On the Upper Semicontinuity of HSL Numbers

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To Phoebe

Abstract

Suppose that B is the quotient of a polinomial ring with coefficients in a field of characteristic p. In the first part of the thesis we suppose that B is Cohen-Macaulay and, with $H_{\mathfrak{p}}$ being the local cohomology module $H_{\mathfrak{p}B_{\mathfrak{p}}}(B_{\mathfrak{p}})$, we study the Frobenius action Θ on $H_{\mathfrak{p}}$. In particular we are interested in computing the smallest integer $e \geq 0$ for which $\Theta^e(\mathrm{Nil}(H_{\mathfrak{p}})) = 0$, where $\mathrm{Nil}(H_{\mathfrak{p}})$ denotes the set of all elements in $H_{\mathfrak{p}}$ killed by a power of Θ. Such a number is called the HSL number of $H_{\mathfrak{p}}$. We prove that, for every e, the set of all prime ideals $\mathfrak{p}$ for which $\mathrm{HSL}(H_{\mathfrak{p}}) < e$ is Zariski open. An application of this result gives a global test exponent for the calculation of the Frobenius closure of parameter ideals in Cohen-Macaulay rings. In the second part of the thesis we drop the assumptions made on B and we let B be *any* quotient of a polynomial ring. Using the notation $H^j_{\mathfrak{p}} := H^j_{\mathfrak{p}B_{\mathfrak{p}}}(B_{\mathfrak{p}})$, we show that every set $V_{j,e} = \left\{\mathfrak{p} \in \mathrm{Spec}(B) \mid \mathrm{HSL}(H^j_{\mathfrak{p}}) < e\right\}$ is Zariski open and so that $\left\{\mathrm{HSL}(H^j_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Spec}(B), j \geq 0\right\}$ is bounded. Both the methods from the first and second part of the thesis are implemented as algorithms in Macaulay2 and are used to give examples.

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Contents

1 Introduction

Local cohomology was introduced by Alexander Grothendieck in the 1960s ([8]) and it has since been a powerful tool to approch many geometric and algebraic problems as it captures several properties of a commutative ring. For example local cohomology modules can be used to measure the depth of a module on an ideal (see Property 2.7.2), and as a way to test the Cohen-Macaulay and Gorenstein properties. In positive characteristic, the Frobenius endomorphism naturally induces Frobenius actions on all the local cohomology modules (see Section 2.6 for the definition). One of the goals of this thesis is to understand when the Frobenius action on local cohomology modules is injective or how far it is from being injective. A way of measuring this is given by the HSL-numbers (see 2.10.4 for the definition).

In the first part of the thesis we consider a Cohen-Macaulay quotient S of a polynomial ring R of positive characteristic which is also a domain. For these rings we prove the following:

Theorem 1.0.1. *(Theorem 3.4.5) For every prime ideal $\mathfrak{p}$ let $S_\mathfrak{p}$ be the localisation of S at $\mathfrak{p}$ and let $H^{\dim S_\mathfrak{p}}_\mathfrak{p}(S_\mathfrak{p})$ denote the $(\dim S_\mathfrak{p})$-th local cohomology module of $S_\mathfrak{p}$ with respect to $\mathfrak{p}$. For every non-negative integer e, the set defined as*

$$\mathcal{B}_e = \left\{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{HSL}\left(H^{\dim S_\mathfrak{p}}_\mathfrak{p}(S_\mathfrak{p})\right) < e\right\}$$

is Zariski open.

Hence HSL is upper semi-continuous. Note that this result generalises the openness of the F-injective locus (see Definition 2.10.6). Moreover, an application of this result gives a global test exponent for the calculation of Frobenius closures of parameter ideals in Cohen-Macaulay rings (see Corollary 3.7.3).

In order to prove Theorem 3.4.5 we show the following.

Theorem 1.0.2. *Let $S = R/I$ be a quotient of a polynomial ring of positive characteristic p. Let $\bar{\omega}$ be an ideal of S which is isomorphic to a canonical module for S. If ω denotes the preimage of $\bar{\omega}$ in R, then the R-module consting of all e^{th}-Frobenius maps acting on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ is of the form*

$$\mathcal{F}^e = \frac{\left(I^{[p^e]} : I\right) \cap \left(\omega^{[p^e]} : \omega\right)}{I^{[p^e]}}.$$

This result, together with the one previously shown by Lyubeznik in [17, Example 3.7] which states that if S is S_2 then $\mathcal{F}^e$ is generated by one element which corresponds to the natural Frobenius map, gives an explicit description of any Frobenius map acting on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$. We will also show with an example that if the ring is not S_2 (Serre's condition, see Definition 2.2.10) then $\mathcal{F}^e$ is not necessarely principal (see Example 3.4.4).

In the second part of the thesis we consider *any* quotient of a polynomial ring and, dropping all the assumptions made previously, we consider the Frobenius action on *all* the local cohomology modules $H^i_{\mathfrak{p}}(S_{\mathfrak{p}})$. Using a different method, we prove that every set $V_{i,e} = \left\{\mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{HSL}(H^i_{\mathfrak{p}}(S_{\mathfrak{p}})) < e\right\}$ is open and as consequences that the injective locus is Zariski open and that the set $\left\{\operatorname{HSL}(H^j_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec}(R), j \geq 0\right\}$ is bounded.

Both the methods have been implemented as algorithms using Macaulay2, [7].

The algorithms are included at the end of the thesis in the appendix and have been used throughout the thesis to compute all the examples. Even though the results obtained in the second part of the thesis are a generalisation of the results obtained in the first one, from a computational point of view the first algorithm is more efficient than the second.

1.1 Outline of Thesis

Chapter 2 consists of preliminary mathematical material which serves the purpose of setting up the vocabulary and the framework for the rest of the thesis. We define regular rings (Section 2.2) and Cohen-Macaulay rings (Section 2.1), give some examples and state some of their main properties. In Section 2.3 we define complex chains and give two useful examples: the Koszul complex and the $\check{C}$ech complex. In Section 2.4 we discuss some basic facts from Category Theory. In Section 2.5 we define what we mean by the injective hull of a module, we introduce the Matlis functor and recall the Matlis Duality Theorem (Theorem 2.5.9). In Section 2.6 we define local cohomology modules and give a few different characterisations for it. Some properties of local cohomology modules will be listed in Section 2.7. In Section 2.8 we present Gorenstein rings and Section 2.9 canonical modules. In the sections 2.10 and 2.11 we introduce some characteristic p tools: the Frobenius endomorphism and the Δ^e- and Ψ^e-functor.

We will start Chapter 3 by defining the operator $I_e(-)$ and showing some of its properties. In particular we will prove that this operator commutes with localisation and completion. We will then consider a quotient of a regular local ring S of dimension d and describe the action of Frobenius on its top local cohomology module $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$. After that we will give an explicit description of the module consisting of all the e-th Frobenius maps acting on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ and compute the HSL-numbers in the local case. In Section 3.4 we will consider a Cohen-Macaulay non-local domain $B = A/J$ and will prove the main result: the sets $\mathcal{B}_e = \left\{\mathfrak{p} \in \operatorname{Spec} B \mid \mathrm{HSL}(H_{\mathfrak{p}}^{\dim S_{\mathfrak{p}}}(S_{\mathfrak{p}})) < e\right\}$ are Zariski open. In Section 3.5 we will describe an algorithm used to compute these sets $\mathcal{B}_e$ and give some examples. The results of Section 3.7 give a global test exponent for the calculation of Frobenius closures of parameter ideals in Cohen-Macaulay rings.

Chapter 4 contains a generalisation of the results presented in Chapter 3 to the non-Cohen-Macaulay case and is translated into a new algorithm in Section 4.5. The purpose of this chapter is to prove that for all $e > 0$, and for all $j \geq 0$ the sets $V_{e,j} = \left\{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathrm{HSL}(H_{\mathfrak{p}}^{j}(R_{\mathfrak{p}})) < e\right\}$ are Zariski open. In Section 4.1 we generalise $I_e(-)$ which we have had previously defined for ideals to submodules of a free module; we prove that this operator commutes with localisation and completion. In Section 4.2 we describe the Frobenius action on the direct sum of α copies of the injective hull E; as every Artinian module M with a Frobenius action can be embedded into E^{α}, it follows that we can give a description of the Frobenius action on M as a restriction of the action on E^{α} to M. In Section 4.3 we will compute the HSL-numbers of a module over a regular local ring with Frobenius action. And in Section 4.4 we will present a new method for the computation of the HSL loci.

The algorithms that have been used throughout the thesis to compute all the examples can be found at the end of the thesis in the appendix.

1.2 Notation

In this section we fix the notation and terminology used throughout the thesis. For further details see [4] and [1].

We denote by $A = \mathbb{K}[x_1, \cdots, x_n]$ the polynomial ring with coefficients in a finite field $\mathbb{K}$ and variables $x_1, \cdots, x_n$. With the notation $\mathbb{K}[[x_1, \cdots, x_n]]$ we will indicate the ring of formal power series. The module of inverse polynomials will be denoted by $\mathbb{K}[x_1^{-1}, \cdots, x_n^{-1}]$; recall that by that we mean the A-module that is the $\mathbb{K}$-vector space with basis $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_i < 0 \ \forall i = 1, \cdots, n\}$ and A-module structure defined as follows; if $x_1^{\beta_1} \cdots x_n^{\beta_n}$ is a monomial then

$$x_1^{\beta_1} \cdots x_n^{\beta_n} \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \begin{cases} 0 & \text{if } \beta_i + \alpha_i \geq 0, \text{ for } i = 1, \cdots, n \\ x_1^{\beta_1+\alpha_1} \cdots x_n^{\beta_n+\alpha_n} & \text{otherwise.} \end{cases}$$

Let R be a commutative ring. We say that an ideal $\mathfrak{p} \subset R$ is a *prime ideal* if whenever $ab \in \mathfrak{p}$ then a or b belongs to $\mathfrak{p}$. The *spectrum* of a ring, denoted $\mathrm{Spec}(R)$, is the set of all prime ideals of R with the *Zariski topology*, which is the topology where the closed sets are

$$V(I) = \{\mathfrak{p} \in R \mid I \subseteq \mathfrak{p}\}, \qquad \text{with } I \subseteq R \text{ ideals.}$$

The *radical* of an ideal I, denoted $\sqrt{I}$, is the set of all $r \in R$ such that $r^n \in I$

for some power n. An ideal I is called *primary* if whenever $xy \in I$ then either $x \in I$ or $y^n \in I$ for some integer n. If $\mathfrak{p} = \sqrt{I}$ then I is called $\mathfrak{p}$-*primary.* A *primary decomposition* of an ideal I is an expression of the form

$$I = \bigcap_i \mathfrak{q}_i$$

where the ideals $\mathfrak{q}_i$ are primary; such a decomposition always exists in Noetherian rings (but not in general). A primary decomposition is *minimal* if $\sqrt{\mathfrak{q}_i}$ are all distinct and if $\bigcap_{i \neq j} \mathfrak{q}_j \nsubseteq \mathfrak{q}_i$.

We write $\mathrm{Ann}_R(M)$ to indicate the annihilator of an R-module M i.e. the kernel of the natural map $R \to \mathrm{End}_R(M)$. If $\mathfrak{p}$ is a prime ideal of R we say that $\mathfrak{p}$ is *associated* to M if it is the annihilator of an element of M. The set of all prime ideals associated to M will be denoted by $\mathrm{Ass}(M)$. The *associated primes of an ideal* I is the set of associated primes of the module R/I. The *minimal associated primes* are the prime ideal in $\mathrm{Ass}(M)$ which are minimal with respect to the inclusion. The primes in $\mathrm{Ass}(M)$ that are not minimal are called *embedded* primes of M.

An element $z \in R$ is called a *zero-divisor* if there exists $r \neq 0$ in R such that $zr = 0$. It is called *non-zero-divisor* otherwise. We say that a ring is a *domain* if it has no zero-divisors. A ring is *local* if it has only one maximal ideal. If R is local with maximal ideal $\mathfrak{m}$ then we call $\mathbb{K} = R/\mathfrak{m}$ the *residue class field* and we often use the notation $(R, \mathfrak{m}, \mathbb{K})$.

A *multiplicatively closed subset* of R is a subset W of R such that $1 \in W$ and such that if $s, t \in W$ then $st \in W$. Define a relation $\sim$ on $R \times W$ as follows: $(a, s) \sim (b, t)$ if and only if $(at - bs)u = 0$ for some $u \in W$. $\sim$ is an equivalence relation and we denote by $W^{-1}R$ the set of equivalence classes and call it the *localisation* of R by W. Denote respectively a/s and b/t the equivalence classes of the elements (a, s) and (b, t) then define the two operations $\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$ and $\frac{a}{s}\frac{b}{t} = \frac{ab}{st}$ that make $W^{-1}R$ into a ring. In particular we write $R_{\mathfrak{p}}$ when $W = R - \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal in R. The localisation of a ring at a prime ideal is a local ring. Analogously, we can define the localisation $W^{-1}M$ of an R-module M; define a relation $\sim$ on $M \times W$ as follows: $(m, s) \sim (n, t)$ if and only if $(mt - ns)u = 0$ for some $u \in W$. We then define $W^{-1}M$ to be the set of equivalent classes and denote m/s the equivalence class of (m, s). Note that $W^{-1}M$ is a $W^{-1}R$-module once we have defined the usual addition and scalar multiplication. We denote $M_{\mathfrak{p}}$ and $M_{\mathfrak{m}}$ the localisation of M at the prime ideal $\mathfrak{p}$ and $\mathfrak{m}$ respectively. For an R-module M it is equivalent that $M = 0$ and that $M_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ (see Proposition 3.8 from [1]). The *support* of M is the set of all prime ideals $\mathfrak{p}$ such that $M_{\mathfrak{p}} \neq 0$ and it is denoted $\mathrm{Supp}(M)$. The support of a Noetherian module is Zariski closed.

A ring R is *Noetherian* if it satisfies the *ascending chain condition on ideals*; this means that given any chain of inclusions of ideals of R

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq \cdots$$

there exists an integer h at which the chain stabilises i.e. $I_h = I_{h+1} = \cdots$. The *polynomial ring* $\mathbb{K}[x_1, \cdots, x_n]$ with coefficients in a field $\mathbb{K}$ and n variables

$x_1, \cdots, x_n$ is an example of a Notherian ring. If W is a multiplicatively closed subset of R and R is Noetherian then $W^{-1}R$ is Noetherian as well. If R is Noetherian then $R[x_1, \cdots, x_n]$ is Noetherian. Every ideal of a Noetherian ring has a primary decomposition. Analogously a module is Noetherian if it satisfies the ascending chain condition on its submodules.

We say that R is *Artinian* if it satisfies the *descending chain condition on ideals* i.e. if $I_1 \supseteq I_2 \supseteq \cdots$ then there exists an integer h at which $I_h = I_{h+1} = \cdots$. In an Artinian ring every prime ideal is maximal and there is only a finite number of maximal ideals. Also every Artinian ring is isomorphic to a finite direct product of Artinian local rings. A module is *Artinian* if it satisfies the descending chain condition on its submodules.

An R-module M is *finitely generated* if there exist $x_1, \cdots, x_n \in M$ such that for every $m \in M$, $m = a_1x_1 + \cdots + a_nx_n$ for some $a_1, \cdots a_n \in R$.

If $I \subset R$ is any ideal then we define the completion of R with respect to I, namely $\widehat{R}$, as

$$\varprojlim(R/I^i) = \left\{ (\cdots, \bar{r}_3, \bar{r}_2, \bar{r}_1) \in \prod_i R/I^i \mid r_{i+1} - r_i \in I^i \right\}.$$

If M is an R-module then we define $\widehat{M} = \widehat{R} \otimes M$. Alternatively consider the natural surjections $\cdots \to M/I^3M \to M/I^2M \to M/IM \to 0$ and define the I-*adic completion* $\widehat{M}$ of M as the *inverse limit*:

$$\varprojlim(M/I^iM) = \left\{ (\cdots, \bar{m}_3, \bar{m}_2, \bar{m}_1) \in \prod_i M/I^iM \mid m_{i+1} - m_i \in I^iM \right\}.$$

A fundamental example is the following.

Example 1.2.1. *The completion of the polynomial ring* $S[x_1, \cdots, x_n]$ *with respect to the ideal* $(x_1, \cdots, x_n)$ *is the ring of* formal power series $S[[x_1, \cdots, x_n]]$.

The completion of a ring (a module) is a ring (a module). The completion of a Noetherian ring is a Noetherian ring. We say that a ring is I-*complete* if $R = \widehat{R}$.

A sequence of R-modules and R-homomorphisms

$$\cdots \to M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \cdots$$

is *exact* if $\mathrm{Ker}(f_{i+1}) = \mathrm{Im}(f_i)$ for all i, where $\mathrm{Ker}(f_{i+1})$ and $\mathrm{Im}(f_i)$ are the kernel of f_{i+1} and the image of f_i, respectively. In particular an exact sequence of the form

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0$$

is called *short exact sequence* and it follows from the definition that f_1 is injective and f_2 surjective.

2 Background

In this chapter we give a brief introduction to the background and required Commutative Algebra tools that we will use throughout the thesis.

2.1 Regular Rings

Definition 2.1.1 (Regular sequences)**.** *Let R be a ring and M an R-module. A* regular sequence *(or M-*sequence*) in an ideal $I \subseteq R$ on M is a sequence of elements $\bar{x} = x_1, \cdots, x_n \in I$ such that $(x_1, \cdots, x_n)M \neq M$ and for every $i = 1, \cdots, n$, x_i is a nonzerodivisor on $M/(x_1, \cdots, x_{i-1})M$.*

Proposition 2.1.2 (Prop. 1.1.6 [2])**.** *Let R be a Noetherian local ring and let M be a finitely generated R-module. If $\bar{x}$ is an M-sequence then every permutation of $\bar{x}$ is an M-sequence.*

In general a permutation of an M-sequence is not an M-sequence.

Assume that R is Noetherian and let M be an R-module.

Definition 2.1.3 (Maximal Regular Sequence)**.** *An M-sequence $\bar{x} = x_1, \cdots, x_n$ is said to be* maximal *if for any x_{n+1} the sequence $x_1, \cdots, x_n, x_{n+1}$ is not an M-sequence.*

Let $\bar{x} = x_1, \cdots, x_n$ be a regular sequence; since R is Noetherian then the ascending chain $(x_1) \subset (x_1, x_2) \subset \cdots$ stabilises and $\bar{x}$ can be extended to a maximal regular sequence.

