Model theory of multidimensional asymptotic classes

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The candidate confirms that the work submitted is his own, except where work which has formed part of jointly authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

Work from the following jointly authored paper, which is listed as [2] in the References, is included in this thesis:

> S. Anscombe, H.D. Macpherson, C. Steinhorn and D. Wolf, 'Multidimensional asymptotic classes and generalised measurable structures', 2016. In preparation.

Chapter 2 of this thesis is based on work contained in this paper. The work in the paper that is directly attributable to the candidate comprises the definition of an R-mac, which all the authors developed together; the results regarding interpretability; the example regarding direct sums; the non-examples relating to the random graph and the random tournament; and the example regarding classes of $\mathcal{L}$-structures with boundedly many 4-types. (Although the last example is the subject of Theorem 4.6.4 of this thesis, the details behind the example do not appear in the paper; indeed, these details are due to appear in [60], a separate, solely authored paper.) While the candidate also provided input into other parts of the paper through discussion, the rest of the paper is predominantly attributable to the other authors. Note that the paper is still in preparation and thus the information given here may be subject to change.

To my wife, to whose love, support and patience this work is a testament

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Abstract

In this PhD thesis we explore the concept of a multidimensional asymptotic class. This is a new notion in model theory, arising as a generalisation of the Elwes–Macpherson–Steinhorn notion of an N-dimensional asymptotic class [22] and thus ultimately as a development of the Lang–Weil estimates of the number of points of a variety in a finite field [47]. We provide the history and motivation behind the topic before developing its basic theory, paying particular attention to multidimensional exact classes, a special kind of multidimensional asymptotic class where the measuring functions provide the precise sizes of the definable sets, rather than only approximations. We describe a number of examples and non-examples and then show that multidimensional asymptotic classes are closed under bi-interpretability. We use results about smoothly approximable structures [35] and Lie coordinatisable structures [18] to prove the following result, as conjectured by Macpherson: For any countable language $\mathcal{L}$ and any positive integer d the class $\mathcal{C}(\mathcal{L}, d)$ of all finite $\mathcal{L}$-structures with at most d 4-types is a polynomial exact class in $\mathcal{L}$; here a polynomial exact class is a multidimensional exact class with polynomial measuring functions. We finish the thesis by posing some open questions, indicating potential further lines of research.

Contents

Chapter 1

Introduction

Dimension theory, so-called, has become a pervasive theme in contemporary model theory.

Macpherson & Steinhorn, p. 411 of [50]

This thesis is on the topic of asymptotic classes, a relatively new area of research lying at the boundary between finite and infinite model theory, where 'model theory' here refers to the study of mathematical structures in the context of first-order classical logic. Accordingly, it has been written for mathematicians familiar with the fundamentals of model theory. Such familiarity can be gained from a number of texts, of which [53] and [58] are two of the present author's favourites.

This introductory chapter has four aims: To convince the reader of the topic's worth, to outline what has been written about the topic already, to outline what is written about the topic in this thesis, and to explain the notation and terminology used to write about the topic. Accomplishing the first three of these aims already requires the fourth – which is really just bookkeeping, at least for the cognoscenti – so the reader may wish to scan over § 1.4 before continuing.

1.1 History and motivation

The story of asymptotic classes begins with the famous theorem of Zoé Chatzidakis, Lou van den Dries and Angus Macintyre regarding definable sets in finite fields, as published in [12]:

Theorem 1.1.1 (CDM, 1992)**.** *Let $\varphi(\bar{x}, \bar{y})$ be a formula in the language of rings $\mathcal{L}_{\text{ring}} := \{0, 1, +, \cdot\}$, where $n := l(\bar{x})$ and $m := l(\bar{y})$. Then there exist a* *constant $C \in \mathbb{R}^+$ and a finite set D of pairs $(d, \mu) \in \{0, \ldots, n\} \times \mathbb{Q}^+$ such that for every finite field $\mathbb{F}_q$ and for every $\bar{a} \in {\mathbb{F}_q}^m$, if $\varphi({\mathbb{F}_q}^n, \bar{a}) \neq \varnothing$, then*

$$\left| |\varphi({\mathbb{F}_q}^n, \bar{a})| - \mu q^d \right| \leq C q^{d-1/2} \tag{1.1}$$

for some pair $(d, \mu) \in D$. Moreover, the parameters are definable; that is, for each $(d, \mu) \in D$ there exists an $\mathcal{L}_{\mathrm{ring}}$-formula $\varphi_{(d,\mu)}(\bar{y})$ such that for every $\mathbb{F}_q$, $\mathbb{F}_q \models \varphi_{(d,\mu)}(\bar{a})$ if and only if $\bar{a}$ satisfies (1.1) for (d, μ).

The proof of the theorem uses the Lang–Weil estimates of the number of points of a variety in a finite field [47], the work of Ax on pseudofinite fields [4] and the work of Kiefe on quantifier elimination in finite fields [37]. See [11] for some useful notes. There are two key aspects to the theorem: Firstly, it says that the sizes of definable sets in finite fields can be described approximately in terms of a dimension d and a measure μ, with the relative error decreasing asymptotically; indeed, in order to gain some intuition, the reader may find it useful to read (1.1) as $|\varphi({\mathbb{F}_q}^n, \bar{a})| \approx \mu q^d$. Let's call this the *dimension–measure aspect*. Secondly, it says that each such size, taken across the class of finite fields, is governed by a finite set of functions, in a definable way. Let's call this the *finite aspect*.

Dugald Macpherson and Charles Steinhorn decided to investigate both apects of CDM-like behaviour further, whereby they developed the notion of an asymptotic class as a generalisation of Theorem 1.1.1. Richard Elwes, a student of Macpherson, quickly generalised this notion to that of an N-dimensional asymptotic class, the original definition of Macpherson and Steinhorn becoming a 1-dimensional asymptotic class under Elwes's new definition (modulo slightly different error terms):

Definition 1.1.2 (Macpherson–Steinhorn [50], Elwes [21], 2007/2008)**.** Let $\mathcal{L}$ be a first-order language, $N \in \mathbb{N}^+$ and $\mathcal{C}$ a class of finite $\mathcal{L}$-structures. Then $\mathcal{C}$ is an *N-dimensional asymptotic class* if for every $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, where $n := l(\bar{x})$ and $m := l(\bar{y})$,

(a) there exist a finite set $D \subset (\{0, \ldots, Nn\} \times \mathbb{R}^+) \cup \{(0, 0)\}$ and a partition $\{\Phi_{(d,\mu)} : (d, \mu) \in D\}$ of the set $\{(\mathcal{M}, \bar{a}) : \mathcal{M} \in \mathcal{C}, \bar{a} \in M^m\}$ such that for each $(d, \mu) \in D$,

$$\left| |\varphi(\mathcal{M}^n, \bar{a})| - \mu |M|^{d/N} \right| = o\left(|M|^{d/N}\right)$$

for all $(\mathcal{M}, \bar{a}) \in \Phi_{(d,\mu)}$ as $|M| \to \infty$; and

(b) for each $(d, \mu) \in D$ there exists an $\mathcal{L}$-formula $\varphi_{(d,\mu)}(\bar{y})$ such that for every $\mathcal{M} \in \mathcal{C}$, $\mathcal{M} \models \varphi_{(d,\mu)}(\bar{a})$ if and only if $(\mathcal{M}, \bar{a}) \in \Phi_{(d,\mu)}$;

where the meaning of the little-o notation is as follows: For every $\varepsilon > 0$ there exists $Q \in \mathbb{N}$ such that for all $(\mathcal{M}, \bar{a}) \in \Phi_{(d,\mu)}$, if $|M| > Q$, then

$$\left| |\varphi(\mathcal{M}^n, \bar{a})| - \mu |M|^{d/N} \right| < \varepsilon |M|^{d/N}.$$

The investigation of asymptotic classes proved to be fruitful, with many interesting examples and theorems given in [21] and [50].

This now brings us to multidimensional asymptotic classes, the topic of this thesis, which are a further generalisation of the notion of an asymptotic class. The details are given in the next chapter, but we outline the idea behind the new notion here. An N-dimensional asymptotic class $\mathcal{C}$ is just that: N-dimensional. The definable subsets of each $\mathcal{M} \in \mathcal{C}$ must have approximate size $\mu |M|^{d/N}$ for some dimension–measure pair (d, μ). Proving that a class of structures is an N-dimensional asymptotic class is a strong result, but being tied to the dimension–measure functions limits the range of examples. What about classes of structures whose definable sets do not fit this particular dimension–measure picture? For instance, consider a multi-sorted structure $\mathcal{M}$ with sorts of different dimensions. In this case the size $|M|$ of the structure is too crude an input for functions approximating the sizes of the definable sets. However, taking the sizes of the individual sorts as inputs may allow the finite aspect of CDM-like behaviour to be demonstrated, as shown by the example of vector spaces over finite fields (Example 2.3.2). It was this example that originally inspired Macpherson and Steinhorn to move beyond the framework of N-dimensional asymptotic classes, initially leading them to the multi-sorted approach. It is also the reason behind the choice of the term 'multidimensional'.

But we can go further: Why not allow functions of any nature? The multi-sorted approach is still limited and allowing more exotic functions further expands the range of examples, as demonstrated by Example 2.3.8 and Theorem 4.6.4. This is the idea behind a multidimensional asymptotic class for R (Definition 2.1.2): The set of functions R can be anything, save the requirement that only a finite number of functions from R are required for any given formula. Dealing with the functions separately allows us to study the finite aspect of CDM-like behaviour in isolation, permitting statements of the form 'There exists R such that $\mathcal{C}$ is a multidimensional asymptotic class for

R'. This is not to say that we disregard the nature of R. To the contrary: For example, the goal of Chapter 4 is to ascertain the nature of R for a particular family of examples.

This then is the motivation behind the topic of this thesis: Drop the dimension–measure aspect of CDM-like behaviour and focus on the finite aspect. Work on multidimensional asymptotic classes has been successful, with a number of interesting examples found and results proved, some of which are given in this thesis. Nonetheless, the research is still in its infancy and there is much more to be investigated (see Chapter 5).

In this thesis we pay particular attention to multidimensional exact classes (Definition 2.1.4), where we have the finite aspect for the exact sizes of the definable sets, not just their approximate sizes.

Before we move on, we remark on the notion of a measurable structure (Definition 3.1 in [22]). We do not study measurable structures in this thesis, but they are important to mention because of their close relation to asymptotic classes: Any infinite ultraproduct of an N-dimensional asymptotic class is a measurable structure (Proposition 3.9 in [22]). Put heuristically: Measurable structures are the infinite counterparts to N-dimensional asymptotic classes. What then are the infinite counterparts to multidimensional asymptotic classes? In [2] it is shown that generalised measurable structures provide the answer. Generalised measurable structures are to measurable structures as multidimensional asymptotic classes are to N-dimensional asymptotic classes: In a measurable structure the sizes of definable sets are measured by dimension–measure pairs lying in $\mathbb{N} \times \mathbb{R}^+ \cup \{(0,0)\}$, with only finitely many such pairs being required for each formula. In a generalised measurable structure one keeps the finiteness requirement but drops the requirement that the measuring functions be of this form, allowing the codomain of the measuring function to be any ordered semiring (as defined in [2]). As one would hope, an infinite ultraproduct of a multidimensional asymptotic class is generalised measurable [2]. (Generalised) measurable structures are interesting things in their own right, e.g. [28] and [33], but they are also a useful tool for proving things about asymptotic classes. For example, since a measurable structure is supersimple (Corollary 3.7 in [22]), an infinite ultraproduct of an N-dimensional aysmptotic class must also be supersimple. In the new context of multidimensional asymptotic classes, one may place conditions on R and see how they affect the ultraproducts of a multidimensional asymptotic class in R, as investigated in [2].

1.2 Literature review

We have already mentioned the main journal articles on N-dimensional asymptotic classes, namely [21] and [50]. Further material is to be found in the PhD thesis of Richard Elwes [20] and there are two survey articles [22] and [51], the latter also containing work on robust classes, which we do not discuss. Asymptotic classes also arise in [25], [30], [55] and [56]. Darío García, one of the authors of [25], has recently developed a new direction of research, namely that of an o-asymptotic class (the o standing for 'order'), which is an attempt to avoid the issue blocking Non-Example 2.3.9 from being an asymptotic class. Nothing has yet been published on this new topic, but there is a paper in preparation [24]. The situation of multidimensional asymptotic classes in the literature is similar: Nothing has yet been published, but two papers are in preparation [2] (the main paper) and [60]. The manuscript [1] of Sylvy Anscombe and Charlotte Kestner is also relevant, as is the work of Ricardo Bello Aguirre [6], [7]. It is hoped that Bello Aguirre's results may lead to further joint work with the present author: see Question 5.6.

1.3 Outline of the thesis

In Chapter 2 we give the definition of a multidimensional asymptotic class and that of a multidimensional exact class. We make some basic observations before going through a number of technical lemmas that are used at various points throughout the thesis. We then provide a number of examples and non-examples of both multidimensional asymptotic classes and multidimensional exact classes. We finish the chapter with a result allowing new examples to be found via interpretations in known examples.

Chapters 3 and 4 build toward our main result, namely a proof of the conjecture of Macpherson, as stated in the Abstract. In Chapter 3 we introduce smooth approximation and show how it provides a generic example of a multidimensional exact class. This is a nice result in itself, but it is of the form 'There exists R such that $\mathcal{C}$ is a multidimensional exact class for R', thus immediately raising the question as to the nature of R. In Chapter 4 we answer this question by using the great technical machinery of Lie coordinatisation, finishing the chapter with a proof of Macpherson's conjecture.

We end with some open questions in Chapter 5. There are also Appendices A and B, which cover the details of some notions used in the main text.

1.4 Notation and terminology

We use the notation $X := Y$ to mean that X is defined to be equal to Y and the notation $X :\Longleftrightarrow Y$ to mean that X is defined to be equivalent to Y. We use the symbol $\equiv$ to denote equality between formulas and $:\equiv$ to mean defined equality between formulas. The notation $A \subseteq B$ means that A is a subset of B, while $A \subset B$ means that A is a proper subset of B.

We refer to [53] and [58] for general model-theoretic notation and terminology. Unless otherwise specified, $\mathcal{L}$ denotes a finitary (i.e. $L_{\omega,\omega}$), first-order language with a symbol $=$ for equality, $\mathcal{M}$ an $\mathcal{L}$-structure, $\mathcal{C}$ a class of finite $\mathcal{L}$-structures, and T an $\mathcal{L}$-theory, where by an '$\mathcal{L}$-theory' we mean a negation-complete set of sentences, i.e. for every $\mathcal{L}$-sentence φ either $\varphi \in T$ or $\neg\varphi \in T$ (and not both!). Note that we identify isomorphic $\mathcal{L}$-structures; that is (equivalently), two $\mathcal{L}$-structures are distinct if and only if they are not isomorphic.

We distinguish between the $\mathcal{L}$-structure $\mathcal{M}$ (calligraphic font) and its underlying set M (roman font), although we won't maintain this distinction pedantically. We use the notation $\mathcal{N} \leq \mathcal{M}$ to mean that $\mathcal{N}$ is an $\mathcal{L}$-substructure of $\mathcal{M}$. The *theory of* $\mathcal{M}$ is the set

$$\mathrm{Th}(\mathcal{M}) := \{\varphi : \varphi \text{ is an } \mathcal{L}\text{-sentence and } \mathcal{M} \models \varphi\}.$$

For a subset $A \subseteq M$, $\mathcal{L}_A$ denotes the language where the elements of A are named by constant symbols; that is, $\mathcal{L}_A$ is the language obtained by adding to $\mathcal{L}$ a constant symbol c_a for each element $a \in A$, where the assignment [1] of c_a in $\mathcal{M}$ is defined to be a, i.e. ${c_a}^{\mathcal{M}} := a$. We conflate the element $a \in A$ and its constant symbol c_a, denoting both by a and calling such an element/constant symbol a *parameter*.

We use $x, y, z, x_1, x_2, \ldots$, etc. for variables and $a, b, c, a_1, a_2, \ldots$, etc. for parameters. We write $\bar{x}$ and $\bar{a}$ to denote finite tuples of variables and parameters respectively and $l(\bar{x})$ to denote the length of a tuple, e.g. if $\bar{x} = (x_1, \ldots, x_n)$, then $l(\bar{x}) = n$. Concatenation may be used to mean union, e.g. $ab := \{a, b\}$, $B\bar{a} := B \cup \{\bar{a}\}$; whether this applies should be clear from the context.

We write $\varphi(x_1, \ldots, x_n)$ to indicate that all the free variables in the $\mathcal{L}$-formula φ are among the x_i and that the x_i are pairwise distinct. Note that writing $\varphi(x_1, \ldots, x_n)$ does *not* mean that each x_i is necessarily a free variable in φ; for instance, $\varphi(x_1, \ldots, x_n)$ could even be an $\mathcal{L}$-sentence. Also note

[1] We use the word 'assignment' to avoid overuse of the word 'interpretation' in § 2.4, the latter perhaps being the more standard term, e.g. Definition 1.1.2 in [53] and Definition 1.1.2 in [58].

that how we label the free variables in a formula is important: For example, $R(x_1, x_2) \wedge P(x_1)$ and $R(x_2, x_1) \wedge P(x_2)$ are (the only) distinct labellings of the free variables in the formula $R(y, z) \wedge P(y)$ by the variables x_1 and x_2. Accordingly, the notation $\varphi(x_1, \ldots, x_n)$ implicitly specifies a particular labelling of the free variables in φ by $x_1, \ldots, x_n$. When we say things such as 'for every $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$', we really mean 'for every $\mathcal{L}$-formula φ and for every possible labelling of the free variables in φ by $x_1, \ldots, x_n$'.

For parameters $a_1, \ldots, a_n$ and an $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$, $\varphi(a_1, \ldots, a_n)$ denotes the $\mathcal{L}_{\{a_1,\ldots,a_n\}}$-formula obtained from $\varphi(x_1, \ldots, x_n)$ by replacing each instance of x_i with a_i, for every i. So, for example, if $\varphi(x_1, x_2)$ denotes the formula $R(x_1, x_2) \wedge P(x_1)$, then $\varphi(a_1, a_2)$ denotes the formula $R(a_1, a_2) \wedge P(a_1)$, but if $\varphi(x_1, x_2)$ denotes the formula $R(x_2, x_1) \wedge P(x_2)$, then $\varphi(a_1, a_2)$ denotes the formula $R(a_2, a_1) \wedge P(a_2)$.

For an $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ with $l(\bar{x}) = n$ and $l(\bar{y}) = m$, an $\mathcal{L}$-structure $\mathcal{M}$ and a tuple $\bar{a} \in M^m$, we define

$$\varphi(\mathcal{M}^n, \bar{a}) := \{\bar{b} \in M^n : \mathcal{M} \models \varphi(\bar{b}, \bar{a})\}.$$

For a subset $A \subseteq M$, $\operatorname{dcl}(A)$ denotes the definable closure of A and $\operatorname{acl}(A)$ denotes the algebraic closure of A. See Exercises 1.4.10 and 1.4.11 in [53] for definitions of these terms.

If E is a $\varnothing$-definable equivalence relation on $\mathcal{M}^n$ and $\bar{a} \in M^n$, then $\bar{a}/E$ denotes the E-equivalence class that contains $\bar{a}$ and $\ulcorner \bar{a}/E \urcorner$ denotes the same E-equivalence class but as a member of $\mathcal{M}^{\mathrm{eq}}$. So $\ulcorner \bar{a}/E \urcorner \in \mathcal{M}^{\mathrm{eq}}$ is a canonical parameter for the $\bar{a}$-definable subset $\bar{a}/E \subseteq M^n$. See §§ 1.3 and 8.2 of [53] or § 8.4 of [58] for an introduction to $\mathcal{M}^{\mathrm{eq}}$ and canonical parameters.

We define $\mathbb{Q}^+ := \{x \in \mathbb{Q} : x > 0\}$, $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}^{\geq 0} := \mathbb{R}^+ \cup \{0\}$, where $\mathbb{Q}$ and $\mathbb{R}$ denote the set of rational numbers and the set of real numbers respectively. It will be useful to distinguish between the set $\mathbb{N} := \{0, 1, 2, \ldots\}$ of natural numbers and the set $\mathbb{N}^+ := \{1, 2, 3, \ldots\}$ of positive natural numbers. We write $n \in \mathbb{N}$ and $n < \omega$ interchangeably.

See Appendix B for notation and terminology regarding types.

Chapter 2

Multidimensional asymptotic classes

So if a man's wit be wandering, let him study the mathematics.

Francis Bacon, *Of Studies*, 1625 edition

In this chapter we do precisely that and go through the mathematical details of multidimensional asymptotic classes. In § 2.1 we state the main definition of the thesis and make some remarks upon it. We then move on to § 2.2, where we go over a number of technical lemmas. With the exception of the Projection Lemmas (Lemmas 2.2.1 and 2.2.2), these lemmas are not particularly interesting in themselves and so the reader may wish to skip this section upon first reading, returning to it as and when each lemma is used. In § 2.3 we consider a number of examples and non-examples. We end the chapter with § 2.4, which deals with interpretability in multidimensional asymptotic classes.

2.1 Basic definitions

Let $\mathcal{L}$ be a language and let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures. For $m \in \mathbb{N}^+$ define

$$\mathcal{C}(m) := \{(\mathcal{M}, \bar{a}) : \mathcal{M} \in \mathcal{C}, \bar{a} \in M^m\}.$$

The elements of $\mathcal{C}(m)$ are sometimes referred to as *pointed structures*. For completeness we further define $\mathcal{C}(0) := \mathcal{C}$.

Definition 2.1.1 (Definable partition)**.** Let Φ be a partition of $\mathcal{C}(m)$. An element $\pi \in \Phi$ is *definable* if there exists a parameter-free $\mathcal{L}$-formula $\psi(\bar{y})$ with

$l(\bar{y}) = m$ such that for every $(\mathcal{M}, \bar{a}) \in \mathcal{C}(m)$, we have $(\mathcal{M}, \bar{a}) \in \pi$ if and only if $\mathcal{M} \models \psi(\bar{a})$. The partition Φ is *definable* if π is definable for every $\pi \in \Phi$.

We now state the fundamental definition of this thesis, as developed jointly by Anscombe, Macpherson, Steinhorn and the present author:

Definition 2.1.2 (R-mac)**.** Let R be a set of functions from $\mathcal{C}$ to $\mathbb{R}^{\geq 0}$ closed under addition and multiplication. Then $\mathcal{C}$ is a *multidimensional asymptotic class for R in $\mathcal{L}$*, or *R-mac in $\mathcal{L}$* for short, if for every $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, where $n := l(\bar{x})$ and $m := l(\bar{y})$, there exists a finite definable partition Φ of $\mathcal{C}(m)$ such that for each $\pi \in \Phi$ there exists $h_\pi \in R$ such that

$$\Big| |\varphi(\mathcal{M}^n, \bar{a})| - h_\pi(\mathcal{M}) \Big| = o(h_\pi(\mathcal{M})) \tag{2.1}$$

for all $(\mathcal{M}, \bar{a}) \in \pi$ as $|M| \to \infty$, where the meaning of the little-o notation is as follows: For every $\varepsilon > 0$ there exists $Q \in \mathbb{N}$ such that for all $(\mathcal{M}, \bar{a}) \in \pi$, if $|M| > Q$, then

$$\Big| |\varphi(\mathcal{M}^n, \bar{a})| - h_\pi(\mathcal{M}) \Big| \leq \varepsilon h_\pi(\mathcal{M}). \tag{2.2}$$

Remark 2.1.3.

(i) We call the functions h_π the *measuring functions* and the $\mathcal{L}$-formulas that define the partition Φ the *defining $\mathcal{L}$-formulas.*

(ii) The stipulation that R be closed under addition and multiplication is stated for convenience: If A and B are definable sets, then their disjoint union $A \sqcup B$ is definable and has size $|A| + |B|$ and their cartesian product $A \times B$ is definable and has size $|A| \cdot |B|$, so even if the stipulation were not made explicit, it would still need to be satisfied, albeit at least asymptotically and ignoring inconsequential exceptions. We suppress the stipulation for finite classes, e.g. Corollary 2.2.8.

Also note that R must contain a constant function $\mathcal{M} \mapsto k$ for each $k \in \mathbb{N}$ (again at least asymptotically), since one can always define a set of any given fixed size, assuming of course that the structures in $\mathcal{C}$ are arbitrarily large.

(iii) If we drop the requirement that the partition Φ be definable, then we call $\mathcal{C}$ a *weak R-mac*. We call (2.1) the *size clause* and the requirement that the partition be definable the *definability clause*. So a weak R-mac need satisfy only the size clause. We sometimes use the term *full R-mac* to emphasise that both the size and definability clauses hold and the term *strictly weak R-mac* to emphasise that only the size clause holds.

(iv) Condition (2.2) holds if π is bounded – that is, if there exists some $N \in \mathbb{N}$ such that $|M| \leq N$ for all $(\mathcal{M}, \bar{a}) \in \pi$ – since we may choose $Q = N$ for any $\varepsilon > 0$, in which case there is no $(\mathcal{M}, \bar{a}) \in \pi$ with $|M| > Q$ and so the condition is vacuously true. It follows that if $\mathcal{C}$ is finite, then $\mathcal{C}$ is vacuously an R-mac for any R.

Also note that (2.2) is equivalent to

$$(1-\varepsilon)h_\pi(\mathcal{M}) \leq |\varphi(\mathcal{M}^n, \bar{a})| \leq (1+\varepsilon)h_\pi(\mathcal{M}).$$

This alternative form perhaps makes the asymptotic nature of the approximation easier to see.

(v) In the $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ it is important to maintain the distinction between the variables $\bar{x}$ and the variables $\bar{y}$. (Although we use the plural *variables*, either of $\bar{x}$ and $\bar{y}$ could denote a single variable.) The variables $\bar{x}$, which we call *object variables*, are slots for solutions in each $\mathcal{M} \in \mathcal{C}$. The variables $\bar{y}$, which we call *parameter variables*, are slots for parameters from each $\mathcal{M} \in \mathcal{C}$. To aid clarity we sometimes demarcate the two kinds of variables with a semicolon, writing $\varphi(\bar{x}; \bar{y})$.

(vi) There is a more precise notion of a *multidimensional exact class for R in $\mathcal{L}$*, or *R-mec in $\mathcal{L}$* for short, where we have the equality $|\varphi(\mathcal{M}^n, \bar{a})| = h_\pi(\mathcal{M})$ instead of the approximation (2.1); see Definition 2.1.4 below. As with point (iii) above, we have the corresponding size and definability clauses and the notions of strictly weak, weak and full R-mecs. Notice that any R-mec in $\mathcal{L}$ is also an R-mac in $\mathcal{L}$.

We often refer to multidimensional exact classes simply as *exact classes*.

(vii) R-macs and R-mecs are closed under taking subclasses of $\mathcal{C}$ and supersets of R: If $\mathcal{C}$ is an R-mac (resp. -mec) in $\mathcal{L}$, then any subclass of $\mathcal{C}$ is also an R'-mac (resp. -mec) in $\mathcal{L}$ for any superset $R' \supseteq R$. Equivalently, if $\mathcal{C}$ is not an R-mac (resp. -mec) in $\mathcal{L}$, then no superclass of $\mathcal{C}$ is an R'-mac (resp. -mec) in $\mathcal{L}$ for any subset $R' \subseteq R$.

Weak R-macs and weak R-mecs are closed under taking reducts of the language: If $\mathcal{C}$ is a weak R-mac (resp. -mec) in $\mathcal{L}$, then $\mathcal{C}$ is also a weak R-mac (resp. -mec) in any reduct of $\mathcal{L}$. Equivalently, if $\mathcal{C}$ is not a weak

R-mac (resp. -mec) in $\mathcal{L}$, then $\mathcal{C}$ is also not a weak R-mac (resp. -mec) in any extension of $\mathcal{L}$. Note that we can't remove the prefix 'weak' here, since taking a reduct of the language may affect the definability clause.

Definition 2.1.4 (R-mec)**.** Let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures and let R be a set of functions from $\mathcal{C}$ to $\mathbb{N}$ closed under addition and multiplication. Then $\mathcal{C}$ is a *multidimensional exact class for R in $\mathcal{L}$*, or *R-mec in $\mathcal{L}$* for short, if for every $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, where $n := l(\bar{x})$ and $m := l(\bar{y})$, there exists a finite definable partition Φ of $\mathcal{C}(m)$ such that for each $\pi \in \Phi$ there exists $h_\pi \in R$ such that

$$|\varphi(\mathcal{M}^n, \bar{a})| = h_\pi(\mathcal{M})$$

for all $(\mathcal{M}, \bar{a}) \in \pi$.

2.2 Useful lemmas

As the title of this section suggests, we now state and prove a number of useful lemmas. We begin with the Projection Lemmas:

Lemma 2.2.1 (Projection Lemma for R-macs)**.** *Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures. Suppose that the definition of an R-mac (Definition 2.1.2) holds for $\mathcal{C}$ and for all $\mathcal{L}$-formulas $\varphi(x, \bar{y})$ with a single object variable x (as opposed to a tuple $\bar{x}$). Then $\mathcal{C}$ is an R-mac in $\mathcal{L}$.*

Lemma 2.2.2 (Projection Lemma for R-mecs)**.** *Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures. Suppose that the definition of an R-mec (Definition 2.1.4) holds for $\mathcal{C}$ and for all $\mathcal{L}$-formulas $\varphi(x, \bar{y})$ with a single object variable x (as opposed to a tuple $\bar{x}$). Then $\mathcal{C}$ is an R-mec in $\mathcal{L}$.*

A proof of Lemma 2.2.1 is given in [2]. It is adapted from the proof of Theorem 2.1 in [50]. We give a proof of Lemma 2.2.2, which is a simplified version of Anscombe's proof.

Proof of Lemma 2.2.2. Consider an arbitrary $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, where $n := l(\bar{x})$ and $m := l(\bar{y})$. We need to prove that it satisfies both the size and definability clauses. We do this by induction on the length of $\bar{x}$. The base case of the induction is the hypothesis of the lemma.

Let $\bar{x} = (x_1, \ldots, x_n)$. By the induction hypothesis we may assume that the size and definability clauses are satisfied by $\varphi(x_1, \ldots, x_{n-1}; x_n, \bar{y})$, where the semicolon is used to indicate the division between the object variables and the parameter variables (see Remark 2.1.3(v)). So we have a finite partition Γ of $\mathcal{C}(1+m) = \{(\mathcal{M}, a, \bar{b}) : \mathcal{M} \in \mathcal{C}, (a, \bar{b}) \in M^{1+m}\}$ with measuring functions $\{f_i : i \in \Gamma\} \subseteq R$ and defining $\mathcal{L}$-formulas $\{\gamma_i(x_n, \bar{y}) : i \in \Gamma\}$.

Consider each $\gamma_i(x_n, \bar{y})$. By the base case of the induction, each $\gamma_i(x_n, \bar{y})$ satisfies the size and definability clauses, so for each $i \in \Gamma$ we have a finite partition $\Phi_i := \{\pi_{i1}, \ldots, \pi_{ir_i}\}$ of $\mathcal{C}(m) = \{(\mathcal{M}, \bar{b}) : \mathcal{M} \in \mathcal{C}, \bar{b} \in M^m\}$ with measuring functions $\{g_{ij} : 1 \leq j \leq r_i\} \subseteq R$ and defining $\mathcal{L}$-formulas $\{\psi_{ij}(\bar{y}) : 1 \leq j \leq r_i\}$. We thus have $k := |\Gamma|$ finite partitions of $\mathcal{C}(m)$. We use them to construct a single finite partition Φ of $\mathcal{C}(m)$. Define

$$\pi_{(j_1,\ldots,j_k)} := \bigcap_{i \in \Gamma} \pi_{ij_i} \text{ and } J := \{(j_1, \ldots, j_k) : 1 \leq j_i \leq r_i, 1 \leq i \leq k\}.$$

Then $\Phi := \{\pi_{(j_1,\ldots,j_k)} : (j_1, \ldots, j_k) \in J\}$ forms a finite partition of $\mathcal{C}(m)$. We now need to show that this partition works.

