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Continuous and Discrete-Time Sliding Mode Control Design Techniques

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For

my wife, daughter and son, Mitra, Mona and Mahyar

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the memory of my dear father and youngest brother, Dr. Hassan Jafari Koshkouei, who was ill when I was researching Chapters 8 and 9, and who died when he was only 25 years old.

Summary

Sliding mode control is a well-known approach to the problem of the control of uncertain systems, since it is invariant to a class of parameter variations. Well-established investigations have shown that the sliding mode controller/observer is a good approach from the point of view of robustness, implementation, numerical stability, applicability, ease of design tuning and overall evaluation.

In the sliding mode control approach, the controller and/or observer is designed so that the state trajectory converges to a surface named the sliding surface. It is desired to design the sliding surface so that the system stability is achieved.

Many *new* methods and design techniques for the sliding controller/observer are presented in this thesis.

LQ frequency shaping sliding mode is a way of designing the sliding surface and control. By using this method, corresponding to the weighting functions in conventional quadratic performance, a compensator can be designed.

The intention of observer design is to find an estimate for the state and, if the input is unknown, estimate a suitable input. Using the sliding control input form, a suitable estimated input can be obtained. The significance of the observer design method in this thesis is that a discontinuous observer for full order systems, including disturbance inputs, is designed. The system may not be ideally in the sliding mode and the uncertainty may not satisfy the matching condition.

In discrete-time systems instead of having a hyperplane as in the continuous case, there is a countable set of points comprising a so-called lattice; and the surface on which these sliding points lie is named the latticewise hyperplane. Control and observer design using the discrete-time sliding mode, the robust stability of the sliding mode dynamics and the problem of stabilization of discrete-time systems are also studied.

The sliding mode control of time-delay systems is also considered. Time-delay sliding system stability is studied for the cases of full information about the delay and also lack of information. The sliding surface is delay-independent as for the traditional sliding surface, and the reaching condition is achieved by applying conventional discontinuous control.

A well-known method of control design is to specify eigenvalues in a region in the left-hand half-plane, and to design the gain feedback matrix to yield these eigenvalues. This method can also be used to design the sliding gain matrix. The regions considered in this thesis are, a sector, an infinite vertical strip, a disc, a hyperbola and the intersection of two sectors. Previous erroneous results are rectified and new theory developed.

The complex Riccati equation, positivity of a complex matrix and the control of complex systems are significant problems which arise in many control theory problems and are discussed in this thesis.

Contents

List of Figures

List of Tables

Notation

$\mathbb{N}$	:	the set of natural numbers
$\mathbb{Z}$	:	the set of integer numbers
$\mathbb{R}$	:	the set of real numbers
$\mathbb{C}$	:	the set of complex numbers
$\mathbb{R}_+$	:	the set of nonnegative real numbers
$\bar{\mathbb{C}}^+$	:	the set of all complex numbers with nonnegative real parts
$\Re(\cdot)$	:	the real part of the complex number $(\cdot)$
$\Im(\cdot)$	:	the imaginary part of the complex number $(\cdot)$
$\mathrm{Tr}(\cdot)$	:	the trace of matrix $(\cdot)$.
$M_n(\mathbb{R})$	:	the real vector space of real $n \times n$ square matrices
$M_n(\mathbb{C})$	:	the real vector space of complex $n \times n$ square matrices
$(\cdot)^\perp$	:	orthogonal complement
$\bar{P}$	:	the complex conjugate P
$\lVert \cdot \rVert$	:	Euclidean norm, the norm of a vector $\cdot$
$\lVert(\cdot)\rVert$	:	the norm of a matrix $(\cdot)$, i.e. the maximum singular value of $(\cdot)$
$\sigma_m(\cdot)$	:	the minimum singular value of $(\cdot)$
$\sigma_M(\cdot)$	:	the maximum singular value of $(\cdot)$
$C([a,b],\mathbb{R}^n)$	:	class of continuous functions $\phi : [a,b] \longrightarrow \mathbb{R}^n$
$C_\circ^1\,[a,b]$	:	class of continuous and differentiable functions f on the interval $[a,b]$ with $f(0)=0, f(a)=-1$ and $f(b)=1$
I	:	Identity matrix
I_m	:	$m \times m$ identity matrix
$\otimes$	:	Kronecker product
$\mathscr{N}(\cdot)$	:	null space of matrix $(\cdot)$
$\sigma(\cdot)$	:	the spectrum of operator $(\cdot)$
$\lVert(\cdot)\rVert_P$	:	P norm of matrix $(\cdot)$ defined as $\lVert(\cdot)\rVert^2{}_P = (\cdot)^T P(\cdot)$ where P is
	:	a p.d. matrix.

$\lambda_{\min}(\cdot)$	:	minimum eigenvalue of matrix $(\cdot)$
$\lambda_{\max}(\cdot)$	:	maximum eigenvalue of matrix $(\cdot)$
$\rho(\cdot)$	:	spectral radius of the operator $(\cdot)$
$\oplus$	:	direct sum
$< v_1, v_2 >$	:	inner product of vectors v_1 and v_2
adj $(\cdot)$	:	the adjoint of $(\cdot)$
ARE	:	(real) algebraic Riccati equation
CARE	:	complex algebraic Riccati equation
DLE	:	discrete Lyapunov equation
p.d.	:	positive definite
p.d.s.	:	positive definite symmetric
span($\mathscr{B}$)	:	linear subspace spanned by the basis $\mathscr{B}$
sgn	:	signum function
u.p.d.s.	:	unique positive definite symmetric

Chapter 1

Introduction to Sliding Mode Control

1.1 Historical Developments

1.1.1 Brief History of the Genesis of Sliding Mode Control

Variable structure control (VSC) or sliding mode control (SMC) was developed extensively in Russia in the early 1960's, the term "variable structure control" (VSC) being first used in the late 1950's. Flügge-Lotz [40] was the first to present the concept of sliding motion. However, Filippov [38] was the first to consider the solution of differential equations with a discontinuous right-hand side. Filippov's pioneering work still serves as the basis for work in sliding mode control which was essentially developed by [117]-[120] and Emel'yanov [35], [36], Draženović [31] and their co-workers. Most early work concentrated on SISO linear systems in phase canonical form with discontinuous feedback gain. The research was undertaken in eastern Europe and permeated elsewhere through work such as Itkis [59] and Utkin [118]. In the 1970's SMC was extended to MIMO systems by Utkin [118]-[120], Itkis [59] and Zinober [144]. Thereafter numerous theoretical results and applications have been reported (see Zinober [145] and [146]) and many survey and tutorial papers have been published e.g. DeCarlo et al [27] and Utkin [120].

The development of SMC theory has been established for different system models including nonlinear systems, discrete-time systems, time-delay systems, stochastic systems, large scale systems and infinite-dimensional systems, and has been extended to many classes of problems such as system stabilization, tracking, adaptive and optimal control, and state observation [26], [29], [33], [34], [55], [68]-[70], [104], [106], [109], [111], [120], [135], [136], [144].

Sira-Ramírez [103] has examined and interpreted the analysis and design of the sliding mode control of affine nonlinear systems by using differential geometry. SMC theory has been developed for the generalized observability canonical form (GOCF) (Sira-Ramírez [105]). Sliding mode control provides a systematic approach to the problem of maintaining stability and consistent performance in the face of modelling and parameter imprecision and uncertainty. A sliding controller is not necessarily discontinuous [27], [111]. However, to ensure that the state of a system crosses the sliding surface the control must be discontinuous.

1.1.2 Introduction to Sliding Mode Control

Variable structure control has proved successful in practical problems. High-speed switching feedback control, switching between two particular values, drives the state trajectory onto a specified surface, the so-called sliding (switching) surface, in the state space at a finite time and maintains the state trajectory on this surface for all subsequent time. This surface is called the sliding (switching) surface because the state trajectory tends to the surface smoothly and repeatedly crosses the surface in the idealised sense. When the state trajectory is on the sliding surface, the control takes a specific value, the so-called equivalent control. When the state trajectory remains on the sliding surface for all subsequent time after a finite time instant, the system is said to be in the ideal sliding mode.

For the generation of the sliding mode, the precise value of the system parameters need not be identified. This is shown in Example 2.2.1. Specification of the sliding surface and appropriate design of the sliding control yield the system response which asymptotically tracks the desired trajectory.

There are two phases for the design of sliding mode control. The first phase is the construction of the sliding surface so that the state trajectory is directed to the sliding surface. The second phase is the construction of a sliding control law which causes the system trajectory to satisfy a set of sliding conditions for the existence and reachability of a sliding mode.

When the ideal sliding mode exists, the state trajectory is along the sliding surface after a finite time instant and remains on it for all subsequent time. This requires infinitely fast switching. In actual systems, some imperfections such as delay, hysteresis and unmodelled dynamics cause the control to display chatter motion, i.e. the control oscillates rapidly between its extreme values. In some practical examples one wishes to reduce the chattering phenomenon. One method is to consider a continuous control which

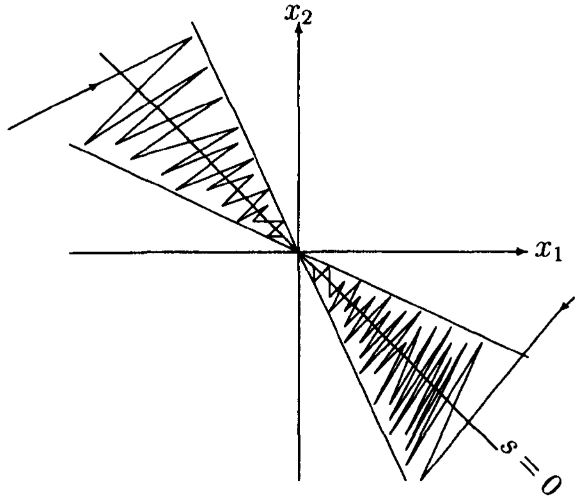

Figure 1.1: Sector region as a neighbourhood of the sliding surface, $s = 0$

has a similar structure to the discontinuous control. The state trajectory then lies within a neighbourhood of the sliding surface, the so-called boundary layer (Slotine and Sastry [109]). Note that, if the frequency of the switching is very high in comparison with the response of the system, this chattering phenomenon is often negligible. More recently, the concept of a sector (cone) layer has been presented by many authors including Furuta and Pan [45]. In the topological sense the cone (or sector) is a neighbourhood of the sliding surface and after a finite time the sliding motion lies inside this cone (sector) (see Fig. 1.1).

1.2 Applications and Methods

Numerous practical applications and theoretical studies of sliding mode control (SMC) have been demonstrated in many laboratories and papers in the last two decades throughout the world. Applications include aircraft flight, helicopter flight, spacecraft control, ship steering, turbogenerators, temperature control of an industrial furnace, robot manipulators, electrical power system, motor control, etc.

Theoretically, as already stated, when the sliding mode occurs, SMC is ideally switched at an infinite frequency, and infinitely fast chattering control preserves the state on the sliding surfaces. In this case, the velocity vectors of the state trajectories always point towards the sliding surfaces. But, in practice, the control is switched at a finite frequency. The trajectories chatter with respect to the sliding surfaces and this causes unwanted chatter motion. For example, this can produce excessive wear of mechanical parts and large heat losses in electrical circuits. Therefore to obviate the chattering, a continuous approximation to SMC in the vicinity of the sliding surfaces may be considered such that the basic VSC and properties of the system are maintained. This idea has been developed by a number of authors including Ambrosino et al [4] and Burton and Zinober [15]. To eliminate this undesired control chattering, Slotine and Sastry [109] proposed a boundary layer approach which approximates the ideal relay characteristics by a linear saturated amplifier. Some conditions guarantee that the state trajectories converge to this boundary layers and remain in the interior of this boundary layer for the subsequent time. Note that during the sliding mode the state trajectories are maintained in these layers.

Sliding control has been applied widely because of its robustness properties. During the sliding mode, the system which satisfies the so-called matching condition, is invariant to parameter variations and independent of certain disturbances. Sufficient conditions of invariance have been proposed by Draženović [31] and reconsidered by El-Ghezawi et al [34].

In the sliding mode, the order of the n order system with m inputs is reduced because the motion of the state is governed by $n-m$ slow modes provided that the sliding surface is designed for the original system. The remaining m modes are the fast modes.

1.2.1 Design Methods

There exist *several* methods for designing a sliding surface, sliding control and observer for continuous and discrete-time systems. The most important methods are:

- Using the reduced order equivalent system approach to find the feedback gain matrix so that all the eigenvalues of the reduced order sliding system are the desired eigenvalues in the left-hand half-plane. Specifying null space eigenvalues within the left-hand half-plane and designing a suitable control yields the sliding eigenvalues relating to the sliding surface (Dorling and Zinober [29]).
- Using pole assignment methods to specify a region in the left-hand half-plane within which these sliding eigenvalues must lie. These regions may have a variety of geometric shapes (Woodham and Zinober [131]).
- Using optimal control laws to yield the sliding surfaces (Young et al [135]).

- By means of the H_∞ technique a generalized system can be specified and then a sliding surface is introduced (Hashimoto and Konno [55]).
- Using output feedback (Żak and Hui [141]).
- A recent method is based on the augmented system where the sliding surface is considered as $\varphi = C(x_1) + x_2$ with $C(\cdot)$ a linear operator of a dynamic system (Young and Ozgüner [137]). In fact there are two classes of compensator for any system; a generalized class of dynamic output controllers and a generalized class of dynamic state variables.
- Another new method of designing SMC is using frequency-shaped weighting functions in a LQ cost functional. This method has been applied to a linear dynamic model of a flexible structure (Young and Ozgüner [137]).

1.3 Outline of the thesis

In Chapter 2 the basic concepts and definitions of the existence of the sliding mode and sliding control are reviewed. Also a class of sliding surface and a method for designing sliding control are presented. A straightforward method for finding the invariance conditions is also proposed. This approach yields some useful information about the system in the sliding mode and the influence of disturbance inputs on the sliding system.

In Chapter 3 the optimal sliding mode and optimal control are discussed. The sliding mode in regulator and tracking problems, and also for a class of servo-mechanism systems and reference signal systems are considered.

In Chapter 4 frequency shaping in the sliding mode is discussed. Frequency shaping is a technique for designing control and the sliding mode by using a conventional functional performance index. A new method of designing sliding mode control is presented when the LQ weighting functions are not constant for all frequencies. Furthermore, conditions for which the spectrum of the original reduced system is a subset of the spectrum of the augmented system are introduced. An iterative constructive procedure for the optimal sliding mode is developed. The sliding mode can be expressed as a linear operator of states in the form of a dynamic system.

In Chapter 5 sufficient conditions for the sliding mode control design of systems with disturbance input, and the sliding dynamics, and a method for the design of asymptotically stable sliding observers, are presented. The stability and ultimate boundedness of state reconstruction error systems via the method of Lyapunov is also studied.

In Chapter 6 the concept of the discrete-time sliding mode is clarified and sufficient conditions for the existence of the sliding mode are presented. Control design using the discrete-time sliding mode is proposed and the robust stability of the sliding mode dynamics is presented. Furthermore, the problem of stabilization of discrete-time systems is studied. The sliding mode observer of linear discrete-time systems is also discussed.

In Chapter 7 the sliding mode control of time-delay systems is considered. Time-delay sliding system stability is studied for the cases of having information about the delay and also lack of information. The sliding surface is delay-independent as for the traditional sliding surface, and the reaching condition is achieved by applying a conventional discontinuous control.

In Chapter 8 complex Lyapunov and complex and generalized Riccati equations are considered. By using these equations the sliding surface and feedback gain matrix can be found such that all the eigenvalues of the closed-loop system lie in specified regions. The work of Shieh et al [101], Woodham [129] and Woodham and Zinober [131] are studied; and errors and inaccuracies are corrected. Various illustrative examples are presented. Several new methods are proposed for all the eigenvalues of the closed-loop system to lie in the specified regions.

In Chapter 9 the real and complex matrix vector spaces are studied and also the relationship between these two matrix vector spaces are clarified. The positivity concept of a matrix is defined such that the meaning in the "real" sense is established. A method for finding the solution of the complex Riccati equation is proposed. Also complex systems (i.e. systems with complex matrices and variables) and their application are briefly considered.

In Chapter 10 conclusions and suggestions for further research are presented.

Chapter 2

Sliding Mode Control Design

2.1 Variable Structure Control Design Using the Sliding Mode

In this chapter, the properties of the sliding mode, some background and basic concepts, definitions of sliding mode control, the design of a new class of stable sliding surface, the design method of a stable sliding surface and the associated control law are studied. Although the control structure and design method are similar to previous work by Ryan and Corless [92] and Dorling and Zinober [29], the design method and structure are somewhat different. This structure is obtained by the properties of sliding mode and guarantees the stability of the sliding mode along or near the sliding surface.

2.2 Conditions for the Existence of a Sliding Mode

The existence of the sliding mode requires the state trajectories to be directed towards the sliding surface in a neighbourhood of the surface [27], [118], [121]. So, for the generation of a sliding mode, the stability of the state trajectory along or to the sliding surface is required to be asymptotic. The largest such neighbourhood is known as the attractive region.

Definition 2.2.1 [118]: The domain D in the manifold $s = 0$ is the sliding mode domain if for any $\epsilon > 0$ there exists a positive real number δ such that any motion starting within the δ-neighbourhood $(B_s(\delta))$ may leave the ϵ-neighbourhood $(B_s(\epsilon))$ only through the

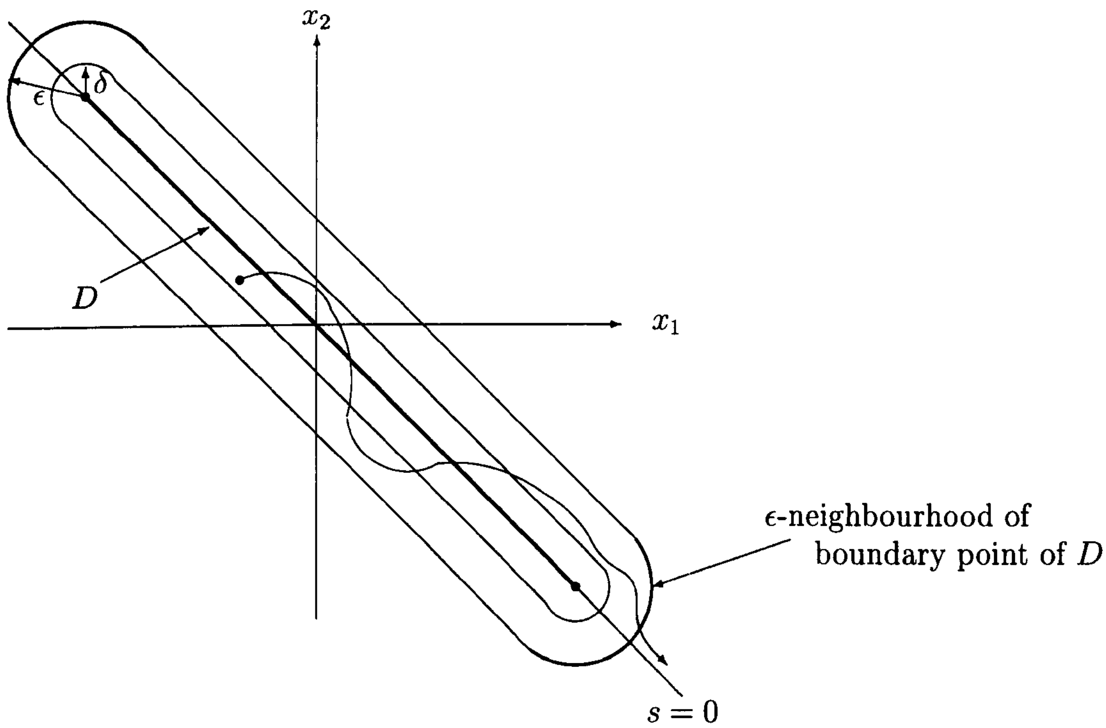

Figure 2.1: Sliding mode domain

ϵ-neighbourhood of the boundary of D (see Fig. 2.1).

The existence of a sliding mode domain can be proved by the second Lyapunov method using a generalized Lyapunov function [121]. Observe that $D \subset \{x \in \mathbb{R}^n | s(x) = 0\}$ and the domain D is an $(n - m)$-dimension domain if the subspace $\{x \in \mathbb{R}^n | s(x) = 0\}$ is an $n - m$-dimensional subspace of $\mathbb{R}^n$. The following theorem gives a sufficient condition based on the Lyapunov method.

Theorem 2.2.1 [118]: *Let Ω and D be n-dimension and $n - m$-dimension domains such that $D \subset \Omega$. A sufficient condition for D to be the sliding mode domain is that there exists an continuously differentiable function $V(t, x, s)$ satisfying the following conditions:*

1. *$V(t, x, s)$ is positive definite with respect to s, i.e. $V(t, x, s) > 0$ for arbitrary $s \neq 0$, t, x; $V(t, x, 0) = 0$ with $s = 0$; and for any real number $r \neq 0$, any t and $x \in \Omega$ the function $V(t, x, s)$ has positive infimum and supremum values on the sphere $||s|| = r$.*

2. *The time derivative V for the system has a negative supremum on Ω except for x on the sliding surface $s = 0$ where the control inputs are undefined.*

A suitable Lyapunov function is $V = \frac{1}{2}s^T s$. So for generation of the sliding mode, a sliding hyperplane with suitable control may be designed such that $\dot{V} = s^T \dot{s} < 0$. However this condition is a sufficient condition for the existence of the sliding mode and may be replaced by another condition, say $s^T P s$, where P is a p.d.s. matrix.

SMC with a discontinuous control law produces a differential equation with discontinuous right-hand side which does not satisfy the conventional theorems on the existence and uniqueness of a solution in differential equation theory. However, for the system with isolated discontinuity points or, more precisely, for the system with zero measure of discontinuity points, some analysis and synthesis methods have been achieved based on classical differential equation theory by means of point to point transformations and averaging at the occurrence of high frequency switching [119], [121]. However, in many practical problems, such as a mechanical system with Coulomb friction, the measure of the discontinuity points set is not zero. So the proof problem of existence and uniqueness of the discontinuous right-hand system arises. One way to consider this problem is that ideal sliding motion is regarded with all nonidealities tending to zero. Then the problem reduces to finding a certain system of differential equation with continuous right-hand sides, despite discontinuous control in the original system, that describes the motion in the sliding mode, i.e the system behaves in a unique way when restricted to $s = 0$ [119], [121].

There are some methods for determining the system motion in a sliding mode including methods proposed by Filippov [39] and Utkin [119]. The method of Filippov [39], which is one of the earliest and purportedly straightforward approaches, is now stated. Consider the n-th order single input system

$$\dot{x}(t) = f(t, x, u) \tag{2.1}$$

with the discontinuous control

$$u = \begin{cases} u^+(t,x) & \text{if } s(x) > 0 \\ u^-(t,x) & \text{if } s(x) < 0 \end{cases} \tag{2.2}$$

It can be shown from Filippov's work in [39] that the state trajectories of (2.1) with discontinuous control strategy (2.2) on $s = 0$ satisfy the equation

$$\dot{x}(t) = \alpha f^+ + (1-\alpha) f^- = f^{eq}, \qquad 0 \leqslant \alpha \leqslant 1$$

where $f^+ = f(t, x, u^+)$ and $f^- = f(t, x, u^-)$ (see Fig. 2.2). Thus there exists α such that $0 < \alpha < 1$, and $f^{eq} = \alpha f^+ + (1-\alpha) f^-$ is tangent to the state trajectory in the sliding mode, i.e. $< \text{grad}(s), f^{eq} > = 0$. Solving the equation $< \text{grad}(s), f^{eq} >= 0$ for

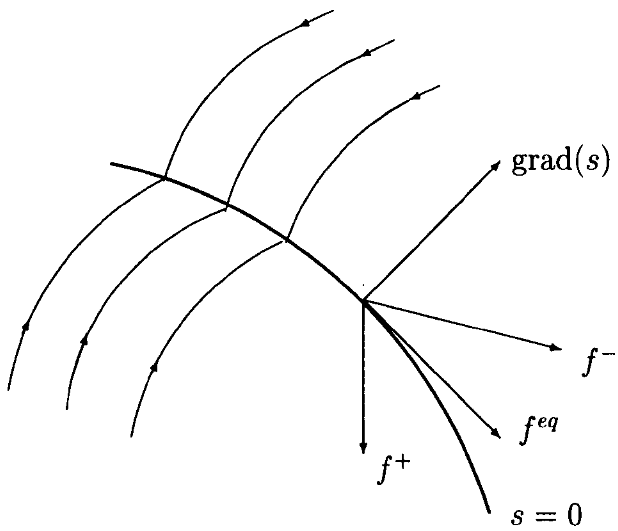

Figure 2.2: The Filippov method for determining the desired velocity vector for sliding mode motion

α yields $\alpha = \dfrac{< \text{grad}(s), f^- >}{< \text{grad}(s), (f^- - f^+) >}$ and then $1 - \alpha = -\dfrac{< \text{grad}(s), f^+ >}{< \text{grad}(s), (f^- - f^+) >}$. Since $0 \leqslant \alpha \leqslant 1$, $0 \leqslant 1 - \alpha \leqslant 1$, there are two cases:

(*i*) For $< \text{grad}(s), (f^- - f^+) > \; > 0$, one has to choose $< \text{grad}(s), f^+ > \; \leqslant 0$ and $< \text{grad}(s), f^-) > \; \geqslant 0$.

(*ii*) For $< \text{grad}(s), (f^- - f^+) > \; < 0$, the only choices are $< \text{grad}(s), f^+ > \; \geqslant 0$ and $< \text{grad}(s), f^- > \; \leqslant 0$.

Therefore the solution to (2.1) with control (2.2) exists and is uniquely defined on $s = 0$. A method of determining the system behaviour in the sliding mode has been proposed by Draženović [31] and Utkin [118]. The following corollary gives the existence of the equivalent control, i.e. control during the the sliding mode, by using the Filippov method. Moreover the relationship between the equivalent control and the actual control is given.

Corollary 2.2.1: *Consider the system (2.1) with control (2.2). Then there exist an* α $(0 < \alpha < 1)$ *such that* $f(t, x, u_{eq}) = \alpha f^+ + (1 - \alpha) f^-$ *where* u_{eq} *is the equivalent control and* $f\,(t, x, u_{eq})$ *is the velocity of the state in the sliding mode.* □

One may conclude that the equivalent control is the average of the control when the state trajectories are on the sliding surface. However, Corollary 2.2.1 shows that the velocity of the state with equivalent control is a convex value of the velocity of the state corresponding to the two different values of the control. However, if the relative degree of an n-input m-output system equals one, the equivalent control is the convex value of two different control values [65].

Example 2.2.1: Consider the following second-order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u \tag{2.3}$$

Let

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

Assume $s(x_1, x_2) = x_1 + cx_2 = 0$ with $c > 0$ and the control law is given by

$$u = \operatorname{sgn}(s) = \begin{cases} 1 & \text{if } s(x) > 0 \\ -1 & \text{if } s(x) < 0 \end{cases}$$

If $s = 0$, $\dot{s} = \dot{x}_1 + c\dot{x}_2 = 0$. Since $\dot{x}_1 = x_2$, $\dot{s} = 0$ yields $x_2(t) = x_2(\tau)e^{-(t-\tau)/c}$ where τ is the initial time and $x_2(\tau)$ is the value of the state x_2 at initial time τ. Hence $\lim_{t\to\infty} x(t) = 0$, i.e. the system in the sliding mode is asymptotically stable. In the sliding mode the equivalent control is given by

$$u_{eq} = -(CB)^{-1}CAx(t) = \frac{a_1x_1 + a_2x_2}{b} - \frac{1}{bc}x_2$$

where $C = \begin{bmatrix} 1 & c \end{bmatrix}$. If $a_2 = ca_1$, the values of a_1 and a_2 do not affect the equivalent control. In this case

$$u_{eq} = \frac{a_1(x_1 + cx_2)}{b} - \frac{1}{bc}x_2 = -\frac{x_2(t)}{bc}$$

and more precisely $u_{eq} = -\frac{1}{bc}x_2(\tau)e^{-(t-\tau)/c}$. Moreover, in this case the behaviour of the motion in the sliding mode is independent of the values of a_1 and a_2. So the system in the sliding mode is governed by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_1 + c\dot{x}_2 &= 0 \end{aligned}$$

which shows that the system in the sliding mode is independent of certain plant parameters or uncertainties.

2.3 System in the Sliding Mode

Consider the linear time-invariant system

$$\dot{x} = Ax + Bu + \Gamma\xi \tag{2.4}$$

where $x \in \mathbb{R}^n$ is the state variable, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ is full rank, $u \in \mathbb{R}^m$ is the input control, $\Gamma \in \mathbb{R}^{n\times m}$ and $\xi \in \mathbb{R}^m$ is the external disturbance input.

It is assumed that (A, B) is completely controllable, and $m < n$. Define the sliding surface as $s = Cx = 0$ where $C \in \mathbb{R}^{m\times n}$. C is selected so that CB is nonsingular, i.e. the relative degree of the system is one. The behaviour of the system in the sliding mode, when the relative degree is greater than one, has been studied in [33]. The ideal sliding mode is said to exist if there is a finite time t_s such that

$$s = Cx = 0 \qquad\qquad t \geqslant t_s \tag{2.5}$$

The sliding surface is the null space of C, i.e.

$$\mathscr{N}(C) = \{x : Cx = 0\} \tag{2.6}$$

Since CB is chosen to be nonsingular, rank(CB) is m and therefore

$$\text{col}(B) \notin \mathscr{N}(C) \tag{2.7}$$

where col(B) indicates the column vectors of the matrix B. Then there exists a nonsingular matrix Γ such that $C = \Gamma B^{\perp}$, i.e.

$$\mathscr{R}(B) \cap \mathscr{N}(C) = \{0\} \tag{2.8}$$

[29]. Since $B(CB)^{-1}C$ is projector operator and the matrices C and B are full rank, $\mathscr{R}(B(CB)^{-1}C) = \mathscr{R}(B)$ and $\mathscr{N}(B(CB)^{-1}C) = \mathscr{N}(C)$. Then

$$\mathscr{R}(B) \subseteq \mathscr{N}^{\perp}(C) \tag{2.9}$$

and

$$X = \mathscr{N}(C) \oplus \mathscr{R}(B)$$

where X is the state space.

In this thesis it is assumed that $C \in \mathbb{R}^{m\times n}$, but if $C \in \mathbb{R}^{l\times n}$ $(l < m)$ the sliding mode may be defined as $s = FCx$ where $F \in \mathbb{R}^{m\times l}$ is called the adaptation (adjustment) matrix and can be found by methods as in [32], [123] and [142]. If $m = l$, take $F = I$.

2.3.1 Invariance Condition (Matching Condition)

During the sliding mode $x \in \mathscr{N}(C)$ and from (2.5) the control does not directly affect the motion. Differentiating (2.5) and inserting (2.4) implies

$$u_{eq} = -(CB)^{-1}(CAx + C\Gamma\xi) \tag{2.10}$$

where u_{eq} is the equivalent control of the system, i.e. the effective control during the sliding mode. The actual control and the equivalent control can be considered approximately to be equal in the neighbourhood of the sliding surface. The motion of the equivalent system is

$$\dot{x} = (A - B(CB)^{-1}CA)x + (I - B(CB)^{-1}C)\Gamma\xi \tag{2.11}$$

and is independent of the disturbance if and only if

$$(I - B(CB)^{-1}C)\Gamma\xi = 0$$

Therefore, in this case it is sufficient that

$$\begin{aligned} \mathrm{col}(\Gamma) \in \mathscr{N}(I - B(CB)^{-1}C) &= \mathscr{R}(B(CB)^{-1}C) \\ &= \mathscr{R}(B) \end{aligned} \tag{2.12}$$

because CB is nonsingular [34]. Equation (2.12) is equivalent to $\mathrm{col}(\Gamma) \notin \mathscr{N}(C)$, and $\mathscr{R}(\Gamma) \subseteq \mathscr{R}(B)$ which is equivalent to $\mathrm{rank}(B,\Gamma)=\mathrm{rank}(B)$ and if Γ is full rank, then $\mathscr{R}(\Gamma) = \mathscr{R}(B)$. This relationship shows that there exists an $m \times m$ matrix Ξ such that $\Gamma = B\Xi$. This condition is known as the matching condition (or matched uncertainty) and was first presented by Draženović [31] and reconsidered by El-Ghezawi et al [33], [34].

A straightforward approach is now presented to yield further information about the system. Assume T is an orthogonal matrix such that

$$TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{2.13}$$

where B_2 is an $m \times m$ nonsingular matrix. Let $Tx = y_T$, then

$$\dot{y}_T(t) = TAT^T y_T(t) + TBu(t) + T\Gamma\xi(t) \tag{2.14}$$

Now assume

$$y_T^T = (y_1^T, y_2^T), \qquad y_1 \in \mathbb{R}^{n-m}, \quad y_2 \in \mathbb{R}^m \tag{2.15}$$

Therefore

$$\dot{y}_1(t) = A_{11}y_1(t) + A_{12}y_2(t) + \Gamma_1\xi \quad (2.16)$$

$$\dot{y}_2(t) = A_{21}y_1(t) + A_{22}y_2(t) + B_2u(t) + \Gamma_2\xi \quad (2.17)$$

where

$$TAT^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad T\Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}$$

The system in the sliding mode is independent of ξ if $\Gamma_1 = 0$. A sufficient condition for the reduced order system (2.16) to be independent of ξ is that there exists an $m \times m$ matrix D such that

$$\Gamma = BD \quad (2.18)$$

We now prove that (2.18) is satisfied if and only if $\Gamma_1 = 0$. Suppose $\Gamma = BD$, then

$$\begin{aligned} T\Gamma &= TBD \\ &= \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} D \\ &= \begin{bmatrix} 0_{(n-m)\times m} \\ B_2D \end{bmatrix} \end{aligned} \quad (2.19)$$

Therefore, $\Gamma_1 = 0$. Conversely, assume that $\Gamma_1 = 0$. Since B_2 is full rank

$$\begin{aligned} \Gamma_2 &= B_2(B_2^{-1}\Gamma_2) \\ &= B_2D \end{aligned}$$

where $D = B_2^{-1}\Gamma_2$. Then

$$\begin{aligned} T\Gamma &= \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \\ &= \begin{bmatrix} 0_{(n-m)\times m} \\ B_2D \end{bmatrix} \\ &= TBD \end{aligned}$$

Since T is an invertible matrix, $\Gamma = BD$.

However, if $m > 1$ the system (2.16) may be independent of ξ but $\Gamma_1 \neq 0$. Therefore the condition (2.18) is not a necessary condition for the independence of the reduced order system (2.16) of ξ. In the general case, the necessary and sufficient condition for independence of the system (2.16) of the perturbation signal ξ is that $\xi \in \mathcal{N}(\Gamma_1)$, where $\mathcal{N}(\Gamma_1)$ is the null space of Γ_1. If $m = 1$, Γ_1 is a real number and $\Gamma_1\xi = 0$ if and only if $\Gamma_1 = 0$. So the system in the sliding mode is independent of ξ if and only if there exists a real number ρ such that

$$\Gamma = \rho B$$

and hence $\text{rank}[B, \Gamma] = 1$.

Let $CT^T = [C_1 \;\; C_2]$. When the ideal sliding mode occurs, $s = 0$, so

$$C_1 y_1(t) + C_2 y_2(t) = 0 \qquad\qquad t \geqslant t_s \tag{2.20}$$

and then

$$y_2(t) = -F y_1(t) \tag{2.21}$$

where $F = C_2{}^{-1} C_1$. The equation (2.16) is given by

$$\dot{y}_1(t) = (A_{11} - A_{12}F) y_1(t) + \Gamma_1 \xi \tag{2.22}$$

which is known as the reduced order system and $A_{11} - A_{12}F$ has $n - m$ eigenvalues, i.e. in the sliding mode m eigenvalues of the system (2.4) are zero [34]. It is desired to find F such that $A_{11} - A_{12}F$ is a stable matrix.

2.4 Design of a Class of Sliding Surface

The sliding surface (2.20) is selected such that the stability of the nominal reduced order system (2.22) is achieved. However, a modified sliding surface (2.20) is required for many purposes such as pole placement in a specified region in the left-hand half-plane, to improve the stability performance and other aims. A sliding surface is defined by introducing a design parameter matrix. The order of the defined hyperplane is the same as (2.20) and the stability of the nominal reduced system (2.22) is preserved. It is clear that for some design parameters the system stability is achieved. The main problem is to establish the class of design matrices to ensure the desired properties. Some design methods will be proposed for obtaining a sliding design parameter r so that all the eigenvalues of $A_{11} - rA_{12}F$ lie in the left-hand half-plane and/or an infinite vertical strip in the left-hand complex plane. This problem is discussed in general in this section and a design method is presented in Section 3.2.

In this section the system (2.4) with $\Gamma = 0$ is studied. Let $C = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}$ where $\tilde{C}_2$ is nonsingular matrix. A class of sliding surfaces

$$\mathscr{S}_\alpha = \{s_\alpha \mid s_\alpha = Cx + \alpha x_2 = 0,\ \sigma_m(\tilde{C}_2) > -\alpha\}$$

is constructed which gives a suitable region in the trajectory phase plane. The design parameter α is a real number in a known range. In [121] the sliding surface has been considered by Utkin as $s = f(x_1) + x_2$ where f is an arbitrary function. So the Utkin definition is a particular case of the above definition, i.e. $s_\alpha = 0$. Assume

$$\begin{aligned} x &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad x_1 \in \mathbb{R}^{n-m}, \quad x_2 \in \mathbb{R}^m \\ A &= \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad B = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \end{aligned} \tag{2.23}$$

Then the system (2.4) is given by

$$\dot{x}_1(t) = \tilde{A}_{11}x_1(t) + \tilde{A}_{12}x_2(t) + \tilde{B}_1 u \tag{2.24}$$

$$\dot{x}_2(t) = \tilde{A}_{21}x_1(t) + \tilde{A}_{22}x_2(t) + \tilde{B}_2 u \tag{2.25}$$

When $s_\alpha = 0, Cx + \alpha x_2 = 0$ and then

$$x_2 = -(\tilde{C}_2 + \alpha I_m)^{-1}\tilde{C}_1 x_1 \tag{2.26}$$

provided that $\tilde{C}_2 + \alpha I_m$ is nonsingular. This condition is satisfied if $\sigma_{\min}(\tilde{C}_2) > -\alpha$. During the sliding mode $\dot{s}_\alpha = 0$ and substituting (2.4) and (2.25) in $\dot{s_\alpha} = C\dot{x} + \alpha\dot{x}_2 = 0$ yields

$$u_{eq} = -(CB + \alpha\tilde{B}_2)^{-1}(CAx + \alpha\tilde{A}_{21}x_1(t) + \alpha\tilde{A}_{22}x_2(t)) \tag{2.27}$$

provided that $CB + \alpha\tilde{B}_2$ is nonsingular. Then

$$u_{eq} = -(CB + \alpha\tilde{B}_2)^{-1}\left[(\tilde{C}_1\tilde{A}_{11} + (\alpha I_m + \tilde{C}_2)\tilde{A}_{21})x_1 + (\tilde{C}_1\tilde{A}_{12} + (\alpha I_m + \tilde{C}_2)\tilde{A}_{22})\ x_2(t)\right]$$

Now consider the system (2.4) and assume T is the orthogonal matrix in (2.13) given by

$$\begin{aligned} T &= \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \\ &= \begin{bmatrix} T_1 & T_2 \end{bmatrix} \end{aligned} \tag{2.28}$$

then

$$y_T = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Tx$$

implies

$$x_1 = T_{11}^T y_1 + T_{21}^T y_2 \tag{2.29}$$

$$x_2 = T_{12}^T y_1 + T_{22}^T y_2 \tag{2.30}$$

Therefore

$$\begin{aligned} x_1 &= \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix}^T \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= T_1^T y_T \\ x_2 &= \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix}^T \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= T_2^T y_T \end{aligned} \tag{2.31}$$

Hence

$$\begin{aligned} s_\alpha &= CT^T y_T + \alpha T_2^T y_T \\ &= (C_1 + \alpha T_{12}^T) y_1 + (C_2 + \alpha T_{22}^T) y_2 \end{aligned} \tag{2.32}$$

where $CT^T = [C_1 \;\; C_2]$. When $s_\alpha = 0$,

$$y_2 = -(C_2 + \alpha T_{22}^T)^{-1}(C_1 + \alpha T_{12}^T) y_1 \tag{2.33}$$

provided that $C_2 + \alpha T_{22}^T$ is nonsingular. Let $F = (C_2 + \alpha T_{22}^T)^{-1}(C_1 + \alpha T_{12}^T)$. The design parameter α should be chosen such that the matrix $A_{11} - A_{12}F$ is a stable matrix.