Proposition 2.1.4 (Rees Theorem; Theorem 1.2.5 [2])**.** *Let R be Noetherian and M finitely generated over R. Let I be an ideal such that $IM \neq M$. Then all maximal M-sequences in I have the same length.*

Definition 2.1.5 (Depth)**.** *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and let* M *be a finitely generated* R*-module. The common length of the maximal* M*-sequences in* $\mathfrak{m}$ *is called the* depth of M*.*

Definition 2.1.6 (Height)**.** *The* height *of a prime ideal* $\mathfrak{p}$*, denoted* $\operatorname{ht}\mathfrak{p}$*, is the supremum of integers* t *such that there exists a chain of prime ideals* $\mathfrak{p} = \mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_t$ *where* $\mathfrak{p}_i \in \operatorname{Spec}(R)$*.*

Definition 2.1.7. *The* Krull dimension *for a ring* R *is*

$$\dim R = \sup\{\operatorname{ht}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

Example 2.1.8. *[4, Corollary 9.1] If* R *is Noetherian then* $\dim R = 0$ *if and only if* R *is Artinian in which case* R *is the direct product of local Artinian rings.*

Example 2.1.9 (Polynomial ring)**.** *[4, Chapter 8] The polynomial ring and the ring of formal power series in* n *variables have dimension* n*.*

Definition 2.1.10. *For an* R*-module* M *the Krull dimension is given by the formula*

$$\dim_R M = \dim\left(\frac{R}{\operatorname{Ann}_R(M)}\right).$$

Definition 2.1.11. *(Regular ring) A* regular local ring *is a Noetherian local ring with the property that the minimal number of generators of its maximal ideal is equal to its Krull dimension. A ring is* regular *if every localisation at every prime ideal is a regular local ring.*

Example 2.1.12. *If $\mathbb{K}$ is a field then $\mathbb{K}[x_1, \cdots, x_n]_{(x_1,\cdots,x_n)}$ and $\mathbb{K}[[x_1, \cdots, x_n]]$ are regular local rings.*

Note the following;

Proposition 2.1.13 (see Proposition 10.16 [4])**.** *If R is a complete regular local ring with residue class field $\mathbb{K}$ and R contains a field then R is isomorphic to $\mathbb{K}[[x_1, \cdots, x_n]]$.*

2.2 Cohen-Macaulay Rings

Definition 2.2.1 (Cohen-Macaulay Rings)**.** *A local* Cohen-Macaulay ring *is defined as a commutative, Noetherian and local ring with Krull dimension equal to its depth. A non-local ring is Cohen-Macaulay if its localisations at prime ideals are Cohen-Macaulay. Similarly, a finitely generated module $M \neq 0$ over a Noetherian local is Cohen-Macaulay if* $\operatorname{depth} M = \dim M$*. If R is non-local then M is Cohen-Macaulay if $M_{\mathfrak{p}}$ is Cohen-Macaulay for every $\mathfrak{p} \in \operatorname{Supp} M$.*

Example 2.2.2. *Artinian rings are Cohen-Macaulay.*

Example 2.2.3. *Regular local rings are Cohen-Macaulay.*

Definition 2.2.4 (Complete Intersection)**.** *A ring R is a* complete intersection *if there is a regular ring S and a regular sequence $x_1, \cdots, x_n \in S$ such that $R \cong S/(x_1, \cdots, x_n)$. R is* locally a complete intersection *if this is true for $R_{\mathfrak{p}}$ for every maximal ideal $\mathfrak{p} \in R$.*

Example 2.2.5 (Prop 18.8 and Prop 18.13 in [4])**.** *Any ring that is locally a complete intersection is Cohen-Macaulay.*

Example 2.2.6 (Theorem 18.18 [4]). *A ring R is determinantal if it possible to write $R = A/I$ where A is a Cohen-Macaulay ring and I is the ideal generated by $n \times n$ minors of an $r \times s$ matrix of indeterminates for some integers n, r, s such that the codimension of I in A is $(r-n+1)(s-n+1)$. Determinantal rings are Cohen-Macaulay.*

We recall that a *system of parameters* for a local ring R with maximal ideal $\mathfrak{m}$ and of Krull dimension n is a set of elements $\underline{x} = x_1, \cdots, x_n$ such that the ideal $(x_1, \cdots, x_n)$ is $\mathfrak{m}$-primary.

Proposition 2.2.7 (Theorem 2.1.2 [2]). *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $M \neq 0$ be a Cohen-Macaulay module. Then*

1. $\dim R/\mathfrak{p} = \operatorname{depth} M$ *for all* $\mathfrak{p} \in \operatorname{Ass} M$.
2. $\bar{x} = x_1, \cdots, x_n$ *is an M-sequence if and only if* $\dim M/\bar{x}M = \dim M - n$.
3. $\bar{x}$ *is an M-sequence if and only if it is a part of a system of parameters of M.*

Also, some properties hold even when R is not local. We recall first the following definition.

Definition 2.2.8 (Unmixed Ideal). *An ideal $I \subseteq R$ is* unmixed *if it has no embedded primes.*

Proposition 2.2.9 (Theorem 2.1.3 [2], Theorem 2.1.6 [2]). *Let R be a Noetherian ring and let M be a finitely generated R-module. Then:*

1. *If M is Cohen-Macaulay then $W^{-1}M$ is Cohen-Macaulay for every multiplicatively closed subset $W \subseteq R$.*

2. *R is Cohen-Macaulay if and only if every ideal I generated by* $\operatorname{ht} I$ *elements is unmixed.*

We now define the Serre's condition for Noetherian rings as follows.

Definition 2.2.10. *A Noetherian ring R has property S_k if* $\operatorname{depth} R_{\mathfrak{p}} \geq \inf\{k, \operatorname{ht}(\mathfrak{p})\}$ *for all primes $\mathfrak{p}$.*

Example 2.2.11. *Any Noetherian ring of dimension 2 which is not Cohen-Macaulay is not S_2.*

Example 2.2.12. *Any Noetherian ring is Cohen-Macaulay if and only if it is S_k for every k.*

2.3 Complexes

In this section we define chain complexes and give two important examples: the Koszul complex and the Čeck complex.

Definition 2.3.1 (Homological complex)**.** *A* homological complex *is a sequence of R-module homomorphisms*

$$M_\bullet = \cdots \xrightarrow{\delta_{i+1}} M_i \xrightarrow{\delta_i} M_{i-1} \xrightarrow{\delta_{i-1}} \cdots$$

such that the composition of each consecutive arrows $\delta_i \delta_{i+1} = 0$.

Because $\operatorname{Im}(\delta_{i+1}) \subseteq \operatorname{Ker}(\delta_i)$, we can give the following definition;

Definition 2.3.2 (Homology modules)**.** *The i^{th}* homology module *of a complex $M_\bullet$ is defined as $H_i(M_\bullet) = \frac{\operatorname{Ker}(\delta_i)}{\operatorname{Im}(\delta_{i+1})}$.*

Roughly speaking the i^{th} homology module of $M_\bullet$ measures how close $M_\bullet$ is to being exact at the i^{th} position.

Definition 2.3.3 (Cohomological complex)**.** *A* cohomological complex *is a sequence of R-module homomorphisms*

$$N^\bullet = \cdots \xrightarrow{\delta^{i-1}} N^i \xrightarrow{\delta^i} N^{i+1} \xrightarrow{\delta^{i+1}} \cdots$$

such that the composition of each consecutive arrows $\delta^{i+1}\delta^i = 0$.

Definition 2.3.4 (Cohomology modules)**.** *The i^{th}* cohomology module *of $N^\bullet$ is defined as $H^i(N^\bullet) = \frac{\mathrm{Ker}(\delta^i)}{\mathrm{Im}(\delta^{i-1})}$.*

An example of complex is the Koszul complex. Let us describe such a complex in a simple situation. Let R be a ring, M an R-module and $r \in R$ an element of R. If $M \xrightarrow{r} M$ is the multiplication by r then its kernel is

$$\mathrm{Ker}(r) = \{m \in M \mid rm = 0\} = \mathrm{Ann}_M(r)$$

and the map is injective if and only if r is a non-zero-divisor on M. Since the image of the map is $\mathrm{Im}(r) = rM$ then $r\colon M \to M$ is surjective if and only if

$$\mathrm{Coker}(r) = \frac{M}{rM} = 0.$$

Consequently the following sequence

$$0 \to M \xrightarrow{r} M \to 0$$

is exact, i.e. the map is injective, if and only if r is an M-sequence (see definition 2.1.1).

Definition 2.3.5 (Koszul complex)**.** *The sequence* $K_\bullet(r;M)\colon = 0 \to M \xrightarrow{r} M \to 0$ *is called the* Koszul complex on the element r*.*

Suppose now to have an M-sequence of elements $\bar{r} = r_1, \cdots, r_n$ of R. We want to generalise the construction above and define the Koszul complex

$$K_\bullet(r_1, \cdots, r_n; M)\colon 0 \to K_n \to \cdots \to K_0 \to 0.$$

Set $K_0 = R$, $K_1 = R^n$ and choose the standard basis $e_1, \cdots, e_n$ for K_1. Then for every other index $i \geq 2$ set $K_i = \wedge^i K_1 = \wedge^i R^n$ which is the free R-module of rank $\binom{n}{i}$ and basis $\{e_{j_1} \wedge \cdots \wedge e_{j_i} \mid 1 \leq j_1 < \cdots < j_i \leq n\}$. Note that $K_n = R$ and that $K_i = 0$ for $i > n$. We define the maps $\delta_i\colon K_i \to K_{i-1}$ as

$$\delta_i(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{h=1}^{i} (-1)^{h+1} r_{j_h} e_{j_1} \wedge \cdots \wedge \hat{e}_{j_h} \wedge \cdots \wedge e_{j_i}$$

where the symbol $\hat{e}_{j_h}$ indicates that the term e_{j_h} is missing.
In particular $\delta_i = 0$ when $i < 1$ and $i > n$; note also that, because the K_i are free modules, the maps δ_i can be represented as matrices.

Definition 2.3.6 (Homological Koszul complex)**.** *The homological complex* $K_\bullet(\bar{r};R)$ *is called the* Koszul complex *on* $\bar{r}$*. If* M *is an* R*-module and* $\bar{r}$ *an* M*-sequence we can define* $K_\bullet(\bar{r};M) = K_\bullet(\bar{r};R) \otimes M$*.*

The cohomological Koszul complex is the dual of the homological Koszul complex i.e. $K^\bullet(r_1, \cdots, r_n; M) \cong \mathrm{Hom}(K_\bullet(r_1, \cdots, r_n; R), M)$.

Another interesting example of complex is the Čech complex. Let x be an element of R and let R_x be the localisation of R at x. Recall that R_x is obtained by inverting the multiplicatively closed set $\{1, x, x^2, \cdots\}$.

Definition 2.3.7. *The Čech complex on x is the complex*

$$\check{C}^\bullet(x; R)\colon 0 \to R \xrightarrow{\eta} R_x \to 0$$

where η is the canonical map sending $r \mapsto [r/1]$.
If $\underline{x} = x_1, \cdots, x_d$ is a sequence of elements in R then the Čech complex on $\underline{x}$ is $\check{C}^\bullet(\underline{x}; R) = \check{C}^\bullet(x_1; R) \otimes \cdots \otimes \check{C}^\bullet(x_n; R)$.

In general we define $\check{C}^k(\underline{\mathrm{x}}; R) = \bigoplus_{1 \leq i_1 < \cdots < i_k \leq d} R_{x_{i_1} \cdots x_{i_k}}$.

2.4 Some Basics in Category Theory

Definition 2.4.1 (Category)**.** *A* category $\mathcal{C}$ *is an algebraic structure consisting of a class $\mathcal{Obj}(\mathcal{C})$ of objects, a class $\mathcal{Hom}(\mathcal{C})$ of morphisms between any two objects and a binary operation* $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ *called* composition *which sends $(f, g) \mapsto g \circ f$ for any three objects A, B and C in $\mathcal{Obj}(\mathcal{C})$. In a category there exists an identity morphism $\mathbb{1}_A \in \mathrm{Hom}(A, A)$ with the property that $f\mathbb{1}_A = \mathbb{1}_B f = f$ for all $f\colon A \to B$ and given three morphisms f, g and h then $f \circ (g \circ h) = (f \circ g) \circ h$.*

Example 2.4.2. *If R is a ring, we define $\mathfrak{Mod}(R)$ to be the category whose objects are R-modules and morphisms are R-homomorphisms and its composition is the usual composition.*

Example 2.4.3. *We define* $\mathfrak{Ab}$ *to be the category whose objects are abelian groups, morphisms are group homomorphisms, and composition is the usual composition.*

Example 2.4.4. *We define* $\mathfrak{Sets}$ *to be the category in which the objects are sets, morphisms are functions, and compositions are the usual compositions.*

Definition 2.4.5 (Monomorphism)**.** *A morphism* $u\colon B \to C$ *in a category* $\mathcal{C}$ *is a* monomorphism *if for all* $A \in \mathcal{O}bj(\mathcal{C})$ *and all morphisms* $f, g\colon A \to B$,

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "u"] & C \end{tikzcd}$$

we have that $uf = ug$ *implies* $f = g$.

Example 2.4.6 (Chaper 5 [24])**.** *Monomorphisms and injections coincide in* $\mathfrak{Mod}(R)$ *and* $\mathfrak{Sets}$.

Definition 2.4.7 (Zero Object)**.** *Let* $\mathcal{C}$ *be a category. An object* $C \in \mathcal{O}bj(\mathcal{C})$ *is said to be* initial *(resp.* final*) if for every object* $X \in \mathcal{C}$ *there exists a unique morphism* $C \to X$ *(resp.* $X \to C$*). An object which is both initial and final is called* zero object.

Definition 2.4.8 (Coproduct)**.** *If* A *and* B *are objects in a category* $\mathcal{C}$ *then their* coproduct *is a triple* $(A \sqcup B, \alpha, \beta)$*, where* $A \sqcup B$ *is an object in* $\mathcal{C}$ *and* $\alpha\colon A \to A \sqcup B$ *and* $\beta\colon B \to A \sqcup B$ *are two morphisms such that for every object* $X \in \mathcal{O}bj(\mathcal{C})$ *and every pair of morphisms* $f\colon A \to X$ *and* $g\colon B \to X$, *there exists a unique morphism* $\theta\colon A \sqcup B \to X$ *making the following diagram* *commute:*

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "f"] & \\ A \sqcup B \arrow[rr, dotted, "\theta"] & & X \\ & B \arrow[ul, "\beta"] \arrow[ur, "g"'] & \end{tikzcd}$$

Example 2.4.9 (Proposition 5.1 [24]). *If R is a ring and A and B are two objects in the category $\mathfrak{Mod}(R)$, then their coproduct exists and is the direct sum $A \oplus B$.*

Example 2.4.10. *The coproduct of two sets in $\mathfrak{Sets}$ is their disjoint union.*

Analogously, we define the product of two objects in a category as follows;

Definition 2.4.11 (Product). *If A and B are objects in $\mathcal{C}$, their* product *is a triple $(A \sqcap B, p, q)$, where $A \sqcap B$ is an object in $\mathcal{C}$ and $p\colon A \sqcap B \to A$ and $q\colon A \sqcap B \to B$ are two morphisms such that, for every object X in $\mathcal{Obj}(\mathcal{C})$ and every pair of morphisms $f\colon X \to A$ and $g\colon X \to B$, there exists a unique morphism $\theta\colon X \to A \sqcap B$ making the following diagram commute:*

$$\begin{tikzcd} & A & \\ A \sqcap B \arrow[ur, "p"] \arrow[dr, "q"'] & & X \arrow[ul, "f"'] \arrow[dl, "g"] \arrow[ll, dotted, "\theta"'] \\ & B & \end{tikzcd}$$

Example 2.4.12 (Proposition 5.8 [24]). *The (categorical) product in $\mathfrak{Mod}(R)$ coincides with the coproduct.*

Example 2.4.13 (Example 5.5[24]). *The (categorical) product of two sets A and B in $\mathcal{O}bj(\mathfrak{Sets})$ is given by the triple $(A \times B, p, q)$ where $A \times B$ is the cartesian product, $p\colon (a, b) \mapsto a$ and $q\colon (a, b) \mapsto b$.*

Definition 2.4.14 (Additive Category). *A category $\mathcal{C}$ is* additive *if the set of all maps from A to B is an additive abelian group for every $A, B \in \mathcal{O}bj(\mathcal{C})$, it has the zero object, it has finite product and coproduct and the distributive laws hold i.e. given two morphisms f and g as in*

$$\begin{tikzcd} X \arrow[r, "a"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "b"] & Y \end{tikzcd}$$

where X and Y are objects in $\mathcal{C}$ then $b(f+g) = bf+bg$ and $(f+g)a = fa+ga$.

Example 2.4.15 (Lemma 2.3 [24]). *The category of R-modules is additive.*

Definition 2.4.16 (Kernel and cokernel). *If $u\colon A \to B$ is a morphism in an additive category $\mathcal{C}$ then its* kernel $\ker u$ *is a morphism $i\colon K \to A$ that satisfies the following universal mapping property: $u \circ i = 0$ and for every $g\colon X \to A$ with $ug = 0$, there exists a unique $\theta\colon X \to K$ with $i \circ \theta = g$.*

$$\begin{tikzcd} X \arrow[d, dashed, "\theta"'] \arrow[dr, "g"] \arrow[drr, "0"] & & \\ K \arrow[r, "i"'] & A \arrow[r, "u"'] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "u"] \arrow[drr, "0"'] & B \arrow[r, "\pi"] \arrow[dr, "h"] & C \arrow[d, dashed, "\theta"] \\ & & Y \end{tikzcd}$$

There is a dual definition for cokernel *(the morphism π in the diagram above).*

Definition 2.4.17 (Abelian Category). *An* abelian category *is an additive* *category in which every morphism has kernel and cokernel; moreover every injective morphism is a kernel and every surjective morphism is a cokernel.*

Definition 2.4.18 (Covariant Functor)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Then we define a* covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ *to be such that:*

1. *if $A \in \mathcal{Obj}(\mathcal{C})$ then $F(A) \in \mathcal{Obj}(\mathcal{D})$;*
2. *if $f\colon A \to B \in \mathcal{Hom}(\mathcal{C})$ then $F(f)\colon F(A) \to F(B) \in \mathcal{Hom}(\mathcal{D})$;*
3. *if $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$ then $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ in $\mathcal{D}$ and $F(g \circ f) = F(g) \circ F(f)$;*
4. *$F(\mathbb{1}_A) = \mathbb{1}_{F(A)}$ for every $A \in \mathcal{Obj}(\mathcal{C})$.*

Definition 2.4.19 (Contravariant functor)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. Then we define a* contravariant functor $F\colon \mathcal{C} \to \mathcal{D}$ *to be a function such that:*

1. *if $A \in \mathcal{Obj}(\mathcal{C})$ then $F(A) \in \mathcal{Obj}(\mathcal{D})$;*
2. *if $f\colon A \to B \in \mathcal{Hom}(\mathcal{C})$ then $F(f)\colon F(B) \to F(A) \in \mathcal{Hom}(\mathcal{D})$;*
3. *if $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$ then $F(C) \xrightarrow{F(g)} F(B) \xrightarrow{F(f)} F(A)$ in $\mathcal{D}$ and $F(g \circ f) = F(g) \circ F(f)$;*
4. *$F(\mathbb{1}_A) = \mathbb{1}_{F(A)}$ for every $A \in \mathcal{Obj}(\mathcal{C})$.*

Example 2.4.20. *Let $\mathcal{C}$ be a category and let $E \in \mathcal{Obj}(\mathcal{C})$, then* $\mathrm{Hom}(-, E)\colon \mathcal{C} \to \mathfrak{Sets}$ *is a contravariant functor defined for all $C \in \mathcal{Obj}(\mathcal{C})$ by sending $C \mapsto$* $\mathrm{Hom}(C, E)$ *and if $f\colon C \to D$ in $\mathcal{C}$ then* $\mathrm{Hom}(f)\colon \mathrm{Hom}(D, E) \to \mathrm{Hom}(C, E)$ *is given by $h \mapsto hf$.*

Definition 2.4.21 (Additive Functor)**.** *Let* $F\colon \mathfrak{Mod} \to \mathfrak{Ab}$ *be a functor.* F *is* additive *if for any* $f, g \in \mathrm{Hom}(\mathcal{C})$ *then* $F(f+g) = F(f) + F(g)$.

Definition 2.4.22 (Exact functor)**.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be two abelian categories and let* $F\colon \mathcal{C} \to \mathcal{D}$ *be a covariant additive functor. We say that* F *is exact if for every short exact sequence* $0 \to A \to B \to C \to 0$ *in* $\mathcal{C}$ *then* $0 \to F(A) \to F(B) \to F(C) \to 0$ *is exact in* $\mathcal{D}$.

Definition 2.4.23 (Injective object)**.** *In an abelian category* $\mathcal{C}$ *an object* E *is injective if for every monomorphism* $i\colon C \to X$ *and every* $f\colon C \to E$ *there exists* $h\colon X \to E$ *such that* $f = hi$.

Definition 2.4.24 (Injective Resolution)**.** *Let* $\mathcal{C}$ *be an abelian category and* $C \in \mathcal{O}bj(\mathcal{C})$*. An* injective resolution *for* C *is an exact sequence*

$$0 \to C \xrightarrow{\gamma} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots$$

where each E^i *is injective.*

Definition 2.4.25 (Injective dimension)**.** *If an* R*-module* M *admits a finite injective resolution, the minimal length among all finite injective resolutions of* M *is called its* injective dimension.