We first consider the size clause. For each $\pi_{(j_1,\ldots,j_k)}$ we need to find a function $h_{(j_1,\ldots,j_k)} \in R$ such that

$$h_{(j_1,\ldots,j_k)}(\mathcal{M}) = |\phi(\mathcal{M}^n, \bar{b})| \tag{2.3}$$

for all $(\mathcal{M}, \bar{b}) \in \pi_{(j_1,\ldots,j_k)}$. So fix some arbitrary $(j_1, \ldots, j_k)$ and consider an arbitrary pair $(\mathcal{M}, \bar{b}) \in \pi_{(j_1,\ldots,j_k)}$. (If $\pi_{(j_1,\ldots,j_k)} = \varnothing$, then any function $h \in R$ would be vacuously suitable, so we can ignore this case.) Let $\chi_i(x_1, \ldots, x_n, \bar{y})$ denote the $\mathcal{L}$-formula

$$\varphi(x_1, \ldots, x_n, \bar{y}) \wedge \gamma_i(x_n, \bar{y}).$$

Then, since the $\mathcal{L}$-formulas $\gamma_i(x_n, \bar{a})$ define the partition Γ, $\varphi(\mathcal{M}^n, \bar{b})$ is partitioned by the $\chi_i(\mathcal{M}^n, \bar{b})$, i.e.

$$\varphi(\mathcal{M}^n, \bar{b}) = \bigcup_{i \in \Gamma} \chi_i(\mathcal{M}^n, \bar{b}), \tag{2.4}$$

where the union is disjoint. Now, for each $i \in \Gamma$ we have

$$\left|\chi_i(\mathcal{M}^n, \bar{b})\right| = \sum_{a \in \gamma_i(\mathcal{M}, \bar{b})} \left|\varphi(\mathcal{M}^{n-1}, a, \bar{b})\right|$$

because $\chi_i(\mathcal{M}^n, \bar{b})$ fibres over $\gamma_i(\mathcal{M}, \bar{b})$. Thus

$$\left|\chi_i(\mathcal{M}^n, \bar{b})\right| = f_i(\mathcal{M}) \cdot \left|\gamma_i(\mathcal{M}, \bar{b})\right|, \tag{2.5}$$

since $\left|\varphi(\mathcal{M}^{n-1}, a, \bar{b})\right| = f_i(\mathcal{M})$ if $\mathcal{M} \models \gamma_i(a, \bar{b})$. But $(\mathcal{M}, \bar{b}) \in \pi_{(j_1,\ldots,j_k)} \subseteq \pi_{ij_i}$ and so $\left|\gamma_i(\mathcal{M}, \bar{b})\right| = g_{ij_i}(\mathcal{M})$, which gives

$$\left|\chi_i(\mathcal{M}^n, \bar{b})\right| = f_i(\mathcal{M}) \cdot g_{ij_i}(\mathcal{M})$$

when put into (2.5). Combining this with (2.4) yields

$$\left|\varphi(\mathcal{M}^n, \bar{b})\right| = \sum_{i \in \Gamma} f_i(\mathcal{M}) \cdot g_{ij_i}(\mathcal{M}).$$

So define

$$h_{(j_1,\ldots,j_k)}(\mathcal{M}) := \sum_{i=1}^{k} f_i(\mathcal{M}) \cdot g_{ij_i}(\mathcal{M})$$

for all $\mathcal{M} \in \mathcal{C}$ and (2.3) is satisfied as required.

We now come to the definability clause. Let $\psi_{(j_1,\ldots,j_k)}(\bar{y})$ denote the formula

$$\bigwedge_{i=1}^{k} \psi_{ij_i}(\bar{y}).$$

Then $(\mathcal{M}, \bar{b}) \in \pi_{(j_1,\ldots,j_k)}$ if and only if $\mathcal{M} \models \psi_{(j_1,\ldots,j_k)}(\bar{b})$. So the definability clause is also satisfied and so we are done. $\square$

The following lemma provides an asymptotic bound for measuring functions:

Lemma 2.2.3. *Suppose that $\mathcal{C}$ is a weak R-mac in $\mathcal{L}$. Let $\varphi(\bar{x}, \bar{y})$ be an $\mathcal{L}$-formula with $n := l(\bar{x})$ and $m := l(\bar{y})$ and, applying the size clause to φ, let Φ be a partition of $\mathcal{C}(m)$ with measuring functions $\{h_\pi : \pi \in \Phi\} \subseteq R$. Then for every $\pi \in \Phi$ and for every $\delta \in (0, 1)$ there exists $Q_{\pi\delta} \in \mathbb{N}$ such that for every $\mathcal{M} \in \pi^1 := \{\mathcal{M} \in \mathcal{C} : (\mathcal{M}, \bar{a}) \in \pi$ for some $\bar{a} \in M^m\}$ with $|M| > Q_{\pi\delta}$,*

$$h_\pi(\mathcal{M}) \leq \frac{|M|^n}{1 - \delta}.$$

Proof. Consider some arbitrary $\pi \in \Phi$ and let $0 < \delta < 1$. Let $Q_{\pi\delta}$ be such that (2.2) holds for $\varepsilon := \delta$, i.e. for every $(\mathcal{M}, \bar{a}) \in \pi$ with $|M| > Q_{\pi\delta}$ we have

$$\Big||\varphi(\mathcal{M}^n, \bar{a})| - h_\pi(\mathcal{M})\Big| \leq \delta h_\pi(\mathcal{M}).$$

This is equivalent to

$$-\delta h_\pi(\mathcal{M}) \leq |\varphi(\mathcal{M}^n, \bar{a})| - h_\pi(\mathcal{M}) \leq \delta h_\pi(\mathcal{M}),$$

which implies
$$(1-\delta)h_\pi(\mathcal{M}) \leq |\varphi(\mathcal{M}^n, \bar{a})|.$$
Thus, since $|\varphi(\mathcal{M}^n, \bar{a})| \leq |M|^n$,
$$(1-\delta)h_\pi(\mathcal{M}) \leq |M|^n.$$
But $1-\delta > 0$ and so
$$h_\pi(\mathcal{M}) \leq \frac{|M|^n}{1-\delta}$$
as required. □

We now introduce the notion of positive-definiteness for R-macs:

Definition 2.2.4 (in the context of Definition 2.1.2)**.** Let $\pi^{\mathcal{M}} := \{\bar{a} \in \mathcal{M}^m : (\mathcal{M}, \bar{a}) \in \pi\}$. The measuring function h_π is *positive-definite* if
$$\varphi(\mathcal{M}, \bar{a}) = \varnothing \ \text{ for all } \ \bar{a} \in \pi^{\mathcal{M}} \iff h_\pi(\mathcal{M}) = 0 \tag{2.6}$$
for all $\mathcal{M} \in \mathcal{C}$ with $\pi^{\mathcal{M}} \neq \varnothing$.

Measuring functions are eventually positive-definite:

Lemma 2.2.5. *Suppose that $\mathcal{C}$ is a weak R-mac in $\mathcal{L}$. Let $\varphi(\bar{x}, \bar{y})$ be an $\mathcal{L}$-formula with $n := l(\bar{x})$ and $m := l(\bar{y})$ and, applying the size clause to φ, let Φ be a partition of $\mathcal{C}(m)$ with measuring functions $\{h_\pi : \pi \in \Phi\} \subseteq R$. Then for each $\pi \in \Phi$ there exists $Q_\pi \in \mathbb{N}^+$ such that (2.6) holds for all $\mathcal{M} \in \mathcal{C}$ with $|M| > Q_\pi$ and $\pi^{\mathcal{M}} \neq \varnothing$.*

Proof. Consider some arbitrary $\pi \in \Phi$.

(i) We first prove that the left-to-right direction of (2.6) eventually holds. For a contradiction, suppose that it never holds, i.e. that for every $Q \in \mathbb{N}^+$ there exists some $\mathcal{M}_Q \in \mathcal{C}$ with $|M_Q| > Q$ such that the left-to-right part of (2.6) fails to hold for $\mathcal{M}_Q$. So $\varphi(\mathcal{M}_Q, \bar{a}) = \varnothing$ for all $\bar{a} \in \pi^{\mathcal{M}_Q} \neq \varnothing$ but $h_\pi(\mathcal{M}_Q) \neq 0$. Let $\varepsilon = \frac{1}{2}$. Then
$$\begin{aligned}\Big| |\varphi({\mathcal{M}_Q}^n, \bar{a})| - h_\pi(\mathcal{M}_Q) \Big| &= |0 - h_\pi(\mathcal{M}_Q)| \\ &= h_\pi(\mathcal{M}_Q) \\ &> \varepsilon h_\pi(\mathcal{M}_Q).\end{aligned}$$

Since this holds for all $Q \in \mathbb{N}^+$, (2.1) does not hold for π and so $\mathcal{C}$ is not a weak R-mac, a contradiction. So there exists $Q_{\pi 1} \in \mathbb{N}^+$ above which the left-to-right direction of (2.6) holds.

(ii) We now prove that the right-to-left direction of (2.6) eventually holds. For a contradiction, suppose that it never holds, i.e. that for every $Q \in \mathbb{N}^+$ there exists some $\mathcal{M}_Q \in \mathcal{C}$ with $|M_Q| > Q$ such that the right-to-left part of (2.6) fails to hold for $\mathcal{M}_Q$. So $h_\pi(\mathcal{M}_Q) = 0$ but there exists $\bar{a} \in \pi^{\mathcal{M}_Q}$ such that $\varphi(\mathcal{M}, \bar{a}) \neq \varnothing$. Let $\varepsilon = 1$. Then

$$\begin{aligned}\Big||\varphi({\mathcal{M}_Q}^n, \bar{a})| - h_\pi(\mathcal{M}_Q)\Big| &= |\varphi({\mathcal{M}_Q}^n, \bar{a})| \\ &> 0 \\ &= \varepsilon h_\pi(\mathcal{M}_Q).\end{aligned}$$

Since this holds for all $Q \in \mathbb{N}^+$, (2.1) does not hold for π and so $\mathcal{C}$ is not a weak R-mac, a contradiction. So there exists $Q_{\pi 2} \in \mathbb{N}^+$ above which the right-to-left direction of (2.6) holds.

Parts (i) and (ii) together imply the result, since we may take $Q_\pi := \max\{Q_{\pi 1}, Q_{\pi 2}\}$. □

R-macs and R-mecs are closed under adding constant symbols:

Lemma 2.2.6.

(i) Suppose that $\mathcal{C}$ is a weak R-mac (resp. -mec) in $\mathcal{L}$. Let $\mathcal{L}'$ be an extension of $\mathcal{L}$ by constant symbols and for $\mathcal{M} \in \mathcal{C}$ let $\mathcal{M}'$ be the $\mathcal{L}'$-expansion of $\mathcal{M}$. Then $\mathcal{C}' := \{\mathcal{M}' : \mathcal{M} \in \mathcal{C}\}$ is a weak R-mac (resp. -mec) in $\mathcal{L}'$.

(ii) Suppose that $\mathcal{C}$ is a full R-mac (resp. -mec) in $\mathcal{L}$. Let $\mathcal{L}'$ be an extension of $\mathcal{L}$ by constant symbols and for $\mathcal{M} \in \mathcal{C}$ let $\mathcal{M}'$ be the $\mathcal{L}'$-expansion of $\mathcal{M}$. Then $\mathcal{C}' := \{\mathcal{M}' : \mathcal{M} \in \mathcal{C}\}$ is a full R-mac (resp. -mec) in $\mathcal{L}'$.

Proof. We simultaneously prove parts (i) and (ii) for R-macs, the proof for R-mecs being all but identical.

Consider an $\mathcal{L}'$-formula $\varphi(\bar{x}, \bar{y})$, where $n := l(\bar{x})$ and $m := l(\bar{y})$. We can write the constant symbols from $\mathcal{L}' \setminus \mathcal{L}$ that occur in $\varphi(\bar{x}, \bar{y})$ as the tuple $\bar{c}$, where $k := l(\bar{c})$. We can thus write φ as $\varphi(\bar{x}, \bar{y}, \bar{c})$, where $\varphi(\bar{x}, \bar{y}, \bar{z})$ is an $\mathcal{L}$-formula. Since $\mathcal{C}$ is an R-mac in $\mathcal{L}$, we have a finite partition Φ of $\mathcal{C}(m+k) = \{(\mathcal{M}, \bar{a}, \bar{b}) : \mathcal{M} \in \mathcal{C}, \bar{a} \in M^m, \bar{b} \in M^k\}$ with measuring functions $\{h_\pi : \pi \in \Phi\} \subseteq R$ and defining $\mathcal{L}$-formulas $\{\psi_\pi(\bar{y}, \bar{z}) : \pi \in \Phi\}$. Define

$$\pi' := \{(\mathcal{M}', \bar{a}) : (\mathcal{M}, \bar{a}, \bar{c}^{\mathcal{M}'}) \in \pi\},$$

where $\bar{c}^{\mathcal{M}'} \in M^k$ denotes the assignment in $\mathcal{M}'$ of the tuple $\bar{c}$ of constant symbols. Then $\Phi' := \{\pi' : \pi \in \Phi\}$ is a finite partition of $\mathcal{C}'(m)$ with measuring functions $\{h_\pi : \pi' \in \Phi'\}$, where $h_\pi(\mathcal{M}') := h_\pi(\mathcal{M})$, since for each $\pi' \in \Phi'$ we have

$$\begin{aligned}\Big||\varphi((\mathcal{M}')^n, \bar{a})| - h_\pi(\mathcal{M}')\Big| &= \Big||\varphi(\mathcal{M}^n, \bar{a}, \bar{c}^{\mathcal{M}'})| - h_\pi(\mathcal{M})\Big| \\ &= o(h_\pi(\mathcal{M})) \\ &= o(h_\pi(\mathcal{M}'))\end{aligned}$$

for all $(\mathcal{M}', \bar{a}) \in \pi'$ as $|M| \to \infty$. So part (i) is proved. Moreover, each π' is defined by the parameter-free $\mathcal{L}'$-formula $\psi_\pi(\bar{y}, \bar{c})$. So part (ii) is proved. □

The following lemma shows that to prove that a class $\mathcal{C}$ is an R-mec in $\mathcal{L}$, it suffices to show that the definition eventually holds for each $\mathcal{L}$-formula:

Lemma 2.2.7. *Suppose that the definition of a multidimensional exact class (Definition 2.1.4) holds for $\varphi(\bar{x}, \bar{y})$, R and the subclass*

$$\mathcal{C}(m)_{>Q} := \{(\mathcal{M}, \bar{a}) : (\mathcal{M}, \bar{a}) \in \mathcal{C}(m) \text{ and } |M| > Q\}$$

of $\mathcal{C}(m)$, where $m := l(\bar{y})$, Q is some positive integer, and R contains the constant function $\mathcal{M} \mapsto k$ for each positive integer $k \leq Q$. Then the definition also holds for $\varphi(\bar{x}, \bar{y})$, R and $\mathcal{C}(m)$.

Proof. By the hypothesis of the lemma there exists a finite partition Φ of $\mathcal{C}(m)_{>Q}$ with measuring functions $\{h_\pi : \pi \in \Phi\}$ and defining $\mathcal{L}$-formulas $\{\psi_\pi(\bar{y}) : \pi \in \Phi\}$. Let

$$\Gamma_i := \{(\mathcal{M}, \bar{a}) : \mathcal{M} \in \mathcal{C}(m) \setminus \mathcal{C}(m)_{>Q} \text{ and } |\varphi(\mathcal{M}^n, \bar{a})| = i\}.$$

Then $\{\Gamma_i : 0 \leq i \leq Q\} \cup \Phi$ is a finite partition of $\mathcal{C}$ with measuring functions $\{g_i : 0 \leq i \leq Q\} \cup \{h_\pi : \pi \in \Phi\}$, where $g_i(\mathcal{M}) := i$ for all $\mathcal{M} \in \mathcal{C}$. So the size clause holds for $\mathcal{C}$.

Let σ_Q be the $\mathcal{L}$-sentence $\exists x_1 \ldots \exists x_Q \forall y \bigvee_{1 \leq i \leq Q} y = x_i$, i.e. σ_Q says that there are at most Q elements, and let $\varphi_i(\bar{y})$ be the $\mathcal{L}$-formula $\exists!_i \bar{x}\, \varphi(\bar{x}, \bar{y})$, i.e. $\varphi_i(\bar{a})$ says that $|\varphi(\mathcal{M}^n, \bar{a})| = i$. Then the partition in the previous paragraph is defined by the $\mathcal{L}$-formulas $\{\varphi_i(\bar{y}) \wedge \sigma_Q : 1 \leq i \leq Q\} \cup \{\psi_\pi(\bar{y}) \wedge \neg\sigma_Q : \pi \in \Phi\}$. □

Corollary 2.2.8. *Let $\mathcal{C}$ be a finite class of finite $\mathcal{L}$-structures, where the largest structure in $\mathcal{C}$ has size Q. If R is the set of functions $\{\mathcal{M} \mapsto k : k \in \mathbb{N}, k \leq Q\}$, then $\mathcal{C}$ is an R-mec in $\mathcal{L}$.*

Proof. Use the proof of Lemma 2.2.7. □

Remark 2.2.9. If we replace ‘multidimensional exact class’ with ‘multidimensional asymptotic class’ in the statement of Lemma 2.2.7, then the proof becomes easier: The size clause is handled by Remark 2.1.3(iv) and σ_Q takes care of the definability clause.

Our last useful lemma is a compactness-like result:

Lemma 2.2.10. *Let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures. For $\mathcal{L}' \subseteq \mathcal{L}$ let $\mathcal{C}_{\mathcal{L}'}$ denote the class of all $\mathcal{L}'$-reducts of structures in $\mathcal{C}$. If $\mathcal{C}_{\mathcal{L}'}$ is an R-mac (resp. -mec) in $\mathcal{L}'$ for every finite $\mathcal{L}' \subseteq \mathcal{L}$, then $\mathcal{C}$ is an R-mac (resp. -mec) in $\mathcal{L}$.*

Proof. This follows from Definition 2.1.2 (resp. Definition 2.1.4), whose first (second-order) quantifier ranges over $\mathcal{L}$-formulas, and the following two facts: Firstly, $\mathcal{L}$-formulas are finite and so any $\mathcal{L}$-formula is an $\mathcal{L}'$-formula for some finite $\mathcal{L}' \subseteq \mathcal{L}$. Secondly, for every $\mathcal{L}'$-formula $\chi(\bar{y})$ (where $m := l(\bar{y})$, for every $\mathcal{L}'$-reduct $\mathcal{M}'$ of an $\mathcal{L}$-structure $\mathcal{M}$ and for every $\bar{a} \in M^m$, $\mathcal{M}' \models \chi(\bar{a})$ if and only if $\mathcal{M} \models \chi(\bar{a})$. □

2.3 Examples and non-examples

We now go through a number of examples and non-examples of R-macs and R-mecs. We start with a generic example that incorporates the previous framework:

Example 2.3.1. Any N-dimensional asymptotic class (Definition 1.1.2) is an R-mac, where the functions in R are of the form $\mathcal{M} \mapsto \mu|M|^{d/N}$ for some dimension–measure pair (d, μ).

A number of new examples have been found that do not fit into the previous framework of N-dimensional asymptotic classes:

Example 2.3.2 (Theorem 4.3.2 in [25]). Let $\mathcal{C}$ be the class of all finite vector spaces, where both the base finite field and the dimension vary freely. We consider these vector spaces as two-sorted structures (V, K), with a sort V in the language of groups with an abelian group structure, a sort K in the language of rings with a field structure, and a function $K \times V \to V$ for scalar multiplication. We call V the *vector sort* and K the *field sort*. (See pp. 5 and 12 of [58] for a summary of multi-sorted structures and languages.) Let $\mathbb{Q}[\mathbf{V}, \mathbf{K}]$ denote the set of polynomials with rational coefficients and $\mathbf{V}$ and $\mathbf{K}$ as indeterminants. Then $\mathcal{C}$ is a $\mathbb{Q}[\mathbf{V}, \mathbf{K}]$-mac, where we define $p((V, K)) := p(|V|, |K|)$ for $p \in \mathbb{Q}[\mathbf{V}, \mathbf{K}]$ and $(V, K) \in \mathcal{C}$.

Remark 2.3.3. Building on the work of Granger in [26], in [2] it is shown that expanding Example 2.3.2 by adding an orthogonal or symplectic bilinear form $V \to K$ still yields a multidimensional asymptotic class (see § 19 of [3] for the definitions of these terms). Note that the work in [1] is also relevant.

In order to prove the next example we'll need to define the notion of a disjoint union of classes and then prove a lemma:

Definition 2.3.4. Consider $\mathcal{C}_1, \dots, \mathcal{C}_k$, where each $\mathcal{C}_i$ is a class of $\mathcal{L}_i$-structures. Define the *disjoint union* of $\mathcal{C}_1, \dots, \mathcal{C}_k$ to be

$$\mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_k := \{\mathcal{M}_1 \sqcup \cdots \sqcup \mathcal{M}_k : \mathcal{M}_i \in \mathcal{C}_i\},$$

where we define a first-order structure on $\mathcal{M}_1 \sqcup \cdots \sqcup \mathcal{M}_k$ as follows: The domain is $M_1 \cup \cdots \cup M_k$, which we make formally disjoint if necessary. The language is $\mathcal{L}_1 \sqcup \cdots \sqcup \mathcal{L}_k$, which has a sort S_i for each M_i and contains all $\mathcal{L}_i$-symbols for every $i \in \{1, \dots, k\}$, with each $\mathcal{L}_i$-symbol being restricted to the sort S_i.

Lemma 2.3.5. *Let $\mathcal{C}_i$ be an R_i-mac (resp. -mec) in $\mathcal{L}_i$. Then $\mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_k$ is an R-mac (resp. -mec) in $\mathcal{L} := \mathcal{L}_1 \sqcup \cdots \sqcup \mathcal{L}_k$, where R is the set generated by $R_1 \cup \cdots \cup R_k$ under addition and multiplication.*

Sketch proof. We restrict our attention to the case $k = 2$, the general case following by induction.

Consider an $\mathcal{L}$-formula $\varphi(\bar{x}_1, \bar{x}_2; \bar{y}_1, \bar{y}_2)$, where $\bar{x}_i$ and $\bar{y}_i$ are of sort S_i. By an induction on the complexity of the formula, one can show that $\varphi(\bar{x}_1, \bar{x}_2; \bar{y}_1, \bar{y}_2)$ is equivalent to a finite disjunction of $\mathcal{L}$-formulas of the form $\chi(\bar{x}_1, \bar{y}_1) \wedge \theta(\bar{x}_2, \bar{y}_2)$, where χ is an $\mathcal{L}_1$-formula, θ is an $\mathcal{L}_2$-formula, and the disjuncts are pairwise inconsistent. Since the domains of $\mathcal{M}_1 \in \mathcal{C}_1$ and $\mathcal{M}_2 \in \mathcal{C}_2$ are disjoint, we have

$$|\chi(\mathcal{M}_1 \sqcup \mathcal{M}_2, \bar{a}_1) \wedge \theta(\mathcal{M}_1 \sqcup \mathcal{M}_2, \bar{a}_2)| = |\chi(\mathcal{M}_1, \bar{a}_1)| \cdot |\theta(\mathcal{M}_2, \bar{a}_2)|.$$

One then proceeds by using the facts that the disjuncts are pairwise inconsistent, thus allowing summation, and that each $\mathcal{C}_i$ is an R-mac/-mec. ☐

Example 2.3.6. Consider the class $\mathcal{C}$ of finite cyclic groups and for arbitrary $k \in \mathbb{N}^+$ define $\mathcal{C}_k := \{C_1 \oplus \cdots \oplus C_k : C_i \in \mathcal{C}\}$. Let $\mathcal{L}$ be the language of groups (with or without a constant symbol for the identity element – recall

Lemma 2.2.6). Then $\mathcal{C}_k$ is a multidimensional exact class in $\mathcal{L}'$, where $\mathcal{L}'$ is $\mathcal{L}$ adjoined with a unary predicate P_i for each part of the direct sum:

$$P_i^{C_1 \oplus \cdots \oplus C_k} := \{(0, \ldots, 0, \underset{\substack{\uparrow \\ i^{\text{th}} \text{ place}}}{a}, 0, \ldots, 0) : a \in C_i\}.$$

Proof. Theorem 3.14 in [50] states that $\mathcal{C}$ is a 1-dimensional asymptotic class in $\mathcal{L}$. Inspection of the proof of this theorem shows that $\mathcal{C}$ is in fact an exact class. So by Lemma 2.3.5, $\underbrace{\mathcal{C} \sqcup \cdots \sqcup \mathcal{C}}_{k \text{ times}}$ is an exact class in $\underbrace{\mathcal{L} \sqcup \cdots \sqcup \mathcal{L}}_{k \text{ times}}$. Since $\mathcal{L}'$ is equipped with the predicates P_i, $\mathcal{C}_k$ and $\mathcal{C} \sqcup \cdots \sqcup \mathcal{C}$ are $\varnothing$-bi-interpretable (Definition 2.4.2). Therefore $\mathcal{C}_k$ is an exact class by Proposition 2.4.6(ii). □

Remark 2.3.7. We comment on Example 2.3.6.

(i) I conjecture that the P_i are necessary for the definability clause; that is, I conjecture that $\mathcal{C}_k$ is a strictly weak exact class in $\mathcal{L}$.

(ii) The disjoint-union technique can be used with other direct sums, e.g. Question 5.6.

(iii) The class $\mathcal{C}$ of finite cyclic groups is a multidimensional exact class, but it is not a 1-dimensional exact class. Indeed, there do not exist N-dimensional exact classes, for the following reason: Consider two disjoint definable sets A and B with $|A| = \alpha|M|^{a/N}$ and $|B| = \beta|M|^{b/N}$, where $a > b$. Then their union $A \cup B$, which is definable, has size $\alpha|M|^{a/N} + \beta|M|^{b/N}$, which cannot be expressed in the form $\mu|M|^{d/N}$ for a dimension–measure pair (d, μ). This is not an issue for an N-dimensional asymptotic class, since $|M|^{a/N}$ swamps $|M|^{b/N}$ as $|M| \to \infty$. It is also not an issue for a multidimensional exact class, where one is not bound to dimension–measure pairs.

Example 2.3.8 (Proposition 4.4.2 in [25])**.** Consider the class of homocyclic groups

$$\mathcal{C} := \{(\mathbb{Z}/p^n\mathbb{Z})^m : p \text{ is prime and } n, m \in \mathbb{N}^+\}$$

in the language $\mathcal{L} := \{+\}$. This class is an R-mec, where R consists of functions of the form

$$\sum_{i=0}^{r} \sum_{j=-rd}^{rd} c_{ij} p^{m(in+j)},$$

where r is the length of the object-variable tuple of the given $\mathcal{L}$-formula (see Remark 2.1.3(v)); d is a positive integer that is constructively determined by the $\mathcal{L}$-formula; and the c_{ij} are integers that depend on the $\mathcal{L}$-formula, with $c_{ij} := 0$ whenever $in + j < 0$. (Each group $(\mathbb{Z}/p^n\mathbb{Z})^m \in \mathcal{C}$ is determined by a triple (p, n, m), so by defining a function on such triples we also define a function on $\mathcal{C}$.)

Further examples will arise as the thesis progresses. We now turn our attention to non-examples, which are often just as interesting.

Non-Example 2.3.9 (Example 3.1 in [50]). The class $\mathcal{C}$ of all finite linear orders in (any extension of) the language $\mathcal{L} = \{<\}$ does not form a weak R-mac for any R.

Proof. Let R be any set of functions from $\mathcal{C}$ to $\mathbb{R}^{\geq 0}$ and let $\varphi(x, y)$ be the formula $x < y$. Consider the finite linear order $\mathcal{M}_k := \{a_0 < \cdots < a_k\}$. Then $|\varphi(\mathcal{M}_k, a_i)| = i$. So as we let k increase and let i vary we define arbitrarily many subsets of distinct sizes. Thus no finite number of functions from R can approximate $|\varphi(\mathcal{M}_k, a_i)|$ for all $k, i \in \mathbb{N}$. Let's make that rigorous.

By way of contradiction, suppose that there exists R such that $\mathcal{C}$ forms a weak R-mac. So for the formula $\varphi(x, y)$ there exists a finite partition Φ of $\mathcal{C}(1)$ with measuring functions $\{h_\pi : \pi \in \Phi\} \subseteq R$. Let $t := |\Phi|$. Consider the sequence $(\mathcal{M}_{dt})_{d \in \mathbb{N}^+}$ of structures from $\mathcal{C}$. We have $|M_{dt}| = |\{a_0, \ldots, a_{dt}\}| = dt + 1$ and thus, since $|\Phi| = t$, there exists $\pi \in \Phi$ such that $|\pi^{\mathcal{M}_{dt}}| \geq d + 1$, where $\pi^{\mathcal{M}_{dt}} := \{a \in M_{dt} : (\mathcal{M}_{dt}, a) \in \pi\}$. Therefore there exist $a_{i(d)}, a_{j(d)} \in \pi^{\mathcal{M}_{dt}}$ such that $|i(d) - j(d)| \geq d$. Now consider the sequence $(\mathcal{M}_{dt}, a_{i(d)})_{d \in \mathbb{N}^+}$. Since Φ is finite, an infinite subsequence S is contained within a single $\pi_0 \in \Phi$.

Set $\varepsilon := \frac{1}{6t}$. Let Q_1 be such that (2.2) holds for π_0, h_{π_0} and ε and, by taking $\delta := \frac{1}{2}$ in Lemma 2.2.3, let Q_2 be such that $h_{\pi_0}(\mathcal{M}) \leq 2|M|$ for all $\mathcal{M} \in \pi_0^1 := \{\mathcal{M} \in \mathcal{C} : (\mathcal{M}, a) \in \pi_0 \text{ for some } a \in M\}$ with $|M| > Q_2$. Define $Q := \max\{Q_1, Q_2, 3\}$. Consider some $(\mathcal{M}_{dt}, a_{i(d)}) \in S$ with $|M_{dt}| > Q$. Then

$$\begin{aligned}
\frac{|M_{dt}|-1}{t} &= d \\
&\leq |i(d)-j(d)| \\
&= \big||\varphi(\mathcal{M}_{dt},a_{i(d)})|-|\varphi(\mathcal{M}_{dt},a_{j(d)})|\big| \qquad \text{(since } |\varphi(\mathcal{M}_k,a_i)|=i\text{)} \\
&= \big||\varphi(\mathcal{M}_{dt},a_{i(d)})|-h_{\pi_0}(\mathcal{M}_{dt}) \\
&\qquad\quad + h_{\pi_0}(\mathcal{M}_{dt})-|\varphi(\mathcal{M}_{dt},a_{j(d)})|\big| \\
&\leq \big||\varphi(\mathcal{M}_{dt},a_{i(d)})|-h_{\pi_0}(\mathcal{M}_{dt})\big| \\
&\qquad\quad + \big|h_{\pi_0}(\mathcal{M}_{dt})-|\varphi(\mathcal{M}_{dt},a_{j(d)})|\big| \quad \text{(triangle inequality)} \\
&\leq \frac{1}{6t}h_{\pi_0}(\mathcal{M}_{dt})+\frac{1}{6t}h_{\pi_0}(\mathcal{M}_{dt}) \\
&= \frac{1}{3t}h_{\pi_0}(\mathcal{M}_{dt}),
\end{aligned}$$

where the penultimate step follows because $(\mathcal{M}_{dt},a_{i(d)},a_{j(d)})\in\pi_0$ and $|\mathcal{M}_{dt}|>Q_1$. Multiplying by $3t$ yields

$$3|\mathcal{M}_{dt}|-3\leq h_{\pi_0}(\mathcal{M}_{dt}).$$

Since $|M_{dt}|>3$, $2|M_{dt}|<3|M_{dt}|-3$ and thus

$$2|M_{dt}|<h_{\pi_0}(\mathcal{M}_{dt}).$$

But $|M_{dt}|>Q_2$ and so $h_{\pi_0}(\mathcal{M}_{dt})\leq 2|M_{dt}|$, a contradiction. □

The following non-example is informative, as it shows that the choice of language in Example 2.3.8 is important:

Non-Example 2.3.10. Let p be prime. Then the class $\{\mathbb{Z}/p^n\mathbb{Z} : n\in\mathbb{N}^+\}$ of multiplicative monoids in (any extension of) the language $\mathcal{L}=\{\times\}$ does not form a weak R-mac for any R.

