Disformal Couplings
& Cosmology

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A thesis submitted for the degree of Doctor of Philosophy

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August 2016

To Carsten. For everything.

Disformally coupled fields are predicted to occur in nature, and cosmology in particular, by fundamental theories of strings and branes. They also arise independently from considerations of the most general relation permissible between two metric tensors of a given theory of gravitation. This work explores the cosmological consequences that arise when such couplings are added to the standard model of cosmology and the disformally coupled field is asked to play the role of dark energy. Among other things, it is shown that disformal interactions modify the angles of light cones and can induce motion damping of the field, similar to the well known Hubble friction, in the cosmological background. In addition, an extension is considered to the theoretical framework whereby the disformal interaction strengths can vary from species to species. Some models based on this generalisation are found to be well constrained by both astronomical and ground based particle experiments (discussed in chapter 3), whereas others (discussed in chapter 4) are actually able to avoid these constraints, while simultaneously offering insight into potential dark energy-dark matter interactions in the cosmos. Finally, a particularly well behaved form of disformal coupling is invoked to address to the cosmological constant problem (chapter 5).

Acknowledgements

This work would not have been possible without the help of a great many people to whom I am indebted. First and foremost I thank my supervisor Carsten. To call him generous with his time, knowledge, and attention would be a drastic understatement; he has guided me when I needed, given me independence when I asked, and I have been carried through the difficulties by his enthusiasm. For this I will always be grateful.

To my collaborators Susan Vu, Clare Burrage, and Jeremy Sakstein: Susan I thank for her invaluable advice and support in the beginning, when I knew nothing, and all her help with my codes, which were illegible; Clare Burrage for her many helpful discussions and, especially, for writing my references; and Jeremy for his insights and careful reading of the many pages of notes and emails I sent. They have each been fantastic to work with, both as scientists and as people.

My thanks also go to the members, past and present, of CRAG for providing a lively and interactive environment: Victor, Wahiba, Supakchai, Sam, Jurgen, Carl, Laura, Peter, Andrew, Chris, and especially to Elizabeth for all her help with my post doc applications. I would like to thank Jake Shipley and David Dempsey for many enlightening office conversations. Whether on black hole shadows, MAS tutorials, or Brexit, it has been a pleasure to share an office with them and I am sad to be leaving.

Finally but most importantly to my family I am, and will always be, immensely grateful for their continued and unfailing support, and for taking interest in dark energy, cosmology, and the things that were important to me.

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'There is this quality,
in things,
of the right way seeming wrong at first'

John Updike; Rabbit run

The crisis of beyond-Horndeski

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Recently I have read an interesting attempt to combine modern cosmology with archaeology, or at least to extract mutual benefit from considering the two together, which I would like to share [1]. "Just as an archaeologist deduces the existence of a previously unknown settlement on the basis of a few objects left in the ground and cross-reference, so the cosmologist detects the existence of a new pulsar when radiation reaches his or her instruments. Both are remote from the events they describe, both are working with sparse and usually fragmentary evidence." Of course, most scientists working at the forefront of their fields must deal often with fragmentary evidence, especially before their field has properly matured, but it is specifically that the discovered pulsar did exist—but likely does not exist still—that lends this comparison its weight.

Cosmology, pre-science, is older than the bible. Indeed the stories of creation in most religious texts are themselves early cosmologies, but it is the modern cosmology of Einstein, Hubble, Lemaître, that develops stories compliant with modern physical law, spun from distant fossils of light catalogued and accurately dated. The idea that we can piece together a coherent and self consistent mathematical model that faithfully represents a system as large as our entire universe from measurements of zero dimensional points of light, images of sources confined by causality to within our Hubble radius and often much nearer, must sound as mad as the claim that we can glean the thoughts and feelings of people living a hundred thousand years ago from the study of the small indestructible tools they leave behind. And of course each enterprise has been far more than moderately successful.

This common problem of fragmentary evidence leads to a severe under-

determination [2]. Cosmology in particular must always be under-determined, because of its unique problem: the universe itself is unique. There is only one universe of ours to study, and so the physical laws we abstract from cosmological observation that would determine how *any* universe must behave will therefore contain a certain bias. It is these two major factors: fragmentary evidence due to the distances to the principal objects of study, and the uniqueness of the universe, and our place in it, that make the field of cosmology look like no other science. This has been true since it was first given concrete foundations—Einstein's general theory of relativity and Hubble's distance-redshift relation—in the 1920's. Today a student of cosmology is likely to drown in a flood of possible universes-that-could-be: hidden extra dimensions and abhorrently complex scalar field theories, and this, as we will see in this chapter, is the reason why. Cosmology looks like no other science because the universe is like no other natural system.

1.1 Radius of our universe

Modern cosmology arguably began at the start of the twentieth century, with publication of the general theory of relativity, and has since grown with momentous pace. Today many, perhaps most, astronomy departments across the world host at least one cosmologist, and multi-million pound experiments delivering precision cosmological data to these groups have made cosmology itself a precision science.

Perhaps the first experimental effort in the field was to measure what many in the 1920's called the radius of the universe. Theorists like Einstein and Willem de Sitter had opposing views on what this parameter actually meant, with similarly opposite cosmology models that contained it, which I will describe in Sec. 1.1.1. The first concrete step forward for cosmology was the marriage of observation and theoretical interpretation of Hubble's redshift relation, and the realisation that this radius was not constant, but a dynamic quantity (Sec. 1.1.2). A briefly held reaction to this idea persisted for a time—the steady state theory—before arguably the second great step forward, the observation of an isotropic background of microwave radiation, silenced it, and a discussion in Sec. 1.1.4 of the third defining result that our universe is not only expanding, but also accelerating, will bring us very near the state of cosmology today.

What we will see scattered throughout this brief historical account is series of problems similar to those encountered today, and with similar reactions from the scientific communities involved; history repeats itself, and we will be tempted to conclude that the crises that, I will argue, many cosmologists currently face (particularly those involved with dark matter and energy) is not without precedent, and perhaps hints to a resolution of these crises may also be found buried in this past [1].

1.1.1 Radius of our universe

Though we could not and should not attribute to modern cosmology a singular instance of genesis, it is certainly true that the field after Einstein published his cosmological field equations (including the cosmological constant term $\lambda g_{\mu\nu}$) in 1917 was almost unrecognisable from such as it was a year before. This date then marks a natural point to begin our story. Before this landmark publication [6], the generally held view was this: the universe was simply the milky way embedded in Euclidean space. Matter density (roughly the numbers of stars and nebulae per unit volume) fell off with with greater and greater distance from the centre of this disk—nebulae had not yet been recognised for what they were, entire milky ways in themselves—and, beyond some distance, matter ceased to exist and speculation took over[2].

Virialization of stars is what stabilised this universe: there was no reason to assume space itself might be unstable. But Einstein's theory of a space-time that was dynamic brought this issue to the fore. If the universe had existed forever, there must be a static solution to his equations that could model the universe. Certainly flat space-time with matter is no solution, and further there was the problem of boundary conditions: what boundary conditions should one impose on a model of the entire universe? A neat solution to the problem, Einstein proposed, was a space-time that was spherical in its spatial sections. Periodicity seemed natural in some undefinable way, as circles and spheres have always seemed to astronomers and philosophers[3].

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[1]The main references for Sec. 1.1.1 and 1.1.2 will be the excellent articles [3–5].

[2]This speculation was not part of science, and no astronomer took it seriously in their professional writings, though no doubt some must have spent sleepless nights worrying about it.

[3]It is curious that, given the milky way picture that so many, Einstein included, then adhered to, that a super-sized Schwarzschild solution with Minkowski asymptotes was not considered.

The radius of this sphere, $\mathcal{R}$, was the first cosmological parameter that astronomers systematically sought to measure and, as the sphere model required tuning the cosmological constant in the Einstein equations to stabilise it, measuring $\mathcal{R}$ amounted to measuring Λ, and, simultaneously, the average matter density, ρ, of the universe [3]:

$$\Lambda = \frac{1}{\mathcal{R}^2} = \frac{8\pi G}{2c^2}\rho$$

In a series of letters, Einstein estimated $\rho/c^2 \simeq 10^{-22}$g/cm^3 from star counts, which gave an estimate of the size of the universe: $\mathcal{R} = 10^7$ light years. The furthest visible stars he noted where about 10^4 light years away.

In a report to the Royal Astronomical Society in 1917 [7], de Sitter demonstrated the existence of another static solution with constant curvature, but his universe was empty. He noted that a solution exists for [4]:

$$\Lambda = \frac{3}{\mathcal{R}^2}, \tag{1.1}$$

provided that $\rho = 0$. This, he argued, could represent an approximation to a universe with negligible density, and $\rho/c^2 \simeq 10^{-22}$g/cm^3 was certainly close enough to zero. Though empty, and hence physically inferior to Einstein's, de Sitter's universe had a very interesting feature. It seemed to be able to explain a mysterious observation by Slipher [8] that the spectra of most nebulae were shifted toward the red[4]. Perhaps due to his use of obfuscating static coordinates, there was much confusion surrounding de Sitter's solution at the time, and though Slipher-like redshifts seemed to appear in it, 'expansion' was very far from people's minds.

De Sitter referred in reverent style to Einstein's universe as model A, and his as model B. A major business of the day was to determine which was more like our universe. Differences between the two were hard distinguish, as 'real' and 'coordinate' effects were (as they still infuriatingly are) often confused, however it it is interesting that both acknowledged the cosmological constant, and both offered up a parameter, the radius of the universe $\mathcal{R}$, to be measured by experimentalists. Indeed as is so common in science, a theory can often fixate a community simply by giving them enough work to do.

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[4]This was the precursor of Hubble's law, but was not precise enough to convince the community, and besides there was little theoretical motivation to entertain it at the time, so it sparked no revolution, as Hubble's observations later would.

de Sitter himself proposed many ways to measure the radius of the universe. As his own system neglected the average density of matter, measurements of this type could not be used to estimate $\mathcal{R}$, as Einstein had done for system A, and so de Sitter was compelled to use geometric arguments. Nebulae (now known to be spiral galaxies) had appreciable angular size on the sky. Andromeda is about 6 times the size of the full moon. In de Sitter's warped geometry, discrepancy between the actual and apparent angular sizes of these objects could determine accurately the curvature radius, if only one knew their actual size, which no-one at the time did. With some hand-waving, he placed an upper bound with this method of $\mathcal{R} \geq 10^{12} AU$ [7].

There were many notable attempts during this period to measure $\mathcal{R}$. For a detailed list and review see [3]. Most involved astrophysical systems with measurable angular size, like the Large Magellanic Clouds. This occupied the community for over a decade, and many estimates were ventured, most around $10^{11} - 10^{14}$AU, for both systems. Estimates for system B were on the whole more consistent, and it is surprising that there was consistency at all, especially across the two competing models, given that such a parameter is not even discussed these days, much less are proposals ventured to measure it. There was certainly a lower limit: the universe had to be larger than the milky way, and to much above this scale was perhaps just inconceivable at the time. A number too large no other astronomer would take seriously.

A remark by Author Eddington in his textbook on the mathematics of relativity [9] in part anticipated what was to follow (quote by way of [3]): "[systems A and B are] two limiting cases, the circumstances of the actual world being intermediate between them. de Sitter's empty world is obviously intended as a limiting case; and the presence of stars and nebulae must modify it, if only slightly, in the direction of Einstein's solution." I would like to point out here that it was the curvature radius, $\mathcal{R}$, on peoples minds and in their published papers, not the cosmological constant, though the latter determined the former in both systems. It is interesting that such attention was paid to $\mathcal{R}$, which was in effect a specific property of each universe solution, while Λ as it appeared in Einstein's cosmological equations was a true fundamental theory parameter, and a property of nature herself. (In section 1.2 I will however attempt to blur this distinction between a fundamental parameter and the property of some solution.)

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1.1.2 Lemaître, Friedmann, Hubble

An announcement in the November 23rd 1924 issue of the New York Times ran: "Finds spiral nebulae are stellar systems. Dr Hubbell confirms world view that they are 'island universes' [Immanuel Kant's term] similar to our own" [10], and the cosmological world view lurched forward into a new era. Hotly debated was the origin of these nebulae. Did they reside within the milky way, as stars do? Or were they milky ways of their own? Hubble settled the question, and recorded distances of about 10^6 light years (recall Einstein's original estimate of $\mathcal{R} \simeq 10^7$ light years). Suddenly the value of $\mathcal{R}$ had to be much bigger than originally thought, indeed the universe needed not have a finite size at all.

Elsewhere, System's A and B were being challenged on mathematical grounds. In 1922, Alexander Friedmann wrote a paper that questioned the very assumption that the universe need be static [11]. He considered a case more general than Einstein and de Sitter had done, where the curvature radius was dynamic with time, $\mathcal{R}(t)^5$. The Einstein equations for Friedman reduced to ordinary differential equations for $\mathcal{R}(t)$, the **Friedmann equations** [4]:

$$\left(\frac{\mathcal{R}'}{\mathcal{R}}\right)^2 + 2\frac{\mathcal{R}''}{\mathcal{R}} + \frac{c^2}{\mathcal{R}^2} - \Lambda c^2 = 0$$

$$3\left(\frac{\mathcal{R}'}{\mathcal{R}}\right)^2 + 3\frac{c^2}{\mathcal{R}^2} - \Lambda c^2 = \kappa \rho c^4.$$

Friedmann approached the result with a mathematician's instinct to abstract— he found the possibility of a singular state $\mathcal{R} = 0$, as well cyclic or bouncing phases where $\mathcal{R}$ oscillates—and made no contact with observation. Slipher's redshifts were not mentioned in the paper. As a professor in St Petersburg, he was perhaps disconnected from western science to some extent, and his paper made no mark for at least a decade. The idea of an expanding universe, and that *our own* could actually be expanding came independently from Lemaître, and some years later.

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[5]By assuming spatial sections at constant time were isotropic and homogeneous, as Einstein and de Sitter had done, global constant-time hyper-surfaces could be defined and so a unique (up to 1 parameter diffeomorphisms) cosmic time t was well-defined everywhere.

With the Hooker telescope on Mount Wilson that Hubble had used to identify cepheid variables, Hubble began the systematic cataloguing of nebulae distances and the quantitative shift in their spectra towards the red. Building on Slipher's findings, he and his assistant convincingly demonstrated a linear relation between redshift and distance of nebulae [12]. The distance-redshift relation he left to theorists to interpret, and is often mis-represented as having discovered the expansion of the universe. Such an interpretation of the data required a theory that he did not posses.

Georges Lemaître was a Belgian priest and astronomer with, like Einstein, a strong head for physical intuition, and a desire to describe real world phenomena, not content with mathematical abstraction [13]. He independently derived Friedmann's differential equations for $\mathcal{R}(t)$ and showed that if a light pulse was emitted at some time t_1, corresponding to a radius $\mathcal{R}(t_1)$ and received (by Hubble) at time t_2, then there would be a spectral shift or **redshift** [14]:

$$z := \frac{\delta\lambda}{\lambda} = \frac{\mathcal{R}(t_1) - \mathcal{R}(t_2)}{\mathcal{R}(t_1)}. \tag{1.2}$$

The idea of an expanding universe was a hard pill to swallow at the time, but this striking marriage of observation and theory convinced the majority that such a mad idea was one worth taking seriously. The idea of an expanding universe, as Friedmann had realised, implied perhaps a beginning, when $\mathcal{R} = 0$, though not necessarily. With these new more general assumptions, one could easily show that Einstein's universe was unstable. A slight perturbation would do one of two things: a contraction perturbation would cause Einstein's system A to contract down to a Friedmann singularity; a nudge in the opposite direction would cause it to expand outward until the matter diluted away—the final state of this latter instability is of course de Sitter's solution. Lemaître proposed that the universe began as as system A, and is expanding, on course to end up in a de Sitter phase. Lemaître found use in Eddington's earlier remark " [systems A and B are] two limiting cases, the circumstances of the actual world being intermediate between them". The intermediate phase he speaks of is the universe in transit, dynamically evolving from one to the other.

To a colloquium in celebration of Einstein's 70th birthday, 1949, Howard Robertson, professor of mathematical physics at Caltech, read an address entitled: On the present state of relativistic cosmology [15]. It began: "It has been aptly remarked by one of you that, for a review of the 'present

state' of relativistic cosmology, the present occasion is either thirteen years too late, or three years too early". At the time of address, the (now standard) model of an expanding universe was in trouble. With Hubble's law came a measurement of the Hubble constant at the present day:

$$\frac{\mathcal{R}'_0}{c\mathcal{R}_0} = 5.37 \times 10^{-10}\text{ly}^{-1},$$

with some error Robertson did not quote, though presumably at least on the order of the value itself. This presented a problem. To quote the same address: "From these data the value $\Lambda = 4.4 \times 10^{-18}(\text{light years})^{-2}$ of the cosmological constant can be obtained, and $\mathcal{R}(t)$ found by numerical integration. The resulting model has a repulsively short time-scale ($< 10^9$ ly), an uncomfortably small radius (4.7×10^8ly) and an excessively high present density (6×10^{-27}gcm^{-3}, as opposed to an estimated 10^{-29}). Altogether, this model is so unpalatable as to lead Hubble to consider calling upon some 'new principle of nature' to obviate it."

The temporal issue described is what was known as the age problem. 10^9 years, it was then known, was less than the ages of some stars. This paradox was unacceptable, and threw the very idea of an expanding homogeneous universe in jeopardy. In particular it motivated the community to scrutinise the theory; to look for assumptions that had been made to throw away. Homogeneity was questioned (a line of mistrust that still continues today) and the astronomy of nebulae and cepheid variables were scrutinised similarly. On homogeneity, it would have been surprising had this not been called into question. It had little experimental validation at the time. Hints were made at a possible **end of greatness**: a length scale that, when quantities like the density of matter and the curvature of space are averaged over, become homogeneous; but these were yet to be placed on solid ground. Nevertheless, the expanding isotropic model found itself in hot water midway through the 20th century, and in the next section we see that this made room for contenders; the most influential was known as the steady state theory.

The combination of Hubble's precise determination of the law that came to bear his name and the interpretation by Lemaître of redshift due to the expansion of space [14] convinced the community that the universe is in a transient state, unstable and evolving, with a possible beginning at a crushing singularity. Such a shift in world view many found hard to accept, and this has always been cosmology's great obstacle: it is a science tantalisingly near to the really big questions: "does time have a beginning?"; "what is the fate

of the universe?"; "was there a first cause?" These border on philosophy and religion; as a result, many attempts at answers within a physical framework can be perceived an attack on a deeply held belief, and resistance to this can be strong.

We saw in the last section that 'naturalness' lead Einstein to think $\mathcal{R}$ must be constant. Data showed otherwise. Today, naturalness is often used to argue that the cosmological constant can not be constant. What data will show this time is not yet clear. It is curious that naturalness has lead to so many mutually exclusive conclusions about the nature of the universe; most mythologies and religions naturally assumed the universe was dynamical and had some sort of beginning: a cosmogony [16]. Why did naturalness lead Einstein to the opposite conclusion?

1.1.3 A steady state controversy

The interpretation of Hubble's law as the redshift of cosmic photons due the expansion of space was doubted by very few, and almost none within the mainstream community. Einstein's static model ceased to be. Yet the age problem could not be ignored, and to extend the age of the universe while retaining this expanding behaviour was no simple task. There were two main types of proposals, both denying the notion that the universe had a finite age (still an metaphysical migraine for many): oscillating universes, and steady state theory [17].

The former, favoured by Friedmann for its connections to Hindu mythology, suffered from an infinite sequence of singularities. Friedmann's equations did not permit solutions where $\mathcal{R}$ could reach a minimum at a finite size and then re-expand, without the ad hoc assumption of exotic matter. The point $\mathcal{R} = 0$ was periodically traversed, and so too the laws of physics would periodically break down.

An alternate way to marry an age-less universe with the observation of cosmic expansion was developed by Fred Hoyle [18], and again by Hermann Bondi and Thomas Gold [19] in 1948. (Both papers in fact appear in the same volume of the Monthly Notices of the Royal Astronomical society.) They extended the cosmological principle to what they called the perfect cosmological principle: that the universe looks the same from place to place, and now also *from time to time*. This required the postulate (perhaps equally unpalatable as an infinite series of singularities) of continual matter creation.

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The rate needed would not be large, three new hydrogen atoms per cubic meter per million years (indeed unmeasurably small) but would be enough to maintain cosmic expansion in a delicate balance for all time. Maintaining this balance was difficult, and almost all model freedom was given over to it. The theory was almost unmodifiable, and nearly uniquely determined by the perfect cosmological principle.

It was never really met with warm feeling. Most thought it just bad physics, and foresaw its falsification by experiment as a quick and painless process. It did not go so quickly, as unambiguous results in experimental cosmology were still a difficult, distant perfection, but the accidental observation published in 1965 of an isotropic background of microwave radiation at a temperature of 2.73K [20], the **cosmic microwave background**, was quickly interpreted as the after glow of a big bang [21], and sounded the death knell for the steady state theory. The age problem would have to be solved some other way, and in fact it was in a dramatically non-revolutionary way, by more precise measurement of Hubble's constant, and careful considerations of the model's details.

From Friedmann's equations came forth a deluge of permissible relativistic cosmologies. Some expanded, some were static, some bounced and some contracted, and with all the variation in rates, timescales and geometries implied by the parameter freedoms. Einstein's relativistic cosmology was not really a model, but a way of producing models. Falsifiability was made very difficult, for any experimental constraint on some particular solution, another could be made to fit. The steady state theory on the other hand was unique, and powerfully predictive. It is by this very quality that its progenitors sought to defend it. Of course this ended up being its downfall, though it must be said that had the cosmic microwave background *not* been discovered, relativistic cosmologies would not have been discounted, just those select few with an initial singularity. This is a recurrent theme in cosmology, and we will meet it again in force in section 1.3.

1.1.4 Acceleration

The third great revolutionary step in this story, the description of which brings us very close to the present day, was that of the acceleration of the expansion of the universe. In 1998 and 1999 respectively, the *Supernova cosmology project* [22] and *Supernova search team* [23] published near identical
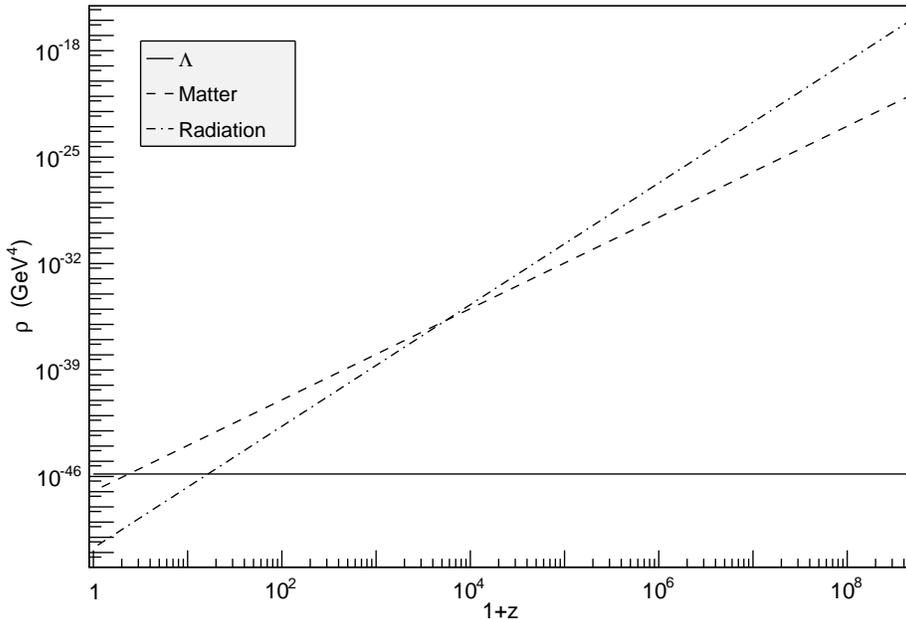
Figure 1.1: Illustration of the coincidence problem: The energy densities of matter and radiation in the universe increase as the universe shrinks (as we look further back in time, to greater redshift, z), but remain roughly similar. Λ on the other hand remains constant, and was wildly different to the rest of the species in the universe until very close to today ($z = 0$). (Boundary value data for the solutions in this figure taken from [24].)

results: the expansion of the universe was speeding up and, in particular, their observations were consistent only with a Friedmann universe who's dynamics are dominated today by the presence of a vacuum energy. The cosmological constant, it seemed, had finally been measured, and its energy scale was small: about 10^{-42} GeV. Saul Permutter, leader of the supernova cosmology project, and Brian P. Schmidt, leader of the High-Z Supernova search team, shared the 2011 Nobel prize for their discovery with Adam Riess.

The accelerated expansion of the universe poses many of its own problems for the modern day cosmologist [5, 25]. I will mention here the famous cosmological problem, which serves in fact to define many subfields of cos-

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mology and direct the attention of huge swathes of cosmologists around the world. If the accelerated expansion is being driven by a constant, Λ, whose energy density does not dilute as the universe expands, and, as observation tells us this density is of roughly the same order of magnitude as the rest of the matter in the universe today, then it must have been wildly different for the majority of the universe's lifetime. Since Copernicus, astronomers have reduced the significance of our place in the universe from the very centre of everything to a humble address: near the edge of a fairly average galaxy in an infinite universe with no preferred spatial origin. Ironic, then, that this observation should now place us at such a special time. See Fig. 1.1 for an illustration.

The history sketched in the above pages shows 'naturalness' can be as helpful a guiding principle as a magnetic compass, but there is something so unnatural (in the loose sense described above) about the nature and value of the cosmological constant that many consider alternatives in which it is dynamical, that is $\Lambda(t)$. We saw that the first measurable constant of cosmology, the radius of the universe, was fixed by natural arguments, but was forced to vary by observation. Here we witness the reverse, a constant deemed too unnatural to remain fixed. It may turn out that the stories are also opposites in their resolutions: data may show Λ to be a true constant, though we wish it not on grounds of naturalness. In the next section I leave the history of cosmology behind to look more generally at this idea of varying the constants of nature.

1.2 How to vary Nature's constants

A hydrodynamicist may call the density of water a constant of nature. We can imagine one of the first scientists to perform physical experiments in a lab with water, and, working perhaps with an as-then undeveloped understanding of thermodynamics she might come to the conclusion that the density of her medium is constant[6]. There is no experiment she is able to perform at first that alters the density of water, as for example a gas may be compressed. She may develop a mathematical basis for her findings, along with a new law of nature: a fixed amount of water cannot be compressed,

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[6]The inspiration for this story came from the opening sentence of [26]

and the ratio of the weight of a sample divided by its volume is a constant of nature.

With subsequent, more precise experimentation she will come to find this law only an approximation: no longer true under more accurate scrutiny, or more extreme conditions. Generalisations of the original laws to allow the variation of density might lead the hydrodynamicist to be able to model gases with the same set of new equations, and in this way perhaps unify scientific understanding of water with the air in the laboratory. Her new theory is in a very tangible sense more fundamental than its precursor, and a natural constant, the density of water, has been explained in terms of more fundamental constructions.

This section will expound the deep connection between fundamental physics and fundamental constants and, hence, how breakthroughs in fundamental physics are very often reflected in a change in our understanding of the latter: an 'explanation' of the values of these constants; the discoveries of new ones; the rejection of old ones as fundamental. Out of attempts to vary these constants, for a range of different motives, came the scalar-tensor paradigm, the topic of Sec. 1.3.

1.2.1 A metrological debate

It is often said there are 3 basic physical dimensions: length, L, time, T, and mass, M [27], an idea attributed to Gauss. All SI units are derivatives (combinations under multiplication) of these three. As Lev Okun states [27]: "In spite of the tremendous changes in physics, three basic dimensions are still necessary and sufficient to express the dimension of any physical quantity. [...] It does not depend on the number and nature of fundamental interactions. For instance, in a world without gravity it would still be three." The same could not be said about fundamental constants: in a world without gravity, what would Newton's constant G signify?

It is said more often that only dimensionless ratios can be measured [27, 28]. The length of my arm is about a metre long, that is the ratio of the length of my arm to the length of a metre stick is close to unity, and a comprehensive system of conventions, the SI unit system, standardises this maxim internationally. This is never a point of contention, and the chapters to follow will stick closely to it. However, the question of wether or not a constant or unit is *fundamental* is hotly debated, and an interesting

discussion between three leading physical theorists can be found here [27].

Like the hydrodynamicist, every scientist must deal with a specific set of constant quantities. The density of water, the mass of an electron, the cosmological constant. The constants defined by so-called fundamental theories of nature, like $\hbar$ in quantum mechanics, are often labelled fundamental, and this implies the following three only are fundamental: the speed of light, c, Planck's constant $\hbar$, and Newton's constant G. Their interpretation as basic units of nature cements their importance: the speed of light is the maximum speed of massless particles, Planck's constant is the minimum unit of action, and the Planck mass (a derivative of G) is the mass of a black hole who's Compton wavelength equals its Schwarzschild radius. In [27] it is frequently argued that Boltzmann's constant, k_B, has no such special status and is merely a conversion factor between energy and temperature.

Michael Duff in [27] takes a pragmatic approach which I find particularly compelling: no dimensionful constants are fundamental, all are simply conversion factors; length, mass, and time are conveniences, and we could chose to work with more dimensions, or less, depending on the job at hand. He provides an illuminating example, which I will now reproduce. The speed of light is merely a conversion factor for those not used to treating time as a fourth dimension. It is the $O(1,3)$ symmetry beneath this speed that is physical. Sailors, in his example, measure distances across the ocean surface (x, y) in nautical miles, and depth (z) in fathoms, out of practical necessity.

We are left with the choice of only dimensionless values and combinations as fundamental, and the only ones truly measurable. Theorists that vary the speed of light (who we will meet later in this section) encounter traps in this way. If only a conversion factor is varying, then, Ellis and Uzan argue [29], one can end up working with standard physics in a poorly chosen, non-static system of units. The controlled setting best suited to vary constants (that is, the dimensionless parameters) is that of scalar tensor theories, whereby a scalar field replaces the constant in some action representation, and is allowed autonomous dynamics of its own. Says Michael Duff [27]: "Indeed, replacing parameters by scalar fields is the only sensible way I know to implement time varying constants of Nature." We now turn to some archetypal examples of this approach in action.

1.2.2 The fine-structure constant

Possible variations in time of what is perhaps the most famous dimensionless constant in physics—the fine-structure constant α—have been extensively explored. There are a number of motivations all different on the surface, but beneath a common drive connects them: the drive to understand electromagnetism more deeply and completely. Returning again to our hydrodynamicist, we see that recognising the constant density of water is but an approximation to a deeper more comprehensive theory can bring unification and, importantly, the ability to *predict* a value that was before just a premise. These same desires run beneath the surface here too, and we will see how early numerological attempts to explain the value of α—dismissed by many as crackpot—evolved into the capable machinery of moduli stabilisation; the modern approach to 'explain' the values of fundamental constants.

α, the constant that Feynman in 1983 called a 'magic number' [30] has had a colourful past. It first turned up in spectroscopy, in a formula for the fine structure of the energy levels of hydrogen-like atoms. Many attempts at the time to connect, for example, electromagnetism and gravity, took the form of 'guessing' numerical relationships between α, G, whole numbers, π, the charge of an electron, e, and electron mass m_e. For example, in a paper submitted in 1913, Gilbert Lewis and Elliot Adams developed a theory that claimed all dimensionless constants of nature could be linked to whole numbers [31]. Among other things, they postulated:

$$\alpha^{-1} = 16\sqrt[3]{\frac{\pi^8}{15}}$$

which rather impressively agrees with today's measured value of α^{-1} to within 0.2%. Their motivations were not completely clear; evidently, they thought whole numbers perfect (natural) in some sense undefinable. Driven by similar ambiguous volition, the physicist with a fitting middle name Arthur Constant Lunn postulated in 1922 a relationship between α and G [32]:

$$\frac{Gm_e^2}{\hbar c} = \frac{\alpha^{18}}{2^{11}\pi^6}. \tag{1.3}$$

(Check it; with modern values it is still very close!)

Many saw these attempts as unscientific. Maxwell's famous unification of the electromagnetic equations lead him to the conclusion: $c = 1/\sqrt{\mu_0\epsilon_0}$,

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but it is worth pointing out that, even though this quantity had been previously noticed to have the dimensions of velocity, it took much more than dimensional analysis and numerical guesses to deliver Maxwell's unifying interpretation that changed the face of physics.

There are many examples of this type of numerical guess work involving α which can be found in [30], along with the reaction of mainstream science to these efforts. These early attempts to understand nature through her constants marks a strong trend in fundamental physics still very much alive today.

In the early 2000s, evidence from the observations of distant quasars began to emerge that the value of α had been lower in the past, at redshift $z \in [1, 3.5]$ [33]. This prompted a sharp spike in interest in the question of varying α, as well of a flood of possible explanations [34]. (Many claimed they already had the answer: theories with extra compact dimensions and constants defined in these higher dimensional spaces naturally exhibit varying α behaviour in the low energy effective limits that we would observe.) However it was realised long before that naïvely replacing α with $\alpha(t)$ for some function to be determined in the model equations was pathological. The main problem was global non-conservation of energy. A system in which $\alpha(t)$ varies mechanically is akin to one that is not closed, and some external applied force drives the coupling strength of electromagnetism up and down, or extracts energy from the system. But if this system is the Universe, such a scenario is impossible—where does the missing energy go? Importantly for general relativity, this missing energy must still gravitate, but how to model this?

The solution is to use a scalar field [33]. Built from an action, the missing energy is transferred to a scalar field sector with its own dynamics, and this energy is fed back into the universe via the Einstein equations. The pathology is thus avoided, and the mechanics become self consistent.

These varying α observations were not reliably replicated. At present, the evidence for varying α is inconsistent, and no longer persuasive. However in that thin interim there was hope, and out of it developed a solid mathematical framework for modelling variation of the constants, the scalar tensor framework.

Linked closely to varying α theories are those that vary the speed of light (indeed if the speed of light varies then α normally will too). The latter were mainly developed for an alternate reason: to solve the problems of early universe inflation without recourse to a potential driven expansion phase.

The horizon problem in particular found a nice solution here: if the speed of light were much greater in the early universe, much larger volumes could be kept in causal contact, and thus thermal equilibrium. The problem that we see no horizons frozen into the cosmic microwave background today can thus be solved by expanding the horizons with varying light speed [35, 36].

George Ellis and Jean-Phillipe Uzan are unimpressed with this approach [29]. They argue that, without an action formulation, a varying speed of light theory is nothing more than standard physics in a time dependent unit system. This is because c has dimension and, as we discussed, only dimensionless quantities are measurable. They also point out that the constant c has many facets, and varying one of these does not imply variation in the rest. For example, c is the speed at which photons travel, but it is further a constant built onto the fabric of spacetime, independent of light, as well as the speed of graviton propagation and more besides. Assuming, they say, the photon has a nonzero mass does not affect the speed of gravitational waves, nor the structure of spacetime. As we will see, disformal couplings can represent a mechanism to vary the space time constant, c, that appears in special relativity from an action in certain carefully constructed models—background cosmology specifically—but this interpretation must be handled with extreme caution.

It is worth pointing out before moving on that the fine-structure constant is not really constant at all: as a dimensionless measure of the electromagnetic interaction strength e it depends on the energy of the interacting particles. It then must have been different, larger, during the very early universe when temperatures were high. Too it can now be derived, with fantastic precision, from quantum electrodynamics. And it is hoped that the electroweak and strong forces are unified at some grand energy scale, where all the coupling 'constants' converge to a single value.

1.2.3 The gravitational constant

To tell the second archetypal varying constant story, that of Newton's constant, we go back to numerology, and Dirac [37]. Again we see early attempts to uncover theories beyond what is known in the form of guessing relationships between natural constants.

Dirac's thesis was to find a relationship between various large dimensionless numbers in science. In a paper somewhat boldly entitled 'A New Basis

for Cosmology' (written supposedly on his honeymoon [33]) he proposed his fundamental principle: *"Any two of the very large dimensionless numbers occurring in Nature are connected by a simple mathematical relation, in which the coefficients are of the order of magnitude unity"*. Thus, as John Barrow explains, "if we form $N \simeq c^3 t / G m_n \simeq 10^{80}$, the number of nucleons in the visible universe, and equate it it to the square of $N_1 \simeq e^2 / G m_n^2 \simeq 10^{40}$, the ratio of electrostatic and gravitational forces between two protons then we are lead to conclude that one of the constants, e, G, c, h, m_n, must vary with time. Dirac chose $G \propto t^{-1}$ to cary the time variation." Though this new basis for cosmology did not last long, the idea of varying G took hold on the collective imagination.

It resurfaced some years later, first in work by Pascual Jordan [38], and then more publicly with a paper by Brans and Dicke in 1961 [39]. Both provided an action foundation for the approach, which Dirac had not, and this variation was induced for a completely different purpose. Brans and Dicke attempted to modify general relativity to accommodate Mach's principle, a task Einstein himself had agonised over in the early days of general relativity before abandoning it, and one which barely resides on the fringes of the collective scientific conscience today—it was most likely considered an obscure academic exercise then, as now. The formulation of the theory is what interests us here, which survived beyond Brans and Dicke's original motivation, and so we'll not give to much thought to Mach, but I refer to [40], [41] for discussions. An academic exercise perhaps, but the work is nonetheless a tour de force; sound insights into the pragmatic aspects of measurement within scalar tensor theirs are utilised in much subsequent work, and perhaps the first mention of an Einstein frame (though not named) is accompanied by hints at the existence of a new symmetry group of the theory, emerging from the interaction between the scalar field and curvature. We will encounter these themes again and again, implicit and explicit, they are woven into the fabric of this thesis.

What emerged, some time after and explosively, from both varying α and G theories, was an abstraction in mathematical terms, a breaking away from the original physical intentions of the two, while retaining the mathematical machinery. Scalar tensor theories became a focal point of gravitation theory research in their own right, not a model but a way of producing models, just as the relativistic cosmology paradigm had been over half a century before. As with the curvature radius, acknowledgement that the constants of nature could be dynamic lead to a massive proliferation of theories which continues

to extend to this day. The driving force behind the explosion was the discovery of the accelerated expansion of the universe, and the subsequent race to find an appropriate mechanism. Fuelled by this enigma, and uninhibited by the constraints of interpretation, the number of scalar tensor theories seemed to grow without bound, and the resulting field today is in a crisis state. In the next section we take a quick detour to cover moduli stabilisation and the other face of scalar tensor theories, before returning to inspect closely this state of crisis.

1.2.4 'Explaining' the constants: a modern viewpoint

When pushed to its logical conclusion, the reductionist idea of explaining things in terms of more basic things delivers us something like string theory. If, at the deepest level, strings are all that exist, and the variety of physical phenomena emerges naturally from the mass motions of their uncountable numbers, then the constants of nature must too emerge from this picture. If our starting point is simply the string, then the only fundamental parameter must be the relaxed length of this string.

6 out of the 10 dimensions of string theory—if it describes our world—must be hidden from us somehow [42, 43]. This is often done through compactification: when building a model of the visible universe from strings in high dimensional space the string theorist often makes 6 out of the 10 dimensions small. They are then compelled to find a way to keep them small. The constants of nature, when defined in terms of strings in high dimensional space, are greatly distorted in the 4D effective theories that are the products of compactification, and, if this compactification is not stabilised by our string theorist, then the effective constants re-distort when the shape of space convulses. If this theorist cannot stabilise these extra dimensions, then they must go in search for signatures left in the cosmos of these convulsions in the form of varying constants.

Compactification of a higher dimensional theory returns an effective one of lower dimension containing new fields relating to the size and shape of compact space. These fields are called moduli fields. This theory need not be string theory, and compactifications of arbitrary high dimensional theories have become objects of study independent of and abstracted from string theory. Practically, they have been used to explain dark energy, and as they can 'explain' the values of constants in terms of the stabilised sizes of their

compact spaces, have been invoked to solve hierarchy problems of particle physics [44].

Dark energy observation has spurred on the abstracted study of higher dimensional spaces [45]. There are many ways to compactify a space. There are many ways to keep it compact. The scalar tensor framework again has proven the ideal in which to consider this proliferation in a coherent and systematic way. A problem like the cosmological constant problem that defines a field can also push that field into crisis if a solution refuses to be found, and here the crisis is manifest in the scope of the scalar tensor paradigm, which encompasses now higher dimensional compactifications, varying constant theories and much besides beneath its umbrella of perverse, directionless generality. We now turn to look at this umbrella, and try, from the lessons of history, to speculate some possible resolutions.

1.3 The floodgates of generality are open

By the late 1980's, the case for inflation had won over just about everyone (see for example [46–49]). Particularly persuasive was its ability to explain the observed homogeneity of matter on the largest scales, which it had done with startling elegance. However for all its success, the paradigm required something in return, which was that the observable universe be as flat as could be measured. This was a problem for the following reason. The Friedmann equations required that, if the universe be flat, then the matter density of the universe must be at a critical value, $\rho_M = \rho_{crit}$, a condition more cleanly expressed in terms of the density contrast $\Omega_M := \rho_M/\rho_{crit} = 1$. However, estimates of the matter density of the universe placed the density fraction at $\Omega_M \simeq 0.2 - 0.4$, and these estimates invariably relied on measuring only clumps of matter: galaxies, galaxy clusters, and dark matter halos.

As inflation was to prized to dismiss, the task of the day was to find candidates for the the hidden component, X, that did not cluster, and was completely homogeneous in space. Several options were considered, the most promising being: a cosmological constant; a homogeneous scalar field; hot dark matter that was too relativistic to clump; and a tangled network of strings. Each choice must have $\Omega_X = 1 - \Omega_M \simeq 0.7$. The different choices were set apart by their equations of state—the ratio of pressure to energy density $w_I := P_I/\rho_I$, which was the particular variable the supernova search

teams aimed to measure and, in 1998, both confirmed a measurement of w consistent with a cosmological constant[7], or slowly varying scalar field. This mysterious X component driving the acceleration, whatever it is, goes now by the name dark energy.

The cosmological constant brought with it the cosmological constant problem: as a constant energy density, it would have an energy density on the order equivalent to that of matter today, but 10^{-100} less just after the production of the cosmic microwave background [48]. A dynamic Λ could evolve with the radiation bath, and the two values could remain comparable. Such was the main motivation for considering a varying cosmological constant at the time, and as we have discussed, this must come with a scalar field.

Carroll argued that, from the point of view of particle physics and effective field theory, a new field meant new interactions [50]. As the low energy effective limit of some high energy fundamental field theory, interactions between all degrees of freedom appear naturally in the Lagrangian for any realistic scalar field, and, if not seen in nature, theorists must find a way to keep them small. To a particle physicist at least, interactions are the default, and the task is to explain the absence of them, not the other way around.

Thus interactions between the dark energy field and matter were considered, notably in a paper by Wetterich [51], expanded on by Amendola, 1999 [52], and in many others. Links were made to other interacting gravitational scalar field theories, in particular the theory of Brans and Dicke discussed in the previous section. In 1987, Wetterich had claimed that a softly broken conformal symmetry in particle physics produces a scalar field with a potential small enough to act as the cosmological constant, but one that also modulates the value of Newton's constant, Brans-Dicke style, making it susceptible to constraint [53, 54]. The idea of varying constants was infiltrating the mainstream cosmology community, before and after the discovery of the universe's accelerated expansion.

In these early days, the scalar field candidate for dark energy was considered a simple sum of kinetic and (minus the) potential energy, with a Lagrangian density: $\mathcal{L}_\Phi = X - V$. But how would such a field interact? Gravitationally, certainly, but perhaps via variation of the constants of na-

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[7]It would be a mistake to suppose no-one was seriously considering Λ, or the resulting accelerated expansion of the universe it would imply, at the time before these announcements.

ture? Perhaps directly to standard model fields. And could its derivatives interact? The dark energy field toes the line between gravitation and particle physics. The former is not well understood in terms of interacting quantum fields, and it was (and still is) not clear if the scalar interacts gravitationally, via curvature, or quantum mechanically, in a standard model fashion. Regardless, the number of ways to interact were many, and the forms the Lagrangian could take, especially once direct interaction with curvature was considered, grew almost it seemed without bound.

But a bound had already been placed some years before. In 1973, at the University of Waterloo, Gregory Horndeski had published the most general Lagrangian that involved a single scalar field and Lorentz signature metric tensor in 4 dimensions whose Euler Lagrange equations are free of instability [55], which has been recently rediscovered in a new context [56–58]. The paper went unnoticed for some years, and the discovery of dark energy has seen interest skyrocket (see Fig. 1.2). It is not simple (see Chapter 2), and I will mention it again a few times in these pages. The hope was that such a Lagrangian (which included a cosmological constant under various parameter choices) must contain, somewhere in its functional freedom, the true dark energy theory.

But there are many loopholes to his theorem, as we will see in later chapters, some leading to the extended Lagrangian of beyond-Horndeski. However, nothing in principle prevents the study of multiple fields and, even worse, the Einstein frame (a construct we will see shortly) allows species-dependant gravity sectors, where each could be some variant of beyond-Horndeski; a veritable nightmare: vector fields, three-form fields, continuous towers of fields, [60–62], see [63] and references therein for a long list. The inexplicable nature of dark energy has seen a proliferation of alternatives to the single number Λ escalate to such heights of complexity, it is sometimes difficult to recognise that a dark energy paper is referring to cosmology at all [64]!

This is not however the first time such a crisis (whose symptom is excessive generality without apparent empirical need) has gripped a scientific community in reaction to a particularly difficult problem. Nor is it the first time a sub-field of astronomy has been gripped so. History is filled with episodes like this one, and how each was finally resolved may give us clues to the potential resolutions that await this present crisis of beyond-Horndeski.

We have already discussed the age problem. Hubble's acute measurement of his famous constant, when combined with Friedmann's equations,
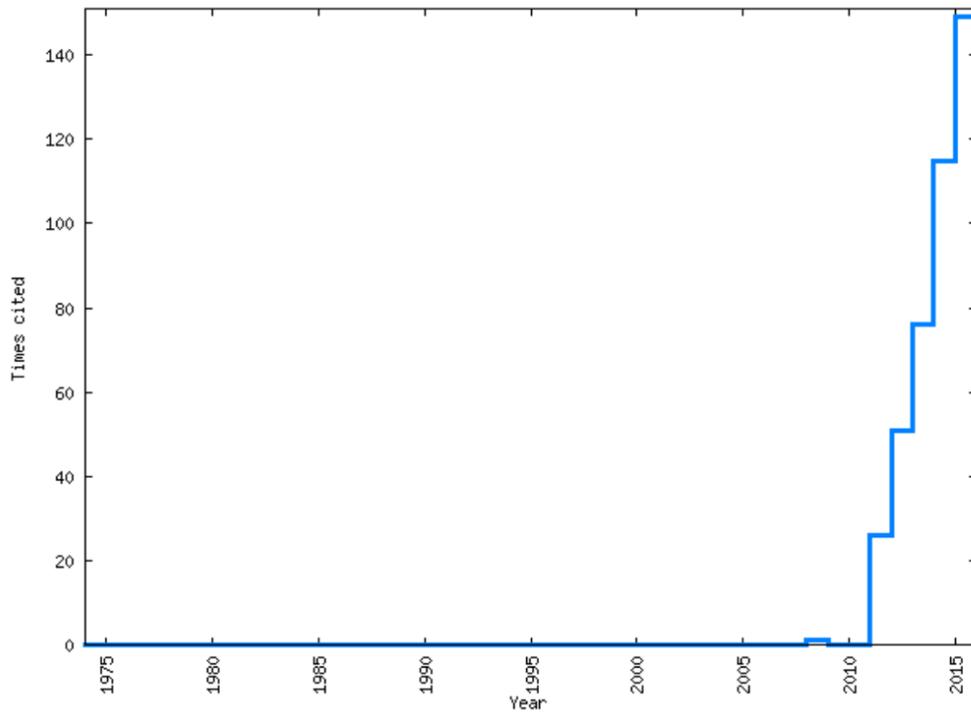
Figure 1.2: Number of citations of Horndeski's original paper, binned by year [59].

predicted that the universe be younger than some stars contained in it. Such a paradox was finally resolved by the measurement of the precise value of cosmological constant, which altered the redshift distance relation, $d(z)$, extending the age of the universe to an acceptable 14 billion years. This resolution came nearly half a century after Robertson's address, described in Sec. 1.1.2, but the age problem was nonetheless resolved within the means of the expanding cosmology paradigm. No drastic revision was necessary. Similarly, the problem of the low mass fraction, Ω_M, described above was resolved in a similarly un-revolutionary way, again by measurement of the cosmological constant. The steady state theory that attempted to solve the age problem failed largely because it is so un-adaptable. The perfect cosmological principle restricted it so far that falsification was relatively simple, whereas relativistic cosmologies as a way of producing models was so malleable that it could accommodate the observation of cosmic expansion and an isotropic background of microwaves. It is similarly true that scalar tensor theories of dark energy will prove very hard to completely discount, as there are many forms they can take to fit almost any data set. That the cosmological constant has lasted so long is nothing short of miraculous, but its time will soon come. Perhaps, it already has: [65].

Not all long standing problems within astronomy have been fitted so neatly into boxes prepared by the prevailing theory. Kuhn describes the Copernican revolution as an important such case [66]. The geocentric model of Ptolemy, though lasting several centuries and remaining very successful in its predictions of the motions of the stars and planets, had become by Copernicus' time cumbersome and unwieldy. Kuhn writes in 1962: "as time went on, a man[8] looking at the net result of the normal research effort of many astronomers could observe that astronomy's complexity was increasing far more rapidly than its accuracy", and: "In the sixteenth century, Copernicus' coworker, Domenico da Novara, held that no system so cumbersome and inaccurate as the Ptolemaic had become could possibly be true of nature." The resolution came of course in a complete revolution within astronomy. The eventual move from geo- to helio-centrism altered the collective conception of our place in the universe, reducing its respective importance, and was a move only to be exacerbated by almost every major astronomical observation that was to follow.

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[8]that Kuhn's entire book is directed exclusively at his own sex is unfortunate and hard to ignore.

Other famous examples of a prolonged crisis preceding paradigm shift include the many and varied attempts to explain the lack of ether detection that facilitated the emergence of special relativity and, more further afield, the proliferation of adapted phlogiston theories from which emerged Lavoisier's oxygen theory of combustion, and the subsequent chemical revolution. Is this crisis of beyond-Horndeski (for a crisis is what it is to those involved in deciphering the nature of dark energy) indicative of a paradigm shift on the horizon, or will the mysteries of dark energy prove particularly difficult problems none-the-less within the grasp of the current cosmological paradigm, as the age problem had been in Fred Hoyle's time? To be sure the former prospect is the more exciting of the two, but it will be some time before we see such a drastic revision of our tools and procedures realised. In Kuhn's own words: "As in manufacture so in science—retooling is an extravagance to be reserved for the occasion that demands it. The significance of crisis is the indication they provide that an occasion for retooling has arrived."

1.4 Disformal couplings: a 'celebrity' interaction

Given this crisis state, there are two directions in which to progress. We can tackle this generality head on and attempt to place constraints on as many candidate theories as possible, usually with disregard for any motivating, more fundamental, physics. For some interesting work in this direction, see for example the parameterised post Friedmann formalism [67,68], or cosmography [69]. The second option is to do the opposite, and inspect closely some special cases to see if anything interesting emerges. This thesis describes an attempt at the latter: I place a magnifying glass over disformal relations in gravity and explore the effects they have on cosmology.

Disformal relations, as I will describe in this final section of the chapter, have a relatively broad base of interest, having emerged practically independently from several distinct sources, and became an entity worthy of study in their own right when they were named by Bekenstein in 1992 [70]. They can only be understood in the context of scalar tensor theories, and to facilitate their introduction I describe first the notion of an Einstein frame. I

then cover two separate origins of disformal theories: varying speed of light, and Bekenstein's two-geometry approach. Finally, to bring the chapter to a close, I will quickly cover the state of the disformal literature today; what questions are being asked in regards to disformal transformations? How have constraints been placed on disformal theories of gravity? In the conclusions to this thesis, chapter 6, I speculate answers to the following question. What does the future look like for disformal theories: are they a flash in the plan, or are they heading for assimilation into our standard cosmological toolbox?