Definition 2.4.26 (Right Derived Functor)**.** *Let* $F\colon \mathcal{B} \to \mathcal{C}$ *be an additive covariant functor between abelian categories and for every* $B \in \mathcal{O}bj(\mathcal{B})$ *fix an injective resolusion*

$$0 \to B \xrightarrow{\alpha} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots$$

then consider the exact sequence E^B

$$0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots$$

and finally take homology

$$\mathrm{H}^i(F(E^B)) = \frac{\ker(Fd^i)}{\operatorname{Im}(Fd^{i-1})}.$$

$\mathrm{H}^i(F(E^B))$*, that we will denote as* $(\mathcal{R}^i F)B$ *is called* right derived functor*.*

Definition 2.4.27 (Ext)**.** *Let* A *be an* R*-module. We define* $\operatorname{Ext}^i_R(A, -) = \mathcal{R}^i F$ *where*

$$F = \operatorname{Hom}_R(A, -) \colon \mathfrak{Mod} \to \mathfrak{Mod}.$$

More precisely, using the notation above

$$\operatorname{Ext}^i_R(A, B) = (\mathcal{R}^i F)B = \mathrm{H}^i(F(E^B)) = \frac{\ker(d^i_\star)}{\operatorname{Im}(d^{i-1}_\star)}$$

where

$$\begin{array}{ccc} \operatorname{Hom}_R(A, E^i) & \xrightarrow{d^i_\star} & \operatorname{Hom}_R(A, E^{i+1}) \\ f & \longmapsto & d^n f \end{array}$$

Property 2.4.28. *If* B *is injective then* $\operatorname{Ext}^i_R(A, B) = 0$*.*

We give now another description of the Ext-functor in terms of projective resolutions. One could prove that the two descriptions are equivalent (see Theorem 6.67 [24]). We will use the second definition in Section 2.6 when we define local cohomology modules.

Definition 2.4.29 (Epimorphism)**.** *A morphism* $f\colon B \to C$ *in a category* $\mathcal{C}$ *is an* epimorphism *if for all objects* D *and all morphisms* $h, k\colon C \to D$ *we have that* $hv = kv$ *implies* $h = k$.

Example 2.4.30. *[24, page 321] In the categories* $\mathfrak{Sets}$ *and* $\mathfrak{Mod}(R)$ *epimorphisms and surjections are the same thing.*

Definition 2.4.31 (Projective object)**.** *In an abelian category* $\mathcal{C}$ *an object* P *is* projective *if for every epimorphism* $g\colon B \to C$ *and every* $f\colon P \to C$ *there exists* $h\colon P \to B$ *such that* $f = gh$.

Definition 2.4.32 (Projective resolution)**.** *Let* $\mathcal{C}$ *be an abelian category and* $C \in \mathcal{Obj}(\mathcal{C})$. *A* projective resolution *for* C *is an exact sequence*

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} C \to 0$$

where each P_i *is projective.*

Definition 2.4.33. *Let* $T\colon \mathcal{A} \to \mathcal{B}$ *be an additive covariant functor between abelian categories and for every* $C \in \mathcal{Obj}(\mathcal{A})$ *fix a projective resolusion*

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} C \to 0$$

then consider the deleted sequence $P_C\colon \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$, *apply the functor and finally take homology*

$$(\mathcal{R}^i T)C = \mathrm{H}^i(T(P_C)) = \frac{\ker(Td_{i+1})}{\mathrm{Im}(Td_i)}.$$

With this notation we have that $\mathrm{Ext}^i_R(-,C) = (\mathcal{R}^i T)C$ where $T = \mathrm{Hom}_R(-,C)\colon \mathfrak{Mod} \to \mathfrak{Mod}$.

$$\mathrm{Ext}^i_R(B,C) = \mathrm{H}^i(\mathrm{Hom}_R(P_C,B)) = \frac{\ker(d^{\star,i})}{\mathrm{Im}(d^{\star,i-1})}$$

where

$$\begin{array}{ccc} \mathrm{Hom}_R(A,E^i) & \xrightarrow{d^{\star,i}} & \mathrm{Hom}_R(A,E^{i+1}). \\ f & \longmapsto & d^n f \end{array}$$

Suppose $\mathcal{C} = \mathfrak{Mod}(R)$ then we have the following.

Property 2.4.34 (Theorem 3.25[12])**.** *Consider an exact sequence of R-modules*

$$0 \to M' \to M \to M'' \to 0.$$

For each R-module N, the following sequences are exact:

$$\begin{aligned} (1)\quad & \cdots \to \mathrm{Ext}^i_R(M,N) \to \mathrm{Ext}^i_R(M',N) \to \mathrm{Ext}^{i+1}_R(M'',N) \to \\ & \qquad \mathrm{Ext}^{i+1}_R(M,N) \to \cdots . \\ (2)\quad & \cdots \to \mathrm{Ext}^i_R(N,M) \to \mathrm{Ext}^i_R(N,M'') \to \mathrm{Ext}^{i+1}_R(N,M') \to \\ & \qquad \mathrm{Ext}^{i+1}_R(N,M) \to \cdots \end{aligned}$$

2.5 Injective Hulls and Matlis Duality

Proposition 2.5.1. *[2, Theorem 3.1.8] Any R-module M can be embedded in an injective R-module E.*

Definition 2.5.2. *The* injective hull *of an R-module M, namely $E_R(M)$*

or E_R, if M is obvious from the context, is the smallest injective R-module which contains M.

A different way to define the injective hull is the following. Let M be an R-module and N an R-submodule of M. M is called *essential extension* of N if for every R-submodule L of M then $L \cap N = \{0\}$ implies $L = \{0\}$. By Zorn's Lemma there exists an essential extension which is maximal (with respect to the inclusion); such an extension is the *injective hull*.

Example 2.5.3. *Let R be the ring of formal series $\mathbb{K}[[x_1, \cdots, x_n]]$ and let $\mathfrak{m}$ be its maximal ideal. We will see more in detail in Example 2.8.5 that the injective hull of $R/\mathfrak{m}$ is the module of inverse polynomials $\mathbb{K}[x_1^{-1}, \cdots, x_n^{-1}]$.*

Property 2.5.4. *[12, Theorem A20] Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring and $\widehat{R}$ its $\mathfrak{m}$-adic completion. Then $E_R(\mathbb{K}) \cong E_{\widehat{R}}(\mathbb{K})$.*

Property 2.5.5. *[12, Theorem A21] Let R be a Noetherian ring and E be an injective R-module. Then*

1. *$E \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} E_R(R/\mathfrak{p})^{\mu_\mathfrak{p}}$ and the numbers $\mu_\mathfrak{p}$ do not depend on the decomposition;*

2. *$E_R(R/\mathfrak{p}) \cong E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$ for every prime ideal $\mathfrak{p} \subset R$.*

Therefore understanding the injective modules over a Noetherian ring R comes down to understanding the injective hulls $E_R(R/\mathfrak{p}) \cong E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$.

Property 2.5.6. *[12, Theorem A25] Let $(R, \mathfrak{m}, \mathbb{K}) \to (S, \mathfrak{n}, \mathbb{L})$ be a homomorphism between two local rings and let S be finitely generated over the image of R. Then $\operatorname{Hom}_R(S, E_R(\mathbb{K})) = E_S(\mathbb{L})$.*

In particular if S is of the form R/I for some ideal I, since $\mathrm{Hom}_R(R/I, E_R) \cong \mathrm{Ann}_{E_R}(I)$, then $\mathrm{Ann}_{E_R}(I) \cong E_S$ (cf [9, Corollary 3.3]).

A special case of Example 2.4.20 is the following;

Definition 2.5.7. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring.*

The functor $\mathrm{Hom}_R(-, E_R)\colon \mathfrak{Mod}(R) \to \mathfrak{Mod}(R)$ which will be denoted by $(-)^\vee$ is called the Matlis dual functor*.*

Example 2.5.8. *Let $(R, \mathfrak{m})$ be a local ring and let $I \subset R$ be any ideal. Then*

$$\mathrm{Ann}_{E_R}(I)^\vee = \mathrm{Hom}_R(\mathrm{Ann}_{E_R}(I), E_R) = \mathrm{Hom}_R(\mathrm{Hom}_R(R/I, E_R), E_R) = \frac{R}{I}.$$

Theorem 2.5.9 (Matlis Duality Theorem)**.** *Let $(R, \mathfrak{m}, \mathbb{K})$ be a complete local ring and E be the injective hull of $\mathbb{K}$. Then $(-)^\vee = Hom_R(-, E)$ is a functor such that:*

1. *$Hom_R(E, E) \cong R$;*

2. *$Hom_R(R, E) \cong E$;*

3. *if M is Noetherian then $(M)^\vee$ is Artinian and $((M)^\vee)^\vee \cong M$;*

4. *if N is Artinian then $N^\vee$ is Noetherian and $((N)^\vee)^\vee \cong N$.*

Corollary 2.5.10. *If R is local then $E_R(\mathbb{K})$ is Artinian.*

2.6 Definitions of Local Cohomology

In this section we give three equivalent definitions for a local cohomology module.

1. Let R be a Noetherian ring and M an R-module. Suppose I is an ideal of R and consider the decreasing sequence of ideals $I \supseteq I^2 \supseteq \cdots \supseteq I^t \supseteq \cdots$ of R; every surjection $\frac{R}{I^{t+1}} \to \frac{R}{I^t}$ induces a map $\mathrm{Ext}^i(R/I^t, M) \to \mathrm{Ext}^i(R/I^{t+1}, M)$.

   The i^{th}-*local cohomology module* of M with support in I is defined to be

$$\mathrm{H}_I^i(M) = \varinjlim_t \mathrm{Ext}^i(R/I^t, M).$$

2. Since $\mathrm{Hom}_R(R/I, M) \cong \mathrm{Ann}_M(I)$, if $I \supseteq J$ is any inclusion of ideals of R then the map $\mathrm{Hom}_R(R/I, M) \to \mathrm{Hom}_R(R/J, M)$ can be identified with the inclusion $\mathrm{Ann}_M(I) \subseteq \mathrm{Ann}_M(J)$. In particular consider the inclusions $I \supseteq I^2 \supseteq \cdots \supseteq I^t \supseteq \cdots$ and define

$$\mathrm{H}_I^0(M) \cong \bigoplus_t \mathrm{Ann}_M(I^t) = \{x \in M | \; xI^t = 0 \textit{ for some } t\}.$$

   The other cohomology modules $\mathrm{H}_I^i(M)$ are defined to be the i^{th} right derived functor of $\mathrm{H}_I^0(M)$. More precisely, let $0 \to E_0 \to \cdots E_i \to \cdots$ be an injective resolution of M, where $M = \mathrm{Ker}(E_0 \to E_1)$, then

$$\mathrm{H}_I^i(M) = \varinjlim_t \left(\mathrm{Hom}_R(R/I^t, E_0) \to \cdots \to \mathrm{Hom}_R(R/I^t, E_i) \to \cdots\right).$$

3. Another characterisation for a local cohomology module can be given in terms of Koszul cohomology. For further details on the construction see [12, Chapter 7, Section 3]

Let $\bar{r} = r_1, \cdots, r_n$ be a sequence of elements in R, consider the sequence

$$\cdots \to K^\bullet(\bar{r}^t; M) \to K^\bullet(\bar{r}^{t+1}; M) \to \cdots$$

and set

$$K^\bullet(\bar{r}^\infty; M) := \varinjlim_t K^\bullet(\bar{r}^t; M).$$

Let I be the ideal generated by the elements $r_1, \cdots, r_n$ then $\mathrm{H}_I^\bullet(M)$ is the same as the Koszul cohomology $\mathrm{H}^\bullet(\bar{r}^\infty; M)$ where

$$\mathrm{H}^\bullet(\bar{r}^\infty; M) := \varinjlim_t(\cdots \to \mathrm{H}^\bullet(\bar{r}^t; M) \to \mathrm{H}^\bullet(\bar{r}^{t+1}; M) \to \cdots).$$

2.7 Properties of Local Cohomology Modules

In this section we recall some of the properties of local cohomology modules. We start with a consequence of Property 2.4.28.

Property 2.7.1. *If M is injective then $\mathrm{H}_I^i(M) = 0$ for all $i > 0$.*

The next property provides a relation between the depth of a module on an ideal and the local cohomology modules.

Property 2.7.2. *[9, Theorem 6.9] Let M be a finitely generated module over a Noetherian ring R and let I be an ideal of R. Then*

$$IM = M \iff H_I^i(M) = 0 \quad \forall i.$$

Otherwise if $IM \neq M$ then

$$\operatorname{depth}_I(M) = \min\{i \mid H_I^i(M) \neq 0\}.$$

Property 2.7.3. *[23, Chapter 3, page 474], [9] Let I and J be two ideals of a Noetherian ring R. If they have the same radical, then $\mathrm{H}_I^i(M) \cong \mathrm{H}_J^i(M)$ for all i and all R-modules M.*

Property 2.7.4. *[18, section 1.1, page 42] Given any short exact sequence $0 \to A \to B \to C \to 0$ of R-modules there is a long exact sequence*

$$0 \to \mathrm{H}_I^0(A) \to \mathrm{H}_I^0(B) \to \mathrm{H}_I^0(C) \to \mathrm{H}_I^1(A) \to \cdots$$

$$\cdots \to \mathrm{H}_I^{i-1}(C) \to \mathrm{H}_I^i(A) \to \mathrm{H}_I^i(B) \to \mathrm{H}_I^i(C) \to \cdots$$

and $\mathrm{H}_I^i(-)$ is a covariant additive functor. This follows from the fact that $\mathrm{Ext}_R^i(R/I^t, M)$ is a covariant additive functor of M that is exact and preserves the direct limit.

Property 2.7.5. *[Proposition 7.4 [9]] If $R \to S$ is a homomorphism between two Noetherian rings, $I \subset R$ is an ideal of R and M is an S-module then $\mathrm{H}_I^i(M) \cong \mathrm{H}_{IS}^i(M)$ as S-modules.*

Consider the case in which $I = \mathfrak{m} \subset R$ is a maximal ideal. Let M be an R-module and set $\widehat{R}$ to be the completion of R with respect to $\mathfrak{m}$. Then the study of local cohomology modules with support in a maximal ideal reduces to the case in which R is local and complete. To be more precise we have:

Property 2.7.6. *With the above notation the followings hold:*

1. $\mathrm{H}^i_{\mathfrak{m}}(M) \cong \mathrm{H}^i_{\mathfrak{m}R_{\mathfrak{m}}}(M_{\mathfrak{m}})$.

2. *If R is local then $\mathrm{H}^i_{\mathfrak{m}}(M) \cong \mathrm{H}^i_{\mathfrak{m}\widehat{R}}(\widehat{R} \otimes M)$ as $\widehat{R}$-modules (and R-modules).*

The next property is particularly useful.

Property 2.7.7. *Let $(R, \mathfrak{m})$ be a complete, Cohen-Macaulay local ring of dimension d. Then $\mathrm{H}^i_{\mathfrak{m}}(R) = 0$ if and only if $i \neq d$.*

2.8 Gorenstein Rings

Definition 2.8.1 (Gorenstein ring)**.** *A Noetherian local ring R is* Gorenstein *if its injective dimension as an R-module is finite. A non local ring is* generically Gorenstein *if each localisation at a minimal prime ideal is a Gorenstein local ring.*

Definition 2.8.2. *A Noetherian local ring such that its completion is the quotient of a regular local ring by an ideal generated by a regular sequence is called* local complete intersection ring*.*

Example 2.8.3. *[12, Proposition 11.19] Complete intersections are Gorenstein.*

For a Noetherian ring there is the following chain of inclusions: Cohen-Macaulay rings $\supset$ Gorenstein rings $\supset$ complete intersections rings $\supset$ regular local rings, [2, Proposition 3.1.20].

Property 2.8.4. *[9, Theorem 11.5] If $(R, \mathfrak{m}, \mathbb{K})$ is a complete, local and Gorenstein ring of dimension d then*

$$E_R(\mathbb{K}) \cong \mathrm{H}^d_{\mathfrak{m}}(R).$$

Example 2.8.5. *[3, Example 12.4.1] Let R be the ring of formal power series $\mathbb{K}[[x_1, \cdots, x_n]]$ with n variables and with coefficients in a field $\mathbb{K}$. R is a Gorenstein, complete and local ring with maximal ideal $\mathfrak{m} = (x_1, \cdots, x_n)$ and canonical module R. Its injective hull $E_R(\mathbb{K})$ is isomorphic to $\mathrm{H}^n_{\mathfrak{m}}(R)$ and can be computed using Čech complex as follows; let $\mathcal{P}$ be the set of monomials in which at least one exponent is non-negative and $\mathcal{N}$ the set of monomials with negative exponents then*

$$\begin{aligned} E_R(\mathbb{K}) &= \mathrm{Coker}\left(\bigoplus_{i=1}^{n} R_{x_1\ldots\hat{x_i}\ldots x_n} \to R_{x_1\ldots x_n}\right) = \\ &= \frac{R_{x_1\cdots x_n}}{\mathrm{Span}_{\mathbb{K}}(\mathcal{P})} = \\ &= \mathrm{Span}_{\mathbb{K}}(\mathcal{N}) = \\ &= \mathbb{K}[x_1^-, \cdots, x_n^-], \end{aligned}$$

the module of inverse polynomials.

2.9 Canonical Modules

Definition 2.9.1 (Canonical Module)**.** *A finitely generated R-module ω over a Cohen-Macaulay local ring $(R, \mathfrak{m}, \mathbb{K})$ of dimension n is a* canonical module *for R if its Matlis dual $\omega^\vee$ is isomorphic to $\mathrm{H}^n_{\mathfrak{m}}(R)$. Given a non-local Cohen-*

Macaulay ring R, a global canonical module Ω *for R is an R-module such that $\Omega_{\mathfrak{p}}$ is a canonical module for $R_{\mathfrak{p}}$ for all prime ideal $\mathfrak{p} \subset R$.*

Property 2.9.2. *[4, Section 21] If R is Gorenstein then a canonical module for R is R itself.*

Property 2.9.3. *[12, Theorem 11.44] Let $(R, \mathfrak{m}, \mathbb{K})$ be a local Cohen-Macaulay ring of dimension n, ω_R a canonical module for R and fix an isomorphism $\omega_R^\vee \cong \mathrm{H}^n_{\mathfrak{m}}(R)$ where $(-)^\vee = \mathrm{Hom}_R(-, \mathrm{E})$. If M is a finitely generated R-module then, for every $0 \leq i \leq n$, there is an isomorphism functorial in M:*

$$\mathrm{H}^i_{\mathfrak{m}}(M) \cong \mathrm{Ext}^{n-i}_R(M, \omega_R)^\vee.$$

Furthermore, we have:

Property 2.9.4. *[12, Theorem 11.46] If $(R, \mathfrak{m}, \mathbb{K})$ is a local Cohen-Macaulay ring then ω_R is a canonical module for R if and only if $\widehat{\omega_R}$ is a canonical module for $\widehat{R}$.*

Property 2.9.5. *[12, Theorem 11.47] If $(R, \mathfrak{m}, \mathbb{K})$ is a local Cohen-Macaulay ring then if ω_R and α_R are two canonical modules for R then $\omega_R \cong \alpha_R$.*

Definition 2.9.6 (Reduced ring)**.** *A reduced ring is a ring that has no non-zero nilpotent elements.*

Every field and every polynomial ring over a field in arbitrarily many variables is a reduced ring. The following result was proved in [9] and a proof can also be found in [21, Proposition 2.4].

Property 2.9.7. *Let $(R, \mathfrak{m})$ be a complete, local ring of dimension d and suppose that R is Cohen-Macaulay with canonical module ω. If R is a domain, or if it is reduced or more in general if the localisation of R at every minimal prime is Gorenstein then ω is isomorphic with an ideal of R that contains a nonzerodivisor.*

We generalised this result to the non-local case. Let us define now canonical modules for non-local rings.

