}(\beta_0 \circ \rho_0)$, and $\beta \circ \rho$ is defined if and only if $\text{top}(\beta_0) = \text{bot}(\rho_0)$. But if $\beta \circ \rho$ is defined then $\text{bot}(\beta_0 \circ \rho_0) = \text{bot}(\beta_0)$. Then $\alpha \circ (\beta \circ \rho)$ is defined if and only if $\text{top}(\alpha_0) = \text{bot}(\beta_0)$ and $\text{top}(\beta_0) = \text{bot}(\rho_0)$. Similarly, $(\alpha \circ \beta) \circ \rho$ is defined if and only if $\text{top}(\alpha_0) = \text{bot}(\beta_0)$ and $\text{top}(\beta_0) = \text{bot}(\rho_0)$. So the composition $\alpha \circ (\beta \circ \rho)$ is defined if and only if $(\alpha \circ \beta) \circ \rho$ is defined, then the product in $\mathbb{P}_{n,2}(\delta_0, \delta_1)$ is an associative when vanishes. Furthermore, if it does not vanish, we have

$$\begin{aligned} \alpha \circ (\beta \circ \rho) =& \big(\alpha_0 \circ (\beta_0 \circ \rho_0), \alpha_1 \circ (\beta_1 \circ \rho_1)\big) \\ =& \big((\alpha_0 \circ \beta_0) \circ \rho_0, (\alpha_1 \circ \beta_1) \circ \rho_1\big) = (\alpha \circ \beta) \circ \rho, \end{aligned}$$

as the composition of partition diagrams is associative. Since the product on $\mathbb{P}_{n,2}(\delta_0, \delta_1)$ will be a linear extension of the multiplication $\circ$, then the product is also an associative when it doesn't vanish. $\square$

Theorem 2.3. *The space* $\mathbb{P}_{n,2}(\delta_0, \delta_1)$ *is an associative* $\mathbb{F}$*-algebra and its identity is the element* $\sum_{A \subseteq \underline{n}} 1_{(A,A^c)}$*, where* $1_{(A,A^c)} := (1_A, 1_{A^c})$ *and* 1_A *is the diagram where each node* $i \in A$ *is only connected with the node* i' *(similarly, we define* 1_{A^c}*).*

Proof. Since the product on $\mathbb{P}_{n,2}(\delta_0,\delta_1)$ is associative, so we only need to show that the element $\sum\limits_{A\subseteq \underline{n}} 1_{(A,A^c)}$ is the identity. Let $\alpha = (\alpha_0,\alpha_1) \in \mathcal{P}_{n,2}$ then

$$\Big(\sum_{A\subseteq \underline{n}} 1_{(A,A^c)}\Big)\alpha = \sum_{A\subseteq \underline{n}} 1_{(A,A^c)}\alpha,$$

but $1_{(A,A^c)}\alpha = 0$ unless $\mathrm{top}(\alpha_0) = A$, so

$$\Big(\sum_{A\subseteq \underline{n}} 1_{(A,A^c)}\Big)\alpha = (\alpha_0 \circ 1_{\mathrm{top}(\alpha_0)}, \alpha_1 \circ 1_{\mathrm{top}(\alpha_1)}) = \alpha.$$

Similarly, we have $\alpha \sum\limits_{A\subseteq \underline{n}} 1_{(A,A^c)} = \alpha$. Thus $\sum\limits_{A\subseteq \underline{n}} 1_{(A,A^c)}$ is the identity. □

We call the algebra $\mathbb{P}_{n,2}(\delta_0,\delta_1)$, or simply $\mathbb{P}_{n,2}$, the *two-colour partition algebra.*

Remark 2.4. We can construct a category which consisting $\{(A,A^c) \mid A \subseteq \underline{n}\}$ as objects and $\mathcal{P}_{n,2}$ as the morphisms. A diagram $d = (d_0,d_1) \in \mathcal{P}_{n,2}$ is an arrow from (A,A^c) to (B,B^c) if $\mathrm{top}(d_0) = A$ and $\mathrm{bot}(d_0) = B$. We define the top and the bottom of the diagram d to be

$$\mathrm{top}(d) = (A,A^c), \qquad \mathrm{bot}(d) = (B,B^c).$$

Also the identity arrow for (A,A^c) is the diagram $1_{(A,A^c)}$ since $1_{(A,A^c)}d = d$. When $n = 1$, this category is represented by the graph in figure 2.2.

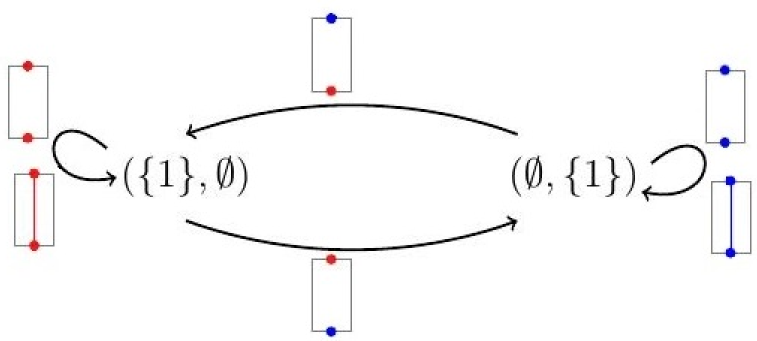


FIGURE 2.2: The set $\mathcal{P}_{1,2}$ as a category.

The number of different partitions of a set with n elements is the *Bell number* B_n, see for example Section 4.2 in [18]. From the definition of the set $\mathcal{P}_{n,2}$, we have

$$\dim \mathbb{P}_{n,2} = |\mathcal{P}_{n,2}| = \sum_{A \subset \underline{n} \cup \underline{n'}} |\mathcal{P}_A| \times |\mathcal{P}_{A^c}| = \sum_{\substack{A \subset \underline{n} \cup \underline{n'} \\ |A|=k}} \mathsf{B}_k \times \mathsf{B}_{2n-k} \ .$$

The number of subsets with k elements of $\underline{n} \cup \underline{n'}$ is $\binom{2n}{k}$, see for example Section 4.2 in [18], then

$$\dim \mathbb{P}_{n,2} = \sum_{k=0}^{2n} \binom{2n}{k} \mathsf{B}_k \times \mathsf{B}_{2n-k} \ . \tag{2.3}$$

We have another formula for the dimension of $\mathbb{P}_{n,2}$. If we think of elements in the set $\mathcal{P}_{n,2}$ as coloured images of elements in $\mathcal{P}_n$, where the only rule of the colouring is that nodes and edges in the same block have the same colour. Let $d \in \mathcal{P}_n$ have l parts, by colouring d we obtain 2^l elements in $\mathcal{P}_{n,2}$, then

$$\dim \mathbb{P}_{n,2} = \sum_{l=1}^{2n} 2^l \left\{ {2n \atop l} \right\}, \tag{2.4}$$

where $\left\{ {n \atop l} \right\}$ is *Stirling number of the second kind* and it is equal to the number of partitions of a set of n elements with l parts, see for example Section 4.2 in [18].

2.1.2 The multi-colour partition algebra

For any positive integer m, let $\mathfrak{C}_0, \ldots, \mathfrak{C}_{m-1}$ be different colours where none of them is white, and $\delta_0, \ldots, \delta_{m-1}$ be scalars corresponding to these colours.

Define the set $\Phi^{n,m}$ to be

$$\{(A_0, \ldots, A_{m-1}) \mid A_i \subseteq \underline{n} \cup \underline{n'} \ \forall\ i \in \mathbb{Z}_m,\ \bigcup_{i=0}^{m-1} A_i = \underline{n} \cup \underline{n'},\ A_i \cap A_j = \emptyset\ \forall i \neq j\}.$$

We construct basis elements of the multi-colour partition algebra in similar way to the algebra $\mathbb{P}_{n,2}$. Let $(A_0, \ldots, A_{m-1}) \in \Phi^{n,m}$ (note that some of these subsets can be an empty set). Define the set $\mathcal{P}_{A_0,\ldots,A_{m-1}}$ to be the set $\prod_{i=0}^{m-1} \mathcal{P}_{A_i}$ and

$$\mathcal{P}_{n,m} := \bigcup_{(A_0,\ldots,A_{m-1})\in\Phi^{n,m}} \mathcal{P}_{A_0,\ldots,A_{m-1}}. \tag{2.5}$$

The element $d = (d_0, \ldots, d_{m-1}) \in \mathcal{P}_{A_0,\ldots,A_{m-1}}$ is represented by the same diagram as the partition $\bigcup_{i=0}^{m-1} d_i \in \mathcal{P}_n$ after *colouring* it as follows. We use the colour $\mathfrak{C}_i$ to draw all the edges and the nodes in the partition d_i.

Similarly, a diagram representing an element in $\mathcal{P}_{n,m}$ is not unique. We say two diagrams are equivalent if they represent the same tuple of partitions. The term *multi-colour partition diagram* will be used to mean an equivalence class of diagrams representing a multi-colour partition.

Let $d = (d_0, \ldots, d_{m-1}) \in \mathcal{P}_{A_0,\ldots,A_{m-1}}$. We define the following sets:

$$\left.\begin{aligned} \operatorname{top}(d_i) &= A_i \cap \underline{n}, \\ \operatorname{bot}(d_i) &= A_i \cap \underline{n}', \\ \operatorname{top}(d) &= (\operatorname{top}(d_0), \ldots, \operatorname{top}(d_{m-1})), \\ \operatorname{bot}(d) &= (\operatorname{bot}(d_0), \ldots, \operatorname{bot}(d_{m-1})). \end{aligned}\right\} \tag{2.6}$$

Definition 2.5. Let $\mathbb{P}_{n,m}(\delta_0, \ldots, \delta_{m-1})$ be the $\mathbb{F}$-vector space with the basis $\mathcal{P}_{n,m}$ and with the composition:

$$(\alpha_i)(\beta_i) = \begin{cases} \prod_{i=0}^{m-1} \delta_i^{c_i}(\beta_i \circ \alpha_i) & \text{if } \operatorname{bot}(\alpha) = \operatorname{top}(\beta), \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

where $\delta_i \in \mathbb{F}$, $\alpha, \beta \in \mathcal{P}_{n,m}$, c_i is the number of removed connected components from the middle row when computing the product $\beta_i \circ \alpha_i$ for each $i = 0, \ldots, m-1$ and $\circ$ is the composition of partition diagrams.

The product on $\mathbb{P}_{n,m}(\delta_0, \ldots, \delta_{m-1})$-sometimes we use $\mathbb{P}_{n,m}$ or $\mathbb{P}_{n,m}(\breve{\delta})$ to refer to this space where $\breve{\delta} = (\delta_0, \ldots, \delta_{m-1})$- is associative and the proof is similar to the one of Theorem 2.2. Then it is an associative algebra with the identity:

$$1_{\mathbb{P}_{n,m}} = \sum_{(A_0,\ldots,A_{m-1})\in\Xi^{n,m}} 1_{(A_0,\ldots,A_{m-1})} := \sum_{(A_0,\ldots,A_{m-1})\in\Xi^{n,m}} (1_{A_0},\ldots,1_{A_{m-1}}),$$

where $\Xi^{n,m} = \{(A_0,\ldots,A_{m-1}) \mid \cup_{i=0}^{m-1} A_i = \underline{n}, A_i \cap A_j = \emptyset \ \forall i \neq j\}$, 1_{A_i} is the partition of the set $A_i \sqcup A_i'$ where any node j is only connected with the node j' for all $j \in A_i$ and $A_i' := \{j' \mid j \in A_i\}$, for all $0 \leq i \leq m-1$. The algebra $\mathbb{P}_{n,m}(\breve{\delta})$ is called the *multi-colour partition algebra*.

Remark 2.6. We can construct a category which consisting the set $\Xi^{n,m}$ as objects and $\mathcal{P}_{n,m}$ as morphisms, where $d = (d_0,\ldots,d_{m-1}) \in \mathcal{P}_{n,m}$ is a morphism from the object top(d) to bot(d).

The diagrams in $\mathcal{P}_{n,m}$ are constructed by colouring elements of $\mathcal{P}_n$, so any partition in $\mathcal{P}_n$ that has k parts can be used to define m^k different diagrams in $\mathcal{P}_{n,m}$. Hence the dimension of the algebra $\mathbb{P}_{n,m}$ is

$$\dim \mathbb{P}_{n,m} = \sum_{k=1}^{2n} m^k \left\{ {2n \atop k} \right\}, \tag{2.8}$$

where $\left\{ {n \atop k} \right\}$ is Stirling number of the second kind. From this equation, we obtain the table 2.1.

TABLE 2.1: The dimension of some low rank examples of the algebra $\mathbb{P}_{n,m}$.

$\dim \mathbb{P}_{n,m}$	$n=0$	1	2	3	4	n
$m=1$	1	2	15	203	4140	B_{2n}
2	1	6	94	2430	89918	$\sum_{k=0}^{2n} \binom{2n}{k} \mathsf{B}_k \times \mathsf{B}_{2n-k}$
3	1	12	309	12351	681870	$\sum_{k=1}^{2n} 3^k \left\{ {2n \atop k} \right\}$
4	1	20	756	42356	3188340	$\sum_{k=1}^{2n} 4^k \left\{ {2n \atop k} \right\}$

2.2 Basic concepts of the algebra $\mathbb{P}_{n,m}(\breve{\delta})$

In this section, we generalise some concepts and definitions of subsets of the monoid $\mathcal{P}_n$ such as the propagating number and the symmetric group, and then use them to define ideals and sub-algebras of the algebra $\mathbb{P}_{n,m}$, where we follow Grimm and Martin's approach [23].

Definition 2.7. [23, Section 2]**.** The propagating number of $\alpha \in \mathcal{P}_{n,m}$, $\#(\alpha)$, is the number of parts which contain nodes from both the top and the bottom rows in any colour, i.e. $\#(\alpha) = \sum_{i=0}^{m-1} \#(\alpha_i)$ or simply $\#(\alpha) = \#(\bigcup_{i=0}^{m-1} \alpha_i)$.

Definition 2.8. [23, Section 2]**.** The $\mathfrak{C}_j$-propagating number of $\alpha = (\alpha_0, \ldots, \alpha_{m-1}) \in \mathcal{P}_{n,m}$, $\#_j(\alpha)$, is the propagating number of α_j.

The propagating number in the algebra $\mathbb{P}_{n,m}$ has a similar property to the propagating number in the algebra $\mathbb{P}_n(\delta)$.

Lemma 2.9. *Let* $\alpha = (\alpha_0, \ldots, \alpha_{m-1}), \beta = (\beta_0, \ldots, \beta_{m-1}) \in \mathcal{P}_{n,m}$ *with* $\alpha\beta \neq 0$*, then*

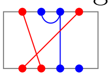
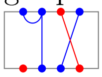
$$\#(\alpha\beta) \leq \min(\#(\alpha), \#(\beta)) \;, \tag{2.9}$$

$$\#_j(\alpha\beta) \leq \min(\#_j(\alpha), \#_j(\beta)). \tag{2.10}$$

Proof. Second part is clear, since $\#_j(\alpha\beta) = \#(\beta_j \circ \alpha_j)$. Finally, from the definition 2.7 we have

$$\begin{aligned}\#(\alpha\beta) &= \#(\beta_0 \circ \alpha_0) + \cdots + \#(\beta_{m-1} \circ \alpha_{m-1}) \\ &\leq \#(\alpha_0) + \cdots + \#(\alpha_{m-1}) = \#(\alpha), \qquad \text{(from equation (1.11))}.\end{aligned}$$

Similarly, we have $\#(\alpha\beta) \leq \#(\beta)$. So we have $\#(\alpha\beta) \leq \min(\#(\alpha), \#(\beta))$. □

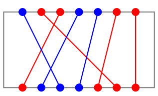
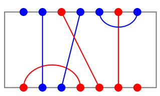
A *planar multi-colour partition* in the set $\mathcal{P}_{n,m}$ is a multi-colour partition represented by a diagram that does not have edge crossings in the same colour. This is the same definition that Grimm and Martin use in [23]. In other words, there can be crossed edges but they don't have the same colour. A *planar multi-colour diagram*, or simply planar diagram, is a diagram representing a planar multi-colour partition. For example,  is not a planar diagram, and  is a planar diagram.

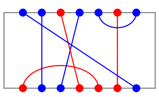
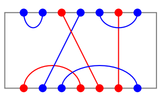
Remark 2.10. This definition of planar diagram is consistent with the definition of a planar diagram in the algebra $\mathbb{P}_n(\delta)$ provided that we consider all the diagrams in $\mathbb{P}_n(\delta)$ to have been coloured by using only one colour.

We define subsets of $\mathcal{P}_{n,m}$ corresponding to those subsets of $\mathcal{P}_n$ which are defined in the equation (1.12) as following:

$$\left.\begin{aligned} \mathfrak{S}_{n,m} &= \{d \in \mathcal{P}_{n,m} \mid \#(d) = n\}, \\ \mathcal{A}_{n,m} &= \{d \in \mathcal{P}_{n,m} \mid d \text{ is planar}\}, \\ \mathcal{Q}_{n,m} &= \{d \in \mathcal{P}_{n,m} \mid \#(d) < n\}, \\ \mathcal{B}_{n,m} &= \{d \in \mathcal{P}_{n,m} \mid \text{ all blocks of } d \text{ have size } 2\}, \\ \mathcal{T}_{n,m} &= \mathcal{A}_{n,m} \cap \mathcal{B}_{n,m}. \end{aligned}\right\} \qquad (2.11)$$

Examples of diagrams in the previous subsets of $\mathcal{P}_{7,2}$ are:

 $\in \mathfrak{S}_{7,2}, \mathcal{B}_{7,2}$,  $\in \mathcal{A}_{7,2}, \mathcal{Q}_{7,2}$,

 $\in \mathcal{B}_{7,2}, \mathcal{Q}_{7,2}$,  $\in \mathcal{T}_{7,2}, \mathcal{B}_{7,2}$.

Remark 2.11. In the next chapter, it will be shown that $\mathfrak{S}_{n,m}$ is a morphisms set of a groupoid, and we call it the *multi-colour symmetric groupoid*. Furthermore, note that all the sets defined in equation (2.11) except the set $\mathcal{Q}_{n,m}$ form morphism sets of categories so they can be used to define algebras. These will be denoted by $\mathbb{F}\mathfrak{S}_{n,m}$, $\mathbb{F}\mathcal{A}_{n,m}$, $\mathbb{B}_{n,m}(\breve{\delta})$, and $\mathbb{T}_{n,m}(\breve{\delta})$.

Diagrams representing multi-colour partitions in the sets $\mathfrak{S}_{n,m}$, $\mathcal{B}_{n,m}$ and $\mathcal{Q}_{n,m}$ can be obtained by colouring elements in the sets $\mathfrak{S}_n$, $\mathcal{B}_n$ and $\mathcal{Q}_n$ respectively (as they are defined by equation (1.12)). But this is not true for diagrams that representing multi-colour partitions in sets $\mathcal{A}_{n,m}$ and $\mathcal{T}_{n,m}$ since some of non-planar diagrams in $\mathbb{P}_n(\delta)$ can be coloured to be planar diagrams in $\mathbb{P}_{n,m}$.

Definition 2.12. [23, Section 2]**.** A *strictly planar multi-colour partition* is a multi-colour partition whose a digram can be formed from colouring an element in $\mathcal{A}_n$. This means inside a strictly planar diagram there is no edge crossing even between different colours. Let $\mathcal{A}^*_{n,m}$ be the set of all strictly planar multi-colour partitions in $\mathcal{P}_{n,m}$. A *strictly planar diagram* is a diagram representing a strictly planar multi-colour partition.

Define the subsets $\mathcal{P}_{n,m}[\lambda_0, \ldots, \lambda_{m-1}]$ and $\mathcal{P}_{n,m}(\lambda_0, \ldots, \lambda_{m-1})$ of $\mathcal{P}_{n,m}$, where $(\lambda_0, \ldots, \lambda_{m-1}) \in \mathbb{Z}^m_{\geq 0}$ such that $\sum\limits_{j=0}^{m-1} \lambda_j \leq n$, to be

$$\mathcal{P}_{n,m}[\lambda_0, \ldots, \lambda_{m-1}] = \{d \in \mathcal{P}_{n,m} \mid \#_j(d) = \lambda_j \text{ for all } j \in \mathbb{Z}_m\}, \tag{2.12}$$

$$\mathcal{P}_{n,m}(\lambda_0, \ldots, \lambda_{m-1}) = \bigcup_{l_j \leq \lambda_j} \mathcal{P}_{n,m}[l_0, \ldots, l_{m-1}]. \tag{2.13}$$

Let $\mathbb{P}_{n,m}(\breve{\delta}; \lambda_0, \ldots, \lambda_{m-1})$, or simply $\mathbb{P}_{n,m}(\breve{\delta}; \lambda)$ where $\lambda = (\lambda_0, \ldots, \lambda_{m-1})$, be the ideal of the algebra $\mathbb{P}_{n,m}(\breve{\delta})$ that is generated by the set $\mathcal{P}_{n,m}[\lambda]$.

Proposition 2.13. *Let* $\lambda \in \mathbb{Z}^m_{\geq 0}$ *such that* $\sum\limits_{j=0}^{m-1} \lambda_j \leq n$*. The ideal* $\mathbb{P}_{n,m}(\breve{\delta}; \lambda)$ *has the set* $\mathcal{P}_{n,m}(\lambda)$ *as a basis.*

Proof. Since $\mathbb{P}_{n,m}(\breve{\delta}; \lambda)$ is generated by $\mathcal{P}_{n,m}[\lambda_0, \ldots, \lambda_{m-1}]$, so it contains all elements of the form $\alpha\beta$ and $\beta\alpha$ where $\alpha \in \mathcal{P}_{n,m}[\lambda]$ and $\beta \in \mathbb{P}_{n,m}$. By using Lemma 2.9 $\#_j(\alpha\beta) \leq \lambda_j$ for all $j \in \mathbb{Z}_m$, so $\mathbb{P}_{n,m}(\breve{\delta}; \lambda)$ is a subset of the ideal that is generated by $\mathcal{P}_{n,m}(\lambda)$. Now we need to show the converse, which is obvious since any element $\alpha \in \mathcal{P}_{n,m}(\lambda)$ can be written in the form $\beta\mu\rho$ (note that this factorization is not unique) where $\beta, \rho \in \mathbb{P}_{n,m}$, $\mu \in \mathcal{P}_{n,m}[\lambda]$, $\text{top}(\alpha) = \text{top}(\beta)$ and $\text{bot}(\alpha) = \text{bot}(\rho)$. □

The set of all ideals of the algebra $\mathbb{P}_{n,m}$ that are of the form $\mathbb{P}_{n,m}(\breve{\delta}; \lambda)$, for some λ, is a lattice with a partial order: $\mathbb{P}_{n,m}(\breve{\delta}; i_0, \ldots, i_{m-1}) \leq \mathbb{P}_{n,m}(\breve{\delta}; j_0, \ldots, j_{m-1})$ if and only if $i_k \leq j_k$ for each $k \in \mathbb{Z}_m$.

Now we define for each $0 \leq k \leq n$ new subsets:

$$\mathcal{P}_{n,m}[k] = \bigcup_{\sum_{j=0}^{m-1} l_j = k} \mathcal{P}_{n,m}[l_0, \ldots, l_{m-1}], \tag{2.14}$$

$$\mathcal{P}_{n,m}(k) = \bigcup_{0 \leq j \leq k} \mathcal{P}_{n,m}[j]. \tag{2.15}$$

It is clear that $\mathcal{Q}_{n,m} = \mathcal{P}_{n,m}(n-1)$ and $\mathfrak{S}_{n,m} = \mathcal{P}_{n,m}[n]$.

Let $\mathbb{P}_{n,m}(\breve{\delta}; k)$ be the ideal that is generated by the set $\mathcal{P}_{n,m}[k]$. Note that $\mathbb{P}_{n,m}(\breve{\delta}; k)$ contains all the diagrams whose propagating number is less than or equal to k, so

$$\mathbb{P}_{n,m}(\breve{\delta}) \supset \mathbb{P}_{n,m}(\breve{\delta}; n-1) \supset \mathbb{P}_{n,m}(\breve{\delta}; n-2) \supset \ldots \supset \mathbb{P}_{n,m}(\breve{\delta}; 0). \tag{2.16}$$

Proposition 2.14. *The set $\mathcal{P}_{n,m}(k)$ is the $\mathbb{F}$-basis of the ideal $\mathbb{P}_{n,m}(\breve{\delta}; k)$.*

The proof of the previous proposition depends on the propagating number property, see (2.9).

2.3 The coloured images of $\mathbb{P}_n(\delta)$-generators

In order to find a generating form of the algebra $\mathbb{P}_{n,m}$, we will use a presentation of the algebra $\mathbb{P}_n(\delta)$ and to do that we need first to define a coloured image of a diagram. As all diagrams that represent multi-colour partitions are obtained from colouring diagrams in the monoid $\mathcal{P}_n$, such that all nodes in a part of a partition and their edges which connect them have the same colour, so any set of generators for $\mathbb{P}_n(\delta)$ can be used to define generators for the algebra $\mathbb{P}_{n,m}(\breve{\delta})$ with relations obtained by modifying the relations of the algebra $\mathbb{P}_n(\delta)$, with keeping in mind the effect of the colours in the multiplication on the algebra $\mathbb{P}_{n,m}$.

Let $x = (x_1, \ldots, x_n) \in \mathbb{Z}_m^n$, define the tuple

$$\tilde{x} := (A_0, \ldots, A_{m-1}), \tag{2.17}$$

where $A_i \subseteq \underline{n}$ such that $j \in A_i$ if and only if $x_j = i$ where $1 \leq j \leq n$. In other words, the colour $\mathfrak{C}_j$ is represented by the element $j \in \mathbb{Z}_m$. Note that $\{A_i\}_{0 \leq i \leq m-1}$ is a partition of $\underline{n}$, and $\tilde{x} = \tilde{y}$ if and only if $x = y$.

Example 2.14.1. *Consider* $x = (0, 1, 0, 0, 1) \in \mathbb{Z}_3^5$ *, so* $\tilde{x} = (\{1, 3, 4\}, \{2, 5\}, \emptyset)$.

Let the node $j \in \underline{n}$ in the top row of a diagram D representing a multi-colour partition have the colour $\mathfrak{C}_i$, define $x := (x_j) \in \mathbb{Z}_m^n$ where $x_j = i$. From this definition, it is evident that $\mathrm{top}(D) = \tilde{x}$.

Remark 2.15. Henceforth we will say a diagram in the set $\mathcal{P}_{n,m}$ instead of a diagram representing a multi-colour partition in the set $\mathcal{P}_{n,m}$, and we will write $D_1 \circ D_2 = 0$ to mean $D_1 \circ D_2$ is undefined as $D_2 D_1 = 0$ in this case.

Definition 2.16. Let $\alpha = \{X_1, \ldots, X_r\} \in \mathcal{P}_n$ and $D = (D_0, \ldots, D_{m-1}) \in \mathcal{P}_{n,m}$ where $D_i \subset \alpha$ for each i, $\bigcup_i D_i = \alpha$ and $D_i \cap D_j = \emptyset$ if $i \neq j$. We say that D is a *coloured image* of α or a diagram of the shape α.

In other words, D is a coloured image of α if we can get α from D by ignoring the colours. We call α the *uncoloured image* of D.

Lemma 2.17. *Let* D_1, D_2 *be diagrams in* $\mathcal{P}_{n,m}$ *of shapes* α_1 *and* α_2*, respectively. If the colours match up, i.e.* $D_1 D_2 \neq 0$*, then the diagram of* $D_2 \circ D_1$ *has the shape* $\alpha_2 \circ \alpha_1$.

Proof. This follows immediately from the definition of the product on $\mathcal{P}_{n,m}$. □

Let $x, y \in \mathbb{Z}_m^n$ and $\alpha = \{X_1, \ldots, X_r\} \in \mathcal{P}_n$. We say that *colouring* α with top and bottom equal to $\tilde{x}$ and $\tilde{y}$ respectively, is *defined* if they satisfy for each $i, j \in \underline{n}$:

- $x_i = x_j$ if there k such that $i, j \in X_k$.

- $y_i = y_j$ if there k such that $i', j' \in X_k$.
- $x_i = y_j$ if there k such that $i, j' \in X_k$.

Simply, this means that any nodes in the same part have the same colour.

From this way of colouring, we obtain the next lemma.

Lemma 2.18. *Let $x, y \in \mathbb{Z}_m^n$ and $\alpha \in \mathcal{P}_n$. If colouring α with top and bottom equal to $\tilde{x}$ and $\tilde{y}$ respectively is defined, then there is a unique coloured image of α, denoted by α_y^x, with top and bottom equal to $\tilde{x}, \tilde{y}$ respectively.* □

Let D_1, D_2 be diagrams in $\mathcal{P}_{n,m}$, then $D_1 = \alpha_y^x$ and $D_2 = \beta_v^u$ for some $\alpha, \beta \in \mathcal{P}_n$ and $x, y, u, v \in \mathbb{Z}_m^n$. By using Lemma 2.17 and Lemma 2.18, we have

$$D_2 \circ D_1 = \begin{cases} 0 & \text{if } \ y \neq u, \\ \big(\beta \circ \alpha\big)_v^x & \text{if } \ y = u, \end{cases} \tag{2.18}$$

since $\operatorname{top}(D_2 \circ D_1) = \operatorname{top}(D_1)$ and $\operatorname{bot}(D_2 \circ D_1) = \operatorname{bot}(D_2)$.

A *decomposition* of a diagram α is a finite sequence of diagrams such that their multiplication equals α.

Proposition 2.19. *Let $D = \alpha_y^x$ for some $\alpha \in \mathcal{P}_n$ and $x, y \in \mathbb{Z}_m^n$. Then every decomposition of α in $\mathcal{P}_n$ can be used to define a decomposition for D in $\mathcal{P}_{n,m}$. The converse also holds.*

Proof. Let $\alpha = t_k \circ t_{k-1} \circ \cdots \circ t_1$ for some $t_1, \ldots, t_k \in \mathcal{P}_n$ and $k \in \mathbb{N}$. From equation (2.18), we have the following decomposition of D:

$$D = (t_k \circ t_{k-1} \circ \cdots \circ t_1)_y^x = (t_k)_y^{u^{(k-1)}} \circ (t_{k-1})_{u^{(k-1)}}^{u^{(k-2)}} \circ \cdots \circ (t_2)_{u^{(2)}}^{u^{(1)}} \circ (t_1)_{u^{(1)}}^x . \tag{2.19}$$

All we need to do, is defining the tuples $u^{(1)}, \ldots, u^{(k-1)}$ such that the colouring will be defined, where $u^{(l)} \in \mathbb{Z}_m^n$ for each l. These tuples are defined as following:

- $u_j^{(1)} = x_i$ when i, j' are contained in a part of t_1 for any $1 \leq i, j \leq n$. So $(t_1)_{u^{(1)}}^x$ is defined.

- $u_i^{(k-1)} = y_j$ when i, j' are contained in a part of t_k for any $1 \leq i, j \leq n$. So $(t_k)_y^{u^{(k)}}$ is defined.

- $u_j^{(l)} = u_i^{(l-1)}$ when i, j' are contained in a part of t_l for any $1 \leq i, j \leq n$, where $2 \leq l \leq k-1$. So $(t_l)_{u^{(l)}}^{u^{(l-1)}}$ is also defined.

- There are maybe some points in the middle rows of the decomposition $t_k \circ t_{k-1} \circ \cdots \circ t_1$, which are not connected to the top row of t_1 and the bottom row of t_k (non-propagating edges), then their colours may be chosen such that two points of them have the same colour if they are connected by an edge. For example, in figure 2.3 the colours of dashed and dotted edges and their nodes may be chosen, such that the nodes that are connected with dashed (dotted) edges have the same colour.

Conversely, when we have a decomposition of D and simply by ignoring the colours we obtain a decomposition of α. □

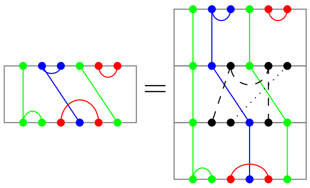
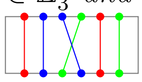


FIGURE 2.3: A decomposition of a digram in the algebra $\mathbb{P}_{6,3}$.

Colouring the elements s_i, q_i, p_j and u_i where $i \in \underline{n-1}$ and $j \in \underline{n}$, as they are defined in Section 1.7, is described as follows. Let $x = (x_1, \ldots, x_n) \in \mathbb{Z}_m^n$, we can colour the element s_i such that the top equals $\tilde{x}$, this coloured image of s_i is denoted by $\mathsf{s}_{(i,x)}$. Note that $\text{bot}(\mathsf{s}_{(i,x)}) = \widetilde{x^{\mathsf{s}_i}}$, where $x^\pi = (x_{\pi^{-1}(i)})$ for all $\pi \in \mathfrak{S}_n$.

In general, colouring any element $\sigma \in \mathfrak{S}_n$ with a top equals $\tilde{x}$ is defined, and the bottom will be $\widetilde{x^{\sigma^{-1}}}$.

Example 2.19.1. *If $x = (0, 1, 1, 2, 0, 2) \in \mathbb{Z}_3^6$ and the colours $\mathfrak{C}_0, \mathfrak{C}_1$ and $\mathfrak{C}_2$ are red, blue and green respectively, then* $\mathsf{s}_{(3,x)} =$ .

We define the diagram 1_x to be the coloured image of $id \in \mathfrak{S}_n$, where the node i is only connected to i' with an $\mathfrak{C}_{x_i}$-edge. Hence, we have $1_{\mathbb{P}_{n,m}} = \sum_{x \in \mathbb{Z}_m^n} 1_x$.

To make colouring the element q_i defined, we need to take in consideration that the nodes $i, i+1, i'$ and $(i+1)'$ have the same colour. So we define an index set, $\Gamma_i \subset \mathbb{Z}_m^n$, to preserve this condition:

$$\Gamma_i := \{x \in \mathbb{Z}_m^n \mid x_i = x_{i+1}\}, \tag{2.20}$$

and define the diagram $\mathsf{q}_{(i,x)}$, where $x \in \Gamma_i$, to be the coloured image of the diagram q_i such that $\text{top}(\mathsf{q}_{(i,x)}) = \tilde{x}$. From the graphical visualization of q_i, it is clear that $\text{top}(\mathsf{q}_{(i,x)}) = \text{bot}(\mathsf{q}_{(i,x)})$. Note that $x^{\mathsf{s}_i} = x$ for all $x \in \Gamma_i$.

To determine a coloured image of the diagram p_j, an index set according to j needed to define:

$$\Omega_j := \{(x, y) \in \mathbb{Z}_m^n \times \mathbb{Z}_m^n \mid \ x_i = y_i \ \forall \ i \neq j\}. \tag{2.21}$$

The diagram $\mathsf{p}_{(j,x,y)}$, where $(x, y) \in \Omega_j$, is the coloured image of the element p_j such that $\text{top}(\mathsf{p}_{(j,x,y)}) = \tilde{x}$ and $\text{bot}(\mathsf{p}_{(j,x,y)}) = \tilde{y}$.

Define a set Ω_i^* to be

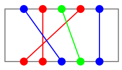
$$\Omega_i^* := \{(x, y) \in \Gamma_i \times \Gamma_i \mid \ x_j = y_j \ \forall \ j \neq i, i+1\}, \tag{2.22}$$

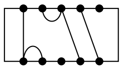
and $\mathsf{u}_{(i,x,y)}$ to be the coloured image of the element u_i such that $\text{top}(\mathsf{u}_{(i,x,y)}) = \tilde{x}$ and $\text{bot}(\mathsf{u}_{(i,x,y)}) = \tilde{y}$, where $(x, y) \in \Omega_i^*$.

Proposition 2.20. *The groupoid $\mathfrak{S}_{n,m}$ (we show that later in Chapter 3) is generated by the diagrams $\mathsf{s}_{(i,x)}$ for all $x \in \mathbb{Z}_m^n, i = 1, \ldots, n-1$. Also each element in $\mathcal{A}_{n,m}^*$ can be written as a sequence of the elements $\mathsf{q}_{(i,y)}$ and $\mathsf{p}_{(j,u,v)}$, where $y \in \Gamma_i$, $(u, v) \in \Omega_j$, $i \in \underline{n-1}$ and $j \in \underline{n}$.*

Proof. Let D be a diagram in $\mathfrak{S}_{n,m}$ of a shape $\sigma \in \mathfrak{S}_n$ and $\text{top}(D) = \tilde{x}$ for some $x \in \mathbb{Z}_m^n$. Since σ is a permutation in the group $\mathfrak{S}_n$, so it can be written as $\mathsf{s}_{i_1}\mathsf{s}_{i_2} \cdots \mathsf{s}_{i_r}$

for some integers $i_1, \dots, i_r$. Now, colouring the edges to get the same diagram D gives us the decomposition $\mathsf{s}_{(i_1,x)}\mathsf{s}_{(i_2,x^{s_{i_1}})}\cdots\mathsf{s}_{(i_r,x^{s_{i_1}s_{i_2}\cdots s_{i_{r-1}}})}$ of D (use (2.19)), so all the diagrams of shapes s_i for some i generate the groupoid $\mathfrak{S}_{n,m}$. The proof for $\mathcal{A}^*_{n,m}$ is similar, since any strictly planar diagram is coloured image of a planar diagram in the monoid $\mathcal{P}_n$. □

Example 2.20.1. *A decomposition of the permutation* (134) *is* $\mathsf{s}_1\mathsf{s}_3\mathsf{s}_2\mathsf{s}_1$ *in* $\mathfrak{S}_5$, *so the corresponding factorization of the diagram*  *is* $\mathsf{s}_{(1,x)}\mathsf{s}_{(3,y)}\mathsf{s}_{(2,u)}\mathsf{s}_{(1,v)}$ *where* $x = (1, 0, 2, 0, 1)$, $y = (0, 1, 2, 0, 1)$, $u = (0, 1, 0, 2, 1)$ *and* $v = (0, 0, 1, 2, 1)$.

Example 2.20.2. *Take* α *to be the diagram* . *One of decompositions of* α *is* $\mathsf{p}_5\mathsf{q}_4\mathsf{p}_4\mathsf{q}_2\mathsf{p}_2\mathsf{q}_3\mathsf{p}_3\mathsf{q}_1$, *so*

$$\alpha^x_y = \mathsf{p}_{(5,x,u)}\mathsf{q}_{(4,u)}\mathsf{p}_{(4,u,v)}\mathsf{q}_{(2,v)}\mathsf{p}_{(2,v,z)}\mathsf{q}_{(3,z)}\mathsf{p}_{(3,z,y)}\mathsf{q}_{(1,y)},$$

where $x = (0, 2, 2, 1, 0)$, $y = (0, 0, 1, 2, 1)$, $u = (0, 2, 2, 1, 1)$, $v = (0, 2, 2, 2, 1)$ *and* $z = (0, 0, 2, 2, 1)$.

The next result shows that there is a natural factorization $\mathcal{P}_{n,m} = \mathfrak{S}_{n,m}\mathcal{A}^*_{n,m}\mathfrak{S}_{n,m}$, which is the first step for finding a presentation for the algebra $\mathbb{P}_{n,m}$.

Lemma 2.21. *Let* $D \in \mathcal{P}_{n,m}$. *Then* $D = \pi_1 t \pi_2$ *for some* $\pi_1, \pi_2 \in \mathfrak{S}_{n,m}$ *and* $t \in \mathcal{A}^*_{n,m}$.

Proof. Assume that D has the shape $\alpha \in \mathcal{P}_n$ and $\text{top}(D) = \tilde{x}$, $\text{bot}(D) = \tilde{y}$ for some $x, y \in \mathbb{Z}^n_m$. By using equation (1.13), we can write α as $\sigma d \theta$ where $\sigma, \theta \in \mathfrak{S}_n$ and $d \in \mathcal{A}_n$, and we are done since the decomposition of D can be gotten by recolouring the previous factorization of α, such that π_1 is the coloured image of σ with top row equal to $\tilde{x}$, π_2 is the coloured image of θ with bottom row equal to $\tilde{y}$ and t is the coloured image of d with top row equal to $\widetilde{x^{\sigma^{-1}}}$ and the bottom row $\widetilde{y^{\theta}}$. For example, see figure 2.4.

Next step is showing that the coloured image t is defined, which follows immediately from Proposition 2.19. □

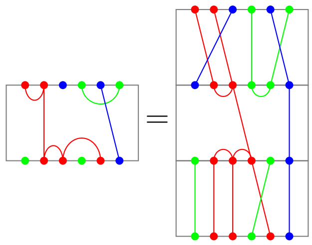


FIGURE 2.4: An example of the property $\mathcal{P}_{n,m} = \mathfrak{S}_{n,m}\mathcal{A}^*_{n,m}\mathfrak{S}_{n,m}$.

Definition 2.22. [23, Section 4]**.** The *white multi-diagram of shape* $\theta \in \mathcal{P}_n$ is the sum of all possible different coloured copies of θ in the algebra $\mathbb{P}_{n,m}$.

Simply, we well say a *white diagram* instead of a white multi-diagram. For instance, the identity of $\mathbb{P}_{n,m}$ is the white diagram of shape $id \in \mathfrak{S}_n$.

2.4 A generating set for the algebra $\mathbb{P}_{n,m}$

In this section we aim to define the multi-colour partition algebra by generators and relations.

Proposition 2.23. *The elements* $\mathsf{s}_{(i,x)}, 1_x, \mathsf{q}_{(i,w)}$ *and* $\mathsf{p}_{(j,u,v)}$*, where* $x \in \mathbb{Z}^n_m$*,* $w \in \Gamma_i$*,* $i \in \underline{n-1}$*,* $(u,v) \in \Omega_j$ *and* $j \in \underline{n}$*, satisfy the following relations:*

1. *For all* $y \in \mathbb{Z}^n_m$*,* $1_x 1_y = \begin{cases} 0 & \text{if } y \neq x, \\ 1_x & \text{if } y = x. \end{cases}$

2. *For all* $y \in \mathbb{Z}^n_m$*,* $1_x \mathsf{s}_{(i,y)} = \left\{ \begin{array}{ll} 0 & \text{if } y \neq x, \\ \mathsf{s}_{(i,y)} & \text{if } y = x. \end{array} \right\} = \mathsf{s}_{(i,y)} 1_{x^{\mathsf{s}_i}}.$

3. $1_x \mathsf{q}_{(i,w)} = \left\{ \begin{array}{ll} 0 & \text{if } w \neq x, \\ \mathsf{q}_{(i,w)} & \text{if } w = x. \end{array} \right\} = \mathsf{q}_{(i,w)} 1_x.$

4. $1_x \mathsf{p}_{(j,u,v)} = \begin{cases} 0 & \text{if } u \neq x, \\ \mathsf{p}_{(j,u,v)} & \text{if } u = x. \end{cases}$

5. $\mathsf{p}_{(j,u,v)} 1_x = \begin{cases} 0 & \text{if } v \neq x, \\ \mathsf{p}_{(j,u,v)} & \text{if } v = x. \end{cases}$

6. *For all* $l \in \underline{n-1}$, $\mathsf{s}_{(i,x)}\mathsf{s}_{(l,y)} = \begin{cases} 0 & \text{if } y \neq x^{\mathsf{s}_i}, \\ 1_x & \text{if } y = x^{\mathsf{s}_i}, i = l, \\ \mathsf{s}_{(l,x)}\mathsf{s}_{(i,x^{\mathsf{s}_l})} & \text{if } y = x^{\mathsf{s}_i}, l \neq i \pm 1. \end{cases}$

7. $\mathsf{s}_{(i,x)}\mathsf{s}_{(i+1,x^{\mathsf{s}_i})}\mathsf{s}_{(i,x^{\mathsf{s}_i\mathsf{s}_{i+1}})} = \mathsf{s}_{(i+1,x)}\mathsf{s}_{(i,x^{\mathsf{s}_{i+1}})}\mathsf{s}_{(i+1,x^{\mathsf{s}_{i+1}\mathsf{s}_i})}$, *for all* $i \in \underline{n-2}$.