1.4.1 The Einstein frame

Consider again the theory of Brans and Dicke from Sec. 1.2. We will meet this theory properly with an action representation in the next chapter, but for now recall that they had allowed the value of Newton's constant, G, to vary in space and time from an action approach [39]. They introduced a scalar field, ϕ, that carried the dimensions of G, and replaced the constant as it appears in the Einstein Hilbert action of general relativity with the scalar field. In the appendix, they present an alternate representation of the theory that has since become known as the Einstein frame. We have already seen that it is not variation in G one is able to measure, but in the dimensionless ratio:

$$\frac{Gm_e^2}{\hbar c}. \tag{1.4}$$

By redefining the gravitational field g of the Brans-Dicke action as:

$$\tilde{g} = C(\phi)g \tag{1.5}$$

for some scalar function C of ϕ, they were able to transfer the variation in G to a variation in m_e. The dimensionless ratio remains the same, and so the physical content of the theory, the observable consequences, are not affected. This second reformulation of the action, now known as the Einstein frame, has been used frequently ever since. In the work of this thesis it figures very heavily indeed. An attempt to formulate a scalar tensor theory that is invariant under the above transformation of the gravitational field can be found in [71].

1.4.2 Bekenstein, variable speed of light

The above transformation, Eq. (1.5), is a well known map between gravitational fields, or metric tensors in general, called a conformal transformation. Bekenstein addressed the question as to whether it was the most general relation permissible between two valid metric tensors involving a single scalar field [72]. After entertaining the idea that the relation may make $\tilde{g}$ Finslerian as opposed to just Riemannian, which he concluded did not produce a geometry conducive to physics, he arrived at the following expression for the most general relation:

$$\tilde{g} = C(\phi, X)g + D(\phi, X)d\phi \otimes d\phi, \tag{1.6}$$

for X the kinetic energy of ϕ, and d the exterior derivative. Such a transformation between metric tensors he christened a **disformal transformation**. It is worth pointing out that no reference is made in his paper to dark energy, which had not at the time been observed, and the question he posed was, rather like Brans and Dicke years before, mostly academic. Since then the transformation has been put to some good use, as we will see below.

Towards the end of the decade, in a distant context, a relation of a similar type was being used to solve the problems of inflation [73–77]. A big issue with the big bang picture was the horizon problem: how could it be that such distant parts of the universe that we can see via the cosmic microwave background are in thermal equilibrium, yet could not, within the measured age of the universe, have been in causal contact? Inflation theory solves this problem with a very fast expansion epoch fractions of a second after the big bang singularity; Moffat, Clayton, Magueijo and Barrow attempted to solve the same problem with a variable speed of light. If the space-time constant c had been much higher in the past, much larger regions of the universe could be kept in thermal contact.

These models are tightly constrained today, and no longer mainstream, but I introduce them because the above disformal transformation was the very ingredient they used to provide this varying c effect. We will see in later chapters how the disformal transformation can vary c, indeed the second half of chapter 3 is devoted entirely to this effect. The pioneers of these theories were not aware for some time of Bekenstein's work, and did not describe their transformation as disformal until some time later.

Disformal transformations as above also appear naturally in brane world

scenarios [78]. However, although important, this aspect of disformal transformations will not concern us in this work.

1.4.3 Today's disformal landscape

The application of disformal transformations to gravitation theories that exist in the literature today are frequent and varied. I will here give a quick overview, and mention that the breadth of the field is too wide to do any justification in these few pages. That being said, I recognise four main themes that roughly divide the literature: using disformals to understand the mathematical structure of scalar tensor theories, particularly with regards to the Horndeski Lagrangian and its extensions [79–81]; placing constraints on theories that in some way involve a disformal transformation [82–88]; phase space analysis and appeals to disformal transforms to alleviate the cosmological constant problem [89–91]; and the disformal invariance of certain actions and theories [92–96].

1. Understanding scalar tensor theories. Given our current state of ignorance as to the true nature of dark energy, isolating every action that could possibly describe it is an imperative. Of course searching for the true theory amongst the false then becomes akin to finding a needle in a haystack, but at least we would know where the haystack *was*; we could then restrict our attention to within its boundaries. The mission would be impossible otherwise. An important first attempt to outline the boundary is the Horndeski action, and so it becomes necessary now to understand better the structure of this theory.

Brans Dicke theory we have seen permits an Einstein frame description by means of a conformal transformation of the metric. It was shown by [97] that, with a disformal transformation above, much of the Horndeski action can be represented by an Einstein frame description; much but not all. They find only a subset of theories covered by the Horndeski functional freedom can be converted by a general disformal transformation into an Einstein frame. This is significant because metric transformations if suitably well defined, as we will make clear in the next chapter, carry actions onto physically equivalent actions. Thus, to find the true physical theory of dark energy, we are interested only in scalar tensor theories *modulo* metric transformations, which is to say that two actions must be physically equivalent if related by a disformal transformation.

At about the same time it was shown in [80] that disformal transformations also provide a loophole to Horndeski's original theorem. The authors of [97] find that only a subset of Horndeski can be converted into an Einstein frame but, surprisingly, in [80] the authors are able to show that some Einstein frame theories are not contained in the Horndeski theory. So disformal transformations have also been used to show the haystack is larger than we initially thought. A general work has pushed this idea further wherein disformal transformations have been used to link many disparate modified gravity theories together, including mimetic and some khronometric theories, under the equivalence relation [79]. This notion of physical equivalence will be questioned in detail in chapters 2 and 3.

2. Constraints. Disformal transformations are relations between some pair of metric tensors; they are of course not theories. Disformal theories of gravity, loosely defined here as a scalar tensor theory involving in some shape or form a disformal transformation, form a wide umbrella, and no constraint can be expected to apply unilaterally. However many such theories have been well constrained, using approaches that are as varied and wide as the applications of disformals themselves. Some notable examples are the following. In [85] it was shown that, for a very generic class of disformal theories, the derivative scalar field terms, on quantisation, produce derivative interactions between quantum modes of the matter and scalar fields. The resulting particle production would show up as missing energy in colliders, and the authors apply LHC run 1 data to tightly constrain their disformal model parameters.

Other examples pertain to astrophysics and cosmology. A similar class of disformal theories were expressed in the post newtonian formalism by the authors of [84], wherein similarly tight constraints were extracted from comparison between, in particular, derived disformal post newtonian parameters and the Cassini bound. The constraints they claim render all disformal effects in the cosmology of their class of theories negligible. Other approaches involve: using the non-detection of light shining through walls, similar to tests applied to axion theory [88]; modelling scalar tensor coupled neutron stars [82]; and analysing the properties of the cosmic microwave background, disformally coupled [87,98]. Variation of the fine structure constant was also shown to be a prediction of some of these disformal theories by the authors of [99], who subsequently place parameter constraints. Constraints on disformal theories, with special regard to cosmology, will be a recurrent theme in this thesis and chapters 3 and 4 deal directly with it.

3. The coincidence problem. A great attractor to disformal theories came from a promise of their potential to trigger late time acceleration of the universe, while being effectively screened from local gravity experiments [100]. The significance of the former property was that such a trigger could be used to alleviate the cosmological constant problem: the energy density in the scalar field could remain similar but subdominant to, say, the cosmic microwave background, and the disformal coupling could trigger the onset of Λ-style acceleration near today. Several recent phase space analyses, for example [91], have cast doubt over this possibility, and a detailed, up-to-date, and comprehensive such investigation can be found in [101]. Updated details of the screening mechanism can be found in [89], in which it is concluded that screening is possible, but for a more complicated scalar field Lagrangian. Chapter 5 of this thesis covers work addressed at finding structure in the phase space of a particularly safe disformal theory capable of alleviating the cosmological constant problem.

4. Disformal invariance. The disformal transformation can be applied to well known actions, and it is interesting to ask whether such an action then remains invariant. It is already known, and I discuss it in chapter 2, that the theory of electromagnetism is independent of scale, that is to say it is conformally invariant. Expressed in metric transformation terms, a conformal transformation of the space-time metric does not affect the electromagnetic action. We will see in later chapters this result does not generalise to disformal transforms. Are there any well known actions that are invariants of these maps, for example that of the Dirac equation? See [92–95] for interesting results in this direction, though I warn that some use a slightly modified version of disformal as defined above.

A final unexpected application worth a mention was to describing electron transport properties on strained graphene [102]. The graphene sheet can be modelled rather like a brane, and the disformal factor describes the curvature of the sheet, as it can describe the curvature of branes in cosmology[9].

Given the case studies just discussed, and the questions posed, *I will now assume that disformal transformations are interesting in and of themselves.* This is the premise underlying all work I will now describe, and chapters 3, 4, and 5 are devoted to painting pictures of universes that involve them.

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[9]I thank Clare Burrage for directing my attention to this work

Cosmology with scalar fields

<div style="text-align: right; font-size: 3em;">2</div>

This chapter forms the mathematical counterpart to chapter 1. Some important ideas introduced there are cemented here as mathematical definitions to be used throughout the rest of the thesis. Choices of symbols and names can often vary from reference to reference, and so I use this chapter to make clear my own. I first begin with relevant parts of general relativity, then cosmology with a cosmological constant. After this comes a description of scalar tensor theories, where I will make some of my own definitions whose scope extends no further than the thesis, though in some cases the words will have meaning that overlaps with those of the literature. The aim is to provide a level of rigour that will let me clear up what I see as a hindering ambiguity in the language of frame transformations, to be discussed in the concluding section of this chapter.

Throughout this thesis I have used index free notation wherever possible. This makes for a clean presentation of the ideas, but it also more clearly displays where metric dependencies are otherwise hidden by upper and lower index notation, which will be important when we start transforming the metrics of theories at the end of this chapter, and then throughout the rest of the thesis.

2.1 Universes with Λ

2.1.1 Space, time, gravity

In this subsection some mathematical definitions, notation, and relevant results related to general relativity that will be used throughout the thesis are quickly introduced. It serves this purpose only; familiarity with general relativity is assumed throughout. Cosmology-specific notation, definitions, and important results can be found in the following subsection. Unless specified otherwise, this section, 2.1, will use the resources: *Primordial cosmology* by Peter and Uzan [103], and *Spacetime and geometry* by Carroll [104].

I will use $\mathcal{M}$ to denote a smooth, 4-dimensional manifold. Depending on the topology of $\mathcal{M}$, we can equip it with any element of some large set of metric tensors, and we are interested here in those with Lorentz signature, which will be fixed throughout this thesis as $(-, +, +, +)$.

As in the context of general relativity and its surrounding paradigm, an event in space-time is defined here as a point, $p \in \mathcal{M}$. The pseudo-Riemannian manifold $(\mathcal{M}, g)$ I refer to as a space-time, and the metric tensor g I will occasionally refer to as a gravitational field.

I will denote an arbitrary action of fields defined on $\mathcal{M}$ as $\mathcal{S}$ which will always be differentiable in the sense of a functional derivative. As an integral, I will often write $\mathcal{S}$ as:

$$\mathcal{S}[g] = \int_{\mathcal{M}} d^4 x L(g) \tag{2.1}$$

for $\mathcal{S}$ a functional of the metric tensor, and some function L. Then:

$$\delta \mathcal{S}[g] = \int_{\mathcal{M}} d^4 x \frac{\delta \mathcal{S}}{\delta g} \delta g \tag{2.2}$$

where following [104] I have chosen to denote the functional derivative, the coefficient of δg, as ${\delta \mathcal{S}}/{\delta g}$. To be clear:

$$\frac{\delta \mathcal{S}}{\delta g} := \frac{\partial L}{\partial g} - \partial_\alpha \frac{\partial L}{\partial(\partial_\alpha g)}. \tag{2.3}$$

I remark that some authors including [103] use ${\delta L}/{\delta g}$ or variants of this, but this notation will not be used here.

In this thesis I will consider more generally an action as any one of the following maps:

$$\mathcal{S}[g], \quad \mathcal{S}[g, \phi], \quad \mathcal{S}[g, A], \quad \mathcal{S}[g, \phi, A] \tag{2.4}$$

for g any metric tensor, ϕ any scalar field, and A any massless one-form. By the Euler-Lagrange equations of an action $\mathcal{S}$, I mean the entire following set of differential equations:

$$\frac{\delta \mathcal{S}}{\delta g}[g', ...] = 0 \quad (\textbf{Einstein equations}) \tag{2.5a}$$

$$\frac{\delta \mathcal{S}}{\delta \phi}[..\phi', .] = 0 \quad (\textbf{Klein Gordon equation}) \tag{2.5b}$$

$$\frac{\delta \mathcal{S}}{\delta A}[..., A'] = 0 \quad (\textbf{Maxwell equations}) \tag{2.5c}$$

or at least as many as can be defined, depending on the specific functional (above) I am using. I will, for example, always refer to the first of this set as the Einstein equations, regardless of the form of the action. Later on we will see nonstandard equations that will come from modifying the standard Ricci term, but I will keep the name Einstein equation, as is standard practice in the modified gravity literature. Similarly, we will come to see modified Klein-Gordon equations and modified Maxwell equations, but they will retain their given names above.

The action and Euler-Lagrange equations provide a way to model physical systems. The stationary points of a particular form of $\mathcal{S}$ will provide a set of differential equations that, by Hamilton's principle, automatically satisfy Newton's laws of mechanical motion. By construction, the action is always invariant under diffeomorphic mappings of the underlying manifold. The form of $\mathcal{S}$ is determined often by the matter content in the physical system one aims to model, and the requirement that $\mathcal{S}$ depend on the underlying manifold only up to diffeomorphisms can be re-expressed in physical terms as: a physical system should be describable in any valid set of coordinates. In the last section of this chapter, I will discuss another type of transformation that will not be a diffeomorphism, and so the action will not be invariant under this new map, but I will introduce a lesser way in which it can be considered equivalent.

Written as an integral, I will refer to the **general relativity action** as the action defined by the following expression:

$$\mathcal{S} := \mathcal{S}_H + \mathcal{S}_\Lambda + \mathcal{S}_M \tag{2.6a}$$

where:

$$\mathcal{S}_H[g] := \int_{\mathcal{M}} d\pi(g) \frac{R(g)}{2\kappa}, \quad \mathcal{S}_\Lambda[g] := \int_{\mathcal{M}} d\pi(g) \frac{-\Lambda}{\kappa},$$

$$\mathcal{S}_M[g] := \int_{\mathcal{M}} d\pi(g) \mathcal{L}_M(g) \tag{2.6b}$$

for R the Ricci scalar of some metric g, $\mathcal{L}_M(g)$ the Lagrangian density for some matter distribution, Λ the familiar cosmological constant, κ has the usual definition:

$$\kappa = \frac{8\pi G}{c^4} \tag{2.7}$$

for G the familiar Newton's constant—I will use κ from here on, so as not to conflict with the Einstein tensor, which becomes simply G in index free notation—and $d\pi(g)$ the covariant 4-dimensional volume measure on $\mathcal{M}$. If we define the coordinate fields $x^0,...,x^3$ on $\mathcal{M}$, then $d\pi(g)$ expressed in terms of these coordinates is:

$$d\pi(g) = \sqrt{-|g|}d^4x, \tag{2.8}$$

for $|g|$ the the determinant of the matrix representation of g in coordinates $x^0,...,x^3$. I will denote the coefficients/components of the matrix representation of g in some basis determined by $x^0,...,x^3$ as $g_{\mu\nu}$. The inverse of g I denote g^{-1}, and the components of g^{-1} in the basis given by coordinates $x^0,...,x^3$ I will denote $g^{\mu\nu}$.

The Einstein equations of the general relativity action are

$$G(g) + \Lambda g^{-1} = \kappa T(g) \tag{2.9a}$$

where:

$$G := -\frac{2\kappa}{\sqrt{-|g|}} \frac{\delta S_H}{\delta g} \tag{2.9b}$$

is the Einstein tensor, and:

$$T := \frac{2}{\sqrt{-|g|}} \frac{\delta \mathcal{S}_M}{\delta g}, \tag{2.9c}$$

the stress energy momentum tensor of matter, which I will refer to for brevity as simply the **stress tensor**, and note that T *does not* here mean the trace

of said tensor, which it can in other texts (including some that I have co-authored [105]). This tensor is conserved:

$$\nabla T = 0 \qquad\qquad (2.9\text{d})$$

for ∇ the covariant derivative metric compatible with g.

A perfect fluid is a matter system that can be modelled adequately by two scalar variables: thermodynamic energy density, $\rho : \mathcal{M} \to \mathbb{R}$ and pressure $P : \mathcal{M} \to \mathbb{R}$; and a velocity field $u : \mathcal{M} \to T\mathcal{M}$, for $T\mathcal{M}$ the tangent bundle of $\mathcal{M}$. The value of u at each point $p \in \mathcal{M}$ is interpreted as the instantaneous velocity of the fluid element at that point. As is commonplace in the field of cosmology, we will assume a great many matter systems to be perfect fluids.

The stress tensor for a perfect fluid is given by:

$$T = (\rho + P)u \otimes u + Pg^{-1} \ . \qquad\qquad (2.10)$$

See also [106] for a more physical motivation to this definition.

2.1.2 Cosmology

A universe in this thesis will be a particular space-time $(\mathcal{M}, g)$ studied by a cosmologist. This is a working, not a precise, definition, but this will in general come to mean it is a space-time that obeys the following cosmological principle. For want of a better word, I will refer to the totality of physical existence as the **Universe**, capitalised. A universe, we will come to see, is then any space-time that attempts to model the physical Universe within some realm of validity—when coarse-grained over large enough distances.

> *The cosmological principle* states that the Universe on largest scales—that is, when averaged over distances larger than about a mega parsec—is isotropic and homogeneous in its spatial sections.

A universe $(\mathcal{M}, g)$ that attempts to model the Universe must then be spatially isotropic and homogeneous, for some definition of space to follow. Isotropy means rotational invariance of the space about some point, $p \in \mathcal{M}$. Homogeneity means translational invariance. The two combined imply that a universe obeying the cosmological principle is isotropic in space about every point.

The next result determines the set of all space-times consistent with the above principle, i.e. all valid universes. In chapter 4 we will consider too perturbed universes as better approximations to the Universe.

If a space-time $(\mathcal{M}, g)$ is consistent with the cosmological principle, then there exists coordinates (ct, r, θ, ϕ) such that the line element of g can be written:

$$ds^2 = -c^2 dt^2 + a^2(t) \left(\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)^2 \right) \qquad (2.11)$$

for some $K \in \mathbb{R}$. Such a space-time I will refer to as a **universe**. If $K = 0$, I will refer to it as a **flat universe**, and will generally use Cartesian coordinates, $(x^1, x^2, x^3) := (x, y, z)$ to describe it. This coordinate system defines a rest frame for the universe.

An observer at rest with respect to the above coordinates in a universe I will refer to as a **comoving observer**. The coordinate t is called the **cosmic time**, and will be the proper time of the comoving observer (by definition of said observer). The above family of line-elements, Eq. (2.11), were those found independently by Friedmann and Lemaître (see Sec. 1.1.2) and so are often referred to as the **Friedmann-Lemaître line elements** (or simply FL line element). The family of metric tensors implicitly defined by Eq. (2.11) are called the **FL metric tensors** (or simply metrics for short). The function a is called the **scale factor** and describes the expansion of a particular universe.

There are two independent Einstein equations of the general relativity action for universes, with matter described as a perfect fluid, which can be written:

$$\frac{\dot{a}^2(t)}{a^2(t)} = \frac{8\pi G}{3} \rho(t) + \frac{\Lambda c^2}{3} - \frac{Kc^2}{a(t)^2} \qquad (2.12a)$$

and:

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3} \left(\rho(t) + \frac{3P(t)}{c^2} \right) + \frac{\Lambda c^2}{3}. \qquad (2.12b)$$

They are called the **Friedmann equations**.

We can see that $[K] = L^{-2}$ and that there exists different universes for each value of K and function a that is a solution to the Friedmann equations. The geometry of the spatial sections, i.e. the manifolds with line element $\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2)^2$ at different cosmic times, t, are uniquely determined by the value of K, a fixed parameter to be measured

if these universes are to approximately describe our Universe. The three qualitatively distinct cases are represented by $K < 0$, $K = 0$, and $K > 0$ corresponding to universes with hyperbolic, flat, and hyperspherical spatial geometries respectively.

Example 2.1 (Einstein's 'model A'). Einstein considered, in a paper submitted to the Prussian academy of sciences in 1917 [6] and a private note to a friend [3] a static universe (that is, a space-time satisfying the cosmological principle, whose geometry is constant in cosmic time) filled with a homogeneous fluid of density $\rho \simeq 10^{-22}\text{gcm}^{-1}$ [4]. This corresponds to a solution of the Friedmann equations where $\ddot{a}(t) = \dot{a}(t) = P(t) = 0 \ \forall t$[1]. Combining $\ddot{a}(t) = 0$ with (2.12b) we get:

$$\frac{8\pi G}{c^2}\rho = 2\Lambda.$$

$\dot{a}(t) = 0$ and (2.12a) imply

$$\Lambda = \frac{K}{a_0^2}$$

For some fixed a_0. Recalling that $[K] = L^{-2}$ and $[a] = 1$, $\sqrt{\frac{a_0^2}{K}}$ has the dimensions of length. It is Einstein's radius of the universe, $\mathcal{R}$, from the previous chapter. It is the quantity $\mathcal{R}(t) = \frac{a(t)}{\sqrt{K}}$ that Lemaître originally realised was the variable size of the expanding Universe, and indeed many cosmology papers still featured the notation $\mathcal{R}(t)$ as late as the 2000s [107]. When Einstein's universe is seen as a constant $a(t)$ solution to the Friedmann equations, one can quite easily check that it is an unstable one.

For a small positive Λ, the curvature parameter K is positive and 3-space in this universe is hyperspherical. ▲

Example 2.2 (de Sitter's 'model B'). Willem de Sitter found another solution, published also in 1917 [4, 7], where $\rho = 0$ and:

$$\Lambda = 3\frac{a_0^2}{K}. \tag{2.13}$$

Certainly from (2.12b) we see this does not yield a static solution in these coordinates. However, the Ricci scalar is:

$$R = 4\Lambda \tag{2.14}$$

---

[1]Of course Einstein did not have the Friedmann equations to hand at that time. We are working anachronistically.

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which is constant for all space (this universe satisfies the cosmological principle) and time (the universe is static). That R is constant means there exists a static coordinate system describing de Sitter's universe, which is the solution he presented in his paper. With hindsight, (and comoving coordinates), we understand de Sitter's solution now to be an expanding one, vey much like the dark-energy dominated epoch of our current physical Universe. ▲

Example 2.3 (Hubble's law)**.** Consider two comoving observers A and B in some universe $(\mathcal{M}, g)$. By definition, the proper time of A and B is the cosmic time, t. Thus the geodesics of A and B we denote $x_A(t), x_B(t) \in \mathcal{M}$. Let the the spatial sections of $\mathcal{M}$ at some time t be Σ_t, with induced metric γ. We are interested in the observers' separation, $d(x_A(t), x_B(t))$ for $x_A(t), x_B(t) \in \Sigma_t$ at an instant t in cosmic time.

First (Σ_t, γ) is a Riemannian manifold. We have that, for a Riemannian manifold [108]:

$$d(x_A(t), x_B(t)) = \inf\{L[\zeta]\} := L \qquad (2.15)$$

such that $\zeta : [s_A, s_B] \to \Sigma_t$ is a smooth curve from $x_A(t)$ to $x_B(t)$, where:

$$L[\zeta] = \int_{[s_A, s_B]} \sqrt{\gamma\left(\frac{d\zeta}{ds}, \frac{d\zeta}{ds}\right)}\, ds. \qquad (2.16)$$

From the FL line element, we can write $\gamma_{ij}(t, x^k)dx^i dx^j := a^2(t)\gamma'_{ij}(x^k)dx^i dx^j$ for some 3-space metric γ'_{ij}, and:

$$L[\zeta](t) \;\; = \;\; a(t)\int_{[s_A, s_B]} \sqrt{\gamma'\left(\frac{d\zeta}{ds}, \frac{d\zeta}{ds}\right)}\, ds \qquad (2.17)$$

$$:= \;\; a(t)l[\zeta]. \qquad (2.18)$$

$l := \inf\{l[\zeta]\}$ is the comoving distance, independent of time, between comoving observers A and B, and it is clear that $\inf\{L[\zeta]\} = a(t)\inf\{l[\zeta]\}$.

Finally then we see that:

$$\dot{L} = \frac{\dot{a}}{a}L, \qquad (2.19)$$

which is Hubble's law: galaxies comoving with the expansion of the universe recede from us with a velocity $\dot{L}$ proportional to their distance L. It is standard in cosmology to define:

$$H(t) := \frac{\dot{a}(t)}{a(t)} \qquad (2.20)$$

as the **Hubble parameter**. Often in the literature it is referred to as the Hubble constant [65], a relic from the past when it was believed to be fixed, but this is confusing for obvious reasons, and I will not adopt this name here. ▲

A photon with wavevector k^μ emitted by some source, will have energy $E_{\text{emit}} = -(g_{\mu\nu}\hbar k^\mu u^\nu)_{\text{emit}}$ in the source's rest frame. If the photon is received by an observer, it will be with energy $E_{\text{rec}} = -(g_{\mu\nu}\hbar k^\mu u^\nu)_{\text{rec}}$ in this observer's rest frame. Their ratio is a very useful quantity in cosmology:

$$1 + z := \frac{(g_{\mu\nu}k^\mu u^\nu)_{\text{emit}}}{(g_{\mu\nu}k^\mu u^\nu)_{\text{rec}}} \tag{2.21}$$

and is called the **redshift** of the photon.

If the source and observer of a photon are both comoving in some universe, $\mathcal{M}$, the photon redshift relates simply to the scale factor a as:

$$1 + z = \frac{a(t_{\text{rec}})}{a(t_{\text{emit}})}. \tag{2.22}$$

It is common to refer to the velocity field u of comoving observers in a universe $\mathcal{M}$ as the Hubble flow, and the motions of galaxies relative to this flow—that is $\delta v := v_{\text{galaxy}} - u(p)$ for some galaxy at $p \in \mathcal{M}$—the peculiar velocities of the galaxies. The last result shows, in cosmology, the redshift of a photon is a direct measure of the expansion of a universe, once peculiar velocities have been accounted for. In fact z and a are diffeomorphic variables, and both are diffeomorphic to cosmic time t in most cases of interest, i.e. when the universe expansion does not stop or change direction (as it would for, say, a cyclic universe, or, pertinently, in the universe we consider in chapter 5). Any one of these three can be used as a time variable to parameterise a universe, so long as the universe size is only strictly increasing / decreasing in time, and indeed all three frequently are. This will break down in chapter 5, as we will see.

For a final example, I present the equation of state of a perfect fluid in a universe, and the three main cases for cosmology.

Example 2.4. For a perfect fluid with stress tensor T in a universe, $(\mathcal{M}, g)$, the conservation equation becomes:

$$\dot{\rho}(t) + 3H(t)\left(\rho(t) + \frac{P(t)}{c^2}\right) = 0. \tag{2.23}$$

This conservation equation, combined with the Friedman equations, Eq. (2.12a) and (2.12b), almost form a closed system that can be solved. All that is needed to close the system is to specify the relation between energy density and pressure. It is standard define the dimensionless **equation of state**, w, as:

$$w := \frac{P}{c^2 \rho} \tag{2.24}$$

which is constant for many systems of interest. By defining w this way, with the constant c not set to unity, some results of section 3.1 will come as less of a surprise. For cosmology there are three important cases as to the value of w, which we will see do not all hold for modified gravity generally.

1. Pressureless matter. If T describes a perfect fluid of particles stationary in the cosmology rest frame, i.e. particles that are comoving, then $w = 0$. In this case, Eq. (2.23) has the solution:

$$\rho(t) \propto a^{-3}(t). \tag{2.25}$$

The energy density of the fluid in the comoving rest frame is just the rest mass density of particles, which decreases as space isotropically expands.

2. Fluid of massless particles. If T describes a perfect fluid of massless particles like photons, then $w = 1/3$. In this case, Eq. (2.23) has the solution:

$$\rho(t) \propto a^{-4}(t). \tag{2.26}$$

The rest energy of the fluid elements are diluted with the expansion of space, but the wavelengths of the particles are stretched with the scale factor: the energy of the fluid is diminished by an extra factor of a^{-1}.

3. Cosmological constant as a fluid. the stress tensor of the cosmological constant we can read off Eq (2.9a):

$$T_\Lambda = -\frac{\Lambda}{\kappa} g^{-1} \tag{2.27}$$

which determines that $\rho_\Lambda c^2 = \Lambda/\kappa$ and $P_\Lambda = \Lambda/\kappa$, thus $w_\Lambda = -1$ and:

$$\rho_\Lambda(t) \propto a^0(t). \tag{2.28}$$

The energy density of the cosmological constant does not dilute as the universe expands. See for example [109] for an clear physical presentation of the above.

▲

2.2 Scalar tensor cosmology

In the last chapter, we saw how a dynamic radius of the universe, $\mathcal{R}(t)$, was able to explain Hubble's observation of galaxy light redshift linearly dependent on galaxy distance. In the last section we used Friedmann's line element for an expanding universe to derive Hubble's famous law. In this section we apply the paradoxical notion of varying constants to some famous ones: the gravitational constant κ, and the cosmological constant Λ. In doing so, we will introduce some of the relevant mathematics of scalar tensor theories.

There are two problems with performing the replacement $\kappa \to \kappa(p)$, for p a space-time point, in all equations. The first is that energy and momentum are not globally conserved [33]. For example, consider performing this action on the Einstein equations. We get $G = \kappa(p)T$ and, by the Bianchi identities, $\nabla T = -T\nabla(\ln\kappa)$. This is not an energy exchange, the energy simply 'vanishes' from the system (or is fed in from nowhere). While being a conceptual headache (where would this energy come from? If the Universe is the totality of physical existence, this precludes by definition any external energy reservoir or driving forces), it is also hard to keep this theory free of pathologies and instabilities in its dynamical system. The second is that, in some cases, we end up just working with standard physics, but one where the units vary needlessly, and without physical consequence [29].

With that in mind, we consider now the archetypal way to vary κ from an action.

Example 2.5 (Brans-Dicke theory)**.** Written as an integral, the **Brans-Dicke action** is defined by the following expression [39]:

$$\mathcal{S}[g, \phi] := \int_{\mathcal{M}} d\pi(g) \left[\frac{\phi R(g)}{2} - \omega \frac{g^{-1}(d\phi, d\phi)}{\phi} + \mathcal{L}_M(g) \right] \tag{2.29}$$

for ω a dimensionless constant. Clearly $[\phi] = [\kappa]^{-1}$.

The Einstein equations of the Brans-Dicke action are:

$$G_{\mu\nu} = \frac{8\pi\phi^{-1}}{c^4} T_{\mu\nu} + \frac{\omega}{\phi^2}(\phi_{,\mu}\,\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\alpha}\,\phi^{,\alpha}) + \phi^{-1}(\phi_{,\mu;\nu} - g_{\mu\nu}\Box\phi) \tag{2.30}$$

and we see that energy conservation is now restored, compared with the naïve example above; it is transferred into the scalar field, ϕ. As well as a

conservation equation for matter, $\mathcal{L}_M(g)$, there is now too a Klein Gordon equation for the scalar field:

$$2\omega\phi^{-1}\Box\phi - \frac{\omega}{\phi^2}\phi^{,\mu}\phi_{,\mu} + R = 0. \tag{2.31}$$

The dynamics are now self consistent, and we are in no danger of simply working with standard physics in funny units. For one thing, a new species has been added to the space-time that, itself, gravitates.

As Ellis and Uzan stress [29], it is not κ that varies (such a statement is a subjective one about unit choices) but the dimensionless ratio:

$$\frac{\kappa c^3 m_e^2}{8\pi\hbar} \tag{2.32}$$

that is varying, where m_e is the mass of the electron. Only if an experiment measures the space-time variation of this ratio can we confidently assert that the Universe is adequately described by something akin to the Brans-Dicke action. The scalar field in the Brans-Dicke action plays a role very similar to the dilaton of higher dimensional theories [110]. ▲

In the above example, for the action to describe the Universe, the ratio $\kappa c^3 m_e^2/8\pi\hbar$ must be constant at and cosmologically near the present day: the constant must be **stabilised**. This condition will apply to any varying constant theory we might like to consider physically viable, including varying Λ.

We now come to what is, for this thesis, by far the most important example presented here: varying Λ.

Example 2.6 (Dark energy). Consider a varying Λ action:

$$\mathcal{S}[g,\phi] := \int_{\mathcal{M}} d\pi(g) \left[\frac{R(g) - 2\Lambda(\phi)}{2\kappa} + X(g) + \mathcal{L}_{\mathrm{M}}(g) \right], \tag{2.33}$$

where $\mathcal{L}_{\mathrm{M}}$ is some unspecified Lagrangian of matter fields, and

$$X(g) := -\frac{1}{2}g^{-1}(d\phi, d\phi) \tag{2.34}$$

is the field's kinetic energy. (I stress here that our field's Lagrangian may change throughout the thesis, and the above will not always be the total energy of the field's motion, nonetheless X defined above will always retain

this definition, despite any change to the field's Lagrangian.) This is the action for a scalar field with a potential:

$$V(\phi) = \frac{\Lambda(\phi)}{\kappa}. \qquad (2.35)$$

▲

In this thesis, the particular scalar field Lagrangian $\mathcal{L}_\Phi(\phi, X(g)) = X(g) - V(\phi)$ will play a very prominent role, due mainly to its simplicity, and will be referred to as the **quintessence Lagrangian**. A scalar field with dynamics described by an action, $\mathcal{S}$, will be said to be **quintessent in S** if the scalar Lagrangian is of the above form. In the next section we will come across theories with multiple actions, and the scalar field quintessent in one but not another. If the scalar field ϕ of some action $\mathcal{S}$ controls the value of the cosmological constant, $\Lambda = \Lambda(\phi)$, as in the above example, this field will be called a **dark energy field**. Much of the work described in this thesis will involve quintessent dark energy fields[2].

A scalar field of course need not be quintessent. Not even a dark energy field need be [111, 112]. In general $\mathcal{L}_\Phi(\phi, X(g))$ can be any function, but the action for a dark energy field may be more general still. The next result which I will not prove for obvious reasons—Horndeski's theorem—is an attempt to define the boundary of this generality based on instability considerations: the most general dark energy action, and so I show it now as more of a curiosity. It provides in effect a benchmark for dark energy research, and many theories, though not intrinsically interesting, become interesting with respect to it.

Result 2.2.1 (Horndeski's theorem). *Let ϕ be a scalar field defined on a space-time $(\mathcal{M}, g)$, with action $\mathcal{S}$, such that the Klein Gordon equation of $\mathcal{S}$ contains derivatives of ϕ that are at most second order. Then $\mathcal{S}$ is given by the relation [55, 56, 96]:*

$$\mathcal{S}[g, \phi] := \int_{\mathcal{M}} d\pi(g) \left(\sum_{i=2}^{5} \mathcal{L}_i + \mathcal{L}_{\mathrm{M}}(g) \right), \qquad (2.36a)$$

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[2]Not all authors use this exact terminology, and the terms quintessence and dark energy do not have fixed meaning in the literature. I am fixing the meaning of these terms for the duration of the thesis only.

where $\mathcal{L}_\mathrm{M}$ is some arbitrary matter field Lagrangian, and

$$\mathcal{L}_2 = K(\phi, X) \tag{2.36b}$$

$$\mathcal{L}_3 = -G_3(\phi, X)\Box\phi \tag{2.36c}$$

$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right] \tag{2.36d}$$

$$\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi) \tag{2.36e}$$

$$-\frac{1}{6}G_{5,X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right. \tag{2.36f}$$

$$\left. + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)\right] \tag{2.36g}$$

for any smooth functions $K, G_3, ...G_5$. We refer to $\mathcal{S}$ as the **Horndeski action**.

By Ostrogradsky's theorem, any dynamical system with third-or-above order derivatives of some degree of freedom is generically unstable [113]. This, in fact, is the mathematical basis for Newton's second law, which states that the equation of motion of a body can not feature time derivatives of higher order than acceleration, $\ddot{r}$:

$$\ddot{r} = F(r, \dot{r}) \tag{2.37}$$

for some force F to be specified by the system under consideration[3].

As we have already discussed however, there are loopholes in Horndeski's approach, which, when exploited, allow most notably the beyond-Horndeski action. The loopholes in the above result are found using the notion of a frame transformation, which we will meet in the next section. Nonetheless, Res. 2.2.1 represents an important step in limiting the number of actions that could possibly describe our Universe in light of the discovery of the accelerated expansion.

2.3 Frame transformations

This section clears up the ambiguous language surrounding *frames* of scalar tensor theories. I appropriate some terms whose meaning will often overlap with the literature, but may sometimes be in conflict with parts of it. The aim is to cement the language of frames—and frame *transformations*—so that

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[3]It is this reason why we never meet $\ddot{r}$ in elementary mechanics, though of course it is perfectly permissible mathematically.

the rest of the thesis is not confused. I begin with a discussion of conformal transformations as maps, and the abelian group they form, and point out the electromagnetic free action is a well known invariant of this group. I then move on to general frame transformations, and show how they can be used to uncover loopholes in Horndeski's theorem, Res. 2.2.1. I will use the following notation convention for the composition of two maps: $g \circ f(x) := f(g(x))$. To ward off confusion, I note that many authors use the reverse convention.

Conformal transformations

Let $\mathcal{G}$ be the set of all metrics permissibly defined on $\mathcal{M}$ by its topology, with Lorentz signature. Then, a **conformal transformation** is a map:

$$
\begin{aligned}
f_C : \mathcal{G} &\rightarrow \mathcal{G} \\
g &\mapsto Cg
\end{aligned}
$$

where ϕ is some scalar field on $\mathcal{M}$, and $C(\phi, X(g))$ a smooth function such that $C(p) > 0 \quad \forall p \in \mathcal{M}$ (see appendix G of [104]).

Result 2.3.1. *A conformal transformation is a well defined map.*

Proof. To be a well defined map, $f_C(g) \in \mathcal{G} \quad \forall g \in \mathcal{G}$, which means $f_C(g)$ must be a pseudo-Riemannian metric tensor. It must be symmetric, bilinear and non-degenerate for every pseudo-Riemannian metric tensor g [114]. To see symmetry note $f_C(g)(A, B) = Cg(A, B) = Cg(B, A) = f_C(B, A)$. Bilinearity of $f_C(g)$ follows in the same way, directly from bilinearity of g. For non-degeneracy, we note that the following ratio is a scalar quantity:

$$
\sqrt{\frac{|f_C(g)|}{|g|}} = C^2 \tag{2.38}
$$

so, if g non-degenerate, and $C(p) > 0 \quad \forall p \in \mathcal{M}$, then $\sqrt{-|f_C(g)|} > 0 \quad \forall p$ and hence the inverse metric $(f_C(g))^{-1}$ exists everywhere. (Note smoothness of $f_C(g)$ guaranteed because C is smooth). ∎

Let $\mathcal{C}$ be the set of all such conformal transformations.

Result 2.3.2. *The set $\mathcal{C}$ forms an abelian group with function composition, $\circ$.*

Proof. The proof is almost trivial. The identity is given by setting $C(\phi) = 1 \quad \forall p \in \mathcal{M}$ in the above definition. Associativity and commutativity follow from associativity and commutativity of multiplication of scalar functions on $\mathcal{M}$. For every conformal transformation f_C, there exists an inverse $f_C^{-1} = f_{C^{-1}}$ as, for any function C, there exists C^{-1} as $C(\phi(p)) \neq 0 \quad \forall p \in \mathcal{M}$ by definition, and, finally, closure is assured by closure of scalar functions under multiplication: $f_{C_1} \circ f_{C_2}(g) = C_2 C_1 g = f_{C_2 C_1}(g)$ [115]. ∎

The next result is also well known, but I repeat it here as it will be useful in Sec. 3.2.

Result 2.3.3. $(\mathcal{C}, \circ)$ *is a symmetry group of the electromagnetic action,* $\mathcal{S}_{\text{EM}}$.

Proof. The definition of S_{EM} is [103]:

$$S_{\text{EM}}[g, A] := -\frac{1}{4\mu_0} \int_{\mathcal{M}} d\pi(g) F^2(g, A), \tag{2.39}$$

where μ_0 is the permittivity of free space, and:

$$F(A) = dA \tag{2.40a}$$
$$F^2(g, A) = g^{-1} \otimes g^{-1}(F, F). \tag{2.40b}$$

Then:
$$f_C \circ d\pi(g) = d\pi(f_C(g)) = d^4 x \sqrt{-|C(\phi)g|} = C^2 d\pi(g)$$

and:
$$f_C \circ F^2(g, A) = (Cg)^{-1} \otimes (Cg)^{-1}(F, F) = C^{-2} F^2(g, A)$$

so:

$$f_C \circ \mathcal{S}_{\text{EM}}[g, A] := \mathcal{S}_{\text{EM}}[Cg, A] \tag{2.41}$$
$$= -\frac{1}{4\mu_0} \int_{\mathcal{M}} d\pi(Cg) F^2(Cg, A) \tag{2.42}$$
$$= -\frac{1}{4\mu_0} \int_{\mathcal{M}} d\pi(g) C^2 C^{-2} F^2(g, A) \tag{2.43}$$
$$= \mathcal{S}_{\text{EM}}[g, A] \quad \forall g \in \mathcal{G}.$$

∎

Disformal transformations

A **disformal transformation** is a map:

$$\begin{aligned} f_D : \mathcal{G} &\rightarrow \mathcal{G} \\ g &\mapsto g + D(\phi, X) d\phi \otimes d\phi \end{aligned}$$

where ϕ is some scalar field, d the exterior derivative, and D a smooth function. This does not complete the definition, however; f_D is not well defined unless we impose a restriction on D.

Result 2.3.4. *The disformal transformation $f_D : \mathcal{G} \rightarrow \mathcal{G}$ is only a well defined map if it can be expressed in the form:*

$$f_D(g) = g + \left(1 - Z^2\right) \frac{d\phi \otimes d\phi}{2X} \tag{2.44}$$

for some smooth function $Z(\phi, X) > 0$.

Proof. As for Res. 2.3.1, symmetry and bilinearity of $f_D(g)$ follow from bilinearity of g, and note that $d\phi \otimes d\phi(A, B) = d\phi(A)d\phi(B) = d\phi(B)d\phi(A) = d\phi \otimes d\phi(B, A)$. Non-degeneracy is not assured this time. We claim that [96]:

$$\sqrt{\frac{|f_D(g)|}{|g|}} = \sqrt{1 - 2XD}. \tag{2.45}$$

Then we must have:

$$\sqrt{\frac{|f_D(g)|}{|g|}} > 0 \tag{2.46}$$

which is a condition we must impose on D to preserve the existence of $(f_D(g))^{-1}$ everywhere. Let $Z = \sqrt{1 - 2XD}$, then $D = (1 - Z^2)/2X$ and, after proving the claim, we are done.

Proof of claim. In index notation we have:

$$\begin{aligned} f_D(g)_{\mu\nu} &= g_{\mu\nu} + D\partial_\mu\phi\partial_\nu\phi & (2.47) \\ &= g_{\mu\nu} + 2DXu_\mu u_\nu & (2.48) \\ &= g_{\mu\alpha}\delta^\alpha_\nu + 2DXg_{\mu\alpha}u^\alpha u_\nu & (2.49) \end{aligned}$$

for the normalised one-form $u := d\phi/\sqrt{2X}$. Then:

$$|f_D(g)_{\mu\nu}| = |g_{\mu\alpha}||\delta^\alpha_\nu + 2DXu^\alpha u_\nu| \tag{2.50}$$

where recall $|.|$ denotes the determinant. Next chose coordinates such that $u = (1, 0, 0, 0)$. In this system we have

$$\frac{|f_D(g)|}{|g|} = 1 - 2XD, \tag{2.51}$$

but this is a scalar equation true in any frame. ∎

Let $\mathcal{D}$ be the set of all well defined disformal transformations, f_D.

Result 2.3.5. *The only subset of $\mathcal{D}$ that forms a group under composition of functions, $\circ$, is the set $\{f_0\}$ where f_0 is the identity transformation.*

Proof. Assume some subset $S \subset \mathcal{D}$ forms a group under $\circ$. Then S is closed under composition. In particular, for some $f_D \in S$, f_D^n is also in S $\forall n \in \mathbb{N}$. Then:

$$\begin{aligned} f_D^n(g) &= g + n(1 - Z^2)\frac{d\phi \otimes d\phi}{2X} \\ &:= g + (1 - Z'^2)\frac{d\phi \otimes d\phi}{2X}. \end{aligned} \tag{2.52}$$

For $f_D^n(g)$ to be well defined we require $1 - Z'^2 < 1 \quad \Rightarrow \quad n(1 - Z^2) < 1$ which implies $Z^2 > 1 - \frac{1}{n}$.

Now if S a group every element has an inverse, so $f_D^{-n}(g) \in S$. Then:

$$\begin{aligned} f_D^{-n}(g) &= g - n(1 - Z^2)\frac{d\phi \otimes d\phi}{2X} \\ &:= g + (1 - Z'^2)\frac{d\phi \otimes d\phi}{2X}. \end{aligned} \tag{2.53}$$

The same argument from before gives us the condition $Z^2 < 1 + \frac{1}{n}$. Putting them together gives:

$$1 - \frac{1}{n} < Z^2 < 1 + \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

Taking the large n limit, but recalling that we cannot take an infinite number of compositions, gives:

$$1 \le Z^2 \le 1 \tag{2.54}$$

and hence the only element in S is the identity, $f_{D=0}$. ∎

We cannot have a closed group of disformal transformations that are always invertible and always well defined. As no disformal group exists, no action can have disformal transformations as a symmetry group. In addition, it is often said that 'actions are invariant under disformal transformations' (for example [97]), but we see this cannot be true. Clearly $\mathcal{S}[g] \neq \mathcal{S}[f_D(g)]$ unless f_D is the identity. As I will show, we can say that two actions are equivalent in a looser sense, but we will see we must be careful even with this language.

Frame transformations

A **frame transformation** is a map:

$$
\begin{aligned}
f : \mathcal{G} &\rightarrow \mathcal{G} \\
g &\mapsto Cg + Dd\phi \otimes d\phi
\end{aligned}
$$

where ϕ is some scalar field, d the exterior derivative, and $C(\phi, X), D(\phi, X)$ are smooth functions, such that $C(p) > 0 \quad \forall p \in \mathcal{M}$. It is clear any frame transformation can be written as a product $f = f_C \circ f_D$ for some $f_C \in \mathcal{C}$ and $f_D \in \mathcal{D}$. The definition is not complete; f is only well defined if extra conditions are imposed.

Result 2.3.6. *A frame transformation $f : \mathcal{G} \to \mathcal{G}$ is only a well defined map if it can be expressed in the form:*

$$
f(g) = Cg + C \left(1 - Z^2\right) \frac{d\phi \otimes d\phi}{2X} \tag{2.55}
$$

for some smooth functions $C(\phi, X), Z(\phi, X) > 0$.

Proof. Again the only contentious issue is of non-degeneracy of the maps and their inverses. For a general frame transformation we have:

$$
\sqrt{\frac{|f(g)|}{|g|}} = C^2 \sqrt{1 - 2X\frac{D}{C}} > 0 \tag{2.56}
$$

where, as before, the last inequality is a constraint imposed on C and D to ensure that $(f(g))^{-1}$ always exists. (note: $(f(g))^{-1}$ is not the same as $f^{-1}(g)$. The latter is the reverse transformation of some f applied to a metric g, while the former is the inverse metric of $f(g)$). Then we require $C > 0$ and $Z = \sqrt{1 - 2X\frac{D}{C}} > 0$ and, rearranging for D, we have the result. ∎

Throughout this thesis I will mostly write frame transformations as $g \mapsto Cg + Dd\phi \otimes d\phi$, leaving it implicit that such a transformation is well defined, and *can* hence be written in terms of some $Z > 0$ as above. However, the scalar function Z associated with any frame transformation will crop up repeatedly in these pages, so I promote it here to the status of a named function: the **disformal scalar**:

$$Z := \sqrt{1 - 2X\frac{D}{C}}. \tag{2.57}$$

Note now the symmetry: we require $C, Z > 0$ for a well behaved map, and $C \to 1$ represents the disformal limit of a transformation, while $Z \to 1$ is the conformal one.

Let $\mathcal{F}$ be the set of all frame transformations. $\mathcal{F}$ clearly does not form a group, however, $\mathcal{C}$ does form a group as is well known, though it is not clear whether $\mathcal{C}$ is the largest subset of $\mathcal{F}$ that does. As the proofs will become much more involved, and we are heading off topic as it is, I will leave this line of questioning for future work.

As with disformal transformations, there is an issue that some maps of the form $g \mapsto Cg + Dd\phi \otimes d\phi$ do not carry healthy metric tensors onto healthy metric tensors. To avoid these undesirable cases, the conditions in result 2.3.6 must be imposed on C and D but, as with the disformal transformations, not all frame transformations satisfying these conditions will be invertible. As an aside, note it is clear from the definitions that conformal and disformal transformations are both special cases of frame transformations, i.e. $\mathcal{C}, \mathcal{D} \subset \mathcal{F}$.

Frame transformations are interesting in part for the following reason due to Bekenstein [70]. Let $(\mathcal{M}, g)$ be a space-time. Any map $f : \mathcal{G} \to \mathcal{G}$ involving a single scalar field ϕ, and no more than first order derivatives of ϕ, such that $(\mathcal{M}, f(g))$ is still a space-time is a well-defined frame transformation.

Although $(\mathcal{F}, \circ)$ is not a group, and so no action can claim it as a symmetry group, we can still ask whether any well known actions are invariants of certain elements of $(\mathcal{F}, \circ)$. See [92–95] for some interesting work in this direction.

Scalar tensor theories

Let $\mathcal{S} : \mathcal{G} \to \mathbb{R}$ be some action. If we compose the action with a frame transformation $f : \mathcal{G} \to \mathcal{G}$, the result is another action: $f \circ \mathcal{S}$. The actions

$\mathcal{S}$ and $f \circ \mathcal{S}$ are not the same: they are not equal as maps, unless f is the identity. I would still like to say they are equivalent, in some looser sense, but this will prove difficult to do.

Result 2.3.7. *Let $\mathcal{S}$, $\mathcal{S}'$ be two actions, and f_C a conformal transformation. Let $\sim$ be the relation: $\mathcal{S} \sim \mathcal{S}' \;\Rightarrow\; \mathcal{S}' = f_C \circ \mathcal{S}$ for some f_C. Then $\sim$ is a valid equivalence relation.*

Proof. This follows directly from the fact that $\mathcal{C}$ is a group: reflexivity follows because $\mathcal{C}$ contains an identity; symmetry follows because each $f_C \in \mathcal{C}$ has an inverse; transitivity follows because $\mathcal{C}$ is closed under function composition. [115] ∎

If $\mathcal{S}$ is intended to describe some physical system, like our Universe, then it is clear that the information content in $\mathcal{S}$ is not lost under a conformal transformation, $f_C \circ \mathcal{S}$, it is simply redressed. There is a sense that both $\mathcal{S}$ and $f_C \circ \mathcal{S}$ are *of the same theory*, though it is not clear in what sense this is meant.

The problem here is clear: $\mathcal{F}$ is not a group where $\mathcal{C}$ is, and so the equivalence relation cannot be nicely generalised. This was my original intention, but it did not pan out; the condition, I found, that a general frame transformation be well defined conflicts with the condition that it be invertible in some cases, and finding a subset S such that $\mathcal{C} \subset S \subset \mathcal{F}$ and $(S, \circ)$ is a group proved much harder than I had hoped.

The lack of a suitable group of general frame transformations spoils our chance to generalise this equivalence relation, but in a more limited way we can still talk about actions being equivalent, if related by a well defined invertible frame transformation f. *From here on, when referring to a frame transformation, I will always implicitly assume it is well defined.* The next result determines the condition that f meet for it to be invertible.