Discussion 2.9.8 (Canonical modules in Macaulay2)**.** *Given a ring R as in Property 2.9.7 we can compute explicitly a canonical ideal for it, i.e an ideal which is isomorphic to a canonical module for R. We start by computing a canonical module as a cokernel of a certain matrix A, say R^n/V. In order to find an ideal isomorphic to it, we look for a vector w such V is the kernel of $w : R^n \to R$ given by multiplication by w on the left. An ad-hoc way to find such w is to look among the generators of the module of syzygies of the rows of V. The algorithm just described has been translated into code using Macaulay2 and the code can be found in the appendix.*

Let A be a polynomial ring and $J \subset A$ be an ideal of A and let B be the quotient ring A/J; if B is Cohen-Macaulay of dimension d, then $\bar{\Omega} = \mathrm{Ext}_A^{\dim A - d}(B, A)$ is a global canonical module for B; morover, if B is generically Gorenstein then $\bar{\Omega}$ is isomorphic to an ideal of B, [2, Prop. 3.3.18 (b)].

Therefore let B be generically Gorenstein and assume $\bar{\Omega} \subseteq B$. Let Ω be the preimage of $\bar{\Omega}$ in A; then the following B-module is well defined:

$$\mathcal{U}_{(e)} = \frac{\left(J^{[p^e]} : J\right) \cap \left(\Omega^{[p^e]} : \Omega\right)}{J^{[p^e]}}. \tag{1}$$

Since A is Noetherian, $\mathcal{U}_{(e)}$ is a finitely generated A-module (and B-module). For every prime ideal $\mathfrak{p} \supseteq J$ write $H_{\mathfrak{p}} = \mathrm{H}^{\dim \widehat{B}_{\mathfrak{p}}}_{\mathfrak{p}\widehat{B}_{\mathfrak{p}}}(\widehat{B}_{\mathfrak{p}})$. It follows from Theorem 3.2.5 that the A-module $\mathcal{F}^e(H_{\mathfrak{p}})$ consisting of the Frobenius maps on $H_{\mathfrak{p}}$ is of the form:

$$\mathcal{F}^e(H_{\mathfrak{p}}) = \frac{\left(J^{[p^e]}\widehat{A}_{\mathfrak{p}} : J\widehat{A}_{\mathfrak{p}}\right) \cap \left(\Omega^{[p^e]}\widehat{A}_{\mathfrak{p}} : \Omega\widehat{A}_{\mathfrak{p}}\right)}{J^{[p^e]}\widehat{A}_{\mathfrak{p}}} \tag{2}$$

and consequently $\mathcal{F}^e(H_{\mathfrak{p}}) \cong \mathcal{U}_{(e)}\widehat{A}_{\mathfrak{p}}$. Since $\mathcal{F}^e(H_{\mathfrak{p}})$ is generated by one element by Theorem 3.2.1, $\mathcal{U}_{(e)}\widehat{A}_{\mathfrak{p}}$ is principal as well.

2.10 The Frobenius Endomorphism

Definition 2.10.1. *Let R be a ring and for every positive integer e define the e^{th}-iterated Frobenius endomorphism $T^e \colon R \to R$ to be the map $r \mapsto r^{p^e}$. For $e = 1$, $R \to R$ is the* natural Frobenius *map on R.*

For any R-module M we define $F^e_* M$ to be the Abelian group M with R-module structure given by $r \cdot m = T^e(r)m = r^{p^e}m$ for all $r \in R$ and $m \in M$. We can extend this construction to obtain the *Frobenius functor* F^e_R from R-modules to R-modules as follows. For any R-module M, we consider the $F^e_* R$-module $F^e_* R \otimes_R M$ and after identifying the rings R and $F_* R$, we may regard $F^e_* R \otimes_R M$ as an R-module and denote it $F^e_R(M)$ or just $F^e(M)$ when R is understood. The functor $F^e_R(-)$ is exact when R is regular, see [2, Corollary 8.2.8], and for any matrix C with entries in R, $F^e_R(\operatorname{Coker} C)$ is the cokernel of the matrix $C^{[p^e]}$ obtained from C by raising its entries to the p^eth power, see [10].

Definition 2.10.2. *For any R-module M an additive map $\varphi : M \to M$ is an* e^{th}-Frobenius map *if it satisfies $\varphi(rm) = r^{p^e}\varphi(m)$ for all $r \in R$ and $m \in M$. Note that there is a bijective correspondence between $\operatorname{Hom}_R(M, F_*M)$ and the Frobenius maps on M.*

For every $e \geq 0$ let $\mathcal{F}^e(M)$ be the set of all Frobenius maps on M. Each $\mathcal{F}^e(M)$ is an R-module: for all $\varphi \in \mathcal{F}^e(M)$ and $r \in R$ the map $r\varphi$ defined as $(r\varphi)(m) = r\varphi(m)$ is in $\mathcal{F}^e(M)$ for all $m \in M$. If $\varphi \in \mathcal{F}^e(M)$ we can define for $i \geq 0$ the R-submodules $M_i = \{m \in M \mid \varphi^i(m) = 0\}$. We define the *submodule of nilpotent elements in M* as

$$\operatorname{Nil}(M) = \cup_{i \geq 0} M_i.$$

Theorem 2.10.3 (see Proposition 1.11 in [5] and Proposition 4.4 in [15])**.** *If $(R, \mathfrak{m})$ is a complete regular ring, M is an Artinian R-module and $\varphi \in \mathcal{F}^e(M)$ then the ascending sequence $\{M_i\}_{i \geq 0}$ above stabilises, i.e., there exists an $e \geq 0$ such that $\varphi^e(\operatorname{Nil}(M)) = 0$.*

Note that if $M_i = M_{i+1}$ then $M_i = M_j$ for every $j \geq i$.

Definition 2.10.4. *We define the* HSL number *or* index of nilpotency *of φ*

on M, denoted $\mathrm{HSL}(M)$*, to be the smallest integer e at which* $\varphi^e(\mathrm{Nil}(M)) = 0$*, or ∞ if no such e exists.*

We can rephrase Theorem 2.10.3 by saying that under the hypothesis of the theorem, $\mathrm{HSL}(M) < \infty$.

Another way of describing a Frobenius map $\varphi \colon M \to M$ on an R-module M is to think of M as a module over a certain skew-commutative ring $R[\theta; f^e]$ where the latter is defined as follows. $R[\theta; f^e]$ is the free R-module $\bigoplus_{i=0}^{\infty} R\theta^i$ endowed with the further non-commutative operation $\theta s = s^{p^e}\theta$ for every $s \in S$. Therefore it is equivalent to say that M is an R-module with a Frobenius action given by φ and that M is an $R[\theta; f^e]$-module with module structure given by $\theta m = \varphi(m)$.

The action of Frobenius on a local cohomology module is constructed as follows. Any R-linear map $M \to N$ induces an R-linear map $\mathrm{H}^i_I(M) \to \mathrm{H}^i_I(N)$ for every i. The map $R \to F_*R$ sending $r \mapsto F_*r^p$ is R-linear because $F_*r^p = r \cdot F_*1$ and so it induces for every i a map $\mathrm{H}^i_I(R) \to \mathrm{H}^i_I(F_*R) = \mathrm{H}^i_{IF_*R}(F_*R) = \mathrm{H}^i_{F_*I^{[p^e]}}(F_*R) = \mathrm{H}^i_{F_*I}(F_*R) = F_*\,\mathrm{H}^i_I(R)$ where in the first equality we used the Independence Theorem for local cohomology [3, Proposition 4.1] and in the third that the ideals I and $I^{[p^e]}$ have same radical see [3, Proposition 3.1.1]. So we get an R-linear map $\mathrm{H}^i_I(R) \to F_*\,\mathrm{H}^i_I(R)$ which is the same as a Frobenius map $\mathrm{H}^i_I(R) \to \mathrm{H}^i_I(R)$.

If $(R, \mathfrak{m})$ is of dimension d and $x_1, \cdots, x_d$ is a system of parameters for

R then we can write $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}}(R)$ as the direct limit

$$\frac{R}{(x_1,\cdots,x_d)R} \xrightarrow{x_1\cdots x_d} \frac{R}{(x_1^2,\cdots,x_d^2)R} \xrightarrow{x_1\cdots x_d} \cdots$$

where the maps are the multiplication by $x_1\cdots x_d$, [10, Theorem 11.5]. Another way to describe the natural Frobenius action on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}}(R)$ is the following. The natural Frobenius map on R induces a natural Frobenius map on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}}(R)$ in the following way; a map $\phi \in \mathcal{F}^e(\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}}(R))$ is defined on the direct limit above by mapping the coset $a + (x_1^t,\cdots,x_d^t)R$ in the t-th component to the coset $a^{p^e} + (x_1^{tp^e},\cdots,x_d^{tp^e})R$ in the tp^e-th component, [13, Section 2].

Definition 2.10.5. *A local ring $(R,\mathfrak{m})$ is F-*injective *if the natural Frobenius map $\mathrm{H}^i_{\mathfrak{m}}(R) \to \mathrm{H}^i_{\mathfrak{m}}(R)$ is injective for all i.*

When the ring is Cohen-Macaulay the only non-zero local cohomology module is the top local cohomology module (see [Property 2.7.7]) therefore a Cohen-Macaulay ring is F-injective if the Frobenius map $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}}(R) \to \mathrm{H}^{\mathrm{d}}_{\mathfrak{m}}(R)$ is injective.

Definition 2.10.6. *We define the Cohen-Macaulay F-injective (CMFI for short) locus of a ring R of characteristic p as follows:*

$$\mathrm{CMFI}(R) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid R_{\mathfrak{p}} \textit{ is CMFI }\}.$$

2.11 The Δ^e- and Ψ^e-functors

Let $(R,\mathfrak{m})$ be a complete and local ring and let $(-)^{\vee}$ denote the *Matlis dual*, i.e. the functor $\mathrm{Hom}_R(-,E_R)$, where $E_R = \mathrm{E}_R(\mathbb{K})$ is the injective hull of the residue field $\mathbb{K}$ of R. In this section we recall the notions of Δ^e-functor and Ψ^e-functor which have been described in more detail in [13, Section 3].

Let $\mathcal{C}^e$ be the category of Artinian $R[\theta, f]$-modules and $\mathcal{D}^e$ the category of R-linear maps $\alpha_M \colon M \to F_R^e(M)$ with M a Noetherian R-module and where a morphism between $M \overset{\alpha_M}{\to} F_R^e(M)$ and $N \overset{\alpha_N}{\to} F_R^e(N)$ is a commutative diagram of R-linear maps:

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[d, "\alpha_M"'] & N \arrow[d, "\alpha_N"'] \\ F_R^e(M) \arrow[r, "F_R^e(h)"] & F_R^e(N). \end{tikzcd}$$

We define a functor $\Delta^e : \mathcal{C}^e \to \mathcal{D}^e$ as follows: given an e^{th}-Frobenius map θ of the Artinian R-module M, we obtain an R-linear map $\phi : F_*^e(R) \otimes_R M \to M$ which sends $F_*^e r \otimes m$ to $r\theta m$. Taking Matlis duals, we obtain the R-linear map

$$M^\vee \to (F_*^e(R) \otimes_R M)^\vee \cong F_*^e(R) \otimes_R M^\vee$$

where the last isomorphism is the functorial isomorphism described in [15, Lemma 4.1]. This construction can be reversed, yielding a functor $\Psi^e : \mathcal{D}^e \to \mathcal{C}^e$ such that $\Psi^e \circ \Delta^e$ and $\Delta^e \circ \Psi^e$ can naturally be identified with the identity functor. See [13, Section 3] for the details of this construction.

3 On the Upper Semicontinuity of HSL Numbers

Let B be a quotient of a polynomial ring with coefficients in a field of characteristic p. Suppose that B is Cohen-Macaulay and for every prime ideal $\mathfrak{p} \subset B$, let $H_\mathfrak{p}$ denote $\mathrm{H}^{\dim B_\mathfrak{p}}_{\mathfrak{p}B_\mathfrak{p}}\left(\widehat{B_\mathfrak{p}}\right)$. Each such $H_\mathfrak{p}$ is an Artinian module endowed with a natural Frobenius map Θ and let $\operatorname{Nil}(H_\mathfrak{p})$ denote the set of all elements in $H_\mathfrak{p}$ killed by some power of Θ. A theorem by Hartshorne-Speiser and Lyubeznik shows that there exists an $e \geq 0$ such that $\Theta^e \operatorname{Nil}(H_\mathfrak{p}) = 0$. The smallest such e is the HSL-number of $H_\mathfrak{p}$ which we denote $\operatorname{HSL}(H_\mathfrak{p})$.

In this chapter we show that for all $e > 0$, the sets

$$\mathcal{B}_e = \{\mathfrak{p} \in \operatorname{Spec} B \mid \operatorname{HSL}(H_\mathfrak{p}) < e\}$$

are Zariski open, hence HSL is upper semi-continuous. This result generalises the openness of the F-injective locus. An application of this result gives a global test exponent for the calculation of Frobenius closures of parameter ideals in Cohen-Macaulay rings.

We will start by defining the operator $I_e(-)$ and showing some of its properties. We will then consider a quotient of a complete, regular local ring S and describe the action of Frobenius on its top local cohomology module $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$. After that we will give an explicit description of the module consisting of all the e-th Frobenius maps acting on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ and compute the

HSL-numbers in the local case. In Section 3.4 we will consider a Cohen-Macaulay non-local domain $B = A/J$ and will prove the main result: the sets $\mathcal{B}_e$ defined above are Zariski open.

3.1 The $I_e(-)$ Operator

In this section we define the operator $I_e(-)$ which has been introduced in [13], and in [3] with the notation $(-)^{[1/p^e]}$. We will show that this commutes with localisations and completions.

For any ideal I of a ring R, we shall denote by $I^{[p^e]}$ the e^{th}-Frobenius power of I, i.e. the ideal generated by $\{a^{p^e} | a \in I\}$.

Definition 3.1.1. *If R is a ring and $J \subseteq R$ an ideal of R we define $I_e(J)$ to be the smallest ideal L of R such that its e^{th}-Frobenius power $L^{[p^e]}$ contains J.*

In general, such an ideal may not exist; however it does exist in polynomial rings and power series rings, cf [13, Proposition 5.3].

Let A be a polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ and W be a multiplicatively closed subset of A and $J \subset A$ an ideal.

Lemma 3.1.2. *If $L \subseteq W^{-1}A$ is any ideal then $L^{[p^e]} \cap A = (L \cap A)^{[p^e]}$.*

Proof. Let $\frac{g_1}{1}, \dots, \frac{g_s}{1}$ be a set of generators for L and let G be the ideal of A generated by $g_1, \dots, g_s$. Then we can write $L^{[p^e]} \cap A = \sum_{w \in W} (G^{[p^e]} :_A w)$ and $(L \cap A)^{[p^e]} = \sum_{w \in W} (G :_A w)^{[p^e]}$. Since A is regular, for any $w \in W$, w^{p^e}

is in W and $(G^{[p^e]} :_A w^{p^e}) = (G :_A w)^{[p^e]}$ so $(L \cap A)^{[p^e]} \subseteq L^{[p^e]} \cap A$. Also $(G^{[p^e]} :_A w) \subseteq (G^{[p^e]} :_A w^{p^e})$ so $L^{[p^e]} \cap A \subseteq (L \cap A)^{[p^e]}$. $\square$

Lemma 3.1.3. *If J is any ideal of A then $I_e(W^{-1}J)$ exists for any integer e and equals $W^{-1}I_e(W^{-1}J \cap A)$.*

Proof. If $L \subseteq W^{-1}A$ is an ideal such that $W^{-1}J \subseteq L^{[p^e]}$, then

$$W^{-1}I_e(W^{-1}J \cap A) \subseteq L.$$

In fact, $W^{-1}J \cap A \subseteq L^{[p^e]} \cap A = (L \cap A)^{[p^e]}$ where the equality follows from Lemma 3.1.2. Thus $I_e(W^{-1}J \cap A) \subseteq L \cap A$ so $W^{-1}I_e(W^{-1}J \cap A) \subseteq W^{-1}(L \cap A) \subseteq L$. Hence $W^{-1}I_e(W^{-1}J \cap A)$ is contained in all the ideals L such that $W^{-1}J \subseteq L^{[p^e]}$. If we show that $W^{-1}J \subseteq (W^{-1}I_e(W^{-1}J \cap A))^{[p^e]}$ then $I_e(W^{-1}J)$ exists and equals $W^{-1}I_e(W^{-1}J \cap A)$. But since $W^{-1}J \cap A \subseteq I_e(W^{-1}J \cap A)^{[p^e]}$ then using Lemma 3.1.2 we obtain $W^{-1}J = W^{-1}(W^{-1}J \cap A) \subseteq W^{-1}(I_e(W^{-1}J \cap A)^{[p^e]}) = (W^{-1}I_e(W^{-1}J \cap A))^{[p^e]}$. $\square$

Proposition 3.1.4. *Let $\widehat{A}$ denote the completion of A with respect to any prime ideal and W any multiplicatively closed subset of A. Then the following hold:*

1. *$I_e(J \otimes_A \widehat{A}) = I_e(J) \otimes_A \widehat{A}$, for any ideal $J \subseteq A$;*
2. *$W^{-1}I_e(J) = I_e(W^{-1}J)$.*

Proof. 1. Write $\widehat{J} = J \otimes_A \widehat{A}$. Since $I_e(\widehat{J})^{[p^e]} \supseteq \widehat{J}$ using [17, Lemma 6.6] we obtain

$$(I_e(\widehat{J}) \cap A)^{[p^e]} = I_e(\widehat{J})^{[p^e]} \cap A \supseteq \widehat{J} \cap A = J.$$

But $I_e(J)$ is the smallest ideal such that $I_e(J)^{[p^e]} \supseteq J$, so $I_e(\widehat{J}) \cap A \supseteq I_e(J)$ and hence $I_e(\widehat{J}) \supseteq (I_e(\widehat{J}) \cap A) \otimes_A \widehat{A} \supseteq I_e(J) \otimes_A \widehat{A}$.
On the other hand, $(I_e(J) \otimes_A \widehat{A})^{[p^e]} = I_e(J)^{[p^e]} \otimes_A \widehat{A} \supseteq J \otimes_A \widehat{A}$ and so $I_e(J \otimes_A \widehat{A}) \subseteq I_e(J) \otimes_A \widehat{A}$.

2. Since $J \subseteq W^{-1}J \cap A$, $I_e(J) \subseteq I_e(W^{-1}J \cap A)$, and so $W^{-1}I_e(J) \subseteq W^{-1}I_e(W^{-1}J \cap A)$. By Lemma 3.1.3, $W^{-1}I_e(W^{-1}J \cap R) = I_e(W^{-1}J)$ hence $W^{-1}I_e(J) \subseteq I_e(W^{-1}J)$.
For the reverse inclusion it is enough to show that

$$W^{-1}J \subseteq (W^{-1}I_e(J))^{[p^e]}$$

because from this it follows that $I_e(W^{-1}J) \subseteq W^{-1}I_e(J)$. Since $J \subseteq I_e(J)^{[p^e]}$ then $W^{-1}J \subseteq W^{-1}(I_e(J)^{[p^e]}) = (W^{-1}I_e(J))^{[p^e]}$ where in the latter equality we have used Lemma 3.1.2.

□

3.2 The Frobenius Action on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$

Let $(R, \mathfrak{m})$ be a complete, regular and local ring, I an ideal of R and write $S = R/I$. Let d be the dimension of S and suppose S is Cohen-Macaulay with canonical module $\bar{\omega}$. Assume that S is generically Gorenstein so that

$\bar{\omega} \subseteq S$ is an ideal of S, (see [20, Proposition 2.4]) and consider the following short exact sequence:

$$0 \to \bar{\omega} \to S \to S/\bar{\omega} \to 0.$$

This induces the long exact sequence

$$\cdots \to \mathrm{H}^{\mathrm{d}-1}_{\mathfrak{m}S}(S) \to \mathrm{H}^{\mathrm{d}-1}_{\mathfrak{m}S}(S/\bar{\omega}) \to \mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(\bar{\omega}) \to \mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S) \to 0.$$

Since S is Cohen-Macaulay, the above reduces to

$$0 \to \mathrm{H}^{\mathrm{d}-1}_{\mathfrak{m}S}(S/\bar{\omega}) \to \mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(\bar{\omega}) \to \mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S) \to 0. \tag{3}$$

A natural Frobenius map acting on S induces a natural Frobenius map acting on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$. The following theorem gives a description of the natural Frobenius (up to a unit) which we will later use in Theorem 3.2.5.