8. *For all* $(z, y) \in \Omega_k$, *and* $k \in \underline{n}$,

$$\mathsf{p}_{(k,z,y)}\mathsf{p}_{(j,u,v)} = \begin{cases} 0 & \text{if } y \neq u, \\ \delta_{y_k}\mathsf{p}_{(k,z,v)} & \text{if } k = j,\ y = u, \\ \mathsf{p}_{(j,z,w)}\mathsf{p}_{(k,w,v)} & \text{if } y = u, \text{ where } w_l = y_l\ \forall\ l \neq k, j, \\ & w_k = z_k, w_j = v_j\ . \end{cases}$$

9. *For all* $y \in \Gamma_l$, $\mathsf{q}_{(i,w)}\mathsf{q}_{(l,y)} = \begin{cases} 0 & \text{if } w \neq y, \\ \mathsf{q}_{(l,w)}\mathsf{q}_{(i,y)} & \text{if } w = y, \\ \mathsf{q}_{(i,w)} & \text{if } i = l,\ w = y. \end{cases}$

10. $\mathsf{p}_{(j,u,v)}\mathsf{q}_{(i,w)} = \begin{cases} 0 & \text{if } v \neq w, \\ \mathsf{q}_{(i,u)}\mathsf{p}_{(j,u,v)} & \text{if } v = w,\ j \neq i, i+1. \end{cases}$

11. $\mathsf{q}_{(i,w)}\mathsf{p}_{(j,u,v)} = \begin{cases} 0 & \text{if } w \neq u, \\ \mathsf{p}_{(j,u,v)}\mathsf{q}_{(i,v)} & \text{if } w = u, j \neq i, i+1. \end{cases}$

12. $\mathsf{s}_{(i,x)}\mathsf{p}_{(j,u,v)} = \begin{cases} 0 & \text{if } u \neq x^{s_i} \\ \mathsf{p}_{(j,x,v^{\mathsf{s}_i})}\mathsf{s}_{(i,v^{\mathsf{s}_i})} & \text{if } u = x^{\mathsf{s}_i}, j \neq i, i+1. \end{cases}$

13. $\mathsf{p}_{(j,u,v)}\mathsf{s}_{(i,x)} = \begin{cases} 0 & \text{if } v \neq x, \\ \mathsf{s}_{(i,u)}p_{(j,u^{\mathsf{s}_i},x)} & \text{if } v = x, j \neq i, i+1. \end{cases}$

14. $\mathsf{s}_{(i,x)}\mathsf{p}_{(i,x^{\mathsf{s}_i},v)}\mathsf{p}_{(i+1,v,z)} = \mathsf{p}_{(i,x,w)}\mathsf{p}_{(i+1,w,z)} = \mathsf{p}_{(i,x,w')}\mathsf{p}_{(i+1,w',z^{\mathsf{s}_i})}\mathsf{s}_{(i,z^{\mathsf{s}_i})}$, *for all* $(x^{s_i}, v) \in \Omega_i$, $(v, z) \in \Omega_{i+1}$, *where* $w_j = x_j = w'_j$ *for all* $j \neq i$ *and* $w_i = z_i$, $w'_j = z_{i+1}$.

15. $\mathsf{s}_{(i,x)}\mathsf{p}_{(i,x^{\mathsf{s}_i},v)}\mathsf{s}_{(i,v)} = \mathsf{p}_{(i+1,x,v^{\mathsf{s}_i})}$, *for all* $(x^{\mathsf{s}_i}, v) \in \Omega_i$.

16. $\mathsf{s}_{(i,x)}\mathsf{q}_{(l,w)} = \begin{cases} 0 & \text{if } w \neq x^{\mathsf{s}_i}, \\ \mathsf{q}_{(l,x)}\mathsf{s}_{(i,x)} & \text{if } w = x^{\mathsf{s}_i}, l \neq i \pm 1. \end{cases}$

17. $\mathsf{q}_{(l,w)}\mathsf{s}_{(i,x)} = \begin{cases} 0 & \text{if } w \neq x, \\ \mathsf{s}_{(i,w)}\mathsf{q}_{(l,w^{\mathsf{s}_i})} & \text{if } w = x, l \neq i \pm 1. \end{cases}$

18. $\mathsf{s}_{(i,x)}\mathsf{q}_{(i,x)} = \mathsf{q}_{(i,x)} = \mathsf{q}_{(i,x)}\mathsf{s}_{(i,x)}$, *for all* $x \in \Gamma_i$.

19. $\mathsf{s}_{(i,x)}\mathsf{s}_{(i+1,x^{\mathsf{s}_i})}\mathsf{q}_{(i,x^{\mathsf{s}_i\mathsf{s}_{i+1}})}\mathsf{s}_{(i+1,x^{\mathsf{s}_i\mathsf{s}_{i+1}})}\mathsf{s}_{(i,x^{\mathsf{s}_i})} = \mathsf{q}_{(i+1,x)}$, *for all* $x \in \Gamma_{i+1}$.

20. $\mathsf{q}_{(i,w)}\mathsf{p}_{(i,w,w)}\mathsf{q}_{(i,w)} = \mathsf{q}_{(i,w)} = \mathsf{q}_{(i,w)}\mathsf{p}_{(i+1,w,w)}\mathsf{q}_{(i,w)}$.

21. $\mathsf{p}_{(i,u,v)}\mathsf{q}_{(i,v)}\mathsf{p}_{(i,v,z)} = \mathsf{p}_{(i,u,z)}$, *for all* $(u,v),(v,z) \in \Omega_i$, $v \in \Gamma_i$.

22. $\mathsf{p}_{(i,u,v)}\mathsf{q}_{(i-1,v)}\mathsf{p}_{(i,v,z)} = \mathsf{p}_{(i,u,z)}$, *for all* $(u,v),(v,z) \in \Omega_i$, $v \in \Gamma_{i-1}$.

Proof. Let $D_1, D_2 \in \mathcal{P}_{n,m}$. The element D_1D_2 will be zero if $\mathrm{bot}(D_1) \neq \mathrm{top}(D_2)$. Since $\mathrm{top}(1_x) = \mathrm{bot}(1_x) = \tilde{x}$, $\mathrm{top}(\mathsf{s}_{(i,x)}) = \tilde{x}$, $\mathrm{bot}(\mathsf{s}_{(i,x)}) = \widetilde{x^{\mathsf{s}_i}}$, $\mathrm{top}(\mathsf{q}_{(i,w)}) = \tilde{w} = \mathrm{bot}(\mathsf{q}_{(i,w)})$, $\mathrm{top}(\mathsf{p}_{(j,u,v)}) = \tilde{u}$ and $\mathrm{bot}(\mathsf{p}_{(j,u,v)}) = \tilde{v}$, so whenever the bottom of one of the previous elements does not equal the top of another element, the product of the first element and the second will be zero.

When the product does not vanish, all these relations can be verified by drawing the diagram products that they refer to. To prove the relation 14, we will write x_i over the node i to say that the node has the colour $\mathfrak{C}_{x_i}$. The proof of relation 14 equalities is given in figure 2.5. □

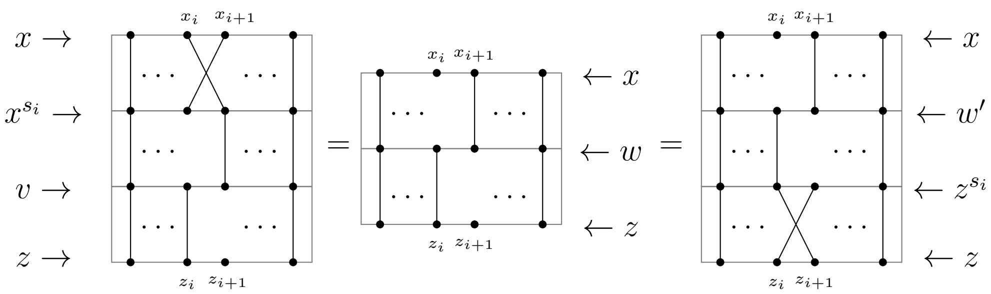


FIGURE 2.5: $\mathsf{s}_{(i,x)}\mathsf{p}_{(i,x^{\mathsf{s}_i},v)}\mathsf{p}_{(i+1,v,z)} = \mathsf{p}_{(i,x,w)}\mathsf{p}_{(i+1,w,z)} = \mathsf{p}_{(i,x,w')}\mathsf{p}_{(i+1,w',z^{\mathsf{s}_i})}\mathsf{s}_{(i,z^{\mathsf{s}_i})}$.

Corollary 2.24. *The elements* $\mathsf{s}_{(i,x)}$, $\mathsf{q}_{(i,y)}$ *and* $\mathsf{p}_{(j,u,v)}$, *where* $x \in \mathbb{Z}_m^n$, $i \in \underline{n-1}$, $y \in \Gamma_i$, $(u,v) \in \Omega_j$ *and* $j \in \underline{n}$, *satisfy the following relations:*

1. $\mathsf{q}_{(i,x)}\mathsf{s}_{(i-1,x)}\mathsf{q}_{(i,x)} = \mathsf{q}_{(i,x)}\mathsf{q}_{(i-1,x)}$ *for all* $x \in \Gamma_i \cap \Gamma_{i-1}, 2 \leq i \leq n-1$.

2. $\mathsf{p}_{(i,u,v)}\mathsf{s}_{(i,v)}\mathsf{p}_{(i,v^{\mathsf{s}_i},w)} = \mathsf{p}_{(i+1,u,v')}\mathsf{p}_{(i,v',w)}$ *for all* $(u,v),(v^{\mathsf{s}_i},w) \in \Omega_i$ *where* $v_l = x_l$ *for all* $l \neq i$ *and* $v_i = w_i$.

3. $\mathsf{p}_{(i,u,v)}\mathsf{q}_{(i,v)}\mathsf{p}_{(i+1,v,w)} = \mathsf{p}_{(i,u,w^{\mathsf{s}_i})}\mathsf{s}_{(i,w^{\mathsf{s}_i})}$ *for all* $v \in \Gamma_i, (v,w) \in \Omega_{i+1}$.

4. $\mathsf{p}_{(i+1,u,v)}\mathsf{q}_{(i,v)}\mathsf{p}_{(i,v,w)} = \mathsf{s}_{(i,u)}\mathsf{p}_{(i,u^{\mathsf{s}_i},w)}$ *for all* $(u,v) \in \Omega_{i+1}, v \in \Gamma_i, (v,w) \in \Omega_i$.

Proof. This can be proved by using the relations in previous proposition. We are going to show only the first part.

$$\begin{aligned}
\mathsf{q}_{(i,x)}\mathsf{s}_{(i-1,x)}\mathsf{q}_{(i,x)} &= \big(\mathsf{q}_{(i,x)}\mathsf{s}_{(i,x)}\big)\mathsf{s}_{(i-1,x)}\mathsf{q}_{(i,x)} && \textit{(From the relation } 18) \\
&= \mathsf{q}_{(i,x)}\mathsf{s}_{(i,x)}\mathsf{s}_{(i-1,x)}\mathsf{q}_{(i,x)}\big(\mathsf{s}_{(i-1,x)}\mathsf{s}_{(i,x)}\mathsf{s}_{(i,x)}\mathsf{s}_{(i-1,x)}\big) \\
& && \textit{(Note that } x = x^{\mathsf{s}_i} = x^{\mathsf{s}_{i-1}}) \\
&= \mathsf{q}_{(i,x)}\Big(\mathsf{s}_{(i,x)}\mathsf{s}_{(i-1,x)}\mathsf{q}_{(i,x)}\mathsf{s}_{(i-1,x)}\mathsf{s}_{(i,x)}\Big)\mathsf{s}_{(i,x)}\mathsf{s}_{(i-1,x)} \\
&= \mathsf{q}_{(i,x)}\mathsf{q}_{(i-1,x)}\mathsf{s}_{(i,x)}\mathsf{s}_{(i-1,x)} && \textit{(From the relation } 19) \\
&= \mathsf{q}_{(i-1,x)}\mathsf{q}_{(i,x)}\mathsf{s}_{(i,x)}\mathsf{s}_{(i-1,x)} && \textit{(From the relation } 9) \\
&= \mathsf{q}_{(i-1,x)}\mathsf{q}_{(i,x)}\mathsf{s}_{(i-1,x)} && \textit{(From the relation } 18) \\
&= \mathsf{q}_{(i,x)}\mathsf{q}_{(i-1,x)}\mathsf{s}_{(i-1,x)} && \textit{(From the relation } 9) \\
&= \mathsf{q}_{(i,x)}\mathsf{q}_{(i-1,x)}. && \textit{(From the relation } 18)
\end{aligned}$$

□

Remark 2.25. Every relation between the elements $\mathsf{s}_i, \mathsf{q}_i, \mathsf{p}_j$ in $\mathbb{P}_n(\delta)$ corresponds to a relation in $\mathbb{P}_{n,m}$ between the elements $\mathsf{s}_{(i,x)}, \mathsf{q}_{(i,y)}, \mathsf{p}_{(j,u,v)}$ when the colours match up with all possible choices of colours, where $i = 1, \ldots, n-1$ and $j = 1, \ldots, n$. For example, the relation $\mathsf{s}_i^2 = 1$ is corresponding to the relation $\mathsf{s}_{(i,x)}\mathsf{s}_{(i,x^{\mathsf{s}_i})} = 1_x$. All the corresponding relations to the relations in Theorem 1.23 exist by Proposition 2.23. Since any other relation in $\mathbb{P}_n(\delta)$ can be computed by these relations in Theorem 1.23, so from equations (2.18) and (2.19) and Lemma 2.18 the corresponding relation in $\mathbb{P}_{n,m}(\breve{\delta})$ can be computed by the relations in Propositions 2.23.

Lemma 2.26. *Let $t, b \in \mathcal{A}^*_{n,m}$ and $\pi \in \mathfrak{S}_{n,m}$ such that $t \circ \pi \circ b$ is defined. Then there is $d \in \mathcal{A}^*_{n,m}$ and $\sigma \in \mathfrak{S}_{n,m}$ such that*

$$b \circ \pi \circ t = \sigma \circ d \circ t. \tag{2.23}$$

Proof. Since any element in $\mathcal{A}^*_{n,m}$ is a coloured image of an element in $\mathcal{A}_n$, by ignoring the colours and using equation (1.14), we obtain a non-coloured copy of our equation and all what we need to do is recolour the diagrams. Since we use the relations in Theorem 1.23 to compute the decomposition of non-coloured partitions, thus we need to use the relations in Proposition 2.23 and Corollary 2.24 to compute the corresponding decomposition of multi-colour partitions. For example, see figure 2.6. □

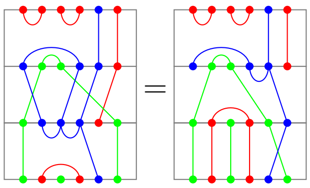


FIGURE 2.6: An example of the relation (2.23)

Theorem 2.27. *The algebra $\mathbb{P}_{n,m}(\breve{\delta})$ is generated by the elements $\mathsf{s}_{(i,x)}, 1_x$, $\mathsf{q}_{(i,y)}$ and $\mathsf{p}_{(j,u,v)}$, where $x \in \mathbb{Z}^n_m$, $y \in \Gamma_i$, $i \in \underline{n-1}$, $(u,v) \in \Omega_j$ and $j \in \underline{n}$, with all the relations in Proposition 2.23.*

Proof. It has been shown that these elements satisfy the relations and every partition in $\mathcal{P}_{n,m}$ can written as sequence products of these generators in Proposition 2.23, Proposition 2.20 and Lemma 2.21. Then we only need to show that any product in $\mathbb{P}_{n,m}$ can be computed by using the relations in Proposition 2.23.

Let $D_1, D_2 \in \mathcal{P}_{n,m}$. By using the decomposition $\mathcal{P}_{n,m} = \mathfrak{S}_{n,m}\mathcal{A}^*_{n,m}\mathfrak{S}_{n,m}$, we have $D_1 = \pi_1 t \pi_2$, $D_2 = \sigma_1 b \sigma_2$ for some $t, b \in \mathcal{A}^*_{n,m}$ and $\pi_1, \pi_2, \sigma_1, \sigma_2 \in \mathfrak{S}_{n,m}$. Assume that $\mathrm{top}(D_1) = \tilde{x}$, $\mathrm{bot}(D_1) = \tilde{y}$, $\mathrm{top}(D_2) = \tilde{w}$ and $\mathrm{bot}(D_2) = \tilde{z}$, thus $\mathrm{bot}(\pi_2) = \tilde{y}$ and $\mathrm{top}(\sigma_1) = \tilde{w}$ in the previous decompositions of D_1 and D_2. If $y \neq w$, the relations 1, 2 and 6 in Proposition 2.23 lead to $D_1D_2 = 0$. On the other hand, if $y = w$, by using the equation (2.19) we have

$$D_2 \circ D_1 = \left(\sigma_2' \circ b' \circ \sigma_1' \circ \pi_2' \circ t' \circ \pi_1' \right)_z^x ,$$

where π_1', π_2', t', σ_1', σ_2' and b' are diagrams in $\mathcal{P}_n$ obtained from π_1, π_2, t, σ_1, σ_2 and b after ignoring the colours. Recall that $D_1 D_2 = \big(\prod_{i=0}^{m-1} \delta_i^{c_i} \big) D_2 \circ D_1$ where c_i is the number of removed connected components that have the colour $\mathfrak{C}_i$, so

$$\pi_1' t' \pi_2' \sigma_1' b' \sigma_2' = \delta^{\sum_{i=0}^{m-1} c_i} \big(\sigma_2' \circ b' \circ \sigma_1' \circ \pi_2' \circ t' \circ \pi_1' \big).$$

Now the relations in Theorem 1.23 can compute the element $\pi_1' t' \pi_2' \sigma_1' b' \sigma_2'$, and the relations in Proposition 2.23 corresponding to the relations in Theorem 1.23, then they are sufficient to compute $D_1 D_2$. □

2.5 The bubble algebra $\mathbb{T}_{n,m}(\breve{\delta})$

In this section we will define the bubble algebra, which is introduced in [23], as a sub-algebra of the multi-colour partition algebra and determine its dimension and find a generating set for it.

The diagrams in the bubble algebra in the case of two colours can be constructed by drawing two Kauffman diagrams (or just one) with no internal loops, using different colours in the same frame with n nodes on the northern face and n nodes on the southern face, such that if a node is contained in first Kauffman diagram, it will not be contained in the second. This means that at these diagrams the nodes are connected in pairs with different colours where an intersection is just allowed between different colour edges.

There is another way to describe these diagrams, as Grimm and Martin[23] did, as a sheet of bubble wrap (bubble wrap made from two sheets of polythene welded together along certain lines to trap bubbles) where we are allowed to draw red lines only on the back sheet and blue lines are only in the front sheet, see figure 2.7. In this realisation, edges are not allowed to cross on the same sheet, but they may deformed isotopically as before.

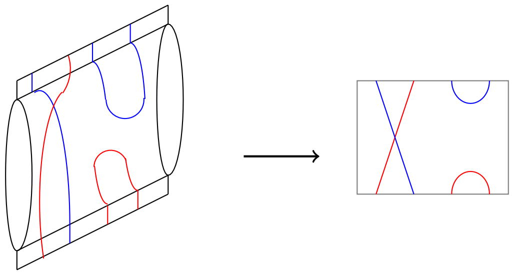


FIGURE 2.7: Representing a diagram as bubble wrap.

The composite of two diagrams is defined if the two diagrams have the same number of end points. In this case the composite is zero unless the colours match up precisely. If they do match up the composite is a multi-colour partition which is obtained by linking the diagrams together as for the Temperley-Lieb algebra replacing any $\mathfrak{C}_j$ loop appearing inside the diagram by the scalar δ_j times the rest of the diagram.

The *bubble algebra* $\mathbb{T}_{n,2}(\delta_0, \delta_1)$ -it is denoted by $T_n^2(\delta_r, \delta_b)$ in [23], or simply $\mathbb{T}_{n,2}$- is the $\mathbb{F}$-linear extension of the set of these diagrams which are isotopy classes of bubble diagrams and composition, with internal closed loop replacement. The loop replacement scalar here depends on the colour. The identity of the bubble algebra is the summation of all the diagrams which connect i only to i' with any colour for each $1 \leq i \leq n$.

From the description of diagrams in the bubble algebra $\mathbb{T}_{n,2}$, we can identify bubble diagrams with multi-colour partitions, and hence it is a sub-algebra of the algebra $\mathbb{P}_{n,2}$.

Theorem 2.28. *The bubble algebra* $\mathbb{T}_{n,2}(\delta_0, \delta_1)$ *is the sub-algebra of* $\mathbb{P}_{n,2}(\delta_0, \delta_1)$ *spanned by the set* $\mathcal{T}_{n,2}$*, which is defined in equation* (2.11)*.*

Proof. We are going to show that $\mathcal{T}_{n,2}$ is a sub-category of $\mathcal{P}_{n,2}$, see Remark 2.4, and then the rest follows immediately from the algebra $\mathbb{P}_{n,2}$ and from bubble diagrams realisation. Since $1_x \in \mathcal{T}_{n,2}$ for each $x \in \mathbb{Z}_m^n$, we only need to show that the set $\mathcal{T}_{n,2}$ is closed under the composition when it is defined. Let $D = (D_0, D_1)$ and $B = (B_0, B_1)$ be two-colour partition diagrams in $\mathcal{T}_{n,2}$ such that $D \circ B$ is defined, so from (2.7), we have $D_i \circ B_i$ also defined as partition diagrams. Now from the definition of $\mathcal{T}_{n,2}$, all the diagrams D_i and B_i are Kauffman's diagrams, but then $D_i \circ B_i$ is also Kauffman's diagram for each i. Thus $D \circ B \in \mathcal{T}_{n,2}$, and we are done. $\square$

Remark 2.29. The algebra $\mathbb{T}_{n,m}(\delta_0, \dots, \delta_{m-1})$, or $\mathbb{T}_{n,m}$ and $\mathbb{T}_{n,m}(\breve{\delta})$ for simplicity, which the bubble algebra with m colours, similarly can be defined to be a sub-algebra of the algebra $\mathbb{P}_{n,m}(\breve{\delta})$ generated by the subset $\mathcal{T}_{n,m}$, which is defined in equation (2.11).

2.5.1 The dimension of bubble algebra

Proposition 2.30. *For each $n \in \mathbb{N}$, the dimension of the algebra $\mathbb{T}_{n,2}(\delta_0, \delta_1)$ is given by the formula*

$$\dim \mathbb{T}_{n,2} = \dim \mathsf{TL}_n \dim \mathsf{TL}_{n+1}, \tag{2.24}$$

where TL_n is the Temperley-Lieb algebra.

Proof. In order to compute the dimension of $\mathbb{T}_{n,2}$, we will compute the number of diagrams with k red edges, where $k = 0, 1, \dots, n$.

Drawing diagram with k red edges needs $2k$ red nodes, there are $\binom{2n}{2k}$ options to choose $2k$ nodes from $2n$ without repetition and the order does not matter. Next, connecting $2k$ red points in pairs without crossing gives us $\dim \mathsf{TL}_k$ probabilities, see Section 1.8.1. On the other hand, there are $\dim \mathsf{TL}_{n-k}$ ways to connect $2(n-k)$ blue nodes. Thus the total number of diagrams in this case is $\binom{2n}{2k} \dim \mathsf{TL}_k \dim \mathsf{TL}_{n-k}$.

By taking all the possibilities of k, we obtain the following formula:

$$\dim \mathbb{T}_{n,2} = \sum_{k=0}^{k=n} \binom{2n}{2k} \dim \mathsf{TL}_k \dim \mathsf{TL}_{n-k} = \frac{\mathsf{C}_n}{(n+1)} \sum_{k=0}^{n} \binom{n+1}{k} \binom{n+1}{n-k},$$

as $\dim \mathsf{TL}_n = \mathsf{C}_n = \frac{2n!}{n!(n+1)!}$, where C_n is Catalan number see Section 1.8. Next, we use the formula $\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}$, which is known as *Vandermonde's convolution formula* (see for example Theorem 4.2 in [34]), to finish our calculation:

$$\dim \mathbb{T}_{n,2} = \frac{\mathsf{C}_n}{(n+1)} \binom{2n+2}{n} = \mathsf{C}_n \mathsf{C}_{n+1}. \qquad \square$$

Proposition 2.31. *For each $n \in \mathbb{N}$, the dimension of the algebra $\mathbb{T}_{n,m}(\breve{\delta})$ is given by the formula*

$$\dim \mathbb{T}_{n,m} = \sum_{\sum_{j=0}^{m-1} k_j = n} \frac{(2n)!}{\prod_{i=0}^{m-1} k_i!(k_i+1)!}. \tag{2.25}$$

Proof. In order to compute the dimension of $\mathbb{T}_{n,m}$, we will compute the number of diagrams with k_j edges of the colour $\mathfrak{C}_j$, where $j \in \mathbb{Z}_m$ and $\sum_{j=0}^{m-1} k_j = n$.

Drawing diagram with k_0 edges of the colour $\mathfrak{C}_0$ needs $2k_0$ nodes, there are $\binom{2n}{2k_0}$ choices for these nodes. Next, drawing diagram with k_j strings of the colour $\mathfrak{C}_j$ needs $2k_j$ nodes, there are $\binom{2n-2\sum_{i=0}^{j-1} k_i}{2k_j}$ choices for these nodes, where $j = 1, \dots, m-2$ (the last colour takes the rest of nodes).

Next, connecting $2k_j$ points in pairs without crossing gives us $\dim \mathsf{TL}_{k_j} = \mathsf{C}_{k_j}$ possibilities. Thus the total number of diagrams is

$$\begin{aligned}
\dim \mathbb{T}_{n,m} &= \sum_{\sum k_j = n} \binom{2n}{2k_0} \times \cdots \times \binom{2(n - \sum_{i=0}^{m-3} k_i)}{2k_{m-2}} \prod_{i=0}^{m-1} \mathsf{C}_{k_i}, \\
&= \sum_{\sum k_j = n} \frac{(2n)!}{(2k_0)!(2k_1)! \cdots (2k_{m-1})!} \prod_{i=0}^{m-1} \mathsf{C}_{k_i}, \\
&= \sum_{\sum k_j = n} \frac{(2n)!}{k_0!(k_0+1)!k_1!(k_1+1)! \cdots k_{m-1}!(k_{m-1}+1)!}.
\end{aligned} \qquad \square$$

In table 2.2, it has been listed, up to rank $n = 5$, the dimension of the algebra $\mathbb{T}_{n,m}$ where $m = 1, 2, 3, 4$.

Table 2.2: Examples of dimensions of the bubble algebra.

n	$\dim \mathsf{TL}_n$	$\dim \mathbb{T}_{n,2}$	$\dim \mathbb{T}_{n,3}$	$\dim \mathbb{T}_{n,4}$
0	1	1	1	1
1	1	2	3	4
2	2	10	24	44
3	5	70	285	740
4	14	588	4242	16016
5	42	5544	73206	410928

2.5.2 A generating set of the bubble algebra $\mathbb{T}_{n,m}(\breve{\delta})$

We use a presentation of the Brauer algebra to derive a generating set for the bubble algebra. As Grimm and Martin[23] mentioned, the diagram basis of the algebra $\mathbb{T}_{n,m}$ is like the Brauer diagram basis of the Brauer algebra $\mathbb{B}_n(\delta)$ after ignoring the colours. Therefore, colouring a Brauer diagram (if it is possible, since if there are $m+1$ or more edges in the diagram such that each one cross the others, colouring it will be undefined since there must be at least two crossing edges having the same colour) gives a diagram representing a multi-colour partition in the set $\mathcal{T}_{n,m}$.

We follow the same idea of defining a generating set of the algebra $\mathbb{P}_{n,m}$ to obtain one of the algebra $\mathbb{T}_{n,m}$, by rewriting the relations in Theorem 1.24 and colouring the generators in the same theorem.

Theorem 2.32. *The algebra $\mathbb{T}_{n,m}(\breve{\delta})$ is generated by the diagrams $1_x, \mathsf{s}_{(i,y)}$ and $\mathsf{u}_{(i,z,w)}$ where $x \in \mathbb{Z}_m^n$, $y \in \Upsilon_i = \{y \in \mathbb{Z}_m^n \mid y_i \neq y_{i+1}\}$, $(z,w) \in \Omega_i^*$ (see equation (2.22)) and $i \in \underline{n-1}$, subject to the relations:*

1. *For all $y \in \mathbb{Z}_m^n$,* $1_x 1_y = \begin{cases} 0 & \text{if } y \neq x, \\ 1_x & \text{if } y = x. \end{cases}$

2. *For all $y \in \mathbb{Z}_m^n$,* $1_x \mathsf{s}_{(i,y)} = \left\{ \begin{array}{ll} 0 & \text{if } y \neq x, \\ \mathsf{s}_{(i,y)} & \text{if } y = x. \end{array} \right\} = \mathsf{s}_{(i,y)} 1_{x^{\mathsf{s}_i}}.$

3. $1_x \mathsf{u}_{(i,z,w)} = \begin{cases} 0 & \text{if } z \neq x, \\ \mathsf{u}_{(i,z,w)} & \text{if } z = x. \end{cases}$

4. $\mathsf{u}_{(i,z,w)}1_x = \begin{cases} 0 & \textit{if } \; w \neq x, \\ \mathsf{u}_{(i,z,w)} & \textit{if } \; w = x. \end{cases}$

5. For all $l \in \underline{n-1}$, $\mathsf{s}_{(i,u)}\mathsf{s}_{(l,y)} = \begin{cases} 0 & \textit{if } \; y \neq u^{\mathsf{s}_i}, \\ 1_u & \textit{if } \; y = u^{\mathsf{s}_i}, i = l, \\ \mathsf{s}_{(l,u)}\mathsf{s}_{(i,u^{\mathsf{s}_l})} & \textit{if } \; y = u^{\mathsf{s}_i}, l \neq i \pm 1. \end{cases}$

6. $\mathsf{s}_{(k,y)}\mathsf{s}_{(k+1,y^{\mathsf{s}_k})}\mathsf{s}_{(k,y^{\mathsf{s}_k\mathsf{s}_{k+1}})} = \mathsf{s}_{(k+1,y)}\mathsf{s}_{(k,y^{\mathsf{s}_{k+1}})}\mathsf{s}_{(k+1,y^{\mathsf{s}_{k+1}\mathsf{s}_k})}$ *for all* $k \in \underline{n-2}$.

7. $\mathsf{u}_{(i,u,v)}\mathsf{u}_{(j,w,z)} = \begin{cases} 0 & \textit{if } \; w \neq v, \\ \delta_{v_i}\mathsf{u}_{(i,u,z)} & \textit{if } \; i = j, \; w = v, \\ \mathsf{u}_{(j,u,v)}\mathsf{u}_{(i,w,z)} & \textit{if } \; w = v, j \neq i \pm 1. \end{cases}$

8. $\mathsf{u}_{(k,u,v)}\mathsf{u}_{(k+1,v,z)}\mathsf{u}_{(k,z,w)} = \mathsf{u}_{(k,u,w)}$.

9. $\mathsf{u}_{(k+1,u,v)}\mathsf{u}_{(k,v,z)}\mathsf{u}_{(k+1,z,w)} = \mathsf{u}_{(k+1,u,w)}$.

10. $\mathsf{s}_{(i,y)}\mathsf{u}_{(l,w,z)} = \begin{cases} 0 & \textit{if } \; i = l, \\ 0 & \textit{if } \; i \neq l, w \neq y^{\mathsf{s}_i}, \\ \mathsf{u}_{(l,y,z^{\mathsf{s}_i})}\mathsf{s}_{(i,z^{\mathsf{s}_i})} & \textit{if } \; w = y^{\mathsf{s}_i}, l \neq i, i \pm 1. \end{cases}$

11. $\mathsf{u}_{(l,w,z)}\mathsf{s}_{(i,y)} = \begin{cases} 0 & \textit{if } \; i = l, \\ 0 & \textit{if } \; i \neq l, z \neq y, \\ \mathsf{s}_{(i,w)}\mathsf{u}_{(l,w^{\mathsf{s}_i},z^{\mathsf{s}_i})} & \textit{if } \; z = y, l \neq i, i \pm 1. \end{cases}$

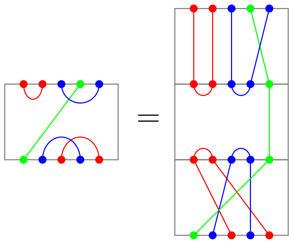
12. $\mathsf{s}_{(k,y)}\mathsf{u}_{(k+1,y^{\mathsf{s}_k},u)}\mathsf{u}_{(k,u,v)} = \mathsf{s}_{(k+1,y)}\mathsf{u}_{(k,y^{\mathsf{s}_{k+1}},v)}$.

13. $\mathsf{s}_{(k+1,y)}\mathsf{u}_{(k,y^{\mathsf{s}_{k+1}},u)}\mathsf{u}_{(k+1,u,v)} = \mathsf{s}_{(k,y)}\mathsf{u}_{(k+1,y^{\mathsf{s}_k},v)}$.

Proof. We need first to check that these elements generate our algebra. As we said before, if we ignore the colours in any diagram D in the set $\mathcal{T}_{n,m}$ we obtain a Brauer diagram which can be written as word of the elements u_i , s_j, see Theorem 1.24. We may recolour this factorization (Note that we may not recolour all the decompositions) as in (2.19) to obtain a decomposition of the diagram D in the elements $\mathsf{u}_{(i,u,v)}$ and $\mathsf{s}_{(j,x)}$.

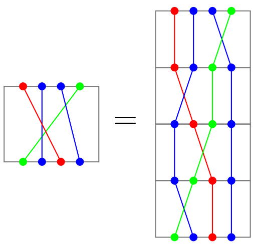
Let $D \in \mathcal{T}_{n,m}$, then it can be written on the form $\sigma D'\theta$ (see Lemma 2.21) where $D' \in \mathcal{A}^*_{n,m}$, σ is the unique diagram with n propagating lines and $\mathrm{top}(D) = \mathrm{top}(\sigma)$

and bot(σ) is on the form $\big(\{1,\ldots,l_1\},\{l_1+1,\ldots,l_2\},\ldots,\{l_{m-1}+1,\ldots,n\}\big)$ for some non-negative integers $l_1,\ldots,l_{m-1}$, and θ is the unique diagram with n propagating lines that rearrange the nodes in the bottom row to have a diagram whose top equals $\big(\{1,\ldots,l'_1\},\{l'_1+1,\ldots,l'_2\},\ldots,\{l'_{m-1}+1,\ldots,n\}\big)$ for some non-negative integers $l'_1,\ldots,l'_{m-1}$ (note that both σ and θ are defined such that there are not any crossing lines whose the same colour). For example, see the following figure.



Any decomposition of the uncoloured image of D' in the elements u_i (which is existed since the uncoloured image is a Temperley-Lieb diagram) can be recoloured to have a decomposition of D'. We still need to show that σ and θ can be written as words in $\mathsf{s}_{(j,x)}$, it is enough to show one of them.

Let θ be a diagram whose n propagating lines and top equals $\big(\{1,\ldots,l_1\},\{l_1+1,\ldots,l_2\},\ldots,\{l_{m-1}+1,\ldots,n\}\big)$ for some non-negative integers $l_1,\ldots,l_{m-1}$. To obtain a decomposition of θ, we begin by checking the node that is connected to n'. If $\{n,n'\}$ is a part of θ, we go to the next node. Otherwise, If $\{h,n'\}$ is a part of θ (note that h will be one of the nodes $l_1,\ldots,l_{m-1}$ since there is no crossed edges that have the same colour) we move the node h step by step until we connect it to the node n'. After that we check the node that is connected to $(n-1)'$ and do what we did with h. Since we start by moving last node of each colour and then move the next node, there will be no crossed edges that have the same colour. For example, see the following figure. Hence we have a decomposition of θ in the elements $\mathsf{s}_{(j,x)}$.



The elements $1_x, \mathsf{u}_{(i,u,v)}$ and $\mathsf{s}_{(i,y)}$ clearly satisfy the previous relations, it can be verified by drawing the diagram products that they refer to, see figure 2.8 for example. Furthermore, any product of elements in $\mathbb{T}_{n,m}$ can be computed by using these relations, since they contain in somewhat all the relations in Theorem 1.24, which define the Brauer algebra $\mathbb{B}_n(\delta)$ although it is not immediately obvious, for example the relation $\mathsf{u}_i^2 = \delta \mathsf{u}_i$ in $\mathbb{B}_n(\delta)$ corresponds to the relation

$$\mathsf{u}_{(i,u,v)}\mathsf{u}_{(i,w,z)} = \begin{cases} 0 & \textit{if } \ w \neq v, \\ \delta_{v_i}\mathsf{u}_{(i,u,z)} & \text{if } \ w = v, \end{cases}$$

in the algebra $\mathbb{T}_{n,m}$, and the relations in Theorem 1.24 are enough to compute any product in the algebra $\mathbb{B}_n(\delta)$, and we are done. □

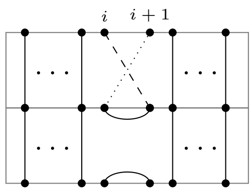


FIGURE 2.8: The proof of $\mathsf{s}_{(i,y)}\mathsf{u}_{(i,w,z)} = 0$ since $y_i \neq y_{i+1}$.

2.6 Some useful idempotent sub-algebras

In this section we will discuss certain special idempotent elements in the algebras $\mathbb{T}_{n,m}(\breve{\delta})$, $\mathbb{P}_{n,m}(\breve{\delta})$ and $\mathbb{F}\mathfrak{S}_{n,m}$.

The diagrams of shape $id \in \mathfrak{S}_n$ are orthogonal idempotents, since

$$1_x 1_y = \begin{cases} 0 & \text{if } \ y \neq x, \\ 1_x & \text{if } \ y = x, \end{cases}$$

for all $x, y \in \mathbb{Z}_m^n$. Thus we have a decomposition of the identity as a sum of orthogonal idempotents since $1_{\mathbb{P}_{n,m}} = 1_{\mathbb{T}_{n,m}} = 1_{F\mathfrak{S}_{n,m}} = \sum_{x \in \mathbb{Z}_m^n} 1_x$.

For each $\lambda = (\lambda_0, \ldots, \lambda_{m-1}) \in \Gamma_{(n,m)}$, $\Gamma_{(n,m)}$ is the set of all m-compositions of n (see Section 1.3.1), we define $\underline{\underline{\lambda}} \in \mathbb{Z}_m^n$ to be

$$\underline{\underline{\lambda}} = (\underbrace{0, \ldots, 0}_{\lambda_0-\text{times}}, \underbrace{1, \ldots, 1}_{\lambda_1-\text{times}}, \ldots, \underbrace{m-1, \ldots, m-1}_{\lambda_{m-1}-\text{times}}). \tag{2.26}$$

Theorem 2.33. *Let $\#_i(1_y) = \lambda_i$ for each i where $y \in \mathbb{Z}_m^n$, then the elements 1_y and $1_{\underline{\underline{\lambda}}}$ are conjugate in the algebras $\mathbb{T}_{n,m}$, $\mathbb{F}\mathfrak{S}_{n,m}$ and $\mathbb{P}_{n,m}$.*

Proof. Note that $\#(1_y) = n$ for any $y \in \mathbb{Z}_m^n$, so $\lambda := (\lambda_0, \ldots, \lambda_{m-1}) \in \Gamma_{(n,m)}$, which means the tuple $\underline{\underline{\lambda}}$ is defined.

First it will be shown in the algebra $\mathbb{T}_{n,m}$. We need to define an invertible element $D \in \mathbb{T}_{n,m}$ such that $D^{-1} 1_y D = 1_{\underline{\underline{\lambda}}}$. We claim that the element

$$\theta^y + \sum_{\substack{u \in \mathbb{Z}_m^n, \\ u \neq y}} 1_u$$

satisfies the previous equation, where θ^y is the coloured image of a permutation θ with top equals $\tilde{y}$, and θ will be defined later to be a specific permutation that changes the order of nodes to obtain $\underline{\underline{\lambda}}$ from y (such that there are no crossing lines whose the same colour in θ^y).

Let's define the map $\theta \in \mathfrak{S}_n$ as follows: Assume that $i \in \underline{n}$ and $y_i = j \in \mathbb{Z}_m$, and define $\theta(i)$ to be $\nu_{i,j} + \sum\limits_{k<j} \lambda_k$, where $\nu_{i,j}$ be the number of integers $l \in \underline{n}$ that are strictly smaller than i and $y_l = j$.

We are going to show that $\theta \in \mathfrak{S}_n$, by proving that θ is an injective map. It is obvious that θ is well-defined. Assume that $i_1, i_2 \in \underline{n}$ without loss of generality we can say that $i_1 < i_2$. Now there are two possibilities: $y_{i_1} = y_{i_2} = j$ or $y_{i_1} = j_1 \neq j_2 = y_{i_2}$. If $y_{i_1} = y_{i_2}$, then $\nu_{i_1,j} < \nu_{i_2,j}$ so $\theta(i_1) < \theta(i_2)$. On the other side $y_{i_1} \neq y_{i_2}$, then if $j_1 < j_2$, so $\theta(i_1) = \nu_{i_1,j_1} + \sum\limits_{k<j_1} \lambda_k \leq \sum\limits_{k<j_1+1} \lambda_k < \theta(i_2)$. Similarly, if $j_2 < j_1$, thus $\theta(i_2) < \theta(i_1)$. Therefore θ is injective.

From the way that we define θ, it is evident that $\theta^y \in \mathbb{T}_{n,m}$ since if $y_i = y_j$ where $i < j$, so $\theta(i) < \theta(j)$ this implies that there is no crossing lines with the same colour (note there will be a crossing lines of the same colour if and only if there exist $y_i = y_j$ for some $i < j$ and $\theta(i) > \theta(j)$). Similarly, the diagram $\left(\theta^{-1}\right)_y$, the coloured image of θ^{-1} with bottom equals $\tilde{y}$, is contained in $\mathbb{T}_{n,m}$ because by flipping the diagram $\left(\theta^{-1}\right)_y$ we obtain θ^y. Also note that $\text{bot}(\theta^y) = \widetilde{\underline{\underline{\lambda}}}$.

Finally, take $D = \theta^y + \sum_{\substack{u\in\mathbb{Z}^n_m,\\ u\neq y}} 1_u$ and $D' = \left(\theta^{-1}\right)_y + \sum_{\substack{u\in\mathbb{Z}^n_m,\\ u\neq \underline{\underline{\lambda}}}} 1_u$. Note that $D, D' \in \mathbb{T}_{n,m}$, $DD' = 1_{\mathbb{T}_{n,m}} = D'D$ and $D1_yD' = 1_{\underline{\underline{\lambda}}}$.

The element D is also contained in $\mathbb{F}\mathfrak{S}_{n,m}$ and in $\mathbb{P}_{n,m}$, so the elements 1_y and $1_{\underline{\underline{\lambda}}}$ are conjugate in both of them. □

The next theorem is proved in the same fashion which Jegan has followed in Theorem 3.1.4 in [28], which says:

$$1_{\underline{\underline{\lambda}}}\mathbb{T}_{n,m}(\delta_0, \ldots, \delta_{m-1})1_{\underline{\underline{\lambda}}} \cong \mathsf{TL}_{\lambda_0}(\delta_0) \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathsf{TL}_{\lambda_{m-1}}(\delta_{m-1}). \tag{2.27}$$

Theorem 2.34. *Let* $\lambda = (\lambda_0, \ldots, \lambda_{m-1}) \in \Gamma_{(n,m)}$*, then*

$$1_{\underline{\underline{\lambda}}}\mathbb{P}_{n,m}(\delta_0, \ldots, \delta_{m-1})1_{\underline{\underline{\lambda}}} \cong \mathbb{P}_{\lambda_0}(\delta_0) \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathbb{P}_{\lambda_{m-1}}(\delta_{m-1}), \tag{2.28}$$

$$1_{\underline{\underline{\lambda}}}\mathbb{F}\mathfrak{S}_{n,m}1_{\underline{\underline{\lambda}}} \cong \mathbb{F}\mathfrak{S}_{\lambda_0} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathbb{F}\mathfrak{S}_{\lambda_{m-1}}. \tag{2.29}$$

Proof. We will prove the first part, and the proof of the second one is similar. Let $\mathbb{A} = \mathbb{P}_{n,m}(\breve{\delta})$ and $\mathsf{e} = 1_{\underline{\underline{\lambda}}}$. For any diagram $d \in \mathsf{e}\mathbb{A}\mathsf{e}$, top$(d)$ and bot(d) are equal to $\left(\{1, \ldots, \lambda_0\}, \{\lambda_0 + 1, \ldots, \lambda_1 + \lambda_0\}, \ldots, \{\sum_{i=0}^{m-2} \lambda_i + 1, \ldots, n\}\right) = \widetilde{\underline{\underline{\lambda}}}$, because all of the other elements of $\mathbb{A}$ will be killed by e.

Define a linear map $\psi : \mathsf{e}\mathbb{A}\mathsf{e} \to \bigotimes_{i=0}^{m-1} \mathbb{P}_{\lambda_i}(\delta_i)$ as follows: let $a \in \mathbb{A}$ and $\mathsf{e}a\mathsf{e} \neq 0$. By ignoring all nodes and edges that do not have the colour $\mathfrak{C}_j$, we will obtain a partition D_j of the set $X_j \cup X'_j$, where $X_j = \{\sum_{h=0}^{j-1} \lambda_h + 1, \ldots, \sum_{h=0}^{j} \lambda_h\}$. By replacing $\sum_{h=0}^{j-1} \lambda_h + k$ by k in the partition D_j, we obtain a partition of the set $\underline{\lambda_j} \cup \underline{\lambda_j}'$, say D'_j, where $\underline{\lambda_j} = \{1, \ldots, \lambda_j\}$. Thus we have $D'_j \in \mathbb{P}_{\lambda_j}(\delta_j)$ for each $j \in \mathbb{Z}_m$. Let's define $\psi(\mathsf{e}a\mathsf{e})$ to be

$$\psi(\mathsf{e}a\mathsf{e}) = D'_0 \otimes D'_1 \otimes \cdots \otimes D'_{m-1}.$$

We will show that ψ is an algebra homomorphism.