Proof. Let R be any set of functions from $\mathcal{C}$ to $\mathbb{R}^{\geq 0}$ and let $\varphi(x,y)$ be the formula $\exists z\,(x=z\times y)$. Then $|\varphi(\mathbb{Z}/p^n\mathbb{Z},p^i)|=p^{n-i}$. So as we let n increase and let i vary we define arbitrarily many subsets of distinct sizes. Thus, by a very similar argument to that given in the proof of Non-Example 2.3.9, no finite number of functions from R can approximate $|\varphi(\mathbb{Z}/p^n\mathbb{Z},p^i)|$ for all $n,i\in\mathbb{N}$. □

Remark 2.3.11.

(i) Non-Examples 2.3.9 and 2.3.10 are special cases of the general fact that an ultraproduct of a multidimensional asymptotic class cannot have the strict order property [2]. (See Definition 2.14 in [10] or Exercise 8.2.4 in [58] for a definition of the strict order property.)

(ii) The issue preventing Non-Example 2.3.10 from being an R-mac is the unbounded exponent n. If the exponent is bounded, then one can have an R-mac, as shown by the work of Bello Aguirre in [6] and [7], some of which is discussed in Question 5.6.

The following two non-examples are due to appear in [2]. They concern ultraproducts, the random graph and the random tournament,[1] which are covered extensively in the literature, e.g. [5], Exercise 2.5.19 in [53] and Exercise 1.2.4 in [58] (ultraproducts) and p. 232 of [14], p. 17 of [16], §§ 1–2 of [23], p. 435 of [43], pp. 50–52 of [53] and Exercise 3.3.1 in [58] (the random graph and the random tournament).

Non-Example 2.3.12. The random graph is not elementarily equivalent to an ultraproduct of a multidimensional exact class.

Proof. Let $\mathcal{L} := \{E\}$ be the language of graphs. By way of contradiction, suppose that there exists a class $\mathcal{C}$ of finite $\mathcal{L}$-structures with an ultraproduct $\mathcal{U}$ such that $\mathcal{C}$ is an R-mec in $\mathcal{L}$ for some R and $\mathcal{U}$ is elementarily equivalent to the random graph.

By a thinning-out argument (see e.g. Lemma 2.1 in [52]), we may find a subclass of $\mathcal{C}' \subseteq \mathcal{C}$ such that every infinite ultraproduct of $\mathcal{C}'$ is elementarily equivalent to $\mathcal{U}$. We may assume that $\mathcal{C}' = \mathcal{C}$, since $\mathcal{C}'$ is also an R-mec by Remark 2.1.3(vii). Thus for every $\mathcal{L}$-sentence σ, σ is true in the random graph if and only if σ is cofinitely true in $\mathcal{C}$. Thus, since the random graph has quantifier elimination (Exercise 3.3.1 in [58]), $\mathcal{C}$ has eventual quantifier elimination; that is, for every $\mathcal{L}$-formula φ there exist $Q \in \mathbb{N}$ and a quantifier-free $\mathcal{L}$-formula χ such that for all $G \in \mathcal{C}$ with $|G| > Q$, φ and χ are equivalent in G.

A graph is said to be *d-regular* if for any set A of at most d vertices, the number of vertices that are adjacent to every vertex in A depends only on the isomorphism type of the induced subgraph on A.

Let $\varphi(x, y_1, \ldots, y_5) :\equiv \bigwedge_{1 \leq i \leq 5} E(x, y_i)$, i.e. φ says that x is adjacent to each y_i. Since $\mathcal{C}$ is an R-mec, φ gives rise to a finite partition Φ of $\mathcal{C}(5)$ with defining

[1] Due to its different guises, the random graph goes by various names, including the 'Rado graph' and 'the generic (countable homogeneous) graph'. The random tournament has similar aliases.

$\mathcal{L}$-formulas $\{\psi_\pi(y_1, \ldots, y_5) : \pi \in \Phi\}$. By eventual quantifier elimination and Remark 2.1.3(vii) we may assume that each ψ_π is quantifier-free.

We use the ψ_π to show that every $G \in \mathcal{C}$ is 5-regular. So consider some arbitrary $G \in \mathcal{C}$ and let $\bar{y} := (y_1, \ldots, y_5)$. Since $\psi_\pi(\bar{y})$ is quantifier-free, whether or not a tuple $\bar{a} := (a_1, \ldots, a_5) \in G^5$ of vertices satisfies $\psi_\pi(\bar{y})$ is determined entirely by the $\mathcal{L}$-substructure on $\bar{a}$. But $|\varphi(G, \bar{a})|$ is precisely determined by which ψ_π holds for $\bar{a}$, since $\mathcal{C}$ is an R-mec (not just an R-mac). Moreover, $\varphi(G, \bar{a})$ is the same for every ordering of the a_i in $\bar{a}$. Thus $|\varphi(G, \bar{a})|$ depends only on the isomorphism type of the induced subgraph on $\bar{a}$ and so G is indeed 5-regular.

As discussed at the end of [9], a finite 5-regular graph is either (a) the pentagon, (b) the line graph of $K_{3,3}$ or (c) a complete multipartite graph. Recall the characteristic property of the random graph, which is expressible as a first-order axiom schema in $\mathcal{L}$: For any two finite sets A and B of vertices, there exists a vertex adjacent to every vertex in A and to no vertex in B. So for any given $|A|$ and $|B|$, this property is cofinitely true in $\mathcal{C}$. But this cannot be the case, since every graph in $\mathcal{C}$ is of the form (a), (b) or (c), a contradiction. $\square$

Remark 2.3.13. This argument also shows that neither the generic countable K_n-free graph nor its complement is equivalent to an ultraproduct of an exact class. We will discuss the relationship between homogeneous structures and exact classes further in Question 5.3.

Non-Example 2.3.14. The random tournament is not elementarily equivalent to an ultraproduct of a multidimensional exact class.

Proof. Let $\mathcal{L} := \{\dot{}\}$ be the language of tournaments. We define the *out-neighbourhood* of a vertex a in a tournament T to be $\mathrm{Out}(a) := \{b \in T : a \,\dot{}\, b\}$ and the *in-neighbourhood* to be $\mathrm{In}(a) := \{b \in T : b \,\dot{}\, a\}$.

By way of contradiction, suppose that there exists a class $\mathcal{C}$ of finite $\mathcal{L}$-structures with an ultraproduct $\mathcal{U}$ such that $\mathcal{C}$ is an R-mec in $\mathcal{L}$ for some R and $\mathcal{U}$ is elementarily equivalent to the random tournament.

We can use the same argument from the proof of Non-Example 2.3.12 to assume that for every $\mathcal{L}$-sentence σ, σ is true in the random tournament if and only if σ is cofinitely true in $\mathcal{C}$. Thus, since the random tournament has quantifier elimination (see [43]), $\mathcal{C}$ has eventual quantifier elimination.

A tournament T is said to be *d-regular* if $|\mathrm{Out}(a)| = d$ for all $a \in T$. A tournament T is *regular* if T is d-regular for some d. We claim that any finite regular tournament has an odd number of vertices.

Let T be a d-regular tournament. Then $\sum_{a\in T}|\mathrm{Out}(a)| = nd$, where n is the number of vertices in T. We also have $\sum_{a\in T}|\mathrm{Out}(a)| = \frac{1}{2}n(n-1)$, since $\binom{n}{2} = \frac{1}{2}n(n-1)$ is the number of edges in the complete graph on n vertices. (Recall that a tournament on n vertices is obtained from the complete graph on n vertices by assigning a direction to each edge.) So $nd = \frac{1}{2}n(n-1)$, which implies $n = 2d+1$. So n is odd, as claimed. Notice that we thus also have $|\mathrm{In}(a)| = d$ for all $a \in T$, since $|\mathrm{In}(a)| = (n-1) - |\mathrm{Out}(a)| = (2d+1-1) - d = d$.

Now consider the $\mathcal{L}$-formula $\varphi(x,y) :\equiv y \,\dot{}\, x$. So for a tournament T, $\varphi(T,a) = \mathrm{Out}(a)$. Since $\mathcal{C}$ is an R-mec, $\varphi(x,y)$ gives rise to a finite partition Φ of $\mathcal{C}(1)$ with defining $\mathcal{L}$-formulas $\{\psi_\pi(y) : \pi \in \Phi\}$. By eventual quantifier elimination and Remark 2.1.3(vii) we may assume that the ψ_π are quantifier-free.

We use the ψ_π to show that every $T \in \mathcal{C}$ is regular. So consider some arbitrary $T \in \mathcal{C}$. Since $\psi_\pi(y)$ is quantifier-free, whether or not a vertex $a \in T$ satisfies $\psi_\pi(y)$ is determined entirely by the subtournament substructure on a. But a is a single vertex and so there is only one tournament structure on a, namely the empty structure. So we must have $|\Phi| = 1$. Since $\mathcal{C}$ is an R-mec, $|\varphi(T,a)|$ is precisely determined by which ψ_π holds for a and thus, since there is only one ψ_π, every out-neighbourhood has the same size. So T is indeed regular.

For a pair $a_1 \,\dot{}\, a_2 \in T \in \mathcal{C}$ we define the following subtournaments of T:

- $X_1(a_1,a_2) := \mathrm{Out}(a_1) \cap \mathrm{Out}(a_2)$
- $X_2(a_1,a_2) := \mathrm{Out}(a_1) \cap \mathrm{In}(a_2)$
- $X_3(a_1,a_2) := \mathrm{In}(a_1) \cap \mathrm{Out}(a_2)$
- $X_4(a_1,a_2) := \mathrm{In}(a_1) \cap \mathrm{In}(a_2)$

Recall the characteristic property of the random tournament, which is expressible as a first-order axiom schema in $\mathcal{L}$: For any finite sets A and B of vertices there exists a vertex c such that $c \in \mathrm{Out}(a)$ for every $a \in A$ and $c \in \mathrm{In}(b)$ for every $b \in B$. Thus, taking suitable A and B, we see that there exists a cofinite subclass $\mathcal{C}' \subseteq \mathcal{C}$ such that each $X_i(a_1,a_2)$ is non-empty for every $a_1 \,\dot{}\, a_2 \in T \in \mathcal{C}'$. By Remark 2.1.3(vii) we may assume that $\mathcal{C}' = \mathcal{C}$. We claim that each $X_i(a_1,a_2)$ is a regular tournament.

We prove the claim for X_2, the proofs for the other X_i being all but the same. Consider the $\mathcal{L}$-formula

$$\chi(x; z, y_1, y_2) :\equiv y_1 \,\dot{}\, z \wedge z \,\dot{}\, y_2 \wedge y_1 \,\dot{}\, x \wedge x \,\dot{}\, y_2 \wedge z \,\dot{}\, x.$$

Since $\mathcal{C}$ is an R-mec, we have a partition Γ of $\mathcal{C}(3)$ with defining $\mathcal{L}$-formulas $\{\psi_\gamma(z, y_1, y_2) : \gamma \in \Gamma\}$. By eventual quantifier elimination and Remark 2.1.3(vii) we may assume that the ψ_γ are quantifier-free. Thus $|\chi(T, b, a_1, a_2)|$ depends only on the subtournament substructure on (b, a_1, a_2). But if we fix a_1 and a_2 and let $b \in X_2(a_1, a_2)$ vary, then the subtournament structure on (b, a_1, a_2) remains constant. So $|\mathrm{Out}(b)| = |\chi(T, b, a_1, a_2)|$ is constant for $b \in X_2(a_1, a_2)$, i.e. $X_2(a_1, a_2)$ is regular, as claimed.

For any $a_1 \,^{\cdot}\, a_2 \in T \in \mathcal{C}$ we have

$$T = X_1(a_1, a_2) \cup X_2(a_1, a_2) \cup X_3(a_1, a_2) \cup X_4(a_1, a_2) \cup \{a_1, a_2\},$$

where the union is disjoint. So the number of vertices in T is equal to

$$|X_1(a_1, a_2)| + |X_2(a_1, a_2)| + |X_3(a_1, a_2)| + |X_4(a_1, a_2)| + 2. \tag{2.7}$$

But each $X_i(a_1, a_2)$ is a regular tournament and thus has odd size, which by (2.7) implies that T has even size, a contradiction because T is also a regular tournament. $\square$

Remark 2.3.15. The situation is quite different for asymptotic classes: The random graph is elementarily equivalent to any infinite ultraproduct of the class of Paley graphs, which is a 1-dimensional asymptotic class (Example 3.4 in [50]), and the random tournament is elementarily equivalent to any infinite ultraproduct of the class of Paley tournaments, which is also a 1-dimensional asymptotic class (Example 3.5 in [50]). This is an interesting phenomenon, especially in light of Theorem 7.5.6 in [18] and Theorem 4.6.4. We will discuss it further in Question 5.3.

2.4 Interpretability

In this section we show that R-macs and R-mecs are closed under bi-interpretability (Proposition 2.4.6), adapting work by Elwes in [21] on interpretations in N-dimensional asymptotic classes.

Definition 2.4.1 (Interpretation of a structure)**.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $\mathcal{N}$ an $\mathcal{L}'$-structure. Let $A \subseteq M$. The structure $\mathcal{N}$ is *A-interpretable* in $\mathcal{M}$ if there exist

(i) $r \in \mathbb{N}^+$;

(ii) an $\mathcal{L}_A$-definable set $X \subseteq M^r$;

(iii) an $\mathcal{L}_A$-definable equivalence relation E on X;

(iv) a function $f\colon N \to X/E$; and

(v) an $\mathcal{L}'$-structure on X/E, i.e. for each $\mathcal{L}'$-symbol P there exists an $\mathcal{L}_A$-definable, E-invariant subset of X^k (see below), where k is equal to the arity of P,

such that f is an $\mathcal{L}'$-isomorphism. We write $\mathcal{N}^*$ for the induced $\mathcal{L}'$-structure on X/E. We say that $\mathcal{N}$ is *parameter-interpretable* in $\mathcal{M}$ if $\mathcal{N}$ is A-interpretable in $\mathcal{M}$ for some $A \subseteq M$.

Some further explanation of point (v) is in order. Consider an n-ary relation symbol $P(y_1, \ldots, y_n)$ in $\mathcal{L}'$. Point (v) says that there exists an $\mathcal{L}_A$-formula $\hat{P}(\bar{x}_1, \ldots, \bar{x}_n)$, where $l(\bar{x}_i) = r$ for each $i \in \{1, \ldots, n\}$, that defines an E-invariant subset $\hat{P}((\mathcal{M}^r)^n) \subseteq X^n$, where E-invariance means the following: If $E(\bar{a}_1, \bar{b}_1), \ldots, E(\bar{a}_n, \bar{b}_n)$, then $\mathcal{M} \models \hat{P}(\bar{a}_1, \ldots, \bar{a}_n)$ if and only if $\mathcal{M} \models \hat{P}(\bar{b}_1, \ldots, \bar{b}_n)$. Put another way: $\hat{P}(X^n)$ is a union of E-equivalence classes.[2] We may thus define the assignment of P in X/E to be $\hat{P}(X^n)/E$. So, making the necessary minor adjustments for function and constant symbols, we see that point (v) gives rise to an assignment in X/E for every symbol in $\mathcal{L}'$, which is the definition of an $\mathcal{L}'$-structure on X/E. Some terminology: We call $\hat{P}(\bar{x}_1, \ldots, \bar{x}_n)$ the *$\mathcal{L}$-translation* of P.

Now suppose that $\mathcal{N}$ is parameter-interpretable in $\mathcal{M}$ via an interpretation $f\colon \mathcal{N} \to \mathcal{N}^*$ and that $\mathcal{M}$ is parameter-interpretable in $\mathcal{N}$ via an interpretation $g\colon \mathcal{M} \to \mathcal{M}^*$; in such a situation we say that $\mathcal{M}$ and $\mathcal{N}$ are *mutually parameter-interpretable*. Then f induces an $\mathcal{L}$-isomorphism $f^*\colon \mathcal{M}^* \to \mathcal{M}^{**}$ for an $\mathcal{L}$-structure $\mathcal{M}^{**}$ interpreted in $\mathcal{N}^*$ and hence also in $\mathcal{M}$. Similarly, g induces an $\mathcal{L}'$-isomorphism $g^*\colon \mathcal{N}^* \to \mathcal{N}^{**}$ for an $\mathcal{L}'$-structure $\mathcal{N}^{**}$ interpreted in $\mathcal{M}^*$ and hence also in $\mathcal{N}$. We say that $\mathcal{M}$ and $\mathcal{N}$ are *parameter-bi-interpretable* if there exist interpretations f and g such that the isomorphisms $f^*g\colon \mathcal{M} \to \mathcal{M}^{**}$ and $g^*f\colon \mathcal{N} \to \mathcal{N}^{**}$ are parameter-definable[3] in $\mathcal{M}$ and $\mathcal{N}$ respectively.

We say that $\mathcal{M}$ and $\mathcal{N}$ are *$\varnothing_{\mathcal{M}}$-bi-interpretable* if they are parameter-bi-interpretable in such a way that no parameters from M are needed, i.e. we further require that $\mathcal{N}$ be $\varnothing$-interpretable in $\mathcal{M}$ (via f) and that the isomorphism f^*g be $\varnothing$-definable in $\mathcal{M}$.

[2] The equivalence relation E on X naturally gives rise to an equivalence relation on any cartesian power X^s of X, namely $E((\bar{a}_1, \ldots, \bar{a}_s), (\bar{b}_1, \ldots, \bar{b}_s)) :\Longleftrightarrow \bigwedge_{i=1}^{s} E(\bar{a}_i, \bar{b}_i)$.

[3] 'Parameter-definable' means that parameters can be used, not that they must be: Any $\varnothing$-definable set is parameter-definable.

We say that $\mathcal{M}$ and $\mathcal{N}$ are *$\varnothing$-bi-interpretable* if they are parameter-bi-interpretable in such a way that neither parameters from M nor parameters from N are needed, i.e. we require that $\mathcal{N}$ be $\varnothing$-interpretable in $\mathcal{M}$ (via f), that $\mathcal{M}$ be $\varnothing$-interpretable in $\mathcal{N}$ (via g) and that the isomorphisms f^*g and g^*f be $\varnothing$-definable in $\mathcal{M}$ and $\mathcal{N}$ respectively.

Definition 2.4.2 (Interpretation of a class of structures)**.** Let $\mathcal{C}$ and $\mathcal{C}'$ be classes of $\mathcal{L}$- and $\mathcal{L}'$-structures respectively. We say that $\mathcal{C}'$ is *parameter-interpretable* in $\mathcal{C}$ if there exists an injection $\alpha\colon \mathcal{C}' \to \mathcal{C}$ such that each $\mathcal{N} \in \mathcal{C}'$ is parameter-interpretable in $\alpha(\mathcal{N})$ in such a way that $\mathcal{N}^*$ is uniformly parameter-defined across $\mathcal{C}$ (see below for an explanation). We say that $\mathcal{C}$ and $\mathcal{C}'$ are *parameter-bi-interpretable* if there exists a bijection $\alpha\colon \mathcal{C}' \to \mathcal{C}$ such that for each $\mathcal{N} \in \mathcal{C}'$, $\mathcal{N}$ and $\alpha(\mathcal{N})$ are parameter-bi-interpretable in such a way that $\mathcal{N}^*$ and f^*g are uniformly parameter-defined across $\mathcal{C}$ and that $\alpha(\mathcal{N})^*$ and g^*f are uniformly parameter-defined across $\mathcal{C}'$.

We say that $\mathcal{C}$ and $\mathcal{C}'$ are *$\varnothing_{\mathcal{C}}$-bi-interpretable* if they are parameter-bi-interpretable in such a way that no parameters from structures in $\mathcal{C}$ are needed.

We say that $\mathcal{C}$ and $\mathcal{C}'$ are *$\varnothing$-bi-interpretable* if they are parameter-bi-interpretable in such a way that neither parameters from structures in $\mathcal{C}$ nor parameters from structures in $\mathcal{C}'$ are needed.

By the expression 'the $\mathcal{N}^*$ are uniformly parameter-defined across $\mathcal{C}$' we mean that the $\mathcal{L}$-formulas that define $\mathcal{N}^*$ – i.e. that define the subset $X = X_{\mathcal{N}} \subseteq \alpha(\mathcal{N})^r$, the equivalence relation E on $X = X_{\mathcal{N}}$, and the $\mathcal{L}$-translations of the $\mathcal{L}'$-symbols in $X_{\mathcal{N}}/E$, as laid out in Definition 2.4.1 – are the same for each $\mathcal{N} \in \mathcal{C}'$. So, for example, the $\mathcal{L}$-formula that defines each subset $X_{\mathcal{N}}$ is of the form $\chi(\bar{x}, \bar{y})$, where $r = l(\bar{x})$, $q := l(\bar{y})$ and for every $\mathcal{N} \in \mathcal{C}'$ there exists $\bar{c} \in \alpha(\mathcal{N})^q$ such that $X_{\mathcal{N}} = \chi(\alpha(\mathcal{N})^r, \bar{c})$. We call $\bar{c}$ the *translation parameter(s).* If no translation parameters are needed, then $q = 0$.

Elwes proved a result regarding interpretability in N-dimensional asymptotic classes:

Proposition 2.4.3 (Lemma 3.7 in [21])**.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be classes of $\mathcal{L}$- and $\mathcal{L}'$-structures respectively.*

(i) If $\mathcal{C}'$ is parameter-interpretable in $\mathcal{C}$ and $\mathcal{C}$ is a full N-dimensional asymptotic class in $\mathcal{L}$, then $\mathcal{C}'$ is a weak N'-dimensional asymptotic class in $\mathcal{L}'$ for some N'.

(ii) If $\mathcal{C}$ and $\mathcal{C}'$ are $\varnothing$-bi-interpretable and $\mathcal{C}$ is a full N-dimensional asymptotic class in $\mathcal{L}$, then $\mathcal{C}'$ is a full N'-dimensional asymptotic class in $\mathcal{L}'$ for some N'.

Remark 2.4.4. Elwes actually stated a slightly stronger version of Proposition 2.4.3(ii), assuming only $\varnothing_{\mathcal{C}'}$-bi-interpretability. However, it would seem that both his proof and that of Proposition 2.4.6(ii) require the stronger hypothesis of $\varnothing$-bi-interpretability. We will explain this point later on when it arises in our proof.

We state and prove a version of Proposition 2.4.3 for R-macs and R-mecs:

Definition 2.4.5. Let R be a set of functions from a class $\mathcal{C}$ to $\mathbb{R}^{\geq 0}$. We define

$$\mathrm{Frac}(R) := \left\{ \sum_{i=1}^{n} \frac{g_i}{h_i} : g_i, h_i \in R, n \in \mathbb{N}^+ \text{ and } h_i(\mathcal{M}) \neq 0 \text{ for all } \mathcal{M} \in \mathcal{C} \right\},$$

where we define

$$\left(\sum_{i=1}^{n} \frac{g_i}{h_i} \right)(\mathcal{M}) := \sum_{i=1}^{n} \frac{g_i(\mathcal{M})}{h_i(\mathcal{M})}$$

for all $\mathcal{M} \in \mathcal{C}$.

Proposition 2.4.6. *Let $\mathcal{C}$ and $\mathcal{C}'$ be classes of $\mathcal{L}$- and $\mathcal{L}'$-structures respectively.*

(i) If $\mathcal{C}'$ is parameter-interpretable in $\mathcal{C}$ and $\mathcal{C}$ is a full R-mac (resp. -mec) in $\mathcal{L}$, then $\mathcal{C}'$ is a weak $\mathrm{Frac}(R)$*-mac (resp. -mec) in $\mathcal{L}'$.*

(ii) If $\mathcal{C}$ and $\mathcal{C}'$ are $\varnothing$-bi-interpretable and $\mathcal{C}$ is a full R-mac (resp. -mec) in $\mathcal{L}$, then $\mathcal{C}'$ is a full $\mathrm{Frac}(R)$*-mac (resp. -mec) in $\mathcal{L}'$.*

We prove parts (i) and (ii) separately, although the proof of the latter builds on the proof of the former. We prove the result only for R-macs, but the proof for R-mecs is just a simpler version of the proof for R-macs. Many of the key ideas in the proof are due to Elwes, although our approach is slightly more direct than that in [21]. We use the notation and terminology given in Definitions 2.4.1 and 2.4.2 throughout.

Proof of Proposition 2.4.6(i). Let $\varphi'(x_1, \ldots, x_n; y_1, \ldots, y_m)$ be an arbitrary $\mathcal{L}'$-formula, where the semicolon separates the object and parameter variables (Remark 2.1.3(v)). We need to show that this formula satisfies the size clause.

We translate the $\mathcal{L}'$-formula φ' into an $\mathcal{L}$-formula φ: As discussed in Definition 2.4.1 and Definition 2.4.2, each $\mathcal{L}'$-symbol has a uniform $\mathcal{L}$-translation. So we replace each $\mathcal{L}'$-symbol in φ' with its $\mathcal{L}$-translation. We leave any boolean connectives and adapt any quantifiers in accordance with the new variables; for example, if a variable x in φ' is in the scope of a quantifier $\forall x$ and x becomes $\bar{x}$ in φ, then the quantifier $\forall x$ becomes $\forall \bar{x}$ in φ. The resulting $\mathcal{L}$-formula is $\varphi(\bar{x}_1, \ldots, \bar{x}_n, \bar{y}_1, \ldots, \bar{y}_m)$, where $l(\bar{x}_i) = l(\bar{y}_i) = r$, r being such that $X_{\mathcal{N}} \subseteq \alpha(\mathcal{N})^r$ (see Definition 2.4.2 for definitions of α and $X_{\mathcal{N}}$).

Since the $\mathcal{L}$-translation of each $\mathcal{L}'$-symbol defines an E-invariant subset of $X_{\mathcal{N}}$ (Definition 2.4.1(v)), $\varphi(\alpha(\mathcal{N})^{r\cdot n + r\cdot m}) \subseteq X_{\mathcal{N}}^{n+m}$ is a union of E-equivalence classes (recall footnote 2). So for all $\mathcal{N} \in \mathcal{C}'$ and for all $a_1, \ldots, a_m \in N$ we have

$$f(\varphi'(\mathcal{N}^n, a_1, \ldots, a_m)) = \varphi(\alpha(\mathcal{N})^{r\cdot n}, \tilde{f}(a_1), \ldots, \tilde{f}(a_m))/E, \tag{2.8}$$

where $\tilde{f}$ is a choice function on the set $\{f(a) : a \in N\}$, i.e. $\tilde{f}(a)$ is some arbitrary element of the equivalence class $f(a)$, and where

$$f(\varphi'(\mathcal{N}^n, a_1, \ldots, a_m)) := \{(f(b_1), \ldots, f(b_n)) : \mathcal{N} \models \varphi'(b_1, \ldots, b_n, a_1, \ldots, a_m)\}.$$

Note that, under the assumption of $\varnothing_{\mathcal{C}'}$-bi-interpretability, the $\mathcal{L}$-translation might require translation parameters from the structures in $\mathcal{C}$; that is, the $\mathcal{L}$-translation of φ' might actually be of the form $\varphi(\bar{x}_1, \ldots, \bar{x}_n, \bar{y}_1, \ldots, \bar{y}_m, \bar{c}_{\mathcal{N}})$, where $\bar{c}_{\mathcal{N}}$ is a tuple of parameters from $\alpha(\mathcal{N})$. However, by Lemma 2.2.6(ii) we can extend $\mathcal{L}$ to include constant symbols for these translation parameters, so we can ignore this issue (although it will come up again in the proof of Proposition 2.4.6(ii)).

Our goal is to show that $\varphi'(x_1, \ldots, x_n; y_1, \ldots, y_m)$ satisfies the size clause, which means that we need to calculate the approximate size of $\varphi'(\mathcal{N}^n, a_1, \ldots, a_m)$. By (2.8) we have

$$|\varphi'(\mathcal{N}^n, a_1, \ldots, a_m)| = |\varphi(\alpha(\mathcal{N})^{r\cdot n}, f(a_1), \ldots, f(a_m))/E|. \tag{2.9}$$

It thus suffices to calculate the right-hand side of (2.9), which, as we shall see, we can do because $\mathcal{C}$ is an R-mac.

Note that we may safely assume that α is surjective, i.e. that $\mathcal{C} = \{\alpha(\mathcal{N}) : \mathcal{N} \in \mathcal{C}'\}$, since $\{\alpha(\mathcal{N}) : \mathcal{N} \in \mathcal{C}'\}$ is a subclass of $\mathcal{C}$ and thus is also an R-mac by Remark 2.1.3(vii).

We now calculate the approximate size of $\varphi(\alpha(\mathcal{N})^{r\cdot n}, \bar{b}_1, \ldots, \bar{b}_m)/E$ for varying $\alpha(\mathcal{N}) \in \mathcal{C}$ and $(\bar{b}_1, \ldots, \bar{b}_m) \in \alpha(\mathcal{N})^{r\cdot m}$. For brevity we henceforth write $\bar{y} := (\bar{y}_1, \ldots, \bar{y}_m)$ and $\bar{b} := (\bar{b}_1, \ldots, \bar{b}_m)$. The equivalence classes in $\varphi(\alpha(\mathcal{N})^{r\cdot n}, \bar{b})$

are uniformly defined by the $\mathcal{L}$-formula

$$\tilde{\varphi}(\bar{x}_1,\ldots,\bar{x}_n;\bar{b},\bar{d}_1,\ldots,\bar{d}_n) :\equiv \varphi(\bar{x}_1,\ldots,\bar{x}_n,\bar{b}) \wedge \bigwedge_{1\leq i\leq n} E(\bar{x}_i,\bar{d}_i)$$

for tuples $(\bar{d}_1,\ldots,\bar{d}_n)\in X^n_{\mathcal{N}}$ varying between equivalence classes. Note that the $\mathcal{L}$-formula $E(\bar{v}_1,\bar{v}_2)$ that defines the equivalence relation E on $X^n_{\mathcal{N}}$ may require parameters from each $\alpha(\mathcal{N})\in\mathcal{C}$, but we can subsume these into $\bar{c}_{\mathcal{N}}$.

Since $\mathcal{C}$ is an R-mac, the formula $\tilde{\varphi}(\bar{x}_1,\ldots,\bar{x}_n;\bar{y},\bar{v}_1,\ldots,\bar{v}_n)$ gives rise to a finite partition $\Gamma_1,\ldots,\Gamma_k$ of $\mathcal{C}(r\cdot m+r\cdot n)=\{(\alpha(\mathcal{N}),\bar{b},\bar{d}_1,\ldots,\bar{d}_n):\mathcal{N}\in\mathcal{C}',\bar{b}\in\alpha(\mathcal{N})^{r\cdot m},\bar{d}_i\in\alpha(\mathcal{N})^r\}$ with measuring functions $h_1,\ldots,h_k\in R$ and defining $\mathcal{L}$-formulas $\{\theta_j(\bar{y},\bar{v}_1,\ldots,\bar{v}_n):1\leq j\leq k\}$.

Note that since φ is E-invariant, it is safe to assume that the h_j respect the equivalence relation – i.e. that for every $\bar{b}\in\alpha(\mathcal{N})^{r\cdot m}$, if $(\bar{d}_1,\ldots,\bar{d}_n)$ and $(\bar{d}'_1,\ldots,\bar{d}'_n)$ lie in the same equivalence class, then $(\alpha(\mathcal{N}),\bar{b},\bar{d}_1,\ldots,\bar{d}_n)$ and $(\alpha(\mathcal{N}),\bar{b},\bar{d}'_1,\ldots,\bar{d}'_n)$ belong to the same Γ_i – for if they do not, then we can rearrange the partition so that they do, since $\tilde{\varphi}(\alpha(\mathcal{N})^{r\cdot n},\bar{b},\bar{d}_1,\ldots,\bar{d}_n)=\tilde{\varphi}(\alpha(\mathcal{N})^{r\cdot n},\bar{b},\bar{d}'_1,\ldots,\bar{d}'_n)$.