Result 2.3.8. *Let $f \in \mathcal{F}$ be a general frame transformation. If the disformal scalar $Z(\phi, X)$ of f satisfies:*

$$0 < Z^2 < 1 + \frac{1}{C} \quad \forall p \in \mathcal{M} \tag{2.58}$$

for $C(\phi, X) > 0$ the conformal scalar of f, then f is invertible.

Proof. A general frame transformation can be written:

$$f(g) = Cg + C \left(1 - Z^2\right) \frac{d\phi \otimes d\phi}{2X}$$

51

for some smooth functions $C(\phi, X), Z(\phi, X) > 0$. Then, its inverse is:

$$
\begin{aligned}
f^{-1}(g) &= C^{-1}g - (1 - Z^2)\frac{d\phi \otimes d\phi}{2X} \\
&:= C'g + C'(1 - Z'^2)\frac{d\phi \otimes d\phi}{2X}.
\end{aligned}
$$

For f^{-1} to be well defined, we require $C' > 0 \quad \Rightarrow \quad C > 0$ which is not new information, and $(1 - Z'^2) < 1 \quad \Rightarrow \quad -C(1 - Z^2) < 1 \quad \Rightarrow \quad Z^2 < 1 + \frac{1}{C}$. Recalling that we already require $Z^2 > 0$ for f, we have the result. $\blacksquare$

I will sometimes call such a map a **frame isomorphism**, and refer to two actions related by such a map as frame-isomorphic.

An interesting choice would be to define a theory as an equivalence class of actions defined by an equivalence relation of frame transformations: $[\mathcal{S}] = [f \circ \mathcal{S}]$. Though perhaps not a useful definition in practice, it would have allowed me to refer to something that exists independent of frame; a theory could be represented by many frame-related actions[4]. While falling short of my original aim, I can still provide a working definition of a theory. If a frame transformation f is well defined and invertible (and its inverse is well defined), then we can say that $f \circ \mathcal{S}$ and $\mathcal{S}$ are two (unequal but equivalent) actions describing the same **theory**. When referring to a theory in future pages, I will mean it in this sense.

The next example highlights two very important actions of a given theory.

Example 2.7. Consider again the Brans-Dicke action from the previous section:

$$
\mathcal{S}[g, \phi] := \int_{\mathcal{M}} d\pi(g) \left[\frac{\phi R(g)}{2\kappa_*} - \omega \frac{g^{-1}(d\phi, d\phi)}{\phi} + \mathcal{L}_{\mathrm{M}}(g) \right] \tag{2.59}
$$

where we recall that $\mathcal{L}_{\mathrm{M}}(g)$ is an arbitrary Lagrangian for matter fields, perhaps the standard model, and I have rescaled ϕ by some constant value κ_* that will carry the dimensions of the gravitational constant (but of course will not be said constant, unless $\phi = 1$).

---

[4]A far better approach would be to write actions that are manifestly frame invariant, as is already the case for diffeomorphisms, but, as diffeomorphism symmetry is mathematically well understood, whereas frame symmetry is not, I will not attempt this here.

Chose an f such that $f(g) = \phi g$. Then the action $f \circ \mathcal{S}$ becomes (not proven here, see Sec. 10.1 of [103]):

$$
\begin{aligned}
f \circ \mathcal{S}[g, \phi] &= \mathcal{S}[f(g), \phi] \\
&= \int_{\mathcal{M}} d\pi(f(g)) \left[\frac{\phi R(f(g))}{2\kappa_*} - \omega \frac{(f(g))^{-1}(d\phi, d\phi)}{\phi} + \mathcal{L}_{\mathrm{M}}(f(g)) \right] \\
&= \int_{\mathcal{M}} d\pi(g) \left[\frac{R(g)}{2\kappa_*} + \mathcal{L}_{\Phi}(g, \phi) \right] + \int_{\mathcal{M}} d\pi(\phi g) \mathcal{L}_{\mathrm{M}}(\phi g)
\end{aligned}
$$

for some $\mathcal{L}_{\Phi}(g, \phi, d\phi)$ whose form is not important. The new action $f \circ \mathcal{S}$ has the attractive property that the gravitational part (the R term) looks like it does in the general relativity action. This choice simplifies the Einstein equations (which I will not show, but see e.g. Sec. 10.1 of [103]), and for this reason, this action is used often in calculations. ▲

Generally, for some action $\mathcal{S}$, if there exists a frame transformation f such that the R term in $f \circ \mathcal{S}$ is the same as it is in the general relativity action, then I refer to $f \circ \mathcal{S}$ as the **Einstein frame action** of the theory. I then refer to $\mathcal{S}$—the action in which the matter Lagrangian term appears as it does in the general relativity action—as the **Jordan frame action**. If a theory has an Einstein frame action, then I refer to the theory as a **scalar tensor theory**[5]

Induced frame maps

We now come to the important notion of induced frame maps. A frame transformation f carries metric tensors onto other metric tensors. This is the greatest change we can make to an action, and will induce many changes in the variables that depend on the metric for a definition. For a simple example, the scalar field kinetic energy:

$$
X(f(g)) = -\frac{1}{2}(f(g))^{-1}(d\phi, d\phi) \neq X(g). \tag{2.60}
$$

---

[5]Again I remind the reader that this language is not quite standard and the scope of these definitions is just the pages of this thesis, in particular it is not always assumed a scalar tensor theory has an Einstein frame action. One could however read much of the literature and not notice the discrepancy.

There are other more subtle examples. The 4-velocity of a free falling observer:

$$u(f(g)) = \frac{d}{d\tau(f(g))} = \frac{d\tau(g)}{d\tau(f(g))} u(g), \tag{2.61}$$

and there are other examples besides. In general I will call a map between variables defined by the frame transformation f an **induced frame map**, f_*. In the four velocity example above, this corresponds to:

$$f_*(u(g)) = u(f(g)) = \frac{d\tau(g)}{d\tau(f(g))} u(g). \tag{2.62}$$

I will collect a list of induced frame maps between Jordan and Einstein frame quantities of theories of particular interest in the appendix. I will often use the concept of an induced frame map implicitly, and not refer to some f_* directly in the text.

In general, I will use the phrase **X is a Y frame quantity** to describe the derived quantities defined by various frame actions. X will come to stand for 4-velocity, momentum, stress tensor, e.t.c. Y will come to stand for Jordan, Einstein, radiation, matter, e.t.c. In this language, 'frame' becomes an empty term. It is now descriptive, a mathematical adjective: there is no mathematical object called a frame. It thus makes no sense to refer specifically to a frame.

A final note to make is that, in the literature, disformal transformations and frame transformations grew out of conformal ones, as extensions. Indeed frame transformations were often considered interesting because conformal ones are; they are interesting in their opposition. It is for this reason—and, as we have seen, conformal transformations form a group with all the benefits that follow—that many conformal notions, like that of frame and physical equivalence, were assumed to carry over to the disformal case; they do not. These past four years have been a painful process in uncovering those aspects of the conformal literature where conformal intuition does not carry over, as we had too often hoped it would.

I will close this chapter with the example of beyond-Horndeski.

Example 2.8 (beyond-Horndeski)**.** Consider some scalar tensor theory whose Einstein frame action $\mathcal{S}$ is given by:

$$\mathcal{S}[g, \phi] := \int_{\mathcal{M}} d\pi(g) \left[\frac{R(g)}{2\kappa_*} + \mathcal{L}_\Phi(g, \phi) \right] + \int_{\mathcal{M}} d\pi(f(g)) \mathcal{L}_{\mathrm{M}}(f(g))$$

where f is some general *invertible* frame transformation:

$$f(g) = C(\phi, X)g + C(\phi, X)(1 - Z(\phi, X)^2)\frac{d\phi \otimes d\phi}{2X}. \qquad (2.63)$$

The Euler Lagrange equations are obtained by taking first order space-time derivatives of the Lagrangian of $\mathcal{S}$, which itself contains only up to first order derivatives of the field ϕ—we thus expect the theory to be stable in the sense of Ostrogradsky, and indeed it will be.

But f is assumed to be invertible. And it is not hard to find invertible frame transformations (it is hard to find a closed group of them). It has been shown that the Jordan frame action of the theory, $f^{-1} \circ \mathcal{S}$, is not any special case of the Horndeski action in general, and the Euler-Lagrange equations of $f^{-1} \circ \mathcal{S}$ contain third and fourth order derivatives of the metric and scalar field. We can see this because the Ricci scalar R, as a operator, involves second order derivatives, and so the Ricci term $R(f^{-1}(g))$ will contain third order space-time derivatives of ϕ. In the Jordan frame representation it seems the theory is Ostrogradsky *un*-stable.

This apparent paradox was resolved by Zumalacárrequi and García-Bellido [80], who have shown that hidden constraints in the theory imposed by the condition that f be invertible can be used to solve away higher derivatives of ϕ in the Euler-Lagrange equations of $f^{-1} \circ \mathcal{S}$, producing a viable Ostrogadsky-stable dynamical system that is not part of the Horndeski set.   ▲

Thus, frame transformations have been used to expand the (already unmanageable) set of permissible theories involving scalar fields on curved spaces that aim to solve the dark energy problem. In the next chapter we will see how frame transformations provide yet another way to expand this set.

Couplings to radiation

<div style="text-align: right; font-size: 3em;">3</div>

With this chapter I begin to describe the work that I have published with Carsten and other collaborators. The chapter divides neatly down the middle. The first half, Sec. 3.1, covers my very first project with Carsten, and his then PhD student Susan Vu, and the second, 3.2, describes the collaboration of Carsten, Clare Burrage, and myself in the third and final years of my time at Sheffield. They are presented together here as they share an electromagnetic theme but, as they effectively bookend my course, they represent quite different attitudes toward what are in fact very similar theories. When beginning the first, we knew and understood very little about disformal transformations in scalar tensor theories, and our rocky start is evident from the slightly more erratic nature of the first section, however by the third project we had found the light switch, and the work was much more directed; indeed we knew what we were looking for before we found it: vacuum Cherenkov radiation induced by frame transformations.

3.1 Cosmic microwave background distortions

This project began as an attempt to combine in a theory the cosmic microwave background with disformal transformations. However, the disformal relation did not mix well with the rest of the theory, and many aspects of conformal transformations, we found, did not carry over to the more general case. The big problem was: an invertible frame transformation must preserve

the physical content of a theory, but it is not immediately clear how, and, for the Einstein frame stress tensor, it was not clear what definitions could be made, and what had to be derived to preserve the physical content. A publication of these results can be found at [105].

The investigation into the existence of (at least) two frame actions of a disformal theory lead us to two startling conclusions, which I will describe using a cosmological toy model of a single fluid in a homogeneous isotropic universe disformally coupled to a scalar field. I describe next how we developed a more general model that we then placed in a more realistic cosmology setting, and extracted real observables that could be compared to current cosmological data. These comparisons allowed us to place constraints on our general model, which I will also discuss. In this section, Greek indices I chose to run from 0 to 3 (over all space-time coordinates) and Latin indices, 1 to 3 (just the spatial ones).

Consider a scalar tensor theory on a 4-dimensional smooth manifold $\mathcal{M}$. By the definition of chapter 2 the scalar tensor theory has an Einstein frame action, $\mathcal{S}$, which, from the definition of an Einstein frame, we can always write as:

$$\mathcal{S}[g, \phi] \quad = \quad \int_{\mathcal{M}} d\pi(g) \left[\frac{R(g)}{2\kappa} + \mathcal{L}_{\Phi}(\phi, X(g)) \right] \qquad (3.1)$$

$$+ \int_{\mathcal{M}} d\pi(f(g)) \mathcal{L}_{\mathrm{M}}(f(g)) \qquad (3.2)$$

for some frame transformation $f \in \mathcal{F}$. If f is invertible, the Jordan frame action exists, and can be written $f^{-1} \circ \mathcal{S}$. The metric tensor g that minimises $\mathcal{S}$, (that is, the metric solution of the Einstein equations of $\mathcal{S}$ subject to some boundary conditions) I will call the **Einstein frame metric**. The metric tensor $f^{-1}(g) := \tilde{g}$ that minimises $f^{-1} \circ \mathcal{S}$ I will call the **Jordan frame metric**.

The Jordan frame and Einstein frame actions give us frame-isomorphic descriptions of a scalar tensor theory on the pseudo-Riemannian manifolds $(\mathcal{M}, \tilde{g})$ and $(\mathcal{M}, g)$ respectively. As f is invertible, we obviously have that the physical content of the theory is contained completely in both descriptions.

However, it is not straightforward to see this in practice. Indeed f is just a change of variables, but it is a *huge* change; it is the greatest reversible change we could make to an action. In this section we search for variables at the lower levels of thermodynamics and the kinetic theory of gasses of some theory defined on both $(\mathcal{M}, \tilde{g})$ and $(\mathcal{M}, g)$ that are invariant under the action

of a frame isomorphism, and so in a sense physically meaningful. The Jordan and Einstein frame actions are special, in that they are diametrically opposite in the following sense. The Jordan frame curvature tensor is a function of ϕ while the Jordan frame stress tensor is not. On the other hand, the Einstein frame stress tensor is a function of the scalar, whereas the Einstein frame curvature tensor is not is not. To address physical equivalence, I look now only at these two frames. *In this section, I will set everything I can to one, i.e.* $c = \kappa = k_b = \hbar = 1$.

3.1.1 Frame transformations & physical equivalence

In this section, I will explain the main theoretical results we uncovered concerning the dynamics of perfect fluids in a disformal cosmology setting. To keep things simple, the results are derived in the context of a toy model, with only a single perfect fluid in a flat universe—a space-time with coordinates given by the FL line element such that $K = 0$—disformally coupled to a scalar field. The next section will use these results heavily when I present our cosmological model involving 2 species: matter and radiation, both disformally coupled, but with different strengths to the scalar.

I begin in section 3.1.1 with this toy model expressed in terms of the theory's Einstein Frame action, and derive the equations of motion for the fields and matter fluid. Then, in section 3.1.1, using an induced frame map from Jordan Frame variables (such as pressure and energy density e.t.c) to Einstein frame ones, I show how we re-derived the conservation equation for the Einstein frame matter fluid using simple arguments about Jordan frame matter conservation. I'll then present two consistent kinetic theory descriptions of the same fluid in each frame, again using the maps, and show how we reproduced the conservation equation for the Einstein frame fluid using the kinetic approach. The point of all this was originally to see if both frames were in fact physically equivalent, but it turned out to have important unseen consequences, as we will see. The main textbook references for this section are *Modern cosmology* [109] and *The cosmic microwave background* [116].

Action

This toy model is of a single perfect fluid in a disformal cosmology. For simplicity we chose the scalar field to be quintessent in the Einstein frame action, $\mathcal{S}$, which can thus be written:

$$\mathcal{S} := \mathcal{S}_H + \mathcal{S}_\Phi + \mathcal{S}_M, \tag{3.3a}$$

where:

$$\mathcal{S}_H := \int d\pi(g)\frac{R(g)}{2}, \quad \mathcal{S}_\Phi := \int d\pi(g)\left[X(g) - V(\phi)\right] \tag{3.3b}$$

$$\mathcal{S}_M := \int d\pi(fg)\mathcal{L}_{\mathrm{M}}(fg), \tag{3.3c}$$

and f is the frame transformation:

$$fg := C(\phi)g + D(\phi)d\phi \otimes d\phi, \tag{3.3d}$$

and we neglect that C and D can depend on the field's kinetic energy for simplicity. Then, the Jordan frame action is $\tilde{\mathcal{S}} := f^{-1} \circ \mathcal{S}$.

The Einstein equations of $\mathcal{S}$ are:

$$G(g) = T_\Phi(g, \phi, d\phi) + T(g, \phi, d\phi) \tag{3.4a}$$

where:

$$T_\Phi \quad := \quad \frac{2}{\sqrt{-|g|}}\frac{\delta \mathcal{S}_\Phi}{\delta g} \tag{3.4b}$$

$$T \quad := \quad \frac{2}{\sqrt{-|g|}}\frac{\delta \mathcal{S}_M}{\delta fg}\left(\frac{\partial fg}{\partial g}\right)_{\phi, d\phi}, \tag{3.4c}$$

and G is the standard Einstein tensor of the Einstein frame metric, g:

$$G = -\frac{2}{\sqrt{-|g|}}\frac{\delta \mathcal{S}_H}{\delta g}. \tag{3.4d}$$

The Einstein equations of $\tilde{\mathcal{S}}$ are:

$$\tilde{G}(\tilde{g}, \phi, d\phi) = \kappa\tilde{T}_\Phi(\tilde{g}, \phi, d\phi) + \kappa\tilde{T}(\tilde{g}) \tag{3.5a}$$

where:

$$\tilde{T}_\Phi \ := \ \frac{2}{\sqrt{-|\tilde{g}|}}\frac{\delta\tilde{\mathcal{S}}_\Phi}{\delta\tilde{g}} \tag{3.5b}$$

$$\tilde{T} \ := \ \frac{2}{\sqrt{-|\tilde{g}|}}\frac{\delta\tilde{\mathcal{S}}_M}{\delta\tilde{g}}, \tag{3.5c}$$

and:

$$\tilde{G} := -\frac{2}{\sqrt{-|\tilde{g}|}}\frac{\delta\tilde{\mathcal{S}}_H}{\delta f^{-1}\tilde{g}}\left(\frac{\partial f^{-1}\tilde{g}}{\partial\tilde{g}}\right)_{\phi,d\phi}, \tag{3.5d}$$

were $\tilde{g} = fg$ is the Jordan frame metric. I stress that $\tilde{G}$ is *not* in general the Einstein tensor of the Jordan frame metric, but it does play an analogous role in the modified Einstein equations, as can be seen from (3.5a).

The Einstein frame and Jordan frame stress tensors for matter relate as [96]:

$$T(g,\phi,d\phi) = \sqrt{\frac{|fg|}{|g|}}\tilde{T}(fg)\left(\frac{\partial fg}{\partial g}\right)_{\phi,d\phi} \tag{3.6}$$

and the G tensors as:

$$\tilde{G}(\tilde{g},\phi,d\phi) = \sqrt{\frac{|f^{-1}\tilde{g}|}{|\tilde{g}|}}G(f^{-1}\tilde{g})\left(\frac{\partial f^{-1}\tilde{g}}{\partial\tilde{g}}\right)_{\phi,d\phi}. \tag{3.7}$$

We will not have much use for $\tilde{G}$. One can perhaps infer from its definition that it will be very complicated and whenever work with the gravitational sector of a theory is to be done we work with the Einstein frame action exclusively. We do however care about the Jordan frame stress tensor, and for the frame transformation specified above the two stress tensors T and $\tilde{T}$ relate as:

$$T = C^3 Z\tilde{T} \tag{3.8}$$

where I recall Z is the disformal scalar:

$$Z = \sqrt{1 - 2X\frac{D}{C}} \tag{3.9}$$

of the previous chapter.

The Klein-Gordon equation of $\mathcal{S}$ is:

$$
\begin{aligned}
\frac{1}{\sqrt{-|g|}} \frac{\delta \mathcal{S}_\Phi}{\delta \phi} &= -\frac{T}{2}\left(\frac{\partial f g}{\partial g}\right)^{-1}_{\phi,d\phi}\left(\frac{\partial f g}{\partial \phi}\right)_{g,d\phi} \\
&\quad + \nabla\left(\frac{T}{2}\left(\frac{\partial f g}{\partial g}\right)^{-1}_{\phi,d\phi}\left(\frac{\partial f g}{\partial (d\phi)}\right)_{g,\phi}\right) \\
&:= Q
\end{aligned}
\tag{3.10}
$$

where, for the above frame transformation, we have in index notation [117]:

$$
Q = \frac{C'}{2C}T^{\mu\nu}g_{\mu\nu} + \frac{D'}{2C}\phi_{,\mu}\,\phi_{,\nu}\,T^{\mu\nu} - \nabla_\mu\left(\frac{D}{C}\phi_{,\nu}\,T^{\mu\nu}\right),
\tag{3.11}
$$

and primes here denote derivatives with respect to ϕ. For our quintessent field this simplifies to:

$$
\Box\phi - V' + Q = 0.
\tag{3.12}
$$

We propose that the matter considered here is a perfect fluid, and assume that, for the Einstein frame T, the standard definition can be made:

$$
T = (\rho + P)u \otimes u + Pg^{-1}
\tag{3.13}
$$

where each element of the fluid is assumed at rest in the cosmic rest frame, that is: u is also the 4-velocity field of all comoving observers. We obtain an equation of motion for this fluid by virtue of the Bianchi identities:

$$
\begin{aligned}
\nabla T &= -\nabla T_\Phi \\
&= -Qg^{-1}d\phi.
\end{aligned}
\tag{3.14}
$$

As I have stated at the beginning of this section, the space-time we consider is a flat universe, and so we set the Einstein frame metric g to FL form, though this is by no means a unique choice. The Einstein frame line element is then:

$$
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2\delta_{ij}dx^i dx^j.
\tag{3.15}
$$

u we can now compute properly as:

$$
u = \frac{d}{d\tau} = \frac{dx^\mu}{dt}\partial_\mu = (1,\ 0,\ 0,\ 0).
\tag{3.16}
$$

How the pressure relates to the energy density (the equation of state) will depend on the type of matter we are describing, so to remain general I leave it open:

$$P = w\rho. \tag{3.17}$$

For a flat universe to *remain* flat, the matter fields in a universe will also have to obey the cosmological principle. This means, in the above coordinates, the scalar field ϕ can have no spatial gradients, $\phi_{,i} = 0$, which implies that $\tilde{g} = fg$ can have none also, and thus the Jordan frame space-time $(\mathcal{M}, \tilde{g})$ is *also* a universe.

Using the ingredients listed above, we can derive the equation of motion for our perfect fluid:

$$\dot{\rho} + 3H(\rho + P) = -Q\dot{\phi} \tag{3.18}$$

and a simplified Klein-Gordon equation for the scalar:

$$\ddot{\phi} + 3H\dot{\phi} + V' = Q \tag{3.19}$$

with the usual Hubble parameter $H = \dot{a}/a$, and dots cosmic time derivatives.

To glean information about our disformal theory, we now look at some simplified special cases. As this work focuses on disformal effects, we can isolate the disformal term in Eq. (3.3d), setting $C(\phi) = 1$. We will consider in chapter 4 a non-trivial conformal function and explore its effects in comparison with a purely disformal case, but this simple ansatz is sufficient for a preliminary study. For the same reason, we treat D as a constant energy scale, $D(\phi) = M^{-4}$. As long as the derivatives of ϕ are kept small—as they must be for dark energy and cosmology models—then variation of D with ϕ will produce effects subdominant to the disformal interaction itself.

The *disformal* line element then becomes:

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = -Z^2 dt^2 + a^2 \delta_{ij} dx^i dx^j, \tag{3.20}$$

where I have used the definition of the disformal scalar to get:

$$Z = \left(1 - \frac{\dot{\phi}^2}{M^4}\right)^{\frac{1}{2}}. \tag{3.21}$$

The condition that f be well defined, $Z > 0$, is here manifest as the condition that $\mathcal{O}(1,3)$ symmetry, or causality, be preserved under f.

This line element has great significance. Matter in our theory lives in a disformal geometry, so matter particles and elements of fluids will follow geodesics given by this disformal line element. This is the crux of this theory; all the results I will describe follow from this. Q now greatly simplifies to:

$$Q = \frac{3H\dot{\phi}(\rho + P) + V'(\phi)\rho}{M^4 + \rho - \dot{\phi}^2}. \tag{3.22}$$

None of the above is new to this project. In fact these equations were derived well before in [117], but I introduce them here as they are vital for a discussion of what follows. Now, as a last note, for what follows it is useful to re-arrange (3.18) into a more suggestive form:

$$\dot{\rho} + 3H(\rho + \frac{P}{Z^2}) = -\frac{\dot{Z}}{Z}\rho. \tag{3.23}$$

Thermodynamics

We now look at the Jordan frame action $\tilde{\mathcal{S}}$ in some detail. As a frame transformation of this sort introduces no torsion, viscosity or any other effects that null a perfect fluid description, we assume we can express this stress energy tensor, though still ignorant of the exact form of $\mathcal{S}_M$, as a perfect fluid:

$$\tilde{T} = (\tilde{\rho} + \tilde{P})\tilde{u} \otimes \tilde{u} + \tilde{P}\tilde{g}^{-1}, \tag{3.24}$$

where:

$$\tilde{u} = \frac{d}{d\tilde{\tau}} = \frac{d\tau}{d\tilde{\tau}}u = (Z^{-1},\ 0,\ 0,\ 0)\ . \tag{3.25}$$

It is not particularly useful to know what the full form of the Jordan frame curvature tensor $\tilde{G}$ actually is for our purposes. It will no doubt be long and very complicated; we can do perfectly well without it. The use of considering this frame action lies in the virtue that the matter sector of this theory looks standard: it is uncoupled from the scalar, ϕ. This means $\tilde{T}$ is conserved with respect to the disformal geometry. Defining the disformal covariant derivative, $\tilde{\nabla}$, that is metric compatible with $\tilde{g}$, we can turn this idea of conservation into a precise mathematical statement [118]:

$$\tilde{\nabla}\tilde{T} = 0. \tag{3.26}$$

This is, in fact, a very powerful statement. We can use it to then derive the Jordan frame continuity equation in cosmology. Equation (3.26) becomes:

$$\dot{\rho} + 3H(\tilde{\rho} + \tilde{P}) = 0. \tag{3.27}$$

Recalling the relation between the frame tensors, which for $C = 1$ becomes:

$$T = Z\,\tilde{T}, \tag{3.28}$$

we can now derive the relation between Jordan and Einstein frame pressure (the induced frame maps):

$$\rho = \frac{\tilde{\rho}}{Z}, \quad P = Z\tilde{P}, \tag{3.29}$$

which when substituted back into equation (3.27), recovers (3.23). Thus everything is consistent. Thermodynamics in the Jordan frame can be mapped to the Einstein frame via (3.28). But there is a caveat. Note that (3.29) implies the equation of state of the fluid is *not frame invariant*; in fact:

$$\boxed{\frac{P}{Z^2\rho} = \frac{\tilde{P}}{\tilde{\rho}}.} \tag{3.30}$$

This result, at the time of publication [105], was completely new. If constructing a disformal model using perfect fluid definitions for the Einstein frame stress tensor, one must always bear in mind that the equation of state will be given by (3.30). Now as the Jordan frame matter sector is uncoupled to the scalar field, $\tilde{P}/\tilde{\rho}$ should not contain ϕ derivatives. It is then the Jordan frame equation of state that will be standard: if the fluid is of radiation, $\tilde{P} = \tilde{\rho}/3$.

Note here that if I momentarily reinstate the speed of light constant c, We see that the equation of state result becomes:

$$\frac{P}{c^2 Z^2 \rho} = \frac{\tilde{P}}{c^2 \tilde{\rho}}. \tag{3.31}$$

We can see too from the above disformal line element that massless particles moving along null geodesics of $\tilde{g}$ will too move with velocity

$$c_s := cZ. \tag{3.32}$$

This final observation suggests to us that it is helpful to think of the Einstein frame action describing a theory in which the speed of light varies. We should be careful with this language; the theory exists independently of frame, and, certainly, the speed of massless particles when defined with respect to the Jordan frame space-time $(\mathcal{M}, fg)$ is constant. Too, in the derivation of Eq. (3.30) we did not *prove* that the speed of light was c_s, we found it might be sensible to work in units where this was true. What does not change under a unit re-definition is the dimensionless quantity w, which is not however frame invariant.

And these results tell us something else: we must be careful, when working with Einstein frame actions, with naïve matter physics definitions. We were able to impose the prefect fluid definition on both T and $\tilde{T}$ above, but the caveat was the equation of state of T had to be derived from considerations of physics in the Jordan frame action. This is key. Physical equivalence of two frame-isomorphic actions depends on the correct definitions being used. To maintain frame equivalence at the lower level of thermodynamics some choices, e.g. the form of w, are forced and not free. The Jordan frame action, we have seen, throws up formidably complex Euler Lagrange equations and, as in the case of coordinate freedom, a sensible choice of frame can significantly reduce the difficulty of the problem. Just as polar coordinates simplify the equations of circles, the Einstein frame action can greatly simplify the differential equations that describe a given scalar tensor theory.

But—and this is perhaps the crux of the chapter—when working with the Einstein frame action, that trouble one might think they have avoided turns up in other places. In the above, we had to work quite hard to find the correct form of the Einstein frame stress tensor, and in the next section we will work harder to find the correct Einstein frame Boltzmann equation. When we remember that, for example, the cosmic microwave background is modelled today by very detailed Boltzmann equation processes, and we realise that every aspect of those models will have to be checked and modified for an Einstein frame description, suddenly all that work saved by simplifying the Euler-Lagrange equations seems much less worthwhile. It is a message I will need to repeat often.

Kinetic theory

In Sec. 3.1.1 we found that the equation of state for a disformally coupled fluid was frame dependent. We can continue to play this game, and ask what else is frame dependent? Sec. 3.1.1 was concerned with the macroscopic behaviour of the fluid. What about the microscopic? Are there frame discrepancies in the kinetic descriptions of a perfect fluid also? In short: yes. To see why, we need to build a statistical picture of the particle constituents of the perfect fluid considered in the last section. The behaviour of a statistical system is encoded in the dynamics of its distribution function, $\mathtt{f}$, so to see how the kinetic theory of a perfect fluid changes under a frame transformation, we need to see how its distribution function evolves in each frame: we want to look at the Boltzmann equation of the fluid in each frame [109, 116].

From a microscopic perspective, the well known definitions of the stress tensor of a perfect fluid can be written as integrals over phase space [109, 116]. *We assume the standard definition can be applied to the Einstein and Jordan frame stress tensors simultaneously.* Then:

$$T := \int d\omega \mathtt{f} \, p \otimes p \tag{3.33}$$

$$\tilde{T} := \int d\tilde{\omega} \mathtt{f} \, \tilde{p} \otimes \tilde{p} \tag{3.34}$$

for p, $\tilde{p}$ the momenta coordinates of some Einstein and Jordan frame phase spaces respectively, and $d\omega$, $d\tilde{\omega}$ volume measures of the same respective spaces. We find the two definitions above determine, to some degree, the underlying phase spaces. The relation $T = Z\tilde{T}$ requires that:

$$\tilde{p} = \alpha p \tag{3.35}$$

for all momenta coordinates, and some α to be found, which follows when we compare components of the relation above, i.e. $T^{\mu\nu} = Z\tilde{T}^{\mu\nu}$ for every μ and ν, and recall that any distribution function $\mathtt{f}$ is a non-negative function. For massive comoving particles, we have already found that:

$$u = Z\tilde{u} \tag{3.36}$$

and so for these same massive particles:

$$p = mu = mZ\tilde{u} = Z\tilde{p} \tag{3.37}$$

where we have chosen to remain in units where the mass of particles is constant. Z is a scalar quantity, and so not only must the above relation be true in any coordinate system, but also for massive and massless particles alike.

I now derive the Boltzmann equation for $\mathtt{f}$ and $\tilde{\mathtt{f}}$ defined by their stress tensors above.

Combing Eq.s (3.37) with (3.33) and (3.34), and noting that

$$d\tilde{\omega}\tilde{\mathtt{f}} = d\tilde{p}_1 d\tilde{p}_2 d\tilde{p}_3 \tilde{\mathtt{f}}, \quad d\omega\mathtt{f} = dp_1 dp_2 dp_3 \mathtt{f} \tag{3.38}$$

for each $\tilde{p}_\mu := \tilde{g}_{\mu\nu}\tilde{p}^\nu$, we can derive a relation between the distribution functions as:

$$d\tilde{p}_1 d\tilde{p}_2 d\tilde{p}_3 \tilde{\mathtt{f}} = dp_1 dp_2 dp_3 \mathtt{f} Z. \tag{3.39}$$

Total particle number can be computed directly from integrals over the phase space. In each frame representation:

$$\tilde{N} := \int d\tilde{\omega} d\tilde{\pi}\tilde{\mathtt{f}}, \quad N := \int d\omega d\pi \mathtt{f}. \tag{3.40}$$

As, in the Jordan frame case, the matter sector is uncoupled from the scalar field there is no particle creation and so $d\tilde{\mathtt{f}}/dt = 0$. Then:

$$
\begin{aligned}
\frac{d\tilde{N}(t + \Delta t) - d\tilde{N}(t)}{\Delta t} &= d^3 x d^3 \tilde{p} \left[\frac{\tilde{\mathtt{f}}(t + \Delta t) - \tilde{\mathtt{f}}(t)}{\Delta t} \right] \\
&= d^3 x d^3 p \left[\frac{(Z\mathtt{f})(t + \Delta t) - (Z\mathtt{f})(t)}{\Delta t} \right] \\
&= 0.
\end{aligned}
$$

Letting $\Delta t \to dt$, the time evolution of the Einstein frame distribution function becomes:

$$\boxed{\frac{d}{dt}\left(Z\mathtt{f} \right) = 0.} \tag{3.41}$$

The Einstein frame Boltzmann equation can then be derived from equation (3.41). Let λ be an affine parameter such that $p^\mu = dx^\mu/d\lambda$. Generically $\mathtt{f} = \mathtt{f}(p^\mu, x^\mu)$ for an arbitrary statistical system, so:

$$\frac{d\mathtt{f}}{d\lambda} = \frac{\partial \mathtt{f}}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda} + \frac{\partial \mathtt{f}}{\partial p^\alpha}\frac{dp^\alpha}{d\lambda} . \tag{3.42}$$

In this theory, the particle constituents of the disformally coupled fluid travel along disformal geodesics, so we can write down the Jordan frame geodesic equation:

$$\frac{d\tilde{p}^{\alpha}}{d\tilde{\lambda}} + \tilde{\Gamma}^{\alpha}_{\beta\gamma}\tilde{p}^{\beta}\tilde{p}^{\gamma} = 0 \tag{3.43}$$

where $\tilde{\lambda}$ is a new affine parameter such that $\tilde{p}^{\mu} = dx^{\mu}/d\tilde{\lambda}$, and we cannot here assume that $d\lambda/d\tilde{\lambda}$ is constant; in general we will see that it can depend on ϕ. We can use Eq. (3.43) to compute the Einstein frame geodesic equation:

$$
\begin{aligned}
0 &= \frac{d\tilde{p}^{\alpha}}{d\tilde{\lambda}} + \tilde{\Gamma}^{\alpha}_{\beta\gamma}\tilde{p}^{\beta}\tilde{p}^{\gamma} \\
&= \frac{d\tilde{p}^{\alpha}}{d\lambda}\frac{d\lambda}{d\tilde{\lambda}} + \tilde{\Gamma}^{\alpha}_{\beta\gamma}p^{\beta}p^{\gamma}\left(\frac{d\lambda}{d\tilde{\lambda}}\right)^{2} \\
&= \frac{d}{d\lambda}\left(p^{\alpha}\frac{d\lambda}{d\tilde{\lambda}}\right)\frac{d\lambda}{d\tilde{\lambda}} + \tilde{\Gamma}^{\alpha}_{\beta\gamma}p^{\beta}p^{\gamma}\left(\frac{d\lambda}{d\tilde{\lambda}}\right)^{2} \\
&= \frac{dp^{\alpha}}{d\lambda}\left(\frac{d\lambda}{d\tilde{\lambda}}\right)^{2} + p^{\alpha}\frac{d^{2}\lambda}{d\tilde{\lambda}^{2}} + \tilde{\Gamma}^{\alpha}_{\beta\gamma}p^{\beta}p^{\gamma}\left(\frac{d\lambda}{d\tilde{\lambda}}\right)^{2} \\
\rightarrow \frac{dp^{\alpha}}{d\lambda} &= -\tilde{\Gamma}^{\alpha}_{\beta\gamma}p^{\beta}p^{\gamma} - p^{\alpha}\frac{d^{2}\lambda}{d\tilde{\lambda}^{2}}\left(\frac{d\tilde{\lambda}}{d\lambda}\right)^{2}.
\end{aligned}
$$

Where I have used the disformal Christoffel symbols, $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ [96]. Finally, in terms of a Liouville operator:

$$
\begin{aligned}
\hat{\mathcal{L}}\mathtt{f} &\equiv \frac{d\mathtt{f}}{dt} \\
&= p^{\alpha}\frac{\partial\mathtt{f}}{\partial x^{\alpha}} - \left\{\tilde{\Gamma}^{\alpha}_{\beta\gamma}p^{\beta}p^{\gamma} + p^{\alpha}\frac{d^{2}\lambda}{d\tilde{\lambda}^{2}}\left(\frac{d\tilde{\lambda}}{d\lambda}\right)^{2}\right\}\frac{\partial\mathtt{f}}{\partial p^{\alpha}} \\
&= -\frac{d}{dt}\left(\ln\frac{d\tilde{\lambda}}{d\lambda}\right)\mathtt{f}. \tag{3.44}
\end{aligned}
$$

From the definition of λ we get that:

$$\frac{d\tilde{\lambda}}{d\lambda} = \alpha^{-1} = Z. \tag{3.45}$$

In cosmology at the background level, no statistical average quantity can depend on spatial direction (as the universe is homogeneous and isotropic),

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so $\mathtt{f} = \mathtt{f}(p^0, t)$ only and:

$$\begin{aligned}
\hat{\mathcal{L}}\mathtt{f} &= p^0 \frac{\partial \mathtt{f}}{\partial t} - \frac{H}{Z^2}\delta_{ij}p^i p^j \frac{\partial \mathtt{f}}{\partial p^0} \\
&= -\frac{\dot{Z}}{Z}p^0 \mathtt{f},
\end{aligned} \tag{3.46}$$

where I have used that:

$$\begin{aligned}
\frac{d^2\lambda}{d\tilde{\lambda}^2}\left(\frac{d\tilde{\lambda}}{d\lambda}\right)^2 &= \frac{d(Z^{-1})}{d\tilde{\lambda}}Z^2 \\
&= Z^{-1}\frac{dx^\mu}{d\lambda}\frac{\partial(Z^{-1})}{\partial x^\mu}Z^2 \\
&= \frac{dx^0}{d\lambda}\frac{\partial(Z^{-1})}{\partial x^0}Z + 0 + 0 + 0 \\
&= -p^0\frac{\dot{Z}}{Z}.
\end{aligned}$$

Integrating (3.46) over all momenta space gives:

$$\int \frac{d^3p}{\sqrt{-g}}p^0\frac{\partial \mathtt{f}}{\partial t} - \frac{H}{Z^2}\int \frac{d^3p}{\sqrt{-g}}a^2\delta_{ij}p^i p^j \frac{\partial \mathtt{f}}{\partial p^0} = -\frac{\dot{Z}}{Z}\int \frac{d^3p}{\sqrt{-g}}p^0\mathtt{f}. \tag{3.47}$$

Integrating the middle term once by parts and neglecting the surface term gives:

$$\int \frac{d^3p}{\sqrt{-g}}p^0\frac{\partial \mathtt{f}}{\partial t} - \frac{H}{Z^2}\int \frac{d^3p}{\sqrt{-g}}\left\{3Z^2 p^0 + \frac{a^2\delta_{ij}p^i p^j}{p^0}\right\}\mathtt{f} = -\frac{\dot{Z}}{Z}\int \frac{d^3p}{\sqrt{-g}}p^0\mathtt{f}. \tag{3.48}$$

Comparing definitions (3.33) and (3.34) to their macroscopic counterparts, (3.13) and (3.24), the kinetic can be connected to the thermodynamic in a straightforward way:

$$T^{00} = \rho = \int \frac{d^3p}{\sqrt{-g}}p^0\mathtt{f} \tag{3.49}$$

$$T^{ij} = a^{-2}\delta^{ij}P = \int \frac{d^3p}{\sqrt{-g}}\frac{p^i p^j}{p^0}\mathtt{f} \tag{3.50}$$

Then, at last, putting it all together, we recover equation (3.23). Finally note that particle number here is *not* a frame invariant quantity:

$$\tilde{N} = \int d\tilde{\pi}d\tilde{\omega}\tilde{\mathtt{f}} = \int Zd\pi Zd\omega \mathtt{f} = \int d\mu \mathtt{f}Z^2 \neq N.$$

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Again, we have developed a perfect fluid model consistent with the action approach. Note again, we have another caveat: *the Boltzmann equation, (3.41), is now frame dependent also.* This result was at the time also completely new. It seems, at the particle level, there is some sort of particle production in the Einstein frame phase space where there wasn't in the Jordan frame phase space. On comparison with standard Boltzmann transport theory, it seems we must interpret the last term on the RHS of (3.46), $\frac{\dot{Z}}{Z} \int \frac{d^3p}{\sqrt{-g}} p^0 \mathtt{f}$, as an effective collision term, as it contains no derivatives of $\mathtt{f}$ [119].

Recently [83] have, for a related theory involving disformal transformations, taken a different approach whereby the phase space and Boltzmann equation are held fixed under the disformal transformation. It would be interesting to investigate in future work how these two approaches link together.

Now one might raise the objection that perhaps the thermodynamic picture really doesn't fit with the action picture, and that we just exploited a degree of freedom, the equation of state, in order to force a reconciliation. Equation (3.30) can be re-derived, however, independently via this kinetic theory model. Consider for simplicity the case of a radiation fluid. The photon constituents travel on disformal null geodesics, i.e $\tilde{g}_{\mu\nu}\tilde{p}^\mu\tilde{p}^\nu = 0$. We known too that $\tilde{p}^\mu = p^\mu d\lambda/d\tilde{\lambda}$, i.e. the two *contravariant* momenta are related by a non-zero conformal relation, so $\tilde{g}_{\mu\nu}p^\mu p^\nu = 0$ and hence we can compute that:

$$
\begin{aligned}
g_{\mu\nu}T^{\mu\nu} &= \int \frac{d^3p}{\sqrt{-g}} \left(\tilde{g}_{\mu\nu} - \frac{\phi_\mu\phi_\nu}{M^4} \right) \frac{p^\mu p^\nu}{p^0} \mathtt{f} \\
-\rho + 3P &= \int \frac{d^3p}{\sqrt{-g}} \frac{\tilde{g}_{\mu\nu}p^\mu p^\nu}{p^0} \mathtt{f} - \int \frac{d^3p}{\sqrt{-g}} \frac{\phi_{,\mu}\,\phi_{,\nu}\,p^\mu p^\nu}{M^4 p^0} \mathtt{f} \\
&= 0 - \frac{\phi_{,0}^2}{M^4} \int \frac{d^3p}{\sqrt{-g}} p^0 \mathtt{f} \\
&= -\frac{\dot{\phi}^2}{M^4}\rho \\
\rightarrow 3P &= Z^2\rho.
\end{aligned}
\tag{3.51}
$$

It seems inevitable that the equation of state must be modified for Einstein frame thermodynamics, when the kinetic picture is also considered.

The main goal of considering what happens in each frame was actually to answer the question: to what variables do we attach physical meaning?

Quantities can be defined in either frame: Jordan or Einstein pressure. Jordan or Einstein 4-velocity. Jordan or Einstein momenta. Which is the one we observe? The answer is the Jordan frame variable set. The reasons are simple. Any true observer will belong to the matter sector. Any experiment that has ever been conducted, at least to our knowledge, used equipment made of matter. The proper time with which these measurements were made was then the disformal proper time, $\tilde{\tau}$ for $-d\tilde{\tau}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu$. All quantities relevant to this time are the Jordan frame variables. Note this does not mean one must work solely with those of the Jordan frame, just that they must remember to convert to Jordan frame variables on comparison with measurement. So while the non-standard equation of state and Boltzmann equations for a fluid are not observable signatures in this disformal theory (the distortions vanish in the Jordan frame quantities), we must bear them in mind when exploring Einstein frame model dynamics. Note that many observables in conformal theories are frame independent [120]. For lack of a similar systematic study of frame invariance of all cosmological observables for the disformal case we assume the above, that the Jordan frame is closest to observation. This is important work that the disformal literature still lacks and would be an interesting avenue to consider for future research.

There is certainly a lesson to be learnt from these last few pages, and I have said it before: the simplicity of the Einstein frame Euler Lagrange equations is a deceptive simplicity; the hours of labour one saves are not saved at all, they are simply transferred, and one hard problem is swapped for another. With tools like mathematica's xAct so readily available these days, many tediously long tensor calculations can be automated by computer, and this functionality will only get better with time, so the drawbacks of the Jordan frame equations are looking less and less like actual drawbacks. What a computer is not likely to able to do any time soon is work out the best definition of Einstein frame matter quantities that do not lead to contradictions, unpalatable predictions, or headaches.

3.1.2 Multi-coupled scalar tensor theories

In this section I consider two advances in tandem: the concept of a scalar tensor theory is generalised in some direction, and the consequences of using a dark energy field as the scalar that underpins the frame transformation are explored. We will see first of all that this particular abstraction of the scalar

tensor theory defined in the previous pages represents yet another loophole in Horndeski's theorem.

Consider again the Einstein frame action $\mathcal{S}$ of some scalar tensor theory defined on a dimension 4 manifold $\mathcal{M}$:

$$\mathcal{S}[g, \phi] = \int_{\mathcal{M}} d\pi(g) \left[\frac{R(g)}{2\kappa} + \mathcal{L}_\phi(g, \phi, d\phi) \right] + \int_{\mathcal{M}} d\pi(fg)\mathcal{L}(fg) \qquad (3.52)$$

for f some frame isomorphism. One way to extend this theory, while retaining the existence of an Einstein frame action, is to split the matter Lagrangian up into pieces: $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ . . . and, for each term, introduce a new frame isomorphism f_1, f_2, f_3 . . . The action for this generalised theory will look, in the Einstein frame, like this:

$$\mathcal{S}[g, \phi] = \int_{\mathcal{M}} d\pi(g) \left[\frac{R(g)}{2\kappa} + \mathcal{L}_\phi(g, \phi, d\phi) \right] + \sum_{I=1}^{N} \int_{\mathcal{M}} d\pi(f_I g)\mathcal{L}_I(f_I g). \quad (3.53)$$

If a theory has an Einstein frame action, and it can be expressed in the form above, then I will refer to it as a **multi-coupled scalar tensor theory (MCSTt)**. As each f_I is by definition invertible, there will exist a frame action of the MCSTt for every frame isomorphism, and they will not be equal if the maps are not. As the matter Lagrangian is now split over several, in general distinct, Riemannian manifolds $(\mathcal{M}, f_I(g))$ there is no longer a clear cut choice as to which is the Jordan frame manifold. The Jordan frame action hence ceases to be a meaningful label for MCSTts and I will not use it in this context. I will use the phrase X is a Y frame action, where Y will come to stand for baryon, radiation, etc. A very early example, a simple conformal MCSTt, was considered in [47].

It is not guaranteed that the operation which splits the matter Lagrangian up into terms and places each on a distinct geometry is a well defined one. If $\mathcal{L}_1$ were to be the standard model, but with, say, $\mathcal{L}_2$ the quark sector, the question as to whether flavour physics would survive such an operation is unsettled (and indeed I do not settle it in this thesis) however—and this is the main reason for considering ϕ a dark energy field at the same time—if the discrepancies induced in the geometries by the frame maps are small, and for a dark energy field they will always be close to the Hubble scale at 10^{-42}GeV [121], then of course the standard theory of flavour physics will hold approximately away from these energy scales, and we need not worry too much about the health and safety of the standard model under such a

mildly invasive operation. In physics you can get away with anything, so long as it is negligible.

Frame indicies, species indicies

In general the MCSTt defined above can be written: $\mathcal{S} := \mathcal{S}_H + \mathcal{S}_\Phi + \sum_I \mathcal{S}_I$, where:

$$\mathcal{S}_H[g] := \int_\mathcal{M} d\pi(g) \frac{R(g)}{2\kappa}, \quad \mathcal{S}_\Phi[g,\phi] := \int_\mathcal{M} d\pi(g) \mathcal{L}_\phi(g,\phi,d\phi),$$

$$\mathcal{S}_I[g,\phi] = \int_\mathcal{M} d\pi(f_I g) \mathcal{L}_I(f_I g) \quad (3.54)$$

To streamline presentation, and facilitate some calculations to follow, I will now introduce some notation: capital subscripts, e.g. $\mathcal{S}_H$, are labels of either frame or species. If only one letter is present with no vertical bar, the quantity is an Einstein frame quantity, and the subscript labels the species. For example T_Φ is the Einstein frame stress tensor of the scalar field. If two labels are present, the letter to the left of the vertical bar, $H|J$ labels the species, H, while the letter to the right labels the frame, J. Some quantities, like the total action, precede species and so only the frame needs to be specified, which is done by an empty species label: $|J$. $G_{|J}$ is the curvature tensor in the J frame. To be pedantic I could label this $T_{H|J}$ but I think the meaning is clear. Note also the vertical bar | is not to be confused for a covariant derivative.

The Einstein equations of $\mathcal{S}$ are:

$$G(g) = \kappa T_\Phi(g,\phi,d\phi) + \kappa \sum_I T_I(g,\phi,d\phi)$$

where:

$$G = -\frac{2\kappa}{\sqrt{-|g|}} \frac{\delta \mathcal{S}_H}{\delta g}$$

$$T_\Phi = \frac{2}{\sqrt{-|g|}} \frac{\delta \mathcal{S}_\Phi}{\delta g}$$

$$T_I = \frac{2}{\sqrt{-|g|}} \frac{\delta \mathcal{S}_I}{\delta f_I g} \left(\frac{\partial f_I g}{\partial g} \right)_{\phi,d\phi}$$

such that G is the standard Einstein tensor, and g the Einstein frame metric.

The Einstein equations of the J frame action $\mathcal{S}_{|J} := f_J^{-1} \circ \mathcal{S}$ are:

$$G_{|J}(g_J, \phi, d\phi) = \kappa T_{\Phi|J}(g_J, \phi, d\phi) + \kappa \sum_{I \neq J} T_{I|J}(g_J, \phi, d\phi) + \kappa T_{J|J}(g_J)$$

where:

$$
\begin{aligned}
G_{|J} &:= -\frac{2\kappa}{\sqrt{-|g_J|}} \frac{\delta \mathcal{S}_H}{\delta f_J^{-1} g_J} \left(\frac{\partial f_J^{-1} g_J}{\partial g_J} \right)_{\phi, d\phi} \\
T_{\Phi|J} &:= \frac{2}{\sqrt{-|g_J|}} \frac{\delta \mathcal{S}_\Phi}{\delta f_J^{-1} g_J} \left(\frac{\partial f_J^{-1} g_J}{\partial g_J} \right)_{\phi, d\phi} \\
T_{I|J} &:= \frac{2}{\sqrt{-|g_J|}} \frac{\delta \mathcal{S}_I}{\delta f_I f_J^{-1} g_J} \left(\frac{\partial f_I f_J^{-1} g_J}{\partial g_J} \right)_{\phi, d\phi} \\
T_{J|J} &:= \frac{2}{\sqrt{-|g_J|}} \frac{\delta \mathcal{S}_J}{\delta g_J}
\end{aligned}
$$

and g_J is the J frame metric: the metric tensor that minimises the action $f_J^{-1} \circ \mathcal{S}$, and $\mathcal{S}_{J|I} = f_J^{-1} \circ \mathcal{S}_I$ for some I term in the Einstein frame action $\mathcal{S}$. The stress tensors relate between arbitrary frames as:

$$T_{I|K} = \sqrt{\frac{|f_J f_K^{-1} g_K|}{|g_K|}} T_{I|J} \left(\frac{\partial f_J f_K^{-1} g_K}{\partial g_K} \right)_{\phi, d\phi} \tag{3.57}$$

and the Klein Gordon equation of $\mathcal{S}$ is:

$$
\begin{aligned}
\frac{1}{\sqrt{-|g|}} \frac{\delta \mathcal{S}_\Phi}{\delta \phi} &= -\sum_I \frac{T_I}{2} \left(\frac{\partial f_I g}{\partial g} \right)_{\phi, d\phi}^{-1} \left(\frac{\partial f_I g}{\partial \phi} \right)_{g, d\phi} \\
&\quad + \sum_I \nabla \left(\frac{T_I}{2} \left(\frac{\partial f_I g}{\partial g} \right)_{\phi, d\phi}^{-1} \left(\frac{\partial f_I g}{\partial (d\phi)} \right)_{g, \phi} \right).
\end{aligned}
$$

Can we generalise any of the results from the previous section to MCSTts? Should we? Certainly if the Jordan frame equations of the previous section were enough to put you off, then likely any attempt presented here to unravel the Einstein equations above will see you ripping the thesis up right here and burning it. But the equation of state as a frame dependent quantity is a result that can be easily generalised, and if the scalar field is a dark energy field that we posit as the driving force behind the universe's accelerated expansion, this

result can be used to place constraints on such a theory, as we see in the next section.