Let $\mathsf{e}a_1\mathsf{e}, \mathsf{e}a_2\mathsf{e} \in \mathsf{e}\mathbb{A}\mathsf{e}$ for some $a_1, a_2 \in \mathbb{A}$. If $\mathsf{e}a_i\mathsf{e} \neq 0$ for $i = 1, 2$, we have $(\mathsf{e}a_1\mathsf{e})(\mathsf{e}a_2\mathsf{e}) = \mathsf{e}a_1\mathsf{e}a_2\mathsf{e} = \mathsf{e}a_1a_2\mathsf{e}$, so

$$\begin{aligned}\psi(\mathsf{e}a_1\mathsf{e})\psi(\mathsf{e}a_2\mathsf{e}) =&(D'_0 \otimes \cdots \otimes D'_{m-1})(B'_0 \otimes \cdots \otimes B'_{m-1})\\ =&D'_0B'_0 \otimes \cdots \otimes D'_{m-1}B'_{m-1} = \psi(\mathsf{e}a_1a_2\mathsf{e}) = \psi((\mathsf{e}a_1\mathsf{e})(\mathsf{e}a_2\mathsf{e})).\end{aligned}$$

Checking the other axioms of an algebra homomorphism is easy. This implies that ψ is an algebra homomorphism. Also, ψ is injective and surjective by the way it is defined. Therefore, ψ is an algebra isomorphism. □

Chapter 3

The Multi-Colour Symmetric Groupoid Algebra

In this chapter we find an isomorphism between the algebra $\mathbb{F}\mathfrak{S}_{n,m}$ and a finite direct sum of cellular algebras and we use this to determine the complete set of non-isomorphic simple $\mathbb{F}\mathfrak{S}_{n,m}$-modules, which is the goal of this chapter. We will use this to find an index set of all simple modules of the algebra $\mathbb{P}_{n,m}(\breve{\delta})$. In Section 3.3, we show that the generalized symmetric group algebra is isomorphic to the algebra $\mathbb{F}\mathfrak{S}_{n,m}$ when m is invertible and $\mathbb{F}$ is an algebraically closed field.

3.1 The multi-colour symmetric groupoid

In this section we show that the set $\mathfrak{S}_{n,m} = \{d \in \mathcal{P}_{n,m} \mid \#d = n\}$ is a groupoid, see section 1.2, and view its elements as m-tuples of permutations.

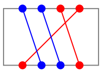
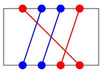
The set $\mathfrak{S}_{n,2}$ considered as morphisms is a groupoid with a set of objects $\{(A, A^c) \mid A \subseteq \underline{n}\}$ and the maps:

$$s(d) = \text{top}(d), \qquad t(d) = \text{bot}(d)\ , \qquad id((A, A^c)) = 1_{(A,A^c)},$$

$$(d)^{-1} = d^*, \qquad d_1 \bullet d_2 = d_1 \circ d_2,$$

where $\circ$ is the multiplication on $\mathcal{P}_{n,2}$ and d^* corresponds to the reflecting of d in the horizontal axis passing through the middle of the diagram d. Note that

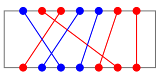
$$d \circ d^* = 1_{\text{bot}(d)}, \qquad d^* \circ d = 1_{\text{top}(d)}.$$

For example, the diagram  is a morphism from $\big(\{3,4\},\ \{1,2\}\big)$ to $\big(\{1,4\},\{2,3\}\big)$, and its inverse is  .

In general, the set $\mathfrak{S}_{n,m}$ is a subcategory of $\mathcal{P}_{n,m}$, see Remark 2.6, and each element in $\mathfrak{S}_{n,m}$ is invertible, so $\mathfrak{S}_{n,m}$ is a groupoid and it is called the multi-colour symmetric groupoid.

Diagrams in the multi-colour symmetric groupoid $\mathfrak{S}_{n,m}$ are constructed by colouring the diagrams in the symmetric group $\mathfrak{S}_n$. Each permutation in $\mathfrak{S}_n$ forms m^n diagrams in $\mathfrak{S}_{n,m}$ since each element in $\mathfrak{S}_n$ has n blocks, so $|\mathfrak{S}_{n,m}| = m^n n!$ where $n, m \in \mathbb{Z}^+$.

A partition in the partition algebra $\mathbb{P}_n(\delta)$ with n propagating lines can be viewed as a permutation. Similarly, there is another way to describe the diagrams that represent multi-colour partitions in $\mathfrak{S}_{n,2}$ as ordered pairs of bijective functions with the union of their domains and the union of their codomains are equal to $\underline{n}$. Also the intersection of their domains and the intersection of their codomains are empty sets. Let $d = (d_0, d_1) \in \mathfrak{S}_{n,2}$ where $top(d_0) = A_1$ and $bot(d_0) = A_2$. We can consider the partitions d_0, d_1 as bijective maps $d_0 : A_1 \to A_2$ and $d_1 : A_1^c \to A_2^c$. We construct these maps as follows: For all $i \in \underline{n}$ which is connected to $j' \in \underline{n'}$ in the partition d_k, then $d_k(i) = j$ where $k = 0, 1$.

Example 3.0.1. *Consider the diagram* $d =$ . *The bijective functions which are related to* d *are defined by*

$$d_0(2) = 6,\ \ d_0(3) = 1,\ \ d_0(6) = 5,\ \ d_0(7) = 7,$$
$$d_1(1) = 3,\ \ d_1(4) = 2,\ \ d_1(5) = 4.$$

If the objects (A_1, A_1^c) and (A_2, A_2^c) are connected (there is a morphism between them) in the groupoid $\mathfrak{S}_{n,2}$, then $|A_1| = |A_2|$ since drawing a line from the top row to the bottom row needs two nodes, one on the top and the other on the bottom, so it's evident that the number of red (blue) nodes on the top row is equal to the number of red (blue) nodes on the bottom row.

Similarly, the groupoid $\mathfrak{S}_{n,m}$ can be defined to be the set of all tuples $(f_0, \ldots, f_{m-1})$ where $f_i : A_i \to B_i$ is a bijective map for all $i = 0, \ldots, m-1$ such that $\{A_j\}$ and $\{B_j\}$ are partitions of the set $\underline{n}$. Note that objects $(A_0, \ldots, A_{m-1})$ and $(B_0, \ldots, B_{m-1})$ are connected in $\mathfrak{S}_{n,m}$ if and only if $|A_j| = |B_j|$ for all $0 \leq j \leq m-1$.

We will use the same definition of the *type* of an element as that used in [46].

Definition 3.1. For a diagram $d \in \mathfrak{S}_{n,m}$ the *type* of d is defined to be

$$\text{type}(d) := (\#_0(d), \#_1(d), \ldots, \#_{m-1}(d)) \ . \tag{3.1}$$

As $\sum_{j=0}^{m-1} \#_j(d) = n$ and $\#_j(d) \geq 0$ for all j, so type(d) is an m-composition of n. Actually, we can define the set $\Gamma_{(n,m)}$, the set of all m-compositions of n, to be the set of all different types of $\mathfrak{S}_{n,m}$-diagrams.

Definition 3.2. Let $\lambda \in \Gamma_{(n,m)}$, the set $\mathfrak{S}_{\lambda,m}$ is the sub-groupoid of $\mathfrak{S}_{n,m}$ that contains all diagrams of type λ.

The product of two diagrams in $\mathfrak{S}_{n,m}$ will be zero if they have different types. From this we obtain

$$\mathbb{F}\mathfrak{S}_{n,m} = \bigoplus_{\lambda \in \Gamma_{(n,m)}} \mathbb{F}\mathfrak{S}_{\lambda,m}. \tag{3.2}$$

Note that the identity of the algebra $\mathbb{F}\mathfrak{S}_{\lambda,m}$ is the sum of all the coloured images of shape $id \in \mathfrak{S}_n$ of type λ.

3.2 The algebra $\mathbb{F}\mathfrak{S}_{n,m}$ is a cellular algebra

In this section we study the structure of the algebra $\mathbb{F}\mathfrak{S}_{n,m}$, we show that the algebra $\mathbb{F}\mathfrak{S}_{n,m}$ decomposes as a finite direct sum of cellular algebras.

Theorem 3.3. *Let n and m be positive integers and $\lambda \in \Gamma_{(n,m)}$, then*

$$\mathbb{F}\mathfrak{S}_{\lambda,m} \cong \mathbb{F}\big(\prod_{i=0}^{m-1} \mathfrak{S}_{\lambda_i}\big) \otimes_{\mathbb{F}} M_{n_\lambda}(\mathbb{F}), \tag{3.3}$$

where $n_\lambda := \binom{n}{\lambda_0, \ldots, \lambda_{m-1}}$ and we put $\mathfrak{S}_0 = \mathfrak{S}_1$.

Proof. To prove that, we will use Theorem 1.5 since $\mathfrak{S}_{\lambda,m}$ is a connected groupoid. First, we will show that $G \cong \prod_{i=0}^{m-1} \mathfrak{S}_{\lambda_i}$, where G is defined by (1.2) and x_0 is the object $\underline{\underline{\widetilde{\lambda}}}$, see equations (2.17) and (2.26). Since the multiplication on the groups G and $\prod_{i=0}^{m-1} \mathfrak{S}_{\lambda_i}$ have the same rules, all we need to do is define a bijection between them. Let $\pi \in G$, from the previous section π can be viewed as a tuples $(\pi_0, \ldots, \pi_{m-1})$ where π_i is a permutation on the set $\{\sum_{j=0}^{i-1} \lambda_j + 1, \ldots, \sum_{j=0}^{i} \lambda_j\}$. So $\pi_j \in \mathfrak{S}_{\lambda_j}$ for each i, thus the isotropy group G is isomorphic to $\prod_{i=0}^{m-1} \mathfrak{S}_{\lambda_i}$.

Finally, we need to compute the cardinality of the object set of $\mathfrak{S}_{\lambda,m}$, say l, which is equal to the number of ways we can partition the set $\underline{n}$ into m blocks $\{X_0, \ldots, X_{m-1}\}$ where $|X_i| = \lambda_i$ for all i. We choose λ_0 elements from n, so we have $\binom{n}{\lambda_0}$ choices. Then, there are $n - \lambda_0$ elements remaining and we need to choose λ_1 of them, so we have $\binom{n-\lambda_0}{\lambda_1}$. By iteration, we have

$$l = \binom{n}{\lambda_0}\binom{n-\lambda_0}{\lambda_1} \cdots \binom{n - \sum_{i=0}^{m-3} \lambda_i}{\lambda_{m-2}} = n_\lambda.$$

Substituting all previous details into the isomorphism in Theorem 1.5, we obtain

$$\mathbb{F}\mathfrak{S}_{\lambda,m} \cong \mathbb{F}\big(\prod_{i=0}^{m-1} \mathfrak{S}_{\lambda_i}\big) \otimes_{\mathbb{F}} M_{n_\lambda}(\mathbb{F}). \ \square$$

Corollary 3.4. *Let that n and m be positive integers, then*

$$\mathbb{F}\mathfrak{S}_{n,m} \cong \bigoplus_{\lambda \in \Gamma_{(n,m)}} \Big(\mathbb{F}\big(\prod_{i=0}^{m-1} \mathfrak{S}_{\lambda_i} \big) \otimes_{\mathbb{F}} M_{n_\lambda}(\mathbb{F}) \Big), \tag{3.4}$$

where $n_\lambda = \binom{n}{\lambda_0, \ldots, \lambda_{m-1}}$.

Proof. It comes directly from the equation (3.2) and the previous theorem. □

Define the sub-groupoid $\widehat{\mathfrak{S}}_{n,m}$ to be

$$\widehat{\mathfrak{S}}_{n,m} := \mathfrak{S}_{n,m} \cap \mathcal{A}_{n,m}, \tag{3.5}$$

where the set $\mathcal{A}_{n,m}$ is defined by (2.11). So crossing two edges having the same colour is not allowed in $\widehat{\mathfrak{S}}_{n,m}$. Easily it can be shown that $\widehat{\mathfrak{S}}_{n,m}$ is a groupoid.

Theorem 3.5. *Let that n and m be positive integers. Then*

$$\mathbb{F}\widehat{\mathfrak{S}}_{n,m} \cong \bigoplus_{\lambda \in \Gamma_{(n,m)}} M_{n_\lambda}(\mathbb{F}).$$

Proof. From (3.2), we have

$$\mathbb{F}\widehat{\mathfrak{S}}_{n,m} = \bigoplus_{\lambda \in \Gamma_{(n,m)}} \mathbb{F}(\mathfrak{S}_{\lambda,m} \cap \mathcal{A}_{n,m}).$$

Now $\mathfrak{S}_{\lambda,m} \cap \mathcal{A}_{n,m}$ is a groupoid. Note that there is only one morphism from any object to itself in $\mathfrak{S}_{\lambda,m} \cap \mathcal{A}_{n,m}$ since the crossing is not allowed. The cardinality of the set of objects is n_λ, so we are done after substituting into Theorem 1.5. □

As a consequence of the previous theorems and some properties of cellular algebras, we have the following fact.

Corollary 3.6. *The groupoid algebras $\mathbb{F}\mathfrak{S}_{n,m}$, $\mathbb{F}\mathfrak{S}_{\lambda,m}$ and $\mathbb{F}\widehat{\mathfrak{S}}_{n,m}$ are all cellular algebras, where $\lambda \in \Gamma_{(n,m)}$.*

Proof. All the summands in the decomposition of the algebra $\mathbb{F}\mathfrak{S}_{\lambda,m}$ in Theorem 3.3 are cellular, so the algebra $\mathbb{F}\mathfrak{S}_{n,m}$ is also a cellular algebra by Proposition 1.16 and its cell modules have the form $\mathscr{S}_{\boldsymbol{\mu}_0} \otimes \cdots \otimes \mathscr{S}_{\boldsymbol{\mu}_{m-1}} \otimes \mathbb{F}^{n_\lambda}$ where $\boldsymbol{\mu} = (\boldsymbol{\mu}_0, \ldots, \boldsymbol{\mu}_{m-1})$ is a multi-partition of type λ and $\mathscr{S}_{\boldsymbol{\mu}_i}$ is the Specht module of the symmetric group algebra $\mathbb{F}\mathfrak{S}_{\lambda_i}$ corresponding to the partition $\boldsymbol{\mu}_i$, see Section 1.3.2. Similarly, the algebras $\mathbb{F}\mathfrak{S}_{n,m}$ and $\mathbb{F}\widehat{\mathfrak{S}}_{n,m}$ are cellular. □

From the isomorphism (3.4), we obtain an index set of all cell modules of the algebra $\mathbb{F}\mathfrak{S}_{n,m}$, which is

$$\Lambda_{\mathfrak{S}_{n,m}} := \{(\lambda, \boldsymbol{\mu}) \mid \lambda \in \Gamma_{(n,m)},\ \boldsymbol{\mu} \text{ is a multi-partition of } \lambda\}. \tag{3.6}$$

3.3 The relations between the algebras $\mathbb{F}\mathfrak{S}_{n,m}$ and $\mathbb{F}\mathbb{Z}_m \wr \mathfrak{S}_n$

The aim of this section is to show that the generalized symmetric group algebra $\mathbb{F}\mathbb{Z}_m \wr \mathfrak{S}_n$ is isomorphic to the algebra $\mathbb{F}\mathfrak{S}_{n,m}$ when m is invertible in $\mathbb{F}$ and $\mathbb{F}$ is algebraically closed.

Before proving the main theorem, we need to state some propositions and lemmas.

Proposition 3.7. *Let Θ and Ω be white diagrams in $\mathbb{F}\mathfrak{S}_{n,m}$ of shapes θ and ω, respectively. Then $\Theta\Omega$ is the white diagram of the shape $\theta\omega \in \mathfrak{S}_n$.*

Proof. Recall that a white diagram of shape (a diagram) d is the sum of all possible coloured images of this diagram d. Any coloured image of a permutation θ can be identified by its top, where if the node $i \in \underline{n}$ is, for example, a red node on the top row, then the edge which has the nodes i on the top row and $\theta(i)$ on the bottom row as end-points, will also be red. Similarly, we can determine a coloured image of θ by its bottom.

Let θ^A be the coloured image of θ and ω^B be the coloured image of ω, where $\text{top}(\theta^A) = A$ and $\text{top}(\omega^B) = B$. From the definition of the product on $\mathfrak{S}_{n,m}$, the term $\theta^A\omega^B$ will be zero unless $\text{bot}(\theta^A) = B$. If $\text{bot}(\theta^A) = B$, then the nodes i, $\theta(i)$ and $\omega(\theta(i)) = \theta\omega(i)$ have the same colour and they all are connected. In other words, i and $\theta\omega(i)$ are connected in the diagram $\theta^A\omega^B$ for all $i \in \underline{n}$, hence $\theta^A\omega^B$ has the shape $\theta\omega$. Also $\text{top}(\theta^A\omega^B) = A$, so $\theta^A\omega^B = (\theta\omega)^A$.

Now, both of Θ and Ω equal the sum all different coloured copies of θ and ω, respectively. So

$$\Theta\Omega = \sum_A \theta^A \cdot \sum_B \omega^B = \sum_{A,B} \theta^A\omega^B \ ,$$

but $\theta^A\omega^B = 0$ unless $\text{bot}(\theta^A) = B$, and when $\text{bot}(\theta^A) = B$, $\theta^A\omega^B = (\theta\omega)^A$, then

$$\Theta\Omega = \sum_A (\theta\omega)^A.$$

Note that on the right-hand side, it is the white diagram of shape $\theta\omega$, and we are done. □

From the previous proposition, we can see that the product of white diagrams can be computed as products of permutations in the group $\mathfrak{S}_n$.

Definition 3.8. In the algebra $\mathbb{F}\mathfrak{S}_{n,m}$, the element S_i is the white diagram which has the same shape of the element $\mathsf{s}_i \in \mathfrak{S}_n$ where $1 \leq i \leq n-1$.

The elements $S_1, \ldots, S_{n-1}$ satisfy all the relations that the transpositions $\mathsf{s}_1, \ldots,$ s_{n-1} satisfy.

Lemma 3.9. *The following properties are true:*

1. $S_i^2 = 1_{\mathfrak{S}_{n,m}}$ *for all* $1 \leq i < n$.
2. $S_iS_j = S_jS_i$ *if* $|i-j| > 1$.
3. $S_iS_{i+1}S_i = S_{i+1}S_iS_{i+1}$ *for all* $1 \leq i \leq n-2$.

Proof. This follows immediately from Proposition 3.7 and the fact that the elements $\{\mathsf{s}_i\}_{i\in\underline{n-1}}$ satisfy the relations $\mathsf{s}_i^2 = id$ for each $1 \leq i < n$, $\mathsf{s}_i\mathsf{s}_j = \mathsf{s}_j\mathsf{s}_i$ if $|i-j| > 1$ and $\mathsf{s}_i\mathsf{s}_{i+1}\mathsf{s}_i = \mathsf{s}_{i+1}\mathsf{s}_i\mathsf{s}_{i+1}$ for all $1 \leq i \leq n-2$. □

We need to agree on a specific order of both bases of the algebra $\mathbb{F}\mathbb{Z}_m \wr \mathfrak{S}_n$ and the algebra $\mathbb{F}\mathfrak{S}_{n,m}$.

Definition 3.10. [e.g. 4, Section 1.4]**.** The co-lexicographic order on the Cartesian product $\prod_i^n A_i$ of partially ordered sets is defined as

$$(a_1, a_2, \dots, a_n) \leq (b_1, b_2, \dots, b_n) \text{ if and only if } a_i < b_i$$

where i is the greatest number in $\{1, ..., n\}$ for which $a_i \neq b_i$.

For instance, a co-lexicographic order of the set $\mathbb{Z}_m^n$ is given by

$$\overline{0} < e_1 < 2e_1 < \cdots < (m-1)e_1 < e_2 < e_1 + e_2 < \cdots < (m-1)\overline{1}\ ,$$

where $\overline{0} = (0, \dots, 0)$, $\overline{1} = (1, \dots, 1)$, and $e_i = (0, \dots, 1, \dots, 0)$ with 1 at the i^{th} position. This order can be used to define an order of the basis $\{(x; \pi) \mid x \in \mathbb{Z}_m^n, \pi \in \mathfrak{S}_n\}$ of the algebra $\mathbb{F}\mathbb{Z}_m \wr \mathfrak{S}_n$: $(x; \pi) \leq (y; \pi)$ if and only if $x \leq y$ in the set $\mathbb{Z}_m^n$.

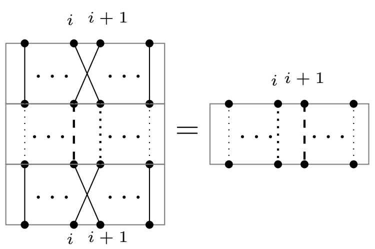
Recall that 1_x is the coloured image of $id \in \mathfrak{S}_n$ whose top equals $\tilde{x}$, where $x \in \mathbb{Z}_m^n$. We say that $1_x < 1_y$ if and only if $x < y$.

Lemma 3.11. *The elements 1_x and S_i satisfy the relation:*

$$S_i 1_x S_i = 1_{x^{\mathsf{s}_i}}\,,$$

where $x^{\mathsf{s}_i} = (x_{\mathsf{s}_i(1)}, \dots, x_{\mathsf{s}_i(n)})$. *In particular,* $S_i 1_{e_j} S_i = 1_{e_{\mathsf{s}_i(j)}}$, $S_i 1_{\overline{0}} S_i = 1_{\overline{0}}$ *and* $S_i 1_{\overline{1}} S_i = 1_{\overline{1}}$.

Proof. From the definition of S_i we have $S_i = \sum_{y \in \mathbb{Z}_m^n} \mathsf{s}_{(i,y)}$. Now we can show that by using either the relations in Theorem 2.32 or the visualization of the product $S_i 1_x S_i$:



,

we obtain that in the product $S_i 1_x S_i$ all that happens is that the order of the edges in the position i and $i+1$ changes, which is the same as $1_{x^{\mathsf{s}_i}}$. □

After proving the next theorem, we found that our next theorem is the same as Theorem 16 in [46] but we proved it independently albeit in a similar fashion.

Theorem 3.12. *The algebras* $\mathbb{F}\mathbb{Z}_m \wr \mathfrak{S}_n$ *and* $\mathbb{F}\mathfrak{S}_{n,m}$ *are isomorphic if* $\mathbb{F}$ *is algebraically closed and* $\gcd(m, \mathrm{Char}(\mathbb{F})) = 1$.

Proof. Let ω be a primitive m th root of the unity (ω exists since $\mathbb{F}$ is an algebraically closed field). From Proposition 1.7, the set $X := \{\overline{e}_1, \ldots, \overline{e}_n, \bar{\mathsf{s}}_1, \ldots, \bar{\mathsf{s}}_{n-1}\}$ generates $\mathbb{F}\mathbb{Z}_m \wr \mathfrak{S}_n$, where $\overline{e}_i = (e_i; id)$ and $\bar{\mathsf{s}}_i = (\overline{0}, \mathsf{s}_i)$. Define the map $f : X \to \mathbb{F}\mathfrak{S}_{n,m}$ by

$$\overline{e}_i \mapsto \sum_{x \in \mathbb{Z}_m^n} \omega^{x_i} 1_x \ , \tag{3.7}$$

$$\bar{\mathsf{s}}_i \mapsto S_i \ .$$

To check that this defines an algebra homomorphism, we need to show that the relations in Proposition 1.7 hold. We already proved the last three relations in Lemma 3.9, so we need just to prove the first four relations:

$$f(\overline{e}_i)^m = \Big(\sum_{x \in \mathbb{Z}_m^n} \omega^{x_i} 1_x \Big)^m = \sum_{x \in \mathbb{Z}_m^n} \omega^{x_i m} 1_x \ ,$$

since $1_x^2 = 1_x$ and $1_x 1_y = 0$ for all $x \neq y$. But $\omega^m = 1$, so

$$f(\overline{e}_i)^m = \sum_{x \in \mathbb{Z}_m^n} (\omega^m)^{x_i} 1_x = \sum_{x \in \mathbb{Z}_m^n} 1_x = 1_{\mathbb{F}\mathfrak{S}_{n,m}} \ .$$

So the first relation in Proposition 1.7 holds. Also, $f(\overline{e}_i) f(\overline{e}_j) = f(\overline{e}_j) f(\overline{e}_i)$, since $1_x 1_y = 1_y 1_x$ and the scalar product is distributive.

Now, from Lemma 3.11, we have $S_i 1_x S_i = 1_{x^{\mathsf{s}_i}} =: 1_y$, so

$$f(\bar{\mathsf{s}}_i) f(\overline{e}_i) f(\bar{\mathsf{s}}_i) = \sum_{x \in \mathbb{Z}_m^n} \omega^{x_i} S_i 1_x S_i = \sum_{y \in \mathbb{Z}_m^n} \omega^{x_i} 1_y \ ,$$

and note that $y_{(i+1)} = x_{\mathsf{s}_i(i+1)} = x_i$, thus

$$f(\bar{\mathsf{s}}_i)f(\bar{e}_i)f(\bar{\mathsf{s}}_i) = \sum_{y\in\mathbb{Z}_m^n} \omega^{y_{(i+1)}} 1_y = f(\bar{e}_{i+1}) \ .$$

Let $i \neq j, j+1$, and by rewriting it we have $j \neq i, i-1$, so

$$f(\bar{\mathsf{s}}_j)f(\bar{e}_i)f(\bar{\mathsf{s}}_j) = \sum_{x\in\mathbb{Z}_m^n} \omega^{x_i} 1_{x^{\mathsf{s}_j}} .$$

Set $y = x^{\mathsf{s}_j}$, so that $y_i = x_{\mathsf{s}_j(i)} = x_i$, since $j \neq i, i-1$, and thus

$$f(\bar{\mathsf{s}}_j)f(\bar{e}_i)f(\bar{\mathsf{s}}_j) = \sum_{y\in\mathbb{Z}_m^n} \omega^{y_i} 1_y = f(\bar{e}_i) \ .$$

Thus all the relations in Proposition 1.7 are satisfied. Hence, we have an algebra homomorphism $f : \mathbb{F}\mathbb{Z}_m \wr \mathfrak{S}_n \to \mathbb{F}\mathfrak{S}_{n,m}$ extending the map f.

Let $\theta = \mathsf{s}_{i_1} \cdots \mathsf{s}_{i_m} \in \mathfrak{S}_n$, then from the properties of an algebra homomorphism we have $f((\bar{0};\theta)) = S_{i_1} \cdots S_{i_m} = \Theta$, where Θ is the white diagram of shape θ. So $f((x;\theta)) = f((x;id))\Theta$ for any $x \in \mathbb{Z}_m^n$. Also

$$\begin{aligned}
f((x;id)) &= f((\sum_{i=1}^n x_i e_i; id)) = f(\prod_{i=1}^n (\bar{e}_i)^{x_i}), \\
&= \prod_{i=1}^n \Big(\sum_{y\in\mathbb{Z}_m^n} \omega^{y_i} 1_y \Big)^{x_i}, \qquad (\text{ from equation (3.7)}) \\
&= \sum_{y\in\mathbb{Z}_m^n} \omega^{\sum_{i=1}^n x_i y_i} 1_y, \qquad (\text{ since } 1_y^2 = 1_y,\ 1_y 1_w = 0 \text{ for all } y \neq w). \qquad (3.8)
\end{aligned}$$

Hence,

$$f((x;\theta)) = \sum_{y\in\mathbb{Z}_m^n} \omega^{\sum_{i=1}^n x_i y_i} \theta^y. \qquad (3.9)$$

Let M be the matrix of the homomorphism f with respect to the basis $\{(x;\theta)\}_{\substack{x\in\mathbb{Z}_m^n \\ \theta\in\mathfrak{S}_n}}$ and $\Big(f((x;id))\Big)_{x\in\mathbb{Z}_m^n}$ be the submatrix of M that is obtained by writes $f((x;id))$ as a column and ignores the zero rows, then $\Big(f((x;id))\Big)_{x\in\mathbb{Z}_m^n} = \Big(\omega^{\sum_{i=1}^n x_i y_i}\Big)_{x,y\in\mathbb{Z}_m^n}$ (from equation (3.8)) which is a $m^n \times m^n$ matrix. From equation (3.9) we have

$$M = \bigoplus_{\theta \in \mathfrak{S}_n} \Big(f((x; id)) \Big)_{x \in \mathbb{Z}_m^n} .$$

We are going to show that $\Big(f((x; id)) \Big)_{x \in \mathbb{Z}_m^n} = \mathbf{F}_m^{(n)}$, where $\mathbf{F}_m^{(n)}$ is a tensor product of Vandermonde matrix as defined by equation (1.9) . Since $\mathbf{F}_m^{(n)}$ is invertible when the field $\mathbb{F}$ is algebraically closed and $\gcd(m, \mathrm{Char}(\mathbb{F})) = 1$, and $\dim \mathbb{F}\mathbb{Z}_m \wr \mathfrak{S}_n = \dim \mathbb{F}\mathfrak{S}_{n,m}$, so f is bijective, thus an isomorphism.

To prove that $\Big(f((x; id)) \Big)_{x \in \mathbb{Z}_m^n} = \mathbf{F}_m^{(n)}$, we will use induction on n: it is clear that it is true when $n = 1$. We will assume that

$$\Big(f((x; id)) \Big)_{x \in \mathbb{Z}_m^{n-1}} = \mathbf{F}_m^{(n-1)}$$

and prove the next step. The term $f((x; id))$ can be written in the form:

$$f((x; id)) = \sum_{y \in \mathbb{Z}_m^n} \omega^{\sum_{i=1}^{n} x_i y_i} 1_y = \sum_{\substack{y \in \mathbb{Z}_m^n \\ y < e_n}} \omega^{\sum_{i=1}^{n} x_i y_i} 1_y + \sum_{\substack{y \in \mathbb{Z}_m^n \\ e_n \le y < 2e_n}} \omega^{\sum_{i=1}^{n} x_i y_i} 1_y + \\ + \sum_{\substack{y \in \mathbb{Z}_m^n \\ 2e_n \le y < 3e_n}} \omega^{\sum_{i=1}^{n} x_i y_i} 1_y + \cdots + \sum_{\substack{y \in \mathbb{Z}_m^n \\ (m-1)e_n \le y}} \omega^{\sum_{i=1}^{n} x_i y_i} 1_y .$$

Now if $ke_n \le y < (k+1)e_n$ for some k then $y_n = k$. By substitution into the last equation, we have

$$f((x; id)) = \sum_{\substack{y \in \mathbb{Z}_m^n \\ y < e_n}} \omega^{\sum_{i=1}^{n-1} x_i y_i} 1_y + \omega^{x_n} \sum_{\substack{y \in \mathbb{Z}_m^n \\ e_n \le y < 2e_n}} \omega^{\sum_{i=1}^{n-1} x_i y_i} 1_y + \omega^{2x_n} \sum_{\substack{y \in \mathbb{Z}_m^n \\ 2e_n \le y < 3e_n}} \omega^{\sum_{i=1}^{n-1} x_i y_i} 1_y + \\ + \cdots + \omega^{(m-1)x_n} \sum_{\substack{y \in \mathbb{Z}_m^n \\ (m-1)e_n \le y}} \omega^{\sum_{i=1}^{n-1} x_i y_i} 1_y .$$

Now if $he_n \leq x < (h+1)e_n$ for some h then $x_n = h$ and

$$f((x;id)) = \sum_{\substack{y\in\mathbb{Z}_m^n \\ y<e_n}} \omega^{\sum_{i=1}^{n-1} x_i y_i} 1_y + \omega^h \sum_{\substack{y\in\mathbb{Z}_m^n \\ e_n\leq y<2e_n}} \omega^{\sum_{i=1}^{n-1} x_i y_i} 1_y + \omega^{2h} \sum_{\substack{y\in\mathbb{Z}_m^n \\ 2e_n\leq y<3e_n}} \omega^{\sum_{i=1}^{n-1} x_i y_i} 1_y + \\ + \cdots + \omega^{(m-1)h} \sum_{\substack{y\in\mathbb{Z}_m^n \\ (m-1)e_n\leq y}} \omega^{\sum_{i=1}^{n-1} x_i y_i} 1_y \,.$$

Furthermore, $x = x' + he_n$ where $x' < e_n$ (so we consider x' as an element in $\mathbb{Z}_m^{n-1}$ since $x'_n = 0$), hence

$$f((x;id)) = f((x';id))\big(f(\overline{e}_i)\big)^h = f((x';id))\big(\sum_{\substack{y\in\mathbb{Z}_m^n \\ y<e_n}} 1_y + \omega^h \sum_{\substack{y\in\mathbb{Z}_m^n \\ e_n\leq y<2e_n}} 1_y + \\ + \cdots + \omega^{(m-1)h} \sum_{\substack{y\in\mathbb{Z}_m^n \\ (m-1)e_n\leq y}} 1_y\big) = \sum_{\substack{y\in\mathbb{Z}_m^n \\ y<e_n}} f((x';id))1_y + \\ + \omega^h \sum_{\substack{y\in\mathbb{Z}_m^n \\ e_n\leq y<2e_n}} f((x';id))1_y + \cdots + \omega^{(m-1)h} \sum_{\substack{x\in\mathbb{Z}_m^n \\ (m-1)e_n\leq y}} f((x';id))1_y \\ = \sum_{\substack{y\in\mathbb{Z}_m^n \\ y<e_n}} \omega^{\sum_{i=1}^{n-1} x'_i y_i} 1_y + \omega^h \sum_{\substack{y\in\mathbb{Z}_m^n \\ e_n\leq y<2e_n}} \omega^{\sum_{i=1}^{n-1} x'_i y_i} 1_y + \cdots + \omega^{(m-1)h} \sum_{\substack{y\in\mathbb{Z}_m^n \\ (m-1)e_n\leq y}} \omega^{\sum_{i=1}^{n-1} x'_i y_i} 1_y.$$

By comparing the last two equation, we have $\omega^{\sum_{i=1}^{n-1} x_i y_i} = \omega^{\sum_{i=1}^{n-1} x'_i y_i}$ and $\omega^{\sum_{i=1}^{n-1} x_i y_i} = \omega^{\sum_{i=1}^{n-1} x'_i y'_i}$, where $y = y' + ke_n$ for some k and $y' < e_n$. Next we break the matrix $\Big(f((x;id))\Big)_{x\in\mathbb{Z}_m^n}$ into sections: take the section $he_n \leq x < (h+1)e_n$ and $ke_n \leq y < (k+1)e_n$, so the sub-matrix corresponds this section (from the previous equations) is

$$\big(\omega^{hk}\omega^{\sum_{i=1}^{n-1} x_i y_i}\big)_{\substack{he_n\leq x<(h+1)e_n \\ ke_n\leq y<(k+1)e_n}} = \omega^{hk}\big(\omega^{\sum_{i=1}^{n-1} x_i y_i}\big)_{\substack{he_n\leq x<(h+1)e_n \\ ke_n\leq y<(k+1)e_n}},$$

but

$$\big(\omega^{\sum_{i=1}^{n-1} x_i y_i}\big)_{\substack{he_n\leq x<(h+1)e_n \\ ke_n\leq y<(k+1)e_n}} = \big(\omega^{\sum_{i=1}^{n-1} x'_i y'_i}\big)_{\substack{x'<e_n \\ y'<e_n}} = \Big(f((x;id))\Big)_{x\in\mathbb{Z}_m^{n-1}} = \mathbf{F}_m^{(n-1)}.$$

Hence,

$$\Big(f((x;id))\Big)_{x\in\mathbb{Z}_m^n} = \begin{bmatrix} \mathbf{F}_m^{(n-1)} & \mathbf{F}_m^{(n-1)} & \cdots & \mathbf{F}_m^{(n-1)} \\ \mathbf{F}_m^{(n-1)} & \omega\mathbf{F}_m^{(n-1)} & \cdots & \omega^{m-1}\mathbf{F}_m^{(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{F}_m^{(n-1)} & \omega^{m-1}\mathbf{F}_m^{(n-1)} & \cdots & \omega^{(m-1)^2}\mathbf{F}_m^{(n-1)} \end{bmatrix}, \\ = \mathbf{F}_m^{(n)} .$$

□

The algebras $\mathbb{F}\mathfrak{S}_{n,m}$ and $\mathbb{F}\mathbb{Z}_m \wr \mathfrak{S}_n$ are not isomorphic in general. For example, let $\operatorname{Char}\mathbb{F} = 2$, by Corollary 3.4 the algebra $\mathbb{F}\mathfrak{S}_{1,2}$ is isomorphic to $\bigoplus^{2}\mathbb{F}$, so it is semi-simple. On the other hand, the algebra $\mathbb{F}\mathbb{Z}_2 \wr \mathfrak{S}_1$ is not semi-simple by Maschke's theorem, so these algebras are not isomorphic in this case.

3.4 Representation theory of the algebra $\mathbb{F}\mathfrak{S}_{n,m}$

Irreducible modules of the algebra $\mathbb{F}\mathfrak{S}_{n,m}$ can be studied by determining the ones of the algebra $\mathbb{F}\mathbb{Z}_m \wr \mathfrak{S}_n$ when they are isomorphic, see for example [7], [47] and [51]. But we will use the isomorphism in Corollary 3.4:

$$\mathbb{F}\mathfrak{S}_{n,m} \cong \bigoplus_{\lambda\in\Gamma_{(n,m)}} \Big(\mathbb{F}\big(\prod_{i=0}^{m-1}\mathfrak{S}_{\lambda_i}\big) \otimes_{\mathbb{F}} M_{n_\lambda}(\mathbb{F})\Big).$$

This leads to the next fact.

A field $\mathbb{F}$ is perfect if every irreducible polynomial over $\mathbb{F}$ has distinct roots, see Section 3.4 in [49].

Corollary 3.13. *Let* $\mathbb{F}$ *is a perfect field. Then the multi-colour symmetric groupoid algebra* $\mathbb{F}\mathfrak{S}_{n,m}$ *is semi-simple if and only if* $\operatorname{Char}\mathbb{F}$ *is zero or strictly greater than* n*.*

Proof. It follows from the facts that the algebras $\mathbb{F}\mathfrak{S}_l$ and $M_l(\mathbb{F})$ are cellular algebras and the rest comes by using Proposition 1.17, Maschke's theorem (see for example Theorem 4.1.1 in [16]) and Corollary 3.4. □

Recall that a partition is called p-regular if it does not have p parts of the same size, see Section 1.3.2. A multi-partition $\boldsymbol{\mu} = (\boldsymbol{\mu}_0, \ldots, \boldsymbol{\mu}_{m-1})$ is called p-regular if $\boldsymbol{\mu}_i$ is p-regular for each i.

Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be finite dimensional algebras over $\mathbb{F}$. As it is proved in Section 3.10 in [16], any simple module of $\mathbb{A}_1 \otimes \mathbb{A}_2$ has the form $M_1 \otimes M_2$ where M_i is a simple module of $\mathbb{A}_i$, $i = 1, 2$. Also, $M_k(\mathbb{F}) \otimes M_l(\mathbb{F}) \cong M_{kl}(\mathbb{F})$ for any integers $k, l > 0$, see for example Section 3.10 in [16]. Hence, the simple modules of $\mathbb{F}\mathfrak{S}_{n,m}$ over any field is completely determined by studying the representations for all symmetric group algebras $\mathbb{F}\mathfrak{S}_k$ where $k \leq n$. Using the fact that the set of all p-regular partitions index the set of all simple modules of the symmetric group, see Theorem 1.9, we have the next theorem.

Theorem 3.14. *Let $\mathbb{F}$ be a field of characteristic p, then the non-isomorphic simple modules of the algebra $\mathbb{F}\mathfrak{S}_{n,m}$ are parametrized by*

$$\Lambda^0_{\mathfrak{S}_{n,m}} := \{(\lambda, \boldsymbol{\mu}) \mid \lambda \in \Gamma_{(n,m)}, \boldsymbol{\mu} \text{ is a } \mathsf{p}\text{-regular multi-partition of type } \lambda\}.$$

Proof. This follows from Theorem 1.9 and the preceding description. □

Example 3.14.1. *Let* $\operatorname{Char}\mathbb{F} \neq 2, 3$, *then the group algebras $\mathbb{F}\mathfrak{S}_2$ and $\mathbb{F}\mathfrak{S}_3$ are semi-simple by Maschke's theorem and*

$$\mathbb{F}\mathfrak{S}_2 \cong \mathbb{F} \oplus \mathbb{F}, \qquad \mathbb{F}\mathfrak{S}_3 \cong \mathbb{F} \oplus \mathbb{F} \oplus M_2(\mathbb{F}).$$

From the isomorphism (3.4) *and the previous decompositions, we obtain*

$$\mathbb{F}\mathfrak{S}_{2,2} \cong M_2(\mathbb{F}) \oplus \bigoplus^{2} \mathbb{F}\mathfrak{S}_2 \cong M_2(\mathbb{F}) \oplus \bigoplus^{4} \mathbb{F},$$

$$\mathbb{F}\mathfrak{S}_{3,2} \cong \bigoplus^{2} \big(\mathbb{F}\mathfrak{S}_2 \otimes M_3(\mathbb{F}) \oplus \mathbb{F}\mathfrak{S}_3\big) \cong \bigoplus^{4} \mathbb{F} \oplus \bigoplus^{2} M_2(\mathbb{F}) \oplus \bigoplus^{4} M_3(\mathbb{F}).$$

Example 3.14.2. *Let* $\operatorname{Char}\mathbb{F} = 2$, *then the algebra $\mathbb{F}\mathfrak{S}_2$ is not semi-simple with radical spanned by the element $id + \mathsf{s}_1$. From the isomorphism* (3.4)*, we have*

$$\mathbb{F}\mathfrak{S}_{2,2} \cong M_2(\mathbb{F}) \oplus \bigoplus^{2} \mathbb{F}\mathfrak{S}_2,$$

so $\mathrm{Rad}\,\mathbb{F}\mathfrak{S}_{2,2} \cong \bigoplus^{2} \mathrm{Rad}\,\mathbb{F}\mathfrak{S}_2$*. Thus the algebra* $\mathbb{F}\mathfrak{S}_{2,2}$ *has three simple modules, two of them are one dimensional and the dimension of third one is* 2*. From the definition of the multiplication on* $\mathbb{F}\mathfrak{S}_{2,2}$ *we have*

$$\mathbb{F}\mathfrak{S}_{2,2} = \mathbb{F}\langle 1_{(1,0)}, 1_{(0,1)}, \mathsf{s}_{(1,(1,0))}, \mathsf{s}_{(1,(0,1))}\rangle \oplus \mathbb{F}\langle 1_{(0,0)}, \mathsf{s}_{(1,(0,0))}\rangle \oplus \mathbb{F}\langle 1_{(1,1)}, \mathsf{s}_{(1,(1,1))}\rangle,$$

as an algebra. The first summand is isomorphic to $M_2(\mathbb{F})$*. Also* $\mathbb{F}\langle 1_{(i,i)}, \mathsf{s}_{(1,(i,i))}\rangle \cong \mathbb{F}\mathfrak{S}_2$ *where* $i = 0, 1$*. Since the radical of* $\mathbb{F}\mathfrak{S}_2$ *is spanned by* $id + \mathsf{s}_1$*, so the radical of* $\mathbb{F}\langle 1_{(i,i)}, \mathsf{s}_{(1,(i,i))}\rangle$ *is spanned by* $1_{(i,i)} + \mathsf{s}_{(1,(i,i))}$*. Hence*

$$\frac{\mathbb{F}\mathfrak{S}_{2,2}}{\langle 1_{(0,0)} + \mathsf{s}_{(1,(0,0))}, 1_{(1,1)} + \mathsf{s}_{(1,(1,1))}\rangle} \cong M_2(\mathbb{F}) \oplus \mathbb{F} \oplus \mathbb{F}.$$

Chapter 4

Representation Theory Of The Algebra $\mathbb{P}_{n,m}(\breve{\delta})$

4.1 Indexing set for the simple $\mathbb{P}_{n,m}$-modules

The aim of this section is to show that $\mathbb{P}_{n-1,m}(\breve{\delta})$ has an embedding into $\mathbb{P}_{n,m}(\breve{\delta})$ and describe an indexing set for the irreducible modules of the algebra $\mathbb{P}_{n,m}(\breve{\delta})$.

There is an inclusion between the algebras $\mathbb{P}_{n-1,m}(\breve{\delta})$ and $\mathbb{P}_{n,m}(\breve{\delta})$ defined by the map $\mathcal{I} : \mathbb{P}_{n-1,m} \to \mathbb{P}_{n,m}$ which is defined on the basis by

$$\mathcal{I}(d) = \mathcal{I}_0(d) + \cdots + \mathcal{I}_{m-1}(d), \tag{4.1}$$

where $d \in \mathcal{P}_{n-1,m}$ and $\mathcal{I}_j(d)$ is defined to be the same diagram except with one more extra non-crossing $\mathfrak{C}_j$-propagating line in the rightmost part. It is easy to check that $\mathcal{I}$ is an algebra homomorphism as the map $\mathcal{I}$ just adds a white line to the $\mathcal{P}_{n-1,m}$-diagrams and this line does not have any effect in the product of elements in $\mathcal{I}(\mathbb{P}_{n-1,m}(\breve{\delta}))$. The map $\mathcal{I}$ is called the *natural inclusion.*

Remark 4.1. The map $\mathcal{I}$ defines also an inclusion $\mathbb{T}_{n-1,m}(\breve{\delta})$ into $\mathbb{T}_{n,m}(\breve{\delta})$. Furthermore, $\mathbb{F}\mathfrak{S}_{n-1,m} \hookrightarrow \mathbb{F}\mathfrak{S}_{n,m}$ by the same map.