Let $Y_j(\alpha(\mathcal{N}),\bar{b})$ denote the union of the equivalence classes in $\varphi(\alpha(\mathcal{N})^{r\cdot n},\bar{b})$ that take the function h_j. By the assumption in the previous paragraph $Y_j(\alpha(\mathcal{N}),\bar{b})\cap Y_{j'}(\alpha(\mathcal{N}),\bar{b})=\varnothing$ for $j\neq j'$. The set $Y_j(\alpha(\mathcal{N}),\bar{b})$ is uniformly defined by the $\mathcal{L}$-formula

$$\tilde{\varphi}_j(\bar{x}_1,\ldots,\bar{x}_n;\bar{b}) :\equiv \varphi(\bar{x}_1,\ldots,\bar{x}_n,\bar{b})\wedge\theta_j(\bar{b},\bar{x}_1,\ldots,\bar{x}_n).$$

Since $\mathcal{C}$ is an R-mac, the formula $\tilde{\varphi}_j(\bar{x}_1,\ldots,\bar{x}_n;\bar{y})$ gives rise to a finite partition $\pi_{j1},\ldots,\pi_{je_j}$ of $\mathcal{C}(r\cdot m)=\{(\alpha(\mathcal{N}),\bar{b}):\mathcal{N}\in\mathcal{C}',\bar{b}\in\alpha(\mathcal{N})^{r\cdot m}\}$ with measuring functions $g_{j1},\ldots,g_{je_j}\in R$. Again, we may assume that this partition respects the equivalence relation, i.e. that if $\bar{b}$ and $\bar{b}'$ lie in the same equivalence class, then $(\alpha(\mathcal{N}),\bar{b})$ and $(\alpha(\mathcal{N}),\bar{b}')$ lie in the same π_{ji}.

The idea now is to show that for $(\alpha(\mathcal{N}),\bar{b})\in\pi_{ji}$, $|Y_j(\alpha(\mathcal{N}),\bar{b})/E|$ is approximately equal to $\frac{g_{ji}(\alpha(\mathcal{N}))}{h_j(\alpha(\mathcal{N}))}$. This is to be expected, since $g_{ji}(\alpha(\mathcal{N}))$ is approximately equal to $|Y_j(\alpha(\mathcal{N}),\bar{b})|$ and $h_j(\alpha(\mathcal{N}))$ is approximately equal to the size of each equivalence class in $Y_j(\alpha(\mathcal{N}),\bar{b})$. See Figure 2.1:

For brevity we henceforth write $\bar{d}:=(\bar{d}_1,\ldots,\bar{d}_n)$. Let $E_{\mathcal{N}}$ be a choice of one $\bar{d}\in X^n_{\mathcal{N}}$ from each equivalence class in $X^n_{\mathcal{N}}$; that is, $X^n_{\mathcal{N}}/E=\{\bar{d}/E:\bar{d}\in E_{\mathcal{N}}\}$ and $|X^n_{\mathcal{N}}/E|=|E_{\mathcal{N}}|$. (We do not claim that $E_{\mathcal{N}}$ is $\mathcal{L}$-definable.) Then, defining $\mathcal{N}_j(\bar{b}):=E_{\mathcal{N}}\cap\theta_j(\bar{b},\alpha(\mathcal{N})^{r\cdot n})$, for every $(\alpha(\mathcal{N}),\bar{b})\in\mathcal{C}(r\cdot m)$ and for

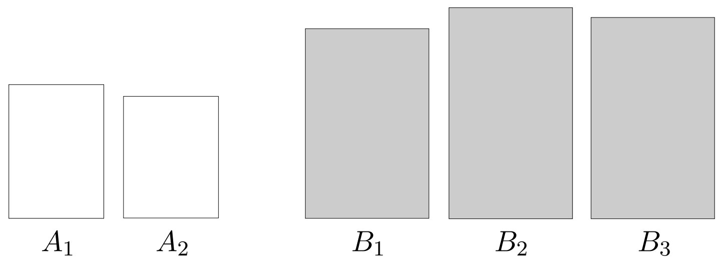

Figure 2.1: Suppose that $\varphi(\alpha(\mathcal{N})^{r\cdot n}, \bar{b}) = A_1 \cup A_2 \cup B_1 \cup B_2 \cup B_3$ for some $(\alpha(\mathcal{N}), \bar{b}) \in \pi_{1i} \cap \pi_{2i'}$, where each A is an equivalence class of approximate size $h_1(\alpha(\mathcal{N}))$ and each B is an equivalence class of approximate size $h_2(\alpha(\mathcal{N}))$. Then $Y_1(\alpha(\mathcal{N}), \bar{b}) = A_1 \cup A_2$ has approximate size $g_{1i}(\alpha(\mathcal{N}))$ and $Y_2(\alpha(\mathcal{N}), \bar{b}) = B_1 \cup B_2 \cup B_3$ has approximate size $g_{2i'}(\alpha(\mathcal{N}))$. Thus $|Y_1(\alpha(\mathcal{N}), \bar{b})/E|$, which is precisely equal to 2, is approximately equal to $\frac{g_{1i}(\alpha(\mathcal{N}))}{h_1(\alpha(\mathcal{N}))}$. Likewise $|Y_2(\alpha(\mathcal{N}), \bar{b})/E|$, which is precisely equal to 3, is approximately equal to $\frac{g_{2i'}(\alpha(\mathcal{N}))}{h_2(\alpha(\mathcal{N}))}$.

each $j \in \{1, \ldots, k\}$ we have

$$Y_j(\alpha(\mathcal{N}), \bar{b}) = \bigcup_{\bar{d} \in \mathcal{N}_j(\bar{b})} \tilde{\varphi}(\alpha(\mathcal{N})^{r\cdot n}, \bar{b}, \bar{d}), \tag{2.10}$$

where the union is disjoint because equivalence classes are disjoint.

Consider some arbitrary $j \in \{1, \ldots, k\}$ and $i \in \{1, \ldots, e_j\}$. By the definitions of Γ_j and h_j we have

$$\Big| |\tilde{\varphi}(\alpha(\mathcal{N})^{r\cdot n}, \bar{b}, \bar{d})| - h_j(\alpha(\mathcal{N})) \Big| = o(h_j(\alpha(\mathcal{N}))) \tag{2.11}$$

for all $(\alpha(\mathcal{N}), \bar{b}, \bar{d}) \in \Gamma_j$ as $|\alpha(\mathcal{N})| \to \infty$. By the definitions of π_{ji} and g_{ji} we have

$$\Big| |Y_j(\alpha(\mathcal{N}), \bar{b})| - g_{ji}(\alpha(\mathcal{N})) \Big| = o(g_{ji}(\alpha(\mathcal{N}))) \tag{2.12}$$

for all $(\alpha(\mathcal{N}), \bar{b}) \in \pi_{ji}$ as $|\alpha(\mathcal{N})| \to \infty$.

Let $\varepsilon > 0$. Recall the ε–Q definition of the little-o notation (see the end of Definition 2.1.2). Let $Q_1 \in \mathbb{N}$ be such that (2.11) holds for $\frac{\varepsilon}{2}$ and let $Q_2 \in \mathbb{N}$ be such that (2.12) holds for $\frac{\varepsilon}{2}$. Set $Q := \max\{Q_1, Q_2\}$ and $t := |Y_j(\alpha(\mathcal{N}), \bar{b})/E|$. Notice that $t = |\mathcal{N}_j(\bar{b})|$. Bear in mind that t depends on $\alpha(\mathcal{N})$, $\bar{b}$ and j.

Let $(\alpha(\mathcal{N}), \bar{b}) \in \pi_{ji}$ be such that $|\alpha(\mathcal{N})| > Q$. (If π_{ji} is bounded, then we have (2.18) by Remark 2.1.3(iv).) Then

$$
\begin{aligned}
&\Big||Y_j(\alpha(\mathcal{N}),\bar{b})| - th_j(\alpha(\mathcal{N}))\Big| \\
&= \left|\sum_{\bar{d}\in\mathcal{N}_j(\bar{b})} |\tilde{\varphi}(\alpha(\mathcal{N})^{r\cdot n},\bar{b},\bar{d})| - th_j(\alpha(\mathcal{N}))\right| && \text{(by (2.10))} \\
&= \left|\sum_{\bar{d}\in\mathcal{N}_j(\bar{b})} \Big[|\tilde{\varphi}(\alpha(\mathcal{N})^{r\cdot n},\bar{b},\bar{d})| - h_j(\alpha(\mathcal{N}))\Big]\right| && \text{(since } t = |\mathcal{N}_j(\bar{b})|) \\
&\le \sum_{\bar{d}\in\mathcal{N}_j(\bar{b})} \Big||\tilde{\varphi}(\alpha(\mathcal{N})^{r\cdot n},\bar{b},\bar{d})| - h_j(\alpha(\mathcal{N}))\Big| && \text{(triangle inequality)} \\
&\le \underbrace{\frac{\varepsilon}{2}h_j(\alpha(\mathcal{N})) + \cdots + \frac{\varepsilon}{2}h_j(\alpha(\mathcal{N}))}_{t\text{ times}} \\
&= \frac{t\varepsilon}{2}h_j(\alpha(\mathcal{N})),
\end{aligned}
$$

where the penultimate step follows by applying (2.11) to Q and $\frac{\varepsilon}{2}$ for each $\bar{d}\in\mathcal{N}_j(\bar{b})$, which we can do because $(\alpha(\mathcal{N}),\bar{b},\bar{d})\in\Gamma_j$ if $\bar{d}\in\mathcal{N}_j(\bar{b})$. Thus

$$-\frac{t\varepsilon}{2}h_j(\alpha(\mathcal{N})) \le |Y_j(\alpha(\mathcal{N}),\bar{b})| - th_j(\alpha(\mathcal{N})) \le \frac{t\varepsilon}{2}h_j(\alpha(\mathcal{N})),$$

which is equivalent to

$$-\frac{t\varepsilon}{2}h_j(\alpha(\mathcal{N})) \le th_j(\alpha(\mathcal{N})) - |Y_j(\alpha(\mathcal{N}),\bar{b})| \le \frac{t\varepsilon}{2}h_j(\alpha(\mathcal{N})). \tag{2.13}$$

Now, since $(\alpha(\mathcal{N}),\bar{b})\in\pi_{ji}$ and $|\alpha(\mathcal{N})| > Q$, we may also apply (2.12) to Q and $\frac{\varepsilon}{2}$, yielding

$$-\frac{\varepsilon}{2}g_{ji}(\alpha(\mathcal{N})) \le |Y_j(\alpha(\mathcal{N}),\bar{b})| - g_{ji}(\alpha(\mathcal{N})) \le \frac{\varepsilon}{2}g_{ji}(\alpha(\mathcal{N})). \tag{2.14}$$

Adding (2.13) and (2.14) gives

$$
\begin{aligned}
-\frac{\varepsilon}{2}g_{ji}(\alpha(\mathcal{N})) - \frac{t\varepsilon}{2}h_j(\alpha(\mathcal{N})) &\le th_j(\alpha(\mathcal{N})) - g_{ji}(\alpha(\mathcal{N})) \\
&\le \frac{\varepsilon}{2}g_{ji}(\alpha(\mathcal{N})) + \frac{t\varepsilon}{2}h_j(\alpha(\mathcal{N})).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
-\varepsilon\max\{g_{ji}(\alpha(\mathcal{N})), th_j(\alpha(\mathcal{N}))\} &\le th_j(\alpha(\mathcal{N})) - g_{ji}(\alpha(\mathcal{N})) \\
&\le \varepsilon\max\{g_{ji}(\alpha(\mathcal{N})), th_j(\alpha(\mathcal{N}))\}
\end{aligned}
\tag{2.15}
$$

for all $(\alpha(\mathcal{N}),\bar{b}) \in \pi_{ji}$ such that $|\alpha(\mathcal{N})| > Q$. There are two cases:

Case 1: $h_j(\alpha(\mathcal{N})) \leq g_{ji}(\alpha(\mathcal{N}))$. Then (2.15) becomes

$$-\varepsilon g_{ji}(\alpha(\mathcal{N})) \leq th_j(\alpha(\mathcal{N})) - g_{ji}(\alpha(\mathcal{N})) \leq \varepsilon g_{ji}(\alpha(\mathcal{N})),$$

which, after rearrangement and applying the identity $t = |Y_j(\alpha(\mathcal{N}),\bar{b})/E|$, is equivalent to

$$\left| |Y_j(\alpha(\mathcal{N}),\bar{b})/E| - \frac{g_{ji}(\alpha(\mathcal{N}))}{h_j(\alpha(\mathcal{N}))} \right| \leq \varepsilon \frac{g_{ji}(\alpha(\mathcal{N}))}{h_j(\alpha(\mathcal{N}))}.$$

Therefore, defining $\frac{g_{ji}}{h_j}(\alpha(\mathcal{N})) := \frac{g_{ji}(\alpha(\mathcal{N}))}{h_j(\alpha(\mathcal{N}))}$, we have

$$\left| |Y_j(\alpha(\mathcal{N}),\bar{b})/E| - \frac{g_{ji}}{h_j}(\alpha(\mathcal{N})) \right| = o\left(\frac{g_{ji}}{h_j}(\alpha(\mathcal{N}))\right) \tag{2.16}$$

for all $(\alpha(\mathcal{N}),\bar{b}) \in \pi^g_{ji} := \{(\alpha(\mathcal{N}),\bar{b}) \in \pi_{ji} : h_j(\alpha(\mathcal{N})) \leq g_{ji}(\alpha(\mathcal{N}))\}$ as $|\alpha(\mathcal{N})| \to \infty$.

Case 2: $g_{ji}(\alpha(\mathcal{N})) < th_j(\alpha(\mathcal{N}))$. Then (2.15) becomes

$$-\varepsilon th_j(\alpha(\mathcal{N})) \leq th_j(\alpha(\mathcal{N})) - g_{ji}(\alpha(\mathcal{N})) \leq \varepsilon th_j(\alpha(\mathcal{N})),$$

which, after rearrangement and applying the identity $t = |Y_j(\alpha(\mathcal{N}),\bar{b})/E|$, is equivalent to

$$\left| |Y_j(\alpha(\mathcal{N}),\bar{b})/E| - \frac{g_{ji}(\alpha(\mathcal{N}))}{h_j(\alpha(\mathcal{N}))} \right| \leq \varepsilon |Y_j(\alpha(\mathcal{N}),\bar{b})/E|.$$

Therefore

$$\left| |Y_j(\alpha(\mathcal{N}),\bar{b})/E| - \frac{g_{ji}}{h_j}(\alpha(\mathcal{N})) \right| = o\left(|Y_j(\alpha(\mathcal{N}),\bar{b})/E|\right)$$

for all $(\alpha(\mathcal{N}),\bar{b}) \in \pi^h_{ji} := \{(\alpha(\mathcal{N}),\bar{b}) \in \pi_{ji} : g_{ji}(\alpha(\mathcal{N})) < th_j(\alpha(\mathcal{N}))\}$ as $|\alpha(\mathcal{N})| \to \infty$. Thus by little-o exchange (Appendix A) we have

$$\left| |Y_j(\alpha(\mathcal{N}),\bar{b})/E| - \frac{g_{ji}}{h_j}(\alpha(\mathcal{N})) \right| = o\left(\frac{g_{ji}}{h_j}(\alpha(\mathcal{N}))\right) \tag{2.17}$$

for all $(\alpha(\mathcal{N}),\bar{b}) \in \pi^h_{ji}$ as $|\alpha(\mathcal{N})| \to \infty$.

Since $\pi_{ji} = \pi^g_{ji} \cup \pi^h_{ji}$, (2.16) and (2.17) together imply that

$$\left| |Y_j(\alpha(\mathcal{N}), \bar{b})/E| - \frac{g_{ji}}{h_j}(\alpha(\mathcal{N})) \right| = o\left(\frac{g_{ji}}{h_j}(\alpha(\mathcal{N}))\right) \tag{2.18}$$

for all $(\alpha(\mathcal{N}), \bar{b}) \in \pi_{ji}$ as $|\alpha(\mathcal{N})| \to \infty$, which is what we wanted.

Before we move on, we need to deal with the case $h_j(\alpha(\mathcal{N})) = 0$, as we can't divide by zero. Since functions in R are eventually positive-definite (Lemma 2.2.5), by taking a larger Q if necessary we may assume that h_j is positive-definite for all $\alpha(\mathcal{N})$ with $|\alpha(\mathcal{N})| > Q$. So for the case $h_j(\alpha(\mathcal{N})) = 0$ we define $\frac{g_{ji}}{h_j} := 0$. Then (2.18) still holds, since for $|\alpha(\mathcal{N})| > Q$ we have that $h_j(\alpha(\mathcal{N})) = 0$ implies $|Y_j(\alpha(\mathcal{N}), \bar{b})/E| = 0$. Defining $\frac{g_{ji}}{h_j}$ selectively like this is slightly out of kilter with the definition of $\mathrm{Frac}(R)$ (Definition 2.4.5), but not in a serious way. Since g_{ji} is also eventually positive-definite, for large enough $\alpha(\mathcal{N})$ we have that $h_j(\alpha(\mathcal{N})) = 0$ implies $g_{ji}(\alpha(\mathcal{N})) = 0$, and so defining $\frac{g_{ji}}{h_j}$ in this way really just amounts to setting a convention within $\mathrm{Frac}(R)$ that $\frac{0}{0} := 0$.

We now need to accomplish our task of calculating the approximate size of $\varphi(\alpha(\mathcal{N})^{r \cdot n}, \bar{b}_1, \ldots, \bar{b}_m)/E$. For each $j \in \{1, \ldots, k\}$ we have a finite partition $\Phi_j := \{\pi_{ji} : 1 \leq i \leq e_j\}$ of $\mathcal{C}(r \cdot m)$. We use these k partitions to construct a single finite partition Φ of $\mathcal{C}(r \cdot m)$. Define

$$\pi_{(i_1, \ldots, i_k)} := \bigcap_{j=1}^{k} \pi_{j i_j} \text{ and } I := \{(i_1, \ldots, i_k) : 1 \leq i_j \leq e_j, 1 \leq j \leq q\}.$$

Then $\Phi := \{\pi_{(i_1, \ldots, i_k)} : (i_1, \ldots, i_k) \in I\}$ forms a finite partition of $\mathcal{C}(r \cdot m)$. We now need to show that this partition works.

We have

$$\varphi(\alpha(\mathcal{N})^{r \cdot n}, \bar{b}) = \bigcup_{j=1}^{k} Y_j(\alpha(\mathcal{N}), \bar{b}).$$

Since each $Y_j(\alpha(\mathcal{N}), \bar{b})$ is a union of E-equivalence classes and is disjoint from the other $Y_{j'}(\alpha(\mathcal{N}), \bar{b})$, we thus have the following partition of $\varphi(\alpha(\mathcal{N})^{r \cdot n}, \bar{b})/E$:

$$\varphi(\alpha(\mathcal{N})^{r \cdot n}, \bar{b})/E = \bigcup_{j=1}^{k} \left(Y_j(\alpha(\mathcal{N}), \bar{b})/E\right).$$

Hence

$$|\varphi(\alpha(\mathcal{N})^{r \cdot n}, \bar{b})/E| = \sum_{j=1}^{k} \left|Y_j(\alpha(\mathcal{N}), \bar{b})/E\right|. \tag{2.19}$$

Let $\varepsilon > 0$ and consider some arbitrary $(i_1, \dots, i_k) \in I$. For each $j \in \{1, \dots, k\}$ let $Q(j) \in \mathbb{N}^+$ be such that (2.18) holds for $\frac{\varepsilon}{k}$, where we take $g_{ji} := g_{ji_j}$. Set $Q' := \max\{Q(j) : 1 \le j \le k\}$. Then for every $(\alpha(\mathcal{N}), \bar{a}) \in \pi_{(i_1,\dots,i_k)}$ with $|M| > Q'$,

$$\begin{aligned}
&\left| |\varphi(\alpha(\mathcal{N})^{r \cdot n}, \bar{b})/E| - \sum_{j=1}^{k} \frac{g_{ji_j}}{h_j}(\alpha(\mathcal{N})) \right| \\
&= \left| \sum_{j=1}^{k} |Y_j(\alpha(\mathcal{N}), \bar{b})/E| - \sum_{j=1}^{k} \frac{g_{ji_j}}{h_j}(\alpha(\mathcal{N})) \right| \quad \text{(by (2.19))} \\
&\le \sum_{j=1}^{k} \left| |Y_j(\alpha(\mathcal{N}), \bar{b})/E| - \frac{g_{ji_j}}{h_j}(\alpha(\mathcal{N})) \right| \quad \text{(triangle inequality)} \\
&\le \sum_{j=1}^{k} \frac{\varepsilon}{k} \frac{g_{ji_j}}{h_j}(\alpha(\mathcal{N})) \\
&= \varepsilon \sum_{j=1}^{k} \frac{g_{ji_j}}{h_j}(\alpha(\mathcal{N})),
\end{aligned}$$

where the penultimate step follows by applying (2.18) to Q' and $\frac{\varepsilon}{k}$ for each $j \in \{1, \dots, k\}$, which we can do because $(\alpha(\mathcal{N}), \bar{b}) \in \pi_{(i_1,\dots,i_k)} \subseteq \pi_{ji_j}$. Therefore

$$\left| |\varphi(\alpha(\mathcal{N})^{r \cdot n}, \bar{b})/E| - \sum_{j=1}^{k} \frac{g_{ji_j}}{h_j}(\alpha(\mathcal{N})) \right| = o\left(\sum_{j=1}^{k} \frac{g_{ji_j}}{h_j}(\alpha(\mathcal{N})) \right) \tag{2.20}$$

for all $(\alpha(\mathcal{N}), \bar{b}) \in \pi_{(i_1,\dots,i_k)}$ as $|\alpha(\mathcal{N})| \to \infty$.

We now pull everything back to $\mathcal{C}'$. We define

$$\pi'_{(i_1,\dots,i_k)} := \{(\mathcal{N}, a_1, \dots, a_m) \in \mathcal{C}'(m) : (\alpha(\mathcal{N}), \tilde{f}(a_1), \dots, \tilde{f}(a_m)) \in \pi_{(i_1,\dots,i_k)}\}.$$

Then $\Phi' := \{\pi'_{(i_1,\dots,i_k)} : (i_1, \dots, i_k) \in I\}$ is a finite partition of $\mathcal{C}'(m)$. Note that by our earlier assumption that each π_{ji} respects the equivalence relation, the set $\pi'_{(i_1,\dots,i_k)}$ does not depend on the choice function $\tilde{f}$. We also define

$$\frac{g_{ji}}{h_j}(\mathcal{N}) := \frac{g_{ji}}{h_j}(\alpha(\mathcal{N}))$$

for $\mathcal{N} \in \mathcal{C}'$. Then (2.9) and (2.20) together imply that for every $(i_1, \dots, i_k) \in I$,

$$\left| |\varphi'(\mathcal{N}^n, a_1, \dots, a_m)| - \sum_{j=1}^{k} \frac{g_{ji_j}}{h_j}(\mathcal{N}) \right| = o\left(\sum_{j=1}^{k} \frac{g_{ji_j}}{h_j}(\mathcal{N}) \right)$$

for all $(\mathcal{N}, a_1, \ldots, a_m) \in \pi'_{(i_1,\ldots,i_k)}$ as $|N| \to \infty$. So $\mathcal{C}'$ is a weak $\mathrm{Frac}(R)$-mac. $\square$

We now come to the second part of the result. We recall the statement:

Proposition 2.4.6(ii). *Let $\mathcal{C}$ and $\mathcal{C}'$ be classes of $\mathcal{L}$- and $\mathcal{L}'$-structures respectively. If $\mathcal{C}$ and $\mathcal{C}'$ are $\varnothing$-bi-interpretable and $\mathcal{C}$ is a full R-mac (resp. -mec) in $\mathcal{L}$, then $\mathcal{C}'$ is a full* $\mathrm{Frac}(R)$*-mac (resp. -mec) in $\mathcal{L}'$.*

Proof. Following on from the proof of Proposition 2.4.6(i), we now need to show that the partition Φ' of $\mathcal{C}'(m)$ is definable. So consider some arbitrary $\pi'_{(i_1,\ldots,i_k)} \in \Phi'$. Since $\mathcal{C}$ is a full R-mac, each π_{ji} is definable and hence the intersection $\pi_{(i_1,\ldots,i_k)}$ is also definable. Thus the partition Φ of $\mathcal{C}(r \cdot m)$ is definable; let $\{\psi_{\pi_{(i_1,\ldots,i_k)}}(\bar{y}) : \pi_{(i_1,\ldots,i_k)} \in \Phi\}$ be the defining $\mathcal{L}$-formulas. Note that this is where Remark 2.4.4 becomes apparent: In the proof of Proposition 2.4.6(i) we could ignore the translation parameters by applying Lemma 2.2.6(ii). We cannot do that here though, since expanding $\mathcal{C}$ by constant symbols might stop it from being $\varnothing$-interpretable in $\mathcal{C}'$. Thus, in order to guarantee that each $\psi_{\pi_{(i_1,\ldots,i_k)}}(\bar{y})$ is parameter-free, it seems that we need to assume that no translation parameters are required in the interpretation of $\mathcal{C}$ in $\mathcal{C}'$.

For brevity we henceforth write π and π' for $\pi_{(i_1,\ldots,i_k)}$ and $\pi'_{(i_1,\ldots,i_k)}$ respectively, since the subscript $(i_1, \ldots, i_k)$ is no longer relevant.

For all $(\mathcal{N}, \bar{a}) \in \mathcal{C}'(m)$ we have

$$\begin{aligned}(\mathcal{N}, \bar{a}) \in \pi' &\iff (\alpha(\mathcal{N}), \tilde{f}(\bar{a})) \in \pi && \text{(by the definition of } \pi'\text{)} \\ &\iff \alpha(\mathcal{N}) \models \psi_\pi(\tilde{f}(\bar{a})) && \text{(since } \psi_\pi \text{ defines } \pi\text{)} \\ &\iff \alpha(\mathcal{N})^* \models \psi_\pi(g\tilde{f}(\bar{a})) && \text{(since } g \text{ is an isomorphism).}\end{aligned} \tag{2.21}$$

Since $\alpha(\mathcal{N})^*$ is the $\varnothing$-interpretation of $\alpha(\mathcal{N})$ in $\mathcal{N}$, we can find a parameter-free $\mathcal{L}'$-translation $\psi'_\pi(\bar{y}')$ of $\psi_\pi(\bar{y})$ such that

$$\alpha(\mathcal{N})^* \models \psi_\pi(g\tilde{f}(\bar{a})) \iff \mathcal{N} \models \psi'_\pi(\tilde{g}\tilde{f}(\bar{a})), \tag{2.22}$$

where $\tilde{g}$ is a choice function for g in the same way that $\tilde{f}$ is a choice function for f. Since the isomorphism $g^* f : \mathcal{N} \to \mathcal{N}^{**}$ is uniformly $\varnothing$-definable across $\mathcal{C}'$, we can find a parameter-free $\mathcal{L}'$-formula $\psi''_\pi(y_1, \ldots, y_m)$ such that

$$\mathcal{N} \models \psi'_\pi(\tilde{g}\tilde{f}(\bar{a})) \iff \mathcal{N} \models \psi''_\pi(\bar{a}). \tag{2.23}$$

Together (2.21), (2.22) and (2.23) yield

$$(\mathcal{N}, \bar{a}) \in \pi' \iff \mathcal{N} \models \psi''_\pi(\bar{a}).$$

So π' is definable, as required. □

Remark 2.4.7. We consider potential strengthenings of Proposition 2.4.6:

(i) If for every $\mathcal{N} \in \mathcal{C}'$ the size of the equivalence classes in $X_\mathcal{N}$ is constant, then in Proposition 2.4.6(i) it suffices to assume that $\mathcal{C}$ is only a weak R-mac (resp. -mec). Note that Lemma 3.2 in [21] is effectively this result for the case where each equivalence class is a singleton.

Sketch proof. Let $|E_\mathcal{N}|$ be the size of each equivalence class in $X_\mathcal{N}$. We may then rewrite (2.9) as

$$|\varphi'(\mathcal{N}^n, a_1, \dots, a_m)| = |\varphi(\alpha(\mathcal{N})^{r \cdot n}, f(a_1), \dots, f(a_m))| / |E_\mathcal{N}|.$$

So if h_π is the measuring function for $\varphi(\alpha(\mathcal{N})^{r \cdot n}, f(a_1), \dots, f(a_m))$, then $h_\pi/|E_\mathcal{N}|$ is the measuring function for $\varphi'(\mathcal{N}^n, a_1, \dots, a_m)$. □

However, in general the hypothesis of Proposition 2.4.6(i) cannot be weakened in this way, as shown by Example 2.4.8.

(ii) One cannot strengthen the conclusion of Proposition 2.4.6(i) to $\mathcal{C}'$ being a full Frac(R)-mac (resp. -mec), even if $\mathcal{C}'$ is $\varnothing$-interpretable in $\mathcal{C}$, as shown by Example 2.4.9.

(iii) The hypothesis of Proposition 2.4.6(ii) can be weakened, since its proof does not require the isomorphism $f^*g \colon \mathcal{M} \to \mathcal{M}^{**}$ to be definable (with or without parameters). The surjectivity of α is also not essential.

(iv) We cannot weaken the hypothesis of $\varnothing$-bi-interpretability in Proposition 2.4.6(ii) to only $\varnothing_\mathcal{C}$-bi-interpretability, as shown by Example 2.4.9. Note that Example 2.4.9 concerns only translation parameters, not parameters defining the isomorphism $g^*f \colon \mathcal{N} \to \mathcal{N}^{**}$. I strongly suspect that it is also the case that one cannot weaken the hypothesis by allowing parameters in the $\mathcal{L}'$-definition of g^*f, but I do not have an explicit counterexample.

Example 2.4.8. We justify Remark 2.4.7(i).

Let $\mathcal{L} := \{E, P\}$, where E and P are both binary relation symbols. We define an $\mathcal{L}$-structure $\mathcal{M}_n$ as follows: The assignments of E and P in $\mathcal{M}_n$ are both equivalence relations. We first partition $\mathcal{M}_n$ into n P-equivalence classes, which we denote by $P_1(\mathcal{M}_n), \ldots, P_n(\mathcal{M}_n)$. We then partition each $P_i(\mathcal{M}_n)$ into $2(n-i)$ E-equivalence classes of size n and i E-equivalence classes of size $2n$. So $|P_i(\mathcal{M}_n)| = 2(n-i)n + i \cdot 2n = 2n^2$ for each i. Let $\mathcal{C} := \{\mathcal{M}_n : n \in \mathbb{N}^+\}$.

One can show that $\mathcal{C}$ has eventual quantifier elimination; that is, if φ is an $\mathcal{L}$-formula, then there exist $Q \in \mathbb{N}$ and a quantifier-free $\mathcal{L}$-formula χ such that if $n > Q$, then φ is equivalent to χ in $\mathcal{M}_n$. Using this, one can then show that $\mathcal{C}$ is a weak R_0-mec in $\mathcal{L}$, where R_0 is generated under addition and multiplication by the functions $\mathcal{M}_n \mapsto n$ and $\mathcal{M}_n \mapsto k$ for each $k \in \mathbb{N}$. It is in fact a strictly weak R_0-mec because $\mathcal{L}$ cannot uniformly distinguish between the different E- and P-equivalence classes.