Theorem 3.2.1 (see [17] Example 3.7)**.** *Let $\mathcal{F}^e := \mathcal{F}_e(\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S))$ be the R-module consisting of all e^{th}-Frobenius maps acting on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$. If S is S_2 then $\mathcal{F}^e$ is generated by one element which corresponds, up to unit, to the natural Frobenius map.*

We aim to give an explicit description of the R-module $\mathcal{F}^e$ and consequently of the natural Frobenius map that generates it.
We will see in Example 3.4.4 that if the ring is not S_2 then $\mathcal{F}^e$ is not necessarely principal.

Remark 3.2.2. *The inclusion $\bar{\omega} \to S$ is $R[\theta, f^e]$-linear where $\theta s = s^{p^e}$*

acts on $\bar{\omega}$ by restriction. This induces an $R[\theta; f^e]$-linear map $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(\bar{\omega}) \xrightarrow{\alpha}$ $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S) \to 0$ where the structure of $R[\theta, f^e]$-module on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(\bar{\omega})$ is obtained from the one on $\bar{\omega} \subseteq S$ and where $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ has a natural structure of $R[\theta; f^e]$-module as we have seen in the introduction.

Since any kernel of an $R[\theta; f^e]$-map is an $R[\theta; f^e]$-module, $\ker(\alpha) = \mathrm{H}^{\mathrm{d}-1}_{\mathfrak{m}S}(S/\bar{\omega})$ *is an $R[\theta; f^e]$-module as well. Hence the sequence $0 \to \mathrm{H}^{\mathrm{d}-1}_{\mathfrak{m}S}(S/\bar{\omega}) \to \mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(\bar{\omega}) \to \mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S) \to 0$ is an exact sequence of $R[\theta; f^e]$-modules.*

Identifying $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(\bar{\omega})$ with $E_S = \mathrm{Ann}_{E_R}(I)$ we get the inclusion $\mathrm{H}^{\mathrm{d}-1}_{\mathfrak{m}S}(S/\bar{\omega}) \subseteq \mathrm{Ann}_{E_R}(I)$ therefore $\mathrm{H}^{\mathrm{d}-1}_{\mathfrak{m}S}(S/\bar{\omega})$ must be of the form $\mathrm{Ann}_{E_S}(J)$ for a certain ideal $J \subseteq R$. More precisely we have the following:

Lemma 3.2.3. *$\mathrm{H}^{\mathrm{d}-1}_{\mathfrak{m}S}(S/\bar{\omega})$ and $\mathrm{Ann}_{E_S}(\bar{\omega})$ are isomorphic.*

Proof. Identify $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(\bar{\omega}) \cong E_S \cong Ann_{E_R}(I)$ then $\mathrm{H}^{\mathrm{d}-1}_{\mathfrak{m}S}(S/\bar{\omega}) \subseteq E_S$ is a submodule. All submodules of E_S are of the form $Ann_{E_R}(J)$ for some $J \supseteq I$. So $\mathrm{H}^{\mathrm{d}-1}_{\mathfrak{m}S}(S/\bar{\omega}) \cong Ann_{E_R}(J)$ for some $J \supseteq I$. Note that $(0 :_R Ann_{E_R}(J)) = (0 :_R (R/J)^\vee) = (0 :_R R/J) = J$. On the other hand, Corollary 3.3.18 in [2] proves that S/ω is Gorenstein and $\mathrm{H}^{\mathrm{d}-1}_{\mathfrak{m}S}(S/\bar{\omega})$ is the injective hull of $T = S/\omega$. The injective hull of T has no T-torsion so $(0 : \mathrm{H}^{\mathrm{d}-1}_{\mathfrak{m}S}(S/\bar{\omega})) = \omega$. Therefore $J = \omega$. □

Remark 3.2.4 (Frobenius action on E_S)**.** *All $R[\theta; f^e]$-module structures on $\mathrm{Ann}_{E_R}(I) = E_S$ have the form uF where F is the natural Frobenius map on E_R and $u \in (I^{[p^e]} : I)$. The identification $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(\bar{\omega})$ with E_S endows E_S with a Frobenius map which then has to be of the form uF with $u \in (I^{[p^e]} : I)$, [13, see Proposition 4.1].*

In general if we start with an $R[\theta; f]$-module M, we can consider M as an $R[\theta_e, f^e]$-module where $f^e\colon R \to R$, $f^e(a) = a^{p^e}$, $\theta_e(m) = \theta^e(m)$. In our case, for $M = E_S$ the action of θ_e on E_S is:

$$\theta_e = \underbrace{\theta \circ \cdots \circ \theta}_{e\ times} = (uF)^e = u^{\nu_e} F^e$$

where $\nu_e = 1 + p + \cdots + p^{e-1}$ when $e > 0$ and $\nu_0 = 0$. Therefore when we apply the Δ^e-functor to $E_S \in \mathcal{C}^e$ we obtain the map

$$R/I \overset{u^{\nu_e}}{\to} R/I^{[p^e]}.$$

Theorem 3.2.5. *Let ω be the preimage of $\bar{\omega}$ in R. The R-module consisting of all e^{th}-Frobenius maps acting on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ is isomorphic to*

$$\mathcal{F}^e = \frac{\left(I^{[p^e]} : I\right) \cap \left(\omega^{[p^e]} : \omega\right)}{I^{[p^e]}}$$

where the isomorphism is given by $u + I^{[p^e]} \mapsto uF$.

Proof. By Lemma 3.2.3 we can rewrite (3) as

$$0 \to \mathrm{Ann}_{E_S}(\bar{\omega}) \to \mathrm{Ann}_{E_R}(I) \to \mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S) \to 0. \tag{4}$$

Apply the Δ^e-functor to the latter short exact sequence. When we apply it to $E_S = \mathrm{Ann}_{E_R}(I)$ and $\mathrm{Ann}_{E_S}(\omega)$ we obtain respectively $\Delta^e(\mathrm{E}_S) = R/I \overset{u^{\nu_e}}{\to} R/I^{[p^e]}$ and $\Delta^e(\mathrm{Ann}_{E_S}(\omega)) = R/\omega \to R/\omega^{[p^e]}$. Thus the inclusion

$\mathrm{Ann}_{E_S}(\omega) \to E_S$ yields the diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S))^{\vee} \arrow[r] \arrow[d] & R/I \arrow[r] \arrow[d, "u^{\nu_e}"'] & R/\omega \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^e_R(\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S))^{\vee} \arrow[r] & R/I^{[p^e]} \arrow[r] & R/\omega^{[p^e]} \arrow[r] & 0.
\end{tikzcd}$$

Now, we can identify $(\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S))^{\vee}$ with ω/I and $F^e_R(\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S))^{\vee}$ with $\omega^{[p^e]}/I^{[p^e]}$. Therefore when we apply Δ^e to the sequence (4) we obtain the short exact sequence in $\mathcal{D}^e$:

$$\begin{tikzcd}
0 \arrow[r] & \bar{\omega}/I \arrow[r] \arrow[d] & R/I \arrow[r] \arrow[d, "u^{\nu_e}"'] & R/\omega \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bar{\omega}^{[p^e]}/I^{[p^e]} \arrow[r] & R/I^{[p^e]} \arrow[r] & R/\omega^{[p^e]} \arrow[r] & 0
\end{tikzcd}$$

where the central vertical map is the multiplication by u^{ν_e}. The only way to make the diagram above commutative is that the other two vertical maps are also the multiplication by u^{ν_e}. It follows that $u \in \left(I^{[p^e]} \colon I\right) \cap \left(\omega^{[p^e]} \colon \omega\right)$. Finally consider the surjection $\varphi\colon \left(I^{[p^e]} \colon I\right) \cap \left(\omega^{[p^e]} \colon \omega\right) \to \mathcal{F}^e(\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S))$; $u \in \operatorname{Ker}\varphi$ if and only if $u\colon \frac{\bar{\omega}}{I} \to \frac{\bar{\omega}^{[p^e]}}{I^{[p^e]}}$ is the zero map which happens if and only if $u\bar{\omega} \subset I^{[p^e]} \subseteq I$ i.e. $u\omega = 0$. $\bar{\omega}$ contains a non-zero-divisor and since $\bigcup \mathrm{Ass}(I) = \bigcup \mathrm{Ass}(I^{[p^e]})$ then $\bar{\omega}$ contains a non-zero-divisor modulo $I^{[p^e]}$, say x. So $ux \in I^{[p^e]}$ implies $u \in I^{[p^e]}$. Therefore $\operatorname{Ker}\varphi = I^{[p^e]}$. □

3.3 HSL Numbers in the Local Case

In this section we give an explicit formula for the HSL-numbers.

Theorem 3.3.1. $\mathrm{HSL}(\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S))$ *is the smallest integer e for which*

$$\frac{I_e(u^{\nu_e}\omega)}{I_{e+1}(u^{\nu_{e+1}}\omega)} = 0$$

where ω is the preimage of $\bar{\omega}$ in R and $\nu_e = 1 + p + \cdots + p^{e-1}$ when $e > 0$, and $\nu_0 = 0$ and where we set $I_0(J) = J$ for all ideals J.

Proof. For all $e \geq 0$ define $M_e = \left\{x \in \mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S) | \theta^e x = 0\right\}$ and note that $\{M_e\}_{e\geq 0}$ form an ascending sequence of $R[\theta; f^e]$-submodules of $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ that stabilises by Theorem 2.10.3. Consider the short exact sequence of $R[\theta; f^e]$-modules $0 \to \mathrm{H}^{\mathrm{d}-1}_{\mathfrak{m}S}(S/\bar{\omega}) \to \mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(\bar{\omega}) \to \mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S) \to 0$ where the action of θ on $E_S = \mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(\bar{\omega})$ is given by $u^{\nu_e}F$ where F is the natural Frobenius on E_R. We have seen we can write this sequence as

$$0 \to \mathrm{Ann}_{E_S}(\bar{\omega}) \to \mathrm{Ann}_{E_R}(I) \to \mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S) \to 0.$$

It follows that

$$\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S) \cong \frac{\mathrm{Ann}_{E_R}(I)}{\mathrm{Ann}_{E_S}(\bar{\omega})}.$$

Since each M_e is a submodule of $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ then it is of the form $\frac{\mathrm{Ann}_{E_S}(L_e)}{\mathrm{Ann}_{E_S}(\bar{\omega})}$ for some ideals $L_e \subseteq R$ contained in I. Apply the Δ^e-functor to the inclusion $M_e \hookrightarrow \mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ to obtain

$$\begin{tikzcd}
\omega/I \arrow[r, two heads] \arrow[d, "u^{\nu_e}"'] & \omega/L_e \arrow[d] \\
\omega^{[p^e]}/I^{[p^e]} \arrow[r, two heads] & \omega^{[p^e]}/L_e^{[p^e]}
\end{tikzcd}$$

where the map $\omega/I \to \omega^{[p^e]}/I^{[p^e]}$ is the multiplication by u^{ν_e} by Remark 3.2.4. It follows that the map $\omega/L_e \to \omega^{[p^e]}/L_e^{[p^e]}$ must be the multiplication by u^{ν_e} because of the surjectivity of the horizontal maps; note that such a map is well defined because $u^{\nu_e}\omega \subseteq \omega^{[p^e]}$, and then $L_e \subseteq \omega$. Moreover $\omega/L_e \to \omega^{[p^e]}/L_e^{[p^e]}$ must be a zero-map by construction of Δ^e. Hence, $u^{\nu_e}\omega \subseteq L_e^{[p^e]}$ and for every L_e with $u^{\nu_e}\omega \subseteq L_e^{[p^e]}$ the action of θ on M_e is zero. We want the largest $M_e \subset \mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ for which θ^e acts as zero. The largest module $\frac{\mathrm{Ann}_{E_S}(L_e)}{\mathrm{Ann}_{E_S}(\bar{\omega})}$ killed by θ^e corresponds to the smallest L_e such that $u^{\nu_e}\omega \subseteq L_e^{[p^e]}$ i.e. $L_e = I_e(u^{\nu_e}\omega)$. $\square$

Corollary 3.3.2. *S is F-injective if and only if $\omega = I_1(u\omega)$.*

Proof. S is F-injective if and only if the index of nilpotency is zero i.e. if and only if $\omega = I_1(u\omega)$. $\square$

3.4 HSL **Loci**

Let A be a polynomial ring $\mathbb{K}[x_1, \cdots, x_n]$ with coefficients in a perfect field $\mathbb{K}$ of positive characteristic p and let M be a finitely generated A-module generated by $g_1, \cdots, g_s$. Let $e_1, \cdots, e_s$ be the canonical basis for A^s and define the map

$$\begin{aligned} A^s &\xrightarrow{\varphi} M \\ e_i &\longmapsto g_i. \end{aligned}$$

φ is surjective and extends naturally to an A-linear map $J\colon A^t \to A^s$ with $\ker\varphi = \operatorname{Im} J$. Let J_i be the matrix obtained from $J \in \mathrm{Mat_{s,t}(A)}$ by erasing the i^{th}-row. With this notation we have the following:

Lemma 3.4.1. *M is generated by g_i if and only if $\operatorname{Im} J_i = A^{s-1}$.*

Proof. Firstly suppose $\operatorname{Im} J_i = A^{s-1}$. We can add to J, columns of $\operatorname{Im} J$ without changing its image so we can assume that J contains the elementary vectors $e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_s$:

$$J = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & 1 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i,1} & a_{i,2} & \cdots & a_{i,n} & b_1 & b_2 & \cdots & b_s \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m-1,1} & a_{m-1,2} & \cdots & a_{m-1,n} & 0 & \cdots & 1 & 0 \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

In this way for every $j \neq i$ we have $g_j - b_j g_i = 0$ i.e. g_i generates M. Viceversa if M is generated by g_i then for all $j \neq i$ we can write $g_j = r_j g_i$ i.e. $g_j - r_j g_i = 0$ and the relation $g_j - r_j g_i$ gives a relation $e_j - r_j e_i$ in the image of J, so $e_j - r_j e_i \in \operatorname{Ker} \varphi = \operatorname{Im} J$. Hence we can assume that J contains a column whose entries are all zeros but in the i-th and j-th positions where there is 1 and r_j respectively. Consequently J_i contains the $(s-1) \times (s-1)$ identity matrix. $\square$

Let W be a multiplicatively closed subset of A. Localise the exact sequence $A^t \to A^s \to M \to 0$ with respect to W obtaining the exact sequence $W^{-1}A^t \to W^{-1}A^s \to W^{-1}M \to 0$. With this notation we have:

Proposition 3.4.2. *$W^{-1}M$ is generated by $\frac{g_i}{1}$ if and only if $W^{-1}J_i = (W^{-1}A)^{s-1}$*

Proof. Apply Lemma 3.4.1 to the localised sequence $W^{-1}A^t \to W^{-1}A^s \to W^{-1}M \to 0$. $\square$

Proposition 3.4.2 is equivalent to saying that $W^{-1}M$ is generated by $\frac{g_i}{1}$ if and only if the intersection of W with the ideal of $(s-1)\times(s-1)$ minors of J_i is not trivial. So we have the following;

Corollary 3.4.3. *Let M be a finitely generated A-module and let $g_1, \cdots, g_s$ be a set of generators for M. If M is locally principal, then for each $i = 1, \cdots, s$*

$$\mathcal{G}_i = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid M\widehat{A}_{\mathfrak{p}} \text{ is generated by the image of } g_i\}$$

is a Zariski open set and $\cup_i \mathcal{G}_i = \operatorname{Spec}(A)$. Moreover, $\mathcal{G}_i = V(J_i)^c$ for every $i = 1, \cdots, s$.

Proof. $\mathfrak{p} \in \mathcal{G}_i$ if and only if $\mathfrak{p} \not\supseteq J_i$. $\square$

Note that Corollary 3.4.3 gives a description of $\mathcal{G}_i$ in terms of minors of the matrix J_i. This description will be used to implement the algorithm in Section 3.5.

In the rest of this section let $J \subset A$ be an ideal of A and let B be the quotient ring A/J; we saw that if B is Cohen-Macaulay of dimension d, then $\bar{\Omega} = \operatorname{Ext}_A^{\dim A - d}(B, A)$ is a global canonical module for B. If B is generically Gorenstein then we can assume $\bar{\Omega} \subseteq B$. Let Ω be the preimage of $\bar{\Omega}$ in A; then the following B-module is well defined:

$$\mathcal{U}_{(e)} = \frac{\left(J^{[p^e]} : J\right) \cap \left(\Omega^{[p^e]} : \Omega\right)}{J^{[p^e]}}. \tag{5}$$

Since A is Noetherian, $\mathcal{U}_{(e)}$ is a finitely generated A-module (and B-module). For every prime ideal $\mathfrak{p} \subset A$ write $H_{\mathfrak{p}} = \mathrm{H}^{\dim \widehat{B}_{\mathfrak{p}}}_{\mathfrak{p}\widehat{B}_{\mathfrak{p}}}(\widehat{B}_{\mathfrak{p}})$. It follows from Theorem 3.2.5 that the A-module $\mathcal{F}^e(H_{\mathfrak{p}})$ consisting of the Frobenius maps on $H_{\mathfrak{p}}$ is of the form:

$$\mathcal{F}^e(H_{\mathfrak{p}}) = \frac{\left(J^{[p^e]}\widehat{A}_{\mathfrak{p}} : J\widehat{A}_{\mathfrak{p}}\right) \cap \left(\Omega^{[p^e]}\widehat{A}_{\mathfrak{p}} : \Omega\widehat{A}_{\mathfrak{p}}\right)}{J^{[p^e]}\widehat{A}_{\mathfrak{p}}} \tag{6}$$

and consequently $\mathcal{F}^e(H_{\mathfrak{p}}) \cong \mathcal{U}_{(e)}\widehat{A}_{\mathfrak{p}}$. Since $\mathcal{F}^e(H_{\mathfrak{p}})$ is generated by one element by Theorem 3.2.1, $\mathcal{U}_{(e)}\widehat{A}_{\mathfrak{p}}$ is principal as well.

From Corollary 3.4.3 with $M = \mathcal{U}_{(e)}$ it follows that for every prime ideal $\mathfrak{p} \in \operatorname{Spec}(A) = \bigcup_i \mathcal{G}_i$ there exists an i such that $\mathfrak{p} \in \mathcal{G}_i$ and the A-module $\mathcal{U}_{(e)}\widehat{A}_{\mathfrak{p}}$ is generated by one element which is precisely the image of g_i.

We saw in Theorem 3.2.1 that if a ring S is S_2 then the module of Frobenius maps acting on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ is principal. The following example given by Karl Schwede shows that if a ring S is not S_2 then the module of Frobenius maps acting on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ is not necessarily generated by one element. All assertions in this example are based on calculations carried out with [7].

Example 3.4.4. *Let R be the polynomial ring $\mathbb{Z}_2[a, b, c, d]$ and let I be the ideal generated by $ac + bd$, $b^3 + c^2$, $ab^2 + cd$ and $a^2b + d^2$. The quotient ring $S = R/I$ has dimension 2 and is a domain as I is a prime ideal. Because S is not Cohen-Macaulay then S is not S_2. A canonical module for S is isomorphic to the ideal $\omega = (d, a, b^3+c^2)$ of S. The R-module of the Frobenius* *maps on* $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ *defined in (5) as*

$$\mathcal{U}_{(1)} = \frac{\left(\Omega^{[2]} : \Omega\right) \cap \left(I^{[2]} : I\right)}{I^{[2]}}$$

is given by the cokernel of the matrix

$$X = \begin{pmatrix} d & a & 0 & ab & d & 0 & b^3 + c^2 \\ c & b & ab & 0 & 0 & d & 0 \\ 0 & 0 & d & c & b & a & 0 \end{pmatrix}.$$

X is generated by the i-th row if and only if the ideal of the matrix X_i (obtained from X by removing the i-th row) is the unit ideal. It turns out that

$$X_1 = (cd, bd, c^2, d^2, bc, b^2, ac, ab)$$

$$X_2 = (d^2, cd, bd, ac, ab, a^2, b^3c + c^3, b^4 + bc^2)$$

$$X_3 = (d, b + c, bc + c, ac, ab^2, a^2b).$$

Therefore the R-module of Frobenius maps acting on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ is not principal.