Theorem 4.2. *Let $\big(m\prod_{j=0}^{m-1}\delta_j\big)\neq 0$. Then $\mathsf{e}=\sum_{(x,y)\in\Omega_n}\frac{1}{m\delta_{y_n}}\mathsf{p}_{(n,x,y)}$, where Ω_n is the set defined by 2.21, is idempotent and there is an isomorphism of algebras*

$$\mathsf{e}\mathbb{P}_{n,m}\mathsf{e}\cong\mathbb{P}_{n-1,m}. \tag{4.2}$$

Proof. As it is mentioned $\mathbb{P}_{n-1,m}\hookrightarrow\mathbb{P}_{n,m}$ by the inclusion $\mathcal{I}$, so $\mathbb{P}_{n-1,m}\cong\mathrm{im}(\mathcal{I})$. The element e is an idempotent since

$$\begin{aligned}\mathsf{e}^2&=\frac{1}{m^2}\sum_{(x,y)\in\Omega_n}\sum_{(z,w)\in\Omega_n}\frac{1}{\delta_{y_n}\delta_{w_n}}\mathsf{p}_{(n,x,y)}\mathsf{p}_{(n,z,w)},\\&=\frac{1}{m^2}\sum_{(x,y)\in\Omega_n}\sum_{(y,w)\in\Omega_n}\frac{1}{\delta_{y_n}\delta_{w_n}}\delta_{y_n}\mathsf{p}_{(n,x,w)},\qquad\text{(from relation 8 in Pro. 2.23)}\\&=\frac{1}{m}\sum_{(x,w)\in\Omega_n}\frac{1}{\delta_{w_n}}\mathsf{p}_{(n,x,w)}=\mathsf{e}.\end{aligned}$$

Also from the graphical visualization, it is evident that $\mathsf{e}\mathcal{I}(d)\mathsf{e}=\mathsf{e}\mathcal{I}(d)=\mathcal{I}(d)\mathsf{e}$ for all $d\in\mathcal{P}_{n-1,m}$.

Now, define the map $f:\mathbb{P}_{n-1,m}\to\mathsf{e}\mathbb{P}_{n,m}\mathsf{e}$ by sending an element d to $\mathsf{e}\mathcal{I}(d)$. The well-definedness of f is clear and also it is a bilinear map since $\mathcal{I}$ is a module homomorphism and the multiplication in $\mathbb{P}_{n,m}$ is distributive, so we only need to check the image of the multiplication of two diagrams. Let $d_1,d_2\in\mathcal{P}_{n-1,m}$, so

$$\begin{aligned}f(d_1d_2)&=\mathsf{e}\mathcal{I}(d_1d_2)=\big(\mathsf{e}\mathcal{I}(d_1)\big)\mathcal{I}(d_2)=\big(\mathcal{I}(d_1)\mathsf{e}\big)\mathcal{I}(d_2)\\&=\big(\mathcal{I}(d_1)\mathsf{e}^2\big)\mathcal{I}(d_2)=\big(\mathcal{I}(d_1)\mathsf{e}\big)\big(\mathsf{e}\mathcal{I}(d_2)\big)=f(d_1)f(d_2).\end{aligned}$$

Then f is an algebra homomorphism. Also $f(d)\neq 0$ unless $d=0$, so f is injective.

Let $d\in\mathcal{P}_{n,m}$. The element $\mathsf{e}d\mathsf{e}$ will be sum of m^2 diagrams. At every diagram of these, the nodes n and n' are not connected to any other node, and the other blocks it will be the same in all m^2 diagrams. Note that those blocks form a partition in $\mathcal{P}_{n-1,m}$, say d', and $f(d')=\mathsf{e}d\mathsf{e}$. Then f is an algebra isomorphism. □

As a consequence of last theorem, the category of $\mathbb{P}_{n-1,m}$-modules and left $\mathbb{P}_{n,m}\mathsf{e}\mathbb{P}_{n,m}$-modules are essentially isomorphic categories, and according to Green [21], there are two functors

$$\mathbb{P}_{n-1,m}\text{-mod} \xrightarrow{\mathbf{G}} \mathbb{P}_{n,m}\text{-mod} \xrightarrow{\mathbf{F}} \mathbb{P}_{n-1,m}\text{-mod}$$

such that **FG** is the identity since $\mathbb{P}_{n,m}$ is an algebra over a field, for more details see Section 1.1.

Proposition 4.3. *For each* $n \in \mathbb{N}$, *the following is an isomorphism of algebras:*

$$\mathbb{P}_{n,m}/\mathbb{P}_{n,m}\mathsf{e}\mathbb{P}_{n,m} \cong \mathbb{F}\mathfrak{S}_{n,m}, \tag{4.3}$$

where $\mathsf{e} = \sum_{(x,y)\in\Omega_n} \frac{1}{m\delta_{y_n}} \mathsf{p}_{(n,x,y)}$ *and* $\big(m \prod_{j=0}^{m-1} \delta_j\big) \neq 0$.

Proof. The ideal $\mathbb{P}_{n,m}\mathsf{e}\mathbb{P}_{n,m}$ contains all diagrams having a propagating number less than or equal to $n-1$, this means $\mathbb{P}_{n,m}\mathsf{e}\mathbb{P}_{n,m} = \mathbb{P}_{n,m}(\breve{\delta}; n-1)$, see equation (2.15). Therefore any diagram in $\mathbb{P}_{n,m}/\mathbb{P}_{n,m}\mathsf{e}\mathbb{P}_{n,m}$ has exactly n propagating lines. □

Let $\Lambda_{\mathfrak{S}_{n,m}}$ be an index set for the cell modules of the algebra $\mathbb{F}\mathfrak{S}_{n,m}$, see (3.6). From the last proposition and the last theorem, we obtain the following useful corollary.

Corollary 4.4. *Let* $\Lambda_{\mathbb{P}_{n,m}}$ *denote an index set for the cell modules of the algebra* $\mathbb{P}_{n,m}(\breve{\delta})$. *If* $\big(m \prod_{j=0}^{m-1} \delta_j\big) \neq 0$, *then* $\Lambda_{\mathbb{P}_{n,m}}$ *is the disjoint union*

$$\begin{aligned} \Lambda_{\mathbb{P}_{n,m}} &= \Lambda_{\mathbb{P}_{n-1,m}} \bigsqcup \Lambda_{\mathfrak{S}_{n,m}} = \bigsqcup_{t=0}^{n} \Lambda_{\mathfrak{S}_{t,m}}, \\ &= \{(t, \lambda, \boldsymbol{\mu}) \mid t \in \mathbb{Z}_{n+1},\ \lambda \in \Gamma_{(t,m)},\ \boldsymbol{\mu} \ \textit{ is a multi-partition of } \lambda\}. \end{aligned} \tag{4.4}$$

Proof. It comes directly from Theorem 1.3 and equation (3.6). □

4.2 The algebra $\mathbb{P}_{n,m}(\breve{\delta})$ is a cellular algebra

In this section we shall prove our main result of this chapter, which is that the multi-partition algebra $\mathbb{P}_{n,m}$ is cellular. We show that the algebra $\mathbb{P}_{n,m}$ satisfies the conditions to be a cellular algebra by using Theorem 1.15.

C. Xi [55] has proved that the partition algebra $\mathbb{P}_n(\delta)$ is a cellular algebra, by using the fact that the symmetric group algebra is a cellular algebra. We will do the same, showing that $\mathbb{P}_{n,m}(\breve{\delta})$ is cellular by using the fact that the tensor product of finitely many symmetric group algebras is a cellular algebra.

Consider the order relation on the set $\underline{n} \cup \underline{n'}$: $1 < \ldots < n < 1' < \ldots < n'$. Let $\rho \in \mathcal{P}_X$ for some $X \subset \underline{n} \cup \underline{n'}$, the partition ρ is said to be written in *standard form* if ρ is written as $\{M_1, \ldots, M_l\}$ where $M_i = \{a_1^{(i)}, \ldots, a_{t_i}^{(i)}\}$ with $a_1^{(i)} < \ldots < a_{t_i}^{(i)}$ and $a_1^{(1)} < \ldots < a_1^{(l)}$. We say $M_i < M_j$ if and only if $a_1^{(i)} < a_1^{(j)}$. There is only one standard form for each partition ρ. Also, we define $|\rho|$ to be l, the number of parts of ρ.

We say that $(d_0, \ldots, d_{m-1}) \in \mathcal{P}_{n,m}$ is written in *standard form* if and only if each d_i is written in standard form.

Let $\lambda = (\lambda_0, \ldots, \lambda_{m-1}) \in \bigsqcup_{l=0}^{n} \Gamma_{(l,m)}$, where $\Gamma_{(l,m)}$ is the set of all m-compositions of l (see Section 1.3.1). Define V_λ to be the vector space with basis:

$$\Omega_\lambda = \{ \big((d_0, D_0), \ldots, (d_{m-1}, D_{m-1})\big) \mid \; d_i \in \mathcal{P}_{A_i}, \;\text{ for some }\; A_i \subseteq \underline{n} \text{ such that } \bigcup_{i=0}^{m-1} d_i \in \mathcal{P}_{\underline{n}}, \; |d_i| \geq \lambda_i \text{ and } D_i \subseteq d_i \;\text{ with }\; |D_i| = \lambda_i \; \}.$$

For example, $\big((\{\{1,3\},\{5\}\},\{\{5\}\}),(\{\{2\},\{4\}\},\{\{2\}\})\big)$ is an element in $\Omega_{(1,1)}$ where $n = 5$.

For each $M \subseteq \underline{n} \cup \underline{n'}$, we define the set M' to be the same elements of M after adding primes to the elements that do not have a prime and removing the prime from the elements that have a prime.

Let $N \subset \underline{n} \cup \underline{n'}$, $M \neq N$ and $\rho \in \mathcal{P}_M$, we denote by $\zeta_N(\rho)$ the partition of $M \setminus N$ obtained from ρ by deleting all elements in N from the parts of ρ, and by $\xi_N(\rho)$ the set of parts of ρ that do not contain any element in N.

Let $x \in \mathcal{P}_M$ and $y \in \mathcal{P}_N$, then we define the partition $x \centerdot y \in \mathcal{P}_{M \cup N}$ to be the smallest partition in $\mathcal{P}_{M \cup N}$ which contains $x \cup y$(this means each part of $x \cup y$ is a subset of a part of $x \centerdot y$). For example $\{\{1\},\{2,3\}\} \centerdot \{\{2,4\}\} = \{\{1\},\{2,3,4\}\}$. From the definition, it is clear that $x \centerdot y = y \centerdot x$.

For a diagram $\rho \in \mathcal{P}_M$, if we interchange the primed element j' with the unprimed element j, then we get a new partition of M'. Let us denote this new partition by $*(\rho)$ or simply ρ^*. In general, if $\rho = (\rho_0, \ldots, \rho_{m-1}) \in \mathcal{P}_{n,m}$ we define $*(\rho)$ to be $(\rho_0^*, \ldots, \rho_{m-1}^*)$.

Example 4.4.1. *Take* $\rho = \{\{2,3'\},\{4\},\{6\}\}$ *and* $N = \{1,2,6\}$*, so* $\zeta_N(\rho) = \{\{3\}, \{4\}\}$*,* $\xi_N(\rho) = \{\{4\}\}$ *and* $\rho^* = \{\{2',3\},\{4'\},\{6'\}\}$.

Lemma 4.5. *The linear map* $*$ *is an anti-involution of the algebra* $\mathbb{P}_{n,m}$*.*

Proof. The map $*$ is defined on $\mathcal{P}_{n,m}$, so $*$ extends to $\mathbb{P}_{n,m}$ by linearity. It is clear that $(\alpha^*)^* = \alpha$ and $(\alpha\beta)^* = (\beta)^*(\alpha)^*$ holds true for all $\alpha, \beta \in \mathcal{P}_{n,m}$. This follows immediately from the graphical realization of the map $*$ and the product in $\mathbb{P}_{n,m}$. $\square$

Let $\mathbb{S}_{n,\lambda}$ be the algebra $1_{\underline{\underline{\lambda}}}\mathbb{F}\mathfrak{S}_{n,m}1_{\underline{\underline{\lambda}}}$, where $\lambda \in \Gamma_{(n,m)}$ and $\underline{\underline{\lambda}} \in \mathbb{Z}_m^n$ is defined by equation (2.26). From equation (2.29), we have

$$\mathbb{S}_{n,\lambda} \cong \mathbb{F}\mathfrak{S}_{\lambda_0} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathbb{F}\mathfrak{S}_{\lambda_{m-1}}. \tag{4.5}$$

Thus $\dim \mathbb{S}_{n,\lambda} = \prod_{i=0}^{m-1} \lambda_i!$, and the set

$$\{(f_0, \ldots, f_{m-1}) \mid f_i \in \mathfrak{S}_{\lambda_i} \text{ for each } i \in \mathbb{Z}_m\}$$

can be regarded as a basis of the algebra $\mathbb{S}_{n,\lambda}$. Let $f_i \in \mathfrak{S}_{\lambda_i}$, where $i \in \mathbb{Z}_m$. Throughout this chapter the element $(f_0, \ldots, f_{m-1})$ is used to denote its image in $\mathbb{S}_{n,\lambda}$, which is simply the diagram in $\mathfrak{S}_{n,m}$ formed by drawing f_0 by the colour $\mathfrak{C}_0$ followed by drawing f_1 by the colour $\mathfrak{C}_1$ in the same frame and so on. By Proposition 1.16 and since symmetric group algebras are cellular, we have that the algebra $\mathbb{S}_{n,\lambda}$ is cellular.

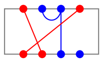
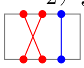
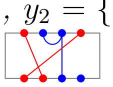
Lemma 4.6. *Each element in* $\mathbb{P}_{n,m}$ *can be written uniquely as an element of* $V_\lambda \otimes_{\mathbb{F}} V_\lambda \otimes_{\mathbb{F}} \mathbb{S}_{l,\lambda}$ *for some* $\lambda \in \Gamma_{(l,m)}$ *and* $l = 0, \ldots, n$*.*

Proof. Consider a diagram $\rho = (\rho_0, \ldots, \rho_{m-1}) \in \mathcal{P}_{n,m}$ and $x_h := \zeta_{\underline{n'}}(\rho_h)$ and $y_h := \zeta_{\underline{n'}}(\rho_h^*)$ for each $h \in \mathbb{Z}_m$. Note that $x_h \in \mathcal{P}_{\text{top}(\rho_h)}$ and $y_h \in \mathcal{P}_{\text{top}(\rho_h^*)}$. Take $\lambda = (\#(\rho_0), \ldots, \#(\rho_{m-1}))$ and $l = |\lambda| := \#(\rho_0) + \cdots + \#(\rho_{m-1})$.

Let S_{ρ_h} be the set of parts of ρ_h containing both primed and unprimed elements. Then $|S_{\rho_h}| = \#(\rho_h) = \lambda_h$. Define $X_h := \zeta_{\underline{n'}}(S_{\rho_h})$ and $Y_h = \zeta_{\underline{n'}}(S^*_{\rho_h})$. It is clear that both X_h and Y_h contain λ_h parts, thus $v = ((x_0, X_0), \ldots, (x_{m-1}, X_{m-1}))$ and $w = ((y_0, Y_0), \ldots, (y_{m-1}, Y_{m-1}))$ are contained in the set Ω_λ.

Now if we write X_h and Y_h in standard form: $\{X_1^h, \ldots, X_{\lambda_h}^h\}$, $\{Y_1^h, \ldots, Y_{\lambda_h}^h\}$. We define $b := (b_0, \ldots, b_{m-1}) \in \mathbb{S}_{l,\lambda}$, where b_h is a bijective map from X_h to Y_h that sends i to j if there is a part $T \in S_{\rho_h}$ containing both X_i^h and Y_j^h. Since v, w and $(b_0, \ldots, b_{m-1})$ are uniquely determined by ρ in a standard form, we can associate with the given ρ a unique element $v \otimes w \otimes b$.

Conversely, each element $\mu \otimes \nu \otimes b$ with $\mu, \nu \in \Omega_\lambda$ and $b \in \mathbb{S}_{l,\lambda}$ corresponds to a unique multi-colour partition $\rho \in \mathbb{P}_{n,m}$. □

Example 4.6.1. *Take the diagram*  *in* $\mathcal{P}_{4,2}$. *So we have* $x_1 = \{\{1\}, \{4\}\} = X_1$, $x_2 = \{\{2, 3\}\} = X_2$, $y_1 = \{\{1\}, \{2\}\} = Y_1$, $y_2 = \{\{3\}, \{4\}\}$, $Y_2 = \{\{3\}\}$ *and* (b_1, b_2) *is the diagram* . *Hence the diagram*  *corresponds to the element* $((x_1, X_1), (x_2, X_2)) \otimes ((y_1, Y_1), (y_2, Y_2)) otimes (b_1, b_2)$.

There is a bilinear map $\phi_\lambda : V_\lambda \otimes_{\mathbb{F}} V_\lambda \to \mathbb{S}_{l,\lambda}$, where $l = |\lambda| = \sum_{j=0}^{m-1} \lambda_j$, defined as follows. Let $v = ((x_0, X_0), \ldots, (x_{m-1}, X_{m-1})) \in \Omega_\lambda$ be fixed and assume that $X_h = \{X_1^h, \ldots, X_{\lambda_h}^h\}$ is written in standard form for each h. Take $w \in \Omega_\lambda$ and assume that $w = ((y_0, Y_0), \ldots, (y_{m-1}, Y_{m-1}))$ where $Y_h = \{Y_1^h, \ldots, Y_{\lambda_h}^h\}$ is also written in standard form for each h. From the definition of Ω_λ, we can assume that $x_h \in \mathcal{P}_{A_h}$ and $y_h \in \mathcal{P}_{B_h}$ where $\cup_{h=1}^m A_h = \underline{n} = \cup_{h=1}^m B_h$. Then we define $\phi_\lambda(v \otimes w)$ to be

$$\begin{cases} \prod_{h=0}^{m-1} \delta_h^{c_h}(b_0, \ldots, b_{m-1}) & \text{if } A_h = B_h, \text{ and each part of } x_h \centerdot y_h \\ & \text{contains only one part of } X_h \text{ and contains} \\ & \text{only part of } Y_h \text{ for each } h \in \mathbb{Z}_m, \\ 0 & \text{otherwise,} \end{cases} \tag{4.6}$$

where $c_h = |\xi_{X_h \sqcup Y_h}(x_h \centerdot y_h)|$, and b_h is defined as follows: since for each i there is a unique part of $x_h \centerdot y_h$ containing both X_i^h and Y_j^h, we define b_h to be the permutation taking i to j. Thus $b_h \in \mathfrak{S}_{\lambda_h}$ and $(b_0, \ldots, b_{m-1}) \in \mathbb{S}_{l,\lambda}$. This element $(b_0, \ldots, b_{m-1})$ is denoted by $\Upsilon_\lambda(v; w)$.

If we extend ϕ_λ linearly to the whole space $V_\lambda \otimes_{\mathbb{F}} V_\lambda$, then we have the following lemma.

Lemma 4.7. *The map $\phi_\lambda : V_\lambda \otimes_{\mathbb{F}} V_\lambda \to \mathbb{S}_{\sum \lambda_j, \lambda}$ is a bilinear form.*

Proof. This holds since any map on a basis of a vector space defines a unique linear map on the vector space and Ω_λ is a basis of V_λ. □

Lemma 4.8. *Let ρ, ω be partitions in $\mathcal{P}_{n,m}$. If $\rho = u \otimes x \otimes b$ and $\omega = y \otimes v \otimes d$ are contained in $V_\lambda \otimes_{\mathbb{F}} V_\lambda \otimes_{\mathbb{F}} \mathbb{S}_{l,\lambda}$, then*

$$\rho\omega = \begin{cases} \Big(u \otimes v \otimes b\phi_\lambda\big(x \otimes y\big)d\Big) \text{ modulo } J_{\lambda<} & \text{if } \operatorname{top}(\omega) = \operatorname{bot}(\rho), \\ 0 & \text{otherwise}, \end{cases} \tag{4.7}$$

where $J_{\lambda<} := \bigoplus_{\substack{\xi \in \bigsqcup_{t=0}^{l} \Gamma_{(t,m)} \\ \xi \neq \lambda \text{ and } \xi_j \leq \lambda_j \ \forall j}} V_\xi \otimes_{\mathbb{F}} V_\xi \otimes_{\mathbb{F}} \mathbb{S}_{\sum \xi_j, \xi}.$

Proof. Let $x = ((x_0, X_0), \ldots, (x_{m-1}, X_{m-1}))$ and $y = ((y_0, Y_0), \ldots, (y_{m-1}, Y_{m-1}))$. From the definitions of the multiplication in $\mathbb{P}_{n,m}$ and of the map $\xi_{X_h \sqcup Y_h}$, we have that $|\xi_{X_h \sqcup Y_h}(x_h \centerdot y_h)|$ equals the number of connected components removed from the middle row when we construct the product $\rho_h\omega_h$, where $\rho = (\rho_0, \ldots, \rho_{m-1})$ and $\omega = (\omega_0, \ldots, \omega_{m-1})$. Hence, it is sufficient to show that the element $u \otimes v \otimes b\phi_\lambda\big(x \otimes y\big)d$ represents the element $\prod_{h=0}^{m-1} \delta_h^{c_h}\big(\omega \circ \rho\big)$ in $\mathbb{P}_{n,m}$ modulo $J_{\lambda<}$, where $|\xi_{X_h \sqcup Y_h}(x_h \centerdot y_h)| = c_h$.

In the second case, if $\phi_\lambda\big(x \otimes y\big) = 0$, from the definition of ϕ_λ we obtain that $\#(\rho_h\omega_h) < \lambda_h$ for some h. This implies that $\rho\omega \in J_{\lambda<}$. So the result holds when $\phi_\lambda\big(x \otimes y\big) = 0$.

Now assume that $\phi_\lambda\big(x \otimes y\big) = (\prod_{h=0}^{m-1} \delta_h^{c_h})s$, where $s = (s_0, \ldots, s_{m-1})$ is defined as above, then we need to show that $u \otimes v \otimes bsd$ represents the element $\omega \circ \rho$.

From the definition of ϕ_λ, we can realize that $\zeta_{\underline{n'}}\big((\omega \circ \rho)_h\big) = \zeta_{\underline{n'}}\big(\rho_h\big) = u_h$ and that $\zeta_{\underline{n'}}\big((\omega_h \circ \rho_h)^*\big) = \zeta_{\underline{n'}}\big(\omega_h^*\big) = v_h$, where $u = ((u_0, U_0), \ldots, (u_{m-1}, U_{m-1}))$ and $v = ((v_0, V_0), \ldots, (v_{m-1}, V_{m-1}))$.

Note that there are only λ_h distinct parts of $x_h \centerdot y_h$, say $P_1^h, \ldots, P_{\lambda_h}^h$, each one containing a single X_j^h and a single $Y_{s_h(j)}^h$. So there is a part $\rho_h\omega_h$ which contains both $U_{b_h^{-1}(j)}^h$ and $Y_{s_h(j)}^h$, where $b = (b_0, \ldots, b_{m-1})$. Since Y_i^h and $V_{d_h(i)}^h$ are contained in the same part of $y_h \centerdot v_h$, where $d = (d_0, \ldots, d_{m-1})$, then $U_{b_h^{-1}(j)}^h$ and $V_{s_h d_h(j)}^h$ are contained in the same part of $\rho_h\omega_h$. Hence $\rho\omega$ is represented by $u \otimes v \otimes bsd$. □

The following corollary is a consequence of the definitions and the previous lemma.

Corollary 4.9. *Let $\alpha = x \otimes y \otimes b$ with $x, y \in \Omega_\lambda$ and $b \in \mathbb{S}_{l,\lambda}$, then $*(\alpha) = y \otimes x \otimes b^*$.*

Lemma 4.10. *Let $\lambda, \mu \in \bigsqcup_{t=0}^{n} \Gamma_{(t,m)}$ with $\lambda \neq \mu$ and $\lambda_j \leq \mu_j$ for each j. Take $\alpha = u \otimes x \otimes b \in V_\mu \otimes V_\mu \otimes \mathbb{S}_{\sum \mu_j, \mu}$ where b is a basis element of $\mathbb{S}_{\sum \mu_j, \mu}$ and $\beta = y \otimes v \otimes s \in V_\lambda \otimes V_\lambda \otimes \mathbb{S}_{\sum \lambda_j, \lambda}$ with s a basis element of $\mathbb{S}_{\sum \lambda_j, \lambda}$. Let $x_i \in \mathcal{P}_{A_i}$ and $y_i \in \mathcal{P}_{B_i}$ for some subsets $A_i, B_i \subseteq \underline{n}$, where $x = ((x_0, X_0), \ldots, (x_{m-1}, X_{m-1}))$ and $y = ((y_0, Y_0), \ldots, (y_{m-1}, Y_{m-1}))$. Then*

- *$A_i \neq B_i$ if and only if $\alpha\beta = 0$, where $i = 0, \ldots, m-1$.*
- *If $0 \neq \alpha\beta = \prod_{h=0}^{m-1} \delta_h^{|\xi_{X_h \sqcup Y_h}(x_h \centerdot y_h)|} w \otimes z \otimes d$, where $w = ((w_0, W_0), \ldots, (w_{m-1}, W_{m-1}))$ and $z = ((z_0, Z_0), \ldots, (z_{m-1}, Z_{m-1}))$, then*

  (1) if $|W_h| = \lambda_h$ for each h, then $z = v$, $d = d's$ for some $d' \in \mathbb{S}_{\sum \lambda_j, \lambda}$, and w and d' do not depend on s (moreover, $d' = b\Upsilon_\lambda(x; y)$);

  (2) if $|W_h| < \lambda_h$ for some h, then $\alpha\big(y \otimes v \otimes s'\big) \in J_{\lambda^<}$ for any $s' \in \mathbb{S}_{\sum \lambda_j, \lambda}$.

Proof. The first part is clear since $\mathrm{bot}(\alpha) = (A_0, \ldots, A_{m-1})$ and $\mathrm{top}(\beta) = (B_0, \ldots, B_{m-1})$. If $|W_h| = \lambda_h$ for each h, then $|Z_h| = \lambda_h$ as $\alpha\beta \neq 0$. Since every Z_i^h is always obtained from V_i^h, we have $z = v$. Hence d is also of the desired form. The other assertions follow immediately from the definition of the multiplication of two basis elements in the set $\mathcal{P}_{n,m}$.

Finally, the proof of last part is obvious since s_h and s'_h can be considered as two bijections from Y_h to V_h and $|W_h| < \lambda_h$ for some h, so there is a part of $x_h \centerdot y_h$ containing more than one element of Y_h, thus we always have $\alpha\big(y \otimes v \otimes s'\big) \in J_{\lambda^<}$ for any $s' \in \mathbb{S}_{\sum \lambda_j, \lambda}$. □

The next corollary is a result of the previous two lemmas.

Corollary 4.11. *$J_\lambda := \bigoplus_{\substack{\mu \in \bigsqcup_{l=0}^{n} \Gamma_{(l,m)} \\ \text{where } \mu_j \leq \lambda_j \ \forall\, j \in \mathbb{Z}_m}} V_\mu \otimes_{\mathbb{F}} V_\mu \otimes_{\mathbb{F}} \mathbb{S}_{\sum \mu_j, \mu}$ is an ideal of $\mathbb{P}_{n,m}$.*

Proof. From the definitions, it is obvious that J_λ contains all the diagrams in which the number of $\mathfrak{C}_j$-propagating lines is less than or equal to λ_j for each $j \in \mathbb{Z}_m$, so $J_\lambda = \mathbb{P}_{n,m}(\breve{\delta}; \lambda)$ (see Proposition 2.13). □

Lemma 4.12. *Let $\sigma : \mathbb{S}_{l,\lambda} \to \mathbb{S}_{l,\lambda}$ be the involution which is defined by $b \mapsto b^*$ for all $b \in \prod_{j=0}^{m-1} \mathfrak{S}_{\lambda_j}$. Then $\sigma\phi_\lambda(x \otimes y) = \phi_\lambda(y \otimes x)$ for all $x, y \in V_\lambda$.*

Proof. Let $x = ((x_0, X_0), \dots, (x_{m-1}, X_{m-1}))$ and $y = ((y_0, Y_0), \dots, (y_{m-1}, Y_{m-1}))$. If we assume $\phi_\lambda(x \otimes y) = 0$, then it follows from the definition of ϕ_λ and $x_h \centerdot y_h = y_h \centerdot x_h$ that $\phi_\lambda(y \otimes x) = 0$. Now assume that $\phi_\lambda(x \otimes y) \neq 0$. In this case, if X_i^h and $Y_{b_h(i)}^h$ with $b = \Upsilon_\lambda(x; y)$ are contained in the same part of $x_h \centerdot y_h$, then Y_i^h and $X_{b_h^*(i)}^h$ are also contained in the same part of $y_h \centerdot x_h$. Thus $\Upsilon_\lambda(y; x) = b^*$. This shows that $\sigma\phi_\lambda(x \otimes y) = \phi_\lambda(y \otimes x)$. □

Now we are ready to prove the main result.

Theorem 4.13. *The multi-colour partition algebra $\mathbb{P}_{n,m}(\breve{\delta})$ is a cellular algebra.*

Proof. Put $J_{-1} = 0$, $\mathfrak{S}_0 = \{1\}$ and $B_\lambda = \mathbb{S}_{l,\lambda}$, where $l = \sum_{i=0}^{m-1} \lambda_i$. Then the multi-colour partition algebra has a decomposition

$$\mathbb{P}_{n,m} = \bigoplus_{\lambda \in \bigsqcup_{l=0}^{n} \Gamma_{(l,m)}} V_\lambda \otimes_{\mathbb{F}} V_\lambda \otimes \mathbb{S}_{l,\lambda}. \tag{4.8}$$

Note that B_λ is a cellular algebra with respect to the involution σ as defined in Lemma 4.12, since it is isomorphic to the tensor product of finitely many cellular algebras (see equation (2.29) and Proposition 1.16). By the lemmas in this section, the above decomposition satisfies all conditions in Theorem 1.15, thus the algebra $\mathbb{P}_{n,m}(\breve{\delta})$ is a cellular algebra. □

4.3 Cell modules of the algebra $\mathbb{P}_{n,m}(\breve{\delta})$

In this section, we describe a complete set of generically simple modules $\mathbb{V}_n(\boldsymbol{\mu})$ of the algebra $\mathbb{P}_{n,m}(\breve{\delta})$, where $\boldsymbol{\mu}$ is a m-multi-partition of an integer less than or equal to n, see Subsection 1.3.1. These are simple modules of $\mathbb{P}_{n,m}$ except for finitely many values of $\breve{\delta}$.

From the definition of the algebra $\mathbb{S}_{|\lambda|,\lambda}$, the set of all cell modules of the algebra $\mathbb{S}_{|\lambda|,\lambda}$ is $\{\mathscr{S}_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \vdash \lambda\}$, where $|\lambda| = \sum \lambda_i$ and

$$\mathscr{S}_{\boldsymbol{\mu}} := \mathscr{S}_{\boldsymbol{\mu}_0} \otimes \cdots \otimes \mathscr{S}_{\boldsymbol{\mu}_{m-1}}, \tag{4.9}$$

where $\mathscr{S}_{\boldsymbol{\mu}_i}$ is the Specht module of the symmetric group $\mathfrak{S}_{\lambda_i}$ associated to a partition $\boldsymbol{\mu}_i$ with a bilinear form $\langle\ ,\ \rangle_{\boldsymbol{\mu}_i}$. We define a bilinear form on $\mathscr{S}_{\boldsymbol{\mu}}$ by $\langle\ ,\ \rangle_{\boldsymbol{\mu}} := \prod_{i=0}^{m-1} \langle\ ,\ \rangle_{\boldsymbol{\mu}_i}$,

From Theorems 4.13 and 1.15, we have the following fact.

Corollary 4.14. *The cell modules of* $\mathbb{P}_{n,m}(\breve{\delta})$ *are* $\mathbb{V}_n(\boldsymbol{\mu}) := V_\lambda \otimes v_\lambda \otimes \mathscr{S}_{\boldsymbol{\mu}}$*, where* $\lambda \in \bigsqcup_{l=0}^{n} \Gamma_{(l,m)}$ *and* $\boldsymbol{\mu} \vdash \lambda$*,* v_λ *is a fixed non-zero element of* V_λ*, and* $\mathscr{S}_{\boldsymbol{\mu}}$ *is the cell module of the algebra* $\mathbb{S}_{|\lambda|,\lambda}$*, as it is defined in* (4.9)*. If* $|\lambda| = 0$*, take* $\lambda = \overline{0} = \boldsymbol{\mu}$ *and* $\mathscr{S}_{\overline{0}} = \mathbb{F}$*, where* $\overline{0} = (\underbrace{0, \ldots, 0}_{m\ \text{times}})$*.(Note that* $\boldsymbol{\mu}$ *determines* λ*.)*

Let $\langle\ ,\ \rangle_{\lambda,\boldsymbol{\mu}}$ be a bilinear form on the module $\mathbb{V}_n(\boldsymbol{\mu})$ defined by (4.7) via its basis of diagrams (we add λ just to make it different from the bilinear form on the module $\mathscr{S}_{\boldsymbol{\mu}}$). Let us write $\mathbb{G}_n(\boldsymbol{\mu})$ for the Gram matrix of the inner product $\langle\ ,\ \rangle_{\lambda,\boldsymbol{\mu}}$ on the cell module $\mathbb{V}_n(\boldsymbol{\mu})$. For $(\lambda, \boldsymbol{\mu}) \in \Lambda^0_{\mathbb{P}_{n,m}}$, the simple $\mathbb{P}_{n,m}$-module $\mathsf{L}_n(\boldsymbol{\mu})$ is the simple quotient of the module $\mathbb{V}_n(\boldsymbol{\mu})$. Remember, the module $\mathbb{V}_n(\boldsymbol{\mu})$ is simple if and only if $\det \mathbb{G}_n(\boldsymbol{\mu}) \neq 0$, for more details see Section 1.5.

Recall that a multi-partition $\boldsymbol{\mu} = (\boldsymbol{\mu}_0, \ldots, \boldsymbol{\mu}_{m-1})$ is called p-regular if $\boldsymbol{\mu}_i$ is regular for each i, see Subsection 1.3.2. If $\mathsf{p} = 0$, then all multi-partitions are p-regular. As a corollary of Theorem 4.13, we classify the simple modules.

Corollary 4.15. *Let* $\mathbb{P}_{n,m}(\breve{\delta})$ *be the multi-partition algebra over a field* $\mathbb{F}$ *of characteristic* p. *If* $\prod_{j=0}^{m-1} \delta_j \neq 0$ *then the non-isomorphic simple modules are parametrized by* $\{(l, \boldsymbol{\mu}) \mid l \in \mathbb{Z}_{n+1}, \boldsymbol{\mu}$ *is a* p*-regular* m*-multi-partition of* $l\}$.

Proof. It follows from Corollary 4.14 that all simple $\mathbb{P}_{n,m}(\breve{\delta})$-modules are parametrized by $\{(l, \boldsymbol{\mu}) \mid \langle\ ,\ \rangle_{\lambda,\boldsymbol{\mu}} \neq 0\}$. Let $A, B \in \mathbb{V}_n(\boldsymbol{\mu})$, from the definition of the module $\mathbb{V}_n(\boldsymbol{\mu})$, these elements can be written as

$$A = a \otimes v_\lambda \otimes \alpha \ \ , B = b \otimes v_\lambda \otimes \beta,$$

where $\alpha = (\alpha_0, \ldots, \alpha_{m-1}), \beta = (\beta_0, \ldots, \beta_{m-1}) \in \mathscr{S}_{\boldsymbol{\mu}}$ and $a, b \in V_\lambda$ where $\boldsymbol{\mu}_i \vdash \lambda_i$ for each i. From Lemma 4.8, we have

$$AB = \Big(a \otimes v_\lambda \otimes \alpha\phi_\lambda\big(v_\lambda \otimes b\big)\beta\Big) \ \ \textit{modulo } J_{\lambda^<},$$

when $\mathrm{top}(B) = \mathrm{bot}(A)$, otherwise it will be zero. Take $b = v_\lambda$, where

$$v_\lambda = ((v_0, V_0), \ldots, (v_{m-1}, V_{m-1})) \in V_\lambda,$$

from the definition of the map ϕ_λ, it is obvious that

$$\phi_\lambda\big(v_\lambda \otimes v_\lambda\big) = \prod_{i=0}^{m-1} \delta_i^{|v_i| - \lambda_i}(id, \ldots, id).$$

So when $b = v_\lambda$, the multiplication AB is

$$AB = \prod_{i=0}^{m-1} \delta_i^{|v_i| - \lambda_i}\Big(a \otimes v_\lambda \otimes \alpha\beta\Big) \ \ \textit{modulo } J_{\lambda^<}.$$

Thus $\langle A, B\rangle_{\lambda,\boldsymbol{\mu}} = 0$ if and only if $\left(\prod_{i=0}^{m-1} \delta_i^{|v_i|-\lambda_i}\right)\langle \alpha, \beta\rangle_{\boldsymbol{\mu}} = 0$ in the module $\mathscr{S}_{\boldsymbol{\mu}}$ (note that $\langle \alpha, \beta\rangle_{\boldsymbol{\mu}} = \prod_{i=1}^{m} \langle \alpha_i, \beta_i\rangle_{\boldsymbol{\mu}_i}$, where $\langle \alpha_i, \beta_i\rangle_{\boldsymbol{\mu}_i}$ is computed in the module $\mathscr{S}_{\boldsymbol{\mu}_i}$). Now, we are going to check each term $\delta_i^{|v_i|-\lambda_i}\langle\ ,\ \rangle_{\boldsymbol{\mu}_i}$ individually. If $\lambda_i > 0$, the partition v_i can be chosen such that $|v_i| = \lambda_i$ so we only need to check when $\langle\ ,\ \rangle_{\boldsymbol{\mu}_i}$ equals zero. From Theorem 1.9, $\langle\ ,\ \rangle_{\boldsymbol{\mu}_i} \neq 0$ if and only if $\boldsymbol{\mu}_i$ is p-regular partition of λ_i. If $\lambda_i = 0$, then $\delta_i^{|v_i|-\lambda_i}\langle\ ,\ \rangle_{\boldsymbol{\mu}_i} = \delta_i^{|v_i|}$ which is non-zero when $\delta_i \neq 0$. This shows that $\langle\ ,\ \rangle_{\lambda,\boldsymbol{\mu}} \neq 0$ if and only if $\boldsymbol{\mu}$ is a p-regular multi-partition of λ and $\prod_{j=0}^{m-1} \delta_j \neq 0$. □

Next, we shall determine for which values of the parameters δ_i the algebra $\mathbb{P}_{n,m}$ is quasi-hereditary.

Corollary 4.16. *The algebra $\mathbb{P}_{n,m}$ is quasi-hereditary if and only if $\delta_i \neq 0$ for all i and the characteristic of $\mathbb{F}$ is either zero or strictly grater than n.*

Proof. It comes directly from previous result and the fact that a cellular algebra is quasi-hereditary when $\Lambda = \Lambda^0$, see Remark 1.20. □

4.4 Semi-simplicity of the algebra $\mathbb{P}_{n,m}$ over the complex field

In this section we shall work towards proving the final results of this chapter. We show that the algebra $\mathbb{P}_{n,m}(\breve{\delta})$ is non-semisimple over the complex field if and only if δ_j is a non-negative integer less than $2n-1$ for some $j \in \mathbb{Z}_m$. From here to the end of this chapter we will assume that $\mathbb{F} = \mathbb{C}$.

Over the complex field we have

$$\mathrm{Rad}(\mathbb{P}_{n,m}(\breve{\delta})) = \mathrm{Rad}(\mathbb{P}_{n,m}(\breve{\delta}; n-1)),$$

and we can prove that as follows. Since $\mathbb{P}_{n,m}/\mathbb{P}_{n,m}(\breve{\delta}; n-1) \cong \mathbb{C}\mathfrak{S}_{n,m}$ and $\mathbb{C}\mathfrak{S}_{n,m}$ is a semi-simple algebra (see Corollary 3.13), so $\mathrm{Rad}(\mathbb{P}_{n,m}) \subseteq \mathrm{Rad}(\mathbb{P}_{n,m}(\breve{\delta}; n-1))$ (view

$\mathbb{P}_{n,m}(\breve{\delta}; n-1)$ as an algebra without identity). Also $\mathbb{P}_{n,m}(\breve{\delta}; n-1)/\operatorname{Rad}(\mathbb{P}_{n,m})$ is an ideal in $\mathbb{P}_{n,m}/\operatorname{Rad}(\mathbb{P}_{n,m})$, so the quotient $\mathbb{P}_{n,m}(\breve{\delta}; n-1)/\operatorname{Rad}(\mathbb{P}_{n,m})$ is a semi-simple algebra. This implies $\operatorname{Rad}(\mathbb{P}_{n,m}(\breve{\delta}; n-1)) \subseteq \operatorname{Rad}(\mathbb{P}_{n,m})$.

Define idempotent elements

$$\mathbf{I}_{k,x} := \prod_{j=k+1}^{n} \frac{1}{\delta_{x_j}} \mathsf{p}_{(j,x,x)}, \qquad k = 0, \dots, n-1, \tag{4.10}$$

where $x \in \mathbb{Z}_m^n$ and $\delta_{x_j} \neq 0$ for each $j > k$.

Lemma 4.17. *The element $\mathbf{I}_{0,x}$ is a primitive idempotent in the algebra $\mathbb{P}_{n,m}$ for each $x \in \mathbb{Z}_m^n$. Furthermore, the left ideal $\mathbb{P}_{n,m}\mathbf{I}_{0,x}$ is an indecomposable module.*

The proof is clear since $\mathbf{I}_{0,x}\mathbb{P}_{n,m}\mathbf{I}_{0,x} = \mathbb{F}\mathbf{I}_{0,x}$. Also $\dim(\mathbb{P}_{n,m}\mathbf{I}_{0,x}) = \sum\limits_{l=1}^{n} m^l \left\{ {n \atop l} \right\}$, where $\left\{ {n \atop l} \right\}$ is the Stirling number of the second kind.

Let $\mathsf{e}_{\boldsymbol{\mu}_j}$ be the primitive idempotent corresponding to the Specht module $\mathscr{S}_{\boldsymbol{\mu}_i}$ in the group $\mathfrak{S}_{\lambda_i}$, then $(\mathsf{e}_{\boldsymbol{\mu}_0}, \dots, \mathsf{e}_{\boldsymbol{\mu}_{m-1}}) \in \mathbb{S}_{|\lambda|,\lambda}$ so we can use the inclusion map $\mathcal{I}$, see (4.1), and define the element

$$\mathsf{e}_{\boldsymbol{\mu}} := \mathcal{I}^{n-|\lambda|}((\mathsf{e}_{\boldsymbol{\mu}_0}, \dots, \mathsf{e}_{\boldsymbol{\mu}_{m-1}})) \in \mathbb{F}\mathfrak{S}_{n,m}.$$

Let $\boldsymbol{\mu} \vdash \lambda \vDash l = |\lambda|$ for some $l = 0, \dots, n$ and define x to be a tuple defined as follows: $x_i = j$ when $\sum\limits_{k=0}^{j-1} \lambda_k + 1 \leq j \leq \sum\limits_{k=0}^{j} \lambda_k$ and x_i for $i > \sum\limits_{k=0}^{m-1} \lambda_k$ takes any value such that $\delta_{x_i} \neq 0$ and that to make $\mathbf{I}_{l,x}$ is defined. Without losing the generality, we can assume that $\delta_{m-1} \neq 0$ and take x to be

$$(\underbrace{0, \dots, 0}_{\lambda_0 \text{ times}}, \underbrace{1, \dots, 1}_{\lambda_1 \text{ times}}, \dots, \underbrace{m-2, \dots, m-2}_{\lambda_{m-2} \text{ times}}, \underbrace{m-1, \dots, m-1}_{n-\sum_{i=0}^{m-2} \lambda_i \text{ times}}).$$

Hence the element the element $\mathbf{I}_{l,x}\mathsf{e}_{\boldsymbol{\mu}}$ represented by the element $w \otimes w \otimes (\mathsf{e}_{\boldsymbol{\mu}_0}, \dots, \mathsf{e}_{\boldsymbol{\mu}_{m-1}})$ where $w = ((w_0, W_0), (w_{m-1}, W_{m-1}))$, $w_i = W_i = \{\{\sum_{j<i} \lambda_j + 1\}, \dots, \{\sum_{j \leq i} \lambda_j\}\}$ when $i < m-1$, and $w_{m-1} = \{\{\sum_{j<m-1} \lambda_j + 1\}, \dots, \{n\}\}$ and $W_{m-1} = \{\{\sum_{j<m-1} \lambda_j + 1\}, \dots, \{\sum_{j \leq m-1} \lambda_j\}\}$. From the definition of cell modules of the algebra $\mathbb{S}_{l,\lambda}$ we have

$\mathbb{S}_{l,\lambda}(\mathsf{e}_{\boldsymbol{\mu}_0}, \ldots, \mathsf{e}_{\boldsymbol{\mu}_{m-1}}) = \mathscr{S}_{\boldsymbol{\mu}}$ (see (4.9)). Since the element v_λ in the definition of the module $\mathbb{V}_n(\boldsymbol{\mu})$ does not have any role, so we can take $w = v_\lambda$ and then by using Lemma 4.6 we can show that

$$\mathbb{V}_n(\boldsymbol{\mu}) \cong \mathbb{P}_{n,m}\mathbf{I}_{l,x}\mathsf{e}_{\boldsymbol{\mu}} \pmod{\mathbb{P}_{n,m}(\breve{\delta}, l-1)}. \tag{4.11}$$

As consequence of the previous paragraph, we have the following fact. Recall that we write $\mu \rhd \lambda$ to denote that μ is a partition obtained from the partition λ by adding a box to λ after regarding them as Young diagrams. Also $\mu \lhd \lambda$ means that μ is a partition obtained from λ by removing a box. Additionally, $\mu \lhd \rhd \lambda$ means that μ is a partition obtained from λ by removing a box after adding a box.