Let $\mathcal{C}' := \{\mathcal{M}_n/E : \mathcal{M}_n \in \mathcal{C}\}$, considered as structures in the language $\mathcal{L}' := \{P\}$. Then $\mathcal{C}'$ is $\varnothing$-interpretable in $\mathcal{C}$. However, $\mathcal{C}'$ is not a weak R-mac in $\mathcal{L}'$ for any R, since $|P_i(\mathcal{M}_n)/E| = 2n - i$ and so we can define arbitrarily many subsets of different sizes by letting n increase and i vary. We can make this rigorous by adapting the argument given in Non-Example 2.3.9. So $\mathcal{C}$ is a strictly weak R_0-mac that interprets a class $\mathcal{C}'$ that is not a weak R-mac for any R, as required.

Note that this example also applies to Proposition 2.4.3(i), since $\mathcal{C}$ is also a strictly weak 1-dimensional asymptotic class.

Example 2.4.9. We justify Remark 2.4.7(ii) and Remark 2.4.7(iv).

Let $\mathcal{L}' = \{I\}$, where I is a binary relation symbol, and let $\mathcal{C}'$ be the class of all finite $\mathcal{L}'$-structures such that each $\mathcal{N} \in \mathcal{C}'$ consists of exactly two I-equivalence classes, one of size n and the other of size $2n$. So each $\mathcal{N} \in \mathcal{C}'$ has size $3n$ for some $n \in \mathbb{N}^+$. Using eventual quantifier elimination (see Example 2.4.8), one can show that $\mathcal{C}'$ is a weak R-mec in $\mathcal{L}'$, where R is generated under addition and multiplication by the functions $\mathcal{N} \mapsto \frac{1}{3}|N|$ and $\mathcal{N} \mapsto k$ for each $k \in \mathbb{N}$. It is in fact a strictly weak R-mec because $\mathcal{L}'$ cannot uniformly distinguish between the two equivalence classes without employing parameters. Notice that $\mathcal{C}'$ is also a strictly weak $\mathrm{Frac}(R)$-mec by Remark 2.1.3(vii), since R contains the function $\mathcal{N} \mapsto 1$ and so $R \subseteq \mathrm{Frac}(R)$.

Now consider the language $\mathcal{L} = \{I, c\}$, where I is a binary relation symbol and c is a constant symbol. Let $\mathcal{C}$ be the same as $\mathcal{C}'$, except that c is assigned as a member of the smaller I-equivalence class in each $\mathcal{M} \in \mathcal{C}$. Since $\mathcal{C}'$ is a weak R-mec, $\mathcal{C}$ is a weak R-mec by Lemma 2.2.6(i). Furthermore, $\mathcal{C}$ is a full R-mec, since the constant symbol allows $\mathcal{L}$ to uniformly distinguish between the two equivalence classes.

We now come to the main point: $\mathcal{C}$ and $\mathcal{C}'$ are $\varnothing_{\mathcal{C}}$-bi-interpretable, since $\mathcal{C}$ is interpretable in $\mathcal{C}'$ *with* parameters and $\mathcal{C}'$ is interpretable in $\mathcal{C}$ *without* parameters. Using the notation from Definition 2.4.1, the isomorphisms f, g, g^*f, etc. are simply identity maps and thus are $\varnothing$-definable. However, while $\mathcal{C}$ is a full R-mec, $\mathcal{C}'$ is a *strictly* weak $\mathrm{Frac}(R)$-mec. So Remark 2.4.7(ii) and Remark 2.4.7(iv) are both justified.

Note that this example also applies to Proposition 2.4.3(ii), since $\mathcal{C}'$ and $\mathcal{C}$ are also strictly weak and full 1-dimensional asymptotic classes respectively.

Chapter 3

Smooth approximation and exact classes

Smoothly approximable structures are $\aleph_0$-categorical structures which can be well approximated by finite structures.

Cherlin & Hrushovski, p. 2 of [18]

The goal of this chapter is to prove Proposition 3.2.1, which states that finite structures smoothly approximating an $\aleph_0$-categorical structure form a multidimensional exact class (Definition 2.1.4). In § 3.1 we define the notion of smooth approximation and then provide some examples. In § 3.2 we state and prove the result.

For the next two chapters we make extensive use of the Ryll-Nardzewski Theorem (Appendix B), so we consider only countable languages.

3.1 Smooth approximation

The notion of smooth approximation was introduced by Lachlan in the 1980s, arising as a generalisation of $\aleph_0$-categorical, $\aleph_0$-stable structures [17], in particular Corollary 7.4 of that paper. [13], [40], [44], [45] and [46] are also relevant, but the key texts on smooth approximation itself are [35] by Kantor, Liebeck and Macpherson and [18] by Cherlin and Hrushovski. A history of the development of the notion is to be found in § 1.1 of [18] and there is a survey article [48], which also contains improvements and errata to [35]. Smooth approximation also arises in the context of asymptotic classes in [20], [21], [50] and [51].

Recall that for $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ we use the notation $\mathcal{N} \leq \mathcal{M}$ to mean that $\mathcal{N}$ is an $\mathcal{L}$-substructure of $\mathcal{M}$.

Definition 3.1.1 (Homogenous substructure)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. $\mathcal{N}$ is a *homogeneous substructure* [1] of $\mathcal{M}$, notationally $\mathcal{N} \leq_{\text{hom}} \mathcal{M}$, if $\mathcal{N} \leq \mathcal{M}$ and for every $k \in \mathbb{N}^+$ and every pair $\bar{a}, \bar{b} \in N^k$, $\bar{a}$ and $\bar{b}$ lie in the same $\mathrm{Aut}(\mathcal{M})$-orbit if and only if $\bar{a}$ and $\bar{b}$ lie in the same $\mathrm{Aut}_{\{N\}}(\mathcal{M})$-orbit, where

$$\mathrm{Aut}_{\{N\}}(\mathcal{M}) := \{\sigma \in \mathrm{Aut}(\mathcal{M}) : \sigma(N) = N\}.$$

Definition 3.1.2 (Smooth approximation)**.** An $\mathcal{L}$-structure $\mathcal{M}$ is *smoothly approximable* if $\mathcal{M}$ is $\aleph_0$-categorical and there exists a sequence $(\mathcal{M}_i)_{i<\omega}$ of finite homogeneous substructures of $\mathcal{M}$ such that $M_i \subset M_{i+1}$ for all $i < \omega$ and $\bigcup_{i<\omega} M_i = M$. We say that $\mathcal{M}$ is *smoothly approximated* by the $\mathcal{M}_i$.

We provide some examples of smoothly approximable structures, starting with a trivial example:

Example 3.1.3. Let $\mathcal{M}$ be a countably infinite set in the language of equality. Enumerate $\mathcal{M}$ as $(a_i : i < \omega)$ and let $\mathcal{M}_i = \{a_0, \ldots, a_i\}$. Then each $\mathcal{M}_i$ is a finite homogeneous substructure of $\mathcal{M}$ and $\mathcal{M} = \bigcup_{i<\omega} \mathcal{M}_i$.

Example 3.1.4. Consider a language $\mathcal{L} := \{I_1, I_2\}$, where I_1 and I_2 are binary relation symbols. Let $\mathcal{M}$ be a countable $\mathcal{L}$-structure where $I_1^{\mathcal{M}}$ and $I_2^{\mathcal{M}}$ are equivalence relations such that $I_1^{\mathcal{M}}$ has infinitely many classes, $I_2^{\mathcal{M}}$ refines $I_1^{\mathcal{M}}$, every I_1-equivalence class contains infinitely many I_2-equivalence classes, and every I_2-equivalence class is infinite; that is, $\mathcal{M}$ is partitioned into infinitely many I_1-equivalence classes, each of which is then partitioned into infinitely many I_2-equivalence classes, each of which is infinite. Note that $\mathcal{M}$ is unique up to isomorphism and hence $\aleph_0$-categorical, since the structure is first-order expressible in $\mathcal{L}$.

Enumerate the I_1-equivalence classes as $(a_i : i < \omega)$ and the I_2-equivalence classes within each a_i as $(a_{ij} : j < \omega)$. Finally, enumerate the elements of each a_{ij} as $(a_{ijk} : k < \omega)$. Let $\mathcal{M}_{(r,s,t)} := \{a_{ijk} : i \leq r, j \leq s, k \leq t\}$. Then each $\mathcal{M}_{(r,s,t)}$ is a finite homogeneous substructure of $\mathcal{M}$ and $\mathcal{M} = \bigcup_{r<\omega} \mathcal{M}_{(r,r,r)}$.

Note that this example straightforwardly generalises to the case of n nested equivalence relations for any $n < \omega$.

[1] 'Homogeneous substructure' is defined as one term, not as the conjunction of two words; that is, 'homogeneous substructure' does not mean a substructure that is homogeneous.

Example 3.1.5. Let $\mathcal{M}$ be the direct sum of ω-many copies of the additive group $\mathbb{Z}/p^2\mathbb{Z}$, where p is some fixed prime. Note that $\mathcal{M}$ is $\aleph_0$-categorical, which can be seen via Szmielew invariants (see Appendix A.2 in [31]). Let $\mathcal{M}_i$ consist of the first i copies of $\mathbb{Z}/p^2\mathbb{Z}$. Then each $\mathcal{M}_i$ is a finite homogeneous substructure of $\mathcal{M}$ and $\mathcal{M} = \bigcup_{i<\omega} \mathcal{M}_i$.

3.2 Smooth approximation is exact

We now state and prove Proposition 3.2.1, leaving some necessary technical lemmas until after the main proof.

Proposition 3.2.1. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure smoothly approximated by finite homogeneous substructures $(\mathcal{M}_i)_{i<\omega}$. Then there exists R such that $\mathcal{C} := \{\mathcal{M}_i : i < \omega\}$ is an R-mec in $\mathcal{L}$.*

Proof. Let $\varphi(\bar{x}, \bar{y})$ be an $\mathcal{L}$-formula with $n := l(\bar{x})$ and $m := l(\bar{y})$.

We first cover the size clause. We use the Ryll-Nardzewski Theorem (Appendix B): Since $\mathcal{M}$ is $\aleph_0$-categorical, $\mathrm{Aut}(\mathcal{M})$ acts oligomorphically on $\mathcal{M}$ and thus $\mathcal{M}^m$ has only finitely many $\mathrm{Aut}(\mathcal{M})$-orbits, say $\Theta_1, \ldots, \Theta_k$. We use these orbits to define a finite partition $\pi_1, \ldots, \pi_k$ of $\mathcal{C}(m) = \{(\mathcal{M}_i, \bar{a}) : i < \omega, \bar{a} \in {M_i}^m\}$:

$$(\mathcal{M}_i, \bar{a}) \in \pi_j :\Longleftrightarrow \bar{a} \in \Theta_j.$$

Define $\pi_j^{\mathcal{M}_i} := \{\bar{a} \in {M_i}^m : (\mathcal{M}_i, \bar{a}) \in \pi_j\}$ and let $\bar{a}, \bar{b} \in {M_i}^m$. Then

$$\begin{aligned} \bar{a}, \bar{b} \in \pi_j^{\mathcal{M}_i} &\Longleftrightarrow \bar{a}, \bar{b} \in \Theta_j \\ &\Longrightarrow \bar{a} \text{ and } \bar{b} \text{ lie in the same } \mathrm{Aut}_{\{M_i\}}(\mathcal{M})\text{-orbit} \\ &\qquad (\text{since } \mathcal{M}_i \leq_{\text{hom}} \mathcal{M}) \\ &\Longrightarrow |\varphi({\mathcal{M}_i}^n, \bar{a})| = |\varphi({\mathcal{M}_i}^n, \bar{b})|. \end{aligned} \tag{3.1}$$

We justify the last implication: Since $\bar{a}$ and $\bar{b}$ lie in the same $\mathrm{Aut}_{\{M_i\}}(\mathcal{M})$-orbit, there is some $\sigma \in \mathrm{Aut}_{\{M_i\}}(\mathcal{M})$ such that $\sigma(\bar{a}) = \bar{b}$. But $\sigma \restriction M_i$ is an automorphism of $\mathcal{M}_i$ (Lemma 3.2.2) and thus $\mathcal{M}_i \models \varphi(\bar{c}, \bar{a})$ if and only if $\mathcal{M}_i \models \varphi(\sigma(\bar{c}), \sigma(\bar{a}))$. Therefore $\sigma\colon \varphi({\mathcal{M}_i}^n, \bar{a}) \to \varphi({\mathcal{M}_i}^n, \bar{b})$ is a bijection and hence $|\varphi({\mathcal{M}_i}^n, \bar{a})| = |\varphi({\mathcal{M}_i}^n, \bar{b})|$.

Define $h_j(\mathcal{M}_i) := |\varphi({\mathcal{M}_i}^n, \bar{a})|$, where $\bar{a}$ is some arbitrary element of $\pi_j^{\mathcal{M}_i}$ (if no such $\bar{a}$ exists, then the value of h_j at $\mathcal{M}_i$ can be chosen to be anything, say 0); this function is well-defined by (3.1). Then $\pi_1, \ldots, \pi_k$ and $h_1, \ldots, h_k$ satisfy the size clause.

We now come to the definability clause. We use the Ryll-Nardzewski Theorem again: Each orbit Θ_j is the solution set of an isolated m-type and so the $\mathcal{L}$-formula isolating this type defines Θ_j in $\mathcal{M}$; let $\psi_j(\bar{y})$ be the isolating formula for Θ_j. So $\mathcal{M} \models \psi_j(\bar{a})$ if and only f $\bar{a} \in \Theta_j$. We claim that the following is eventually true, i.e. there exists $Q \in \mathbb{N}$ such that for each ψ_j, if $i > Q$, then

$$\mathcal{M}_i \models \psi_j(\bar{a}) \iff \bar{a} \in \pi_j^{\mathcal{M}_i} \tag{3.2}$$

for every $\bar{a} \in {M_i}^m$. By Lemma 2.2.7 this suffices to prove the definability clause.

We prove this claim: Apply Lemma 3.2.7 to ψ_j to obtain $Q_j \in \mathbb{N}$ such that if $i > Q_j$ and $\bar{a} \in {M_i}^m$, then

$$\mathcal{M} \models \psi_j(\bar{a}) \iff \mathcal{M}_i \models \psi_j(\bar{a}). \tag{3.3}$$

Let $Q := \max\{Q_j : 1 \leq j \leq k\}$. Consider $\bar{a} \in {M_i}^m$ with $i > Q$. Then

$$\mathcal{M}_i \models \psi_j(\bar{a}) \overset{(3.3)}{\iff} \mathcal{M} \models \psi_j(\bar{a}) \iff \bar{a} \in \Theta_j \iff \bar{a} \in \pi_j^{\mathcal{M}_i}$$

and so (3.2) holds. □

Lemma 3.2.2. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $\mathcal{N} \leq \mathcal{M}$. If $\sigma \in \mathrm{Aut}_{\{N\}}(\mathcal{M})$, then $\sigma \restriction N \in \mathrm{Aut}(\mathcal{N})$, where $\sigma \restriction N$ is the restriction of σ to N.*

Proof. This follows from the definitions of a substructure and of an isomorphism. □

Definition 3.2.3 and Lemmas 3.2.4 to 3.2.7 are all in the context of Proposition 3.2.1.

Definition 3.2.3 (Canonical language)**.** We define the *canonical language* [2] of $\mathcal{M}$ to be

$$\mathcal{L}^* := \mathcal{L} \cup \{P_\Theta : \Theta \text{ is a } \mathrm{Aut}(\mathcal{M})\text{-orbit of } \mathcal{M}\},$$

where each P_Θ is a new unary predicate symbol. We expand $\mathcal{M}$ to an $\mathcal{L}^*$-structure $\mathcal{M}^*$ by defining the assignment of each P_Θ in $\mathcal{M}^*$ to be Θ. We expand each $\mathcal{M}_i$ to an $\mathcal{L}^*$-structure ${\mathcal{M}_i}^*$ by defining the assignment of each P_Θ to be $\Theta \cap M_i$.

Lemma 3.2.4. $\mathrm{Aut}(\mathcal{M}) = \mathrm{Aut}(\mathcal{M}^*)$.

[2] Note that the term *canonical language* is sometimes used to refer to the smaller language $\mathcal{L}^* \setminus \mathcal{L}$. We avoid this usage.

Proof. Since $\mathcal{L} \subseteq \mathcal{L}^*$, $\mathrm{Aut}(\mathcal{M}^*) \subseteq \mathrm{Aut}(\mathcal{M})$. It remains to show that $\mathrm{Aut}(\mathcal{M}) \subseteq \mathrm{Aut}(\mathcal{M}^*)$. So consider some $\sigma \in \mathrm{Aut}(\mathcal{M})$. Suppose that $\mathcal{M}^* \models P_\Theta(a)$ for some $a \in M$. Then $a \in \Theta$ and thus $\sigma(a) \in \Theta$, since Θ is an $\mathrm{Aut}(\mathcal{M})$-orbit. Hence $\mathcal{M}^* \models P_\Theta(\sigma(a))$ and so $\sigma \in \mathrm{Aut}(\mathcal{M}^*)$. □

Lemma 3.2.5. $\mathrm{Th}(\mathcal{M}^*)$ *has quantifier elimination; in particular, any* $\mathcal{L}^*$-*formula is equivalent in* $\mathrm{Th}(\mathcal{M}^*)$ *to a quantifier-free* $(\mathcal{L}^* \setminus \mathcal{L})$-*formula.*

Proof. Consider an $\mathcal{L}^*$-formula $\chi(\bar{y})$ with $m := l(\bar{y})$. By the Ryll-Nardzewski Theorem, $\mathcal{M}^m$ has only finitely many $\mathrm{Aut}(\mathcal{M})$-orbits, say $\Theta_1, \ldots, \Theta_k$. By Lemma 3.2.4, these are also the $\mathrm{Aut}(\mathcal{M}^*)$-orbits of $(\mathcal{M}^*)^m$. Now, if $\mathcal{M}^* \models \chi(\bar{a})$ and $\bar{b}$ lies in the same $\mathrm{Aut}(\mathcal{M}^*)$-orbit as $\bar{a}$, then $\mathcal{M}^* \models \chi(\bar{b})$. So $\chi((\mathcal{M}^*)^m) \cap \Theta_i = \Theta_i$ or $\chi((\mathcal{M}^*)^m) \cap \Theta_i = \varnothing$ and hence $\chi((\mathcal{M}^*)^m)$ must be the union of some of the Θ_i, say

$$\chi((\mathcal{M}^*)^m) = \Theta_{i_1} \cup \cdots \cup \Theta_{i_r}.$$

(We could have $\chi((\mathcal{M}^*)^m) = \varnothing$, in which case $\chi(\bar{y})$ would be equivalent in $\mathrm{Th}(\mathcal{M}^*)$ to the quantifier-free $(\mathcal{L}^* \setminus \mathcal{L})$-formula $\bigwedge_{1 \le i \le k} \neg P_{\Theta_i}(\bar{y})$.) Therefore

$$\mathcal{M}^* \models \forall \bar{y} \left(\chi(\bar{y}) \leftrightarrow \bigvee_{1 \le j \le r} P_{\Theta_{i_j}}(\bar{y}) \right).$$

But $\bigvee_{1 \le j \le r} P_{\Theta_{i_j}}(\bar{y})$ is a quantifier-free $(\mathcal{L}^* \setminus \mathcal{L})$-formula, as required. □

Lemma 3.2.6. $\mathcal{M}^*$ *is smoothly approximated by* $({\mathcal{M}_i}^*)_{i<\omega}$.

Proof. Since $\mathcal{M}$ is $\aleph_0$-categorical, by Lemma 3.2.4 and the Ryll-Nardzewski Theorem, $\mathcal{M}^*$ is also $\aleph_0$-categorical. Also note that each ${\mathcal{M}_i}^*$ is a finite $\mathcal{L}^*$-substructure of $\mathcal{M}^*$. It remains to show that ${\mathcal{M}_i}^* \le_{\mathrm{hom}} \mathcal{M}^*$. If $\bar{a}, \bar{b} \in {\mathcal{M}_i}^*$ lie in the same $\mathrm{Aut}(\mathcal{M}^*)_{\{M_i\}}$-orbit, then $\bar{a}$ and $\bar{b}$ lie in the same $\mathrm{Aut}(\mathcal{M}^*)$-orbit, since $\mathrm{Aut}(\mathcal{M}^*)_{\{M_i\}} \subseteq \mathrm{Aut}(\mathcal{M}^*)$. Now suppose that $\bar{a}, \bar{b} \in {\mathcal{M}_i}^*$ lie in the same $\mathrm{Aut}(\mathcal{M}^*)$-orbit. By Lemma 3.2.4, $\bar{a}$ and $\bar{b}$ lie in the same $\mathrm{Aut}(\mathcal{M})$-orbit. Thus, since $\mathcal{M}_i \le_{\mathrm{hom}} \mathcal{M}$, there exists $\sigma \in \mathrm{Aut}(\mathcal{M})_{\{M_i\}}$ such that $\sigma(\bar{a}) = \bar{b}$. But $\sigma \in \mathrm{Aut}(\mathcal{M}^*)_{\{M_i\}}$, again by Lemma 3.2.4, and so $\bar{a}$ and $\bar{b}$ lie in the same $\mathrm{Aut}(\mathcal{M}^*)_{\{M_i\}}$-orbit. □

Lemma 3.2.7. *Let* $\chi(\bar{y})$ *be an* $\mathcal{L}$-*formula with* $m := l(\bar{y})$. *Then there exists* $Q \in \mathbb{N}$ *such that if* $i > Q$ *and* $\bar{c} \in {M_i}^m$, *then*

$$\mathcal{M} \models \chi(\bar{c}) \iff \mathcal{M}_i \models \chi(\bar{c}).$$

Proof. Consider $\mathcal{M}^*$. By Lemma 3.2.5, $T := \mathrm{Th}(\mathcal{M}^*)$ has quantifier elimination and thus there is a quantifier-free $\mathcal{L}^*$-formula $\delta(\bar{y})$ such that $\forall \bar{y}\,(\chi(\bar{y}) \leftrightarrow \delta(\bar{y})) \in T$. Thus by compactness there is an $\mathcal{L}^*$-sentence $\tau \in T$ such that

$$\tau \models \forall \bar{y}\,(\chi(\bar{y}) \leftrightarrow \delta(\bar{y})). \tag{3.4}$$

By Lemma 3.2.6 and the $\forall\exists$-axiomatisation of T (see the proof of Proposition 5.4 in [35]), there exists $Q \in \mathbb{N}$ such that $\mathcal{M}_i{}^* \models \tau$ for all $i > Q$. Now, consider some arbitrary $\bar{c} \in M_i{}^m$ with $i > Q$. Since δ is quantifier-free and $\mathcal{M}_i{}^* \leq \mathcal{M}^*$,

$$\mathcal{M}^* \models \delta(\bar{c}) \iff \mathcal{M}_i{}^* \models \delta(\bar{c}).$$

Hence by (3.4) we have

$$\mathcal{M}^* \models \chi(\bar{c}) \iff \mathcal{M}_i{}^* \models \chi(\bar{c})$$

because $\mathcal{M}^* \models \tau$ and $\mathcal{M}_i{}^* \models \tau$. But χ is an $\mathcal{L}$-formula and thus

$$\mathcal{M} \models \chi(\bar{c}) \iff \mathcal{M}_i \models \chi(\bar{c}),$$

as required. □

Chapter 4

Lie coordinatisation

This proof is like the Hydra – every time I think I've understood something, a whole bunch of new questions pop up!

A perplexed PhD student ...

The goal of this chapter is to use Lie coordinatisation to prove Theorem 4.6.4, as conjectured by Macpherson. As such, our account of Lie coordinatisation is streamlined for this purpose and we leave some important notions from [18] by the wayside, most notably orientation and orthogonality. That being said, we make explicit a number of details that are only implicit in [18], especially in our proofs of Theorem 4.4.1 and Proposition 4.5.3. Our presentation is based primarily on [18], with input from [19]; the reader may find it helpful to have these texts to hand, especially the former.

The history of Lie coordinatisation is complex and we give only a very brief summary; see § 1 of [15] and §§ 1.1–1.2 of [18] for a more detailed picture. The notion was developed by Cherlin and Hrushovski as (inter alia) an attempt to find a structure theory for smoothly approximable structures, building on the work of Kantor, Liebeck and Macpherson in [35]. Deep links between other model-theoretic notions were discovered through their investigation (§ 1.2 of [18]). In particular, it was shown that Lie coordinatisability and smooth approximation are equivalent (Theorem 2 in [18]). Note that the classification of finite simple groups plays a fundamental role, albeit in the background.

In contrast to its mathematical depth, Lie coordinatisation has made only a shallow footprint in the literature, in part due to the development of simple theories (see pp. 8–10 of [18]). The first publication on the topic was the

... for whom Goodstein's theorem was of little consolation.

paper [32] by Hrushovski, in joint work with Cherlin. Some technical issues were found in this paper (see p. 7 of [18]) and corrected results were published in [15], which is essentially an abridgement of the main text [18]. The paper [19] by Chowdhury, Hart and Sokolović makes significant contributions and Hrushovski has published some further work on quasifiniteness in [34]. There are also some unpublished notes [29] by Hill and Smart. Lie coordinatisation arises in the context of asymptotic classes in [20], [21], [50] and [51].

We now outline the structure of this chapter. In § 4.1 we go over the basic concepts of Lie coordinatisation and in § 4.2 we provide two examples of Lie coordinatisable structures. § 4.3 develops the notion of an envelope, which is fundamental to the rest of the chapter. We then move on to § 4.4, where we state and sketch a proof of a result (Theorem 4.4.1) that allows us to apply Proposition 3.2.1 to obtain a short version of Macpherson's conjecture (Corollary 4.4.2). § 4.5 then provides us with the extra information needed to prove the full version of the conjecture in § 4.6.

4.1 Lie geometries and Lie coordinatisation

In this section we state the definition of Lie coordinatisation. We need to go over a number of preliminaries first, starting with Lie geometries. We refer the reader to chapter 7 of [3] for the terminology and theory of vector spaces with forms.

Definition 4.1.1 (Linear Lie geometry)**.** Let K be a finite field. A *linear Lie geometry* over K is one of the following six kinds [1] of structures:

1. *A degenerate space.* An infinite set in the language of equality.
2. *A pure vector space.* An infinite-dimensional vector space V over K.
3. *A polar space.* Two infinite-dimensional vector spaces V and W over K with a non-degenerate bilinear form $V \times W \to K$.
4. *A symplectic space.* An infinite-dimensional vector space V over K with a symplectic bilinear form $V \times V \to K$.
5. *A unitary space.* An infinite-dimensional vector space V over K with a unitary sesquilinear form $V \times V \to K$.

[1] We use the word 'kind' in order to avoid overuse of the word 'type'.

6. *An orthogonal space.* An infinite-dimensional vector space V over K with a quadratic form $V \to K$ whose associated bilinear form is non-degenerate.

Remark 4.1.2. We comment on Definition 4.1.1.

(i) We consider linear Lie geometries as two-sorted structures (V, K), with a sort V in the language of groups with an abelian group structure, a sort K in the language of rings with a field structure, and a function $K \times V \to V$ for scalar multiplication. We call V the *vector sort* and K the *field sort*. (See pp. 5 and 12 of [58] for a summary of multi-sorted structures and languages.) The elements of K are named by constant symbols.[2] In the polar case, the vector sort is $V \cup W$ in the language of groups equipped with an equivalence relation with precisely two classes V and W, each with an abelian group structure.

(ii) We have ignored quadratic Lie geometries (Definition 2.1.4 in [18]), as we do not need to consider them, save only to rule them out in the proof of Proposition 4.5.3. They arise from the fact that in characteristic 2 every symplectic bilinear form has many associated quadratic forms.

Fact 4.1.3. *Every linear Lie geometry has quantifier elimination and is $\aleph_0$-categorical.*

Proof. Lemmas 2.2.8 and 2.3.19 in [18]. □

Definition 4.1.4 (Projective Lie geometry)**.** Let L be a linear Lie geometry. Define an equivalence relation $\sim$ on $L \setminus \mathrm{acl}(\varnothing)$ by $a \sim b :\Longleftrightarrow \mathrm{acl}(a) = \mathrm{acl}(b)$, where acl denotes the usual model-theoretic algebraic closure in L. The *projectivisation* of L is then the quotient structure $(L \setminus \mathrm{acl}(\varnothing))\, /\!\sim$. A *projective Lie geometry* is a structure that is the projectivisation of some linear Lie geometry.

Remark 4.1.5 (comment after Definition 2.1.7 in [18])**.** By quantifier elimination (Fact 4.1.3), algebraic closure is just linear span and so a projective Lie geometry is a projective geometry in the usual sense.

Definition 4.1.6 (Affine Lie geometry)**.** An *affine Lie geometry* is a structure of the form $(V, A, \oplus, -)$, where V is the vector sort of a linear Lie geometry (but not a degenerate space), A is a set, $\oplus\colon V \times A \to A$ is a regular group action and $-\colon A \times A \to V$ is such that $a = v \oplus b$ implies $a - b = v$. Here 'regular'

[2] Note that this is what the prefix 'basic' refers to in Definition 2.1.6 in [18]. Since we will always name the field elements by constant symbols, we suppress this prefix.

means that for every $a, b \in A$ there exists a unique $v \in V$ such that $a = v \oplus b$. In the polar case the structure is $(V, W, A, \oplus, -)$, where $\oplus\colon V \times A \to A$ is a regular group action and $-\colon A \times A \to V$ is such that $a = v \oplus b$ implies $a - b = v$.

The notions of canonical and stable embeddedness are fundamental to Lie coordinatisation:

Definition 4.1.7. Consider an $\mathcal{L}$-structure $\mathcal{N}$ and an $\mathcal{L}'$-structure $\mathcal{M}$ such that the underlying set M is an $\mathcal{L}_N$-definable subset of N. Let $c \in \mathcal{N}^{\mathrm{eq}}$ be a canonical parameter for M. (See § 8.2 of [53] or § 8.4 of [58] for an introduction to canonical parameters.)

(i) $\mathcal{M}$ is *canonically embedded* in $\mathcal{N}$ if the $\mathcal{L}'_\varnothing$-definable relations of $\mathcal{M}$ are precisely the $\mathcal{L}_c$-definable relations on $\mathcal{M}$; that is, for every $n \in \mathbb{N}^+$, a subset $D \subseteq M^n$ is $\mathcal{L}'_\varnothing$-definable in the structure $\mathcal{M}$ if and only if it is $\mathcal{L}_c$-definable in the structure $\mathcal{N}$. (The notation $\mathcal{L}'_\varnothing$ isn't strictly necessary, since $\mathcal{L}' = \mathcal{L}'_\varnothing$, but the subscript $\varnothing$ is added to emphasise $\varnothing$-definability.)

(ii) $\mathcal{M}$ is *stably embedded* in $\mathcal{N}$ if every $\mathcal{L}_N$-definable relation on $\mathcal{M}$ is $\mathcal{L}_M$-definable in a uniform way; that is, for every $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$, where $n := l(\bar{x})$ and $m := l(\bar{y})$, if $\varphi(\mathcal{N}^n, \bar{a}) \subseteq M^n$ for every $\bar{a} \in N^m$, then there exists an $\mathcal{L}$-formula $\varphi'(\bar{x}, \bar{z})$, where $r := l(\bar{z})$, such that for every $\bar{a} \in N^m$ there exists $\bar{a}' \in M^r$ such that $\varphi(\mathcal{N}^n, \bar{a}) = \varphi'(\mathcal{N}^n, \bar{a}')$. (Note that we need not have $m = r$.)

(iii) $\mathcal{M}$ is *fully embedded* in $\mathcal{N}$ if $\mathcal{M}$ is both canonically and stably embedded in $\mathcal{N}$.

Intuitively, $\mathcal{M}$ is fully embedded in $\mathcal{N}$ if $\mathcal{N}$ cannot place any additional structure on $\mathcal{M}$.

We won't need the following definition until § 4.5, but it follows on from the previous definitions.