Let $T_{J|I}$ and $T_{J|J}$ be J frame stress tensors of the I and J frame actions respectively of some MCSTt. Let the frame isomorphism relating the frames be:

$$f_{I \to J}(g_I) = C(\phi)_{I \to J} g_I + D(\phi)_{I \to J} d\phi \otimes d\phi$$

and $Z_{I \to J}$ is the disformal scalar associated with $f_{I \to J}$. Then we have the following two results.

1. Assume we can simultaneously make the following definitions:

$$T_{J|I} = \left(\rho_{J|I} + \frac{P_{J|I}}{c^2} \right) u_{J|I} \otimes u_{J|I} + P_{J|I} g_I^{-1}$$

   and

$$T_{J|J} = \left(\rho_{J|J} + \frac{P_{J|J}}{c^2} \right) u_{J|J} \otimes u_{J|J} + P_{J|J} g_J^{-1} \ .$$

   Then:

$$\frac{P_{J|J}}{c^2 \ \rho_{J|J}} = \frac{P_{J|I}}{c^2 Z_{I \to J}^2 \ \rho_{J|I}}.$$

2. Assume we can simultaneously make the following definitions:

$$T_{J|I} = \int d\omega_{J|I} \mathtt{f}_{J|I} p_{J|I} \otimes p_{J|I}$$

   and:

$$T_{J|J} = \int d\omega_{J|J} \mathtt{f}_{J|J} p_{J|J} \otimes p_{J|J} \ .$$

   Then:

$$\frac{d}{d\lambda_{I,J}} (\mathtt{f}_{J|I} Z_{I \to J}) = 0,$$

   where $\lambda_{I|J}$ are affine parameters such that $p_{I|J} = \frac{d}{d\lambda_{I|J}}$.

I will use these results when I come to discuss constraints of a particular model in the next section.

The presence of terms like $\partial f_I f_J^{-1} g_J / \partial g_J$ in the above Einstein equations advises caution: we have seen it is difficult to construct closed groups of frame transformations, and while it is not so hard to find the inverse of a

frame map f, it is not always certain that we can compose two invertible maps to produce another that is invertible. This caution is what advised us against considering too many frames of some scalar tensor theory, and it here will advise us similarly to refrain from splitting the matter Lagrangian into too many pieces. Two will be enough for this thesis (which gives three frame actions in total, including the Einstein frame), and more is ill advised, unless one can be sure that inverses for all compositions always exist.

And now another word on Horndeski. (I have noticed my fixation on the theorem here; much work in scalar tensor dark energy paradigm these days gravitates towards it, and though I do not actually use it here directly, it has come to define and exemplify many of the goals and pitfalls of modern dark energy theory.) It is pretty clear that the definition of an MCSTt above represents another departure from the theorem: another loophole. What it allows, in principle, is every different term in some matter Lagrangian split, $\mathcal{L}_i$ to have a Horndeski-like gravitational sector, that is in general distinct from its neighbour, $\mathcal{L}_j$. To see this observe that an MCSTt has, at least, a frame action for every term in the matter Lagrangian split, and each frame action will have a complex gravitational sector with frame-specific Horndeski functions $G_2, ..., G_5$. I will not say more on this now, except to comment that this generality is to me farcical given the simplicity of the cosmological data, which still supports the single number, Λ, from Einstein's original equations of 1917.

Dark energy & the microwave background

I will close this section with an application of multi-coupled scalar tensor theories to cosmology, and cosmological data. To a zeroth order approximation, we can partition the elements of any late-time (essentially post inflation) cosmological model into two sets. The massless free streaming particles, namely the cosmic microwaves and the neutrino background, form a set. Everything else: dark matter, baryons and so on, forms another. The division is essentially between things that move very fast and very far, and things that barely move at all; and it is a sensible one, because since the production of the microwave background, really nothing in the universe has occupied the space in between; things tend to be relativistic with geodesics billions of light years long, or not moving at all with respect to the cosmic rest frame. The first set can be modelled by a single perfect fluid, with pressure $P = \rho/3$, and the second by another, with pressure $P = 0$. The interactions between

each, at this zeroth order level, are negligible.

I now present the model construction. Consider an MCSTt with the matter Lagrangian split into two parts, $\mathcal{L}_1$, $\mathcal{L}_2$, defined over a 4-dimensional smooth manifold $\mathcal{M}$. Associate now $\mathcal{L}_1$ with the Lagrangian of a pressureless perfect fluid that will describe our non-relativistic component. I will mark this species with the species label M. Associate next $\mathcal{L}_2$ with the Lagrangian of a perfect fluid with relativistic particle constituents, and label it R. From the definition of an MCSTt, to each is associated a frame map f_M and f_R, and so the Einstein frame action of this theory, $\mathcal{S}$ is:

$$
\begin{aligned}
\mathcal{S}[g, \phi] \;\; := \;\; & \int_{\mathcal{M}} d\pi(g) \left[\frac{R(g)}{2\kappa} + \mathcal{L}_\Phi(g, \phi, d\phi) \right] \\
& + \int_{\mathcal{M}} d\pi(f_M g) \mathcal{L}_M(f_M g) + \int_{\mathcal{M}} d\pi(f_R g) \mathcal{L}_R(f_R g) \quad (3.58)
\end{aligned}
$$

for disformal transformations f_M, f_R, which are set to be as simple as possible:

$$
f_M g \;\; := \;\; g + \frac{1}{M_M^4} d\phi \otimes d\phi \tag{3.59}
$$

$$
f_R g \;\; := \;\; g + \frac{1}{M_R^4} d\phi \otimes d\phi \tag{3.60}
$$

where recall I have set c and $\hbar$ to 1, and M_M and M_R are energy scales which will be the model parameters that data sets will eventually be enlisted to constrain. We chose ϕ to be a dark energy field, and, by keeping these constants small enough, attempted to make sure f_I be always invertible. However, this was not always possible to guarantee, as we will see. We wished to replicate cosmological data, so the Riemann manifold $(\mathcal{M}, g)$ will be a flat universe, which implies that $(\mathcal{M}, f_R g)$ and $(\mathcal{M}, f_M g)$ too must be. There are two frame actions of special interest for this theory besides the Einstein frame one defined above, which are:

1. *The radiation frame action $f_R^{-1} \circ \mathcal{S}$.* The radiation frame Einstein equations define a stress tensor $T_{R|R}$ for the R fluid that is not dependent on the scalar field—it is uncoupled from the scalar in the Euler Lagrange equations. This stress tensor is conserved with respect to the covariant derivative ∇_R metric compatible with $f_R g$:

$$
\nabla_R T_{R|R} = 0. \tag{3.61}
$$

2. *The matter frame action $f_M^{-1} \circ \mathcal{S}$.* The Einstein equations of $\mathcal{S}_{|M} := f_M^{-1} \circ \mathcal{S}$ define a stress tensor for the pressureless matter (abbreviated: matter) species, $T_{M|M}$, that is uncoupled from the scalar field. This tensor is conserved with respect to ∇_M.

It is the Einstein frame in which the Euler Lagrange equations are simplest, and so it was these Euler Lagrange equations we chose to use when performing numerical calculations. Remember there is no longer a well defined Jordan frame action, so the warnings from the previous section do not apply. An important premise underlying the work was the following:

> *It was assumed that matter frame quantities, with the label $\cdot|M$, were the ones measured directly by experiment.*

We will see the ramifications of this assumption later on in this section. The arguments underlying it were the following. For scalar tensor theories involving conformal transformations only, it is generally accepted that the Jordan frame action defines the quantities that are directly measured: particle masses defined with respect to this action are constant, and particles freely fall along geodesics of the Jordan frame metric, $f_C g$. Using measurement of planetary orbits, for example, to determine the space-time curvature induced by the sun would be a much tougher job if one did not assume the planets followed geodesic orbits. For scalar tensor theories involving general frame transformations the situation is less well understood, but it is still on the whole accepted that Jordan frame quantities are those that are directly measured.

For MCSTts there was no accepted wisdom to follow, however, as the majority of matter in our model described above, that is stars, planets etc reside on one geometry, $(\mathcal{M}, f_M g)$, it seemed plausible to suggest that *this* space-time must take on the role left by the absence of the Jordan frame of scalar tensor theories as the one in which physical measurement is done. I will not say more on this, and I now simply assume the above premise for the rest of this section.

We chose to remain in the Einstein frame to simplify calculations. In all that follows, g will be of FL form:

$$g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2 dx^i dx^i. \tag{3.62}$$

An observer at rest in this coordinate system will be said to be **comoving with g**. The comoving velocity field as always I denote u. The Einstein

equations for $\mathcal{S}$ are:

$$G(g) = \kappa T_\Phi(g, \phi, d\phi) + \kappa T_M(g, \phi, d\phi) + \kappa T_R(g, \phi, d\phi) \tag{3.63}$$

where the Einstein frame stress tensors by construction of the model are:

$$T_M = \rho_M u \otimes u \tag{3.64}$$

$$T_R = (\rho_R + \frac{P_R}{c^2}) u \otimes u + P_R g^{-1}, \tag{3.65}$$

so that every element of both fluids is comoving with g, and:

$$\frac{P_R}{c^2 Z_R^2 \rho_R} = \frac{P_{R|R}}{c^2 \rho_{R|R}} = \frac{1}{3}, \quad P_M = 0 \tag{3.66}$$

for Z_R the disformal scalar associated with f_R. ϕ we chose to be a dark energy field, and for simplicity we assumed that the scalar field was quintessent in $\mathcal{S}$:

$$\mathcal{L}_\Phi(g, \phi, d\phi) := X(g, d\phi) - V(\phi), \tag{3.67}$$

with

$$V(\phi) = M_V^4 e^{\beta_V \phi}, \tag{3.68}$$

where $\beta_V, M_V \in \mathbb{R}$ are constants. $\beta_V = \kappa^{1/2}$, so that $[\phi\beta_V] = [1]$. The slope of V chosen above is small enough that ϕ can play the role of dark energy; V will change gradually only at late times, and act like an approximate cosmological constant: it will be stabilised. Putting this all together, the Einstein equations become the Friedmann equations:

$$H^2 = \frac{1}{3} \sum_I \rho_I \tag{3.69}$$

$$\dot{H} + H^2 = -\frac{1}{6} \left(\sum_I \rho_I + \frac{3}{c^2} \sum_I P_I \right) \tag{3.70}$$

for $I \in \{M, R, \Phi\}$, and:

$$H(t) := \frac{\dot{a}(t)}{a(t)}, \tag{3.71}$$

for H the Hubble parameter, with 'dot' the derivative with respect to cosmic time t.

The conservation equations and Klein Gordon equation of $\mathcal{S}$ in the chosen coordinates are [105]:

$$\ddot{\phi} + 3H\dot{\phi} + V' \ = \ \sum_I Q_I, \tag{3.72}$$

$$\dot{\rho}_I + 3H(\rho_I + P_I) \ = \ -Q_I\dot{\phi}, \tag{3.73}$$

which for an MCSTt with the matter Lagrangian split in two is:

$$Q_1 \ = \ \frac{\mathcal{A}_2}{\mathcal{A}_1\mathcal{A}_2 - D_1 D_2 \rho_1 \rho_2} \left(\mathcal{B}_1 - \frac{\mathcal{B}_2 D_1 \rho_1}{\mathcal{A}_2} \right) \tag{3.74}$$

$$Q_2 \ = \ \frac{\mathcal{A}_1}{\mathcal{A}_1\mathcal{A}_2 - D_1 D_2 \rho_1 \rho_2} \left(\mathcal{B}_2 - \frac{\mathcal{B}_1 D_2 \rho_2}{\mathcal{A}_1} \right), \tag{3.75}$$

where

$$\mathcal{A}_I \ = \ 1 + D_I(\rho_I - \dot{\phi}^2)$$

$$\mathcal{B}_I \ = \ D_I \left\{ 3H \left(1 + \frac{P_I}{\rho_I} \right) \dot{\phi} + V' \right\} \rho_I.$$

Here $\{1, 2\} = \{I, J\} = \{M, R\}$.

We then turned to placing constraints on this model. The distorted equation of state affects the dynamics of the macroscopic energy density and pressure of the cosmic microwave background (CMB). We associated this energy density with a thermodynamic temperature. The modified Boltzmann equation, $d\mathtt{f}/d\lambda \neq 0$, we linked to another cosmological observable: the shape of the CMB intensity spectrum, quantified through μ-*distortions* [122]. See [123] for a related application of μ-distortion measurements to constrain theories involving disformal transformations. (The measured cosmic microwave background spectrum is very close indeed to blackbody form, with very small perturbations. Departures to the background spectrum are often searched for in the form of an effective chemical potential, μ, which could arise from a number of modifications to the standard cosmological model. A measured non zero chemical potential of the background spectrum would be a distortion from what is currently expected from the base theory, and is hence named a μ-distortion.)

The CMB was produced when photons de-coupled from matter in the early universe. When produced, both the CMB and other matter fluids were extremely well thermalised, and so the photon bath was to a very high degree

of precision of blackbody form, with temperature given by the Stephan-Boltzmann law. The evolution of the temperature of the CMB, Θ, with cosmic redshift has been measured by various authors in the redshift range 0 to about 3 [124–127]. These measurements greatly support that the evolution of CMB temperature with redshift is linear, as predicted by general relativity. For the MCSTt defined above we found this is no longer the case.

Assuming any distortions to the blackbody form of the distribution function, $\mathtt{f}_R$, caused by the disformal coupling to be small, compared with the errors in measurement of this temperature evolution, $\Theta(z)$ (this assumption is valid, as we will soon see), the matter frame energy density of the R fluid, $\rho_{R|M}$, we related to a blackbody temperature using the Stephan-Boltzmann law:

$$\rho_{R|M} = \sigma\Theta^4, \tag{3.76}$$

where σ is Stephan's constant. This is the practical content of the above premise, which assumed matter frame quantities are those that are directly measured. Some examples of the $\Theta(z)$ curves for universes modelled by the above dynamical system with different values of D_M, D_R are shown in Fig. 3.1. Details of the numerical procedures can be found below.

Another application of the premise was to the distribution function. If, by the premise, it were $\mathtt{f}_{R|M}$ (matter frame R fluid distribution function) that was directly observed, then an initial blackbody spectrum so observed could not remain so as the universe evolved.

I present here our derivation of the distortions to the shape of $\mathtt{f}_{R|M}$, quantified as effective μ-distortions, induced by disformal couplings to the matter frame cosmic microwave background distribution function. For the R fluid of our model, intended to model the cosmic microwave and neutrino background, we defined the microscopic counterparts to the R fluid stress tensors above as:

$$T_{R|M} = \int d\omega_{R|M}\mathtt{f}_{R|M}p_{R|M} \otimes p_{R|M}, \quad T_{R|R} = \int d\omega_{R|R}\mathtt{f}_{R|R}p_{R|R} \otimes p_{R|R}$$

and so we obtained:

$$\frac{d}{d\lambda}(\mathtt{f}_{R|M}Z_{M\rightarrow R}) = 0. \tag{3.77}$$

Fix some t_{ini} to be some time several billion years ago such that $\mathtt{f}_{R|R}$ was of blackbody form (such a time must exist, and must be far in the past). The Klein Gordon equation above contains the term $3H\dot{\phi}$ which is proportional
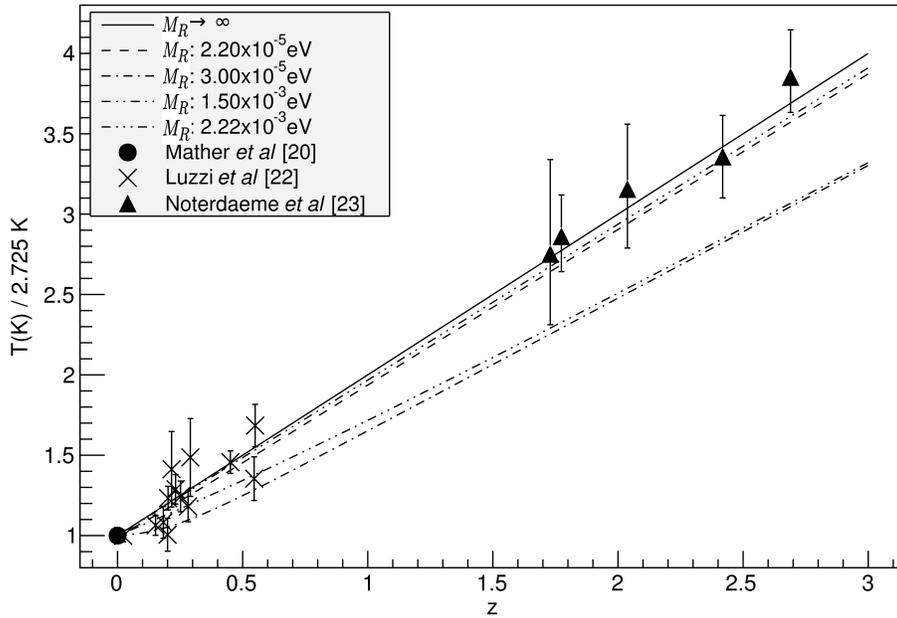
Figure 3.1: Plot of the ratio $\Theta(z)/\Theta(t_{\text{today}}) := T(K)/2.725K$ where $\Theta(t_{\text{today}}) = 2.275K$, against redshift, z, for the exponential potential and fixed $M_M = 0.05eV$. The solid line corresponds to the GR limit, where both M_M and $M_R \to \infty$. Note that both limits $M_R \to \infty$ (corresponding to vanishingly small couplings) and $M_R \to 0$ (corresponding to a damped evolution of the field) lead to a temperature evolution allowed by the data in this figure.

to the first derivative of the field, thus it acts like a friction term proportional to the expansion rate of the universe, the so-called **Hubble friction**, so any change in the value of ϕ is limited to near the present epoch. While ϕ was unable to overcome Hubble friction and had yet to start rolling, we have $d\phi(t_{\text{ini}}) = 0$, which implies $Z_{M \to R}(t_{\text{ini}}) = 1$, and so

$$\mathtt{f}_{R|R}(t_{\text{ini}}) = \mathtt{f}_{R|M}(t_{\text{ini}}) = \frac{1}{e^{\omega/T} - 1} \tag{3.78}$$

Then, from Eq. (3.77) we see $\mathtt{f}_{R|M} Z_{M \to R}$ is a constant of the universe evolution, so:

$$\mathtt{f}_{R|M}(t_{\text{ini}}) Z_{M \to R}(t_{\text{ini}}) = \mathtt{f}_{R|M}(t_{\text{today}}) Z_{M \to R}(t_{\text{today}}). \tag{3.79}$$

Too, at t_{ini}, ϕ derivatives were greatly suppressed by Hubble friction, and so $Z_{M \to R}(t_{\text{ini}}) = 1$. We can, assuming the distortion to be small (as experimental constraints tell us it must be) parameterise it as an effective μ-distortion:

$$\begin{aligned}
\mathtt{f}_{R|M}(t_{\text{today}}) &= \frac{\mathtt{f}_{R|M}(t_{\text{ini}})}{Z_{M \to R}(t_{\text{today}})} \\
\Rightarrow \left(e^{\omega/T - \mu} - 1\right) &= \left(e^{\omega/T} - 1\right) Z_{M \to R}(t_{\text{today}}).
\end{aligned} \tag{3.80}$$

μ is small, so we can Taylor expand:

$$\exp(\mu) \simeq 1 + \mu. \tag{3.81}$$

One can then show that:

$$\mu \simeq \{(Z_{M \to R} - 1\} \left(1 - e^{-\omega/T}\right). \tag{3.82}$$

The COBE satellite has set the current limits on how large this dimensionless μ parameter can be [128]:

$$|\mu| < 9 \times 10^{-5}. \tag{3.83}$$

This small number justifies our treating μ as small in defining an effective blackbody temperature, and for the Taylor expansion. We used both the micro and macroscopic constraints of μ distortions and a modified CMB temperature evolution with redshift respectively to place constraints on this MCSTt, Eq. (3.58), which are shown in figure 3.2.
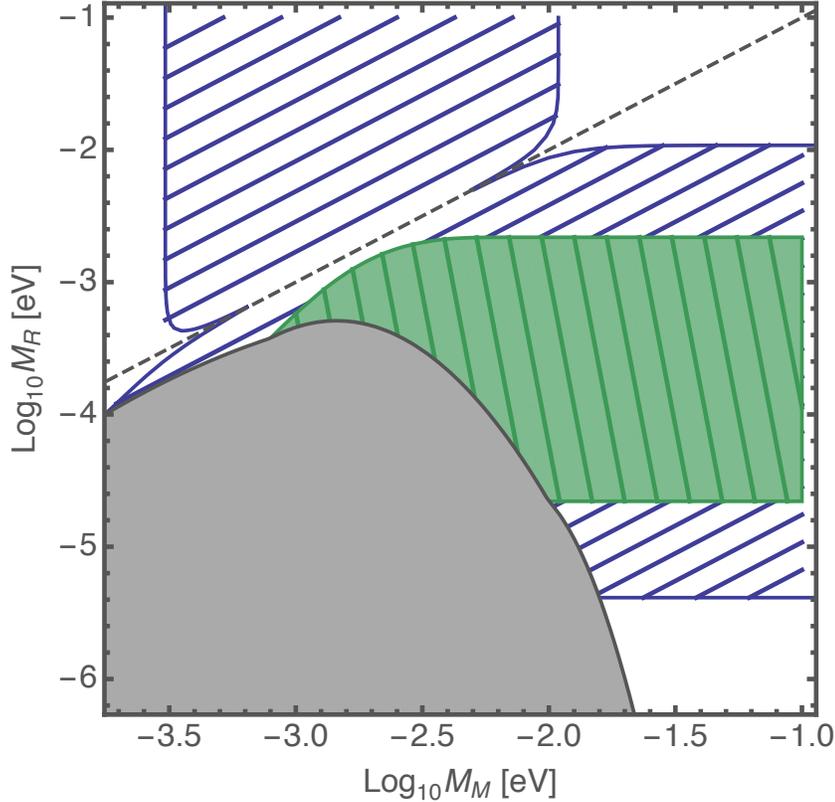
Figure 3.2: Bounds on the $M_M \times M_R$ parameter space. The dashed line represents $M_M = M_R$. The green shaded region is excluded by the CMB temperature evolution alone, to above a 68% C.L. The blue hatch-filled areas are excluded by constraints set by μ-distortions. The solid grey shaded area corresponds to the region of the parameter space we could not integrate in, as the numerics became unreliable here. Note there is a finite separation between the straight line and the μ-distortion constrained regions, i.e. a finite width either side of the line that is not excluded by our methods.

Boundary conditions & numerical details:

For the dynamical system defined above, the dynamical variables (those that were integrated with a numerical routine) were:

$$\rho_M, \rho_R, \phi, \dot{\phi}. \tag{3.84}$$

The system is over-specified, and not all the differential equations are necessary. This is a general property of general relativity systems. We chose to have H a constraint variable, and not integrate it directly, although the $\dot{H}$ equation would have allowed us to.

Each dynamical variable must have a boundary condition specified, and almost invariably in late time cosmology these are set by measurements today, where measurement can be done directly. Then, defining:

$$\Omega_I = \frac{\rho_{I|M}}{3H_{|M}^2} \tag{3.85}$$

we fixed: $\Omega_M = 0.3$, $\Omega_\Phi = 0.7$, $H_{|M}(t_{\text{today}}) = 67.6 \text{kms}^{-1}\text{Mpc}^{-1}$ and $\Theta(t_{\text{today}}) = 2.725$ K in line with the Planck 2015 data release [129]. We fixed the initial value for $\beta_V \phi(t_{\text{ini}}) = 1$, but varied M_V such that the desired cosmological parameters today, Ω_I, H_0 were reached. β_V was held fixed. The theory's free parameters were the disformal energy scales M_D, M_R.

For different values of the two energy scales, (M_D, M_R), the initial values $\rho_I(t_{\text{ini}})$ and M_V were adjusted by an algorithm to match present day values of Ω_I etc. Once the boundary conditions were satisfied, a particular curve $\Theta(z)$ was numerically obtained, and compared to the data shown in Fig. 3.1 with a χ^2 test. Some example $\Theta(z)$ curves are shown alongside the data in Fig. 3.1 for illustration.

If a given pair (M_D^*, M_R^*) gave a χ^2 value corresponding to a confidence level less than %68, that point formed part of the exclusion zone, the blue shaded region in Fig. 3.2. Any point (M_D^*, M_R^*) that produced, after the system was tuned to produce the desired boundary conditions, a value of μ larger than the experimentally allowed upper limit, this point formed part of the green excluded region of Fig. 3.2.

I now make a few important notes about this exclusion plot, Fig. 3.2, before wrapping up the section. First is that the two blue regions and the green one are each bounded from two sides: the top blue 'arm' is bounded from the left and the right; the blue and green overlapping regions (the 'arms'

below the dashed line) are bounded from above and below. This is simply because the disformal coupling of the field to matter has a damping effect on the ϕ field. We are faced with a trade off: couple the field too weakly, and any disformal effects on matter are suppressed; couple the field too strongly, and the matter has a damping back-reaction on the field: if the field cannot evolve, its derivatives vanish, along with all disformal effects.

Second: the straight line is a guide for the eye to show when the species are coupled the same, or, when $f_R = f_M$, our observable signatures vanish. This can be easily understood as follows. If the couplings are the same, the proper times of the two geometries coincide. Notice though that for μ-distortion constraints, the discrepancy in the couplings need not be big at all. We could perhaps imagine such a discrepancy coming from quantum corrections of ϕ couplings to various matter species. Note also, there is always a finite (though perhaps negligibly small) space between the straight line and the edges of the μ-distortion exclusion zones. The straight line and the exclusion zones never overlap.

Third: the grey region was numerically inaccessible because here the model parameters were such that the system got dangerously close to points where $Z \to 0$. The breakdown of invertibility of $\tilde{g}$ was manifest in the dynamical system as a pole in the coupling function Q.

What has been shown in this section is that in an MCSTt, standard matter physics definitions cannot be assumed without consideration. That is, quantities like the equation of state or Boltzmann equation of a perfect fluid are not frame invariant, as they are in conformal theories. In the Einstein frame we severely distort the matter sector, and so cannot transfer all standard results from general relativity across to this new theory without first checking whether or not each still holds. This makes the Einstein frame a bit of a trap. Yes calculations are simpler than they are when working with the Jordan dual (or some I frame counterpart in the MCSTt case), but this simplification comes at the expense of great conceptual opacity. Particles do not follow geodesics in this frame, and to derive any reliable results one has to go back to first principles. It has been shown here that Einstein and Jordan frame actions are mathematically equivalent, if the involved frame transformations are invertible; to say they are also physically equivalent at lower levels is not straightforward, and one must above all be careful with definitions.

Additionally we have seen how the above modified definitions have observable consequences when applied to MCSTts. These can be used to place

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constraints on model parameters, though the damping effect of the disformal coupling lessens the extent to which this endeavour can be successful: there are resonance values of M_M and M_R where interesting behaviour in cosmology can be seen and moving away from these values, in both directions, observable effects diminish.

3.2 Disformal Electrodynamics

In the history of disformal transformations in gravity theories, we have seen in the introduction, there exists a multiplicity of purpose. They were geometry corrections to general relativity in compactifications of higher dimensional brane-world gravity theories, but they were also utilised in the literature to vary the relationship between the speed of light and the speed of gravitational waves, which could solve the horizon and flatness problems of early universe cosmology without recourse to a potential-driven inflationary phase [75] [76] [77]. (Such theories are now very tightly constrained by observations [36].) This second aspect of disformal theories—their tendency to distort the light cones of fundamental fields with respect to each other—will be the focus of this section. See [130] for the publication of this work.

Scalar tensor theories involving a single disformal transformation have been constrained quite severely via global tests in cosmology [131], or local tests in the solar system [84] or the laboratory [85, 132], which has led some to postulate that disformal couplings can, for example, only be between the scalar and dark matter [96, 133]. As the nature of dark matter is poorly understood, the constraints of disformal couplings to it are rather weak. Such a theory in fact falls into the category of multi-coupled scalar tensor theories, the subject of the last section, and we will study such a case in much detail in chapter 4, but here we will place a magnifying glass over the MCSTt of the previous section. Splitting the matter Lagrangian into a relativistic and a non-relativistic piece will invariably lead to observable deviations from standard matter theory. Here I inspect some in detail.

A handful of these deviations must be in the form of novel radiation processes. Due to the variation in the relative speeds of photons and gravitons in disformal theories, it remains an open question as to whether charged particles in an MCSTt can Cherenkov radiate in vacuum. In this section I expound work by Carsten, Clare Burrage and myself, in which we unambigu-

ously demonstrate that this is indeed the case, and deduce the conditions that must be met in order for this to occur. I will describe also how we discovered that another radiative interaction channel will open under those same model conditions, a channel that depends on the dynamics of the theory's speed of light. For reasons that will become clear, we dubbed this interaction *vacuum bremsstrahlung*.

Consider again the MCSTt from the previous section. The matter Lagrangian splitting operation $\mathcal{L} = \mathcal{L}_M + \mathcal{L}_R$ placed the non relativistic elements of the universe, M, on one geometry, and relativistic components, R, on another. The cosmic microwave background formed a major part of latter set and so implicitly contained in $\mathcal{L}_R$ must have been the Lagrangian of electromagnetism, with Maxwell field A. We conveniently ignored interaction terms $\mathcal{S}_{int}$ in the previous formulation, and is not yet clear how one would fit into our picture. With the acknowledgement of these two points, the Einstein frame action $\mathcal{S}$ of the MCSTt from the previous section written more explicitly becomes:

$$
\begin{aligned}
\mathcal{S}[g, \phi, A] \quad &:= \quad \int_{\mathcal{M}} d\pi(g) \left[\frac{R(g)}{2\kappa} + \mathcal{L}_\Phi(g, \phi, d\phi) \right] \\
&+ \int_{\mathcal{M}} d\pi(f_M g) \mathcal{L}_M(f_M g) + \int_{\mathcal{M}} d\pi(f_R g) \mathcal{L}_R(f_R g, A) \\
&+ \mathcal{S}_{int}[g, \phi, A],
\end{aligned}
\tag{3.86}
$$

and we now consider slightly more involved disformal transformations f_M, f_R:

$$
\begin{aligned}
f_M g \quad &:= \quad g + D_M(\phi) d\phi \otimes d\phi \tag{3.87} \\
f_R g \quad &:= \quad g + D_R(\phi) d\phi \otimes d\phi. \tag{3.88}
\end{aligned}
$$

A primary challenge of this section will be to find out whether the MSCTt of the previous section still functions as a coherent theory at this more detailed level: is there a sensible definition we can make for the form of $\mathcal{S}_{int}[g, \phi, A]$? Is there a unique one?

In the next section I refine this action and restrict our attention to a minimal subsystem in which to cleanly explore novel radiative processes but in the meantime this schematic action, Eq. (3.86), highlights a key point: there are three metrics in our theory, all related by disformal transformations, so there are three different frame actions within which to make calculations,

and three representations of each variable. In standard scalar tensor theory it is commonplace to perform computations with the Einstein frame action, where the gravitational term is of Einstein-Hilbert form, and reserve physical interpretation for the Jordan frame, however, for this work, we will find that it is in fact the R frame action, $f_R^{-1} \circ \mathcal{S}$ in which calculations are simplest. This will hopefully become clear as I unveil our calculation.

As I have mentioned above, we find that two radiation channels are open to a disformally coupled charged particle, provided certain radiation conditions are satisfied: vacuum Cherenkov and bremsstrahlung radiation. Both of which I will describe in what follows. In section 3.2.1 I introduce the gravitational part of the action (3.86), and specify a small charged particle-and-field subsystem—adequate to determine the conditions under which vacuum Cherenkov radiation occurs in a disformal theory. In this section I present Maxwell's equations with disformal couplings present, then solutions and finally constraints on model parameters from collider based experiments. In section 3.2.2 I present the case for bremsstrahlung, define the relevant parts of the action, show equations of motion, and discuss the conditions to be met for its presence. We did not use vacuum bremsstrahlung to place theory constraints in this work, but simply offered an illustration as to the scale of the effect in a cosmology setting using cosmic rays. In this section, integrals will always be over the entire manifold $\mathcal{M}$, so I will make this implicit: $\int := \int_{\mathcal{M}}$ to simplify the notation.

3.2.1 Vacuum Cherenkov radiation

Action

The salient feature of the MCSTt defined above is the disformally transformed radiation term in $\mathcal{S}$; we asked what novel changes this detail will introduce into the theory of electromagnetism, and so the gravitational sector is not of importance at this stage. As a suggestive nod toward cosmology, we chose to let the scalar field ϕ play the role of dark energy, but this interpretation is not immediately important.

The electromagnetic free action takes on new life in this MCSTt. The Einstein frame action is:

$$\mathcal{S}_E[g, \phi, A] = -\frac{1}{4\mu_0} \int d\pi(f_R g) F^2(f_R g, dA) \qquad (3.89)$$

where recall that μ_0 is the permittivity of free space, and:

$$\begin{aligned}
F(A) &= dA && \text{(3.90a)} \\
F^2(f_R g, dA) &= f_R g^{-1} \otimes f_R g^{-1}(dA, dA). && \text{(3.90b)}
\end{aligned}$$

We would like to obtain the Maxwell equations of the theory, but we cannot without a properly defined interaction term $\mathcal{S}_{int}$ between the electromagnetic field and particles described by some term of $\mathcal{L}_M$. On first inspection, there seem to be two 'obvious' choices:

$$\text{a) } \mathcal{S}_{int} := \int J_{M|M} \wedge A, \quad \text{or} \quad \text{b) } \mathcal{S}_{int} := \int J_{M|R} \wedge A.$$

where $J_{M|M}$ is the M frame current 3-form of some electromagnetically charged matter, and $J_{M|R}$ the R frame current 3-form of the same matter. The 1-form A is a fundamental field and so does not change under a frame transformation. Note for each choice, imposing gauge invariance $A \rightarrow A + d\theta$ implies the conservation of each current:

$$\mathcal{S}_{int}[A + d\theta] = \mathcal{S}_{int}[A] + B.T. \quad \Rightarrow \quad dJ_{M|I} = 0$$

with $I = M$ or R for choice a) or b) of $\mathcal{S}_{int}$ respectively, and B.T. stands for 'boundary terms'. It will turn out to be sensible in fact to set them equal,

$$J_{M|M} := J_{M|R} := J \qquad \text{(3.91)}$$

as we will see later. As it is given that:

$$J_{M|I} := \frac{1}{6} j^{\alpha}_{M|I} \sqrt{-|f_I g|} \epsilon_{\alpha\beta\gamma\delta} dx^{\beta} dx^{\gamma} dx^{\delta}, \qquad \text{(3.92)}$$

Eq. (3.91) implies that the components of J transform in the following way:

$$j^{\alpha}_{M|M} = j^{\alpha}_{M|R} \sqrt{\frac{|f_R g|}{|f_M g|}}. \qquad \text{(3.93)}$$

For this work we decided primarily to use the matter frame action of the

theory, $\mathcal{S}_{|M} := f_M^{-1} \circ \mathcal{S}$:

$$
\begin{aligned}
\mathcal{S}_{|M}[g, \phi, A] \quad = \quad & \int d\pi(f_M^{-1}g) \left[\frac{R(f_M^{-1}g)}{2\kappa} + \mathcal{L}_\Phi(f_M^{-1}g, \phi, d\phi) \right] \\
& - \frac{1}{4\mu_0} \int d\pi(f_R f_M^{-1}g) F^2(f_R f_M^{-1}g, dA) \\
& + \int d\pi(g)\mathcal{L}_M(g) \\
& + \int J \wedge A
\end{aligned}
\tag{3.94}
$$

(where recall I have defined $\mathcal{S}_{int}$ to be invariant under frame transformations f_R, f_M) and the matter frame Maxwell equations. The composition of frame transformations $f_R(f_M^{-1}(g))$ must be another frame transformation, which will impose extra restrictions on D_M and D_R, (namely that they be suitably small) and to de-clutter presentation of the equations to follow, let:

$$
f_R f_M^{-1} g = g + (D_R - D_M)d\phi \otimes d\phi =: g + B d\phi \otimes d\phi, \tag{3.95}
$$

and the disformal scalar associated with the map $f_M^{-1} \circ f_R$ we will use heavily, so let:

$$
Z := Z_{M \to R} = \sqrt{\frac{|f_R f_M^{-1}g|}{|g|}} = \sqrt{1 - 2XB}, \tag{3.96}
$$

and so, for $f_M^{-1} \circ f_R$ to be well defined and invertible, recall from the proof of Res. 2.3.5 (setting $n = 1$) that this requires:

$$
0 < Z < 2. \tag{3.97}
$$

Finally, let g be the metric tensor such that:

$$
\frac{\delta \mathcal{S}_{|M}}{\delta g}[g, \phi, A] = 0. \tag{3.98}
$$

I will refer to this g as the **matter metric, or M metric**, and $g_R := f_R f_M^{-1} g$ I will refer to as the **radiation metric, or R metric**.

Disformal Maxwell's equations

In this section, I will show our derivation of the Maxwell equations in a simple space-time for the action above. The electromagnetic field equation can be

readily obtained from this action:

$$\partial_\alpha \left(\sqrt{-|g_R|} g_R^{\alpha\mu} g_R^{\beta\nu} F_{\mu\nu} \right) = -\mu_0 \sqrt{-|g|} j_{M|M}^\beta . \tag{3.99}$$

We would like to work with a simple space-time in which to cleanly observe the disformal effects.

Defining the space-time

Let: the M geometry $(\mathcal{M}, g)$ be flat, and select coordinates $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$ such that:

$$g = \eta, \tag{3.100}$$

where η is the Minkowski metric. (I remind the reader that matter particles of the $\mathcal{L}_M$ sector move on geodesics with respect to this metric).

Let: the scalar field ϕ depend on time only in this coordinate system:

$$\phi(x^\mu) = \phi(t). \tag{3.101}$$

Then one can show that these together imply $(\mathcal{M}, g_R)$ is also Ricci flat, a fact we will make use of in the section on vacuum bremsstrahlung.

For the vector potential, $A = A_\mu dx^\mu$, and current three form J we define:

$$A_0 := -\Phi/c, \quad (A_1, A_2, A_3) := \mathbf{A}, \tag{3.102}$$

$$j_{M|M}^0 := c\rho, \quad (j_{M|M}^1, j_{M|M}^2, j_{M|M}^3) := \mathbf{j} \tag{3.103}$$

for some Φ, $\mathbf{A}$, ρ, $\mathbf{j}$. Then, working in the **disformal Lorenz gauge**:

$$\nabla \cdot \mathbf{A} = -\dot{\Phi}/(cZ)^2, \tag{3.104}$$

the dot denoting the derivative with respect to time t, Eq. (3.99) becomes

$$\left(\nabla^2 - \frac{1}{c^2 Z^2} \frac{\partial^2}{\partial t^2} \right) \Phi = -\frac{Z}{\epsilon_0} \rho \tag{3.105a}$$

$$\left(\nabla^2 - \frac{1}{c^2 Z^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} + \frac{1}{c^2} \frac{\dot{Z}}{Z} \left(\nabla\dot{\Phi} + \dot{\mathbf{A}} \right) = -\frac{\mu_0}{Z} \mathbf{j} \tag{3.105b}$$

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where ∇ is for now the flat 3-space derivative operator associated with η. The disformal Lorenz gauge is chosen here because it produces simple equations. Of course we could consider instead something like $\nabla \cdot \mathbf{A} = -\dot{\Phi}/c^2$ and the equations would be equivalent up to a gauge transformation, but they would also be needlessly more complicated. In deriving the last equation, we made use of the identity $\nabla(\nabla \cdot \mathbf{V}) = \nabla^2 \mathbf{V} + \nabla \times (\nabla \times \mathbf{V})$ and defined in the usual way $\epsilon_0 = 1/\mu_0 c^2$. Maxwell's equations read

$$\nabla \cdot \mathbf{E} = \frac{Z}{\epsilon_0}\rho \tag{3.106a}$$

$$\nabla \times \mathbf{B} = \frac{\mu_0}{Z}\mathbf{j} + \frac{\mu_0}{Z}\frac{\partial}{\partial t}\left(\frac{\epsilon_0}{Z}\mathbf{E}\right) \tag{3.106b}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.106c}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \;, \tag{3.106d}$$

where $\mathbf{E}$ and $\mathbf{B}$ are defined in the usual way:

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \quad \text{and} \quad \mathbf{B} = \nabla \times \mathbf{A} \;. \tag{3.107}$$

Momentarily considering a vacuum (i.e. $\rho = 0$, $\mathbf{j} = \mathbf{0}$), and assuming that Z is constant, from Maxwell's equations we can derive the following wave equations for the fields $\mathbf{E}$ and $\mathbf{B}$

$$-\frac{1}{c^2 Z^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \nabla^2 \mathbf{E} = 0 \tag{3.108a}$$

$$-\frac{1}{c^2 Z^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} + \nabla^2 \mathbf{B} = 0 \;, \tag{3.108b}$$

which shows that, in the absence of charges and with Z constant, electromagnetic fields propagate with a modified speed[1]. Prompted by this observation, we define more generally

$$c_s(t) := cZ(t) = \left(c^2 - B\dot{\phi}^2\right)^{1/2}. \tag{3.109}$$

The set of equations (3.106) quite clearly suggest that we can go further; an effective speed of light here arises as a consequence of the fact that the

---

[1]It was shown in [99] that the fine-structure coupling 'constant' is not constant in this theory.

disformal couplings modify space-time geometry and hence distort the electromagnetic vacuum, producing an *effective medium* for the electromagnetic field, whose permeability, μ_0, and permittivity, ϵ_0, of free space are modified by the scalar interaction. We thus also make the definitions

$$\mu(t) := \frac{\mu_0}{Z(t)} \quad \text{and} \quad \epsilon(t) := \frac{\epsilon_0}{Z(t)} \tag{3.110}$$

to physically characterize this new effective vacuum, and, subsequently, the auxiliary fields

$$\mathbf{H} := \frac{1}{\mu(t)}\mathbf{B} \quad \text{and} \quad \mathbf{D} := \epsilon(t)\mathbf{E}. \tag{3.111}$$

Given this effective medium formulation, we can now ask how the energy density will change in the field due to time evolution of our scalar field. In terms of the auxiliary fields the first two Maxwell equations simplify as follows:

$$\nabla \cdot \mathbf{D} = \rho \tag{3.112a}$$

$$\nabla \times \mathbf{H} - \dot{\mathbf{D}} = \mathbf{j}, \tag{3.112b}$$

from which we obtain Poynting's theorem in our theory:

$$\frac{d}{dt}(U_E + U_H) = \frac{\dot{Z}}{Z}(U_E + U_H) - \mathbf{E} \cdot \mathbf{j} - \nabla \cdot (\underbrace{\mathbf{E} \times \mathbf{H}}_{\mathbf{S}}) \ . \tag{3.113}$$

Here we have defined the field energy densities

$$U_E := \frac{1}{2}\epsilon(t)|\mathbf{E}|^2, \qquad U_H := \frac{1}{2}\mu(t)|\mathbf{H}|^2 \ , \tag{3.114}$$

and identified the standard Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, which we will use to compute the energy lost by a charged particle in superluminal flight in the next section.

To summarise, we have found that when the scalar is time dependent only, our field theory with disformal couplings reduces to that of an electromagnetic field in an effective linear medium, whose resistance to the formation and evolution of field disturbances (ϵ, μ) will depend on $Z(t)$: the ratio of the two metric determinants. This establishes an interesting conceptual link between the geometry of space and the physical response of the fields defined

on it. The link should strengthen the reader's intuition that many analogues of electricity in linear media should carry through to this model.

I would like to make the following observation: here we had little trouble repackaging the M frame Maxwell's equations of some MCSTt as a varying constant theory. It in fact presented itself to us during the early stages of the project as the natural formulation of the equations. This is in stark contrast to the last section, especially kinetic theory, where the theory refused to be packaged as such. We ran into contradictions with bad definitions when struggling to do what emerged naturally here. I have my suspicion now, some years later, that the Einstein frame stress tensor T itself is simply a bad definition. Notice it is not mentioned in this work, and everything seems to fit together. I must confess that, after all this time, I still have no idea what T physically means.

Field solutions and the Cherenkov radiation condition

As a first application of the model, we will investigate under which circumstances Cherenkov radiation can occur. We follow the calculation in [134]. The speed of light c_s, given by Eq. (3.109), is smaller than the bare speed of light c if the field evolves in time, i.e. if $\dot{\phi}$ is non-vanishing. A charged particle can then move faster than c_s and this is the situation which we will now study. Let us therefore consider a moving particle with charge q, for which

$$
\begin{aligned}
\rho(\mathbf{x}, t) &= q\delta(\mathbf{x} - \mathbf{x}_p(t)) & \text{(3.115a)} \\
\mathbf{j}(\mathbf{x}, t) &= \rho\mathbf{v} \ , & \text{(3.115b)}
\end{aligned}
$$

with $\mathbf{x}_p(t)$ the time dependent position in 3-space of the moving particle, and $\mathbf{v} = \dot{\mathbf{x}}_p$ the velocity. Furthermore, we assume in this section that $\phi = \phi(t)$ with $c_s = cZ = \text{constant}$.

Then, considering the Fourier space components one obtains from Eq.(3.105a)

$$
\Phi_k(\omega, \mathbf{k}) = \frac{2\pi q}{\epsilon} \frac{\delta(\omega - \mathbf{k} \cdot \mathbf{v})}{k^2 - \dfrac{\omega^2}{c_s^2}}, \tag{3.116}
$$

where ω is the frequency and $\mathbf{k}$ the wave vector of each plane wave mode of

the Fourier decomposition, and Eq.(3.105b) can be solved to find

$$\mathbf{A}_k = 2\pi q\mu \frac{\delta(\omega - \mathbf{k} \cdot \mathbf{v})}{k^2 - \dfrac{\omega^2}{c_s^2}}\mathbf{v} \ . \tag{3.117}$$

As a consistency check, these solutions imply the disformal Lorenz gauge condition $\mathbf{k} \cdot \mathbf{A}_k = \omega\Phi_k/c_s^2$. The Fourier coefficients of $\mathbf{B}$ are related to $\mathbf{A}_k$ via $\mathbf{B}_k = i\mathbf{k} \times \mathbf{A}_k$ and the Fourier coefficients of $\mathbf{E}$ are given by $\mathbf{E}_k = -i\mathbf{k}\Phi_k + i\omega\mathbf{A}_k$. We find

$$\mathbf{B_k}(\mathbf{k}, \omega) = 2\pi i q\mu \frac{\mathbf{k} \times \mathbf{v}}{k^2 - \dfrac{\omega^2}{c_s^2}}\delta(\omega - \mathbf{k} \cdot \mathbf{v}) \tag{3.118}$$

and

$$\mathbf{E}_k(\mathbf{k}, \omega) = -\frac{2\pi i q}{\epsilon} \frac{\mathbf{k} - \dfrac{\omega}{c_s^2}\mathbf{v}}{k^2 - \dfrac{\omega^2}{c_s^2}}\delta(\omega - \mathbf{k} \cdot \mathbf{v}) \ . \tag{3.119}$$

To find the energy loss along the particle's trajectory, we assume without the loss of generality that the particle moves along the z-axis with velocity $\mathbf{v} = (0, 0, v)$, and that the observer is located at a distance r from the z-axis. The energy loss per unit length is then given by the integral

$$-\frac{d\mathcal{E}}{dz} = -2\pi r \int E_z(\mathbf{r}, t)B_\phi(\mathbf{r}, t)dt = -r \int E_z(\mathbf{r}, \omega)H_\phi(\mathbf{r}, \omega)d\omega, \tag{3.120}$$

where

$$
\begin{aligned}
E_z(\mathbf{r}, \omega) &= \frac{1}{(2\pi)^3} \int d^3k E_z(\mathbf{k}, \omega)e^{i\mathbf{k}\cdot\mathbf{r}} \quad \text{and} \\
H_\phi(\mathbf{r}, \omega) &= \frac{1}{(2\pi)^3} \int d^3k H_\phi(\mathbf{k}, \omega)e^{i\mathbf{k}\cdot\mathbf{r}}.
\end{aligned} \tag{3.121}
$$

Evaluating the integrals for $\beta = v/c_s > 1$, we find

$$
\begin{aligned}
E_z(\mathbf{r}, \omega) &= \frac{iq\mu\omega}{2\pi}\left[1 - \frac{1}{\beta^2}\right]e^{i\omega z/\beta c_s}K_0(\alpha r), \tag{3.122} \\
H_\phi(\mathbf{r}, \omega) &= \frac{\alpha q}{2\pi}e^{iz\omega/\beta c_s}K_1(\alpha r) \ , \tag{3.123}
\end{aligned}
$$

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where $\alpha = -(i\omega/c_s)\sqrt{1-\beta^{-2}}$. Note that for large αr

$$K_0(\alpha r) \approx \sqrt{\pi/(2\alpha r)}\exp(-\alpha r), \qquad (3.124)$$

so these represent outgoing waves if $\beta > 1$. The expressions for E_z and H_ϕ are identical to those for electromagnetic waves propagating through a medium, leading to Cherenkov radiation for $v > c_s$. The integral (3.120) can be evaluated for $|\alpha|r \gg 1$, giving

$$-\frac{d\mathcal{E}}{dz} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{c^2}\int \omega\left(1 - \frac{1}{\beta^2}\right)d\omega \ . \qquad (3.125)$$

Constraints

There are direct constraints on isotropic deviations of the speed of light from unity from laboratory experiments [135,136] at the level of $|1 - c_s/c| < 10^{-10}$, however stronger constraints arise from searches for Cherenkov radiation from particles in vacuum. These can be done in terrestrial experiments, with bounds $|1 - c_s/c| < 10^{-11}$ coming from the absence of vacuum Cherenkov radiation from 104.5 GeV electrons and positrons at LEP [137]. Indeed, the energetics of the LEP beam were so well understood that measurements of the synchrotron emission rate indicate that any deviation of the speed of photons is constrained by $|1 - c_s/c| < 5 \times 10^{-15}$, [138]. Observations of high energy cosmic rays provide significantly tighter constraints; the lack of vacuum Cherenkov radiation from high energy electrons and neutrinos propagating over astronomical distances constrains $|1 - c_s/c| < 10^{-20}$ [139, 140], however these constraints come with some uncertainty about the high energy dynamics of the source of the cosmic ray.

To translate these constraints into constraints on disformal electrodynamics, we assume now that the scalar field is slowly evolving and plays the role of dark energy. Firstly, we assume the constraint $|1 - c_s/c| < 5 \times 10^{-15}$. The speed of light c_s should not deviate drastically from one, so we can expand $Z \approx 1 - B\dot{\phi}^2/2c^2$. Now, for simplicity we let the gravitational and matter metrics coincide, that is we set $D_M = 0$.