Proposition 4.18. *The generic restriction from $\mathbb{P}_{n,m}$ to $\mathbb{P}_{n-1,m}$ is*

$$\mathbb{V}_n(\boldsymbol{\mu}) \downarrow_{\mathbb{P}_{n-1,m}} \cong \bigoplus_{h=0}^{m-1} \Bigg(\Big(\bigoplus_{\substack{\boldsymbol{\mu}'_h \rhd \boldsymbol{\mu}_h \\ \boldsymbol{\mu}'_i = \boldsymbol{\mu}_i \forall i \neq h}} \mathbb{V}_{n-1}(\boldsymbol{\mu}')\Big) \bigoplus \Big(\bigoplus_{\substack{\boldsymbol{\mu}'_h \lhd\rhd \boldsymbol{\mu}_h \\ \boldsymbol{\mu}'_i = \boldsymbol{\mu}_i \forall i \neq h}} \mathbb{V}_{n-1}(\boldsymbol{\mu}')\Big) \bigoplus \Big(\bigoplus_{\substack{\boldsymbol{\mu}'_h \lhd \boldsymbol{\mu}_h \\ \boldsymbol{\mu}'_i = \boldsymbol{\mu}_i \forall i \neq h}} \mathbb{V}_{n-1}(\boldsymbol{\mu}')\Big) \Bigg).$$

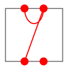
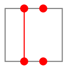
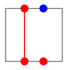
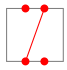
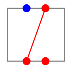
Proof. Any element in $\mathbb{V}_n(\boldsymbol{\mu})$ can be written as m-tuples of partitions, see (4.11). Each one is included in a module $\mathfrak{V}_{\lambda_i+u_i}(\boldsymbol{\mu}_i)$ (see (1.15)) of the partition algebra $\mathbb{P}_{\lambda_i+u_i}(\delta_i)$ for some u_i such that $\sum \lambda_i + u_i = n$ by ignoring all the colours except the one $\mathfrak{C}_i$. We finish the proof by using the inclusion $\mathbb{P}_{n-1,m} \hookrightarrow \mathbb{P}_{n,m}$ and Proposition 1.26. □

By using the Frobenius reciprocity, see (1.1), and the previous fact we can compute the induced modules of the cell modules of the algebra $\mathbb{P}_{n,m}$.

Example 4.18.1. *Let $m = 2$. By last proposition, for generic values δ_0 and δ_1 we have*

$$\mathbb{V}_2((1),(0)) \downarrow_{\mathbb{P}_{1,2}} \cong \Big(\bigoplus^{3} \mathbb{V}_1((1),(0)) \Big) \oplus \mathbb{V}_1((0),(0)). \quad \textit{(see figure 4.1).}$$

We can show that as following: the module $\mathbb{V}_2((1),(0))$ *can be taken to be the module that is spanned by the elements*

$b_1 =$ , $b_2 =$ , $b_3 =$ , $b_4 =$ , $b_5 =$ .

On the other hand the module $\mathbb{V}_1((1),(0)) = \mathbb{F}\langle 1_{(0)}\rangle$ *and the module* $\mathbb{V}_1((0),(0))$ *has a basis containing the elements* $a_1 = \mathsf{q}_{(1,(0),(0))}$ *and* $a_2 = \mathsf{q}_{(1,(1),(0))}$. *Now if* $\delta_0 \neq 0$, *it is easy to show that the spaces* $X_1 = \mathbb{F}\langle b_1 - \delta_0^{-1} b_4\rangle$, $X_2 = \mathbb{F}\langle b_2\rangle$ *and* $X_3 = \mathbb{F}\langle b_3\rangle$ *are* $\mathbb{P}_{1,2}$*-modules and all of them isomorphic to the module* $\mathbb{V}_1((1),(0))$. *Similarly* $X_4 = \mathbb{F}\langle b_4, b_5\rangle$ *is* $\mathbb{P}_{1,2}$*-module and it is isomorphic to the module* $\mathbb{V}_1((0),(0))$,*we can show that by using the map that sends* a_1 *and* a_2 *to* b_4 *and* b_5 *respectively. Note that* $\mathbb{V}_2((1),(0)) = \bigoplus_{i=1}^{4} X_i$, *and we are done.*

The generic Bratteli restriction diagram for the irreducible representations associated to the inclusion $\mathbb{P}_{n,m} \hookrightarrow \mathbb{P}_{n-1,m}$ is shown in figure 4.1. From that we can compute the dimensions of the generic simple modules of $\mathbb{P}_{n,m}$, see table 4.1.

$n \setminus \boldsymbol{\mu}$	$((0),(0))$	$((1),(0))$	$((0),(1))$	$((1),(1))$	$((1^2),(0))$	$((2),(0))$
0	1					
1	2	1	1			
2	6	5	5	2	1	1
3	22	25	25	18	9	9
4	94	133	133	134	67	67
$n \setminus \boldsymbol{\mu}$	$((0),(1^2))$	$((0),(2))$	$((1^2),(1))$	$((2),(1))$	$((1),(1^2))$	$((1),(2))$
2	1	1				
3	9	9	3	3	3	3
4	67	67	42	42	42	42
$n \setminus \boldsymbol{\mu}$	$((3),(0))$	$((2,1),(0))$	$((1^3),(0))$	$((0),(3))$	$((0),(2,1))$	$\ldots$
3	1	2	1	1	2	$\ldots$
4	14	26	14	14	26	$\ldots$

TABLE 4.1: The dimensions of some cell modules of the algebra $\mathbb{P}_{n,m}$.

It is easy to show that the module $\mathbb{V}_n(\overline{0})$ is isomorphic to $\mathbb{P}_{n,m}\mathbf{I}_{0,x}$ for any $x \in \mathbb{Z}_m^n$, such that $\delta_{x_j} \neq 0$ for each j, since x doesn't have any role here. In both of spaces, the basis is in one-to-one correspondence with the elements of the set:

$$\{(d_0,\ldots,d_{m-1}) \mid \ d_i \in \mathcal{P}_{D_i}, \text{ for some } \ D_i \subseteq \underline{n}, \text{ such that } \bigcup_{i=0}^{m-1} d_i \in \mathcal{P}_{\underline{n}}\}.$$

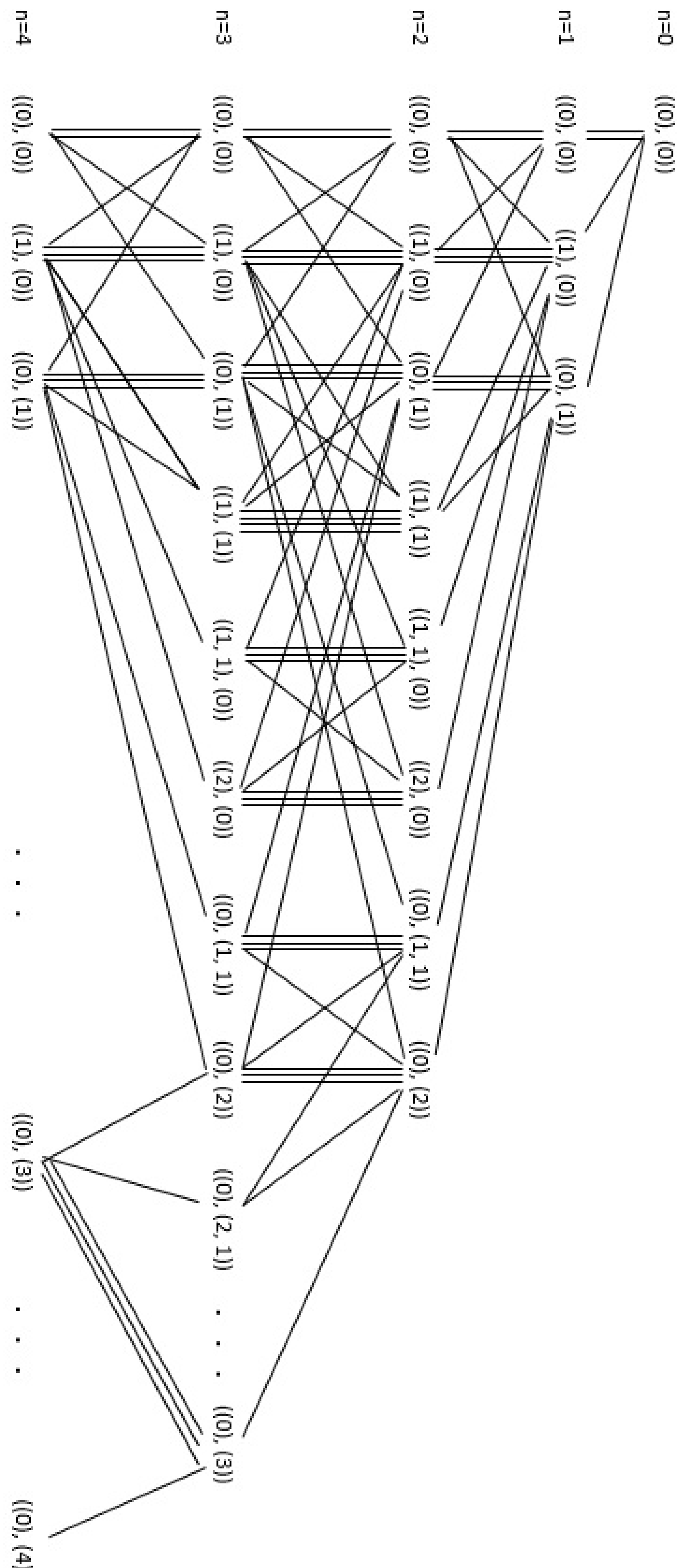


FIGURE 4.1: The Bratteli diagram for a tower of multi-colour partition algebras.

Let $d = (d_0, \ldots, d_{m-1})$ and $b = (b_0, \ldots, b_{m-1})$ be basis elements of $\mathbb{V}_n(\overline{0})$, where $d_i \in \mathcal{P}_{D_i}$ and $b_i \in \mathcal{P}_{B_i}$ for some subsets D_i and B_i. From graphical visualization, we have

$$\langle d, b\rangle_{\overline{0},\overline{0}} = \begin{cases} 0 & \text{if } D_i \neq B_i \text{ for some } i, \\ \prod_{i=0}^{m-1} \langle d_i, b_i\rangle_{0,(0)} & \text{otherwise,} \end{cases}$$

where $\langle d_i, b_i\rangle_{0,(0)}$ is computed in the algebra $\mathbb{P}_{|D_i|}(\delta_i)$.

Let $\mathbb{M}_{n,\delta}((0))$ be the Gram matrix of $\mathbb{P}_n(\delta)$ of the inner product corresponding to the trivial partition of 0. From the last equation, it is obvious that $\mathbb{G}_n(\overline{0})$ is a direct sum of tensor products of matrices $\mathbb{M}_{n_i,\delta_i}((0))$ such that $\sum n_i = n$. Furthermore, the matrix $\mathbb{M}_{n_i,\delta_i}((0))$ is singular over complex field if and only if $\delta_i \in \{0, 1, \ldots, n_i - 1\}$, for more details see [43]. By using the property 1.8, the matrix $\mathbb{G}_n(\overline{0})$ is singular over the complex field if and only if one of the matrices $\mathbb{M}_{n_i,\delta_i}((0))$ is singular, this implies $\mathbb{V}_n(\overline{0})$ is simple unless one of the scalars δ_i is a natural number less than n.

Theorem 4.19. *The algebra $\mathbb{P}_{n,m}(\delta_0, \ldots, \delta_{m-1})$ is semisimple over $\mathbb{C}$ for each integers $n \geq 0$ and $m \geq 1$ if and only if none of the parameters δ_i is a a natural number less than $2n$.*

Proof. By using the induction/restriction rules and Frobenius reciprocity, then the module $\mathbb{V}_{2n}(\overline{0})$ as $\mathbb{P}_{n,m}$-module contains all the modules $\mathbb{V}_n(\boldsymbol{\mu})$ as sub-modules. Hence $\det \mathbb{G}_{2n}(\overline{0}) \neq 0$ if and only if $\det \mathbb{G}_n(\boldsymbol{\mu}) \neq 0$ for $\boldsymbol{\mu}$ in level n. This implies that if $\mathbb{V}_{2n}(\overline{0})$ is simple, then all the modules $\mathbb{V}_n(\boldsymbol{\mu})$ is simple for all $\breve{\delta}$ at level n, so the algebra $\mathbb{P}_{n,m}$ is semi-simple. As it is shown in above $\mathbb{V}_{2n}(\overline{0})$ is simple over the complex field unless one of the scalars δ_i is a natural number less than $2n$. □

4.5 Some low rank calculations

We end this chapter by discussing the above properties on two examples to illustrate the main results in the previous sections over the complex field.

Let us consider $n = 1$. The algebra $\mathbb{P}_{n,2}$ is a 6-dimensional algebra. The vector spaces V_λ are

$$V_{(1,0)} = \mathbb{C}\big\langle\big((\{1\},\{1\}),(\emptyset,\emptyset)\big)\big\rangle \ , \ V_{(0,1)} = \mathbb{C}\big\langle\big((\emptyset,\emptyset),(\{1\},\{1\})\big)\big\rangle$$

$$V_{(0,0)} = \mathbb{C}\big\langle\big((\{1\},\emptyset),(\emptyset,\emptyset)\big),\big((\emptyset,\emptyset),(\{1\},\emptyset)\big)\big\rangle,$$

and also $\mathbb{S}_{1,(1,0)} \cong \mathbb{C}\mathfrak{S}_1 \times \mathfrak{S}_0 \cong \mathbb{C}$, similarly $\mathbb{S}_{1,(0,1)} \cong \mathbb{C}$ and $\mathbb{S}_{0,(0,0)} \cong \mathbb{C}$. It is easy to see that

$$\mathbb{P}_{1,2} = \mathbb{C}\langle 1_{(1)}, \mathsf{p}_{(1,(0),(1))}, \mathsf{p}_{(1,(1),(1))}\rangle \bigoplus \mathbb{C}\langle 1_{(0)}, \mathsf{p}_{(1,(0),(0))}, \mathsf{p}_{(1,(1),(0))}\rangle,$$

as left modules.

To make notation easier, put $a_1 = 1_{(0)}$, $a_2 = 1_{(1)}$, $a_3 = \mathsf{p}_{(1,(0),(1))}$, $a_4 = \mathsf{p}_{(1,(1),(1))}$, $a_5 = \mathsf{p}_{(1,(0),(0))}$ and $a_6 = \mathsf{p}_{(1,(1),(0))}$. The cell modules of $\mathbb{P}_{1,2}$ are

$$\mathbb{V}_1((1),(0)) = \mathbb{C}a_1 \ , \ \mathbb{V}_1((0),(1)) = \mathbb{C}a_2,$$

$$\mathbb{V}_1((0),(0)) = \mathbb{C}\langle a_3, a_4\rangle,$$

Both of $\mathbb{V}_1((1),(0))$ and $\mathbb{V}_1((0),(1))$ are simple, since their dimension is one. On the other hand, $\mathbb{V}_1((0),(0))$ is a simple module unless $\delta_0\delta_1 = 0$, since the Gram matrix of $\mathbb{V}_1((0),(0))$ is

$$\begin{pmatrix} \delta_0 & 0 \\ 0 & \delta_1 \end{pmatrix}.$$

Now the action of the algebra $\mathbb{P}_{1,2}$ on $\mathbb{V}_1((0),(0))$ is given by

$$\big(\sum_{i=1}^{6} \alpha_i a_i\big) a_3 = (\alpha_1 + \delta_0\alpha_5)a_3 + \delta_0\alpha_6 a_4,$$

$$\big(\sum_{i=1}^{6} \alpha_i a_i\big) a_4 = \delta_1\alpha_3 a_3 + (\alpha_2 + \delta_1\alpha_4)a_4.$$

In the case $\delta_0 = 0$ and $\delta_1 \neq 0$, then $\mathbb{V}_1((0),(0))$ is an indecomposable module with sub-module $\mathbb{C}a_3$, and $\mathbb{C}a_3 \cong \mathbb{C}a_1$ as modules. Now, if $\delta_0 = 0$ and $\delta_1 = 0$, then

$\mathbb{V}_1((0),(0))$ is decomposable and it is isomorphic to $\mathbb{V}_1((1),(0)) \oplus \mathbb{V}_1((0),(1))$.

For the second example, take $n = 2$. The algebra $\mathbb{P}_{2,2}$ is 94-dimensional. The corresponding vector spaces V_λ are

$$\begin{aligned}
V_{(2,0)} =& \mathbb{C}\langle \Big((\{\{1\},\{2\}\},\{\{1\},\{2\}\}),(\emptyset,\emptyset) \Big) \rangle,\\
V_{(0,2)} =& \mathbb{C}\langle \Big((\emptyset,\emptyset),(\{\{1\},\{2\}\},\{\{1\},\{2\}\}) \Big) \rangle,\\
V_{(1,0)} =& \mathbb{C}\langle \Big((\{1,2\},\{1,2\}),(\emptyset,\emptyset) \Big), \Big((\{1\},\{1\}),(\{2\},\emptyset) \Big), \Big((\{2\},\{2\}),(\{1\},\emptyset) \Big),\\
& \Big((\{\{1\},\{2\}\},\{1\}),(\emptyset,\emptyset) \Big), \Big((\{\{1\},\{2\}\},\{2\}),(\emptyset,\emptyset) \Big) \rangle,\\
V_{(0,1)} =& \mathbb{C}\langle \Big((\emptyset,\emptyset),(\{1,2\},\{1,2\}) \Big), \Big((\{2\},\emptyset),(\{1\},\{1\}) \Big), \Big((\{1\},\emptyset),(\{2\},\{2\}) \Big),\\
& \Big((\emptyset,\emptyset),(\{\{1\},\{2\}\},\{1\}) \Big), \Big((\emptyset,\emptyset),(\{\{1\},\{2\}\},\{2\}) \Big) \rangle,\\
V_{(1,1)} =& \mathbb{C}\langle \Big((\{1\},\{1\}),(\{2\},\{2\}) \Big), \Big((\{2\},\{2\}),(\{1\},\{1\}) \Big) \rangle,\\
V_{(0,0)} =& \mathbb{C}\langle \Big((\emptyset,\emptyset),(\{1,2\},\emptyset) \Big), \Big((\{1,2\},\emptyset),(\emptyset,\emptyset) \Big), \Big((\{1\},\emptyset),(\{2\},\emptyset) \Big),\\
& \Big((\{2\},\emptyset),(\{1\},\emptyset) \Big), \Big((\{\{1\},\{2\}\},\emptyset),(\emptyset,\emptyset) \Big), \Big((\emptyset,\emptyset),(\{\{1\},\{2\}\},\emptyset) \Big) \rangle.
\end{aligned}$$

To compute the modules $\mathbb{V}_2((2),(0))$, $\mathbb{V}_2((1,1),(0))$, we need to determine the simple modules of $\mathbb{C}\mathfrak{S}_2$, which are only the trivial and sign modules, they are $\mathbb{C}\langle id + \mathsf{s}_1\rangle$ and $\mathbb{C}\langle id - \mathsf{s}_1\rangle$ respectively. This implies that

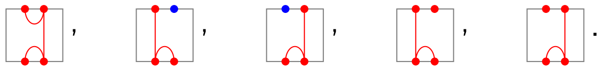
$$\begin{aligned}
\mathbb{V}_2((2),(0)) &= \mathbb{C}\langle 1_{(0,0)} + \mathsf{s}_{(1,(0,0))}\rangle,\\
\mathbb{V}_2((1,1),(0)) &= \mathbb{C}\langle 1_{(0,0)} - \mathsf{s}_{(1,(0,0))}\rangle.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{V}_2((0),(2)) &= \mathbb{C}\langle 1_{(1,1)} + \mathsf{s}_{(1,(1,1))}\rangle,\\
\mathbb{V}_2((0),(1,1)) &= \mathbb{C}\langle 1_{(0,0)} - \mathsf{s}_{(1,(1,1))}\rangle,\\
\mathbb{V}_2((1),(1)) &= \mathbb{C}\langle 1_{(0,1)}, \mathsf{s}_{(1,(1,0))}\rangle.
\end{aligned}$$

All the previous cell modules are simple.

Also, the module $\mathbb{V}_{2,}((1),(0))$ can be spanned by the elements:

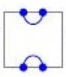
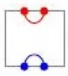
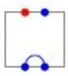
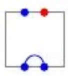
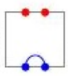
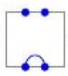


The Gram matrix of this module with respect to the previous basis with the same order is

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & \delta_1 & 0 & 0 & 0 \\ 0 & 0 & \delta_1 & 0 & 0 \\ 1 & 0 & 0 & \delta_0 & 0 \\ 1 & 0 & 0 & 0 & \delta_0 \end{pmatrix},$$

and its determinant is $\delta_1^2\delta_0(\delta_0-2)$. Thus the module $\mathbb{V}_2((1),(0))$ is simple unless $\delta_1\delta_0(\delta_0-2)=0$. When $\delta_0=2$ and $\delta_1\neq 0$, the module $\mathbb{V}_2((1),(0))$ is indecomposable with radical spanned by $2\mathsf{q}_{(1,(0,0))}-\mathsf{p}_{(1,(0,0),(0,0))}\mathsf{q}_{(1,(0,0))}-\mathsf{p}_{(2,(0,0),(0,0))}\mathsf{q}_{(1,(0,0))}$. Similarly, $\mathbb{V}_2((0),(1))$ is simple unless $\delta_0^2\delta_1(\delta_1-2)=0$.

Finally, the module $\mathbb{V}_2((0),(0))$ can be spanned by

$a_1=$ , $a_2=$ , $a_3=$ , $a_4=$ , $a_5=$ , $a_6=$ .

The Gram matrix of $\mathbb{V}_2((0),(0))$ with respect to the previous basis with the same order is

$$\begin{pmatrix} \delta_1 & 0 & 0 & 0 & 0 & \delta_1 \\ 0 & \delta_0 & 0 & 0 & \delta_0 & 0 \\ 0 & 0 & \delta_0\delta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_0\delta_1 & 0 & 0 \\ 0 & \delta_0 & 0 & 0 & \delta_0^2 & 0 \\ \delta_1 & 0 & 0 & 0 & 0 & \delta_1^2 \end{pmatrix}.$$

Thus $\mathbb{V}_2((0),(0))$ is simple unless $\delta_0^4\delta_1^4(\delta_0-1)(\delta_1-1)=0$. When $\delta_0=1=\delta_1$, the radical of this module is $\mathbb{C}\langle a_1-a_6, a_2-a_5\rangle$.

Chapter 5

Representation Theory Of The Algebra $\mathbb{T}_{n,m}$

The notion of the bubble algebra $\mathbb{T}_{n,m}(\breve{\delta})$ has been introduced by Grimm and Martin [23], and they proved various properties of the algebra as it is generically semi-simple. Jegan [28] in Section 2.1 showed that the bubble algebra $\mathbb{T}_{n,m}(\breve{\delta})$ is always a cellular algebra, and it is a tower recollement when all of the parameters δ_i are non-zero. The theory of towers of algebras has been introduced in [11]. In this chapter we provide some further information on the structure of the bubble algebra, for instance calculating its Cartan matrix over the complex field by investigating the head and the radical of each cell module by using the ones of the Temperley-Lieb algebra.

5.1 Introduction

This section introduces some of the basic modules for the algebra $\mathbb{T}_{n,m}(\breve{\delta})$ that are related to cell modules of $\mathbb{T}_{n,m}(\breve{\delta})$ since it is cellular.

The number of propagating lines in any diagram $d \in \mathcal{T}_{n,m}$, see (2.11), has the form $\#(d) = n - 2v$ for some an integer v, where $0 \leq v \leq [^n/_2]$, since making an arc, an edge connects two nodes in the same row (top or bottom) of a diagram, needs to two nodes on this row.

We follow Grimm and Martin [23] and define the subsets $\mathcal{T}_{n,m}[\lambda_0, \dots, \lambda_{m-1}]$, or simply $\mathcal{T}_{n,m}[\lambda]$, $\mathcal{T}_{n,m}[k]$ and $\mathcal{T}_{n,m}(k)$ of the set $\mathcal{T}_{n,m}$ to be

$$\mathcal{T}_{n,m}[\lambda_0, \dots, \lambda_{m-1}] = \{d \in \mathcal{T}_{n,m} \mid \#_j(d) = \lambda_j \text{ for all } j \in \mathbb{Z}_m\}, \tag{5.1}$$

$$\mathcal{T}_{n,m}[k] = \bigcup_{\sum_j \lambda_j = k} \mathcal{T}_{n,m}[\lambda_0, \dots, \lambda_{m-1}], \tag{5.2}$$

$$\mathcal{T}_{n,m}(k) = \bigcup_{l \leq k} \mathcal{T}_{n,m}[l], \tag{5.3}$$

where $\lambda_i \in \mathbb{N}$ and $\sum_{j=0}^{m-1} \lambda_j$, $k \in \{n, n-2, \dots, n-2[{}^n/_2]\}$, and $\#_j(d)$ is the number of propagating lines in a diagram d that have the colour $\mathfrak{C}_j$.

Definition 5.1. Let $\mathbb{T}_{n,m}(\breve{\delta}; \lambda_0, \dots, \lambda_{m-1})$, or simply $\mathbb{T}_{n,m}(\breve{\delta}; \lambda)$, and $\mathbb{T}_{n,m}(\breve{\delta}; k)$ be the ideals of the algebra $\mathbb{T}_{n,m}(\breve{\delta})$ that are generated by the sets $\mathcal{T}_{n,m}[\lambda]$ and $\mathcal{T}_{n,m}[k]$, respectively.

Lemma 5.2. *The ideal* $\mathbb{T}_{n,m}(\breve{\delta}; \lambda)$ *has the set* $\bigcup_{0 \leq l_j \leq [{}^{\lambda_j}/_2]} \mathcal{T}_{n,m}[\lambda - 2l]$ *as basis, where* $\lambda - 2l = (\lambda_0 - 2l_0, \dots, \lambda_{m-1} - 2l_{m-1})$.

Proof. The proof is similar to showing that the ideal $\mathbb{P}_{n,m}(\breve{\delta}; \lambda)$ has the set $\mathcal{P}_{n,m}(\lambda)$ as basis (Proposition 2.13). □

The set of all ideals of the algebra $\mathbb{T}_{n,m}$ that have the form $\mathbb{T}_{n,m}(\breve{\delta}; \lambda)$ is a lattice with a partial order: $\mathbb{T}_{n,m}(\breve{\delta}; \lambda_0, \dots, \lambda_{m-1}) \leq \mathbb{T}_{n,m}(\breve{\delta}; j_0, \dots, j_{m-1})$ if and only if $j_k - \lambda_k$ is an even non-negative number for each k.

Now we define a $\mathbb{T}_{n,m}(\breve{\delta})$-module to be the quotient

$$\mathbb{T}_{n,m}[\breve{\delta}; \lambda_0, \dots, \lambda_{m-1}] = \frac{\mathbb{T}_{n,m}(\breve{\delta}; \lambda_0, \dots, \lambda_{m-1})}{\mathbb{T}_{n,m}(\breve{\delta}; \lambda_0, \dots, \lambda_{m-1}) \bigcap \mathbb{T}_{n,m}(\breve{\delta}; \sum \lambda_j - 2)}\ , \tag{5.4}$$

where $\lambda \in \Gamma_{(n-2v,m)}$ for some $0 \leq v \leq [^n/_2]$ (recall that $\Gamma_{(l,m)}$ is the set of all m-compositions of l, see Section 1.3.1). Note that the ideal $\mathbb{T}_{n,m}(\breve{\delta}; \sum \lambda_j - 2)$ will be taken to be zero when $\sum \lambda_j < 2$.

Lemma 5.3. *The module $\mathbb{T}_{n,m}[\breve{\delta}; \lambda_0, \ldots, \lambda_{m-1}]$ has the set $\mathcal{T}_{n,m}[\lambda]$ as a basis, where $\lambda \in \Gamma_{(n-2v,m)}$ for some $0 \leq v \leq [^n/_2]$.*

Proof. It comes directly from Lemma 5.2, where the image of all the diagrams that have propagating number lesser than $\sum \lambda_j$ will be zero. □

The ideal $\mathbb{T}_{n,m}(\breve{\delta}; k)$, from the definition, contains all the diagrams with at most k propagating lines. We may form a filtration of $\mathbb{T}_{n,m}$ by these ideals:

$$\mathbb{T}_{n,m}(\breve{\delta}) \supset \mathbb{T}_{n,m}(\breve{\delta}; n-2) \supset \mathbb{T}_{n,m}(\breve{\delta}; n-4) \supset \ldots \supset \mathbb{T}_{n,m}\big(\breve{\delta}; n - 2[^n/_2]\big) \supset 0. \quad (5.5)$$

This filtration refines to one with section spanned by the set $\mathcal{T}_{n,m}[\lambda]$, where $n - \sum \lambda_j$ is an even number.

A *half-multi-colour-diagram* is a diagram obtained by cutting horizontally a diagram in the set $\mathcal{T}_{n,m}$ in the middle such that each propagating line is cut once, and that is always possible. As for the Temperley-Lieb algebra, we can form a unique bubble diagram from two half-multi-colour-diagrams providing that they have the same number of propagating lines of each colour. Let $\mathcal{T}^{|\rangle}_{n,m}[\lambda]$ be the set of top pieces obtained by cutting elements of the set $\mathcal{T}_{n,m}[\lambda]$, where $\lambda \in \Gamma_{(n-2v,m)}$ for some $0 \leq v \leq [^n/_2]$. Similarly $\mathcal{T}^{\langle|}_{n,m}[\lambda]$ is the set of bottom pieces obtained by cutting elements of $\mathcal{T}_{n,m}[\lambda]$. Denote by $|D\rangle$ and $\langle D|$ the top half-multi-colour-diagram and the bottom half-multi-colour-diagram respectively obtained from cutting a diagram $D \in \mathcal{T}_{n,m}[\lambda]$.

Note that we shall often refer to an element of $\mathcal{T}^{|\rangle}_{n,m}[\lambda]$ as half diagram or just a diagram when it is clear, although we mean this to be taken as a top half-multi-colour-diagram.

A half-diagram is called a $((n_0, p_0), \ldots, (n_{m-1}, p_{m-1}))$-*link state*, if it contains both n_j nodes and p_j arcs of the colour $\mathfrak{C}_j$ for each $j \in \mathbb{Z}_m$ where $\sum_{j=0}^{m-1} n_j = n$. This means that there are $n_j - 2p_j$ unconnected nodes of the colour $\mathfrak{C}_j$ for each j. We refer them as defects as for the Temperley-Lieb algebra.

Denote by $\mathbb{F}\mathsf{M}_n(\lambda_0, \ldots, \lambda_{m-1})$, or simply $\mathbb{F}\mathsf{M}_n(\lambda)$, the vector space with a basis $\mathsf{M}_n(\lambda_0, \ldots, \lambda_{m-1})$ which contains all the top halves of all diagrams that are contained in $\mathbb{T}_{n,m}(\breve{\delta}; \lambda)$. In other words, $\mathsf{M}_n(\lambda)$ contains all link states that have number of defects of the colour $\mathfrak{C}_j$ on the form $\lambda_j - 2t_j$ for each $j \in \mathbb{Z}_m$ where $0 \leq t_j \leq [{}^{\lambda_j}/_2]$. Note that there is no condition on the colours of arcs.

Lemma 5.4. *Let* $\lambda \in \Gamma_{(n-2v,m)}$ *for some* $0 \leq v \leq [{}^n/_2]$. *The vector space* $\mathbb{F}\mathsf{M}_n(\lambda)$ *is a left* $\mathbb{T}_{n,m}$*- module with the action defined by the concatenation of a diagram with a half-diagram then proceeding as we would with two diagrams in* $\mathbb{T}_{n,m}$ *(remove each loop and replace it by the parameter corresponding to the loop's colour), and finally omit any new bottom arcs.*

Proof. Let $x \in \mathcal{T}_{n,m}$ and d be a half-diagram in $\mathsf{M}_n(\lambda)$. We only need to show that $xd \in \mathbb{F}\mathsf{M}_n(\lambda)$. Without loss of generality, we can assume $xd \neq 0$. Multiplying x with d cannot create any additional propagating lines of any colour. Thus the number of $\mathfrak{C}_j$-defects in xd is of the form $\lambda_j - 2t_j$ where $0 \leq t_j \leq [{}^{\lambda_j}/_2]$, because making an extra $\mathfrak{C}_j$-arc needs two $\mathfrak{C}_j$-nodes. □

Define a subset $\mathsf{M}_n^<(\lambda)$ of $\mathsf{M}_n(\lambda)$ to be

$$\mathsf{M}_n^<(\lambda_0, \ldots, \lambda_{m-1}) = \bigcup_{j=0}^{m-1} \mathsf{M}_n(\lambda_0, \ldots, \lambda_j - 2, \ldots, \lambda_{m-1}). \tag{5.6}$$

Note that $\mathsf{M}_n(\lambda_0, \ldots, \lambda_j - 2, \ldots, \lambda_{m-1})$ is taken to be the empty-set when $\lambda_j < 2$.

Let $\mathbb{F}\mathsf{M}_n^<(\lambda)$ be the module that generated by $\mathsf{M}_n^<(\lambda)$, thus $\mathbb{F}\mathsf{M}_n^<(\lambda)$ is a sub-module of $\mathbb{F}\mathsf{M}_n(\lambda)$.

Lemma 5.5. *Let* $\Delta_n(\lambda_0, \ldots, \lambda_{m-1})$*, or simply* $\Delta_n(\lambda)$*, be the module* $\mathbb{F}\mathsf{M}_n(\lambda)/\mathbb{F}\mathsf{M}_n^<(\lambda)$ *of* $\mathbb{T}_{n,m}$*, where* $\sum \lambda_i = n - 2v$ *for some* $0 \leq v \leq [{}^n/_2]$. *Then the module* $\Delta_n(\lambda)$ *has the set* $\mathcal{T}_{n,m}^{|\rangle}[\lambda]$ *as a basis.*

Proof. In the quotient $\mathbb{F}\mathsf{M}_n(\lambda)/\mathbb{F}\mathsf{M}_n^{<}(\lambda)$, the image of any link state with less than λ_j defects of the colour $\mathfrak{C}_j$ for each j is zero. Thus the left multiplication by any diagram in the set $\mathcal{T}_{n,m}$ will be either zero or a half-diagram with exactly λ_j defects of the colour $\mathfrak{C}_j$ for each j multiplied by a scalar. □

Example 5.5.1. *The module* $\Delta_3(1,0)$ *for the algebra* $\mathbb{T}_{3,2}(\breve{\delta})$ *is spanned by the half-diagrams in figure 5.1.*

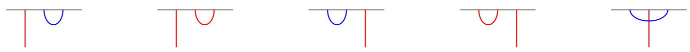


FIGURE 5.1: A basis of the module $\Delta_3(1,0)$.

The module $\mathbb{T}_{n,m}[\breve{\delta};\lambda]$, defined in (5.4), has $\mathcal{T}_{n,m}[\lambda]$ as basis (see Lemma 5.3). Since any two half-diagrams connect in unique way, we obtain a bijection

$$\mathcal{T}_{n,m}[\lambda_0,\dots,\lambda_{m-1}] \leftrightarrow \mathcal{T}^{|\rangle}_{n,m}[\lambda_0,\dots,\lambda_{m-1}] \times \mathcal{T}^{\langle|}_{n,m}[\lambda_0,\dots,\lambda_{m-1}].$$

Thus each module $\mathbb{T}_{n,m}[\breve{\delta};\lambda]$ breaks up as a sum of isomorphic left modules each with basis of the form $\{\ |a\rangle\langle b|\ |\ a \in \mathcal{T}_{n,m}[\lambda]\}$ where $b \in \mathcal{T}_{n,m}[\lambda]$ is fixed. Also the half-diagram $\langle b|$ in the definition of basis elements has no role, so each summand is isomorphic to the module $\Delta_n(\lambda)$.

5.2 Cellularity of the bubble algebra $\mathbb{T}_{n,m}(\breve{\delta})$

In the next few sections, we shall begin studying the representation theory of the algebra $\mathbb{T}_{n,m}(\breve{\delta})$.

Proposition 5.6. *[28, Proposition 1.3.2]. The algebra* $\mathbb{T}_{n,m}(\breve{\delta})$ *is a cellular algebra over any field* $\mathbb{F}$*, with the involution sending each diagram* d *to its reflection* d^* *in the horizontal plane. Also the indexing set is the set*

$$\Lambda_{\mathbb{T}_{n,m}} := \bigcup_{v=0}^{[n/2]} \Gamma_{(n-2v,m)}, \tag{5.7}$$

where $\Gamma_{(l,m)}$ *is the set of all* m*-compositions of* l*, see Section 1.3.1. The order on the set* $\Lambda_{\mathbb{T}_{n,m}}$ *is defined by*

$$(\lambda_0, \dots, \lambda_{m-1}) \geq (\lambda'_0, \dots, \lambda'_{m-1}) \text{ if and only if } \lambda_j \leq \lambda'_j \text{ for each } j. \tag{5.8}$$

The modules $\Delta_n(\lambda)$ *where* $\lambda \in \Lambda_{\mathbb{T}_{n,m}}$ *are cell modules of the algebra* $\mathbb{T}_{n,m}$*.* □

Each cell module $\Delta_n(\lambda)$ comes with a bilinear form via its basis of top half-diagrams (and the dual basis of bottom half-diagrams). Let $d, d' \in \mathcal{T}_{n,m}[\lambda]$, $x = \langle d|$ and $y = |d'\rangle$, so

$$dd' = |d\rangle\langle d|\, |d'\rangle\langle d'| = \langle d||d'\rangle\ |d\rangle\langle d'| =: \langle d||d'\rangle d'',$$

so

$$\langle x, y\rangle = \begin{cases} \langle d||d'\rangle & \text{if } d'' \in \mathcal{T}_{n,m}[\lambda], \\ 0 & \text{otherwise.} \end{cases} \tag{5.9}$$

This form is contravariant, see Proposition 2.4 in [20].

Let $\mathsf{G}_n(\lambda)$ to be the Gram matrix of the inner product defined in (5.9) on the cell module $\Delta_n(\lambda)$ with respect to half-diagrams basis. Since we work over a field, we can check when the module $\Delta_n(\lambda)$ is simple or not by computing $\det \mathsf{G}_n(\lambda)$, since $\Delta_n(\lambda)$ is simple if and only if $\det \mathsf{G}_n(\lambda) \neq 0$ whenever $\langle\ ,\ \rangle \neq 0$ (see Section 1.5). For example, the Gram matrix of $\Delta_3(1,0)$ with respect to the basis in figure 5.1 is

$$\mathsf{G}_3(1,0) = \begin{pmatrix} \delta_0 & 1 & 0 & 0 & 0 \\ 1 & \delta_0 & 0 & 0 & 0 \\ 0 & 0 & \delta_1 & 0 & 0 \\ 0 & 0 & 0 & \delta_1 & 0 \\ 0 & 0 & 0 & 0 & \delta_1 \end{pmatrix}.$$

Note that $|\mathsf{G}_3(1,0)| = \delta_1^3(\delta_0^2 - 1)$, so the module $\Delta_3(1,0)$ is simple if and only if $\delta_0 \neq \pm 1$ and $\delta_1 \neq 0$.

Theorem 5.7. *[23, Theorem 1]. The cell modules* $\Delta_n(\lambda_0, \lambda_1)$ *for the algebra* $\mathbb{T}_{n,2}$ *are generically simple.*

Let $\Lambda^0_{\mathbb{T}_{n,m}}$ be subset of $\Lambda_{\mathbb{T}_{n,m}}$ that contains all $\lambda \in \Lambda_{\mathbb{T}_{n,m}}$ such that $\langle\ ,\ \rangle \neq 0$. Note that if $\delta_j \neq 0$ for some $0 \leq j \leq m-1$, then we have

$$\Lambda^0_{\mathbb{T}_{n,m}} = \Lambda_{\mathbb{T}_{n,m}},$$

as we can take a half diagram with all the arcs of the colours corresponding to non-zero scalars. Even if $\delta_j = 0$ for all j, then for each cell module $\Delta_n(\lambda)$ such that $\sum_{j=0}^{m-1} \lambda_j \neq 0$, the inner product $\langle\ ,\ \rangle \neq 0$ because we can still find diagrams such their product is equal to one. For example, see figure 5.2. Thus $\Lambda^0_{\mathbb{T}_{n,m}} = \Lambda_{\mathbb{T}_{n,m}}$ unless n is an even integer and $\delta_i = 0$ for each $i \in \mathbb{Z}_m$. In the case n is an even integer and $\delta_i = 0$ for each $i \in \mathbb{Z}_m$, then

$$\Lambda^0_{\mathbb{T}_{n,m}} = \Lambda_{\mathbb{T}_{n,m}} \setminus \{(0, \ldots, 0)\}.$$

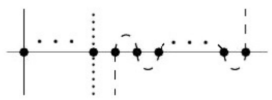


FIGURE 5.2: A non-zero product of two half-diagrams when $\sum \lambda_i \neq 0$.

Proposition 5.8. *The bubble algebra* $\mathbb{T}_{n,m}(\breve{\delta})$ *is a quasi-hereditary if and only if* $\delta_j \neq 0$ *for some* $0 \leq j < m$ *or* n *is an odd integer.*

Proof. It is enough to show that $\Lambda^0_{\mathbb{T}_{n,m}} = \Lambda_{\mathbb{T}_{n,m}}$ as we did in the last paragraph, by applying Remark 1.20. □

5.3 Examples

The following couple of examples illustrate the simplest cases of semi-simple and non-semi-simple bubble algebras.

Let $\{E_{ij}\}_{i,j\in\underline{n}}$ be the standard basis of the matrix algebra $M_n(\mathbb{F})$, and the diagrams $1_x, \mathsf{s}_{(i,x)}$ and $\mathsf{u}_{(i,u,v)}$ are defined as in Section 2.3 for some $i \in \underline{n-1}$, $x \in \mathbb{Z}_m^n$ and $(u,v) \in \Omega_i^*$, see (2.22).

Example 5.8.1. *From the definition of the multiplication on the algebra* $\mathbb{T}_{2,2}$, *we have*

$$\mathbb{T}_{2,2}(\delta_0,\delta_1) = \mathbb{F}\langle 1_x, 1_y, \mathsf{s}_{(1,x)}, \mathsf{s}_{(1,y)}\rangle \oplus \mathbb{F}\langle 1_u, 1_v, \mathsf{u}_{(1,u,u)}, \mathsf{u}_{(1,v,v)}, \mathsf{u}_{(1,u,v)}, \mathsf{u}_{(1,v,u)}\rangle,$$

as an algebra, where $x = (1,0)$, $y = (0,1)$, $u = (0,0)$, $v = (1,1)$. *Also it is easy to show that*

$$\mathbb{F}\langle 1_x, 1_y, \mathsf{s}_{(1,x)}, \mathsf{s}_{(1,y)}\rangle \cong M_2(\mathbb{F}),$$

for any pair (δ_0,δ_1) *and any field* $\mathbb{F}$. *Then the algebra* $\mathbb{F}\langle 1_x, 1_y, \mathsf{s}_{(1,x)}, \mathsf{s}_{(1,y)}\rangle$ *is always semi-simple. Hence the semisimplicity of the algebra* $\mathbb{T}_{2,2}$ *depends only on the algebra* $\mathbb{A} = \mathbb{F}\langle 1_u, 1_v, \mathsf{u}_{(1,u,u)}, \ \mathsf{u}_{(1,v,v)}, \mathsf{u}_{(1,u,v)}, \mathsf{u}_{(1,v,u)}\rangle$.