With the notation above, we prove our main result.

Theorem 3.4.5. *For every $e > 0$, the set $\mathcal{B}_e = \{\mathfrak{p} \in \operatorname{Spec}(A) | \operatorname{HSL}(H_{\mathfrak{p}}) < e\}$ is Zariski open.*

Proof. Let $u_1, \cdots, u_s$ be a set of generators for $\mathcal{U}_{(e)}$ and write for $i = 1, \cdots, s$

$$\mathcal{G}_i = \left\{ \mathfrak{p} \in \operatorname{Spec}(A) \; | \mathcal{U}_{(e)} \widehat{A}_{\mathfrak{p}} \text{ is generated by } \frac{u_i}{1} \right\}.$$

Define

$$\Omega_{i,e} = \frac{I_e(u_i^{\nu_e}(\Omega))}{I_{e+1}(u_i^{\nu_{e+1}}(\Omega))}$$

then it follows from Proposition 3.1.4 that

$$\frac{I_e(\bar{u}_i^{\nu_e}(\widehat{\Omega}_{\mathfrak{p}}))}{I_{e+1}(\bar{u}_i^{\nu_{e+1}}(\widehat{\Omega}_{\mathfrak{p}}))} = (\widehat{\Omega}_{i,e})_{\mathfrak{p}}$$

for every prime ideal $\mathfrak{p}$. Note that for every $i = 1, \cdots, s$ the set $\mathrm{Supp}(\Omega_{i,e})^C = \{\mathfrak{p} \quad | \quad (\widehat{\Omega}_{i,e})_{\mathfrak{p}} = 0\}$ is open.

If $\mathfrak{p}$ is such that $\mathrm{HSL}(H_{\mathfrak{p}}) < e$ then $\mathfrak{p} \in \mathcal{G}_i$ for some i; we can then use $\bar{u}_i$ to compute $\Omega_{i,e}$ and $(\widehat{\Omega}_{i,e})_{\mathfrak{p}} = 0$ i.e. $\mathfrak{p} \in \mathrm{Supp}(\Omega_{i,e})^C$; therefore

$$\mathfrak{p} \in \bigcup_{i=1}^{s} \left(\mathrm{Supp}(\Omega_{i,e})^C \cap \mathcal{G}_i\right).$$

Viceversa, let $\mathfrak{p} \in \bigcup_i (\mathrm{Supp}(\Omega_{i,e})^C \cap \mathcal{G}_i)$ then $\mathfrak{p} \in \mathrm{Supp}(\Omega_{j,e})^C \cap \mathcal{G}_j$ for some j. Compute $\mathrm{HSL}(H_{\mathfrak{p}})$ using u_j. Since $\mathfrak{p} \in \mathrm{Supp}(\Omega_{j,e})^C$ then $(\widehat{\Omega}_{j,e})_{\mathfrak{p}} = 0$ and so $\mathrm{HSL}(H_{\mathfrak{p}}) < e$. In conclusion

$$\{\mathfrak{p} \in \mathrm{Spec}(A) |\, \mathrm{HSL}(H_{\mathfrak{p}}) < e\} = \bigcup_{i=1}^{s} \left(\mathrm{Supp}(\Omega_{i,e})^C \cap \mathcal{G}_i\right)$$

and therefore $\mathcal{B}_e$ is Zariski open. □

Corollary 3.4.6. $\sup\{\mathrm{HSL}(H_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Spec}(S)\} < \infty$.

For $e = 1$ we have the following.

Corollary 3.4.7. *The F-injective locus is open.*

3.5 The Computation of the HSL Loci

In the case where a ring $S = R/I$ of positive characteristic p is a Cohen-Macaulay domain we have an explicit algorithm to compute the F-injective locus $\mathcal{B}_e$ of S for every positive integer e.

With the same notation as in Theorem 3.4.5 we have that $\mathcal{B}_e = \bigcup_{i=1}^{s}(\mathrm{Supp}(\Omega_{i,e})^C \cap \mathcal{G}_i)$. Because $\mathrm{Supp}(\Omega_{i,e}) = V(\mathrm{Ann}_R(\Omega_{i,e}))$ and the sets $\mathcal{G}_i$ are of the form $V(K_i)^c$ for some ideals $K_1, \cdots, K_s \subset R$, we can then write $\mathcal{B}_e$ as

$$\begin{aligned}
&\bigcup_i V(\mathrm{Ann}_R(\Omega_{i,e}))^c \cap V(K_i)^c = \\
&\bigcup_i (V(\mathrm{Ann}_R(\Omega_{i,e})) \cup V(K_i))^c = \\
&\bigcup_i V(\mathrm{Ann}_R(\Omega_{i,e})K_i)^c = \\
&\Big(\bigcap_i V(\mathrm{Ann}_R(\Omega_{i,e})K_i)\Big)^c = \\
&\left[V\left(\sum_i \mathrm{Ann}_R(\Omega_{i,e})K_i\right)\right]^c.
\end{aligned}$$

Therefore, given a positive integer e and a Cohen-Macaulay domain S, an algorithm to find the locus $\mathcal{B}_e$ can be described as follows.

1. Compute a canonical module for S, then find an ideal $\Omega \subseteq S$ which is isomorphic to it.

2. Find the R-module of the Frobenius maps on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ defined in (5) as

$$\mathcal{U}_{(e)} = \frac{\left(\Omega^{[p^e]} : \Omega\right) \cap \left(I^{[p^e]} : I\right)}{I^{[p^e]}}$$

as the cokernel of a matrix $X \in \mathrm{Mat_{s,t}(R)}$.

3. Find the generators $u_1, \cdots, u_s$ of $\mathcal{U}_{(e)}$.

4. Compute the ideals K_i's of the $(s-1) \times (s-1)$-minors of X.

5. For every generator u_i, compute the ideal $\Omega_{i,e} = I_e(u_i^{\nu_e}\Omega)/I_{e+1}(u_i^{\nu_{e+1}}\Omega)$.

6. Compute $\mathcal{B}_e$ as $\left[V(\sum_i \mathrm{Ann}_R(\Omega_{i,e})K_i)\right]^c$.

We now make use of the algorithm above to compute the loci in an example. The algorithm has been implemented in Macaulay2 [7].

Example 3.5.1. *Let R be the polynomial ring $\mathbb{Z}_2[x_1, \cdots, x_5]$ and let I be the ideal $I = (x_2^2+x_1x_3, x_1x_2x_4^2+x_3^3x_5, x_1^2x_4^2+x_2x_3^2x_5)$. The quotient ring $S = R/I$ is a domain because I is prime and it is Cohen-Macaulay of type 2 so it is not Gorenstein. A canonical module for S is given by $\mathrm{Ext}^{\dim R - \dim S}(S, R)$ and can be produced as the cokernel of the matrix*

$$\begin{pmatrix} x_2 & x_1 & x_3^2x_5 \\ x_3 & x_2 & x_1x_4^2 \end{pmatrix}$$

which is isomorphic to an ideal Ω which is the image in S of the ideal $(x_2, x_1, x_3^3x_5)$ in R.

The R-module $\mathcal{U}_{(e)}$ of the Frobenius maps on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ turns out to be given by the cokernel of the one-row matrix

$$X = \begin{pmatrix} x_2^2 + x_1x_3 & x_1x_2x_4^2 + x_3^3x_5 & x_1^2x_4^2 + x_2x_3^2x_5 \end{pmatrix}$$

whose generator is $u = x_1^2x_2^2x_4^2 + x_1^3x_3x_4^2 + x_2^3x_3^2x_5 + x_1x_2x_3^3x_5$. *Since* X *has only one row then the computation of* $\mathcal{B}_e$ *reduces to* $\mathcal{B}_e = \left[V(\mathrm{Ann}_R(\Omega_e)\right]^c$. *It turns out that* $I_1(u^{\nu_1}\Omega) = (x_1x_4, x_2x_3, x_1x_3, x_3^3x_5, x_2^2)$, $I_2(u^{\nu_2}\Omega) = ((x_1x_4, x_2x_3, x_1x_3 + x_2^2, x_3^3x_5)) = I_3(u^{\nu_3}\Omega)$. *Consequently, being* $\Omega_0 = \frac{\Omega}{I_1(u^{\nu_1}(\Omega))}$ *and* $\Omega_1 = \frac{I_1(u^{\nu_1}(\Omega))}{I_2(u^{\nu_2}(\Omega))}$, *we have* $\mathcal{B}_0 = V(x_1, x_2, x_3^2)^c \cup V(x_1, x_2, x_5)^c \cup V(x_2^2, x_3, x_4)^c$, $\mathcal{B}_1 = V(x_2, x_3, x_4)^c$ *and* $\mathcal{B}_e = V(1)^c$ *for every* $e > 1$. *In other words, the* HSL*-number can be at the most 2. More precisely, if we localise* S *at a prime that does not contain the prime ideal* $(x_1, x_2, x_3^2) \cap (x_1, x_2, x_5) \cap (x_2^2, x_3, x_4)$ *then we get an* F*-injective module. Outside* (x_2, x_3, x_4) *the* HSL*-number is less or equal to 1; On* $V(x_2, x_3, x_4)$ *the* HSL *number is exactly 2.*

3.6 An Application: F-injectivity and F-purity

Definition 3.6.1 (F-pure)**.** *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of equal characteristic* $p > 0$. *A map of* R*-modules* $N \to \tilde{N}$ *is* pure *if for every* R*-module* M *the map* $N \otimes_R M \to \tilde{N} \otimes_R M$ *is injective. A local ring (R,m) is called* F-pure *if the Frobenius endomorphism* $F\colon R \to R$ *is pure.*

We note that F-pure implies F-injective [11] and the converse holds if R is Gorenstein [6, Lemma 3.3]. The following is an example constructed by Fedder in [6, Example 4.8] that is an example of ring which is F-injective but not F-pure.

Example 3.6.2. *Let* R *be the local ring* $\mathbb{K}[x_1, \cdots, x_5]_{\mathfrak{m}}$, *where* $\mathfrak{m}$ *is the (maximal) ideal generated by the variables* $x_1, \cdots, x_5$ *and* $\mathbb{K}$ *is a field of charac-*

teristic p. Let I be the ideal of two by two minors of the matrix

$$\begin{pmatrix} x_1^n & x_3 & x_5 \\ x_4 & x_3 & x_2^n \end{pmatrix}.$$

If $p \leq n$ then R/I is F-injective but not F-pure.

We now revisit Theorem 1.1 from [25] which gives some other examples of rings which are not F-pure. Our algorithm shows that for some p and n these rings are F-injective.

Example 3.6.3. *Let R be the polynomial ring $\mathbb{K}[x_1, \cdots, x_5]$ and $J_{\{p,n,m\}}$ the ideal generated by the size two minors of the matrix*

$$\begin{pmatrix} x_1^2 + x_5^m & x_2 & x_4 \\ x_3 & x_1^2 & x_2^n - x_4 \end{pmatrix}$$

where m and n are positive integers satisfying $m - m/n > 2$ and p is the characteristic of the field $\mathbb{K}$; Singh proved in [25, Theorem 1.1] that if p and m are coprime integers then $S_{\{p,n,m\}} := R/J_{\{p,n,m\}}$ is not F-pure. Our algorithm implemented in Macaulay2 [7] shows that in the following cases $S_{\{p,n,m\}}$ are F-injective.

- $p = 3$, $n = 2$, $m = 7, \cdots, 1000$;
- $p = 5$, $n = 2$, $m = 6, \cdots, 100$;
- $p = 7$, $n = 2$, $m = 6, \cdots, 100$;
- $p = 11$, $n = 2$, $m = 6, \cdots, 100$;

- $p = 3$, $n = 3$, $m = 8$.

3.7 An Application: Test Exponents for Frobenius Closures and HSL Numbers

Let S be a ring of characteristic p and $J \subseteq S$ an ideal.

Definition 3.7.1. *The* Frobenius closure *of J is the ideal*

$$J^F = \left\{a \in S \mid a^{p^e} \in J^{[p^e]} \textit{ for some } e > 0\right\}.$$

Note that if $a^{p^{\bar{e}}} \in J^{[p^{\bar{e}}]}$ then $a^{p^e} \in J^{[p^e]}$ for every $e \geq \bar{e}$.

Let $g_1, \cdots g_n$ be a set of generators for J^F. For each generator g_i let e_i be the integer such that $g_i^{p^{e_i}} \in J^{p^{e_i}}$. If we then choose $\bar{e} = \max\{e_1, \cdots, e_n\}$ then $(J^F)^{[p^{\bar{e}}]} \subseteq J^{[p^{\bar{e}}]}$. We say that $\bar{e}$ is a *test exponent for the Frobenius closure of J.*

With the notation introduced in Section 2.2, we have the following.

Theorem 3.7.2. *[14, Theorem 2.5] Let $(S, \mathfrak{m})$ be a local, Cohen-Macaulay ring and let $\underline{x} = x_1, \cdots, x_d$ be a system of parameters. Then the test exponent for the ideal $(\underline{x})$ is $\bar{e} = \mathrm{HSL}(\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S))$.*

Proof.

$$\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S) = \varinjlim_{t} \left(\frac{S}{\underline{\mathrm{x}}} \xrightarrow{x_1 \cdots x_n} \cdots \xrightarrow{x_1 \cdots x_n} \frac{S}{\underline{\mathrm{x}}^t} \xrightarrow{x_1 \cdots x_n} \cdots \right)$$

has a natural Frobenius action T which we can define on a typical element of the direct limit as

$$T[a + \underline{\mathrm{x}}^t] = a^p + \underline{\mathrm{x}}^{pt}.$$

Therefore $a^{p^e} \in \underline{x}^{p^e}$ if and only if $T^e[a + \underline{x}^t] = 0$ i.e. $[a + \underline{x}^t]$ is nilpotent and we can take $\bar{e} = \mathrm{HSL}(\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S))$. $\square$

Corollary 3.7.3. *Let S be the quotient of a polynomial ring and let ϵ be the bound for $\{\mathrm{HSL}(\mathrm{H}^{dimS}_{\mathfrak{m}}(S)) \mid \mathfrak{m}$ is maximal$\}$. If $J \subseteq S$ is locally a parameter ideal (i.e. for every maximal ideal $\mathfrak{m} \supseteq J$, $J_{\mathfrak{m}}$ is a parameter ideal) then $(J^F)^{[p^\epsilon]} = J^{p^\epsilon}$.*

4 The non Cohen-Macaulay Case

In this chapter we generalise the results proved in Chapter 3 to any complete ring by proving that for all $e > 0$, and for all $j \geq 0$ the sets

$$V_{e,j} = \left\{ \mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{HSL}(H^j_{\mathfrak{p}}(R_{\mathfrak{p}})) < e \right\}$$

are Zariski open. Given any local ring $(R, \mathfrak{m})$ (not necessarely Cohen-Macaulay) of dimension d, we define ω to be a canonical module for R if ω is an R-module such that $\omega^\vee \cong \mathrm{H}^d_{\mathfrak{m}}(R)$ (as in the Cohen-Macaulay case).

We will tackle the problem starting by generalising the operator $I_e(-)$ which we have defined in Section 3.1. We will then find a different method to describe the Frobenius action on $\mathrm{H}^i_{\mathfrak{m}}(R)$ making use of the following result proved by G. Lyubeznik.

Proposition 4.0.4 (see Section 2 [19]). *Let $(R, \mathfrak{m})$ be a local ring and $I \subset R$ an ideal and write $S = R/I$. Consider $\mathrm{H}^i_{\mathfrak{m}}(S)$ as an $R[T; f^e]$-module where T is the natural Frobenius map. Write $\delta = \dim R$.*
Then $\Delta^e(\mathrm{H}^i_{\mathfrak{m}}(S))$ is isomorphic to the map

$$\operatorname{Ext}^{\delta-j}_R(R/I, R) \to \operatorname{Ext}^{\delta-j}_R(R/I^{[p]}, R)$$

induced by the surjection $R/I^{[p]} \to R/I$.

Finally we will give an algorithm to compute the HSL-loci in the general case and give some examples.
Although this new method works for every complete ring, the new algorithm

is computationally much slower than the one described in Section 3.5.

4.1 Generalisation of the Operator $I_e(-)$

In this section we generalise the $I_e(-)$ operator, which we have previously defined on ideals, to submodules of free modules. Also, we prove that it commutes with completion and localisation as in Proposition 3.1.4.
We recall that if R is a ring and A is a matrix with entries in R then $A^{[p^e]}$ indicates the matrix obtained from A by raising its entries to the p^e-th power.

Definition 4.1.1. *Let $M \subseteq R^\alpha$ be an R-submodule. We define $M^{[p^e]}$ to be the R-submodule of R^α generated by $\{m^{[p^e]} \mid m \in M\}$.*

Definition 4.1.2. *A ring R is* intersection-flat *if for all $e \geq 0$ and for any family $\{M_\lambda\}$ of R-modules,*

$$F_*^e R \otimes_R \bigcap_\lambda M_\lambda = \bigcap_\lambda F_*^e R \otimes_R M_\lambda.$$

Note that polynomial rings or power series rings with coefficients in an F-finite ring, and complete regular rings are intersection-flat, see [13, Proposition 5.3]. We will assume in this section that R is a regular ring intersection flat for all $e \geq 0$.

Definition 4.1.3. *Let M be a submodule of R^α. We define $I_e(M)$ to be the smallest submodule L of R^α with the property that $M \subseteq L^{[p^e]}$.*

Theorem 4.1.4. *Let R be regular and intersection-flat, then $I_e(M)$ exists for every submodule $M \subseteq R^\alpha$.*

Proof. We want to prove that there exists a submodule L of R^α with the property that L is the smallest submodule with the property that $M \subseteq L^{[p^e]}$. Let L be the intersection of all the submodules L_λ of R^α such that their Frobenius p^e-th power contains M. Since R is regular, we can identify $F^e_*(L)$ and $L^{[p^e]}$ and from the intersection-flatness it follows that $L^{[p^e]} = \bigcap_\lambda L_\lambda^{[p^e]}$. Because M is contained in each $L_\lambda^{[p^e]}$ then $M \subseteq L^{[p^e]}$ and L is minimal with this property. □

Lemma 4.1.5. *Let M be a submodule of $\widehat{R}^\alpha$. Then $M^{[p^e]} \cap R^\alpha = (M \cap R^\alpha)^{[p^e]}$.*

Proof. G. Lyubeznik and K. E. Smith proved in [17, Lemma 6.6] that this is true for local rings. For a non-local ring R, $M^{[p^e]} \cap R^\alpha_{\mathfrak{p}} = (M \cap R^\alpha_{\mathfrak{p}})^{[p^e]}$ for every prime ideal $\mathfrak{p}$. Intersecting with R^α we get:

$$M^{[p^e]} \cap R^\alpha = (M \cap R^\alpha_{\mathfrak{p}})^{[p^e]} \cap R^\alpha = (M \cap R^\alpha_{\mathfrak{p}} \cap R^\alpha)^{[p^e]} = (M \cap R^\alpha)^{[p^e]}.$$

□

We generalise Lemma 3.1.3 with the following:

Lemma 4.1.6. *Let W be a multiplicatively closed subset of R. For any submodule M of R^α the following identity holds:*

$$I_e(W^{-1}M) = W^{-1}I_e(W^{-1}M \cap R^\alpha).$$

Proof. Let $L \subseteq W^{-1}R^\alpha$ be a submodule such that $W^{-1}M \subseteq L^{[p^e]}$. Then $L^{[p^e]} \cap R^\alpha = (L \cap R^\alpha)^{[p^e]}$. Since $W^{-1}M \cap R^\alpha \subseteq L^{[p^e]} \cap R^\alpha$ then $I_e(W^{-1}M \cap$

$R^\alpha) \subseteq L \cap R^\alpha$ and so $W^{-1}I_e(W^{-1}M \cap R^\alpha) \subseteq W^{-1}(L \cap R^\alpha) \subseteq L$. Moreover,

$$\begin{aligned} W^{-1}M &\subseteq W^{-1}(W^{-1}M \cap R^\alpha) \\ &\subseteq W^{-1}I_e(W^{-1}M \cap R^\alpha)^{[p^e]} \\ &= (W^{-1}I_e(W^{-1}M \cap R^\alpha))^{[p^e]} \end{aligned}$$

so $W^{-1}I_e(W^{-1}M \cap R^\alpha)$ is the smallest submodule $K \subseteq W^{-1}R^\alpha$ for which $W^{-1}M \subseteq K^{[p^e]}$. Hence $I_e(W^{-1}M) = W^{-1}I_e(W^{-1}M \cap R^\alpha)$. □

Theorem 4.1.7. $I_e(-)$ *commutes with localisation and completion.*

Proof. Let W be a multiplicatively closed subset of R. For any submodule M of R^α we have $M \subseteq W^{-1}M \cap R^\alpha$ so $W^{-1}I_e(M) \subseteq W^{-1}I_e(W^{-1}M \cap R^\alpha)$ and the latter term is the same as $I_e(W^{-1}M)$ for lemma 4.1.6. So we only need to prove the inclusion $I_e(W^{-1}M) \subseteq W^{-1}I_e(M)$. Since $M \subset I_e(M)^{[p^e]}$ then $W^{-1}M \subset W^{-1}I_e(M)^{[p^e]}$ and $W^{-1}I_e(M)^{[p^e]} = (W^{-1}I_e(M))^{[p^e]}$ because of 4.1.5; since $I_e(W^{-1}M)$ is the smallest such that its p^e-th Frobenius power contains $W^{-1}M$ then we have $I_e(W^{-1}M) \subseteq W^{-1}I_e(M)$ i.e. $I_e(-)$ commutes with localisation.