Definition 4.1.8 (Localisation)**.** Let P be a projective Lie geometry, arising from a linear Lie geometry L. Suppose that P is fully embedded in an $\mathcal{L}$-structure $\mathcal{M}$. The *localisation* P/A of P over a finite set $A \subset M$ is defined as follows: Let f be the bilinear/sesquilinear form on L, where for a degenerate or pure vector space we define $f(v, w) := 0$ for all $v, w \in L$ and for an orthogonal space f is the bilinear form associated to the quadratic form on L. Define

$$L_A^\perp := \{v \in L : f(v, w) = 0 \text{ for all } w \in \mathrm{acl}(A) \cap L\}$$

or, in the polar case,

$$L_A^\perp := \{v \in V : f(v,w) = 0 \text{ for all } w \in \mathrm{acl}(A) \cap W\} \\ \cup \{v \in W : f(v,w) = 0 \text{ for all } w \in \mathrm{acl}(A) \cap V\}.$$

Let $L_A^\perp/(L_A^\perp \cap \mathrm{acl}(A))$ be the quotient space, in the usual sense of a quotient of abelian groups. (This makes sense by Remark 4.1.5.) Then P/A is the projectivisation of $L_A^\perp/(L_A^\perp \cap \mathrm{acl}(A))$; that is, let $\sim$ be as in Definition 4.1.4 and then quotient $L_A^\perp/(L_A^\perp \cap \mathrm{acl}(A))$ by $\sim$.

We are now ready to state the definition of Lie coordinatisation itself:

Definition 4.1.9 (Lie coordinatisation)**.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure. A *Lie coordinatisation* of $\mathcal{M}$ is an $\mathcal{L}_\varnothing$-definable partial order $<$ of M that forms a tree of finite height with an $\mathcal{L}_\varnothing$-definable root w such that the following condition holds: For every $a \in M \setminus \{w\}$ either the immediate predecessor u of a has only finitely many immediate successors (which implies $a \in \mathrm{acl}(u)$) or, if $a \notin \mathrm{acl}(u)$, then there exist $b < a$ and an $\mathcal{L}_b$-definable projective Lie geometry J fully embedded in $\mathcal{M}$ such that either

(i) $a \in J$ or, if $a \notin J$, then

(ii) there exist $c \in M$ with $b < c < a$ and an $\mathcal{L}_c$-definable affine Lie geometry (V, A) fully embedded in $\mathcal{M}$ such that $a \in A$, the projectivisation of V is J, and $J < V < A$,

where for subsets $X, Y \subset M$ the notation $X < Y$ means that every element of X lies in a lower level of the tree than every element of Y. We call the Lie geometries J and (V, A) *coordinatising geometries*. By a *Lie coordinatised structure* we mean a structure equipped with a Lie coordinatisation.

Definition 4.1.10 (Lie coordinatisability)**.** An $\mathcal{L}$-structure $\mathcal{M}$ is *Lie coordinatisable* if it is $\varnothing$-bi-interpretable (Definition 2.4.1) with a Lie coordinatised structure that has finitely many 1-types.

Remark 4.1.11.

(i) We have actually defined so-called 'weak Lie coordinatisability' (p. 17 of [18]), since in Definition 4.1.9 we did not stipulate the orientation condition relating to quadratic coordinatising geometries (Definition 2.1.10 in [18]). This condition is important and cannot be ignored in general, but we can ignore it because we do not need to consider quadratic Lie geometries (Remark 4.1.2(ii)). For brevity we thus suppress the prefix 'weak', the sketch proof of Theorem 4.4.1 being an exception. Note that the orientation condition is also ignored in [19] for the same reason (p. 517 of [19]).

Fact 4.1.12. *If $\mathcal{M}$ is Lie coordinatisable, then $\mathcal{M}$ is $\aleph_0$-categorical.*

Proof. Lemma 2.3.19 in [18]. □

Remark 4.1.13. The distinction between Lie coordinatisation and Lie coordinatisability is important to maintain in general, but we freely move from the latter to the former by adding finitely many sorts from $\mathcal{M}^{\mathrm{eq}}$ to $\mathcal{M}$.

The following is one of the deep results of [18]:

Fact 4.1.14. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Then $\mathcal{M}$ is Lie coordinatisable if and only if $\mathcal{M}$ is smoothly approximable.*

Proof. Theorem 2 in [18]. □

4.2 Examples

We give two examples of Lie coordinatisable structures, returning to Examples 3.1.4 and 3.1.5. This is no coincidence, as shown by Fact 4.1.14.

Example 4.2.1. Consider a language $\mathcal{L} := \{I_1, I_2\}$, where I_1 and I_2 are binary relation symbols. Let $\mathcal{M}$ be a countable $\mathcal{L}$-structure where $I_1^{\mathcal{M}}$ and $I_2^{\mathcal{M}}$ are equivalence relations such that $I_1^{\mathcal{M}}$ has infinitely many classes, $I_2^{\mathcal{M}}$ refines $I_1^{\mathcal{M}}$, every I_1-equivalence class contains infinitely many I_2-equivalence classes, and every I_2-equivalence class is infinite; that is, $\mathcal{M}$ is partitioned into infinitely many I_1-equivalence classes, each of which is then partitioned into infinitely many I_2-equivalence classes, each of which is infinite. We claim that $\mathcal{M}$ is Lie coordinatisable.

We first outline the tree structure. At the root we place $\ulcorner\mathcal{M}\urcorner$ (the canonical parameter of $\mathcal{M}$ in $\mathcal{M}^{\mathrm{eq}}$, which is $\varnothing$-definable), above which we place the I_1-classes, as imaginary elements of $\mathcal{M}^{\mathrm{eq}}$. Above each I_1-class we then place the I_2-classes, again as imaginary elements of $\mathcal{M}^{\mathrm{eq}}$, with every I_2-class above the I_1-class in which the I_2-class is contained. Finally, above each I_2-class we place the elements of $\mathcal{M}$ contained in that I_2-class. So this tree has height 3 and infinite width at each level.

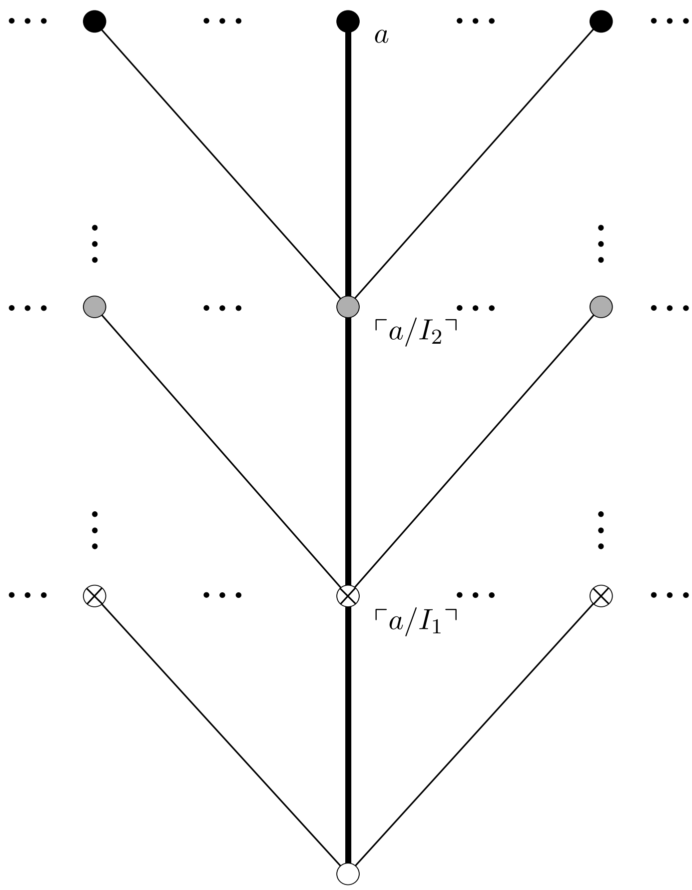

Figure 4.1: A finite fragment of the tree from Example 4.2.1, with the branch leading to the element a in bold. The nodes are shaded according to membership: The white node is $\ulcorner \mathcal{M} \urcorner$, the crossed nodes are elements of $\mathcal{M}/I_1$, the grey nodes are elements of $(a/I_1)/I_2$, and the black nodes are elements of a/I_2. The small dots represent the rest of the tree.

Let's explain the notation used in Figure 4.1. So consider some arbitrary $a \in M$. For $j = 1$ or 2, let a/I_j denote the I_j-class that contains a and let $\ulcorner a/I_j \urcorner$ denote the same I_j-class but as a member of $\mathcal{M}^{\mathrm{eq}}$; so $\ulcorner a/I_j \urcorner \in \mathcal{M}^{\mathrm{eq}}$ is a canonical parameter for the a-definable subset $a/I_j \subset M$. We define $(a/I_1)/I_2$ and $\ulcorner (a/I_1)/I_2 \urcorner$ similarly.

We now use this notation to check that Definition 4.1.9 holds for the tree. The imaginary element $\ulcorner a/I_1 \urcorner$ lies in the $\ulcorner \mathcal{M} \urcorner$-definable degenerate projective geometry $\mathcal{M}/I_1$ and $\ulcorner \mathcal{M} \urcorner < \ulcorner a/I_1 \urcorner$. The imaginary element $\ulcorner a/I_2 \urcorner$ lies in the $\ulcorner a/I_1 \urcorner$-definable degenerate projective geometry $(a/I_1)/I_2$ and $\ulcorner a/I_1 \urcorner < \ulcorner a/I_2 \urcorner$. Finally, the real element a lies in the $\ulcorner a/I_2 \urcorner$-definable degenerate projective geometry a/I_2 and $\ulcorner a/I_2 \urcorner < a$. Adjoining a finite number of sorts from $\mathcal{M}^{\mathrm{eq}}$ (recall Remark 4.1.13), each of these geometries is fully embedded in $\mathcal{M}$. (Note that $\mathcal{M}/I_2$ is *not* fully embedded, since I_1 defines extra structure on $\mathcal{M}/I_2$ that is not definable within $\mathcal{M}/I_2$ using equality alone.) So $\mathcal{M}$ is indeed Lie coordinatisable.

Remark 4.2.2. This example generalises to the case where we have n equivalence relations $I_1, \dots, I_n$ such that there are infinitely many I_1-classes, I_{j+1} refines I_j and every I_j-class contains infinitely many I_{j+1}-classes (for $1 \le j \le n-1$), and every I_n-class is infinite. At the base of the tree (the 0^{th} level) we place $\ulcorner \mathcal{M} \urcorner$. At the j^{th} level (for $1 \le j \le n-1$) we place the I_j-classes, as imaginary elements of $\mathcal{M}^{\mathrm{eq}}$, with every I_j-class above the I_{j-1}-class in which the I_j-class is contained. Finally, at the top of the tree (the n^{th} level) we place the elements of M, with each $a \in M$ placed above $\ulcorner a/I_n \urcorner$.

Example 4.2.3 (Example 2.1.11 in [18])**.** Let $\mathcal{L} := \{0, +\}$ and let p be a fixed prime number. (The case $p = 2$ *is* allowed.) We define M to be the direct sum of ω-many copies of $\mathbb{Z}/p^2\mathbb{Z}$, i.e.

$$M := \{(a_i)_{i<\omega} : a_i \in \mathbb{Z}/p^2\mathbb{Z} \text{ and } a_i = 0 \text{ for all but finitely many } i\}.$$

(We specify the direct sum because it is countable, unlike the direct product.) The set M naturally forms an $\mathcal{L}$-structure $\mathcal{M}$, the $\mathcal{L}$-structure arising component-wise from the $\mathcal{L}$-structure of the group $\mathbb{Z}/p^2\mathbb{Z}$. Explicitly: $0^{\mathcal{M}} := (0)_{i<\omega}$ and $(a_i)_{i<\omega} + (b_i)_{i<\omega} := (a_i + b_i)_{i<\omega}$. For brevity we write 0 for $0^{\mathcal{M}}$. We claim that $\mathcal{M}$ is Lie coordinatisable.

We first introduce some notation: For $v \in M$ let $\mathcal{M}_v := \{a \in M : pa = v\}$, where $pa := \underbrace{a + a + \cdots + a}_{p \text{ times}}$. Observe that $\mathcal{M}_0$ has a vector space structure over $\mathbb{F}_p$ and thus is a linear Lie geometry over $\mathbb{F}_p$. Let $P(\mathcal{M}_0)$ be the projectivisation of $\mathcal{M}_0$ (Definition 4.1.4). Then $P(\mathcal{M}_0) = \mathcal{M}_0 \setminus \{0\}/{\sim}$, where $a \sim b$ if and only if $a = rb$ for some $r \in \mathbb{F}_p$ (recall Remark 4.1.5). So $|a/{\sim}| = p - 1$ for all $a \in \mathcal{M}_0$. Adjoining a sort for $P(\mathcal{M}_0)$ (recall Remark 4.1.13), we also have that $P(\mathcal{M}_0)$ is fully embedded in $\mathcal{M}$.

We now outline the tree structure. At the root we place 0, above which we place the elements of $P(\mathcal{M}_0)$, considered as imaginary elements of $\mathcal{M}^{\mathrm{eq}}$. On the next level we place the elements of $\mathcal{M}_0 \setminus \{0\}$, with each a placed above $\ulcorner a/{\sim} \urcorner$. Finally, the top level contains the elements of $\mathcal{M} \setminus \mathcal{M}_0$, with each $b \in \mathcal{M}_a$ placed above a. So we have a tree of height 3 and infinite width at each level, although the second level comprises an infinite amount of finite

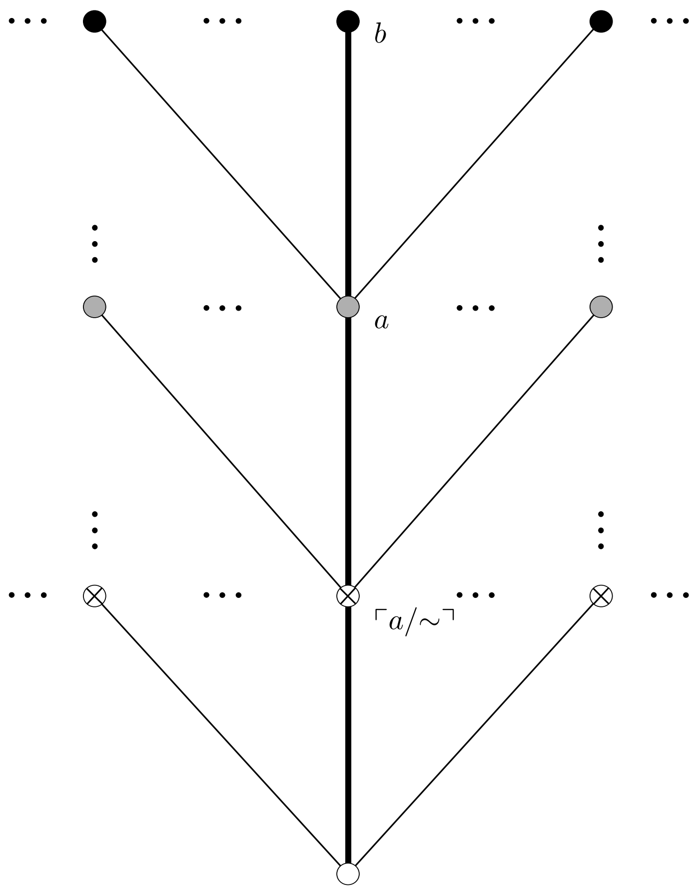

Figure 4.2: A finite fragment of the tree from Example 4.2.3, with the branch leading to the element $b \in \mathcal{M}_a$ in bold. The nodes are shaded according to membership: The white node is the zero vector, the crossed nodes are elements of $P(\mathcal{M}_0)$, the grey nodes are elements of $a/{\sim}$, and the black nodes are elements of $\mathcal{M}_a$. Note that there are only finitely many (in fact $p-1$) nodes immediately above each crossed node. The small dots represent the rest of the tree.

branching. Note that we're using the fact here that if $b \in \mathcal{M} \setminus \mathcal{M}_0$, then $b \in \mathcal{M}_a$ for some $a \in \mathcal{M}_0$. Proof: Suppose $pb \neq 0$. So $pb = a$ for some $a \in M$. Then $pa = p(pb) = p^2 b = 0$, since $p^2 c = 0$ for all $c \in M$. So $b \in \mathcal{M}_a$ and $a \in \mathcal{M}_0$, as required.

Let's check that Definition 4.1.9 holds for this tree. So consider some arbitrary non-zero $a \in \mathcal{M}_0$ and $b \in \mathcal{M}_a$. See Figure 4.2 for an illustration. The imaginary element $\ulcorner a/{\sim}\urcorner$ lies in the 0-definable projective geometry $P(\mathcal{M}_0)$, which is fully embedded, as noted in the previous paragraph, and $0 < \ulcorner a/{\sim}\urcorner$. The real element a is algebraic over $\ulcorner a/{\sim}\urcorner$, since $a/{\sim}$ is $\ulcorner a/{\sim}\urcorner$-definable and finite, again as noted in the previous paragraph, and $\ulcorner a/\sim \urcorner < a$. This leaves us with the top level of the tree, which we deal with in the next paragraph.

Firstly, observe that $0 < a < b$. The real element a defines an affine geometry $(\mathcal{M}_0, \mathcal{M}_a)$, where $\mathcal{M}_0$ is the $\mathbb{F}_p$-vector space, $\mathcal{M}_a$ is the $\mathcal{M}_0$-affine space, and the action $\mathcal{M}_0 \times \mathcal{M}_a \to \mathcal{M}_a$ is given by $(u, v) \mapsto u + v$. (This action is well-defined, since $p(u+v) = pu + pv = 0 + a = a$ and so $u + v \in \mathcal{M}_a$.) As we have already noted, the projectivisation of $\mathcal{M}_0$ is $P(\mathcal{M}_0)$, which is a fully embedded, 0-definable projective geometry, and we have $b \in \mathcal{M}_a$ by assumption. So the tree structure does indeed satisfy the definition of Lie coordinatisation.

Remark 4.2.4. This example generalises to the direct sum of ω-many copies of $\mathbb{Z}/p^n\mathbb{Z}$, for any $n \in \mathbb{N}^+$. When $n = 1$, the tree structure is the same as in the case $n = 2$, except that $\mathcal{M}_0 \setminus \{0\}$ forms the top level, since $\mathcal{M} \setminus \mathcal{M}_0 = \varnothing$. When $n \geq 3$, the first three levels (0, $P(\mathcal{M}_0)$ and $\mathcal{M}_0$) are the same, but at the third level one places the elements of $\{b \in M : b \in \mathcal{M}_a \text{ for some } a \in \mathcal{M}_0\}$, instead of simply $\mathcal{M} \setminus \mathcal{M}_0$, and at the $(j+1)^{\text{th}}$ level (for $1 \leq j \leq n$) one places $\{c \in M : c \in \mathcal{M}_b \text{ for some } b \text{ in the } j^{\text{th}} \text{ level}\}$. The $(n+1)^{\text{th}}$ level is the uppermost level.

Remark 4.2.5. In both Example 4.2.1 and Example 4.2.3 the tree is nicely stratified, namely root–degenerate–degenerate–degenerate in the former and root–projective–algebraic–affine in the latter. This need not be the case, however: There are Lie coordinatising trees containing maximal chains of different lengths. For example, one could take the disjoint union (in a suitable language, with a common root) of two Lie coordinatising trees of different heights.

4.3 Standard systems of geometries and envelopes

In this section we develop the key notion of an envelope of a Lie coordinatised structure. Our presentation is a simplified version of that given in [18], streamlined for the purpose of stating and proving Proposition 4.5.3. We begin with the notion of a standard system of geometries:

Definition 4.3.1 (Standard system of geometries)**.** Let $\mathcal{M}$ be a Lie coordinatised $\mathcal{L}$-structure. A *standard system of geometries* on $\mathcal{M}$ is a $\varnothing$-definable function $J \colon A \to \mathcal{P}(M)$ whose domain A is the set of realisations of a 1-type over $\varnothing$ in $\mathcal{M}$ and whose image is a set of projective coordinatising Lie geometries of the same kind, i.e. $J(a)$ and $J(b)$ are isomorphic for every $a, b \in A$. By '$\varnothing$-definable' we mean that there is an $\mathcal{L}$-formula $\varphi(x, y)$ such that $\varphi(\mathcal{M}, a) = J(a)$ for every $a \in A$. We call A the *domain* of J, which we denote by $\mathrm{dom}(J)$.

Definition 4.3.2 (Dimension function)**.**

(i) Let J be a Lie geometry over a field K. An *approximation* of J is a finite-dimensional geometry over K of the same kind as J. For example, if J is the projectivisation of a pure vector space over a finite field K, then an approximation of J is a finite-dimensional projective space over K, or if J is a degenerate space, then an approximation of J is a finite set in the language of equality.

(ii) Let $\mathcal{M}$ be a Lie coordinatised structure. A *dimension function* is a function μ on a finite set of S of standard systems of geometries on $\mathcal{M}$ that assigns an approximation to each $J \in S$, i.e. $\mu(J)$ is an approximation of $J(a)$ for some $a \in \mathrm{dom}(J)$. (This is independent of the choice of a, since $J(a)$ is the same kind of projective Lie geometry for every $a \in \mathrm{dom}(J)$.) We call S the *domain* of μ, which we denote by $\mathrm{dom}(\mu)$.

Definition 4.3.3 (μ-Envelope)**.** Let $\mathcal{M}$ be a Lie coordinatised structure. Then a μ-*envelope* is a pair (E, μ) consisting of a finite subset $E \subset M$ and a dimension function μ for which the following three conditions holds:

(i) E is algebraically closed in $\mathcal{M}$. (Note that this implies that E is a substructure of $\mathcal{M}$.)

(ii) For every $a \in M \setminus E$ there exist $J \in \mathrm{dom}(\mu)$ and $b \in \mathrm{dom}(J) \cap E$ such that $\mathrm{acl}(E) \cap J(b)$ is a proper subset of $\mathrm{acl}(E, a) \cap J(b)$.

(iii) For every $J \in \mathrm{dom}(\mu)$ and for any $b \in \mathrm{dom}(J) \cap E$, $J(b) \cap E$ and $\mu(J)$ are isomorphic.

Remark 4.3.4.

(i) We often denote a μ-envelope by E, rather than (E, μ), leaving the dimension function as implicit. We similarly often use the term 'envelope', rather than 'μ-envelope'.

(ii) It may help the reader's intuition to know that envelopes form homogeneous substructures of $\mathcal{M}$ (Lemma in 3.2.4 [18]). Indeed, this is how the left-to-right direction of Fact 4.1.14 is proved (pp. 61–62 of [18]).

(iii) In general one can have countably infinite approximations and envelopes, but we do not need to consider them.

The following definition is fundamental to the work in §4.5 and arises from Definitions 3.1.1.4(5) and 5.2.1 in [18]:

Definition 4.3.5. Consider a μ-envelope (E, μ), where $\mathrm{dom}(\mu) = \{J_1, \dots, J_s\}$. For each J_i we define $d_E(J) := \dim \mu(J)$, where $\dim \mu(J) := |\mu(J)|$ if $\mu(J)$ is a pure set. We further define $d^*_E(J) := (-\sqrt{q})^{d_E(J)}$, where q is the size of the base finite field of $\mu(J)$, or, if $\mu(J)$ is a pure set, then we define $d^*_E(P) := d_E(P)$. (Taking $-\sqrt{q}$, rather than just q, does initially look strange. It is done purely for unitary spaces: see the end of the proof of Proposition 4.5.3.) Finally, we define $\bar{d}^*(E) := (d^*_E(J_1), \dots, d^*_E(J_s))$.

We illustrate the preceding definitions by returning to Examples 4.2.1 and 4.2.3:

Example 4.3.6 (continuation of Example 4.2.1)**.** Put simply, an example of an envelope in this case is a subset $E \subseteq \mathcal{M}$ that intersects a fixed number (n_1) of I_1-classes, a fixed number (n_2) of I_2-classes within each of these I_1-classes, and a fixed number (n_3) of elements within each of these I_2-classes. So, up to isomorphism, an envelope is given by a triple (n_1, n_2, n_3). Using the enumerations from Example 3.1.4, two examples of envelopes are

$$E_1 := \{a_{ijk} : 1 \le i \le 3, 1 \le j \le 6, 1 \le k \le 1\}$$

$$\text{and } E_2 := \{a_{ijk} : 19 \le i \le 21, 3 \le j \le 8, 2015 \le k \le 2015\}.$$

The triple for both E_1 and E_2 is $(n_1, n_2, n_3) = (3, 6, 1)$. Let's explain this in terms of standard systems of geometries and dimension functions.

Consider the following three standard systems of geometries:

- $J_\alpha \colon \mathcal{M} \to \mathcal{M}^{\mathrm{eq}}$, $J_\alpha(a) := \ulcorner \mathcal{M}/I_1 \urcorner$;
- $J_\beta \colon \mathcal{M} \to \mathcal{M}^{\mathrm{eq}}$, $J_\beta(a) := \ulcorner (a/I_1)/I_2 \urcorner$; and
- $J_\gamma \colon \mathcal{M} \to \mathcal{M}^{\mathrm{eq}}$, $J_\gamma(a) := \ulcorner a/I_2 \urcorner$.

A dimension function μ on $\{J_\alpha, J_\beta, J_\gamma\}$ assigns an approximation to each of $J_\alpha(a)$, $J_\beta(a)$ and $J_\gamma(a)$, where $a \in M$ is arbitrary. An approximation of a given geometry is determined by the dimension of the approximation, which in this case is equal to the size of the approximation, since all the projective geometries are degenerate. Thus μ is determined by a choice of triple (n_1, n_2, n_3), as mentioned in the previous paragraph. So, if $\mu = (n_1, n_2, n_3)$, then a μ-envelope is an envelope with associated triple (n_1, n_2, n_3). Furthermore, again because all the projective geometries in this example are degenerate, we have $\bar{d}^*(E) = (n_1, n_2, n_3)$.

Example 4.3.7 (continuation of Example 4.2.3)**.** Consider the standard system of geometries $J\colon \{0\} \to \mathcal{M}^{\mathrm{eq}}$, where $J(0) := P(\mathcal{M}_0)$. A dimension function μ on J assigns an approximation to $P(\mathcal{M}_0)$, i.e. a finite-dimensional subspace of $P(\mathcal{M}_0)$. So, as in Example 4.2.1, μ is determined by an integer. A μ-envelope is then a finite power of $\mathbb{Z}/p^2\mathbb{Z}$; that is, a subset $E := \{(a_i)_{i<\omega} \in M : a_i \neq 0$ only if $i = t_j$ for some $j\}$ given by n integers $t_1, \ldots, t_n \in \mathbb{N}^+$, where $n := \dim \mu(J)$. Thus E is determined, up to isomorphism, by n. Since the base field is $\mathbb{F}_p$, which has size p, we have $\bar{d}^*(E) = (-\sqrt{p})^n$. This is a 1-tuple because there is only one standard system in the domain of μ.

4.4 Macpherson's conjecture, short version

We now take a big step towards proving Theorem 4.6.4 by proving a shorter version, namely Corollary 4.4.2, where the existence of a multidimensional exact class is asserted but the nature of the measuring functions is not specified. We first provide a sketch proof of part 2 of Theorem 6 in [18], as this result is crucial to our proof of Corollary 4.4.2. The key ingredients needed to prove the result are contained in [18], namely Propositions 4.4.3, 4.5.1 and 8.3.2 and their proofs, but the (non-trivial) argument putting them together is not made completely explicit. We state the result in a way that is convenient for our present purposes, but it is essentially the same as the original statement in [18], the only significant difference being the use of the equivalence of Lie coordinatisation and smooth approximation (Fact 4.1.14).

Theorem 4.4.1. *Let $\mathcal{L}$ be a finite language and let $d \in \mathbb{N}^+$. Define $\mathcal{C}(\mathcal{L}, d)$ to be the class of all finite $\mathcal{L}$-structures with at most d 4-types. Then there is a finite partition $\mathcal{F}_1, \ldots, \mathcal{F}_k$ of $\mathcal{C}(\mathcal{L}, d)$ such that the $\mathcal{L}$-structures in each $\mathcal{F}_i$ smoothly approximate an $\mathcal{L}$-structure $\mathcal{F}_i^*$. Moreover, the $\mathcal{F}_i$ are definably distinguishable: For each $\mathcal{F}_i$ there exists an $\mathcal{L}$-sentence χ_i such that for all $\mathcal{M} \in \mathcal{C}(\mathcal{L}, d)$ above some minimum size, $\mathcal{M} \models \chi_i$ if and only if $\mathcal{M} \in \mathcal{F}_i$.*

Sketch proof.[3] We first show that there cannot exist infinitely many pairwise elementarily inequivalent Lie coordinatisable $\mathcal{L}$-structures with the same skeletal type, where a skeletal type is, roughly speaking, a full description of the

[3] The main argument was given by Hrushovski in email correspondence and Macpherson provided essential input by working out key details. The contribution of the present author lay in working through further details and writing up the proof.

Lie coordinatising tree structure in an extended language $\mathcal{L}_{sk}$; see § 4.2 of [18] for the full definition. So, for a contradiction, suppose that there are in fact infinitely many such $\mathcal{L}$-structures $\{\mathcal{N}_i : i < \omega\}$ with the same skeletal type S. Working in $\mathcal{L}_{sk}$, by a judicious choice of ultrafilter we can take a non-principal ultraproduct $\mathcal{N}^*$ of the $\mathcal{N}_i$ such that $\mathcal{N}^* \not\equiv \mathcal{N}_i$ for all $i < \omega$. We may assume that $\mathcal{N}^*$ is countable by moving to a countable elementary substructure. Since the skeletal type S is expressible in $\mathcal{L}_{sk}$ (this is a general fact of skeletal types, not just S) and true in each $\mathcal{N}_i$, by Łos's theorem $\mathcal{N}^*$ is Lie coordinatised and has skeletal type S. Work in chapter 4 of [18], especially Proposition 4.4.3 and its proof, shows that every Lie coordinatised structure is quasifinitely axiomatised, and thus in particular $\mathcal{N}^*$ is quasifinitely axiomatised. Put roughly, this means that $\mathrm{Th}(\mathcal{N}^*)$ is axiomatised by a sentence σ and an axiom schema of infinity, where we consider $\mathrm{Th}(\mathcal{N}^*)$ as an $\mathcal{L}'$-theory in a finite language $\mathcal{L}'$ containing $\mathcal{L}_{sk}$. This axiom schema of infinity holds for all the $\mathcal{N}_i$ because they each have the same skeletal type as $\mathcal{N}^*$. Furthermore, again by Łos's theorem, there exists $j < \omega$ such that $\mathcal{N}_j \models \sigma$. Thus $\mathcal{N}^* \equiv \mathcal{N}_j$, a contradiction.

We now return to the original class $\mathcal{C} := \mathcal{C}(\mathcal{L}, d)$. We take an infinite ultraproduct $\mathcal{U}^*$ of the structures in $\mathcal{C}$. We take this ultraproduct in a non-standard model of set theory, working with some suitable Gödel coding of formulas, which allows us to consider $\mathcal{U}^*$ as an $\mathcal{L}^*$-structure, where $\mathcal{L}^*$ is the ultrapower of the language $\mathcal{L}$; that is, $\mathcal{L}^*$ extends $\mathcal{L}$ by including infinitary formulas with nonstandard Gödel numbers, although the number of free variables in any given formula remains finite. We may again assume that $\mathcal{U}^*$ is countable by moving to a countable elementary substructure. $\mathcal{U}^*$ is 4-quasifinite (Definition 2.1.1 in [18]) and thus by Theorem 3 in [18] is weakly Lie coordinatisable. So by Proposition 7.5.4 in [18] the $\mathcal{L}$-reduct $\mathcal{U}$ of $\mathcal{U}^*$ is also weakly Lie coordinatisable. The $\mathcal{L}$-structure $\mathcal{U}$ thus has a skeletal type. By the first part of the proof there can be only finitely many pairwise elementarily inequivalent Lie coordinatisable $\mathcal{L}$-structures with this skeletal type, say $\mathcal{F}_1^*, \ldots, \mathcal{F}_k^*$. By Proposition 4.4.3 each $\mathcal{F}_i^*$ has a characteristic sentence, say χ_i. The χ_i yield a partition $\mathcal{C} = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_k$, where each χ_i is true in all $\mathcal{M} \in \mathcal{F}_i$ and false in all $\mathcal{M} \in \mathcal{F}_j$ for $j \neq i$, potentially with the exception of some small structures. Moreover, again by Proposition 4.4.3, this partition is such that each $\mathcal{M} \in \mathcal{F}_i$ is an envelope of $\mathcal{F}_i^*$ and so by work in chapter 3 of [18] the structures in $\mathcal{F}_i$ smoothly approximate $\mathcal{F}_i^*$.

