Reducts of $\aleph_0$-Categorical Structures

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The candidate confirms that the work submitted is their own, except where work which has formed part of jointly authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given where reference has been made to the work of others.

The work in Chapter 4 of the thesis has appeared in work that has been submitted but not yet published:

$2^{\aleph_0}$ pairwise non-isomorphic maximal-closed subgroups of $\mathrm{Sym}(\mathbb{N})$ via the classification of the reducts of the Henson digraphs, L. Agarwal and M. Kompatscher.

The authors are equally responsible for the work in Chapter 4.

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Abstract

Given two structures $\mathcal{M}$ and $\mathcal{N}$ on the same domain, we say that $\mathcal{N}$ is a reduct of $\mathcal{M}$ if all $\emptyset$-definable relations of $\mathcal{N}$ are $\emptyset$-definable in $\mathcal{M}$. In this thesis, the reducts of the generic digraph, the Henson digraphs, the countable vector space over $\mathbb{F}_2$ and of the linear order $\mathbb{Q}.2$ are classified up to first-order interdefinability. These structures are $\aleph_0$-categorical, so classifying their reducts is equivalent to classifying the closed groups that lie in between the structures' automorphism groups and the full symmetric group.

Contents

Chapter 1

Introduction

This thesis contributes to the large body of work concerning the two intimately related topics of reducts of countable structures and of closed subgroups of $\mathrm{Sym}(\mathbb{N})$. Motivation for this work comes from both areas.

In the topic of reducts, the reducts of the generic digraph, the Henson digraphs, the countable vector space over $\mathbb{F}_2$ and of the linear order $2.\mathbb{Q}$ are classified up to first-order interdefinability. In all cases only finitely many reducts appear, supporting a conjecture of Thomas in [Tho91] which says that all countable homogeneous structures in a finite relational language have only finitely many reducts. Evidence for this conjecture is building as there have been numerous classification results, e.g. [Cam76], [Tho91], [Tho96], [JZ08], [PPP$^+$14], [BPP15], [Aga16]. This conjecture is unresolved and continues to provide motivation for study.

As a corollary to the classification of the reducts of the Henson digraphs, this thesis answers positively a question of Macpherson, Question 5.10 in [BM15], which asked whether there are uncountably many pairwise non-conjugate maximal-closed subgroups of $\mathrm{Sym}(\mathbb{N})$ with $\mathrm{Sym}(\mathbb{N})$ bearing the pointwise convergence topology. Several related questions have recently been tackled. Independently, [BM15] and [BR13] showed that

there exist non-oligomorphic maximal-closed subgroups of $\mathrm{Sym}(\mathbb{N})$, the existence of which was asked in [JZ08]. Also, independently, [KS15] and [BR13] positively answered Macpherson's question of whether there are countable maximal-closed subgroups of $\mathrm{Sym}(\mathbb{N})$. One question that remains open is whether every proper closed subgroup of $\mathrm{Sym}(\mathbb{N})$ is contained in a maximal-closed subgroup of $\mathrm{Sym}(\mathbb{N})$, (Question 7.7 in [MN96] and Question 5.9 in [BM15]).

The main tool used in the classification of the reducts of the generic digraph and the Henson digraphs is that of the so-called 'canonical functions'. This Ramsey-theoretic tool was developed by Bodirsky and Pinsker to help analyse certain closed clones in relation to constraint satisfaction problems, a topic in theoretical computer science. With further developments ([BP11], [BPT13]), canonical functions have become powerful tools in studying reducts. The robustness and relative ease of the methodology is becoming more evident as several classifications have been achieved by their use, e.g. [BB13], [PPP$^+$14], [LP15], [BPP15], [Aga16], [BBWPP16].

We outline the structure of the thesis. In Chapter 2, we provide the necessary prerequisites and background. This includes notational conventions that we use.

In Chapter 3, we prove the classification of the reducts of the generic digraph. Section 3.1 provides basic definitions and facts on the generic digraph. Section 3.2 contains the definition of the reducts and the statement of the classification. In Section 3.3 we describe the reducts, establishing notation and important lemmas that are used in the rest of the chapter. In Section 3.4 we carry out the combinatorial analysis of the possible behaviours of canonical functions. Section 3.5 contains the proof of the classification, putting together the pieces from the previous sections.

Chapter 4 contains the classification of the reducts of the Henson digraphs. Surprisingly, perhaps, the proof is almost exactly the same as that of the generic digraph and thus the structure of the chapter is the same. The chapter ends, however, with Section 4.6 where we use the classification to show that there exist $2^{\aleph_0}$ maximal-closed subgroups of $\mathrm{Sym}(\mathbb{N})$.

Chapter 5 collects the remaining material of the thesis. Section 5.1 contains the classification of the reducts of the countable vector space over $\mathbb{F}_2$. Section 5.2 contains the classification of the reducts of $2.\mathbb{Q}$. In Section 5.3, we provide a summary of the thesis and some potential areas of further study.

Chapter 2

Prerequisites and Background

We assume familiarity with several core notions in first-order logic and model theory: languages, formulas, theories, structures, substructures, isomorphisms, types and quantifier elimination. See [Hod97] for details. We choose not to define them as we believe it is necessary to have an intuition for these notions to be able to follow the thesis. For those notions we do choose to define, the intention is that prior intuition or familiarity is not required and that we provide the relevant facts and details needed to understand the thesis.

2.1 Notational Conventions

Structures are first-order structures and they are countably infinite, unless stated otherwise. $\mathcal{M}$ and $\mathcal{N}$ will denote structures, and their domains are M and N respectively. $\mathrm{Th}(\mathcal{M})$ denotes the theory of $\mathcal{M}$. If T is a theory, then $\mathcal{S}(T)$ is the set of types of T. $\mathcal{S}(\mathcal{M}) = \mathcal{S}(\mathrm{Th}(\mathcal{M}))$. If $\bar{a}$ is a tuple in a structure, $\mathrm{tp}(\bar{a})$ denotes the type of $\bar{a}$. M^M is the set of all functions $M \to M$, $\mathrm{Sym}(M)$ is the set of all bijections $M \to M$ and $\mathrm{Aut}(\mathcal{M})$ is the set of all automorphisms of $\mathcal{M}$. As structures are countably infinite, we

implicitly assume that M equals $\mathbb{N}$ as a set so that automorphism groups can be considered subgroups of $\mathrm{Sym}(\mathbb{N})$.

Given a formula $\phi(x, y)$, we use $\phi^*(x, y)$ to denote the formula $\phi(y, x)$. If f has domain A and $\bar{a} \in A$, then $f(a_1, \ldots, a_n) := (f(a_1), \ldots, f(a_n))$. Let $\bar{a}, \bar{b} \in M^n$, for some $n \in \mathbb{N}$, be tuples in an $\mathcal{L}$-structure $\mathcal{M}$; we say $\bar{a}$ and $\bar{b}$ are isomorphic, and write $\bar{a} \cong \bar{b}$, to mean that the function $a_i \mapsto b_i$ for all i such that $1 \leq i \leq n$ is an $\mathcal{L}$-isomorphism. We say $\bar{a}$ is a *proper* tuple if all the elements of $\bar{a}$ are pairwise distinct.

There will be instances where we do not adhere to strictly correct notational usage, however, the meaning should be clear from the context. We list some examples. We write '$a \in (a_1, \ldots, a_n)$' instead of '$a = a_i$ for some i such that $1 \leq i \leq n$'. We write 'Let $\bar{a} \in A$' instead of 'Let $\bar{a} \in A^n$, where n is the length of the $\bar{a}$'. Another example is that we sometimes use c to represent the singleton set $\{c\}$ containing it. A fourth example is that we do not distinguish between relation symbols and the interpretations of those symbols in a model.

2.2 $\aleph_0$-categorical structures

Definition 2.2.1. We say a theory T is $\aleph_0$-categorical if it has only one countable model up to isomorphism. We say a structure $\mathcal{M}$ is $\aleph_0$-categorical if $\mathrm{Th}(\mathcal{M})$ is $\aleph_0$-categorical.

Examples and non-examples:

- $(\mathbb{N}, =)$ is $\aleph_0$-categorical. Proving this is a straightforward exercise.

- Let K be a finite field. Then the countably infinite vector space over K is $\aleph_0$-categorical, because every countably infinite vector space over K has dimension $\aleph_0$.

- $(\mathbb{Q}, <)$ is $\aleph_0$-categorical. This follows by the proto-typical back-and-forth argument.

- Peano Arithmetic is not $\aleph_0$-categorical. This is a consequence of Gödel's Incompleteness Theorems.

Theorem 2.2.2. *(i) (Engeler, Ryll-Nardzewski and Svenonius Theorem) Let $\mathcal{M}$ be countable. The following are equivalent:*

- *$\mathcal{M}$ is an $\aleph_0$-categorical structure.*

- *For all $n \in \mathbb{N}$, $\mathcal{M}$ has finitely many n-types.*

- *For all $n \in \mathbb{N}$, the action of $Aut(\mathcal{M})$ on M^n has finitely many orbits.*

- *For every type $p \in \mathcal{S}(\mathcal{M})$, there is a formula ϕ such that for all $\bar{a} \in M$, $\bar{a}$ realises p if and only if $\mathcal{M} \models \phi(\bar{a})$.*

(ii) Let $\mathcal{M}$ be $\aleph_0$-categorical and let $\bar{a}, \bar{b} \in M$. If $tp(\bar{a}) = tp(\bar{b})$, then there exists an automorphism of $\mathcal{M}$ mapping $\bar{a}$ to $\bar{b}$.

A proof of (i) can be found in [Hod97, Theorem 6.3.1]. This theorem is often referred to as the Ryll-Nardzewski Theorem. We continue this unfair tradition for the sake of conciseness. A proof of (ii) can be found in [Hod97, Corallary 6.3.3].

Important Remark. As a result of (ii), there is bijective correspondence between n-types of $\mathcal{M}$ and orbits of n-tuples of $\mathcal{M}$. Given a type $p(\bar{x})$ you obtain the orbit $\{\bar{x} \in M^n : tp(\bar{x}) = p\}$, and given an orbit $A \subseteq M^n$ you obtain the type $p(\bar{a})$, where $\bar{a} \in A$. In this light, and as has become customary in modern model theory, we sometimes blur the distinction between a type and the set of tuples that realise that type.

2.3 Homogeneous structures

Let $\mathcal{M}$ be a structure. We say $\mathcal{M}$ is *homogeneous* if every isomorphism $f : A \to B$ between finitely generated substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$.

Examples and non-examples:

- $(\mathbb{N}, =)$ is homogeneous. After unravelling the definitions, this statement says that for any $\bar{a}, \bar{b} \in \mathbb{N}$ there is a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $f(\bar{a}) = \bar{b}$. Proving this is straightforward.

- $(\mathbb{N}, <)$ is not homogeneous. $\{0\}$ and $\{1\}$ are isomorphic as linear orders, but the only automorphism of $(\mathbb{N}, <)$ is the identity.

- $(\mathbb{Z}, <)$ is not homogeneous. $\{0, 1\}$ and $\{0, 2\}$ are isomorphic as linear orders, but every automorphism of $(\mathbb{Z}, <)$ preserves distances.

- $(\mathbb{Q}, <)$ is homogeneous. Let $a_1 < \ldots < a_n$ and $b_1 < \ldots < b_n$ be elements of $\mathbb{Q}$. We need to find an isomorphism $f : (\mathbb{Q}, <) \to (\mathbb{Q}, <)$ such that $f(a_i) = b_i$ for $i = 1, \ldots, n$. This amounts to showing that we can find an order-preserving bijection from any open interval of $\mathbb{Q}$ to any other. This can be done by an appropriate stretching and translation of the interval. Alternatively, you can use a back-and-forth argument in which the two enumerations begin with the $a_i's$ and $b_i's$.

Possibly the most important result concerning homogeneous structures is the Fraïssé correspondence between homogeneous structures and amalgamation classes.

Definition 2.3.1. Let $\mathcal{L}$ be a language and let $\mathcal{C}$ be a set of finite $\mathcal{L}$-structures. We say $\mathcal{C}$ is an *amalgamation class* if it satisfies the following conditions.

(i) It is downward closed i.e. if $A \in \mathcal{C}$ and B is a substructure of A, then $B \in \mathcal{C}$.

(ii) It has the joint embedding property i.e. if $B_1, B_2 \in \mathcal{C}$ then there exists $C \in \mathcal{C}$ such that B_1 and B_2 are embeddable in C.

(iii) It has the amalgamation property i.e. if $A, B_1, B_2 \in \mathcal{C}$ and $f_i : A \to B_i$ are embeddings, then there exists $C \in \mathcal{C}$ and embeddings $g_i : B_i \to C$ such that $g_1 f_1 = g_2 f_2$.

When trying to show a class is an amalgamation class, the most difficult property to check is the amalgamation property. In practice, when checking the amalgamation property one may assume without loss that A is a substructure of B_1 and B_2 and that $A = B_1 \cap B_2$. For the classes we consider, one can choose the domain of C to be $B_1 \cup B_2$ (the alternatives are to add new elements to the union and/or to identify elements of B_1 and B_2). What remains then is adding the appropriate structure onto the domain to ensure C is in $\mathcal{C}$. Naturally we ensure that B_1 and B_2 are substructures, so what is left to determine is how the elements of $B_1 \backslash A$ and $B_2 \backslash A$ are related.

Examples:

- The set of all finite $\emptyset$-structures, i.e. of all finite sets, is an amalgamation class. For the amalgamation property, after letting $C = B_1 \cup B_2$, there is nothing left to be done.

- The set of all finite graphs is an amalgamation class. For the amalgamation property, simply add no edges between $B_1 \backslash A$ and $B_2 \backslash A$ to obtain an appropriate graph C.

- The set of all finite linear orders is an amalgamation class. This time there is some work to be done. Given linear orders $(A, <) \subseteq (B_1, <'), (B_2, <'')$, consider the partial order C whose domain is $B_1 \cup B_2$ and whose order relation $<_C$ is the union of the relations $<'$ and $<''$. To fulfill the amalgamation property, we need to extend the partial order $<_C$ to a linear order. This is possible by the Szpilrajn extension theorem, [Szp30], that every partial order can be extended to a linear order.

Definition 2.3.2. The age of a structure $\mathcal{M}$, Age($\mathcal{M}$), is the set of all finite substructures of $\mathcal{M}$.

Theorem 2.3.3. *[Fra53]. Let $\mathcal{L}$ be a countable language.*

(i) *Let $\mathcal{M}$ be a countable homogeneous $\mathcal{L}$-structure. Then Age($\mathcal{M}$) is an amalgamation class.*

(ii) *Let $\mathcal{C}$ be an amalgamation class. Then there exists a unique, up to isomorphism, countable homogeneous $\mathcal{M}$ whose age is $\mathcal{C}$. This structure is known as the* Fraïssé limit *of $\mathcal{C}$.*

Examples:

- The Fraïssé limit of the set of all finite sets is just a countable set.

- The Fraïssé limit of the set of all finite linear orders is $(\mathbb{Q}, <)$.

- The Fraïssé limit of the set of all finite graphs is the random graph. (This is also known as the Erdős-Rado graph.)

Homogeneous structures are of particular interest to us because of the following theorem, a proof of which can be found in [Hod97, Theorem 6.4.1].

Theorem 2.3.4. *Let $\mathcal{L}$ be a finite relational language and $\mathcal{M}$ be a homogeneous $\mathcal{L}$-structure. Then $\mathcal{M}$ is $\aleph_0$-categorical and has quantifier elimination.*

2.4 The topology of $\mathbf{Sym}(\mathbb{N})$ and $\mathbb{N}^{\mathbb{N}}$

Definition 2.4.1. Let $F \subseteq \mathbb{N}^{\mathbb{N}}$.

(i) Let $g \in \mathbb{N}^{\mathbb{N}}$. We say g is in the *closure* of F, cl(F), if for all finite $A \subset \mathbb{N}$ there is $f \in F$ such that $f(a) = g(a)$ for all $a \in A$.

(ii) We say F is *closed* if $F = $cl($F$).

This defines the so-called pointwise convergence topology on $\mathbb{N}^{\mathbb{N}}$. Equipped with this topology, and with function composition as the binary operation, $\mathbb{N}^{\mathbb{N}}$ becomes a topological monoid. As Sym($\mathbb{N}$) is a subset of $\mathbb{N}^{\mathbb{N}}$ it inherits this topology, via the subspace topology, and becomes a topological group (in fact, a Polish group).

Notation.

- If $F \subseteq$ Sym($\mathbb{N}$), $\langle F \rangle$ denotes the smallest closed subgroup of Sym($\mathbb{N}$) containing F.

- If $F \subseteq \mathbb{N}^{\mathbb{N}}$, $\mathrm{cl}_{\mathrm{tm}}(F)$, the topological monoid closure of F, denotes the smallest closed submonoid of $\mathbb{N}^{\mathbb{N}}$ containing F.

- If $g \in \mathrm{cl}_{\mathrm{tm}}(F)$ we say g is *generated* by F.

Remarks and Examples.

- The reason for choosing the notation '$\langle F \rangle$' over the arguably more natural choice '$\mathrm{cl}_{\mathrm{tg}}(F)$' is mainly aesthetic - we believe it is easier to read and is more pleasing to the eye.

- $\langle F \rangle \subseteq \mathrm{cl}_{\mathrm{tm}}(F)$. In fact, $\langle F \rangle = \mathrm{cl}_{\mathrm{tm}}(F) \cap$ Sym($\mathbb{N}$).

- $\mathrm{cl}_{\mathrm{tm}}($Sym($\mathbb{N}$)$)$ is the set of injections $\mathbb{N} \to \mathbb{N}$.

The main reason for introducing this topology is because of the connection between closed groups and structures.

Fact 2.4.2. *Let G be a subgroup of $Sym(\mathbb{N})$. Then G is closed if and only if there is a relational structure $\mathcal{M}$ such that $G = Aut(\mathcal{M})$.*

This is well known and can be found in, for example, [Cam90, Statement (2.6)]. Showing that the automorphism group of a structure is closed is a straightforward exercise in unravelling definitions. In the other direction, given a closed group G, the structure $\mathcal{M}$ is obtained by adding predicates for all the orbits of G's action on $\mathbb{N}^k$ for all k. The reason you need G to be closed is to ensure that G is indeed the automorphism group of this resulting structure.

2.5 Reducts

Definition 2.5.1. (i) Let $\mathcal{M}$ be an $\mathcal{L}$-structure. A relation $P \subseteq M^k$ is *$\emptyset$-definable* in $\mathcal{M}$ if there exists an $\mathcal{L}$-formula $\phi(x_1, \ldots, x_k)$ such that $P = \{(x_1, \ldots, x_k) \in M^k : \mathcal{M} \models \phi(x_1, \ldots, x_k)\}$.

(ii) Let $\mathcal{M}, \mathcal{N}$ be two structures on the same domain M. We say $\mathcal{N}$ is a *reduct* of $\mathcal{M}$ if all relations $\emptyset$-definable in $\mathcal{N}$ are $\emptyset$-definable in $\mathcal{M}$.

Intuitively, $\mathcal{N}$ is a reduct of $\mathcal{M}$ if $\mathcal{N}$ is a less detailed version of $\mathcal{M}$ or if $\mathcal{N}$ is obtained by discarding information from $\mathcal{M}$.

If $\mathcal{M}$ is a structure and $P_1, P_2, \ldots$ are $\emptyset$-definable relations in $\mathcal{M}$, then it is a straightforward exercise to show that $(M, P_1, P_2, \ldots)$ is a reduct of $\mathcal{M}$. We will use this fact implicitly throughout without further mention.

Consider the structure $(\mathbb{Q}, <)$. The following are examples of reducts of it.

- $(\mathbb{Q}, =)$.

- $(\mathbb{Q}, P_1)$ where $P_1 = \{(x, y, z) \in \mathbb{Q}^3 : x < y < z\}$.

- $(\mathbb{Q}, <_w)$ where $<_w = \{(x, y, a, b) \in \mathbb{Q}^4 : x < y \leftrightarrow a < b\}$.

- $(\mathbb{Q}, cyc)$ where $cyc = \{(x, y, z) \in \mathbb{Q}^3 : x < y < z \text{ or } y < z < x \text{ or } z < x < y\}$.

Given a structure $\mathcal{M}$ a natural question to ask is: How many reducts does $\mathcal{M}$ have? I claim this is a natural question, however, as stated this question is completely meaningless. This is because there would be arbitrarily many reducts: Let $\mathcal{M}$ be any structure, let I be any indexing set and for all $i \in I$ let P_i be the equality relation on M, so that $(M, (P_i)_{i \in I})$ is a reduct of $\mathcal{M}$. By varying I as we wish, this construction gives universe-many reducts.

The issue with this naive approach is that we are distinguishing structures based on the syntactical formalities of first-order logic. Instead, and this is a major theme in model theory, we should consider two structures which contain the same information to be the same structure. This is formalised in the following definition.

Definition 2.5.2. Let $\mathcal{M}$ and $\mathcal{N}$ be two structures on the same domain.

(i) $\mathcal{N}$ is *(first-order) definable* in $\mathcal{M}$ if for all named relations P of N, P is $\emptyset$-definable in M.

(ii) $\mathcal{N}$ and $\mathcal{M}$ are *(first-order) interdefinable* if $\mathcal{N}$ is definable in $\mathcal{M}$ and vice-versa.

(iii) $\mathcal{N}$ is a *proper reduct* of $\mathcal{M}$ if $\mathcal{N}$ is a reduct of $\mathcal{M}$ but not interdefinable with $\mathcal{M}$.

Remarks.

(i) **IMPORTANT.** From now on, two structures which are interdefinable will be considered to be the same structure.

(ii) $\mathcal{N}$ being definable in $\mathcal{M}$ is equivalent to $\mathcal{N}$ being a reduct of $\mathcal{M}$.

(iii) There are refined notions of interdefinability where one puts restrictions on the formulas that one can use, e.g. the study of reducts up to pp-interdefinability

gives information on the computational complexity of certain CSPs (constraint satisfaction problems).

Let us have a second look at the examples of reducts of $(\mathbb{Q}, <)$. The binary relation $<$ is definable in $(\mathbb{Q}, P_1)$, so $(\mathbb{Q}, <)$ and $(\mathbb{Q}, P_1)$ are inter-definable and thus considered to be the same structure. The binary relation $<$ is not definable in $(\mathbb{Q}, =)$, $(\mathbb{Q}, <_w)$ or $(\mathbb{Q}, cyc)$, so these three structures are proper reducts of $(\mathbb{Q}, <)$. We can show that $<$ is not definable in these structures by finding an automorphism that does not preserve $<$. For example, $x \mapsto -x$ is an automorphism of $(\mathbb{Q}, <_w)$ which does not preserve $<$.

With the caveat that two inter-definable structures are considered the same, the question of how many reducts a structure has becomes interesting. The first result of this kind is (a consequence of) a theorem of Cameron. Note that cyc_w is the 6-ary relation $\{(a, b, c, x, y, z) \in \mathbb{Q}^6 : cyc(a, b, c) \leftrightarrow cyc(x, y, z)\}$.

Theorem 2.5.3. *([Cam76]).* $(\mathbb{Q}, <)$ *has exactly 5 reducts, namely* $(\mathbb{Q}, <)$, $(\mathbb{Q}, <_w)$, $(\mathbb{Q}, cyc)$, $(\mathbb{Q}, cyc_w)$ *and* $(\mathbb{Q}, =)$.

In the 90s, Thomas and their PhD student Bennett obtained similar results.

- The random graph has 5 reducts. ([Tho91]).

- The Henson graphs have 2 reducts (so no non-trivial reducts.) ([Tho91]).

- The random k-hypergraph has $2^k + 1$ reducts. ([Tho96]).