The Friedmann equation evaluated today reads

$$3H_0^2 = \kappa\left(\rho c^4 + \frac{1}{2}\dot{\phi}^2 + c^2V\right) \qquad (3.126)$$

for the bare speed c: the constant light speed as defined with respect to the radiation frame. If we assume that the scalar ϕ plays the role of dark energy then we have

$$\Omega_{\mathrm{DE}} = \frac{\kappa}{6}\left(\frac{\dot{\phi}}{H_0}\right)^2 + \frac{\kappa c^2 V}{3H_0^2} \simeq 0.7, \qquad (3.127)$$

where Ω_{DE} is the dark energy density parameter. The equation of state of dark energy is

$$w_{\mathrm{DE},0} = \frac{\dot{\phi}^2 - 2c^2 V}{\dot{\phi}^2 + 2c^2 V} \qquad (3.128)$$

which, combined with equation (3.127) gives

$$\frac{\kappa\dot{\phi}^2}{2c^2} = \frac{3}{2}\Omega_{\mathrm{DE}}H_0^2(1 + w_{\mathrm{DE},0}). \qquad (3.129)$$

Hence, the constraint can be written as $B\dot{\phi}^2/2c^2 < 5 \times 10^{-15}$ or, expressed as a dimensionless ratio:

$$\frac{BH_0^2}{\kappa} < \frac{10^{-14}}{3\Omega_{\mathrm{DE}}(1 + w_{\mathrm{DE},0})}. \qquad (3.130)$$

In Fig. 3.3 we show the constraint on the energy scale:

$$M := \left(\frac{c\hbar^3}{B}\right)^{1/4} \qquad (3.131)$$

as a function of the dark energy equation of state $w_{\mathrm{DE},0}$, measured today, setting $\Omega_{\mathrm{DE}} = 0.7$. We remind the reader that constraints of this type will always place limits on the difference between the disformal couplings to matter and radiation, since $B = D_R - D_M$, though we have set $D_M = 0$ here.

3.2.2 Vacuum Bremsstrahlung

We have seen that the particle will emit Cherenkov radiation in vacuum, if the effective speed of light c_s drops below the particle speed v. A natural question to ask, given the close resemblance at the classical level our model has with that of a linear dielectric medium, is whether or not other radiative channels are open in the presence of a disformal coupling.

The aim of the derivation here presented is singularly to 'prove' the following result statement:
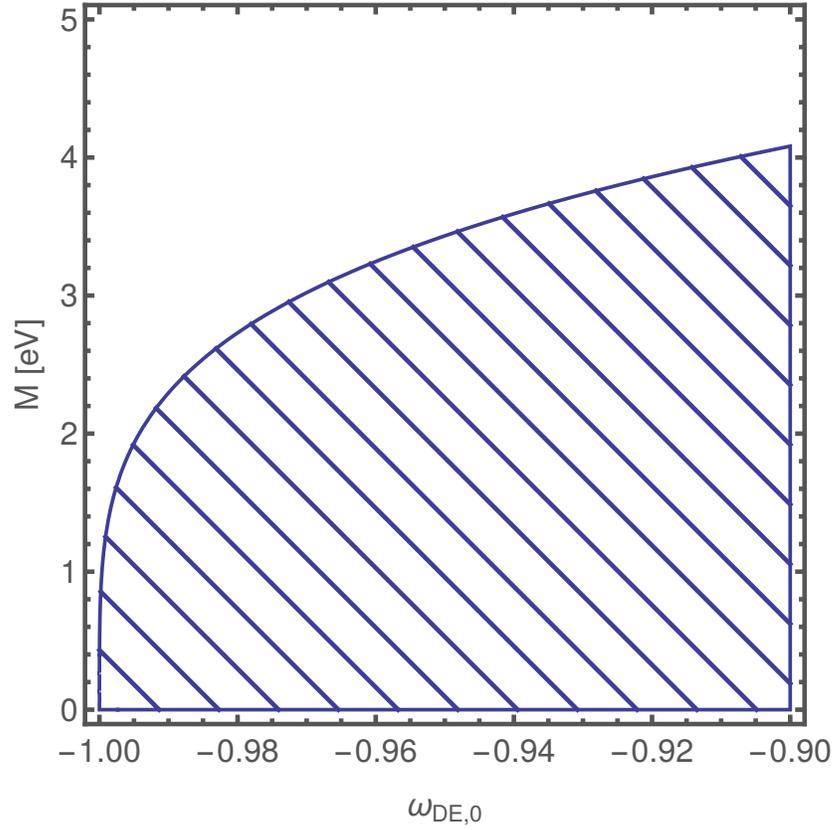
Figure 3.3: Cherenkov radiation in vacuum constraints the energy scale M, defined in Eq. (3.131), as a function of the current dark energy equation of state $\omega_{\text{DE},0}$. The shaded region is ruled out by bounds coming from the LEP constraint $|1 - c_s/c| < 5 \times 10^{-15}$. As the dark energy equation of state approaches -1, $\dot{\phi}$ approaches 0 and hence $c_s \to c$ and the constraint on M vanishes in this limit.

> A charged particle on an expanding background in motion—
> uniform as observed in the particle's own frame—will in general
> radiate if the electromagnetic field couples to a second, distinct
> geometry, disformally related to the first.

Much about the language used here is at this stage unclear. I write this result
statement as more of a referral for later, to anchor the section's purpose in
case it gets confused in all the detail, and its precise meaning should become
clear as the mathematics are introduced. It is this effect that we named
vacuum bremsstrahlung and here the derivation will provide the conditions
for its presence: the radiation condition.

We are particularly interested in the possibility that charged cosmic rays
might emit bremsstrahlung due to the evolution of the scalar ϕ in the cosmo-
logical background. I therefore now show how we generalised our calculations
to an expanding background with c_s now time dependent.

Disformal Maxwell's equations in expanding space

As before, I will now show the derivation of the Maxwell equations, but
this time for an expanding universe. The covariant M frame field equations
derived from $\mathcal{S}_{|M}$ are as before:

$$\partial_\alpha \left(\sqrt{-|g_R|} g_R^{\alpha\mu} g_R^{\beta\nu} F_{\mu\nu} \right) = -\mu_0 \sqrt{-|g_M|} j_{M|M}^\beta, \tag{3.132}$$

and with a view to applying what we learn to cosmic rays in cosmology, we
chose the following space-time. Once defined, I will show that the Maxwell
equations become tractable with a well chosen set of field and coordinate
re-definitions. These, it will turn out, are exactly the transformations to
deliver us the R frame Maxwell equations. I will highlight the presence of
a radiative term in the electric field solution of these equations. I will then
reverse the map between solutions of R and M frames, and demonstrate that
presence of the radiation is not an artefact of frame choice, but a physical
phenomenon.

Defining the space-time II

Let: the M geometry $(\mathcal{M}, g_M)$ be a flat universe, and select now comoving
coordinates $(x^0, x^1, x^2, x^3) = (c\tau, x, y, z)$ such that:

$$g_M(\tau) = a^2(\tau)\eta, \tag{3.133}$$

where $\eta = diag(-1, 1, 1, 1)$ is again the Minkowski metric, and τ a conformal time. This implies:

$$g_R = a^2 \left(\eta + \frac{B}{a^2} d\phi \otimes d\phi \right) = a^2 h_R \tag{3.134}$$

where the metric h_R has been defined:

$$h_R := \eta + \frac{B}{a^2} d\phi \otimes d\phi. \tag{3.135}$$

$h_R^{\mu\nu}$ will denote the components of h_R in some basis, and $|h_R|$ the determinant of h_R, again in some specified basis.

Let: the scalar field ϕ depend on coordinate time only in this coordinate system:

$$\phi(x^\mu) = \phi(\tau). \tag{3.136}$$

Given these coordinates, we have that:

$$\begin{aligned}
\partial_\alpha \left(\sqrt{-|g_R|} g_R^{\alpha\mu} g_R^{\beta\nu} F_{\mu\nu} \right) &= \partial_\alpha \left(\sqrt{-|a^2 h_R|} a^{-2} h_R^{\alpha\mu} a^{-2} h_R^{\beta\nu} F_{\mu\nu} \right) \\
&= \partial_\alpha \left(h_R^{\alpha\mu} h_R^{\beta\nu} F_{\mu\nu} \right),
\end{aligned} \tag{3.137}$$

and for the interaction term, we define the *comoving current*

$$\sqrt{-|\eta|} J^\mu := \sqrt{-|g|} j^\mu_{M|M}, \tag{3.138}$$

so that

$$J = \frac{1}{6} J^\mu \epsilon_{\mu\beta\gamma\delta} dx^\beta \wedge dx^\gamma \wedge dx^\delta, \tag{3.139}$$

and as $dJ = 0$ (see section 3.2.1), we have that the comoving current is conserved with respect to the flat metric η, i.e.

$$\begin{aligned}
dJ = 0 \quad \Rightarrow \quad & \frac{1}{6} \left(\frac{\partial J^\mu}{\partial x^\alpha} dx^\alpha \right) \wedge \epsilon_{\mu\beta\gamma\delta} dx^\beta \wedge dx^\gamma \wedge dx^\delta \\
& = \frac{\partial J^\mu}{\partial x^\mu} dx^0 \wedge ... \wedge dx^3 = 0 \\
\Rightarrow \quad & \frac{\partial J^\mu}{\partial x^\mu} = 0. \tag{3.140}
\end{aligned}$$

Lastly, we consider a point-like charged particle whose motion can be described by a curve $\mathbf{x}_p(\tau)$, and, since $\partial_\mu J^\mu = 0$, we can define:

$$J^0 := c\Omega, \quad (J^1, J^2, J^3) := \mathbf{J}, \tag{3.141}$$

such that

$$\Omega(\mathbf{x}, \tau) \;\; := \;\; Q\delta(\mathbf{x} - \mathbf{x}_p(\tau)) \tag{3.142a}$$
$$\mathbf{J}(\mathbf{x}, \tau) \;\; := \;\; \Omega\mathbf{V} \tag{3.142b}$$

for $\mathbf{V} := d\mathbf{x}_p/d\tau$ and the constant Q the charge of the particle. By construction this ansatz satisfies the continuity equation. Comparing this to the physical current, expressed in terms of the physical time, t, it is straightforward to show that $j^{\mu\prime} = (c\Omega/a^3, \mathbf{v}\Omega/a^3)$, and hence the charge density dilutes as a^{-3}, as it must in isotropically expanding space. It is also clear that, for $a(\tau)$ an arbitrary function, light still propagates with velocity

$$c_s(\tau) = Z(\tau)c. \tag{3.143}$$

3.2.3 Disformal Maxwell's equations in expanding space

The electromagnetic field equations in these coordinates can be readily obtained from the covariant form above as before; they are the expanding-space counterpart to Eq.s (3.106):

$$\nabla \cdot \mathbf{E} \;\; = \;\; \frac{Z}{\epsilon_0}\Omega \;, \tag{3.144a}$$

$$\nabla \times \mathbf{B} \;\; = \;\; \frac{\mu_0}{Z}\mathbf{J} + \frac{\mu_0}{Z}\frac{\partial}{\partial\tau}\left(\frac{\epsilon_0}{Z}\mathbf{E}\right) \;, \tag{3.144b}$$

$$\nabla \cdot \mathbf{B} \;\; = \;\; 0 \;, \tag{3.144c}$$

$$\nabla \times \mathbf{E} + \frac{\partial\mathbf{B}}{\partial\tau} \;\; = \;\; 0 \;. \tag{3.144d}$$

In these equations, ∇ is the *flat* 3-space derivative operator. Even though space is expanding, this is valid, as the dependance of the system on the scale factor a was absorbed by the field redefinitions in the previous section.

From definition (3.109) we see that if the speed of light were to vary in time in some coordinate system with time t, there would naturally exist some new system of coordinates such that this speed remains constant. If

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the particle were to travel with fixed velocity in the original system, it would appear to accelerate with respect to these new coordinates in which c_s is constant. The electric field thus 'sees' an accelerating charge. We would expect such a field to radiate accordingly, and indeed this is what we will find.

To make this intuition mathematically precise, we must consider a case more general than the previous sections, whereby $Z(t)$ becomes now an arbitrary function of time. Some suitable field and coordinate redefinitions will help us find solutions in this new case. Considering again the disformal Maxwell equations, (3.144), the following redefinitions are useful:

$$\widetilde{\mathbf{E}} := \frac{\mathbf{E}}{Z(\tau)}, \quad \tilde{\mathbf{J}} := \frac{\mathbf{J}}{Z(\tau)}, \quad d\tilde{\tau} := Z(\tau)d\tau. \tag{3.145}$$

These fields obey the following equations:

$$\nabla \cdot \widetilde{\mathbf{E}} = \frac{\Omega}{\epsilon_0}, \tag{3.146a}$$

$$\nabla \times \mathbf{B} = \mu_0\tilde{\mathbf{J}} + \mu_0\epsilon_0\frac{\partial}{\partial\tilde{\tau}}\left(\widetilde{\mathbf{E}}\right), \tag{3.146b}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3.146c}$$

$$\nabla \times \widetilde{\mathbf{E}} + \frac{\partial\mathbf{B}}{\partial\tilde{\tau}} = 0. \tag{3.146d}$$

This set of equations allows us make the standard gauge field definitions: $\mathbf{B} = \nabla \times \mathbf{A}$ as before, and now

$$\widetilde{\mathbf{E}} = -\nabla\tilde{\Phi} - \mathring{\mathbf{A}} \tag{3.147}$$

where '$\circ$' denotes the derivative with respect to $\tilde{\tau}$. Then, working again in the disformal Lorenz gauge: $\nabla \cdot \mathbf{A} = -\mathring{\tilde{\Phi}}/c^2$, we arrive at the field-potential equations of motion:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial\tilde{\tau}^2}\right)\tilde{\Phi} = -\frac{\Omega}{\epsilon_0} \tag{3.148a}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial\tilde{\tau}^2}\right)\mathbf{A} = -\mu_0\tilde{\mathbf{J}}. \tag{3.148b}$$

The system (3.148) is closed, and is now instantly recognisable from classical electrodynamics, hence easily solvable. Important to note here is that these

tilde variables and coordinates just defined are exactly those we would have, had we started with $\mathcal{S}_{|R}$ and continued to work on the $(\mathcal{M}, g_R)$ space-time—we are now working with the R frame Maxwell equations.

Field solutions and the bremsstrahlung condition

The system of equations, (3.148), is readily satisfied by the Liénard-Wiechert potentials [141]. In terms of our electromagnetic frame quantities—recalling that our coordinates are all comoving—these solutions read

$$\tilde{\Phi}(\mathbf{x}, \tilde{\tau}) = \frac{Q}{4\pi\epsilon_0} \frac{1}{[1 - \mathbf{n}(\tilde{\tau}') \cdot \boldsymbol{\beta}(\tilde{\tau}')]} \frac{1}{X(\tilde{\tau}')} \tag{3.149a}$$

$$\mathbf{A}(\mathbf{x}, \tilde{\tau}) = \mu_0 \epsilon_0 \mathbf{V}(\tilde{\tau}') \tilde{\Phi} \tag{3.149b}$$

where $\tilde{\tau}'$ is the retarded R frame time, defined by the implicit equation

$$(\tilde{\tau}' - \tilde{\tau})c + X(\tilde{\tau}') = 0, \tag{3.150}$$

and we have made the following definitions

$$X(\tau) := |\mathbf{x} - \mathbf{x}_p(\tau)|, \quad \mathbf{n}(\tau) := \frac{\mathbf{x} - \mathbf{x}_p(\tau)}{X(\tau)},$$

$$\boldsymbol{\beta}(\tau) := \frac{\tilde{\mathbf{V}}(\tau)}{c} = \frac{\mathbf{V}(\tau)}{c_s(\tau)}. \tag{3.151}$$

We can see that $\boldsymbol{\beta}$ is a frame invariant quantity![2] Combining (3.147) with (3.149) and reversing the field redefinitions gives the following electric field profile in the M frame:

$$\mathbf{E}(\mathbf{x}, \tau) = \frac{Q}{4\pi\epsilon(\tau)} \left[\frac{(1 - \beta^2)(\mathbf{n} - \boldsymbol{\beta})}{X^2[1 - \mathbf{n} \cdot \boldsymbol{\beta}]^3} + \frac{\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{c_s X[1 - \mathbf{n} \cdot \boldsymbol{\beta}]^3} \right]_{\text{ret}}, \tag{3.152}$$

and, also in the M frame:

$$\mathbf{B}(\mathbf{x}, \tau) = \left[\mathbf{n} \times \frac{\mathbf{E}}{c_s} \right]_{\text{ret}} \tag{3.153}$$

---

[2]Note that $\mathbf{n}$ is frame invariant almost trivially: we do not coordinate transform when we perform a frame transformation, and further $\mathbf{n}$ is only defined on the constant time hypersurfaces. The specific frame transform employed here distorts the relation between these hypersurfaces but leaves each individually untouched.

where we have reverted back to the τ time derivative, ''', and quantities enclosed in the square brackets, $[...]_{\text{ret}}$, are to be evaluated at the retarded time τ' given implicitly by equation (3.150) together with the relationship between τ and $\tilde{\tau}$

$$\tau = \int \frac{d\tilde{\tau}}{Z(\tilde{\tau})} \ , \tag{3.154}$$

which is non-local, and not analytically solvable in general. In the M frame, the Poynting vector as obtained from Poynting's theorem is:

$$\mathbf{S} = \mathbf{E} \times \frac{\mathbf{B}}{\mu(\tau)} \tag{3.155}$$

and so, for a charged particle on a straight trajectory ($\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} = 0$) the comoving power radiated, i.e. the power radiated per unit conformal time, is:

$$\mathcal{P} = \frac{1}{4\pi\epsilon(\tau)} \frac{2Q^2}{3c_s(\tau)} \frac{1}{(1 - \beta^2)^3} |\dot{\boldsymbol{\beta}}|^2, \tag{3.156}$$

obtained from (3.155).

The second (radiative) term in equation (3.152) will be non-zero if $|\dot{\boldsymbol{\beta}}| \neq 0$. We can clearly see that, from the definition of $\boldsymbol{\beta}$ in equation (3.151), this can be true even when the particle is not accelerating. If the comoving velocity $\mathbf{V}$ is constant, then $\dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \, \dot{c_s}/c_s$ and electromagnetic radiation will still carry energy outward, away from the particle. We sum this result up in the following radiation condition: a charged particle on an expanding background in motion—uniform as seen by a stationary observer on the same background—will in general radiate if the electromagnetic field couples to a second, distinct geometry, disformally varying with respect to the first: that is if $\dot{c_s} \neq 0$.

We see in this setup that if the scalar field ϕ evolves in time (with $\dot{Z}$ non-zero), the particle appears to the electric field as one that is accelerating, even when in vacuum and hence radiates; to this phenomenon we attach the name vacuum bremsstrahlung. All this effect requires, really, is the condition that the speed of light vary with time. In fact, though we have demonstrated the presence of vacuum bremsstrahlung in a disformally coupled field setting, it will no doubt be more widely applicable. We expect any theory in which the speed of light is dynamical in this sense to exhibit this phenomena, and hence to be testable in this way. We shall see in the next section, however, that for our theory, the effect is much smaller than Cherenkov radiation.
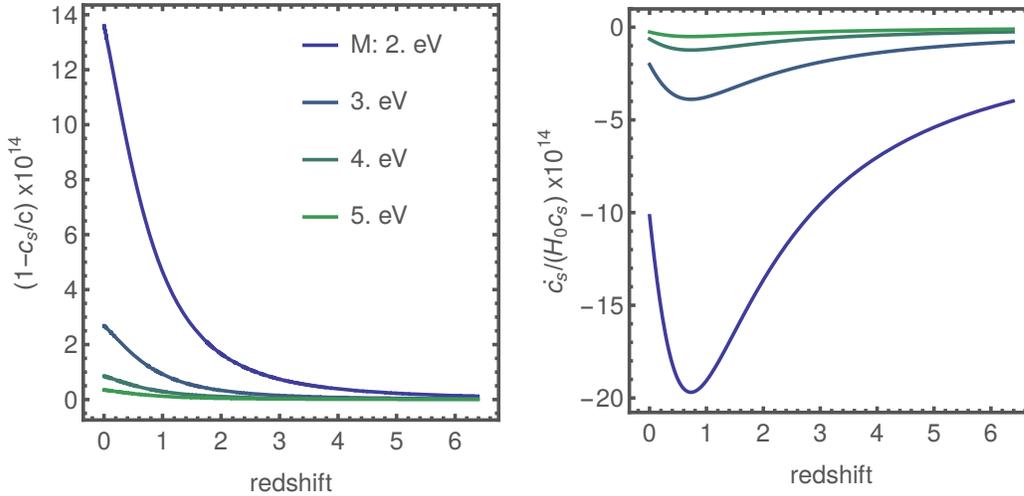
Figure 3.4: Cosmological evolution of the observed speed of light, c_s, with redshift for values within LEP constraints (see Sec. 3.2.1). M is the energy scale associated to the disformal coupling—defined in Eq. (3.131)—between light and a quintessence scalar field with exponential potential: $V(\phi) = V_0\exp[-\phi\kappa^{1/2}]$.

It was pointed out to us by Robert Blaga after posting our first draft to the arXiv that the condition $\mathbf{V}$ is constant is not trivial: it is not a geodesic of an expanding space time. As he pointed out, geometric bremsstrahlung, the radiation emitted by a particle whose velocity is not constant due to the curved background as described by general relativity [142], has been studied elsewhere, and he directed us to some interesting papers on the subject [143–146]. I will neglect this effect, and refer the reader to these references for more information.

Energy lost from a coupled cosmic ray

We will consider an ultra-high energy cosmic ray (a ray with energy in excess of about 10^{15} eV) in what follows (for an extensive review of cosmic rays, see [147]). This means we can safely assume the cosmic ray travels along a straight line geodesic, that is $\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} = 0$; intergalactic magnetic fields are

extremely weak and, while a non-negligible effect on cosmic rays is to be expected [148], it is certainly small enough to ignore for this preliminary study. The radiation condition for expanding space is thus: vacuum bremsstrahlung will occur if $\dot{c}_s \neq 0$, even when the comoving velocity—and hence the physical velocity—is constant. In this case, we have that $\dot{\boldsymbol{\beta}} = \boldsymbol{\beta}\dot{Z}/Z$ and so both the square of the factor $\dot{Z}/Z$ and the sixth power of the Lorentz factor, $(1 - \boldsymbol{\beta}^2)^{-3}$, will determine the magnitude of energy lost by the particle through this process.

If the scalar field ϕ is responsible for the late time accelerated expansion of the universe, then the cosmic ray's bremsstrahlung energy loss will be suppressed by the Hubble scale as measured in the present epoch (within a few redshift). Further, our model must obey the LEP constraints imposed on it in Sec. 3.2.1 which translates to an energy scale M of roughly eV or above. Both factors drive the allowed values of $\dot{Z}/Z$ down to very small values indeed for any viable cosmology scenario. I show in Fig. 3.4 several allowed evolution histories of the speed of light for a simple extension to the standard cosmological model, whereby the dark energy field is driven by an exponential potential with mild negative slope: $V(\phi) = V_0\exp[-\phi\kappa^{1/2}]$.

We see from the left panel of Fig. 3.4 that $\dot{c}_s/c_s$ must be very small—many orders of magnitude *less* than the Hubble scale at $H_0 \simeq 10^{-42}$ GeV! Observations of ultra high energy cosmic rays [149, 150] tell us we can not neglect cases of charged particles with energy in excess of, say, a PeV, and bounds on M from LEP mean that the expression for radiated power, Eq. (3.156) is valid up to very high velocity, but not above that for a few PeV, when vacuum Cherenkov radiation radically alters the nearby electric field behaviour. In these cases the Lorentz factor is huge, and Eq. (3.156) shows the amount of power radiated by the ray is highly sensitive to the size of the Lorentz factor, yet, in Fig. 3.5 it is clear that this is not enough to beat the Hubble scale suppression. See also Fig. 3.6.

For most of these cosmic rays a source within the Milky Way is highly unlikely. More probable: they were accelerated by jets protruding from the active nuclei of quasars, some of which have been recorded by the Sloan Digital Sky Survey at cosmic distances of redshift up to about $z \simeq 6$ [151]. However, the right panel of Fig. 3.5 shows that even the integrated energy loss across a distance this large is suppressed by the Hubble scale (as could perhaps be inferred from dimensional analysis.) I conclude this section by remarking that such an effect will never be practically measurable if the disformal coupling is to dark energy. The Hubble scale today is so far from
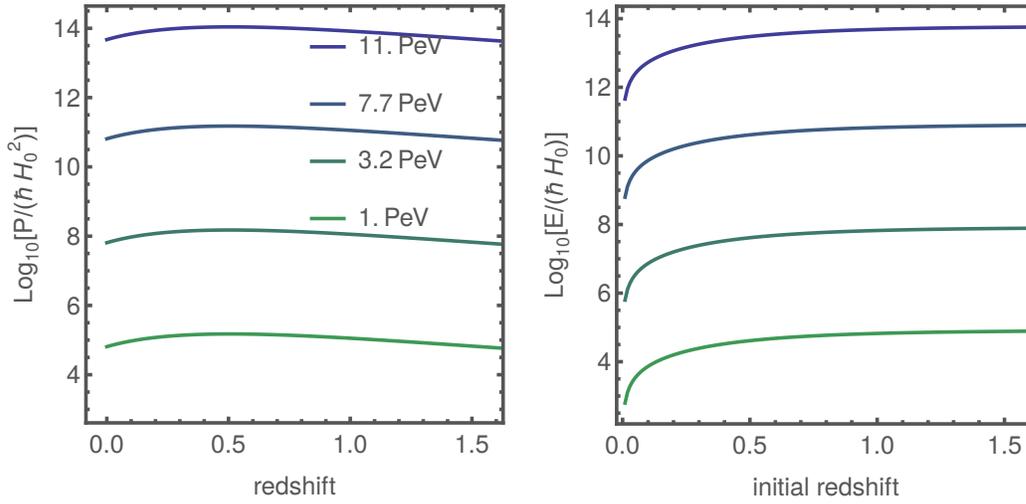
Figure 3.5: (Left:) Energy radiated per unit cosmic time $P = \mathcal{P}a$ at each redshift, where $\mathcal{P}$ is defined in Eq. (3.156). (Right:) Integrated energy loss by the particle for its entire trajectory, beginning at some initial redshift, arriving at earth today. In both plots, each curve corresponds to a cosmic ray with relativistic energy shown in the legend; PeV= 10^{15} eV. The disformal energy scale, M, is fixed at 2 eV for the left and right panels.
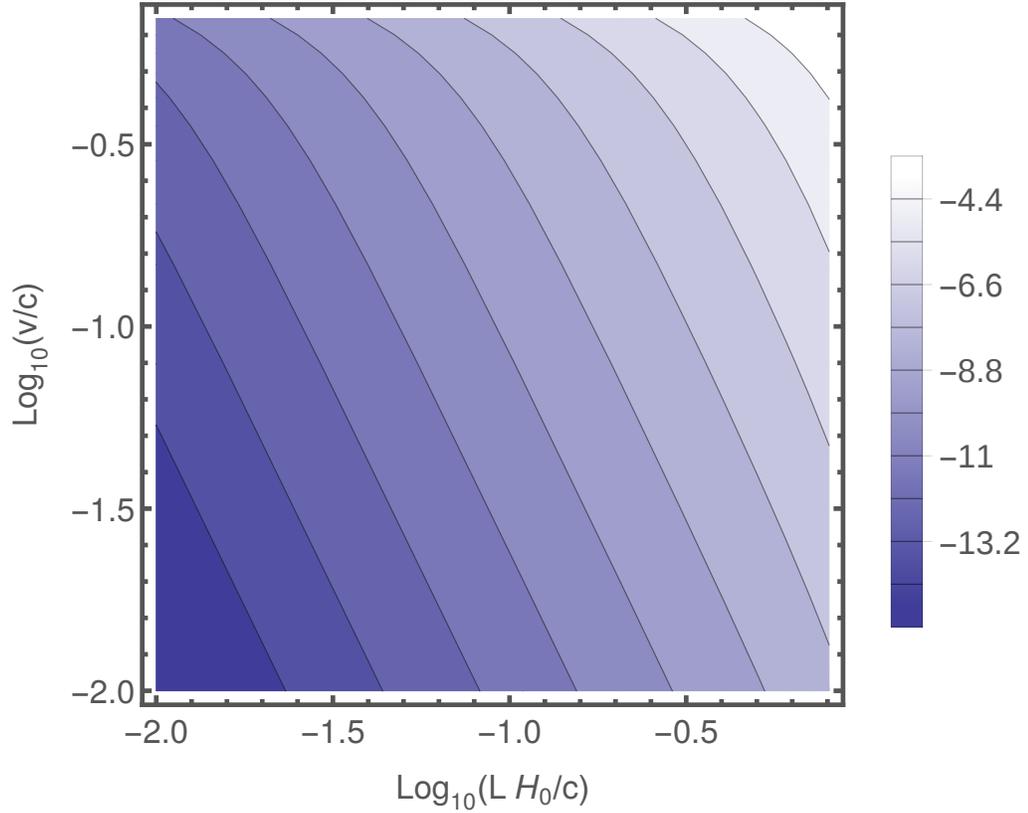
Figure 3.6: How bremsstrahlung radiated power depends on particle speed and disformal coupling strength. (Right colour bar) $max_{z \in [0,1.5]} [\log_{10}(\mathcal{P})]$ which is the maximum power radiated by a cosmic ray in flight due to vacuum bremsstrahlung (see Eq. (3.156)) over some cosmic distance given by $z \in [0, 1.5]$ (see figure 3.5 above). The dependency of $max(\mathcal{P})$ on the particle speed, v, (Y-axis) and disformal interaction length, L, (X-axis) is shown.

any of those observed in particle physics that a second order effect in a dynamic speed of light theory like vacuum bremsstrahlung will be negligible. For any such coupling or dynamic light speed in inflation is, however, a different story. The scale of some disformal inflation models (see for example [152]) may be large enough that during or just after reheating these effects must be taken into account.

3.2.4 Conclusions

Carsten, Clare and I have shown that disformal couplings allow charged particles to emit Cherenkov radiation and bremsstrahlung in vacuum. The distortion of causal structure by the scalar field, a characteristic consequence of these interactions, can cause the speed of photons to be lower than that of a charged particle—and even to vary in time—mimicking a dielectric medium. Such a varying constant interpretation was what allowed us to conjecture the result in the first place, and indeed the expression for the power radiated now became a result we could have even plausibly guessed! (Of course it would not have been advisable to do so without the support of a calculation).

To demonstrate this, we developed a theory of electrodynamics in which a scalar field couples disformally to photons and charged particles, on both flat and expanding backgrounds. Unless the coupling strengths to each species are forced to be equal, two distinct frame actions appear in the theory, each with a specific role: working out observational quantities, such as the observed speed of light, required use of the frame in which matter is uncoupled from the scalar (i.e. the matter, or M, frame), but photons in general are not. Calculations were found to be simplest however, especially for a time dependent coupling, in the R frame, where freely falling photons always follow geodesics.

Working in flat space, we determined the constraints on dark energy models with disformal couplings that arise from the non-observation of vacuum Cherenkov radiation by the LEP collaboration. These parameter-space bounds are complementary to those obtained from spectral distortions of the CMB [105] (and section 3.1 of this thesis); they each cover different regions and agree across their intersection. Finally, we were able to show that the dark energy fine tuning problem is a problem for vacuum bremsstrahlung detection also: suppression of this particle physics interaction by the Hubble scale is unbeatable for any conceivable measurement one could dream of

making on the earth's cosmic ray flux.

That these results cease to be a surprise when formulated in terms of varying constants indicates this interpretation is the natural one for the theory described above: indeed it is the hallmark of any natural formulation of a theory that novel results become almost tautologies. This, to me, validates our choice to absorb field dependence into μ_0 and ϵ_0.

In this study, we have converted the bounds on maximum attainable velocities of particles obtained by the LEP group to those on the scalar field coupling interaction M via the Friedman equation. Explicitly, we: a) assumed our gravity sector was as simple as possible (quintessence with exponential potential, uncoupled to matter) and: b) produced constraints that are dependent on the measured dark energy equation of state today. This work should thus be extended along these two lines. How sensitive are these limits to changes in the gravitational sector of the theory? And can a more in-depth cosmological analysis remove the dependence of the bounds on the equation of state? The interplay between particle physics and cosmology has so far been exceedingly rich, and constraining cosmological parameters using results from ground-based particle experiments in this fashion remains a surprisingly fruitful venture.

Couplings to dark matter

<div style="text-align: right; font-size: 3em;">4</div>

Observations of the cosmic microwave background radiation (CMB) and large scale structures have allowed cosmologists to formulate a model of cosmology in which the standard model particles are a subdominant matter form. The model predicts the existence of dark matter which only interacts very weakly with itself and the other matter particles. In addition, the model requires an energy form with negative pressure, dubbed dark energy, which is responsible for the accelerated expansion of the universe in the present epoch (see [153] for the 2013 results of the Planck mission). A major task of present day cosmology is to illuminate the properties of dark matter and dark energy.

In the preceding chapters I have built up the idea of scalar tensor theories and their multi-coupled extensions, and they are here applied to the dark sector of cosmology; cosmologists do not know a priori the gravitational interactions of the dark sector are well described by general relativity, and so a separate geometry for dark matter is not unthinkable, indeed some brane world theories predict it so [78]. MCSTts give us a controlled way in which to model scenarios where dark matter's gravitational interactions are more general than general relativity, while at the same time avoiding stringent bounds of modified gravitational interactions that apply to the visible constituents of the universe, for example solar system tests [86]. For the publication of the results described in this section, see [133].

Motivated by the many drawbacks of cosmological constant dark energy, some of them we have already discussed, cosmologists have studied other possibilities, such as dynamical scalar fields, or modified theories of gravity. In the work described here, we focused on a union of the two: the case of a scalar field as a dark energy candidate, which modifies the force of gravity.

In such models, couplings to all matter/energy forms are expected unless symmetries exist which forbid or suppress interactions, yet, problematically, a scalar field coupled to matter would mediate a long range fifth force between the different particles, a force which is not observed in nature [154]. Such non-detection implies that the coupling to baryons must be very small, whereas constraints on coupling to neutrinos and dark matter are substantially weaker, and must be obtained from cosmological observations. And, very recently, evidence has emerged to suggest that an interaction between elements of the dark sector is not just plausible, but actually favoured by current data ([155] and [156]). The analysis was in each case purely phenomenological, assuming a minimal amount of underlying theory, yet it is a progressive step toward understanding the nature of these invisible elements of our Universe.

In light of these facts, we dedicated this work to the investigation of dark energy as a very light scalar field coupled to dark matter only, and assume all interactions between the standard model and the dark sector are negligible. Theories with an interacting dark sector have been discussed in the literature extensively, and the cosmologist's ignorance of this sector's physical nature is reflected in the wide variety of interaction types considered; see e.g. [51–54, 157–169] and references therein. In many of these works, the gravity sector of the theory is of scalar-tensor form, the scalar plays the role of dark energy, and the coupling of the scalar field to dark matter is described via an effective Newton's constant that depends on the local value of the scalar field (conformal coupling).

As an extension of this idea, we allowed the additional possibility of disformal couplings between the two dark elements. Disformal models of gravity, initiated by Bekenstein [72], have been attracting much attention recently, particularly with regards to cosmology [78,80,86,91,96,117,170–173]. These disformal factors have been used in stabilising scaling solutions in massive gravity [174], modifying the speed of gravitational wave propagation during inflation [175], even, as I mentioned at the end of chapter 1, describing electron transport theory in strained graphene, and many other ways besides. One of the central issues we addressed in this work is whether or not cosmological observations will allow us to disentangle the effects of conformal and disformal couplings.

To demonstrate clearly what I mean by conformal and disformal couplings to dark matter, let me now write down the Einstein frame action for the

theory we considered:

$$\mathcal{S}[g,\phi] = \int d\pi(g) \left[\frac{\mathcal{R}(g)}{2\kappa} + \mathcal{L}_M(g) + \mathcal{L}_\Phi(g, \phi, d\phi) \right]$$
$$+ \int d\pi(fg) \mathcal{L}_{\mathsf{C}}(fg), \tag{4.1}$$

where M now corresponds to the visible sector (i.e. the standard model particles) and ϕ is a dark energy field, quintessent in $\mathcal{S}$:

$$\mathcal{L}_\Phi(g, \phi, d\phi) = X(g, d\phi) - V(\phi). \tag{4.2}$$

Cold dark matter, described by the Lagrangian $\mathcal{L}_{\mathsf{C}}$, depends on the metric

$$fg = C(\phi)g + D(\phi)d\phi \otimes d\phi := \tilde{g}. \tag{4.3}$$

The functions V, C and D encapsulate our theory's remaining freedom, that will be specified in later sections. C and D go by the names **conformal factor** and **disformal factor** respectively. In our chosen theory dark matter particles follow geodesics determined by fg, and various aspects of these particles, for instance their mass, will now depend on the dark energy field. The scalar field couples to dark matter via products of the functions C and D with dark matter variables in the Lagrangian—hence in the Euler Lagrange equations—and these interactions I refer to as conformal / disformal couplings. From my previous focus on transformations I now shift to couplings. Couplings and transformations are of course linked words, one always implies the other, but they are not synonymous. The shift represents a shift in the type of work now being described. I will be dealing with differential equations and mathematical models, and the question of frame equivalence will finally take a back seat.

The chapter is organised as follows: in the next section I discuss the evolution of the background system, and specify the different choices of free function (V, C and D) forms and parameters that we used consistently throughout our analysis. In section 3 I will turn our attention to the evolution of cosmological perturbations in the presence of disformal and conformal couplings, and compute both matter and angular power spectra for various cases. All numerical work, including background simulations and both power spectra, is the output of a modified version of the publicly available Boltzmann code CLASS [176], an open source program exemplar: transparent,

efficient, and superbly well documented. Throughout the work, we aimed to emphasise coupling-type discernibly; can one actually 'observe' a purely disformal phenomenon? Finally note that, specific to this chapter, I will here set only the speed of light, c, and reduced Planck's constant, $\hbar$, to unity.

4.1 Background Cosmology

This section is split into several parts. Firstly, I write down the background equations for $\mathcal{S}$ (Eq. (5.5) above). The background dynamics are then described in detail. Finally the effective coupling to dark matter and the effective equation of state of the dark energy scalar field are discussed.

4.1.1 Equations of motion

The background space-time we consider $(\mathcal{M}, g)$ is a flat universe. The standard Friedmann-Lemaître (FL) metric solution to the Einstein equations for the metric $g_{\mu\nu}$ with flat spatial hypersurfaces is:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a^2(\tau)[-d\tau^2 + \delta_{ij}dx^i dx^j]. \qquad (4.4)$$

Here τ is the conformal time and $a(\tau)$ is the scale factor. For the rest of the chapter, dots denote derivatives with respect to τ. Note that the disformal metric, $\tilde{g}$, which dark matter particles "feel" is given by Eq.s (4.3) and (4.4) as

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = Ca^2(\tau)[-Z^2d\tau^2 + \delta_{ij}dx^i dx^j], \qquad (4.5)$$

where we are by now well acquainted with the disformal scalar Z. For our chosen frame isomorphism f, Z becomes:

$$Z = \sqrt{1 - 2X\frac{D}{C}}. \qquad (4.6)$$

The background value of the scalar field depends only on τ. We assumed that neutrinos are massless in our analysis, and hence are combined with the photons in a single relativistic species. Frame indices I will be using are the following: the different sectors of the theory are specified by a massless relativistic component, R, and baryon component, B, a cold dark matter component C, and the scalar field Φ. The relativistic species as well as the

baryons are assumed to be uncoupled from the scalar and hence the evolution of their energy densities is described by standard conservation equations:

$$\dot{\rho}_R + 3\mathcal{H}(\rho_R + P_R) = 0 \tag{4.7a}$$
$$\dot{\rho}_B + 3\mathcal{H}\rho_B = 0 \tag{4.7b}$$

where $P_R = \rho_R/3$, and $\mathcal{H} = \dot{a}/a$ is the conformal Hubble parameter. On the other hand, the scalar field obeys the Klein Gordon equation, and is now coupled to dark matter via a coupling function Q:

$$\ddot{\phi} + 2\mathcal{H}\dot{\phi} + a^2\frac{dV}{d\phi} = a^2Q \ , \tag{4.8}$$

where the background form of Q is given by

$$Q = -\frac{a^2C' - 2D(3\mathcal{H}\dot{\phi} + a^2V' + \frac{C'}{C}\dot{\phi}^2) + D'\dot{\phi}^2}{2(a^2C + D(a^2\rho_{\mathsf{c}} - \dot{\phi}^2))}\rho_{\mathsf{c}} \ . \tag{4.9}$$

The non-conservation of the dark energy stress tensor implies subsequent non-conservation of dark matter; energy loss from one species must be mirrored by energy gain in the other, and so for the cold dark matter species, we obtain

$$\dot{\rho}_{\mathsf{c}} + 3\mathcal{H}\rho_{\mathsf{c}} = -Q\dot{\phi} \ . \tag{4.10}$$

Finally, from Einstein's equations I present the Friedmann equation, which takes the standard form:

$$\mathcal{H}^2 = \frac{\kappa a^2}{3}\sum_I \rho_I \ , \tag{4.11}$$

for $I \in \{R, B, \mathsf{C}, \Phi\}$, and $\rho_\Phi = \dot{\phi}^2/2a^2 + V(\phi)$.

4.1.2 Analysis of the dynamics

The dark energy-dark matter interaction encoded in the equations above describes a peculiar scenario. For the dark energy field, the coupling contributes to an effective potential which depends in general on ϕ and $\dot{\phi}$. The dark matter then gains and loses energy as the geometry described by $\tilde{g}$ is stretched and distorted; the conformal factor dilutes the dark matter over

space-time by modifying the isotropic expansion it feels, $C^{1/2}a$, while the disformal factor distorts dark matter particles' light cones. The nature of this energy transfer process will depend crucially on how we specify our three free functions $V(\phi)$, $C(\phi)$ and $D(\phi)$. Suffice to say for now that we require our cosmology to be empirically plausible—observation tells us our dark energy field must roughly resemble a cosmological constant, and the scalar field must evolve very slowly to account for the accelerated expansion. To be specific, in this work we studied the following forms for V, C and D:

$$V = M_V^4 e^{\beta_V \phi} , \tag{4.12a}$$

$$C = C_0 e^{\beta_C \phi}, \tag{4.12b}$$

$$D = M_D^{-4} e^{\beta_D \phi} . \tag{4.12c}$$

where $M_V, C_0, M_D, \beta_I \in \mathbb{R}$ are all constants $\forall I \in \{V, C, D\}$. We will also find it convenient for plotting and so on to work with the following dimensionless ratio:

$$\Gamma_0 := \left(\frac{M_V}{M_D}\right)^4 . \tag{4.13}$$

We can, without loss of generality, set $C_0 = 1$, as this parameter simply corresponds to a global redefinition of units. Such a choice does not affect the dynamics. The dark energy scale, M_V, is taken to be a fitting parameter that must be tuned such that our final time boundary conditions agree with measurement of the universe today, that is to say $\Omega_\Phi = 0.68$, $\Omega_c = 0.27$ and $\Omega_B = 0.049$ [153]. Typically we find $M_V \sim \mathcal{H}_0 \sim \text{meV}$. The conformal coupling is dimensionless, but the disformal factor introduces a new scale into the model. The case $M_D \to \infty$ corresponds to the standard coupled quintessence scenario, and the opposite limit where $M_D \to 0$, it turns out, is actually a ΛCDM limit, regardless of the form of C. This unexpected feature is a consequence of a suppression effect to be clarified shortly, though we have already met it at the end of Sec. 3.1. In between these limits, we found disformal effects leave an observable imprint on cosmological observables that is maximal if $M_D \approx M_V$.

We studied a variety of models with different values for M_D, β_C and β_D, and the parameter combinations we considered are summarised in table 4.1, where the meaning of the final column I will specify in the next section. For this entire chapter, β_V is fixed at -2. The field ϕ must be slowly rolling in its potential such that the cosmological observables are close to their measured values, and so the potential must be shallow. Small deviations from $\beta_V = -2$

#	Name	β_C	β_D	M_D	x behaviour
1	Uncoupled	0	0	∞	$x = 1 \; \forall \; \tau$
2	Conformal	-0.2	0	∞	decreasing with τ
3	Disformal	0	0	M_V	increasing with τ
4	Mixed	-2	2	M_V	single stationary minimum

Table 4.1: Description of the four models used throughout this chapter. The model parameters are defined in Eqs. (4.12). The function x is defined in Eq. (4.15), and 'Name' corresponds to the model's label in plot legends. Note that in the fifth column, $M_D = \infty$ simply represents the limit for which the disformal coupling vanishes.

induce small deviations in the system solutions, and do not qualitatively change the picture. A more complete follow up study should of course explore different values of β_V, and indeed it must be marginalized over when fitting model predictions to data.

In Fig. 4.1 I show the evolution of the coupling function, in which the most prominent feature is an early time suppression induced by the disformal factor. The coupling effectively 'switches on' during some past epoch, quite late in the universe's evolution, and before this time it is in fact completely negligible. We can see the effect this has on the evolution of the dark sector's energy densities in the next figure. Compared to the purely conformal case, the scalar field receives no great kick at early times when a disformal factor is included, and though β_C is here a factor of about 11 larger than current experimental upper bounds—$\beta_C < 0.17$ to 95% confidence [177] as of 2009[1]— the field mimics a cosmological constant throughout the majority of this universe's simulated lifetime.

Probing different free functions and parameters, we found this suppression to be no lucky coincidence of the mixed model, but seems a general property of a disformal factor included in almost any cosmology, and I recall we have seen it already in chapter 3. We can see why this must be the case by examining Eq. (4.9). A non-zero D means the presence of a term proportional to ρ_c appears in the denominator. As long as the disformal scale M_D is of the same order as the dark energy scale or less, this term will continue

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[1]the β parameter defined in [177] is twice the one used in this chapter.
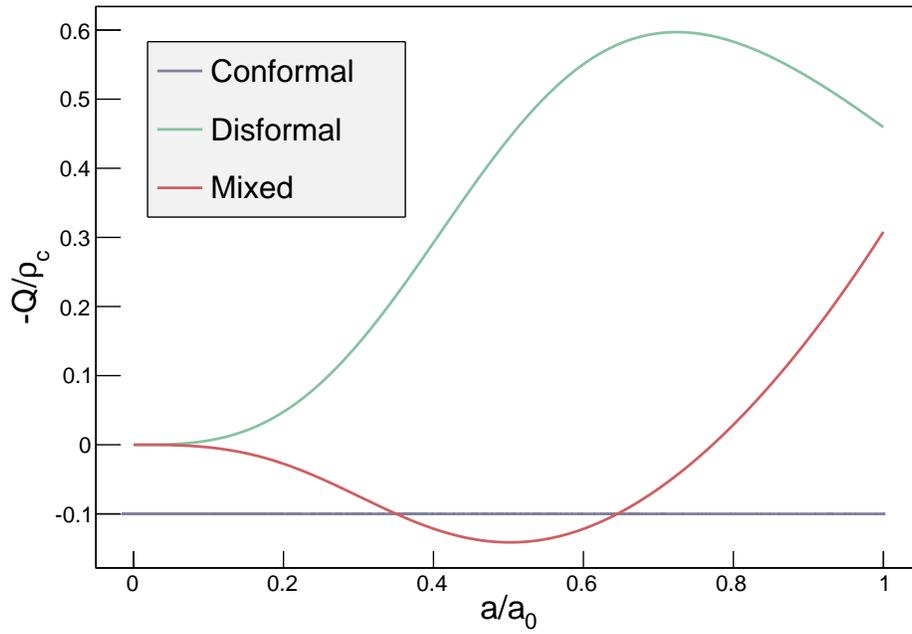
Figure 4.1: The scalar coupling function, Q, for background solutions of various models, plotted against the scale factor a where a_0 is the value of a today. The free functions are defined in Eq.s (4.12), and $\beta_V = -2$ for all curves. The rest of the model parameters are given in table 4.1.
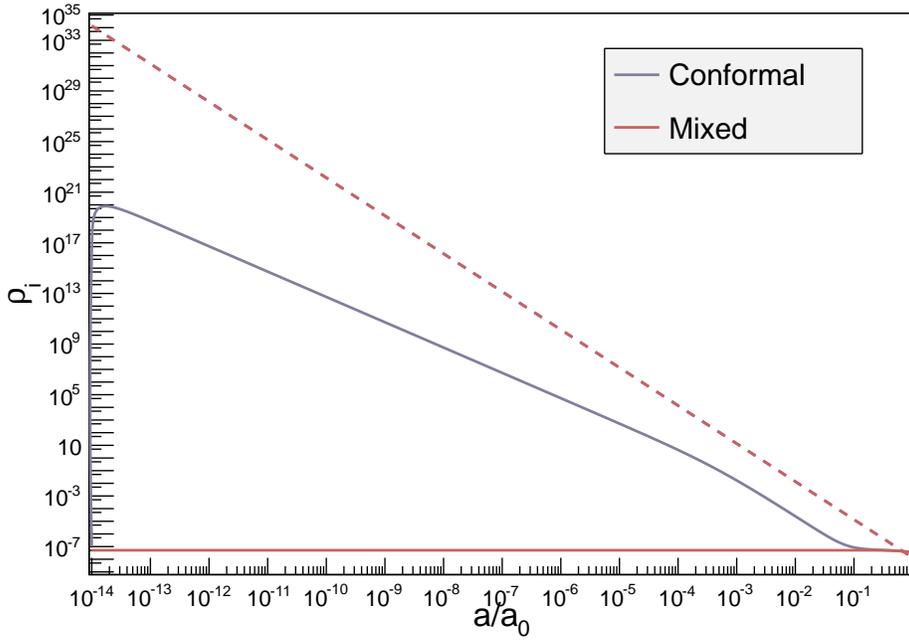
Figure 4.2: Evolution of energy densities of: dark matter (dashed lines) and dark energy (solid lines) for background solutions of various models, plotted against the scale factor a where a_0 is the value of a today. The free functions are defined in Eq.s (4.12), and $\beta_V = -2$ for all curves. The rest of the model parameters are given in table 4.1. Here the dark matter curves for both models coincide.

to make the coupling negligible until dark matter is roughly of that scale, i.e. today. We will show in later sections that the same screening effect holds too for linear perturbations, for the very same reason.

We have arrived at the first main result of the chapter: in a cosmological setting, a disformal factor can suppress a conformal contribution at early times. The key point is that, for certain orders of magnitude of the disformal factor, the value of Q and its linear perturbation, δQ (whose exact form is given in section 4.2), are very much diminished for the majority of the universe's evolution—in the presence of disformal couplings, significant conformal ones are not necessarily in disagreement with cosmological observations.

4.1.3 The characteristic coupling function, x

The expression of Q is not simple to analyse, and collecting all our coupling effects under the obfuscated umbrella Q has somewhat obscured the physics. Its form is necessary for finding numerical solutions, but for the analytics we can do better. To elucidate the effects conformal and disformal couplings have on cold dark matter, I now define a new variable x that greatly simplifies the analysis, and, as it turns out, the dark matter equation (4.10) will become easily solvable. In fact, this remains true for any species whose exact solution can be found in ΛCDM, for example photons. I relegate the details and general case to appendix B, but for pressure-less dark matter we obtained

$$\rho_{\mathsf{c}} \propto a^{-3} x \ , \tag{4.14}$$

where I have defined the quantity

$$x(\tau) := \frac{C^{1/2}(\tau)/Z(\tau)}{C(\tau_0)^{1/2}/Z(\tau_0)} \ , \tag{4.15}$$

such that τ_0 is the conformal time of the present epoch. Another useful quantity will be the derivative of x:

$$\frac{\dot{x}}{x} = -\frac{Q\dot{\phi}}{\rho_{\mathsf{c}}} \ , \tag{4.16}$$

which I express in terms of a rate. Then, looking again at Eq. (4.10), we see now the evolution of ρ_{c} as a competition of rates: that of the Hubble expansion rate, $\mathcal{H}$, and the rate of interaction with the scalar field, $\dot{x}/x$.

The positivity of x follows naturally from the fact that both C and Z separately must be positive. This condition is defined by the metric (4.5); we must preserve causality, or suffer the consequences. Throughout the course of this work, it will become clear that the whole system can be characterised by x and its derivative—at least, in terms of observables—at both the background and perturbative level. We can already see this to be true at the zeroth order. As x contains only background quantities however, it is quite remarkable that this remains true at first order.