Next we are going to determine when the algebra $\mathbb{A}$ *is semi-simple and compute the Jacobson radical of* $\mathbb{A}$ *when it is not semi-simple.*

Let $\mathbb{A}_1$ *be the algebra*

$$\left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & e & f \end{pmatrix} \mid a,b,c,d,e,f \in \mathbb{F} \right\}.$$

Consider the map $f : A_1 \to \mathbb{A}$ *which is defined by*

$$\begin{aligned} f(E_{11}) &= 1_u - \delta_0^{-1}\mathsf{u}_{(1,u,u)}, & f(E_{22}) &= 1_v - \delta_1^{-1}\mathsf{u}_{(1,v,v)}, \\ f(E_{33}) &= \delta_0^{-1}\mathsf{u}_{(1,u,u)}, & f(E_{44}) &= \delta_1^{-1}\mathsf{u}_{(1,v,v)}, \\ f(E_{34}) &= \delta_0^{-1}\mathsf{u}_{(1,u,v)}, & f(E_{43}) &= \delta_1^{-1}\mathsf{u}_{(1,v,u)}, \end{aligned}$$

where δ_0 *and* δ_1 *are invertible in* $\mathbb{F}$. *Showing that* f *is a homomorphism is easy by checking all the relations of* $\mathbb{A}_1$. *One of these relations is* $E_{ii}^2 = E_{ii}$ *for each*

$i = 1, \ldots, 4$ and we have also $f(E_{ii})^2 = f(E_{ii})$ for each i, for example

$$f(E_{11})^2 = \left(1_u - \delta_0^{-1}\mathsf{u}_{(1,u,u)}\right)^2 = 1_u - 2\delta_0^{-1}\mathsf{u}_{(1,u,u)} + \delta_0^{-2}\mathsf{u}_{(1,u,u)}^2 = f(E_{11}).$$

Moreover, the determinant of the corresponding matrix of f with respect to the standard basis is $\delta_0^{-2}\delta_1^{-2}$, so it is an isomorphism if and only if $\delta_0\delta_1 \neq 0$. Therefore the algebra $\mathbb{T}_{2,2}$ is semi-simple when $\delta_0\delta_1 \neq 0$, and in this case $\mathbb{T}_{2,2}(\breve{\delta}) \cong \bigoplus^2 \mathbb{F} \oplus \bigoplus^2 M_2(\mathbb{F})$.

In the case when $\delta_0 = 0$ and $\delta_1 \neq 0$, it can be shown that $\mathbb{A} \cong \mathbb{A}_2$ and so $\mathbb{A}$ is not semi-simple, where

$$\mathbb{A}_2 = \left\{ \begin{pmatrix} a & d & e & 0 & 0 \\ 0 & b & f & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & c \end{pmatrix} \mid a, b, c, d, e, f \in \mathbb{F} \right\}.$$

To prove that we use the map which is defined by

$$\begin{aligned} 1_u &\mapsto E_{11} + E_{33} + E_{44}, & 1_v &\mapsto E_{22} + E_{55}, \\ \mathsf{u}_{(1,u,u)} &\mapsto E_{13}, & \mathsf{u}_{(1,v,v)} &\mapsto \delta_1 E_{22}, \\ \mathsf{u}_{(1,u,v)} &\mapsto E_{12}, & \mathsf{u}_{(1,v,u)} &\mapsto \delta_1 E_{23}. \end{aligned}$$

This map satisfies all the relations that defined the algebra $\mathbb{A}$, so it defines a homomorphism. The corresponding matrix to this homomorphism with respect to the basis $\{E_{11} + E_{33} + E_{44},\ E_{22},\ E_{12},\ E_{13},\ E_{23},\ E_{55}\}$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \delta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta_1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which is invertible when $\delta_1 \neq 0$, so $\mathbb{A} \cong \mathbb{A}_2$. Note that the Jacobson radical of the algebra $\mathbb{A}_2$ is the ideal $I = \mathbb{F}\langle E_{12}, E_{13}, E_{23}\rangle$ since I is a nilpotent ideal and $\mathbb{A}_2/I \cong$

$\bigoplus^{3} \mathbb{F}$ *(see for example Corollary 1.4 in [2]). Hence, the Jacobson radical of the algebra* $\mathbb{T}_{2,2}$ *is* $J = \mathbb{F}\langle \mathsf{u}_{(1,u,v)}, \mathsf{u}_{(1,u,u)}, \mathsf{u}_{(1,v,u)} \rangle$*, and* $\mathbb{T}_{2,2}(\breve{\delta})/J \cong \bigoplus^{3} \mathbb{F} \oplus M_2(\mathbb{F})$*.*

Similarly, when $\delta_1 = 0$ *and* $\delta_0 \neq 0$ *the algebra* $\mathbb{T}_{2,2}$ *is not semi-simple and its radical is the ideal* $\mathbb{F}\langle \mathsf{u}_{(1,u,v)}, \mathsf{u}_{(1,v,v)}, \mathsf{u}_{(1,v,u)} \rangle$*.*

Finally, let $\delta_0 = 0 = \delta_1$*. We are going to show that the algebra*

$$\mathbb{A}_3 = \left\{ \begin{pmatrix} a & d & 0 & e \\ 0 & a & 0 & 0 \\ 0 & c & b & f \\ 0 & 0 & 0 & b \end{pmatrix} \mid a, b, c, d, e, f \in \mathbb{F} \right\}$$

is isomorphic to $\mathbb{A}$*, so* $\mathbb{A}$ *is not semi-simple. This can be shown by using the following map, which is an isomorphism:*

$$1_u \mapsto E_{11} + E_{22}, \qquad 1_v \mapsto E_{33} + E_{44}, \qquad \mathsf{u}_{(1,u,v)} \mapsto E_{14},$$
$$\mathsf{u}_{(1,u,u)} \mapsto E_{12}, \qquad \mathsf{u}_{(1,v,v)} \mapsto E_{34}, \qquad \mathsf{u}_{(1,v,u)} \mapsto E_{32}.$$

Now the algebra $\mathbb{A}_3$ *has Jacobson radical* $I' = \langle E_{12}, E_{14}, E_{32}, E_{34} \rangle$*, as* I' *is a nilpotent ideal and* $\mathbb{A}_3/I'$ *is semi-simple. Thus the algebra* $\mathbb{T}_{2,2}$ *is not semi-simple with a quotient isomorphic to* $\bigoplus^{2} \mathbb{F} \oplus M_2(\mathbb{F})$*.*

For a general field $\mathbb{F}$, we have the following fact as a generalization of the previous example.

Proposition 5.9. *Let* $\mathbb{F}$ *be an arbitrary field. Then the algebra* $\mathbb{T}_{2,m}(\breve{\delta})$ *is semi-simple over* $\mathbb{F}$ *provided that* δ_j *is invertible in* $\mathbb{F}$ *for each* $j = 0, \ldots, m-1$*.*

Proof. We are going to show that

$$\mathbb{T}_{2,m}(\breve{\delta}) \cong \bigoplus^{m(m-1)/2} M_2(\mathbb{F}) \oplus \bigoplus^{m} \mathbb{F} \oplus M_m(\mathbb{F}),$$

when δ_j is invertible in $\mathbb{F}$ for each j. Let Φ be the set $\{(i,j)|i,j \in \mathbb{Z}_m, i<j\}$. From the definition of the multiplication on $\mathbb{T}_{2,m}$, we have

$$\mathbb{T}_{2,m}(\breve{\delta}) \cong \bigoplus_{(i,j)\in\Phi} \mathbb{F}\langle 1_{(i,j)}, 1_{(j,i)}, \mathsf{s}_{(1,(i,j))}, \mathsf{s}_{(1,(j,i))}\rangle \oplus \mathbb{A},$$

where $\mathbb{A} = \mathbb{F}\langle 1_{(i,i)}, \mathsf{u}_{(1,(i,i),(j,j))} : i,j \in \mathbb{Z}_m\rangle$.

It is obvious that $\mathbb{F}\langle 1_{(i,j)}, 1_{(j,i)}, \mathsf{s}_{(1,(i,j))}, \mathsf{s}_{(1,(j,i))}\rangle \cong M_2(\mathbb{F})$ for each $(i,j) \in \Phi$. Also we can show that the cardinality of the set Φ is $\frac{m(m-1)}{2}$.

Let B be the algebra that contains all matrices of the form

$$\begin{pmatrix} a_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_m & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & b_{11} & \dots & b_{1m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & b_{m1} & \dots & b_{mm} \end{pmatrix},$$

where $a_i, b_{ij} \in \mathbb{F}$ for each $i,j \in \{1,\dots,m\}$. Define the map $f: B \to \mathbb{A}$ that sends

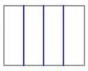
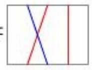
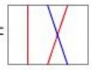
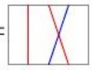
$$\begin{aligned} E_{i+1\,i+1} &\mapsto 1_{(i,i)} - \delta_i^{-1}\mathsf{u}_{(1,(i,i),(i,i))}, \quad \text{if } i = 0,\dots,m-1, \\ E_{i+1\,j+1} &\mapsto \delta_{i-m}^{-1}\mathsf{u}_{(1,(i-m,i-m),(j-m,j-m))}, \quad \text{if } i,j = m,\dots,2m-1. \end{aligned}$$

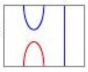
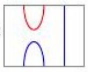
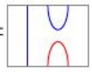
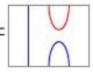
It is easy to check that the map f defines a homomorphism and the determinant of the corresponding matrix of this homomorphism is $\prod_{j=0}^{m-1} \delta_j^{-m}$, so it is an isomorphism if and only if $\prod_{j=0}^{m-1} \delta_j \neq 0$, which is satisfied when δ_j is invertible in $\mathbb{F}$ for each $j \in \mathbb{Z}_m$. □

The algebra $\mathbb{T}_{3,2}$ can be written as direct sum of two algebras $\mathbb{A}_1$ and $\mathbb{A}_2$, where $\mathbb{A}_1$ (similarly $\mathbb{A}_2$) is the algebra that spanned by all the diagrams in the set $\mathcal{T}_{3,2}$ with blue (red) propagating number is equal to 3 or 1. The proof of that follows from the fact that $\mathbb{A}_1 \cap \mathbb{A}_2 = \{0\}$ and the product of two elements of $\mathbb{A}_i$ is also contained in

$\mathbb{A}_i$ for each i and $xy = 0 = yx$ for each $x \in \mathbb{A}_1$ and each $y \in \mathbb{A}_2$. Then each simple module of $\mathbb{T}_{3,2}$ will be a simple module either of $\mathbb{A}_1$ or $\mathbb{A}_2$.

The algebra $\mathbb{A}_1$ is 35-dimensional algebra and it is generated by the following diagrams:

$D_1 =$  , $D_2 =$  , $D_3 =$  , $D_4 =$  , $D_5 =$  ,

$D_6 =$  , $D_7 =$  , $D_8 =$  , $D_9 =$  .

Note that if we change red to blue and blue to red in the previous diagrams, we will obtain a generator set for the algebra $\mathbb{A}_2$.

In order to study the semi-simplicity of the algebra $\mathbb{A}_1$, we define a homomorphism $f : \mathbb{A}_1 \to B$ where B is the sub-algebra of the matrix algebra $M_9(\mathbb{F})$ that contains all matrices of the form:

$$\begin{pmatrix} a_{11} & \cdots & a_{15} & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{51} & \cdots & a_{55} & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & a_{66} & a_{67} & a_{68} & 0 \\ 0 & \cdots & 0 & a_{76} & a_{77} & a_{78} & 0 \\ 0 & \cdots & 0 & a_{86} & a_{87} & a_{88} & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & a_{99} \end{pmatrix},$$

and f is the map that sends

$$\begin{array}{lll} D_1 \mapsto E_{11} + E_{22} + E_{99}, & D_2 \mapsto E_{43} + E_{76}, & D_3 \mapsto E_{34} + E_{67}, \\ D_4 \mapsto E_{45} + E_{78}, & D_5 \mapsto E_{54} + E_{87}, & D_6 \mapsto \delta_0 E_{15}, \\ D_7 \mapsto \delta_1 E_{51} + E_{52}, & D_8 \mapsto \delta_0 E_{23}, & D_9 \mapsto E_{31} + \delta_1 E_{32}. \end{array}$$

It is easy to check that f can be extended to define an algebra homomorphism, by checking all the relations that connected them, see Theorem 2.32. For example, we have $D_i^2 = 0$ for each $i \neq 1$, and on the other hand we also have $f(D_i)^2 = 0$ for each $i \neq 1$. Furthermore, the determinant of the matrix that corresponding to this linear transform is $\delta_0^{15}(\delta_1^2-1)^5$. Hence, the algebra $\mathbb{A}_1$ is semi-simple when $\delta_0(\delta_1^2-1)\neq 0$. It is similar for the algebra $\mathbb{A}_2$ replacing δ_0, δ_1 by δ_1, δ_0 respectively. Then we have the next lemma.

Lemma 5.10. *If $\delta_0^2 \neq 1 \neq \delta_1^2$ and $\delta_0\delta_1 \neq 0$, then the algebra $\mathbb{T}_{3,2}$ is semi-simple and*

$$\mathbb{T}_{3,2}(\delta_0,\delta_1) \cong \bigoplus^{2} \left(F \oplus M_3(\mathbb{F}) \oplus M_5(\mathbb{F})\right).$$

In the case $\delta_0(\delta_1^2-1)=0$, the algebra $\mathbb{A}_1$ is not semi-simple, since $\mathbb{A}_1$ has a non-zero nilpotent ideal J, thus from Corollary 1.4 in [2] the ideal J is contained in the Jacobson radical of the algebra $\mathbb{A}_1$. For example, when $\delta_0=0$, take J to be the two-sided ideal that is generated by the diagram $\mathsf{u}_{(2,x,x)}$ where $x=(1,0,0)$. Note that J has a basis that contains 9 diagrams that have a red arc on both top and bottom faces and one blue propagating line, so $J^2=0$. Also when $\delta_1^2=1$, we can take J to be the two-sided ideal that is generated by the element $\mathsf{u}_{(2,y,y)} - \mathsf{u}_{(1,y,y)}\mathsf{u}_{(2,y,y)}$ where $y=(1,1,1)$, which is 5-dimensional vector space and $J^2=0$.

Similarly, when $\delta_1(\delta_0^2-1)=0$, the algebra $\mathbb{A}_2$ is not semi-simple. Thus the algebra $\mathbb{T}_{3,2}$ is not semi-simple when $\delta_0\delta_1(\delta_0^2-1)(\delta_1^2-1)=0$.

Let $\mathbb{T}^+_{n,2}(\breve{\delta})$ be the subspace of $\mathbb{T}_{n,2}(\breve{\delta})$ that is spanned by all the diagrams in $\mathcal{T}_{n,2}$ which have an even number of blue-nodes on the top face. Since making an arc needs two nodes on the same face, thus the number of blue-nodes on the bottom face of the diagrams in $\mathbb{T}^+_{n,2}(\breve{\delta})$ will be also an even number. The composition of two diagrams in $\mathbb{T}_{n,2}(\breve{\delta})$ does not change the number of blue-nodes on top face of the first diagram, then $\mathbb{T}^+_{n,2}(\breve{\delta})$ is an algebra with an identity equal to the sum of all coloured images of $id \in \mathfrak{S}_n$ that have an even number of blue-propagating lines. Similarly, define $\mathbb{T}^-_{n,2}(\breve{\delta})$ to be the subspace of $\mathbb{T}_{n,2}(\breve{\delta})$ that is spanned by all the diagrams in $\mathcal{T}_{n,2}$ which have an odd number of blue-nodes on the top face. Also, $\mathbb{T}^-_{n,2}(\breve{\delta})$ is an algebra with identity equal to the sum of all coloured images of $id \in \mathfrak{S}_n$ that have an odd number of blue-propagating lines.

Lemma 5.11. *For any* $n \geq 1$*, we have*

$$\mathbb{T}_{n,2}(\breve{\delta}) = \mathbb{T}^{+}_{n,2}(\breve{\delta}) \oplus \mathbb{T}^{-}_{n,2}(\breve{\delta}), \tag{5.10}$$

as an algebra.

Proof. This come from the fact any diagram in $\mathcal{T}_{n,2}$ will have an even number or an odd number of blue-nodes on the top face, a diagram which have an even number of blue-nodes will be contained in $\mathbb{T}^{+}_{n,2}(\breve{\delta})$ and the diagrams that have an odd number of blue-nodes are in $\mathbb{T}^{-}_{n,2}(\breve{\delta})$, so $\mathbb{T}_{n,2}(\breve{\delta}) = \mathbb{T}^{+}_{n,2}(\breve{\delta}) + \mathbb{T}^{-}_{n,2}(\breve{\delta})$. Furthermore, it is clear that $\mathbb{T}^{+}_{n,2}(\breve{\delta}) \cap \mathbb{T}^{-}_{n,2}(\breve{\delta})$ is zero and the product of any two diagrams from $\mathbb{T}^{+}_{n,2}(\breve{\delta})$ and $\mathbb{T}^{-}_{n,2}(\breve{\delta})$ respectively will be zero. □

It is obvious that the algebra $\mathbb{T}^{+}_{n,2}(\breve{\delta})$ is a cellular with the same cell modules $\Delta_n(\lambda_0, \lambda_1)$ of the algebra $\mathbb{T}_{n,2}(\breve{\delta})$ such that λ_1 is an even number. Similarly, the algebra $\mathbb{T}^{-}_{n,2}(\breve{\delta})$ is cellular with cell modules $\Delta_n(\lambda_0, \lambda_1)$ such that λ_1 is an odd number.

As consequence of the last lemma, to study the representations of the algebra $\mathbb{T}_{n,2}(\breve{\delta})$, it is enough to study the representations of the algebras $\mathbb{T}^{+}_{n,2}(\breve{\delta})$ and $\mathbb{T}^{-}_{n,2}(\breve{\delta})$.

5.4 Further properties of cell modules of $\mathbb{T}_{n,m}$

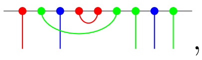
As for the Temperley-Lieb algebra, a basis of $\Delta_n(\lambda)$ is the set that contains all $((\lambda_0 + 2p_0, p_0), \ldots, (\lambda_{m-1} + 2p_{m-1}, p_{m-1}))$-link states where $p_0, \ldots, p_{m-1}$ are non-negative integers such that $\sum_{j\in\mathbb{Z}_m} (\lambda_j + 2p_j) = n$.

Definition 5.12. Let $a = |D\rangle \in \Delta_n(\lambda)$ for some $D \in \mathcal{T}_{n,m}[\lambda]$. The distribution of the colours of a is the set top(D). This set will be denoted by top(a).

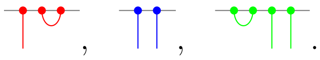
Let a be a $\big((\lambda_j + 2p_j, p_j)\big)_{j\in\mathbb{Z}_m}$-link state and b be a $\big((\lambda_j + 2p'_j, p'_j)\big)_{j\in\mathbb{Z}_m}$-link state where $\sum_{j\in\mathbb{Z}_m} p_j = \sum_{j\in\mathbb{Z}_m} p'_j$. It is evident that $\langle a, b\rangle = 0$ unless $p_j = p'_j$ for each j and the distributions of the colours of a and b are same.

Each $((n_0, p_0), \ldots, (n_{m-1}, p_{m-1}))$-link state determines a collection of (n_j, p_j)-link states as they are defined in Section 1.8.2, where each j represents the colour $\mathfrak{C}_j$, by omitting all the parts that have colour not $\mathfrak{C}_j$.

Example 5.12.1. *Take α to be the following $((3,1),(2,0),(4,1))$-link state:*



so α can be considered as a collection of the following link states:

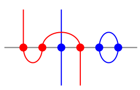
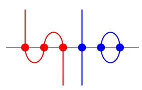


Let a and b be $\big((n_j, p_j)\big)_{j\in\mathbb{Z}_m}$-link states with the same distribution of colours, and a_j be the (n_j, p_j)-link state which is obtained from a by omitting all the parts that have colour not $\mathfrak{C}_j$. Similarly, we define b_j. From the graphical visualization of the product on the algebra $\mathbb{T}_{n,m}$, we obtain

$$\langle a, b\rangle = \langle a_0, b_0\rangle_{n_0,p_0,\delta_0} \times \cdots \times \langle a_{m-1}, b_{m-1}\rangle_{n_{m-1},p_{m-1},\delta_{m-1}} , \qquad (5.11)$$

where $\langle a_j, b_j\rangle_{n_j,p_j,\delta_j}$ denotes the standard bilinear form on V_{n_j,p_j} as $\mathsf{TL}_{n_j}(\delta_j)$-module, see Section 1.8.2.

Note that distribution of colours, if it matches up, does not play any rule. In other words, if a, b, c and d be $\big((n_j, p_j)\big)_{j\in\mathbb{Z}_m}$-link states such that $a_j = c_j$ and $b_j = d_j$, then $\langle a, b\rangle = \langle c, d\rangle$ if $\mathrm{top}(a) = \mathrm{top}(b)$ and $\mathrm{top}(c) = \mathrm{top}(d)$. Note that a and c may have different distributions of colours. Actually, if they have the same distribution, then $a = c$. For example,

 $= \delta_1 =$  .

The next step is the computation of $\dim \Delta_n(\lambda)$ for all $\lambda \in \Lambda_{\mathbb{T}_{n,m}}$.

Proposition 5.13. *1. If $\sum_{j\in\mathbb{Z}_m} \lambda_j = n$, then $\dim \Delta_n(\lambda) = n_\lambda := \binom{n}{\lambda_0,\ldots,\lambda_{m-1}}$.*

2. If $\sum_{j\in\mathbb{Z}_m} \lambda_j = n-2v$ for some $v \in \{1,\ldots,[^n/_2]\}$, then $\dim \Delta_n(\lambda)$ *is given by the formula*

$$\frac{n!\prod_{j\in\mathbb{Z}_m}(\lambda_j+1)}{(n+m)!}\sum_{u\in\Gamma_{(v,m)}}\binom{n+m}{u_0,\lambda_0+u_0+1,\ldots,u_{m-1},\lambda_{m-1}+u_{m-1}+1}.$$

Proof. The first statement can be proved by using the fact that the dimension of $\Delta_n(\lambda)$ is just the number of different permutations of n objects where there are λ_j objects that have the colour $\mathfrak{C}_j$ for all j, which is equal to the multinomial coefficient n_λ, see for example Theorem 4.3 in [34].

When $\sum_{j\in\mathbb{Z}_m} \lambda_j = n-2v$, the basis elements of $\Delta_n(\lambda)$ are $\big((\lambda_j+2u_j,u_j)\big)_{j\in\mathbb{Z}_m}$- link states where $\sum u_j = v$. We compute $\dim \Delta_n(\lambda)$ in three steps. First choose natural numbers $u_0,\ldots,u_{m-1}$ such that $\sum u_j = v$, then distribute λ_j+2u_j nodes of the colour $\mathfrak{C}_j$ for all j on a line. There are $n_{\lambda+2u}$ different distributions of these nodes. Each distribution makes $\prod_{j\in\mathbb{Z}_m} \mathsf{d}_{\lambda_j+2u_j,u_j}$ different $\big((\lambda_j+2u_j,u_j)\big)_{j\in\mathbb{Z}_m}$- link states, since each $\big((\lambda_j+2u_j,u_j)\big)_{j\in\mathbb{Z}_m}$- link state can be considered as a collection that contains a (λ_j+2u_j,u_j)- link state for each j and the number of different (λ_j+2u_j,u_j)- link states is $\dim \mathsf{V}_{\lambda_J+2u_j,u_j} = \mathsf{d}_{\lambda_j+2u_j,u_j}$, see equation (1.18). So

$$\dim \Delta_n(\lambda) = \sum_{u\in\Gamma_{(v,m)}} n_{\lambda+2u}\prod_{j\in\mathbb{Z}_m}\mathsf{d}_{\lambda_j+2u_j,u_j}. \tag{5.12}$$

From the definition of $\mathsf{d}_{\lambda_j+2u_j,u_j}$, we obtain the dimension of $\Delta_n(\lambda)$ is equal to

$$\sum_{u\in\Gamma_{(v,m)}}\frac{n!}{\prod_{j=0}^{m-1}(\lambda_j+2u_j)!}\prod_{j\in\mathbb{Z}_m}\frac{(\lambda_j+2u_j)!(\lambda_j+1)}{u_j!(\lambda_j+u_j+1)!} =$$

$$\frac{n!\prod_{j\in\mathbb{Z}_m}(\lambda_j+1)}{(n+m)!}\sum_{u\in\Gamma_{(v,m)}}\binom{n+m}{u_0,\lambda_0+u_0+1,\ldots,u_{m-1},\lambda_{m-1}+u_{m-1}+1}. \qquad \square$$

Corollary 5.14. *Let $m=2$, then*

$$\dim \Delta_n(\lambda_0,\lambda_1) = \frac{n!(\lambda_0+1)(\lambda_1+1)}{(\lambda_0+v+1)!(\lambda_1+v+1)!}\binom{n+2}{v},$$

where $\lambda_0+\lambda_1 = n-2v$ for some v. Therefore, $\dim \Delta_n(\lambda_0,\lambda_1) = \dim \Delta_n(\lambda_1,\lambda_0)$.

Proof. From the previous proposition, the dimension of $\Delta_n(\lambda_0, \lambda_1)$ equals

$$n!(\lambda_0+1)(\lambda_1+1)\sum_{u_0=0}^{v}\frac{1}{u_0!(\lambda_0+u_0+1)!(v-u_0)!(\lambda_1+v-u_0+1)!} =$$
$$\frac{n!(\lambda_0+1)(\lambda_1+1)}{(\lambda_0+v+1)!(\lambda_1+v+1)!}\sum_{u_0=0}^{v}\binom{\lambda_0+v+1}{v-u_0}\binom{\lambda_1+v+1}{u_0},$$

by using *Chu-Vandermonde identity* $\binom{x+a}{v} = \sum_{k=0}^{v}\binom{x}{k}\binom{a}{v-k}$, see for example Theorem 4.2 in [34], we obtain

$$\dim\Delta_n(\lambda_0,\lambda_1) = \frac{n!(\lambda_0+1)(\lambda_1+1)}{(\lambda_0+v+1)!(\lambda_1+v+1)!}\binom{n+2}{v}.$$ □

5.5 Idempotent localization

In this section we compute the radical and Gram matrix of each cell module $\Delta_n(\lambda)$ where $\lambda \in \Lambda_{\mathbb{T}_{n,m}}$.

Lemma 5.15. *[28, Lemma 3.1.6]. Let $\lambda \in \Lambda_{\mathbb{T}_{n,m}}$ and $\mu \in \Gamma_{(n,m)}$, then*

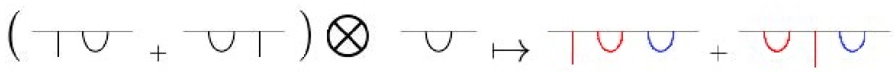
$$1_{\underline{\underline{\mu}}}\Delta_n(\lambda) \cong \begin{cases} \bigotimes_{j=0}^{m-1}\mathsf{V}_{\mu_j,t_j} & \text{if } \mu_j-\lambda_j = 2t_j \text{ for each } j \\ & \text{for some } t_j\in\mathbb{N}, \\ 0 & \text{otherwise,}\end{cases} \tag{5.13}$$

as $1_{\underline{\underline{\mu}}}\mathbb{T}_{n,m}1_{\underline{\underline{\mu}}}$-module, where $\underline{\underline{\mu}}$ is defined by the relation 2.26 and the modules V_{μ_j,t_j} are the cell modules of $\mathsf{TL}_{\mu_j}(\delta_j)$, see (1.17).

Remark 5.16. As it is mentioned before in the end of Chapter 2, for any $\mu \in \Gamma_{(n,m)}$ the algebras $\bigotimes_{i=0}^{m-1}\mathsf{TL}_{\mu_i}(\delta_i)$ and $1_{\underline{\underline{\mu}}}\mathbb{T}_{n,m}(\breve{\delta})1_{\underline{\underline{\mu}}}$ are isomorphic with a map sending any tuple of diagrams in $\bigotimes_{i=0}^{m-1}\mathsf{TL}_{\mu_i}(\delta_i)$ to the diagram in $1_{\underline{\underline{\mu}}}\mathbb{T}_{n,m}(\breve{\delta})1_{\underline{\underline{\mu}}}$ formed by drawing these diagrams in one frame one by one using different colours such that the diagram from $\mathsf{TL}_{\mu_i}(\delta_i)$ is drawn in the colour $\mathfrak{C}_i$. Similarly, if $\mathsf{V}_{\mu_0,p_0},\dots,\mathsf{V}_{\mu_{m-1},p_{m-1}}$ are cell modules for the algebras $\mathsf{TL}_{\mu_0}(\delta_0),\dots,\mathsf{TL}_{\mu_{m-1}}(\delta_{m-1})$ respectively, then elements of the module $\bigotimes_{i=0}^{m-1}\mathsf{V}_{\mu_i,p_i}$ can be represented by $\big((\mu_i,p_i)\big)_{i\in\mathbb{Z}_m}$- link states by using the same map which it is the same isomorphism that was used in the previous lemma. For example, see figure 5.3.

Important convention : whenever we write $\bigotimes_{i=0}^{m-1} \mathsf{V}_{\mu_i,p_i}$ or $\bigotimes_{i=0}^{m-1} M_i$ where M_i is a sub-module or quotient module of V_{μ_i,p_i}, we mean their image in $1_{\underline{\underline{\mu}}}\Delta_n(\lambda)$ under the isomorphism in Remark 5.16.

FIGURE 5.3: Illustration of a map from $\mathsf{V}_{3,1} \otimes \mathsf{V}_{2,1}$ to $1_{(0,0,0,1,1)}\Delta_5(1,0)$



The conclusion of the next theorem is the same as Jegan [28] made in Lemma 3.2.10 and our proofs are closely related.

Theorem 5.17. *Let $\mathsf{G}_n(\lambda)$ denote the Gram matrix of the module $\Delta_n(\lambda)$ of the inner product that defined by (5.9) and with respect to the half-diagrams basis. If $\sum_{j\in\mathbb{Z}_m} \lambda_j = n - 2v$ for some v, then*

$$\mathsf{G}_n(\lambda) = \bigoplus_{u\in\Gamma_{(v,m)}} \bigoplus^{n_{\lambda+2u}} \mathsf{G}_{\lambda_0+2u_0,u_0,\delta_0} \otimes \cdots \otimes \mathsf{G}_{\lambda_{m-1}+2u_{m-1},u_{m-1},\delta_{m-1}},$$

where $\mathsf{G}_{\lambda_j+2u_j,u_j,\delta_j}$ is the Gram matrix of the cell $\mathsf{TL}_{\lambda_j+2u_j}(\delta_j)$-module $\mathsf{V}_{\lambda_j+2u_j,u_j}$ with a specific bilinear form and half-diagrams basis.

Proof. To compute the Gram matrix of the module $\Delta_n(\lambda)$, note that the inner product will be zero if the two link states have a different distribution or a different number of coloured points. Also the value of the inner product does not depend on the distribution of colour as long the two links states have the same distribution, so

$$\mathsf{G}_n(\lambda) = \bigoplus_{u\in\Gamma_{(v,m)}} \bigoplus^{n_{\lambda+2u}} A_{u_0,\ldots,u_{m-1}},$$

where $A_{u_0,\ldots,u_{m-1}}$ a matrix computed by using all the $\big((\lambda_j + 2u_j, u_j)\big)_{j\in\mathbb{Z}_m}$-link states that have the following distribution of colour:

$$(\underbrace{0,\ldots,0}_{\lambda_0+2u_0 \text{ times}}, \underbrace{1,\ldots,1}_{\lambda_1+2u+1 \text{ times}}, \ldots, \underbrace{m-1,\ldots,m-1}_{\lambda_{m-1}+2u_{m-1} \text{ times}}).$$

From equation (5.11), we have $A_{u_0,\ldots,u_{m-1}} = \mathsf{G}_{\lambda_0+2u_0,u_0,\delta_0} \otimes \cdots \otimes \mathsf{G}_{\lambda_{m-1}+2u_{m-1},u_{m-1},\delta_{m-1}}$.

□

Example 5.17.1. *Let $n = 7$, $m = 3$ and $\lambda = (1, 0, 2)$. From the previous theorem, we obtain*

$$\mathsf{G}_7(1,0,2) = \Big(\bigoplus^{21} \mathsf{G}_{5,2,\delta_0}\Big) \oplus \Big(\bigoplus^{105} \mathsf{G}_{4,2,\delta_1}\Big) \oplus \Big(\bigoplus^{210} \delta_1 \mathsf{G}_{3,1,\delta_0}\Big) \oplus \Big(\bigoplus^{7} \mathsf{G}_{6,2,\delta_2}\Big) \oplus \Big(\bigoplus^{35} \mathsf{G}_{3,1,\delta_0} \otimes \mathsf{G}_{4,1,\delta_2}\Big) \oplus \Big(\bigoplus^{105} \delta_1 \mathsf{G}_{4,1,\delta_2}\Big).$$

Example 5.17.2. *Let $\sum_j \lambda_j = n$. From the last theorem we have $\mathsf{G}_n(\lambda) = \bigoplus^{n_\lambda}(1) = I_{n_\lambda \times n_\lambda}$, where $I_{n_\lambda\times n_\lambda}$ is the identity matrix, so the module $\Delta_n(\lambda)$ is simple whenever $\sum_j \lambda_j = n$. Also when $\sum_j \lambda_j = n-2$, then*

$$\mathsf{G}_n(\lambda) = \bigoplus_{i=0}^{m-1} \bigoplus^{n_{(\lambda_0,\ldots,\lambda_{i-1},\lambda_i+2,\lambda_{i+1},\ldots,\lambda_{m-1})}} \mathsf{G}_{\lambda_i+2,1,\delta_i}.$$

The following corollaries are immediate consequences.

Corollary 5.18. *[28, Lemma 3.2.12]. The determinant of Gram matrix is*

$$\det \mathsf{G}_n(\lambda) = \prod_{u\in\Gamma_{(v,m)}} \left(\prod_{j=0}^{m-1} (\det \mathsf{G}_{\lambda_j+2u_j,u_j,\delta_j})^{\mathsf{d}^{-1}_{\lambda_j+2u_j,u_j}} \right)^{(\prod_{j=0}^{m-1} \mathsf{d}_{\lambda_j+2u_j,u_j})\cdot n_{\lambda+2u}},$$

for each $\lambda \in \Gamma_{(n-2v,m)}$, where $\mathsf{d}_{\lambda_j+2u_j,u_j}$ is defined by (1.18) and $n_\mu := \binom{n}{\mu_0,\ldots,\mu_{m-1}}$ for each $\mu \in \Gamma_{(n,m)}$.

Proof. By using relations (1.6), (1.8) and Theorem 5.17, we obtain this formula. □

The previous result shows that $\det \mathsf{G}_n(\lambda) \neq 0$ if and only if $\det \mathsf{G}_{\lambda_j+2u_j,u_j,\delta_j} \neq 0$ for all $j \in \mathbb{Z}_m$ and all $u \in \Gamma_{(v,m)}$, then the following fact is straightforward, which is a generalization of Proposition 6 in [23].

Corollary 5.19. *Let $\delta_j = q_j + q_j^{-1} \neq 0$ for all $j \in \mathbb{Z}_m$. If q_j is not a root of unity for any j, then the algebra $\mathbb{T}_{n,m}(\breve{\delta})$ is semi-simple algebra and the modules $\Delta_n(\lambda)$, where $\lambda \in \Lambda_{\mathbb{T}_{n,m}} = \bigcup_{v=0}^{[n/2]} \Gamma_{(n-2v,m)}$, form a complete set of non-isomorphic irreducible modules of $\mathbb{T}_{n,m}$, and the algebra $\mathbb{T}_{n,m}(\breve{\delta})$ decomposes as*

$$\mathbb{T}_{n,m}(\breve{\delta}) \cong \bigoplus_{\lambda \in \Lambda_{\mathbb{T}_{n,m}}} \bigoplus^{\dim \Delta_n(\lambda)} \Delta_n(\lambda),$$

as a left module.

Proof. The proof comes directly from Corollary 5.18, Theorems 1.29 and 1.14 since the algebra $\mathbb{T}_{n,m}$ is a cellular algebra. The last statement appears as consequence of Wedderburn's theorem (see for example Theorem 1.3.5 in [3]). □

Proposition 5.20. *The head of the module $\Delta_n(\lambda)$ where $\lambda \in \Gamma_{(n-2v,m)}$ for some v such that $\lambda \in \Lambda^0_{\mathbb{T}_{n,m}}$, denoted by $\mathsf{L}_n(\lambda)$, satisfies the relation*

$$\dim \mathsf{L}_n(\lambda) = \sum_{u \in \Gamma_{(v,m)}} n_{\lambda+2u} \prod_{i=0}^{m-1} \dim \mathsf{L}_{\lambda_i+2u_i,u_i,\delta_i}, \tag{5.14}$$

where $\mathsf{L}_{\lambda_i+2u_i,u_i,\delta_i}$ is the head of the $\mathsf{TL}_{\lambda_i+2u_i}(\delta_i)$-module $\mathsf{V}_{\lambda_i+2u_i,u_i}$.

Proof. This follows from the fact that $\dim \mathsf{L}_n(\lambda) = \operatorname{rank}(\mathsf{G}_n(\lambda))$ as the algebra is over a field and $\lambda \in \Lambda^0_{\mathbb{T}_{n,m}}$, see Section 1.5. From Theorem 5.17 and the relations (1.5) and (1.7), we obtain

$$\operatorname{rank}(\mathsf{G}_n(\lambda)) = \sum_{u \in \Gamma_{(v,m)}} n_{\lambda+2u} \prod_{i=0}^{m-1} \operatorname{rank}(\mathsf{G}_{\lambda_i+2u_i,u_i,\delta_i}).$$

On the other hand, we also have $\operatorname{rank}(\mathsf{G}_{\lambda_i+2u_i,u_i,\delta_i}) = \dim \mathsf{L}_{\lambda_i+2u_i,u_i,\delta_i}$ for each i, thus

$$\dim \mathsf{L}_n(\lambda) = \sum_{u\in\Gamma_{(v,m)}} n_{\lambda+2u} \prod_{i=0}^{m-1} \dim \mathsf{L}_{\lambda_i+2u_i,u_i,\delta_i}. \qquad \square$$

Corollary 5.21. *The module $\mathsf{L}_n(\lambda)$ decomposes as*

$$\bigoplus_{u\in\Gamma_{(v,m)}} \bigoplus^{n_{\lambda+2u}} \mathsf{L}_{\lambda_0+2u_0,u_0,\delta_0} \otimes \cdots \otimes \mathsf{L}_{\lambda_{m-1}+2u_{m-1},u_{m-1},\delta_{m-1}},$$

as a vector space, where $\lambda \in \Gamma_{(n-2v,m)}$ for some v such that $\lambda \in \Lambda^0_{\mathbb{T}_{n,m}}$.

Proof. It comes directly from the fact that any two vector spaces which have the same dimension are isomorphic, and by the last proposition they have the same dimension. $\square$

Lemma 5.22. *Let $(\lambda_0, \lambda_1) \in \Lambda^0_{\mathbb{T}_{n,2}}$. The dimensions of* $\operatorname{Rad}(\Delta_n(\lambda_0, \lambda_1))$ *is*

$$\sum_{u\in\Gamma_{(v,2)}} n_{\lambda+2u}\Big(\dim \mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} \dim \mathsf{V}_{\lambda_1+2u_1,u_1} + \dim \mathsf{V}_{\lambda_0+2u_0,u_0} \dim \mathsf{R}_{\lambda_1+2u_1,u_1,\delta_1} - \dim \mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} \dim \mathsf{R}_{\lambda_1+2u_1,u_1,\delta_1}\Big),$$

where $\lambda \in \Gamma_{(n-2v,2)}$ and $\mathsf{R}_{\lambda_i+2u_i,u_i,\delta_i}$ is the radical of the $\mathsf{TL}_{\lambda_i+2u_i}(\delta_i)$-module $\mathsf{V}_{\lambda_i+2u_i,u_i}$.

Proof. Since $\dim \operatorname{Rad}(\Delta_n(\lambda)) = \dim \Delta_n(\lambda) - \dim \mathsf{L}_n(\lambda)$, from equations (5.12) and (5.14) we obtain that $\dim \operatorname{Rad}(\Delta_n(\lambda))$ equals

$$\sum_{u\in\Gamma_{(v,2)}} n_{\lambda+2u} \dim \mathsf{V}_{\lambda_0+2u_0,u_0} \dim \mathsf{V}_{\lambda_1+2u_1,u_1} - \sum_{u\in\Gamma_{(v,2)}} n_{\lambda+2u} \prod_{i=0}^{1} \dim \mathsf{L}_{\lambda_i+2u_i,u_i,\delta_i}.$$

But $\dim \mathsf{L}_{n,p,\delta} = \dim \mathsf{V}_{n,p} - \dim \mathsf{R}_{n,p,\delta}$, so $\dim \operatorname{Rad}(\Delta_n(\lambda))$ is

$$\sum_{u\in\Gamma_{(v,2)}} n_{\lambda+2u}\Big(\dim \mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} \dim \mathsf{V}_{\lambda_1+2u_1,u_1} + \dim \mathsf{V}_{\lambda_0+2u_0,u_0} \dim \mathsf{R}_{\lambda_1+2u_1,u_1,\delta_1} - \dim \mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} \dim \mathsf{R}_{\lambda_1+2u_1,u_1,\delta_1}\Big). \qquad \square$$

Theorem 5.23. *Let $\lambda \in \Gamma_{(n-2v,2)}$ for some v such that $\lambda \in \Lambda^0_{\mathbb{T}_{n,2}}$. Then the radical* $\mathrm{Rad}(\Delta_n(\lambda_0, \lambda_1))$ *decomposes as*

$$\bigoplus_{u \in \Gamma_{(v,2)}} \bigoplus^{n_{\lambda+2u}} \big(\mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} \otimes \mathsf{V}_{\lambda_1+2u_1,u_1} + \mathsf{V}_{\lambda_0+2u_0,u_0} \otimes \mathsf{R}_{\lambda_1+2u_1,u_1,\delta_1}\big),$$

as a vector space, and it is equal to

$$\sum_{u \in \Gamma_{(v,2)}} \sum_{\sigma \in \widehat{\mathfrak{S}}_{n,2}} \sigma\big(\, \mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} \otimes \mathsf{V}_{\lambda_1+2u_1,u_1} + \mathsf{V}_{\lambda_0+2u_0,u_0} \otimes \mathsf{R}_{\lambda_1+2u_1,u_1,\delta_1}\,\big).$$

For the definition of $\widehat{\mathfrak{S}}_{n,2}$ see equation (3.5)*. Remember by $\mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} \otimes \mathsf{V}_{\lambda_1+2u_1,u_1}$ and $\mathsf{V}_{\lambda_0+2u_0,u_0} \otimes \mathsf{R}_{\lambda_1+2u_1,u_1,\delta_1}$ we mean their images in $1_{\underline{\underline{\lambda+2u}}}\Delta_n(\lambda)$, see Remark 5.16.*

Proof. First part comes directly from the last lemma, since they have the same dimension, note that $(\mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} \otimes \mathsf{V}_{\lambda_1+2u_1,u_1}) \cap (\mathsf{V}_{\lambda_0+2u_0,u_0} \otimes \mathsf{R}_{\lambda_1+2u_1,u_1,\delta_1}) = \mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} \otimes \mathsf{R}_{\lambda_1+2u_1,u_1,\delta_1}$.