Definition 2.4.1: Define the sliding surface as

$$s_{\tilde{M}} = Cx + \tilde{M} x_2 = 0 \tag{2.34}$$

where $\tilde{M} \in \mathbb{R}^{m \times m}$ is an arbitrary matrix. Matrix $\tilde{M}$ is called the modification matrix of the sliding surface. For $\tilde{M} = \alpha I$, α is called the coefficient of the sliding surface. When $\tilde{M}$=0 (or α=0) the sliding surface is said to be a principal sliding surface or simply the sliding surface. □

Definition 2.4.1 yields

$$s_{\tilde{M}} = \tilde{C}_1 x_1 + \tilde{C}_2 x_2 + \tilde{M} x_2 = 0$$

and if $\tilde{C}_2 + \tilde{M}$ is nonsingular, $s_{\tilde{M}} = 0$ implies

$$x_2 = -(\tilde{C}_2 + \tilde{M})^{-1} \tilde{C}_1 x_1 \tag{2.35}$$

which for $\tilde{M} = 0$ coincides with Utkin definition [121]. Here $\tilde{M}$ is an arbitrary matrix such that $\tilde{C}_2 + \tilde{M}$ is nonsingular. In this case the (2.33) becomes

$$y_2 = -(C_2 + \tilde{M}T_{22}^T)^{-1}(C_1 + \tilde{M}T_{12}^T)y_1$$

provided that $C_2 + \tilde{M}T_{22}^T$ is nonsingular. The set

$$\mathscr{S}_{\tilde{M}} = \left\{ s_{\tilde{M}} \mid s_{\tilde{M}} = Cx + \tilde{M}x_2 = 0,\ \tilde{C}_2 + \tilde{M} \text{ is nonsingular} \right\}$$

consists of a class of sliding surface.

2.5 Control Design Technique

The procedure of designing sliding control using the properties of the sliding mode is developed. The technique includes a mild modification of the technique in [29] and [129]. In previous work the structure of the switching part of the control law is prespecified and then a control is found with this structure such that the state approaches the sliding surface. By selecting a suitable transformation the sliding surface is converted to the intersection of the m-coordinate surfaces with $n-m$ dimension. In this way the structure of the sliding mode is simplified.

Using the transformation (2.13) the system is converted to two subsystems (2.16) and (2.17). Now it is desired to change the state coordinate such that one of coordinates surfaces is on the sliding surface. Consider a second transformation

$$z = Sy_T \tag{2.36}$$

where

$$S = \begin{bmatrix} I_{n-m} & 0 \\ F & I_m \end{bmatrix}$$

then

$$S^{-1} = \begin{bmatrix} I_{n-m} & 0 \\ -F & I_m \end{bmatrix}.$$

If

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

where $z_1 \in \mathbb{R}^{n-m}$ and $z_2 \in \mathbb{R}^m$, then

$$\dot{z}_1 = (A_{11} - A_{12}F)z_1 + A_{12}z_2 \tag{2.37}$$

$$\dot{z}_2 = (A_{21} - A_{22}F + FA_{11} - FA_{12}F)z_1 + (A_{22} + FA_{12})z_2 + B_2u \tag{2.38}$$

Now let

$$\begin{aligned} \Sigma &= A_{11} - A_{12}F \\ \Psi &= A_{22} + FA_{12} \\ \chi &= A_{21} - A_{22}F + F\Sigma \end{aligned} \tag{2.39}$$

Then the system can be represented by

$$\dot{z}_1 = \Sigma z_1 + A_{12}z_2 \tag{2.40}$$

$$\dot{z}_2 = \chi z_1 + \Psi z_2 + B_2u \tag{2.41}$$

Since C_2 is nonsingular

$$\begin{aligned} s &= Cx \\ &= CT^TTx \\ &= CT^TS^{-1}Sy_T \\ &= CT^TS^{-1}z \\ &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} I_{n-m} & 0 \\ -F & I_m \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ &= C_2z_2 \end{aligned} \tag{2.42}$$

and it is concluded that $s = 0$ if and only if $z_2 = 0$. So if $s = 0$, $z_2 = 0$ and when the sliding mode occurs, $\dot{s}$=0. Since

$$\dot{s} = C_2\dot{z}_2 \tag{2.43}$$

the ideal sliding mode exists if and only if $z_2 = 0$ and $\dot{z}_2 = 0$. Hence for generation of the sliding mode, z_2 should be a function such that: (i) for all $t \geqslant t_s$, $\dot{z}_2(t) = 0$ and $z_2(t) = 0$; or (ii) after a finite time instant $\dot{z}_2^T z_2 < 0$ in the neighbourhood of $z_2 = 0$. One can select

$$\dot{z}_2 = \Psi_* z_2 - K\text{sgn}(Mz_2) \tag{2.44}$$

where Ψ_* is an arbitrary negative definite real matrix, and M and $K = K(t, z)$ are nonsingular matrices such that $z_2^T K\text{sgn}(Mz_2) > 0$. This choice is now considered. Another suitable and more general choice will be studied in Remark 2.5.1. Assume Φ is any

$m \times m$ nonsingular matrix and $\Phi\Phi^T$ is positive definite. The matrix $\Psi_* = -\Phi\Phi^T$ and substituting (2.44) into (2.41) gives

$$\Psi_* z_2 - K\text{sgn}(Mz_2) = \chi z_1 + \Psi z_2 + B_2 u \tag{2.45}$$

Hence

$$z_2(t) = e^{\Psi_* t} z_2(0) + \Psi_*^{-1}(e^{\Psi_* t} - 1)K\text{sgn}(Mz_2(t)) \tag{2.46}$$

From (2.45)

$$u = -B_2^{-1}(\chi z_1 + (\Psi - \Psi_*)z_2 + K\text{sgn}(Mz_2)) \tag{2.47}$$

which can be considered as the control law, including continuous (linear) and discontinuous (switching or nonlinear) parts. In [29] M has been chosen as a p.d.s. matrix, but here M is only a nonsingular matrix which also includes p.d.s. matrices. The positivity of M may require the positivity of K. For simplicity, K may be chosen to be a diagonal matrix with positive elements. The matrix M actually depends on matrix C_2 in (2.42) and could be selected as $M = C_2$. The choice may be physically meaningful in some practical systems. Since there are five design parameters in control formula, this control law type is very useful, flexible and more confidence than previously. The choice of design parameters depend upon the conditions, and structure of the system and should be obtained so that the stability of the sliding mode is guaranteed. So the previous work is a particular case of this work. Thus

$$u = u_{eq} + u_s \tag{2.48}$$

where

$$u_s = -B_2^{-1}K\text{sgn}(Mz_2)$$

is the nonlinear (switching or discontinuous) part of the control and

$$u_{eq} = -B_2^{-1}(\chi z_1 + (\Psi - \Psi_*)z_2) \tag{2.49}$$

is the linear (equivalent or continuous) part of the control. But

$$z_2 = [0 \quad I_m]\, STx \tag{2.50}$$

hence

$$u_s = -B_2^{-1}K\text{sgn}\left([0 \quad M]\ STx\right) \tag{2.51}$$

and the sliding control law is

$$\begin{aligned} u &= u_{eq} + u_s \\ &= -B_2{}^{-1}\left\{(\chi \quad \Psi - \Psi_*)\, STx - K\mathrm{sgn}\left([0 \quad M]\ STx\right)\right\} \end{aligned} \tag{2.52}$$

which ensures that the state trajectory converges to the sliding surface $[0 \quad I_m]\, STx = 0$ and remains on it thereafter.

2.5.1 Reduction of Chattering Phenomenon

For reduction of the chattering, which is produced by the discontinuous control, one can modify u to be a smooth continuous control. Let

$$Mz_2 = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{bmatrix}$$

Take

$$\zeta_k = \begin{cases} 1 & \text{if} \quad \eta_k > \epsilon \\ f_k & \text{if} \quad |\eta_k| \leqslant \epsilon, \\ -1 & \text{if} \quad \eta_k < -\epsilon \end{cases} \qquad 1 \leqslant k \leqslant m$$

where ϵ is a small positive real number and $f_k \in C^1_\circ\, [-\epsilon \ , \ \epsilon]$ belongs to the class of functions which are continuous and differentiable on the interval $[-\epsilon \ , \ \epsilon]$ and $f_k(0) = 0,\ f_k(\epsilon) = 1,$ $f_k(-\epsilon) = -1$. For example,

$$f_k(t) = \sin\left(\frac{(4m+1)\pi t}{2\epsilon}\right), \qquad m \in \mathbb{N} \tag{2.53}$$

Consider the control law (2.48) with

$$u_s = -B_2^{-1} K \zeta$$

where K is an $m \times m$ non-singular matrix and

$$\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_m \end{bmatrix}$$

The function u_s is now continuous because for any $1 \leqslant k \leqslant m$, ζ_k is a continuous function. In (2.53) since $f_k(t)$ oscillates between -1 and 1, chattering may exist with a sliding boundary width of 2ℓ where ℓ is sufficiently small.

Consider an approximation of the discontinuous part of control (2.48) as

$$u_s = -K\frac{Mz_2}{\|Mz_2\| + \delta}$$

where $\delta > 0$ is sufficiently small [29]. The performance of this approximation in practical problems yields acceptable results. However, there may exist unacceptance chattering if δ is close to 0. For large δ, this function is no longer an admissible approximation of the discontinuous part of control. Consider Example 2.2.1 with $a_1 = a_2 = 0$, $b = 1$, $c = 1$ and the control law

$$u = \frac{x_1 + x_2}{|x_1 + x_2| + \delta} \tag{2.54}$$

Simulation results are shown in Fig 2.3.

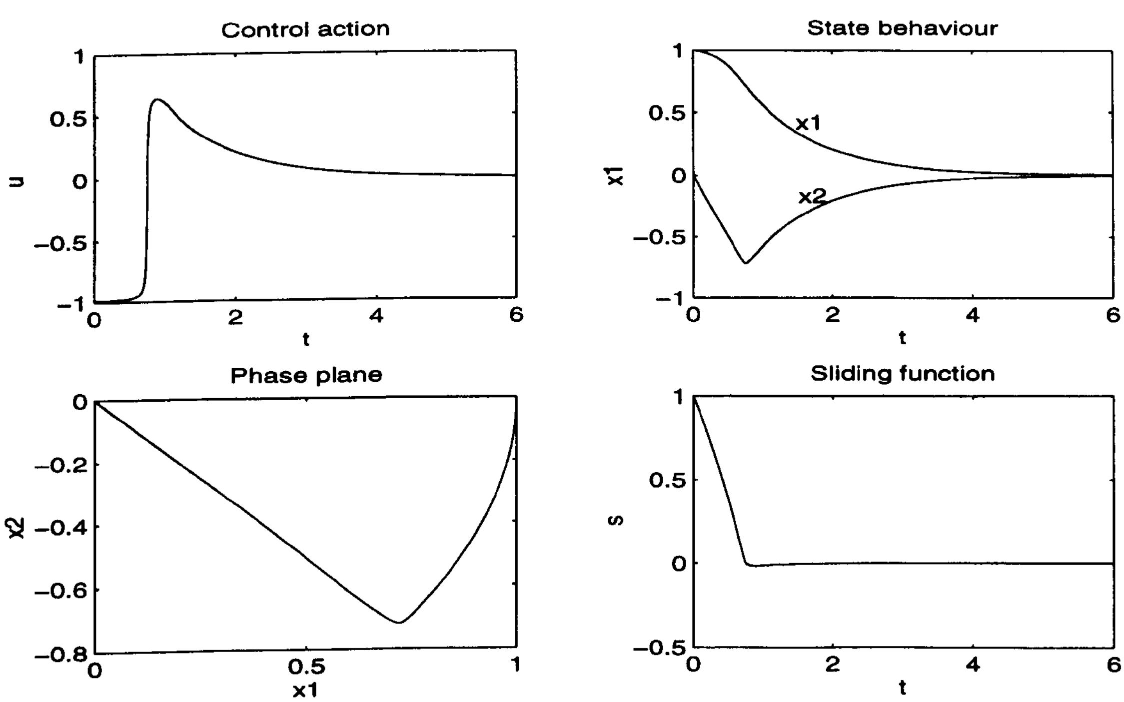

Figure 2.3: Phase plane of the system (2.3) with control (2.54) for $\delta = 0.01$

Hence the continuous control removes the undesired chattering behaviour. The boundary layer, as already stated is a way to reduce the chattering. The width of the boundary layer can be chosen arbitrarily small.

Let $K = \rho(x,t)K_1\Gamma$ where ρ is a real bounded function and K_1 is a nonsingular $m \times m$

matrix. Then

$$\begin{aligned} u &= u_{eq} + u_s \\ &= -B_2^{-1}\left\{\chi z_1 + (\Psi - \Psi_*)z_2 - \rho(x,t)K_1\Gamma \,\mathrm{sgn}\left(\begin{bmatrix}0 & M\end{bmatrix} STx\right)\right\} \end{aligned}$$

For example, let $K_1 = B_2$,

$$\mathrm{sgn}(Mz_2) = \begin{bmatrix} \mathrm{sgn}(\eta_1) \\ \mathrm{sgn}(\eta_2) \\ \vdots \\ \mathrm{sgn}(\eta_m) \end{bmatrix}$$

where

$$\mathrm{sgn}(\eta_k) = \begin{cases} 1 & \text{if} \quad \eta_k > 0 \\ 0 & \text{if} \quad \eta_k = 0\ , \\ -1 & \text{if} \quad \eta_k < 0 \end{cases} \qquad 1 \leqslant k \leqslant m$$

and $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \cdots, \gamma_m)$ with

$$\gamma_k = \begin{cases} \dfrac{|\eta_k|}{\|Mz_2\|} & \text{if} \quad Mz_2 \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

then,

$$u = -B_2^{-1}(\chi z_1 + (\Psi - \Psi_*)z_2) - \rho(x,t)\frac{\begin{bmatrix}0 & M\end{bmatrix} STx}{\|\begin{bmatrix}0 & M\end{bmatrix} STx\|}$$

When $\Gamma = I_m$, the control is the same as (2.52)

$$u = -B_2^{-1}(\chi z_1 + (\Psi - \Psi_*)z_2 - \rho(x,t)K_1\mathrm{sgn}\left(\begin{bmatrix}0 & M\end{bmatrix} STx\right)$$

Therefore this control design method is more general than that considered previously, e.g. [29]. The sliding mode is governed by (2.16) and (2.21). The design of the sliding mode requires the determination of the gain matrix F such that all the eigenvalues of $A_{11} - A_{12}F$ lie in the left-half complex plane.

Remark 2.5.1 It is desired to select u_s to be a discontinuous function on the sliding surface. This function can be chosen in many ways. One choice has been already studied; another approach is now presented which may be useful in many practical problems because there are five design parameter functions in the discontinuous part of the control law. The designer can choose these functions based on the desired properties so that the sliding mode starts at a finite time. Assume that the nonlinear (switching) part control law (2.48) is in the form of

$$u_s = -\rho(x,t)\frac{\alpha(t,x)Mz_2 + \beta(t,x)Nz_2}{|\vartheta(t,x)|.\|Mz_2\| + |\eta(t,x)|.\|Nz_2\|}$$

where $\rho > 0$, α, β, ϑ and η are real bounded functions of t and x such that $\vartheta \neq 0$ or $\eta \neq 0$. Matrices M and N are choose such that

$$\mathscr{N}(N) = \mathscr{N}(M) = \{0\}$$

A sufficient condition for the existence of the sliding mode is that $\alpha M + \beta N$ is a negative definite matrix. Let P_D be the u.p.d. solution of the Lyapunov equation

$$P_D\psi_* + \psi_*^T P_D = -I_m \tag{2.55}$$

where ψ_* is an arbitrary negative definite matrix. From (2.42) $P_D z_2 = 0$ if and only if $z_2 = 0$. So in the sliding mode $z_2 = 0$ and u_s should be discontinuous at these points. In fact the sliding surface is the coordinate plane $z_2 = 0$. Consider the discontinuous part of the control as

$$u_s = -\rho(x,t)\frac{\left[\alpha(t,x)B_2^{-1} - \beta(t,x)I_m\right] P_D z_2}{\left(|\vartheta(t,x)| + |\eta(t,x)|.\|B_2^{-1}\|\right)\|P_D z_2\|}$$

For generation of the sliding mode it is sufficient to choose the functions α and β such that $\alpha B_2{}^{-1}P_D - \beta P_D$ is a p.d. matrix. Since $z = STx$,

$$u_s = -\rho(x,t)\frac{\left[\alpha(t,x)B_2^{-1} - \beta(t,x)I_m\right][0 \quad P_D]STx}{\left(|\vartheta(t,x)| + |\eta(t,x)|.\|B_2^{-1}\|\right)\|[0 \quad P_D]STx\|}$$

and

$$u = -B_2{}^{-1}\left[\chi \quad \Psi - \Psi_*\right]STx - \rho(x,t)\frac{\left[\alpha(t,x)B_2^{-1} - \beta(t,x)I_m\right][0 \quad P_D]STx}{\left(|\vartheta(t,x)| + |\eta(t,x)|.\|B_2^{-1}\|\right)\|[0 \quad P_D]STx\|}$$

By choosing $\beta = 0$, $\alpha = 1$, $\vartheta = 0$ and $\eta = 1/\|B_2^{-1}\|$, the control law is as in [29]

$$u = -B_2{}^{-1}\left\{\left[\chi \quad \Psi - \Psi_*\right]STx + \rho(x,t)\frac{[0 \quad P_D]STx}{\|[0 \quad P_D]STx\|}\right\}$$

Since all the functions ρ, α, β, ϑ and η are bounded and also ϑ and η are not zero simultaneously, the discontinuous part of the control law is bounded

$$\|u_s\| \leqslant \rho\frac{\max\{|\alpha|, |\beta|\}}{\min\{|\vartheta|, |\eta|\}}$$

if $\min\{|\vartheta|, |\eta|\} \neq 0$. Also instead of ρ, α, β, ϑ and η being real functions they can be chosen as $m \times m$ matrices. Then the norms of these matrices replace the absolute values. In this case, if $\min\{\|\vartheta\|, \|\eta\|\} \neq 0$ then

$$\|u\| \leqslant \rho\frac{\max\{\|\alpha\|, \|\beta\|\}}{\min\{\|\vartheta\|, \|\eta\|\}}$$

For existence of a feedback gain sliding matrix F in (2.22) such that the sliding system (2.22) is stable, (A, B) must be a stabilizable pair.

Lemma 2.5.1: *Let λ_k $(1 \leqslant k \leqslant n)$ be the eigenvalues of A. A necessary and sufficient condition for the controllability of (A, B) is that for all k,* rank $[\lambda_k I - A \quad B] = n$.

Proof: See [11, page 87]. □

Since only the eigenvalues of the state matrix A are required to test the controllability of the system, Lemma 2.5.1 is very useful in practical problems.

For existence of a feedback gain sliding matrix F satisfying (2.22), (A, B) must be a stabilizable pair. However, the reduced order system (2.16) is the system in the sliding mode, which shows that the matrices A_{11} and A_{12} influence the reduced order system is similar to the state and control matrices A and B in (2.4). So, it seems that the existence of the feedback gain sliding matrix F also depends on the stabilizability of (A_{11}, A_{12}). The following lemma shows that the controllability (stabilizability) of the system implies the controllability (stabilizability) of the reduced order system, and vice versa. So, for establishing the stabilizability of the system, it is preferable to prove the stabilizability of the reduced order system. Our proof is new and benefits from Lemma 2.5.1. This lemma can also be directly applied to test the controllability of the system.

Lemma 2.5.2 [118]: *(A_{11}, A_{12}) is controllable (stabilizable) if and only if (A, B) is controllable (stabilizable).*

Proof: The proof in [135] is for $s \in \mathbb{C}$ while the proof below is for $s \in \sigma(A)$.

Necessity. Assume (A, B) is a controllable pair and λ is an eigenvalue of A. Lemma 2.5.1 yields rank$[\lambda I - A \quad B] = n$. So

$$\text{rank}\,[\lambda I - A \quad B] = \text{rank}\begin{bmatrix} \lambda I - A_{11} & A_{12} & 0 \\ A_{21} & \lambda I - A_{22} & B_2 \end{bmatrix} = \text{rank}\begin{bmatrix} \lambda I - A_{11} & A_{12} \end{bmatrix} + m$$

since rank$(B_2) = m$. Thus, rank$[\lambda I - A_{11} \quad A_{12}] = n - m$

Sufficiency. Suppose (A_{11}, A_{12}) is a controllable pair. Then for any eigenvalue λ of A_{11}, rank$[\lambda I - A \quad B] = n - m$. Since B_2 is a nonsingular matrix

$$\text{rank}\begin{bmatrix} \lambda I - A_{11} & A_{12} & 0 \\ A_{21} & \lambda I - A_{22} & B_2 \end{bmatrix} = n$$

This shows that if B is full rank then for the proof of the controllability pair (A, B) it is sufficient that for all the eigenvalues λ of A_{11}, $\text{rank}([\lambda I - A_{11} \quad A_{12}] = n - m$. So if (A, B) is only a stabilizable pair, then all the uncontrollable stable eigenvalues are the eigenvalues of A_{11} and vice versa. Therefore (A_{11}, A_{12}) is a stabilizable pair if and only if (A, B) is a stabilizable pair. □

The proof of Lemma 2.5.2 establishes that the controllability of (A, B) depends only on the eigenvalues of A_{11} which means the system is controllable if and only if for all the eigenvalues of A_{11}, $\text{rank}\begin{bmatrix} \lambda I - A & B \end{bmatrix} = n$. Since (A, B) is a controllable pair, (A_{11}, A_{12}) is also controllable and the Riccati equation

$$A_{11}^T \hat{P} + \hat{P} A_{11} - \hat{P} A_{12} R^{-1} A_{12}^T \hat{P} = -Q \tag{2.56}$$

where Q and R are arbitrary $(n-m) \times (n-m)$ semi-positive and $m \times m$ p.d.s. respectively, has a u.p.d.s. solution $\hat{P}$, and $A_{11} - A_{12} R^{-1} A_{12}^T \hat{P}$ is a stable matrix. Then the sliding surface is given by (2.21) where $F = R^{-1} A_{12}^T \hat{P}$. A design procedure for sliding control is as follows:

- Find T in (2.13) and then the system (2.4) is converted to (2.16) and (2.17).
- Solve the Riccati equation (2.56) to obtain F.
- Consider the transformation S in (2.36) and then the system (2.4) is converted to (2.40) and (2.41). The sliding surface is $[0 \quad I_m] STx = 0$.
- Select the $m \times m$ nonsingular matrices K and M such that $z_2^T K \text{sgn}(M z_2) > 0$ where $z_2 = [0 \quad I_m] STx$.

This method guarantees the stability of the sliding system in the absence of disturbance and also for the case when the matching condition is satisfied. If $\hat{P}$ is the u.p.d.s solution of the Riccati equation

$$(A_{11}^T - \alpha I_{n-m}) \hat{P} + \hat{P}(A_{11} - \alpha I_{n-m}) - \hat{P} A_{12} R^{-1} A_{12}^T \hat{P} = -Q$$

where R and Q are arbitrary $(n-m) \times (n-m)$ semi-p.d.s. and $m \times m$ p.d.s. respectively and α is a negative real number, then all the eigenvalues of $A_{11} - A_{12} F$ where $F = R^{-1} A_{12}^T \hat{P}$ lie to the left of the vertical line $x = \alpha$. Further discussion of the ARE will be presented in Chapter 8.

2.6 Summary and Discussion

In this chapter, the basic concept of the sliding mode, a method for designing a class of sliding surface and a method for control design have been presented. Although the control design method is similar to previous work (Ryan and Corless [92], Dorling and Zinober [29]), the design method and the control structure have been somewhat expanded. Using the sliding mode properties a suitable control can be constructed. In the new revised method, by applying an appropriate Riccati equation, the sliding gain matrix can be obtained such that the stability of the reduced order system is achieved.

Two different methods of sliding control design for SISO and MIMO systems including a disturbance input will be presented in Chapter 5. Further results may be obtained by investigating the dynamical behaviour of s and $\dot{s}$ in the neighbourhood of $s = 0$ to find a more general formulation of control.

Chapter 3

Optimal Sliding Mode Control

The LQ method is a suitable method for designing a sliding gain matrix yielding a stable reduced order system. In this chapter the design of the optimal sliding mode and related problems are studied. Our design ensures that in a specific parameter range, the stability of the system in the sliding mode is preserved. Some results about the sliding mode for a class of servo-mechanism systems, reference signal systems and tracking problems are also obtained.

3.1 Optimal Control and the Sliding Mode

Consider the system (2.4) with $\Gamma = 0$ and the quadratic LQ cost functional index

$$J = \int_0^\infty (x^T Q x + u^T R u) dt \tag{3.1}$$

where Q and R are semi-p.d.s and p.d.s matrices, respectively [11, page 280]. The optimal control is given by

$$u_o = -R^{-1} B^T P x$$

where P is u.p.d.s matrix solution of the Riccati equation

$$A^T P + PA - PBR^{-1} B^T P = -Q \tag{3.2}$$

The index (3.1) is not suited to find the optimal sliding surface, because the sliding mode is control-independent and during the sliding mode the system is governed by the reduced order system. Utkin [121, page 140] considered in the sliding mode

$$J = \int_0^\infty (x^T Q x + u_{eq}^T R u_{eq}) dt \tag{3.3}$$

where $u_{eq} = -(CB)^{-1}CAx$. The index (3.3) can be applied to find the sliding surface. By defining the sliding surface as $y_2 = -Fy_1$, then the equivalent control is

$$u_{eq} = -B_2^{-1}(FA_{11} - A_{22}F - FA_{12}F + A_{21})y_1$$

Therefore, (3.3) is converted to

$$J = \int_0^\infty (y_1^T W y_1)dt \tag{3.4}$$

where

$$\begin{aligned} W &= Q_{11} + \Gamma^T(B_2{}^{-1})^T R B_2^{-1}\Gamma - 2Q_{12}F + F^T Q_{22}F \\ \Gamma &= FA_{11} - A_{22}F - FA_{12}F + A_{21} \end{aligned}$$

and

$$TQT^T = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

Hence, to find F the index (3.4) should be minimized with respect to F on the trajectories (2.16).

Some new insight is now provided into the above approach. If the index (3.3) is minimized with respect to u_{eq}, then from (3.3) $u_{eq} = u_o$. This is impossible, because all the eigenvalues of $A - BR^{-1}B^TP$ lie in the left-hand half-plane, but at least m eigenvalues of $A - B(CB)^{-1}CA$ of the sliding mode are zero. In fact, the rank of $A - BR^{-1}B^TP$ is n but the rank of $I - B(CB)^{-1}C$ is $n - m$. Therefore, for all systems u_{eq} cannot have the form u_o. In particular, assume that the matrix A is a p.d.s matrix, then for all weighting matrices Q and R, A is not a solution of the Riccati equation (3.2). Therefore the following theorem has been proved.

Theorem 3.1.1: *Assume B is $n \times m$ matrix. Then for all semi-p.d.s Q*

$$2P^2 - PB(CB)^{-1}B^TP = -Q \tag{3.5}$$

does not have a p.d solution. □

A state feedback gain control can also be found by considering other quadratic cost functional expressions rather than (3.1). Consider

$$\begin{aligned} ||\dot{x}||^2 + ||y||^2 &= ||Ax + Bu||^2 + ||Cx||^2 \\ &= (Ax + Bu)^T(Ax + Bu) + (Cx)^T(Cx) \\ &= x^T(A^TA + C^TC)x + 2x^TA^TBu + u^TB^TBu \\ &= x^TQx + 2x^TS^Tu + u^TRu \end{aligned}$$

where $Q = A^TA + C^TC$, $S = B^TA$ and $R = B^TB$. Then

$$\begin{aligned} J &= \int_0^\infty \left(\|\dot{x}\|^2 + \|y\|^2\right) dt \\ &= \int_0^\infty \left(x^TQx + 2x^TS^Tu + u^TRu\right) dt \end{aligned} \tag{3.6}$$

So the minimization of (3.6) yields the optimal feedback control

$$u_o = -R^{-1}(B^TP + S)x(t) = -\left(R^{-1}B^TP + (B^TB)^{-1}B^TA\right)x(t) \tag{3.7}$$

where P the is u.p.d.s matrix solution of the Riccati equation

$$(A - BR^{-1}S)^TP + P(A - BR^{-1}S) - PBR^{-1}B^TP = -Q + S^TR^{-1}S \tag{3.8}$$

If $C = B^T$ then an optimal control is

$$u_{on} = -(B^TB)^{-1}B^TPx(t) \tag{3.9}$$

where P is the u.p.d.s. solution of the ARE

$$A^TP + PA - PBR^{-1}B^TP = S^TR^{-1}B^TP + PBR^{-1}S - Q + S^TR^{-1}S = Q_e \tag{3.10}$$

On the other hand, the equivalent control is

$$u_{eq} = -(B^TB)^{-1}B^TAx(t) \tag{3.11}$$

Therefore in this case, the optimal control law given by (3.7) is the summation of the equivalent control (3.11) and the control law (3.9), i.e.

$$u_o = u_{eq} + u_{on}$$

Therefore the equivalent control never equals the optimal control but it can be a part of the optimal control law by selecting appropriate weighting functions or more precisely by minimizing the functional index (3.6).

3.2 Sliding Mode Using the LQ Approach

The design of the sliding surface using the linear quadratic (LQ) approach has been considered by Young et al [135]. The basic idea is that y_2 is the input control of the subsystem (2.16), and LQ methods can be used to find the optimal control or more precisely the optimal sliding mode. Consider the singular quadratic cost functional

$$J = \int_{t_s}^\infty x^TQxdt \tag{3.12}$$

where Q is a p.d.s. matrix. Assume T is the transformation (2.13)

$$TQT^T = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}$$

Then (3.12) and $y = Tx$ yield

$$J = \int_{t_s}^{\infty} (y_1^T Q_{11} y_1 + 2y_1^T Q_{12} y_2 + y_2^T Q_{22} y_2) dt \tag{3.13}$$

Suppose

$$\begin{aligned} \hat{Q}_{11} &= Q_{11} - Q_{12} Q_{22}^{-1} Q_{12}^T \\ \hat{A}_{11} &= A_{11} - A_{12} Q_{22}^{-1} Q_{12}^T \\ v(t) &= y_2(t) + Q_{22}^{-1} Q_{12}^T y_1(t) \end{aligned} \tag{3.14}$$

Since (A, B) is a controllable pair, $(\hat{A}, A_{12})$ is also a controllable pair. Moreover, Q is a p.d.s. matrix, so $\hat{Q}$ is a p.d.s. matrix. Then (3.13) is converted to

$$J = \int_{t_s}^{\infty} \left(y_1^T(t) \hat{Q}_{11} y_1(t) + v(t)^T Q_{22} v(t) \right) dt \tag{3.15}$$

Therefore the Riccati equation

$$\hat{A}_{11}^T P + P \hat{A}_{11} - P A_{12} Q_{22}^{-1} A_{12}^T P = -\hat{Q}_{11}$$

has the u.d.s.p. solution P and

$$v(t) = -Q_{22}^{-1} A_{12}^T P y_1(t)$$

So

$$y_2(t) = -Q_{22}^{-1} \left(Q_{12}^T + A_{12}^T P \right) y_1(t)$$

Thus, without loss of generality it is assumed $Q_{12} = 0$ and then (3.13) becomes

$$J = \int_{t_s}^{\infty} (y_1^T Q_{11} y_1 + y_2^T Q_{22} y_2) dt \tag{3.16}$$

and

$$y_2 = -Q_{22}^{-1} A_{12}^T P y_1 = -F y_1 \tag{3.17}$$

where P is the u.p.d.s. matrix solution of the following Riccati equation

$$A_{11}^T P + P A_{11} - P A_{12} Q_{22}^{-1} A_{12}^T P = -Q_{11} \tag{3.18}$$

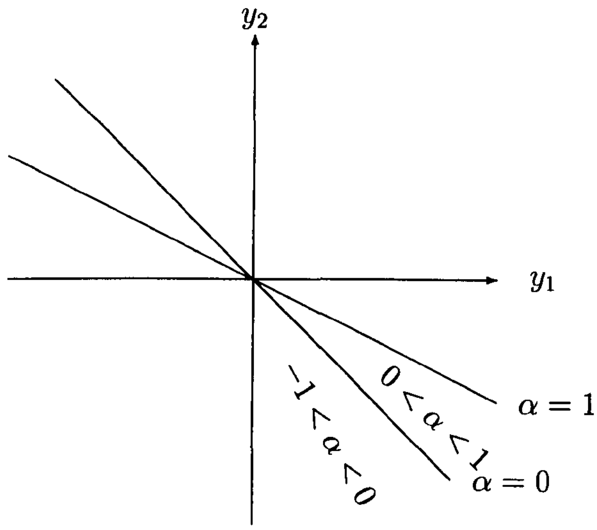

Figure 3.1: The stability region of the sliding system

So $y_2 + Fy_1 = 0$ is the sliding surface for system (2.14), and

$$Cx = 0, \qquad C = [F \quad I]T$$

is the sliding surface for the original system (2.4).

Define the sliding surface as

$$S_M = y_2 + Fy_1 + My_2 = 0$$

where M is an $m \times m$ nonsingular matrix such that $\sigma_m(M) > -1$ and $F = Q_{22}^{-1}A_{12}^TP$. Then

$$y_2 = -(I_m + M)^{-1}Q_{22}^{-1}A_{12}^TPy_1 \tag{3.19}$$

For the particular case when M$=\alpha I_m$

$$y_2 = -\frac{1}{1+\alpha}Q_{22}^{-1}A_{12}^TPy_1, \quad \alpha > -1 \tag{3.20}$$

For stability of the reduced order system, it is necessary that all the eigenvalues of the closed-loop matrix $A_{11} - A_{12}(I_m + M)^{-1}Q_{22}^{-1}A_{12}^TP$ lie in the left-hand complex plane. Since $1/(1+\alpha) > 1/2$, the stability of the reduced order system is preserved [93]. Therefore, the closed-loop matrix $A_{11} - A_{12}Q_{22}^{-1}A_{12}^TP/(1+\alpha)$ is stable if $|\alpha| < 1$. In Fig. 3.1 the appropriate region of the sliding surface for (2.14) is shown.

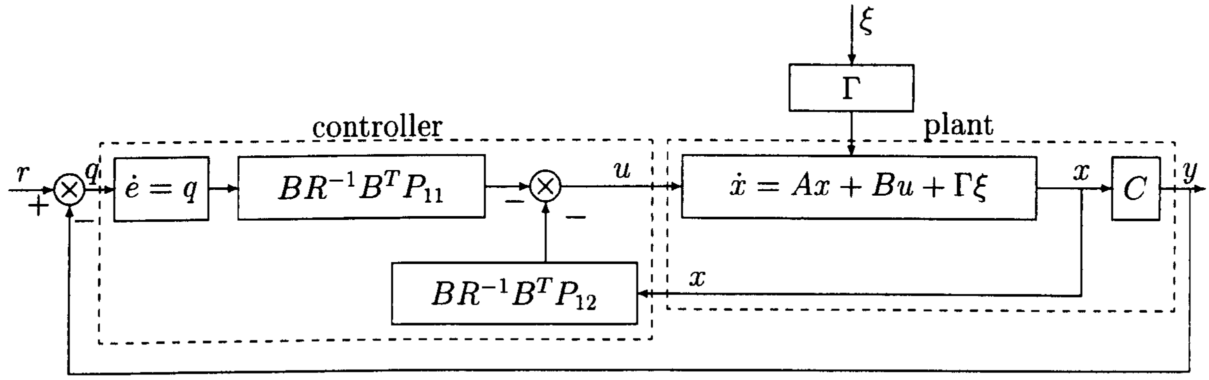

Figure 3.2: Block-diagram of servo-mechanism system with optimal feedback

3.3 Output Tracking Problems

3.3.1 Sliding Mode Servo-Mechanism Systems

The linear quadratic optimal control problem was studied in Section 3.1. The LQ method yields a full linear state feedback controller. The optimal control law to achieve tracking or regulation has been presented by Wang and Munro [125], Mahalanabis and Pal [89], Saif [97], amongst others. The sliding mode of servo-mechanisms and regulators are discussed in this section. Recall the system (2.4) with output

$$y = Cx \tag{3.21}$$

where $C \in \mathbb{R}^{m\times n}$. Assume $r \in \mathbb{R}^m$, the reference input, is a time-varying bounded piecewise continuous function. Assume the rank of

$$\hat{A} = \begin{bmatrix} A & B \\ -C & 0 \end{bmatrix}$$

is $n+m$. The output y is required to remain as close as possible to the reference input r. Define

$$\dot{e} = r - y \tag{3.22}$$

where e is the output of the servo compensator given by (3.22) (see Fig. 3.2).

Then

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u + \bar{T}r + \bar{\Gamma}\xi \tag{3.23}$$

$$\bar{y} = \bar{C}\bar{x} \tag{3.24}$$

where

$$\bar{x} = \begin{bmatrix} x \\ e \end{bmatrix},\ \bar{A} = \begin{bmatrix} A & 0 \\ -\mathcal{C} & 0 \end{bmatrix},\ \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix},\ \bar{\Gamma} = \begin{bmatrix} \Gamma \\ 0 \end{bmatrix},\ \bar{T} = \begin{bmatrix} 0 \\ I \end{bmatrix},\ \bar{\mathcal{C}} = \begin{bmatrix} \mathcal{C} & 0 \\ 0 & I \end{bmatrix}$$

From Lemma 2.5.1 the pair $(\bar{A}, \bar{B})$ is completely controllable if, for any eigenvalues λ of $\bar{A}$,

$$\text{rank}\left[\bar{A} - \lambda I_{(n+m)} \quad \bar{B}\right] = \begin{bmatrix} A - \lambda I_n & 0 & B \\ -\mathcal{C} & -\lambda I_m & 0 \end{bmatrix} = n + m \tag{3.25}$$

In order to prove (3.25), first suppose $\lambda \neq 0$. Since (A, B) is a controllable pair

$$\text{rank}\left[A - \lambda I_n \quad B\right] = n$$

and then (3.25) is satisfied for $\lambda \neq 0$. If one eigenvalue of $\bar{A}$ is zero, i.e. $\lambda = 0$, equation (3.25) is converted to

$$\text{rank}\begin{bmatrix} A & 0 & B \\ -\mathcal{C} & 0 & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} A & B \\ -\mathcal{C} & 0 \end{bmatrix}$$

where rank is $n + m$.

The minimization of the performance index

$$J = \int_0^\infty (\bar{x}^T Q \bar{x} + u^T R u) dt \tag{3.26}$$

where R and Q are p.d. and semi-p.d.s. matrices respectively, yields the optimal control

$$u_o = -R^{-1} \bar{B}^T \bar{P} \bar{x} \tag{3.27}$$

where P is the u.p.d. matrix solution of the Riccati equation

$$\bar{A}^T \bar{P} + \bar{P} \bar{A} - \bar{P} \bar{B} R^{-1} \bar{B}^T \bar{P} = -\bar{Q} \tag{3.28}$$

Substituting (3.27) in (3.23) yields

$$\dot{\bar{x}} = (\bar{A} - \bar{B} R^{-1} \bar{B}^T \bar{P})\bar{x} + \bar{\Gamma}\xi + \bar{T} r \tag{3.29}$$

Then

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} &= \left(\begin{bmatrix} A & 0 \\ -\mathcal{C} & 0 \end{bmatrix} - \begin{bmatrix} B \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} B^T & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \right) \begin{bmatrix} x \\ e \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ I \end{bmatrix} r + \begin{bmatrix} \Gamma \\ 0 \end{bmatrix} \xi \\ &= \begin{bmatrix} (A - BR^{-1}B^T P_{11})x - BR^{-1}B^T P_{12} e + \Gamma\xi \\ r - \mathcal{C}x \end{bmatrix} \end{aligned} \tag{3.30}$$

where

$$\bar{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$$

Therefore the optimal control is

$$u_o = -BR^{-1}B^T P_{11}x - BR^{-1}B^T P_{12}e$$

where

$$e = \int_0^t (r - Cx)d\tau$$

The gain matrix $-BR^{-1}B^T P_{11}$ is a linear state feedback matrix for the original system (2.4) and the gain matrix $-BR^{-1}B^T P_{12}$ provides the integral control action to improve the accuracy (see Fig. 3.2). The system satisfies

$$\dot{x} = (A - BR^{-1}B^T P_{11})x - BR^{-1}B^T P_{12}e + \Gamma\xi \tag{3.31}$$

The closed-loop system (3.29) is asymptotically stable when the disturbance and reference inputs are step functions. Its poles are the roots of the characteristic equation

$$\det\left(sI_{(n+m)} - \bar{A} + \bar{B}R^{-1}\bar{B}^T P\right) = \begin{bmatrix} sI_n - A + BR^{-1}B^T P_{11} & -BR^{-1}B^T P_{12} \\ -C & sI_m \end{bmatrix} = 0$$

In the presence of disturbances, the state is ultimately bounded and the boundedness width depends upon the disturbance bound.

3.4 Sliding Control Design

Consider the system (3.23). Define the sliding surface as

$$s = Cx + C_e e = 0$$

where $C_e \in \mathbb{R}^{m\times m}$ is a design matrix. The sliding function s contains a term proportional to the integral of the error. This term yields the ideal sliding mode. The virtual equivalent control is

$$u_{eq} = -(CB)^{-1}\left((CA - C_e C)x + C_e r + C\Gamma\xi\right)$$

Consider the discontinuous servo-control

$$u = -(CB)^{-1}\left((CA - C_e C)x + C_e r + K\text{sgn}(s)\right)$$

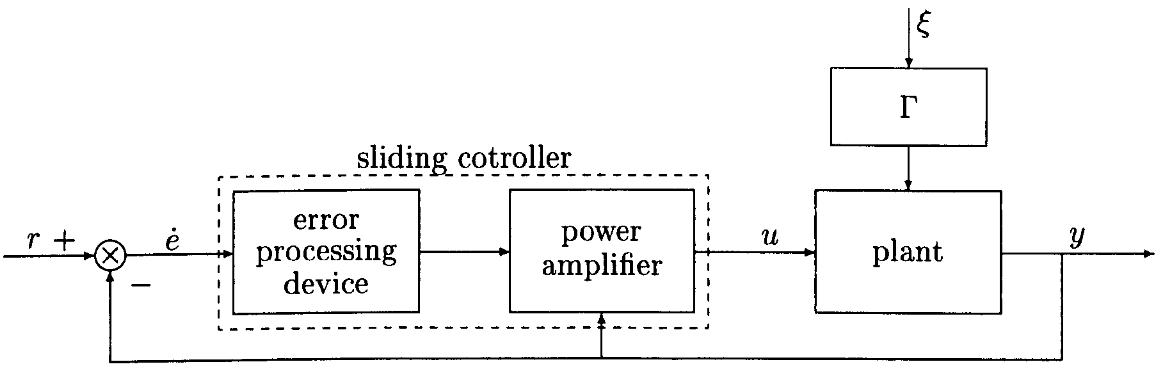

Figure 3.3: A block-diagram of sliding servo-mechanism system with feedback

where K is a diagonal matrix with positive entries and $\lambda_{\min}(K) > \|C\Gamma\|M$. This control guarantees the existence of the sliding mode because $\lambda_{\min}(K) > \|C\Gamma\|M$ yields

$$s^T\dot{s} = s^T\left(C\Gamma\xi - K\text{sgn}(s)\right) < 0$$

This is in Section 5.2.2. The sliding surface should be designed so that in the absence of disturbance and in the case of the matching conditions being satisfied, sliding system stability is achieved (see Fig. 3.3).