- The random tournament has 5 reducts. ([Ben97]).

Based on their initial results, Thomas conjectured in [Tho91] that any countable structure homogeneous in a finite relational language has only finitely many reducts. Evidence for this conjecture has been building, for example:

- $(\mathbb{Q}, <, P_1, \ldots, P_k)$ has finitely many reducts, where the P_i are unary relations which are intervals. In particular, $(\mathbb{Q}, <, 0)$ has 116 reducts. ([JZ08]).

- The generic partial order has 5 reducts. ([PPP$^+$14]).

- The generic ordered graph has 44 reducts. ([BPP15]).

Thomas' conjecture remains unsolved and provides motivation for further study. In particular, two of the main results of this thesis are that the generic digraph and the Henson digraphs have finitely many reducts.

The lattice of reducts

As well as counting the number of reducts for some structure $\mathcal{M}$, we could also ask how the reducts themselves compare with one another. In general, the set of reducts of a structure $\mathcal{M}$ form a lattice, where we say $\mathcal{N} < \mathcal{N}'$ if $\mathcal{N}$ is a proper reduct of $\mathcal{N}'$. The top element is always the original structure $\mathcal{M}$ and the bottom element is the trivial structure $(M, =)$. The meet (respectively join) of two structures $\mathcal{N}$ and $\mathcal{N}'$ will be the structure whose named relations are precisely the $\emptyset$-definable relations that are definable in both (respectively in at least one of) $\mathcal{N}$ and $\mathcal{N}'$. Intuitively, the meet contains the intersection of the information in the two structures, and the join contains the union of the information.

For example, the reducts of $(\mathbb{Q}, <)$ form the following lattice.

$$
\begin{array}{ccc}
& (\mathbb{Q}, <) & \\
\diagup & & \diagdown \\
(\mathbb{Q}, <_w) & & (\mathbb{Q}, cyc) \\
\diagdown & & \diagup \\
& (\mathbb{Q}, cyc_w) & \\
& | & \\
& (\mathbb{Q}, =) &
\end{array}
$$
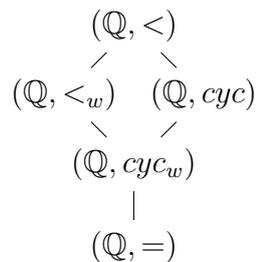
Reducts and closed groups

The proofs of many, but not all (e.g. [JZ08]), of these results use the correspondence between countable structures and closed subgroups of $\mathrm{Sym}(\mathbb{N})$. This correspondence is useful because to study the reducts of a structure $\mathcal{M}$, we can instead look at the closed groups lying in between $\mathrm{Aut}(\mathcal{M})$ and $\mathrm{Sym}(\mathbb{N})$. However, this approach does not always succeed. For example, consider the graph $(\mathbb{Z}, E)$ where vertices u, v are adjacent if $|u - v| = 1$. Then let P be the equivalence relation where $u \sim v$ if the distance between u and v is even. Now $\mathrm{Aut}(\mathbb{Z}, E) < \mathrm{Aut}(\mathbb{Z}, P)$, but $(\mathbb{Z}, P)$ is *not* a reduct of $(\mathbb{Z}, E)$ - we would require an infinite disjunction to define P from E. Fortunately, this kind of issue does not arise in the context of $\aleph_0$-categorical structures.

Lemma 2.5.4. *Let $\mathcal{M}$ be an $\aleph_0$-categorical structure. Then the mapping $\mathcal{N} \mapsto Aut(\mathcal{N})$ is an anti-isomorphism from the lattice of reducts of $\mathcal{M}$ to the lattice of closed groups G such that $Aut(\mathcal{M}) \leq G \leq Sym(\mathbb{N})$.*

Remark: This is well known, however, the only proof we have yet found of this in the literature is 'It follows from the Ryll-Nardzewski Theorem.'

Proof. First we show this function reverses the ordering on the lattices. Suppose $\mathcal{N} \leq \mathcal{N}'$ and that $f \in \mathrm{Aut}(\mathcal{N}')$ - we need to show that f must also be in $\mathrm{Aut}(\mathcal{N})$. Now, $f \in \mathrm{Aut}(\mathcal{N}')$ implies f preserves all $\emptyset$-definable relations in $\mathcal{N}'$. As $\mathcal{N}$ is a reduct of $\mathcal{N}'$, this implies that f preserves all $\emptyset$-definable relations in $\mathcal{N}$. Hence $f \in \mathrm{Aut}(\mathcal{N})$, as required.

Next, we show that this mapping is surjective, so let G be a closed group such that $\mathrm{Aut}(\mathcal{M}) \leq G \leq \mathrm{Sym}(\mathbb{N})$. We claim that the structure $\mathcal{N}$ corresponding to G (from Fact 2.4.2) is suitable. All that needs to be checked is that $\mathcal{N}$ is a reduct of $\mathcal{M}$. To do this, it suffices to show that all the named relations of $\mathcal{N}$ are definable in $\mathcal{M}$. By the construction of $\mathcal{N}$, it suffices to show that all the orbits of G are definable in $\mathcal{M}$.

Since $G \geq \text{Aut}(\mathcal{M})$, any orbit of G is a union of orbits of $\text{Aut}(\mathcal{M})$. By the Ryll-Nardzewski Theorem, we can say that this union is a finite union and also that the orbits of $\text{Aut}(\mathcal{M})$ are themselves definable in $\mathcal{M}$. But a finite union of definable sets is plainly definable.

Lastly, we show that the mapping is injective, so let $\mathcal{N}$ and $\mathcal{N}'$ be two reducts of $\mathcal{M}$ such that $\text{Aut}(\mathcal{N}) = \text{Aut}(\mathcal{N}')$. This means that $\mathcal{N}$ and $\mathcal{N}'$ have the same orbits, so by Theorem 2.2.2 (ii) they have the same types. Then by the Ryll-Nardzewski Theorem, this implies $\mathcal{N}$ and $\mathcal{N}'$ have the same definable sets. By definition, this implies that $\mathcal{N}$ and $\mathcal{N}'$ are reducts of each other, hence they are equal, as required. $\qquad\square$

IMPORTANT. In this thesis, all the structures we study are $\aleph_0$-categorical, so by the above fact reducts correspond bijectively to closed groups. For this reason, we identify the notion of a reduct of a structure $\mathcal{M}$ and a closed group containing $\text{Aut}(\mathcal{M})$.

2.6   Structural Ramsey Theory

Structural Ramsey Theory forms a small, though crucial, ingredient for our work on the generic digraph and Henson digraphs. Unfortunately, its significance may not be evident as its use is buried inside the proof of a lemma from a published paper. Nevertheless, for completeness' sake and because it is a fun notion, we provide some background.

We start by recalling the original (Finite) Ramsey's Theorem. To do this, we introduce some non-standard notation (which we will shortly generalise): for natural numbers $A \leq B$, we let $\binom{B}{A}$ denote the set of subsets $\{X \subseteq \{1, 2, \ldots, B\} : |X| = A\}$.

Theorem 2.6.1. *([Ram30]). For all $A \leq B \in \mathbb{N}$, there is $C \geq B \in \mathbb{N}$ such that for all colourings $\chi : \binom{C}{A} \to \{ \text{red, blue} \}$, there exists $B' \in \binom{C}{B}$ such that the restriction of χ to $\binom{B'}{A}$ is a constant map.*

Terminology: Subsets B' on which the restriction of a colouring is constant are known as monochromatic ('single-coloured') subsets.

Motto: Can you find a big thing (C) so that when you colour in copies of little things (A) you can find a monochromatic medium sized thing (B)?

It will probably take several minutes just to parse the statement, and it may be worthwhile to consider special cases. For example, when $A = 1$ this theorem corresponds to the pigeonhole principle, where for all B, $C = 2B - 1$ will work; if I have 19 letters to place into 2 pigeonholes, one of the pigeonholes must contain at least 10 letters. When $A = 2$ and $B = 3$, this theorem corresponds to the famous puzzle of how many people are sufficient at a party to guarantee finding three people who all know each other or three people who all do not know each other.

The theorem is actually true for any finite number of colours; the proof is a simple (and cute!) induction so the mathematical difficulty is concentrated on the case of two colours. For the sake of conciseness we (and frequently the literature) avoid introducing another variable for the number of colours.

To generalise this theorem to structures, we first generalise the notation. Let A, B be $\mathcal{L}$-structures.

- We write $A \leq B$ to mean that there is a subset of B isomorphic to A, i.e., that A is embeddable in B.

- $\binom{B}{A} := \{X \subseteq B : X$ is $\mathcal{L}$-isomorphic to $A\}$.

With one more change (replacing $\mathbb{N}$ with a set of $\mathcal{L}$-structures), the statement of Ramsey's Theorem above becomes the definition of a Ramsey class.

Definition 2.6.2. Let $\mathcal{C}$ be a set of finite $\mathcal{L}$-structures. We say that $\mathcal{C}$ is a *Ramsey class* if: For all $A \leq B \in \mathcal{C}$, there is $C \geq B \in \mathcal{C}$ such that for all colourings $\chi : \binom{C}{A} \to \{$ red, blue $\}$, there exists $B' \in \binom{C}{B}$ which is monochromatic.

Examples:

- The set of all finite sets.

- The set of all finite linear orders.

- The set of all finite complete graphs.

The statement that each of these are Ramsey classes is equivalent to Ramsey's Theorem above, just re-phrased. A non-example is the set of finite graphs; a proof that this is not a Ramsey class can be found in [Prö13, Page 137, Proposition 12.11] and we note these pages are available on the online Google preview of the book. A non-trivial example of a Ramsey class is the set of all finite linearly-ordered graphs, a proof of which can be found on the next page of the same book.

That adding a linear order to relational structures creates a Ramsey class is no coincidence, and the next theorem we describe will make this evident. The theorem is due to Nešetřil and Rödl, [NR83], who use non-standard terminology. I will introduce their terminology and then immediately translate it using standard notions.

An indexed family $\Delta = (\delta_i : i \in I)$ of natural numbers is called a *type*. A *set system of type* Δ is a pair $(X, \mathcal{M})$ where X is a linearly ordered finite set and $\mathcal{M} = (\mathcal{M}_i : i \in I)$ where for every $M \in \mathcal{M}_i, M \subseteq X$ and $|M| = \delta_i$.

I will now translate this. An n-ary relation R of X is *symmetric* if for all $x_1, \ldots, x_n \in X$ and all σ in the symmetric group S_n, we have $R(x_1, \ldots, x_n)$ if and only if $R(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. We say the relation R is *irreflexive* if $R(x_1, \ldots, x_n)$ implies $(x_1, \ldots, x_n)$ is a proper tuple. Now, let $\mathcal{L} = \{R_i : i \in I\} \cup \{<\}$ be a relational language. Then the type, in the sense of the above paragraph, is the set of arities of the relation symbols and a set system is a finite $\mathcal{L}$-structure where $<$ is interpreted as a linear order and where R_i is symmetric and reflexive for all i. We replace the phrase 'a set system of type Δ' with the phrase 'an $\mathcal{L}$ set system'.

To state the main result of [NR83] we need one more definition. A set system X is called *irreducible* if for for all $x, y \in X$ there is R_i, of arity n say, and $x_3, \ldots, x_n$ such that $R_i(x, y, x_3, \ldots, x_n)$.

Theorem 2.6.3. *([NR83, Theorem A]) Let $\mathcal{L} = \{R_i : i \in I\} \cup \{<\}$ be a relational language and let $\mathfrak{a}$ be a set of irreducible $\mathcal{L}$ set systems. Let $\mathcal{C}$ be the set of $\mathcal{L}$ set systems which do not embed any set system from $\mathfrak{a}$. Then $\mathcal{C}$ is a Ramsey class.*

Remark: Elements of $\mathfrak{a}$ are known as forbidden systems.

To end, we need one final definition.

Definition 2.6.4. A structure $\mathcal{M}$ is *Ramsey* if the age of $\mathcal{M}$ is a Ramsey class.

2.7 Canonical functions

Definition 2.7.1. Let $\mathcal{M}, \mathcal{N}$ be any structures. Let $f : \mathcal{M} \to \mathcal{N}$ be any function between the domains of the structures.

(i) The *behaviour* of f is the relation $\{(p, q) \in S(\mathcal{M}) \times S(\mathcal{N}) : \exists \bar{a} \in M, \bar{b} \in N$ such that $\mathrm{tp}(\bar{a}) = p$, $\mathrm{tp}(\bar{b}) = q$ and $f(\bar{a}) = \bar{b}\}$.

(ii) If the behaviour of f is a function $S(\mathcal{M}) \to S(\mathcal{N})$, then we say f is *canonical*. Rephrased, we say f is canonical if for all $\bar{a}, \bar{a}' \in M$, $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{a}') \Rightarrow \mathrm{tp}(f(\bar{a})) = \mathrm{tp}(f(\bar{a}'))$.

(iii) If f is canonical, we use the same symbol f to denote its behaviour.

Examples.

- For any structure $\mathcal{M}$, every automorphism $f \in \mathrm{Aut}(\mathcal{M})$ is a canonical function and for all types $p \in S(\mathcal{M})$, $f(p) = p$.

- Let $\mathcal{M} = (M, =)$. Then $f : \mathcal{M} \to \mathcal{M}$ is canonical if and only if f is constant or injective.

- $- : (\mathbb{Q}, <) \to (\mathbb{Q}, <), x \mapsto -x$, is canonical.

Lemma 2.7.2. *(i) The composition of canonical functions is canonical: If $f : \mathcal{M} \to \mathcal{M}'$ and $g : \mathcal{M}' \to \mathcal{M}''$ are canonical, then $g \circ f : \mathcal{M} \to \mathcal{M}''$ is canonical.*

(ii) If $f : \mathcal{M} \to \mathcal{N}$ is canonical and $\mathcal{N}'$ is a reduct of $\mathcal{N}$, then f remains canonical when considered as a function $\mathcal{M} \to \mathcal{N}'$.

Proof. (i) Let $\bar{a}, \bar{b} \in \mathcal{M}$ have the same type; we want to show that $g \circ f(\bar{a})$ and $g \circ f(\bar{b})$ have the same type. Since f is canonical, tp$(f(\bar{a}))$ =tp$(f(\bar{b}))$. Then since g is canonical, tp$(g(f(\bar{a})))$ =tp$(g(f(\bar{b})))$, as required.

(ii) Let $\bar{a}, \bar{b} \in \mathcal{M}$ have the same type; we want to show that $f(\bar{a})$ and $f(\bar{b})$ have the same type in $\mathcal{N}'$. By assumption we know that $f(\bar{a})$ and $f(\bar{b})$ have the same type in $\mathcal{N}$. This means that $\bar{a}$ and $\bar{b}$ satisfy the same formulas i.e. they are elements of the same definable sets of $\mathcal{N}$. By the definition of a reduct, this means they lie in the same definable sets of $\mathcal{N}'$, i.e. they have the same type in $\mathcal{N}'$, as required. $\square$

The benefit of canonical functions is that they are particularly well-behaved and can be easily manipulated and analysed. Furthermore, the next result essentially reduces the task of determining reducts to the task of analysing the behaviours of canonical functions.

Lemma 2.7.3. *([BPT13, Lemma 14]). Let $\mathcal{M} = (M, <, R_1, \ldots, R_k)$ be a countable structure in a finite relational language. Suppose that:*

- *$<$ is interpreted as a linear order.*

- *$\mathcal{M}$ is homogeneous.*

- *$\mathcal{M}$ is Ramsey.*

- $g : M \rightarrow M$ *is any function.*

- $c_1, \ldots, c_n$ *are any elements of* M.

Then there exists a function $f : M \rightarrow M$ *such that:*

(i) $f \in cl_{tm}(Aut(\mathcal{M}) \cup \{g\})$.

(ii) $f(c_i) = g(c_i)$ *for* $i = 1, \ldots n$.

(iii) *When regarded as a function from* $(M, <, R_1, \ldots, R_k, \bar{c})$ *to* $(M, <, R_1, \ldots, R_k)$, f *is a canonical function.*

This lemma is the reason for introducing structural Ramsey theory and the topological monoid $\mathbb{N}^{\mathbb{N}}$. We highly recommend understanding the proof of this lemma as the argument is an elegant combination of several notions and powerful tools.

For our purposes, we actually require a slightly weaker conclusion than what is stated, namely that f is canonical when regarded as a function from $(M, <, R_1, \ldots, R_k, \bar{c})$ to $(M, R_1, \ldots, R_k)$. (The linear order has been dropped from the codomain.) This is indeed a weaker statement by Lemma 2.7.2 (ii).

Chapter 3

The Generic Digraph

3.1 Basic definitions and facts

Definition 3.1.1. (i) A directed graph (V, E) consists of a set V and an irreflexive, antisymmetric relation $E \subseteq V^2$. V represents the set of vertices and E represents the set of directed edges. If $(a, b) \in E$, we visualise it as an edge going out of a and into b. We abbreviate 'directed graph' by 'digraph'.

(ii) By an empty digraph we mean a digraph whose edge set is empty.

(iii) The generic digraph is the Fraïssé limit of the set of all finite digraphs. In this chapter, (D, E) will denote the generic digraph.

(iv) We let $\bar{E}(x, y)$ denote the underlying graph relation $E(x, y) \vee E(y, x)$. We let $N(x, y)$ denote the non-edge relation $\neg \bar{E}(x, y)$.

That the set of all finite digraphs is an amalgamation class follows exactly the same argument as for undirected graphs - see Section 2.3. This is a common theme: many properties of the generic digraph are straightforward adaptations from the random graph.

For this reason, we do not always provide full details and instead we will direct the reader to the relevant part of Cameron's excellent survey on the random graph, [Cam97].

The following lemma collects several useful properties of the generic digraph.

Lemma 3.1.2. *(i) (D, E) is $\aleph_0$-categorical and has quantifier elimination.*

(ii) Let $\bar{a}, \bar{b} \in D$. If $tp(\bar{a}) = tp(\bar{b})$, then there exists an automorphism mapping $\bar{a}$ to $\bar{b}$.

(iii) The generic digraph (D, E) is the unique, up to isomorphism, countable digraph satisfying the following extension property: for all finite pairwise disjoint subsets $U, V, W \subset D$ there exists $x \in D \backslash (U \cup V \cup W)$ such that $(\forall u \in U) E(x, u)$, $(\forall v \in V) E(v, x)$ and $(\forall w \in W) N(x, w)$.

(iv) All countable digraphs can be embedded into the generic digraph.

(v) Let $A \subseteq D$ and $B = A^c$. Then $(A, E|_A)$ or $(B, E|_B)$ is isomorphic to the generic digraph.

(vi) Let $A \subset D$ be finite and of size at least 2, let $a_1, a_2 \in A$ and let $(b_1, \ldots, b_n) \in D$ such that $(a_1, a_2) \cong (b_1, b_2)$. Then there exists $f \in Aut(D, E)$ such that $f(b_1, b_2) = (a_1, a_2)$ and $f(b_i) \in D \backslash A$ for $i = 3, \ldots, n$.

Remark. Due to the importance of the property in (iii), we give it the name 'the extension property'.

Proof. (i) and (ii) are instances of the general theory from Chapter 2, Theorem 2.3.4 and Theorem 2.2.2.

(iii) First we show that (D, E) has the extension property, so let U, V and W be finite disjoint subsets of D. Let $A = U \cup V \cup W$. Consider the finite digraph B obtained from A by adding a new vertex c so that we have edges from c to all elements of U, edges from V to c and non-edges between c and W. The age of (D, E) is the set of all finite

digraphs, thus B is embeddable in D; let B' be the copy of B inside D, let A' be the copy of A inside B' and let c' be the copy of c. Let f be an isomorphism from A' to A. By homogeneity f can be extended to an automorphism g of (D, E). Then $g(c')$ witnesses the extension property, as required.

Next we need to show uniqueness, i.e., that two countable digraphs which satisfy the extension property are isomorphic. To do this one uses a standard back-and-forth argument, identical to that for the random graph - see [Cam97, Fact 2].

(iv) This is proved using only the forth part of a back-and-forth argument. Again, the argument is identical to that for the random graph - see [Cam97, Proposition 6].

(v) By (iii), it suffices to show that (A, E_A) or (B, E_B) satisfies the extension property. Suppose for contradiction that both fail the extension property. Let $U_1, V_1, W_1 \subset A$ and $U_2, V_2, W_2 \subset B$ witness this failure. Now let $U = U_1 \cup U_2, V = V_1 \cup V_2$ and $W = W_1 \cup W_2$. These are finite pairwise disjoint subsets of D. By (iii), we know that D satisfies the extension property, so we can find an appropriate witness x in D. Now observe that x is also a witness for U_1, V_1, W_1 and for U_2, V_2, W_2. But this means we have a contradiction, because x must be in A or in B.

(vi) Let $a_1, \ldots, a_k$ be an enumeration of A. By the extension property, there are $a'_3, \ldots, a'_k \in D \backslash \{b_1, \ldots, b_n\}$ such that $(b_1, b_2, a'_3, \ldots, a'_k) \cong (a_1, \ldots, a_k)$. By homogeneity, there is $f \in \mathrm{Aut}(D, E)$ mapping $(b_1, b_2, a'_3, \ldots, a'_k)$ to $(a_1, \ldots, a_k)$, and such an f satisfies the requirements of the lemma. $\qquad \square$

In order to use the canonical functions machinery (in particular, Lemma 2.7.3) we need to expand the generic digraph to include a linear order. This is described in the following definition.

Definition 3.1.3. (i) An *ordered digraph* is a digraph which is also linearly ordered. Formally, it is a structure $(V, E, <)$ where (V, E) is a digraph and $(V, <)$ is a linear order.

(ii) We let $(D, E, <)$ be the Fraïssé limit of the set of all finite ordered digraphs.

To prove that the set of all finite ordered digraphs is an amalgamation class, one takes the union of the arguments that show finite digraphs and finite linear orders form an amalgamation class - see Section 2.3.

Lemma 3.1.4. $(D, E, <)$ *is a Ramsey structure.*

Remark: We note that this is not an original result, but the proofs we have seen in the literature are brief and lack detail.

Proof. We want to use Theorem 2.6.3, however, it requires symmetric relations and by definition the edge relations of digraphs are not symmetric. To solve this problem, one encodes the edge relation of a (finite) ordered digraph $(D, E, <)$ using two symmetric relations E_1, E_2 - one relation for each possible direction of the edge. Explicitly, given $(D, E, <)$, we draw an E_1-edge between $x, y \in D$ if $x < y$ and $E(x, y)$, and draw an E_2-edge if $x < y$ and $E(y, x)$. This gives a mapping, Θ say, from finite ordered digraphs to $\{E_1, E_2, <\}$ set systems. Then the inverse mapping is forced: given an $\{E_1, E_2, <\}$ set system $(D, E_1, E_2, <)$ and $x < y \in D$, we draw an E-edge from x to y if $E_1(x, y)$ and draw an E-edge from y to x if $E_2(x, y)$.

Now we are done: by Theorem 2.6.3 the set of $\{E_1, E_2, <\}$ set systems is a Ramsey class, so by the correspondence given by Θ the set of finite ordered digraphs is also a Ramsey class.

This may be convincing, but there are two details we need to check. The first is whether this mapping Θ is a bijection between the set of finite ordered digraphs and the set of $\{E_1, E_2, <\}$ set systems. This is actually false, and the (only) reason is that an $\{E_1, E_2, < \}$ set system can have E_1 and E_2 holding on the same tuple, which corresponds to there being a directed edge in both directions, which is not allowed. To fix this we simply forbid this phenomenon: The 2-element $\{E_1, E_2, <\}$ set system with domain $A = \{a, b\}$

and where both $E_1(a, b)$ and $E_2(a, b)$ hold is an irreducible set system, so forbidding A still allows us to use Theorem 2.6.3.