With x defined, it is now time for me to address the issue of our large free function space. What we were looking for in this study was general characteristics of conformal and disformally coupled dark matter, not idiosyncrasies corresponding to specific choices of the functional forms and parameters of V, C and D. To make this step toward more comprehensive conclusions, we first notice that—as I have already stated—both C and D do not actually work independently, but affect the system jointly through x. I will then partition observationally distinct models based on the behaviour of their respective x function in conformal time.

As previously stated, we aimed to keep the models presented here realistic, with observables like the CMB anisotropies close to their measured values, and so we worked in the slow roll regime. This means that not only should V be a relatively shallow function, but so too C, as it is also able to drive the field. The disformal factor, however, induces a damping in the field dynamics, and we found it can not push the field by itself, but rather hinders its movement. This damping gave us some more leeway in how shallow C and V can be. What this ultimately meant was that we did not consider scaling solutions or attractors; this work was not aimed at solving the coincidence problem, rather we found an alternative notion of naturalness is manifest here: the general inclusion of a disformal factor serves to push an arbitrary coupled cosmology toward one with an effective cosmological constant.

Given what has just been said, we suggested that for a qualitative first study, it will be enough to consider just four distinct models:

1. uncoupled quintessence, where $x = 1 \; \forall \; \tau$,

2. x is a decreasing function of time,

3. x an increasing function of time,

4. x has a single minimum.

I can now comment on the final column of table 4.1 mentioned earlier. In particular, it is not of vital importance that V is of exponential form, a power law e.t.c. What does matter is whether x is pushed upward, to larger values, or made to roll down to smaller ones. So, only the direction of the slope of C relative to V affects the evolution of cold dark matter.

I show the x behaviour for the four models in figure 4.3. Note again the defining feature of the disformal term is that $\dot{x}$—and hence the coupling—vanishes for early times, while for the conformal model it rapidly diverges. At early times then the distinction between conformal and disformal effects is strikingly clear, but at late times however, this is not so. We used here a conformal only model to produce a decreasing x function model, but we could have achieved this by other means. For example, were we to pick a more complicated choice of D function, and a different potential, we would get qualitatively the same late time behaviour. At least at the background level, it is this late time x behaviour that is observable, as long as dark energy remains sub-dominant, and we will see in the next section why we categorised our models based on this criteria.

4.1.4 An effective equation of state

The dynamics of our gravitationally coupled system are in general quite complex. There is energy transfer between the elements of the dark sector that depends not only on the dark energy field, but also its first and second derivatives. We anticipated that when interpreting data, cosmologists will use a much simpler parameterization. This assumed model is most often of a non-interacting dark sector, where dark matter is pressureless dust and dark energy some fluid with an open equation of state. Following [178] we reformulated our theory at the level of the zeroth order equations of motion to fit this neat picture, and defined an effective, or apparent, dark energy equation of state w_{eff}.

In the Friedman equation we first performed an effective splitting between the two dark components:

$$\mathcal{H}^2 = \frac{8\pi G a^2}{3} \left(\rho_{B,0} a^{-3} + \rho_{\mathsf{C},0} a^{-3} + \rho_{\Phi,\text{eff}} \right) \tag{4.17}$$

with

$$\rho_{\Phi,\text{eff}} := \rho_\Phi - \rho_{\mathsf{C},0} a^{-3} + \rho_{\mathsf{C}}, \tag{4.18}$$
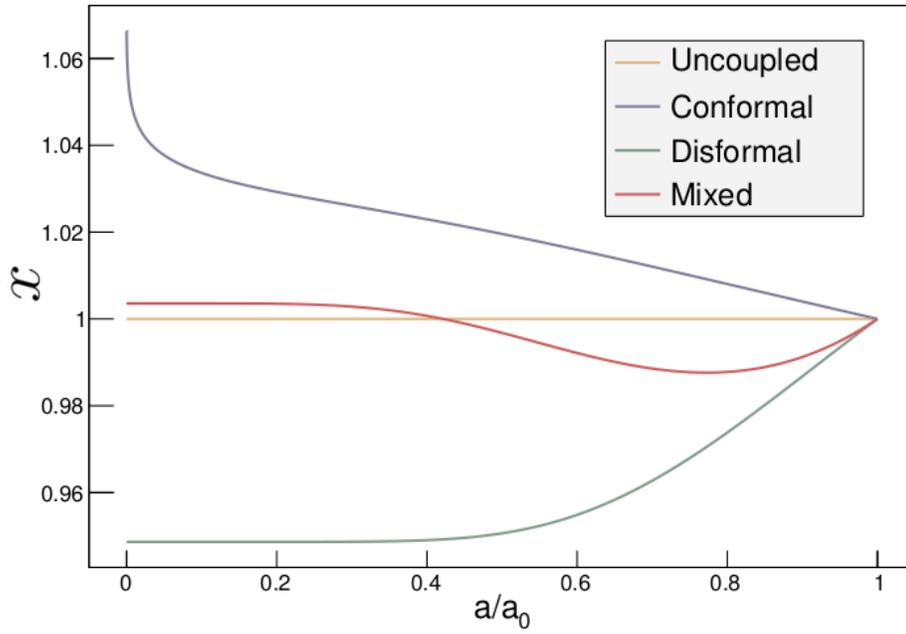
Figure 4.3: Evolution of the characteristic x function defined in Eq. (4.15) for the four models described in table 4.1, with $\beta_V = -2$ for all curves. The model parameters are defined in Eq.s (4.12).

and $\rho_{\mathsf{c},0}$ the measured dark matter energy density at the current epoch. Then, taking the time derivative of $\rho_{\Phi,\mathrm{eff}}$ and defining w_{eff}

$$w_{\mathrm{eff}} := -\frac{\dot{\rho}_{\Phi,\mathrm{eff}}}{3\mathcal{H}\rho_{\Phi,\mathrm{eff}}} - 1 \; , \qquad (4.19)$$

we found

$$w_{\mathrm{eff}} = \frac{p_{\Phi}}{\rho_{\Phi,\mathrm{eff}}} \qquad \text{with} \qquad p_{\Phi} = X - V, \qquad (4.20)$$

where $X = \dot{\phi}^2/2a^2$ I recall is the kinetic energy of the field.

To physically motivate our above definition consider the following scenario. Suppose that some cosmological observations, for example the supernova redshift data of the supernova cosmology project, are interpreted assuming a model where dark matter particles are cold and do not interact— and so the dark matter fluid energy density in this assumed model dilutes with cosmic expansion as a^{-3}—and dark energy evolves with some unspecified dynamics, but the *true* underlying reality is accurately described by the interacting dark sector model of the action (5.5) above. In this scenario the dark energy equation of state is a free function to be determined by the data, and is exactly the function w_{eff} we have defined above. This is a commonly assumed model for data fitting (see for example [179] where evidence for phantom dark energy is claimed to be present in the Planck data.)

What can we expect to observe in the behaviour of this new effective system? An interesting first question to ask is: will we see phantom behaviour? Using Eq. (4.14), it is simple for one to derive the following *phantom condition*:

$$w_{\mathrm{eff}} < -1 \qquad \Leftrightarrow \qquad 2X < \rho_{\mathsf{c},0}a^{-3}(1-x) \qquad (4.21)$$

where I recall that x is normalised to unity today. It is clear that the evolution of x will dictate whether or not we see the effective dark energy cross the phantom line, and this is directly related to the coupling of the underlying true model: if x is an decreasing function phantom behaviour is impossible, and in this scenario energy flow is from dark energy to dark matter. Conversely, energy flow in the opposing direction (x is an increasing function) will propel the universe toward even greater expansion, as the relative contribution to the cosmic inventory from the vacuum energy will grow. Clearly, this system should not exhibit the standard instabilities expected from true phantom dark energy models, as it is simply a phenomenological re-parameterisation.
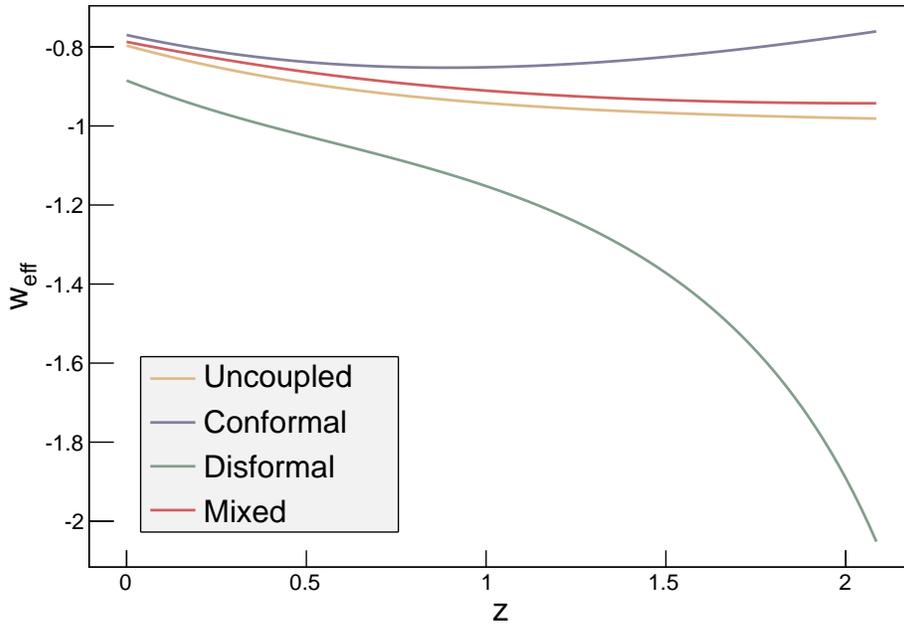
Figure 4.4: Evolution of the apparent equation of state, w_{eff}, defined in Eqn (4.20) for the four models described in table 4.1. The model parameters are defined in Eq.s (4.12).

For our four chosen models in table 4.1, I display the effective equation of state as discussed above in the redshift range accessible to the proposed Euclid satellite [180], roughly $z \in [0.5, 2]$. While the disformal model here goes divergently phantom, for the others, the phantom line is never crossed. In the pure conformal case, the effective system tends further from ΛCDM, toward the boundary between acceleration and staticity. Das et al. [178] however find a conformal model that replicates our disformal model's divergently phantom behaviour, and so the take home message is emphatically *not* that disformal couplings induce effective phantom behaviour while conformal ones do not. Rather the point is that if x is increasing with time, phantom behaviour will likely ensue—a direct result of the phantom condition (4.21).

At the background level then, the coupling has a nice interpretation as a variable dark energy equation of state. This correspondence is best illustrated through its effect on luminosity distances, d_L.

The observed flux, F, of an object with fixed intrinsic luminosity L will decrease with increasing separation between observer and object. If the space between object and observer is not curved and static (so not expanding) the flux decreases with the increasing surface area of the sphere at distance R from the luminous object:

$$F = \frac{L}{4\pi R^2}.$$
(4.22)

This connection between flux and luminosity does not hold in expanding space but, if an astronomer knows both F and L for some fixed luminous object, they can compute the quantity:

$$d_L := \sqrt{\frac{L}{4\pi F}}.$$
(4.23)

called the **luminosity distance**, which gives a measure of the separation. Knowing both d_L and the true separation distance R (or of some other distance measure, e.g. angular diameter distance) between some fixed object and earth tells the astronomer much about the dynamics of the space in between.

In figure 4.5 I show the luminosity distance difference ratio for our four models, defined as:

$$\left.\frac{\Delta d_{\mathrm{L}}}{d_{\mathrm{L}}}\right|_{\mathrm{i}} := \frac{d_{\mathrm{L,i}} - d_{\mathrm{L,uncoupled}}}{d_{\mathrm{L,uncoupled}}}.$$
(4.24)
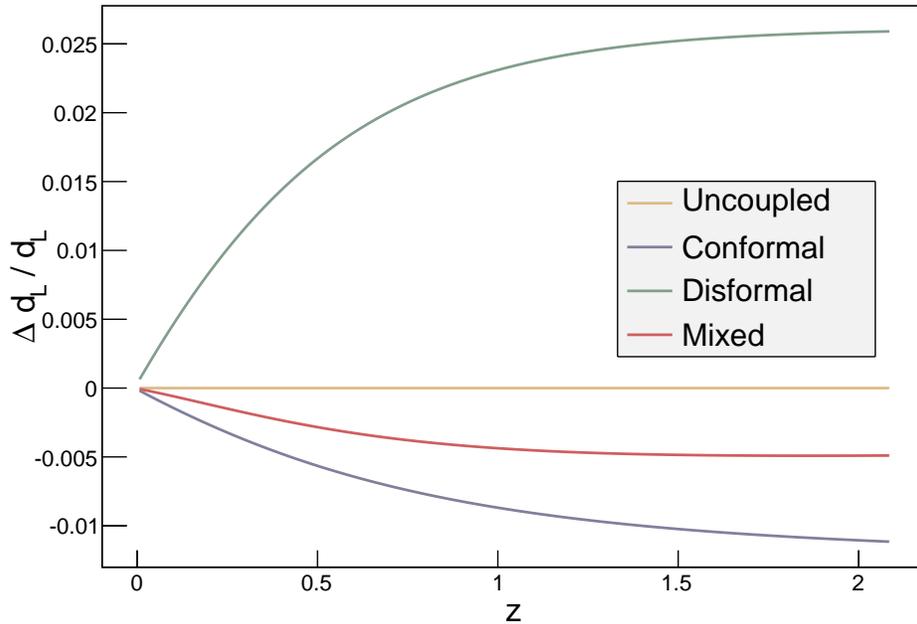
Figure 4.5: Evolution with redshift of the fractional difference in luminosity distance, Eq. (4.24), between the four models described in table 4.1 and the uncoupled case. The model parameters are defined in Eq.s (4.12).

As our intuition suggests, energy flow into the scalar field accelerates expansion, causing observed objects such as supernova at a fixed redshift to appear further from us than for the uncoupled case.

To conclude this section, we have seen that conformal and disformal effects can not always be distinguished when only dark matter is coupled. Whether they can or not typically depends on the epoch in question: at early times the distinction is clear, as disformal contributions in general suppress conformal ones; at late times the two act together through the x function, and whether energy flow is into or out of dark matter tells us nothing about the underlying behaviour of C and D, nor will any observed phantom behaviour. What defines early and late times in this context is the new scale introduced by the disformal factor, M_D. When the dark matter energy density becomes comparable to that scale, the coupling switches on and begins

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to influence the field.

The reason why the two types of coupling become indistinguishable at late times is ultimately because dark matter is pressureless. What sets disformal factors apart, we have seen, is that they warp light cones and shift causal structure, but this has little effect when the particle constituents of the coupled fluid is cold dark matter.

4.2 Evolution of cosmological perturbations

I now direct our attention toward the evolution of cosmological perturbations in our theory. Up until now, for the last three and a half chapters, I have dealt only with universes: space-times that satisfy the cosmological principle. The physics of the everyday of course does not satisfy this principle, the room that I write in is no more homogeneous than the room in which you now sit, reading. On the largest scales, beyond the end of greatness, homogeneity is a sound assumption to make, but as the scales under consideration shrink, lumps and bumps and perturbations appear. At this point in the thesis it is time to consider **perturbed universes**, space-times where in space-like averages they are homogeneous, but in their details, not. As is customary, we assumed no back-reaction: the background equations were numerically integrated independent of the perturbations (though of course not vice-versa).

I begin by first writing down the perturbation equations and subsequently discuss predictions for cosmological observables, such as the CMB anisotropies and matter power spectrum. Along the way I will try to be categorical about the various effects induced by the couplings for the different models; will there be anything about these spectra that is characteristically disformal?

To be concrete, we chose to work in the Newtonian gauge. To avoid confusion, I will reserve δ to denote matter density contrast: $\delta := \frac{\delta\rho}{\rho}$, δP the pressure perturbation and $\delta\phi$ is the perturbation of the scalar field. I denote by $\hat{\delta}$ a general perturbation operator. The perturbed Einstein frame line element in the chosen gauge is:

$$ds^2 = a^2(\tau)\left[-(1+2\Psi)d\tau^2 + (1-2\Phi)\delta_{ij}dx^i dx^j\right] \qquad (4.25)$$

which means that dark matter particles follow geodesics described by the following perturbed space-time:

$$d\tilde{s}^2 = Ca^2(\tau)[-(1+2A)Z^2d\tau^2 + 2(\partial_i B)Zd\tau dx^i + (1-2E)\delta_{ij}dx^i dx^j] \quad (4.26)$$

where recall Z is the disformal scalar, and A, B, and E are functions of the dark energy field background and perturbation values. Their exact forms are:

$$A = \Psi + \frac{\hat{\delta}C}{2C} + \frac{\hat{\delta}Z}{Z} \quad (4.27a)$$

$$B = \left(\frac{1}{Z} - Z\right)\frac{\delta\phi}{\phi'} \quad (4.27b)$$

$$E = \Phi - \frac{\hat{\delta}C}{2C} \quad (4.27c)$$

It is now clear that only disformal factors induce off diagonal perturbations in the metric which, we will see, affect the velocity field perturbations.

The perturbation of an arbitrary fluid's stress tensor can be computed as:

$$\hat{\delta}T^{\mu\nu} = (1 + \frac{\delta P}{\delta\rho})\delta\rho u^\mu u^\nu + 2(\rho + P)\hat{\delta}u^{(\mu}u^{\nu)} + \delta P g^{\mu\nu} + P\hat{\delta}g^{\mu\nu} + \Pi^{\mu\nu}, \quad (4.28)$$

where $\Pi^{\mu\nu}$ is the anisotropic stress of the given fluid that, while it is not the perturbation of any back ground quantity, for example δP, it is small enough in realistic cosmological models to be considered only at the level of linear perturbations. (It is the only term in the above expansion (4.28) able to source gravitational waves.) For the visible sector we again neglected that neutrinos have mass, hence we could neglect anisotropic stress, i.e. $\Pi^{\mu\nu} = 0$. In general, the velocity perturbation $\hat{\delta}u^\mu$ could be split into scalar and vector parts:

$$\hat{\delta}u^\mu = \partial^\mu\theta + v^\mu. \quad (4.29)$$

The vector term v^μ will source vector modes of the gravitational field which we here ignored, and so only the dynamics of θ for each species is considered below. Finally, to solve the differential equations, it is exceedingly helpful, and routinely done in cosmological perturbation theory, to go to Fourier space, with wave vectors $\mathbf{k}$, and keep the same notation for the Fourier modes

of each perturbation variable—this means we expand some perturbation variable X in Fourier modes:

$$X(\mathbf{x}, t) = \int d^3 k X_k(\mathbf{k}, t) e^{2\pi i \mathbf{k} \cdot \mathbf{x}} \tag{4.30}$$

and obtain evolution equations for each mode. Then we relabel X_k as simply X (i.e. drop the k subscripts) for brevity.

The evolution equations for the Fourier modes of the relativistic, R, and baryonic, B, matter perturbation variables follow from the energy–momentum conservation equations and are given by

$$\dot{\delta}_R = -(1 + w_R)(\theta_R - 3\dot{\Phi}) - 3\mathcal{H}\left(\frac{\delta P_R}{\delta \rho_R} - w_R\right)\delta_R \tag{4.31a}$$

$$\dot{\theta}_R = -\mathcal{H}(1 - 3w_R)\theta_R - \frac{\dot{w}_R}{1 + w_R}\theta_R + k^2\Psi + \frac{\delta P_R/\delta \rho_R}{1 + w_R}k^2\delta_R \tag{4.31b}$$

$$\dot{\delta}_B = -\theta_B + 3\dot{\Phi} \tag{4.31c}$$

$$\dot{\theta}_B = -\mathcal{H}\theta_B + k^2\Phi. \tag{4.31d}$$

Recall the equation of state defined for background quantities of an arbitrary fluid is defined as $\omega_I := P_I/\rho_I$.

Perturbations in the dark energy field, $\delta\phi$, evolve according to the perturbed Klein Gordon equation

$$\ddot{\delta\phi} + 2\mathcal{H}\dot{\delta\phi} + (k^2 + a^2 V'')\delta\phi = \dot{\phi}(\dot{\Psi} + 3\dot{\Phi}) - 2a^2(V' - Q)\Psi + a^2\delta Q \tag{4.32}$$

and perturbation of Q is given by the cumbersome expression [117]

$$\delta Q = -\frac{\rho_{\mathsf{c}}}{a^2 C + D(a^2\rho_{\mathsf{c}} - \dot{\phi}^2)}[\mathcal{B}_1\delta_{\mathsf{c}} + \mathcal{B}_2\dot{\Phi} + \mathcal{B}_3\Psi + \mathcal{B}_4\dot{\delta\phi} + \mathcal{B}_5\delta\phi], \tag{4.33}$$

where

$$\mathcal{B}_1 = \frac{a^2 C'}{2} - 3DH\dot{\phi} - Da^2(V'-Q) - D\dot{\phi}^2\left(\frac{C'}{C} - \frac{D'}{2D}\right), \quad (4.34a)$$

$$\mathcal{B}_2 = 3D\dot{\phi}, \quad (4.34b)$$

$$\mathcal{B}_3 = 6DH\dot{\phi} + 2D\dot{\phi}^2\left(\frac{C'}{C} - \frac{D'}{2D} + \frac{Q}{\rho_\mathsf{c}}\right), \quad (4.34c)$$

$$\mathcal{B}_4 = -3DH - 2D\dot{\phi}\left(\frac{C'}{C} - \frac{D'}{2D} + \frac{Q}{\rho_\mathsf{c}}\right), \quad (4.34d)$$

$$\mathcal{B}_5 = \frac{a^2 C''}{2} - Dk^2 - Da^2 V'' - D'a^2 V' - 3D'H\dot{\phi}$$
$$- D\dot{\phi}^2\left(\frac{C''}{C} - \left(\frac{C'}{C}\right)^2 + \frac{C'D'}{CD} - \frac{D''}{2D}\right)$$
$$+ (a^2 C' + D'a^2\rho_\mathsf{c} - D'\dot{\phi}^2)\frac{Q}{\rho_\mathsf{c}}. \quad (4.34e)$$

The non-conservation of dark matter induces, at the perturbative level, factors of Q and δQ in its conservation equations, which become

$$\dot{\delta}_\mathsf{c} = -\theta_\mathsf{c} + 3\dot{\Phi} + \frac{Q}{\rho_\mathsf{c}}\dot{\phi}\,\delta_\mathsf{c} - \frac{\dot{\phi}}{\rho_\mathsf{c}}\delta Q - \frac{Q}{\rho_\mathsf{c}}\dot{\delta\phi}, \quad (4.35a)$$

$$\dot{\theta}_\mathsf{c} = -H\theta_\mathsf{c} + k^2\Psi + \frac{Q}{\rho_\mathsf{c}}\dot{\phi}\theta_\mathsf{c} - \frac{Q}{\rho_\mathsf{c}}k^2\delta\phi. \quad (4.35b)$$

An important point has to be clarified before we continue: is it justified to set the Einstein frame pressure and shear of the coupled dark matter to zero? After all, we have seen in the previous chapter that disformal couplings can transform the equation of state of a coupled species to one dependent on the field, ϕ (see [105], [181] and appendix B). In the appendix I present the full set of frame transformations between Jordan and Einstein frame matter variables (or equivalently, between uncoupled and coupled variables

respectively)[2]. In particular, for a general coupled species:

$$\delta \tilde{P} = \frac{1}{C^2 \gamma} \left[\delta P - \left(\frac{2\hat{\delta} C}{C} + \frac{\hat{\delta} \gamma}{\gamma} \right) P \right] , \tag{4.36a}$$

$$\tilde{\Pi}^i_j = \Pi^i_j . \tag{4.36b}$$

So, as dark matter has vanishing pressure in the frame for which it is uncoupled, one may indeed set δP_{C}, $\Pi^{ij}_{\mathsf{C}} = 0$.

The perturbed Einstein equations in our theory take the form as in standard ΛCDM. I don't quote them all here, but instead present just those relevant to our analysis of structure growth. For a full list, see appendix C of [103]. In particular, we made use of the $00-$component of the Einstein equations

$$k^2 \Phi + 3\mathcal{H} \left(\dot{\Phi} + \mathcal{H}\Psi \right) = -4\pi G a^2 \delta\rho \tag{4.37}$$

and the $i \neq j-$component, which leads to

$$\Phi - \Psi = 0 \quad \forall \ C, D, \tag{4.38}$$

as we ignored anisotropic stress (I remind the reader that we ignored neutrino masses in our analysis). Our theory thus predicts no gravitational slip ($\eta :=$ $\Phi/\Psi = 1$), independent of the coupling.

With the perturbed equations written down, you might first ask whether the disformal factor suppresses a conformal one at the level of linear perturbations. Before getting to the figures, we can already guess that it will be the case from Eq. (4.33); looking at the denominator we see the same $Da^2\rho_{\mathsf{C}}$ term that was responsible for the suppression at zeroth order. Still, its important to be concrete, and so I demonstrate this suppression in Fig. 4.6 and Fig. 4.7.

If we compare figures 4.1 and 4.6 a similarity immediately leaps out: one plot appears to resemble the negative of the other. Additionally, the curves for δQ appear to rise and fall gradually, over background timescales, rather than, for example, the fast oscillations in the field perturbations (figure 4.7). This simplicity may come as a surprise when juxtaposed with the arduous

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[2]Note the reappearance of the Jordan frame; it may be a multi-coupled scalar tensor theory we are dealing with but, as there are only two metrics in this theory, the term we felt could be reinstated.
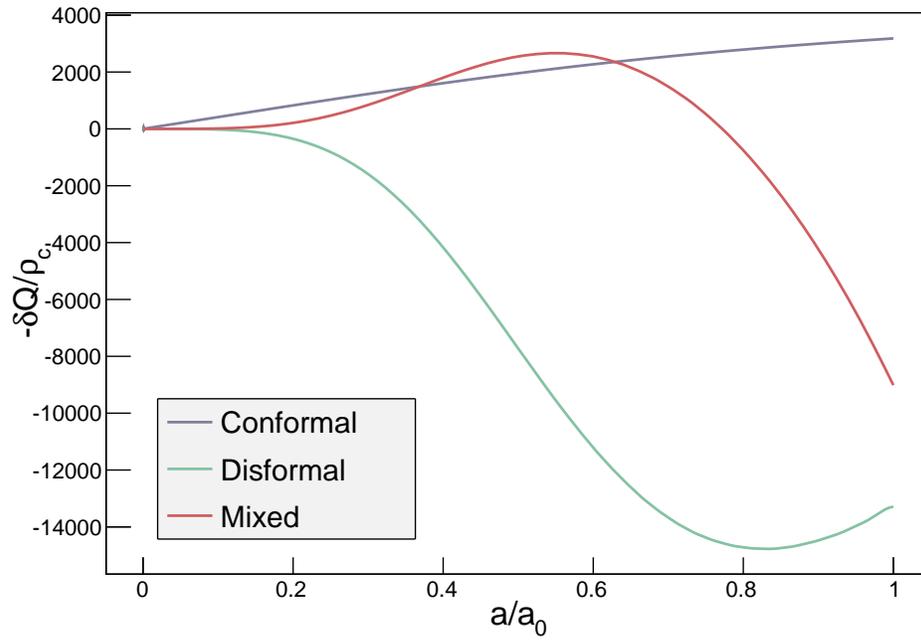
Figure 4.6: Dependence on the scale factor of the linear perturbation in the coupling function, δQ, given by Eq. (4.33). The three models shown are described in table 4.1, with equations (4.12).
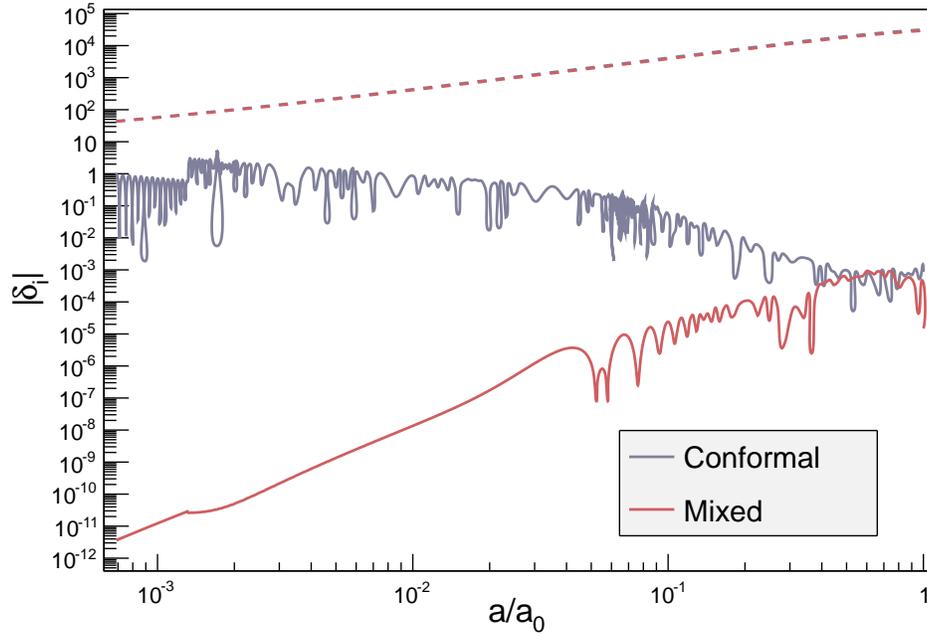
Figure 4.7: Evolution with the scale factor of density contrast absolute values in: the dark matter energy density (dashed lines), and dark energy density (solid lines). These curves correspond to modes with wavenumber $k = 0.3 Mpc^{-1}$. The models shown are described in table 4.1, with Eq.s (4.12).
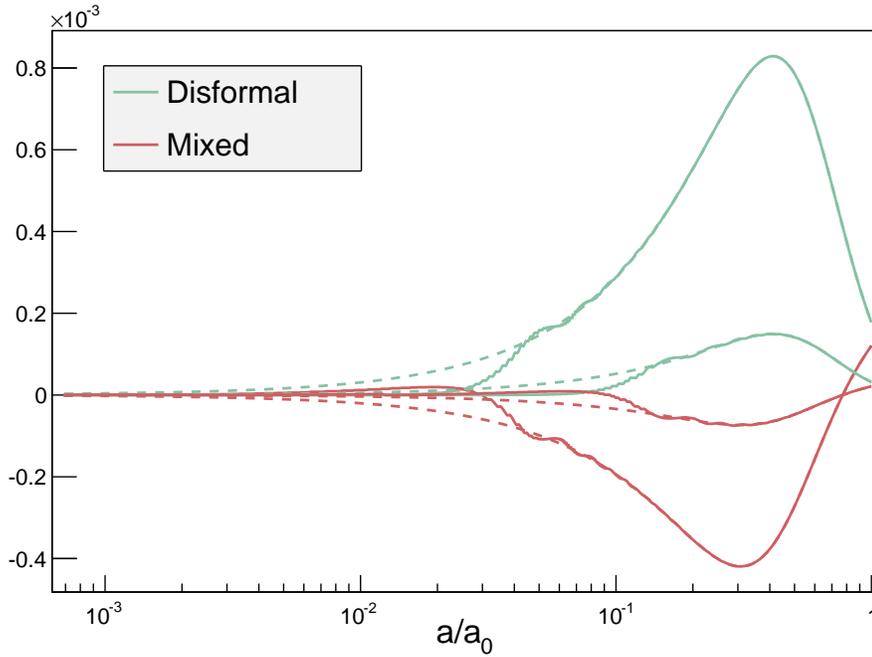
Figure 4.8: I show here, for models only including a disformal term, how the system approaches Eq. (4.39). Dashed lines: $Q\delta_{\mathsf{C}}$, where Q is defined by Eq.s (4.9), and δ_C is the dark matter density contrast. Solid lines: δQ, given by (4.33). The models shown are described in table 4.1, along with Eq.s (4.12).
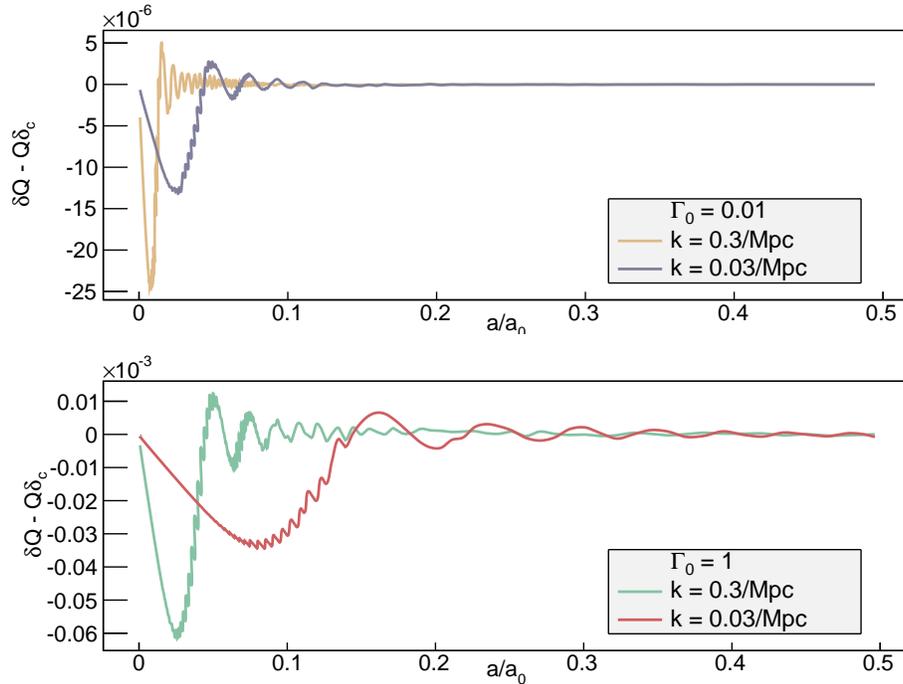
Figure 4.9: Oscillations induced by a disformal term in the perturbed coupling function (4.33) for models where $\beta_V = -2$, $\beta_C = 0$, $\beta_D = 0$ and $M_D = M_V/\Gamma_0^{1/4}$ where Γ_0 is introduced in Eq. (4.13). Terms in δQ proportional to dark matter perturbations only, $Q\delta_{\rm c}$, have been subtracted to isolate the dark energy perturbation fluctuations.

complexity of the δQ equation from which the curves have sprung. In fact there is a strong relationship between the background and perturbed coupling function Q, and the damped oscillatory behaviour of δQ at $a/a_0 \sim 2 \times 10^{-2}$ (figure 4.8) betrays a important aspect of the coupling—the system at late times is drawn to a solution where [96]

$$\delta Q \simeq Q\delta_{\rm c}, \tag{4.39}$$

which we identified with the limit in which all dark energy perturbations in δQ vanish. The approximate equivalence, Eq. (4.39), does not hold at very early times but becomes progressively more accurate later on. In Fig. 4.9 I present the oscillating part of δQ for an exclusively disformal model. We

see clearly that both the oscillation and damping time scales depend on the mode's wavenumber, k, which we expect, and the new scale introduced by the disformal factor, M_D. In the conformal limit, characterised by $M_D \to \infty$, we found in general they disappear as their period tends also to ∞: a conformal factor in general introduces no scale, so a purely conformally coupled theory does not exhibit these oscillations.

To briefly summarise by way of comparison, at the background level the new disformal scale determined an epoch in which the coupling is effectively 'turned on'; at the perturbation level, M_D now sets a damping and oscillating timescale for perturbations in the dark energy field. The introduction of this scale, we can now see, is primarily what sets apart the effects of conformal and disformal factors on the scalar field dynamics at the zeroth and linear order levels of the model.

4.2.1 Growth of large scale structure

We have seen some evidence to support that, at least in principle, conformal and disformal effects *can* be separated. The new important feature is of course the new scale, M_D. We now turn to the pressing question of observables: does the new scale also leave recognisable imprints on the formation and subsequent growth of structure?

The key quantity we are interested in is the growth factor, $f(z)$, which is a convenient parameterisation of linear growth. In the literature, the growth rate is defined as:

$$f(z) := \frac{\mathrm{d} \ln \delta_M}{\mathrm{d} \ln a} = \frac{\dot{\delta}_M}{\mathcal{H} \delta_M},\tag{4.40}$$

where we notice the definition is with respect to *all* pressureless matter, M:

$$\rho_M := \rho_\mathsf{c} + \rho_B, \quad \delta_M := \delta \rho_M / \rho_M \tag{4.41}$$

not just dark matter. As our theory only couples dark matter to the scalar, we are now faced with an interesting question: how will the composite fluid of dark matter and baryons cluster into structure, if both species feel different (effective) gravitational forces in general? We would like to calculate the growth equation for not just cold dark matter, but for all matter. It is this total matter quantity that influences the gravitational potentials, which lens distant galaxy and CMB light. As a first step then, I now define a 'baryon
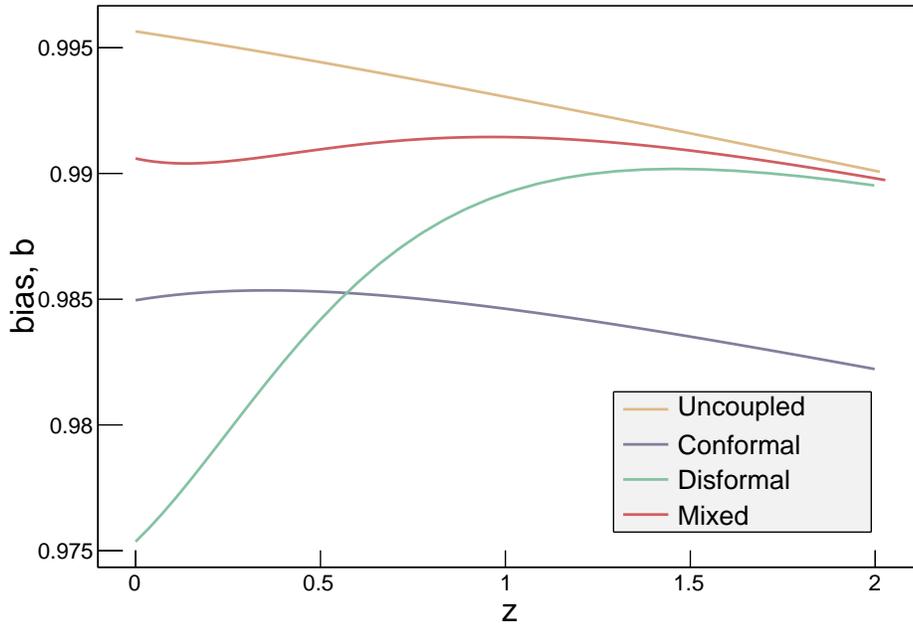
Figure 4.10: How the late time bias, as defined in Eq. (4.42), varies with redshift, z, for the models described in table 4.1. The free functions that define the parameters displayed in the table are given by Eq.s (4.12).

bias' parameter b, as:

$$\delta_B = b\delta_M \qquad (4.42)$$

where b will in general depend on time (and possibly scale, but since the scalar field is nearly massless, we found that b in our model does not depend on the wave number k; the situation would be different for different potentials which are not of quintessence form).

When I plot the late time evolution of this bias (figure 4.10) we see as we might expect that dark matter and baryons cluster at different rates, reflecting the underlying variations in each species' experience of gravity. What, primarily, we glean from the plots is that to reasonable accuracy, $\delta_B \simeq \delta_M$; from this we directly infer also that $\delta_c \simeq \delta_M$. I will return to address the validity of this assumption very soon, but for now, using this, along with sub-horizon approximations and Eq. (4.39) I show our derivation of the linearised

growth equation for matter when cold dark matter is gravitationally coupled to dark energy.

Let me take for granted momentarily that $\delta_C \simeq \delta_M$. By the additivity of the stress energy tensors and baryon conservation, the evolution of all-matter perturbations (dark matter + baryons) can be derived from:

$$\hat{\delta}\left(\nabla_\mu T_M^{\mu\nu}\right) = \hat{\delta}\left(\nabla_\mu T_B^{\mu\nu} + \nabla_\mu T_C^{\mu\nu}\right) = \hat{\delta}\left(Q\phi^\nu\right) \tag{4.43}$$

which gives the growth equation:

$$\ddot{\delta}_M + \mathcal{H}_{\text{eff}}\dot{\delta}_M = 4\pi G_{\text{eff}}a^2\rho_M\delta_M \tag{4.44a}$$

where I have defined:

$$\mathcal{H}_{\text{eff}} := \mathcal{H} + \frac{\dot{x}}{x}\frac{\rho_C}{\rho_M} \tag{4.44b}$$

$$G_{\text{eff}} := G + \frac{1}{4\pi\dot{\phi}^2}\left(\frac{\dot{x}}{x}\frac{\rho_C}{\rho_M}\right)^2 = G + \frac{1}{4\pi}\frac{Q^2}{\rho_M^2}, \tag{4.44c}$$

which is valid for $k \gg \mathcal{H}$. I note that while the error $|\frac{\delta_M - \delta_B}{\delta_M}|$ is of the order $\sim 1-3\%$ for the models considered, the error this induces in the growth equation (4.44a) turns out to be only ever as large as $\sim 0.1\%$ in general, and usually substantially smaller. The error propagates through the derivation in a favourable way, which afforded us valuable comparison between the true evolution of δ_M and our simplified growth equation to an accuracy sufficient for this study. I stress that, of course, evolution of cosmological perturbations is inextricably linked to evolution of the background.

We are now in a position to examine the growth rate, (4.40) for our various models. First to note is that both $\mathcal{H}_{\text{eff}}$ and G_{eff} contain only background quantities that have no k dependence—any departure from general relativity here will be scale invariant. This is due to our choice of potential for the scalar field, which is of quintessence form and the field is nearly massless. With this choice, our theory predicts that, like ΛCDM, measuring the growth at different length scales (within the quasi-static regime of course) will not lay bare the novel features of the coupling presented here. If the oscillations depicted in Fig. 4.9 were perhaps to have survived till today, this would change the story: a major observational test to distinguish disformal couplings would then be to see how these fluctuations depend on scale, k, and thus probe the value of M_D itself. In general though this is not the case; the
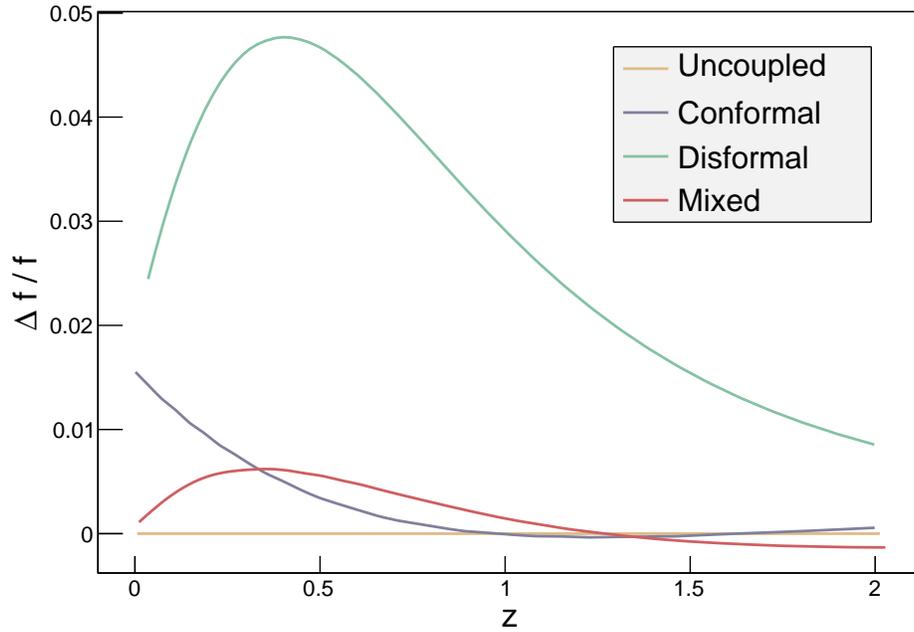
Figure 4.11: The fractional difference in the growth rate $f(z)$, Eq. (4.40), for the models of table 4.1, against redshift, z. The difference is with respect to the uncoupled case, i.e. $\Delta f/f := (f_i - f_{\text{uncoupled}})/f_{\text{uncoupled}}$ where i runs over the four models.

severe damping present in all models we have considered show that sustaining these oscillations long enough to observe them is difficult to achieve in practice, and so highly unlikely in reality. What looked in previous sections like a tool to measure disformal couplings turned out to be just an fleeting fluctuation, completely intractable empirically.

Looking at the curve for the purely conformal case in Fig. 4.11, it seems that, because both $\mathcal{H}$ and $\mathcal{H}_{\mathrm{eff}}$ are suppressed by the coupling (the background expansion rate is slowed as energy is transferred from dark energy to dark matter, exemplified by the effective equation of state for this model) growth is enhanced by the coupling. For the disformal only model, cosmic expansion *and* the extra friction felt by δ_M ($\mathcal{H}_{\mathrm{eff}}$) is enhanced but we see that the growth rate is largest for the purely disformal model.

In general relativity, the growth rate defined in Eq. (4.40) is simply related to the growth of the gravitational potential Φ. This relation is slightly modified in the coupled quintessence scenario, as I will now show. I begin by using Eq. (4.37) in the quasi-static regime (valid in the sub-horizon limit, deep inside the matter dominated epoch):

$$k^2 \Phi = -4\pi G a^2 \left(\delta\rho_B + \delta\rho_{\mathsf{c}} + \delta\rho_\Phi \right). \tag{4.45}$$

It turns out that the dark energy perturbation is negligible compared to the contributions from the baryons and dark matter (we checked this numerically). Then, remembering that $\delta\rho_M = \delta\rho_B + \delta\rho_{\mathsf{c}}$ and using the background equations, we could derive the following equation for Φ:

$$\frac{\dot{\Phi}}{\Phi} = -\mathcal{H} + \frac{\dot{\delta}_M}{\delta_M} + \frac{\dot{x}}{x} \frac{\rho_{\mathsf{c}}}{\rho_M}. \tag{4.46}$$

Here, x is the quantity defined in Eq. (4.15). We defined

$$f_{\mathrm{eff}} := \frac{\mathrm{d}\ln(a\Phi)}{\mathrm{d}\ln a} = \frac{(a\Phi)^{\cdot}}{\mathcal{H}(a\Phi)}, \tag{4.47}$$

and so we found

$$f_{\mathrm{eff}} = f + \frac{\rho_C}{\rho_M} \frac{\mathrm{d}\ln x}{\mathrm{d}\ln a}. \tag{4.48}$$

We see that in the uncoupled case, for which $x = \mathrm{const}$, f_{eff} and f coincide. I plot the behaviour of f_{eff} in Fig. 4.12. Whereas f characterises the growth of the density contrast in matter, f_{eff} characterises the growth in the
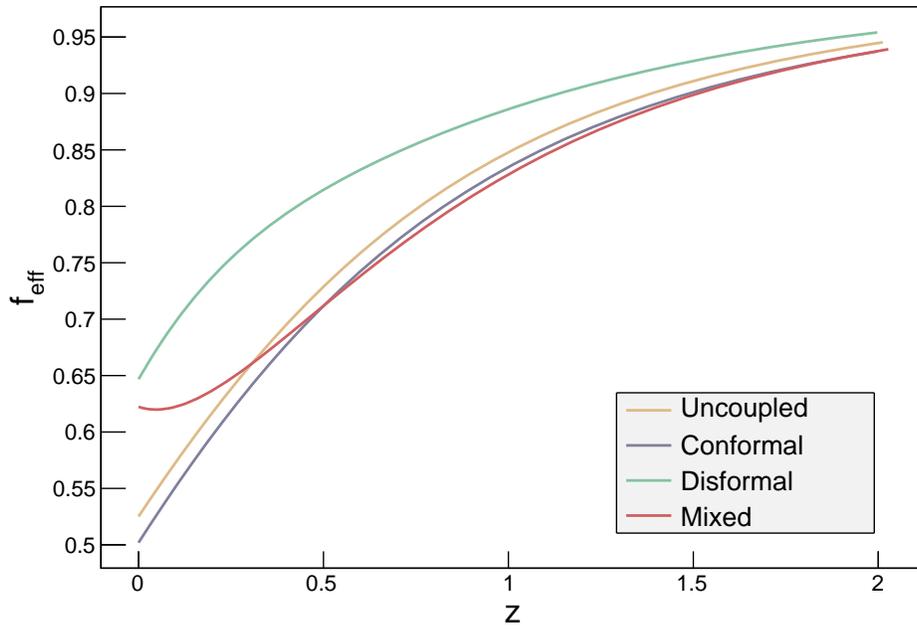
Figure 4.12: Evolution of the growth rate f_{eff}, as defined in Eq. (4.47), as a function of redshift, z, for the models of table 4.1.

gravitational potential Φ. For f_{eff} now, the departure from the uncoupled case is significant, especially at very late times—well within the reach of current redshift galaxy surveys. Crucially, this new growth rate gives us a direct measure of the gravitational potential Φ which describes the shapes of gravity wells into which galaxies fall, and the CMB is lensed.

Having diverged slightly, I now return to our central question: are conformal and disformal effects at all separable observationally at the perturbation level? Unfortunately, it would appear not. Just as for the effective equation of state, it seems that what x is doing at very late times (when the coupling suppression has already been lifted) is what dictates how either growth rate, f or f_{eff}, will behave: the disformal oscillations, characteristic of a newly introduced scale, die out long before the present day, along with any hopes of discerning between the conformal and disformal factors. Again, just as at the background level, the dynamics of late time growth are determined by the late time behaviour of the function x—a degeneracy between C and D—*no matter how it comes about*. Whether the two coupling types can be separated, again, depends on the epoch. Earlier on, a distinction is manifest, but at the later stages of universe evolution, the two act together, and the distinction blurs. As before, 'later' is defined by the scale M_D relative to the evolving Hubble scale.

4.2.2 The power spectra

I will now discuss the predicted CMB anisotropies and the matter power spectra. In Fig. 4.13 I show the angular power spectrum for the CMB anisotropies. For the type of models discussed in this chapter, if a particular model has a reduction in power at low l values it will have an enhancement at large ones, and vice versa, with respect to the uncoupled case. At small multipole values the angular power spectrum is dominated by the integrated Sachs-Wolfe (ISW) effect, which is dependent on the late time behaviour of the large scale gravitational potential. If a particular model undergoes enhanced expansion at latest times, quantified in an earlier section by an effective dark energy equation of state crossing the phantom line, then the gravitational potential Φ on large scales decays. The corresponding low l anisotropies in the CMB are reduced as a consequence. For the same model the opposite happens for large multipoles: the anisotropies are enhanced. Since we have fixed the boundary conditions for the cosmological parameters
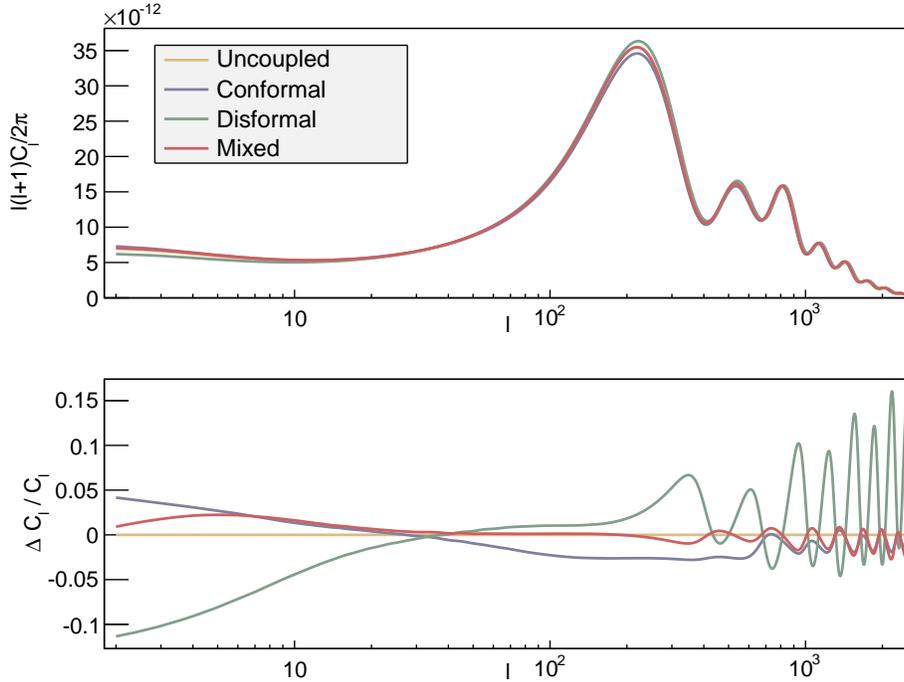
Figure 4.13: Top: angular power spectrum, C_l, against angular scale, l, for the four models described in table 4.1 with parameters defined in equations (4.12). Bottom: fractional difference between the various models and the uncoupled case, i.e. $\Delta C_l/C_l := (C_{l,\mathrm{i}} - C_{l,\mathrm{uncoupled}})/C_{l,\mathrm{uncoupled}}$.