The sliding surface of the system (2.4) is $Cx = 0$, but the sliding surface of system (3.23) is $s = Cx + C_e e = 0$ which is a proportional integral sliding surface.

When $y = 0$, the sliding function is converted to $s = C_e\int_0^t r(\tau)d\tau$. So if $y = 0$ after a finite time, the servo system is in the sliding mode if $C_e\int_0^t r(\tau)d\tau = 0$ after a finite time, i.e. $e \in \mathcal{N}(C_e)$.

If $C_e r(t)$ is a uniformly continuous function after a finite time and $\lim_{t\to\infty}\int_0^t C_e r(\tau)d\tau$ exists (with finite limit), then the Barbalat Lemma (see Appendix C) yields $\lim_{t\to\infty} C_e r(t) = 0$.

For some particular cases the behaviour of the tracking system is now considered. Define $r(t)$ as

$$r(t) = \begin{cases} r_1(t) & \text{if} \quad 0 \leqslant t < t_1 \\ 0 & \text{if} \quad\quad t \geqslant t_1 \end{cases} \tag{3.32}$$

where $r_1(t)$ is bounded. Assume $t_2 = \max\{t_1,\ t_s\}$ where t_s is the time when the sliding mode of the system (2.4) is reached. Therefore, when the system is in the sliding mode and $t \geqslant t_2$, the system (3.23) is also in the sliding mode. However, the starting times of the sliding mode of these two systems are different. The sliding surface of the system (2.4) is a subspace of the sliding surface of system (3.23). The ideal equivalent control of the system (2.4) is given by (2.10). Note that the control of the two systems (2.4) and (3.23) are not the same. But after $t \geqslant t_2$ the control of both systems is the identical equivalent control. Therefore, in the sliding mode after a certain time instant, the input reference signal does not affect the system. It is therefore not necessary that the reference input be injected for all the time; the reference input may be cut off or kept constant after a certain time as in the regulator.

Remark 3.4.1 Define

$$r(t) = \begin{cases} r_1(t) & \text{if} \quad 0 \leqslant t < t_1 \\ \alpha & \text{if} \quad t \geqslant t_1 \end{cases} \tag{3.33}$$

where $\alpha \in \mathbb{R}^{m\times 1}$ is a constant vector and $r_1(t) \in \mathbb{R}^{m\times 1}$ is bounded. In this case the sliding surface is transformed to $s - \alpha = 0$ or

$$\tilde{C}_1 x_1 + \tilde{C}_2 x_2 - \alpha = 0 \tag{3.34}$$

where $C = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}$. □

3.4.1 Optimal Sliding Mode for the Tracking Problem

Consider the tracking system (2.4) and assume T is an orthogonal transformation matrix which is satisfied by (2.13). Then the system (2.4) can be written as

$$\dot{y}_1(t) = A_{11} y_1(t) + A_{12} y_2(t) + \Gamma_1 \xi_1 \tag{3.35}$$

$$\dot{y}_2(t) = A_{21} y_1(t) + A_{22} y_2(t) + B_2 u(t) + \Gamma_2 \xi_2 \tag{3.36}$$

where y^T and TAT^T are given by the equations (2.14)-(2.17) and

$$CT^T = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad T\Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}$$

In (3.35) y_2 is an input of the system. Let $\dot{e} = r - y$, with $r(t)$ bounded and

$$\bar{y}_1 = \begin{bmatrix} y_1 \\ e \end{bmatrix}$$

Therefore

$$\dot{\bar{y}}_1 = \begin{bmatrix} \dot{y}_1 \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ -C_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ e \end{bmatrix} + \begin{bmatrix} A_{12} \\ -C_2 \end{bmatrix} y_2 + \begin{bmatrix} \Gamma_1 \\ 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ I \end{bmatrix} r$$

and

$$\dot{\bar{y}}_1 = \bar{A}_{11}\bar{y}_1 + \bar{A}_{12}\bar{y}_2 + \bar{\Gamma}_1\xi + \bar{T}r \quad (3.37)$$

Since $(\bar{A}, \bar{B})$ is a controllable pair, $(\bar{A}_{11}, \bar{A}_{12})$ is a controllable pair [121] where

$$\bar{A}_{11} = \begin{bmatrix} A_{11} & 0 \\ -C_1 & 0 \end{bmatrix}, \bar{A}_{12} = \begin{bmatrix} A_{12} \\ -C_2 \end{bmatrix}$$

The performance measure

$$J = \int_0^\infty (\bar{y_1}^T\bar{Q}\bar{y_1} + 2\bar{y_1}^T\bar{N}y_2 + y_2^T\bar{R}y_2)dt \quad (3.38)$$

is to be minimized where

$$TQT^T = \begin{bmatrix} \bar{Q} & \bar{N} \\ \bar{N}^T & \bar{R} \end{bmatrix}$$

The optimal sliding surface is

$$S(\bar{y}_1, y_2) = y_2 + K\bar{y}_1 = 0, \quad K = \bar{R}^{-1}(\bar{A}_{12}^T P + \bar{N}^T) \quad (3.39)$$

or

$$S(y_1, y_2, e) = K_1 y_1 + y_2 + K_2 e = 0, \quad K = [K_1 \;\; K_2] \quad (3.40)$$

where P is the u.p.d.s. matrix solution of the Riccati equation

$$\bar{A}_{11}^T P + P\bar{A}_{11} - (P\bar{A}_{12} + \bar{N})\bar{R}^{-1}(\bar{A}_{12}^T + \bar{N}^T)P = -Q \quad (3.41)$$

The sliding surface (3.40) ensures that the system (3.37) is stable if the disturbance input is a step function and $\Gamma_1 = 0$.

Suppose $\Gamma_1 = 0$ and r_∞ is the limiting value of r. The solution of system (3.37) is

$$\bar{y}_1(t) = e^{\bar{A}_l t}\bar{y}_1(0) + \int_0^t e^{\bar{A}_l(t-\tau)}\bar{T}r(\tau)d\tau$$

where $\bar{A}_l = \bar{A}_{11} - \bar{A}_{12}K$, $\lim_{t\to\infty} \bar{y}_1(t) = \bar{A}_l^{-1}\bar{T}r_\infty$. If $\bar{A}_l^{-1}\bar{T}r_\infty = 0$, the tracking problem is perfectly achieved.

The sliding surface can be defined as

$$S(\bar{y}_1, y_2) = (\tilde{M} + I_m)y_2 + K\bar{y}_1 = 0$$

where $\tilde{M}$ is an arbitrary matrix with $\sigma_m(\tilde{M}) > -1$ and K defined by (3.39). The modification matrix $\tilde{M}$ is selected so that the nominal system (3.37) is stable.

Remark 3.4.2 The orthogonal matrix $\tilde{T} = \text{diag}(T, I)$ converts the system (3.23) to (3.37).

3.4.2 Dynamical Reference Input Signal

The design of the sliding manifold for a nonlinear tracking systems has been proposed by many authors including Davies et al [26]. The basic idea is based on the fact that in practical cases, a system may need an input signal for switching, working, tracking and/or starting. This signal can be given various forms depending on what signal is necessary as the input of the system. A system may need a reference signal

(*i*) only for switching for a certain interval of time. In this case the reference signal is given by (3.32).

(*ii*) for all time, but the reference signal is constant or piecewise. The reference signal in the form of (3.33) where α is constant or a piecewise function. When α is a piecewise function, the switching surface is (3.34), but in different intervals the switching surface is different.

(*iii*) for all time, but the reference signal is the output signal of another system, pre-system or reference system. The reference system operates as a signal generator. This case is a generalization of cases of above (*i*) and (*ii*). In this case, the original system can be considered as a subsystem of an augmented system consisting of the original system and a reference system.

A generalized approach for linear systems is presented which can be applied to the non-linear case including the nonlinear affine system and nonlinear systems where the nonlinearity appears only in the disturbance term.

Consider the system (2.4) and also assume $R(t) \in \mathbb{R}^m$ is a reference signal defined as

$$\begin{aligned} \dot{R}(t) &= \Lambda R(t) + B_r r(t) \\ y_r &= C_R R(t) \end{aligned} \tag{3.42}$$

where $\Lambda \in \mathbb{R}^{m\times m}$ is a stable matrix, $C_R \in \mathbb{R}^{m\times m}$, $B_r \in \mathbb{R}^{m\times p}$, $y_r \in \mathbb{R}^m$, $r(t) \in \mathbb{R}^p$ is bounded. Define

$$\dot{e} = \Upsilon(y_r - y(t)) \tag{3.43}$$

where $\Upsilon \in \mathbb{R}^{m\times m}$ is a nonsingular matrix. Taking

$$\tilde{x} = \begin{bmatrix} x \\ R \\ e \end{bmatrix}$$

then the system has the form

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u + \tilde{\Gamma}\xi + \tilde{B}_r r \tag{3.44}$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & \Lambda & 0 \\ -\Upsilon C & C_R\Upsilon & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{\Gamma} = \begin{bmatrix} \Gamma \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{B}_r = \begin{bmatrix} 0 \\ B_r \\ 0 \end{bmatrix}$$

The stability of Λ is required because R is uncontrollable via u. This assumption ensures the existence of linear feedback in order to satisfy system stability in the absence of the disturbance input. So the stability of Λ is sensible and important, and a system with an unstable model is not usually driven. The controllability matrix is

$$\begin{aligned} \mathscr{C} &= \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \tilde{A}^2\tilde{B} & \dots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix} \\ &= \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \\ 0 & 0 & 0 & \dots & 0 \\ 0 & -\Upsilon CB & -\Upsilon CAB & \dots & -\Upsilon CA^{n-1}B \end{bmatrix} \\ &= \begin{bmatrix} A & 0 & B \\ 0 & 0 & 0 \\ -\Upsilon C & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & B & AB & \dots & A^{n-2}B \\ 0 & 0 & 0 & \dots & 0 \\ I & 0 & 0 & \dots & 0 \end{bmatrix} \end{aligned} \tag{3.45}$$

with rank $\mathscr{C} = n + m$. So $(\tilde{A}, \tilde{B})$ is not a controllable pair but it is a stabilizable pair because of the the stability of Λ. This condition ensures the existence of a linear feedback gain matrix which yields the stability of the nominal servo system (see Appendix C). An optimal controller results from the minimization of the cost functional

$$J = \int_0^\infty (\tilde{x}^T Q \tilde{x} + u^T R u) dt$$

where Q, R are semi-p.d.s. and p.d.s. matrices respectively. Then

$$u_o = -R^{-1}\tilde{B}^T P x = -Kx$$

where P is an u.p.d.s. matrix solution of the Riccati equation

$$\tilde{A}^T P + P\tilde{A} - P\tilde{B}R^{-1}\tilde{B}^T P = -Q$$

Assume

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix}$$

then

$$K = R^{-1}B^T\left[P_{11} \quad P_{12} \quad P_{13}\right]$$

and the optimal control is

$$u_o = -R^{-1}B^T(P_{11}x + P_{12}R + P_{13}e)$$

This control is proportional integral control where $-R^{-1}B^TP_{11}x$ and $-R^{-1}B^TP_{12}R$ are ordinary linear state feedbacks and the term $-R^{-1}B^TP_{13}e$ provides integral control action to improve the static accuracy.

3.4.3 Tracking Sliding Control

Define the switching surface as

$$s(\tilde{x}) = \tilde{C}\tilde{x} \tag{3.46}$$

where

$$\tilde{C} = \begin{bmatrix} C & C_R & C_e \end{bmatrix}$$

and $C_e \in \mathbb{R}^{m\times m}$ is an arbitrary matrix. Then the switching surface is

$$s = Cx + C_RR + C_ee = 0 \tag{3.47}$$

and

$$x_2 = -C_2^{-1}(C_1x_1 + C_RR + C_ee)$$

where $C = [C_1 \quad C_2]$ and C_2 is nonsingular matrix. When $s = 0, \dot{s} = 0$ and

$$C\dot{x} + C_R\dot{R} + C_e\dot{e} = 0 \tag{3.48}$$

inserting (2.4), (3.42) and (3.43) in (3.48) gives

$$u_{eq} \quad = \quad -(CB)^{-1}((CA - C_e\Upsilon C)x + (C_R\Lambda + C_e\Upsilon C_R)R + C_RB_rr + C\Gamma\xi) \tag{3.49}$$

Consider the discontinuous control

$$u = -(CB)^{-1}\left((CA - C_e\Upsilon C)x + (C_R\Lambda + C_e\Upsilon C_R)R + C_RB_rr + K\text{sgn } s\right)$$

where K is a diagonal p.d. matrix $\lambda_{\min}(K) > \|C\Gamma\|M$ and s is defined by (3.46). Similarly to Section 3.4.1, this control guarantees the existence of the sliding mode. To achieve perfect tracking of the desired output y_r, the reduced order augmented system must be stable. So the sliding gain matrix $\tilde{C}$ should be selected such that the stability of the sliding augmented system is guaranteed. Section 3.4.4 yields an optimal sliding surface which ensures the stability of the nominal sliding augmented system.

3.4.4 Sliding Surface Design

Consider the system (3.35) and let

$$\tilde{y}_1 = \begin{bmatrix} y_1 \\ R \\ e \end{bmatrix}$$

Then the system has the form

$$\dot{\tilde{y}}_1 = \tilde{A}_{11}\tilde{y}_1 + \tilde{A}_{12}y_2 + \tilde{\Gamma}_1\xi + \tilde{B}_r r \tag{3.50}$$

where

$$\tilde{A}_{11} = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & \Lambda & 0 \\ -\Upsilon C_1 & C_R\Upsilon & 0 \end{bmatrix}, \quad \tilde{A}_{12} = \begin{bmatrix} A_{12} \\ 0 \\ -\Upsilon C_2 \end{bmatrix}, \quad \tilde{\Gamma}_1 = \begin{bmatrix} \Gamma_1 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{B}_r = \begin{bmatrix} 0 \\ B_r \\ 0 \end{bmatrix}$$

The optimal sliding surface is obtained from the minimization of the cost functional

$$J = \int_0^\infty (\tilde{y}_1^T Q_{11}\tilde{y}_1 + y_2^T Q_{22} y_2)dt$$

where $Q_{11} \in \mathbb{R}^{(n+m)\times(n+m)}$ and $Q_{22} \in \mathbb{R}^{m\times m}$ are arbitrary matrices. Then

$$y_2 = -Q_{22}^{-1}\tilde{A}_{12}^T\hat{P}\tilde{y}_1 = -K\tilde{y}_1$$

where $\hat{P}$ is a semi-p.d.s. matrix solution of the Riccati equation

$$\tilde{A}_{11}^T\hat{P} + \hat{P}\tilde{A}_{11} - \hat{P}\tilde{A}_{12}Q_{22}^{-1}\tilde{A}_{12}^T\hat{P} = -Q_{11}$$

Assume

$$\hat{P} = \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} & \hat{P}_{13} \\ \hat{P}_{12} & \hat{P}_{22} & \hat{P}_{23} \\ \hat{P}_{13} & \hat{P}_{23} & \hat{P}_{33} \end{bmatrix}$$

then $K = Q_{22}^{-1}\tilde{A}_{12}^T \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} & \hat{P}_{13} \end{bmatrix}$ and

$$y_2 = -K\tilde{y}_1 = -Q_{22}^{-1}\tilde{A}_{12}^T(\hat{P}_{11}y_1 + \hat{P}_{12}R + \hat{P}_{13}e)$$

is the sliding surface of the augmented system. Take $\tilde{C} = [K \quad I]$. With this choice of $\tilde{C}$, the nominal reduced order augmented system is stable. When $y = 0$, the sliding function is converted to $s = C_R R + C_e\Upsilon C_R \int_0^t R(\tau)d\tau$. So if $y = 0$ after a finite time, the augmented system is also in the sliding mode if $C_R R = 0$, i.e. $R \in \mathcal{N}(C_R)$.

3.5 Summary and Discussion

In this chapter LQ, LQR (regulator), LQT (tracking) problems and the sliding mode for these problems have been studied. These methods yield suitable sliding mode control. Using the linear quadratic cost functional guarantees the stability of the system in the absence of disturbance and also when the disturbance is a step function. When the system has a reference signal input, the design of the sliding surface is different. In this case the reference input and its effect on the system must be considered. By using both output tracking and regulator problems the sliding surface has been obtained. The relationship between the two systems (2.4) and (3.23) in the sliding mode has been clarified. The reference input has been considered as a dynamic system, and the control and the sliding surface obtained. In fact, in this case the reference system is a dynamic system independent of the original system and operates as a signal reference controller or generator. Further results could be obtained for the nonlinear optimal control case to generalize the work by Davies et al [26], or by using statistical methods for stochastic processes. If the system is an m-input l-output system, Υ is an $m \times l$ matrix and the sliding surface can be defined as already stated. More results may be obtained by investigating the sliding condition regarding the boundary layer and also extending the results to nonlinear and discrete-time systems.

Chapter 4

Frequency Shaping in Sliding Mode Control

4.1 Prologue

The frequency shaping approach to linear quadratic (LQ) design has been proposed in recent years [5], [41]-[55], [66], [67], [86], [114], [137]. For example, Moore and Mingori [86] discussed frequency-shaped LQ and spectral factorization. They proposed techniques for the construction of optimal controllers which preserve the robustness properties of standard LQ state feedback. Tharp et al [114] discussed the parameterization of LQ frequency weightings, the associated dynamic controller and a two-phase procedure for the design of controllers for systems utilizing frequency weighting. They developed a technique to retain the spectrum of closed-loop design model, resulting from a conventional LQ problem, as a subset of eigenvalues of the closed-loop augmented system.

In [137] a method for control and sliding mode design using the frequency domain techniques was presented, but considered only the case when the control weighting matrix is dependent on frequency. In this chapter all the possible cases are considered for which the weighting functions may be frequency-dependent. The frequency shaping of sliding mode control and design compensators for the reduced order system are studied. Furthermore, the conditions that the poles of original LQ reduced order system remain the poles of the reduced order system with compensator are obtained.

The frequency-dependent weight functions may have a penalty on the control at high frequencies, e.g. the frequency response may drop off slowly as $1/\omega$ at high frequencies

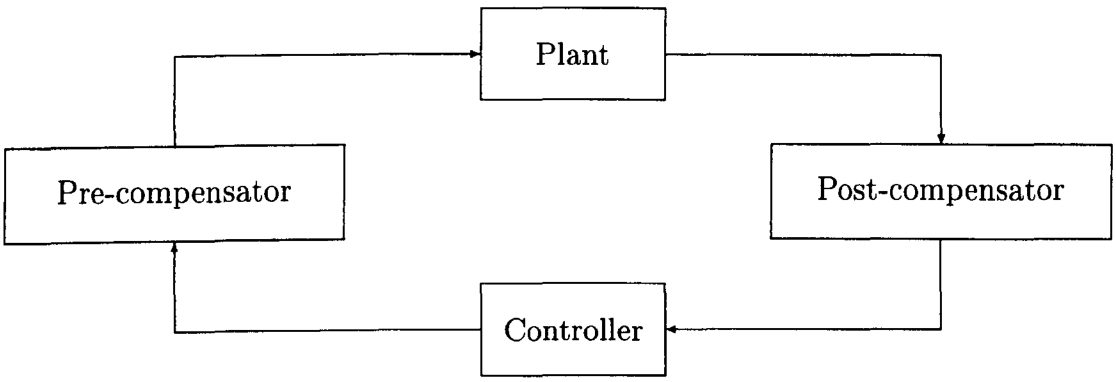

Figure 4.1: A block-diagram of the augmented system

[48]. For the case of state feedback in LQ design, the controller has 60° phase margin and $(\frac{1}{2}, \infty)$ gain margin. Using appropriate frequency weight functions yields augmented control systems which can be applied to obtain the sliding mode and optimal control. Frequency shaping is a way of dealing with plant uncertainties and links linear quadratic optimal control and sliding mode control.

In Section 4.2 the main problem and some new results are discussed. The main problem arises when the weighting functions in the LQ performance functional are frequency-dependent. In this way a pre-compensator, post-compensator or both can be designed for the system in the sliding mode (see Fig. 4.1).

4.2 Frequency Shaping of the Sliding Mode

In this section, methods are presented for finding the sliding surface when the weighting matrices are functions of frequency. The quadratic cost (3.16) can be written in the frequency domain using Parseval's Theorem

$$J = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(y_1^*(i\omega) Q_{11}(i\omega) y_1(i\omega) + y_2^*(i\omega) Q_{22}(i\omega) y_2(i\omega) \right) d\omega \tag{4.1}$$

where the matrices $Q_{11}(i\omega)$ and $Q_{22}(i\omega)$ are frequency-dependent Hermitian weighting matrices. They are p.d. matrices for all frequencies except a set of frequencies with zero measure, i.e. for almost every frequency the weighting functions are p.d. matrices. Assume the weight functions are proper rational functions of ω^2 [48]. This assumption guarantees that the optimal sliding solution is causal. Note that any real function can be approximated by a rational function. There exist four cases:

(*i*) both Q_{11} and Q_{22} are constant for all frequencies

(*ii*) Q_{22} is a function of ω^2 and Q_{11} is constant for all frequencies

(*iii*) Q_{11} is a function of ω^2 and Q_{22} is constant for all frequencies

(*iv*) both Q_{11} and Q_{22} are functions of ω^2

Case (*i*) was considered in Section 3.2 and (*ii*) has been presented by Young and Ozgüner [137] and Hashimoto and Konno [55]. It has been shown by Anderson et al [5] that when $Q_{11}(i\omega)$ and $Q_{22}(i\omega)$ are the inverse of each other, then identical closed-loop poles and an optimal sliding mode are obtained. Therefore, except for this case, the sliding gain matrices are not identical, i.e. the sliding surfaces are no longer the same.

Cases (*ii*), (*iii*) and (*iv*) are considered here. Case (*ii*) is considered first, i.e. Q_{11} is constant for all frequencies and Q_{22} is a function of ω^2. Assume that $W_2(s)$ is a spectral factor of Q_{22}, i.e.

$$Q_{22}(i\omega) = W_2{}^*(i\omega)W_2(i\omega) \tag{4.2}$$

Then the quadratic cost (4.1) can be replaced by

$$\begin{aligned} J &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(y_1^*(i\omega)Q_{11}(i\omega)y_1(i\omega) + (W_2(i\omega)y_2(i\omega))^* W_2(i\omega)y_2(i\omega)\right) d\omega && (4.3)\\ &= \int_{t_s}^{\infty} (y_1^* Q_{11} y_1 + \tilde{u}^*\tilde{u})dt && (4.4)\end{aligned}$$

where

$$\tilde{u}(s) = W_2(s)y_2(s)$$

This implies that $\tilde{u}$ is the output of a filter or dynamic system with transfer function $W_2(s)$ and input y_2. If $W_2(s)$ is considered to be a transfer function, then $W_2(s)$ represents the pre-compensator transfer function of the system

$$\begin{aligned} \dot{x}_{w_2} &= A_{w_2}x_{w_2} + B_{w_2}y_2 \\ \tilde{u}_{w_2} &= C_{w_2}x_{w_2} + D_{w_2}y_2 \end{aligned} \tag{4.5}$$

The optimal sliding surface for the augmented system, which is the original system with the dynamic compensator (4.5), is now studied. Consider

$$\dot{x}_e = A_e x_e + B_e y_2 \tag{4.6}$$

$$A_e = \begin{bmatrix} A_{w_2} & 0 \\ 0 & A_{11} \end{bmatrix}, \quad x_e = \begin{bmatrix} x_{w_2} \\ y_1 \end{bmatrix}, \quad B_e = \begin{bmatrix} B_{w_2} \\ A_{12} \end{bmatrix},$$

$$Q_e = \begin{bmatrix} C_{w_2}^T C_{w_2} & 0 \\ 0 & Q_{11} \end{bmatrix}, \quad N_e = \begin{bmatrix} C_{w_2}^T D_{w_2} \\ 0 \end{bmatrix}$$

and $R_e = D_{w_2}^T D_{w_2}$. The quadratic cost (4.1) is converted to

$$J = \int_{t_s}^{\infty} (x_e^T Q_e x_e + 2x_e^T N_e y_2 + y_2^T R_e y_2) dt \tag{4.7}$$

Therefore, the sliding surface is

$$s = y_2 + K x_e = 0, \qquad K = R_e^{-1}(B_e^T P_e + N_e^T) \tag{4.8}$$

where P_e is the u.p.d.s. matrix solution of ARE

$$A_e^T P_e + P_e A_e - (P_e B_e + N_e) R_e^{-1} (B_e^T P_e + N_e^T) = -Q_e \tag{4.9}$$

For $K = [K_1 \quad K_2]$ equation (4.8) becomes

$$s = y_2 + K_1 x_{w_2} + K_2 y_1 = 0 \tag{4.10}$$

which is not the sliding surface of the original system, but is a linear operator of states. Assume

$$\eta = K_1 x_{w_2} + y_2 \tag{4.11}$$

Then the system

$$\begin{aligned} \dot{x}_{w_2} &= A_{w_2} x_{w_2} + B_{w_2} y_2 \\ \eta &= K_1 x_{w_2} + y_2 \end{aligned} \tag{4.12}$$

is a filter for y_2 which is obtained by designing the sliding surface (4.12) [137]. Therefore there is a filter for y_2 corresponding to the sliding surface.

4.3 Iterative Constructive Procedure for the Optimal Sliding Surface

One way to obtain various sliding surfaces is to alter the weighting functions in the functional performance index (4.1). The problem is how should the weighting functions be selected. An iterative method enables one to consider various sliding surfaces and choose the desired sliding surface. This method may be applied a finite number times and the sliding surface chosen by comparing the eigenvalues of the reduced order systems corresponding to each sliding surface. Example (a) in Section 4.6 illustrates this approach.

An iterative method for designing the sliding surface is now presented. Conventional weighting matrices are considered which yield a new compensator and augmented system. In this way, various sliding surfaces can be designed. Equations (4.5) and (4.11) yield the new system.

Consider

$$\tilde{u}_1 = C_1 x_{w_2} + D_1 y_2$$

where

$$C_1 = \begin{bmatrix} C_0 \\ K_{01} \end{bmatrix}, \quad D_1 = \begin{bmatrix} D_0 \\ I_m \end{bmatrix}$$

with $C_0 = C_{w_2}$, $D_0 = D_{w_2}$ and $K_{01} = K_1$. Suppose

$$\tilde{Q}_1(s) = D_1 + C_1(sI - A_{w_2})^{-1} B_{w_2}$$

i.e. $\tilde{Q}_1$ is the transfer function of the system

$$\begin{aligned} \dot{x}_{w_2} &= A_{w_2} x_{w_2} + B_{w_2} y_2 \\ \tilde{u}_1 &= C_1 x_{w_2} + D_1 y_2 \end{aligned} \tag{4.13}$$

Now consider (4.6) with weighting matrices

$$Q_{1e} = \begin{bmatrix} C_1^T C_1 & 0 \\ 0 & Q_{11} \end{bmatrix}, \quad N_{1e} = \begin{bmatrix} C_1^T D_1 \\ 0 \end{bmatrix}$$

and $R_{1e} = D_1^T D_1$. The quadratic cost (4.1) is converted to

$$J = \int_{t_s}^{\infty} (x_e^T Q_{1e} x_e + 2x_e^T N_{1e} y_2 + y_2^T R_{1e} y_2) dt \tag{4.14}$$

Therefore, the sliding surface is

$$s_1 = y_2 + K_{1e} x_e = 0, \qquad K_{1e} = R_{1e}^{-1}(B_e^T P_{1e} + N_{1e}^T) \tag{4.15}$$

where P_{1e} is the u.p.d.s. matrix solution of the appropriate Riccati equation. For $K_{1e} = [K_{11} \quad K_{12}]$ equation (4.15) is

$$s_1 = y_2 + K_{11} x_{w_2} + K_{12} y_1 = 0 \tag{4.16}$$

Since the quadratic cost functionals (4.7) and (4.14) are different, the sliding surfaces (4.8) and (4.15) are not the same, i.e. a new compensator and a new sliding surface are designed. For the next step, consider

$$\tilde{u}_2 = C_2 x_{w_2} + D_2 y_2$$

where

$$C_2 = \begin{bmatrix} C_1 \\ K_{11} \end{bmatrix}, \quad D_2 = \begin{bmatrix} D_1 \\ I_m \end{bmatrix}$$

Suppose

$$\tilde{Q}_2(s) = D_2 + C_2(sI - A_{w_2})^{-1} B_{w_2}$$

Consider (4.6) with weighting matrices

$$Q_{2e} = \begin{bmatrix} C_2^T C_2 & 0 \\ 0 & Q_{11} \end{bmatrix}, \quad N_{2e} = \begin{bmatrix} C_2^T D_2 \\ 0 \end{bmatrix}$$

and $R_{2e} = D_2^T D_2$. The quadratic cost (4.1) now becomes

$$J = \int_{t_s}^{\infty} (x_e^T Q_{2e} x_e + 2x_e^T N_{2e} y_2 + y_2^T R_{2e} y_2) dt \tag{4.17}$$

Therefore, the sliding surface is

$$s_2 = y_2 + K_{2e} x_e = 0, \qquad K_{2e} = R_{2e}^{-1}(B_e^T P_{2e} + N_{2e}^T) \tag{4.18}$$

where P_{1e} is the u.p.d.s. matrix solution of the appropriate Riccati equation. For $K_{2e} = [K_{21} \quad K_{22}]$ equation (4.18) is given by

$$s_2 = y_2 + K_{21} x_{w_2} + K_{22} y_1 = 0 \tag{4.19}$$

By proceeding iteratively for a given positive integer number $N \geqslant 1$ it is obtained that

$$\tilde{u}_N = C_N x_{w_2} + D_N y_2$$

where

$$C_N = \begin{bmatrix} C_{N-1} \\ K_{(N-1)1} \end{bmatrix}, \quad D_N = \begin{bmatrix} D_{N-1} \\ I_m \end{bmatrix}$$

Suppose

$$\tilde{Q}_N(s) = D_N + C_N(sI - A_{w_2})^{-1} B_{w_2}$$

i.e. $\tilde{Q}_N$ is the transfer function of the system

$$\begin{aligned} \dot{x}_{w_2} &= A_{w_2}x_{w_2} + B_{w_2}y_2 \\ \tilde{u}_N &= C_N x_{w_2} + D_N y_2 \end{aligned} \tag{4.20}$$

Now consider (4.6) with weighting matrices

$$Q_{Ne} = \begin{bmatrix} C_N^T C_N & 0 \\ 0 & Q_{11} \end{bmatrix}, \quad N_{Ne} = \begin{bmatrix} C_N^T D_N \\ 0 \end{bmatrix}$$

and $R_{Ne} = D_N^T D_N$. The quadratic cost (4.1) is converted to

$$J = \int_{t_s}^{\infty} (x_e^T Q_{Ne} x_e + 2x_e^T N_{Ne} y_2 + y_2^T R_{Ne} y_2) dt \tag{4.21}$$

Therefore, the sliding surface is

$$s_N = y_2 + K_{Ne}x_e = 0, \qquad K_{Ne} = R_{Ne}^{-1}(B_e^T P_{Ne} + N_{Ne}^T) \tag{4.22}$$

where P_{Ne} is the u.p.d.s. matrix solution of the Riccati equation

$$A_e^T P_{Ne} + P_{ne}A_e - (P_{Ne}B_e + N_{Ne})R_{Ne}{}^{-1}(B_e^T P_{Ne} + N_{Ne}^T) = Q_N \tag{4.23}$$

For $K_N = [K_{N1} \quad K_{N2}]$ equation (4.22) is given by

$$s_N = y_2 + K_{N1}x_{w_2} + K_{N2}y_1 = 0 \tag{4.24}$$

Therefore, for all N there exists an augmented system, an optimal sliding surface and a filter for y_2 relating to the sliding surface.

4.3.1 Design of a Post-Compensator

Next Case (*iii*) is discussed. Suppose $Q_{11}(s)$ is a function of ω^2 but $Q_{22}(s)$ is constant for all ω,

$$Q_{11}(i\omega) = W_1{}^*(i\omega)W_1(i\omega) \tag{4.25}$$

where $W_1(i\omega)$ is the spectral factor $Q_{11}(s)$. Then (4.1) becomes

$$J = \frac{1}{2\pi}\int_{-\infty}^{\infty} \{(W_1(i\omega)y_1(i\omega))^* W_1(i\omega)y_1(i\omega) + y_2^*(i\omega)Q_{22}y_2(i\omega)\}\, d\omega \tag{4.26}$$

Assume $\tilde{u}_{w_1}(i\omega) = W_1(i\omega)y_1(i\omega)$ and the transfer function of the system

$$\dot{x}_{w_1} = A_{w_1}x_{w_1} + B_{w_1}y_1 \tag{4.27}$$

$$\tilde{u}_{w_1} = C_{w_1}x_{w_1} + D_{w_1}y_1 \tag{4.28}$$

is $W_1(s)$. Consider

$$\dot{\hat{y}}_1 = \hat{A}\hat{y}_1 + \hat{B}y_2 \tag{4.29}$$

$$\hat{y}_1 = \begin{bmatrix} y_1 \\ x_{w_1} \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} A_{11} & 0 \\ B_{w_1} & A_{w_1} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} A_{12} \\ 0 \end{bmatrix}$$

$$\hat{Q} = \begin{bmatrix} D_{w_1}^T D_{w_1} & D_{w_1}^T C_{w_1} \\ C_{w_1}^T D_{w_1} & C_{w_1}^T C_{w_1} \end{bmatrix}$$

For $C_{w_1}^T C_{w_1} > 0$ and $D_{w_1}^T D_{w_1} - D_{w_1}^T C_{w_1}(C_{w_1}^T C_{w_1})^{-1} C_{w_1}^T D_{w_1} \geqslant 0$, $\hat{Q}$ is semi-p.d.s. matrix (see Appendix C.2). The quadratic cost (4.26) is now

$$J = \int_{t_s}^{\infty} \left(\hat{y}_1^T \hat{Q} \hat{y}_1 + y_2^T Q_{22} y_2 \right) dt \tag{4.30}$$

and the sliding surface is

$$s = y_2 + K\hat{y}_1 = 0, \quad K = Q_{22}^{-1}\hat{B}^T\hat{P} \tag{4.31}$$

where $\hat{P}$ is the u.p.d.s. matrix solution of the ARE

$$\hat{A}^T\hat{P} + \hat{P}\hat{A} - \hat{P}\hat{B}Q_{22}^{-1}\hat{B}^T\hat{P} = -\hat{Q} \tag{4.32}$$

Assume $K = [K_1 \quad K_2]$ then the sliding surface is given by

$$y_2 + K_1y_1 + K_2x_{w_1} = 0 \tag{4.33}$$

The equations (4.10) and (4.33) are basically similar, i.e. the frequency shaping control weighting function and frequency shaping state weighting have similar effects on the closed-loop system. The equations (4.5) and (4.27)-(4.28) describe a pre-compensator and a post-compensator for the system (2.16), respectively. The equations (4.10) and (4.33) indicate that the sliding surface can be considered as a linear operator corresponding to the pre- and post-compensator, respectively.

Assume $\mathcal{T}$ is the linear operator

$$\mathcal{T}(y_1) = K_1y_1 + K_2x_{w_1}$$

Define the sliding surface $\varphi = \mathcal{T}(y_1) + y_2$. For the system

$$\begin{aligned} \dot{x}_{w_1} &= A_{w_1} x_{w_1} + B_{w_1} y_1 \\ z &= K_1 y_1 + K_2 x_{w_1} \end{aligned} \tag{4.34}$$

the augmented system is (4.29). Hence, the sliding surface is

$$\varphi = y_2 + K_1 y_1 + K_2 x_{w_1} = 0$$

The system (4.34) is a filter for y_1 which has been obtained by designing the sliding surface (4.33). Also note that a linear operator has a realization as a dynamical system.

4.4 Relationship between LQ Reduced System and Augmented Sliding System

The augmented system is a new system which is a combination of the original LQ reduced order system and a compensator. Generally, the poles of the reduced order system are not the poles of the augmented system. Since allocation of the system poles is very important for designing control, it is desirable to design a compensator such that the poles of the LQ system remain the poles of the augmented system. In this way some of the properties of the LQ system are conserved. In this section conditions are stated for the poles of the LQ reduced order system to be preserved as the poles of the reduced order augmented system. Note that in the following discussion, the general case of decomposition of the weighting matrix is considered; in particular when the weighting matrix is not strictly proper. Consider the performance index (4.1). Assume

$$Q_{11}(i\omega) = Q_0 + \hat{Q}_{11}(i\omega) \tag{4.35}$$

where Q_0 is constant matrix. Suppose

$$J_2 = \int_{t_s}^{\infty} (y_1^T Q_0 y_1 + y_2^T Q_{22} y_2) dt \tag{4.36}$$

where Q_{22} is constant for almost all frequencies. Then an optimal sliding surface for the original system is

$$y_2 = -Q_{22}^{-1} A_{12}^T P y_1 = -K y_1 \tag{4.37}$$

where P is the u.p.d.s. solution of the Riccati equation

$$A_{11}^T P + P A_{11} - P A_{12} Q_{22}^{-1} A_{12}^T P = -Q_0 \tag{4.38}$$

Suppose one selects $Q_0 = 0$. Denote $\lambda_1, \lambda_2, \ldots, \lambda_{n_1}$ and $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_{n_2}$ as the eigenvalues of A_{11} in the closed left-hand half plane and open right-hand half plane, respectively. Then the eigenvalues of $A_{11} - A_{12}K$ are $\lambda_1, \lambda_2, \ldots, \lambda_{n_1}, -\hat{\lambda}_1, -\hat{\lambda}_2, \ldots, -\hat{\lambda}_{n_2}$ [64]. When $Q_0 = 0$, for the stability of the reduced order system it is necessary that matrix A_{11} has no eigenvalues on the imaginary axis.

For

$$\hat{Q}_{11} = \hat{W}_1^* \hat{W}_1$$

the transfer function of the system

$$\begin{aligned} \dot{\hat{x}}_{w_1} &= \hat{A}_{w_1}\hat{x}_{w_1} + +\hat{B}_{w_1}y_1 \\ \hat{w} &= \hat{C}_{w_1}\hat{x}_{w_1} + \hat{D}_{w_1}y_1 \end{aligned} \tag{4.39}$$

is $\hat{W}_1(s)$.

Consider the augmented system

$$\dot{\hat{y}}_e = \hat{A}_e \hat{y}_e + \hat{B}_e y_2 \tag{4.40}$$

where

$$\hat{y} = \begin{bmatrix} y_1 \\ \hat{x}_{w_1} \end{bmatrix}, \quad \hat{A}_e = \begin{bmatrix} A_{11} & 0 \\ \hat{B}_{w_1} & \hat{A}_{w_1} \end{bmatrix}, \quad \hat{B}_e = \begin{bmatrix} A_{12} \\ 0 \end{bmatrix}$$

Then the index (4.1) is converted to

$$J = \int_{t_s}^{\infty} (\hat{y}_e^T \tilde{Q}_{11} \hat{y}_e + y_2^T Q_{22} y_2) dt \tag{4.41}$$

where

$$\tilde{Q}_{11} = \begin{bmatrix} Q_0 + \hat{D}_{w_1}^T \hat{D}_{w_1} & \hat{D}_{w_1}^T \hat{C}_{w_1} \\ \hat{C}_{w_1}^T \hat{D}_{w_1} & \hat{C}_{w_1}^T \hat{C}_{w_1} \end{bmatrix}$$

Hence the optimal sliding surface is

$$y_2 = -Q_{22}^{-1} \hat{B}_e^T P_e \hat{y}_e = -K_e \hat{y}_e \tag{4.42}$$

where P_e is the u.p.d.s. solution of the Riccati equation

$$\hat{A}_e^T P_e + P_e \hat{A}_e - P_e \hat{B}_e Q_{22}^{-1} \hat{B}_e^T P_e = -\tilde{Q}_{11} \tag{4.43}$$

The system in the sliding mode is governed by

$$\dot{\hat{y}}_1 = (\hat{A}_e - \hat{B}_e K_e)\hat{y}_1 \tag{4.44}$$

Theorem 4.4.1 *If*

$$\begin{aligned}\hat{A}_{w_1} &= (A_{11} - A_{12}K) + \hat{B}_{w_1} \\ \hat{C}_{w_1} &= \hat{D}_{w_1}\end{aligned} \tag{4.45}$$

then the eigenvalues of $A_{11} - A_{12}K$ are also the eigenvalues of $\hat{A}_e - \hat{B}_e K_e$.