Using Lemma 4.1.5, the same argument proves that $I_e(-)$ commutes with completion. □

4.2 The Frobenius Action on $E_R^\alpha(\mathbb{K})$

We have seen in Example 1.2.1 and Example 2.8.5 that if R is a regular local ring then R is isomorphic to a power series ring $\mathbb{K}[[x_1, \cdots, x_n]]$ for some field $\mathbb{K}$ of characteristic p and that the injective hull E_R is then isomorphic to the module of inverse polynomials $\mathbb{K}[x_1^-, \cdots, x_n^-]$ which has a Frobenius map given by $F\lambda x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \lambda^p x_1^{p\alpha_1} \cdots x_n^{p\alpha_n}$ for all $\alpha_1, \cdots, \alpha_n < 0$.

Let F be the natural Frobenius map on E_R; for any $\alpha \times \alpha$-matrix U we can define a Frobenius map $\Theta_{U,e}$ on E_R given by

$$\Theta_{U,e} \begin{pmatrix} z_1 \\ \vdots \\ z_\alpha \end{pmatrix} = U^t \begin{pmatrix} F^e z_1 \\ \vdots \\ F^e z_\alpha \end{pmatrix}.$$

Theorem 4.2.1. *Let M be an Artinian $R[\Theta; f^e]$-module. Then M can be embedded into E_R^α for some $\alpha > 1$ and the Frobenius action on M is given by $\Theta_{U,e}$ for some $\alpha \times \alpha$-matrix U. Furthermore, M is isomorphic to $\operatorname{Ann}_{E_R^\alpha}(A^t)$ for some $\beta \times \alpha$-matrix A and the Frobenius action on $\operatorname{Ann}_{E_R^\alpha}(A^t)$ is the restriction of the action on E_R^α to $\operatorname{Ann}_{E_R^\alpha}(A^t)$.*

Proof. Let M be an Artinian $R[\Theta; f^e]$-module, then M can be embedded into E_R^α for some $\alpha \geq 1$. An application of the Matlis dual to the inclusion $M \subseteq E_R^\alpha$ gives the map $R^\alpha \to M^\vee = \operatorname{Coker} A$ and if we apply the Δ^e-functor to M we obtain the map $\operatorname{Coker} A \to \operatorname{Coker} A^{[p^e]}$ and this must be given by the multiplication by some $\alpha \times \alpha$-matrix U. If we make E_R^α into an $R[\Theta; f^e]$-module by taking $\Theta = \Theta_{U,e}$ then we get the following commutative diagram

$$\begin{tikzcd} R^\alpha \arrow[r] \arrow[d, "U"] & \operatorname{Coker} A \arrow[d, "U"] \\ R^\alpha \arrow[r] & \operatorname{Coker} A^{[p^e]}. \end{tikzcd}$$

Note that U must satisfy $\operatorname{Im} UA \subseteq \operatorname{Im} A^{[p^e]}$.

Apply now the Ψ^e-functor to the latter diagram to get the inclusion of $R[\Theta; f^e]$-modules $M \subseteq E_R^\alpha$ where the action of Θ on E_R^α is given by $\Theta_{U,e}$. As a submodule of E_R^α, $M \cong \operatorname{Ann}_{E_R^\alpha}(A^t)$ and the action on $\operatorname{Ann}_{E_R^\alpha}(A^t)$ is given by the restriction of the action of Θ on E^α to $\operatorname{Ann}_{E_R^\alpha}(A^t)$. $\square$

4.3 HSL **Numbers**

In this section let R be a regular local ring and M be an $R[\Theta; f^e]$-module. Using the same notation as in the previous section write $\Delta^1(M) = \operatorname{Coker} A \xrightarrow{U} \operatorname{Coker} A^{[p]}$ where A is an $\alpha \times \beta$-matrix with coefficients in R and U is an $\alpha \times \alpha$-matrix with coefficients in R. Note that M is an $R[\Theta^e; f^e]$-module and $\Delta^e(M) = \operatorname{Coker} A \xrightarrow{U^{[p^{e-1}]} \cdots U^{[p]} U} \operatorname{Coker} A^{[p^e]}$.

For all $e \geq 0$ define $M_e = \{m \in M \mid \Theta^e(m) = 0\}$ and note that $\{M_e\}_{e \geq 0}$ form an ascending sequence of $R[\Theta^e, f^e]$-submodules of M that stabilises by Theorem 2.10.3.

Apply now the Δ^e-functor to the inclusion $M_e \subseteq M$ obtaining the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Coker} A \arrow[r] \arrow[d, "U^{[p^{e-1}]} \cdots U^{[p]} U"'] & \operatorname{Coker} B_e \arrow[d, "U^{[p^{e-1}]} \cdots U^{[p]} U"] \\
\operatorname{Coker} A^{[p^e]} \arrow[r] & \operatorname{Coker} B_e^{[p^e]}.
\end{tikzcd}$$

for some $\alpha \times \gamma$-matrix B_e.

Since $\Delta^e(M_e) = 0$ then the map $\operatorname{Coker} B_e \xrightarrow{U^{[p^{e-1}]} \cdots U^{[p]} U} \operatorname{Coker} B_e^{[p^e]}$ must be zero too. Since for every $e \geq 0$ M_e is the biggest submodule of M which is killed by Θ^e, then B_e must be the matrix with the smallest image which contains the image of A making the latter diagram commute. This means that

$$\operatorname{Im} B_e = \operatorname{Im} A + I_e(\operatorname{Im}(U^{[p^{e-1}]} \cdots U^{[p]}U))$$

and the descending chain $\{B_e\}_{e\geq 0}$ stabilises (when the chain $\{M_e\}_{e\geq 0}$ stabilises too) if and only if $\frac{B_e}{B_{e+1}} = 0$.

We have proved the following.

Theorem 4.3.1. *Let R be a regular local ring and M be an $R[\Theta; f]$-module. If $\Delta^1(M) = \operatorname{Coker} A \xrightarrow{U} \operatorname{Coker} A^{[p]}$ then the* HSL*-number of M is the smallest e such that*

$$\frac{\operatorname{Im} A + I_e(\operatorname{Im}(U^{[p^{e-1}]} \cdots U^{[p]}U))}{\operatorname{Im} A + I_{e+1}(\operatorname{Im}(U^{[p^{e}]} \cdots U^{[p]}U))} = 0.$$

4.4 A New Method for the Computation of the HSL Loci

In this section let R be a polynomial ring over a field of prime characteristic p, let $I \subset R$ be an ideal of R and write $S = R/I$. Note that we are not requiring any further assumptions on the quotient S.

For every ideal $\mathfrak{p} \subset R$ consider the local cohomology module $\mathrm{H}^{\dim \widehat{S}_{\mathfrak{p}}}_{\mathfrak{p}\widehat{S}_{\mathfrak{p}}}(\widehat{S}_{\mathfrak{p}})$ which we will denote for short $H_{\mathfrak{p}}$. Each $H_{\mathfrak{p}}$ is an Artinian $\widehat{R}[\Theta; f]$-module. Using 2.9.3 write $\delta_{\mathfrak{p},j} = \dim R_{\mathfrak{p}} - j$; we have:

$$\Delta^e(H^j_{\mathfrak{p}}) = \operatorname{Ext}^{\delta_{\mathfrak{p},j}}_R(R/I, R) \otimes \widehat{R} \xrightarrow{\varphi_{\mathfrak{p},j}} F(\operatorname{Ext}^{\delta_{\mathfrak{p},j}}_R(R/I, R) \otimes \widehat{R})$$

and $F(\mathrm{Ext}_R^{\delta_{\mathfrak{p},j}}(R/I,R)\otimes\widehat{R})\cong\mathrm{Ext}_R^{\delta_{\mathfrak{p},j}}(R/I^{[p]},R)\otimes\widehat{R}$. Note that the map $\varphi_{\mathfrak{p},j}$ is induced by the surjection $\varphi\colon R/I^{[p]}\to R/I$ by Proposition 4.0.4.

Theorem 4.4.1. *For all $j\geq 0$ fix a presentation $\mathrm{Coker}(R^{\beta_j}\xrightarrow{A_j}R^{\alpha_j})=\mathrm{Ext}_R^j(R/I,R)$ and a matrix $U_j\colon\ \mathrm{Coker}\,A_j\to\mathrm{Coker}\,A_j^{[p]}$ which is isomorphic to the map $\mathrm{Ext}^j(R/I,R)\to\mathrm{Ext}^j(R/I^{[p]},R)$ induced by the surjection $\varphi\colon R/I^{[p]}\to R/I$. For all $j\geq 0$ and $e>0$ write*

$$B_{j,e}=\mathrm{Im}\,A_j+I_e(\mathrm{Im}(U_j^{[p^{e-1}]}\cdots U_j^{[p]}U_j).$$

Then $\mathrm{HSL}(H_{\mathfrak{p}}^j)<e$ if and only if $\mathfrak{p}$ is not in the support of $B_{j,e-1}/B_{j,e}$.

Proof. Note that $\Delta^1(H_{\mathfrak{p}}^j)$ is the completion at $\mathfrak{p}$ of the map $U_j\colon\ \mathrm{Coker}\,A_j\to\mathrm{Coker}\,A_j^{[p]}$ so from Theorem 4.3.1 it follows that $\mathrm{HSL}(H_{\mathfrak{p}})<e$ if and only if

$$\frac{\widehat{R_{\mathfrak{p}}}\otimes\mathrm{Im}\,A_j+I_e(\mathrm{Im}(U_j^{[p^{e-1}]}\cdots U_j^{[p]}U_j)}{\widehat{R_{\mathfrak{p}}}\otimes\mathrm{Im}\,A_j+I_{e+1}(\mathrm{Im}(U_j^{[p^{e}]}\cdots U_j^{[p]}U_j)}=0.$$

By Proposition 4.1.7, the latter equality can be written as $\widehat{R_{\mathfrak{p}}}\otimes(B_{j,e-1}/B_{j,e})=0$ i.e $\mathrm{HSL}(H_{\mathfrak{p}}^j)<e$ if and only if $\mathfrak{p}\notin\mathrm{Supp}(B_{j,e-1}/B_{j,e})$. $\square$

Corollary 4.4.2. *The F-injective locus of S, i.e. $\cap_j V_{1,j}$, is open.*

Proof. The Frobenius map is injective on $H_{\mathfrak{p}}^j$ if and only if

$$\mathrm{Im}\,A_j+I_1(\mathrm{Im}\,U_j)=\mathrm{Im}\,A_j+R^{\alpha_j}$$

for some α_j. Set $B_{j,0}=\mathrm{Im}\,A_j+R^{\alpha_j}$ then the F-injective locus of S is given by the intersection on j of the complements of the supports of $B_{j,0}/B_{j,1},\cdots$,

$B_{\dim S,0}/B_{\dim S,1}$ therefore it is open. $\square$

Since every set $V_{j,e} = \{\mathfrak{p} \in \mathrm{Spec}(R) | \,\mathrm{HSL}(H^j_{\mathfrak{p}}) < e\}$ is open and $\mathrm{Spec}(R) = \bigcup_{j\geq 0, e>0} V_{j,e}$ then we also have:

Corollary 4.4.3. *The set* $\{\mathrm{HSL}(H^j_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{Spec}(R), j \geq 0\}$ *is bounded.*

4.5 A New Algorithm

Let R be a polynomial ring over a field of prime characteristic p, let $I \subset R$ be an ideal of R and let $S = R/I$ be a quotient ring which is not necessarely Cohen-Macaulay. Write $\Delta^e(H^i_{\mathfrak{p}})$ as $\mathrm{Coker}(A_i) \stackrel{U_i}{\to} \mathrm{Coker}(A_i^{[p^e]})$ for some matrices A_i and U_i as we did in Section 4.3. Then we can compute such matrices explicitly as follows.
Let $\cdots \stackrel{\bar{A}_{j+1}}{\to} F_j \stackrel{\bar{A}_j}{\to} \cdots \stackrel{\bar{A}_2}{\to} F_1 \stackrel{\bar{A}_1}{\to} F_0 \stackrel{\bar{A}_0}{\to} R/I \to 0$ be a free resolution of R/I and apply the Frobenius functor to it obtaining

$$\begin{tikzcd}
\cdots \arrow[r] & F_j \arrow[r, "\bar{A}_j"] & \cdots \arrow[r] & F_1 \arrow[r, "\bar{A}_1"] & F_0 \arrow[r, "\bar{A}_0"] & R/I \arrow[r] & 0 \\
\cdots \arrow[r] & F_j \arrow[r, "\bar{A}_j^{[p]}"] \arrow[u, dashed, "\bar{U}_j"] & \cdots \arrow[r] & F_1 \arrow[r, "\bar{A}_1^{[p]}"] \arrow[u, dashed, "\bar{U}_1"] & F_0 \arrow[r, "\bar{A}_0^{[p]}"] \arrow[u, dashed, "\bar{U}_0"] & R/I^{[p]} \arrow[r] \arrow[u, two heads, "\varphi"] & 0
\end{tikzcd}$$

where the maps $\bar{U}_i$'s are constructed in this way; knowing φ and $\bar{A}_0$, we construct the map $\bar{U}_0$ in the commutative diagram above in such a way to make

$$\begin{tikzcd}
F_0 \arrow[r, "\bar{A}_0"] & R/I \arrow[r] & 0 \\
F_0 \arrow[r, "\bar{A}_0^{[p]}"] \arrow[u, dashed, "\bar{U}_0"] & R/I^{[p]} \arrow[r] \arrow[u, two heads, "\varphi"] & 0.
\end{tikzcd}$$

commutative. Once we know $\bar{U}_0$ we can construct $\bar{U}_1$ and so on. In general the computation of $\bar{U}_i$ requires that we know the previous maps $\varphi, \bar{U}_0, \cdots, \bar{U}_{i-1}$.

Therefore, given a positive integer e, an algorithm to find the HSL-loci $V_{i,e}$ of S is the following;

1. Find $\bar{U}_i$ and $\bar{A}_j$ as explained above.
2. Find the induced maps U_i and A_i by applying $\mathrm{Hom}(-, R)$ to the diagram above (the matrices get transposed and the arrows reversed), then take the cohomology.
3. Compute $B_{i,e} = \mathrm{Im}\, A_i + I_e(\mathrm{Im}(U_i^{[p^{e-1}]} \cdots U_i^{[p]} U_i))$;
4. Compute

$$\bigcap_j \mathrm{Supp}(B_{j-1,e}/B_{j,e})^c =$$

$$\bigcap_j V(\mathrm{Ann}_R(B_{j-1,e}/B_{j,e}))^c.$$

The fact that to compute each U_i we have to calculate i Ext's makes the algorithm just described significantly slower than the one in Section 3.5. On the other hand, this algorithm always succedes in the computation of the HSL-loci as it does not require assumptions on S.

We compute now the HSL-loci of the ring from Example 3.4.4 where the algorithm described in Section 3.5 could not be used as the ring was not Cohen-Macaulay. The algorithm has been implemented in Macaulay2 [7].

Example 4.5.1. *Let S be as in Example 3.4.4, i.e. let S be the quotient R/I with $R = \mathbb{Z}_2[a,b,c,d]$ and $I =< ac+bd, b^3+c^2, ab^2+cd, a^2b+d^2 >$. We saw that S is a domain which is not S_2 and the R-module of Frobenius maps acting on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ is not principal.*
Compute a free resolution for S then apply the Frobenius functor to it obtaining the following commutative diagram;

$$\begin{tikzcd}
R^1 \arrow[r, "\bar{A}_2"] & R^4 \arrow[r, "\bar{A}_3"] & R^4 \arrow[r, "\bar{A}_1"] & R^1 \arrow[r, "\bar{A}_0"] & R/I \arrow[r] & 0 \\
R^1 \arrow[r, "\bar{A}_3^{[p]}"] \arrow[u, dashed, two heads, "\bar{U}_3"] & R^4 \arrow[r, "\bar{A}_2^{[p]}"] \arrow[u, dashed, two heads, "\bar{U}_2"] & R^4 \arrow[r, "\bar{A}_1^{[p]}"] \arrow[u, dashed, two heads, "\bar{U}_1"] & R^1 \arrow[r, "\bar{A}_0^{[p]}"] \arrow[u, dashed, two heads, "\bar{U}_0"] & R/I^{[p]} \arrow[r] \arrow[u, two heads, "\varphi"] & 0.
\end{tikzcd}$$

where the maps $\bar{U}_i$'s are:

$$\bar{U}_0 = \begin{pmatrix} 1 \end{pmatrix},$$

$$\bar{U}_1 = \begin{pmatrix} ac+bd & a^2d & bd+cd & 0 \\ 0 & ad+d^2 & b^3 & 0 \\ 0 & a^3 & 0 & 0 \\ 0 & 0 & ad+d^2 & b^3+bc+c^2 \end{pmatrix},$$

.

$$\bar{U}_2 = \begin{pmatrix} a^2b^2+abd+bd^2 & b^4 & b^2d^2 & b^2c^2 \\ a^2d+ad^2 & ab^3+abc+ac^2+bcd & a^2cd+abd^2 & b^2cd+c^2d \\ 0 & 0 & acd+cd^2 & 0 \\ 0 & ab^2+abc+b^2d & a^3c+a^2bd+ad^2+d^3 & ab^2c+b^3d+bcd \end{pmatrix},$$

$$\bar{U}_3 = \begin{pmatrix} abcd + bcd^2 \end{pmatrix},$$

$$\bar{U}_4 = \begin{pmatrix} 0 \end{pmatrix}.$$

We then compute the generating maps $U_j\colon \operatorname{Coker} A_j \to \operatorname{Coker} A_j^{[p]}$ induced by $\bar{U}_j$; it turns out that $H^0_{\mathfrak{p}}$ is zero for every $\mathfrak{p}$.

$$U_3 = \begin{pmatrix} abcd + bcd^2 \end{pmatrix} \qquad\qquad A_3 = \begin{pmatrix} d & c & b & a \end{pmatrix}$$

therefore $\operatorname{Im} A_3 + I_1(\operatorname{Im} U_3) = (d \;\; c \;\; b \;\; a \;\; 1) = \operatorname{Im} A_3 + R$*, i.e. the Frobenius map is injective on $H^1_{\mathfrak{p}}$ for all $\mathfrak{p}$. .*

$$U_2 {=} \begin{pmatrix} ab^2{+}ab{+}ac{+}bd{+}cd & ab{+}ac{+}bd \\ ab^3{+}bcd & abc{+}b^2d{+}cd \end{pmatrix} \qquad A_2 {=} \begin{pmatrix} a{+}d & b{+}c & a & b \\ ab & b^2 & d & c \end{pmatrix}$$

therefore

$$\operatorname{Im} A_2 + I_1(\operatorname{Im} U_2) = \begin{pmatrix} a+d & b+c & a & b & 1 & 0 \\ ab & b^2 & d & c & 0 & 1 \end{pmatrix} = \operatorname{Im} A_2 + R^2$$

i.e. the Frobenius map is injective on $H^2_{\mathfrak{p}}$ for all $\mathfrak{p}$;
finally $\bar{U}_1 = 0$ so the Frobenius action on $H^3_{\mathfrak{p}}$ is injective as well. It follows that S is F-injective.