Note that the work cited from chapter 4 of [18] is written in terms of Lie coordinatisability, but inspection of the proofs shows that weak Lie coordinatisability suffices. $\square$

Corollary 4.4.2 (Macpherson's conjecture, short version)**.** *For any countable language $\mathcal{L}$ and any $d \in \mathbb{N}^+$ there exists R such that the class $\mathcal{C}(\mathcal{L}, d)$ of all finite $\mathcal{L}$-structures with at most d 4-types is an R-mec in $\mathcal{L}$.*

Proof. Let $\mathcal{C} := \mathcal{C}(\mathcal{L}, d)$. The reader should recall Remark 2.1.3(vii), as we will use it at various points in this proof.

First suppose that $\mathcal{L}$ is finite. By Theorem 4.4.1, $\mathcal{C}$ can be finitely partitioned into subclasses $\mathcal{F}_1, \ldots, \mathcal{F}_k$ such that the structures in each $\mathcal{F}_i$ smoothly approximate an $\mathcal{L}$-structure $\mathcal{F}_i^*$. Thus by Proposition 3.2.1 each $\mathcal{F}_i$ is an R_i-mec in $\mathcal{L}$ for some R_i. Let $R_{\mathcal{L}} := R_1 \cup \cdots \cup R_k$. We claim that $\mathcal{C}$ is an $R_{\mathcal{L}}$-mec in $\mathcal{L}$.

We prove this claim: Let $\varphi(\bar{x}, \bar{y})$ be an $\mathcal{L}$-formula with $n := l(\bar{x})$ and $m := l(\bar{y})$. Since each $\mathcal{F}_i$ is an R_i-mec, we have a suitable finite partition Φ_i of each $\mathcal{F}_i(m)$. Then $\Phi_1 \cup \cdots \cup \Phi_k$ is a finite partition of $\mathcal{C}(m)$ and so $\mathcal{C}$ is a weak $R_{\mathcal{L}}$-mec in $\mathcal{L}$. It remains to show that the definability clause holds. We again use Theorem 4.4.1: For each $\mathcal{F}_i$ there is an $\mathcal{L}$-sentence χ_i such that $\mathcal{M} \models \chi_i$ if and only if $\mathcal{M} \in \mathcal{F}_i$, for sufficiently large $\mathcal{M}$. So, by conjoining χ_i to the defining $\mathcal{L}$-formulas of each Φ_i, we satisfy the definability clause, using Lemma 2.2.7 to deal with the finite number of potential exceptions. So the claim is proved.

Now suppose that $\mathcal{L}$ is infinite. Consider some arbitrary finite $\mathcal{L}' \subset \mathcal{L}$ and let $\mathcal{C}_{\mathcal{L}'}$ denote the class of all $\mathcal{L}'$-reducts of structures in $\mathcal{C}$. Each structure in $\mathcal{C}_{\mathcal{L}'}$ has at most d 4-types, since a reduct cannot have more types than the original structure. Thus, by the first part of the proof, $\mathcal{C}_{\mathcal{L}'}$ is an $R_{\mathcal{L}'}$-mec in $\mathcal{L}'$. (It could be the case that $\mathcal{C}_{\mathcal{L}'}$ is a proper subclass of the class of all finite $\mathcal{L}'$-structures with at most d 4-types, but that wouldn't matter, since a subclass of an R-mec is also an R-mec.) Let $\mathbb{L}$ be the set of all finite subsets of $\mathcal{L}$ and define

$$R := \bigcup_{\mathcal{L}' \in \mathbb{L}} R_{\mathcal{L}'}.$$

Then each $\mathcal{C}_{\mathcal{L}'}$ is an R-mec in $\mathcal{L}'$ by Remark 2.1.3(vii). Therefore $\mathcal{C}$ is an R-mec in $\mathcal{L}$ by Lemma 2.2.10. $\square$

Remark 4.4.3. The reader may well be wondering what's so special about 4-types. Well, firstly, if there is a bound on the number of n-types, then there is a bound on the number of k-types for all $k \leq n$. So in the statement of Theorem 4.6.4 we could replace 4-types with n-types for any $n > 4$ and the result would still go through, since there would still be a bound on the number of 4-types. As for 4 itself, this is harder to explain. One explanation is that the projective linear group preserves the cross-ratio, which is a projective invariant on 4-tuples of colinear points. The classification of finite simple groups also plays a role. The full details can be found in [35] and [48], the latter improving the original bound on 5-types in [35] to one on 4-types. Macpherson has made conjectures concerning bounds on 2- and 3-types: see Question 5.5.

4.5 Definable sets in envelopes

Corollary 4.4.2 provides no information about the structure of R, only its existence. In this section we use Lie geometries to ascertain information about the nature of R. We first need to define a notion of rank:

Definition 4.5.1 (Definition 2.2.1 in [18])**.** Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $D \subseteq M$ be a parameter-definable set. We define the *CH-rank* of D as follows:

(i) $\mathrm{rk}(D) > 0$ if and only if D is infinite.

(ii) For $n \in \mathbb{N}$, $\mathrm{rk}(D) \geq n+1$ if and only if there exist parameter-definable subsets $D_1, D_2 \subset M$ and functions $\pi\colon D_1 \to D$ and $f\colon D_1 \to D_2$ such that

- $\mathrm{rk}(\pi^{-1}(d)) = 0$ for all $d \in D$;
- $\mathrm{rk}(D_2) > 0$; and
- $\mathrm{rk}(f^{-1}(d)) \geq n$ for all $d \in D_2$.

If $\mathrm{rk}(D) > n$ for all $n \in \mathbb{N}$, then we define $\mathrm{rk}(D) = \infty$.

Remark 4.5.2. It seems likely that CH-rank is equal to SU-rank (Definition 8.6.1 in [58]). I haven't been able to prove this and there doesn't appear to be a clear answer in the literature, but note that the original paper [32] on Lie coordinatisation predates the work of Kim and Pillay on simple theories [39] and that SU-rank is used in [19]. Whether or not the two ranks are equal is a moot point for our purposes, but imagining the two ranks to be equal may help the reader's intuition.

The following result provides us with information about the sizes of definable sets in envelopes, which we will then use in § 4.6 to shed light on the structure of R in Corollary 4.4.2. It uses Definition 4.3.5 and is a generalisation of Proposition 5.2.2 in [18]. Proposition 5.2.2 is essentially about the formula $x = x$, since it concerns the sizes of envelopes, rather than the sizes of definable subsets of envelopes, and arbitrary formulas with parameters arise only as part of the proof. In contrast, Proposition 4.5.3 concerns arbitrary formulas with parameters from the outset and so more complexity arises. We also go into considerably more detail on certain points than in the proof given in [18].

Proposition 4.5.3. *Let $\mathcal{E}$ be an ordered family of envelopes of a Lie coordinatised $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathrm{dom}(\mu) = \mathrm{dom}(\mu')$ for all $(E,\mu),(E',\mu') \in \mathcal{E}$ and such that the parity and signature of orthogonal spaces are constant on the family, where by 'ordered family' we mean that for all $(E,\mu),(E',\mu') \in \mathcal{E}$ either $E \subseteq E'$ or $E' \subseteq E$. Let $\bar{a} \in M^m$ (where m is arbitrary), let $D_{\bar{a}} \subseteq M$ be an $\mathcal{L}_{\bar{a}}$-definable set and let s be the size of the common domian of the dimension functions. Then there exists a polynomial $\rho \in \mathbb{Q}[\mathbf{X}_1,\ldots,\mathbf{X}_s]$ and an integer $Q \in \mathbb{N}$ such that $|D_{\bar{a}} \cap E| = \rho(\bar{d}^*(E))$ for all $(E,\mu) \in \mathcal{E}$ with $|E| > Q$ and $\bar{a} \in E^m$.*

Remark 4.5.4. We offer a brief explanation of the parity/signature assumption; full details can be found in § 21 of [3]. The parity of a finite-dimensional orthogonal space V refers to $\dim(V)$, distinguishing between odd and even dimension. The signature refers to the quadratic form on V, there being only two possibilities (up to equivalence); in the even-dimensional case this is determined by the Witt index and in the odd-dimensional case by the hyperbolic hyperplane. The assumption is important, but its use is restricted to the calculations at the end of the proof, and there only in the orthogonal case.

Proof of Proposition 4.5.3. Let $\varphi(x,\bar{a})$ be the $\mathcal{L}_{\bar{a}}$-formula that defines $D_{\bar{a}}$. So $D_{\bar{a}} = \varphi(\mathcal{M},\bar{a})$. By Fact 4.1.12 and the Ryll-Nardzewski Theorem we may assume without loss of generality that $\varphi(x,\bar{a})$ defines the set of realisations of a 1-type $r(x)$ over $\bar{a}$ in $\mathcal{M}$; see Appendix B for the details. So $D_{\bar{a}} = r(\mathcal{M})$. Also note that since $\mathcal{E}$ is ordered by $\subseteq$, either $|D_{\bar{a}} \cap E| = \varnothing$ for all $(E,\mu) \in \mathcal{E}$ or there exists $Q \in \mathbb{N}$ such that $|D_{\bar{a}} \cap E| \neq \varnothing$ for all $(E,\mu) \in \mathcal{E}$ with $|E| > Q$. In the former case we can set $Q := 0$ and $\rho := 0$. So we henceforce assume that we are in the latter case. With these two assumptions in hand, we are now in a position to start the main line of argument. We proceed by induction on CH-rank.

First suppose that $\mathrm{rk}(D_{\bar{a}}) = 0$. Then $D_{\bar{a}}$ is finite. Let $k := |D_{\bar{a}}|$. Since $D_{\bar{a}}$ is both finite and $\bar{a}$-definable, $D_{\bar{a}} \subseteq \mathrm{acl}(\bar{a})$. Thus, since envelopes are algebraically closed (Definition 4.3.3), $D_{\bar{a}} \subseteq E$ for all $E \in \mathcal{E}$ with $\bar{a} \in E^m$. So $|D_{\bar{a}} \cap E| = |D_{\bar{a}}| = k$ for all $E \in \mathcal{E}$ with $\bar{a} \in E^m$. Hence the constant polynomial $\rho := k$ suffices.

Now consider the case $\mathrm{rk}(D_{\bar{a}}) > 0$. Then $D_{\bar{a}}$ is infinite. Assume as the induction hypothesis that the result holds for any parameter-definable subset of M with CH-rank strictly less than $\mathrm{rk}(D_{\bar{a}})$. Let $d \in D_{\bar{a}}$. For a contradiction, suppose that every step in the tree below d is algebraic; that is, if $c_0 < c_1 < \cdots < c_t = d$ is the chain leading to d, where c_0 is the root of the tree, then each c_{i+1} is algebraic over its immediate predecessor c_i. We claim that $d \in \mathrm{acl}(\varnothing)$.

We prove this claim. We proceed by induction on i to show that $c_i \in \mathrm{acl}(\varnothing)$ for every i, and so in particular $d = c_t \in \mathrm{acl}(\varnothing)$. Since the root is $\varnothing$-definable (Definition 4.1.9), $c_0 \in \mathrm{dcl}(\varnothing) \subseteq \mathrm{acl}(\varnothing)$. Now suppose that $c_i \in \mathrm{acl}(\varnothing)$. Then $c_{i+1} \in \mathrm{acl}(\mathrm{acl}(\varnothing))$, since $c_{i+1} \in \mathrm{acl}(c_i)$ by our supposition. But $\mathrm{acl}(\mathrm{acl}(\varnothing)) = \mathrm{acl}(\varnothing)$, since algebraic closure is idempotent, and hence $c_{i+1} \in \mathrm{acl}(\varnothing)$. So the claim is proved.

We now use the claim to derive a contradiction. Since $d \in \mathrm{acl}(\varnothing)$, there exists some $\mathcal{L}$-formula $\chi(x)$ such that $\mathcal{M} \models \chi(d)$ and $\chi(\mathcal{M})$ is finite. So $\chi(x) \in \mathrm{tp}(d/\varnothing) \subseteq \mathrm{tp}(d/\bar{a}) = r(x)$ and hence $D_{\bar{a}} = r(\mathcal{M}) \subseteq \chi(\mathcal{M})$ is finite, a contradiction.

So by the contradiction there exists $c \leq d$ such that c is not algebraic over its immediate predecessor. Take c to be minimal, i.e. lowest in the tree. By Definition 4.1.9 the non-algebraicity of c implies that c lies in a coordinatising geometry J, where J is b-definable for some $b < c$. The minimality of c implies that J is a projective Lie geometry, since the vector and affine parts of a coordinatising affine Lie geometry lie above the projectivisation of the vector part. Recalling Remark 4.1.2(ii), the same argument applies to quadratic geometries: The affine part Q of a coordinatising quadratic geometry, namely the set of quadratic forms on which the vector part V acts by translation, lies above V in the tree, V being a symplectic space. So the minimality of c implies that J is the projectivisation of V.

Case 1: The element b is the root. Then $b \in \mathrm{dcl}(\varnothing)$ and so J is $\varnothing$-definable. We define a set that is central to our argument:

$$S := \{(c', d') \in M^2 : \mathrm{tp}((c', d')/\bar{a}) = \mathrm{tp}((c, d)/\bar{a})\}.$$

Let S_i be the projection of S to the i^{th} coordinate. Then S_1 is the set of realisations of $\mathrm{tp}(c/\bar{a})$ and S_2 is the set of realisations of $\mathrm{tp}(d/\bar{a})$, as proved in the next paragraph. Then $S_1 \subseteq J$, since $c \in J$ and J is $\varnothing$-definable, and $S_2 = D_{\bar{a}}$, since $\mathrm{tp}(d/\bar{a}) = r(x)$.

We prove the claim that S_1 is the set of realisations of $\mathrm{tp}(c/\bar{a})$: If $c' \in S_1$, then it is immediate from the definition of S that $c' \models \mathrm{tp}(c/\bar{a})$. Now suppose that $c' \models \mathrm{tp}(c/\bar{a})$. By the Ryll-Nardzewski Theorem, $\mathcal{M}$ is saturated and thus there exists $\sigma \in \mathrm{Aut}(\mathcal{M}/\bar{a})$ such that $\sigma(c) = c'$ (see Remark B.6; we'll use this remark several more times and henceforth won't cite it explicitly). Thus $(c', \sigma(d)) = (\sigma(c), \sigma(d)) \in S$ and hence $c' \in S_1$. So the claim is proved. The proof of the claim that S_2 is the set of realisations of $\mathrm{tp}(d/\bar{a})$ proceeds symmetrically.

Let's now consider the intersection of $D_{\bar{a}}$ with an envelope. So take some arbitrary $(E, \mu) \in \mathcal{E}$ with $|E| > Q$ and $\bar{a} \in E$. Since $D_{\bar{a}} \cap E \neq \varnothing$, we may assume without loss of generality that $d \in E$, for if $d \notin E$, then we may take some $d' \in D_{\bar{a}} \cap E$ and repeat the previous arguments for this new element d'.

Define
$$S_E := \{(c', d') \in S : d' \in E\}.$$

We will use this set to calculate the size of $D_{\bar{a}} \cap E$, but we first need to go over some preliminaries. Let S_{Ei} be the projection of S_E to the i^{th} coordinate. Then $S_{E2} = S_2 \cap E = D_{\bar{a}} \cap E$. We claim that $S_{E1} = S_1 \cap E$.

We prove this claim. Let $c' \in S_{E1}$. Then $(c', d') \in S_E$ for some $d' \in E$. Now, $c' \leq d'$ and so $c' \in \mathrm{dcl}(d')$. Thus, since envelopes are algebraically closed (by definition), $c' \in E$. So $c' \in S_1 \cap E$ (since $S_{E1} \subseteq S_1$), as required. Now let $c' \in S_1 \cap E$. Let $d'' \in D \cap E$. Since $\mathrm{tp}(d''/\bar{a}) = \mathrm{tp}(d/\bar{a})$, there exists $\sigma \in \mathrm{Aut}(\mathcal{M}/\bar{a})$ such that $\sigma(d) = d''$. Let $c'' := \sigma(c)$. Then $(c'', d'') \in S_E$. By the same argument used earlier in this paragraph, $c'' \in E$. Now, $\mathrm{tp}(c''/\bar{a}) = \mathrm{tp}(c'/\bar{a})$ and so there exists $\sigma' \in \mathrm{Aut}(\mathcal{M}/\bar{a})$ such that $\sigma'(c'') = c'$. Now, since envelopes are homogeneous substructures (Lemma 3.2.4 in [18] and Definition 3.1.2) and $c', c'' \in E$, we may assume that $\sigma(E) = E$. Let $d' := \sigma'(d'')$. Then $d' \in E$, since $d'' \in E$. Hence $(c', d') \in S_E$ and so $c' \in S_{E1}$, as required. So the claim is proved.

We introduce some further definitions: For $c' \in S_1$ let $c'/S_2 := \{d' : (c', d') \in S\}$ and $c'/S_{E2} := \{d' : (c', d') \in S_E\}$, and for $d' \in S_2$ let $d'/S_1 := \{c' : (c', d') \in S\}$ and $d'/S_{E1} := \{c' : (c', d') \in S_E\}$. The sizes of the c'/S_{E2} and the d'/S_{E1} are in fact independent of c' and d', as we now show.

First consider some arbitrary $c' \in S_{E1}$. Let $D_{\bar{a}c}$ be the set of realisations of $\mathrm{tp}(d/\bar{a}c)$. Then, by the definition of S, $D_{\bar{a}c} = c/S_2$. Let $d' \in c'/S_{E2}$. Then, since $\mathrm{tp}((c', d')/\bar{a}) = \mathrm{tp}((c, d)/\bar{a})$, there exists $\sigma \in \mathrm{Aut}(\mathcal{M}/\bar{a})$ such that $\sigma(c', d') = (c, d)$. We claim that $\sigma\colon c'/S_2 \to c/S_2$ is a bijection. Injectivity is immediate. It is well-defined, since if $d'' \in c'/S_2$, then $\sigma(c', d'') = (c, \sigma(d'')) \in S$ and so $\sigma(d') \in c/S_2$. It is surjective, since if $d'' \in c/S_2$, then $\sigma^{-1}(c, d'') = (c', \sigma^{-1}(d'')) \in S$ and so $\sigma^{-1}(d'') \in c'/S_2$. So the claim is proved. Now, as mentioned previously, envelopes are homogeneous substructures. So, since $d, d' \in E$, we may assume that $\sigma(E) = E$. Thus

$$\begin{aligned} |c'/S_{E2}| &= |c'/S_2 \cap E| \\ &= |c/S_2 \cap E| \\ &= |D_{\bar{a}c} \cap E| \end{aligned} \tag{4.1}$$

for all $c' \in S_{E1}$.

Now consider some arbitrary $d' \in S_{E2}$. Since $c \leq d$, $c \in \mathrm{dcl}(d)$. Thus, since $\mathrm{tp}(d'/\bar{a}) = \mathrm{tp}(d/\bar{a})$, there exists a unique $c' \in M$ such that $(c', d') \in S$. But $d' \in E$ and so $(c', d') \in S_E$. Hence

$$|d'/S_{E1}| = 1 \tag{4.2}$$

for all $d' \in S_{E2}$.

We are now in a position to calculate the size of S_E and thereby also that of $D_{\bar{a}} \cap E$. Let's first calculate $|S_E|$ in terms of $|S_{E1}|$:

$$\begin{aligned} |S_E| &= \sum_{c' \in S_{E1}} |c'/S_{E2}| \\ &= |S_{E1}| \cdot |D_{\bar{a}c} \cap E| \quad \text{(by (4.1))}. \end{aligned} \tag{4.3}$$

And now in terms of $|S_{E2}|$:

$$\begin{aligned} |S_E| &= \sum_{d' \in S_{E2}} |d'/S_{E1}| \\ &= |S_{E2}| \qquad\qquad \text{(by (4.2))}. \end{aligned} \tag{4.4}$$

So, since $S_{E2} = D_{\bar{a}} \cap E$, (4.3) and (4.4) yield

$$|D_{\bar{a}} \cap E| = |S_{E1}| \cdot |D_{\bar{a}c} \cap E|. \tag{4.5}$$

First consider S_{E1}. We previously proved that $S_{E1} = S_1 \cap E$. We also showed that S_1 is the set of realisations of $\mathrm{tp}(c/\bar{a})$ and that S_1 is a subset of J. By the Ryll-Nardzewski Theorem, $\mathrm{tp}(c/\bar{a})$ is isolated and so S_1 is $\bar{a}$-definable. So S_1 is an $\bar{a}$-definable subset of a projective geometry. Thus, as we will show later (after Case 2), there exists a polynomial $\rho_1 \in \mathbb{Q}[\mathbf{X}_1, \ldots, \mathbf{X}_s]$ such that $\rho_1(\bar{d}^*(E)) = |S_1 \cap E|$.

Now consider $D_{\bar{a}c}$, which is a parameter-definable subset of M, again by the Ryll-Nardzewski Theorem. We have $\mathrm{rk}(D_{\bar{a}c}) < \mathrm{rk}(D_{\bar{a}})$, as proved in the following paragraph, and thus by the induction hypothesis there exists a polynomial $\rho_2 \in \mathbb{Q}[\mathbf{X}_1, \ldots, \mathbf{X}_s]$ such that $|D_{\bar{a}c} \cap E| = \rho_2(\bar{d}^*(E))$.

We prove the claim that $\mathrm{rk}(D_{\bar{a}c}) < \mathrm{rk}(D_{\bar{a}})$. Let $n := \mathrm{rk}(D_{\bar{a}c})$. We previously showed that $D_{\bar{a}c} = c/S_2$. We also showed that for every $c' \in S_1$ there exists $\sigma \in \mathrm{Aut}(\mathcal{M}/\bar{a})$ such that $\sigma(c/S_2) = c'/S_2$, which thus means $\mathrm{rk}(c'/S_2) = n$ for every $c' \in S_1$. Define $f\colon D_{\bar{a}} \to S_1$ by $f(d') := c'$, where c' is such that $(c', d') \in S$. As we showed earlier, for every $d \in S_2$ there is precisely one c' such that $(c', d') \in S$, so f is well-defined. Then, since $f^{-1}(c') = c'/S_2$, we have $\mathrm{rk}(f^{-1}(c')) = n$ for every $c' \in S_1$. Also note that $\mathrm{rk}(S_1) > 0$, since S_1 is infinite (because c is not algebraic over its immediate predecessor). Thus, taking $D := D_1 := D_{\bar{a}}$, $D_2 := S_1$ and π to be the identity map in Definition 4.5.1, we see that $\mathrm{rk}(D_{\bar{a}}) \geq n + 1 > \mathrm{rk}(D_{\bar{a}c})$. So the claim is proved.

Define $\rho := \rho_1 \cdot \rho_2$. Then (4.5) gives us the desired result:

$$\begin{aligned}|D_{\bar{a}} \cap E| &= |S_{E1}| \cdot |D_{\bar{a}c} \cap E| \\ &= \rho_1(\bar{d}^*(E)) \cdot \rho_2(\bar{d}^*(E)) \\ &= \rho(\bar{d}^*(E)).\end{aligned}$$

End of Case 1.

Case 2: The element b is not the root. Since c is minimal, b and each element below b (except the root) is algebraic over its immediate predecessor. Thus, by the same induction used earlier in the proof, $b \in \mathrm{acl}(\varnothing)$. Thus, by inspection of Definition 4.1.9, we see that we may add to $\mathcal{L}$ a constant symbol for b without affecting the Lie coordinatising tree. Adding the new constant symbol preserves the inequality $\mathrm{rk}(D_{\bar{a}c}) < \mathrm{rk}(D_{\bar{a}})$, again since $b \in \mathrm{acl}(\varnothing)$, but it makes J $\varnothing$-definable. We may thus simply repeat the argument given in Case 1 in the extended language $\mathcal{L}_b$. *End of Case 2.*

We now prove our earlier claim of the existence of a polynomial $\rho_1 \in \mathbb{Q}[\mathbf{X}_1, \ldots, \mathbf{X}_s]$ such that $\rho_1(\bar{d}^*(E)) = |S_1 \cap E|$. The set S_1 is an $\mathcal{L}_{\bar{a}}$-definable subset of J and thus, since J is fully embedded in $\mathcal{M}$, S_1 is $\bar{a}$-definable in the language of J; we may assume that $\bar{a}$ lies in J by stable embeddedness. We now consider the localisation $J/\bar{a}$ of J at $\bar{a}$ (Definition 4.1.8). J fibres over $J/\bar{a}$, where two elements lie in the same fibre if and only if they have the same algebraic closure over $\bar{a}$. These fibres all have the same finite size, where this size is determined by $\mathrm{tp}(\bar{a})$. Now, S_1 might not respect these fibres; that is, the intersection of S_1 with each fibre might vary in size. However, since the fibres are finite, there are only finitely many possible sizes for these intersections and so we can $\bar{a}$-definably partition the set of fibres according to size. We then consider the intersection of each part of the partition with E: We calculate the size of the base of the fibres, which is a $\varnothing$-definable subset of $J/\bar{a}$, and then multiply this result by the size of the fibre. We then sum these results to obtain $|S_1 \cap E|$. So, in short, by localising J at $\bar{a}$, it suffices to consider $\varnothing$-definable subsets of projective Lie geometries. It remains to do the explicit calculations in each kind of projective Lie geometry. We use quantifier elimination (Fact 4.1.3).

A projectivisation of a degenerate space. Projectivisation in this case is trivial. The only $\varnothing$-definable set is the whole space itself. (We can rule out $\varnothing$ because $D_{\bar{a}} \cap E \neq \varnothing$.) So $S_1 = J$. Thus, since $J \cap E = \mu(J)$ (Definition 4.3.3), where μ is the dimension function of E, we have $|S_1 \cap E| = d_E(J)$, as required.

A projectivisation of a pure vector space. The only $\varnothing$-definable set is again the whole space itself. So $S_1 = J$. Thus, going via the approximation of the linear space, which has dimension $\dim \mu(J) + 1$, we have

$$|S_1 \cap E| = \frac{q^{\dim \mu(J)+1} - 1}{q-1} = q^{\dim \mu(J)} + 1 = (-\sqrt{q})^{2 \dim \mu(J)} + 1 = d_E(J)^2 + 1,$$

as required.

A projectivisation of a polar space. This is the same as the vector space case, except that we can define either half of the space or the whole space. If the former, then the answer is the same as that in the vector space case. If the latter, then we multiply this answer by 2.

A projectivisation of a symplectic space. Since there is only one 1-type, this case is the same as the pure vector space case.

A projectivisation of a unitary space. The calculations can be found in the proof of Proposition 5.2.2 in [18]. Note that it is this case that forces us to consider $(-\sqrt{q})^{\dim \mu(J)}$, rather than just $q^{\dim \mu(J)}$.

An projectivisation of an orthogonal space. The calculations can again be found in the proof of Proposition 5.2.2 in [18]. Note that this is where the assumption regarding constant signature and parity is used (Remark 4.5.4). Also note that there is a small typographical error in the calculations: On p. 91 of [18] it should state $n(2i+j, \alpha) = q^i n(j, \alpha) + q^{j-1}(q^{2i} - q^i)$, the original term $q^i n(i, \alpha)$ being incorrect.

One final note: The calculations for unitary and orthogonal spaces in [18] are actually done in the linear Lie geometry, rather than in the projectivisation. However, by a similar fibering argument to the one used earlier with the localisation, this is sufficient. □

4.6 Macpherson's conjecture, full version

In this section, the last of the chapter, we introduce the notion of a polynomial exact class, enabling us to state and prove Theorem 4.6.4, the main result of this thesis.

Definition 4.6.1 (Polynomial exact class)**.** Let $\mathcal{L}$ be a language and $\mathcal{C}$ a class of finite $\mathcal{L}$-structures. Then $\mathcal{C}$ is a *polynomial exact class* in $\mathcal{L}$ if there exist

- $R \subseteq \mathbb{Q}[\mathbf{X}_1, \ldots, \mathbf{X}_k]$ for some $k \in \mathbb{N}^+$,
- $\mathcal{L}$-formulas $\delta_1(\bar{x}_1, \bar{y}_1), \ldots, \delta_k(\bar{x}_k, \bar{y}_k)$ and
- $\bar{a}_1 \in M^{l(\bar{y}_1)}, \ldots, \bar{a}_k \in M^{l(\bar{y}_k)}$ for each $\mathcal{M} \in \mathcal{C}$

such that $\mathcal{C}$ is an R-mec in $\mathcal{L}$ where

$$h(\mathcal{M}) = h\left(|\delta_1(\mathcal{M}^{l(\bar{x}_1)}, \bar{a}_1)|, \ldots, |\delta_k(\mathcal{M}^{l(\bar{x}_1)}, \bar{a}_k)|\right)$$

for every $h \in R$ and for every $\mathcal{M} \in \mathcal{C}$.

Remark 4.6.2.

(i) If we replace 'R-mec' with 'R-mac' in Definition 4.6.1, then we define a *polynomial asymptotic class*. In this case we allow polynomials with irrational coefficients.

Note that any 1-dimensional asymptotic class is a polynomial asymptotic class, since we may take δ to be the $\mathcal{L}$-formula $x = x$ and h to be the polynomial $\mu\mathbf{X}^d$, where (d, μ) is the dimension–measure pair.

(ii) Definition 4.6.1 is a working definition and may be subject to change before its final version appears in [60]. The idea is that only finitely many δ_i are needed and that they do not depend on each $\mathcal{L}$-formula.

Example 4.6.3. The class $\mathcal{C}$ of finite vector spaces in Example 2.3.2 is a polynomial asymptotic class.

Theorem 4.6.4 (Macpherson's conjecture, full version)**.** *For any countable language $\mathcal{L}$ and for any $d \in \mathbb{N}^+$ the class $\mathcal{C}(\mathcal{L}, d)$ of all finite $\mathcal{L}$-structures with at most d 4-types is a polynomial exact class in $\mathcal{L}$.*

Proof. By Corollary 4.4.2 we know that $\mathcal{C} := \mathcal{C}(\mathcal{L}, d)$ is a multidimensional exact class. It remains to show that the measuring functions are polynomial in the sense of Definition 4.6.1.

Recall our use of Theorem 4.4.1 in the proof of Corollary 4.4.2: We partitioned $\mathcal{C}$ into subclasses $\mathcal{F}_1, \ldots, \mathcal{F}_k$ such that the $\mathcal{L}$-structures in each $\mathcal{F}_i$ smoothly approximate an $\mathcal{L}$-structure $\mathcal{F}_i^*$. By the work in [18] each $\mathcal{F}_i$ is a class of envelopes for $\mathcal{F}_i^*$, which is Lie coordinatisable. So Proposition 4.5.3 implies that $\mathcal{C}$ is a polynomial exact class, since each coordinatising Lie geometry is fully embedded in and thus (by definition) also definable in $\mathcal{F}_i^*$.

We address some details arising from this proof. Firstly, by the Projection Lemma (Lemma 2.2.2) it suffices to consider $\mathcal{L}$-formulas in one object variable, as we do in Proposition 4.5.3. Secondly, by Lemma 3.2.7 the intersection $\varphi(\mathcal{F}_i^*, \bar{a}) \cap \mathcal{M}$ is equal to the relativisation $\varphi(\mathcal{M}, \bar{a})$ for all $\mathcal{M} \in \mathcal{F}_i$ above some minimum size, so by Lemma 2.2.7 it suffices to consider the intersection. Thirdly, since $\mathcal{C}$ is an exact class, rather than just an asymptotic class, the measuring functions are determined by the formula and thus it is not necessary to show that the polynomials given by Proposition 4.5.3 are uniform in the parameter $\bar{a}$; this point is important because the measuring functions cannot depend on the parameters. Lastly, the hypothesis of constant parity and signature in the statement of Proposition 4.5.3 can be satisfied by partitioning each $\mathcal{F}_i$ into (up to) four subclasses, each with constant parity and signature. $\square$

Remark 4.6.5. Theorem 4.6.4 generalises Theorem 3.8 in [50] and Proposition 4.1 in [21].