Proof: Assume $\lambda_1, \lambda_2, \dots, \lambda_{n-m}$ are the eigenvalues of $A_{11} - A_{12}K$ and $v_1, v_2, \dots, v_{n-m}$ the corresponding eigenvectors, respectively. Therefore

$$A_{11} - A_{12}K = V^{-1}\Lambda V$$

where $V = \begin{bmatrix} v_1 & v_2 & \dots & v_{n-m} \end{bmatrix}$ and $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{n-m})$. The Hamiltonian matrix for system (4.40) and index (4.41) is

$$\begin{aligned} H_e &= \begin{bmatrix} \hat{A}_e & -\hat{B}_e Q_{22}^{-1} \hat{B}_e^T \\ -\tilde{Q}_{11} & -\hat{A}_e^T \end{bmatrix} \\ &= \begin{bmatrix} A_{11} & 0 & -A_{12}Q_{22}^{-1}A_{12}^T & 0 \\ \hat{B}_{w_1} & \hat{A}_{w_1} & 0 & 0 \\ -Q_0 - \hat{D}_{w_1}^T \hat{D}_{w_1} & -\hat{D}_{w_1}^T \hat{C}_{w_1} & -A_{11}^T & -\hat{B}_{w_1}^T \\ -\hat{C}_{w_1}^T \hat{D}_{w_1} & -\hat{C}_{w_1}^T \hat{C}_{w_1} & 0 & -\hat{A}_{w_1}^T \end{bmatrix} \end{aligned} \tag{4.46}$$

Since the eigenvalues of $\hat{A}_e - \hat{B}_e K_e$ are the eigenvalues of H_e (see Theorem C.1.1), it is sufficient to prove that the eigenvalues of $A_{11} - A_{12}K$ are the eigenvalues of H_e. Assume

$$\hat{V} = \begin{bmatrix} V \\ -V \\ PV \\ 0 \end{bmatrix}$$

Then

$$\begin{aligned} H_e\hat{V} &= \begin{bmatrix} (A_{11} - A_{12}Q_{22}^{-1}A_{12}^T P)V \\ (\hat{B}_{w_1} - \hat{A}_{w_1})V \\ -\{(Q_0 + A_{11}^T P) + \hat{D}_{w_1}^T \hat{D}_{w_1} - \hat{C}_{w_1}^T \hat{D}_{w_1}\}V \\ -(\hat{D}_{w_1}^T \hat{C}_{w_1} - \hat{C}_{w_1}^T \hat{C}_{w_1})V \end{bmatrix} \\ &= \begin{bmatrix} (A_{11} - A_{12}K)V \\ -(A_{11} - A_{12}K)V \\ (PA_{11} - PA_{12}K)V \\ 0 \end{bmatrix} \end{aligned} \tag{4.47}$$

$$= \begin{bmatrix} V\Lambda \\ -V\Lambda \\ PV\Lambda \\ 0 \end{bmatrix}$$

$$= \hat{V}\Lambda \tag{4.48}$$

Therefore $\lambda_1, \lambda_2, \ldots, \lambda_{n-m}$ are also the eigenvalues of $\hat{A}_e - \hat{B}_e K_e$. □

Lemma 4.4.1 *Assume that the conditions of Theorem 4.4.1 are satisfied and P is the u.p.d.s. solution of ARE (4.38). The u.p.d.s. solution of ARE (4.43) is*

$$P_e = \begin{bmatrix} P + \hat{P} & \hat{P} \\ \hat{P} & \hat{P} \end{bmatrix}$$

where $\hat{P}$ is the u.p.d.s. matrix solution of the Riccati equation

$$A_{w_1}^T \hat{P} + \hat{P} A_{w_1} - \hat{P} A_{12} Q_{22}^{-1} A_{12}^T \hat{P} = -\hat{D}_{w_1}^T \hat{D}_{w_1} \tag{4.49}$$

Proof: Substituting P_e into the left-hand side of (4.43) and using Theorem 4.4.1 and (4.38), yields

$$\hat{A}_e^T P_e + P_e \hat{A}_e - P_e \hat{B}_e Q_{22}^{-1} \hat{B}_e^T = \begin{bmatrix} A_{11} & 0 \\ \hat{B}_{w_1} & \hat{A}_{w_1} \end{bmatrix}^T \begin{bmatrix} P + \hat{P} & \hat{P} \\ \hat{P} & \hat{P} \end{bmatrix} + \begin{bmatrix} P + \hat{P} & \hat{P} \\ \hat{P} & \hat{P} \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ \hat{B}_{w_1} & \hat{A}_{w_1} \end{bmatrix}$$

$$- \begin{bmatrix} P + \hat{P} & \hat{P} \\ \hat{P} & \hat{P} \end{bmatrix} \begin{bmatrix} A_{12} \\ 0 \end{bmatrix} Q_{22}^{-1} \begin{bmatrix} A_{12} \\ 0 \end{bmatrix}^T \begin{bmatrix} P + \hat{P} & \hat{P} \\ \hat{P} & \hat{P} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{matrix} Q_o + \hat{P}(A_{11} - A_{12}Q_{22}^{-1}A_{12}^T P + \hat{B}_{w_1}) + (A_{11} - \\ A_{12}Q_{22}^{-1}A_{12}^T P + \hat{B}_{w_1})^T \hat{P} - \hat{P}A_{12}Q_{22}^{-1}A_{12}\hat{P} \end{matrix} & \begin{matrix} (A_{11} + \hat{B}_{w_1} - A_{12}Q_{11}^{-1}A_{12}^T P)^T \hat{P} + \\ \hat{P}\hat{A}_{w_1} - A_{12}Q_{22}^{-1}A_{12}^T \hat{P} \end{matrix} \\ \begin{matrix} \hat{P}(A_{11} + \hat{B}_{w_1} - A_{12}Q_{22}^{-1}A_{12}^T P)^T + \hat{A}_{w_1}\hat{P} - \\ \hat{P}A_{12}Q_{22}^{-1}A_{12}^T \hat{P} \end{matrix} & \hat{A}_{w_1}\hat{P} + \hat{P}\hat{A}_{w_1} - \hat{P}A_{12}Q_{22}^{-1}A_{12}^T \hat{P} \end{bmatrix}$$

$$= \begin{bmatrix} Q_0 + \hat{D}_{w_1}^T \hat{D}_{w_1} & \hat{C}_{w_1}^T \hat{D}_{w_1} \\ \hat{D}_{w_1}^T \hat{C}_{w_1} & \hat{C}_{w_1}^T \hat{C}_{w_1} \end{bmatrix}$$

$$= \tilde{Q}_{11}$$

□

Therefore, the sliding surface is

$$\begin{aligned} y_2 &= -K_e\hat{y}_1 \\ &= -Q_{22}^{-1}B_e^T P_e\hat{y}_1 \\ &= -Q_{22}^{-1}A_{12}^T(\hat{P}+P)y_1 - Q_{22}^{-1}A_{12}^T P\hat{x}_{w_1} \end{aligned} \tag{4.51}$$

and the reduced order system is

$$\begin{aligned} \dot{\hat{y}}_1 &= (\hat{A}_e - \hat{B}_e K_e)\hat{y}_1 \\ &= \begin{bmatrix} A_{11} - A_{12}Q_{22}^{-1}A_{12}^T(\hat{P}+P) & -A_{12}Q_{22}^{-1}A_{12}^T P \\ \hat{B}_{w_1} & \hat{A}_{w_1} \end{bmatrix} \begin{bmatrix} y_1 \\ \hat{x}_{w_1} \end{bmatrix} \end{aligned} \tag{4.52}$$

4.5 Design of Two Compensators for Frequency-Dependent Weighting Functions

Now consider Case (iv), i.e. both Q_{11} and Q_{22} are functions of ω^2. Let $W_\ell(s)$ be the spectral factor of $Q_{\ell\ell}$, $\ell = 1, 2$. Then $W_\ell(s)$, $\ell = 1$, 2, are the transfer functions of the systems (4.27)-(4.28) and (4.5), respectively. W_1 and W_2 are post- and pre- compensators (see Fig. 4.2). Then the quadratic cost (4.1) is transformed to

$$\begin{aligned} J &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(y_1^*(i\omega)W_1^*(i\omega)W_1(i\omega)y_1(i\omega) + y_2^*(i\omega)W_2^*(i\omega)W_2(i\omega)y_2(i\omega)\right) d\omega \\ &= \int_{t_s}^{\infty} \left(\tilde{v}^T\tilde{v} + \tilde{u}^T\tilde{u}\right) dt \end{aligned} \tag{4.53}$$

where

$$\begin{aligned} \tilde{v} &= W_1(i\omega)y_1(i\omega) \\ \tilde{u} &= W_2(i\omega)y_2(i\omega) \end{aligned} \tag{4.54}$$

are frequency shaped state variables and inputs, respectively. Consider

$$\dot{\tilde{x}}_1 = \tilde{A}\tilde{x}_1 + \tilde{B}x_2 \tag{4.55}$$

$$\tilde{x}_1 = \begin{bmatrix} y_1 \\ x_{w_1} \\ x_{w_2} \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A_{11} & 0 & 0 \\ B_{w_1} & A_{w_1} & 0 \\ 0 & 0 & A_{w_2} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} A_{12} \\ 0 \\ B_{w_2} \end{bmatrix}$$

$$\tilde{Q}_{11} = \begin{bmatrix} D_{w_1}^T D_{w_1} & D_{w_1}^T C_{w_1} & 0 \\ C_{w_1}^T D_{w_1} & C_{w_1}^T C_{w_1} & 0 \\ 0 & 0 & C_{w_2}^T C_{w_2} \end{bmatrix}, \quad \tilde{Q}_{12} = \begin{bmatrix} 0 \\ 0 \\ C_{w_2}^T D_{w_2} \end{bmatrix}, \quad \tilde{Q}_{22} = D_{w_2}^T D_{w_2}$$

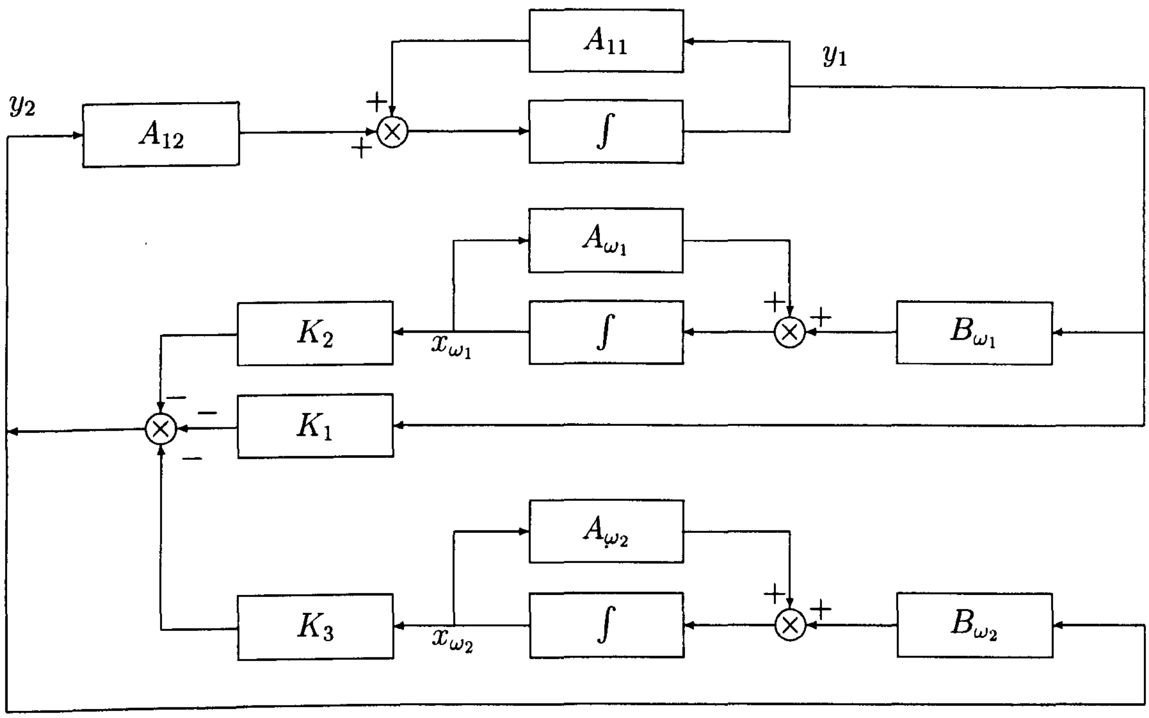

Figure 4.2: Structure of generalized system in the sliding mode

Since $C_{w_1}^T C_{w_1} > 0$ and $D_{w_1}^T D_{w_1} - D_{w_1}^T C_{w_1}(C_{w_1}^T C_{w_1})^{-1} C_{w_1}^T D_{w_1} \geqslant 0$ then

$$\begin{bmatrix} D_{w_1}^T D_{w_1} & D_{w_1}^T C_{w_1} \\ C_{w_1}^T D_{w_1} & C_{w_1}^T C_{w_1} \end{bmatrix}$$

and $\tilde{Q}_{11}$ are semi-p.d.s. matrices which guarantee the existence of an u.p.d.s. solution of the conventional ARE. The quadratic cost (4.1) can be replaced by

$$J = \int_{t_s}^{\infty} (\tilde{x}_1^T \tilde{Q}_{11} \tilde{x}_1 + 2\tilde{x}_1^T \tilde{Q}_{12} y_2 + y_2^T \tilde{Q}_{22} y_2) dt \tag{4.56}$$

which is minimized with respect to y_2. The optimal sliding surface is

$$s = y_2 + K\tilde{x}_1 = 0, \quad K = \tilde{Q}_{22}^{-1}(\tilde{B}^T \tilde{P} + \tilde{Q}_{12}^T) \tag{4.57}$$

where $\tilde{P}$ is a p.d.s. matrix solution of ARE

$$\tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} + (\tilde{P}\tilde{B} + \tilde{Q}_{12})\tilde{Q}_{22}^{-1}(\tilde{B}^T \tilde{P} + \tilde{Q}_{12}^T) = -\tilde{Q}_{11} \tag{4.58}$$

Let K $= [K_1 \quad K_2 \quad K_3]$. Then the sliding surface is given by

$$y_2 + K_1 y_1 + K_2 x_{w_1} + K_3 x_{w_2} = 0 \tag{4.59}$$

Note that in this case the sliding surface is a linear operator of states. Unlike both the previous cases, the sliding is no longer in the phase plane (y_1, y_2). When both Q_{11} and Q_{22} are frequency-dependent, their effect on the system is similar to only one of them being frequency-dependent. However, the sliding surfaces in this case differ, but the structure of the sliding surfaces are similar.

Consider

$$\eta_1 = K_2 x_{w_1} + K_1 y_1$$

$$\eta_2 = K_3 x_{w_2} + y_2$$

Then the systems

$$\begin{aligned} \dot{x}_{w_1} &= A_{w_1} x_{w_1} + B_{w_1} y_1 \\ \eta_1 &= K_2 x_{w_1} + K_1 y_1 \end{aligned} \tag{4.60}$$

and

$$\begin{aligned} \dot{x}_{w_2} &= A_{w_2} x_{w_2} + B_{w_2} y_2 \\ \eta_2 &= K_3 x_{w_2} + y_2 \end{aligned} \tag{4.61}$$

are filters for y_2 and y_1, respectively. These filters are given by the design of the sliding surface.

Remark 4.5.1: All the methods in this chapter can be used for designing control. It is sufficient that Q_{11}, Q_{22}, y_1, y_2 are replaced by Q, R, x and u respectively, using the usual LQ control notation, since y_2 is the input control of the subsystem (2.16).

4.6 Example: Two-Link Robot Manipulator

Robot manipulators are controllable nonlinear mechanical systems. The task of a two-link robot is to move to a given final position as specified by two constant given joint angles. Each link joint has a motor for providing input torque, an encoder for measuring joint position and a tachometer measuring joint velocity. It is desired to design the control or a sliding surface such that the joint positions θ_r and ϕ_r tend to the desired positions θ_{rd} and ϕ_{rd}, which are specified by a motion planning system. When a robot hand is required to moved along a specified path, there is a tracking problem.

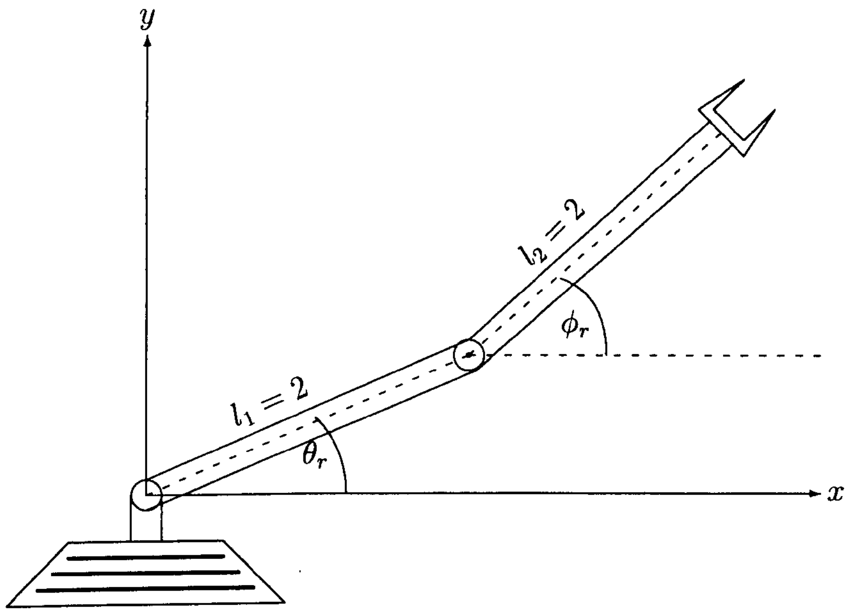

Figure 4.3: Robot manipulator with two link arms

Consider the robot manipulator with two link arm, which moves in a horizontal plane. The nonlinear equations governing its movements are

$$\ddot{\theta}_r = g\left[-J_2 v_1 \dot{\theta}_r + a v_2 \dot{\phi}_r + ab\dot{\theta}_r^2 + bJ_2\dot{\phi}_r^2 + 2bJ_2\dot{\theta}_r\dot{\phi}_r + J_2 u_1 - a u_2\right] \quad (4.62)$$

$$\ddot{\phi}_r = g\left[a v_1 \dot{\theta}_r - h v_2 \dot{\phi}_r - bh\dot{\theta}_r^2 + abJ_2\dot{\phi}_r^2 + 2ab\dot{\theta}_r\dot{\phi}_r - a u_1 + h u_2\right] \quad (4.63)$$

with

$$a = J_2 + 2m_2 l_1 l_2 \cos\phi_r \qquad b = 2m_2 l_1 l_2 \sin\phi_r$$

$$h = J_1 + J_2 + 4m_2 l_1^2 + \tilde{I} + 4m_2 l_2 l_2 \cos\phi_r$$

$$g = = \frac{1}{J_2 h - a^2}$$

where l_i is the length of link i, J_i is the inertia moment of link i about axis i, m_i is the mass of link i, v_i is the viscous friction constant for axis i and $\tilde{I}$ is the moment of inertia of the axis motor [129] (see Fig. 4.3). For simplicity, the cross terms in the equations (4.62) and (4.63) will be ignored in the design, but can be included in the model simulation. This strategy enables suitable results to be obtained [129]. If the cross terms are ignored, equations (4.62) and (4.63) become

$$\begin{bmatrix} \ddot{\theta}_r \\ \ddot{\phi}_r \end{bmatrix} = \begin{bmatrix} -gJ_2 v_1 & g a v_2 \\ g a v_1 & -g h v_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_r \\ \dot{\phi}_r \end{bmatrix} + \begin{bmatrix} gJ_2 & -ga \\ -ga & gh \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (4.64)$$

Let

$$\theta_r = x_1 \qquad \dot{\theta}_r = x_2 \qquad \phi_r = x_3 \qquad \dot{\phi}_r = x_4$$

then

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = \ddot{x}_1 \qquad \dot{x}_3 = x_4 \qquad \dot{x}_4 = \ddot{x}_3$$

and

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -gJ_2v_1 & 0 & gav_2 \\ 0 & 0 & 0 & 1 \\ 0 & gav_1 & 0 & -ghv_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ gJ_2 & -ga \\ 0 & 0 \\ -ga & gh \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.65}$$

Consider the robot manipulator system (4.65)

$$\dot{x} = Ax + Bu$$

with

$$A = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0 & -0.3320 & 0 & 0.0187 \\ 0 & 0 & 0 & 1.0000 \\ 0 & 0.7830 & 0 & -0.1914 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 130.8 & -308.3 \\ 0 & 0 \\ -308.3 & 3155.4 \end{bmatrix}$$

Assume

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -0.3906 & 0 & 0.9206 \\ 0 & -0.9206 & 0 & -0.3906 \end{bmatrix}$$

is the transformation matrix given by (2.13) then

$$A_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -0.3906 & -0.9206 \\ 0.9206 & -0.3906 \end{bmatrix}$$

(*a*) Consider the functional (4.1) with $Q_{11} = I_2$ and $Q_{22}(s) = W_2^*(s)W_2(s)$. Assume

$$W_2(s) = D_{w_2} + C_{w_2}(sI_2 - A_{w_2})^{-1}B_{w_2}$$

with

$$A_{w_2} = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.6 \end{bmatrix}, \qquad B_{w_2} = \begin{bmatrix} 0 & 0 \\ 0.5 & 1 \end{bmatrix}$$

$$C_{w_2} = \begin{bmatrix} 0.3 & 0.9 \\ 0.3 & 0 \end{bmatrix}, \qquad D_{w_2} = \begin{bmatrix} 0.28 & 0.90 \\ 0.30 & -0.02 \end{bmatrix}$$

Then the augmented system is (4.6) with

$$A_e = \begin{bmatrix} 0.3 & 0.1 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_e = \begin{bmatrix} 0 & 0 \\ 0.5000 & 1.0000 \\ -0.3906 & -0.9206 \\ 0.9206 & -0.3906 \end{bmatrix}$$

$$N_e = \begin{bmatrix} 0.1740 & 0.2640 \\ 0.2520 & 0.8100 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad R_e = \begin{bmatrix} 0.1684 & 0.2460 \\ 0.2460 & 0.8104 \end{bmatrix}$$

$$Q_e = \begin{bmatrix} 0.18 & 0.27 & 0 & 0 \\ 0.27 & 0.81 & 0 & 0 \\ 0 & 0 & 1.00 & 0 \\ 0 & 0 & 0 & 1.00 \end{bmatrix}$$

The u.p.d.s solution of ARE (4.9) is

$$P_e = \begin{bmatrix} 685.3062 & -43.8132 & -67.5632 & 2.4334 \\ -43.8132 & 8.3965 & 8.8390 & -0.5894 \\ -67.5632 & 8.8390 & 10.8023 & -0.5776 \\ 2.4334 & -0.5894 & -0.5776 & 0.3376 \end{bmatrix}$$

So

$$K = \begin{bmatrix} 15.2715 & 0.5819 & -0.6878 & 3.1932 \\ 16.8784 & 0.4274 & -0.8771 & -1.2032 \end{bmatrix}$$

and the sliding surface is

$$s = y_2 + \begin{bmatrix} 15.2715 & 0.5819 \\ 16.8784 & 0.4274 \end{bmatrix} x_{w_2} + \begin{bmatrix} -0.6878 & 3.1932 \\ -0.8771 & -1.2032 \end{bmatrix} y_1 = 0$$

The eigenvalues of $A_e - B_e K$ are $-0.3537 \pm 1.0688i$, -0.1495, -3.4473. So the augmented system in the sliding mode is stable. The locations of the resulting eigenvalues depend upon the choice of the proper weighting function Q_{22}. The sliding function $s(t)$ is not the sliding function for the LQ system but is a linear operator of the augmented states and has the form of a dynamic compensator. Simulation results are shown in Fig. 4.4.

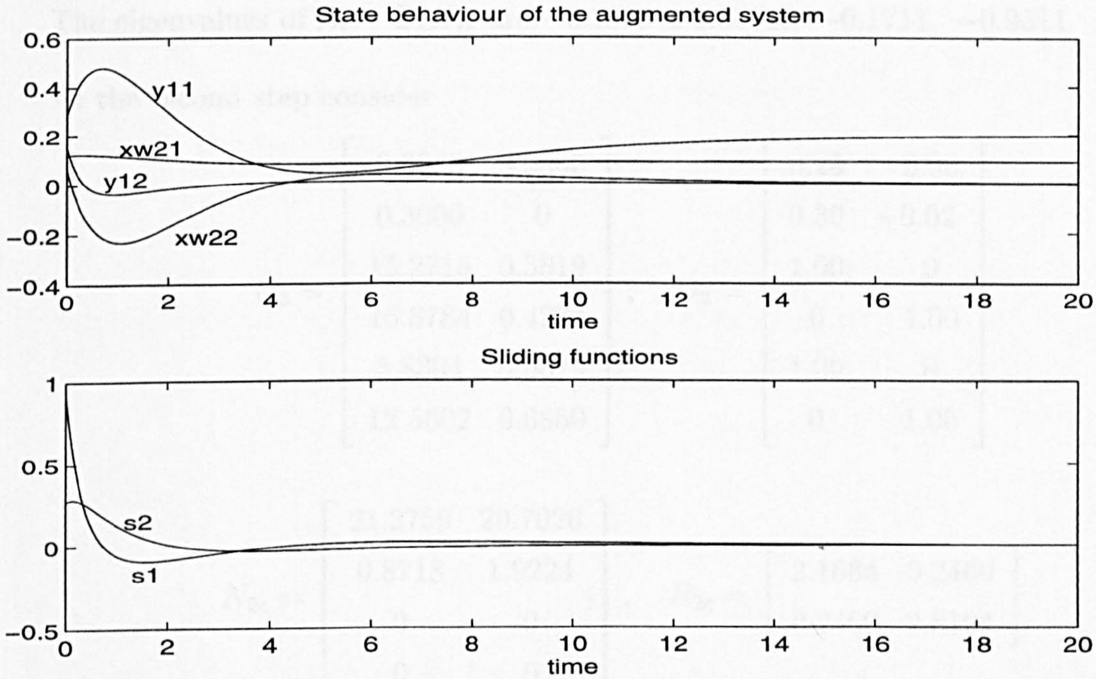

Figure 4.4: Responses of Example (*a*)

The iterative constructive method is now applied. At the first step consider

$$C_1 = \begin{bmatrix} 0.3000 & 0.9000 \\ 0.3000 & 0 \\ 15.2715 & 0.5819 \\ 16.8784 & 0.4274 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.28 & 0.90 \\ 0.30 & -0.02 \\ 1.00 & 0 \\ 0 & 1.00 \end{bmatrix}$$

$$N_{1e} = \begin{bmatrix} 15.4455 & 17.1424 \\ 0.8339 & 1.2374 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad R_{1e} = \begin{bmatrix} 1.1684 & 0.2460 \\ 0.2460 & 1.8104 \end{bmatrix}$$

$$Q_{1e} = \begin{bmatrix} 518.2784 & 16.3706 & 0 & 0 \\ 16.3706 & 1.3313 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

At the first step the gain matrix is

$$K_{1e} = \begin{bmatrix} 5.8304 & 0.0379 & -0.4473 & 0.8252 \\ 12.5602 & 0.6850 & -0.5926 & -0.4663 \end{bmatrix}$$

The eigenvalues of $A_e - B_e K_{1e}$ are $-0.1819 \pm 0.8691i$, -0.1711, -0.9311.

At the second step consider

$$C_2 = \begin{bmatrix} 0.3000 & 0.9000 \\ 0.3000 & 0 \\ 15.2715 & 0.5819 \\ 16.8784 & 0.4274 \\ 5.8304 & 0.0379 \\ 12.5602 & 0.6850 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.28 & 0.90 \\ 0.30 & -0.02 \\ 1.00 & 0 \\ 0 & 1.00 \\ 1.00 & 0 \\ 0 & 1.00 \end{bmatrix}$$

$$N_{2e} = \begin{bmatrix} 21.2759 & 29.7026 \\ 0.8718 & 1.9224 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad R_{2e} = \begin{bmatrix} 2.1684 & 0.2460 \\ 0.2460 & 2.8104 \end{bmatrix}$$

So the second step yields the gain matrix

$$K_{2e} = \begin{bmatrix} 31.1239 & 0.2781 & -3.3756 & 15.7457 \\ 26.4561 & 2.0299 & -0.8349 & -5.6859 \end{bmatrix}$$

The eigenvalues of $A_e - B_e K_{2e}$ are $-0.1031 \pm 0.7789i$, -0.1749, -0.6789. At the third step consider

$$C_3 = \begin{bmatrix} 0.3000 & 0.9000 \\ 0.3000 & 0 \\ 15.2715 & 0.5819 \\ 16.8784 & 0.4274 \\ 5.8304 & 0.0379 \\ 12.5602 & 0.6850 \\ 31.1239 & 0.2781 \\ 26.4561 & 2.0299 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 0.28 & 0.90 \\ 0.30 & -0.02 \\ 1.00 & 0 \\ 0 & 1.00 \\ 1.00 & 0 \\ 0 & 1.00 \\ 1.00 & 0 \\ 0 & 1.00 \end{bmatrix}$$

$$N_{3e} = \begin{bmatrix} 52.3997 & 56.1587 \\ 1.1499 & 3.9523 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad R_{3e} = \begin{bmatrix} 3.1684 & 0.2460 \\ 0.2460 & 3.8104 \end{bmatrix}$$

So the third step yields the gain matrix

$$K_{3e} = \begin{bmatrix} 3.0334 & 0.1311 & -0.2920 & 0.4816 \\ 11.7485 & 0.6377 & -0.4192 & -0.2967 \end{bmatrix}$$

The eigenvalues of $A_e - B_e K_{3e}$ are $-0.1074 \pm 0.8902i$, $-0.1232, -0.5245$.

Therefore, various sliding surfaces and augmented systems with different poles can be found. One can then select a sliding surface with suitable eigenvalues taking into account the actual system requirements.

(b) The following illustrates the case when $\tilde{Q}_{11}$ is a function of frequency but Q_{22} is constant. Assume $Q_0 = 0.05$. Then the u.p.d.s. solution of ARE (4.38) is

$$P = \begin{bmatrix} 0.2236 & 0 \\ 0 & 0.2236 \end{bmatrix}$$

Therefore,

$$K = \begin{bmatrix} -0.0873 & 0.2058 \\ -0.2058 & -0.0873 \end{bmatrix}$$

and $A_{11} - A_{12}K$ has just one eigenvalue -0.2236 (repeated twice). Consider

$$Q_{11}(i\omega) = Q_0 + \hat{Q}_{11}(i\omega)$$

and (4.39) with

$$\hat{A}_{w_1} = \begin{bmatrix} -0.2236 & 0 \\ 0.4000 & 0.2764 \end{bmatrix}, \quad \hat{B}_{w_1} = \begin{bmatrix} 0 & 0 \\ 0.4 & 0.5 \end{bmatrix}, \quad \hat{C}_{w_1} = \begin{bmatrix} 1.5 & 4.50 \\ 1.5 & 0.45 \end{bmatrix}$$

$$\hat{D}_{w_1} = \hat{C}_{w_1}$$

Then

$$\hat{A}_e = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.2236 & 0 \\ 0.4000 & 0.5000 & 0.4000 & 0.2764 \end{bmatrix}, \quad \hat{B}_e = \begin{bmatrix} -0.3906 & -0.9206 \\ 0.9206 & -0.3906 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Taking $Q_{22} = I_2$ and

$$\tilde{Q}_{11} = \begin{bmatrix} 4.5500 & 7.4250 & 4.5000 & 7.4250 \\ 7.4250 & 20.5025 & 7.4250 & 20.4525 \\ 4.5000 & 7.4250 & 4.5000 & 7.4250 \\ 7.4250 & 20.4525 & 7.4250 & 20.4525 \end{bmatrix}$$

The u.p.d.s solution of ARE (4.43) is

$$P_e = \begin{bmatrix} 1.8676 & 1.5054 & 1.6440 & 1.5054 \\ 1.5054 & 4.7733 & 1.5054 & 4.5497 \\ 1.6440 & 1.5054 & 7.1020 & 1.5054 \\ 1.5054 & 4.5497 & 1.5054 & 4.5497 \end{bmatrix}$$

and

$$K_e = \begin{bmatrix} 0.6564 & 3.8063 & 0.7437 & 3.6005 \\ -2.3073 & -3.2503 & -2.1014 & -3.1629 \end{bmatrix}$$

The eigenvalues of $\hat{A}_e - \hat{B}_e K_e$ are -4.8346, -1.3068, and -0.2236 (repeated). So the repeated eigenvalues -0.2236 are the eigenvalues of $A_{11} - A_{12}K$. The sliding surface is

$$y_2 = -\begin{bmatrix} 0.6564 & 3.8063 \\ -2.3073 & -3.2503 \end{bmatrix} y_1 - \begin{bmatrix} 0.7437 & 3.6005 \\ -2.1014 & -3.1629 \end{bmatrix} x_{w_1} \tag{4.66}$$

The reduced order system is

$$\begin{bmatrix} \dot{\hat{x}}_{w_1} \\ \dot{\tilde{y}}_1 \end{bmatrix} = \begin{bmatrix} -1.8677 & -1.5055 & -1.6441 & -1.5055 \\ -1.5055 & -4.7736 & -1.5055 & -4.5500 \\ 0 & 0 & -0.2236 & 0 \\ 0.4000 & 0.5000 & 0.4000 & 0.2764 \end{bmatrix} \begin{bmatrix} \hat{x}_{w_1} \\ \tilde{y}_1 \end{bmatrix}$$

Since $A_{11} = 0$, the eigenvalues of the LQ closed-loop $A_{11} - A_{12}K$ are zero if $Q_0 = 0$. On the other hand, it is desired to find K_e so that the eigenvalues of the closed-loop $A_{11} - A_{12}K$ remain the eigenvalues of the closed-loop $A_e - B_e K_e$. So Q_0 should be selected as a nonzero matrix. Here select $Q_0 = 0.05$. Then $A_{11} - A_{12}K$ is a stable matrix and the stability is given by $A_e - B_e K_e$. If (A_{11}, A_{12}) is not a stabilizable pair then Theorem 4.4.1 fails, because there is not a feedback gain K such that $A_{11} - A_{12}K$ is a stable matrix. Simulation results are shown in Fig. 4.5.

(c) Now consider the case (iv) in which two weighting matrices are frequency-dependent. Suppose

$$A_{w_1} = \begin{bmatrix} 0 & 2 \\ 0.3 & 2 \end{bmatrix}, \quad B_{w_1} = \begin{bmatrix} 0 & 0 \\ 0.4 & 0.5 \end{bmatrix}$$

$$C_{w_1} = \begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix}, \quad D_{w_1} = \begin{bmatrix} 0.95 & 3.00 \\ 1.00 & -0.05 \end{bmatrix}$$

$$A_{w_2} = \begin{bmatrix} 0 & 2.0 \\ 2.0 & 0.5 \end{bmatrix}, \quad B_{w_2} = \begin{bmatrix} 0 & 1.0 \\ 7.0 & 0.4 \end{bmatrix}$$

$$C_{w_2} = \begin{bmatrix} 1.0 & 0 \\ -0.61.0 & \end{bmatrix}, \quad D_{w_2} = \begin{bmatrix} 0.7 & 0 \\ -0.6 & 0.7 \end{bmatrix}$$

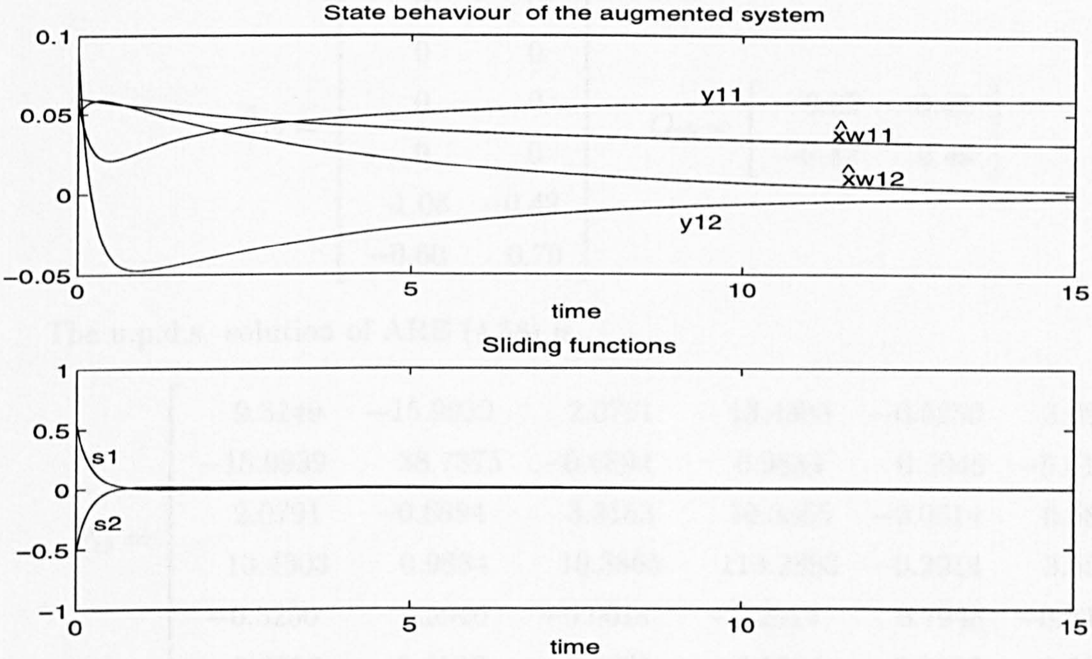

Figure 4.5: Responses of Example (*b*)

Therefore

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0.4 & 0.5 & 0.3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0.2 & 0.5 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} -0.3906 & -0.9206 \\ 0.9206 & -0.3906 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1.0000 \\ 7.0000 & 0.4000 \end{bmatrix}$$

$$\tilde{Q}_{11} = \begin{bmatrix} 1.9025 & 2.8000 & 1.9500 & 2.8500 & 0 & 0 \\ 2.8000 & 9.0025 & 2.9500 & 9.0000 & 0 & 0 \\ 1.9500 & 2.9500 & 2.0000 & 3.0000 & 0 & 0 \\ 2.8500 & 9.0000 & 3.0000 & 9.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.3600 & -0.6000 \\ 0 & 0 & 0 & 0 & -0.6000 & 1.0000 \\ 0 & 0 & 0 & 0 & -0.6000 & 1.0000 \end{bmatrix}$$

$$\tilde{Q}_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1.06 & -0.42 \\ -0.60 & 0.70 \end{bmatrix}, \quad \tilde{Q}_{22} = \begin{bmatrix} 0.85 & -0.42 \\ -0.42 & 0.49 \end{bmatrix}$$

The u.p.d.s. solution of ARE (4.58) is

$$\tilde{P}_{12} = \begin{bmatrix} 9.3149 & -15.9939 & 2.0791 & 13.4303 & -0.5250 & 3.0916 \\ -15.9939 & 38.7375 & -0.6894 & 0.9834 & 0.5946 & -5.6197 \\ 2.0791 & -0.6894 & 3.3153 & 10.3865 & -0.0614 & 0.5821 \\ 13.4303 & 0.9834 & 10.3865 & 113.2893 & -0.2914 & 3.5534 \\ -0.5250 & 0.5946 & -0.0614 & -0.2914 & 0.7948 & -0.2406 \\ 3.0916 & -5.6197 & 0.5821 & 3.5534 & -0.2406 & 1.0681 \end{bmatrix}$$

and

$$K = \begin{bmatrix} 3.8636 & 1.6441 & 2.7856 & 21.5820 & -0.2407 & 1.4231 \\ 0.0128 & -2.7951 & -0.6191 & -5.2115 & 1.7316 & 0.2719 \end{bmatrix}$$

So the sliding surface is given by (4.59) with

$$K_1 = \begin{bmatrix} 3.8636 & 1.6441 \\ 0.0128 & -2.7951 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 2.7856 & 21.5820 \\ -0.6191 & -5.2115 \end{bmatrix}$$

$$K_3 = \begin{bmatrix} -0.2407 & 1.4231 \\ 1.7316 & 0.2719 \end{bmatrix}$$

Simulation results are shown in Fig. 4.6.

So in Case (*c*) two compensators corresponding to the weighting matrices Q_{11} and Q_{22} have been designed. The order of the augmented system is much higher than the LQ system.

Cases (*ii*), (*iii*) and (*iv*) can be applied to practical problems as illustrated by the examples in Section 4.6. However, if it is desired to apply the iterative method, Case (*ii*) (Example (*a*)) is more suitable than (*iii*) and (*iv*) (Examples (*b*) and (*c*)). To apply Theorem 4.4.1, Case (*iii*) (as in Example (*b*)) is recommended, while for Case (*iv*) (Example (*c*)) improves the stability of the system more strongly than the other cases.