($\Omega_{\Phi,0} = 0.7$, $\mathcal{H}_0$, etc) at the present time, the cosmological parameters differ (slightly) for the individual models at the time of decoupling. This results in different relative heights for the peaks at high multipoles.

The matter power spectra are shown in Fig. 4.14. For all models discussed here, there is an enhancement of power on small scales (large wave numbers), which is a direct result of the new scalar interaction between dark matter particles, which is always attractive. The peaks of the baryonic acoustic oscillations are shifted, which is due to fact that the cosmological parameters are different at the time of decoupling. The couplings do not directly influence the position of the peaks. Strikingly, as for the C_l's, the mixed model lies very close to the uncoupled case. As we found before, the conformal contribution to the effective coupling is suppressed by the disformal factor.
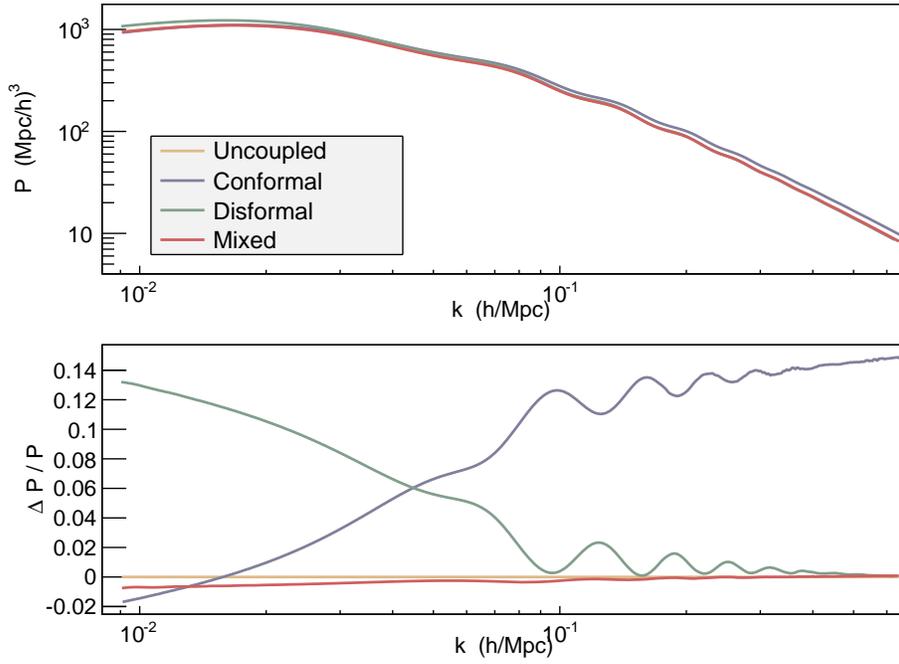
Figure 4.14: Top: matter power spectrum, P, against mode scale, k, for the four models described in table 4.1 with parameters defined in equations (4.12). Bottom: fractional difference between the various models and the uncoupled case, i.e. $\Delta P/P := (P_{\mathrm{i}} - P_{\mathrm{uncoupled}})/P_{\mathrm{uncoupled}}$. h is the dimensionless reduced Hubble constant, defined as $H_0 = 100 h Mpc^{-1}$..

4.2.3 Disformal Instabilities

Up to this point we have studied cases for which $D > 0$. I will now discuss models in which D is negative. As we will see, we find runaway growth of perturbations in the scalar field, which we feel justify our choice to neglect this case for the entire preceding study. To see how a disformal theory can become unstable in its perturbations, we can first re-write Eq. (4.32) in a more suggestive form, for simplicity treating the case where $C = 1$, and D is constant:

$$\ddot{\delta\phi} + 2\mathcal{H}\left[1 - \left(\frac{3}{2} + \frac{\dot{x}/x}{\dot{a}/a}\right)\xi\right]\dot{\delta\phi} + (k^2 + a^2 V'')[1 - \xi]\delta\phi = S(a,k) \quad (4.49)$$

where, using that $|1 + (\rho_{\mathsf{c}}D)^{-1}| \gg |\frac{(\dot{\phi}/a)^2}{\rho_{\mathsf{c}}D}|$, we have defined the parameter ξ as

$$\xi := \frac{1}{1 + \frac{1}{\rho_{\mathsf{c}}D}}, \quad (4.50)$$

and all terms that do not contain $\delta\phi$ or its derivatives are collected in the source term $S(a,k)$. The homogeneous solution evolves according to

$$\ddot{\delta\phi}_a + 2\mathcal{H}\left[1 - \left(\frac{3}{2} + \frac{\dot{x}/x}{\dot{a}/a}\right)\xi\right]\dot{\delta\phi}_a + \omega^2\delta\phi_a = 0, \quad (4.51)$$

with $\omega^2 := (k^2 + a^2 V'')[1 - \xi]$. We see that ω^2 is always positive if D is positive, and will become negative if D is negative. Eq. (4.51) and Fig. 4.15 demonstrate this clearly: for positive D, $\xi < 1$ and the effective oscillator frequency ω is real; for negative D, $\xi > 1$ and the frequency can become imaginary—the $\delta\phi_a$ solution becomes an exponentially growing function. So much for perturbations, but the background system is also unstable here. An epoch where $(\rho_{\mathsf{c}}D)^{-1} \sim -1$, and hence the system traverses a pole, we can see is almost guaranteed to occur for negative D at *some* time τ, whether this happens at higher redshift or in our future.

4.3 Conclusions

In this chapter I have described our study of the observational consequences of an extension to the coupled quintessence scenario, incorporating disformal
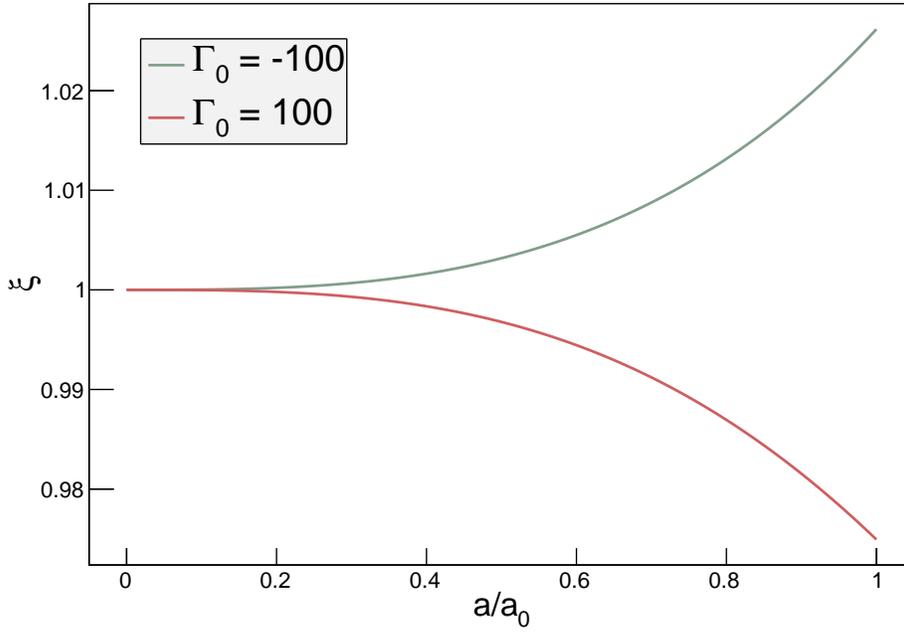
Figure 4.15: The instability function defined in (4.50) for two purely disformal models. For both curves, $\beta_V = -2$, $\beta_C = 0$, $\beta_D = 0$ and $M_D = M_V/\Gamma_0^{1/4}$ where Γ_0 is introduced in (4.13)

terms. By keeping dark energy in the slow roll regime, we have focused not on solving the cosmological coincidence problem, but rather on searching for observable signatures of realistic (near ΛCDM) coupled dark sector theories. Studies like these are an imperative when so little about the dark sector is known and so little can be assumed.

An important result of our study was that tension between a model with large conformal coupling (β_C in our notation) and data can be alleviated by the addition of a disformal interaction. This is because the disformal contribution very effectively suppresses the coupling function Q and it's linear perturbation δQ for a significant portion of the universe's lifetime. The suppression is clearly manifest in both power spectra, as the predictions of the mixed model are very close to those of ΛCDM, although β_C is of order one. We also found that, reformulating the theory in terms of the disformal scalar, the conservation equation for a coupled matter species is solvable, and we have used the solution to derive a condition indicating whether or not a given coupled dark energy model could be interpreted as exhibiting phantom behaviour.

Our analysis of the perturbations tells us that, as in the standard coupled quintessence model, the disformal term does not affect the gravitational slip. Additionally the growth rate of the matter density contrast and the growth rate of the gravitational potential no longer coincide in models with couplings—we expect this will provide an observational key to breaking degeneracies between information contained in the CMB, gravitational lensing and LSS. We furthermore found that a negative disformal coupling D generically induces dark energy instabilities: perturbations in the scalar field will eventually grow quasi-exponentially, which is bad news in general for frame isomorphisms.

The results of the paper described in this chapter suggest that it is very difficult to discern between conformal and disformal effects when using only background observables and first order cosmological perturbations (e.g. CMB anisotropies or the matter power spectrum); an analysis of the non-linear regime on small scales it seems will be necessary to look for a 'disformal smoking gun'.

This preliminary study, far from complete, still must make direct contact with data. I leave this task for future work, where I intend to use CLASS's Monte Carlo multi-nest algorithm Monte Python to confront the expansive data set open to cosmologists today. Further, more of the theory's functional freedom ($\mathcal{L}_\Phi$, C, and D) must be explored. How robust will our conclusions

be under relocations within this function space? Models with both conformal and disformal couplings between dark matter and dark energy are motivated from string theory (see [78]), and in these models, the scalar field is a DBI field where the functions C and D are specified by the extra-dimensional space. As a consequence, the effective coupling Q has a different form and behaviour. I will turn my attention to such models in future work.

Universal couplings & pathology protection

<div style="text-align: right; font-size: 3em;">5</div>

In this penultimate chapter I describe the final project, a collaboration with Jeremy Sakstein that progressed through a long-term exchange of emails and responses, embellished here and drawn together in a somewhat coherent narrative. We began with high hopes, and twists and hidden turns kept us baffled and committed but, ultimately, what we found was not worth a publication. Some ideas will lead to exciting results and some will lead to nowhere; this was the latter, and as I write an unfinished paper draft sits in my file system that will probably never see the arXiv. Nevertheless, the journey was interesting so worth recounting in these pages, and it comes last, perhaps because to me it sounded a death knell for some of the disformal theories of cosmology studied here which I thought it best not to awkwardly follow with the description of another, but perhaps simply because it came last chronologically. In Sec. 5.1 I motivate the initial idea, and spell out the model in equations. In 5.2 I discuss the first problem we encountered, a hidden bounce, and in Sec. 5.3 I present an attempt to understand the bounce and the global dynamics more generally. The second problem with the theory we encountered comes last and, unfortunately, I end the chapter on this unresolved low.

5.1 Guarding against pathology with a conformal extension

It has been recently pointed out in [90] that disformal transformations in theories of cosmology can sometimes be pathological; we have already seen that general frame maps are not always assured to be well defined. For a covariant demonstration of this problem, recall the ratio of the determinants of the two metrics, g and fg, in terms of the disformal scalar:

$$\sqrt{\frac{|fg|}{|g|}} = \sqrt{1 - 2DX} = Z \tag{5.1}$$

where recall $X = -\frac{1}{2}g^{-1}(d\phi, d\phi)$ is the scalar field kinetic energy, a positive quantity. If the field dynamics are such that $2DX \to 1$—a limit not a priori forbidden—the disformal transformation becomes ill defined and the model breaks down. In Sec. 3.1 we saw a part of the model's disformal-parameter space numerically inaccessible, due primarily to the system approaching poles in the coupling function derived directly from the frame transformation. The coupling function Q was an Einstein frame quantity, defined by non-conservation of the Einstein frame stress tensor, and in [90] they were able to relate this pathology to a phantom instability in the Jordan frame system of the same theory with $\omega_\phi = -3$.

Of course their results do not imply that any theory involving a disformal transformation is necessarily faulty, and it was shown in [99] for a very similar cosmological model that such a point is a repeller in the background dynamical system, but it does motivate us to look at frame transformations with a resistance to such a scenario built in.

In this work we proposed a minimal extension to the frame transformation that guards against this pathology in a covariant way via a simple cancellation. Consider the following ansatz:

$$f(g) = B^2(\phi)A^2(X)g + B^2(\phi)L^2 d\phi \otimes d\phi, \tag{5.2}$$

where:

$$A^2(X) := 1 + 2L^2 X, \tag{5.3}$$

$B^2(\phi)$ an arbitrary conformal factor, and L a parameter with the dimensions of length. Then if we recompute the determinant ratio:

$$\sqrt{\frac{|fg|}{|g|}} = B^4 A^3 \sqrt{A^2 - 2L^2 X} = B^4 A^3, \tag{5.4}$$

and recall that $B, A > 0$, it appears that $A(X)$ guarantees safety from the disformal metric singularities described above. This will indeed be the case for cosmology, though this safety comes at the cost of rather more involved equations of motion[1]. In fact, this pathology protection mechanism has taken our theory into the choppy waters of beyond-Horndeski: the Einstein equations, we will see, contain third order derivatives of ϕ, but in a way that exploits a loophole of Horndeski's original theorem—our theory remains Ostrogradsky stable.

The added conformal factor B does nothing for the pathology protection argument above, but it is an added free function that we will find use for later on, when a point is to be made about bouncing universes, and the conformal limit $L \to 0$ makes the maths clean and the explanation concise.

With the frame transformation (5.2) above, now a guaranteed frame isomorphism, the Einstein frame action $\mathcal{S}$ of the full theory we considered is:

$$S[g, \phi] := \frac{1}{\kappa} \int d\pi(g) \left[\frac{R(g)}{2} + X(g, \phi, d\phi) - V(\phi) \right]$$
$$+ \int d\pi(fg)\mathcal{L}(fg), \tag{5.5}$$

where R is the standard scalar kinetic term for what I label the gravitational metric, g, and κ the standard gravitational constant. The speed of light, $c = 1$. We associated ϕ with a dark energy scalar field, though it will not always be $V(\phi)$ that drives expansion of the universe. Here I retreat from multi-coupled extensions back to the safer shores of scalar tensor, and so frame indices will no longer be necessary; tildes, ~, will denote again Jordan frame quantities, where 'Jordan' again makes sense, and the Jordan frame

---

[1]A second way to avoid the metric singularity is to consider a scalar field Lagrangian that is DBI. This works because The DBI Lagrangian imposes an effective speed limit on ϕ, preventing the geometry going singular. As the theory of branes in extra dimensions involves disformal terms and DBI Lagrangians, this speed limit can be seen as a protection of Lorentz symmetry in the bulk.

action is $\tilde{\mathcal{S}} := f^{-1} \circ \mathcal{S}$, and $\tilde{g} := fg$ the Jordan metric. I refer the reader to the start of Sec. 3.1.2 for a discussion of this specific use of the term Jordan.

Our setting is again a cosmological one, with $\mathcal{S}$ defined on a flat universe (no perturbations considered here), and further we considered only the late time behaviour: the cosmic microwave and neutrino background we could thus ignore, and so $\mathcal{L}$ we took to be the Lagrangian for pressureless matter (dark matter and baryons, though no distinction was necessary for our purposes) and considered only the matter dominated epoch and its future in this study. *The action written in this form makes ϕ a dimensionless field.* This is a parameterisation apart from previous chapters so caution is advised when making comparisons.

Euler Lagrange equations

The Einstein equations of $\mathcal{S}$ above are:

$$G(g) = \kappa T_\Phi(g, \phi, d\phi) + \kappa T(g, \phi, d\phi)$$

where G is the usual Einstein tensor, and T_Φ, T are the Einstein frame stress tensors with their usual definitions.

The Einstein and Jordan frame stress tensors of matter, T and $\tilde{T}$ respectively, relate in index notation as:

$$T^{\alpha\beta} = B^4 A^3 \tilde{T}^{\mu\nu}(L^2 \partial^\alpha \phi \partial^\beta \phi g_{\mu\nu} + A^2 \delta^{\alpha\beta}_{\mu\nu}). \tag{5.6}$$

The Klein Gordon equation is:

$$\Box\phi - V(\phi) = \kappa Q, \tag{5.7}$$

where:

$$\begin{aligned}
Q &= -\beta(\phi)A^2(X)T - \beta(\phi)T^{\mu\nu}L^2\partial_\mu\phi\partial_\nu\phi - TL^2\Box\phi \\
&\quad + T^{\mu\nu}L^2\nabla_\mu\nabla_\nu\phi - 2\beta(\phi)L^2 XT \\
&\quad + \frac{2L^4}{A^2(X)}\left(T^{\mu\nu}\nabla_\nu\phi\nabla_\mu\nabla_\alpha\phi\nabla^\alpha\phi - T\nabla^\mu\phi\nabla_\mu\nabla_\nu\phi\nabla^\nu\phi\right) \\
&\quad - L^2\nabla^\mu\phi\nabla_\mu T_\mathrm{m},
\end{aligned} \tag{5.8}$$

and:

$$\beta(\phi) := \frac{B_{,\phi}}{B} \tag{5.9}$$

And finally the conservation equation is:

$$\nabla T = -Q g^{-1} d\phi. \tag{5.10}$$

I will now specialise the above system to a flat universe, where T describes a perfect fluid with vanishing pressure. From the complexity of Q above, you might justifiably expect a complicated specialised system. In fact, it is identical to conformally coupled quintessence; the system dependence on L vanishes, as we now see.

Considering a flat universe, we can chose coordinates such that the line element for g becomes:

$$ds^2 = -dt^2 + a^2 \delta_{ij} dx^i dx^j \tag{5.11}$$

and let (recall that $c = 1$ for this chapter):

$$T := \rho u \otimes u \tag{5.12}$$

where u is, as in the previous chapters, also the velocity field of comoving observers:

$$u = (1,\ 0,\ 0,\ 0). \tag{5.13}$$

Then, with these choices, we find:

$$Q = \beta(\phi)\rho \tag{5.14}$$

which is surprisingly simple, given the general covariant form of Q above.

The dynamical system in these coordinates for pressureless matter is then:

1. *Friedmann equations:*

$$3H^2 = \kappa\rho + \frac{\dot{\phi}^2}{2} + V(\phi) \tag{5.15}$$

$$\dot{H} = -\frac{\kappa}{2}\rho - \frac{\dot{\phi}^2}{2}. \tag{5.16}$$

2. *Klein Gordon equation:*

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = -\kappa\beta(\phi)\rho. \tag{5.17}$$

3. *Conservation equation:*

$$\dot{\rho} + 3H\rho = \beta(\phi)\rho\dot{\phi}. \tag{5.18}$$

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For completeness, I now show the Jordan frame equivalent of the above system, which we can derive without the Jordan frame action by using the above equations and the relation between the frame stress tensors. Consider now the coordinate system where the matter metric, $\tilde{g}$, is Friedmann:

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = -dt_J^2 + a_J^2(t)dx^i dx_i, \tag{5.19}$$

where the index "J" demarcates this time and scale factor from the distinct Einstein frame one above. It is straightforward to show that:

$$dt_J = Bdt \tag{5.20a}$$
$$a_J = B(\phi)A(X)a. \tag{5.20b}$$

Then the Jordan frame conservation equation can be derived as:

$$\mathring{\rho}_J + 3H_J\rho_J = 0, \tag{5.21a}$$

where H_J has the obvious definition:

$$H_J := \frac{\mathring{a}_J}{a_J} \tag{5.21b}$$

and the '$\circ$' is a derivative with respect to J time, t_J. The scalar field obeys a modified Klein Gordon equation:

$$\mathring{\psi}_J + 3\left[H_J - \frac{(\mathring{AB})}{AB}\right]\psi_J + \frac{V_{,\phi}}{B^2} = -\beta(\phi)(\psi_J^2 + \kappa\rho_J A^3 B^2) \tag{5.21c}$$

for $\psi_J := \mathring{\phi}$, and the Friedmann equation is:

$$3\left[H_J - \frac{(\mathring{AB})}{AB}\right]^2 = \kappa\rho_J A^3 B^2 + \frac{1}{2}\psi_J^2 + \frac{V}{B^2} \tag{5.21d}$$

The second, or acceleration, Friedmann equation for $\mathring{H}_J$ is much more involved, and contains third derivatives of ϕ. We will not concern ourselves with it. The above equations are not in a suitable form to be numerically integrated; the Friedmann and Klein Gordon equations both contain second order derivatives of ϕ which must be eliminated from either one for solutions to be tractable.

Compact phase space

Following [182] we can compactify the Einstein frame Euler Lagrange equations. We will learn much about the global behaviour of our system in both frames this way, as I show in the next sections. I define the following compact variables:

$$\Omega := \frac{\kappa \rho}{3H^2}, \quad x := \frac{\dot{\phi}}{\sqrt{6}H}, \quad y := \frac{\sqrt{V}}{\sqrt{3}H}, \tag{5.22}$$

and chose:

$$V(\phi) = m_0^2 e^{-\lambda \phi} \quad \text{and} \quad B(\phi) = e^{\beta \phi}, \tag{5.23}$$

with m_0, λ and β constants [2]. A very comprehensive analysis of the above compact phase space can be found in [183]. The (x, Ω) plane through the compact space forms an autonomous subsystem, whose dynamical equations are:

$$x' = 3\left(-\frac{(\lambda + \beta)\Omega}{\sqrt{6}} + \frac{\lambda}{\sqrt{6}} + x^3 - \frac{\lambda x^2}{\sqrt{6}} + \frac{x\Omega}{2} - x\right) \tag{5.24a}$$

$$\Omega' = 6x^2\Omega + 3\Omega^2 - 3\Omega + \sqrt{6}\beta\Omega x \tag{5.24b}$$

subject to the constraints:

$$x^2 + \Omega \leq 1, \quad \Omega \geq 0, \tag{5.24c}$$

and 'prime' denotes time derivative with respect to $N := \ln a$. Any compact two dimensional phase space has the useful property that it can be faithfully represented in graphic form, flat on the page. We will see some illustrations of the (x, Ω) plane in the next section. It is important to note that, while our Einstein frame Euler Lagrange equations are indentical to a coupled quintessence model, the more involved frame isomorphism f used here is not a simple conformal transformation, and so the Jordan frame equations will *not* be the same as the Jordan frame equations of a standard conformally coupled quintessence theory. The Jordan frame equations *do* in our case depend non-trivially on the disformal length scale L, and so with these assumptions and in this coordinate system, the theories are not carbon copies, though the Einstein frame Euler Lagrange equations are.

---

[2]note that the choice of B above is equivalent to setting $\beta(\phi)$ to a constant in definition (5.9), up to a constant scaling $B_0 e^{\beta\phi}$, which has no consequence.

As the ins and outs of this particular set of equations have been explored in detail in [183] it would seem, in order to fully understand the global behaviour of our theory, all that is left to do is look up the critical points in the literature, along with their stability tables, and map them over to the Jordan frame.

5.2 Bounces hidden by frame maps

The critical points of the compact dynamical system above are well documented, as are the stability and existence conditions of each. However it was an unexpected stroke of luck that lead the disformal length scale, L, of our action to drop out of the Einstein frame equations, and so the Jordan representation of our theory is not equivalent to the Jordan frame of the documented cases. To look for solutions of our theory capable of addressing the cosmological constant problem, we would like to use this happy accident that the Einstein frame is already well described to look for these solutions by mapping the critical points across to a Jordan frame dynamical system. However, for a critical point to be mapped to a critical point, the dimension of the source and target phase spaces should, in general, be equal, and, for the points to always map to finite points, the target space must be compact. This requires the construction of a compact two dimensional Jordan frame phase space. In this section we will see that, for our particular theory, such a space may be impossible to construct.

5.2.1 Dynamical systems & completeness

The following result is well known, but I repeat it here to facilitate a description of our work that used it. Consider a general autonomous dynamical system, valid over some interval I:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}), \quad I : \alpha < t < \beta \tag{5.25}$$

where we assume that t spans the space of solutions: that is, any integral curve in the space can be completely parameterised by t. For the sake of the proof, I will now call this property of t **completeness**. Then, if we make a

change of time variable:

$$t'(t), \quad \frac{dt'}{dt}\frac{d\mathbf{x}}{dt'} = \mathbf{F}(\mathbf{x}), \tag{5.26}$$

we quite clearly demand the smooth curve $t'(t)$ be a diffeomorphism that is regular, in particular:

$$\frac{dt'}{dt} \neq 0, \quad \frac{dt'}{dt} \nrightarrow \infty \quad \forall t \in I. \tag{5.27}$$

These requirements of a time transformation map make perfect sense. We obviously need that our new time variable has one and only one value corresponding to each value of a complete time variable for it too to be complete, and if this were not true, the transformation would leave holes of information in the new system. Requiring $t'(t)$ be regular also removes the possibility of the map being multiple valued, which can happen when the universe goes through a bounce, as we will see in the next section.

5.2.2 Application to our model

We can use condition (5.27) to reveal whether a particular choice of time coordinate will or will not be complete, and I now apply this to the various potential possibilities of time parameterising our disformal cosmological system. We will see that, for our chosen disformal theory, the universe can go through expanding and contracting phases—bounces—in terms of the Jordan frame variables, yet the Einstein frame scale factor will always be strictly increasing with time. Such a result has been discussed elsewhere in the context of other models [184], but it will be important for Sec. 5.3 where we will argue that it precludes the existence of a compact dimension two Jordan frame dynamical system for our disformal theory.

For this section, 5.2.2, I will use the following conventions and notation. Subscript J labels Jordan frame quantities, E Einstein frame, and:

$$\tilde{g}_{\mu\nu} \; := \; B^2(\phi)A^2(X)g_{\mu\nu} + B^2(\phi)L^2\phi_{,\mu}\,\phi_{,\nu} \tag{5.28a}$$

$$\tag{5.28b}$$

$$ds_E^2 \; = \; -dt_E^2 + a_E dx^i dx_i \tag{5.28c}$$

$$ds_J^2 \; = \; -dt_J^2 + a_J dx^i dx_i, \tag{5.28d}$$

where:

$$A^2(X) = 1 + 2L^2 X \tag{5.28e}$$
$$B(\phi) = B_0 e^{\beta\phi} \tag{5.28f}$$
$$dt_J = B(\phi)dt_E \tag{5.28g}$$
$$a_J = B(\phi)A(X)a_E , \tag{5.28h}$$

and it will be useful to define:

$$N_E := \ln(a_E) \tag{5.28i}$$
$$N_J := \ln(a_J) \tag{5.28j}$$
$$V(\phi) := V_0 e^{-\lambda\phi} . \tag{5.28k}$$

Is t_J complete? - Condition (5.27) requires we first assume that a given coordinate is complete, so let us start by assuming that the Einstein frame cosmic time, t_E, is complete. Then,

$$\frac{dt_J}{dt_E} = B_0 e^{\beta\phi} \tag{5.29}$$

which is never 0 or goes to ∞ in finite time if B_0 is non zero and ϕ contains no poles. We thus have the following implication:

$$t_E \text{ complete} \iff t_J \text{ complete.} \tag{5.30}$$

The implication is biconditional; we could initially have assumed either t_E or t_J was complete and got that the other was for free.

Is a_E complete? - Repeating the above process, we find a known result. First we note:

$$\frac{da_E}{dt_E} = a_E H_E \tag{5.31}$$

$$\frac{dN_E}{dt_E} = H_E \tag{5.32}$$

$$H_E \propto \rho^{1/2} \tag{5.33}$$

from which we conclude (with no surprises) that, assuming the weak energy condition in the Einstein frame[3], da_E/dt_E is finite in finite time, and so $a_E(t_E)$

---

[3]It might at first seem that it should be in the Jordan frame, not the Einstein frame, that we enforce the weak energy condition, placing this above argument on uneven ground, but as any coupling preserves the sign of the energy density under a frame transformation, we get that $\rho_J \geq 0 \Rightarrow \rho_E \geq 0$ so it amounts to the same thing.

is a regular diffeomorphism. We can now extend the above implications:

$$t_E \text{ complete} \iff t_J \text{ complete} \iff a_E \text{ complete} \iff N_E \text{ complete.} \quad (5.34)$$

To state this result nicely in words: if and only if any single one of these time variables is complete, then it follows that they all must be.

Is a_J complete? - Picking one of the complete time variables above, t_J, let us use it and condition (5.27), again, to answer this question. First we can compute the Jordan frame Hubble parameter:

$$H_J := \frac{1}{a_J} \frac{da_J}{dt_J} = \frac{1}{B} \left[H_E + \frac{1}{A} \frac{dA}{dt_E} + \frac{1}{B} \frac{dB}{dt_E} \right] \quad (5.35)$$

such that:

$$\frac{da_J}{dt_J} = a_J H_J. \quad (5.36)$$

From the definition of the Jordan frame scale factor (5.28h) we notice $a_J > 0$, however we have no such guarantee for H_J. In fact we observe, quite generically, the presence of a Jordan frame bounce (see Fig. 5.1), characterised by:

$$\exists t_J \in I \;\; \text{s.t.} \;\; H_J(t_J) = 0 \quad (5.37)$$

and hence the subsequent breakdown of injectivity between t_J and a_J. We conclude that, even though t_J is always complete here, a_J is not; no phase space description of the system expressed completely in terms of Jordan frame quantities can use a_J or N_J as a time variable. The argument presented here utilised the Einstein frame, but we see that the conclusion does not depend on it. That (5.37) can sometimes occur is a fact independent of whether or not we used the Einstein frame, and hence that $N_J(t_J)$ is incomplete is a conclusion independent of this also.

This, we concluded, is bad news in general for a two dimensional Jordan frame phase space construction. We cannot scale H_J out of the system in the usual way (to be discussed in Sec. 5.3.1), and so not reduce the space dimension in this manner. We cannot introduce variables analogous to those of the Einstein frame ($\kappa \rho_J / 3 H_J^2$ e.t.c. or some variant of this), due primarily to the zeros of H_J, and so a clear comparison can not be made; whether or not we pick up more critical points on mapping the phase plane over to the Jordan frame is now an ill defined question, as a straightforward map does not even exist! This conclusion suggests that, instead of looking for any phase plane map, we should search for observationally suitable model trajectories in alternate ways.

5.2.3 A closer look at the bounce

Will the bounce always occur? How generic is the shape in Fig. 5.1, and what causes it? Can it be plausibly avoided to produce cosmologies of interest? Let us explore these questions now. I will use bounce in a loose sense to mean: a point in the expansion history of some universe in which the expansion changes direction, i.e. a transition from expansion to contraction or vice-versa.

A Jordan frame bounce is always characterised by a stationary point in the function $a_J(t_J)$[4]. If a bounce occurs, there will be a resulting breakdown of injectivity between any one of the complete time parameters, and any incomplete one. Let us use this as the mathematical characterisation of a bounce to explore whether or not it will always happen, and if not, why not. Consider the relation:

$$a_J = B(\phi)A(X)a_E \qquad (5.38)$$

From what has just been said, if a bounce occurs, then:

$$\exists t_J \in I \quad \text{s.t.} \quad \frac{dN_J}{dN_E}(:= N'_J) = 0. \qquad (5.39)$$

To simplify the analysis, we notice from Eq. (5.38) that condition (5.39) can be satisfied even in the limit where $L \to 0$, and so we consider only the conformal limit of the theory. Then, using Eq. (5.38), we get a bounce when:

$$
\begin{aligned}
\frac{dN_J}{dN_E} &= \frac{1}{B}\frac{dB}{dN_E} + 1 \\
&= \sqrt{6}\beta x + 1 \\
&= 0 \\
\Rightarrow \quad x &= -\frac{1}{\sqrt{6}\beta}, \qquad (5.40)
\end{aligned}
$$

Thus, if at any point on some universe trajectory, $x = -\frac{1}{\sqrt{6}\beta}$, then that universe traverses a Jordan frame bounce. This is depicted in Fig.s 5.1 and 5.2. Note of course that a bounce can be avoided entirely if $|\sqrt{6}\beta| < 1$, i.e. weak coupling, as then the grey vertical line falls outside the basin of

---

[4]I stress that this does not include stationary points in phase space variables x, Ω etc which I refer to here exclusively as critical points
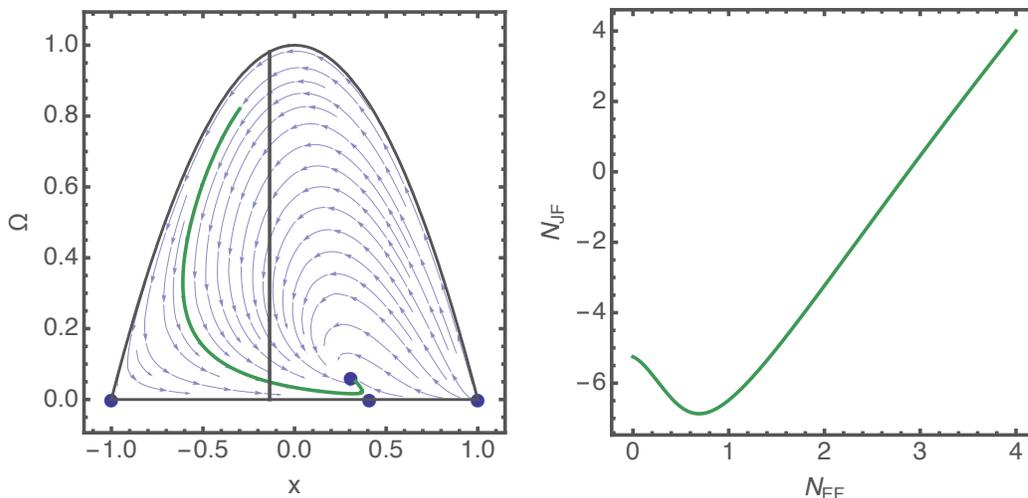
Figure 5.1: Ω vs x for a bouncing universe. Model parameters: $\lambda = 1$, $\beta = 3$, $L = 0$. (Left) the closed grey curve, $\Omega + x^2 = 1$ OR $\Omega = 0$, outlines the basin of attraction. Blue filled in circles are the critical points. The green curve is an integral curve for some initial condition, and the grey vertical line is given by $x = -1/(\sqrt{6}\beta)$. If an integral curve crosses the grey line, $x = -1/(\sqrt{6}\beta)$, the universe goes through a Jordan frame bounce. (Right) the Jordan frame scale factor $N_J = \ln(a_J)$ as a function of the Einstein frame scale factor for the integral curve shown in the left panel.

attraction. Less concretely, we can say that the larger β is, the closer the grey line will lie to the centre of the basin, and hence the more integral curves it will intersect; as we make the coupling stronger, a smaller and smaller proportionate region of the space of initial conditions will produce integral curves that can avoid a bounce.

The phase portraits in figures 5.1 and 5.2 depict systems unlikely to produce viable universes with matter dominated epochs preceding de sitter phases. These choices of model parameters serve more to demonstrate the prevalence of a bounce, and though the models become more complicated with $A(X)$ non-trivial, I found bounces are then much more prone, with sensible looking universes in the Einstein frame becoming bouncing ones in the other.
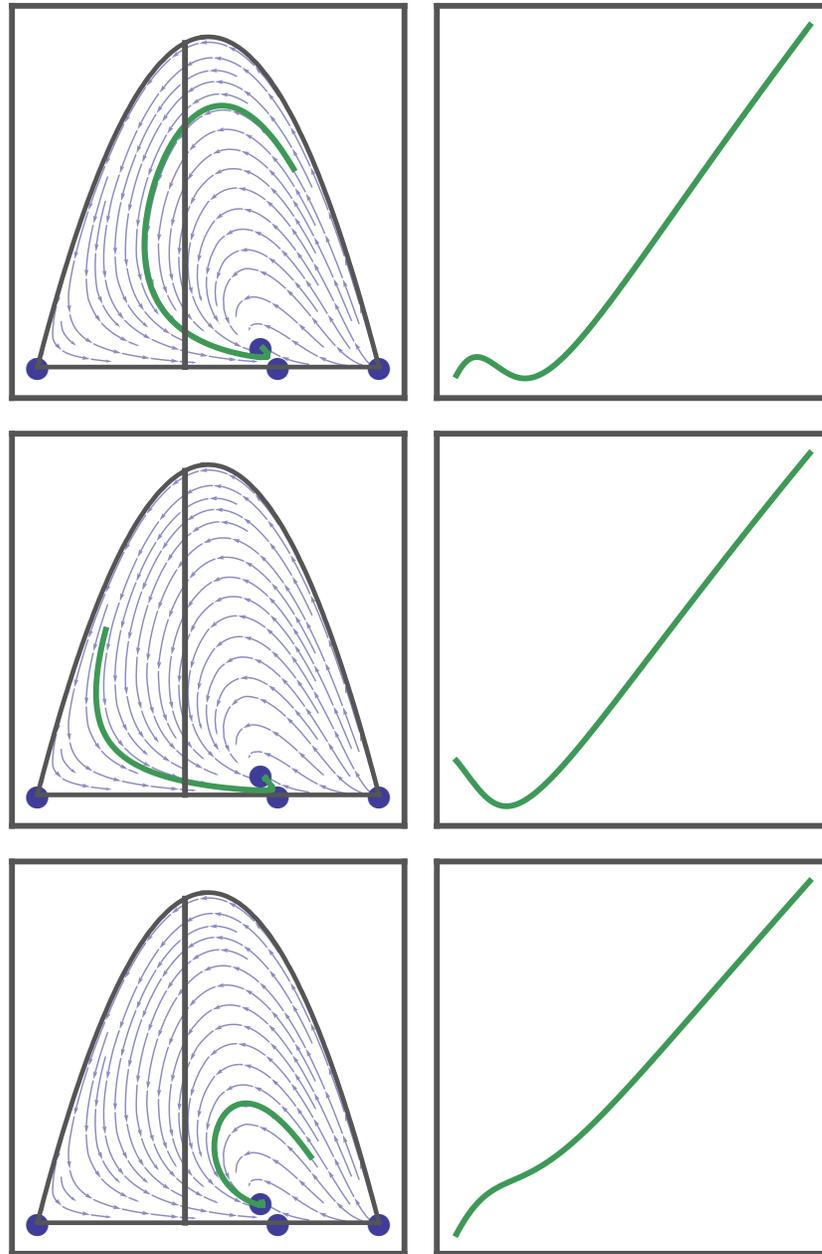
Figure 5.2: The number of intersections between the green and grey lines in the phase portraits (left) equals the number of stationary points in the function $N_J(N_E)$ (right).

5.3 Critical curves

What has just been claimed, namely that the presence of a bounce in the Jordan frame is indicative of a Jordan frame system that a) cannot be compactified, and b) does not contain a two dimensional autonomous phase plane to map the published critical points [183] into, will be expounded here.

We have seen that an attracting critical point in the Einstein frame equations where $\ddot{a}_E > 0$ does not necessarily mean acceleration in the Jordan frame system. This problem persists even in the far more benign, and much studied, conformal limit of our theory, a point which has already been made elsewhere in the literature [184]. Can we actually learn anything concrete about the Jordan frame from an Einstein frame analysis? Here I describe how we reinstated our Einstein frame phase space as a useful interpretive tool by extending the notion of a critical point. Then in the final section I show how we put these findings to work in searching for viable cosmological solutions that address the cosmological constant problem.

In the first sub section, I look for a suitable closed Jordan frame dynamical system, then go on to explain and utilise critical curves in the succeeding sub sections. In the text I will often abbreviate dynamical systems, defined in general as:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t),\tag{5.41}$$

using the following semicolon notation:

$$\{t; \mathbf{x}\}.\tag{5.42}$$

5.3.1 A minimal closure of the Jordan frame

How many dynamical variables does it take to close a Jordan frame system? In this section I will address this question, and in the process define the distinct closed systems describing our model that will be used throughout this section. I will first recap how the phase space dimension is normally reduced in the Einstein frame, and then highlight why this procedure fails for the Jordan frame.

For our model, at the background level, the full system of equations in

the Einstein frame can be written (for pressure, $P = 0$):

$$
\begin{aligned}
\dot{a} &= aH \\
\dot{H} &= -\frac{1}{2}(\rho + \psi^2) \\
\dot{\phi} &= \psi \\
\dot{\psi} &= -3H\psi - V_\phi(\phi) - \kappa\beta(\phi)\rho \\
\dot{\rho} &= -3H\rho + \beta(\phi)\rho,
\end{aligned}
\tag{5.43}
$$

where the dot is derivative with respect to Einstein frame time, t_E. The Friedmann constraint reduces the phase space dimension to four:

$$
3H^2 = \kappa\rho + \frac{1}{2}\psi^2 + V(\phi),
\tag{5.44}
$$

and it is usually most convenient to remove the $\dot{H}$ equation; we no longer treat H as a dynamical variable. Next, we see that the last three equations are independent of a. We can use this scaling property of the system to ignore the evolution equation for a—the phase space is now just three dimensional: $\{t_E; \phi, \psi, \rho\}$. As a last step, clever choices of V and B as exponential, along with a specific time redefinition $t_E \to N_E$ and appropriate division by factors of H, mean we can also remove the ϕ dependence from the last two equations. As for a, ϕ too now scales out nicely, and the effective system dimension reduces to two: $\{N_E; x, \Omega\}$.

The mathematical simplicity of the Einstein frame is palpable; the Jordan frame equivalent of (5.43) is a horrible mess. The other great benefit of the Einstein frame is that the critical point analysis, including stability, has been done some time ago. Physical interpretation of solutions is the Einstein frame's main drawback, and we can not say with certainty what a_J will be doing on inspection of the Einstein frame solutions. What we really need is a Jordan frame phase space whose variables we can easily link to observation.

The problem arises when we try to replicate the scaling down process of the Einstein frame phase space as described above. Naïvely we might start with a system $\{t_J; a_J, H_J, \phi, \psi_J, \rho_J\}$ and dimension reduce by finding an appropriate Friedmann constraint (solving away higher derivatives of ϕ in the equation for H_J using the Klein Gordon equation) and then scale out a_J and ϕ. This will fail for two reasons. First is that H_J is occasionally zero, and so we cannot define new variables where it appears in the denominator. Second is that N_J is incomplete (a fact entailed by the first point) and so

cannot be used as a time variable. ϕ therefore does not decouple in the Jordan frame. a_J however remains decoupled. I stress that the failure of a Jordan frame time variable to be complete is not a consequence of the failure of a frame transformation to be well defined; they are completely unrelated. Recall the frame transformation used in this chapter is always well defined by construction.

Now, since the maps between frame variables involve functions of ϕ and ψ only, it is clear that, using the closed system $\{t_E; \phi, \psi, \rho\}$ and the coordinate transformation:

$$\Pi : \{t_E; \phi, \psi, \rho\} \rightarrow \{t_J; \phi, \psi_J, \rho_J\} \tag{5.45}$$

where:

$$
\begin{aligned}
dt_J &= B(\phi)dt_E & \tag{5.46}\\
\phi &= \phi \\
\psi_J &= \frac{\psi}{B(\phi)} \\
\rho_J &= \frac{\rho}{A^3(\psi)B^4(\phi)}, & \tag{5.47}
\end{aligned}
$$

the evolution of any Jordan frame quantity at any one time can be specified. Here $\psi_J := d\phi/dt_J$. (5.46) shows how ϕ is mixed up with the Jordan frame variables—it is tightly re-coupled back into the system. The scale factor a is not explicit in the maps, which is the reason why a_J is still decoupled. This has just shown that the Jordan frame closes with a minimum of three dynamical variables $\{t_J; \phi, \psi_J, \rho_J\}$. I will refer to this as the **Jordan system** from now on.

In this section I have introduced several closed systems of which I will refer to in later sections. For this reason I have named them:

- the **Wands system**: $\{N_E; x, \Omega\}$

- **extended Wands system**: $\{N_E; \phi, x, \Omega\}$

- **Einstein system**: $\{t_E; \phi, \psi, \rho\}$

- Jordan system: $\{t_J; \phi, \psi_J, \rho_J\}$

5.3.2 From critical points to critical curves

It is in the Wands system $\{N_E; x, \Omega\}$ only that critical points and their stability conditions have been calculated, which fill various tables in the literature, illustrated by phase portraits [90,182,183]. We would like to use this published information to tell us about our own system, the Jordan system $\{t_J; \phi, \psi_J, \rho_J\}$, which is closest to observables. The main problem is immediately apparent: Wands is two dimensional, while the Jordan system has dimension three. The critical points in Wands often become trajectories—or critical curves—in the Jordan system. In what follows, I will work in units where $8\pi G = c = 1$.

Consider a critical point (x_0, Ω_0) in Wands. Integrating the decoupled ϕ equation gives $\phi = \sqrt{6}x_0 N + \phi_0$ at this point (ϕ_0 an arbitrary constant of integration), where $N := N_E = \ln a_E$, and so the corresponding critical curve in extended Wands is:

$$X_0 : I \quad \rightarrow \quad \{N_E; \phi, x, \Omega\} \tag{5.48}$$
$$N \quad \mapsto \quad (\sqrt{6}x_0 N + \phi_0, x_0, \Omega_0) \tag{5.49}$$

Note from this equation that, if $x_0 = 0$, a critical point in Wands is mapped onto a critical point in extended Wands. Unfortunately no critical points of interest in our model will be of this type, but this demonstrates that points are not always mapped to curves under a dimension increase.

This notion of a critical curve is the missing link: there is now one and only one object (point or curve) in any of the three dimensional systems (extended Wands, Einstein, Jordan) associated to each critical point in Wands— we can now map the critical points found by [90, 182, 183] over to objects in the Jordan frame bijectively. What is more, quantities in the Jordan frame (like H_J, q_J e.t.c.) are easily computable in exact form, in terms of N or t_J for example, along these curves which is not possible in general for the rest of the phase space. Attracting critical points in Wands will map onto attracting trajectories in the Jordan system, and it can be shown that any curve that runs close to an attracting critical curve in the Jordan system will get asymptotically near as $t_J \rightarrow \infty$ (see figure 5.3).

Analogous to how one normally obtains information about a nonlinear dynamical system by studying the critical points, we can now study a system of dimension three using a judicious combination of the dimension two Einstein frame phase plane critical points and exact solutions along each corresponding critical curve. These curves thread the Jordan frame space, and

all trajectories in the space end up asymptotically near an attracting one, up to curve parameterisations, in the far future.

5.3.3 Analytic solutions along an arbitrary critical curve

In this section we derive exact expressions for the critical curves of any critical point in parametric form, and important cosmological quantities along them. In what follows, we will stick to the following free function choices:

$$V(\phi) = m_0^2 e^{-\lambda\phi} \quad \text{and} \quad B(\phi) = e^{\beta\phi}, \tag{5.50}$$

Consider again the critical point (x_0, Ω_0) in Wands. I have already noted the corresponding critical curve in extended Wands:

$$X_0(N) = (\sqrt{6}x_0 N + \phi_0, x_0, \Omega_0), \tag{5.51}$$

and I also reiterate that $X_0(N)$ inherits the stability of its corresponding critical point (x_0, Ω_0). Mapping from Wands to extended Wands involves integration and dimension increase. Going from extended Wands to Einstein is simply a change of coordinates, given by:

$$\Theta : \{\phi, x, \Omega\} \to \{\phi, \psi, \rho\} \tag{5.52}$$

where:

$$
\begin{aligned}
\phi &= \phi \\
\psi &= \sqrt{6}H(\phi, x, \Omega)x \\
\rho &= 3H^2(\phi, x, \Omega)\Omega,
\end{aligned} \tag{5.53}
$$

and:

$$H = \frac{\sqrt{V(\phi)}}{\sqrt{3}y} \tag{5.54}$$

and $y = \sqrt{1 - \Omega - x^2}$ is the Friedmann constraint. From now on I will continue to use N as a time coordinate, to keep the equations concise, and so suppress time in the system notation. For example the Einstein system becomes $\{\phi, \psi, \rho\}$. This will be enough to present our conclusions at the end, though I note more complete future work should incorporate time re-parameterisations.
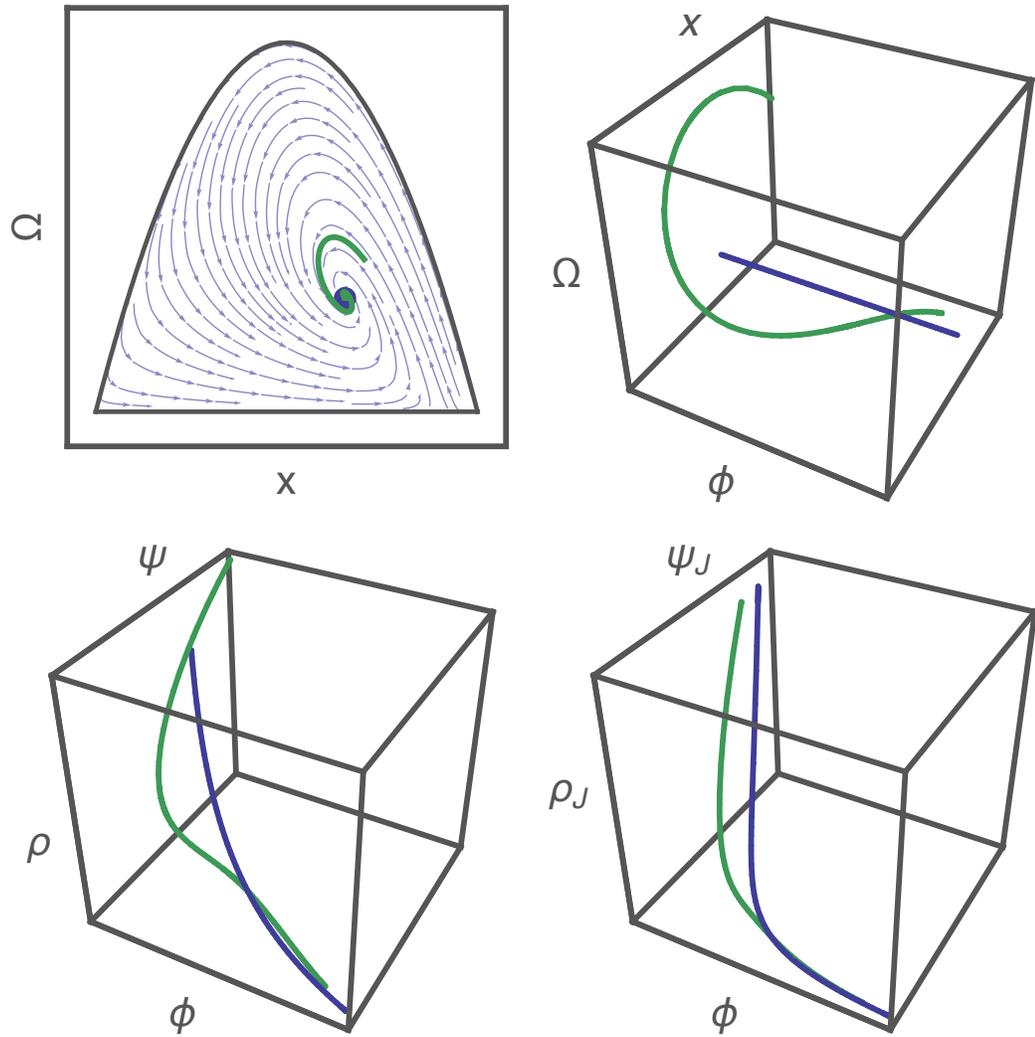
Figure 5.3: Illustration of an attracting critical curve. (Top left) critical point (blue circle) and approaching trajectory in the Wands system. (Top Right) critical curve, blue, corresponding to an attracting critical point, with generic integral curve, green, approaching it for increasing N in the extended Wands system. (Bottom left & right) the same system, now expressed in the Einstein and Jordan system coordinates respectively.