The second detail we need to check is that embeddings between digraphs correspond to embeddings between set systems. A priori, this might look false: an embedding between set systems cannot map E_1-edges to E_2-edges but an embedding between digraphs can change the direction of an edge. However, this is actually not an issue as we are dealing with *ordered* digraphs, so embeddings cannot change the direction of an edge. □

Remark: To highlight the need of the second detail, note that one can encode a (non-ordered) digraph using two symmetric relations: one can enumerate the elements of D and use this enumeration to provide the sense of direction. In this case, you really can have embeddings which change the direction of an edge, because the enumeration is added 'from the outside' so does not need to be preserved by the embeddings.

3.2 Statement of classification

There are two main ways of defining a reduct of (D, E). The first is to find a definable relation P and let the reduct be (D, P). The second is to add a function $f \in \text{Sym}(D)$ to $\text{Aut}(D, E)$ and close under group operations and close under the topology. In view of this, we establish some (abuse of) notation.

Notation. Recall that for $F \subseteq \text{Sym}(D)$, we let $\langle F \rangle$ denote the smallest closed subgroup of $\text{Sym}(D)$ containing F. For brevity, we abuse notation and in this chapter write $\langle F \rangle$ to mean $\langle F \cup \text{Aut}(D, E) \rangle$.

We begin by showing that three particular functions $-, sw$ and rot exist. These functions will give us the three reducts $\langle - \rangle, \langle sw \rangle$ and $\langle rot \rangle$.

Lemma 3.2.1. *There exists $f \in Sym(D)$ such that for all $x, y \in D$, $E(f(x), f(y))$ iff $E(y, x)$.*

Remark. For the rest of this chapter, we choose one such bijection and denote it by $-$.

Proof. The idea is to construct a structure (D, E') which is isomorphic to (D, E), in such a way that any function $f \in \text{Sym}(D)$ witnessing this fact has the desired property. For this lemma, we let $E'(x, y) = E^*(x, y)$. (The star notation is defined in Section 2.1). We need to show that (D, E^*) is isomorphic to (D, E).

By Lemma 3.1.2, it suffices to show that (D, E^*) satisifes the extension property. So let U, V, W be finite disjoint subsets of D. By the definition of E^*, we need to find $x \in D \backslash (U \cup V \cup W)$ such that $\forall u \in U, E(u, x), \forall v \in V, E(x, v)$ and $\forall w \in W, N(x, w)$. This is simply the extension property for (D, E) with the role of U and V swapped, so we know such an x exists (again by Lemma 3.1.2). Thus, (D, E) and (D, E^*) are isomorphic, as required. $\square$

Lemma 3.2.2. *There exists $f \in Sym(D)$ and $a \in D$ such that:*

$$E(f(x), f(y)) \text{ if and only if } \begin{cases} E(x, y) \text{ and } x, y \neq a, \text{ OR,} \\ E^*(x, y) \text{ and } x = a \vee y = a \end{cases}$$

Remark. For the rest of this chapter, we choose one such bijection and denote it by sw.

Remark. In words, sw changes the direction of those edges adjacent to a.

Proof. As in the previous lemma, the idea is to find an appropriate structure (D, E') isomorphic to (D, E). Let $a \in D$ and define $E'(x, y)$ as follows:

$$E'(x, y) \text{ if and only if } \begin{cases} E(x, y) \text{ and } x, y \neq a, \text{ OR,} \\ E^*(x, y) \text{ and } x = a \vee y = a. \end{cases}$$

As before, Lemma 3.1.2 tells us that we need to establish the extension property for

(D, E'). Let $U, V, W \subset D$ be finite and pairwise disjoint, and without loss assume that $a \in U \cup V \cup W$. This time the proof splits into three cases.

Case 1: $a \in U$. Let $U' = U \backslash \{a\}$ and $V' = V \cup \{a\}$. Then the extension property of (D, E) applied to U', V', W gives an appropriate x.

Case 2: $a \in V$. Let $U' = U \cup \{a\}$ and $V' = V \backslash \{a\}$. Then again the extension property of (D, E) gives us an appropriate x.

Case 3: $a \in W$. Then applying the extension property of (D, E) gives us an appropriate x, without needing to modify U, V or W.

Thus, (D, E') satisfies the extension property and hence is isomorphic to (D, E). We end by letting f be an isomorphism $(D, E') \to (D, E)$. $\qquad\square$

Lemma 3.2.3. *There exists $f \in Sym(D)$ and $a \in D$ such that:*

$$
E(f(x), f(y)) \text{ if and only if } \begin{cases} x, y \neq a \text{ and } E(x, y) \\ x = a \text{ and } N(x, y) \\ y = a \text{ and } E(y, x) \end{cases}
$$

Remark. For the rest of this chapter, we fix such a bijection and denote it by rot.

Remark. In words, rot maps edges going out of a to edges going into $rot(a)$, edges going into a to non-edges, and non-edges adjacent to a to edges going out of $rot(a)$.

Proof. The strategy is the same as for $-$ and sw. Let $a \in D$ and define $E'(x, y)$ as follows:

$$
E'(x, y) \text{ if and only if } \begin{cases} x, y \neq a \text{ and } E(x, y) \\ x = a \text{ and } N(x, y) \\ y = a \text{ and } E(y, x) \end{cases}
$$

As before, Lemma 3.1.2 tells us that we need to establish the extension property for (D, E'). Let $U, V, W \subset D$ be finite and pairwise disjoint, and without loss assume that $a \in U \cup V \cup W$.

Case 1: $a \in U$. Let $U' = U \backslash \{a\}$ and $V' = V \cup \{a\}$. The extension property of (D, E) applied to U', V', W gives an appropriate x.

Case 2: $a \in V$. Let $V' = V \backslash \{a\}$ and $W' = W \cup \{a\}$. The extension property of (D, E) applied to U, V', W' gives us an appropriate x.

Case 3: $a \in W$. Let $W' = W \backslash \{a\}$ and $U' = U \cup \{a\}$. The extension property of (D, E) applied to U', V, W' gives us an appropriate x.

Thus, (D, E') satisfies the extension property and hence is isomorphic to (D, E). We end by letting f witness this isomorphism. $\qquad \square$

Definition 3.2.4. (i) We let $\Gamma = (D, \bar{E})$; recall that $\bar{E} := E(x, y) \vee E^*(x, y)$. Γ is a graph and, as will be proved later, is in fact (isomorphic to) the random graph.

(ii) We let $-_\Gamma \in \mathrm{Sym}(D)$ be a function which interchanges the sets of edges and non-edges in Γ.

(iii) Let $a \in D$. We let $sw_\Gamma \in \mathrm{Sym}(D)$ be a function which interchanges the sets of edges and non-edges adjacent to a, and preserves all other edges and non-edges.

Remarks. $(D, \bar{E})$ is interdefinable with (D, N). The existence of $-_\Gamma$ and sw_Γ follows from the same argument as was used for $-, sw$ and rot.

We are now ready to state the classification.

Theorem 3.2.5. *The closed groups lying between Aut(D, E) and Sym(D) are given by the following lattice:*
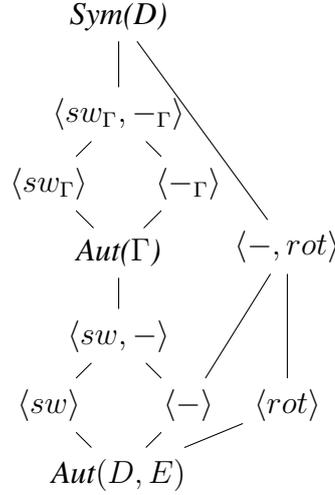
The generic digraph is $\aleph_0$-categorical, so recalling Lemma 2.5.4 we can identify the notions of closed groups and reducts. Thus the theorem above is legitimately a classification of the reducts of the generic digraph.

3.3 Understanding the reducts

In this section we establish several useful lemmas. The first few lemmas will provide some concrete information about the elements of $\langle sw \rangle, \langle - \rangle$ and $\langle rot \rangle$.

Definition 3.3.1. Let $f, g : D \to D$ and $A \subseteq D$. We say f *behaves like g on A* if for all finite tuples $\bar{a} \in A$, $f(\bar{a})$ is isomorphic (as a finite digraph) to $g(\bar{a})$. If $A = D$, we simply say f *behaves like g*.

Example. All automorphisms of (D, E) behave like the identity $id : D \to D$. Conversely, all $f \in \mathrm{Sym}(D)$ which behave like id are automorphisms.

Useful Remark. If $f : D \to D$ is any function and $g \in \mathrm{Aut}(D, E)$, then $h := g \circ f$ behaves like f. The converse is also true if f, h are bijections: if $h \in \mathrm{Sym}(D)$ behaves like $f \in \mathrm{Sym}(D)$, then there is $g \in \mathrm{Aut}(D, E)$ such that $h = g \circ f$. In particular,

two bijections which behave like each other are elements of the same supergroups of $\mathrm{Aut}(D, E)$.

We start with the simplest of the three groups, $\langle - \rangle$.

Lemma 3.3.2. $\langle - \rangle \supseteq \{f \in Sym(D) : f \text{ behaves like } - \}$.

Proof. Let $f \in \mathrm{Sym}(D)$ behave like $-$. Then observe that $g := - \circ f$ behaves like id so $g \in \mathrm{Aut}(D, E)$. Hence, $f = -^{-1} \circ g \in \langle - \rangle$. $\qquad\qquad\square$

In the lemma just proved, the '$\supseteq$' can in fact be replaced by equality. This follows from the classification of the reducts of the generic digraph: $\{f \in \mathrm{Sym}(D) : f \text{ behaves like } - \}$ is a closed group and it is not equal to $\mathrm{Aut}(D, E)$ so it must equal $\langle - \rangle$. The same will be true, and the same argument works, for the descriptions of $\langle sw \rangle$ and $\langle rot \rangle$.

For those curious, it *is* possible to prove equality directly and proofs can be found in [Aga16, Section 3]. We chose not to include them here for a few reasons. The first is that it is not necessary for proving the classification (though it was necessary in [Aga16] as parts of the classification were proved using a different method). The second is that the direct arguments in [Aga16] do not work for the Henson digraphs and we want to present a uniform approach. The third is that the proof we have provided is shorter and simpler, and works for all three groups $\langle - \rangle$, $\langle sw \rangle$ and $\langle rot \rangle$.

Next we look at $\langle sw \rangle$. For $A \subseteq D$, we let $sw_A : D \to D$ denote a function that behaves like id on A and A^c, and that switches the direction of all edges between A and A^c. For example, $sw = sw_a$ for some $a \in D$ and $sw_\emptyset$ is just an automorphism. We first need to check that sw_A even exists.

Lemma 3.3.3. *For all $A \subseteq D$, sw_A exists.*

Proof. The strategy is the same as for proving sw exists, except it is easier as we do not require sw_A to be a bijection.

Consider a new digraph (D, E') where

$$E'(x, y) \text{ if and only if } \begin{cases} E(x, y) \text{ and } (x \in A \leftrightarrow y \in A), \text{ OR,} \\ E^*(x, y) \text{ and } (x \in A \leftrightarrow y \notin A). \end{cases}$$

This is a countable digraph, so by Lemma 3.1.2 we can embed (D, E') into (D, E). Let sw_A be such an embedding to complete the proof. □

If possible, we choose sw_A to be a bijection. Note that there are cases where sw_A cannot be a bijection, namely when the image of sw_A is not isomorphic to the generic digraph. For example, let a be any element of D and let $A = \{x \in D : E(a, x)\}$, so then $sw_A(a)$ will not have any outward edges in its image.

Lemma 3.3.4. $\langle sw \rangle \supseteq \{f \in \text{Sym}(D) : \text{there is } A \subseteq D \text{ such that } f \text{ behaves like } sw_A\}.$

Proof. For this proof, let a denote the element in D so that sw behaves like sw_a.

Let $f \in \text{Sym}(D)$ and suppose there is $A \subseteq D$ such that f behaves like sw_A. We want to show that $f \in \langle sw \rangle$. To do this, it is sufficient to show that *some* function which behaves like sw_A is in $\langle sw \rangle$; this is because of the useful remark provided earlier.

We first prove this for finite A. If $A = \emptyset$, then $f \in \text{Aut}(D, E)$ and so $f \in \langle sw \rangle$. We continue by using induction on $|A|$. For the base case, let $A = \{a'\}$ and let $h \in \text{Aut}(D, E)$ map a' to a. Then $sw \circ h$ behaves like $sw_{a'}$ and is in $\langle sw \rangle$, completing the base case. Now suppose $|A| > 1$ and let $A = \{a_1, \ldots, a_n\}$. Let $A' = \{a_2, \ldots, a_n\}$ and let $a' = sw_{A'}(a_1)$. Then consider the function $sw_{a'} \circ sw_{A'}$. By the inductive hypothesis this function is in $\langle sw \rangle$. Also, this function behaves like sw_A (best seen by drawing a diagram), as required, completing the induction.

Now suppose A is infinite. Since $\langle sw \rangle$ is (topologically) closed, it suffices to show that f is in the closure of $\langle sw \rangle$. So let $B \subset D$ be finite and $\bar{b}$ enumerate its elements. We want to

find a function $g \in \langle sw \rangle$ such that $g(\bar{b}) = f(\bar{b})$. Since (D, E) is homogeneous, it suffices to find a g so that $g(\bar{b}) \cong f(\bar{b})$, i.e., so that g behaves like f on B. A suitable function is $sw_{A \cap B}$, which we know is in $\langle sw \rangle$ because $A \cap B$ is finite. $\qquad \square$

The intuition behind the previous proof is that to get a function which behaves like sw_A, you just switch about each of the elements of A.

The next reduct we analyse is $\langle rot \rangle$. For what follows, $A, B \subseteq D$ are disjoint and $C :=$ $(A \cup B)^c$. For the ordered pair (A, B), an *outward edge* is an edge going from A to B and an *inward edge* is one going from B to A. We say $f : D \to D$ behaves like rot between (A, B), or say 'between A and B', if f maps outward edges of (A, B) to inward edges of $(f(A), f(B))$, inward edges of (A, B) to non-edges between $f(A)$ and $f(B)$, and non-edges between A and B to outward edges of $(f(A), f(B))$.

We let $rot_{A,B,C}$ be a function $D \to D$ which behaves like id on A, B and C and behaves like rot between $(A, B), (B, C)$ and (C, A). We often omit C from the subscript as its role is implicitly determined by A and B. If $C = \emptyset$, so that $B = A^c$, we just write rot_A. If possible we choose $rot_{A,B,C}$ to be a bijection.

The fact that $rot_{A,B}$ exists follows the same argument as for sw: Consider a new digraph (D, E') which is exactly isomorphic to the image of the proposed function, and then let $rot_{A,B}$ be any embedding of (D, E') into (D, E). We do not provide a formal proof because we cannot think of a tidy method of defining the new edge relation E', so we regrettably leave the details to the reader.

Before reading the next proof we recommend the following tasks to internalise $rot_{A,B,C}$:

- Show that rot behaves like rot_a for some $a \in D$ and that $rot_{B,C,A}$ and $rot_{C,A,B}$ both behave like $rot_{A,B,C}$.

- Show that if f behaves like $rot_{A,B,C}$ and fixes A, B, C setwise, then f^{-1} and f^2 behave like $rot_{C,B,A}$, and f^3 behaves like id.

- Determine the behaviour of $rot_a \circ rot_{A,B,C}$, for various possibilities of a.

Lemma 3.3.5. $\langle rot \rangle \supseteq \{f \in Sym(D) : f \text{ behaves like } rot_{A,B} \text{ where } A, B \text{ are disjoint subsets of } D\}$.

Proof. For this proof, let $a \in D$ be the element so that rot behaves like rot_a.

Let A, B be disjoint subsets of D and let $f \in \text{Sym}(D)$ behave like $rot_{A,B}$. We want to show that $f \in \langle rot \rangle$, and to do this it suffices by the useful remark earlier to show that *some* function which behaves like $rot_{A,B}$ is in $\langle rot \rangle$.

We start by showing that $rot_{a'} \in \langle rot \rangle$ for all $a' \in D$. Let $h \in \text{Aut}(D, E)$ map a' to a then consider $rot \circ h$. This is an element of $\langle rot \rangle$ and behaves like $rot_{a'}$, as required.

We next consider the cases where A and B are both finite. We do this by induction on $|A \cup B|$. The base case is when both A and B are empty, which implies that f is an automorphism, so we are done. Now suppose $|A \cup B| \geq 1$. First suppose $|B| \geq 1$ and let $B = \{b_1, \ldots, b_n\}$. Let $B' = \{b_2, \ldots, b_n\}$. By the inductive hypothesis, $rot_{A,B'} \in \langle rot \rangle$. Now let $b' = rot_{A,B'}(b_1)$. Then consider $rot_{b'} \circ rot_{A,B'}$. This is in $\langle rot \rangle$. Less immediately, this behaves like $rot_{A,B}$ - to show this one goes through all cases and determines that we have the correct behaviour. E.g. let us look at the behaviour between b_1 and b_2, and suppose we have an edge from b_2 to b_1. Then applying $rot_{A,B'}$ maps this to an edge from the (current) image of b_1 to the image of b_2. Then applying $rot_{b'}$ maps this to an edge from the (final) image of b_2 to b_1. The result is behaving like id between b_1 and b_2, as required. One similarly checks the other (eight) possibilities, which we leave to the reader.

Now suppose that $|A| \geq 1$. Let $A = \{a_1, \ldots, a_n\}$, $A' = \{a_2, \ldots, a_n\}$, $a' = rot_{A',B}(a_1)$ and $a'' = rot_{a'}(a')$. Then consider $rot_{a''} \circ rot_{a'} \circ rot_{A',B}$. This is in $\langle rot \rangle$ by the inductive hypothesis. As before, after doing the relevant case checking, one will see that this behaves like $rot_{A,B}$. This completes the induction.

Now suppose $A \cup B$ is infinite. Then for all finite $X \subset D$, $rot_{A \cap X, B \cap X}$ behaves like $rot_{A,B}$

on X, so because $\langle rot \rangle$ is closed, we conclude that $rot_{A,B} \in \langle rot \rangle$, as required. $\qquad \square$

The intuition is that to obtain a function which behaves like $rot_{A,B}$, you rotate twice for each element of A and once for each element of B. (We realise that this intuition is off when $C = \emptyset$, in which case you rotate once about each element of A.)

The remaining lemmas will give us conditions on a group G to be equal to $\mathrm{Sym}(D)$ or to contain $\mathrm{Aut}(\Gamma)$. First we need a couple of definitions.

Definition 3.3.6. Let G be a subgroup of $\mathrm{Sym}(D)$ and $n \in \mathbb{N}$.

- G is *n-transitive* if for all proper tuples $\bar{a}, \bar{b} \in D^n$, there exists $g \in G$ such that $g(\bar{a}) = \bar{b}$.

- G is *n-homogeneous* if for all subsets $A, B \subset D$ of size n, there exists $g \in G$ such that $g(A) = B$.

Lemma 3.3.7. *Let $G \leq Sym(D)$ be a closed supergroup of $Aut(D, E)$.*

(i) *If G is n-transitive for all $n \in \mathbb{N}$, then $G = Sym(D)$.*

(ii) *If G is n-homogeneous for all $n \in \mathbb{N}$, then $G = Sym(D)$.*

(iii) *Suppose that whenever $A \subset D$ is finite and has at least one edge, there exists $g \in G$ such that $g(A)$ has fewer edges than in A. Then $G = Sym(D)$.*

(iv) *Suppose that there exists a finite $A \subset D$ and $g \in G$ such that g behaves like id on $D \backslash A$, g behaves like id between A and $D \backslash A$, and, g deletes at least one edge in A. Then $G = Sym(D)$.*

Proof. (i) Let G be an n-transitive closed group and let $f \in \mathrm{Sym}(D)$. Since G is closed, it suffices to show that $f \in \mathrm{cl}(G)$. So let $\bar{a} \in D$ be a finite tuple - we need to find $g \in G$ such that $g(\bar{a}) = f(\bar{a})$. Such a g exists by n-transitivity. Thus $f \in \mathrm{cl}(G)$, as required.

(ii) We will show that G is n-transitive, so let $\bar{a}, \bar{b} \in D$ be proper tuples of length n. Then by n-homogeneity we can map $\bar{a}$ and $\bar{b}$ to the empty digraph using some functions in G, say f and g respectively. By homogeneity, there is an automorphism h of (D, E) mapping $f(a_i)$ to $g^{-1}(b_i)$ for all i. Then $g^{-1}hf$ is a function in G that maps $\bar{a}$ to $\bar{b}$, as required.

(iii) By repeatedly using the assumptions given in the lemma, we can map any finite set A to the empty digraph. This implies that G is n-homogeneous so we are done by (ii).

(iv) Let A and g be as in the lemma. We will show that for all finite $B \subset D$, if B contains at least one edge then there is $f \in G$ such that $f(B)$ has less edges than B has - this suffices by (iii). So let $B \subset D$ be finite, let $b_1 b_2$ be an edge in B and let $\{b_3, \ldots, b_n\}$ enumerate its remaining vertices. Let $a_1 a_2 \in A$ be an edge that is deleted by g. By Lemma 3.1.2 (vi), there is $h \in \mathrm{Aut}(D, E)$ such that $h(b_1, b_2) = (a_1, a_2)$ and $h(b_i) \notin A$ for $i = 3, \ldots, n$. Then $gh \in G$ and $gh(B)$ contains less edges than in B, as required. $\quad\square$

Lemma 3.3.8. Γ *is isomorphic to the random graph.*

Proof. It is well known that the random graph is the unique, up to isomorphism, countable graph G satisfying the following extension property: for all finite disjoint $U, V \subset G$, there is $x \in G$ such that x is adjacent to all vertices in U and no vertices in V. See for example [Cam97] for details.

The fact that $\Gamma := (D, \bar{E})$ satisfies this property of the random graph follows immediately from the extension property of the generic digraph. $\quad\square$

Terminology. Let $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$ be proper tuples in D. We say $\bar{a}$ and $\bar{b}$ are *isomorphic as graphs* if $\bar{E}(a_i, a_j) \leftrightarrow \bar{E}(b_i, b_j)$ for all i, j.

Lemma 3.3.9. *Let $G \leq Sym(D)$ be a closed supergroup of $Aut(D, E)$.*

(i) *Suppose that whenever $\bar{a}$ and $\bar{b}$ are isomorphic as graphs, there exists $g \in G$ such that $g(\bar{a}) = \bar{b}$. Then $G \geq Aut(\Gamma)$.*

(ii) *Suppose that for all $A = \{a_1, \ldots, a_n\} \subset D$, there exists $g \in G$ such that for all edges $a_i a_j$ in A, $E(g(a_i), g(a_j))$ if $i < j$ and $E(g(a_j), g(a_i))$ if $i > j$. (Intuitively, such a g is aligning the edges so they all point in the same direction.) Then, $G \geq Aut(\Gamma)$.*

(iii) *Suppose that for all finite $A \subset D$ and all edges $aa' \in A$ there is $g \in G$ such that g changes the direction of aa' and behaves like id on all other edges and non-edges of A. Then $G \geq Aut(\Gamma)$.*

(iv) *Suppose there is a finite $A \subset D$ and a $g \in G$ such that g behaves like id on $D \backslash A$, g behaves like id between A and $D \backslash A$, and g switches the direction of some edge in A. Then, $G \geq Aut(\Gamma)$.*

Proof. (i) Let $f \in \text{Aut}(\Gamma)$. We want to show that $f \in G$. Since G is closed, it suffices to show that for all finite $\bar{a} \in D$ there exists $g \in G$ such that $g(\bar{a}) = f(\bar{a})$. So let $\bar{a} \in D$. Then $\bar{a}$ and $f(\bar{a})$ are isomorphic as graphs, so by the assumption given in the lemma, there exists $g \in G$ such that g maps $\bar{a}$ to $f(\bar{a})$, as required.