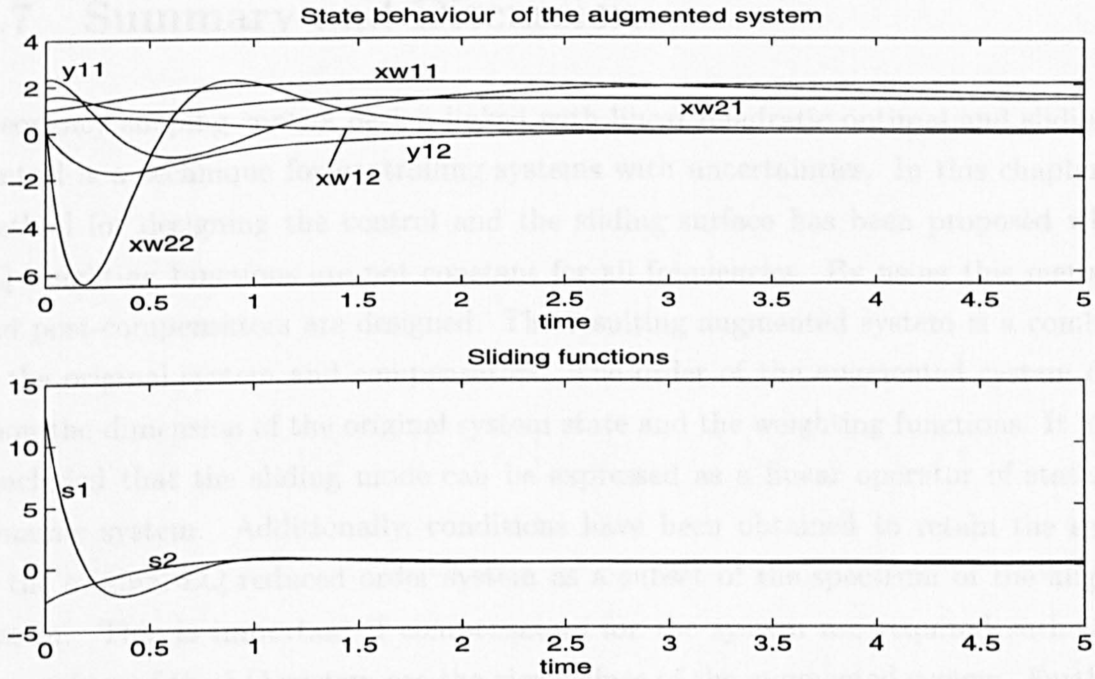

Figure 4.6: Responses of Example (*c*)

In all cases the sliding function has the form of a dynamic compensator. The poles of Q_{22} in Case (*ii*) and the zeros of Q_{11} in Case (*iii*) correspond to the compensator poles and zeros, respectively. In Case (*iv*) there are both. All these compensators influence directly the reduced order closed-loop transfer function. Example 4.6 shows that the frequency shaping of the sliding mode can provide additional flexibility for the design of the sliding mode using LQ techniques. Suitable design of the sliding surface may depend on the (*i*) model validity, (*ii*) actuator/sensor characteristics, (*iii*) sliding function objectives and (*iv*) disturbance spectrum [48].

It is assumed that Q_{11} and Q_{22} are proper weighting functions. Generally, there may be cancellations between the poles of one compensator and the transmission zeros of the other compensator system, when Q_{11} and Q_{22} are not proper functions. A modified method for this case has been presented in [48].

4.7 Summary and Discussion

Frequency shaping control design linked with linear quadratic optimal and sliding mode control is a technique for controlling systems with uncertainties. In this chapter a new method for designing the control and the sliding surface has been proposed when the LQ weighting functions are not constant for all frequencies. By using this method pre- and post-compensators are designed. The resulting augmented system is a combination of the original system and compensators. The order of the augmented system depends upon the dimension of the original system state and the weighting functions. It has been concluded that the sliding mode can be expressed as a linear operator of states i.e. a dynamic system. Additionally, conditions have been obtained to retain the spectrum of the original LQ reduced order system as a subset of the spectrum of the augmented system. This is important if compensators for the system are required such that the eigenvalues of the LQ system are the eigenvalues of the augmented system. Furthermore an iterative constructive procedure has been developed to obtain the optimal sliding mode. This method enables one to find various sliding surfaces and by comparing the eigenvalue locations in the left-hand half-plane a sliding surface can then be selected to suit.

Further research should address H_∞ and the sliding mode, and extend the work in [55]. Possibly a generalized system can be found and then one can use H_∞ methods. Similarly to [55], by using the H_∞ approach, the sliding gain matrix could be found for Cases (*iii*) and (*iv*) of Section 4.1. However, some H_∞ methods to obtain the feedback gain matrix could be adapted to those augmented systems which have been discussed in this chapter.

Chapter 5

Sliding Mode Controller-Observer Design

5.1 Introduction

All state variables may not be measurable in many practical problems. Then using knowledge of the output of the system a suitable estimate of the state is required. Many new state observation techniques for linear and nonlinear systems have been proposed in recent years. The topic of control of nonlinear systems using feedback linearization can be found in [14], [76], [123], [142] and the extension of linearization in [13] amongst others. The method of Lyapunov-based observer design (Thau [113]) has been extended in [75], [140] and [134]. There is a fundamental limiting condition in sliding mode control to guarantee robustness despite the presence of the uncertainty in the system; namely the 'matching condition', i.e. the range space of the disturbance input distribution matrix must be a subspace of the range space of the control input distribution matrix. Matching conditions for linear and nonlinear systems have been considered in many papers including [29], [32], [107], [108], [121], [142].

Sliding mode observers as well as sliding mode controllers are known for their robustness and insensitivity with respect to unknown parameter variations [19], [28]. The fundamental difference between a sliding mode observer and other observer approaches is that the sliding mode observer is usually a discontinuous (or a continuous approximation to a discontinuous observer in the sense of the bounded layer) such that the state error trajectories move onto a specified attractive hyperplane. Robustness, insensitivity properties and simplicity of design make sliding observers a powerful approach. Analysis and comparison of several kinds of observers can be found in [115] showing that the sliding mode observer is a good approach from the point of view of robustness, implementation, numerical stability, applicability, ease of design tuning and overall evaluation.

Walcott and Żak [123], [142] discussed the state observation of nonlinear dynamic systems with bounded nonlinearities/uncertainties. They advocated an observer design method using Lyapunov and min-max methods. Their approach requires the matching condition and is linked to the strictly positive real condition. Yaz and Azemi [134] presented a method for designing an observer for nonlinear deterministic and stochastic systems, and applied the continuous (boundary layer) gain given by Walcott and Żak [123].

Emel'yanov et al [37] considered the problem of output stabilization for uncertain linear time-invariant SISO systems and proposed a method for designing a reduced order observer, i.e. designing an observer for the reduced order system. Almost all of these techniques are based on Lyapunov and min-max methods, i.e. minimizing the derivative of the Lyapunov function and obtaining suitable sufficient conditions.

Edwards and Spurgeon [32] modified the Utkin observer [121] and extended the discontinuous observer to nonlinear systems. They developed a robust discontinuous observer. Sira-Ramírez et al [108] discussed matching conditions of the sliding mode observer for linear systems and also studied the generalized observer canonical form. Koshkouei and Zinober [69] have presented methods for designing an asymptotically stable observer, the existence of the sliding mode and stability of state reconstruction systems of MIMO linear systems including disturbance input, as presented below.

The most popular and well-known observer approach is that of the Luenberger observer [78]. The full order Luenberger observer uses a gain observer matrix so that the state error decays suitably fast. In practical problems, the gain of the observer should be chosen to give eigenvalues of the error system matrix not too far into the left half-plane, to avoid excessive noise amplification [28]. A sliding mode observer yielding insensitivity to unknown parameter variations and noise, has been proposed by Utkin [121]. The reduced order system (slow subsystem) is included in the constant feedback gain matrix, whereas both the slow and fast subsystems relate to the discontinuous vector. Dorling and Zinober [28] compared the full and reduced order Luenberger observers with the Utkin observer. They reported some difficulty in the selection of an appropriate constant switched gain to ensure that the sliding mode occurs and discussed the elimination of chattering. However, the unmatched uncertainty was shown to affect the ideal dynamics prescribed by the chosen sliding surface.

In this chapter some results about sliding dynamics are first presented and then an observer for a system which may not satisfy the *matching condition*, is developed. This type of observer for a SISO system has been considered in [107], [108] for matched uncertainty (matching condition). However, in these papers the feedforward injection input map, the external compensation signal gain, the stability of reconstruction error system and the condition of the existence of the sliding mode control were not studied. In [107], [108] the sliding mode observer was extended to the generalized observer canonical form.

In Section 5.2.1 some sufficient conditions for the existence of the sliding mode for a SISO system with disturbance input are presented, and then the results are extended to a MIMO system in Section 5.2.2. These conditions ensure that the state trajectory approaches the sliding surface in the presence of unmatched uncertainty. In this case, the disturbance rejection problem for the sliding system may not be completely satisfied, but when the sliding mode occurs, the state trajectory moves within a neighbourhood of the sliding surface to the origin.

An approach for designing a sliding observer and the proof of the stability of the state reconstruction error system for linear time-invariant multivariable systems using the Lyapunov method, is given in Section 5.3. Methods are established to find the feedforward injection map and the external feedforward compensation signal, which correspond to the control input distribution map and the input of the reconstruction error system, respectively. Sufficient conditions for the existence of the sliding mode for the reconstruction error system are proposed such that ultimate boundedness or asymptotic stability of the error system is assured. Sufficient conditions are derived to ensure error system stability and the existence of the sliding mode. When there is unmatched uncertainty, the stability of the system may not be achieved. However, a region exists in which the state error trajectory converges to the sliding surface after a finite time and remains on this surface to the origin.

The significance of our method is that a discontinuous observer for full order systems with disturbance input is designed. This system may not be ideally in the sliding mode and the uncertainty may not satisfy the matching condition. Similar to discontinuous controllers, there are many methods to eliminate observer chattering including a continuous approximation for discontinuous feedforward compensation signals [29].

The basic aim of observer design is to find an estimate for the state and, if the input is unknown, estimate a suitable input. Using the sliding control input form, a suitable estimated input can be obtained. Before observer design is studied, a technique for designing a controller using sliding mode properties is stated and some aspects of the behaviour of the sliding dynamics are studied. Then sufficient conditions are presented for the existence of the sliding mode in the face of unmatched uncertainty.

To establish the stability of the error system, suitable conditions on the disturbance input are needed; (*i*) the matching condition, (*ii*) the convergency of the norm of disturbance input signal to zero, (*iii*) the norm of the disturbance signal bounded on the norm of the output error, i.e. there exists a real function M (or a real number M) such that $||\xi|| \leqslant M||C(x-\hat{x})||$ where ξ is the disturbance input and $\hat{x}$ is the estimate of the state x. Otherwise, the asymptotic stability of the error system may not exist in the presence of the disturbance input. Note that since the output is accessible, so is the estimated output.

Suitable examples regarding the results and conclusions are presented in Section 5.4.

5.2 Sufficient Conditions in Sliding Mode for Systems with Unmatched Uncertainty

In this section the condition for the existence of the sliding mode for control systems including disturbance input, and some results about the sliding dynamics and the reaching time to the sliding surface, are presented. If the range of the distribution disturbance input map is not in the range of the distribution control input map, the disturbance affects the system in the sliding mode. However, the existence of the sliding mode guarantees the state to lie in the vicinity of the sliding surface. The control design using the sliding mode technique, when the constant design gain matrix is a diagonal matrix, is considered.

5.2.1 The SISO System

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + bu(t) + \gamma\xi(t) \tag{5.1}$$

$$y(t) = cx(t) \tag{5.2}$$

where $x \in \mathbb{R}^n$ is the state variable, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ is nonzero vector, u is the scalar input control, $c \in \mathbb{R}^n$ such that $cb \neq 0$, $y \in \mathbb{R}$ is the scalar output and $\gamma \in \mathbb{R}^n$ is the perturbation input map. The map $\xi \in \mathbb{R}$ is the bounded scalar disturbance input, i.e. there exists a nonnegative real number M such that $|\xi| \leqslant M$. For suitable performance the real number M is chosen as small as possible and if $\sup|\xi(t)|$ is known, set $M = \sup|\xi(t)|$. It is assume that (A, b) is completely controllable and (A, c) is completely observable.

A technique of control design using the sliding mode is presented, and then the sliding mode dynamical behaviour is studied. The reaching sliding mode condition is $\dot{y}$ sgn $y < 0$. Then

$$\begin{aligned} \dot{y} \text{ sgn } y &= (cAx + cbu + c\gamma\xi) \text{ sgn } y \\ &= (-cbu_{eq} + cbu) \text{ sgn } y \\ &= -cb(u_{eq} - u) \text{ sgn } y \end{aligned} \tag{5.3}$$

For the right-hand side of (5.3) to be negative

$$\begin{aligned} u < u_{eq} \quad & \text{if } \ cby > 0 \\ u > u_{eq} \quad & \text{if } \ cby < 0 \end{aligned} \tag{5.4}$$

Therefore

$$u = \begin{cases} u_{eq} - K_1 & \text{if } \ cby > 0 \\ u_{eq} & \text{if } \ \ y = 0 \\ u_{eq} + K_2 & \text{if } \ cby < 0 \end{cases} \tag{5.5}$$

where K_1 and K_2 are positive real function design gains. For simplicity, assume

$$K_1 = K_2 = \hat{K}$$

then (5.5) becomes

$$u = u_{eq} - \hat{K} \text{ sgn } (cby) \tag{5.6}$$

The control u has two parts, a linear part u_{eq} and a discontinuous part $u_s = -\hat{K}$ sgn (cby), i.e.

$$u = -\frac{1}{cb}(cAx + c\gamma\xi + K \text{ sgn } y) \tag{5.7}$$

where $K = |cb|\hat{K} > 0$. The right-hand side of (5.7) is known except for the disturbance input ξ which is unknown and is not accessible. So it is necessary to replace ξ by an estimate $\hat{\xi}$ so that the reaching sliding mode condition is satisfied. Choosing

$$\hat{\xi} = M \text{ sgn } (c\gamma y)$$

$$\begin{aligned} u &= -\frac{1}{cb}\left(cAx + (|c\gamma|M + K) \text{ sgn } y\right) \\ &= -\frac{1}{cb}\left(cAx + K_1 \text{sgn } y\right), \qquad K_1 = |c\gamma|M + K \end{aligned} \tag{5.8}$$

Then

$$\dot{y}\ \mathrm{sgn}\ y = c\gamma\xi \mathrm{sgn}\ y - M|c\gamma| - K < 0 \tag{5.9}$$

since $c\gamma\xi$ sgn $y \leqslant |c\gamma\xi| \leqslant M|c\gamma|$. The output dynamics satisfies

$$\dot{y} = c\gamma\xi - (|c\gamma|M + K)\ \mathrm{sgn}\ y$$

If (5.1) is an undisturbed system (i.e. $\xi = 0$), the control is given by (5.7) with $\xi = 0$. In this case, differentiating (5.2) and inserting (5.7) yields

$$\dot{y} = -K\mathrm{sgn}\ y \tag{5.10}$$

and

$$y = -K(t - t_s)\ \mathrm{sgn}\ y$$

which implies that

$$t_s = \frac{|y(0)|}{K}$$

where $y(0)$ is the arbitrary initial condition. Note that for $t \in [0, t_s]$ the state moves from $y(0)$ to the sliding surface $y = 0$. The output dynamics (5.10) shows that y converges asymptotically to $y = 0$, and the rate of change of y is guaranteed to be either $-K$ (or K) for y positive (or negative), i.e. the velocity to the sliding surface $y = 0$ is K.

As already stated, using the reaching sliding mode condition for a system including disturbance signal gives a control (5.7) which depends on the perturbation and disturbance inputs. Therefore, an estimate of the disturbance signal in the control law is needed such that the reaching sliding mode condition is achieved. The control law (5.8) shows that if K_1 is chosen sufficiently large, i.e. $K_1 \geqslant |c\gamma|M$, the control can be chosen independently of ξ. In this case, the condition $K_1 \geqslant |c\gamma|M$ on the control feedback gain K_1 is necessary to satisfy the reaching sliding mode condition. Consider

$$u = -\frac{1}{cb}(cAx + K\ \mathrm{sgn}\ y) \tag{5.11}$$

i.e. the input control is independent of the perturbation signal. The output signal is given by

$$\dot{y} = c\gamma\xi - K\ \mathrm{sgn}\ y \tag{5.12}$$

and

$$y(t) = c\gamma\left(\int_{t_s}^{t} \xi\ dt\right) - K(t - t_s)\ \mathrm{sgn}\ y$$

Multiplying (5.12) by sgn y gives

$$\dot{y} \text{ sgn } y = c\gamma\xi \text{ sgn } y - K \tag{5.13}$$

Hence, a sufficient condition for the existence of the sliding mode is

$$c\gamma\xi \text{ sgn } y < K$$

and a sufficient condition for the existence of the sliding mode is

$$|c\gamma|M < K \tag{5.14}$$

The output y satisfies

$$|y| \begin{cases} \geqslant (|c\gamma|M - K)(t - t_s) & \text{if } \ t < t_s \\ = 0 & \text{if } \ t \geqslant t_s \end{cases}$$

and

$$t_s \leqslant \frac{|y(0)|}{K - M|c\gamma|}$$

5.2.2 The MIMO System

Now the results of Section 5.2.1 are extended to MIMO systems. Recall the time-invariant system (2.4)

$$\dot{x}(t) = Ax(t) + Bu(t) + \Gamma\xi(t) \tag{5.15}$$

$$y(t) = Cx(t) \tag{5.16}$$

where $\Gamma \in \mathbb{R}^{n \times m}$ is the perturbation input map and $\xi \in \mathbb{R}^m$ is the bounded disturbance input, i.e. there exists a positive real number M such that $||\xi|| \leqslant M$. The real number M is chosen as small as possible and if $\sup ||\xi(t)||$ is known, $M = \sup ||\xi(t)||$. Assume that (A, B) is completely controllable and (A, C) completely observable.

Now the sufficient conditions for the existence of the sliding mode in the presence of uncertainty are investigated. Choose the control

$$u = -(CB)^{-1}(CAx + C\Gamma\xi + K_1 \text{sgn } y) \tag{5.17}$$

where the design gain matrix K_1 is a diagonal matrix with positive elements

$$K_1 = \text{diag}(k_1, k_2, \ldots, k_m)$$

Then, for $y^T = \begin{bmatrix} y_1 & y_2 & \dots & y_m \end{bmatrix}$

$$\begin{aligned} y^T\dot{y} &= y^T(CAx + CBu + C\Gamma\xi) \\ &= -y^T K_1 \text{ sgn } y \\ &= -(k_1|y_1| + k_2|y_2| + \cdots + k_m|y_m|) \\ &< 0 \end{aligned} \tag{5.18}$$

Differentiating (5.16) and using (5.17) yields the output signal dynamics

$$\dot{y} = -K_1 \text{sgn } y \tag{5.19}$$

Hence, for any $1 \leqslant i \leqslant m$

$$\dot{y}_i = -k_i \text{sgn } y_i \tag{5.20}$$

and then

$$y_i = -k_i(t - t_{is}) \text{ sgn } y_i \tag{5.21}$$

where t_{is} is the time to reach the surface $y_i = 0$. Therefore, the output behaviour is governed by

$$y = -K_1(t - t_s) \text{ sgn } y \tag{5.22}$$

From (5.21)

$$t_s = \max_{1 \leqslant i \leqslant m} \frac{|y_i(0)|}{k_i} \tag{5.23}$$

Since for all i, $1 \leqslant i \leqslant m$, $|y_i(0)| \leqslant \|y(0)\|$ and $\sigma_m(K_1) \leqslant k_i$, (5.23) yields

$$t_s \leqslant \frac{\|y(0)\|}{\sigma_m(K_1)}$$

Note that for $t \in [0, t_s]$ the state variable x moves to the the sliding surface $y = 0$. The output dynamics (5.22) shows that y asymptotically converges to $y = 0$ and for any i, $1 \leqslant i \leqslant m$, the rate of change of y_i is guaranteed to be $-k_i$ (k_i) for y_i positive (negative), i.e. the velocity of y to the sliding surface $y = 0$ is $[k_1 \; k_2 \; \dots \; k_m]$.

If ξ is unknown, the control law (5.17) cannot be implemented. So an estimate $\hat{\xi}$ of ξ is required. Let $\hat{\xi} = M\text{sgn}\,(Cx)$. The control law (5.17) is converted to

$$\begin{aligned} u &= -(CB)^{-1}(CAx + C\Gamma\hat{\xi} + K_1\text{sgn } y) \\ &= -(CB)^{-1}(CAx + (C\Gamma M + K_1)\text{sgn } y) \\ &= -(CB)^{-1}(CAx + K\text{sgn } y), \qquad K = C\Gamma M + K_1 \end{aligned} \tag{5.24}$$

Therefore the control is given by

$$u = -(CB)^{-1}(CAx + K\text{sgn } y) \tag{5.25}$$

where the matrix $K - C\Gamma M$ is a p.d. matrix. For simplicity, consider the matrix K to be a diagonal matrix with positive real entries such that

$$M\sigma_M(C\Gamma) < \sigma_m(K)$$

Then the output signal is given by

$$\dot{y} = C\Gamma\xi - K\text{sgn } y \tag{5.26}$$

for

$$C\Gamma\xi = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{bmatrix}$$

The reaching condition of the sliding mode is

$$\dot{y}_i y_i < 0, \qquad \forall i \quad 1 \leqslant i \leqslant m \tag{5.27}$$

On the other hand

$$y^T\dot{y} = y^T(C\Gamma\xi - K\text{sgn } y)$$

Hence

$$y^T\dot{y} = y^T \begin{bmatrix} \gamma_1 - k_1 \text{ sgn } y_1 \\ \gamma_2 - k_2 \text{ sgn } y_2 \\ \vdots \\ \gamma_m - k_m \text{ sgn } y_m \end{bmatrix}$$

Therefore, if for all i, $1 \leqslant i \leqslant m$

$$y_i(\gamma_i - k_i \text{ sgn } y_i) < 0$$

then $y^T\dot{y} < 0$. Thus, a sufficient condition for the existence of the sliding mode is that

$$|\gamma_i| < k_i \qquad \forall i \quad 1 \leqslant i \leqslant m \tag{5.28}$$

Hence, if

$$\begin{aligned} \gamma_1^2 + \gamma_2^2 + \ldots + \gamma_m^2 &< \min_{1 \leqslant i \leqslant m} k_i^2 \\ &= \sigma_m^2(K) \end{aligned} \tag{5.29}$$

then (5.28) is true. From (5.29)

$$\|C\Gamma\xi\| < \sigma_m(K) \tag{5.30}$$

But

$$\|C\Gamma\xi\| \leqslant M\sigma_M(C\Gamma)$$

so, if

$$M\sigma_M(C\Gamma) < \sigma_m(K) \tag{5.31}$$

then (5.28) is true, i.e. (5.31) is a sufficient condition for (5.28) and for the existence of the sliding mode control.

If (5.26) is satisfied then

$$\dot{y}_i = \gamma_i - k_i \text{ sgn } y_i \qquad \forall i,\ 1 \leqslant i \leqslant m$$

and

$$y_i = \left(\int_{t_{is}}^{t} \gamma_i \, dt\right) - k_i(t - t_{is}) \text{ sgn } y_i$$

where t_{is} is the time to reach the surface $y_i = 0$. Therefore, (5.27) implies

$$\dot{y}_i \text{ sgn } y_i = \gamma_i \text{ sgn } y_i - k_i < 0$$

Hence, a sufficient condition for the state trajectories to converge to the surface $y_i = 0$ is

$$\gamma_i \text{ sgn } y_i < k_i$$

which holds if (5.28) is satisfied. For any $1 \leqslant i \leqslant m$, there is a real number η_i such that $|\gamma_i| \leqslant \eta_i < k_i$ and $\eta_i \leqslant \|C\Gamma\|M$. Hence the i-th output y_i satisfies

$$|y_i| \begin{cases} \geqslant (\eta_i - k_i)(t - t_{is}) & \text{if} \quad t < t_{is} \\ = 0 & \text{if} \quad t \geqslant t_{is} \end{cases}$$

Then

$$t_{is} \leqslant \frac{|y_i(0)|}{k_i - \eta_i}, \qquad \forall i \quad 1 \leqslant i \leqslant m$$

Assume the condition (5.31) is true, then

$$\begin{aligned} t_s &= \max_{1 \leqslant i \leqslant m} t_{is} \\ &\leqslant \max_{1 \leqslant i \leqslant m} \frac{y_i(0)}{k_i - \eta_i} \\ &\leqslant \frac{\|y(0)\|}{\sigma_m(K) - M\sigma_M(C\Gamma)} \end{aligned}$$

5.3 Sliding Mode Observer Design

Sliding observers potentially offer advantages similar to those of sliding controllers; in particular, inherent robustness to parametric uncertainty and straightforward application to important classes of systems.

Here state estimation for the system (5.15)-(5.16) is considered so that the estimate of the state is close to the actual state. This yields a reconstruction error system which is asymptotically stable or ultimately bounded. A method for sliding observer design and sufficient conditions for the existence of the sliding mode and the sliding region, are proposed.

A robust observer for the system (5.15)-(5.16) with an estimate of the disturbance input $\xi(t)$ is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + H(y(t) - \hat{y}(t)) + \Gamma\hat{\xi}(t) \tag{5.32}$$

where $\hat{x}$ is the state estimate, $\hat{\xi}$ is an estimate of the disturbance ξ, and $H \in \mathbb{R}^{n\times m}$ is the observer gain matrix. In the absence of uncertainty the observer will be asymptotically stable if H is selected such that $A - HC$ is a stable matrix. Clearly if ξ is known, set $\hat{\xi} = \xi$. The general form of the sliding observer (5.32) for the system (5.15)-(5.16) may be selected as

$$\dot{\hat{x}} = A\hat{x} + Bu + H(y - \hat{y}) + \Lambda v \tag{5.33}$$

$$\hat{y} = C\hat{x} \tag{5.34}$$

where $v \in \mathbb{R}^m$ is an external discontinuous feedforward compensation signal and $\Lambda \in \mathbb{R}^{n\times m}$ is the feedforward injection map such that $C\Lambda$ is a nonsingular matrix. The state reconstruction error is defined as $e = x - \hat{x}$. Subtracting (5.15) from (5.33) gives the dynamical reconstruction error system

$$\dot{e} = (A - HC)e + \Gamma\xi - \Lambda v \tag{5.35}$$

$$e_y = Ce \tag{5.36}$$

where $e_y = y - \hat{y}$ is the output reconstruction error. The initial state $x_0 = x(t_0)$ is unknown and $\hat{x}_0 = \hat{x}(t_0)$ can be arbitrary assigned. A suitable value for $\hat{x}_0$ is a point on the sliding surface $y = 0$, i.e. $c\hat{x}_0 = 0$.

If the control u is not directly accessible, the conventional estimate is in the form (5.25), i.e.

$$u = -(CB)^{-1}(CA\hat{x} + K\operatorname{sgn}(C\hat{x})) \tag{5.37}$$

where the matrix K is a diagonal matrix with positive entries such that

$$M\sigma_M(C\Gamma) < \sigma_m(K)$$

The ideal sliding mode for the system (5.35)-(5.36) satisfies $e_y = 0$, $\dot{e}_y = 0$ [29]. The virtual equivalent feedforward input is given by

$$v_{eq} = (C\Lambda)^{-1}(CAe + C\Gamma\xi) \tag{5.38}$$

Substituting (5.38) in the state reconstruction error system (5.35) gives the reduced order system

$$\dot{e} = (I - \Lambda(C\Lambda)^{-1}C)Ae + (I - \Lambda(C\Lambda)^{-1}C)\Gamma\xi \tag{5.39}$$

with m of the eigenvalues (5.39) zero and the $n-m$ remaining eigenvalues to be assigned [34]. The reduced order system is independent of the disturbance input signal if there exists an $m \times m$ matrix D such that

$$\Gamma = \Lambda D \tag{5.40}$$

The error system in the sliding mode is now studied. Using the transformation T (2.13), the system (2.4) is converted to the system (2.15). Let $CT^T = [C_1 \;\; C_2]$. Consider a second transformation

$$T_s = \begin{bmatrix} I_{n-m} & 0 \\ C_1 & C_2 \end{bmatrix} \tag{5.41}$$

Then

$$T_sTx = \begin{bmatrix} y_1 \\ y \end{bmatrix} \tag{5.42}$$

and the system (5.15)-(5.16) is converted to

$$\dot{y}_1(t) = \hat{A}_{11}y_1(t) + \hat{A}_{12}y(t) + \hat{\Gamma}_1\xi \tag{5.43}$$
$$\dot{y}(t) = \hat{A}_{21}y_1(t) + \hat{A}_{22}y(t) + CBu(t) + \hat{\Gamma}_2\xi \tag{5.44}$$

where

$$\begin{aligned} T_sTAT^TT_s^{-1} &= \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \\ &= \begin{bmatrix} A_{11} - A_{12}C_2^{-1}C_1 & A_{12}C_2^{-1} \\ C_1A_{11} + C_2A_{21} - (C_1A_{12} + C_2A_{22})C_2^{-1}C_1 & (C_1A_{12} + C_2A_{22})C_2^{-1} \end{bmatrix} \end{aligned}$$

and

$$T_sT\Gamma = \begin{bmatrix} \hat{\Gamma}_1 \\ \hat{\Gamma}_2 \end{bmatrix} = \begin{bmatrix} \Gamma_1 \\ C_1\Gamma_1 + C_2\Gamma_2 \end{bmatrix}$$

Using the transformation T_sT the observer (5.33)-(5.34) is given by

$$\dot{\hat{y}}_1(t) = \hat{A}_{11}\hat{y}_1(t) + \hat{A}_{12}\hat{y}(t) + H_1e_y + \Lambda_1 v \quad (5.45)$$

$$\dot{\hat{y}}(t) = \hat{A}_{21}\hat{y}_1(t) + \hat{A}_{22}\hat{y}(t) + CBu(t) + H_2e_y + \Lambda_2 v \quad (5.46)$$

where

$$T_sT\hat{x} = \begin{bmatrix} \hat{y}_1 \\ \hat{y} \end{bmatrix}, \quad T_sTH = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \quad (5.47)$$

Subtracting (5.43)-(5.44) from (5.45)-(5.46), the error system is given by

$$\dot{e}_1(t) = \hat{A}_{11}e_1(t) + \hat{A}_{12}e_y(t) + \hat{\Gamma}_1\xi - \Lambda_1 v - H_1e_y \quad (5.48)$$

$$\dot{e}_y(t) = \hat{A}_{21}e_1(t) + \hat{A}_{22}e_y(t) + \hat{\Gamma}_2\xi - \Lambda_2 v - H_2e_y \quad (5.49)$$

The sliding mode occurs if $e_y = 0$ and $\dot{e}_y = 0$. Assume that Λ_2 is a nonsingular matrix. The equivalent feedforward input (5.38) is obtained from the subsystem (5.49)

$$v_{eq} = \Lambda_2^{-1}\left(A_{12}e_1 + \hat{\Gamma}_2\xi\right) \quad (5.50)$$

The subsystem (5.48) yields the error system in the sliding mode

$$\dot{e}_1(t) = \hat{A}_{11}e_1(t) + \hat{\Gamma}_1\xi - \Lambda_1 v_{eq} \quad (5.51)$$

where v_{eq} is the equivalent feedforward input. Substituting (5.50) in (5.51), the reduced order error system is

$$\dot{e}_1(t) = (\hat{A}_{11} - \Lambda_1\Lambda_2^{-1}\hat{A}_{21})e_1(t) + (\hat{\Gamma}_1 - \Lambda_1\Lambda_2^{-1}\hat{\Gamma}_2)\xi \quad (5.52)$$

Since (A, C) is observable, the pair $(\hat{A}_{11}, \hat{A}_{21})$ is also observable and $\hat{A}_{11} - \Lambda_1\Lambda_2^{-1}\hat{A}_{21}$ can be assigned arbitrary eigenvalues with negative real parts by a suitable choice of Λ. The bounded inputs $\Gamma\xi$ guarantee the bounded error e, but the asymptotic stability (5.52) is not guaranteed in general. However, some sufficient conditions that ensure the stability of the reduced order error systems. A sufficient condition for the reduced order system to be free of the influence of the disturbance ξ is that

$$\hat{\Gamma}_1 - \Lambda_1\Lambda_2^{-1}\hat{\Gamma}_2 = 0 \quad (5.53)$$

If condition (5.40) holds, then (5.53) is also satisfied.

Remark 5.3.1 It is possible to find the reduced order system (5.52) directly. Consider the transformation

$$T_\Lambda = \begin{bmatrix} I_{n-m} & -\Lambda_1\Lambda_2^{-1} \\ 0 & I_m \end{bmatrix} \tag{5.54}$$

Then the error system (5.48)-(5.49) is converted to

$$\dot{\tilde{e}}_1(t) = \tilde{A}_{11}e_1(t) + \tilde{A}_{12}e_y - (H_1 - \Lambda_1\Lambda_2^{-1}H_2)e_y(t) + (\hat{\Gamma}_1 - \Lambda_1\Lambda_2^{-1}\hat{\Gamma}_2)\xi \tag{5.55}$$

$$\dot{e}_y(t) = \hat{A}_{21}\tilde{e}_1(t) + \tilde{A}_{22}e_y(t) + \hat{\Gamma}_2\xi - H_2e_y - \Lambda_2 v \tag{5.56}$$

where $\tilde{A}_{11} = \hat{A}_{11} - \Lambda_1\Lambda_2^{-1}\hat{A}_{21}$, $\tilde{A}_{12} = \hat{A}_{12} - \Lambda_1\Lambda_2^{-1}\hat{A}_{22} + \tilde{A}_{11}\Lambda_1\Lambda_2^{-1}$ and $\tilde{A}_{22} = \hat{A}_{22} + \hat{A}_{21}\Lambda_1\Lambda_2^{-1}$, $\tilde{e}_1 = e_1 - \Lambda_1\Lambda_2^{-1}e_y$. In the sliding mode

$$\dot{\tilde{e}}_1 = (\hat{A}_{11} - \Lambda_1\Lambda_2^{-1}\hat{A}_{21})\tilde{e}_1 + (\hat{\Gamma}_1 - \Lambda_1\Lambda_2^{-1}\hat{\Gamma}_2)\xi \tag{5.57}$$

Since in the sliding mode $\tilde{e}_1 = e_1$, the reduced order system (5.57) coincides with (5.52).

Now it is desired to obtain H, Λ and v such that the stability of the observer system is preserved. The observer gain H can be found in two ways; pole assignment methods, i.e. assigning n prespecified eigenvalues to the matrix $A - HC$; and the LQ method. The eigenvalues of $A - HC$ and $A^T - C^TH^T$ are the same, so the problem is that of finding a feedback for the dual system corresponding to (5.15) such that the eigenvalues are the prespecified values. (A, C) is observable if and only if (A^T, C^T) is controllable. Therefore, the observability of (A, C) guarantees the existence H.

Now the LQ method is utilized to find H. Therefore the algebraic Riccati equation (ARE)

$$AP + PA^T - PC^TR^{-1}CP = -Q \tag{5.58}$$

with Q, R are arbitrary semi-p.d.s. and p.d.s. matrices respectively, has a u.p.d.s. matrix solution P. Then $A^T - C^TH^T$ is stable with

$$H = PC^TR^{-1} \tag{5.59}$$

which is equivalent to the stability of $A - HC$.

The matrix Λ can be found in several ways:

1. Take
$$\Lambda = PC^T\hat{R}^{-1}$$
where $\hat{R}$ is an arbitrary matrix. If $\hat{R} = R$ then $\Lambda = H$. This choice of Λ may be suitable if there exists uncertainty in the output.

2. Let (A, Λ) be a controllable pair. The vector Λ in (5.35) should be obtained so that the stability of the reduced order system or the eigenvalue allocation of the reduced order system

$$\begin{aligned} \dot{e} &= \left(I - \Lambda \left(C\Lambda\right)^{-1} C\right) Ae \\ Ce &= 0 \end{aligned} \tag{5.60}$$

is achieved. The system (5.60) has m zero eigenvalues and the $n - m$ remaining stable eigenvalues can be freely selected [29], [34], [34], [141].

3. Let D be a nonzero matrix. Take $\Lambda D = \Gamma$ if $\Gamma \neq 0$. In this case, the error system (5.35) in the sliding mode is independent of the perturbation signal.

4. In the sliding mode the systems (5.15) and (5.35) are independent of ξ only if there exist matrices D and $\hat{D}$ such that $\Gamma = BD$ and $\Gamma = \Lambda\hat{D}$, respectively. Therefore if both the systems are independent of ξ, both these condition are simultaneously satisfied. In this case the ideal sliding mode dynamics take place simultaneously on $y = 0$ and $e_y = 0$.

5. The vector Λ can generally be found such that the state reconstruction error system (5.35) is asymptotically stable, i.e.

$$\lim_{t \to \infty} e(t) = 0 \tag{5.61}$$

To ensure that the state approaches and crosses the sliding surface sufficiently fast, v should be a discontinuous function. Let P_f be the u.p.d.s. solution of the Lyapunov equation

$$(A - HC)P_f + P_f(A - HC)^T = -Q_f \tag{5.62}$$

where Q_f is an arbitrary p.d.s. matrix. Consider the discontinuous feedforward input

$$v = W\frac{Ce}{\|Ce\|} \tag{5.63}$$

where W is an $m \times m$ diagonal p.d. matrix with

$$\lambda_{\min}(W) \geqslant M\|D\|\frac{\lambda_{\max}(CP_f^{-1}C^T)}{\lambda_{\min}(CP_f{}^{-1}C^T)} \tag{5.64}$$

Assume the condition (5.40) is satisfied. It is desired to find conditions such that

$$\lim_{t \to \infty} e(t) = 0$$

Set

$$\Lambda = P_f^{-1} C^T W^{-1} \tag{5.65}$$

Note that $C\Lambda$ is a nonsingular matrix and W is a p.d. matrix. Therefore $C\Lambda W$ is nonsingular and

$$\lambda_{\min}(C\Lambda W) = \lambda_{\min}(CP_f^{-1}C^T) \neq 0$$

The quadratic stability of the reconstruction error system is guaranteed by (5.64) and (5.65). A Lyapunov function candidate for (5.35) is

$$V(e) = e^T P_f e \tag{5.66}$$

If $Ce \neq 0$, then

$$\begin{aligned}
\dot{V} &= e^T((A-HC)P_f + P_f(A-HC)^T)e + e^T P_f \Gamma\xi + (\Gamma\xi)^T P_f e - (\Lambda v)^T P_f e - e^T P_f \Lambda v \\
&= e^T((A-HC)P_f + P_f(A-HC)^T)e + 2e^T P_f \Gamma\xi - 2e^T P_f \Lambda v \\
&= e^T((A-HC)P_f + P_f(A-HC)^T)e + 2e^T P_f \Lambda D\xi - 2e^T P_f \Lambda v \\
&= e^T((A-HC)P_f + P_f(A-HC)^T)e + 2e^T P_f P_f^{-1} C^T W^{-1} D\xi - 2e^T P_f P_f^{-1} C^T W^{-1} v \\
&= e^T((A-HC)P_f + P_f(A-HC)^T)e + 2e^T C^T W^{-1} D\xi - 2e^T C^T \frac{Ce}{\|Ce\|} \\
&\leqslant -e^T Q_f e + 2\,\|e^T C^T\| \left(\|W^{-1} D\| M - 1\right) \\
&\leqslant -e^T Q_f e + 2\,\|e^T C^T\| \left(\frac{1}{\lambda_{\min}(W)}\|D\| M - 1\right) \\
&< 0
\end{aligned} \tag{5.67}$$

since

$$\lambda_{\min}(W) \geqslant M\|D\| \frac{\lambda_{\max}(CP_f^{-1}C^T)}{\lambda_{\min}(CP_f^{-1}C^T)} \geqslant M\|D\|$$

If $Ce = 0$, $v = v_{eq}$ and

$$\begin{aligned}
\dot{V} &= e^T((A-HC)P_f + P_f(A-HC)^T)e + 2e^T P_f \Gamma\xi - 2e^T P_f \Lambda v \\
&= -e^T Q_f e + 2e^T P_f P_f^{-1} C^T W^{-1} D\xi - 2e^T P_f P_f^{-1} C^T W^{-1} v_{eq} \\
&= -e^T Q_f e + 2e^T C^T W^{-1} D\xi - 2e^T C^T W^{-1} v_{eq} \\
&= -e^T Q_f e \\
&< 0
\end{aligned} \tag{5.68}$$

Therefore

$$\lim_{t\to\infty} e(t) = 0$$

Now a condition for the existence of the sliding mode is found. Since the system is stable, the convergent sliding mode exists, i.e. $\lim_{t\to\infty} e_y(t) = 0$. The problem is that it is necessary to find a region, the so-called sliding region, so that after a finite time, the state error trajectory lies in the vicinity of the sliding surface and inside the region. Consider

$$
\begin{aligned}
e_y^T \dot{e}_y &= e_y^T\left((CA - CHC)e + C\Gamma\xi\right) - e_y^T C\Lambda W \frac{Ce}{\|Ce\|} \\
&\leqslant \|Ce\|\ \|C(A-HC)e\| + e^T C^T C\left(P_f^{-1}C^T W^{-1}\right) D\xi - e^T C^T C P_f^{-1} C^T W^{-1} W \frac{Ce}{\|Ce\|} \\
&\leqslant \|Ce\|\ \|C(A-HC)e\| + e^T C^T \left(CP_f^{-1}C^T\right) W^{-1} D\xi - e^T C^T \left(CP_f^{-1}C^T\right) \frac{Ce}{\|Ce\|} \\
&\leqslant \|Ce\|\ \|C(A-HC)e\| + \|e^T C^T\| \lambda_{\max}(CP_f^{-1}C^T)\|W^{-1}D\| M \\
&\qquad - e^T C^T \lambda_{\min}\left(CP_f^{-1}C^T\right) \frac{Ce}{\|Ce\|} \\
&\leqslant \|Ce\| \left[\|C(A-HC)\| . \|e\| + \lambda_{\max}(CP_f^{-1}C^T)\|W^{-1}D\|M - \lambda_{\min}(CP_f^{-1}C^T)\right] \quad (5.69)
\end{aligned}
$$

A sufficient condition for the sliding mode is that the right-hand side of (5.69) be non-positive. Thus

$$
\|e\| \leqslant \frac{\lambda_{\min}(CP_f^{-1}C^T) - \lambda_{\max}\left(CP_f^{-1}C^T\right)\|W^{-1}D\|M}{\sigma_M(C(A-HC))} = r_s \tag{5.70}
$$

or

$$
\|e\| \leqslant \frac{\lambda_{\min}(CP_f^{-1}C^T) - \lambda_{\max}(CP_f^{-1}C^T)\|W^{-1}D\|M}{\sigma_M(C)\sigma_M(A-HC)} = \hat{r}_s \tag{5.71}
$$

Remark 5.3.2: One may choose $\Lambda = P_f^{-1}C^T$, then all the conditions for the stability of the system and sliding mode remain intact, and only the velocity of the state approaching the origin and the state trajectory dynamics on the sliding surface may differ.