Now we are going to prove the second part. Note that a basis elements of $\Delta_n(\lambda)$ are $((\lambda_0+2u_0, u_0), (\lambda_1+2u_1, u_1))$-link states where $u \in \Gamma_{(v,2)}$. For fixed $u \in \Gamma_{(v,2)}$, all link states that have the first $\lambda_0 + 2u_0$ nodes red and the following ones blue can be obtained from $\mathsf{V}_{\lambda_0+2u_0,u_0} \otimes \mathsf{V}_{\lambda_1+2u_1,u_1}$, and any other $((\lambda_0+2u_0, u_0), (\lambda_1+2u_1, u_1))$-link state b with different colour distribution can be written in the form σa where a is the link state with the same components as b and its top is $(\{1, \dots, \lambda_0 + 2u_0\}, \{1 + \lambda_0 + 2u_0, \dots, n\})$, and $\sigma \in \widehat{\mathfrak{S}}_{n,2}$ is the coloured permutation that changing the colour order of a to get the same colour distribution of b, then

$$\Delta_n(\lambda) = \sum_{u \in \Gamma_{(v,2)}} \sum_{\sigma \in \widehat{\mathfrak{S}}_{n,2}} \sigma\big(\, \mathsf{V}_{\lambda_0+2u_0,u_0} \otimes \mathsf{V}_{\lambda_1+2u_1,u_1}\,\big).$$

Let y be a $((\lambda_0+2u'_0, u'_0), (\lambda_1+2u'_1, u'_1))$-link state for some $u' \in \Gamma_{(v,2)}$, so from the last equation we can assume that $y = \pi(y_0 \otimes y_1)$ for some $\pi \in \widehat{\mathfrak{S}}_{n,2}$ and y_i is a $(\lambda_i + 2u'_i, u'_i)$-link state for each i. Let x be an element in $\sigma\big(\, \mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} \otimes \mathsf{V}_{\lambda_1+2u_1,u_1}\,\big)$ or in $\sigma\big(\, \mathsf{V}_{\lambda_0+2u_0,u_0} \otimes \mathsf{R}_{\lambda_1+2u_1,u_1,\delta_1}\,\big)$ for some $u \in \Gamma_{(v,2)}$ and some $\sigma \in \widehat{\mathfrak{S}}_{n,2}$, so we can assume that $x = \sigma(x_0 \otimes x_1)$ where $x_0 \in \mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0}$ or $x_1 \in \mathsf{R}_{\lambda_1+2u_1,u_1,\delta_1}$. If $u \neq u'$ or $\sigma \neq \pi$, this means the colour distributions of x and y are different, so from the definition of the multiplication on $\mathbb{T}_{n,m}$ we have $\langle y, x\rangle = 0$. On the other hand, if $u = u'$ and $\sigma = \pi$, from equation (5.11) we have $\langle y, x\rangle = \langle y_0, x_0\rangle_{\lambda_0+2u_0,u_0,\delta_0}\langle y_1, x_1\rangle_{\lambda_1+2u_1,u_1,\delta_1}$. But $x_i \in \mathsf{R}_{\lambda_i+2u_i,u_i,\delta_i}$ for some i, so $\langle y_i, x_i\rangle_{\lambda_i+2u_i,u_i,\delta_i} = 0$ for some i. Hence $\langle y, x\rangle = 0$ for each $y \in \Delta_n(\lambda)$, so $x \in \mathrm{Rad}(\Delta_n(\lambda))$. Thus

$$\sum_u \sum_\sigma \sigma\big(\mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} \otimes \mathsf{V}_{\lambda_1+2u_1,u_1} + \mathsf{V}_{\lambda_0+2u_0,u_0} \otimes \mathsf{R}_{\lambda_1+2u_1,u_1,\delta_1}\big) \subseteq \mathrm{Rad}(\Delta_n(\lambda)),$$

but both of them have the same dimension thus they are identical. □

Example 5.23.1. *Take* $\mathbb{F} = \mathbb{C}$, $n = 3$ *and* $\lambda = (1, 0)$, *so there are two choices of* u *which are* $(1, 0)$ *and* $(0, 1)$. *From equation* (5.14), *we obtain*

$$\dim \mathsf{L}_3(1, 0) = \dim \mathsf{L}_{3,1,\delta_0} + 3 \dim \mathsf{L}_{2,1,\delta_1}.$$

Let $\delta_0 = -1$ *and* $\delta_1 = 0$. *From Corollary 1.35, we have* $\dim \mathsf{L}_{3,1,\delta_0} = 1$ *and* $\dim \mathsf{L}_{2,1,\delta_1} = 0$, *thus* $\dim \mathsf{L}_3(1, 0) = 1$. *Hence,* $\dim \mathrm{Rad}(\Delta_3(1, 0)) = 5 - 1 = 4$. *Also by computing* $\mathsf{R}_{3,1,\delta_0}$ *and* $\mathsf{R}_{2,1,\delta_1}$ *and using the last theorem, we obtain that* $\mathrm{Rad}(\Delta_3(1, 0))$ *is spanned by the following elements:*

Theorem 5.24. *Let* $\lambda \in \Gamma_{(n-2v,m)}$ *for some* v *such that* $\lambda \in \Lambda^0_{\mathbb{T}_{n,m}}$. *Then*

$$\begin{aligned}\mathrm{Rad}(\Delta_n(\lambda)) = \sum_{u\in\Gamma_{(v,m)}} \sum_{\sigma\in\widehat{\mathfrak{S}}_{n,m}} \sigma\Big(&\mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} \otimes \mathsf{V}_{\lambda_1+2u_1,u_1} \otimes \cdots \otimes \mathsf{V}_{\lambda_{m-1}+2u_{m-1},u_{m-1}} \\ &+ \mathsf{V}_{\lambda_0+2u_0,u_0} \otimes \mathsf{R}_{\lambda_1+2u_1,u_1,\delta_1} \otimes \mathsf{V}_{\lambda_2+2u_2,u_2} \otimes \cdots \otimes \mathsf{V}_{\lambda_{m-1}+2u_{m-1},u_{m-1}} + \\ &\cdots + \mathsf{V}_{\lambda_0+2u_0,u_0} \otimes \cdots \otimes \mathsf{V}_{\lambda_{m-2}+2u_{m-2},u_{m-2}} \otimes \mathsf{R}_{\lambda_{m-1}+2u_{m-1},u_{m-1},\delta_{m-1}}\Big).\end{aligned}$$

For the definition of $\widehat{\mathfrak{S}}_{n,m}$ *see equation* (3.5). *Remember by the tensor product of the modules in the last equation we mean their images in* $1_{\underline{\underline{\lambda+2u}}}\Delta_n(\lambda)$, *see Remark 5.16.*

Proof. We can show that by using induction on m and Theorem 5.23. □

Corollary 5.25. *Let* $\lambda \in \Gamma_{(n-2v,m)}$ *such that* $\lambda \in \Lambda^0_{\mathbb{T}_{n,m}}$, *then*

$$\mathsf{L}_n(\lambda) = \sum_{u \in \Gamma_{(v,m)}} \sum_{\sigma \in \widehat{\mathfrak{S}}_{n,m}} \sigma\big(\mathsf{L}_{\lambda_0+2u_0,u_0,\delta_0} \otimes \cdots \otimes \mathsf{L}_{\lambda_{m-1}+2u_{m-1},u_{m-1},\delta_{m-1}}\big).$$

By $\bigotimes_{i=0}^{m-1} \mathsf{L}_{\lambda_i+2u_i,u_i,\delta_i}$ *we mean its images in the module* $1_{\underline{\underline{\lambda+2u}}}\Delta_n(\lambda)$ *under the isomorphism in Remark 5.16.*

Proof. As it is mentioned in Theorem 5.23, we have

$$\Delta_n(\lambda) = \sum_{u \in \Gamma_{(v,m)}} \sum_{\sigma \in \widehat{\mathfrak{S}}_{n,m}} \sigma\big(\mathsf{V}_{\lambda_0+2u_0,u_0} \otimes \cdots \otimes \mathsf{V}_{\lambda_{m-1}+2u_{m-1},u_{m-1}}\big),$$

thus

$$\mathsf{L}_n(\lambda) = \frac{\sum_{u \in \Gamma_{(v,m)}} \sum_{\sigma \in \widehat{\mathfrak{S}}_{n,m}} \sigma\big(\mathsf{V}_{\lambda_0+2u_0,u_0} \otimes \cdots \otimes \mathsf{V}_{\lambda_{m-1}+2u_{m-1},u_{m-1}}\big)}{\mathrm{Rad}(\Delta_n(\lambda))}.$$

From the last theorem, we obtain $\mathsf{L}_n(\lambda)$ equals

$$\sum_{u \in \Gamma_{(v,m)}} \sum_{\sigma \in \widehat{\mathfrak{S}}_{n,m}} \sigma\Big(\frac{\mathsf{V}_0 \otimes \cdots \otimes \mathsf{V}_{m-1}}{\mathsf{R}_0 \otimes \mathsf{V}_1 \otimes \cdots \otimes \mathsf{V}_{m-1} + \cdots + \mathsf{V}_0 \otimes \cdots \otimes \mathsf{V}_{m-2} \otimes \mathsf{R}_{m-1}}\Big),$$

where we put $\mathsf{R}_i := \mathsf{R}_{\lambda_i+2u_i,u_i,\delta_i}$ and $\mathsf{V}_i := \mathsf{V}_{\lambda_i+2u_i,u_i}$ for simplicity.

Let $x_i \in \mathsf{L}_{\lambda_i+2u_i,u_i,\delta_i} := L_i$ for each i, so $x_i = a_i + \mathsf{R}_i$ for some $a_i \in \mathsf{V}_i$ and from that we have

$$\bigotimes_{i=0}^{m-1} x_i = \bigotimes_{i=0}^{m-1} a_i + \mathsf{R}_0 \otimes \mathsf{V}_1 \otimes \cdots \otimes \mathsf{V}_{m-1} + \cdots + \mathsf{V}_0 \otimes \cdots \otimes \mathsf{V}_{m-2} \otimes \mathsf{R}_{m-1},$$

it follows that

$$\frac{\mathsf{V}_0 \otimes \cdots \otimes \mathsf{V}_{m-1}}{\mathsf{R}_0 \otimes \mathsf{V}_1 \otimes \cdots \otimes \mathsf{V}_{m-1} + \cdots + \mathsf{V}_0 \otimes \cdots \otimes \mathsf{V}_{m-2} \otimes \mathsf{R}_{m-1}} = \mathsf{L}_0 \otimes \cdots \otimes \mathsf{L}_{m-1},$$

for each $u \in \Gamma_{(v,m)}$, and we are done. $\square$

Example 5.25.1. *Let $\delta_0^2 = 1$, $\delta_1^2 = 2$ and $m = 2$. For the dimension of some low rank examples of the modules $\Delta_n(\lambda)$ and their radicals over the complex field see table 5.1.*

$n \setminus \lambda$	$(0,0)$	$(1,0)$	$(0,1)$	$(2,0)$	$(1,1)$	$(0,2)$	$(3,0)$	$(2,1)$
1		1	1					
		0	0					
2	2			1	2	1		
	0			0	0	0		
3		5	5				1	3
		1	0				0	0
4	10			9	16	9		
	1			0	4	1		
5		35	35				14	35
		14	6				0	0
6	70			84	140	84		
	20			0	70	35		

$n \setminus \lambda$	$(1,2)$	$(0,3)$	$(4,0)$	$(3,1)$	$(2,2)$	$(1,3)$	$(0,4)$	$(5,0)$
3	3	1						
	0	0						
4			1	4	6	4	1	
			0	0	0	0	0	
5	35	14						1
	15	0						0
6			20	64	90	64	20	
			2	0	15	20	0	

TABLE 5.1: The dimensions of some low rank examples of the modules $\Delta_n(\lambda_0, \lambda_1)$ and their radicals over the complex field when $\delta_0^2 = 1$ and $\delta_1^2 = 2$.

5.6 Homomorphisms between cell $\mathbb{T}_{n,m}$-modules

It was shown in Theorem 5.19 that the algebra $\mathbb{T}_{n,m}(\breve{\delta})$ is a semi-simple algebra when q_j is not root of unity and $0 \neq q_j + q_j^{-1} = \delta_j$ for each $j \in \mathbb{Z}_m$. Therefore in what follows, it will be assumed that q_j is a root of unity for some j, and let $\mathbf{l}_j$ be the minimal positive integer satisfying $q_j^{2\mathbf{l}_j} = 1$.

The first part of the next proposition is Lemma 4.1.1 in [28].

Proposition 5.26. *Let $\lambda, \mu \in \Lambda_{\mathbb{T}_{n,m}}$ and $\theta : \Delta_n(\lambda) \to \Delta_n(\mu)$ be a homomorphism defined by*

$$\theta(a) = \sum_i \alpha_i b_i, \tag{5.15}$$

where $\alpha_i \in \mathbb{F}$, $a \in \mathcal{T}_{n,m}^{|\rangle}[\lambda]$ and $b_i \in \mathcal{T}_{n,m}^{|\rangle}[\mu]$ for each i. Then the following hold:

1. $\text{top}(a) = \text{top}(b_i)$ *for each i.*
2. $\mu_j = \lambda_j - 2t_j$*, for some $t_j \in \{0, \dots, [\lambda_j/2]\}$.*
3. *If δ_j is invertible and a contains an $\mathfrak{C}_j$-arc, then b_i contains an $\mathfrak{C}_j$-arc in the same position. This means that θ preserves arcs when $\delta_j \neq 0$ for each $j \in \mathbb{Z}_m$.*

Proof. Assume that $\lambda \leq \mu$ where $\leq$ is the order from (5.8), which means $\lambda_j \geq \mu_j$ for all $j \in \mathbb{Z}_m$. Otherwise, the homomorphism θ will be zero, by Theorem 1.14 since $\mathbb{T}_{n,m}$ is a cellular algebra.

As $\theta(xa) = x\theta(a)$ for all $x \in \mathbb{T}_{n,m}$, so we obtain

$$\text{top}(a) = \text{top}(b_i),$$

for each b_i in equation (5.15). We can show that by taking $x = 1_{\text{top}(a)}$.

From the previous relation, we have the number of $\mathfrak{C}_j$-nodes in a and b_i are fixed for each j and each i. Let a be a $\big((\lambda_j + 2p_j, p_j)\big)_{j \in \mathbb{Z}_m}$- link state where $\sum_{j=0}^{m-1} (\lambda_j + 2p_j) = n$, then from previous explanation b_i is a $\big((\lambda_j + 2p_j, p'_j)\big)_{j \in \mathbb{Z}_m}$- link state. Since $\lambda_j \geq \mu_j$ for all j, so the number of arcs in b_i is greater than or equals the number of arcs a of each colour, so $\mu_j = \lambda_j + 2p_j - 2p'_j = \lambda_j - 2(p'_j - p_j)$, this means

$$\mu_j = \lambda_j - 2t_j,$$

for some $t_j \in \{0, \dots, [\lambda_j/2]\}$.

Assume that a contains h arcs of the colour $\mathfrak{C}_j$ and $\delta_j \neq 0$. Take $x \in \mathbb{T}_{n,m}$ to be the diagram defined as follows: $\text{top}(x) = \text{bot}(x) = \text{top}(a)$ and if any two nodes

$k, l \in \underline{n}$ are connected in a by a $\mathfrak{C}_j$-arc, then these nodes will also be connected in x by a $\mathfrak{C}_j$-arc and k', l' will be connected by the same colour, otherwise the nodes will be connected to their projection in the bottom row. Note that $xa = \delta_j^h a$, so

$$\theta(a) = \delta_j^{-h} \sum_i \alpha_i x b_i = \sum_i \alpha_i b_i.$$

The $\mathfrak{C}_j$-arcs on the top row will not be affected by the product, so they will be in xb_i in the same positions of a for each i. □

Proposition 5.27. *If δ_j is invertible for each j, then for each n and $\lambda \in \Lambda_{\mathbb{T}_{n,m}}$*

$$\mathrm{End}_{\mathbb{T}_{n,m}}\left(\Delta_n(\lambda)\right) \cong \mathbb{F}. \tag{5.16}$$

Proof. Let $\theta \in \mathrm{End}_{\mathbb{T}_{n,m}}\left(\Delta_n(\lambda)\right)$ and $a \in \mathcal{T}_{n,m}^{|\rangle}[\lambda]$ where $\theta(a) = \sum\limits_l \alpha_i b_i$ for some $b_i \in \mathcal{T}_{n,m}^{|\rangle}[\lambda]$. Since δ_i is invertible for each i, by Proposition 5.26 the homomorphism θ preserves each arc in a at the same position, then the unique choice for b_i is a itself. This implies to θ is diagonal. Take x be a diagram in $\mathcal{T}_{n,m}[\lambda]$ where $|x\rangle = b$ and $\langle x| = a$, so

$$\theta(xa) = x\theta(a) = x(\alpha a) = \alpha\langle a, a\rangle b.$$

Since $\langle a, a\rangle \neq 0$, this implies to $\theta(b) = \alpha b$ since $xa = \langle a, a\rangle b$, so all the entries of θ are equal. In other words, θ corresponds to a scalar $\alpha \in \mathbb{F}$. □

This implies that each module $\Delta_n(\lambda)$ is indecomposable when $\delta_j \neq 0$ for each j, and if the algebra $\mathbb{T}_{n,m}(\breve{\delta})$ is semi-simple then each module $\Delta_n(\lambda)$ is simple.

Lemma 5.28. *Let $\theta : \Delta_n(\lambda) \to \Delta_n(\lambda')$ be a homomorphism where $\lambda, \lambda' \in \Lambda_{\mathbb{T}_{n,m}}$, then θ is completely defined by the values $\theta(a)$ where $a \in X_n(\lambda) := \bigoplus\limits_{\mu \in \Gamma_{(n,m)}} 1_{\underline{\underline{\mu}}} \mathcal{T}_{n,m}^{|\rangle}[\lambda]$, note that $X_n(\lambda)$ is the subset of $\mathcal{T}_{n,m}^{|\rangle}[\lambda]$ such that an element will be in $X_n(\lambda)$ if its top equals* $\mathrm{top}(1_{\underline{\underline{\mu}}})$ *for some $\mu \in \Gamma_{(n,m)}$. Furthermore, if $\sum\limits_i \lambda_i = n$, then θ is determined by the image of $|1_{\underline{\underline{\lambda}}}\rangle$.*

Proof. Let $b \in \mathcal{T}_{n,m}^{|\rangle}[\lambda]$ and $b \notin X_n(\lambda)$. There is an element $x \in \mathfrak{S}_{n,m}^*$, the set of all strictly planar diagrams whose propagating number equals n, such that $xb \in X_n(\lambda)$, the diagram x is the map which reorders the nodes. Then $\theta(b) = x^*\theta(xb)$ where x^* is the reflection of x. $\square$

Defining a non-zero homomorphism between $\Delta_n(\lambda)$ and $\Delta_n(\mu)$ depends on finding scalars $\{\alpha_i\}$ in equation (5.15), that satisfy the axiom

$$\theta(xa) = x\theta(a), \tag{5.17}$$

for all $x \in \mathcal{T}_{n,m}$ and $a \in X_n(\lambda)$, see Lemma 5.28.

5.6.1 The trivial case: only one parameter is a root of unity

In this subsection, it will be assumed that there is a unique element $j \in \mathbb{Z}_m$ such that $q_j^{2\mathbf{l}_j} = 1$ for some $\mathbf{l}_j$ and the other parameters are generic. Without losing the generality we can assume that $j = 0$.

Theorem 4.1.10 in [28] is the same as the next theorem when $m = 2$ but it has been proved in different fashion.

Theorem 5.29. *Let $\sum_j \lambda_j = n$ where $\lambda_0 \geq 2$, $\lambda' = (\lambda_0 - 2, \lambda_1, \ldots, \lambda_{m-1})$ and q_0 is the unique parameter that is a root of unity, then*

$$\mathrm{Hom}_{\mathbb{T}_{n,m}}(\Delta_n(\lambda), \Delta_n(\lambda')) \neq 0$$

if and only if $\det \mathsf{G}_{\lambda_0,1,\delta_0} = 0$.

Proof. Assume that $\theta : \Delta_n(\lambda) \to \Delta_n(\lambda')$ be a homomorphism, so it could be defined by $\theta(a) = \sum_i \alpha_i b_i$, where $a \in \mathcal{T}^{|\rangle}_{n,m}[\lambda]$ and $b_i \in \mathcal{T}^{|\rangle}_{n,m}[\lambda']$ for each i. From Proposition 5.26, b_i is formed from a by connecting two $\mathfrak{C}_0$-defects and $\mathrm{top}(a) = \mathrm{top}(b_i)$ for each i.

Now, we need to find scalars α_i that satisfy equation (5.17) for all $x \in \mathcal{T}_{n,m}$. If $\mathrm{bot}(x) \neq \mathrm{top}(a)$, then $xa = 0$ and $xb_i = 0$ for each i as $\mathrm{top}(a) = \mathrm{top}(b_i)$. So the equation (5.17) is verified when $\mathrm{bot}(x) \neq \mathrm{top}(a)$. Assume $\mathrm{bot}(x) = \mathrm{top}(a)$

and $\#(x) \leq n-4$, this implies to $xa = 0 = xb_i$ because of $\#(xa) \leq \#(x)$ and $\#(xb_i) < \#(x)$. So the equation (5.17) holds for all scalars in this case.

There are only two possibilities to check $\#(x) = n$ or $\#(x) = n-2$ with $\mathrm{bot}(x) = \mathrm{top}(a)$. Firstly, if $\#(x) = n$, from the graphical visualization we have $xa = |x\rangle$. In the other side, we have xb_i obtained from $|x\rangle$ by connecting two $\mathfrak{C}_0$-defects, which the same definition of the terms in $\theta(|x\rangle)$. So the equation (5.17) also holds for all scalars in this case.

Finally, if $\#(x) = n-2$ with $\mathrm{bot}(x) = \mathrm{top}(a)$, then $xa = 0$ and $xb_i = \langle x||b_i\rangle|x\rangle \in \Delta_n(\lambda')$. Let $b_{i,k}$ be the link state obtained from b_i by omitting all the parts that have colour not $\mathfrak{C}_k$. Let $y := |x\rangle$. Thus $b_{i,k}$ and y_k are $(\lambda_k, 0)$-link states when $k \neq 0$ and they are $(\lambda_0, 1)$-link state when $k = 0$. Then $\langle x||b_i\rangle = \langle y_0, b_{i,0}\rangle_{\lambda_0,1,\delta_0}$ from (5.11). Hence $0 = \sum_i \alpha_i \langle y_0, b_{i,0}\rangle_{\lambda_0,1,\delta_0}|x\rangle$ since $\theta(xa) = 0$, so $\sum_i \alpha_i \langle y_0, b_{i,0}\rangle_{\lambda_0,1,\delta_0} = 0$. Thus

$$\mathsf{G}_{\lambda_0,1,\delta_0}\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_{\lambda_0}\end{pmatrix} = \begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}.$$

If $\det \mathsf{G}_{\lambda_0,1,\delta_0} = 0$, so there is non-trivial solution for previous equation, this means that there is a non-zero homomorphism between $\Delta_n(\lambda)$ and $\Delta_n(\lambda')$. Otherwise, θ will be zero. $\square$

Let $\lambda' = (\lambda_0 - 2t, \lambda_1, \ldots, \lambda_{m-1})$ and $\theta : \Delta_n(\lambda) \to \Delta_n(\lambda')$ be a homomorphism defined by equation (5.15), where $\sum_{j=0}^{m-1} \lambda_j = n-2v$ and δ_j is invertible for each j. From Proposition 5.26, $\theta\big(1_{\underline{\underline{\mu}}}\Delta_n(\lambda)\big) \subseteq 1_{\underline{\underline{\mu}}}\Delta_n(\lambda')$ for each $\mu \in \Gamma_{(n,m)}$. Also from Lemma 5.15, if $1_{\underline{\underline{\mu}}}\Delta_n(\lambda) \neq 0$ we have

$$1_{\underline{\underline{\mu}}}\Delta_n(\lambda) \cong \mathsf{V}_{\mu_0,p_0} \otimes \cdots \otimes \mathsf{V}_{\mu_{m-1},p_{m-1}}, \quad (\text{ as } 1_{\underline{\underline{\mu}}}\mathbb{T}_{n,m}1_{\underline{\underline{\mu}}} - \text{ modules})$$

where $p_j = \frac{\mu_j-\lambda_j}{2}$ and V_{μ_j,p_j} is the cell module of $\mathsf{TL}_{\mu_j}(\delta_j)$, defined by (1.17). In this case, the homomorphism θ can be restricted to define a homomorphism

$$\theta_\mu : \mathsf{V}_{\mu_0,p_0} \otimes \cdots \otimes \mathsf{V}_{\mu_{m-1},p_{m-1}} \to \mathsf{V}_{\mu_0,p_0+t} \otimes \mathsf{V}_{\mu_1,p_1} \otimes \cdots \otimes \mathsf{V}_{\mu_{m-1},p_{m-1}} \tag{5.18}$$

as $\bigotimes_{i=0}^{m-1} \mathsf{TL}_{\mu_i}(\delta_i)$-modules. Since δ_j is invertible for each j, the map θ_μ defines a homomorphism

$$f_\mu : \mathsf{V}_{\mu_0,p_0} \to \mathsf{V}_{\mu_0,p_0+t} \tag{5.19}$$

that sends a_0 to $\sum_i \alpha_i b_{i,0}$, where a_0 is the (μ_0, p_0)-link state which is obtained from $a \in 1_{\underline{\underline{\mu}}}\mathcal{T}^{|\rangle}_{n,m}[\lambda]$ by omitting all the parts that have colour not $\mathfrak{C}_0$. Similarly, we define $b_{i,0}$.

The proof f_μ is a homomorphism is not difficult, since the action of $x \in \mathsf{TL}_{\mu_0}(\delta_0)$ on $a_0 \in \mathsf{V}_{\mu_0,p_0+t}$ is the same action of the diagram $\mathcal{I}^{\mu_{m-1}}_{m-1} \circ \cdots \circ \mathcal{I}^{\mu_1}_1(x)$ on a, where $\mathcal{I}^{\mu_j}_j(w)$ is defined to be w except with more μ_j defects in the rightmost part of the colour $\mathfrak{C}_j$, and considering that x has the colour $\mathfrak{C}_0$. One part of the next theorem is contained in Theorem 6.2.2 in [28].

Theorem 5.30. *Let δ_j be invertible for each j, $\lambda' = (\lambda_0 - 2t, \lambda_1, \dots, \lambda_{m-1})$ where $\sum_{j=0}^{m-1} \lambda_j = n$, $\lambda_0 > 2t$ and* $\operatorname{Char}\mathbb{F} = 0$, *then*

$$\dim \operatorname{Hom}_{\mathbb{T}_{n,m}}(\Delta_n(\lambda), \Delta_n(\lambda')) = \begin{cases} 1 & \text{if } \mathbf{l}_0 > t > 0 \text{ and} \\ & \lambda_0 - t + 1 = 0 \pmod{\mathbf{l}_0}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.20}$$

Proof. We are going to show that

$$\dim \operatorname{Hom}_{\mathbb{T}_{n,m}}(\Delta_n(\lambda), \Delta_n(\lambda')) = \dim \operatorname{Hom}_{\mathsf{TL}_{\lambda_0}(\delta_0)}(\mathsf{V}_{\lambda_0,0}, \mathsf{V}_{\lambda_0,t}),$$

by finding a bijection between them and then the rest follows from Lemma 1.30. Let

$$\tau : \operatorname{Hom}_{\mathbb{T}_{n,m}}(\Delta_n(\lambda), \Delta_n(\lambda')) \to \operatorname{Hom}_{\mathsf{TL}_{\lambda_0}(\delta_0)}(\mathsf{V}_{\lambda_0,0}, \mathsf{V}_{\lambda_0,t})$$

and

$$\sigma : \mathrm{Hom}_{\mathsf{TL}_{\lambda_0}(\delta_0)}(\mathsf{V}_{\lambda_0,0}, \mathsf{V}_{\lambda_0,t}) \to \mathrm{Hom}_{\mathbb{T}_{n,m}}(\Delta_n(\lambda), \Delta_n(\lambda'))$$

be maps, $\tau(\theta)$ and $\sigma(f)$ defined as follows, where $f \in \mathrm{Hom}(\mathsf{V}_{\lambda_0,0}, \mathsf{V}_{\lambda_0,t})$ and $\theta \in \mathrm{Hom}(\Delta_n(\lambda), \Delta_n(\lambda'))$.

From Lemma 5.28, the map θ can be determined by the value $\theta(|1_{\underline{\underline{\lambda}}}\rangle) := \sum_i \alpha_i b_i$. But from what we showed in the paragraph before this theorem, θ defines a homomorphism $f_\lambda : \mathsf{V}_{\lambda_0,0} \to \mathsf{V}_{\lambda_0,t}$ which defined by

$$|1_{\mathsf{TL}_{\lambda_0}}\rangle \mapsto \sum_i \alpha_i b_{i,0}.$$

Take $\tau(\theta) = f_\lambda$. It is clear that $\theta = 0$ if and only if $\alpha_i = 0$ for each i and this happens if and only if $f_\lambda = 0$. Thus the map τ is injective.

Now let $f : \mathsf{V}_{\lambda_0,0} \to \mathsf{V}_{\lambda_0,t}$ be a homomorphism defined by $f(|1_{\mathsf{TL}_{\lambda_0}}\rangle) = \sum_i \alpha_i v_i$ where v_i is (λ_0, t)-link state for each i. This homomorphism can be used to define a homomorphism $\sigma(f)$ (or simply σ_f) from $\Delta_n(\lambda)$ to $\Delta_n(\lambda')$ which defined by

$$\sigma_f(|1_{\underline{\underline{\lambda}}}\rangle) = \sum_i \alpha_i \mathcal{I}_{m-1}^{\lambda_{m-1}} \circ \cdots \circ \mathcal{I}_1^{\lambda_1}(v_{\mathfrak{C}_0,i}),$$

where $v_{\mathfrak{C}_0,i}$ is the diagram of shape v_i with all its nodes have the colour $\mathfrak{C}_0$, and $\mathcal{I}_j^{\lambda_j}(w)$ is defined to be the same diagram except with more λ_j defects in the rightmost part of the colour $\mathfrak{C}_j$.

We need only to prove that σ_f is well-defined, i.e. if $y|1_{\underline{\underline{\lambda}}}\rangle = 0$ where $y \in \mathcal{T}_{n,m}$, then also $y\sigma_f(|1_{\underline{\underline{\lambda}}}\rangle) = 0$. Let $y|1_{\underline{\underline{\lambda}}}\rangle = 0$ for some y. If $\mathrm{bot}(y) \neq \tilde{\underline{\underline{\lambda}}}$(see equations (2.17) and (2.26)) or $\#_j(y) \neq \lambda_j$ for some $j \neq 0$ or $\#_0(y) < \lambda_0 - 2t$, it is clear that the product will be zero in both of them. Assume that $\mathrm{bot}(y) = \tilde{\underline{\underline{\lambda}}}$, $\#_j(y) = \lambda_j$ for all $j \neq 0$ and $\lambda_0 > \#_0(xy) \geq \lambda_0 - 2t$(note that $y|1_{\underline{\underline{\lambda}}}\rangle$ still equals zero), since $\mathrm{top}(y)$ and arcs in the top half of y don't have any effect on the product, without losing the generality we can assume that $y = \mathcal{I}_{m-1}^{\lambda_{m-1}} \circ \cdots \circ \mathcal{I}_1^{\lambda_1}(d)$ for some $d \in \mathcal{T}_{\lambda_0,m}$ where all the bottom nodes in d have the colour $\mathfrak{C}_0$. Assume that that $y\sigma_f(|1_{\underline{\underline{\lambda}}}\rangle) \neq 0$. So

$$y\sigma_f(|1_{\underline{\underline{\lambda}}}\rangle) = \sum_i \alpha_i \langle d, v_i\rangle_{\lambda_0,t,\delta_0} \mathcal{I}_{m-1}^{\lambda_{m-1}} \circ \cdots \circ \mathcal{I}_1^{\lambda_1}(dv_{\mathfrak{C}_0,i}).$$

Now in both sides of last equation, there are λ_j propagating lines of the colour $\mathfrak{C}_j$ for each $j \neq 0$, by omitting all of them and then ignoring the colours, we obtain

$$df(|1_{\mathsf{TL}_{\lambda_0}}\rangle) = \sum_i \alpha_i \langle d, v_i\rangle_{\lambda_0,t,\delta_0} dv_i \neq 0,$$

But this a contradiction with the fact f is a homomorphism. Thus σ_f is well-defined.

Finally from the previous details, it is clear that $\tau(\sigma_f) = f$, we are done. □

Corollary 5.31. *Let δ_j is invertible for each j, $\lambda' = (\lambda_0 - 2t, \lambda_1, \ldots, \lambda_{m-1})$ where $\sum_{j=0}^{m-1} \lambda_j = n - 2v$ for some v, $\lambda_0 > 2t$ and $\operatorname{Char}\mathbb{F} = 0$, then there is a non-trivial homomorphism*

$$\theta : \Delta_n(\lambda) \longrightarrow \Delta_n(\lambda')$$

if and only if $\lambda_0 - t + 1 = 0 \pmod{\mathbf{l}_0}$ with $\mathbf{l}_0 > t > 0$.

Proof. Since the algebra $\mathbb{T}_{n,m}$ is a tower of recollement (Jegan [28] showed that in Chapter 2), we have

$$\dim \operatorname{Hom}(\Delta_n(\lambda), \Delta_n(\lambda')) = \dim \operatorname{Hom}(\Delta_{n-2v}(\lambda), \Delta_{n-2v}(\lambda')),$$

see Theorem 2.1.27 in [28], and the rest follows directly from the previous theorem. □

The next theorem is a positive characteristic version of the previous ones.

Theorem 5.32. *Let δ_j is invertible for each j, $\lambda' = (\lambda_0 - 2t, \lambda_1, \ldots, \lambda_{m-1})$ where $\sum_{j=0}^{m-1} \lambda_j = n - 2v$ and $\operatorname{Char}\mathbb{F} = \mathsf{p}$, then there is a non-trivial homomorphism*

$$\theta : \Delta_n(\lambda) \longrightarrow \Delta_n(\lambda')$$

if and only if $\lambda_0 - t + 1 = 0 \pmod{\mathsf{l}_0\mathsf{p}^j}$ *for some non-negative integer* j *with* $\mathsf{l}_0\mathsf{p}^j > t \geq 0$.

Proof. The proof is the same proof as that of Theorem 5.30 and the previous corollary replacing Lemma 1.30 by Theorem 1.32 . □

5.6.2 The general case: several parameters are roots of unity

Let $\lambda \in \Gamma_{(n-2v,m)}$ for some v, and $\theta : \Delta_n(\lambda) \to \Delta_n(\lambda - 2t)$ be a homomorphism, where $\lambda - 2t = (\lambda_0 - 2t_0, \ldots, \lambda_{m-1} - 2t_{m-1})$. The homomorphism θ will be non-zero if and only if there is $\mu \in \Gamma_{(n,m)}$ of the form $\mu = \lambda + 2p$ for some $p \in \Gamma_{(v,m)}$ such that $\theta(1_{\underline{\underline{\mu}}}\Delta_n(\lambda)) \neq \{0\}$, see Lemma 5.28. From Lemma 5.15, we have $1_{\underline{\underline{\mu}}}\Delta_n(\lambda) \cong \bigotimes_{i=0}^{m-1} \mathsf{V}_{\lambda_i+2p_i,p_i}$. Thus we can restrict θ to define a non-trivial homomorphism

$$\theta_\mu : \bigotimes_{i=0}^{m-1} \mathsf{V}_{\mu_i,p_i} \longrightarrow \bigotimes_{i=0}^{m-1} \mathsf{V}_{\mu_i,p_i+t_i}.$$

Note that if $\delta_i \neq 0$ for each i, so p does not have any important role since it is corresponding to number of arcs which are actually preserved, see Proposition 5.26. Furthermore, if we have a homomorphism from $\bigotimes_{i=0}^{m-1} \mathsf{V}_{\lambda_i+2p_i,p_i}$ to $\bigotimes_{i=0}^{m-1} \mathsf{V}_{\lambda_i+2p_i,p_i+t_i}$, we can extend it to get a homomorphism from $\Delta_n(\lambda)$ to $\Delta_n(\lambda - 2t)$. Thus

$$\mathrm{Hom}_{\mathbb{T}_{n,m}}(\Delta_n(\lambda), \Delta_n(\lambda - 2t)) = \{0\}$$

if and only if

$$\mathrm{Hom}_{\bigotimes_{i=0}^{m-1} \mathsf{TL}_{\mu_i}(\delta_i)}\Big(\bigotimes_{i=0}^{m-1} \mathsf{V}_{\mu_i,p_i}, \bigotimes_{i=0}^{m-1} \mathsf{V}_{\mu_i,p_i+t_i}\Big) = \{0\}$$

for each $p \in \Gamma_{(v,m)}$.

Now, if there is a non-zero homomorphism $f_i \in \mathrm{Hom}_{\mathsf{TL}_{\mu_i}(\delta_i)}(\mathsf{V}_{\mu_i,p_i}, \mathsf{V}_{\mu_i,p_i+t_i})$ for each i, then $\otimes f_i \in \mathrm{Hom}_{\bigotimes_{i=0}^{m-1} \mathsf{TL}_{\mu_i}(\delta_i)}(\bigotimes_{i=0}^{m-1} \mathsf{V}_{\mu_i,p_i}, \bigotimes_{i=0}^{m-1} \mathsf{V}_{\mu_i,p_i+t_i})$ is also non-zero. From the previous details we have the following propositions.

Proposition 5.33. *[28, Theorem 6.2.2]. Let δ_j is invertible for each j, $\lambda' = \lambda - 2t$ where $\sum_{j=0}^{m-1} \lambda_j = n - 2v$ for some v. Suppose there exist non-zero homomorphisms from $\mathsf{V}_{\lambda_i,0}$ to $\mathsf{V}_{\lambda_i,t_i}$ as $\mathsf{TL}_{\lambda_i}(\delta_i)$-modules for each i. Then there exists a non-trivial homomorphism from $\Delta_n(\lambda)$ to $\Delta_n(\lambda')$.*

5.7 The ordinary representation theory of the algebra $\mathbb{T}_{n,m}$ at roots of unity

Throughout this section we assume that $\mathbb{F} = \mathbb{C}$, $\delta_i = q_i + q_i^{-1} \in \mathbb{C}$ for each i, $\lambda \in \Lambda^0_{\mathbb{T}_{n,m}}$ and at least one of the parameters is a root of unity other than ± 1 (this means $\delta_i \neq 0$ for some i). We aim to compute the decomposition matrix of the algebra $\mathbb{T}_{n,m}$, then by using Theorem 1.14 the Cartan matrix for $\mathbb{T}_{n,m}$ can be found.

Proposition 5.34. *Let $\lambda \in \Gamma_{(n-2v,m)}$ for some $0 \leq v \leq [{}^n/_2]$. The module $\Delta_n(\lambda)$ is simple if and only if $\lambda_i + 1 = 0 \pmod{\mathbf{l}_i}$ whenever q_i is a root of unity where $i \in \mathbb{Z}_m$.*

Proof. If q_i is not a root of unity for some $i \in \mathbb{Z}_m$, Proposition 1.29 implies to $\mathsf{L}_{\lambda_i+2u_i,u_i,\delta_i} = \mathsf{V}_{\lambda_i+2u_i,u_i}$ for any $u \in \Gamma_{(v,m)}$. On the other hand, if q_i is a root of unity for some, recall that $\dim \mathsf{L}_{n_i,u_i,\delta_i} = \dim \mathsf{V}_{n_i,u_i}$ whenever $n_i - 2u_i + 1 = 0 \pmod{\mathbf{l}_i}$, see Corollary 1.35. Since $(\lambda_i + 2u_i) - 2u_i + 1 = 0 \pmod{\mathbf{l}_i}$, so $\mathsf{L}_{\lambda_i+2u_i,u_i,\delta_i} = \mathsf{V}_{\lambda_i+2u_i,u_i}$. Now, by substituting in equation (5.14) and then from equation (5.12), we obtain $\dim \mathsf{L}_n(\lambda) = \dim \Delta_n(\lambda)$, we are done. □

Lemma 5.35. *Let $\sum \lambda_i = n - 2v$, $\lambda_0 + t + 1 = 0 \pmod{\mathbf{l}_0}$ where $0 < t < \mathbf{l}_0$ and for each $i \neq 0$ we have q_i is not a root of unity or $\lambda_i + 1 = 0 \pmod{\mathbf{l}_i}$ when q_i is a root of unity, then*

$$\dim \mathrm{Rad}(\Delta_n(\lambda)) = \dim \mathsf{L}_n(\lambda_0 + 2t, \lambda_1, \ldots, \lambda_{m-1}). \tag{5.21}$$

If $\sum_{i=0}^{m-1} \lambda_i + 2t > n$, then $\dim \mathrm{Rad}(\Delta_n(\lambda)) = 0$.

Proof. When q_i is not a root of unity or $\lambda_i + 1 = 0 \pmod{\mathbf{l}_i}$ when q_i is a root of unity, we have

$$\mathsf{L}_{\lambda_i+2u_i,u_i,\delta_i} = \mathsf{V}_{\lambda_i+2u_i,u_i}$$

for any u_i, see Proposition 1.29 and Corollary 1.35, thus $\dim \mathsf{R}_{\lambda_i+2u_i,u_i,\delta_i} = 0$ for $i \neq 0$. Hence the dimension of $\mathrm{Rad}(\Delta_n(\lambda))$, from Theorem 5.24, is

$$\sum_{u\in\Gamma_{(v,m)}} n_{\lambda+2u} \dim \mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} \prod_{i=1}^{m-1} \dim \mathsf{L}_{\lambda_i+2u_i,u_i,\delta_i}. \tag{5.22}$$

Since $\lambda_0 + t + 1 = 0 \pmod{\mathbf{l}_0}$, then we can assume that $\lambda_0 + t = k\mathbf{l}_0 + \mathbf{l}_0 - 1$ for some $k \in \mathbb{N}$. Hence

$$(\lambda_0 + 2u_0) - 2(u_0) + 1 = k\mathbf{l}_0 + \mathbf{l}_0 - t,$$

so $r_{(\lambda_0+2u_0,u_0)} = \mathbf{l}_0 - t$ since $0 < t < \mathbf{l}_0$, see Proposition 1.36, for any u_0. Thus from the same proposition we obtain

$$\dim \mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} = \begin{cases} \dim \mathsf{L}_{\lambda_0+2u_0,u_0-t,\delta_0} & \text{if } u_0 - t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now if $\sum_{i=0}^{m-1} \lambda_i + 2t > n$, this happens when $v < t$, so $u_0 - t < 0$. Thus from the previous relation we have $\dim \mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} = 0$ for any u_0, from (5.22) we obtain $\dim \mathrm{Rad}(\Delta_n(\lambda)) = 0$. On the other hand, if $\sum_{i=0}^{m-1} \lambda_i + 2t \leq n$, thus $\dim \mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} \neq 0$ when $t \leq u_0 \leq v$. By substituting in equation (5.22), we obtain

$$\begin{aligned}
\dim \mathrm{Rad}(\Delta_n(\lambda)) &= \sum_{\substack{u\in\Gamma_{(v,m)},\\ t\leq u_0\leq v}} n_{\lambda+2u} \dim \mathsf{L}_{\lambda_0+2u_0,u_0-t,\delta_0} \prod_{i=1}^{m-1} \dim \mathsf{L}_{\lambda_i+2u_i,u_i,\delta_i}, \\
&= \sum_{x\in\Gamma_{(v-t,m)}} n_{\lambda'+2x} \dim \mathsf{L}_{\lambda_0+2t+2x_0,x_0,\delta_0} \prod_{i=1}^{m-1} \dim \mathsf{L}_{\lambda_i+2x_i,x_i,\delta_i}, \\
&= \dim \mathsf{L}_n(\lambda'),
\end{aligned}$$

from equation (5.14), where $\lambda' = (\lambda_0 + 2t, \lambda_1, \ldots, \lambda_{m-1})$. □

Remark 5.36. The same happens when we change the colour in the previous lemma. Let $\sum \lambda_i = n - 2v$, $\lambda_j + t + 1 = 0 \pmod{\mathbf{l}_j}$ where $0 < t < \mathbf{l}_j$ and for each $i \neq j$ we have q_i is not a root of unity or $\lambda_i + 1 = 0 \pmod{\mathbf{l}_i}$ when q_i is a root of unity, then

$$\dim \mathrm{Rad}(\Delta_n(\lambda)) = \dim \mathsf{L}_n(\lambda_0, \dots, \lambda_{j-1}, \lambda_j + 2t, \lambda_{j+1}, \dots, \lambda_{m-1}),$$

If $\sum \lambda_i + 2t > n$, then $\dim \mathrm{Rad}(\Delta_n(\lambda)) = 0$.

Proposition 5.37. *Let* $\sum \lambda_i = n - 2v$, $\lambda_0 + t + 1 = 0 \pmod{\mathbf{l}_0}$ *where* $0 < t < \mathbf{l}_0$ *and for each* $i \neq 0$ *we have* q_i *is not a root of unity or* $\lambda_i + 1 = 0 \pmod{\mathbf{l}_i}$ *when* q_i *is a root of unity, then*

$$\mathrm{Rad}(\Delta_n(\lambda)) \cong \mathsf{L}_n(\lambda_0 + 2t, \lambda_1, \dots, \lambda_{m-1}),$$

where $\sum \lambda_i + 2t \leq n$.