(ii) Let $\bar{a}_1, \bar{a}_2 \in D$ be isomorphic as graphs and let $g_1, g_2 \in G$ be the functions as described in the lemma for these tuples. Then $g_1(\bar{a}_1)$ is isomorphic to $g_2(\bar{a}_2)$ as digraphs. Thus we can get from $\bar{a}_1$ to $\bar{a}_2$ using functions in G, so we are done by (i).

(iii) Let $\bar{a}, \bar{b} \in D$ be isomorphic as graphs. Then by repeatedly using the condition in the lemma, we can switch the appropriate edges in $\bar{a}$ to end up with the digraph $\bar{b}$. Thus we are done by (i).

(iv) Let A and g be as stated in the lemma, and let $aa' \in A$ be an edge whose direction is switched by g. Now let $B \in D$ be finite and let bb' be any edge in B. Then let $h \in \text{Aut}(D, E)$ map bb' to aa' and all other elements of B to $D \backslash A$; such an h exists by Lemma 3.1.2 (vi). Then applying $g \circ h$ to B switches the edge bb' and behaves like id everywhere else. Thus we are done by (iii). $\qquad \square$

3.4 Analysis of canonical functions

To help motivate the analysis we are about to undertake, we sketch a part of the proof of the classification. One task will be to show that if $G > \mathrm{Aut}(D, E)$ then $G \geq \langle - \rangle, \langle sw \rangle$ or $\langle rot \rangle$. Since $G > \mathrm{Aut}(D, E)$, G does not preserve the relation E, so there exist $g \in G$ and $c_1, c_2 \in D$ witnessing this. Then by Lemma 2.7.3, we find a canonical function $f : (D, E, <, c_1, c_2) \to (D, E)$ that agrees with g on (c_1, c_2) and which is generated by G. The behaviour of f will give us information about G. Importantly, there are only finitely many possible behaviours for f, by [BPT13, Proposition 17], and we can check each case and show that G must contain $\langle - \rangle, \langle sw \rangle$ or $\langle rot \rangle$.

The reference which explains why there are only finitely many behaviours may be cryptic so we provide a sketch: By quantifier elimination, the type of a tuple is determined by its quantifier-free type. As all relations of $(D, E <, c_1, c_2)$ have arity ≤ 2, the quantifier-free type of a tuple is determined by the quantifier-free type of all sub-2-tuples. This implies that the behaviour of f is determind by its behaviour on 2-types. Then finally, by $\aleph_0$-categoricity there are only finitely many 2-types, and thus only finite many possible behaviours for f.

Important Convention. Before continuing, please note a convention that we will use for the remainder of the thesis. There will be proofs where we map some digraph A to a digraph B by composing a sequence of functions $f_1, f_2, \ldots$, where the definition of each one depends on those defined earlier. For example, we may have defined f_1 and f_2, and f_3 is going to be a switching function. The convention is that we write 'Let f_3 be $sw_{A'}$' (where A' will be a particular subset of A), instead of the strictly correct 'Let f be $sw_{f_2 f_1(A')}$'.

There are two main benefits. First, the proofs will be easier to follow and will better match the intuition behind the argument. Second, we can avoid naming the functions altogether. We can use phrases like 'First switch about the subset A_1, then apply rot about the point

a', whereas without the convention we would have to say '...then apply rot about the point which is the current image of a'.

Canonical functions from $(D, E, <)$

We start our analysis of the behaviours with the simplest case, which is when no constants are added.

Notation and facts.

- Let $\phi_1(x, y), \ldots, \phi_n(x, y)$ be formulas. We let $p_{\phi_1,\ldots,\phi_n}(x, y)$ denote the (partial) type determined by the formula $\phi_1(x, y) \wedge \ldots \wedge \phi_n(x, y)$.

- There are three 2-types in (D, E): p_E, p_{E^*} and p_N.

- There are six 2-types in $(D, E, <)$: $p_{<,E}, p_{<,E^*}, p_{<,N}, p_{>,E}, p_{>,E^*}$ and $p_{>,N}$.

To determine these facts, there are two main strategies. One is to use quantifier elimination, which implies that any type is determined by quantifier-free formulas. With quantifier-free formulas, one can only express how the two elements are related to each other with respect to the edge and order relation, and we have simply listed all the possibilities. The other strategy is to determine the orbits of $\text{Aut}(D, E, <)$ with the help of homogeneity: given two pairs (a, a') and (b, b') when can we map one to the other?

The following lemma contains a little 'trick' that proves useful during the analysis of the behaviours.

Lemma 3.4.1. *Let $\{a_1, \ldots, a_n\} \in (D, E, <)$ and let $\sigma \in S_n$. Then there exists $g \in \text{Aut}(D, E)$ such that $\bar{a} \cong g(\bar{a})$ as digraphs and $g(a_i) < g(a_j)$ if and only if $\sigma(i) < \sigma(j)$.*

Proof. Follows straightforwardly from the fact that $(D, E, <)$ embeds every finite ordered graph and from the homogeneity of (D, E). $\square$

Roughly, this lemma allows us to manipulate freely how finitely many elements are ordered. The benefits will be seen in the next lemma.

Lemma 3.4.2. *Let G be a closed supergroup of $Aut(D, E)$ and let $f \in cl_{tm}(G)$ be a canonical function from $(D, E, <)$ to (D, E).*

(i) *If $f(p_{<,N}) = p_N, f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_E$, then $- \in G$.*

(ii) *If $f(p_{<,N}) = p_N, f(p_{<,E}) = p_E$ and $f(p_{<,E^*}) = p_E$, then $G \geq Aut(\Gamma)$.*

(iii) *If $f(p_{<,N}) = p_N, f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_{E^*}$, then $G \geq Aut(\Gamma)$.*

(iv) *If $f(p_{<,N}) = p_E$ or $p_{E^*}, f(p_{<,E}) = p_N$ and $f(p_{<,E^*}) = p_N$, then $G \geq Aut(\Gamma)$.*

(v) *If f has any other non-identity behaviour then $G =Sym(D)$.*

Remark on (v): The identity behaviour is the behaviour $p_{<,N} \mapsto p_N, p_{<,E} \mapsto p_E$ and $p_{<,E^*} \mapsto p_{E^*}$.

Proof. (i) We want to show $- \in G$. Since G is closed, it suffices to show that for all finite $\bar{a} \in D$ there exists $g \in G$ such that $g(\bar{a}) = -(\bar{a})$. So let $\bar{a} \in D$ be finite. By the conditions given on f in the lemma, we have $f(\bar{a}) \cong -(\bar{a})$. By homogeneity, there exists $g_1 \in Aut(D, E)$ mapping $f(\bar{a})$ to $-(\bar{a})$. Since $f \in cl_{tm}(G)$, there is $g_2 \in G$ such that $g_2(\bar{a}) = f(\bar{a})$. Letting $g = g_1 \circ g_2$ completes the argument.

(ii) We will use Lemma 3.3.9 (ii). Let $(a_1, \ldots, a_n) \in D$. By Lemma 3.4.1, we may assume that $a_1 < a_2 < \ldots < a_n$ by composing with an automorphism of $Aut(D, E)$ if necessary. Then observe that f aligns the edges of $(a_1, \ldots, a_n)$ to point in the same direction. As $f \in cl_{tm}(G)$, there exists $g \in G$ which agrees with f on $\bar{a}$, completing the argument.

For the remaining arguments, we will no longer comment explicitly on the fact that $f \in cl_{tm}(G)$ implies that f can be imitated on a finite set by a function in G.

(iii) The same argument as (ii) works, except that one instead ensures that $a_1 > \ldots > a_n$.

(iv) Let $\bar{a}$ be any tuple. Then consider $f(\bar{a})$. By Lemma 3.4.1 and by composing with an automorphism of (D, E) if necessary, we may assume that $f(\bar{a}) \cong (\bar{a})$ as linear orders. Now apply f again. The result of this procedure matches the behaviour in (ii) or (iii), so we have reduced this case to those.

Terminology. In future, we use the phrase *applying f twice* to abbreviate the procedure of applying f, re-ordering the elements to match the ordering of the initial tuple, and applying f again.

(v) Case 1: $f(p_{<,N}) = p_N$. We are left with the behaviours where $f(p_{<,E}) = p_N$ or $f(p_{<,E^*}) = p_N$ (or both), as all the other possibilities have been dealt with above. Suppose without loss that $f(p_{<,E}) = p_N$ - the other case is dealt with a symmetric argument. We will use Lemma 3.3.7 (iii) so let A be a finite subset of D that has at least one edge, aa' say. By composing with an automorphism of $\text{Aut}(D, E)$ if necessary we may assume that $a < a'$ (using Lemma 3.4.1), so that $\text{tp}(a, a') = p_{<,E}$. Then by the behaviour of f, $f(A)$ has fewer edges than A, as required.

Case 2: $f(p_{<,N}) = p_E$.
Case 2a: $f(p_{<,E}) = p_E$ and $f(p_{<,E^*}) = p_E$. Let $\bar{a}, \bar{b} \in D^n$ be proper tuples. Then $f(\bar{a}) \cong f(\bar{b})$. Hence using elements in G we can map $\bar{a}$ to $f(\bar{a})$ to $f(\bar{b})$ to $\bar{b}$. Hence G is n-transitive for all n so by Lemma 3.3.7 (i) $G = \text{Sym}(D)$.

Case 2b: $f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_{E^*}$. Apply f twice and use the same argument as in Case 2a to show that $G = \text{Sym}(D)$.

Case 2c: $f(p_{<,E}) = p_E$ and $f(p_{<,E^*}) = p_{E^*}$. Let A be a finite digraph with at least one edge, say $a_1 a_2$. Let A_1 be an ordered digraph obtained from A by adding a linear order so that $a_1 < a_2$ are the two smallest elements in the order; in particular so that $\text{tp}_{A_1}(a_1, a_2) = p_{<,E}$. Now let A_2 be obtained from A_1 by changing the edge $a_1 a_2$ to a non-edge, so $\text{tp}_{A_2}(a_1, a_2) = p_{<,N}$. By construction, $f(A_1) \cong f(A_2)$ as digraphs. Thus

using functions in G, we can get from A_1 to $f(A_1)$ to $f(A_2)$ to A_2. In other words, we can delete any edge from a finite digraph using functions in G, so $G = \text{Sym}(D)$ by Lemma 3.3.7 (iii).

Case 2d: $f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_E$. Applying f twice reduces to a case that is dual to Case 2c.

Case 2e: $f(p_{<,E}) = p_E$ and $f(p_{<,E^*}) = p_N$. Applying f twice reduces to Case 2a.

Case 2f: $f(p_{<,E}) = p_N$ and $f(p_{<,E^*}) = p_E$. Applying f twice reduces to Case 1.

Case 2g: $f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_N$. Let A be a finite digraph with at least one edge, say $a_1 a_2$. Let $A_1 \subset (D, E, <)$ be an ordered digraph obtained from A by adding a linear order so that $a_2 < a_1$ are the two smallest elements in the order; in particular so that $\text{tp}_{A_1}(a_2, a_1) = p_{<,E^*}$. Now apply f twice to A_1; note that the type of (a_2, a_1) is now $p_{<,E}$. Using homogeneity of (D, E), map A_1 to an ordered digraph A_2 where the only change is that a_1 and a_2 are swapped in the linear order, so that $\text{tp}(a_1, a_2) = p_{<,E^*}$. Now apply f for the last time. By construction, the resulting digraph is the same as A except that the edge $a_1 a_2$ is deleted. Thus $G = \text{Sym}(D)$ by Lemma 3.3.7 (iii).

Case 2h: $f(p_{<,E}) = p_N$ and $f(p_{<,E^*}) = p_{E^*}$. This case can be dealt with by using an argument similar to that for 2g.

Case 3: $f(p_{<,N}) = p_{E^*}$. This case is symmetric to Case 2. $\square$

Canonical functions from $(D, E, <, \bar{c})$

We now move on to the general situation where we have added constants $\bar{c} \in D$ to the structure. For convenience, we assume that $c_i < c_j$ for all $i < j$. Since (D, E) is $\aleph_0$-categorical, $(D, E, \bar{c})$ is also $\aleph_0$-categorical. Thus, as discussed in Chapter 2, the n-types of $(D, E, <, \bar{c})$ correspond to the orbits of $\text{Aut}(D, E, <, \bar{c})$, so we continue to conflate the notion of types and orbits.

$(D, E, <, \bar{c})$ has two kinds of 1-types, i.e. two kinds of orbits: singleton orbits, which are of the form $\{c_i\}$, and infinite orbits, which are determined by how their elements are related to the c_i. For example, $\{x \in D : x < c_1 \wedge \bigwedge_i E(x, c_i)\}$ is an infinite orbit.

In order to describe the 2-types, we extend the notation from the previous section.

Notation Let A, B be definable subsets of $(D, E, <, \bar{c})$ and let $\phi_1(x, y), \dots, \phi_n(x, y)$ be formulas. We let $p_{A, B, \phi_1, \dots, \phi_n}(x, y)$ denote the (partial) type determined by the formula $x \in A \wedge y \in B \wedge \phi_1(x, y) \wedge \dots \wedge \phi_n(x, y)$.

Now let $X, Y \subset D$ be orbits of $(D, E, <, \bar{c})$, $\phi \in \{<, >\}$ and $\psi \in \{E, E^*, N\}$. Then all the 2-types of $(D, E, <, \bar{c})$ are of the form $p_{X, Y, \phi, \psi} = \{(a, b) \in D : a \in X, b \in Y, \phi(a, b)$ and $\psi(a, b)\}$.

As for the case without constants, this description of the types can be obtained either by thinking about what can be expressed with quantifier-free formulas (which is probably easier in this case) or by determining orbits of $\text{Aut}(D, E, <, \bar{c})$. Note we have been careful with our wording for description of the 2-types: we say that all 2-types are of a certain form but we are not saying that all objects of that form are 2-types. The reason for this is that some choices of X, Y, ϕ and ψ may be inconsistent, For example, if X is less than c_1 and Y is bigger than c_1, then setting ϕ to be $>$ would create an inconsistency.

The next lemma provides two key properties about infinite orbits that are crucial to our analysis.

Lemma 3.4.3. *Let X be an infinite orbit of $(D, E, <, \bar{c})$.*

(i) *Let $v \in D \backslash (X \cup \bar{c})$. Let $A = (a_0, \dots, a_n)$ be a finite ordered digraph. Then there are $x_1, \dots, x_n \in X$ such that $(a_0, a_1, \dots, a_n) \cong (v, x_1, \dots, x_n)$ as tuples in $(D, E, <, \bar{c})$.*

(ii) *(ii) Let $v, v' \in D \backslash (X \cup \bar{c})$. Let $A = (a_0, \dots, a_n)$ be a finite ordered digraph such that (a_0, a_1) and (v, v') are isomorphic as ordered digraphs. Then there are $x_2, \dots, x_n \in$*

X *such that* $(a_0, a_1, \dots, a_n) \cong (v, v', x_2, \dots, x_n)$ *in* $(D, E, <, \bar{c})$.

(iii) X *is isomorphic to* $(D, E, <)$.

Proof. (i) Let k be the length of the tuple $\bar{c}$ and let x be any element of X. Consider the finite ordered digraph A' which is constructed as follows: start with A, add new vertices $c'_1, \dots, c'_k$ and then add edges and an ordering so that we have $(a_0, c'_1, \dots, c'_k) \cong (v, c_1, \dots, c_k)$ and so that $(a_i, c'_1, \dots, c'_k) \cong (x, c_1, \dots, c_k)$ for all $i > 0$.

A' is embeddable in $(D, E, <)$ so let f be such an embedding. By composing with an automorphism of $(D, E, <)$ if necessary, we can assume that $f(c'_j) = c_j$ for $j = 1, \dots, k$. Then letting $x_i = f(a_i)$ for $i = 1, \dots, n$ completes the proof.

(ii) This is proved using the same argument as in (i): Draw the finite ordered digraph which captures all the requirements, embed it into $(D, E, <)$ and then apply an automorphism of $(D, E, <)$ to position it correctly within $(D, E, <, \bar{c})$.

(iii) From (i), we know that the age of X is the set of all finite ordered digraphs, so it suffices to show that X is homogeneous. Let $(a_1, \dots, a_n), (b_1, \dots, b_n) \in X$ be isomorphic tuples, as ordered digraphs. Then $(c_1, \dots, c_k, a_1, \dots, a_n) \cong (c_1, \dots, c_k, b_1, \dots, b_n)$. By the homogeneity of $(D, E, <)$, there is $f \in \mathrm{Aut}(D, E, <)$ mapping $(c_1, \dots, c_k, a_1, \dots, a_n)$ to $(c_1, \dots, c_k, b_1, \dots, b_n)$. Since f fixes $\bar{c}$, f fixes X setwise, and so $f|_X$ is an automorphism of X mapping $\bar{a}$ to $\bar{b}$, as required. $\square$

Our task now is to analyse the possibilities for $f(p_{X,Y,\phi,\psi})$, where f is a canonical function. The analysis is split into cases depending on how the orbits X and Y relate. The first lemma deals with the situation when $X = Y$.

Lemma 3.4.4. *Let G be a closed supergroup of $\mathrm{Aut}(D, E)$, let $\bar{c} \in D$, let $f \in cl_{tm}(G)$ be a canonical function from $(D, E, <, \bar{c})$ to (D, E). Let X be an infinite orbit of $\mathrm{Aut}(D, E, \bar{c})$.*

(i) If $f(p_{X,X,<,N}, p_{X,X,<,E}, p_{X,X,<,E^*}) = (p_N, p_{E^*}, p_E)$, then $- \in G$.

(ii) If $f(p_{X,X,<,N}, p_{X,X,<,E}, p_{X,X,<,E^*}) = (p_N, p_E, p_E)$ or (p_N, p_{E^*}, p_{E^*}), then $G \geq Aut(\Gamma)$.

(iii) If $f(p_{X,X,<,N}, p_{X,X,<,E}, p_{X,X,<,E^*}) = (p_E, p_N, p_N)$ or (p_{E^*}, p_N, p_N), then $G \geq Aut(\Gamma)$.

(iv) If f has any other non-identity behaviour then $G = Sym(D)$.

Proof. The proof is identical to that of Lemma 3.4.2 because $X \cong (D, E, <)$. □

Next we look at the behaviour of f between two infinite orbits X and Y. This task is split depending on how X and Y relate with regard to the linear order.

Facts and Notation There are two ways that two infinite orbits X and Y of $\mathrm{Aut}(D, E, <, \bar{c})$ can relate to each other with respect to the linear order $<$:

- All of the elements of one orbit, X say, are smaller than all of the elements of Y. This is abbreviated by '$X < Y$'.

- X and Y are interdense: $\forall x < x' \in X, \exists y \in Y$ such that $x < y < x'$ and vice versa.

To prove this, suppose that $X \not< Y$ and $Y \not< X$. Then without loss there exists $x_1, x_2 \in X$ and $y \in Y$ such that $x_1 < y < x_2$. Now let $x_1' < x_2' \in X$; we want to find $y' \in Y$ lying in between these elements. To do this, construct a finite ordered digraph $A = \{c_1', \ldots, c_k', a_1, \ldots, a_5\}$ as follows: ensure $(c_1', \ldots, c_k', a_i) \cong (c_1, \ldots, c_k, x_1)$ for $i = 1, 2, 4, 5$, $(c_1', \ldots, c_k', a_3) \cong (c_1, \ldots, c_k, y)$, $(a_1, a_5) \cong (x_1', x_2')$, $(a_2, a_3, a_4) \cong (x_1, y, x_2)$ and $a_1 < a_2 < a_3 < a_4 < a_5$. This is embeddable in $(D, E, <)$ and by homogeneity we can assume the c_i' are mapped to c_i and (a_1, a_5) is mapped to (x_1', x_2'). Then the image of a_3 in this embedding is a suitable choice for y'. This shows that Y is dense in X. In particular we can find $y_1 < x < y_2$, so repeating the argument shows that X is dense in Y.

We are now in a position to continue analysing the canonical functions. The next lemma contains the analysis for the case where $X < Y$ or $X > Y$.

Lemma 3.4.5. *Let G be a closed supergroup of $Aut(D, E)$, let $\bar{c} \in D$ and let $f \in cl_{tm}(G)$ be a canonical function from $(D, E, <, \bar{c})$ to (D, E). Let $X, Y \subset D$ be infinite orbits such that f behaves like id on X and such that $X < Y$ or $X > Y$.*

(i) *If $f(p_{X,Y,N}, p_{X,Y,E}, p_{X,Y,E^*}) = (p_N, p_{E^*}, p_E)$, then $sw \in G$.*

(ii) *If $f(p_{X,Y,N}, p_{X,Y,E}, p_{X,Y,E^*}) = (p_E, p_{E^*}, p_N)$ or (p_{E^*}, p_N, p_E), then $rot \in G$.*

(iii) *If $f(p_{X,Y,N}, p_{X,Y,E}, p_{X,Y,E^*}) = (p_N, p_E, p_E)$ or (p_N, p_{E^*}, p_{E^*}), then $G \geq Aut(\Gamma)$.*

(iv) *If $f(p_{X,Y,N}, p_{X,Y,E}, p_{X,Y,E^*}) = (p_E, p_N, p_N)$ or (p_{E^*}, p_N, p_N), then $G \geq Aut(\Gamma)$.*

(v) *If f has any other non-identity behaviour, then $G = Sym(D)$.*

Remark: We do not need to include $<$ or $>$ in the subscripts of the types because it is automatically determined by whether $X < Y$ or $Y < X$.

Proof. Assume that $X < Y$. The proof for the case $Y < X$ is symmetric. Let $y_0 \in Y$ be any element.

(i) We want to show that $sw \in G$. Let $a \in D$ be the element such that $sw = sw_a$ and let $(a, a_1, \ldots, a_n) \in D$ be a finite tuple. By Lemma 3.4.3, we can map this tuple to $(y_0, x_1, \ldots, x_n)$ where $x_i \in X$ for all i. Then observe that $f(y_0, x_1, \ldots, x_n) \cong sw(a, a_0, \ldots, a_n)$. This suffices as G is closed.

(ii) Use the same argument as in (i).

(iii) Using Lemma 3.3.9 (ii), it suffices to show that for any finite tuple $\{a_1, \ldots, a_n\} \in D$ we can align all its edges by using functions in G. First we map a_{n-1} to y_0 and the rest of A into X (using Lemma 3.4.3), and then apply f. Then we repeat but with a_{n-2} instead of a_{n-1}, then with a_{n-3}, and so on until a_1.

(iv) The same argument as in (iii) works but with a slight modification: the intuition is that whenever f was applied to some tuple $(a_0, \ldots, a_n)$ in (iii), here we apply f twice to get the same effect. To be more precise, the modification is as follows. Let $(a_0, \ldots, a_n) \in D$. We first map this to an isomorphic copy $(y_0, x_1, \ldots, x_n)$ for some $x_i \in X$, using Lemma 3.4.3. Then apply f. Then again we map this to an isomorphic tuple $(y_0, x'_1, \ldots, x'_n)$ for some $x'_i \in X$. Then apply f a second time. The total effect of this procedure is the same as one application of the canonical function in (iii). Thus we have reduced this case to (iii).

Remark: For the rest of this proof, we will use the phrase "by applying f twice" to refer to the procedure described above.

(v) Case 1: $f(p_{<,N}) = p_N$. The same argument as in Case 1 of Lemma 3.4.2 shows that we can reduce the number of edges from any non-empty finite digraph $A \subset D$, so $G = \text{Sym}(D)$.