In [142] it has been shown that when $V(e) = e^T P_f e$ and $\dot{V}(e) \leqslant -e^T Q_f e$, then the norm $\|e(t)\|$ tends to zero at least as fast as a certain exponential function, so

$$
-\frac{\dot{V}}{V} \geqslant \frac{e^T Q_f e}{e^T P_f e} \tag{5.72}
$$

Let

$$
\mu = \min_{e} \left\{e^T Q_f e \mid e^T P_f e = 1\right\} \tag{5.73}
$$

then

$$
-\frac{\dot{V}}{V} \geqslant \mu \tag{5.74}
$$

To determine the minimum of (5.73), consider the Lagrange multiplier function

$$
G(e,\lambda) = e^T Q_f e - \lambda(e^T P_f e - 1)
$$

Then $e_{\min}$, the minimum of e, satisfies

$$\begin{aligned} \frac{\partial G}{\partial e} = Q_f e_{\min} - \lambda P_f e_{\min} &= 0 \\ e_{\min}^T P_f e_{\min} - 1 &= 0 \end{aligned} \tag{5.75}$$

From (5.75) one can conclude that

$$(P_f^{-1} Q_f - \lambda) e_{\min} = 0$$

i.e. λ is an eigenvalue of $P_f^{-1} Q_f$ and $e_{\min}$ is the corresponding eigenvector. Multiplying (5.75) by $e_{\min}^T$ gives

$$e_{\min}^T Q_f e_{\min} - \lambda e_{\min}^T P_f e_{\min} = 0$$

with $\mu = \lambda$. From (5.74)

$$V(e(t)) \leqslant V(e(t_0))\ e^{-\mu(t-t_0)}$$

so

$$\begin{aligned} \|e\|^2 &\leqslant \frac{e^T P_f e}{\lambda_{\min}(P_f)} \\ &\leqslant \frac{V(e(t_0))}{\lambda_{\min}(P_f)}\ e^{-\mu(t-t_0)} \end{aligned} \tag{5.76}$$

Hence $e(t)$ is bounded and approaches zero at least as fast as $e^{-\frac{\mu t}{2}}$.

Remark 5.3.3: Consider the discontinuous feedforward input

$$v = W \text{sgn}\ e_y, \qquad \text{if } e_y \neq 0$$

where W is an $m \times m$ p.d. matrix which satisfies in the condition (5.64) and sgn e_y indicates the signum function of e_y. Then all the conditions for the existence of the sliding mode and stability (5.64)-(5.70) are satisfied for this feedforward compensation signal.

5.3.1 Sliding Error System with Unmatched Uncertainty

We now develop new theory relating to unmatched uncertainty. If the error system in the face of uncertainty does not satisfy the matching condition, the error system stability is not generally guaranteed. However, if some conditions on the uncertainty rather than boundedness are available, an asymptotic observer is achieved. The behaviour of the system depends upon the norm of a matrix which is named the 'unmatched uncertainty matrix', (or portion [10]). The norm of this matrix, ϵ, is called the 'unmatched uncertainty distance'. If the 'unmatched uncertainty distance' is zero, the matching condition is completely satisfied. Let $D \in \mathbb{R}^{m\times m}$ be a matrix such that

$$\begin{bmatrix} 0_{m\times(n-m)} & I_m \end{bmatrix} \Gamma = \begin{bmatrix} 0_{m\times(n-m)} & I_m \end{bmatrix} \Lambda D \tag{5.77}$$

Set $E = \Gamma - \Lambda D$ and $||E|| = \epsilon$.

Definition 5.3.1: Consider the system (5.15). The uncertainty $\Gamma\xi$ is said to satisfy a matching condition with ϵ-approximation (or briefly ϵ-matching condition) if there exists a matrix D such that

(*i*) condition (5.77) holds;

(*ii*) $||E|| = ||\Gamma - \Lambda D|| = \epsilon$.

If $\epsilon = 0$ the uncertainty is said to satisfy the matching condition.

Sometimes the term 'matching condition with ϵ distance' is utilized instead of the ϵ-matching condition. When an ϵ-matching condition holds, $\Gamma = \Lambda D + E$ where $\epsilon = ||E||$. Matrix ΛD is the matched uncertainty matrix (portion) and E is the unmatched uncertainty matrix (portion) of the matrix Γ. So the unmatched distance is $||E|| = \epsilon$. If $\epsilon \neq 0$, the error system (5.35) may no longer be generally asymptotically stable. But as already stated, if a condition on the disturbance input like the convergence of the norm of disturbance input signal to zero or the norm of the disturbance signal is bounded on the norm of the output error, the system is asymptotically stable. Otherwise, only ultimate boundedness results. So the state error trajectory enters a region centred on the origin and thereafter remains within this region. See Appendix A for the definitions of ultimate boundedness and uniform ultimate boundedness.

Consider the Lyapunov function (5.66). Then similarly to (5.67) one obtains

$$\begin{aligned} \dot{V} &= e^T\left((A-HC)P_f + P_f(A-HC)^T\right)e + 2e^TP_f(\Lambda D + E)\xi - 2e^TP_f\Lambda v \\ &\leqslant -||e||^2\lambda_{\min}(Q_f) + 2||e||||P_fE||M \\ &= -||e||\left\{(1-\theta)\lambda_{\min}(Q_f)||e|| + \theta\lambda_{\min}(Q_f)||e|| - 2||P_fE||M\right\} \end{aligned} \tag{5.78}$$

where $0 < \theta \leqslant 1$. So if

$$||e|| \geqslant \frac{2||P_fE||M}{\theta\lambda_{\min}(Q_f)} \tag{5.79}$$

$\dot{V} < 0$. Since V is a monotonically decreasing function on the outside of the set

$$\Omega_E = \left\{ e \in \mathbb{R}^n \mid \|e\| \leqslant \frac{2\|P_f E\| M}{\theta \lambda_{\min}(Q_f)} = r_\Omega \right\} \tag{5.80}$$

for $\|e\| = \dfrac{2\|P_f E\| M}{\theta \lambda_{\min}(Q_f)}$, the maximum value of V on the compact set Ω_E is attained and the state error trajectory enters the ellipsoid

$$\mathscr{E}_E = \{ e \in \mathbb{R}^n \mid V(e(t)) \leqslant r^2 \} \tag{5.81}$$

where

$$r = \frac{2M\|P_f E\| \sqrt{\lambda_{\max}(P)}}{\theta \lambda_{\min}(Q_f)}$$

The set $\mathscr{E}_E$ is closed and bounded. So according to the Heine-Borel theorem the set $\mathscr{E}_E$ is a compact set. The solution of the error system is uniformly ultimately bounded with ultimate boundedness ratio

$$b_u = \frac{2\|P_f E\| M}{\theta \lambda_{\min}(Q_f)} \sqrt{\frac{\lambda_{\max}(P_f)}{\lambda_{\min}(P_f)}} \tag{5.82}$$

The ultimate boundedness radius r_u is

$$r_u = \inf \{ r \in \mathbb{R} \mid \mathscr{E}_E \subset B_r \}$$

where B_r is a ball with radius r centred on the origin. Note that all trajectories starting inside $\mathscr{E}_E$, remain within this set for all future time, and all trajectories starting outside $\mathscr{E}_E$ enter this compact set within a finite time and remain inside thereafter. Hence $e(t)$ is bounded by

$$\|e(t)\| \leqslant \max \{ \|e(0)\|, b_u \} \tag{5.83}$$

If $\|e(0)\|$ is sufficiently small, then $\|e(t)\| \leqslant b_u$ for all t.

Remark 5.3.4: Since $\|P_f E\| \leqslant \|P_f\|.\|E\| = \epsilon \|P_f\|$, one can conclude that

$$\dot{V} \;\leqslant\; -\|e\| \{ (1-\theta)\lambda_{\min}(Q_f)\|e\| + \theta \lambda_{\min}(Q_f)\|e\| - 2\epsilon \|P_f\| M \} \tag{5.84}$$

So V is a monotonically decreasing function on the outside of the set

$$\Omega_\epsilon = \left\{ e \in \mathbb{R}^n \mid \quad \|e\| \leqslant \frac{2\epsilon M \lambda_{\max}(P_f)}{\theta \lambda_{\min}(Q_f)} \right\} \tag{5.85}$$

The ratio $\lambda_{\max}(P_f)/\lambda_{\min}(Q_f)$ is minimized by the choice $Q = I$ (see Corollary 6.8.1). Consider

$$\mathscr{E}_\epsilon = \{ e \in \mathbb{R}^n \mid V(e(t)) \leqslant {r_1}^2 \} \tag{5.86}$$

where

$$r_1 = \frac{2\epsilon M \lambda_{\max}^{3/2}(P_f)}{\theta \lambda_{\min}(Q_f)}$$

The state error trajectory enters the ellipsoid $\mathscr{E}_\epsilon$ in a finite time and remains inside thereafter.

To obtain explicit bounds on $e(t)$ and $e(t)^T P_f e(t)$ and then show that the state trajectory enters the set $\mathscr{E}_E$ in a finite time and remain inside thereafter, consider

$$\begin{aligned} \frac{d(e(t)^T P_f e(t) e^{\mu t})}{dt} &= \frac{d\left(e^T(t) P_f e(t)\right)}{dt} e^{\mu t} + \mu e^T(t) P_f e(t) e^{\mu t} \\ &\leqslant \left(-\|e(t)\|^2 \lambda_{\min}(Q_f) + 2\|e(t)\| \|P_f E\| M\right) e^{\mu t} + \lambda_{\min}(Q_f) \|e(t)\|^2 e^{\mu t} \\ &= 2\|e(t)\| . \|P_f E\| M e^{\mu t} \end{aligned} \quad (5.87)$$

where $\mu = \lambda_{\min}(Q_f)/\lambda_{\max}(P_f)$. Integrating both sides over the interval $[0, t]$ yields

$$e(t)^T P_f e(t) e^{\mu t} - e^T(0) P_f e(0) \leqslant 2 \int_0^t \|e(t)\| . \|P_f E\| M e^{\mu \tau} d\tau$$

Multiplying both sides by $e^{-\mu t}$ gives

$$e^T P_f e \leqslant e^T(0) P_f e(0) e^{-\mu t} + 2 b_u \|P_f E\| M e^{-\mu t} \int_0^t e^{\mu \tau} d\tau$$

since $\|e\| \leqslant b_u$. Therefore

$$e^T P_f e \leqslant e^T(0) P_f e(0) e^{-\mu t} + \frac{2 b_u M \|P_f E\| \lambda_{\max}(P_f)}{\lambda_{\min}(Q_f)} \left(1 - e^{-\mu t}\right) \quad (5.88)$$

Since after a finite time, $\|e\| \leqslant 2\|P_f E\| M / \theta \lambda_{\min}(Q_f)$, equation (5.88) shows that the state error $e(t)$ converges to the compact set $\mathscr{E}_E$ defined in (5.81), i.e.

$$\lim_{t \to \infty} e^T P_f e \leqslant 4M^2 \|P_f E\|^2 \frac{\lambda_{\max}(P_f)}{\theta \lambda_{\min}^2(Q_f)} \sqrt{\frac{\lambda_{\max}(P_f)}{\lambda_{\min}(P_f)}}$$

and then

$$\lim_{t \to \infty} d(e(t), \mathscr{E}_E) = 0 \quad (5.89)$$

where d denotes the Euclidean metric on $\mathbb{R}^n$ and

$$d(e(t), \mathscr{E}_E) = \inf_{\alpha \in \mathscr{E}_E} d\left(e(t), \alpha)\right)$$

The reaching time to $\mathscr{E}_E$ is finite, otherwise for all $t > 0$, $\dot{V}(e(t)) < 0$ and $V(e(t))$ tends to zero asymptotically. The result (5.89) shows that the state error trajectory enters $\mathscr{E}_E$

at finite time t_e and after this time remains inside $\mathscr{E}_E$. Hence, boundedness is guaranteed in the presence of a bounded disturbance with possibly unknown bound. But the size of $\mathscr{E}_E$ cannot be estimated *a priori* if no bound on the disturbance input is given.

Equation (5.88) also yields

$$\|e\|^2 \leqslant \left(\|e(0)\|^2 e^{-\mu t} + \frac{2b_u M\|P_f E\|}{\lambda_{\min}(Q_f)}(1 - e^{-\mu t})\right) \frac{\lambda_{\max}(P_f)}{\lambda_{\min}(P_f)}$$

Then

$$\begin{aligned} \|e\| &\leqslant \left(\|e(0)\|e^{-\frac{1}{2}\mu t} + \sqrt{\frac{2b_u M\|P_f E\|}{\lambda_{\min}(Q_f)}(1 - e^{-\mu t})}\right)\sqrt{\frac{\lambda_{\max}(P_f)}{\lambda_{\min}(P_f)}} \\ &= \|e(0)\|e^{-\frac{1}{2}\mu t}\sqrt{\frac{\lambda_{\max}(P_f)}{\lambda_{\min}(P_f)}} + \frac{2M\|P_f E\|}{\sqrt{\theta}\lambda_{\min}(Q_f)}\sqrt{1 - e^{-\mu t}}\left(\frac{\lambda_{\max}(P_f)}{\lambda_{\min}(P_f)}\right)^{3/4} \end{aligned}$$

When $t \to \infty$

$$\|e\| \leqslant \frac{2M\|P_f E\|}{\sqrt{\theta}\lambda_{\min}(Q_f)}\left(\frac{\lambda_{\max}(P_f)}{\lambda_{\min}(P_f)}\right)^{3/4} = \hat{b}_u = \sqrt{\theta}\left(\frac{\lambda_{\max}(P_f)}{\lambda_{\min}(P_f)}\right)^{1/4} b_u$$

So

$$\hat{b}_u \begin{cases} > b_u & \text{if } \ 1 \geqslant \theta > \eta \\ = b_u & \text{if } \ \theta = \eta \\ < b_u & \text{if } \ \theta < \eta \end{cases} \tag{5.90}$$

where $\eta = \sqrt{\dfrac{\lambda_{\min}(P_f)}{\lambda_{\max}(P_f)}}$. On the other hand

$$\hat{b}_u \begin{cases} > r_\Omega & \text{if } \ 1 \geqslant \theta > \eta^3 \\ = r_\Omega & \text{if } \ \theta = \eta^3 \\ < r_\Omega & \text{if } \ \theta < \eta^3 \end{cases} \tag{5.91}$$

However $b_u \geqslant r_\Omega$ and the equality holds if $P = pI$ $(p > 0)$. Although Ω_E is not the smallest ultimately bounded set, when Ω_E is a small neighbourhood about the origin, the concept of uniform ultimate boundedness is tantamount to 'practical' asymptotic stability.

Note that for suitable performance one can choose $\theta = 1$ or certainly close to 1.

Remark 5.3.5: In the case of the ϵ-matching condition not being satisfied, the gain matrix W should be chosen so that

$$\lambda_{\min}(W) \geqslant \frac{M\|D\|\lambda_{\max}(CP_f^{-1}C^T) + M\|CE\|}{\lambda_{\min}(CP_f{}^{-1}C^T)} \tag{5.92}$$

In this case the sliding region is

$$\mathscr{S}_r = \{e \in \mathbb{R}^n \mid \|e\| \leqslant r_2\} \tag{5.93}$$

where

$$r_2 \leqslant r_s - \frac{M\|CE\|}{\sigma_M(C(A-HC))} \tag{5.94}$$

and r_s was defined in (5.70).

If the disturbance norm $\|\xi(t)\|$ converges to zero in addition to being bounded, the convergence of $e(t)$ to zero results. To prove this, let $\eta(t)$ be a continuous nonnegative monotonically decreasing real function such that $\|\xi(t)\| < \eta(t)$ and $\lim_{t\to 0} \eta(t) = 0$. Consider (5.87) with $\|\xi(t)\|$ instead of M. Then

$$\begin{aligned}
e^T P_f e &\leqslant e^T(0)P_f e(0)e^{-\mu t} + b_u\|P_f E\|e^{-\mu t}\int_0^t \|\xi(\tau)\|e^{\mu\tau}d\tau \\
&\leqslant e^T(0)P_f e(0)e^{-\mu t} + b_u\|P_f E\|e^{-\mu t}\int_0^t \eta(\tau)e^{\mu\tau}d\tau \\
&\leqslant e^T(0)P_f e(0)e^{-\mu t} + b_u\|P_f E\|e^{-\mu t}\left(\int_0^{t/2} \eta(\tau)e^{\mu\tau}d\tau + \int_{t/2}^t \eta(\tau)e^{\mu\tau}d\tau\right) \\
&\leqslant e^T(0)P_f e(0)e^{-\mu t} + 2\|P_f E\|b_u\left\{\eta(0)e^{-\frac{1}{2}\mu t}(1-e^{-\frac{1}{2}\mu t}) + \eta(t/2)(1-e^{-\frac{1}{2}\mu t})\right\}
\end{aligned} \tag{5.95}$$

Since $\lim_{t\to\infty} \eta(t) = 0$, $\lim_{t\to\infty} e^T(t)P_f e(t) = 0$ and then $\lim_{t\to\infty} e(t) = 0$.

5.4 Examples

The examples below illustrate our results regarding the sliding mode, stability of error system and observer design.

Example 5.4.1: Consider the system

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} -1 & 2.25 & -0.4 \\ 0 & -1.50 & -2.0 \\ 0 & 0.50 & -1.0 \end{bmatrix} x + \begin{bmatrix} 0.30 & 0 \\ 0 & 1 \\ 0.12 & 0 \end{bmatrix} u + \begin{bmatrix} -0.0007 & 0.0015 \\ 0.0103 & 0.0041 \\ 0.0188 & 0.0188 \end{bmatrix} \xi \\
y &= \begin{bmatrix} 0.2 & 0.4 & 0.89 \\ 0.2 & 0 & 1.00 \end{bmatrix} x
\end{aligned}$$

Suppose ξ is a bounded random signal satisfying $||\xi|| < 0.1$. Choosing $R = I_2$ and $Q = I_3$, the u.p.d.s. solution of ARE (5.58) is

$$P = \begin{bmatrix} 1.8831 & 0.6396 & -0.1829 \\ 0.6396 & 0.5534 & -0.1750 \\ -0.1829 & -0.1750 & 0.3449 \end{bmatrix}$$

From (5.59)

$$H = \begin{bmatrix} 0.4697 & 0.1938 \\ 0.1936 & -0.0470 \\ 0.2004 & 0.3083 \end{bmatrix}$$

Note that the eigenvalues of $A - HC$ are $-1.7043 \pm 1.0365i$, -0.7881. Let $Q_f = 5I_3$. The u.p.d.s. solution matrix of the Lyapunov equation (5.62) is

$$P_f = \begin{bmatrix} 8.6608 & 3.0202 & -1.0694 \\ 3.0202 & 2.6000 & -0.7952 \\ -1.0694 & -0.7952 & 1.5302 \end{bmatrix}$$

Let $W = \text{diag}\,(2.5, 2.5)$ and

$$v = W\frac{Ce}{||Ce||} \tag{5.96}$$

From (5.65)

$$\Lambda = \begin{bmatrix} -0.0117 & 0.0244 \\ 0.1709 & 0.0676 \\ 0.3133 & 0.3136 \end{bmatrix}$$

The reduced order error system is independent of ξ since $\Gamma = 0.06\Lambda$. In fact $D = 0.06I_2$. So the error system is quadratically stable which means the estimated state error quadratically converges to the actual state.

The minimum eigenvalue of $C\Lambda W$ is 0.0510. The value of the right-hand side of (5.70) is $r_s = 0.0156$ and the value of the right-hand side of (5.71) is $\hat{r}_s = 0.0119$. After respective short times τ_1 and τ_2 conditions (5.70) and (5.71) are both true. Noting that $\tau_1 \leqslant \tau_2$, when (5.70) is valid for $t \in [\tau_1, \tau_2]$ (5.71) may not be valid, i.e. the condition (5.70) is weaker than (5.71). Simulation results are shown in Fig. 5.1 with $e(0) = [0.1 \quad 0.2 \quad 0.325]^T$ as the initial state error.

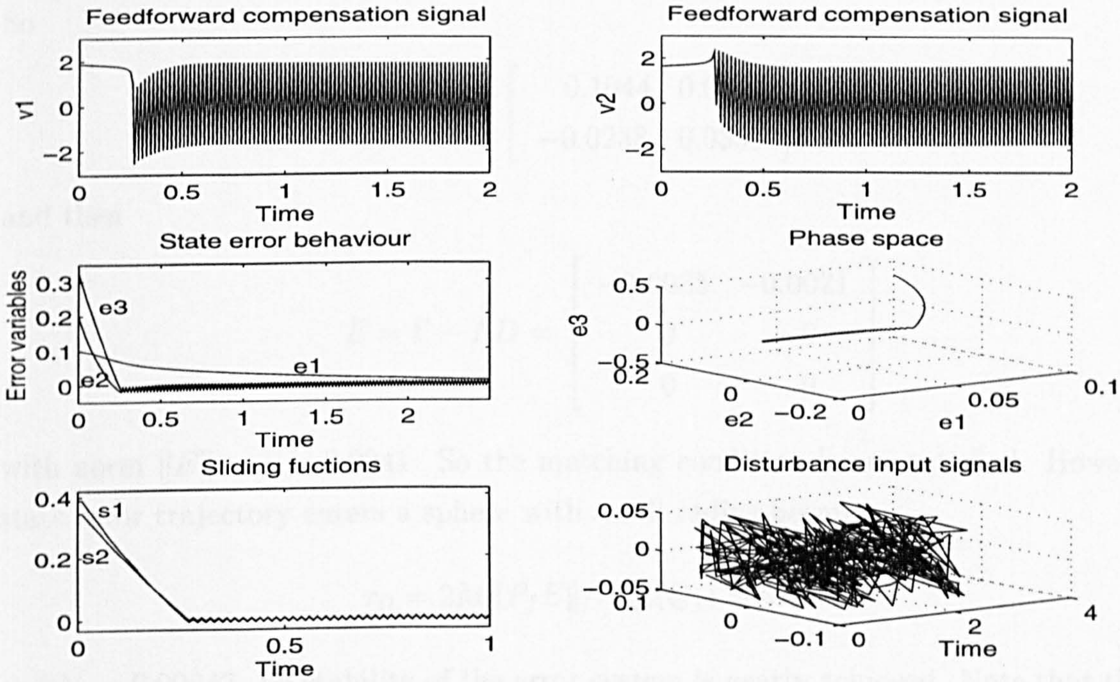

Figure 5.1: Responses of Example 5.4.1 when the LQ method is applied.

To find H such that the eigenvalues of $A - HC$ have specific values, one can apply pole assignment techniques. Suppose it is desired that the eigenvalues of $A - HC$ are $-1,\ -1 \pm 0.15i$. The existence of H is guaranteed by the observability of (A, C). Using the MATLAB Control Toolbox

$$H = \begin{bmatrix} 5.7656 & -5.5452 \\ -1.3486 & -0.7121 \\ 1.2296 & -1.0990 \end{bmatrix}$$

For v as in (5.96) and $Q_f = 3I_3$, the u.p.d.s. matrix solution of the Lyapunov equation (5.62) is

$$P_f = \begin{bmatrix} 1.4234 & 0.2455 & -0.0052 \\ 0.2455 & 1.6728 & -0.0650 \\ -0.0052 & -0.0650 & 1.5066 \end{bmatrix}$$

Equation (5.65) gives

$$\Lambda = \begin{bmatrix} 0.0400 & 0.0568 \\ 0.0991 & 0.0020 \\ 0.2407 & 0.2658 \end{bmatrix}$$

So

$$D = \begin{bmatrix} 0.1044 & 0.0407 \\ -0.0238 & 0.0339 \end{bmatrix}$$

and then

$$E = \Gamma - \Lambda D = \begin{bmatrix} -0.0035 & -0.0021 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

with norm $||E|| = \epsilon = 0.0041$. So the matching condition is not satisfied. However, the state error trajectory enters a sphere with small radius because

$$r_\Omega = 2M||P_f E||/\lambda_{\min}(Q_f) = 0.00039$$

and $b_u = 0.00047$. So stability of the error system is nearly achieved. Note that the value of the right-hand side of (5.70) is 0.0329 and the value of the right-hand side of (5.71) equals 0.0271. In the time interval $[0, 3]$ the maximum value of $||e||$ is 0.3945 and the minimum value is 0.02. Hence the conditions (5.70) and (5.71) are true, after short times τ_1 and τ_2, respectively. It is clear that $\tau_1 \leqslant \tau_2$. However, the time until the start of the sliding mode t_s should be shorter than the time τ_1 when (5.70) is satisfied, i.e. $t_s \leqslant \tau_1$. The sliding region radius (5.93) is $r_2 = 0.0328$. Simulation results are shown in Fig.5.2

Example 5.4.2: The observer design procedure is now illustrated by another example. To design an sliding observer, it is necessary to find H, Λ and select suitable W such that the appropriate conditions are satisfied. Consider the system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 2.25 & -0.4 \\ 0 & -1.50 & -2.0 \\ 0 & 0.50 & -1.0 \end{bmatrix} x + \begin{bmatrix} 0.30 \\ 0 \\ 0.12 \end{bmatrix} u + \begin{bmatrix} -0.0025 \\ 0.0404 \\ 0.0644 \end{bmatrix} \xi \\ y &= \begin{bmatrix} 0.20 & 0.40 & 0.89 \end{bmatrix} x \end{aligned}$$

Assume ξ is a bounded random signal satisfying $|\xi| < 0.101$. Taking $Q = I_3$ and $R = 1$, the u.p.d.s. solution of ARE (5.58) is

$$P = \begin{bmatrix} 1.9640 & 0.6722 & -0.1863 \\ 0.6722 & 0.5879 & -0.2001 \\ -0.1863 & -0.2001 & 0.3763 \end{bmatrix}$$

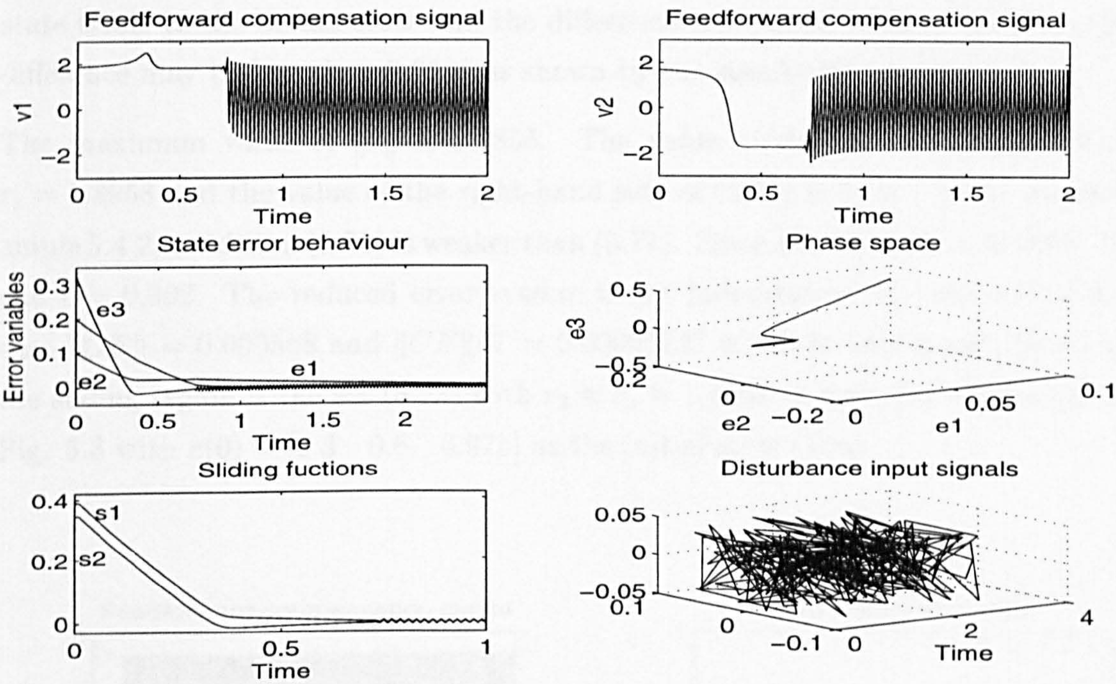

Figure 5.2: Responses of Example 5.4.1 when the pole assignment technique is applied.

From (5.59) H is

$$H = \begin{bmatrix} 0.4959 \\ 0.1915 \\ 0.2176 \end{bmatrix}$$

The eigenvalues of $A - HC$ are $-1.4812 \pm 0.9921i$, -0.9070. Let $Q_f = I_3$. The u.p.d.s. solution matrix of the Lyapunov equation (5.62) is

$$P_f = \begin{bmatrix} 1.8681 & 0.6640 & -0.2271 \\ 0.6640 & 0.5781 & -0.2012 \\ -0.2271 & -0.2012 & 0.3575 \end{bmatrix}$$

Consider

$$v = 3.34 \text{ sgn } e_y$$

Then (5.65) gives

$$\Lambda = \begin{bmatrix} -0.0742 \\ 0.6660 \\ 1.0730 \end{bmatrix}$$

$r_\Omega = 0.00088$ and $b_u = 0.0028$. These results show that there exists a finite time such that after that time, the norm of the state error is less than 0.0028. So the estimated state tends to the actual state and the difference is at most 0.0028. However, the actual difference may be less than 0.0028 as shown by the simulation results.

The maximum value of $\|e\|$ is 1.1855. The value of the right-hand side of (5.70) is $r_s = 1.8958$ and the value of the right-hand side of (5.71) is $\hat{r}_s = 1.4105$. As seen in Example 5.4.2, condition (5.70) is weaker than (5.71). Since $D = 0.06$, $E = [0.0020 \quad 0.0004 \quad 0]$ and $\epsilon = 0.002$. The reduced error system is not independent of ξ since $E \neq 0$. In this case $\|CE\| = 0.000568$ and $\|CE\|M = 0.00005737$ which is very small. So $r_2 \approx r_s$ and the sliding region is the set (5.93) with $r_2 \approx r_s = 1.8958$. Simulation results are shown in Fig. 5.3 with $e(0) = [0.3 \quad 0.6 \quad 0.975]$ as the initial state error.

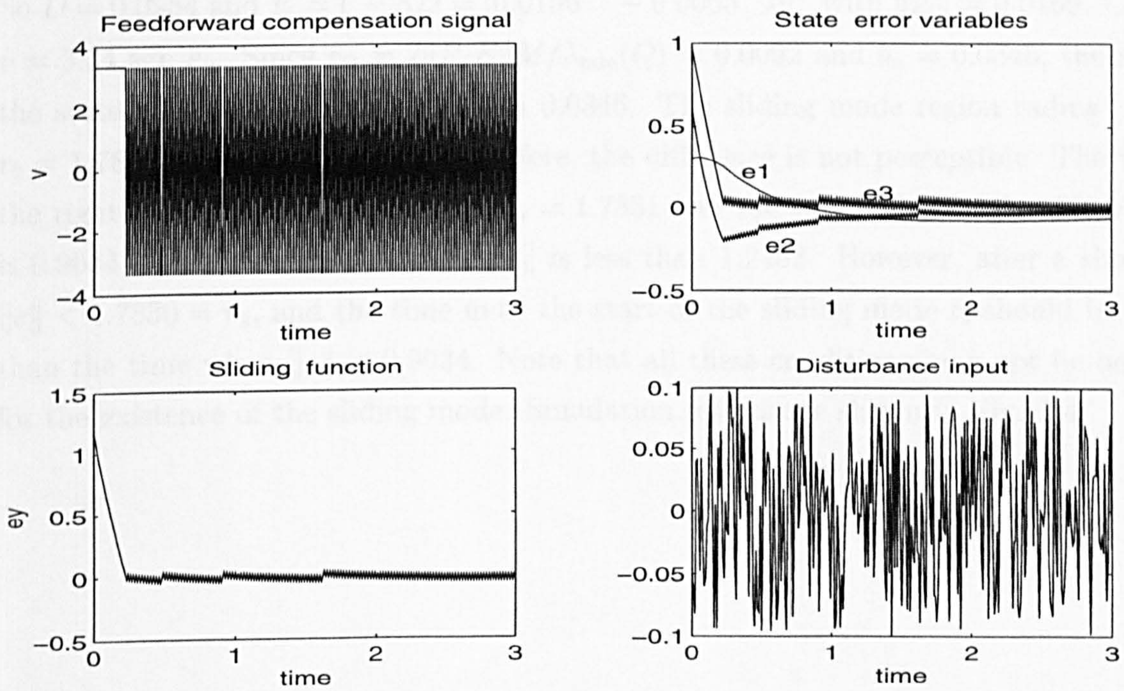

Figure 5.3: Responses of Example 5.4.2 when the LQ method is applied.

Similarly to Example 5.4.1 the gain matrix H can be found such that the eigenvalues of $A - HC$ have specific values. Suppose the eigenvalues of $A - HC$ are -1, $-1 \pm 0.15i$. Using the MATLAB Control Toolbox one determines

$$H = \begin{bmatrix} -0.2912 \\ -1.0677 \\ -0.0165 \end{bmatrix}$$

Let $Q_f = I_3$ and $W = 3.34$. Then the u.p.d.s. matrix solution of the Lyapunov equation (5.62) is

$$P_f = \begin{bmatrix} 3.2614 & 1.0988 & 0.2036 \\ 1.0988 & 0.7288 & -0.0451 \\ 0.2036 & -0.0451 & 0.4850 \end{bmatrix}$$

Therefore (5.65) implies

$$\Lambda = \begin{bmatrix} -0.1894 \\ 0.4916 \\ 0.6747 \end{bmatrix}$$

So $D = 0.0954$ and $E = \Gamma - \Lambda D = [0.0156 \quad -0.0065 \quad 0]^T$ with $||E|| = 0.0169$. Consider $v = 3.34 \text{ sgn } e_y$. Since $r_\Omega = 2||P_f E||M/\lambda_{\min}(Q) = 0.0092$ and $b_u = 0.0346$, the norm of the state error is eventually less than 0.0346. The sliding mode region radius (5.93) is $r_2 = 1.7831$ while $r_s = 1.7831$. Therefore, the difference is not perceptible. The value of the right-hand side of (5.70) equals $r_s = 1.7831$ and the value of the right side of (5.71) is 0.9034. The maximum value of $||e||$ is less than 1.2453. However, after a short time $||e|| < 1.7830 = r_2$, and the time until the start of the sliding mode t_s should be shorter than the time when $||e|| < 0.9034$. Note that all these conditions may not be necessary for the existence of the sliding mode. Simulation results are shown in Fig. 5.4.

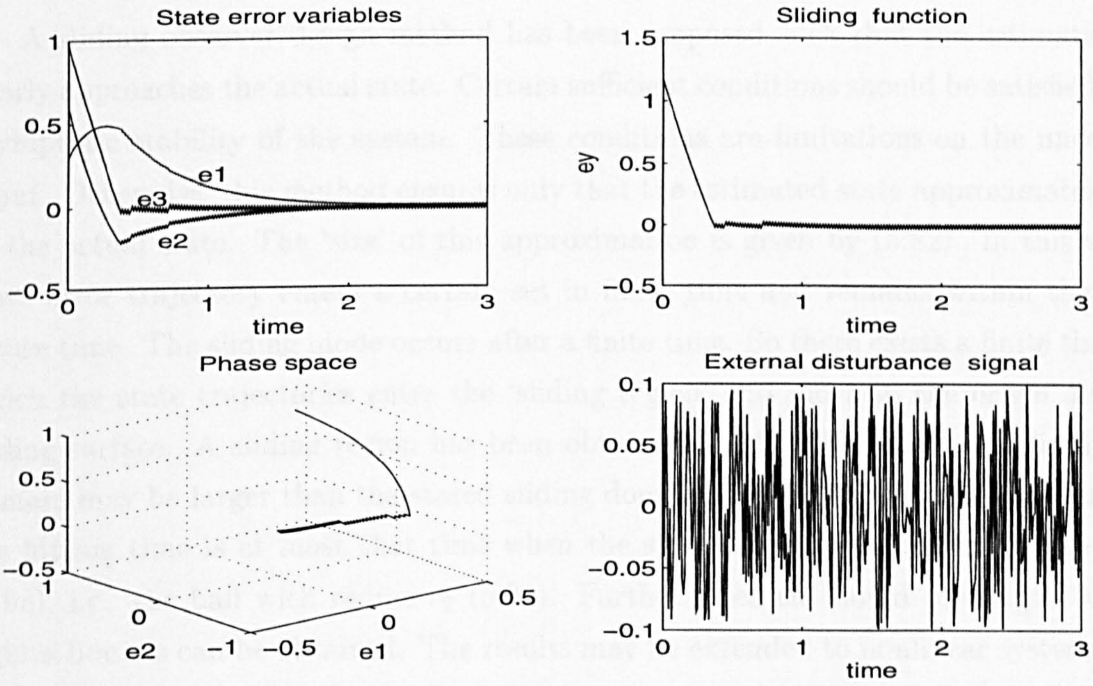

Figure 5.4: Responses of Example 5.4.2 when the pole assignment technique is applied

5.5 Summary and Discussion

In this chapter the sliding dynamics for SISO and MIMO linear systems, and conditions for the existence of the sliding mode in the presence of uncertainty have been studied. The existence of the sliding mode guarantees that the state trajectories converge to a sliding surface at a finite time and then move along the surface to the origin. However, the system may generally not be stable. For the system to be asymptotically stable, some further conditions may be needed. An interesting problem is to study the system with unmatched uncertainty and find some relaxed sliding conditions to impose asymptotic stability.

A sliding observer design method has been proposed such that the estimated state nearly approaches the actual state. Certain sufficient conditions should be satisfied for the asymptotic stability of the system. These conditions are limitations on the uncertainty input. Otherwise, this method ensures only that the estimated state approximately tends to the actual state. The 'size' of this approximation is given by (5.82). In this way the state error trajectory enters a certain set in finite time and remains within the set for future time. The sliding mode occurs after a finite time. So there exists a finite time after which the state trajectories enter the 'sliding region' and move to the origin along the sliding surface. A sliding region has been obtained (5.94). Of course, the sliding mode domain may be larger than the stated sliding domain (5.93). The value r_2 indicates that the hitting time is at most that time when the state error trajectory enters the set $\mathscr{S}_r$ (5.93), i.e. the ball with radius r_2 (5.93). Further research should investigate whether tighter bounds can be obtained. The results may be extended to nonlinear systems where the nonlinearity appears only in the disturbance term.

Chapter 6

Discrete-Time Sliding Mode Control

6.1 Sliding Lattice Design for Discrete-Time Systems

Sliding mode control design is well established for continuous control systems [29]-[34], [118], [121]. Sliding mode control of discrete-time systems has been not studied as much as its continuous counterpart. There are relatively few papers about discrete sliding mode control and most of them discuss SISO discrete-time systems [9], [12], [21], [30], [44], [70]-[72], [84], [98], [105], [111], [112], [122], [139].

Discrete-time sliding mode control (DSMC) has been defined in numerous ways. DSMC of discrete-time systems has been considered by Milosavljević [84] in the context of sampled-data systems and he named the discrete sliding mode *the quasi-sliding mode.* Sarpturk et al [98] presented a new sufficient condition for the existence of DSMC and discussed the stability. The main problem of discrete sliding is to find a suitable reaching condition such that when the sample period tends to zero, the continuous sliding mode reaching condition is satisfied. Some authors have applied the ideal sliding mode conditions for designing control [12], [122]; others sliding mode reaching conditions for SISO systems [44], [98], [105], [138]. In [84] a reaching sliding condition is presented which is only a necessary condition for the existence of the sliding mode.

Utkin [122] and Bartolini et al [12] considered a discrete-time system which is obtained from the linear time-invariant continuous system, and presented a method for designing the control. They considered two cases: (*i*) when complete information of the plant parameters is available, (*ii*) when the system operates under uncertainty conditions. Their method is based upon the definition of ideal sliding and the selection of a suitable real number bounding the control. This method guarantees the existence of a boundary layer with width twice the sample period.

Yu [138] presented an algorithm to calculate the upper and lower bounds for DSMC. These upper and lower bounds are independent of the distance of the system state from the sliding surface. Yu [138] used the sliding mode condition as stated by Milosavljević [84] and defined a control structure based upon linear feedback with switched gain DSMC. Additional conditions ensure the state approaches and/or crosses the sliding hyperplane without divergence from the sliding surface. To eliminate the zigzagging behaviour which appears with this control, he modified the control structure. In spite of using this modified method which ensures the sliding mode does not diverge from the sliding surface, the distance of the sliding motion from the sliding hyperplane is not specified. However, his algorithm shows how to calculate the upper and lower bounds for DSMC that are independent of the distance of the system state from the sliding surface.