Proof. Let $\lambda' = (\lambda_0 + 2t, \lambda_1, \dots, \lambda_{m-1})$. From Theorem 5.30, we have

$$\mathrm{Hom}(\Delta_n(\lambda'), \Delta_n(\lambda)) \neq \{0\}.$$

Let $\Psi : \Delta_n(\lambda') \to \Delta_n(\lambda)$ be a non-zero homomorphism. Its kernel is a proper sub-module of $\Delta_n(\lambda')$ and since the radical of a cell module is a maximal sub-module, so $\mathrm{Ker}\,\Psi \subseteq \mathrm{Rad}(\Delta_n(\lambda'))$. Similarly, $\mathrm{im}\,\Psi \subseteq \mathrm{Rad}(\Delta_n(\lambda))$. It follows that

$$\begin{aligned}\dim \mathrm{Rad}(\Delta_n(\lambda)) &\geq \dim \mathrm{im}\,\Psi = \dim \Delta_n(\lambda') - \dim \mathrm{Ker}\,\Psi, \\ &\geq \dim \Delta_n(\lambda') - \dim \mathrm{Rad}(\Delta_n(\lambda')), \\ &= \dim \mathsf{L}_n(\lambda'). \end{aligned} \tag{5.23}$$

But $\dim \mathrm{Rad}(\Delta_n(\lambda)) = \dim \mathsf{L}_n(\lambda')$ by Lemma 5.35, so $\dim \mathrm{im}\,\Psi = \dim \mathrm{Rad}(\Delta_n(\lambda))$ and $\dim \mathrm{Ker}\,\Psi = \dim \Delta_n(\lambda') - \dim \mathrm{im}\,\Psi = \dim \Delta_n(\lambda') - \dim \mathsf{L}_n(\lambda') = \dim \mathrm{Rad}(\Delta_n(\lambda'))$. Thus $\mathrm{Ker}\,\Psi = \mathrm{Rad}(\Delta_n(\lambda'))$ and $\mathrm{im}\,\Psi = \mathrm{Rad}(\Delta_n(\lambda))$, and by using the first isomorphism theorem(see for example Corollary 3.7.1 in [1]), the proof is concluded. □

Similarly, we have $\mathrm{Rad}(\Delta_n(\lambda)) \cong \mathsf{L}_n(\lambda')$ whenever $\lambda_j + t + 1 = 0 \pmod{\mathbf{l}_j}$ and q_i is not a root of unity or $\lambda_i + 1 = 0 \pmod{\mathbf{l}_i}$ when q_i is a root of unity for each $i \neq j$, where $0 < t < \mathbf{l}_j$, $\sum \lambda_i + 2t \leq n$ and $\lambda' = (\lambda_0, \dots, \lambda_{j-1}, \lambda_j + 2t, \lambda_{j+1}, \dots, \lambda_{m-1})$.

Lemma 5.38. *Let* $\lambda_0 + \lambda_1 = n - 2v$, $\lambda_i + t_i + 1 = 0 \pmod{\mathbf{l}_i}$ *where* $0 < t_i < \mathbf{l}_i$, $i = 0, 1$, *then* $\dim \mathrm{Rad}(\Delta_n(\lambda_0, \lambda_1))$ *equals*

$$\dim \mathsf{L}_n(\lambda_0 + 2t_0, \lambda_1) + \dim \mathsf{L}_n(\lambda_0, \lambda_1 + 2t_1) - \dim \mathsf{L}_n(\lambda + 2t), \tag{5.24}$$

where $t = (t_0, t_1)$. *Whenever* $x_0 + x_1 > n$, *we put* $\dim \mathsf{L}_n(x_0, x_1) = 0$ *for any* $x_0, x_1 \in \mathbb{N}$.

Proof. From Corollary 5.22, we have

$$\dim \mathrm{Rad}(\Delta_n(\lambda)) = I_1 + I_2 - I_3, \tag{5.25}$$

where

$$I_1 = \sum_{u \in \Gamma_{(v,2)}} n_{\lambda+2u} \dim \mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} \cdot \dim \mathsf{V}_{\lambda_1+2u_1,u_1},$$
$$I_2 = \sum_{u \in \Gamma_{(v,2)}} n_{\lambda+2u} \dim \mathsf{V}_{\lambda_0+2u_0,u_0} \cdot \dim \mathsf{R}_{\lambda_1+2u_1,u_1,\delta_1},$$

and

$$I_3 = \sum_{u \in \Gamma_{(v,2)}} n_{\lambda+2u} \dim \mathsf{R}_{\lambda_0+2u_0,u_0,\delta_0} \cdot \dim \mathsf{R}_{\lambda_1+2u_1,u_1,\delta_1}.$$

First, it will be shown that

$$I_1 = \begin{cases} 0 & \text{if } \sum\limits_{i=0,1} \lambda_i + 2t_0 > n, \\ \dim \mathsf{L}_n(\lambda_0 + 2t_0, \lambda_1) & \text{if } \sum\limits_{i=0,1} \lambda_i + 2t_0 \leq n \\ & \text{and } \sum\limits_{i=0,1} (\lambda_i + 2t_i) > n, \\ \dim \mathsf{L}_n(\lambda_0 + 2t_0, \lambda_1) + \dim \mathsf{L}_n(\lambda + 2t) & \text{if } \sum\limits_{i=0,1} (\lambda_i + 2t_i) \leq n. \end{cases} \tag{5.26}$$

Now, since $\lambda_i + t_i + 1 = 0 \pmod{\mathbf{l}_i}$, so we can assume that $\lambda_i + t_i = k_i \mathbf{l}_i + \mathbf{l}_i - 1$ for some $k_i \in \mathbb{N}$, so $(\lambda_i + 2u_i) - 2(u_i) + 1 = k_i \mathbf{l}_i + \mathbf{l}_i - t_i$. From Proposition 1.36, for any

u_i we have

$$\dim \mathsf{R}_{\lambda_i+2u_i,u_i,\delta_i} = \begin{cases} \dim \mathsf{L}_{\lambda_i+2u_i,u_i-t_i,\delta_i} & \text{if } u_i - t_i \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.27}$$

Thus

$$\begin{aligned} I_1 &= \sum_{u_0=t_0}^{v} n_{\lambda+2u} \dim \mathsf{L}_{\lambda_0+2u_0,u_0-t_0,\delta_0} \cdot \dim \mathsf{V}_{\lambda_1+2u_1,u_1}, \\ &= \sum_{x\in\Gamma_{(v-t_0,2)}} n_{(\lambda+2t_0,\lambda_1)+2x} \dim \mathsf{L}_{\lambda_0+2t_0+2x_0,x_0,\delta_0} \cdot \dim \mathsf{V}_{\lambda_1+2x_1,x_1}. \end{aligned}$$

Now if $\lambda_0 + 2t_0 + \lambda_1 > n$, clearly this happens when $v - t_0 < 0$, so $I_1 = 0$. On the other hand, when $\lambda_0 + 2t_0 + \lambda_1 \leq n$, we have

$$\begin{aligned} I_1 = \sum_{x\in\Gamma_{(v-t_0,2)}} & n_{(\lambda+2t_0,\lambda_1)+2x} \dim \mathsf{L}_{\lambda_0+2t_0+2x_0,x_0,\delta_0} \cdot \big(\dim \mathsf{L}_{\lambda_1+2x_1,x_1,\delta_1} + \\ & \dim \mathsf{R}_{\lambda_1+2x_1,x_1,\delta_1} \big), \end{aligned}$$

since $\dim \mathsf{V}_{n,p} = \dim \mathsf{L}_{n,p,\delta} + \dim \mathsf{R}_{n,p,\delta}$ for any n, p and δ. Hence

$$I_1 = \dim \mathsf{L}_n(\lambda') + \sum_{x\in\Gamma_{(v-t_0,2)}} n_{\lambda'+2x} \dim \mathsf{L}_{\lambda_0+2t_0+2x_0,x_0,\delta_0} \dim \mathsf{R}_{\lambda_1+2x_1,x_1,\delta_1},$$

where $\lambda' = (\lambda_0 + 2t_0, \lambda_1)$. From equation (5.27), we have

$$\begin{aligned} I_1 &= \dim \mathsf{L}_n(\lambda') + \sum_{x_1=t_1}^{v-t_0} n_{\lambda'+2x} \dim \mathsf{L}_{\lambda_0+2t_0+2x_0,x_0,\delta_0} \dim \mathsf{L}_{\lambda_1+2x_1,x_1-t_1,\delta_1}, \\ &= \dim \mathsf{L}_n(\lambda') + \sum_{w\in\Gamma_{(v-t_0-t_1,2)}} n_{\lambda+2t+2w} \dim \mathsf{L}_{\lambda_0+2t_0+2w_0,w_0,\delta_0} \dim \mathsf{L}_{\lambda_1+2t_1+2w_1,w_1,\delta_1}. \end{aligned}$$

If $\sum(\lambda_i + 2t_i) > n$, thus $v - t_0 - t_1 < 0$ and this implies to $I_1 = \dim \mathsf{L}_n(\lambda')$. If $\sum(\lambda_i + 2t_i) \leq n$, so $v - t_0 - t_1 \geq 0$, then $I_1 = \dim \mathsf{L}_n(\lambda') + \dim \mathsf{L}_n(\lambda + 2t)$ by Proposition 5.20.

Similarly, we can prove that

$$I_2 = \begin{cases} 0 & \text{if } \sum\limits_{i=0,1} \lambda_i + 2t_1 > n, \\ \dim \mathsf{L}_n(\lambda_0, \lambda_1 + 2t_1) & \text{if } \sum\limits_{i=0,1} \lambda_i + 2t_1 \le n \\ & \text{and } \sum(\lambda_i + 2t_i) > n, \\ \dim \mathsf{L}_n(\lambda_0, \lambda_1 + 2t_1) + \dim \mathsf{L}_n(\lambda + 2t) & \text{if } \sum\limits_{i=0,1} (\lambda_i + 2t_i) \le n. \end{cases} \tag{5.28}$$

Finally, because of equation (5.27) we obtain

$$I_3 = \sum_{u_0=t_0}^{v-t_1} n_{\lambda+2u} \dim \mathsf{L}_{\lambda_0+2u_0,u_0-t_0,\delta_0} \cdot \dim \mathsf{L}_{\lambda_1+2u_1,u_1-t_1,\delta_1}.$$

Now, if $\sum\limits_{i=0,1} (\lambda_i + 2t_i) > n$, this implies $I_3 = 0$. If $\sum\limits_{i=0,1} (\lambda_i + 2t_i) \le n$, we have

$$I_3 = \sum_{x\in\Gamma_{(v-t_0-t_1,2)}} n_{\lambda+2t+2x} \dim \mathsf{L}_{\lambda_0+2t_0+2x_0,x_0,\delta_0} \cdot \dim \mathsf{L}_{\lambda_1+2t_1+2x_1,x_1,\delta_1}.$$

Hence

$$I_3 = \begin{cases} \dim \mathsf{L}_n(\lambda + 2t) & \text{if } \sum\limits_{i=0,1} (\lambda_i + 2t_i) \le n, \\ 0 & \text{otherwise.} \end{cases} \tag{5.29}$$

By substituting I_1, I_2 and I_3 into (5.25), we obtain the formula (5.24). □

Lemma 5.39. *Let* $\lambda \in \Gamma_{(n-2v,m)}$*,* $0 \le s < m$*,* q_i *is not a root of unity or* $\lambda_i + 1 = 0 \pmod{\mathbf{l}_i}$ *when* q_i *is a root of unity for each* $i > s$*, and* $\lambda_j + t_j + 1 = 0 \pmod{\mathbf{l}_j}$ *and* $0 < t_j < \mathbf{l}_j$ *for each* $j \le s$*. Then*

$$\dim \operatorname{Rad}(\Delta_n(\lambda)) = \sum_{\lambda'\in\Xi} \dim \mathsf{L}_n(\lambda'),$$

where $\Xi = \{\lambda' | \lambda'_i = \lambda_i$ *for each* $i > s$ *and* $\lambda'_j = \lambda_j$ *or* $\lambda'_j = \lambda_j + 2t_j$ *for each* $j \le s\}$*. We put* $\mathsf{L}_n(\lambda') = \{0\}$ *whenever* $\sum \lambda'_i > n$*.*

Proof. When q_i is not a root of unity or $\lambda_i + 1 = 0 \pmod{\mathbf{l}_i}$ when q_i is a root of unity, we have $\mathsf{L}_{\lambda_i+2u_i,u_i,\delta_i} = \mathsf{V}_{\lambda_1+2u_i,u_i}$ for any u_i, see Proposition 1.29 and Corollary 1.35, thus $\dim \mathsf{R}_{\lambda_i+2u_i,u_i,\delta_i} = 0$. Hence the dimension of $\mathrm{Rad}(\Delta_n(\lambda))$, from Theorem 5.24, is

$$\sum_{u\in\Gamma_{(v,m)}} n_{\lambda+2u} \dim \Big(\mathsf{R}_0 \otimes \mathsf{V}_1 \otimes \cdots \otimes \mathsf{V}_s + \mathsf{V}_0 \otimes \mathsf{R}_1 \otimes \mathsf{V}_2 \otimes \cdots \otimes \mathsf{V}_s + \cdots + \mathsf{V}_0 \otimes \cdots \otimes \mathsf{V}_{s-1} \otimes \mathsf{R}_s \Big) \prod_{i=s+1}^{m-1} \dim \mathsf{L}_i, \tag{5.30}$$

where we put $\mathsf{V}_i = \mathsf{V}_{\lambda_i+2u_i,u_i}$, $\mathsf{R}_i = \mathsf{R}_{\lambda_i+2u_i,u_i,\delta_i}$ and $\mathsf{L}_i = \mathsf{L}_{\lambda_i+2u_i,u_i,\delta_i}$ for simplicity and $n_{\lambda+2u}$ comes from the different colour distributions that can be obtained from $\mathsf{V}_0 \otimes \cdots \otimes \mathsf{V}_i \otimes \mathsf{R}_{i+1} \otimes \mathsf{V}_{i+2} \cdots \otimes \mathsf{V}_s$. From Theorem 4.1 in [18], we have

$$\dim \big(\mathsf{R}_0 \otimes \mathsf{V}_1 \otimes \cdots \otimes \mathsf{V}_s + \cdots + \mathsf{V}_0 \otimes \cdots \otimes \mathsf{V}_{s-1} \otimes \mathsf{R}_s \big) = \sum_{i=0}^{s} (-1)^i I_{i+1},$$

where

$$\begin{aligned} I_1 &= \sum_{i=0}^{s} \dim \mathsf{R}_i \prod_{j\neq i} \dim \mathsf{V}_j, \\ I_2 &= \sum_{0\leq i,j\leq s, i\neq j} \dim \mathsf{R}_i \dim \mathsf{R}_j \prod_{k\neq i,j} \dim \mathsf{V}_k, \\ &\vdots \\ I_s &= \prod_{i=0}^{s} \dim \mathsf{R}_i. \end{aligned}$$

Now, since $\dim \mathsf{V}_i = \dim \mathsf{R}_i + \dim \mathsf{L}_i$, we obtain $\sum_{i=0}^{s} (-1)^i I_{i+1} = \sum_{i=0}^{s} I'_{i+1}$, where

$$I'_1 = \sum_{i=0}^{s} \dim \mathsf{R}_i \prod_{j\neq i} \dim \mathsf{L}_j,$$

$$I'_2 = \sum_{0\le i,j\le s, i\neq j} \dim \mathsf{R}_i \dim \mathsf{R}_j \prod_{k\neq i,j} \dim \mathsf{L}_k,$$

$$\vdots$$

$$I'_s = \prod_{i=0}^{s} \dim \mathsf{R}_i.$$

Since $\lambda_i + t_i + 1 = 0 \pmod{\mathbf{l}_i}$ for each $i \le s$, then we can assume that $\lambda_i + t_i = k_i\mathbf{l}_i + \mathbf{l}_i - 1$ for some $k_i \in \mathbb{N}$. Hence form Proposition 1.36, for any u_i we have

$$\dim \mathsf{R}_{\lambda_i+2u_i,u_i,\delta_i} = \begin{cases} \dim \mathsf{L}_{\lambda_i+2u_i,u_i-t_i,\delta_i} & \text{if } u_i - t_i \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

By substituting by that in I'_j for each j and then in equation (5.30), we obtain the formula $\dim \mathrm{Rad}(\Delta_n(\lambda)) = \sum_{\lambda'\in\Xi} \mathsf{L}_n(\lambda')$. We are done. □

As a consequence of Theorem 5.33 and Lemma 5.39, we can determine all the simple modules that are included in the Loewy structure (see for example Section 5.1 in [2]) of any module $\Delta_n(\lambda)$, where $\lambda \in \Lambda_{\mathbb{T}_{n,m}}$, and the number of copies of each one occurring in the Loewy structure. Next we are going to compute the Loewy length and Loewy layers for each module.

Theorem 5.40. *Let $\mathbb{T}_{n,2}(\delta_0, \delta_1)$ be the bubble algebra over the complex field and $\lambda_0 + \lambda_1 = n - 2v$, $\lambda_i + t_i + 1 = 0 \pmod{\mathbf{l}_i}$ where $i = 0, 1$ and $0 < t_i < \mathbf{l}_i$, then*

$$\mathsf{L}_n(\lambda + 2t) \hookrightarrow \mathrm{Rad}(\Delta_n(\lambda)) \twoheadrightarrow \mathsf{L}_n(\lambda_0 + 2t_0, \lambda_1) \oplus \mathsf{L}_n(\lambda_0, \lambda_1 + 2t_1),$$

is an exact sequence, where $t = (t_0, t_1)$. Whenever $x_0 + x_1 > n$, we put $\mathsf{L}_n(x_0, x_1) = \{0\}$ for any $x_0, x_1 \in \mathbb{N}$.

Proof. From Theorem 5.23, we know

$$\mathrm{Rad}(\Delta_n(\lambda)) = \sum_{u\in\Gamma_{(v,2)}} \sum_{\sigma\in\widehat{\mathfrak{S}_{n,2}}} \sigma\big(\mathsf{R}_{u_0,0} \otimes \mathsf{V}_{u_1} + \mathsf{V}_{u_0} \otimes \mathsf{R}_{u_1,1}\big),$$

where $\mathsf{R}_{u_i,i} := \mathsf{R}_{\lambda_i+2u_i,u_i,\delta_i}$ and $\mathsf{V}_{u_i} := \mathsf{V}_{\lambda_i+2u_i,u_i}$. Define W_1, W_2 and $W_{1,2}$ to be

$$W_1 := \sum_{u\in\Gamma_{(v,2)}} \sum_{\sigma\in\widehat{\mathfrak{S}_{n,2}}} \sigma\big(\ \mathsf{R}_{u_0,0}\otimes\mathsf{V}_{u_1}\ \big), \qquad W_2 := \sum_{u\in\Gamma_{(v,2)}} \sum_{\sigma\in\widehat{\mathfrak{S}_{n,2}}} \sigma\big(\ \mathsf{V}_{u_0}\otimes\mathsf{R}_{u_1,1}\ \big),$$

$$W_{1,2} := \sum_{u\in\Gamma_{(v,2)}} \sum_{\sigma\in\widehat{\mathfrak{S}_{n,2}}} \sigma\big(\ \mathsf{R}_{u_0,0}\otimes\mathsf{R}_{u_1,1}\ \big).$$

Note that $\mathrm{Rad}(\Delta_n(\lambda)) = W_1 + W_2$ and $W_{1,2} = W_1\cap W_2$. To prove our theorem we are going to show that $\mathsf{L}_n(\lambda+2t)\cong W_{1,2}$ and $(W_1+W_2)/W_{1,2}\cong \mathsf{L}_n(\lambda_0+2t_0,\lambda_1)\oplus \mathsf{L}_n(\lambda_0,\lambda_1+2t_1)$.

First, we need to show that W_1 and W_2 are sub-modules of the module $\mathrm{Rad}(\Delta_n(\lambda))$. This implies to $W_{1,2}$ is also sub-module. Let $x = \sigma(a_0\otimes a_1)\in W_1$ where $\sigma\in\widehat{\mathfrak{S}_{n,2}}$, $a_0\in \mathsf{R}_{u_0,0}$ and a_1 is a link state in V_{u_1} for some u, and let $D=(D_0,D_1)\in\mathcal{T}_{n,2}$. Since $\mathrm{Rad}(\Delta_n(\lambda)) = W_1+W_2$, so x and Dx is also contained in $\mathrm{Rad}(\Delta_n(\lambda))$. Without losing the generality, we can assume $Dx\neq 0$, then from the graphical visualization we have $Dx = \zeta\big((D_0'a_0)\otimes(D_1'a_1)\big)$ for some $\zeta\in\widehat{\mathfrak{S}_{n,2}}$ and D_i' is the diagram D_i after ignoring the colour, for example see the next example. Note that there exists D_1 such that $D_1'a_1$ will be not contained $\mathsf{R}_{u_1',1}$ for some u_1' since $\delta_1\neq 0$(as $\mathsf{V}_{u_1'}\neq \mathsf{R}_{u_1',1}$), thus $D_0'a_0\in \mathsf{R}_{u_0',0}$ for some $u'\in\Gamma_{(v,2)}$ since we can fix D_0' and change D_1', from this we have $Dx\in W_1$, this implies to W_1 is a sub-module. Similarly, W_2 is a sub-module of $\mathrm{Rad}(\Delta_n(\lambda))$.

From Theorems 1.31 and 1.33, there is an $\mathsf{TL}_{\lambda_i+2u_i}(\delta_i)$-isomorphism

$$f_{u_i,i} : \mathsf{L}_{\lambda_i+2u_i,u_i+t_i,\delta_i} \to \mathsf{R}_{u_i,i}$$

since $\lambda_i + t_i + 1 = 0 \pmod{\mathbf{l}_i}$ and $0 < t_i < \mathbf{l}_i$ for each $u\in\Gamma_{(v,2)}$. By using one of these isomorphisms we can define a non-zero homomorphism from $\mathsf{L}_n(\lambda+2t)$ to $W_{1,2}$ as follows: Fix $u\in\Gamma_{(v,2)}$ and $f_{u_0,0}$ and $f_{u_1,1}$. By using $f_{u_0,0}$ and $f_{u_1,1}$ we obtain an $\mathsf{TL}_{\lambda_0+2u_0}(\delta_0)\otimes\mathsf{TL}_{\lambda_1+2u_1}(\delta_1)$-module isomorphism $f_{u_0,0}\otimes f_{u_1,1}$ from $\mathsf{L}_{u_0,0}\otimes\mathsf{L}_{u_1,1}$ to $\mathsf{R}_{u_0,0}\otimes\mathsf{R}_{u_1,1}$, where $\mathsf{L}_{u_i,i} := \mathsf{L}_{\lambda_i+2u_i,u_i,\delta_i}$. Hence we can define the map $\Psi : \mathsf{L}_n(\lambda+2t)\to$

$W_{1,2}$ to be the extension of $f_{u_0,0} \otimes f_{u_1,1}$ since $\mathsf{L}_n(\lambda+2t) = \sum_{w \in \Gamma_{(v,2)}} \sum_{\sigma \in \widehat{\mathfrak{S}_{n,2}}} \sigma\big(\mathsf{L}_{w_0,0} \otimes \mathsf{L}_{w_1,1} \big)$. To prove that Ψ is a non-zero $\mathbb{T}_{n,2}$-homomorphism, it is enough to show that Ψ is well-defined, i.e. if $D(a_0 \otimes a_1) = 0$ where $D \in \mathcal{T}_{n,2}$ and $a_i \in \mathsf{L}_{u_i,i}$ for each i, then $D(f_{u_0,0}(a_0) \otimes f_{u_1,1}(a_1)) = 0$. From the definitions of $\mathsf{R}_{u_0,0} \otimes \mathsf{R}_{u_1,1}$ and $\mathsf{L}_{u_0,0} \otimes \mathsf{L}_{u_1,1}$, we have $a_0 \otimes a_1$ and $f_{u_0,0}(a_0) \otimes f_{u_1,1}(a_1)$ have the same top which is $\underline{\underline{\widetilde{\lambda+2u}}}$. As the set top$(D)$ and the arcs in top half of D don't have any effect on the product, so without losing the generality we can assume that $D = D_0 \otimes D_1$ where $D_i \in \mathsf{TL}_{\lambda_i+2u_i}$. Hence the well-definedness of Ψ comes directly from the fact $f_{u_0,0} \otimes f_{u_1,1}$ is an $\mathsf{TL}_{\lambda_0+2u_0}(\delta_0) \otimes \mathsf{TL}_{\lambda_1+2u_1}(\delta_1)$- module isomorphism, also Ψ is a non-zero homomorphism since $f_{u_i,i} \neq 0$ for each i.

Now as $\mathsf{L}_n(\lambda+2t)$ is a simple module and the fact that $\mathsf{L}_n(\lambda+2t)$ and $W_{1,2}$ have the same dimension, see Corollary 5.25 and Proposition 1.36, finally by using the first isomorphism theorem we obtain that they are isomorphic.

Now since $\mathrm{Rad}(\Delta_n(\lambda)) = W_1 + W_2$ and $W_{1,2} = W_1 \cap W_2$ so it is clear that

$$\frac{W_1 + W_2}{W_{1,2}} \cong \frac{W_1}{W_{1,2}} \oplus \frac{W_2}{W_{1,2}}.$$

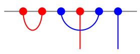
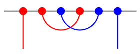
Also

$$\begin{aligned} \frac{W_1}{W_{1,2}} &= \frac{\sum_{u \in \Gamma_{(v,2)}} \sum_{\sigma \in \widehat{\mathfrak{S}_{n,2}}} \sigma\big(\mathsf{R}_{u_0,0} \otimes \mathsf{V}_{u_1} \big)}{\sum_{u \in \Gamma_{(v,2)}} \sum_{\sigma \in \widehat{\mathfrak{S}_{n,2}}} \sigma\big(\mathsf{R}_{u_0,0} \otimes \mathsf{R}_{u_1,1} \big)}, \\ &\cong \sum_{u \in \Gamma_{(v,2)}} \sum_{\sigma \in \widehat{\mathfrak{S}_{n,2}}} \sigma\big(\mathsf{R}_{u_0,0} \otimes \mathsf{L}_{u_1,1} \big). \end{aligned}$$

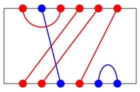
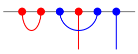
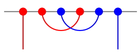
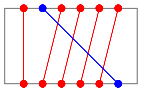
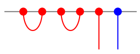
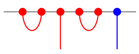
Now as we did to prove the isomorphism between $W_{1,2}$ and $\mathsf{L}_n(\lambda+2t)$, we can show that $\sum_{u \in \Gamma_{(v,2)}} \sum_{\sigma \in \widehat{\mathfrak{S}_{n,2}}} \sigma\, \mathsf{R}_{u_0,0} \otimes \mathsf{L}_{u_i,i} \cong \mathsf{L}_n(\lambda_0 + 2t_0, \lambda_1)$. Similarly,

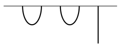
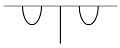
$$\frac{W_2}{W_{1,2}} \cong \mathsf{L}_n(\lambda_0, \lambda_1 + 2t_1). \qquad \square$$

Example 5.40.1. *Let* $\delta_0 = \delta_1 = 1$. *It is easy to show that* $(|\mathsf{u}_1\rangle - |\mathsf{u}_2\rangle)$ *is an element in the radical* $\mathsf{R}_{3,1,\delta_0}$, *so the element*

 $-$ 

is contained in the radical $\mathrm{Rad}(\Delta_6(1,1))$*, since it is an element in* $\sigma\big(\mathsf{R}_{3,1,\delta_0} \otimes \mathsf{V}_{3,1}\big)$ *for some* $\sigma \in \widehat{\mathfrak{S}_{6,2}}$. *Also*

 $\Big($  $-$  $\Big) =$  $\Big($  $-$  $\Big),$

note that the element  $-$  *is an element in* $\mathsf{R}_{5,2,\delta_0}$.

Example 5.40.2. *Let* $\breve{\delta} = (0, \sqrt{2})$*, then* $\mathbf{l}_0 = 2$ *and* $\mathbf{l}_1 = 4$ *and the critical lines are* $\lambda_0 = 1, 3, 5, \ldots$ *and* $\lambda_1 = 3, 7, \ldots$ *which are represented by coloured lines in figure 5.4. Also the arrows in the figure represent non-zero homomorphisms between the cell modules that are indexed by the nodes in the figure. Two nodes will be in the same block if and only if there is an arrow between them. Then decomposition matrix of the algebra* $\mathbb{T}_{6,2}(0, \sqrt{2})$ *is*

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \bigoplus \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \bigoplus \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \bigoplus \bigoplus^{4} (1),$$

we order the basis as following $\{(0,0), (2,0), (0,6), (4,0), (6,0), (1,1), (1,5), (0,2), (2,2), (0,4), (4,2), (2,4), (3,1), (1,3), (5,1), (3,3)\}$. *Then by Theorem 1.14 the Cartan matrix of* $\mathbb{T}_{6,2}(\breve{\delta})$ *is*

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \bigoplus \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \bigoplus \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 2 & 2 & 1 & 4 \end{pmatrix} \bigoplus \bigoplus^{4} (1).$$

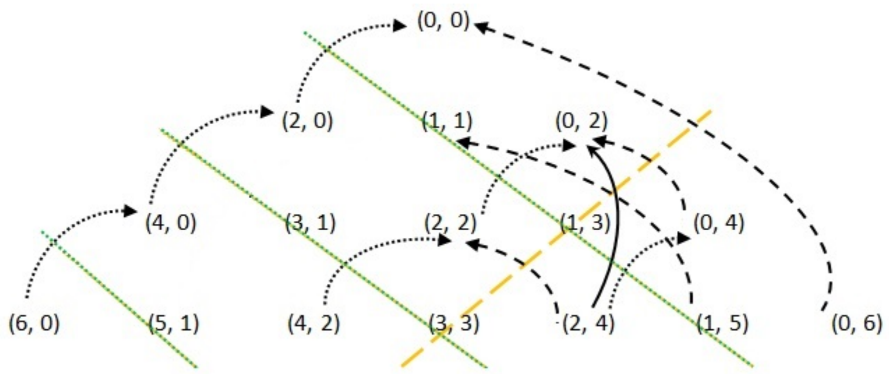


FIGURE 5.4: The Bratteli diagram of the algebra $\mathbb{T}_{6,2}(\breve{\delta})$ when $\mathbf{l}_0 = 2$ and $\mathbf{l}_1 = 4$.

Next theorem is a generalization of last theorem in the case $m > 2$ with several parameters roots of unity.

Theorem 5.41. *Let* $\mathbb{T}_{n,m}(\breve{\delta})$ *be the bubble algebra over the complex field and* $\lambda \in \Gamma_{(n-2v,m)}$, $0 \leq s < m$. *For each* $i > s$, *suppose either* q_i *is not a root of unity or* $\lambda_i + 1 = 0 \pmod{\mathbf{l}_i}$ *when* q_i *is a root of unity, and for each* $j \leq s$ *we have* $\lambda_j + t_j + 1 = 0 \pmod{\mathbf{l}_j}$ *and* $0 < t_j < \mathbf{l}_j$. *Then the length of the radical series of* $\Delta_n(\lambda)$ *is less than or equal to* $s + 1$, *and the radical layers are*

$$\operatorname{Rad}^k(\Delta_n(\lambda))/\operatorname{Rad}^{k+1}(\Delta_n(\lambda)) \cong \bigoplus_{\lambda' \in \Xi_k} \mathsf{L}_n(\lambda'),$$

where $\Xi_k = \{\lambda' |$ *there are exactly* k *values of* j *where* $0 \leq j \leq s$ *such that* $\lambda'_j = \lambda_j + 2t_j$ *and for the other values we have* $\lambda'_i = \lambda_i\}$ *and* $0 \leq k \leq s + 1$. *We put* $\mathsf{L}_n(\lambda') = \{0\}$ *whenever* $\sum \lambda'_i > n$.

Proof. From Theorem 5.24, we have

$$\operatorname{Rad}(\Delta_n(\lambda)) = \sum_{u \in \Gamma_{(v,m)}} \sum_{\sigma \in \widehat{\mathfrak{S}_{n,m}}} \sum_{i=0}^{s} \sigma\big(\mathsf{V}_0 \otimes \cdots \otimes \mathsf{V}_{i-1} \otimes \mathsf{R}_i \otimes \mathsf{V}_{i+1} \otimes \cdots \otimes \mathsf{V}_{m-1}\big),$$

where $\mathsf{R}_i := \mathsf{R}_{\lambda_i + 2u_i, u_i, \delta_i}$ and $\mathsf{V}_i := \mathsf{V}_{\lambda_i + 2u_i, u_i}$, since $\mathsf{R}_i = \{0\}$ for each $i > s$, see Proposition 1.34.

Define W_i, where $0 \le i \le s$, to be

$$W_i = \sum_{u \in \Gamma_{(v,m)}} \sum_{\sigma \in \widehat{\mathfrak{S}_{n,m}}} \sigma\big(\mathsf{V}_0 \otimes \cdots \otimes \mathsf{V}_{i-1} \otimes \mathsf{R}_i \otimes \mathsf{V}_{i+1} \otimes \cdots \otimes \mathsf{V}_{m-1} \big).$$

Note that W_i is a sub-module of $\mathrm{Rad}(\Delta_n(\lambda))$ for each i, the proof is similar to the one in Theorem 5.40. Also define the modules $W_{i_1,\dots,i_k}$ and W^k, where $0 \le i_h \le s$ for each h and $i_h \ne i_{h'}$ for each $h \ne h'$ where $k = 1, \dots, s+1$, to be

$$W_{i_1,\dots,i_k} = \bigcap_{h=1}^{k} W_{i_h},$$
$$W^k = \sum_{(i_1,\dots,i_k)} W_{i_1,\dots,i_k}.$$

Since $W_{i_1,\dots,i_k}$ is an intersection of sub-modules, so $W_{i_1,\dots,i_k}$ is also a sub-module and from their definitions it is clear that $\sum_{i_k} W_{i_1,\dots,i_k} \subseteq W_{i_1,\dots,i_{k-1}}$, thus $W^k \subseteq W^{k-1}$.

We are going to prove that $\mathrm{Rad}^k(\Delta_n(\lambda)) = W^k$, by using induction where it is clear that $\mathrm{Rad}(\Delta_n(\lambda)) = W^1$ and $W^{k+1} \subseteq W^k$, we only need to show $W^k/W^{k+1} \cong \bigoplus_{\lambda' \in \Xi_k} \mathsf{L}_n(\lambda')$:

$$\begin{aligned} \frac{W^k}{W^{k+1}} &= \frac{\sum_{(i_1,\dots,i_k)} W_{i_1,\dots,i_k}}{\sum_{(j_1,\dots,j_{k+1})} W_{j_1,\dots,j_{k+1}}}, \\ &\cong \bigoplus_{(i_1,\dots,i_k)} \frac{W_{i_1,\dots,i_k}}{\sum_{i_{k+1}} W_{i_1,\dots,i_k,i_{k+1}}}. \end{aligned}$$

Without loss generality, we will just compute $W_{0,\dots,k-1}/(\sum_{i=k}^{s} W_{0,\dots,k-1,i})$ which equals

$$\frac{\sum_u \sum_\sigma \sigma\big(\bigotimes_{j=0}^{k-1} \mathsf{R}_j \otimes \bigotimes_{h=k}^{m-1} \mathsf{V}_h \big)}{\sum_{i=k}^{s} \sum_u \sum_\sigma \sigma\big(\bigotimes_{j=0}^{k-1} \mathsf{R}_j \otimes \mathsf{V}_k \otimes \cdots \otimes \mathsf{V}_{i-1} \otimes \mathsf{R}_i \otimes \mathsf{V}_{i+1} \otimes \cdots \otimes \mathsf{V}_{m-1} \big)},$$

it is clear that it is isomorphic to

$$Z = \sum_{u} \sum_{\sigma} \sigma\Big(\bigotimes_{j=0}^{k-1} \mathsf{R}_j \otimes \bigotimes_{l=k}^{s} \mathsf{L}_h \otimes \bigotimes_{l=s+1}^{m-1} \mathsf{V}_l \Big),$$

where $\mathsf{L}_i := \mathsf{L}_{\lambda_i+2u_i,u_i,\delta_i}$. Since $\mathsf{V}_l \cong \mathsf{L}_l$ for each $l > s$, see Corollary 1.35, and from Theorems 1.31 and 1.33 there is a non-zero homomorphism from $\mathsf{L}_{\lambda_i+2t_i+2u_i,u_i,\delta_i}$ to $\mathsf{R}_{\lambda_i+2u_i,u_i,\delta_i}$ for each $i > k$. Hence we can define a non-zero homomorphism from $\mathsf{L}_n(\lambda')$ to Z, also we can show that they have the same dimension and $\mathsf{L}_n(\lambda')$ is simple, so they are isomorphic by using the first isomorphism theorem, where $\lambda' = (\lambda_0 + 2t_0, \ldots, \lambda_{k-1} + 2t_{k-1}, \lambda_k, \ldots, \lambda_{m-1})$. It is clear that $\lambda' \in \Xi_k$ and by taking all the possibilities of the tuple $(i_1, \ldots, i_k)$ we will obtain all the elements in the set Ξ_k, we are done. □

Although it is not covered here, the work in thesis can be continued further by attempting to compute the Cartan matrix of the algebra $\mathbb{T}_{n,m}(\delta_0, \ldots, \delta_{m-1})$ over a field with a positive characteristic.

Chapter 6

Conclusion

In this short final chapter, we summarise what has been achieved so far, and also give some suggestions for further exploration of this topic.

We have determined the generic representation theory of the the multi-colour partition algebra $\mathbb{P}_{n,m}$ over the complex field. In order to understand the representation theory of the algebra $\mathbb{P}_{n,m}$ over $\mathbb{C}$, we have to study the representation theory of the partition algebras. Also, it has been showed that the multi-colour symmetric groupoid algebra is isomorphic to a generalized symmetric group algebra over $\mathbb{C}$.

It was worth studying the representation theory of the Temperley-Lieb algebra as it is closely tied to the representation theory of the bubble algebra. In Chapter 5, we have studied the connection between the cell modules of both the algebra $\mathsf{TL}_n(\delta)$ and the bubble algebra. Although the main results are over the complex field, but many of them are still true over any field.

The relation between the representation theory of the algebras $\mathbb{A}$ and $\mathsf{e}\mathbb{A}\mathsf{e}$, where e is an idempotent, has been used a lot for example see [21], [41], [1] and [39]. Future work lies on generalizing the technique that was used to study both the multi-colour partition algebra and the bubble algebra. It should be possible to apply the same technique if $\mathbb{A}$ is a cellular algebra with an orthogonal decomposition of the identity providing that this decomposition satisfies the same conditions that exist in [53].

Bibliography

[1] F. Anderson and K. Fuller. *Rings and categories of modules.* Springer-Verlag, 1992.

[2] I. Assem, D. Simson, and A. Skowronski. *Elements of the Representation Theory of Associative Algebras 1*, volume 65. Cambridge University Press, 2006.

[3] D. Benson. *Representations and cohomology I.* Cambridge University Press, 1998.

[4] F. Bergeron. *Algebraic combinatorics and coinvariant spaces.* CRC Press, 2009.

[5] D. Bernstein. *Matrix mathematics: theory, facts, and formulas with applications to linear systems theory.* Princeton University Press, 2005.

[6] M. Brešar. *Introduction to noncommutative algebra.* Springer, 2014.

[7] H. Can. Representations of the generalized symmetric groups. *Beitrage zur Algebra und Geometrie*, 37(2):289–308, 1996.

[8] E. Cline, B. Parshall, and L. Scott. Finite dimensional algebras and highest weight categories. *J. Reine Angew. Math.*, 391(2):85–99, 1988.

[9] E. Cline, B. Parshall, and L. Scott. Generic and q-rational representation theory. *Publ. RIMS, Kyoto University*, 35(1):31–90, 1999.

[10] A. Cox, J. Graham, and P. Martin. The blob algebra in positive characteristic. *J. Algebra*, 266(2):584–635, 2003.

[11] A. Cox, P. Martin, A. Parker, and C. Xi. Representation theory of towers of recollement: theory, notes, and examples. *J. Algebra*, 302(1):340–360, 2006.

[12] J. Davies and A. Morris. The Schur multiplier of the generalized symmetric group. *J. the London Mathematical Society*, 2(4):615–620, 1974.

[13] M. Dokuchaev, R. Exel, and P. Piccione. Partial representations and partial group algebras. *J. Algebra*, 226(1):505–532, 2000.

[14] W. Doran and D. Wales. The partition algebra revisited. *J. Algebra*, 231(1):265–330, 2000.

[15] J. East. Generators and relations for partition monoids and algebras. *J. Algebra*, 339(1):1–26, 2011.

[16] P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwender, D. Vaintrob, and E. Yudovina. *Introduction to representation theory*, volume 59. American Mathematical Society, 2011.

[17] W. Fulton and J. Harris. *Representation theory: a first course*, volume 129. Springer Science & Business Media, 1991.

[18] J. Gallier. *Discrete Mathematics.* Springer, 2011.

[19] J. De Gier. Loops, matchings and alternating-sign matrices. *Discrete Mathematics*, 298(1):365–388, 2005.

[20] J. Graham and G. Lehrer. Cellular algebras. *Inventiones Mathematicae*, 123(1):1–34, 1996.

[21] J. Green. *Polynomial representations of GL_n, Lecture Notes in Math.*, volume 830. Springer, 1980.

[22] R. Green and P. Martin. Constructing cell data for diagram algebras. *J. Pure and Applied Algebra*, 209(2):551–569, 2007.

[23] U. Grimm and P. Martin. The bubble algebra: structure of a two-colour Temperley– Lieb algebra. *J. Physics*, 36(42):10551 – 10571, 2003.

[24] T. Halverson. Characters of the partition algebras. *J. Algebra*, 238(2):502–533, 2001.

[25] T. Halverson and A. Ram. Partition algebras. *European J. Combinatorics*, 26(6):869–921, 2005.

[26] G. James. *Representation Theory of the Symmetric Groups, Lecture Notes in Mathematics*, volume 682. Springer- Verlag, 1978.

[27] G. James and A. Kerber. *The representation theory of the symmetric group.* Addison-Wesley Publishing Co., 1981.

[28] M. Jegan. *Homomorphisms between bubble algebra modules.* PhD thesis, City University, 2013.

[29] V. Jones. The Potts model and the symmetric group. *Subfactors: Proceedings of the Taniguchi Symposium on Operator Algebras*, pages 259–267, 1994.

[30] L. Kauffman. State models and the jones polynomial. *Topology*, 26(3):395–407, 1987.

[31] M. Khalkhali. *Basic noncommutative geometry.* European Mathematical Society, 2009.

[32] S. König and C. Xi. On the structure of cellular algebras. *Algebras and Modules II*, 24:365–386, 1998.

[33] S. König and C. Xi. Cellular algebras: inflations and Morita equivalences. *J. The London Mathematical Society*, 60(03):700–722, 1999.

[34] T. Koshy. *Catalan numbers with applications.* Oxford University Press, 2008.

[35] I. Macdonald. *Symmetric functions and Hall polynomials.* Oxford University Press, 1998.

[36] N. Macon and A. Spitzbart. Inverses of Vandermonde matrices. *American Mathematical Monthly*, 65(2):95–100, 1958.

[37] P. Martin. *Potts models and related problems in statistical mechanics*, volume 5. World Scientific, 1991.

[38] P. Martin. Temperley– Lieb algebras for non-planar statistical mechanics : the partition algebra construction. *J. Knot Theory and its Ramifications*, 3(01):51–82, 1994.

[39] P. Martin. The structure of the partition algebras. *J. Algebra*, 183(2):319–358, 1996.

[40] P. Martin. The partition algebra and the Potts model transfer matrix spectrum in high dimensions. *J. Physics*, 33(19):3669–3695, 2000.

[41] P. Martin, R. Green, and A. Parker. Towers of recollement and bases for diagram algebras: planar diagrams and a little beyond. *J. Algebra*, 316(1):392–452, 2007.

[42] P. Martin and H. Saleur. On an algebraic approach to higher dimensional statistical mechanics. *Communications in Mathematical Physics*, 158(1):155–190, 1993.

[43] P. Martin and H. Saleur. Algebras in higher-dimensional statistical mechanics-the exceptional partition (mean field) algebras. *Letters in Mathematical Physics*, 30(3):179–185, 1994.

[44] P. Martin and D. Woodcock. On central idempotents in the partition algebra. *J. Algebra*, 217(1):156–169, 1999.

[45] A. Mathas. *Iwahori–Hecke algebras and Schur algebras of the symmetric group*, volume 15. American Mathematical Society, 1999.

[46] V. Mazorchuk and C. Stroppel. G (l, k, d)–modules via groupoids. *J. Algebraic Combinatorics*, 43(1):11–32, 2016.

[47] M. Osima. On the representations of the generalized symmetric group. *Mathematical J. Okayama University*, 4(1):39–56, 1954.

[48] D. Ridout and Y. Saint. Standard modules, induction and the structure of the Temperley–Lieb algebra. *Advances in Theoretical and Mathematical Physics*, 18(5):957–1041, 2014.

[49] S. Roman. *Field theory*, volume 158. Springer Science & Business Media, 2005.

[50] H. Rui and M. Si. A criterion on the semisimple Brauer algebras II. *J. Combinatorial Theory*, 113(6):1199–1203, 2006.

[51] M. Saeed-Ul-Islam. Irreducible representations of the generalized symmetric group B_n^m. *Glasgow mathematical J.*, 29(01):1–6, 1987.

[52] H. Temperley and E. Lieb. Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 322(1549):251–280, 1971.

[53] K. Wada. A cellular algebra with certain idempotent decomposition. *arXiv preprint arXiv:0805.1147*, 2008.

[54] B. Westbury. The representation theory of the Temperley– Lieb algebras. *Mathematische Zeitschrift*, 219(1):539–565, 1995.

[55] C. Xi. Partition algebras are cellular. *Compositio Mathematica*, 119(1):107–118, 1999.

[56] C. Xi and D. Xiang. Cellular algebras and Cartan matrices. *Linear Algebra and Its Applications*, 365:369–388, 2003.