Case 2: $f(p_{X,Y,N}) = p_E$

Case 2a: $f(p_{<,E}) = p_E$ and $f(p_{<,E^*}) = p_E$. Let $\bar{a} \in D$ be a finite tuple. We can map $\bar{a}$ to a linear order by using the exact same procedure as in (iii). Thus G is n-transitive for all n and so $G = \text{Sym}(D)$.

Case 2b: $f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_{E^*}$. By applying f twice, we reduce to Case 2a.

Case 2c: $f(p_{<,E}) = p_E$ and $f(p_{<,E^*}) = p_{E^*}$. The same argument as in Case 2c of Lemma 3.4.2 shows that $G = \text{Sym}(D)$.

Case 2d: $f(p_{<,E}) = p_{E^*}$ and $f(p_{<,E^*}) = p_E$. By applying f twice, we reduce to a case that is dual to Case 2c.

Case 2e: $f(p_{<,E}) = p_E$ and $f(p_{<,E^*}) = p_N$. By applying f twice, we reduce to Case 2a.

Case 2f: $f(p_{<,E}) = p_N$ and $f(p_{<,E^*}) = p_E$. By applying f twice, we reduce to Case 1. $\square$

In the proof above we only had to study the behaviour of f on $\{y_0\} \cup X$ for one element $y_0 \in Y$. They key property that allowed this was Lemma 3.4.3. This feature allows us to use these arguments with minimal modification to prove the subsequent lemmas.

The next lemma deals with the case where X and Y are interdense.

Lemma 3.4.6. *Let G be a closed supergroup of Aut(D, E), let $\bar{c} \in D$ and let $f \in cl_{tm}(G)$ be a canonical function from $(D, E, <, \bar{c})$ to (D, E). Let $X, Y \subset D$ be infinite orbits such that f behaves like id on X and such that X and Y are interdense. Then at least one of the following holds.*

  (i) f behaves like id, sw or rot between X and Y.

  (ii) $G \geq$Aut(Γ).

Proof. First just consider the increasing tuples from X to Y. With the same arguments as in Lemma 3.4.5 one can show that either

(a) $f(p_{X,Y,N,<}, p_{X,Y,E,<}, p_{X,Y,E^*,<}) = (p_N, p_E, p_{E^*})$,

(b) $f(p_{X,Y,N,<}, p_{X,Y,E,<}, p_{X,Y,E^*,<}) = (p_N, p_{E^*}, p_E)$,

(c) $f(p_{X,Y,N,<}, p_{X,Y,E,<}, p_{X,Y,E^*,<}) = (p_E, p_{E^*}, p_N)$ or (p_{E^*}, p_N, p_E), or

(d) $G \geq$ Aut(Γ).

If (d) is true we are done, so assume (a), (b) or (c) is true. Similarly we can assume that f behaves like id, sw or rot between decreasing tuples from X to Y. If the behaviours between increasing and decreasing tuples are the same, then we are done. Thus it remains to check what happens if we have different behaviours on increasing tuples and decreasing tuples.

Case 1. f behaves like id on decreasing tuples and sw on increasing tuples. Explicitly we are asssuming that:

$$f(p_{X,Y,N,<}, p_{X,Y,E,<}, p_{X,Y,E^*,<}) = (p_N, p_{E^*}, p_E), \text{ and}$$
$$f(p_{X,Y,N,>}, p_{X,Y,E,>}, p_{X,Y,E^*,>}) = (p_N, p_E, p_{E^*}).$$

Let $\bar{a} = (a_0, a_1, \ldots, a_n)$ be a finite digraph with at least one edge $E(a_0, a_1)$. Consider $\bar{a}$ as an ordered digraph by setting $a_i < a_j \leftrightarrow i < j$. Then by homogeneity, $\bar{a}$ has an isomorphic copy $\bar{b} = (b_0, b_1, \ldots, b_n)$ such that $b_1 \in Y$ and $b_i \in X$ for $i \neq 1$. All the edges of $\bar{b}$ are preserved under f, except for the edge $E(b_0, b_1)$ whose direction is switched. By Lemma 3.3.9 (iii), we conclude that $G \geq \text{Aut}(\Gamma)$.

Case 2. f behaves like id on decreasing tuples and rot on increasing tuples. The same argument as in Case 1 works here.

Case 3. f behaves like sw on decreasing tuples and rot on increasing tuples. By applying f twice, we are reduced to Case 2.

The remaining cases are all symmetric to the cases already dealt with, thus completing the proof. $\qquad\square$

We end by looking at how f can behave between the constants $\bar{c}$ and the rest of the structure.

Lemma 3.4.7. *Let G be a closed supergroup of Aut(D, E), let $(c_1, \ldots, c_n) \in D$ and let $f \in cl_{tm}(G)$ be a canonical function from $(D, E, <, \bar{c})$ to (D, E). Suppose that f behaves like id on $D^- := D \backslash \{c_1, \ldots, c_n\}$. Then at least one of the following holds.*

  (i) For all $i, 1 \leq i \leq n$, f behaves like id, sw or rot between c_i and D^-.

 (ii) $G \geq Aut(\Gamma)$.

Proof. Fix some i, $1 \leq i \leq n$. Let $X_{out} = \{x \in D : x < c_1 \wedge E(c_i, x) \wedge \bigwedge_{j \neq i} N(c_j, x)\}$. Define X_{in} and X_N similarly, with $E(c_i, x)$ replaced with $E(x, c_i)$ and $N(x, c_i)$ respectively. Then for any finite digraph $(a_0, a_1, \ldots, a_n)$, there exist $x_1, \ldots, x_n \in$

$X_{out} \cup X_{in} \cup X_N$ such that $(a_0, a_1, \ldots, a_n) \cong (c_i, x_1, \ldots, x_n)$. So by replicating the proof of Lemma 3.4.5 we can assume that f behaves like id, sw or rot between c_i and $X_{out} \cup X_{in} \cup X_N$. Without loss, we can assume f behaves like id, because we can compose f with sw_{c_i} or rot_{c_i} as necessary.

If f behaves like id between c_i and D^- we are done, so suppose there is an infinite orbit X such that f does not behave like id between c_i and X. Assume that there are edges from c_i into X (as opposed to edges from X into c_i, or non-edges) - the arguments for the other two cases are similar.

Let A be any finite digraph containing an edge, ab say. Then observe that there is an embedding of A into D such that a is mapped to c_i, b is mapped into X, and the rest of A is mapped into $X_{out} \cup X_{in} \cup X_N$. Then applying f changes exactly the one edge ab in A, so by Lemma 3.3.7 (iii) or Lemma 3.3.9 (iii) as appropriate, we are done. $\qquad \square$

3.5 Proof of the classification

We are now in a position to prove the classification of the reducts of the generic digraph. We recall the statement for convenience.

Theorem 3.2.5. *The closed groups lying between* $Aut(D, E)$ *and* $Sym(D)$ *are given by the following lattice:*

$$
\begin{array}{c}
Sym(D) \\
\langle sw_\Gamma, -_\Gamma \rangle \\
\langle sw_\Gamma \rangle \qquad \langle -_\Gamma \rangle \\
Aut(\Gamma) \qquad \langle -, rot \rangle \\
\langle sw, - \rangle \\
\langle sw \rangle \qquad \langle - \rangle \quad \langle rot \rangle \\
Aut(D, E)
\end{array}
$$
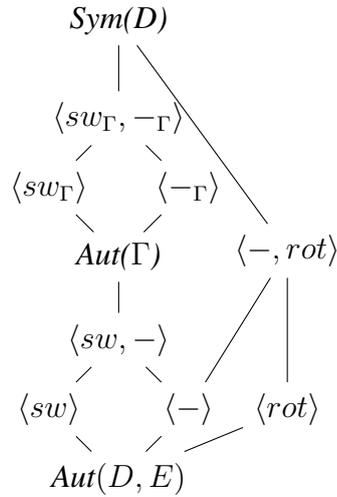
Proof. Let G be a closed group such that $\mathrm{Aut}(D, E) \leq G \leq \mathrm{Sym}(D)$. The proof will be split into five separate tasks.

(i) Showing that $G > \mathrm{Aut}(\Gamma)$, $G \leq \mathrm{Aut}(\Gamma)$ or $G \geq \langle rot \rangle$.

(ii) Showing that if $G > \mathrm{Aut}(\Gamma)$, then $G = \langle sw_\Gamma \rangle, \langle -_\Gamma \rangle, \langle sw_\Gamma, -_\Gamma \rangle$ or $\mathrm{Sym}(D)$.

(iii) Showing that if $G < \mathrm{Aut}(\Gamma)$, then $G = \mathrm{Aut}(D, E), \langle sw \rangle, \langle - \rangle$ or $\langle sw, - \rangle$.

(iv) Showing that if $G \geq \langle rot \rangle$, then $G = \langle rot \rangle, \langle rot, - \rangle$ or $\mathrm{Sym}(D)$.

(v) Showing that the lattice is arranged correctly (e.g. showing that meets and joins are drawn correctly).

(i) Suppose for contradiction that $G \ngeq \mathrm{Aut}(\Gamma)$, $G \nleq \mathrm{Aut}(\Gamma)$ and $G \ngeq \langle rot \rangle$.

Since G is not contained in $\mathrm{Aut}(\Gamma)$, G does not preserve the graph relation $\bar{E}$. Hence, there is $g \in G$ and an edge $c_1 c_2 \in D$ such that $g(c_1 c_2)$ is a non-edge. We apply Lemma 2.7.3 to obtain a canonical $f : (D, E, <, c_1, c_2) \to (D, E)$ which agrees with g on c_1 and c_2.

For any infinite orbit X, f behaves like id or $-$ on it, because otherwise G would contain $\mathrm{Aut}(\Gamma)$ by Lemma 3.4.4. In particular, f preserves non-edges on all infinite orbits. By

Lemma 3.4.5 and the subsequent lemmas, f behaves like id or sw between orbits because otherwise G would contain either Aut(Γ) or $\langle rot \rangle$; again, this implies that f preserves non-edges between all orbits.

But now we have a function $f \in \text{cl}_{\text{tm}}(G)$ which deletes the edge c_1c_2 and maps all non-edges to non-edges. Thus we can use G to delete edges from any finite digraph, so by Lemma 3.3.7 (iii), we conclude that G equals Sym(D). This contradicts that G does not contain Aut(Γ).

(ii) This is exactly the statement of the classification of the reducts of the random graph ([Tho91]).

(iii) First we show that if $G > \text{Aut}(D, E)$, then $G \geq \langle - \rangle$ or $\langle sw \rangle$. Since $G > \text{Aut}(D, E)$, and by using Lemma 2.7.3, there exists an edge $c_1c_2 \in D$ and a canonical $f : (D, E, <, c_1, c_2) \to (D, E)$ such that f switches the direction of the edge c_1c_2. Let X be an infinite orbit.

Claim 1. We may assume f behaves like id on X.

By Lemma 3.4.4 we know that f behaves like id or $-$ on X, otherwise G would contain Aut(Γ). If f behaves like $-$ on X, then $- \in G$ and we are done. Thus we assume that f behaves like id on X.

Claim 2. We may assume that f behaves like id between X and every other infinite orbit Y.

Let Y be another infinite orbit. By Lemma 3.4.5 and Lemma 3.4.6, f behaves like id or sw between X and Y, as otherwise G would contain Aut(Γ). If f behaves like sw between them, then $sw \in G$, so we are done. Note that rot is not a possibility because $G < \text{Aut}(\Gamma)$ implies that G preserves non-edges.

Claim 3. f behaves like id on every infinite orbit and between every pair of infinite orbits.

Suppose not, so there are infinite orbits Y_1 and Y_2 (possibly the same) and there are

distinct $y_1, y_2 \in Y_1, Y_2$, respectively, such that $(y_1, y_2) \ncong f(y_1, y_2)$. Now for any finite digraph $(a_1, a_2, \ldots, a_n)$ with $(y_1, y_2) \cong (a_1, a_2)$, we can find $x_3, \ldots, x_n \in X$ such that $(y_1, y_2, x_3, \ldots, x_n) \cong (a_1, \ldots, a_n)$, by Lemma 3.4.3 (ii). By assumption and the previous two claims, f has the effect of only changing what happens between y_1 and y_2. Using Lemma 3.3.7 (iii) or Lemma 3.3.9 (iii), as appropriate, we conclude that $G \geq \mathrm{Aut}(\Gamma)$.

Claim 4. We may assume that f behaves like id between $\{c_1, c_2\}$ and the union of all infinite orbits.

This follows from Lemma 3.4.7.

Conclusion. We can assume that f behaves everywhere like the identity, except that it changes the direction of the edge $(c_1 c_2)$. But then we get that $G \geq \mathrm{Aut}(\Gamma)$ by Lemma 3.3.9 (iv). Thus we have shown that if $G > \mathrm{Aut}(D, E)$ then $G \geq \langle - \rangle$ or $\langle sw \rangle$.

Next, we show that if $G > \langle - \rangle$ then $G \geq \langle -, sw \rangle$. This follows exactly the same template as above, so we go through the proof in a sketchier manner, highlighting the minor changes.

Let $\bar{c}$ and f canonical witness that $G > \langle - \rangle$. Let X be an infinite orbit. Then by Lemma 3.4.4, we can assume that f behaves like id or $-$ on X. If $-$, then replace f with $- \circ f$. Thus we can assume f behaves like id on X. (Note that one needs to check that $- \circ f$ satisfies all the assumptions that f does: being canonical, being generated by G and witnessing that $G > \langle - \rangle$. These are all straightforward to check.) Then we continue in exactly the same way as before, reaching the conclusion that we may assume f behaves like the id everywhere except on $\bar{c}$. But then by Lemma 3.3.9 (iv) we are done.

Lastly, one needs to show that if $G > \langle sw \rangle$ then $G \geq \langle -, sw \rangle$, and that if $G > \langle -, sw \rangle$ then $G \geq \mathrm{Aut}(\Gamma)$. Again, these follow exactly the same template as above and are left as exercises to the reader.

(iv) We need to show two things. The first is that if $G > \langle rot \rangle$ then $G \geq \langle -, rot \rangle$, and the second is that if $G > \langle -, rot \rangle$ then $G = \mathrm{Sym}(D)$.

First we show that $\langle sw, rot \rangle = \mathrm{Sym}(D)$. By Lemma 3.3.7 (iii), it suffices to show that for all finite $A \subset D$ we can delete edges from A, if it has any, by applying functions from $\langle sw, rot \rangle$. So let $A \subset D$ be finite, let $a \in A$ be a point adjacent to at least one edge and let $A_1 = \{a' \in A : E(a, a')\}$, $A_2 = \{a' \in A : E(a', a)\}$ and $A_3 = \{a' \in A : N(a, a')\}$.

First switch about the subset A_1, so all the edges adjacent to a are now inward edges. Then apply rot_a^2: the edges between a and $A_1 \cup A_2$ become outward edges, and the non-edges between a and A_3 become inward edges. Then apply $sw_{A_1 \cup A_2}$: between a and $A\backslash\{a\}$ we now only have inward edges. Applying rot_a results in all these edges becoming non-edges. The number of edges within $A\backslash\{a\}$ is unchanged, so we have reduced the number of edges in A, as required.

Now that we know $\langle sw, rot \rangle = \mathrm{Sym}(D)$, the proof is exactly the same as those in (iii), so we leave it as an exercise for the reader.

(v) To clarify, we now know that if $G \leq \mathrm{Sym}(D)$ is a closed supergroup of $\mathrm{Aut}(D, E)$, then it must be equal to one of those in the given lattice. We still need check that the lattice has been drawn correctly. Breaking it down, we need to show the following:

(a) $\langle - \rangle$, $\langle sw \rangle$ and $\langle rot \rangle$ are proper reducts of $\mathrm{Aut}(D, E)$.

(b) $\langle - \rangle$, $\langle sw \rangle$ and $\langle rot \rangle$ are not reducts of each other.

(c) $\langle -, sw \rangle$ is a proper reduct of $\langle - \rangle$ and $\langle sw \rangle$.

(d) $\mathrm{Aut}(\Gamma)$ is a proper reduct of $\langle -, sw \rangle$.

(e) $\langle -, rot \rangle$ is a proper reduct of $\langle - \rangle$ and $\langle rot \rangle$, and is not equal to $\mathrm{Sym}(D)$.

(f) The join of $\langle rot \rangle$ and $\langle sw \rangle$ is $\mathrm{Sym}(D)$.

(g) The meet of $\langle sw \rangle$ and $\langle - \rangle$ is $\mathrm{Aut}(D)$.

(h) The meet of $\langle rot \rangle$ and $\langle sw_\Gamma, -_\Gamma \rangle$ is $\mathrm{Aut}(D)$.

(i) The meet of $\langle -, rot \rangle$ and $\langle sw_\Gamma, -_\Gamma \rangle$ is $\langle - \rangle$.

Before proving these, recall the discussion after Lemma 3.3.2. The arguments presented there are now valid, so we now know that $\langle - \rangle = \{f \in \mathrm{Sym}(D) : f$ behaves like $-\}$, $\langle sw \rangle = \{f \in \mathrm{Sym}(D) :$ there is $A \subseteq D$ such that f behaves like $sw_A\}$ and $\langle rot \rangle = \{f \in \mathrm{Sym}(D) : f$ behaves like $rot_{A,B}$ where A, B are disjoint subsets of $D\}$.

(a) This is immediate from the definition of '$\langle \cdots \rangle$'.

(b) Every element of $\langle - \rangle$ and $\langle sw \rangle$ preserves non-edges, but rot does not, so $\langle rot \rangle$ is not a subgroup of either of those. No element of $\langle sw \rangle$ or $\langle rot \rangle$ changes the direction of every edge in (D, E), so $\langle - \rangle$ is not a subgroup of either of those. Every element of $\langle - \rangle \backslash \mathrm{Aut}(D, E)$ changes the direction of every edge but sw does not, so $\langle sw \rangle$ is not a subgroup of $\langle - \rangle$. Lastly, every element of $\langle rot \rangle \backslash \mathrm{Aut}(D, E)$ does not preserve non-edges, but sw does, so $\langle sw \rangle$ is not a subgroup of $\langle rot \rangle$.

(c) Using the same argument as for the three groups already described, we can show that $\langle -, sw \rangle = \{f \in \mathrm{Sym}(D) : f = g$ or $- \circ g$ for some $g \in \langle sw \rangle\}$. Thus, $\langle -, sw \rangle$ is a proper reduct of $\langle - \rangle$ and $\langle sw \rangle$.

(d) Every element of $\langle -, sw \rangle$ preserves non-edges so it is a subgroup of $\mathrm{Aut}(D, N) = \Gamma$. Γ is a proper reduct because there exists a function in Γ which changes the direction of exactly one edge, and no such function can be found in $\langle -, sw \rangle$.

(e) Using the same argument as for previous groups, we can show that $\langle -, rot \rangle = \{f \in \mathrm{Sym}(D) : f = g$ or $- \circ g$ for some $g \in \langle rot \rangle\}$. Thus, $\langle -, rot \rangle$ is a proper reduct of $\langle - \rangle$ and $\langle rot \rangle$. Next, no element of $\langle -, rot \rangle$ can map a tournament on three vertices to an empty digraph, thus $\langle -, sw \rangle \neq \mathrm{Sym}(D)$.

(f) This was proved at the end of (iv).

(g) This follows from (iii) and (b).

(h) We first establish some notation. We say $f : D \to D$ graph-behaves like $g : D \to D$ if for all $\bar{a} \in D$, $f(\bar{a})$ is isomorphic to $g(\bar{a})$ as *undirected* graphs. Let $A \subseteq D$. We say $f : D \to D$ graph-behaves like $sw_{\Gamma,A}$ if f graph-behaves like id on A and on A^c and if f swaps edges and non-edges between A and A^c. By folklore (or by duplicating previous arguments), $\langle sw_\Gamma \rangle = \{f \in \mathrm{Sym}(D) : f$ graphs-behaves like $sw_{\Gamma,A}$ for some $A \subseteq D\}$, and $\langle -_\Gamma, sw_\Gamma \rangle = \{f \in \mathrm{Sym}(D) : \exists g \in \langle sw_\Gamma \rangle$ such that $f = g$ or $f = -_\Gamma \circ g\}$.

The only possibilities for the meet of $\langle rot \rangle$ and $\langle sw_\Gamma, -_\Gamma \rangle$ are $\langle rot \rangle$ or $\mathrm{Aut}(D, E)$. To rule out the former option, it suffices to show that $rot \notin \langle -_\Gamma, sw_\Gamma \rangle$. But given the description of this group, it is easy to see that rot is not an element of it.

(i) Let $f \in \langle -, rot \rangle \cap \langle sw_\Gamma, -_\Gamma \rangle$. From the description of $\langle -, rot \rangle$ in (e), there exists $g \in \langle rot \rangle$ such that $f = g$ or $f = - \circ g$. Since $f \in \langle - \rangle \Leftrightarrow - \circ f \in \langle - \rangle$, without loss we may assume that $f = g$, i.e. that $f \in \langle rot \rangle$. By (h), it follows that $f \in \mathrm{Aut}(D, E)$.

This completes the proof. $\qquad\square$

Chapter 4

The Henson Digraphs

In this chapter we determine the reducts of the Henson digraphs. This is the first time the reducts of uncountably many structures have been classified. A consequence of this classification is a positive answer to a question of Macpherson that asked whether there are uncountably many maximal-closed subgroups of $\mathrm{Sym}(\mathbb{N})$.

Remarkably, perhaps, the proof of this classification is essentially the same as the proof of the classification of reducts of the generic digraph. For this reason many details will not be provided. In particular we do not reproduce arguments that are exactly the same as those in Chapter 3.

We note that the work in this chapter is joint work with Michael Kompatscher but has not yet been published.

4.1 Basic definitions and facts

Definition 4.1.1. (i) A *tournament* is a digraph in which there is an edge between every pair of (distinct) vertices. Given a digraph, a *source*, respectively *sink*, of the digraph

is a vertex which has outward edges to, respectively inward edges from, every other vertex of the digraph.

(ii) Throughout this chapter, $\mathcal{T}$ will denote a set of finite tournaments. We often refer to elements of $\mathcal{T}$ as forbidden tournaments.

(iii) Let $\text{Forb}(\mathcal{T})$ be the set of finite digraphs D such that for all $T \in \mathcal{T}$, D does not embed T.

(iv) We let $(D_{\mathcal{T}}, E_{\mathcal{T}})$ be the unique (up to isomorphism) countable homogeneous digraph whose age is $\text{Forb}(\mathcal{T})$.

(v) A *Henson digraph* is a digraph isomorphic to $(D_{\mathcal{T}}, E_{\mathcal{T}})$ where $\mathcal{T}$ is non-empty and does not contain the 1- or 2-element tournament.

To justify the existence and uniqueness of $(D_{\mathcal{T}}, E_{\mathcal{T}})$, we need to show that $\text{Forb}(\mathcal{T})$ is an amalgamation class. We demonstrate the amalgamation property so let $B_1, B_2 \in \text{Forb}(\mathcal{T})$ and $A = B_1 \cap B_2$ - we need to determine how the elements of $B_1 \backslash A$ and $B_2 \backslash A$ are related to ensure that $B_1 \cup B_2 \in \text{Forb}(\mathcal{T})$. We do this by simply adding no edges between $B_1 \cup B_2$. How does this ensure $B_1 \cup B_2 \in \text{Forb}(\mathcal{T})$? Suppose for contradiction that $T \in \mathcal{T}$ were embeddable in $B_1 \cup B_2$. Since $B_1, B_2 \in \text{Forb}(\mathcal{T})$, T is not embeddable in B_1 or in B_2. Thus the embedding of T must intersect both $B_1 \backslash A$ and $B_2 \backslash A$. But by the definition of a tournament, this implies there is an edge between $B_1 \backslash A$ and $B_2 \backslash A$, which contradicts how we defined the digraph $B_1 \cup B_2$.