In the next example we consider a quotient of a ring of characteristic p. For $p = 2$ such a ring is not Cohen-Macaulay and we compute its HSL-loci by using the algorithm described in this section. Then we change the characteristic of the ring to $p = 3$ and what we get is a Cohen-Macaulay ring so we use the algorithm described in Section 3.5 instead.

Example 4.5.2. *Let $R = \mathbb{K}[x_1 \cdots, x_5]$, with $\mathbb{K}$ a field of positive characteristic p, let $I =< x_2x_4 + x_1x_5, x_4^3 + x_3x_5^2, x_1x_4^2 + x_2x_3x_5, x_2^2x_3 + x_1^2x_4 > \subset R$ be an ideal and write $S = R/I$.*

- *If $p = 2$ then S is a 3-dimensional not Cohen-Macaulay ring whose non-Cohen-Macaulay locus consists of $\mathfrak{p} =< x_1, x_2, x_4, x_5 >$.*

  A free resolution for S has length 3. Since $\mathrm{pd}(R/I) = \dim R - \mathrm{depth}(R/I)$, $\mathrm{depth}(R/I)$ cannot be 0 or 1 otherwise $\mathrm{pd}(R/I)$ would be greater than the length of the free resolution. Hence $H^0_{\mathfrak{p}} = 0$ and $H^1_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$. Consider the local cohomology module $H^2_{\mathfrak{p}}$; with the usual notation we have

$$U_3 = \begin{pmatrix} x_2^2x_3x_4x_5 + x_1x_2x_3x_5^2 \end{pmatrix} \quad \textit{and} \quad A_3 = \begin{pmatrix} x_5 & x_4 & x_2 & x_1 \end{pmatrix}$$

therefore $\mathrm{Im}\, A_3 + I_1(\mathrm{Im}\, U_3) = \begin{pmatrix} x_5 & x_4 & x_2 & x_1 & x_5 & x_2 \end{pmatrix} \neq \mathrm{Im}\, A_3 + R.$

It follows that $\mathrm{HSL}(H^2_{\mathfrak{p}}) < 1$ (i.e. the Frobenius action is injective) if and only if $\mathfrak{p} \notin \mathrm{Supp}\left(\frac{\mathrm{Im}\, A_2 + R}{\mathrm{Im}\, A_3 + I_1(\mathrm{Im}\, U_3)}\right) = V(< x_2, x_5 >)$.

$$\mathrm{Im}\, A_3 + I_2(\mathrm{Im}\, U_3^{[p]} U_3) = \begin{pmatrix} x_5 & x_4 & x_2 & x_1 & x_5 & x_2 \end{pmatrix} = \mathrm{Im}\, A_3 + I_1(\mathrm{Im}\, U_3)$$

consequently $\mathrm{Supp}\left(\frac{\mathrm{Im}\, A_3 + I_1(\mathrm{Im}\, U_3)}{\mathrm{Im}\, A_3 + I_2(\mathrm{Im}\, U_3^{[p]} U_3)}\right) = V(1)^c$ i.e. $HSL(H^2_{\mathfrak{p}}) < 2$ for all $\mathfrak{p}$. In conclusion if we localise at a prime $\mathfrak{q}$ that does not contain *the prime* $\mathfrak{r} =< x_2, x_5 >$ *then the action of Frobenius on* $S_\mathfrak{q}$ *is injective and in particular the* $HSL(H^2_\mathfrak{r}(S_\mathfrak{r})) = 1$.

Finally consider the Frobenius action on $H^3_\mathfrak{p}$.

$$U_2 = \begin{pmatrix} x_2x_3x_4 + x_1x_3x_5 & x_1x_3x_4 & 0 \\ x_1x_4x_5 & x_2x_3x_5 & x_2x_3x_4 + x_1x_3x_5 \\ x_2x_5^2 & x_2x_4x_5 + x_1x_5^2 & x_2x_4^2 + x_1x_4x_5 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} x_4 & x_1 & 0 & x_3 & 0 \\ x_5 & x_2 & x_4 & 0 & x_1 \\ 0 & 0 & x_5 & x_4 & x_2 \end{pmatrix}$$

so we have

$$\operatorname{Im} A_2 + I_1(\operatorname{Im} U_2) = \begin{pmatrix} x_4 & x_1 & 0 & x_3 & 0 & 0 & 1 & 0 \\ x_5 & x_2 & x_4 & 0 & x_1 & 0 & 0 & 1 \\ 0 & 0 & x_5 & x_4 & x_2 & 1 & 0 & 0 \end{pmatrix} = \operatorname{Im} A_2 + R^3$$

therefore the Frobenius action is injective on $H^3_\mathfrak{p}$ *for every prime ideal* $\mathfrak{p}$.

In conclusion we have the following HSL-*loci:* $\mathcal{B}_0 = V(< x_2, x_5 >)^c$ *and* $\mathcal{B}_1 = V(1)^c$.

- *Let now* $p = 3$. *Then* S *is Cohen-Macaulay of dimension* 2 *and a canonical ideal for* S *is the image in* S *of the ideal*

$$\Omega =< x_3x_5 + x_4, x_4^2 + x_1 + x_2 - x_5, x_1x_5, x_2x_4, x_1x_4, x_2x_3, x_4^3 + x_3x_5^2 >$$

in R. A generator for the R-module of the Frobenius maps on $\mathrm{H}^{\mathrm{d}}_{\mathfrak{m}S}(S)$ *is*

$$u = -\,x_1^2x_2^2x_3^2x_4^6x_5^2 + x_1^2x_2^2x_3^3x_4^3x_5^4 - x_1^2x_2^2x_3^4x_5^6 + x_1^2x_2^2x_3x_4^7x_5 - x_1^2x_2^2x_3^2x_4^4x_5^3 + x_1^2x_2^2x_3^3x_4x_5^5 - x_1^2x_2^2x_4^8 - x_1^4x_2^2x_3^2x_4^2x_5^2 + x_1^3x_2^3x_3^2x_4^2x_5^2 - x_1^2x_2^4x_3^2x_4^2x_5^2 + x_1^2x_2^2x_3x_4^5x_5^2 - x_1^2x_2^3x_3^2x_4^2x_5^3 - x_1^2x_2^2x_3^2x_4^2x_5^4$$

and it turns out that

$$I_1(u\Omega) = < x_1x_5, x_4^2 + x_1 + x_2 - x_5, x_2x_4, x_1x_4, x_2x_3, x_3x_5^2 + x_4x_5, x_3x_4x_5 - x_1 - x_2 + x_5, x_4^3 + x_3x_5^2 >= I_2(u^4\Omega)$$

so that $\mathcal{B}_0 = \mathrm{Supp}\left(\frac{\Omega}{I_1(u\Omega)}\right) = V(x_1, x_2, x_4, x_5)^c$ *and* $\mathcal{B}_1 = V(1)^c$.

5 Conclusions

In this thesis we computed the HSL-loci of quotients of polynomial rings. We approached the problem at first in a purely theoretical way and then we translated the results into an algorithm that allowed us to do our computations explicitly on concrete rings. We first treated the case of Cohen-Macaulay domains for which the algorithm was quite fast and then we described another strategy that works for *any* ring. As we noted previously, the only disadvantage of the second method is that the algorithm for it is much slower than the algorithm for the Cohen-Macaulay case. The common ingredient used in both the Cohen-Macaulay case and the non-Cohen-Macaulay case is that they make use of an operation (more specifically the $I_e(-)$ operator) that commutes with localisation and completion.
We gave a constructive description of loci and invariants defined by various properties of a Frobenius action on some modules. There may be other loci and invariants defined by properties of Frobenius maps that could be described using similar methods.

Karen Smith gave the following characterization of F-rational rings using the Frobenius action F on $H^d_{\mathfrak{m}}(R)$:

Theorem 5.0.3. *[26] Let $(R, \mathfrak{m})$ be a d-dimensional excellent local ring of characteristic $p > 0$. Then R is F-rational if and only if R is Cohen-Macaulay and $H^d_{\mathfrak{m}}(R)$ has no proper nontrivial submodules stable under the Frobenius action F.*

One could consider a non-local ring $S = R/I$ and find an explicit description for the F-rational locus in terms of the simplicity of the top local cohomology module i.e find an object G_i such that $G_i = 0$ if and only if $H^i_{\mathfrak{m}}(S)$ is simple.

6 Appendix: Code Used

This appendix includes the code used throughout the thesis. For completeness all the code is included but only the code marked by a * was written by myself, the rest was written by Moty Katzman. The algorithms have been implemented in Macaulay2, [7].

1*) The next two functions tell you whether a certain ring is Cohen-Macaulay and compute the Cohen-Macaulay type respectively.

```
CMR=(M)->(
  R:=ring(M);
  cc:=res coker M;
  pd:=length(cc);
  isCM:=((dim(R)-pd)==dim(coker M));
  return isCM;
)
```

```
TypeR=(M)->(
  R:=ring(M);
  pd:=pdim (coker M);
  depthOfRmodI:=dim(R)-pd;
  B:=res coker M;
  type:=rank (B_pd);
  return type;
)
```

2*) The function OMEGA computes a canonical ideal for a ring R/I where I is defined from the matrix M.

```
Allnumbers=(n)->(
    L:=0; local i;
     for i from 1 to (n-1) do(
      L=flatten toList(L,i););
    return L;
    )

Allzeros=(n)->(
    L:=0;  local i;
     for i from 1 to (n-1) do(
       L=flatten toList(L,0););
     return L;
     )

OMEGA=(M)->(
   R:=ring(M);p:=char(R);answer:=0;T:=TypeR(M);
   if (T==1) then(
   if (CMR(M)) then return R;);
   delta:=dim(R)-dim(coker M);
      Omega0:=relations prune Ext^delta(coker M, R^1);
      S:=R/ideal(M);
      s1:=syz transpose substitute(Omega0,S);
      s2:=entries transpose s1;
      s=#s2_1;
      L:=rank source s1;
      ALL:=Allnumbers(p);
      ALL=set ALL;
      Comb:=ALL^**L;
      n:=#Comb;
      ListComb:=elements Comb;
      for i from 0 to (n-1) do(
         c:=ListComb_i;
         App:=Allzeros(s);
         for j from 0 to (L-1) do (
         App=c_j*(s2)_j +App;
         j=j+1;);
         s3:=App;
         s3=syz gens ideal(s3);
         s3=gens substitute(image(s3),R);
         news3=matrix entries s3;
         newOmega0=matrix entries Omega0;
         z:=(news3%newOmega0);
         if ((z)==0) then  {
          answer=substitute(mingens ideal (App),R);
         break;}
         else print"error";
         i=i+1;);
       substitute(answer,R);
       answer=ideal(answer)+ideal(M);
       answer=gens answer;
       return answer;
)
```

3*) The following function takes an ideal $I = ideal(M)$ (where M is a matrix) and a canonical ideal Om for a ring $R/ideal(M)$ and computes the generators of a matrix X whose cokernel is isomorphic to the R-module $\mathcal{U}_{(1)} = \frac{(J^{[p]}:J)\cap(\Omega^{[p]}:\Omega)}{J^{[p]}}$ defined in Section 3.4.

```
FindGeneratorsX=(M,Om)->(
R:=ring(M);
p:=char R;
GenTop:=ideal 0_R;
O1:=first entries M;
O2:=apply(O1, u->u^p);
O3:=ideal(O2):ideal(O1);
Omega1:=first entries Om;
Omega2:=apply(Omega1, t->t^p);
Omega3:=ideal(Omega2):ideal(Omega1);
if (isRing Om) then GenTop=generators(O3)
else GenTop=generators intersect(Omega3,O3);
I4:=generators(ideal(apply(first entries M, u->u^p)));
X:=relations prune subquotient(GenTop,I4);
v:=(coker X).cache.pruningMap;
v1:=matrix entries v;
ListGen:=first entries(GenTop *v1);
return ListGen;
)
```

4*) The following function takes an ideal I and produces $I^{[p^e]}$.

```
FrobeniusPower= (I,e) ->(
R:=ring I;
p:=char R;
local u;
local answer;
G:=first entries gens I;
if (#G==0) then answer=ideal(0_R)
else answer=ideal(apply(G, u->u^(p^e)));
answer
);
```

5*) The following takes a matrix M, an integer e and produces $M^{[p^e]}$ i.e.

the matrix obtained by raising to p^e the entries of the matrix M.

```
FrobeniusPowerMatrix= (M,e) ->(
R:=ring M;
p:=char R;
G:=entries M;
local i;
local j;
L:={};
apply(G, i->{  L=append(L,apply(i, j->j^(p^e)));});
substitute(matrix L, ring(M))
)
```

6 Given two ideals A and B and an intereger e, the following function gives the ideal $I_e(A) + B$ as output.

```
ethRoot = (A,B,e) ->(
R:=ring(A);
pp:=char(R);
F:=coefficientRing(R);
n:=rank source vars(R);
vv:=first entries vars(R);
R1:=F[v, Y_1..Y_n, MonomialOrder=>
ProductOrder{n,n},MonomialSize=>16];
J0:=apply(1..n, i->Y_i-substitute(v#(i-1)^(pp^e),R1));
S:=toList apply(1..n, i->Y_i=>substitute(v#(i-1),R1));
GG:=(gens substitute(A,R1))%gens(ideal(J0))
G:=first entries compress(GG);
L:=ideal 0_R1;
apply(G, t->
    { L=L+ideal((coefficients(t,Variables=>v))#1);});
L1:=L+substitute(B,R1);
L2:=mingens L1;
L3:=first entries L2;
L4:=apply(L3, t->substitute(t,S));
use(R);
substitute(ideal L4,R)
)
```

7*) The following function calculates ν_e;

```
nu=(e,p)->(
if e==0 then return 0 else
return sum(0 .. e-1, i -> p^i)
)
```

8*) Given a canonical module ω (Om in the function) and a generator u (computed using the function FindGeneratorsX as in 6), the following computes $I_e(u^{\nu_e}\omega)$.

```
Li=(e,u,Om,M)->(
if e==0 then return ideal Om else (
if isRing Om then (
q:=char(Om);
return ethRoot( ideal(u^(nu(e,q))), ideal(0_Om),e);)
else  (
R:=ring(Om);
p:=char(R);
J:=ideal(u^(nu(e,p)))*ideal Om;
return ethRoot(J,ideal(0_R),e)+ideal(M););им);
)
```

9*) Given a matrix X and a positive integer j, it returns the ideal of the minors of the matrix obtained from X by erasing the j^{th}-row.

```
mathcalGj=(j,X)->(
rows:=(rank target X)-1;
return minors(rows,submatrix'(X,{j},));
)
```

10*) The following computes the HSL-loci of $R/ideal(M)$. In particular for $e = 0$ we have the F-injective locus.

```
Loci= (M,Om,e) ->(
j:=0;
R:=ring(M);
FG:=FindGeneratorsX(M,Om);
L:= #FG-1;
D:=ideal 0_R;
App:=ideal 0_R;
X:=ComputeX(M,Om);
for j from 0 to L do(
u:=FG_j;
W1:=Li(e,u,Om,M);
W2:=Li(e+1,u,Om,M);
J:=mathcalGj(j,X);
ANNIH:= W2:W1;
D=ANNIH*J;
D=radical ideal mingens D;
App=D+App;);
App=radical ideal mingens App;
<< "Locus( "<<e <<")="<< App << "\n";print"";
return App;
)
```

11) The following function computes a generating morphism for $H_I^{(\dim R-i)}(R)$ in Chapter 4. The output is (A, U) where $U\colon$ Coker$(A)- > F($Coker $A)$ is the generating morphism.

```
generatingMorphism= (I,i) ->(
local F1; local K; local C;
local F1p; local Kp; local Cp;
R:=ring(I);
Ip:=frobeniusPower(I,1);
M:=coker gens I;
Mp:=coker gens Ip;
resM:=res M;
resMp:=res Mp;
f:=inducedMap(M,Mp);
resf:=res f;
G:=resf#i; G=transpose(G);
F0:=(resM.dd)#(i); F0=transpose(F0);
if (resM.dd)#?(i+1) then(
F1=(resM.dd)#(i+1); F1=transpose(F1);
K=ker F1;)else(K=target(F0););
temp1:=substitute(gens K,R);
```

```
if (temp1==0) then C=coker(F0) else
C=subquotient(substitute(gens K,R),F0);
C=subquotient(substitute(gens K,R),F0);
C1:=prune(C);
h:=C1.cache.pruningMap;
generatingMorphism0:=G*gens(K)*matrix(entries h);
F0p:=(resMp.dd)#(i); F0p=transpose(F0p);
if (resMp.dd)#?(i+1) then
(F1p=(resMp.dd)#(i+1); F1p=transpose(F1p);
Kp=ker F1p;)else(Kp=target(F0p););
temp1=substitute(gens Kp,R);
if (temp1==0) then Cp=coker(F0p) else
Cp=subquotient(substitute(gens Kp,R),F0p);
Cp=subquotient(gens Kp,F0p);
C1p:=prune(Cp);
hp:=C1p.cache.pruningMap;
A0:=gens(Kp)*matrix(entries hp); A=A0| F0p;
gbA:=gb(A, ChangeMatrix => true) ;
B:=generatingMorphism0// A;
--- Now generatingMorphism0=A*B
k:=rank source A0;
(relations(C1), submatrix(B,toList(0..(k-1)),))
)
```

12) The function mEthRoot computes $I_e(-)$ of submodules of free modules as defined in Section 4.1.

```
getExponents=(f)->(
answer:={};
t:=terms(f);
apply(t, i->
{
exps:=first exponents(i);
c:=(coefficients(i))#1;
c=first first entries c;
answer=append(answer,(c,exps));
});
answer
)
```

```
mEthRootOfOneElement= (v,e) ->(
local i; local j;
local d;
local w;
local m;
local answer;
R:=ring(v); p:=char R; q:=p^e;
F:=coefficientRing(R);
n:=rank source vars(R);
V:=ideal vars(R);
vv:=first entries vars(R);
T:=new MutableHashTable;
alpha:=rank target matrix(v);
B:={};
for i from 1 to alpha do
{
vi:=v_(i-1);
C:=getExponents(vi);
apply(C, c->
{
lambda:=c#0;
beta:=c#1;
gamma:=apply(beta, j-> (j%q));
B=append(B,gamma);
key:=(i,gamma);
data:=apply(1..(#beta), j-> vv_(j-1)^((beta#(j-1))//q));
data=lambda*product(toList data);
if (T#?key) then
{
T#key=(T#key)+data;
}
else
{
T#key=data;
};
});
};
B=unique(B);
TT:=new MutableHashTable;
apply(B, b->
{
ww:={};
for i from 1 to alpha do if T#?(i,b) then
ww=append(ww,T#(i,b)) else ww=append(ww,0_R);
ww=transpose matrix {ww};
TT#b=ww;
});
KEYS:=keys(TT);
answer=TT#(KEYS#0);
for i from 1 to (#KEYS)-1 do answer=answer | TT#(KEYS#i);
answer
)
```

```
mEthRoot = (A,e) ->(
local i;
local answer;
answer1:=apply(1..(rank source A),
i->mEthRootOfOneElement (A_(i-1),e));
if (#answer1==0) then
{
answer=A;
}
else
{
answer=answer1#0;
apply(2..(#answer1), i->answer=answer | answer1#(i-1));
};
mingens( image answer )
)
```

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