The method presented in [74] indicates that the lower and upper bounds depend upon the distance of the system state from the sliding surface. Baida [9] studied discrete-time sliding modes based on the definition of Drakunov and Utkin [30] by using unit control methods under uncertainly conditions and minimization of control efficiency. Gao et al [46] defined a quasi-sliding mode band. They used an equivalent form of a continuous so-called reaching law to give a discrete-time reaching law. Sarpturk et al [98] presented the reaching condition for SISO systems

$$|s(k+1)| < |s(k)| \tag{6.1}$$

with s the sliding function, and Sira-Ramírez [105] proposed the following reaching condition

$$|s(k+1)s(k)| < s^2(k) \tag{6.2}$$

which is equivalent to (6.1). Furuta [44] used the Lyapunov function $V(k) = \frac{1}{2}s^2(k)$ and considered the condition

$$s(k)\Delta s(k+1)| < -\frac{1}{2}(\Delta s(k+1))^2 \tag{6.3}$$

with $\Delta s(k+1) = s(k+1) - s(k)$ which is also equivalent to (6.1) and (6.2). Almost all authors have stated the same condition. Spurgeon [111] and Yu and Potts [139] showed that the condition $|s(k+1)| < |s(k)|$ is only a sufficient condition for existence of the discrete sliding mode. Sarpturk et al [98] and Sira-Ramírez [105] presented the necessary condition for the existence of the sliding mode as stated by Milosavljević [84]

$$s(k+1)s(k) < s^2(k) \tag{6.4}$$

yielding an unstable sliding mode along the sliding surface $s = 0$. This condition is not a sufficient condition for the existence of the discrete sliding mode and only guarantees the sliding points approach and/or cross the sliding hyperplane. It is not sufficient for convergence to the sliding latticewise hyperplane [138], [139].

Equivalent control in the Furuta approach [44] is obtained by setting

$$s(k) = s(k+1) \tag{6.5}$$

but it should be emphasized that, when the sliding mode occurs, $s(k) = 0$ and the equivalent control in the sense of Furuta is the same as in the traditional case [121]. Assume for all $k \geqslant k_s$ (6.5) is satisfied. Then, for all $k \geqslant k_s$, $s(k) = s(k_s)$. So according to Furuta's definition, the discrete-time sliding mode occurs if there exists a finite time k_s such that after this time the value of the sliding function is constant. So for all $k \geqslant k_s$, $x(k) - x(k_s) \in \mathcal{N}(C)$, that means the state belongs to the right coset $\mathcal{N}(C)_{+x(k_s)}$ of the null space C. Only when $x(k_s) = 0$ does the state belongs to the null space of C.

Koshkouei and Zinober [70]-[72] have presented a condition which is weaker than the above conditions (6.1), (6.2) and (6.3) and is detailed in Section 6.2.

In discrete-time systems instead of having a hyperplane as in the continuous case, a countable set of points is defined comprising a so-called lattice; and the surface on which these sliding points lie is named the latticewise hyperplane (Koshkouei and Zinober [70]-[72]. The sliding lattice is defined as

$$s(k) = 0 \tag{6.6}$$

One way to design a sliding lattice hyperplane for MIMO (m-input m-output) systems is to consider the intersection of the m sliding lattice surfaces. Let

$$s = \begin{bmatrix} s_1 \\ s_2 \\ \cdots \\ s_m \end{bmatrix}$$

The i-th sliding lattice is $s_i = 0$ and

$$\bigcap_{i=1}^{i=m} \{x(k) : \; s_i(k) = 0\}$$

is a sliding lattice for the system. The i-th component of $u_i(k)$ $(1 \leqslant i \leqslant m)$ of the state feedback control vector $u(k)$ is selected such that the state lies on the i-th sliding lattice. The sliding mode of discrete-time systems is completely different from SMC in continuous systems. In continuous systems the sliding variable is a linear transformation $s : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with $x \longrightarrow Cx$ and when the sliding mode occurs $x \in \text{Ker}(s)$ (or $x \in \mathcal{N}(C)$) where Ker(s) is the kernel of epimorphism (surjective homomorphism) of s in the sense of vector space. So $\frac{\mathbb{R}^n}{\text{Ker}(s)}$ is isomorphic to $\mathbb{R}^m$, i.e. $\frac{\mathbb{R}^n}{\text{Ker}(s)} \cong \mathbb{R}^m$. Therefore, the dimensions of $\frac{\mathbb{R}^n}{\text{Ker}(s)}$ and $\mathbb{R}^m$ are the same. Hence, the dimension of $\mathcal{N}(C)$ (the nullity of linear transformation of s) is $n - m$. In contrast, for discrete-time systems the sliding function consists of a sequence which can be considered as the restriction of the function $s(x(t))$ on $\mathbb{N}$. When the sequence $\{s(k)\}_{k=1}^{\infty}$ is a null sequence, i.e. $\lim s(k) = 0$, the convergent sliding mode exists.

In this chapter a new reaching condition for the existence of the sliding mode is presented and the behaviour of the sliding dynamics is studied. The definition of the discrete sliding mode is clarified and techniques for designing the control by applying the sliding mode properties are presented. There are two ways for designing a control via the sliding mode technique:

(*i*) First the form of the conventional control law is chosen such that the state reaches the desired sliding lattice after some finite time. This method is suitable for designing a control for systems which can take only specific values, like the boost power converter and the quantum boost series resonant converter (SRC).

(*ii*) The precise control is found by using the reaching or ideal sliding mode conditions. This method can be applied to systems which have no particular restrictions on the control law or the control bounds.

For the generation of the sliding mode in continuous systems such that the state trajectory crosses the sliding surface, the control should be discontinuous. As stated in Chapter 2 the sufficient condition for the existence of the sliding mode in a continuous system on a manifold, $s = 0$, is that $s^T \dot{s} < 0$ in the neighbourhood of the manifold [118],[121]. So in continuous sliding systems the control law and/or sliding surface is chosen such that this sliding condition is satisfied.

Some authors have considered a control law for discrete-time systems like the discontinuous control law in continuous systems [44], [105], [122]. Since the control in the discrete-time systems is defined only at the sample points, the structure of the control is not required to be the same as in continuous systems. This is so because the discrete sliding mode condition differs fundamentally from the continuous sliding mode condition.

In discrete-time sliding systems the control can be chosen as a simple linear control [111]. This control guarantees the state to converge onto the sliding hyperplane and the state trajectories may not cross the sliding surface. In the discrete sliding mode the effect of external disturbances on the discrete-time system is reduced, but to eliminate the disturbance completely may need an additional condition like the invariance condition. So, in the presence of disturbances the state may not lie precisely on the sliding lattice hyperplane [82]. However, when certain conditions like the cone inequality for the norm of the state and disturbance input, and the matched uncertainty condition hold, the sliding points may lie on the prespecified sliding lattice.

Problems of the stabilization of linear and nonlinear discrete-time systems have been studied by many authors including [24] via the difference equation, and [82], [99], [132], [133] using the Lyapunov min-max method. In Section 6.2 the concept of the discrete-time sliding mode is clarified and new conditions for the existence of the discrete-time sliding mode (DSM) are suggested. In Section 6.3 a control design procedure is presented such that the robust stability of the sliding mode motion is achieved. Furthermore, the reduced order discrete-time system and the stability of this system are studied.

The problem of the stabilization of discrete-time dynamic systems using the direct method of Lyapunov is addressed in Section 6.4. In Section 6.5 optimal sliding mode control is considered. In fact the sliding lattice gain matrix is found such that all the eigenvalues of the reduced order discrete-time system lie either inside the unit circle or inside a specified circle enclosed within the unit circle with centre on the real axis. In Section 6.6 some examples are considered to illustrate the results of the discrete-time control design theory.

In Section 6.7 an asymptotically stable observer for discrete-time systems is designed by using the properties of the sliding mode such that the stability of the nominal error system in the sliding mode is maintained. Various techniques for finding the feedforward injection map are proposed. A technique for observer design and methods for finding the feedforward injection map and the external feedforward compensation signal are proposed. The stability of the reconstruction error system when the perturbation signal has bounded magnitude proportional to the norm of the state error, is studied in Section 6.8. In Section 6.9 some examples are considered to illustrate the results of the observer theory.

6.2 Discrete-Time Systems

Consider the discrete-time linear time-invariant system

$$x(k+1) = Ax(k) + Bu(k) + \Gamma\xi(k) \tag{6.7}$$
$$y(k) = Cx(k) \tag{6.8}$$

where $k \geqslant 0$ is an integer, $x(k) \in \mathbb{R}^n$ is the state, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ is full rank, $u(k) \in \mathbb{R}^m$ is the input control, $C \in \mathbb{R}^{m\times n}$ such that CB is a nonsingular matrix, $y(k) \in \mathbb{R}^m$ is the output and $\Gamma \in \mathbb{R}^{n\times m}$ is the perturbation input map. The function $\xi \in \mathbb{R}^m$ is the reference or measurable external input and there exists a positive real number M such that $\|\xi\| \leqslant M$. If ξ is unknown or not directly measurable, a suitable estimate of ξ should be selected [112]. Assume that (A, B) is completely controllable. The sliding dynamical sequence is defined by

$$s(k) = Cx(k)$$

Definition 6.2.1: The set of all points $x(k) \in \mathbb{R}^n$, which lie on the hyperplane $Cx(k) = 0$, is said to be the *sliding latticewise hyperplane* or more concisely the "*sliding lattice*".

In fact, the sliding latticewise manifold is an infinite countable subset of the manifold $Cx = 0$ (see Fig. 6.1 (a)).

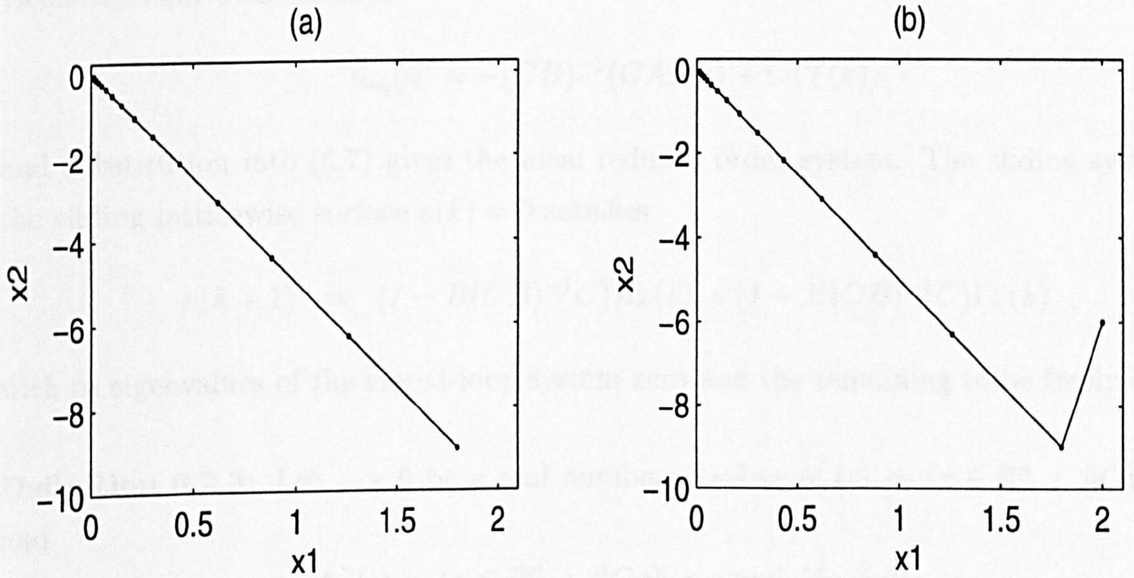

Figure 6.1: (a) Sliding lattice hyperplane; (b) Ideal discrete sliding mode

Therefore, the sliding lattice is formed by the points of the state satisfying

$$s(k) = Cx(k) = 0 \tag{6.9}$$

Definition 6.2.2: The ideal discrete sliding mode is generated if there exists a positive integer k_s such that for all integers $k \geqslant k_s$, $s(k) = 0$. The time instant k_s is the time when the sliding mode is reached (see Fig. 6.1 (b)).

States on the sliding latticewise hyperplane lie in the null space of C, i.e.

$$\mathscr{N}(C) = \{x \in \mathbb{R}^n : Cx = 0\} \tag{6.10}$$

The sliding lattice is a sequence of points on the manifold $Cx = 0$

$$\mathscr{N}_d(C) = \{x : Cx(k) = 0\} \tag{6.11}$$

So $\mathscr{N}_d(C) \subset \mathscr{N}(C)$. Since the dimension of $\mathscr{N}(C)$ is $n - m$, so the dimension of the state space is $n - m$, i.e. the discrete-time system in the sliding mode is converted to a closed-loop system with m zero eigenvalues. Therefore the discrete-time system in the sliding mode as in the case of continuous systems yields two subsystems: a slow and a fast subsystem. The slow subsystem is the system in the sliding mode. The fast subsystem only involves the control and with (6.9) gives the equivalent control. Equivalently, the equivalent control can be obtained by (6.7) and (6.9). Substituting (6.7) in $Cx(k+1) = 0$ yields the equivalent control

$$u_{eq}(k) = -(CB)^{-1}(CAx(k) + C\Gamma\xi(k)) \tag{6.12}$$

and substitution into (6.7) gives the ideal reduced order system. The sliding system on the sliding latticewise surface $s(k) = 0$ satisfies

$$x(k+1) \;=\; (I - B(CB)^{-1}C)Ax(k) + (I - B(CB)^{-1}C)\Gamma\xi(k) \tag{6.13}$$

with m eigenvalues of the closed-loop system zero and the remaining to be freely chosen.

Definition 6.2.3: Let $\epsilon > 0$ be a real number. Define $\mathcal{N}_s(\epsilon) = \{x \in \mathbb{R}^n \ : \ \|Cx\| < \epsilon\}$ and

$$\mathcal{N}_s'(\epsilon) = \{x \in \mathbb{R}^n \ : \ \|Cx\| < \epsilon \text{ and } Cx \neq 0\}$$

$\mathcal{N}_s(\epsilon)$ and $\mathcal{N}_s'(\epsilon)$ are said to be a neighbourhood and a deleted neighbourhood of $s = Cx = 0$, respectively.

Definition 6.2.4: The convergent discrete sliding mode exists if for any real number $\epsilon > 0$ there is a positive integer number k_0 such that for all integers $k \geqslant k_0$, $x(k) \in \mathcal{N}_s(\epsilon)$.

If the sliding mode takes the form of the convergent sliding mode and not the ideal sliding mode, then for a given boundary layer of $s = 0$, the sliding points lie inside the layer after a certain time instant depending upon the layer width.

Definition 6.2.5: The discrete sliding mode exists if there is a real number $\epsilon > 0$ and a positive integer k_0 such that for all integers $k \geqslant k_0$, $x(k) \in \mathcal{N}_s(\epsilon)$.

This guarantees that the sliding points lie in the boundary layer with width 2ϵ.

Definition 6.2.6: Let the sequence $\{\hat{s}_i(k)\}_{k=0}^{\infty}$ be a rearrangement of the nonzero terms of the sequence $\{s_i(k)\}_{k=0}^{\infty}$. The discrete-time system (6.7) is said to exhibit a sliding mode if there exists an integer number k_0 such that for all integers $k \geqslant k_0$ and for all integer $1 \leqslant i \leqslant m$

$$\hat{s}_i(k)(\hat{s}_i(k+1) - \hat{s}_i(k)) < 0 \tag{6.14}$$

Corollary 6.2.1: *Let $s = [s_1 \ s_2 \ \ldots \ s_m]^T$. The discrete-time system (6.7) exhibits a sliding mode if there exists a deleted neighbourhood of $s = 0$, $\mathcal{N}_s'(\epsilon)$ such that for all $x \in \mathcal{N}_s'(\epsilon)$ and for all $1 \leqslant i \leqslant m$*

$$s_i(k)(s_i(k+1) - s_i(k)) < 0 \tag{6.15}$$

Milosavljević [84] defined the sliding mode for SISO systems similarly to Corollary 6.2.1. This definition of the existence of a sliding mode guarantees only that the state approaches and/or crosses the sliding surface, and allows an unstable sliding mode [105], [138].

Corollary 6.2.2: *Let the sequence $\{\hat{s}(k)\}_{k=0}^{\infty}$ be a rearrangement (with the same order) of the nonzero terms of the sequence $\{s(k)\}_{k=0}^{\infty}$. The discrete-time system (6.7) exhibits a sliding mode if there exists a neighbourhood of $s = 0$, $\mathcal{N}_s(\epsilon)$, such that for all $x(k) \in \mathcal{N}_s(\epsilon)$*

$$\hat{s}_i(k)(\hat{s}_i(k+1) - \hat{s}_i(k)) < 0 \tag{6.16}$$

Corollary 6.2.3: *A sufficient condition for the existence of the discrete sliding mode is that there exists a positive integer k_0 such that*

$$\|s(k+1)\| < \|s(k)\|, \qquad k \geqslant k_0 \tag{6.17}$$

in a deleted neighbourhood of $s = 0$.

Proof: If this condition is satisfied, then the sequence $\{\|s(k)\|\}_{k=k_0}^{\infty}$ is monotonically decreasing and converges to zero. □

Since the finite integer k has no effect on the behaviour of the sequence $s(k)$, it is sufficient that (6.17) is satisfied after a certain time k_0.

For SISO systems, Corollary 6.2.3 or its equivalent has been stated as a definition of the convergent sliding mode [21], [44], [46], [98], [105], [111] and [138]. When a convergent sliding mode exists, the sliding points converge to the sliding surface. Therefore Corollary 6.2.3 is a strong sufficient condition for the existence of the discrete sliding mode.

Alternatively, to prove Corollary 6.2.3, one can show that the Lyapunov function $V(k) = \|s(k)\|$ satisfies

$$\Delta V(k) = V(k+1) - V(k) = \|s(k+1)\| - \|s(k)\| < 0$$

The Lyapunov method also guarantees the stability of the sliding mode.

Corollary 6.2.4: *A sufficient condition for the existence of the discrete sliding mode is that there exists a positive integer k_0 such that for all $1 \leqslant i \leqslant m$*

$$|s_i(k+1)| < |s_i(k)|, \qquad k \geqslant k_0 \tag{6.18}$$

in a deleted neighbourhood of $s = 0$.

Proof: This is an immediately result of Corollary 6.2.3. □

When for all $1 \leqslant i \leqslant m$, $s_i(k+1)s_i(k) > 0$, (6.15) and (6.18) are identical, but if there exists i $(1 \leqslant i \leqslant m)$ such that $s_i(k+1)s_i(k) < 0$, they are different. On the other hand, (6.18) gives (6.17), but the converse is true only if $m = 1$. The following corollary yields weaker conditions than (6.18) for the existence of the sliding mode.

Corollary 6.2.5: *Let $s = [s_1 \ s_2 \ \ldots \ s_m]^T$. Assume for any $1 \leqslant i \leqslant m$, $\{s_i{}^+(k)\}_{k=1}$ and $\{s_i{}^-(k)\}_{k=1}$ are the positive and negative subsequences of $\{s_i(k)\}_{k=0}^{\infty}$, respectively. The sufficient condition for the existence of the discrete sliding mode is that the positive subsequence $\{s_i{}^+(k)\}_{k=1}$ and negative subsequence $\{s_i{}^-(k)\}_{k=1}$ starting from some integer number $k_0 \geqslant 0$, are monotonically decreasing and increasing sequences, respectively, i.e.*

$$\begin{aligned} s_i{}^+(k+1) &< s_i{}^+(k) \qquad (k \geqslant k_0) \\ s_i{}^-(k+1) &> s_i{}^-(k) \qquad (k \geqslant k_0) \end{aligned} \tag{6.19}$$

When conditions (6.19) are satisfied, then $\lim_{k\to\infty} s(k) = 0$ which guarantees the existence and stability of the sliding mode on the surface $s = 0$. It is clear that conditions (6.19) may be satisfied, but not necessarily condition (6.17). Also conditions (6.19) give (6.15) for the both sequences $\{s_i{}^+(k)\}_{k=1}$ and $\{s_i{}^-(k)\}_{k=1}$, but the converse may not be true.

Theorem 6.2.1: *Let $\Delta s(k+1) = s(k+1) - s(k)$. Consider the following conditions:*

(*a*) $\|s(k+1)\| < \|s(k)\|$

(*b*) $s^T(k)\Delta s(k+1) < -\frac{1}{2}\|\Delta s(k+1)\|^2$

(*c*) $|s^T(k)s(k+1)| < \|s(k)\|^2$

Conditions (a) and (b) are equivalent, while conditions (a) and (b) imply (c) and the converse is true if $m = 1$.

Proof: $(a) \Leftrightarrow (b)$. (a) Suppose (a) is true, then

$$\begin{aligned}\|s(k+1)\|^2 - \|s(k)\|^2 &= s^T(k+1)s(k+1) - s^T(k)s(k)\\ &= (s^T(k) + (\Delta s(k+1))^T)(s(k) + \Delta s(k+1)) - s^T(k)s(k)\\ &= 2s^T(k)\Delta s(k+1) + \|\Delta s(k+1)\|^2\end{aligned}$$

(a) $\Rightarrow$(c). Suppose (a) is true, then the Cauchy-Schwartz inequality gives

$$|s^T(k)s(k+1)| \leqslant \|s(k)\|\ \|s(k+1)\| < \|s(k)\|^2$$

The converse may not be true. □

As already stated, Sarpturk et al [98], Furuta [44] and Sira-Ramírez [105] defined the convergent quasi-sliding mode as (a), (b) and (c) respectively for SISO systems. Theorem 6.2.1 shows that for SISO systems all these conditions are equivalent. Note that both the Lyapunov functions $V(k) = \|s(k)\|^2$ and $V(k) = \|s(k)\|$ give the condition (a), i.e. these Lyapunov functions essentially have the same effect and the Lyapunov function can be selected in the form $V(k) = \sqrt{\|s(k)\|}$. A sufficient condition for the existence of the sliding mode can be stated as $\|s(k+1)\| < \eta\|s(k)\|$ where $0 < \eta < 1$ is a real number indicating the velocity of motion to reach the sliding mode. In this case, $\|s(k)\| < \|s(0)\|\eta^k$ where $s(0)$ is an arbitrary initial condition. So in this case, the velocity of the state moving onto the sliding lattice hyperplane depends on the value of η and the initial condition $x(0)$ influences the reaching time of the sliding mode. It is clear that the condition (a) in

Theorem 6.2.1 is weaker than the condition $\|s(k+1)\| < \eta\|s(k)\|$, because this condition implies the condition (a) but the converse may not true. Some authors have used the terms quasi-sliding mode [84], [98] and pseudo-sliding mode for the discrete-time sliding mode [138], [139].

It is possible to consider a suitable Lyapunov function to find a more general sliding mode condition than (6.17). Consider the Lyapunov equation $V(k) = s^T(k)Ps(k)$ where P is a p.d. matrix. Since

$$V(k+1) - V(k) = s^T(k+1)Ps(k+1) - s^T(k)Ps(k) = \|s(k+1)\|_P^2 - \|s(k)\|_P^2$$

a sufficient condition for the discrete-time system state to converge onto the sliding latticewise hyperplane is that

$$\|s(k+1)\|_P < \|s(k)\|_P$$

Then the conditions (a), (b) and (c) of Theorem 6.2.1 are converted to the following conditions respectively:

(a') $\|s(k+1)\|_P < \|s(k)\|_P$

(b') $s^T(k)P\Delta s(k+1) < -\frac{1}{2}\|\Delta s(k+1)\|_P^2$

(c') $|s^T(k)Ps(k+1)| < \|s(k)\|_P^2$

and Theorem 6.2.1 also holds.

Theorem 6.2.2: *A sufficient condition for the existence of a convergent sliding mode is that the system (6.7) is asymptotically stable. The converse is not true.*

Proof: Assume the system (6.7) is asymptotically stable, i.e. $\lim_{k\to\infty} x(k) = 0$. Then the sequence $\{x(k)\}_{k=1}^{\infty}$ is a Cauchy sequence. So, for a given real number ϵ there exists an integer k_0 such that for all integers $k \geqslant k_0$

$$\|x(k+1) - x(k)\| < \frac{\epsilon}{\|C\|}$$

On the other hand

$$\|s(k+1) - s(k)\| \leqslant \|C\|.\|x(k+1) - x(k)\|$$

Therefore, for all integers $k \geqslant k_0$, $\|s(k+1) - s(k)\| < \epsilon$, i.e. $\{s(k)\}_{k=1}^{\infty}$ is a Cauchy sequence. Therefore, $\lim_{k\to\infty} s(k) = 0$. □

If there exists k_0 such that, for all $k \geqslant k_0$, $x(k) = 0$, then $s(k) = Cx(k) = 0$. This is a trivial case of Theorem 6.2.2. The proof of Theorem 6.2.2 indicates that if the distance between two consecutive state points is less than ϵ, the distance between two consecutive sliding points is less than $\|C\|\epsilon$, and $s = 0$ is the accumulation point set of the sliding points, i.e. for any boundary layer of $s = 0$, all the sliding points except for a finite number lie on the sliding latticewise surface. So, stability of the system guarantees the existence of a boundary layer with a given width.

6.3 Discrete-Time Control Design

In this section a technique for the design of a controller is presented which guarantees the stability of the sliding mode and yields desired sliding dynamical behaviour. The reaching condition (6.18) and the technique stated in Section 6.1 are utilized. Let $s_i(k)$ be the i-th row of vector $s(k)$, so $s_i = C_i x(k)$ where C_i is the i-th row of matrix C. A sufficient condition for the existence of the sliding mode is that for all $1 \leqslant i \leqslant m$, $|s_i(k+1)| < |s_i(k)|$, which is equivalent to $-|s_i(k)| < s_i(k+1) < |s_i(k)|$. Therefore

$$-|s_i(k)| < C_i B(-u_{eq}(k) + u(k)) < |s_i(k)|$$

So for all $1 \leqslant i \leqslant m$

$$-1 < \frac{C_i B(-u_{eq}(k) + u(k))}{|s_i(k)|} < 1$$

Suppose

$$W_i(k) = \frac{C_i B(-u_{eq}(k) + u(k))}{|s_i(k)|}$$

Then $|W_i(k)| < 1$ and

$$|s_i(k)| W_i(k) = C_i B(-u_{eq}(k) + u(k))$$

Hence

$$W(k)|s(k)| = CB(-u_{eq}(k) + u(k))$$

where $W(k) = \text{diag}(W_1(k), W_2(k), \ldots, W_m(k))$. Therefore, the control law is

$$\begin{aligned} u(k) &= u_{eq}(k) + (CB)^{-1} W(k)|s(k)| \\ &= -(CB)^{-1}\left(CAx(k) + C\Gamma\xi(k) - W(k)|s(k)|\right) \end{aligned} \quad (6.20)$$

where $|s(k)| = \begin{bmatrix} |s_1(k)| & |s_2(k)| & \dots & |s_m(k)| \end{bmatrix}^T$. Substituting (6.20) in (6.7) gives the dynamic state equation

$$\begin{aligned} x(k+1) &= (I - B(CB)^{-1}C)Ax(k) + (I - B(CB)^{-1}C)\Gamma\xi(k) + \\ &\quad +B(CB)^{-1}W(k)|s(k)| \end{aligned} \tag{6.21}$$

This control guarantees the existence of the sliding mode. Inserting (6.21) in

$$s(k+1) = Cx(k+1)$$

yields the dynamic motion of the sliding mode

$$\begin{aligned} s(k+1) &= Cx(k+1) \\ &= C\left((I - B(CB)^{-1}C)Ax(k) + (I - B(CB)^{-1}C)\Gamma\xi(k) + B(CB)^{-1}W(k)|s(k)|\right) \\ &= W(k)|s(k)| \end{aligned} \tag{6.22}$$

Since for any i, $|W_i| < 1$, therefore $||W|| < 1$. From (6.22)

$$||s(k+1)|| \leqslant ||W||.||s(k)||$$

The function $||W(k)||$ indicates the velocity at which the sliding mode occurs and can be constant. Moreover, $||s(k)|| = ||W||^k||s(0)||$ where $s(0)$ is an arbitrary initial condition, i.e. the sliding dynamics is only dependent upon the initial conditions and selection of W. This condition is true for all $k < k_s$, k_s a finite number, and for all $k \geqslant k_s \;\; s(k) = 0$. Thus (6.22) guarantees the existence and stability of the sliding mode. If ξ is unknown, the control (6.20) is no longer accessible because of the uncertainty. So it is necessary to estimate ξ. An estimate for $\xi(k)$ is $\xi(k-1)$ [112] which guarantees that the sliding points lie inside a boundary layer. If $\xi(k) - \xi(k-1)$ is a decreasing sequence (or sufficiently small) after a finite time instant, then the convergence sliding mode occurs and the state lies nearly on the sliding hyperplane.

6.3.1 Discrete-Time System in the Sliding Mode

As stated in Section 6.1, the system in the sliding mode is converted to a subsystem with dimension $n - m$. Therefore, the state in the discrete sliding mode belongs to a subspace with dimension $n - m$. Thus m eigenvalues of the closed-loop system are zero and the remaining $n - m$ eigenvalues can be selected such that the reduced order system is stable. Assume T is an orthogonal matrix (2.13). So $TB = \begin{bmatrix} 0 & B_2 \end{bmatrix}^T$ where B_2 is a nonsingular matrix. Let $Tx = z$, then

$$z(k+1) = TAT^Tz(k) + TBu(k) + T\Gamma\xi(k) \tag{6.23}$$

Now assume $z^T = ({z_1}^T,\ {z_2}^T)$ where $z_1 \in \mathbb{R}^{n-m}$ and $z_2 \in \mathbb{R}^m$, then

$$z_1(k+1) = A_{11}z_1(k) + A_{12}z_2(k) + \Gamma_1\xi(k) \tag{6.24}$$

$$z_2(k+1) = A_{21}z_1(k) + A_{22}z_2(k) + B_2u(k) + \Gamma_2\xi(k) \tag{6.25}$$

where

$$TAT^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad T\Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}$$

The system in the sliding mode is independent of ξ if and only if $\Gamma_1 = 0$. As stated in Chapter 2 the system in the sliding mode is independent of ξ if there exists a matrix D such that $\Gamma = BD$. If $m = 1$ this condition is also necessary. Suppose $CT^T = [C_1\ C_2]$ then

$$s(k) = C_1z_1(k) + C_2z_2(k)$$

When $s(k) = 0$

$$z_2(k) = -Kz_1(k), \qquad K = C_2^{-1}C_1 \tag{6.26}$$

Substituting (6.26) into (6.24) yields the reduced order system

$$z_1(k+1) = (A_{11} - A_{12}K)z_1(k) + \Gamma_1\xi(k) \tag{6.27}$$

and then

$$z_1(k+1) = (A_{11} - A_{12}K)^{k+1-k_s}z_1(k_s) + v \tag{6.28}$$

where

$$v = \sum_{i=k_s}^{k} (A_{11} - A_{12}K)^{i-k_s}\Gamma_1\xi(k - i + k_s)$$

Consider the nominal reduced order system

$$z_1(k+1) = (A_{11} - A_{12}K)^{k+1-k_s}z_1(k_s) \tag{6.29}$$

Then

$$\|z_1(k+1)\| \leqslant \|(A_{11} - A_{12}K)\|^{k+1-k_s}\|z_1(k_s)\|$$

For $\lim_{k\longrightarrow\infty} z_1(k) = 0$ it is sufficient that

$$\|(A_{11} - A_{12}K)\| < 1 \tag{6.30}$$

i.e. all the eigenvalues of $A_{11} - A_{12}K$ lie in the unit circle centred at the origin. When all the eigenvalues of $A_{11} - A_{12}K$ are in the unit circle and $\Gamma_1 \neq 0$, the state variables are bounded since

$$\begin{aligned} \|z_1(k+1)\| &\leqslant \|(A_{11} - A_{12}K)\|^{k+1-k_s}\|z_1(k_s)\| \\ &\quad + \sum_{i=k_s}^{k} \|(A_{11} - A_{12}K)\|^{i-k_s}\|M\|.\|\Gamma_1\| \\ &\leqslant \|(A_{11} - A_{12}K)\|^{k+1-k_s}\|z_1(k_s)\| \\ &\quad + \|M\|.\|\Gamma_1\|\frac{1-\|(A_{11}-A_{12}K)\|^{k-k_s+1}}{1-\|A_{11}-A_{12}K\|} \end{aligned} \tag{6.31}$$

From (6.31) for $k \to \infty$, the norm of $z_1(k)$ is eventually less than

$$\|M\|.\|\Gamma_1\|/(1 - \|A_{11} - A_{12}K\|)$$

which shows that the states during the sliding mode are bounded and if $\|A_{11} - A_{12}K\|$ is small, the width of this bound becomes small. So if the condition (6.30) is satisfied, i.e. the nominal sliding system is stable, the sliding system state is bounded. Otherwise, the system may be unstable despite the existence of the sliding mode.

6.4 System Stability

The problem of system stability will be studied which can appear in several ways: (i) the stability of the system in the sliding mode, (ii) when the system is independent of the perturbation signal, (iii) when the perturbation signal is bounded proportionally to $x(k)$, (iv) when there is no knowledge of the bounded perturbation signal.

Let $A_{eq} = (I - B(CB)^{-1}C)A$ and $B_{eq} = I - B(CB)^{-1}C$. Assume that all the eigenvalues of A_{eq} lie in the unit circle. Then there is a real number r such that $\rho(A_{eq}) < r \leqslant 1$ where $\rho(A_{eq})$ indicates the spectral radius of A_{eq}. Then the discrete Lyapunov equation

$$A_{eq}^T\ PA_{eq} - r^2P = -Q \tag{6.32}$$

with Q an arbitrary p.d.s. matrix, has a p.d.s. matrix solution P. Suppose that the perturbation signal is bounded by a multiplier factor of the norm of the state, i.e. there is a positive real number ξ_0 such that $\|\xi(k)\| \leqslant \xi_0\|x(k)\|$. Now it is shown that the system is stable if

$$\|W\| < \min\left\{1, -\|\Gamma\|\xi_0 + \frac{\mu-\|A_{eq}\|}{\|B(CB)^{-1}\|.\|C\|}\right\} \tag{6.33}$$

where $\mu = \sqrt{||A_{eq}||^2 + \frac{(\lambda_{\min}(Q)+(1-r^2)\lambda_{\min}(P)}{\lambda_{\max}(P)}}$, provided that

$$\mu > ||A_{eq}|| + ||\Gamma||\xi_0||B(CB)^{-1}||.||C||$$

A suitable Lyapunov candidate function is $V(k) = x(k)^T P x(k)$ where P is the u.p.d.s. solution of Lyapunov equation (6.32). Then the triangle and Cauchy-Schwartz inequalities give

$$\begin{aligned}
\Delta V(k) &= V(k+1) - V(k) \\
&\leqslant -||x(k)||^2(\lambda_{\min}(Q) + (1-r^2)\lambda_{\min}(P)) + ||\xi(k)||^2||\Gamma||^2||B_{eq}||^2\lambda_{\max}(P) + \\
&\quad 2||x(k)||||A_{eq}||\lambda_{\max}(P)||B_{eq}||.||\Gamma\xi|| + \\
&\quad 2||x(k)||^2||A_{eq}||\lambda_{\max}(P)||B(CB)^{-1}||.||C||.||W|| + \\
&\quad 2||x(k)||.||B(CB)^{-1}||.||C||\lambda_{\max}(P)||B_{eq}||.||W||.||\Gamma\xi|| \\
&\quad +||W||^2||x(k)||^2||B(CB)^{-1}||^2||C||^2\lambda_{\max}(P) \\
&\leqslant ||x(k)||^2\left[-\lambda_{\min}(Q) - (1-r^2)\lambda_{\min}(P) + \right. \\
&\quad \xi_0^2||\Gamma||^2\lambda_{\max}(P)||B(CB)^{-1}||^2.||C||^2 + \\
&\quad 2\lambda_{\max}(P)||\Gamma||\ ||B(CB)^{-1}||.||C||.||A_{eq}||\xi_0 + \\
&\quad 2\lambda_{\max}(P)||A_{eq}||.||B(CB)^{-1}||.||C||.||W|| + \\
&\quad 2\lambda_{\max}(P)||||B(CB)^{-1}||^2.||C||^2\xi_0||\Gamma||.||W|| + \\
&\quad \left. ||W||^2||B(CB)^{-1}||^2.||C||^2\lambda_{\max}(P)\right] \\
&< 0
\end{aligned} \tag{6.34}$$

from (6.33). Let $\mu_0 = \sqrt{1 + \frac{\lambda_{\min}(Q)+(1-r^2)\lambda_{\min}(P)}{\lambda_{\max}(P)}}$. If $||A_{eq}|| \leqslant 1$ and

$$\mu_0 > 1 + ||\Gamma||\xi_0||B(CB)^{-1}||.||C||$$

the condition

$$||W|| < \min\left\{1, -||\Gamma||\xi_0 + \frac{\mu_0 - 1}{||B(CB)^{-1}||.||C||}\right\} \tag{6.35}$$

yields the stability of the system. Now suppose that $\text{rank}(\Gamma, B) = m$. Then the system in the sliding mode is independent of the external input signal and the system is stable if

$$||W|| < \min\left\{1, \frac{\mu - ||A_{eq}||}{||B(CB)^{-1}||.||C||}\right\} \tag{6.36}$$

If $||A_{eq}|| \leqslant 1$, the condition

$$||W|| < \min\left\{1, \frac{\mu_0 - 1}{||B(CB)^{-1}||.||C||}\right\} \tag{6.37}$$

is a sufficient condition for the stability of the system. When there is insufficient knowledge of $||A_{eq}||$, the condition (6.37) may prove useful.

6.5 Optimal Discrete-Time Sliding Mode

Minimizing the conventional quadratic discrete cost functional is a way to design the feedback gain such that the closed-loop system is stable. Control design using the optimal method for discrete-time and continuous systems has been studied by many authors in recent years [11, page 287], [90, Chapter 8]. However, the method presented next for designing the optimal discrete sliding mode is new. Here the optimal discrete sliding mode and optimal discrete control are studied. A method to ensure all the eigenvalues of the reduced order system lie within a specified circle, is also presented.

6.5.1 Optimal Discrete-Time Control

Consider the discrete cost quadratic functional for the system (6.7)

$$J = \sum_{k=0}^{\infty} \left(x^T(k)Qx(k) + u^T(k)Ru(k)\right) \tag{6.38}$$

where Q and R are arbitrary semi-p.d.s. an p.d.s. matrices, respectively. Minimizing J in (6.38) with respect to u yields the optimal controller

$$u(k) = -Kx(k), \qquad K = (R + B^TPB)^{-1}B^TPA \tag{6.39}$$

where P is the u.p.d.s. solution of the standard algebraic discrete Riccati equation (ADRE)

$$A^TPA - A^TPB(R + B^TPB)^{-1}B^TPA - P = -Q \tag{6.40}$$

All the eigenvalues of $A - BK$ lie in the unit circle U with centre at the origin, i.e. this feedback guarantees the stability of the nominal closed-loop discrete-time system (6.7).

For all the eigenvalues of the closed-loop system to lie in a specified circle D with centre $\alpha \in \mathbb{R}$, $|\alpha| < 1$, and radius $r < 1 - |\alpha|$, consider the shifted system

$$\hat{x}(k+1) = (\frac{A - \alpha I}{r})\hat{x}(k) + \frac{B}{r}\hat{u}(k) + \frac{\Gamma}{r}\hat{\xi}(k) \tag{6.41}$$

[43]. When all the eigenvalues of $A-BK$ lie inside D, the eigenvalues of $(A - BK - \alpha I)/r$ lie within U. So all the eigenvalues of the nominal closed-loop system (6.7) are in D if and only if all the eigenvalues of the shifted system (6.41) are inside the unit circle U. Therefore, when all of the eigenvalues of the closed-loop system (6.7) lie within the open circle D, the control is

$$\hat{u}(k) = -(R + \frac{B^T}{r}P\frac{B}{r})^{-1}\frac{B^T}{r}P\frac{A - \alpha I}{r}\hat{x}(k)$$

and then

$$u(k) = -Kx(k), \qquad K = (r^2R + B^TPB)^{-1}B^TP(A - \alpha I) \tag{6.42}$$

where P is a p.d.s. solution of the DRE

$$(A - \alpha I)^TP(A - \alpha I) - (A - \alpha I)^TPB(r^2R + B^TB)^{-1}B^TP(A - \alpha I) - r^2P = -Q \tag{6.43}$$

6.5.2 Optimal Sliding Lattice

Similarly to continuous systems [135], the optimal sliding lattice can be found by minimizing the conventional quadratic index. The basic idea is that z_2 is the input control of the subsystem (6.24) and the LQ method can be applied for finding the optimal control, or more precisely the optimal sliding lattice. Consider the linear discrete quadratic cost functional

$$J = \sum_{k=k_s}^{\infty} (z_1^T(k)Qz_1(k) + 2z_1^T(k)Nz_2(k) + z_2^T(k)Rz_2(k)) \tag{6.44}$$

where Q, N and R are arbitrary matrices such that

$$\begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \geqslant 0, \qquad R > 0 \tag{6.45}$$

and also Q and R are symmetric matrices. Define

$$\hat{z}_2(k) = z_2(k) + R^{-1}N^Tz_1(k) \tag{6.46}$$

$$\hat{A}_{11} = A_{11} - A_{12}R^{-1}N^T \tag{6.47}$$

$$\hat{Q} = Q - NR^{-1}N^T \tag{6.48}$$

The positivity and symmetry of (6.45) ensure that $\hat{Q}$ is a p.d.s. matrix, and the controllability of (A, B) ensures the controllability of $(\hat{A}, A_{12})$ [121]. Then (6.44) is converted to the standard linear discrete quadratic optimal regulator

$$J = \sum_{k=0}^{\infty} \left(z_1^T(k)\hat{Q}z_1 + \hat{z}_2^T(k)R\hat{z}_2(k)\right) \tag{6.49}$$

Minimizing J in (6.49) with respect to $\hat{z}_2$ yields the optimal gain sliding matrix

$$\hat{z}_2(k) = -(R + A_{12}^TPA_{12})^{-1}A_{12}^TP\hat{A}_{11}z_1(k) \tag{6.50}$$

where P is a p.d.s. solution of the algebraic discrete Riccati equation (DARE)

$$\hat{A}_{11}^T P \hat{A}_{11} - \hat{A}_{11}^T P A_{12}(R + A_{12}^T P A_{12})^{-1} A_{12}^T P \hat{A}_{11} - P = -\hat{Q} \tag{6.51}$$

Using (6.46), (6.47) may be transformed to

$$z_2(k) = -K z_1(k), \qquad K = (R + A_{12}^T P A_{12})^{-1}(A_{12}^T P A_{11} + N^T) \tag{6.52}$$

Matrix K guarantees that all the eigenvalues of $A_{11} - A_{12}K$ lie inside the unit circle U. Similarly to Section 6.5.1, the gain matrix K can be found such that all the eigenvalues of the reduced order system lie inside the specified circle D.