If $\mathcal{T} = \emptyset$ then $(D_{\mathcal{T}}, E_{\mathcal{T}})$ is the generic digraph. If $\mathcal{T}$ contains the 1-element tournament, then $\text{Forb}(\mathcal{T}) = \emptyset$. If $\mathcal{T}$ contains the 2-element tournament, then $(D_{\mathcal{T}}, E_{\mathcal{T}})$ is the countable empty digraph. These are degenerate cases which is why we defined the term Henson digraph to exclude these options.

What is the relevance of 'Henson'? Henson is the mathematician who discovered this construction of forbidding tournaments to create an amalgamation class. Furthermore, as

we will describe in Section 4.6, he used this construction in [Hen72] to find uncountably many countable homogeneous digraphs.

Lemma 4.1.2. *Let (D, E) be a Henson digraph.*

(i) *(D, E) is $\aleph_0$-categorical and has quantifier elimination.*

(ii) *Let $\bar{a}, \bar{b} \in D$. If $tp(\bar{a}) = tp(\bar{b})$, then there exists an automorphism mapping $\bar{a}$ to $\bar{b}$.*

(iii) *Let (D', E') be a digraph such that $Age(D', E') \subseteq Age(D, E)$. Then (D', E') is embeddable in (D, E).*

(iv) *(D, E) is connected: for every distinct $a, b \in D$, there is a path from a to b or from b to a.*

Proof. (i) and (ii) are instances of the general theory from Chapter 2, Theorem 2.3.4 and Theorem 2.2.2.

(iii) Let $(d_1, d_2, \ldots)$ be an enumeration of the elements of D'. We construct partial embeddings inductively, so suppose we have an embedding $f_n : (d_1, \ldots, d_n) \to D$. We need to define an embedding $f_{n+1} : (d_1, \ldots, d_{n+1}) \to D$ that extends f_n. By assumption, $(d_1, \ldots, d_{n+1})$ is in the age of (D, E) and so is embeddable in (D, E), let g be such an embedding. By homogeneity, there is $h \in \mathrm{Aut}(D, E)$ mapping $(g(d_1), \ldots, g(d_n))$ to $(f_n(d_1), \ldots, f_n(d_n))$. Then we let $f_{n+1} = h \circ g$. To complete the proof, take the union of these partial embeddings to obtain an embedding of (D', E') into (D, E).

(iv) Let $a, b \in D$ be distinct. If there is an edge between a and b we are done, so assume ab is a non-edge. Consider the finite digraph $\{a', b', c'\}$ such that there is no edge between a' and b', and there is an edge from a' to c' and from c' to b'. Observe that $\{a', b', c'\}$ lies in $\mathrm{Forb}(\mathcal{T})$, so is embeddable in (D, E). By the homogeneity of (D, E), we map a' to a and b' to b to obtain a $c \in D$ with $E(a, c)$ and $E(c, b)$. $\qquad\square$

As for the generic digraph, we need to add a linear order in order to use the canonical functions machinery.

Definition 4.1.3. (i) We let $(D_{\mathcal{T}}, E_{\mathcal{T}}, <)$ be the unique (up to isomorphism) countable homogeneous ordered digraph such that a finite ordered digraph $(D, E, <)$ is embeddable in $(D_{\mathcal{T}}, E_{\mathcal{T}}, <)$ iff $(D, E) \in \mathrm{Forb}(\mathcal{T})$.

(ii) We say $(D, E, <)$ is a *Henson ordered digraph* if $(D, E, <) \cong (D_{\mathcal{T}}, E_{\mathcal{T}}, <)$ for some non-empty $\mathcal{T}$ that does not contain the 1- or 2-element tournament.

Lemma 4.1.4. *All Henson ordered digraphs are Ramsey structures.*

Proof. The proof is the same as for the generic digraph, except with the additional observation that forbidding tournaments still allows us to use Theorem 2.6.3 because tournaments are irreducible set systems. (Strictly speaking, I should say the set systems that encode tournaments are irreducible.) $\square$

4.2 Statement of classification

For the remainder of the chapter, $(D, E, <)$ will denote some Henson ordered digraph and $\mathcal{T}$ will be its set of forbidden tournaments.

There are two main differences between the classification for the Henson digraphs and for the generic digraph. The first is that $\langle rot \rangle$ does not appear at all. The second is that different Henson digraphs can have different reducts. For this reason, the wording of the definitions and statements need to be tweaked for them to make sense.

Definition 4.2.1. (i) We let $\bar{E}(x, y)$ denote the underlying graph relation $E(x, y) \lor E(y, x)$. We let $N(x, y)$ denote the non-edge relation $\neg \bar{E}(x, y)$.

(ii) Assume (D, E) is isomorphic to the digraph obtained by changing the direction of all its edges. In this case $- \in \text{Sym}(D)$ will denote a bijection such that for all $x, y \in D$, $E(-(x), -(y))$ iff $E(y, x)$.

(iii) Assume (D, E) is isomorphic to the digraph obtained by changing the direction of all the edges adjacent to one particular vertex of D. In this case $sw \in \text{Sym}(D)$ will denote a bijection such that for some $a \in D$:

$$E(sw(x), sw(y)) \text{ if and only if } \begin{cases} E(x, y) \text{ and } x, y \neq a, \text{ OR,} \\ \\ E(y, x) \text{ and } x = a \vee y = a \end{cases}$$

(iv) A *Henson graph* is the Fraïssé limit of the set of all finite K_n-free graphs, for some integer $n \geq 3$. (These are also referred to as the generic K_n-free graphs).

(v) Let G be a closed subgroup of $\text{Sym}(\mathbb{N})$. We say that G is *maximal-closed* if $G \neq \text{Sym}(\mathbb{N})$ and there are no closed groups G' such that $G < G' < \text{Sym}(\mathbb{N})$.

The existence of $-$ or sw depends on which tournaments are forbidden, see Lemma 4.3.1. This explains the wording of Theorem 4.2.2 (iii) below: if, for example, $-$ exists but sw does not, then $\max\{\text{Aut}(D, E), \langle - \rangle, \langle sw \rangle, \langle -, sw \rangle\} = \langle - \rangle$.

Theorem 4.2.2. *Let (D, E) be a Henson digraph and let $G \leq Sym(D)$ be a closed supergroup of Aut(D, E). Then:*

(i) $G \leq Aut(D, \bar{E})$ or $G \geq Aut(D, \bar{E})$

(ii) *If $G < Aut(D, \bar{E})$ then $G = Aut(D, E), \langle - \rangle, \langle sw \rangle$ or $\langle -, sw \rangle$.*

(iii) *$(D, \bar{E})$ is the random graph, $(D, \bar{E})$ is a Henson graph or $(D, \bar{E})$ is not homogeneous. In the last case Aut$(D, \bar{E})$ is equal to $\max\{Aut(D, E), \langle - \rangle, \langle sw \rangle, \langle -, sw \rangle\}$ and is a maximal-closed subgroup of Sym(D).*

The reducts of the random graph and the Henson graphs were classified by Thomas in [Tho91]. The reducts of the random graph were described in Chapter 3 and the Henson graphs all have exactly two reducts, themselves and $\text{Sym}(D)$. Immediately then we get the following corollary of Theorem 4.2.2:

Corollary 4.2.3. *Let* (D, E) *be a Henson digraph. Then its lattice of reducts is a sublattice of the lattice below. In particular, the lattice of reducts of* (D, E) *is (isomorphic to) a sublattice of the lattice of reducts of the generic digraph.*

$$
\begin{array}{c}
\textit{Sym(D)} \\
| \\
\langle sw_\Gamma, -_\Gamma \rangle \\
\diagup \quad \diagdown \\
\langle sw_\Gamma \rangle \qquad \langle -_\Gamma \rangle \\
\diagdown \quad \diagup \\
\textit{Aut}(D, \bar{E}) \\
| \\
\langle sw, - \rangle \\
\diagup \quad \diagdown \\
\langle sw \rangle \qquad \langle - \rangle \\
\diagdown \quad \diagup \\
\textit{Aut}(D, E)
\end{array}
$$
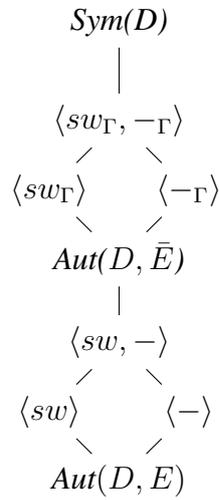
4.3 Understanding the reducts

In this section, we establish several important lemmas that play prominent roles in the proof of the main theorem. This section is similar to the corresponding section in Chapter 3 and so we omit proofs if they are identical to to their counterparts in Chapter 3.

We overload the symbols $-$ and sw by letting them denote actions on finite tournaments. We say $\mathcal{T}$ is closed under $-$ if for every $T \in \mathcal{T}$, the tournament obtained from T by changing the direction of all its edges is in $\mathcal{T}$. We say $\mathcal{T}$ is closed under sw if for every

$T \in \mathcal{T}$ and $t \in T$, the tournament obtained by changing the direction of those edges adjacent to t is in $\mathcal{T}$.

Lemma 4.3.1. *(i) $- : D \to D$ exists if and only if $\mathcal{T}$ is closed under $-$.*

(ii) $sw : D \to D$ exists if and only if $\mathcal{T}$ is closed under sw.

(iii) If sw exists, then for all $A \subseteq D$, sw_A exists.

(iv) $\langle - \rangle \supseteq \{f \in Sym(D) : f$ behaves like $-\}$.

(v) $\langle sw \rangle \supseteq \{f \in Sym(D) :$ there is $A \subseteq D$ such that f behaves like $sw_A\}$.

Proof. (i) 'LHS $\Rightarrow$ RHS': Suppose $-$ exists but $\mathcal{T}$ is not closed under $-$. Then there is $T \in mt$ such that $-(T) \notin \mathcal{T}$. $-(T) \notin \mathcal{T}$ implies there is an embedding of $-(T)$ into (D, E), so let the image of this embedding be T'. Then apply $-$ to (D, E) and consider the image of T': the image of T' is isomorphic to T, so T is embeddable in (D, E), contradicting $T \in \mathcal{T}$.

'RHS $\Rightarrow$ LHS': By duplicating the argument in Lemma 3.2.1, to show that $-$ exists it suffices to show that (D, E) is isomorphic to (D, E^*). To show this, it suffices to show that (D, E^*) is homogeneous and that $\text{Age}(D, E^*) = \text{Age}(D, E)$, due to the uniqueness of Fraïssé limits. That the ages are equal follows from the assumption that $\mathcal{T}$ is closed under $-$. That (D, E^*) is homogeneous follows from the observation that for all $A, B \subseteq D$ and $f : A \to B$, $f : (A, E) \to (B, E)$ is an isomorphism if and only if $f : (A, E^*) \to (B, E^*)$ is an isomorphism.

(ii) 'LHS $\Rightarrow$ RHS': Apply the same argument as in (i) to prove this.

'RHS $\Rightarrow$ LHS': Let $a \in D$, $X_{\text{out}} = \{x \in D : E(a, x)\}$ and $X_{\text{in}} = \{x \in D : E(x, a)\}$. Suppose we found an isomorphism $f : (X_{\text{out}}, E) \to (X_{\text{in}}, E)$. Then we can define sw as the function which maps a to a, maps elements of X_{out} using f and maps elements of X_{in} using f^{-1}. Thus to complete this proof, we need to prove that X_{out} and X_{in} are isomorphic

digraphs. To do this, we will show that they are both homogeneous and have the same age, because by Theorem 2.3.3 we know that Fraïssé limits are unique.

First we show that X_{out} is homogeneous. Note in advance that the same argument shows that X_{in} is homogeneous. Let $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in X_{\text{out}}$ be isomorphic tuples. Then $(a, a_1, \ldots, a_n)$ and $(a, b_1, \ldots, b_n)$ are isomorphic in (D, E), so by homogeneity of (D, E) there is an automorphism g of (D, E) mapping $(a, a_1, \ldots, a_n)$ to $(a, b_1, \ldots, b_n)$. Since g fixed a, g fixes X_{out} setwise. Then conclude by observing that the restriction of g to X_{out} is an automorphism of (X_{out}, E) mapping $(a_1, \ldots, a_n)$ to $(b_1, \ldots, b_n)$.

Next we show that $\text{Age}(X_{\text{out}}) = \text{Age}(X_{\text{in}})$. Let A be a finite sub-digraph of X_{out}. Then let A' be the digraph obtained by adding the vertex a, so that a is a source of A'; this is in $\text{Age}((D, E))$ so is an element of $\text{Forb}(\mathcal{T})$. Now let A'' be the digraph obtained from A' by changing a to a sink. Since $\mathcal{T}$ is closed under sw and $A' \in \text{Forb}(\mathcal{T})$, A'' is also in $\text{Forb}(\mathcal{T})$, so A'' is embeddable in (D, E). By homogeneity, we may assume that the embedding maps $a \in A''$ to $a \in (D, E)$. Since a is a sink in A'', this implies that $A'' \backslash \{a\}$, which is just A, is embedded in X_{in}. Thus we have shown that $\text{Age}(X_{\text{out}}) \subseteq \text{Age}(X_{\text{in}})$. But a symmetric argument shows that $\text{Age}(X_{\text{in}}) \subseteq \text{Age}(X_{\text{out}})$, so we are done.

(iii) First recall precisely what is entailed by the claim that sw_A exists: sw_A exists if there is a function $D \to D$ (not necessarily a bijection) which behaves like id on A and A^c, and which changes the direction of all the edges between A and A^c.

Consider the digraph (D, E') obtained from (D, E) by changing the direction of the edges between A and A^c and leaving all other edges unchanged. If (D, E') is embeddable in (D, E), then sw_A exists as any embedding $(D, E') \to (D, E)$ has the desired property.

We will prove the contrapositive of the statement in the lemma, so suppose sw_A does not exist . By the above discussion, this implies that the digraph (D, E') is not embeddable in (D, E), which by Lemma 4.1.2 implies that $\text{Age}(D, E') \not\subseteq \text{Age}(D, E)$. This implies there exists $T \in \mathcal{T}$ which is embeddable in (D, E'); let g be such an embedding. Let

$B = g^{-1}(g(T) \cap A)$, so B is a subset of T. Now consider the tournament, T' say, obtained by applying the switch operation on T about every element of B. By choice of T and B, T' is isomorphic to $(g(T), E|_{g(T)})$. Hence T' is in the age of (D, E) and so $T' \notin \mathcal{T}$. To summarise, we have $T \in \mathcal{T}, T' \notin \mathcal{T}$ and T' is obtained from T by switching. This means $\mathcal{T}$ is not closed under sw and so by (ii) sw does not exist, as required.

(iv), (v) The proofs of these are identical to those for the generic digraph, Lemma 3.3.2 and Lemma 3.3.4. One does need to read through the arguments and check that the age of the structure does not affect the validity of the argument. □

Lemma 4.3.2. *Let $G \leq Sym(D)$ be a closed supergroup of $Aut(D, E)$.*

 (i) *If G is n-transitive for all $n \in \mathbb{N}$, then $G = Sym(D)$.*

 (ii) *If G is n-homogeneous for all $n \in \mathbb{N}$, then $G = Sym(D)$.*

 (iii) *Suppose that whenever $A \subset D$ is finite and has at least one edge, there exists $g \in G$ such that $g(A)$ has fewer edges than in A. Then $G = Sym(D)$.*

 (iv) *Suppose that there exists a finite $A \subset D$ and $g \in G$ such that g behaves like id on $D \backslash A$, g behaves like id between A and $D \backslash A$, and, g deletes at least one edge in A. Then, $G = Sym(D)$.*

Proof. The proofs are identical to those for the generic digraph, Lemma 3.3.7. □

Lemma 4.3.3. *Let $G \leq Sym(D)$ be a closed supergroup of $Aut(D, E)$.*

 (i) *Suppose that whenever $\bar{a}$ and $\bar{b} \in D$ are isomorphic as graphs, there exists $g \in G$ such that $g(\bar{a}) = \bar{b}$. Then $G \geq Aut(D, \bar{E})$.*

 (ii) *Suppose that for all $A = \{a_1, \ldots, a_n\} \subset D$, there exists $g \in G$ such that for all edges $a_i a_j$ in A, $E(g(a_i), g(a_j))$ if $i < j$ and $E(g(a_j), g(a_i))$ if $i > j$. (Intuitively, such a g is aligning the edges so they all point in the same direction.) Then, $G \geq Aut(\Gamma)$.*

(iii) *Suppose that for all finite $A \subset D$ and all edges $aa' \in A$ there is $g \in G$ such that g changes the direction of aa' and behaves like id on all other edges and non-edges of A. Then $G \geq Aut(D, \bar{E})$.*

(iv) *Suppose there is a finite $A \subset D$ and a $g \in G$ such that g behaves like id on $D\backslash A$, g behaves like id between A and $D\backslash A$, and g switches the direction of some edge in A. Then, $G \geq Aut(D, \bar{E})$.*

Furthermore, *in all of these cases we can also conclude that the underlying graph $(D, \bar{E})$ is homogeneous.*

Proof. The proofs are identical to those for the generic digraph, Lemma 3.3.9. To prove that $(D, \bar{E})$ is homogeneous in all cases, note that the supposition in (i) *is* the definition of $(D, \bar{E})$ being a homogeneous graph and that the other cases are proved by using (i). $\qquad \square$

There is a minor observation worth mentioning here regarding the difference between Henson digraphs and the generic digraph. This is that (ii), (iii) and (iv) in the above lemma are vacuously true for certain choices of $\mathcal{T}$. For example, suppose the tournament $\{\{a, b, c\}, \{ab, bc, ac\}\}$ (the 3-element linear order) is forbidden but $\{\{a, b, c\}, \{ab, bc, ca\}\}$ (the 3-cycle) is not. Then letting $A \subset D$ be a 3-cycle, there is no function g which satisfies the assumptions in (ii) or (iii), as such a g would introduce a 3-element linear order. This phenomenon of a certain function being impossible depending on $\mathcal{T}$ will occur several more times. In each instance, it simply makes the relevant statement vacuously true (as it did here) and thus is not a cause for concern.

4.4   Analysis of canonical functions

We can now undertake the analysis of the canonical functions. This is mostly identical to that of the generic digraph. One difference is that we need to rule out the behaviour that

gave *rot*. Another difference is that we need to keep track of when we know that $(D, \bar{E})$ is homogeneous as a graph - the statement of the classification may give some indication for why this is needed. The last main difference is that unlike the generic digraph, infinite orbits of (D, E) are not necessarily isomorphic to (D, E). We will discuss this more later.

Like we did for the generic digraph, we start by looking at canonical functions when no constants have been added to the language.

Lemma 4.4.1. *Let G be a closed supergroup of Aut(D, E) and let $f \in cl_{tm}(G)$ be a canonical function from $(D, E, <)$ to (D, E).*

(i) *If $f(p_{<,N}, p_{<,E}, p_{<,E^*}) = (p_N, p_{E^*}, p_E)$, then $-$ exists and $- \in G$.*

(ii) *If $f(p_{<,N}, p_{<,E}, p_{<,E^*}) = (p_N, p_E, p_E)$ or (p_N, p_{E^*}, p_{E^*}) then $(D, \bar{E})$ is a homogeneous graph and $G \geq Aut(D, \bar{E})$.*

(iii) *If $f(p_{<,N}, p_{<,E}, p_{<,E^*}) = (p_E, p_N, p_N)$ or (p_{E^*}, p_N, p_N) then $(D, \bar{E})$ is a homogeneous graph and $G \geq Aut(D, \bar{E})$.*

(iv) *If f has any other non-identity behaviour then $G = Sym(D)$.*

Proof. The proofs are all the same to those for the generic digraph, Lemma 3.4.2, except that for (ii) and (iii) we make the additional observation that $(D, \bar{E})$ is homogeneous whenever we make use of Lemma 4.3.3. $\qquad \square$

We now move on to the general situation where we have added constants $\bar{c} \in D$ to the structure. For convenience, we assume that $c_i < c_j$ for all $i < j$. $(D, E, <, \bar{c})$ is $\aleph_0$-categorical, so as before we continue to identify types with orbits.

As in the generic digraph, we have two kinds of 1-types: singleton orbits and infinite orbits. However, unlike in the generic digraph, the infinite orbits will not necessarily be isomorphic to the original structure.

- Let $\mathcal{T} = \{L_3\}$ (L_3 is the 3-element linear order) and let $\bar{c} = (c_1)$. Then consider the orbit $X = \{x \in D : x < c_1 \wedge E(x, c_1)\}$. If there was an edge, ab say, in X then $\{c_1, a, b\}$ would be a copy of L_3. However, L_3 is forbidden, so X contains no edges, so in particular X is not isomorphic to $(D_\mathcal{T}, E_\mathcal{T}, <)$.

Fortunately, there are some orbits that are isomorphic to the original structure. For example, regardless of $\mathcal{T}$, the orbit $X = \{x \in D : x < c_1 \wedge \bigwedge_i N(x, c_i)\}$ is isomorphic to $(D, E, <)$ (a proof of this is provided below). These orbits form a central part of the argument so we give them a definition.

Definition 4.4.2. Let $\bar{c} \in D$ and $X \subset D$ be an orbit of $(D, E, <, \bar{c})$. We say X is *independent* if X is infinite and there are no edges between $\bar{c}$ and X.

The following lemma highlights the key feature of independent orbits that makes them useful. It is the analogue of Lemma 3.4.3 from Chapter 3.

Lemma 4.4.3. *Let X be an independent orbit of $(D, E, <, \bar{c})$.*

   (i) *Let $v \in D \backslash (X \cup \bar{c})$. Let $A = (a_0, \ldots, a_n)$ be a finite ordered digraph embeddable in D. Then there are $x_1, \ldots, x_n \in X$ such that $(a_0, a_1, \ldots, a_n) \cong (v, x_1, \ldots, x_n)$ in $(D, E, <, \bar{c})$.*

  (ii) *Let $v, v' \in D \backslash (X \cup \bar{c})$. Let $A = (a_0, \ldots, a_n)$ be a finite ordered digraph embeddable in D such that (a_0, a_1) and (v, v') are isomorphic as ordered digraphs. Then there are $x_2, \ldots, x_n \in X$ such that $(a_0, a_1, \ldots, a_n) \cong (v, v', x_2, \ldots, x_n)$ in $(D, E, <, \bar{c})$.*

(iii) *X is isomorphic to $(D, E, <)$.*

Proof. (i) Let k be the length of the tuple $\bar{c}$ and let x be any element of X. Consider the finite ordered digraph A' which is constructed as follows: start with A, add new vertices $c'_1, \ldots, c'_k$ and then add edges and an ordering so that we have $(a_0, c'_1, \ldots, c'_k) \cong$

$(v, c_1, \ldots, c_k)$ and $(a_i, c'_1, \ldots, c'_k) \cong (x, c_1, \ldots, c_k)$ for all $i > 0$. (These isomorphisms are isomorphisms of ordered digraphs).

By construction, there are only non-edges between a_j and c'_i for all i and all $j > 0$. As a result of these non-edges, any tournament embeddable in A' must be embeddable in either $\{a_0, a_1, \ldots, a_n\}$ or $\{a_0, c'_1, \ldots, c'_k\}$, which are both in the age of $(D, E, <)$. Therefore A' is also in the age of $(D, E, <)$. This means A' is embeddable in $(D, E, <)$. By homogeneity, we can assume the embedding maps c'_i to c_i and a_0 to v. Then the image of $(a_1, \ldots, a_n)$ is an appropriate choice for the required $(x_1, \ldots, x_n)$.