From (5.54) follows our first result: along any critical curve, $y = y_0$ is constant, and so $H^2(N)$ will evolve identically to the scalar field potential, with the same scale. Then, along the critical curve, H will have an exponential form in N whose scale is set by m_0:

$$H(N) = H_0 e^{-\frac{\lambda}{2}\sqrt{6}x_0 N}, \tag{5.55a}$$

where:

$$H_0 = \frac{m_0}{\sqrt{3}y_0} e^{-\frac{\lambda}{2}\phi_0}. \tag{5.55b}$$

Using (5.53) and (5.54) to express X_0 in the Einstein system coordinates, $X_{0E} := X_0 \circ \Theta$ gives:

$$X_{0E}(N) = (\phi(N), \psi(N), \rho(N)), \tag{5.56a}$$

where:

$$\phi = \sqrt{6}x_0 N + \phi_0 \tag{5.56b}$$
$$\psi = \sqrt{6}H_0 e^{-\frac{\lambda}{2}\sqrt{6}x_0 N} \tag{5.56c}$$
$$\rho = 3H_0^2 \Omega_0 e^{-\lambda\sqrt{6}x_0 N} \tag{5.56d}$$

and, finally, with (5.46) we can express X_0 in the Jordan system coordinates, $X_{0J} := X_0 \circ \Theta \circ \Pi$, as:

$$X_{0J}(N) = (\phi(N), \psi_J(N), \rho_J(N)), \tag{5.57a}$$

where:

$$\phi = \sqrt{6}x_0 N + \phi_0 \tag{5.57b}$$
$$\psi_J = \sqrt{6}\frac{H_0}{B_0} e^{-(\frac{\lambda}{2}+\beta)\sqrt{6}x_0 N} \tag{5.57c}$$
$$\rho_J = 3\frac{H_0}{B_0^2 A^3(N)} e^{-(\lambda+4\beta)\sqrt{6}x_0 N} \tag{5.57d}$$

and:

$$B_0 := e^{\beta\phi_0} \tag{5.57e}$$
$$A(N) = \sqrt{1 + L^2\psi^2(N)}. \tag{5.57f}$$

I have continued to use $N := N_E$ in all the above expressions purely to keep them simple. In the Jordan system, N loses its meaning as log of the size of the universe, but it is still a complete time parameter, so a perfectly valid way to keep track of the system evolution. We could of course re-write X_{0J} as a function of t_J instead, but I will leave this job for later work.

Armed with exact solutions of critical curves (any critical curve), we can now compute interesting quantities along them. I will pick the Jordan frame Hubble parameter H_J and deceleration parameter q_J.

We have that:

$$H_J = \frac{1}{B} \left(H_E + \frac{1}{AB} \frac{d(AB)}{dt_E} \right) \tag{5.58}$$

which, along X_0, gives:

$$H_J(N) = \frac{H_0}{B_0} e^{-(\frac{\lambda}{2}+\beta)\sqrt{6}x_0 N} \left[1 + \sqrt{6}x_0 \left(\beta - \frac{\lambda}{2} g(N) \right) \right] \tag{5.59}$$

where:

$$g(N) := \frac{L^2 \psi(N)}{1 + L^2 \psi(N)} = \frac{e^{\sqrt{6}\lambda x_0(N_* - N)}}{1 + e^{\sqrt{6}\lambda x_0(N_* - N)}} \tag{5.60}$$

(which, from Fig. 5.4 we see $g(N) \in (0,1)$ $\forall N$) and:

$$N_* = \frac{1}{\sqrt{6}\lambda x_0} \ln \left[6L^2 H_0^2 \right]. \tag{5.61}$$

Recall that H_0 is the value of the Einstein frame Hubble parameter at $N = 0$, so while it may seem from the above equations that quantities depend on the absolute, not relative, value of N, and so the Jordan system is no longer autonomous with respect to time, this is untrue; a redefinition $N \mapsto N + const$ would redefine H_0, ϕ_0, B_0 (which we remember are not all free parameters but linked by the relations given above) in such a way that the system variables ρ_J, ψ_J, H_J and so on are all invariant.

To wrap up this section, let us compute the deceleration parameter q_J. The natural definition must be:

$$\frac{1}{H_J} \frac{dH_J}{dN_J} := -(1 + q_J). \tag{5.62}$$

Looking at (5.64) we see that the N dependance is only in the exponential factor (which is easily differentiated) and the $g(N)$ term, which is either approximately 0 or 1 $\forall N$ except near N_* (see figure 5.4).
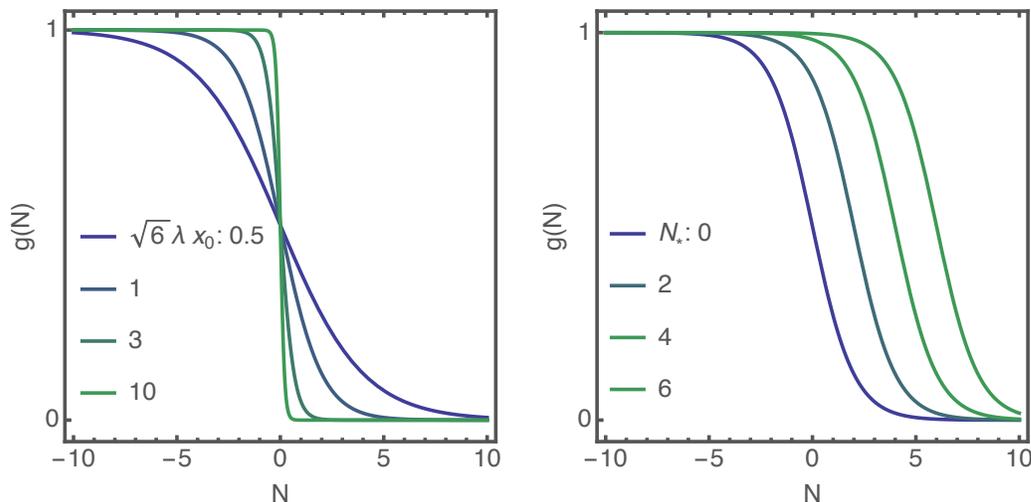
Figure 5.4: $g(N)$ function in expression for H_J, equation (5.60). N_* shifts the position of the step, while increasing λx_0 sharpens the step slope. The parameter values used here are solely to illustrate their effect on $g(N)$, and should not be taken as realistic.

So, solving for q_J in regimes far from N_* along each critical curve gives:

$$
q_J = \begin{cases}
\dfrac{\sqrt{6}x_0\lambda - 1}{\sqrt{6}x_0(\beta - \lambda/2) + 1} & \text{if } N \ll N_* \\[2ex]
\dfrac{\sqrt{6}x_0\lambda/2}{\sqrt{6}\beta x_0 + 1} & \text{if } N \gg N_*.
\end{cases}
\tag{5.63}
$$

These expressions are valid for the lengths of the curves, though, surprisingly, q_J is independent of N for all times except near the step. This lucky coincidence (it would not have been true for an inverse power law potential) heightens the utility of the critical curve approach; any Jordan frame trajectory near these curves (if they are attracting) will approach asymptotically, and inherit the critical curve behaviour as they draw near—by tuning λ, β and Λ to make q_J negative, we can ensure they will be drawn towards de Sitter-like epochs with constant q_J! Note the absence of L in the expressions for q_J. I will comment on this in the next section. I will develop this in the next section for the two attracting critical points in table 5.1.

Number	x_0	y_0
1	$\frac{\lambda}{\sqrt{6}}$	$\sqrt{1 - \frac{\lambda^2}{6}}$
2	$\frac{\sqrt{\frac{3}{2}}}{(\beta+\lambda)}$	$\sqrt{\frac{2\beta(\beta+\lambda)+3}{2(\beta+\lambda)^2}}$

Table 5.1: Stable critical points in the Wands system. Both were derived assuming $\lambda > 0$ for each, and 1 exists for $\lambda < \sqrt{6}$. The existence of 2 is more complicated.

5.3.4 Analysis along the stable curves

The expressions above for X_0 e.t.c. are for any critical point (x_0, Ω_0), and do not depend on stability. The points of particular physical interest are the stable ones, so we explore these in some detail now. The conditions for existence and stability are presented in Fig. 5.5.

We are, first and foremost, concerned about the behaviour of H_J along each critical curve for different regions of the (λ, β, L) parameter space: for what values of (λ, β, L) is H_J negative? Positive? Bouncing? Accelerating? Let us now turn to classifying the system in this way. The goal will be to augment the existence and stability plots in Fig. 5.5 with lines dividing accelerating sections from bouncing ones e.t.c.

First recall the form of H_J along X_0:

$$H_J(N) = \frac{H_0}{B_0} e^{-(\frac{\lambda}{2}+\beta)\sqrt{6}x_0 N} \left[1 + \sqrt{6}x_0 \left(\beta - \frac{\lambda}{2}g(N)\right)\right] \tag{5.64}$$

where schematically for $g(N)$:

$$g(N) = \begin{cases} \simeq 1 & \text{if } N \ll N_* \\ \simeq 0 & \text{if } N \gg N_*. \end{cases} \tag{5.65}$$

Note that the exponential factor is always non-zero and positive, as is H_0/B_0. The interesting behaviour is contained in the Hubble 'discriminator':

$$\Delta := \left[1 + \sqrt{6}x_0 \left(\beta - \frac{\lambda}{2}g(N)\right)\right]. \tag{5.66}$$

Recalling that $g(N)$ steps down from 1 to 0 in some thin window of N (figure 5.4) we can outline only three possible scenarios:
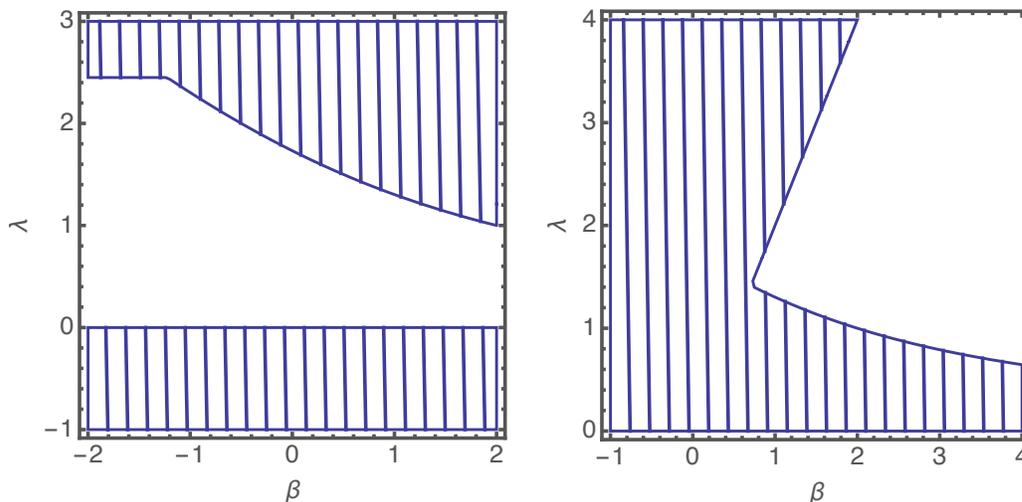
Figure 5.5: For parameter values (β, λ) in the white regions of the above plots, the critical points in table 5.1 of the Wands system both exist and are stable attractors. (Left) critical point 1, (right) critical point 2.

1. $\Delta < 0 \ \forall N$,

2. $\Delta = 0$ for some single value of N (Δ flips sign once),

3. $\Delta > 0 \ \forall N$.

The disformal scale L only appears in $g(N)$ of Eq. (5.66), so does not determine which of the above scenarios will occur; it only affects scenario 2, and, further, only sets the value of N at which Δ flips sign (very close to N_*). The big conclusion here is that only λ and β will decide the fate of the universe along a critical curve: contracting, bouncing or expanding, which we can also infer from the absence of L in the q_J expressions above.

Setting $\Delta = 0$ for $g = 1$ and $g = 0$ separately gives algebraic relations between λ and β that form the boundaries separating scenarios 1 2 and 3 explained above. This is shown in Fig. 5.6. The take home message of Fig. 5.6 is that along critical curve 1 (corresponding to critical point 1 in table 5.1 and the left panel of Fig. 5.6) the universe can be contracting, bouncing or expanding, depending on the choices of λ and β; L does not affect this fate. We cannot even guarantee N_J is complete along this critical curve! (See Fig.
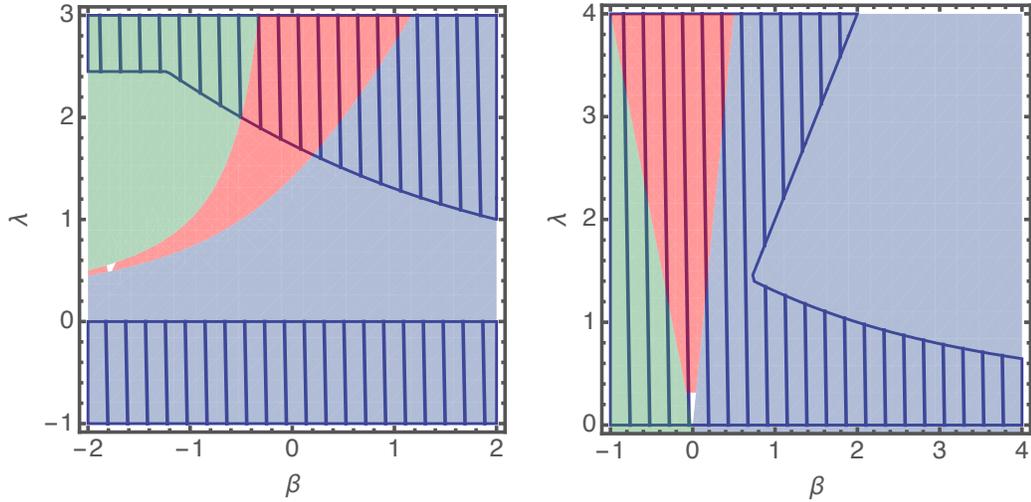
Figure 5.6: Regions without vertical hatching: the critical points in table 5.1 both exist and are stable for these values of (β, λ): (left) critical point 1; (right) critical point 2. Coloured regions correspond to: (green) H_J always negative; (red) $H_J = 0$ for a single value of N; (blue) H_J always positive, along each critical curve.

5.7) On the other hand, for the second critical curve, the universe is only ever expanding; the red and green regions are out of the allowed parameter space region. N_J is always complete along critical curve 2, which is thus expanding in the Jordan frame!

5.3.5 Summary & Conclusions

The aim of this section was primarily to show how the notion of a critical curve can be used to connect Einstein frame critical points in the literature to observables in the Jordan frame. To do this I first demonstrated that any Jordan frame quantity can be expressed as a function of as few as three dynamical Jordan frame variables, but no fewer. Given a minimal Jordan frame dynamical system found in this way, we have shown that, due to the increase in dimension, critical points in the original Einstein frame two dimensional phase plane must be connected to either points or curves, depending on the value of x_0 at each point. Due to the constancy of variables at each Einstein
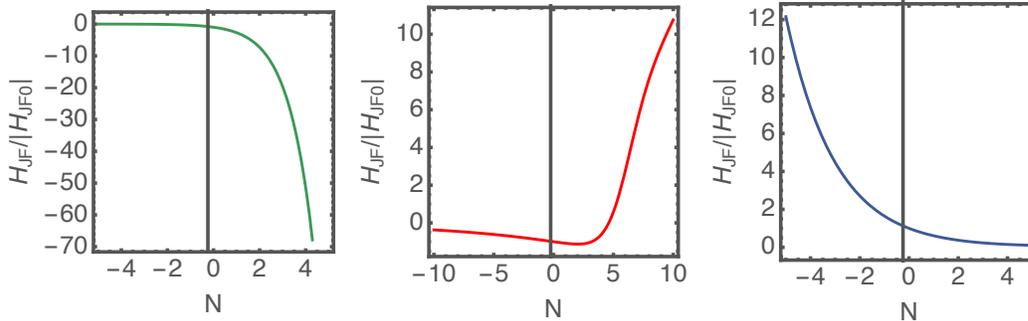
Figure 5.7: Jordan frame Hubble parameter normalised to 1 at $N = 0$ for model parameters (β, λ) in: (left) the green region of Fig. 5.6, left panel; (middle) the red region of Fig. 5.6, left panel; and (right) the blue region of Fig. 5.6, left panel. Vertical grey lines: the value of N_* for each model.

frame critical point, cosmological quantities like H_J were easily computed along each of these curves, and we saw that the evolution of H_J along the curves is not always desirable; bouncing and contracting can sometimes occur.

More importantly, this section has shown critical curves to offer a route towards understanding the Jordan frame dynamics of a given theory, without having to refer to the horrible mess that is the full set of Jordan frame equations. These curves can tell us much more than they have done here, such as which regions of the (β, λ) space produce accelerating attractors, even Minkowski points or matter saddle points, in the Jordan frame.

This may all seem a bit—to be generous—obscure, or (more likely) useless. What, indeed, was the point? The issue as we saw it was: locating critical points and determining their stability for the Einstein frame of these scalar tensor theories has already been done, yet, that information is of practically no use unless it can be connected to observables, and observables, in this theory, are invariably Jordan frame quantities. Unless we can build a bridge between that published work and observables, it is of little use to us here. The intention was that critical curves can form this bridge.

I note here that Einstein frame quantities are not always unobservable. In the conformal limit, as we have discussed, dimensionless ratios are always frame invariant and so, for example, the CMB anisotropy power spectrum as discussed in chapter 4 is conformal transformation invariant [185]. However,

no such proof yet exists for disformals, and we have already seen disformal transformations defy conformal expectations, so to be safe here we will assume Jordan frame quantities only are observable.

5.4 A nasty surprise

In this section we aim to understand in more detail the behaviour of any system simulated by the Jordan frame Euler Lagrange equations described in Sec. 5.1, in the asymptotic limit of large N. In other words, we are trying to find out what a given universe in our theory will look like at some very late time, regardless of the initial conditions. For the purpose of this section, so as not to cloud the central message, I now neglect the conformal factor. Recall that the Einstein frame phase space equations for our model with $B = 1$ are:

$$x' = 3\left(-\frac{\lambda\Omega}{\sqrt{6}} + \frac{\lambda}{\sqrt{6}} + x^3 - \frac{\lambda x^2}{\sqrt{6}} + \frac{x\Omega}{2} - x\right) \tag{5.67a}$$

$$\Omega' = 6x^2\Omega + 3\Omega^2 - 3\Omega \tag{5.67b}$$

subject to the constraints:

$$x^2 + \Omega \leq 1, \quad \Omega \geq 0, \tag{5.67c}$$

where primes denote time derivatives with respect to $N :=$ln$a$, and we have defined compact phase space variables:

$$x := \frac{\psi}{\sqrt{6}H_E}, \quad \Omega := \frac{\kappa\rho_E}{3H_E^2} \tag{5.68}$$

as well as choosing an exponential potential for the scalar field:

$$V(\phi) = m^2 e^{-\lambda\phi}. \tag{5.69}$$

As I have mentioned above, the full critical point analysis for (5.67) can be found in [183]. In the search for viable cosmology solutions, we will only be interested in three:

(I). Matter dominated saddle point, defined by $x = 0$, $\Omega = 1$. Exists $\forall\lambda$.

(II). Dark energy dominated late time attracting point, defined by $x = \frac{\lambda}{\sqrt{6}}$, $\Omega = 0$. Only an attractor for $\lambda^2 < 3$.

(III). Attracting stable spiral, given by $x = \frac{\sqrt{\frac{3}{2}}}{\lambda}$, $\Omega = 1 - \frac{3}{\lambda^2}$. Exists for $\lambda^2 > 3$.

Together, (II) and (III) cover any eventuality of a test system; the asymptotic future of the universe will be described by one or the other, depending on the potential slope, for any value of $\lambda \in \mathbb{R}$.

At critical point (I), $x = 0$ and hence $A = 1$: disformal effects vanish. Here, then, our system is the same in each frame and so will be matter dominated in the Jordan frame too—$a_J \propto t_J^{2/3}$. In our search for solutions to the above equations that will yield viable cosmologies, we will release our system here, and watch it fall toward either of the attractors (II) or (III), depending on the potential slope (see figure 5.8). Once we understand the system's asymptotic behaviour, the nature of the system in the Jordan frame at (II) or (III), viable cosmological solutions will be those that interpolate between point (I) and either (II) or (III) (see Fig. 5.8). I devote what follows to an exploration of the two attracting critical points.

As we have seen in the previous section, each point in fact corresponds to a critical curve for which all cosmological quantities can be found in analytic closed form. We stress that, though the Einstein frame phase portraits in figure 5.8 appear to give us insight into the physical behaviour of the system, they are misleading; we will see for example that what appears to be a matter dominated epoch in the Einstein frame is actually a phantom type universe with total equation of state $\omega_T = -3$. We now use what we have learned from the last sections, the notion of critical curves will be put to work to look for viable cosmologies in our theory.

Recall in the last section we derived the analytic expression for the Jordan frame Hubble parameter H_J along some critical curve, Eq. (5.64). Let us now consider a more intuitive quantity: the total equation of state. We begin by defining the total energy density in the Jordan frame as:

$$3H_J^2 = \kappa \rho_T \tag{5.70}$$

where, for a full expression, $\kappa \rho_T$ is obtained from (5.21d). Then, we can define some ω_T such that:

$$\dot{\rho}_T = -3H_J(1 + \omega_T)\rho_T, \tag{5.71}$$

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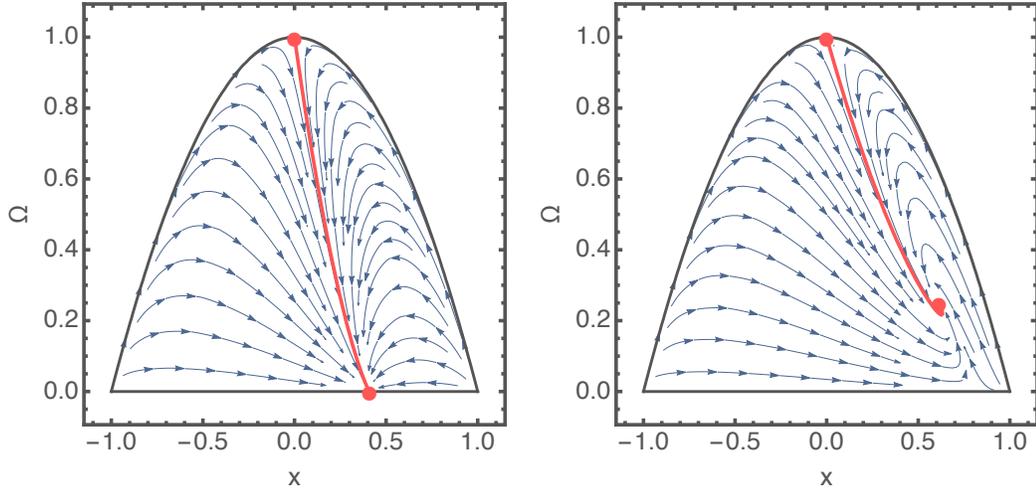
Figure 5.8: Phase portraits for the theory described by equations (5.67). In both panels, the system begins at the matter dominated saddle critical point (I) and then: (left) the system falls toward point (II) with $\lambda = 1$; (right) the system ends at point (III) with $\lambda = 2$. A numerical solution (red curve) shows the path the system takes in each case. The grey closed curve borders the basin of attraction. Points in the plane outside this boundary do not satisfy the Friedmann constraint so are unphysical. For all models considered in this section, $B = 1$.

which gives:

$$\omega_T = -\frac{2\dot{H}_J}{3H_J^2} - 1. \tag{5.72}$$

This equation of state parameter ω_T tells us clearly the details of the Jordan frame expansion. If $\omega_T = 0$ for example, we can infer that $a_J \propto t^{2/3}$, and the scale factor evolution mimics a matter domination epoch. Recall also that $g(N)$ is similar to a step function, that is near 1 before the step, and near 0 after. During the step transition the dynamics will be too complicated to work with analytically, so we will consider instead the two constant regimes before and after this step.

As we increase the disformal length, L, we can push this step in $g(N)$ forward, into the far future, or if we decrease L, backward, into the far past.

In each asymptotic regime on the L scale, $g(N)$ is approximately constant, either 0 or 1, and so ω_T too will be approximately constant, which we see from the exact form of H_J above, combined with: $\dot{N} = H(N)$, for H the Einstein frame Hubble parameter, which has the same exponential dependence on N as H_J.

Then, as we vary L from one limit to the other, from small to large values, the universe at some fixed value of N smoothly transitions from one phase ($g(N) = 1$) to the other ($g(N) = 0$). This is depicted in figure 5.9, where we have defined redshift in the usual way (which is complete for the system parameters we have chosen):

$$1 + z = \frac{a_{J0}}{a_J}. \tag{5.73}$$

If $N >> N_*$ where we recall N_* depends logarithmically on L, the system in the Jordan frame reverts to uncoupled quintessence (Einstein and Jordan frames coincide), whereas if $N << N_*$, things are much more interesting. As ω_T is very close to constant here, we can plot this constant value of ω_T for various values of λ to see how it depends on the potential slope. This is shown in figure 5.10.

Some comments are in order. Both the blue and red curves of figure 5.10 pass through the point $(0, -1)$: the theory is capable of reproducing a universe with a cosmological constant in the Jordan frame on both sides of the step in $g(N)$, in fact $\forall N$. The reason is obvious. If $\lambda = 0$ the field is not driven (see equation (5.21c)) and so $\psi = 0$, hence $A = 1$ and the system mimics a universe with a cosmological constant. This is in stark contrast to a dilatonic coupling of the scalar field, where the coupling function drives the field regardless of the potential slope. In a particularly bland sense, we have thus found cosmological solutions within our theory that fit cosmological background data: if a system with $\lambda = 0$ begins at point (I) and falls toward (II) (clearly (III) does not exist for $\lambda = 0$), then its history will be identical to that of ΛCDM for any value of the disformal length. This, we can see, remains approximately true for small perturbations in λ around 0.

The surprising feature of figure 5.10 is not the physically viable part. It is to the left and right of the $\lambda = 0$ axis. On the graph, as we smoothly travel away from this axis to larger values of $|\lambda|$, ω_T first blows up positive, resurfaces from below, and then stabilises at $\omega_T = -3$. The asymptote lines mark the theory's transition from point (II) as the global attractor
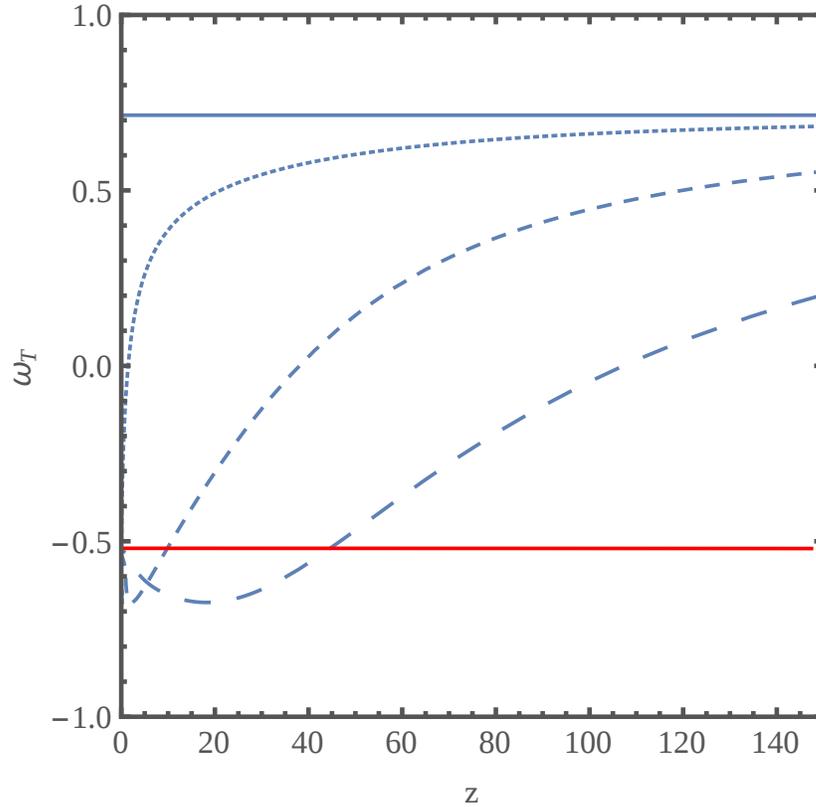
Figure 5.9: Evolution of the total equation of state, ω_T with redshift z for a system placed at critical point (II) (see text) with $\lambda = 1.2$ fixed for all curves, and varying values of L: top curve (solid blue) $L = 10^3/H_{E0}$; middle curves (respectively small, medium, large dashing) $L = 0.5/H_{E0}$, $0.2/H_{E0}$, $0.1/H_{E0}$; bottom curve (solid red) $L = 10^{-3}/H_{E0}$.

to (III), but it is that value $\omega_T = -3$, the nasty surprise, that deserves special mention. In [90] a phantom instability was uncovered in the Jordan frame of a disformal theory, again with $\omega_T = -3$, yet the theories are very different; there is no corresponding Einstein frame metric singularity that our system in this work approaches, as the frame map here was constructed to be a guaranteed isomorphism. In fact, the phantom phase here is not even an instability, but a transient semi-stable state that collapses back into uncoupled quintessence as N grows larger than N_*. As to *why* the value -3
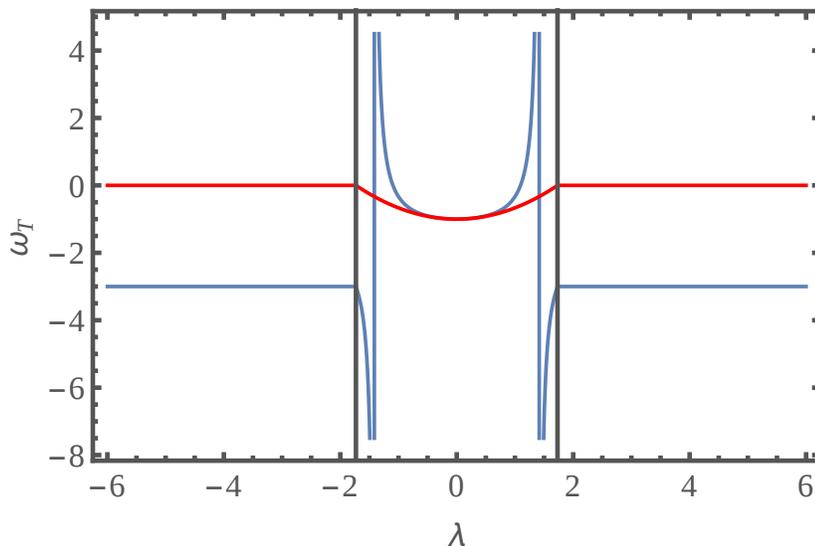
Figure 5.10: Blue curve: $\omega_T(N = 0)$ for $L = 10^3/H_{E0}$ as a function of potential slope λ along the system late time attractor ((II) or (III)). Red curve: $\omega_T(N = 0)$, for $L = 10^{-3}/H_{E0}$ as a function of λ (again at either (II) or (III)). For $\lambda^2 < 3$ the late time attractor scaling solution is point (II), and for $\lambda^2 > 3$ it is point (III). The grey vertical lines outline these three regions.

reappears in this different context—with a now healthy Einstein frame—we offer no substantial explanation. It will certainly be an issue for future work and further scrutiny.

The aim of this work was to map out the global structure of our chosen theory's dynamics in the Jordan frame, and we have succeeded, but only in part. The critical curves thread the phase space, and cover the eventuality of a system placed initially anywhere in this space, but we could only find analytic solutions along these curves in approximate regimes either side of a step. This was enough, though, to show that there exist no stable solutions where the expansion of the simulated universe is driven for suitably long periods of time, with ω_T close to -1, to mimic dark energy observation today, unless λ is very near 0, in which case the disformal part adds nothing and may as well not be included. This model as it stands thus offers no useful new insight into the cosmological constant problem, and is as good as a cosmological constant in presenting a solution to it. It is far more complex

than that with a cosmological constant and so, with an appeal to Occam's razor or, for the technical minded, the application of some Bayesian test, can sadly be ruled out.

Conclusions 6

This thesis describes how I have explored various aspects of disformal couplings in the context of cosmology. In particular, we have seen how they can be used to vary the constants of nature in ways that simple conformal couplings can not, and that they affect photons and massless particles where conformal ones do not. In other contexts we have seen how a disformal coupling in a dynamical system can act as a friction that suppresses unwanted distortions in cosmological observables due to gravity modifications, and we have witnessed an especially healthy transformation be especially useless when applied to the problem of the cosmological constant. In this final chapter I will be brief; the main results are properly summarised first, and an attempt to draw large-scale conclusions from their aggregation comes second. Disformal theories I have mentioned form a wide umbrella, and I will try not to overgeneralise, or speak outside the scope these results define. Finally, I will speculate on the future of disformals and, again, I will try not to paint with too broad a brush.

The main results as I see them, by chapter, are the following.

3. Couplings to radiation. The first section of this chapter has first and foremost shown that, though we may consider two frame isomorphic actions equivalent, they are only physically equivalent if consistent definitions for well known quantities are consistently applied. Satisfying the requirement of physical invariance under some isomorphism f means we must give up some freedom in our definitions. In particular, if standard definitions of two frame related stress tensors of some fixed species in a flat universe are made, then the equation of state, w, is no longer frame invariant, and the choice in one frame determines completely the choice in the other. Similarly, in the

microscopic context, if the standard definitions for each stress tensor T, $\tilde{T}$ are made, then the Boltzmann equation derived from the former is forced and not free. It takes an unusual form that must be derived for consistency across frames and with thermodynamics.

We also in this section saw that, in the generalised framework of multi-coupled scalar tensor theories, the above results have directly measurable consequences. A modified equation of state of the cosmic microwave background alters the cosmic evolution of its temperature, and this in turn must be a small modification for the theory to satisfy experimental bounds. A modified Boltzmann equation for these cosmic microwaves induces, among other things, effective μ-distortions in their spectra, which again must be exceedingly small for the model to remain viable. A region of our disformal-parameter space was held out of bounds to us by pathologies produced on the failure of the frame transformation to be well defined at a certain point on the manifold.

In the second section of this chapter, a disformal transformation's tendency to distort light cone shapes between source and target metrics was, in the context of a specific multi-coupled scalar tensor model, shown to open vacuum Cherenkov interaction channels between charged particles and photons. Particle collider experiments boast some of the most precise measurement possible in physics, and non detection of such an interaction at the LEP collider was used to place tough bounds on the model considered.

Another channel, vacuum bremsstrahlung, we found had been opened by the chosen disformal interactions, and an appeal to place constraints was made to cosmic rays in this case, with a significantly diminished return. Though direct measurement of cosmic ray particles lacks much of the precision of their ground based collider induced counterparts, the energies of cosmic rays can reach many orders of magnitude more than LEP's records. However this proved inconsequential, as vacuum bremsstrahlung induced by a dark energy field, whose variation time scale is on the order of the Hubble scale, was so negligible as to be unmeasurable in any conceivable future experiment.

4. Couplings to dark matter. For the work described in this chapter the disformal transformation was used as a basis to study interactions between the dark sector elements. The full perturbed universe dynamics were explored, and key cosmological observables such as the modified cosmic microwave background anisotropies and large scale structure power spectrum were computed for various disformal parameter values. The prominent re-

sult was the damping nature of disformal couplings in the dynamical system, and the interplay between conformal and disformal factors was found to be an empirically viable one: experimental tension placed on large conformal factors in cosmology was alleviated by the presence of the disformal term. As many brane world scenarios come with both coupling types together, this may lead to the emancipation of many brane world scenarios from once quite stringent bounds. It is interesting that disformal terms as geometry corrections in compactifications of brane worlds were often assumed negligible and so ignored, when they may in fact turn out to be what save these models from exclusion.

We also found in this chapter that the imprints left by conformal and disformal couplings on these observables mentioned were hard to disentangle. No smoking gun signatures of either were found, and while each's effect on the underlying system dynamics were indeed polar opposite in many cases, their combined effect blurred this distinction in observation, and it seems unlikely that present day measurement will be able to discern the two in the sky. We also uncovered that two distinct growth rates can be defined in our modified gravity theory, one a direct measure of the gravitational field perturbations, Φ, and the other a measure of clumped matter δ_M. While the first is sensitive to cosmic microwave light lensing, the second can be measured by luminous galaxy counts, and the two together provide an observational key to break down these degeneracies in the modified gravity models discussed. Finally, we were able to solve the conservation equation for coupled dark matter in our model, and derive a corresponding phantom condition that indicated when phantom-like dark energy behaviour would or would not be inferred by an observer making certain specified assumptions about the dark sector.

5. Universal couplings & pathology protection. In this last chapter we chose to bar the aforementioned singularities induced by ill-defined disformal transformations by constructing a watertight frame isomorphism. We were led through a thicket of hidden bounces and incomplete time parameters to the conclusion that this transformation offered no insight into the cosmological constant problem and, as salt added to wound, that the total equation of state parameter in the Jordan frame held fixed at $\omega_T = -3$ for a large range of parameter values. Not close enough to -1 to celebrate, this unusual value was made worse by the coincidence that it appears in another disformal theory, quite different in structure to our own, for reasons we could not discern. Though a negative note to end on, this by no means invalidates disformal theories as a whole, and indeed there are many other frame isomorphisms we

could consider, each with a unique cosmological potential just waiting to be explored.

The work above shares disformal transformations as its theme, but it would be a mistake to ask that conclusions drawn be applied to all disformal theories. Nevertheless, at the core of these results, and indeed the property at the heart of disformal transformations themselves, is the disformal map's distortion in light cone shapes between both of the metrics at each space time point, and, when applied to universes, the lack of spatial gradients in any field on the space-time meant this was always manifest as a change in light speed between the two geometries. In chapter 3 we saw this modified the equation of state of a perfect fluid, and even induced vacuum Cherenkov radiation. Had we worked with black holes for example, the light cone distortion would have most likely been in the radial direction. In chapter 1 I introduced scalar tensor theories as abstractions of varying constant theories, and also flagged the difficulty that Ellis and Uzan describe in modifying the speed of light [29]. Here we find that disformal transformations in cosmology transform the space-time constant c, when one works in FL coordinates.

Another issue to resurface repeatedly was the notion of observability. The provisional solution presented in all chapters here has been to connect the frame in which the matter Lagrangian is uncoupled from the scalar to observables. In chapter 3 section 1 the matter was the non-relativistic part of the Lagrangian split, in chapter 4 it was the visible matter frame (as opposed to the dark matter frame) that carried the observable torch, and in chapter 5, were the Jordan frame was reinstated, it was this Jordan frame we sought to connect to observation. This provisional solution was offered due to the lack of a complete general-frame invariant description of physics, and observational physics in particular, available to us at the time of each project. The Einstein frame may have its traps, but with multi coupled scalar tensor theories, every frame is a trap in some way or another.

It seems safe to suggest also that the damping behaviour observed in the dynamical systems involving disformal couplings presented in these chapters is a typical phenomenon. Certainly if the Einstein frame of some scalar tensor theory involves matter on a disformal background then, at the level of the Euler Lagrange equations, terms proportional to a product of second order derivatives of the scalar with some matter energy density ρ will appear in some form and, in the Klein Gordon equation, these add to the inertia of the scalar field. In chapter 4 we saw that at early times, when the dark matter energy density was very large compared to the scalar field potential, the

inertia of the scalar field depended on this energy density and the field was held firmly fixed. In chapter 3 we notice this also, it is the reason the bounds of Fig. 3.2 are double sided: couple the field too weakly and no disformal effects are discernible; couple the field too strongly and the matter has a damping back-reaction on the field which is then unable to move. Of course I reiterate that these conclusions become speculation when applied beyond the work these pages describe, but in this instance I expect a large class of disformal theories to exhibit some form of the disformal damping seen here. We have seen this damping can be quite a boon to some brane worlds—tight parameter constraints can be eased significantly by its presence.

Avenues to continue this work have already been anticipated in the above paragraphs. Perhaps the most significant aspect the disformal literature is lacking is the systematic exploration of frame invariant quantities in disformally related actions. Such as already exists for the conformal case, invariance of scalar tensor physics under general frame transformations, while not contested, still must be worked out. A particularly compelling approach is to formulate actions that are manifestly frame invariant. The feasibility of constructions like, perhaps, a 'frame scalar' that is invariant under general frame transformations would be worthwhile to explore, and many parallels between diffeomorphism and frame symmetry may be drawn. The extended symmetry of scalar tensor theories would require a frame transformation symmetry group, which may be hard to construct, but the boon would be that observables would be much better understood, once frame invariance is built right in to the model construction.

A second avenue would be to explore the disformal damping more systematically, and to investigate how bounds on brane world theories are altered when disformal terms are accounted for. How general is this damping mechanism? When does it work and when does it fail? And what utility does it have outside that of branes? A short term goal to extract from the previous work is to apply Bayesian methods—specifically CLASS's Monte Python—to the models of chapter 4. This is an incomplete project in that we show modified gravity distortions to key cosmological observables are suppressed by the disformal interaction, but to what extent this alleviates tension on the model parameters is not yet known. Exactly how large a conformal coupling strength can we now get away with?

Finally, are there any frame isomorphisms that can induce interesting cosmologies? In chapter 2 we saw that, for a frame transformation to be an

isomorphism we required that:

$$0 < Z(\phi, X) < 2, \tag{6.1}$$

For Z a free function. Certainly there are many functional forms Z could take and satisfy this constraint, and a systematic exploration for those with something to say on the cosmological constant problem would be worthwhile.

Despite the 5 preceding chapters it is hard to speculate here on the future of disformal theories, given their breadth of application, but it is also hard not to notice their growing prevalence in the literature. I expect soon that papers will cease to be published with 'disformal' in their titles, but this will happen as they are assimilated into the cosmologist's standard repertoire and cease to be an advertisement, not because they will fall out of fashion and out of favour. Cosmology is a fashionable field and intensely subject to the boom and bust of trends, but disformal transformations are too malleable to disappear altogether. Whether it is a resurgence of brane worlds or the future observation of a galileon in the sky that keeps future theorists at work, disformal transformations will always have their place, at least until the next paradigm comes along. I cannot guess what cosmology will look like after the problems of dark energy are solved, but if there are scalar fields in this future, then there will likely be disformal transformations too.

Index for terms introduced in the text A

INDEX FOR TERMS INTRODUCED IN THE TEXT

Frame transformations

<div style="text-align: right; font-size: 3em;">B</div>

B.1 Background

I here present the set of induced frame maps for various quantities defined by the Einstein and Jordan frame actions of the generally coupled theory of chapter 4 (see also [105] and for a discussion in considerable detail including perturbations see [181]). This will allow me to solve the background conservation equation for a coupled species in the Einstein frame. I begin with the definitions of the stress tensors, and from there, using the map between them, compute the transformation rules. For this appendix $c = 1$. In the Jordan frame:

$$\tilde{T} := \frac{2}{\sqrt{-|\tilde{g}|}} \frac{\delta \tilde{\mathcal{S}}_M}{\delta \tilde{g}}, \tag{B.1a}$$

for which we can now impose a perfect fluid description, hence defining a Jordan frame energy density, $\tilde{\rho}$, velocity field $\tilde{u}$, and pressure, $\tilde{P}$:

$$\tilde{T} := (\tilde{\rho} + \tilde{P})\tilde{u} \otimes \tilde{u} + \tilde{P}\tilde{g}^{-1}. \tag{B.1b}$$

In the Einstein frame:

$$T := \frac{2}{\sqrt{-|g|}} \frac{\delta \mathcal{S}_M}{\delta g}, \tag{B.2a}$$

where we then define:

$$T = (\rho + P)u \otimes u + Pg^{-1}. \tag{B.2b}$$

A map between the two objects can readily be derived [96]:

$$
\begin{aligned}
T &= \sqrt{\frac{\tilde{g}}{g}\frac{\partial \tilde{g}}{\partial g}}\tilde{T} \\
&= C^3\sqrt{1 - 2X\frac{D}{C}}\tilde{T} \\
&= C^3 Z\tilde{T},
\end{aligned}
\tag{B.3}
$$

where we recognize the disformal scalar Z which parameterises the relative contribution of the disformal factor. Note as $D \to 0$, $Z \to 1$. Now, choosing the Einstein frame line element to be of FL form:

$$
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a^2(\tau)[-d\tau^2 + \delta_{ij}dx^i dx^j],
\tag{B.4}
$$

means that, using Eq. (4.3), we get, in terms of the disformal scalar

$$
d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = Ca^2(\tau)[-Z^2 d\tau^2 + \delta_{ij}dx^i dx^j].
\tag{B.5}
$$

Given the background metric choice, the fluids are homogeneous, hence no forces are exerted between elements of the fluid—each element follows a geodesic dictated by the metric. This means the 4-velocity field can be computed directly from it: $\tilde{u} = {}^d\!/\!d\tilde{\tau}$ and $u = {}^d\!/\!d\tau$. Using this and the map (B.3) we get the full list of variable transformations between the Jordan and Einstein frame background quantities:

$$
\tilde{u} = \frac{1}{C^{1/2}Z}u = \frac{1}{C^{1/2}Za}(1,\ 0,\ 0,\ 0)
\tag{B.6a}
$$

$$
\tilde{\Theta} = \frac{1}{C^{1/2}Z}\left[\Theta + \frac{3C_{,\phi}\,\psi}{2C}\right]
\tag{B.6b}
$$

$$
\tilde{\rho} = \frac{Z}{C^2}\rho
\tag{B.6c}
$$

$$
\tilde{P} = \frac{1}{C^2 Z}P
\tag{B.6d}
$$

$$
\tilde{w} = \frac{1}{Z^2}w
\tag{B.6e}
$$

where:

$$
\Theta := \nabla_\mu u^\mu = 3\frac{\dot{a}}{a^2} = 3H, \quad \text{and} \quad \psi := \sqrt{2X} = \dot{\phi}/a.
\tag{B.7}
$$

We are now in a position to solve the conservation equation for coupled matter in the Einstein frame. The Jordan frame stress tensor is conserved, as the matter it describes is uncoupled in this frame, so we can instantly write down:

$$\tilde{\nabla}\tilde{T} = 0, \tag{B.8}$$

where $\tilde{\nabla}$ is the covariant derivative metric compatible with $\tilde{g}$. Given the transformations (B.6), (B.8) reduces to:

$$\dot{\rho} + a\left[\Theta + \frac{3C_{,\phi}\,\psi}{2C}\right]\left(\rho + \frac{P}{Z^2}\right) = \partial_0\left(\ln\frac{Z}{C^2}\right)\rho, \tag{B.9}$$

which, as long as $\tilde{w}$ is constant, is exactly solvable:

$$\rho \propto \frac{C^2}{Z}(C^{1/2}a)^{-3(1+w/Z^2)}. \tag{B.10}$$

That $\tilde{w}$ be constant is not as restrictive a requirement as it sounds. In fact, as matter in the Jordan frame is uncoupled from the scalar, we expect $\tilde{w}$ to be constant wherever it is in ΛCDM. For example, one can show that for any relativistic species (photons, massless neutrinos. . .), $\tilde{w} = 1/3$.

B.2 Perturbations

Derivation of the transformations between perturbation variables of the two frames proceeds in the exact same way, though this time we will not be able to solve the equations exactly, as it can not be done for the uncoupled case.

As discussed in section 4.2, δ will denote the matter density contrast: $\delta := \frac{\delta\rho}{\rho}$, δP the pressure perturbation, $\delta\phi$ is the perturbation of the scalar field and $\hat{\delta}$ a general perturbation operator. Then, working in the Newtonian gauge to first order:

$$ds^2 = a^2(\tau)\left[-(1+2\Psi)d\tau^2 + (1-2\Phi)\delta_{ij}dx^i dx^j\right] \tag{B.11}$$

which means that:

$$d\tilde{s}^2 = Ca^2(\tau)[\; - \; (1+2A)Z^2 d\tau^2$$
$$+ \; 2(\partial_i B)Z d\tau dx^i$$
$$+ \; (1-2E)\delta_{ij}dx^i dx^j] \tag{B.12}$$

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where we recall the definitions of A, B, and E:

$$A \;=\; \Psi + \frac{\hat{\delta} C}{2C} + \frac{\hat{\delta} Z}{Z} \tag{B.13}$$

$$B \;=\; \left(\frac{1}{Z} - Z\right)\frac{\delta\phi}{\phi'} \tag{B.14}$$

$$E \;=\; \Phi - \frac{\hat{\delta} C}{2C}\;, \tag{B.15}$$

where

$$\hat{\delta} Z \;=\; -\frac{D}{ZC}\left[\left(\frac{D'}{D} - \frac{C'}{C}\right)X\delta\phi + \hat{\delta} X\right] \tag{B.16a}$$

$$\hat{\delta} C \;=\; C'\delta\phi\;. \tag{B.16b}$$

Note that $\Psi - \Phi = A - E - \frac{\hat{\delta} C}{C} - \frac{\hat{\delta} Z}{Z}$. As a consequence, if in the Einstein frame $\Phi - \Psi = 0$, implying that the gravitational slip $\eta = \Phi/\Psi = 1$, the slip in the Jordan frame $\tilde{\eta} = E/A$ will, in general, depend on the coupling.

Perturbations to the tensors given in (B.2b) and (B.1b) respectively gives:

$$T^{\mu\nu} \;=\; (\delta\rho + \delta P)u^\mu u^\nu + 2(\rho + P)\delta u^{(\mu} u^{\nu)}$$
$$+ \delta P g^{\mu\nu} + P\delta g^{\mu\nu} + P\Pi^{\mu\nu} \tag{B.17a}$$

$$\tilde{T}^{\mu\nu} \;=\; (\tilde{\delta\rho} + \delta\tilde{P})\tilde{u}^\mu\tilde{u}^\nu + 2(\tilde{\rho} + \tilde{P})\delta\tilde{u}^{(\mu}\tilde{u}^{\nu)}$$
$$+ \delta\tilde{P}\tilde{g}^{\mu\nu} + \tilde{P}\tilde{\delta g}^{\mu\nu} + \tilde{P}\tilde{\Pi}^{\mu\nu}, \tag{B.17b}$$

where we have denoted the fluid's anisotropic stress $\Pi^{\mu\nu}$, which parameterises higher moments of the fluid decomposition. And, just as before, by perturbing the map (B.3) we can compute the transformations between perturbed matter variables:

$$\tilde{\delta} \;=\; \delta + -\frac{2\hat{\delta} C}{C} + \frac{\hat{\delta} Z}{Z} \tag{B.18}$$

$$(\tilde{\rho} + \tilde{P})\tilde{\theta} \;=\; \frac{1}{C^2}\left[(\rho + P)\theta + \left(\frac{P}{Z^2} - P\right)\theta_\phi\right] \tag{B.19}$$

$$\delta\tilde{P} \;=\; \frac{1}{C^2 Z}\left[\delta P - \left(\frac{2\hat{\delta} C}{C} + \frac{\hat{\delta} Z}{Z}\right)P\right] \tag{B.20}$$

$$\tilde{\Pi}^{\mu\nu} \;=\; \Pi^{\mu\nu}\;. \tag{B.21}$$

These equations agree with [181]. θ is the velocity divergence field, defined as $\theta := \partial_i(a\hat{\delta}u^i)$, and quantifies discrepancies between the fluid's velocity field and the underlying geodesic field. We have also defined for dark energy $\theta_\phi := \frac{k^2\delta\phi}{\phi'}$. Mathematically, every field permits a fluid description under a change of variables, and θ_ϕ is the scalar's velocity divergence. Of course such a description may make no sense physically, if dark energy is not an ensemble of particles with a temperature.

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