Chapter 5

Open questions

> *In re mathematica ars proponendi quaestionem pluris facienda est quam solvendi. (In mathematics the art of asking questions is more valuable than solving problems.)*
>
> Georg Cantor

This was the last of the three theses defended by Cantor at his doctoral disputation at the Humboldt University of Berlin in 1867 [54]. Whether or not the assertion is true is a matter of debate or perhaps just taste, but the importance of insightful questions in driving mathematical progress cannot be denied. In this vein, we pose a number of questions arising from the present work.

In this chapter we refer to the important model-theoretic notions of stability and (super)simplicity, which we have so far only mentioned in passing. We do not define these notions, but instead direct the reader to the vast literature on them, [10], [38] and [58] being good introductions. We also consider the notion of homogeneity, which is easier to define:

Definition 5.1. An $\mathcal{L}$-structure $\mathcal{M}$ is *homogeneous* if $\mathcal{M}$ is countable and every isomorphism between substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$.

Note that the word 'homogeneous' is overused in mathematics, especially in model theory. What we call 'homogeneous' might be called 'ultrahomogeneous' by other authors. See the comment after Definition 2.1.1 in [49].

Fact 5.2. *Let $\mathcal{L}$ be a finite relational language.*

(i) If $\mathcal{M}$ is a homogeneous $\mathcal{L}$-structure, then $\mathcal{M}$ is $\aleph_0$-categorical.

(ii) *If $\mathcal{M}$ is an $\aleph_0$-categorical $\mathcal{L}$-structure, then* $\mathrm{Th}(\mathcal{M})$ *has quantifier elimination if and only if $\mathcal{M}$ is homogeneous.*

(iii) *If $\mathcal{M}$ is a stable homogeneous $\mathcal{L}$-structure, then $\mathcal{M}$ is $\aleph_0$-stable.*

Question 5.3. By Fact 5.2 and Corollary 7.4 in [17], if $\mathcal{L}$ is a finite relational language and $\mathcal{M}$ is a stable homogeneous $\mathcal{L}$-structure, then $\mathcal{M}$ is smoothly approximable and thus by Proposition 3.2.1 is elementarily equivalent to an ultraproduct of a multidimensional exact class. Does the converse hold? That is, if $\mathcal{L}$ is a finite relational language and $\mathcal{M}$ is a homogeneous $\mathcal{L}$-structure that is elementarily equivalent to an ultraproduct of a multidimensional exact class, then is $\mathcal{M}$ necessarily stable?

Recalling Remark 2.3.15, answering this question might shed some light on the role in Theorem 7.5.6 in [18] of the generic bipartite graph, which is neither stable nor smoothly approximable.

Question 5.4. Is an infinite ultraproduct of a multidimensional asymptotic class necessarily simple?

We cannot guarantee supersimplicity: Consider the subclass

$$\mathcal{C}' := \{(\mathbb{Z}/p^n\mathbb{Z})^n : p \text{ is prime and } n \in \mathbb{N}^+\} \subset \mathcal{C},$$

where $\mathcal{C}$ is the class of homocyclic groups in Example 2.3.8. The class $\mathcal{C}'$ is an R-mac by Remark 2.1.3(vii), but Theorem 4.4.1 in [25] states that any infinite ultraproduct of $\mathcal{C}'$ is stable but not superstable. (Note that a structure is superstable if and only if it is both stable and supersimple.) This is in contrast to the context of N-dimensional asymptotic classes, as shown by Corollary 2.8 in [21], which states that any infinite ultraproduct of an N-dimensional asymptotic class is supersimple of D-rank at most N.

Question 5.5. Macpherson conjectures that if $\mathcal{L}$ is a finite relational language and $\mathcal{C}$ is a class of finite $\mathcal{L}$-structures with a bound on the number of 2-types, then $\mathcal{C}$ is a multidimensional asymptotic class. Can we prove this?

This conjectured variation of Theorem 4.6.4 is based on the work in [35], [48] and [18], as discussed in Remark 4.4.3. The change from exact classes to asymptotic classes is due to the potential emergence of finite fields, which form only an asymptotic class, not an exact class. We sketch an example: Consider the projective special linear group $\mathrm{PSL}_3(q)$. This acts on $\mathrm{PG}_2(q)$, the set of 1-spaces of $\mathbb{F}_q^3$, via its action on $\mathbb{F}_q^3$. The action is 2-transitive, i.e. for any two pairs $(a, b), (c, d) \in \mathrm{PG}_2(q)$ with $a \neq b$ and $c \neq d$ there exists $g \in \mathrm{PSL}_3(q)$ such that $g(a, b) = (c, d)$. Moreover, the action also preserves the ternary relation of collinearity; that is, if three 1-spaces lie in a 2-space, then their images also lie in a 2-space. So elements of $\mathrm{PSL}_3(q)$ are automorphisms of $\mathrm{PG}_2(q)$ and thus, by 2-transitivity, there is only one 2-type in $\mathrm{PG}_2(q)$. Now consider $\mathrm{PG}_2(q)$ as a structure with the ternary collinearity relation. In this structure one can interpret the set of 2-spaces and hence the projective plane, namely the set of 1-spaces, the set of 2-spaces and an incidence relation between them, given by containment. The projective plane is Desarguesian and so by a classical result the field is uniformly parameter-interpretable in the projective plane (see pp. 222–223 of [31]).

The following weaker conjecture of Macpherson should be easier to prove: If $\mathcal{L}$ is a finite relational language and $\mathcal{C}$ is a class of finite $\mathcal{L}$-structures with a bound on the number of 3-types such that the automorphism group is primitive for cofinitely many structures in $\mathcal{C}$, then $\mathcal{C}$ is a multidimensional asymptotic class, where by 'primitive' we mean that the group acts transitively and preserves no proper non-trivial partition.

Question 5.6. How does the work of Bello Aguirre on finite residue rings in [6] and [7] generalise to R-macs?

Bello Aguirre has shown that for any $l \in \mathbb{N}^+$ the class $\{(\mathbb{Z}/p^l\mathbb{Z}) : p \text{ prime}\}$ is an l-dimensional asymptotic class in the language of rings. He does this by coordinatising each $(\mathbb{Z}/p^l\mathbb{Z})$ by $(\mathbb{Z}/p\mathbb{Z})$ and an asymptotic fragment, a notion he developed for this purpose, and then applying Theorem 1.1.1. Using the method of disjoint unions employed in the proof of Example 2.3.6, his result thus shows that for any $l_1, \ldots, l_k \in \mathbb{N}^+$ the class $\{(\mathbb{Z}/p^{l_1}\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/p^{l_k}\mathbb{Z}) : p_1, \ldots, p_k \text{ prime}\}$ is a multidimensional asymptotic class in the language of rings adjoined with a unary predicate for each part of the direct sum. His notions of coordinatisation and asymptotic fragments appear to generalise to multidimensional asymptotic classes, which he and the present author hope to prove.

The following two questions were suggested to the present author by Ivan Tomašić.

Question 5.7. What is the relationship between the work of Krajíček, Scanlon and others on Euler characteristics and R-macs and R-mecs? [42], [57], [41], [56], [59]

The notion of a generalised measurable structure, as developed in [2], also appears to be related, but a thorough investigation has yet to be carried out.

Question 5.8. What are the interactions between polynomial exact classes and varieties with a polynomial number of points over finite fields?

The work of Brion and Peyre in [8] would be a good starting point for research into this question, as it suggests that algebraic varieties homogeneous under a linear algebraic group may provide a generic example of a polynomial exact class.

Appendix A

Little-o exchange

Lemma A.1. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$. If $|f(x)-g(x)| = o(g(x))$, then $|f(x)-g(x)| = o(f(x))$.*

Proof. By way of contradiction, suppose that $|f(x)-g(x)| \neq o(f(x))$. Then there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ there exists $x > \delta$ such that $|f(x)-g(x)| > \varepsilon_0|f(x)|$. Now, since $|f(x)-g(x)| = o(g(x))$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x)-g(x)| \leq \varepsilon|g(x)|$ for all $x > \delta$. So, in particular, for every $n \in \mathbb{N}^+$ we can find δ_n such that $|f(x)-g(x)| \leq \frac{\varepsilon_0}{n}|g(x)|$ for all $x > \delta_n$. Since $|f(x)-g(x)| \neq o(f(x))$, for every $n \in \mathbb{N}^+$ we can find $x_n > \delta_n$ such that $|f(x_n)-g(x_n)| > \varepsilon_0|f(x_n)|$. So we have

$$\varepsilon_0|f(x_n)| < |f(x_n)-g(x_n)| \leq \frac{\varepsilon_0|g(x_n)|}{n} \tag{A.1}$$

for every $n \in \mathbb{N}^+$. Multiplying by $\frac{1}{\varepsilon_0|g(x_n)|}$ yields

$$\frac{|f(x_n)|}{|g(x_n)|} < \frac{|f(x_n)-g(x_n)|}{\varepsilon_0|g(x_n)|} \leq \frac{1}{n}.$$

(We haven't divided by zero because (A.1) implies $|g(x_n)| > 0$.) Thus $\frac{|f(x_n)|}{|g(x_n)|} \to 0$ and $\frac{|f(x_n)-g(x_n)|}{|g(x_n)|} \to 0$ and hence

$$\frac{|f(x_n)|}{|g(x_n)|} + \frac{|f(x_n)-g(x_n)|}{|g(x_n)|} \to 0 + 0 = 0. \tag{A.2}$$

But $|g(x_n)| - |f(x_n)| \leq |f(x_n)-g(x_n)|$ by the triangle inequality and so

$$1 = \frac{|f(x_n)|}{|g(x_n)|} + \frac{|g(x_n)|-|f(x_n)|}{|g(x_n)|} \leq \frac{|f(x_n)|}{|g(x_n)|} + \frac{|f(x_n)-g(x_n)|}{|g(x_n)|},$$

which contradicts the limit in (A.2). $\square$

Remark A.2. For simplicity I have stated the lemma in terms of functions from $\mathbb{R}$ to $\mathbb{R}$, rather than in terms of the functions in the proof of Proposition 2.4.6(i), which is where little-o exchange is used. However, the lemma and its proof can be straightforwardly adapted by taking the domain of both f and g to be $\mathcal{C}$ (partially ordered by size) and the codomain to be $\mathbb{R}^{\geq 0}$.

Appendix B

The Ryll-Nardzewski Theorem

In this appendix we discuss types, orbits and the Ryll-Nardzewski Theorem.[1] Our goal is to explain how this theorem is used in § 3.2 and in the proof of Proposition 4.5.3. Our treatment will be succinct and most proofs will be omitted or only sketched, as the material is well established in the model-theoretic literature, e.g. § 1.3 of [23], § 5 of [36], § 4.4 of [53] and § 4.3 of [58].

Definition B.1. An $\mathcal{L}$-theory T is $\aleph_0$-*categorical*[2] if T has exactly one countable model up to isomorphism. An $\mathcal{L}$-structure $\mathcal{M}$ is $\aleph_0$-*categorical* if $\mathrm{Th}(\mathcal{M})$ is $\aleph_0$-*categorical*.

There are many equivalent ways of stating the Ryll-Nardzewski Theorem. We use Theorem 4.3.1 in [58]:

Definition B.2. Let T be an $\mathcal{L}$-theory. Two $\mathcal{L}$-formulas $\varphi(x_1, \ldots, x_n)$ and $\psi(x_1, \ldots, x_n)$ are *equivalent in* T if $\forall x_1 \ldots \forall x_n\, (\varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)) \in T$. Note that the terminology *equivalent modulo* T is also used.

Theorem B.3 (Ryll-Nardzewski)**.** *Let T be a countable $\mathcal{L}$-theory. Then T is $\aleph_0$-categorical if and only if for every $n \in \mathbb{N}$ there are only finitely many $\mathcal{L}$-formulas $\varphi(x_1, \ldots, x_n)$ up to equivalence in T.*

One can use this result to prove a number of equivalences to $\aleph_0$-categoricity (Corollary B.5 below), but we first state several prerequisite definitions:

[1] The theorem is sometimes called the 'Engeler–Ryll-Nardzewski–Svenonius Theorem', since each author independently published a characterisation of countable $\aleph_0$-categorical theories in 1959; see p. 541 of [27] or Theorem 7.3.1 in [31]. We will use the term 'Ryll-Nardzewski Theorem' for brevity and to follow the theorem's most common name in the literature.

[2] The terms '$\aleph_0$-categorical' and 'ω-categorical' are used completely synonymously by many model theorists (no doubt to many other logicians' frustration).

Definition B.4. Let T be an $\mathcal{L}$-theory and $\mathcal{M}$ an $\mathcal{L}$-structure.

(i) Let Ψ be a set of $\mathcal{L}$-formulas in free variables $x_1, \ldots, x_n$. Define $\bar{x} := (x_1, \ldots, x_n)$. Ψ is *satisfiable* if there exist an $\mathcal{L}$-structure $\mathcal{M}$ and $\bar{a} \in M^n$ such that $\mathcal{M} \models \psi(\bar{a})$ for every $\psi(\bar{x}) \in \Psi$. For a given $\mathcal{L}$-structure $\mathcal{M}$, Ψ is *satisfiable in* $\mathcal{M}$ if there exists $\bar{a} \in M^n$ such that $\mathcal{M} \models \psi(\bar{a})$ for every $\psi(\bar{x}) \in \Psi$. Ψ is *finitely satisfiable* (*in* $\mathcal{M}$) if every finite subset of Ψ is satisfiable (in $\mathcal{M}$).

Recall that the compactness theorem states that satisfiable and finitely satisfiable are equivalent and that the completeness theorem states that satisfiable and consistent (= no contradiction can be derived in a formal proof system) are equivalent.

(ii) An *n-type in T* is a finitely satisfiable set p of $\mathcal{L}$-formulas with free variables among $x_1, \ldots, x_n$ such that the following two conditions hold: (1) $T \subseteq p$; and (2) for every $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$, either $\varphi(x_1, \ldots, x_n) \in p$ or $\neg\varphi(x_1, \ldots, x_n) \in p$. We call condition (2) *maximal consistency*. For $A \subseteq M$, an *n-type over A in $\mathcal{M}$*, or just an *n-type in $\mathcal{M}$* if $A = \varnothing$, is an n-type in the $\mathcal{L}_A$-theory $\mathrm{Th}(\mathcal{M}, a)_{a \in A}$. Note that we often write $p(x_1, \ldots, x_n)$ or $p(\bar{x})$ to emphasise the free variables and that we may sometimes omit the prefix n, writing only *type*, if n is understood. The suffixes *in T*, *over A* and *in $\mathcal{M}$* may also be omitted.

Claim. An n-type over A in $\mathcal{M}$ is finitely satisfiable if and only if it is finitely satisfiable in $\mathcal{M}$.

Proof. The right-to-left direction is immediate. We prove the contrapositive of the left-to-right direction: Suppose that an n-type $p(\bar{x})$ over A in $\mathcal{M}$ is not finitely satisfiable in $\mathcal{M}$. Then there exists a finite subset $\{\psi_1(\bar{x}), \ldots, \psi_k(\bar{x})\} \subset p$ such that for every $\bar{a} \in M^n$, $\mathcal{M} \models \neg \bigwedge_i \psi_i(\bar{a})$. So $\neg\exists\bar{x} \bigwedge_i \psi_i(\bar{x}) \in \mathrm{Th}(\mathcal{M}, a)_{a \in A}$. Thus, since $\mathrm{Th}(\mathcal{M}, a)_{a \in A} \subset p$, $\{\psi_1(\bar{x}), \ldots, \psi_k(\bar{x}), \neg\exists\bar{x} \bigwedge_i \psi_i(\bar{x})\}$ is a finite subset of p that is not satisfiable. $\square$

(iii) $\mathrm{S}_n(T)$ denotes the set of all n-types in T.

(iv) If $A \subseteq M$ and $\bar{c} \in M^n$, then the *type of $\bar{c}$ over A in $\mathcal{M}$* is the set

$$\mathrm{tp}^{\mathcal{M}}(\bar{c}/A) := \{\psi(\bar{x}) : \psi(\bar{x}) \text{ is an } \mathcal{L}_A\text{-formula and } \mathcal{M} \models \psi(\bar{c})\},$$

which is straightforwardly shown to be an n-type. If $A = \varnothing$, then we may write $\mathrm{tp}^{\mathcal{M}}(\bar{c})$ and refer to this type as the *type of $\bar{c}$ in $\mathcal{M}$*. We may drop the suffix *in $\mathcal{M}$* and use the notation $\mathrm{tp}(\bar{c}/A)$ if $\mathcal{M}$ is understood.

(v) An n-type $p(\bar{x})$ is *realised* in $\mathcal{M}$ if there exists $\bar{c} \in M^n$ such that $\mathcal{M} \models p(\bar{c})$, i.e. $\mathcal{M} \models \varphi(\bar{c})$ for every $\varphi(\bar{x}) \in p(\bar{x})$. We say that such a tuple $\bar{c}$ *realises* p and write $\bar{c} \models p$. We call $p(\mathcal{M}^n) := \{\bar{c} \in M^n : \bar{c} \models p\}$ the *set of realisations of* p *in* $\mathcal{M}$ or the *locus of* p *in* $\mathcal{M}$.[3] The suffix *in* $\mathcal{M}$ may be omitted if $\mathcal{M}$ is understood. If a type p is not realised in $\mathcal{M}$, then we say that $\mathcal{M}$ *omits* p.

Claim. Let $p(\bar{x})$ be a type over A in $\mathcal{M}$. If $\bar{b} \models p$, then $p(\bar{x}) = \mathrm{tp}(\bar{b}/A)$.

Proof. Let $\psi(\bar{x}) \in p(\bar{x})$. Since $\bar{b} \models p$, $\mathcal{M} \models \psi(\bar{b})$ and so $\psi(\bar{x}) \in \mathrm{tp}(\bar{b}/A)$. Thus $p(\bar{x}) \subseteq \mathrm{tp}(\bar{b}/A)$. Hence $p(\bar{x}) = \mathrm{tp}(\bar{b}/A)$, since p is maximally consistent. □

(vi) $\mathcal{M}$ is *κ-saturated* if for every $A \subseteq M$ with $|A| < \kappa$, every n-type over A is realised in $\mathcal{M}$. $\mathcal{M}$ is *saturated* if $\mathcal{M}$ is $|M|$-saturated.

(vii) An $\mathcal{L}$-formula $\chi(\bar{x})$ *isolates* a type $p(\bar{x})$ in T if for every $\mathcal{L}$-formula $\psi(\bar{x})$, $\psi(\bar{x}) \in p(\bar{x})$ if and only if $\forall \bar{x}\, (\chi(\bar{x}) \to \psi(\bar{x})) \in T$. We say that $p(\bar{x})$ is *isolated* if such an $\mathcal{L}$-formula exists. Note that if $\chi(\bar{x})$ isolates $p(\bar{x})$, then $\chi(\bar{x}) \in p(\bar{x})$.

(viii) $\mathrm{Aut}(\mathcal{M})$ denotes the group of automorphisms of $\mathcal{M}$, where the group operation is composition. For each $n \in \mathbb{N}^+$ define a binary relation $\sim_n$ on $\mathcal{M}^n$ by $\bar{a} \sim_n \bar{b}$ if and only if there exists $\sigma \in \mathrm{Aut}(\mathcal{M})$ such that $\sigma(\bar{a}) = \bar{b}$; this is an equivalence relation on $\mathcal{M}^n$ and we call the resulting equivalence classes *orbits*. $\mathrm{Aut}(\mathcal{M})$ acts *oligomorphically* on $\mathcal{M}$ if for every $n \in \mathbb{N}^+$, $\mathrm{Aut}(\mathcal{M})$ has only finitely many orbits on $\mathcal{M}^n$.

For $A \subseteq M$ we define two subsets of $\mathrm{Aut}(\mathcal{M})$:

- $\mathrm{Aut}(\mathcal{M}/A) := \{\sigma \in \mathrm{Aut}(\mathcal{M}) : \sigma(a) = a \text{ for all } a \in A\}$; and
- $\mathrm{Aut}_{\{A\}}(\mathcal{M}) := \{\sigma \in \mathrm{Aut}(\mathcal{M}) : \sigma(A) = A\}$,

where $\sigma(A)$ denotes the image of A under σ. So $\mathrm{Aut}(\mathcal{M}/A)$ is the set of automorphisms that fix A pointwise and $\mathrm{Aut}_{\{A\}}(\mathcal{M})$ is the set of automorphisms that fix A setwise. Note that both $\mathrm{Aut}(\mathcal{M}/A)$ and $\mathrm{Aut}_{\{A\}}(\mathcal{M})$ are subgroups of $\mathrm{Aut}(\mathcal{M})$.

[3] On page 18 of [18] the *locus* of an element $a \in M$ over a set $B \subseteq M$ is defined to be the smallest B-definable subset of M containing a. In the present context of $\aleph_0$-categoricity the two definitions are equivalent – for finite B, which is the case that we're concerned with – since $\mathrm{tp}^{\mathcal{M}}(a/B)$ is isolated and thus its set of realisations in $\mathcal{M}$ is B-definable. Note that we are using the fact that adding a finite number of parameters preserves $\aleph_0$-categoricity (Corollary 4.3.7 in [58]).

Corollary B.5. *Let $\mathcal{L}$ be a countable language, $\mathcal{M}$ a countable $\mathcal{L}$-structure and $T := \mathrm{Th}(\mathcal{M})$. Then the following are equivalent:*

(i) *T is $\aleph_0$-categorical.*

(ii) *$\mathrm{S}_n(T)$ is finite for every $n \in \mathbb{N}$.*

(iii) *For every $n \in \mathbb{N}$, $\mathcal{M}$ realises only finitely many n-types in T.*

(iv) *All types in T are isolated.*

(v) *$\mathrm{Aut}(\mathcal{M})$ acts oligomorphically on $\mathcal{M}$.*

Furthermore, (i) implies that $\mathcal{M}$ is saturated, although the converse does not hold in general.

Remark B.6. Since we make extensive use of Corollary B.5(v), especially in § 3.2, we outline the key idea: It follows from the definition of an isomorphism that in any $\mathcal{L}$-structure $\mathcal{M}$, if $\bar{a}$ and $\bar{b}$ lie in the same $\mathrm{Aut}(\mathcal{M}/A)$-orbit, then $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$. If $\mathcal{M}$ is saturated (and $|A| < |M|$), then the converse also holds (Propositions 4.2.13 and 4.3.3 in [53]). Thus in a saturated $\mathcal{L}$-structure types and orbits are in natural bijection. Saturation follows from Theorem B.3: Consider an n-type $p(\bar{x})$ in a countable $\aleph_0$-categorical $\mathcal{L}$-structure $\mathcal{M}$. By Theorem B.3 there are only a finite number of inequivalent $\mathcal{L}$-formulas in $p(\bar{x})$, say $\psi_1(\bar{x}), \ldots, \psi_k(\bar{x})$. Since types are finitely satisfiable, there exists $\bar{a} \in M^n$ such that $\mathcal{M} \models \bigwedge_i \psi_i(\bar{a})$. Thus, since every $\mathcal{L}$-formula in $p(\bar{x})$ is equivalent to one of the $\psi_i(\bar{x})$, $\bar{a} \models p$.

We will now justify the claim made in the proof of Proposition 4.5.3, namely that since the structure $\mathcal{M}$ is Lie coordinatised and hence by Fact 4.1.12 is $\aleph_0$-categorical, we may assume without loss of generality that the $\mathcal{L}_{\bar{a}}$-formula $\varphi(x, \bar{a})$ defines the locus of a 1-type over $\bar{a}$ in $\mathcal{M}$. (We're abusing notation here by conflating $\{\bar{a}\}$ and $\bar{a}$.)

Let $p_1(x, \bar{y}), \ldots, p_k(x, \bar{y})$ be the $(1+m)$-types in $\mathcal{M}$ that contain the $\mathcal{L}$-formula $\varphi(x, \bar{y})$; there are only finitely many by Corollary B.5(ii). Let $\chi_1(x, \bar{y})$, $\ldots, \chi_k(x, \bar{y})$ be the $\mathcal{L}$-formulas that isolate these types, which exist by Corollary B.5(iv).

Lemma B.7. *Let $\bar{a} \in M^m$. Then $\varphi(\mathcal{M}, \bar{a}) = \chi_1(\mathcal{M}, \bar{a}) \cup \cdots \cup \chi_k(\mathcal{M}, \bar{a})$. Furthermore, the union is disjoint: If $i \neq j$, then $\chi_i(\mathcal{M}, \bar{a}) \cap \chi_j(\mathcal{M}, \bar{a}) = \varnothing$.*

Proof. We first show that $\chi_1(\mathcal{M},\bar{a}) \cup \cdots \cup \chi_k(\mathcal{M},\bar{a}) \subseteq \varphi(\mathcal{M},\bar{a})$. So let $b \in \chi_1(\mathcal{M},\bar{a}) \cup \cdots \cup \chi_k(\mathcal{M},\bar{a})$. Then $\mathcal{M} \models \chi_i(b,\bar{a})$ for some i. Thus $\mathcal{M} \models \varphi(b,\bar{a})$, since $\varphi(x,\bar{y}) \in p_i(x,\bar{y})$ and $\chi_i(x,\bar{y})$ isolates $p_i(x,\bar{y})$. So $b \in \varphi(\mathcal{M},\bar{a})$.

We now show that $\varphi(\mathcal{M},\bar{a}) \subseteq \chi_1(\mathcal{M},\bar{a}) \cup \cdots \cup \chi_k(\mathcal{M},\bar{a})$. So let $b \in \varphi(\mathcal{M},\bar{a})$. Then $\mathcal{M} \models \varphi(b,\bar{a})$ and so $\varphi(x,\bar{y}) \in \mathrm{tp}(b,\bar{a})$. Thus $\mathrm{tp}(b,\bar{a}) = p_i(x,\bar{y})$ for some i. Hence $\mathcal{M} \models \chi_i(b,\bar{a})$, since $\chi_i(x,\bar{y}) \in p_i(x,\bar{y})$. So $b \in \chi_1(\mathcal{M},\bar{a}) \cup \cdots \cup \chi_k(\mathcal{M},\bar{a})$.

Finally, we show that the union is disjoint. We prove the contrapositive. So suppose that there exists $b \in \chi_i(\mathcal{M},\bar{a}) \cap \chi_j(\mathcal{M},\bar{a})$. Then $(b,\bar{a})$ realises both $p_i(x,\bar{y})$ and $p_j(x,\bar{y})$, since $\chi_i(x,\bar{y})$ and $\chi_j(x,\bar{y})$ isolate these types. Hence $p_i(x,\bar{y}) = \mathrm{tp}(b,\bar{a}) = p_j(x,\bar{y})$ and so $i = j$. □

This lemma partly justifies the claim: To prove that Proposition 4.5.3 holds for $\varphi(x,\bar{a})$, it suffices to prove that it holds for each $\chi_i(x,\bar{a})$, since for every $\bar{a} \in M^m$ we have

$$|\varphi(\mathcal{M},\bar{a})| = |\chi_1(\mathcal{M},\bar{a})| + \cdots + |\chi_k(\mathcal{M},\bar{a})|.$$

It remains to show that each $\chi_i(x,\bar{a})$ defines the locus of a 1-type over $\bar{a}$:

Lemma B.8. *Let $\bar{a} \in M^m$ and suppose that $\chi_i(\mathcal{M},\bar{a}) \neq \varnothing$. Then $p_i(x,\bar{a}) := \{\psi(x,\bar{a}) : \psi(x,\bar{y}) \in p_i(x,\bar{y})\}$ is a 1-type over $\bar{a}$ in $\mathcal{M}$ and is isolated by $\chi_i(x,\bar{a})$.*

Proof. We first show that $p_i(x,\bar{a})$ is finitely satisfiable. Since $\chi_i(\mathcal{M},\bar{a}) \neq \varnothing$, there exists $b \in M$ such that $\mathcal{M} \models \chi_i(b,\bar{a})$. Thus, since $\chi_i(x,\bar{y})$ isolates $p_i(x,\bar{y})$, $(b,\bar{a}) \models p_i(x,\bar{y})$ and so $p_i(x,\bar{y}) = \mathrm{tp}(b,\bar{a})$. Thus b satisfies every formula in $p_i(x,\bar{a})$ and so a fortiori $p_i(x,\bar{a})$ is finitely satisfiable.

We now show that $\mathrm{Th}(\mathcal{M},\bar{a}) \subset p_i(x,\bar{a})$. This follows from the previous paragraph: Let $\sigma(\bar{a}) \in \mathrm{Th}(\mathcal{M},\bar{a})$. Then $\sigma(\bar{y}) \in \mathrm{tp}(b,\bar{a}) = p_i(x,\bar{y})$ and hence $\sigma(\bar{a}) \in p_i(x,\bar{a})$.

To prove that $p_i(x,\bar{a})$ is a type, it remains to show that $p_i(x,\bar{a})$ is maximally consistent. So consider an arbitrary $\mathcal{L}_{\bar{a}}$-formula $\psi(x,\bar{a})$. Since $p_i(x,\bar{y})$ is maximally consistent, either $\psi(x,\bar{y}) \in p_i(x,\bar{y})$ or $\neg\psi(x,\bar{y}) \in p_i(x,\bar{y})$. Thus, by the definition of $p_i(x,\bar{a})$, either $\psi(x,\bar{a}) \in p_i(x,\bar{a})$ or $\neg\psi(x,\bar{a}) \in p_i(x,\bar{a})$, as required.

Finally, we show that $\chi_i(x,\bar{a})$ isolates $p_i(x,\bar{a})$. Since $\chi_i(x,\bar{y})$ isolates $p_i(x,\bar{y})$,

$$\psi(x,\bar{y}) \in p_i(x,\bar{y}) \iff \mathcal{M} \models \forall x \forall \bar{y}\, (\chi_i(x,\bar{y}) \to \psi(x,\bar{y})). \tag{B.1}$$

We want to prove that $\psi(x,\bar{a}) \in p_i(x,\bar{a})$ if and only if $\mathcal{M} \models \forall x\, (\chi_i(x,\bar{a}) \to \psi(x,\bar{a}))$.

First suppose that $\psi(x,\bar{a}) \in p_i(x,\bar{a})$. Then $\psi(x,\bar{y}) \in p_i(x,\bar{y})$, since $p_i(x,\bar{y})$ and $p_i(x,\bar{a})$ are types (if $\neg\psi(x,\bar{y}) \in p_i(x,\bar{y})$, then $\neg\psi(x,\bar{a}) \in p_i(x,\bar{a})$, a contradiction). Hence, by applying (B.1) to the case $\bar{y} = \bar{a}$, we have $\mathcal{M} \models \forall x\,(\chi_i(x,\bar{a}) \rightarrow \psi(x,\bar{a}))$.

Now suppose that $\mathcal{M} \models \forall x\,(\chi_i(x,\bar{a}) \rightarrow \psi(x,\bar{a}))$. For a contradiction, further suppose that $\psi(x,\bar{a}) \notin p_i(x,\bar{a})$. Then $\neg\psi(x,\bar{a}) \in p_i(x,\bar{a})$, since $p_i(x,\bar{a})$ is maximally consistent. Hence $\neg\psi(x,\bar{y}) \in p_i(x,\bar{y})$ (by the argument given in the previous paragraph) and so $\mathcal{M} \models \forall x \forall \bar{y}\,(\chi_i(x,\bar{y}) \rightarrow \neg\psi(x,\bar{y}))$ by (B.1). So in particular $\mathcal{M} \models \forall x\,(\chi_i(x,\bar{a}) \rightarrow \neg\psi(x,\bar{a}))$. But $\mathcal{M} \models \chi_i(b,\bar{a})$ and thus $\mathcal{M} \models \psi(b,\bar{a}) \wedge \neg\psi(b,\bar{a})$, a contradiction. $\square$

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