(ii) Use the same strategy as in (i): draw the finite ordered digraph capturing all the requirements for $x_2, \ldots, x_n$ and check this does not embed any of the forbidden tournaments.

(iii) The proof is exactly the same for the corresponding lemma for generic digraphs, Lemma 3.4.3 (ii). $\qquad\square$

As for the generic digraph, the 2-types of (D, E) are of the form $p_{X,Y,\phi,\psi} = \{(a, b) \in D : a \in X, b \in Y, \phi(a, b) \text{ and } \psi(a, b)\}$, where X and Y are orbits of $(D, E, <, \bar{c})$, $\phi \in \{<, >\}$ and $\psi \in \{E, E^*, N\}$.

We now move to the task of analysing the canonical functions. It turns out that it is sufficient to study those cases where we assume X is an independent orbit. This is not self-evident and will only become clear when proving the classification. The first lemma deals with the situation when $X = Y$.

Lemma 4.4.4. *Let G be a closed supergroup of $Aut(D, E)$, let $\bar{c} \in D$, let $f \in cl_{tm}(G)$ be a canonical function from $(D, E, <, \bar{c})$ to (D, E). Let X be an independent orbit of $Aut(D, E, \bar{c})$.*

(i) If $f(p_{X,X,<,N}, p_{X,X,<,E}, p_{X,X,<,E^*}) = (p_N, p_{E^*}, p_E)$, then $-$ exists and $- \in G$.

(ii) *If* $f(p_{X,X,<,N}, p_{X,X,<,E}, p_{X,X,<,E^*}) = (p_N, p_E, p_E)$ *or* (p_N, p_{E^*}, p_{E^*}) *then* $(D, \bar{E})$ *is a homogeneous graph and* $G \geq Aut(D, \bar{E})$.

(iii) *If* $f(p_{X,X,<,N}, p_{X,X,<,E}, p_{X,X,<,E^*}) = (p_E, p_N, p_N)$ *or* (p_{E^*}, p_N, p_N) *then* $(D, \bar{E})$ *is a homogeneous graph and* $G \geq Aut(D, \bar{E})$.

(iv) *If* f *has any other non-identity behaviour then* $G = Sym(D)$.

Proof. The proof is identical to that of Lemma 4.4.1 because $X \cong (D, E, <)$. $\qquad \square$

Next we look at the behaviour of f between an independent orbit X and any other orbit Y. The next lemma contains the analysis for the case where $X < Y$ or $X > Y$.

Lemma 4.4.5. *Let G be a closed supergroup of Aut(D, E), let $\bar{c} \in D$ and let $f \in cl_{tm}(G)$ be a canonical function from $(D, E, <, \bar{c})$ to (D, E). Let $X \subset D$ be an independent orbit on which f behaves like* id *and let Y be an infinite orbit such that $X < Y$ or $X > Y$.*

(i) *If* $f(p_{X,Y,N}, p_{X,Y,E}, p_{X,Y,E^*}) = (p_N, p_{E^*}, p_E)$, *then* sw *exists and* sw $\in G$.

(ii) *If* $f(p_{X,Y,N}, p_{X,Y,E}, p_{X,Y,E^*}) = (p_N, p_E, p_E)$ *or* (p_N, p_{E^*}, p_{E^*}), *then* $(D, \bar{E})$ *is a homogeneous graph and* $G \geq Aut(D, \bar{E})$.

(iii) *If* $f(p_{X,Y,N}, p_{X,Y,E}, p_{X,Y,E^*}) = (p_E, p_N, p_N)$ *or* (p_{E^*}, p_N, p_N), *then* $(D, \bar{E})$ *is a homogeneous graph and* $G \geq Aut(D, \bar{E})$.

(iv) *If* f *has any other non-identity behaviour, then* $G = Sym(D)$.

Remark: We do not need to include $<$ or $>$ in the subscripts of the types because it is automatically determined by whether $X < Y$ or $Y < X$.

Proof. Assume that $X < Y$. The proof for the case $Y < X$ is symmetric. Let $y_0 \in Y$ be any element.

The proof is almost all the same as the proof for the generic digraph, Lemma 3.4.5. The only difference is that we need to show that the behaviour which gave rot is not possible for Henson digraphs. So assume that $f(p_{X,Y,N}, p_{X,Y,E}, p_{X,Y,E^*}) = (p_E, p_{E^*}, p_N)$. We will show that this behaviour is not possible.

Let $T \in \mathcal{T}$ be of minimal cardinality and enumerate T as $(t_0, t_1, \ldots, t_n)$ so that t_0 has at least one edge going into it. Construct a digraph $A = (a_0, a_1, \ldots, a_n)$ as follows: start with A being equal to T and then replace edges into a_0 with non-edges, replace edges out of a_0 with incoming edges, and leave all other edges of A the same.

Since T was minimal, $A \in \text{Forb}(\mathcal{T})$ so A can be embedded in D. Furthermore, by Lemma 4.4.3 there are $x_i \in X$ such that $(a_0, a_1, \ldots, a_n) \cong (y_0, x_1, \ldots, x_n)$. Now apply f. By construction of A, $f(y_0, x_1, \ldots, x_n) \cong (t_0, \ldots, t_n)$. Thus, T is embeddable in D, contradicting $T \in \mathcal{T}$. $\qquad\square$

The arguments for the remaining analyses are the same as the corresponding analyses for the generic digraph, Lemma 3.4.6 and Lemma 3.4.7. Thus we state the lemmas without proof.

Lemma 4.4.6. *Let G be a closed supergroup of Aut(D, E), let $\bar{c} \in D$, let $f \in cl_{tm}(G)$ be a canonical function from $(D, E, <, \bar{c})$ to (D, E). Let $X \subset D$ be an independent orbit on which f behaves like id and let Y be an infinite orbit such that X and Y are interdense. Then at least one of the following holds.*

  (i) *f preserves all the edges and non-edges between X and Y.*

 (ii) *f switches the direction of all the edges between X and Y and sw exists.*

(iii) *$G \geq$Aut$(D, \bar{E})$ and $(D, \bar{E})$ is a homogeneous graph.*

(iv) *$G = Sym(D)$.*

Lemma 4.4.7. *Let G be a closed supergroup of $Aut(D, E)$, let $(c_1, \ldots, c_n) \in D$, let $f \in cl_{tm}(G)$ be a canonical function from $(D, E, <, \bar{c})$ to (D, E). Suppose that f behaves like id on $D^- := D \backslash \{c_1, \ldots, c_n\}$. Then at least one of the following holds.*

(i) *For all $i, 1 \leq i \leq n$, f behaves like id or like sw between c_i and D^-.*

(ii) *$G \geq Aut(D, \bar{E})$ and $(D, \bar{E})$ is a homogeneous graph.*

(iii) *$G = Sym(D)$.*

4.5 Proof of the classification

Theorem 4.2.2. *Let (D, E) be a Henson digraph and let $G \leq Sym(D)$ be a closed supergroup of $Aut(D, E)$. Then:*

(i) *$G \leq Aut(D, \bar{E})$ or $G \geq Aut(D, \bar{E})$*

(ii) *If $G < Aut(D, \bar{E})$ then $G = Aut(D, E), \langle - \rangle, \langle sw \rangle$ or $\langle -, sw \rangle$.*

(iii) *$(D, \bar{E})$ is the random graph, $(D, \bar{E})$ is a Henson graph or $(D, \bar{E})$ is not homogeneous. In the last case $Aut(D, \bar{E})$ is equal to $\max\{Aut(D, E), \langle - \rangle, \langle sw \rangle, \langle -, sw \rangle\}$ and is a maximal-closed subgroup of Sym(D).*

Remark: Recall the meaning of the notation used in (iii): If, for example, $-$ exists but sw does not, then $\max\{Aut(D, E), \langle - \rangle, \langle sw \rangle, \langle -, sw \rangle\} = \langle - \rangle$.

Proof. (i) Suppose for contradiction that $G \not\geq Aut(D, \bar{E})$ and $G \not\leq Aut(D, \bar{E})$. Because of the second assumption G violates the relation $\bar{E}$. By Lemma 2.7.3 this can be witnessed by a canonical function. Precisely, this means there are $c_1, c_2 \in D$ and $f \in cl_{tm}(G)$ such that $f : (D, E, <, c_1, c_2) \to (D, E)$ is a canonical function, $\bar{E}(c_1, c_2)$ and $N(f(c_1), f(c_2))$.

Now let X be an independent orbit of $(D, E, <, c_1, c_2)$.

Claim 1. f behaves like id on X.

By Lemma 4.4.4 we know that f behaves like id or $-$ on X, otherwise G would contain $\text{Aut}(D, \bar{E})$. If f behaves like $-$ on X, then we continue by replacing f by $- \circ f$.

Claim 2. f behaves like id between X and every other infinite orbit Y.

Let Y be another infinite orbit. By Lemma 4.4.5 and Lemma 4.4.6, f behaves like id or sw between X and Y, as otherwise G would contain $\text{Aut}(D, \bar{E})$. If f behaves like sw between them, then we simply replace f by $sw_Y \circ f$. Note that one needs to check sw_Y is a legitimate function, but this has already been done in Lemma 4.3.1 (iii).

Claim 3. f behaves like id on every infinite orbit and between every pair of infinite orbits.

Suppose not, so there are infinite orbits Y_1 and Y_2 (possibly the same) and there are distinct $y_1, y_2 \in Y_1, Y_2$, respectively, such that $(y_1, y_2) \not\cong f(y_1, y_2)$. Now for any finite digraph $(a_1, a_2, \ldots, a_n) \in \text{Forb}(\mathcal{T})$ with $(y_1, y_2) \cong (a_1, a_2)$, we can find $x_3, \ldots, x_n \in X$ such that $(y_1, y_2, x_3, \ldots, x_n) \cong (a_1, \ldots, a_n)$, by Lemma 4.4.3 (ii). Then f has the effect of only changing what happens between y_1 and y_2, since we know f behaves like id on X and between X and all infinite orbits. In short, given any finite digraph, we can use f to change what happens between exactly two of the vertices of the digraph.

There are three options. If f creates an edge from a non-edge, then we we can use f to introduce a forbidden tournament, which gives a contradiction. If f deletes the edge or changes the direction of the edge, then by Lemma 4.3.2 or Lemma 4.3.3, as appropriate, we obtain $G \geq \text{Aut}(D, \bar{E})$.

Claim 4. f behaves like id between $\{c_1, c_2\}$ and the union of all infinite orbits.

The follows immediately from Lemma 3.4.7, composing with sw_{c_i} if necessary.

Conclusion. We can assume that f behaves everywhere like the identity, except on (c_1, c_2), where it maps an edge to a non-edge. But then we get that $G = \text{Sym}(D)$ by

Lemma 4.3.2, completing the proof of (i).

(ii) The proof follows exactly the same series of claims as in part (i) but with minor adjustments to how one starts and concludes. We go through one case as an example, leaving the rest to the reader. We will show that if $\mathrm{Aut}(D, E) < G \le \mathrm{Aut}(D, \bar{E})$, then $G \ge \langle - \rangle$ or $G \ge \langle sw \rangle$ (if they exist). By assumption G preserves non-edges but not the relation E. By Lemma 2.7.3, there is an edge $c_1 c_2$ and a canonical function $f : (D, E, <, c_1, c_2) \to (D, E)$ which changes the direction of the edge $c_1 c_2$. Suppose for contradiction that $G \not\ge \langle - \rangle$ and that $G \not\ge \langle sw \rangle$

Let X be an independent orbit. By Lemma 4.4.4, f must behave like id on X and then by Lemma 4.4.5 and Lemma 4.4.6, f must behave like id between X and all other infinite orbits. By repeating the argument of Claim 3 above, f must behave like id on the union of infinite orbits and so by Lemma 4.4.7 f must behave like id between the constants and the union of infinite orbits. Now we are in the situation of Lemma 4.3.3 (iv), so we conclude that $G \ge \mathrm{Aut}(D, \bar{E})$, so $G \ge \langle - \rangle, \langle sw \rangle$.

(iii) $(D, \bar{E})$ embeds every finite empty graph and is connected (Lemma 4.1.2 (ii)). Hence, if $(D, \bar{E})$ is a homogeneous graph then $(D, \bar{E})$ has to be the random graph or a Henson graph, by the classification of countable homogeneous graphs ([LW80]).

Thus assume that $(D, \bar{E})$ is not a homogeneous graph. Let $G' := \max\{\mathrm{Aut}(D, E), \langle - \rangle, \langle sw \rangle, \ \langle -, sw \rangle\}$. Now let G be a closed group such that $G' < G \le \mathrm{Sym}(D)$. We want to show that $G = \mathrm{Sym}(D)$. By Lemma 2.7.3, there are $\bar{c} \in D$ and a canonical $f : (D, E, <, \bar{c}) \to (D, E)$ generated by G such that f cannot be imitated by any function of G' on $\bar{c}$. To be precise, we mean that for all $g \in G'$, $g(\bar{c}) \neq f(\bar{c})$.

Now we continue as in (i), proving that we may assume f behaves like id on the union of all infinite orbits and like id between $\bar{c}$ and the union of infinite orbits. Hence, we are in the situation of either Lemma 4.3.2 (iv) or Lemma 4.3.3 (iv). Thus, either $G = \mathrm{Sym}(D)$

and we are done, or $(D, \bar{E})$ is a homogeneous graph - contradiction.

Note that in this argument, we may have replaced f with $- \circ f$ or $sw_A \circ f$ for some A. We have to check such a replacement satisfies all the requirements that f did: being canonical, being generated by G, and not being imitated G' on $\bar{c}$. The first requirement is met by Lemma 2.7.2.

If we had to replace f with $- \circ f$, then this is because f behaved like $-$ on an independent X, which implies $-$ exists and hence is in G' and so also in G. Since $- \in G$, $- \circ f$ is generated by G. Since $- \in G'$, $- \circ f$ cannot be imitated by G' on $\bar{c}$.

If we had to replace f with $sw_A \circ f$, then this is because f behaved like sw between X and an infinite orbit Y, which implies sw exists and hence is in G' and G. Since sw exists, sw_A also exists (by Lemma 4.3.1). Since $sw \in G$, then sw_A is generated by G so $sw_A \circ f$ is generated by G. Lastly, because $sw \in G'$, $sw \circ f$ cannot be imitated by G' on $\bar{c}$.

Thus we have shown there are no closed groups in between G' and $\text{Sym}(D)$. Since $\text{Aut}(D, \bar{E})$ contains G' and is a proper subgroup of $\text{Sym}(D)$, we must conclude that $G' = \text{Aut}(D, \bar{E})$, as required.

This completes the proof. $\qquad \square$

4.6 $2^{\aleph_0}$ pairwise non-isomorphic maximal-closed subgroups of $\text{Sym}(\mathbb{N})$

Definition 4.6.1. Let G be a closed subgroup of $\text{Sym}(\mathbb{N})$. We say that G is *maximal-closed* if $G \neq \text{Sym}(\mathbb{N})$ and there are no closed groups G' such that $G < G' < \text{Sym}(\mathbb{N})$.

We construct $2^{\aleph_0}$ maximal-closed subgroups of $\text{Sym}(\mathbb{N})$ such that no two are isomorphic as abstract groups. This is done by modifying Henson's construction of $2^{\aleph_0}$ pairwise non-isomorphic homogeneous countable digraphs and taking their automorphism groups. The

modification is needed to ensure that the groups are maximal. A short argument will show that the automorphism groups are pairwise non-conjugate. That these groups are pairwise non-isomorphic follows from Rubin's work on reconstruction: in [Rub94], it is shown that the automorphism groups of two Henson digraphs are isomorphic as abstract groups if and only if they are conjugate.

Henson's construction in [Hen72] centres on finding an infinite anti-chain of finite tournaments. (An anti-chain of tournaments is a set of tournaments such that no element of the set can be embedded in any other element of the set.)

Definition 4.6.2. Let $n \in \mathbb{N}\backslash\{0\}$. I_n denotes the n-element tournament obtained from the linear order L_n by changing the direction of the edges $(i, i + 1)$ for $i = 1, \ldots, n - 1$ and of the edge $(1, n)$.

By counting 3-cycles, Henson showed that $\{I_n : n \geq 6\}$ is an anti-chain. It is a short exercise to show that the 3-cycles in I_n are $(1, 3, n), (1, 4, n), \ldots, (1, n - 2, n), (1, 2, 3), (2, 3, 4), \ldots, (n - 2, n - 1, n)$. In particular, observe that I_n has at most two vertices through which there are more than four 3-cycles, namely the vertices 1 and n; this observation is useful in our modification.

The automorphism groups of the Henson digraphs constructed by forbidding any subset of these I_n's are not maximal: $\langle - \rangle$ and the automorphism group of the random graph are closed supergroups. By forbidding a few extra tournaments, however, we can ensure that the automorphism groups are maximal.

Let T be a finite tournament that is not embeddable in I_n for any n and that contains a source, s say, but no sink. Such a T can be found, for example, by ensuring there are at least three vertices through which there are more than four 3-cycles.

Let $k = |T|$. Let $\mathcal{T} = \{T' : |T| = k + 1, T \text{ is embeddable in } T'\}$. Then for $A \subseteq \mathbb{N}\backslash\{1, \ldots, k + 1\}$, let $\mathcal{T}_A = \{I_n : n \in A\} \cup \mathcal{T}$. Then let D_A be the Henson digraph whose

set of forbidden tournaments is $\mathcal{T}_A$. The automorphism groups of these D_A is the set of groups we want.

Theorem 4.6.3. $\{Aut(D_A) \ : \ A \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}\}$ *is a set of* $2^{\aleph_0}$ *pairwise non-isomorphic maximal-closed subgroups of Sym*$(\mathbb{N})$.

Proof. **Claim 1.** For all $A \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}, \mathcal{T}_A$ is not closed under $-$.

Let T' be obtained as follows: Change the direction of all the edges of T and then add a new vertex t which is a sink. Since T has no sinks, T can not be embedded into T', hence $T' \notin \mathcal{T}_A$. Now consider $-(T')$. By construction, T is embeddable in $-(T')$, so $-(T') \in \mathcal{T}_A$. Thus $\mathcal{T}_A$ is not preserved under $-$.

Claim 2. For all $A \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}, \mathcal{T}_A$ is not closed under sw.

Let T' be obtained as follows: Change the source s in T to a sink, and then add a new vertex which is a sink. Since T has no sinks, T can not be embedded into T', hence $T' \notin \mathcal{T}_A$. Now consider switching T' about s, to obtain T''. By construction, T is embeddable in T'', so $T'' \in \mathcal{T}_A$. Thus $\mathcal{T}_A$ is not preserved under sw.

Claim 3. For all $A \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}, (D_A, \bar{E})$ is not a Henson graph nor the random graph.

Finite linear orders do not embed any element of $\mathcal{T}_A$, thus are embeddable in D_A. Removing the direction of the edges in a finite linear order gives a complete graph, so $(D_A, \bar{E})$ is not K_n-free for any n, so $(D_A, \bar{E})$ is not a Henson graph.

Now let $U \subset D_A$ be isomorphic to T - this is possible as T has not been forbidden. Then there is no vertex $x \in D$ such that for all $u \in U$, $E(x,u) \vee E(u,x)$, because all tournaments containing T are forbidden. Hence $(D_A, \bar{E})$ does not satisfy the extension property of the random graph and so is not isomorphic to the random graph.

Claim 4. For all $A \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}$, Aut$(D_A)$ is a maximal-closed subgroup of Sym$(\mathbb{N})$.

This follows from the classification Theorem 4.2.2 and the previous three claims.

Claim 5. For all $A \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}$, $\mathcal{T}_A$ is an anti-chain.

Let $T_1, T_2 \in \mathcal{T}_A$ and suppose for contradiction that T_1 is embeddable in T_2. By considering cardinalities and noting that $|T_1|$ must be smaller than $|T_2|$, it follows that $|T_2| \notin \mathcal{T}$. Hence, $T_2 = I_n$ for some $n \in A$. By Henson's arguments, T_1 cannot equal I_m for any $m \in A$. Thus $T_1 \in \mathcal{T}$, which implies that T is embeddable in I_n, contradicting our choice for T.

Claim 6. If $A, B \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}$ are not equal, $D_A \ncong D_B$.

Suppose, without loss, that there is some n in A but not in B. Then I_n is not embeddable in D_A. To prove the claim, it suffices to show that I_n *is* embeddable in D_B. Suppose for contradiction that it is not. Hence, $I_n \notin \text{Forb}(\mathcal{T}_B)$ which means that I_n embeds an element of $\mathcal{T}_B$. But this implies that $\mathcal{T}_{B \cup \{n\}}$ is not an anti-chain, contrary to Claim 5.

Claim 7. If $A, B \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}$ are not equal, then $\text{Aut}(D_A)$ and $\text{Aut}(D_B)$ are not conjugate.

We prove the contrapositive so suppose $\text{Aut}(D_A)$ and $\text{Aut}(D_B)$ are conjugate. Let $f : D_A \to D_B$ be a bijection witnessing this, so that $\text{Aut}(D_A) = f^{-1}\text{Aut}(D_B)f$. In particular this means that f maps orbits of $\text{Aut}D_A$ to orbits of $\text{Aut}D_B$, i.e., that f is canonical. f cannot map edges to non-edges or vice-versa, because non-edges are symmetric and edges are not. This leaves only two options: f behaves like id or f behaves like $-$. We can rule out the latter option because we know from (the proof of) Claim 1 that $\mathcal{T}$ is not closed under $-$. Hence, f behaves like id, which means f is an isomorphism, so by Claim 6 we conclude that $A = B$.

Claim 8. If $A, B \subseteq \mathbb{N}\backslash\{1,\ldots,k+1\}$ are not equal, then $\text{Aut}(D_A)$ and $\text{Aut}(D_B)$ are not isomorphic as pure groups.

This follows from Claim 7 and Rubin's reconstruction results [Rub94].

Together, Claim 4 and Claim 8 prove the theorem. □

Chapter 5

Miscellany

5.1 The Countable Vector Space over $\mathbb{F}_2$

In this section we provide a classification of the reducts of the countable vector space over $\mathbb{F}_2$, using the classification of $\aleph_0$-categorical strictly minimal sets. We let $(V, +, 0)$ denote the vector space and let Aff denote the 4-ary relation which holds if $x + y = z + w$.

Theorem 5.1.1. *The reducts of $(V, +, 0)$ are $(V, +, 0), (V, \mathrm{Aff}), (V, 0)$ and $(V, =)$.*

We note that there are two other proofs of this result and that they use different methodologies. In [BKS15], the result is obtained by hands-on combinatorics, using no machinery apart from basic linear algebra. In [BB13], the proof uses the same canonical functions machinery that we used for the digraphs.

5.1.1 Background

For this section, we assume familiarity with a couple more notions from model theory. Explicitly, we assume familiarity with compactness arguments and elementary extensions.

Definition 5.1.2. Let V be the $\aleph_0$-dimensional vector space over a finite field $\mathbb{F}_q$.

- $GL(\aleph_0, q)$ is the set of invertible linear maps $V \to V$.

- (V, Aff), where Aff is the 4-ary relation which holds if $x + y = z + w$, is known as an *affine space*. $AGL(\aleph_0, q)$ is the automorphism group of (V, Aff). Alternatively, one can define $AGL(\aleph_0, q)$ as the set of permutations obtained by adding every translation, $v \mapsto v + a$, to $GL(\aleph_0, q)$.

- $P(V)$, the *projective space*, is the set of 1-dimensional subspaces of V along with a 3-ary relation which holds if u, v, w all lie in a single 2-dimensional space. $PGL(\aleph_0, q)$ is the automorphism group of $P(V)$. Alternatively, one can define $PGL(\aleph_0, q)$ as the induced action of $GL(\aleph_0, q)$ on the set of 1-dimensional subspaces of V.

We then obtain the semilinear versions of the above groups by letting automorphisms of $\mathbb{F}_q$ act on the spaces.

Definition 5.1.3. Let V be the $\aleph_0$-dimensional vector space over a finite field $\mathbb{F}_q$.

- $\Gamma L(\aleph_0, q)$ is the set of bijections $T : V \to V$ which preserve addition and such that there exists $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$ so that for all $\lambda \in \mathbb{F}_q, v \in V, T(\lambda v) = \sigma(\lambda)T(v)$.

- $A\Gamma L(\aleph_0, q)$ is the group obtained by adding every translation to $\Gamma L(\aleph_0, q)$.

- $P\Gamma L(\aleph_0, q)$ is the group obtained by the induced action of $\Gamma L(\aleph_0, q)$ on $P(V)$.

Example: If q is prime, then $\mathbb{F}_q$ has no non-trivial automorphisms, so $\Gamma L(\aleph_0, q) = GL(\aleph_0, q)$.

Definition 5.1.4. Let $\mathcal{M}$ be a structure.

(i) We say $\mathcal{M}$ is *minimal* if every definable (with parameters) subset of M is finite or cofinite.

(ii) We say $\mathcal{M}$ is *strongly minimal* if every elementary extension of $\mathcal{M}$ is minimal.

(iii) We say $\mathcal{M}$ is primitive if there is no non-trivial $\emptyset$-definable equivalence relation on $\mathcal{M}$, i.e. if $\phi(x, y)$ defines an equivalence relation, then either $\phi(x, y)$ holds iff $x = y$ or $\phi(x, y)$ holds for all $x, y \in M$.

(iv) We say $\mathcal{M}$ is strictly minimal if it is strongly minimal and primitive.

Examples:

- $(M, =)$ is strictly minimal.

- Let K be a finite field of characteristic greater than two. Then a vector space over K is strongly minimal but not primitive. To see that it is not primitive, consider the formula $\phi(x, y)$ which says that there exists $k \in K \backslash \{0\}$ such that $x = ky$.

Theorem 5.1.5. *Let $\mathcal{M}$ be $\aleph_0$-categorical and strictly minimal. Let $G = Aut(\mathcal{M})$. Then one of the following is true:*

(i) *$PGL(\aleph_0, q) \leq G \leq P\Gamma L(\aleph_0, q)$, for some (prime power) q.*

(ii) *$AGL(\aleph_0, q) \leq G \leq A\Gamma L(\aleph_0, q)$, for some q.*

(iii) *$G = Sym(M)$.*

This classification was proved independently by Cherlin [CHL85], Mills, and Zil'ber [Zil84]. A model-theoretic proof was later found by Hrushovski [Hru92] and a geometric proof was found by Evans [Eva86]. This statement of the classification and this history is taken from [EMI97, Page 13].

5.1.2   Proof of the classification

Lemma 5.1.6. *Let $\mathcal{M}$ be $\aleph_0$-categorical and let $\mathcal{N}$ be a reduct of $\mathcal{M}$.*

(i) *If $\mathcal{M}$ is minimal (resp. strongly minimal, primitive, strictly minimal) then $\mathcal{N}$ is minimal (resp. strongly minimal, primitive, strictly minimal).*

(ii) *If $Aut(\mathcal{M}) = PGL(\aleph_0, 2)$, then $Aut(\mathcal{N}) = PGL(\aleph_0, 2)$ or $Sym(M)$.*

Proof. (i) As $\mathcal{N}$ is a reduct of $\mathcal{M}$, every set definable in $\mathcal{N}$ is definable in $\mathcal{M}$. Thus minimality and primitivity are preserved by going to reducts.

All that remains to be checked is strong minimality, so let $\mathcal{M}$ be strongly minimal, $\mathcal{N}$ be a reduct of $\mathcal{M}$ and let $\mathcal{N}' \succ \mathcal{N}$ be an elementarily extension. We want to show that $\mathcal{N}'$ is minimal. Ideally, we would like to say that $\mathcal{N}'$ is the reduct of an elementary extension $\mathcal{M}'$ of $\mathcal{M}$, because $\mathcal{M}$ strongly minimal implies $\mathcal{M}'$ minimal implies $\mathcal{N}'$ minimal. However, it is feasible that the domain of $\mathcal{N}'$ cannot be the domain of some $\mathcal{M}'$ (e.g. if $\mathcal{M}$ is a vector space $(V, +), \mathcal{N} = (V, =)$ and $\mathcal{N}' = (V \cup \{c\}, =)$). Instead, we must find an $\mathcal{M}' \succ \mathcal{M}$ such that its reduct $\mathcal{N}''$ is an elementary extension of $\mathcal{N}'$. To find such an $\mathcal{M}'$, one uses a standard compactness argument which we leave as an exercise to the reader.

(ii) It suffices to show that if $\mathcal{N}$ is a proper reduct of $\mathcal{M}$, then $Aut(\mathcal{N})$ must equal $Sym(M)$.

Since $Aut(\mathcal{M}) = PGL(\aleph_0, 2)$, $\mathcal{M}$ is strictly minimal, so by (i) $\mathcal{N}$ is also strictly minimal. Then, by Theorem 5.1.5, we know that $\mathcal{N}$ must be a projective or affine group over q or must be equal to $Sym(M)$.

Claim: $Aut(\mathcal{N})$ is 3-transitive.

Consider the orbits on (proper) 3-tuples of $PGL(\aleph_0, 2)$. Observe there are two orbits: one consisting of linearly independent triples and the other consisting of dependent triples. Furthermore, if a function preserves these orbits then it must be linear and hence an

element of $\mathrm{PGL}(\aleph_0, 2)$. Hence, since $\mathcal{N}$ is a proper reduct, $\mathrm{Aut}(\mathcal{N})$ contains a function which does not preserve these orbits, so $\mathrm{Aut}(\mathcal{N})$ has only one orbit on 3-tuples, as required.

The only 3-transitive groups in Theorem 5.1.5 are $\mathrm{Sym}(M)$ and $\mathrm{AGL}(\aleph_0, 2)$, so our final task is ruling out $\mathrm{AGL}(\aleph_0, 2)$.

Claim: $\mathrm{Aut}(\mathcal{N})$ is 4-transitive.

Let v_1, v_2 be distinct elements of the projective space. Then observe that in $\mathrm{PGL}(\aleph_0, 2)$, the pointwise stabiliser of $\{v_1, v_2, v_1 + v_2\}$ is transitive on the rest of the projective space. But then because $\mathcal{N}$ is 3-transitive, this observation implies that $\mathcal{N}$ is 4-transitive, as required.

Since $\mathrm{AGL}(\aleph_0, 2)$ is not 4-transitive, we are done. □

Before proving the classification, we introduce one final notion.

Definition 5.1.7. Let $\mathcal{M}$ be a structure and $A \subseteq M$. Then $\mathrm{dcl}_\mathcal{M}(A)$, the *definable closure of A*, is the set $\{x \in M : f(x) = x$ for every $f \in \mathrm{Aut}(\mathcal{M})$ such that $f(a) = a$ for all $a \in A\}$.

In words, the definable closure of A is the set of elements that are fixed by every pointwise stabiliser of A. A simple observation is that if $\mathcal{N}$ is a reduct of $\mathcal{M}$ then $\mathrm{dcl}_\mathcal{N}(A) \subseteq \mathrm{dcl}_\mathcal{M}(A)$.

Proof of Theorem 5.1.1. Let $\mathcal{N}$ be a reduct of $(V, +)$.

First suppose that 0 is definable in $\mathcal{N}$ - we need to show that $\mathcal{N} = (V, +)$ or $(V, 0)$. Let $\mathcal{N}'$ be the structure obtained from $\mathcal{N}$ by deleting the element 0 and retaining all the other information. (Alternatively, $\mathcal{N}'$ is the structure associated to the induced action of $\mathrm{Aut}(\mathcal{N})$ on $V \backslash \{0\}$.) Similarly, let $\mathcal{M}'$ be the structure obtained by deleting 0 from $(V, +)$, and note that $\mathcal{N}'$ is a reduct of $\mathcal{M}'$. Since we are in characteristic 2, non-zero vectors correspond

bijectively to 1-dimensional subspaces, so $\mathcal{M}'$ is effectively the projective space $P(V)$, so Aut($\mathcal{M}'$)=PGL($\aleph_0, 2$). Then by Lemma 5.1.6, Aut($\mathcal{N}'$)= PGL($\aleph_0, 2$) or Sym($V \setminus \{0\}$), which implies that $\mathcal{N} = (V, +)$ or $(V, 0)$, as required.

Now suppose that 0 is not definable in $\mathcal{N}$ - we need to show that $\mathcal{N} = (V, \mathrm{Aff})$ or $(V, =)$. We start by showing that Aut($\mathcal{N}$) is 3-transitive. First note that Aut($V, +$) is 2-transitive on $V \setminus \{0\}$. Then, because 0 is not definable Aut($\mathcal{N}$) we can map some, and hence by transitivity, any non-zero element to 0. So Aut($\mathcal{N}$) is transitive and the pointwise stabilisers of single elements are 2-transitive, which implies Aut($\mathcal{N}$) is 3-transitive.

Now, 3-transitivity implies that $\mathcal{N}$ is primitive and hence strictly minimal. Applying Theorem 5.1.5, we conclude that Aut($\mathcal{N}$)=AGL($\aleph_0, 2$) or Sym(V) as they are the only 3-transitive groups in the list.

It is tempting to say we are done, however there is a subtlety which is easy to miss: Aut($\mathcal{N}$)=AGL($\aleph_0, 2$) only implies that $\mathcal{N}$ is isomorphic to (V, Aff), not that $\mathcal{N}$ equals (V, Aff). So let $\mathrm{Aff}' \subset \mathcal{N}^4$ be such that $\mathcal{N} = (V, \mathrm{Aff}')$. We want to show that $\mathrm{Aff}' = \mathrm{Aff}$.

Because $\mathcal{N} \cong (V, \mathrm{Aff})$, Aff' satisfies the same first-order properties that Aff does. In particular, it satisfies the following:

- For all x, $\mathrm{Aff}'(x, x, x, x)$.

- For all x, y, $\mathrm{Aff}'(x, x, y, y)$.

- For all x_1, x_2, x_3, there exists a unique x_4 such that $\mathrm{Aff}'(x_1, x_2, x_3, x_4)$. In addition, $\mathrm{dcl}_{\mathcal{N}}(x_1, x_2, x_3) = \{x_1, x_2, x_3, x_4\}$.

- Aff' is permutation invariant, i.e. for all $\bar{x} \in \mathcal{N}^4$ and all $\sigma \in S_4$, $\mathrm{Aff}'(\bar{x})$ iff $\mathrm{Aff}'(\sigma(\bar{x}))$, where $\sigma(x_1, x_2, x_3, x_4) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$.

So we want to show that $\mathrm{Aff} = \mathrm{Aff}'$. We know that for all x_1, x_2, x_3, $\mathrm{Aff}(x_1, x_2, x_3, x_1 + x_2 + x_3)$. Thus, by the third point above it suffices to show that for all x_1, x_2, x_3, we have

$\text{Aff}'(x_1, x_2, x_3, x_1 + x_2 + x_3)$. We do this by a case analysis.

Case 1: x_1, x_2 and x_3 are not pairwise distinct. Then we are done by the first, second and fourth point above.

Case 2: x_1, x_2 and x_3 are pairwise distinct and linearly dependent. Hence $x_1 + x_2 = x_3$ and $\text{dcl}_{(V,+)}(x_1, x_2, x_3) = \{x_1, x_2, x_3, 0\}$. Thus we must have $\text{Aff}'(x_1, x_2, x_3, 0)$, so we are done because $x_1 + x_2 + x_3 = x_1 + x_2 + x_1 + x_2 = 0$.

Case 3: x_1, x_2 and x_3 are pairwise distinct and linearly independent. Then by considering the definable closure of these x_i in $(V, +)$, the only options for the fourth element are linear combinations of x_1, x_2 and x_3. The fourth element cannot be $x_1 + x_2$ because then by permutation-invariance and Case 2, we would conclude that $x_3 = 0$, contradicting linear independence. Similarly, we can rule out the other options, leaving us to conclude that $\text{Aff}'(x_1, x_2, x_3, x_1 + x_2 + x_3)$, as wanted.

This completes the proof. $\qquad\qquad\square$

5.2 $2.\mathbb{Q}$

Let $M = \mathbb{Q} \times \{0, 1\}$ and let $<$ be the lexicographic linear order on $\mathbb{Q} \times \{0, 1\}$; then $2.\mathbb{Q} = (M, <)$. Intuitively, $2.\mathbb{Q}$ is obtained by summing $\mathbb{Q}$ copies of the 2-element linear order. This is an $\aleph_0$-categorical linear order. Note that $2.\mathbb{Q}$ is (first-order) interdefinable with $(M, A, B, <_A, <_B, E)$, where $A = \mathbb{Q} \times \{0\}, B = \mathbb{Q} \times \{1\}$ and E is the equivalence relation $E((x, i), (y, j))$ iff $x = y$. In this language, $2.\mathbb{Q}$ becomes homogeneous and in fact Ramsey.

Theorem 5.2.1. $2.\mathbb{Q}$ *has finitely many reducts.*

Note that there is not a concise list of all the reducts, which is why we have stated the theorem as it is.

We have decided not to provide a proof of this result as we believe the details and arguments are dull, fiddly and uninteresting. Instead, we attempt to give a flavour of the arguments and some of the key ideas involved. We apologise in advance to the keen reader who does want the full details and we ask such readers to contact us - we will be happy to oblige!

Lemma 5.2.2. $(M, A, B, <_A, <_B)$ *has finitely many reducts.*

Proof. Let f_1 be an isomorphism (of linear orders) from $\mathbb{Q} \times \{0\} \to \{x \in \mathbb{Q} : x < 0\}$ and let f_2 be an isomorphism from $\mathbb{Q} \times \{1\} \to \{x \in \mathbb{Q} : x < 0\}$. Then let f be the union of f_1 and f_2 to obtain a bijection $M \to \mathbb{Q} \backslash \{0\}$. Then this function f witnesses the fact that $\mathrm{Aut}(M, A, B, <_A, <_B)$ and $\{g|_{\mathbb{Q} \backslash \{0\}} : g \in \mathrm{Aut}(\mathbb{Q}, <), g(0) = 0\}$ are conjugate. This implies that the reducts of $(M, A, B, <_A, <_B)$ correspond to the reducts of $(\mathbb{Q}, <, 0)$ in which 0 is definable. Since the reducts of $(\mathbb{Q}, <, 0)$ were classified in [JZ08], and there are finitely many, this completes the proof. $\qquad\square$

Lemma 5.2.3. *If $\mathcal{N}$ is a reduct of $2.\mathbb{Q}$ in which E is not definable, then $\mathcal{N}$ is a reduct of $(M, A, B, <_A, <_B)$.*

Sketch of proof. We do not provide a proof of this, as the only proof known to us is a tedious case analysis, and we believe there ought to be a conceptual reason why this is true.

However, we will make some comments on the proof. The strategy is the same as that used for the digraphs. We find a function which witnesses that $\mathrm{Aut}(\mathcal{N})$ does not preserve E. Then there are finitely many 'behaviours' to consider, and in each one we can conclude that $\mathrm{Aut}(\mathcal{N})$ contains $\mathrm{Aut}(M, A, B, <_A, <_B)$. We used the quotation marks because we mean 'behaviour' in the non-technical everyday-language sense, as opposed to the precise notion defined earlier in the thesis.

But why do we not want to use the precise notion? After all, the black box Lemma 2.7.3 is applicable and we can obtain canonical functions. The reason is that there are

certain behaviours of canonical functions which do not provide enough information about $\text{Aut}(\mathcal{N})$, at least as far as we could determine. The reason for this is that obtaining a function which is canonical comes at the cost of losing surjectivity, and the arguments we know rely on the functions being surjective. Roughly, the nuisance canonical functions behave as follows: they embed M into A so that the images of $(A, <_A)$ and of $(B, <_B)$ are interdense. No matter how we composed such a function with automorphisms of $2.\mathbb{Q}$, we were not able to manipulate the orbits of tuples as we needed.

But then how do we break down the task into finitely many cases? The answer is to look inside the black box! The proof of Lemma 2.7.3 goes roughly like this: Let f be any function and let $\bar{c}$ be any constants. By the correspondence between Ramsey structures and strongly-amenable groups ([KPT05]), adding constants to the language does not change that the structure is Ramsey. By Ramsey-ness, $\aleph_0$-categoricity and the fact the structure is ordered, we can assume that f behaves canonically on arbitrarily large finite subsets of the domain. Then by homogeneity, we can use automorphisms to align the finite subsets to get a chain whose union is the whole domain, and such that the restrictions of f to these finite subsets are nested functions and are canonical. Then the limit of these functions is the canonical function the blackbox spits out. The construction cannot guarantee the final function is surjective, which is why we need to consider the topological *monoid* generated by f. Intuitively, the canonical function is just a neat way of packaging the information that f has provided.

Coming back to $2.\mathbb{Q}$, what we do is stop at the point where we have found out that f behaves canonically on arbitrarily large finite sets. This way we retain the fact that f is surjective but still get to control the behaviour of f enough to have a finite and manageable list of cases to go through. $\qquad\qquad\square$

What is left to discuss is what happens when E is definable in a reduct $\mathcal{N}$.

Lemma 5.2.4. *There are finitely many reducts of* $2.\mathbb{Q}$ *in which* E *is definable.*

Sketch of proof. Let $\mathcal{N}$ be a reduct in which E is definable. Then consider the quotient $\mathcal{N}' = \mathcal{N}/E$. Observe that $\mathcal{N}'$ must be a reduct of $(\mathbb{Q}, <)$. Hence, there are only 5 options for $\mathcal{N}'$, by the classification of the reducts of the rationals in [Cam76].

What is left to determine is how $\mathrm{Aut}(\mathcal{N})$ can act on the equivalance classes of E and whether there is any interaction between this action and $\mathrm{Aut}(\mathcal{N}')$. On an individual class, $\mathrm{Aut}(\mathcal{N})$ can act like the trivial group, which we denote as I, or it can act like S_2. To describe the possible actions of $\mathrm{Aut}(\mathcal{N}')$ on all the classes, we establish some notation.

Let $f \in \mathrm{Aut}(\mathcal{N})$. Since f preserves E we can consider the quotient function f_E as being the induced function on $\mathbb{Q}$. We let the *switched pairs* of f, $sp(f)$, be the subset $\{q \in \mathbb{Q} : f(q, 0) = (f_E(q), 1)\}$. Then we let $sp(\mathcal{N}) = \{sp(f) : f \in \mathrm{Aut}(\mathcal{N})\}$. By a short and relatively straightforward argument, one can show there are three possibilities for $sp(\mathcal{N}) : \{\emptyset\}, \{\emptyset, \mathbb{Q}\}$ and $P(\mathbb{Q})$, where 'P' denotes the powerset function.

If $sp(\mathcal{N}) = \{\emptyset\}$, then $\mathrm{Aut}(\mathcal{N}) = \mathrm{Aut}(\mathcal{N}') \times I$.

If $sp(\mathcal{N}) = P(\mathbb{Q})$, then $\mathrm{Aut}(\mathcal{N})$ is the wreath product of $\mathrm{Aut}(\mathcal{N}')$ and S_2.

If $sp(\mathcal{N}) = \{\emptyset, \mathbb{Q}\}$ then there are two cases. For three of the possibilities of $\mathrm{Aut}(\mathcal{N}')$, there is one option, namely, $\mathrm{Aut}(\mathcal{N}') \times S_2$, where the non-identity element of S_2, s say, acts by mapping (q, i) to $(q, 1 - i)$.

The other two possibilities of $\mathrm{Aut}(\mathcal{N}')$ have order 2 subgroups - let the cosets be G and G'. Then there are two options. The first is the same as above, $\mathrm{Aut}(\mathcal{N}) = \mathrm{Aut}(\mathcal{N}') \times S_2$. The second option is that $\mathrm{Aut}(\mathcal{N}) = G \times I \cup G' \times \{s\}$.

This completes the 'proof'. □

5.3 Final Thoughts

Summary

In Chapter 3, we proved the classification of the reducts of the generic digraph, which is the Fraïssé limit of the set of all finite digraphs. The main idea of the proof is fundamentally unsophisticated: given an unknown closed group $G \leq \mathrm{Sym}(\mathbb{N})$ properly containing a known group G', we let $f \in G \backslash G'$ and look at $\langle G' \cup \{f\} \rangle$. It just so happens that without any other information, there are so many possibilities for f that it is impractical, if not impossible, to study $\langle G' \cup \{f\} \rangle$ in general. However, extending the generic digraph to an ordered Ramsey structure, we could use the black box Lemma 2.7.3 to show that there are only finitely many simple possibilities for f to consider, allowing us to follow through with the unsophisticated idea.

In Chapter 4, we proved the classification of the reducts of the Henson digraph. Once *somebody* has checked the details, it is arguably sufficient to replace Chapter 4 with the phrase "Same as the the generic digraph". I exaggerate of course, if only because of the corollary in Section 4.6, in which the automorphism groups of a certain subset of the (original) Henson digraphs are shown to be continuum-many pairwise non-isomorphic maximal-closed subgroups of $\mathrm{Sym}(\mathbb{N})$.

In Section 5.1, the classification of $\aleph_0$-categorical strictly minimal sets was used to give an alternative proof of the classification of the reducts of the vector space over $\mathbb{F}_2$. In Section 5.2, we sketched the argument for the classification of the reducts of $2.\mathbb{Q}$. The bulk of the work is in showing that the case where the equivalence relation is not definable reduces to work done by Junker and Ziegler, with the other case following with less difficulties from Cameron's classification.

Future Work

We end by discussing a few possible avenues for future endeavours.

First, there are open questions that have been asked previously:

- (Thomas' Conjecture): If a structure is homogeneous in a finite relational language, then it only has finitely many reducts. ([Tho91])

- Which lattices can be realised as the lattice of reducts of some structure? ([JZ08])

- Is there always a maximal closed group between a closed group G and $\mathrm{Sym}(\mathbb{N})$? ([MN96])

One question that arises from our work is whether there is a direct way of using the classification of the generic digraph to prove the classification for the Henson digraphs, instead of having to go through the whole proof again and checking that all the details work. This phenomenon occurred previously: the same proof worked for both the classification of the random graph and for the generic K_n-free graphs ([Tho91]). We direct the interested reader to [Aga16], Section 5.1. There, the classification of the reducts of the random tournament ([Ben97]) was transferred to help prove some of the classification of the generic digraph.

Note that an initial conjecture was that if you have two homogeneous structures $\mathcal{M}$ and $\mathcal{N}$ such that $\mathrm{Age}(\mathcal{N}) \subseteq \mathrm{Age}(\mathcal{M})$, then the lattice of reducts of $\mathcal{N}$ is a sublattice of the reducts of $\mathcal{M}$. However, this is false. We leave it is an exercise for anybody interested, with the hint that the Henson digraphs provide a counter-example.

Another goal is to determine the lattice of reducts of the countable vector space over $\mathbb{F}_p$. We note that the classification of strictly minimal sets does provide a lot of information about this, but not enough to get a complete description. This is probably not an easy

problem as it is closely related to the investigation of finite covers of projective/affine spaces, which have been studied previously in some depth.

A similar goal would be to study the reducts of other $\aleph_0$-categorical linear orders and determine whether they all have finitely many reducts. These were characterised by Rosenstein ([Ros69]), so the task is slightly less daunting than on first appearance. Note that a positive answer to this question will complete the task, started in [JZ08], of showing that all $\aleph_0$-categorical extensions of $(\mathbb{Q}, 0)$ by unary predicates have finitely many reducts.

Or you could just determine the reducts of your favourite structure.

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