6.6 Examples

Now some examples are considered to demonstrate the sliding lattice theory.

Example 6.6.1 (Chan **[21]**): Consider the discrete-time system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.24 & 0.20 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ d(k) \end{bmatrix} \xi$$

where ξ is a external input signal signal and $d(k)$ is uncertain. The system in the sliding mode is independent of $d(k)$ because $d(k)b = [0 \quad d(k)]^T$. This is achieved via feedforward of the disturbance. Choose the sliding matrix $C = [0 \quad 1]$. The eigenvalues of A_{eq} are zero, so the spectral radius of A_{eq} is 0. Let $Q = I_2$ and $r = 0.9$. A p.d.s. solution P of the Lyapunov equation (6.32) is $P = \text{diag}(1.2346, 2.7587)$ with the eigenvalues 1.2346 and 2.7587 and then $\mu_0 = 1.2031$. Take $W = 0.2021 \sin(k\pi/6)$. The system is stable since condition (6.36) is satisfied. Simulation results are shown in Fig. 6.2 with $x(0) = [0.2 \quad 0.2]^T$.

Example 6.6.2 (Sira-Ramírez **[105]**): Consider the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

Define the sliding sequence as $s(k) = Cx(k) = c_1 x_1(k) + x_2(k)$. Then

$$I - BC/CB = \begin{bmatrix} 1 & 0 \\ -c_1 & 0 \end{bmatrix} \quad \text{and} \quad (I - BC/CB)A = \begin{bmatrix} 0 & 0 \\ 0 & -c_1 \end{bmatrix}$$

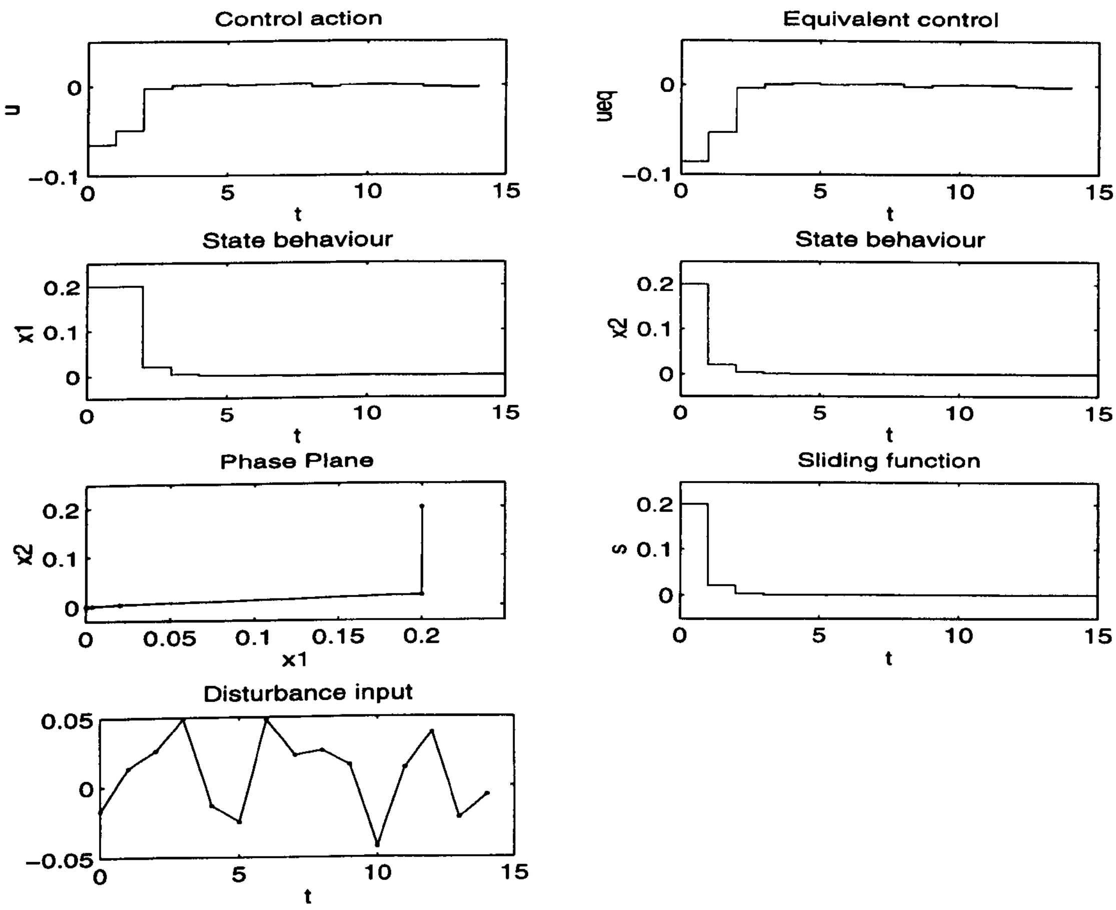

Figure 6.2: The responses of Example 6.6.1

Therefore the eigenvalues of A_{eq} are 0 and $-c_1$. For the stability of the reduced order system take $|c_1| < 1$. The control is given by (6.20) and now

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & -c_1 \end{bmatrix} x(k) + W \begin{bmatrix} 0 & 0 \\ c_1 & 1 \end{bmatrix} x(k) \operatorname{sgn} s(k)$$

Choosing $|c_1| < r \leqslant 1$ and W such that (6.36) is satisfied, then the system is stable. Let $r = 0.97$, $c_1 = 0.7$, $Q = I$ and $W = 0.075\cos(k\pi/4)$. For $c_1 = 0.7$ or $c_1 = -0.7$, a p.d.s. solution P of the Lyapunov equation (6.32) has eigenvalues 1.0628, 4.5749. Obviously (6.36) is satisfied which guarantees the stability of the system. Simulation results are shown in Fig. 6.3.

Example 6.6.3 (Gao et al **[46]**): Consider the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1.2 & 0.1 \\ 1 & 0.6 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

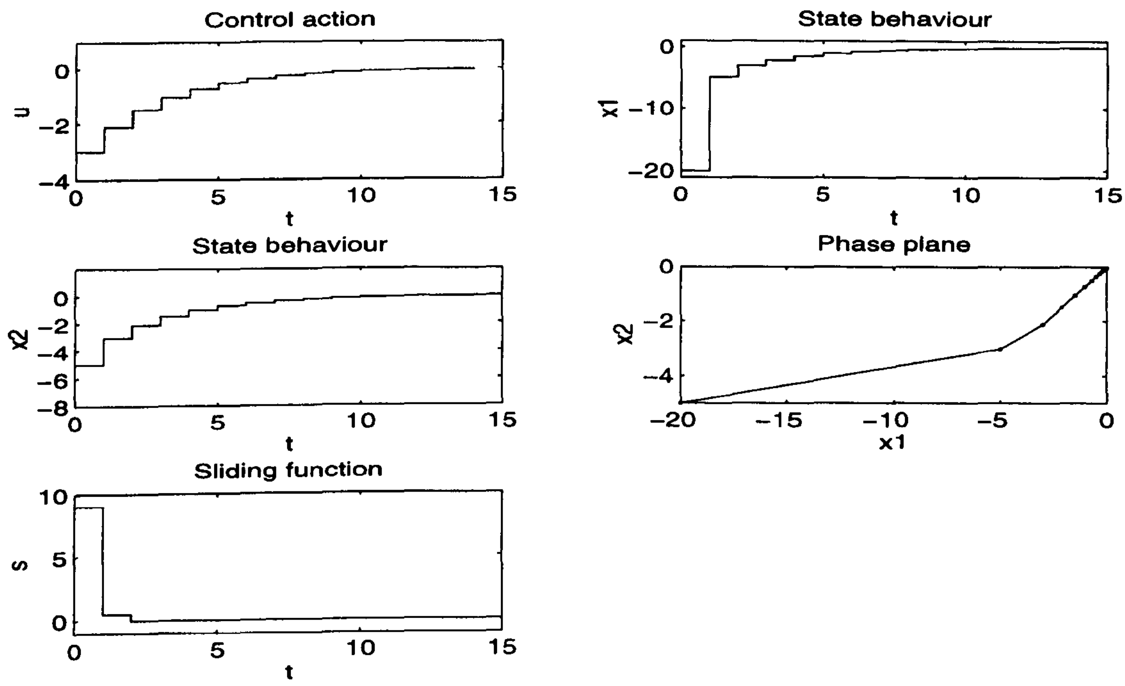

Figure 6.3: The responses of Example 6.6.2

Define the sequence of sliding mode as

$$s(k) = Cx(k) = c_1x_1(k) + c_2x_2(k)$$

Then

$$(I - BC/CB)A = \begin{bmatrix} 1.2 & 0.1 \\ -1.2c_2^{-1}c_1 & -0.1c_2^{-1}c_1 \end{bmatrix}$$

The eigenvalues of A_{eq} are 0 and $1.2 - 0.1c_2^{-1}c_1$. The system is stable in the sliding mode if $|1.2 - 0.1c_2^{-1}c_1| < 1$ which simplifies to $2 < c_2^{-1}c_1 < 22$. By applying the control (6.20)

$$x(k+1) = \begin{bmatrix} 1.2 & 0.1 \\ -1.2c_2^{-1}c_1 & -0.1c_2^{-1}c_1 \end{bmatrix} x(k) + W \begin{bmatrix} 0 & 0 \\ -c_2^{-1}c_1 & 1 \end{bmatrix} x(k)\ \mathrm{sgn}(c_1x_1 + c_2x_2)$$

Choosing $c_2 = 1$ and $c_1 = 5$ the eigenvalues of A_{eq} are 0 and 0.7. Therefore, $r \leqslant 1$ can be selected only greater than 0.7. Let $r = 0.998$ and $Q = I$. A p.d.s. solution P of the Lyapunov equation (6.32) has eigenvalues 75.8083, 1.0040 and $\mu = 6.1411$. Choosing $W = 0.0001(\sin(k\pi/4) + \cos(k\pi/6))$, (6.36) is satisfied which implies that the system is stable. Simulation results are shown in Fig. 6.4 with $x(0) = [2 \quad -6]^T$.

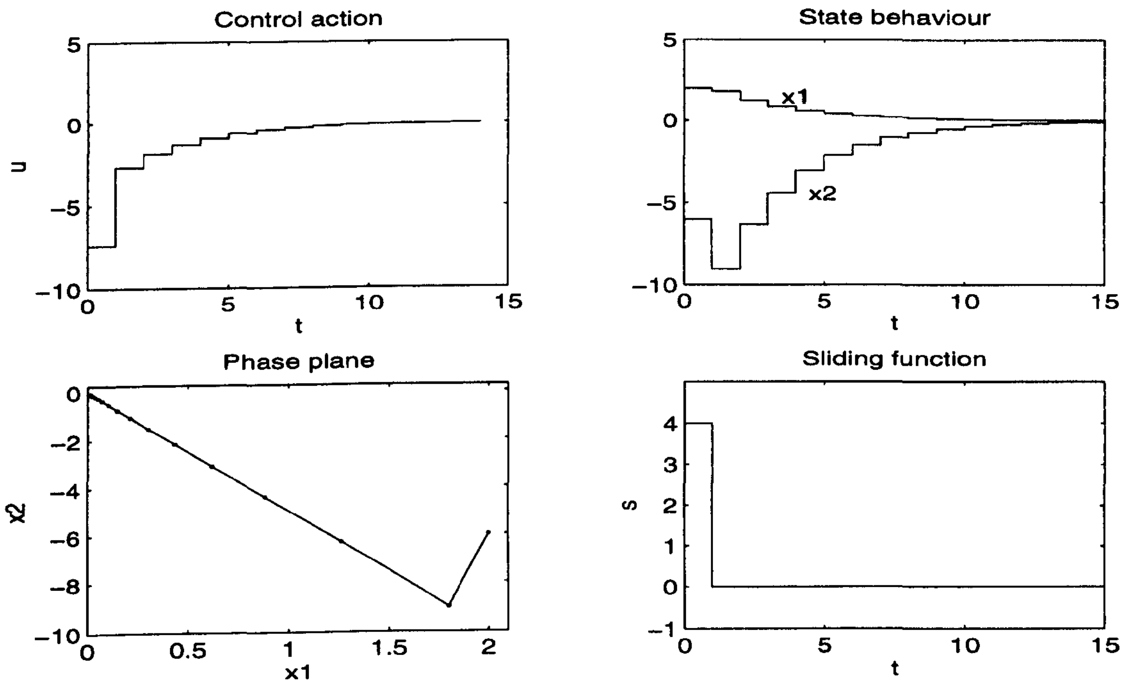

Figure 6.4: The responses of Example 6.6.3

Example 6.6.4 (Spurgeon **[110]**): Consider the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.60 & 0.04 \\ -0.01 & 0.90 & 0.12 \\ -0.16 & -1.25 & 0.57 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0.02 \\ 0.09 \\ 0.99 \end{bmatrix} u(k) + \begin{bmatrix} 0.002 \\ 0.009 \\ 0.099 \end{bmatrix} \xi$$

where ξ is random noise. Define the sequence of the sliding mode as

$$s(k) = Cx(k) = 0.44x_1(k) + 1.7x_2(k) + 0.85x_3$$

Then

$$(I - BC/CB)A = \begin{bmatrix} 0.9943 & 0.5854 & 0.0259 \\ -0.0357 & 0.8344 & 0.0567 \\ -0.4432 & -1.9718 & -0.1267 \end{bmatrix}$$

with the eigenvalues $0.8510 \pm 0.1336i$, 0 guarantees the stability of the reduced order system. The system in the sliding mode is independent of the perturbation input because $\Gamma = 0.1B$. Choosing $r = 0.97$ and $Q = I$, (6.32) gives the p.d.s solution P

$$P = \begin{bmatrix} 8.6950 & 14.5886 & 0.8786 \\ 14.5886 & 47.0455 & 2.8979 \\ 0.8786 & 2.8979 & 1.2460 \end{bmatrix}$$

with eigenvalues 52.1441, 3.7795, 1.0628 and then

$$\left(-\|A_{eq}\| + \sqrt{\|A_{eq}\|^2 + \frac{\lambda_{\min}(Q) + (1-r^2)\lambda_{\min}(P)}{\lambda_{\max}(P)}}\right) / \left(\|B(CB)^{-1}\|.\|C\|\right) = 0.0023$$

Therefore for the stability of the system the function W should be bounded by 0.0023. Take $W = 0.0022\sin(0.1k\pi)$, then

$$u(k) = \begin{bmatrix} 0.9943 & 0.5854 & 0.0259 \\ -0.0357 & 0.8344 & 0.0567 \\ -0.4432 & -1.9718 & -0.1267 \end{bmatrix} x(k) + 0.0091\sin(0.1k\pi)\begin{bmatrix} 0.0199 \\ 0.0897 \\ 0.9867 \end{bmatrix} |s(k)|$$

Simulation results are shown in Fig. 6.5 with $x(0) = [0.80 \; -0.50 \; -0.02]^T$.

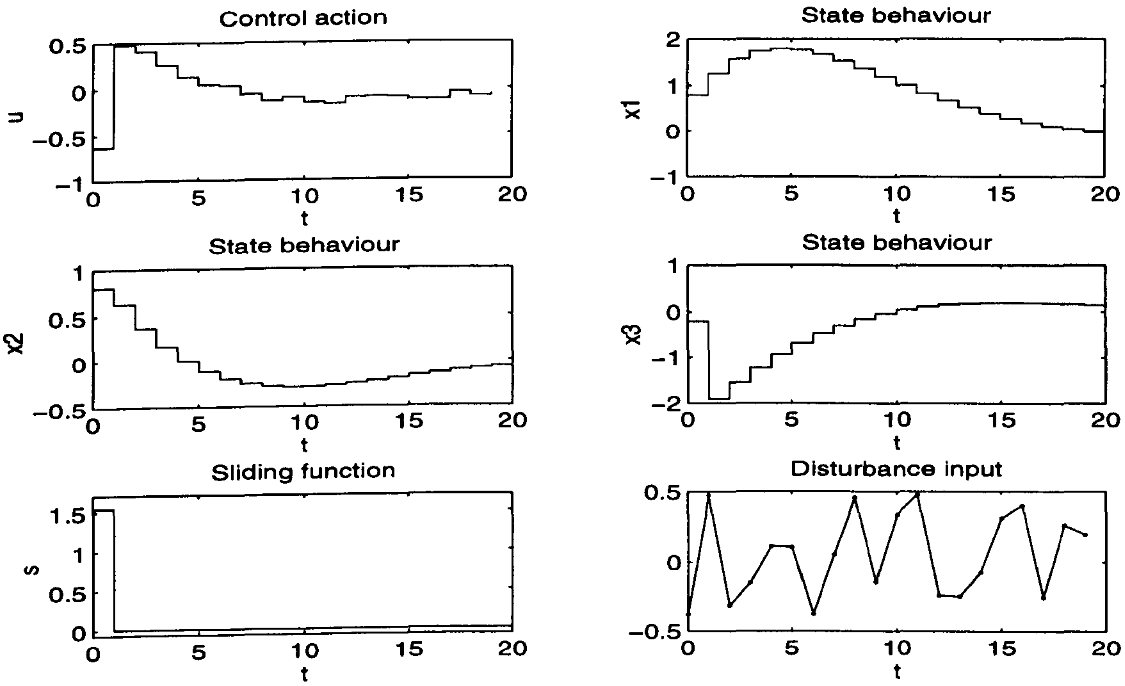

Figure 6.5: The responses of Example 6.6.4

Now an example is considered to demonstrate the MIMO theory.

Example 6.6.5: Consider the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 6 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 10 & 9 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(k) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \xi$$

where

$$\xi = \begin{bmatrix} 0 \\ 0.5\sin(0.1\pi k) \end{bmatrix}$$

The system in the sliding mode is independent of ξ because $B = \Gamma$. Choose the sliding matrix [111]

$$C = \begin{bmatrix} 0.3565 & 3.0000 & 0.3417 & 0.2157 \\ 0.0918 & 0.2157 & 1.0767 & 3.0000 \end{bmatrix}$$

The eigenvalues of A_{eq} are -0.3606, -0.1092 and 0. The spectral radius of A_{eq} is 0.3606. Let $Q = I_4$ and $r = 0.5$. A p.d.s. solution P of the Lyapunov equation (6.32) is

$$P = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 21.3107 & 0 & 2.6746 \\ 0 & 0 & 4 & 0 \\ 0 & 2.6746 & 0 & 42.4283 \end{bmatrix}$$

with the eigenvalues 20.9772, 4, 42.7617 and $\mu = 1.1083$. Hence

$$\frac{\mu - \|A_{eq}\|}{\|B(CB)^{-1}\|.\|C\|} = 0.0353$$

Take $W = -0.0352\sin(k\pi/10)I_2$. The system is stable since condition (6.36) is satisfied. Simulation results are shown in Fig. 6.6.

6.7 Sliding Mode State Observers for Linear Discrete-Time Systems

Discrete-time sliding mode observer design is a new topic. Discrete observer design using sliding mode control has been developed by Koshkouei and Zinober [71]. In Section 6.1 the concept of the discrete sliding mode and the sliding lattice were defined and a method for designing a controller presented such that the stability of the system and sliding mode are conserved. Now discrete sliding mode observer design is considered. Techniques for finding the feedforward injection map and the external feedforward compensation signal will be developed. The discrete-time linear time-invariant system

$$x(k+1) = Ax(k) + bu(k) + \gamma\xi(k) \tag{6.53}$$

$$y(k) = cx(k) \tag{6.54}$$

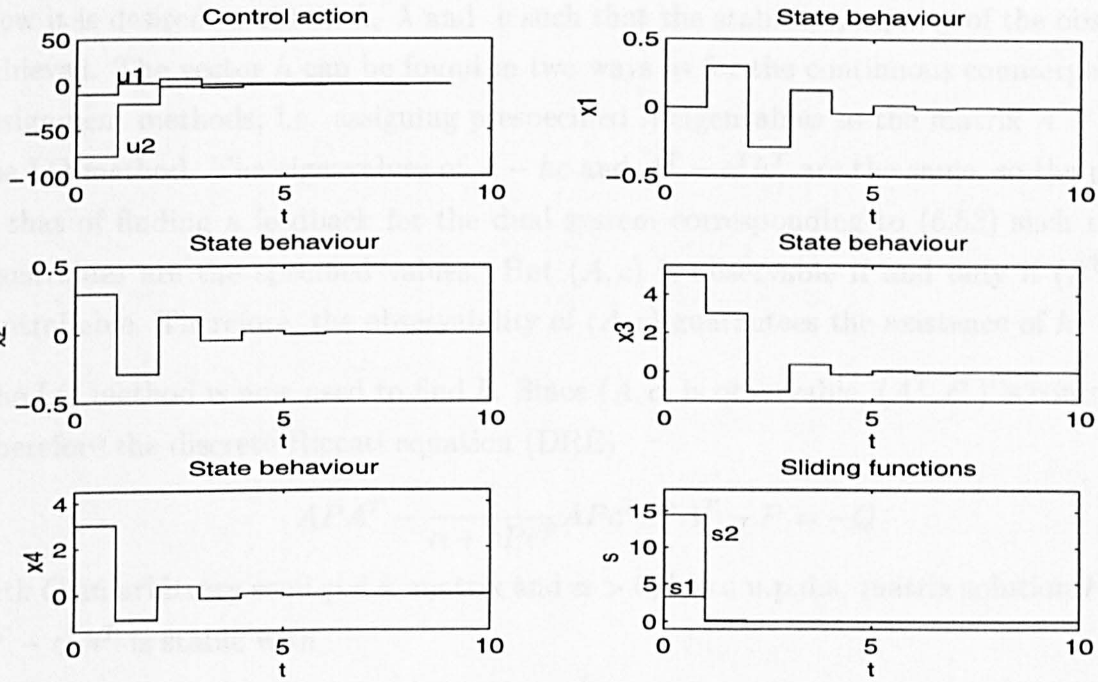

Figure 6.6: The responses of Example 6.6.5

is considered where $k \geqslant 0$ is an integer, $x(k) \in \mathbb{R}^n$ is the state, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ is nonzero vector, $u(k)$ is the scalar input control, $c \in \mathbb{R}^n$ such that $cb \neq 0$, $y(k) \in \mathbb{R}$ is the scalar output, $\gamma \in \mathbb{R}^n$ is the perturbation input map and $\xi \in \mathbb{R}$ is the bounded scalar disturbance input. It is assumed that (A, b) is completely controllable and (A, c) completely observable.

A sliding observer for the system (6.53) is

$$\begin{aligned}\hat{x}(k+1) &= A\hat{x}(k) + bu(k) + h(y(k) - \hat{y}(k)) + \lambda v(k)\\ \hat{y}(k) &= c\hat{x}(k)\end{aligned} \tag{6.55}$$

where $v \in \mathbb{R}$ is an external discontinuous feedforward compensation signal, $\lambda \in \mathbb{R}^n$ is the feedforward injection map such that $c\lambda \neq 0$ and $h \in \mathbb{R}^n$ is the observer gain vector. By choosing h such that $A - hc$ is stable, the observer will be asymptotically stable. The state reconstruction error is defined as $e = x - \hat{x}$. Subtracting (6.53) from (6.55) gives the reconstruction error system

$$e(k+1) = (A - hc)e(k) + \gamma\xi(k) - \lambda v(k) \tag{6.56}$$

$$e_y(k) = ce(k) \tag{6.57}$$

where $e_y(k) = y(k) - \hat{y}(k)$ is the output reconstruction error. Assume that for all k, $|\xi(k)| \leqslant \xi_0 \|e(k)\|$, where ξ_0 is a positive real number.

Now it is desired to obtain h, λ and v such that the stability property of the observer is achieved. The vector h can be found in two ways as for the continuous counterpart; pole assignment methods, i.e. assigning prespecified n eigenvalues to the matrix $A - hc$; and the LQ method. The eigenvalues of $A - hc$ and $A^T - c^T h^T$ are the same, so the problem is that of finding a feedback for the dual system corresponding to (6.53) such that the eigenvalues are the specified values. But (A, c) is observable if and only if (A^T, c^T) is controllable. Therefore, the observability of (A, c) guarantees the existence of h.

The LQ method is now used to find h. Since (A, c) is observable, (A^T, c^T) is controllable. Therefore the discrete Riccati equation (DRE)

$$APA^T - \frac{1}{\alpha + cPc^T} APc^T cPA^T - P = -Q \tag{6.58}$$

with Q an arbitrary semi-p.d.s. matrix and $\alpha > 0$, has a u.p.d.s. matrix solution P. Then $A^T - c^T h^T$ is stable with

$$h^T = \frac{1}{\alpha + cPc^T} cPA^T \tag{6.59}$$

which is equivalent to the stability of $A - hc$. So

$$h = \frac{1}{\alpha + cPc^T} APc^T \tag{6.60}$$

The ideal sliding mode for the system (6.56)-(6.57) is obtained if $e_y(k) = 0$ after a finite integer k. The equivalent feedforward input is given by

$$v_{eq}(k) = (c\lambda)^{-1}(cAe(k) + c\gamma\xi(k)) \tag{6.61}$$

Substituting (6.61) into the state reconstruction error system (6.56) gives the reduced order system

$$e(k+1) = (I - \lambda(c\lambda)^{-1}c)Ae + (I - \lambda(c\lambda)^{-1}c)\gamma\xi \tag{6.62}$$

One of the eigenvalues of matrix $(I-\lambda(c\lambda)^{-1}c)A$ is zero and the $n-1$ remaining eigenvalues can be assigned. The reduced order system is independent of the disturbance input signal if there exists a real number η such that $\gamma = \eta\lambda$. As for continuous systems the vector λ can be found in several ways (see Section 5.3):

1. Take $\lambda = A^T P c^T A/(\beta + cPc^T)$ where β is an arbitrary positive real number. If $\beta = \alpha$ then $\lambda = h$.

2. The vector λ in (6.55) can be obtained so that the stability of the reduced order system and the allocation of the $n - 1$ nonzero stable eigenvalues of the reduced order system

$$e(k+1) = [I - \lambda c/(c\lambda)]Ae(k)$$

are achieved.

3. The vector λ can be found such that the nominal system in the sliding mode is stable. When the reduced order system (6.62) is independent of the perturbation signal, this approach yields the stability of the matrix $[I - \lambda c/(c\lambda)]A$, and $\lim_{t\to\infty} e(t) = 0$. Generally, LQ methods for finding the vector λ cannot be achieved directly. However, the existence of λ under some conditions is guaranteed.

Suppose A is a nonsingular matrix. Let $\lambda = [\lambda_1\ \lambda_2\ \ldots \lambda_n]^T$, $c = [c_1\ c_2 \ldots\ c_n]$, $cA = M$ and $\lambda(c\lambda)^{-1} = F$. Hence $cF = 1$. Therefore, vector λ can be found if there exists a vector F such that $cF = 1$. Then $(I - \lambda(c\lambda)^{-1}c)A = A - FM$. Since (A, c) is observable and A is a nonsingular matrix, (A, M) is also observable and the discrete Riccati equation

$$APA^T - \frac{1}{\beta + MPM^T} APM^T MPA^T - P = -Q \tag{6.63}$$

where Q is an arbitrary semi-p.d.s. matrix and β a positive real number, has a u.p.d.s. matrix solution P. Taking

$$F = \frac{1}{\beta + MPM^T} APM^T \tag{6.64}$$

yields all the eigenvalues of $A - FM$ located inside the unit circle. If $cF = 1$, λ exists such that $\lambda(c\lambda)^{-1} = F$ and all the eigenvalues of $(I - \lambda(c\lambda)^{-1}c)A$ lie within the unit circle. Let $F = [f_1, f_2, \ldots, f_n]^T$. Then $\lambda(c\lambda)^{-1} = F$ gives

$$\begin{array}{ccccccccc} (c_1 f_1 - 1)\lambda_1 & + & c_2 f_1 \lambda_2 & + & \ldots & + & c_n f_1 \lambda_n & = & 0 \\ c_1 f_2 \lambda_1 & + & (c_2 f_2 - 1)\lambda_2 & + & \ldots & + & c_n f_2 \lambda_n & = & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ c_1 f_n \lambda_1 & + & c_2 f_n \lambda_2 & + & \ldots & + & (c_n f_n - 1)\lambda_n & = & 0 \end{array} \tag{6.65}$$

which has an infinity of solutions for which all the eigenvalues of $(I - \lambda(c\lambda)^{-1}c)A$ lie inside the unit circle. Equation (6.65) can be written as $(c \otimes F - I)\lambda = 0$ and then

$$(c \otimes F)\lambda = \lambda \tag{6.66}$$

where $\otimes$ indicates the Kronecker product. The determinant of matrix $c \otimes F - I$ is $\pm(cF - 1)$. Thus the system of equations (6.65) has a nonzero solution λ if and only if $cF = 1$. In this case the vector λ is not unique. Generally such a λ may not exist, but this method can be modified so that it is applicable for many practical problems. The above yields the following lemma which clarifies the conditions of the existence of vector λ.

Lemma 6.7.1: *Assume F is given by (6.64) and $\hat{F} = F/(cF)$. If all the eigenvalues of $A - \hat{F}M$ are inside the unit circle, then there exists a real number λ such that all the eigenvalues of $(I - \lambda(c\lambda)^{-1}c)A$ lie within the unit circle.*

4. In the sliding mode the systems (6.53) and (6.56) are independent of ξ if and only if there exist real numbers ρ and μ such that $\gamma = \rho b$ and $\gamma = \mu\lambda$. Therefore if both systems are independent of ξ, there exists a real number ν such that $\lambda = \nu b$. Take $\lambda = \nu b$ where ν is a real number.

5. The vector λ can be found such that the state reconstruction error system (6.56) is asymptotically stable, i.e. $\lim_{k\to\infty} e(k) = 0$.

The external feedforward compensation signal v is an input of the reconstruction error system (6.56), so the structure is similar to the input control. Therefore, utilizing the discrete-time sliding mode properties yields the control law

$$v(k) = v_{eq}(k) + (c\lambda)^{-1}W|ce(k)|$$

which guarantees the existence the sliding mode and the stability of the error system (6.56) [70]. Since the equivalent control v_{eq} is not accessible, the feedforward compensation input v should be chosen independently of the state error. consider the feedforward compensation input

$$v(k) = -W(k)|ce(k)| \qquad \text{if} \quad ce(k) \neq 0$$

where $W(k)$ is a real function with $|W(k)| < 1$. This feedforward compensation signal ensures that when there exists an integer k_0 such that $ce(k_0) = 0$, then for all integers $k \geqslant k_0$, $ce(k) = 0$. Suppose k_0 is an integer such that $ce(k_0) = 0$, then $v(k_0) = v_{eq}(k_0)$. Substituting (6.56) in $ce(k_0 + 1)$ gives

$$\begin{aligned} s(k_0 + 1) &= ce(k_0 + 1) \\ &= cAe(k_0) + c\gamma\xi(k_0) - c\lambda v_{eq} = 0 \end{aligned}$$

So if $ce(k) = 0$, then $ce(k+1) = 0$. Hence, for the existence of the ideal sliding mode, it is sufficient that there exists some integer k_0 such that $ce(k_0) = 0$, i.e. if a state point is on the sliding surface then the ideal sliding mode exists. As stated in Theorem 6.2.2, when the system (6.53) is asymptotically stable, sliding convergence exists. On the other hand, if the state error trajectory lies on the sliding surface at some time instant, after that time the sliding points remain on the sliding surface and consist of a sliding lattice. Therefore, it is clear that method 5 for finding the vector λ yields stability of the reconstruction error system.

6.8 The Stability of the Reconstruction Error System

Let us now study the stability of the error system (6.56) which is very important because the stability of the reconstruction error system ensures the state observer tends to the actual state. Since $A - hc$ is stable, all the eigenvalues of $A - hc$ lie within the unit circle. Let r be a real number such that $\rho(A - hc) < r \leqslant 1$ where $\rho(A - hc)$ is the spectral radius of $A - hc$. Then the discrete Lyapunov equation (DLE)

$$(A - hc)^T P_g (A - hc) - r^2 P_g = -Q_g \tag{6.67}$$

where Q_g is an arbitrary p.d.s. matrix, has a u.p.d.s. solution matrix P_g. A conventional Lyapunov function candidate for (6.56) is $V(k) = e^T(k) P_g e(k)$. Then

$$\begin{aligned}
\Delta V(k) &= V(k+1) - V(k) \\
&\leqslant e(k)^T \left(-Q_g - (1 - r^2) P_g\right) e(k) + 2|W| \left[e(k)^T (A - hc)^T P_g (A - hc) e(k)\right]^{\frac{1}{2}} \\
&\quad \times \left(\lambda^T P_g \lambda^T\right)^{\frac{1}{2}} |ce(k)| + 2|\xi| (\gamma^T P_g \gamma)^{\frac{1}{2}} \left[e(k)^T (A - hc)^T P_g (A - hc) e(k)\right]^{\frac{1}{2}} \\
&\quad +2|\xi|.|W|.|ce(k)| (\lambda^T P_g \lambda)^{\frac{1}{2}} (\gamma^T P_g \gamma)^{\frac{1}{2}} + \ |\xi|^2 \gamma^T P_g \gamma + W^2 |ce(k)|^2 \lambda^T P_g \lambda \\
&\leqslant \|e(k)\|^2 \{ - \left[\lambda_{\min}(Q_g) + (1 - r^2) \lambda_{\min}(P_g)\right] + \\
&\quad 2|W|.\|c\| \left[-\lambda_{\min}(Q_g) + r^2 \lambda_{\max}(P_g)\right]^{\frac{1}{2}} \|\lambda\| \left(\lambda_{\max}(P_g)\right)^{\frac{1}{2}} + \\
&\quad 2\xi_0 \left[-\lambda_{\min}(Q_g) + r^2 \ \lambda_{\max}(P_g)\right]^{\frac{1}{2}} \|\gamma\| \left(\lambda_{\max}(P_g)\right)^{\frac{1}{2}} \\
&\quad +2\xi_0 |W|.\|c\|.\|\lambda\|.\|\gamma\| \lambda_{\max}(P_g) + {\xi_0}^2 \|\gamma\|^2 \lambda_{\max}(P_g) + W^2 \|c\|^2 \|\lambda\|^2 \lambda_{\max}(P_g) \} \\
&\leqslant \|e(k)\|^2 \{ - \left[\lambda_{\min}(Q_g) + (1 - r^2) \lambda_{\min}(P_g)\right] + \\
&\quad 2 \left[-\lambda_{\min}(Q_g) + r^2 \lambda_{\max}(P_g)\right]^{\frac{1}{2}} \lambda_{\max}(P_g)^{\frac{1}{2}} \left(\|\gamma\| \xi_0 + |W|.\|c\|.\|\lambda\|\right) + \\
&\quad \lambda_{\max}(P_g) \left(\|\gamma\| \xi_0 + \ |W|.\|c\|.\|\lambda\|\right)^2 \}
\end{aligned} \tag{6.68}$$

For the right-hand side (6.68) to be less than zero

$$\|\gamma\| \xi_0 + |W|.\|c\|.\|\lambda\| \ < \ \sqrt{r^2 + (1 - r^2) \frac{\lambda_{\min}(P_g)}{\lambda_{\max}(P_g)}} - \sqrt{r^2 - \frac{\lambda_{\min}(Q_g)}{\lambda_{\max}(P_g)}} = \mu_0 \tag{6.69}$$

Then

$$|W| < \frac{1}{\|c\|.\|\lambda\|} (\mu_0 - \xi_0 \|\gamma\|) \tag{6.70}$$

subject to $\xi_0 < \mu_0 / \|\gamma\|$. Therefore, stability of (6.56), requires that

$$|W| < \min \left\{ 1, \frac{1}{\|c\|.\|\lambda\|} (\mu_0 - \xi_0 \|\gamma\|) \right\} \tag{6.71}$$

Let η be a real number. Taking $\lambda = \frac{\eta}{\|c\|}\gamma$ the error system in the sliding mode is independent of the perturbation signal, and (6.70) becomes

$$|W| < \frac{1}{|\eta|}\left(\frac{\mu_0}{\|\gamma\|} - \xi_0\right) \tag{6.72}$$

subject to $\xi_0 < \frac{\mu_0}{\|\gamma\|}$. Therefore, for the stability of the system (6.56) it is necessary that

$$|W| < \min\left\{1, \frac{1}{|\eta|}\left(\frac{\mu_0}{\|\gamma\|} - \xi_0\right)\right\} \tag{6.73}$$

The above and Theorem 6.2.2 imply the following theorem:

Theorem 6.8.1: *If condition (6.70) is satisfied, the error system (6.56) is asymptotically stable and a convergent sliding mode exists.*

Let us now consider a special case of (6.67). Choosing $r = 1$, condition (6.70) yields

$$|W| < \frac{1}{\|c\|.\|\lambda\|}(\mu_1 - \xi_0\|\gamma\|) \tag{6.74}$$

where $\mu_1 = 1 - \sqrt{1 - \frac{\lambda_{\min}(Q_g)}{\lambda_{\max}(P_g)}}$ with $\xi_0 < \mu_1/\|\gamma\|$. It can be proved that μ_1 is maximal if $Q_g = I$, but first a useful lemma is stated.

Lemma 6.8.1: *Let Q_g be an arbitrary p.d.s. matrix, and P_{g_Q} and P_{g_I} be the u.p.d.s. solutions of the DLE (6.67) corresponding to Q_g and I, respectively. Then*

$$P_{g_Q} \geqslant \lambda_{\min}(Q_g)P_{g_I}$$

Proof: Suppose $w \in \mathbb{R}^n$ is a given vector. Then

$$P_{g_Q} = \sum_{i=0}^{\infty} r^{-2(i+1)}((A - hc)^T)^i Q_g (A - hc)^i$$

$$P_{g_I} = \sum_{i=0}^{\infty} r^{-2(i+1)}((A - hc)^T)^i (A - hc)^i$$

and

$$\begin{aligned} w^T(P_{g_Q})w &= w^T\left(\sum_{i=0}^{\infty} r^{-2(i+1))}((A - hc)^T)^i Q_g (A - hc)^i\right) w \\ &\geqslant \lambda_{\min}(Q_g) w^T \left(\sum_{i=0}^{\infty} r^{-2(i+1)}((A - hc)^T)^i (A - hc)^i\right) w \\ &= \lambda_{\min}(Q_g) w^T P_{g_I} w \end{aligned} \tag{6.75}$$

Since for all real vectors w, $w^T P_{g_Q} w \geqslant \lambda_{\min}(Q_g) w^T P_{g_I} w$, then $P_{g_Q} \geqslant \lambda_{\min}(Q_g) P_{g_I}$. □

The following corollary is obtained immediately from Lemma 6.8.1.

Corollary 6.8.1 [99]: *Let P_g be the p.d.s. solution (6.67). The maximum value of $\frac{\lambda_{\min}(Q_g)}{\lambda_{\max}(P_g)}$ is obtained if $Q_g = I$.*

Therefore, for $Q_g = I$, μ_1 is maximal. But μ_0 may not take the maximum value for $Q_g = I$. In fact, the relationship between Q_g and the ratio $\frac{\lambda_{\min}(P_g)}{\lambda_{\max}(P_g)}$ is unknown [53]. The following theorem is now stated:

Theorem 6.8.2: *Let Q_g be an arbitrary p.d.s. matrix, and P_{g_Q} and P_{g_I} be the u.p.d.s. solutions of the DLE (6.67) corresponding to Q_g and I, respectively. Then*

$$r^2 P_{g_Q} - Q_g \geqslant \lambda_{\min}(Q_g)(r^2 P_{g_I} - I)$$

with the equality satisfied if $Q_g = qI$ for any positive real number q.

Proof: Suppose $w \in \mathbb{R}^n$ is a given vector. The DLE (6.67) can be written as

$$(A - hc)P_{g_Q}(A - hc)^T = r^2 P_{g_Q} - Q_g$$

and the u.p.d. solution is

$$P_{g_Q} = \sum_{i=0}^{\infty} r^{-2(i+1)}((A - hc)^T)^i Q_g (A - hc)^i$$

Thus

$$\begin{aligned} w^T(r^2 P_{g_Q} - Q_g)w &= w^T(A - hc)^T\Big(\sum_{i=0}^{\infty} r^{-2(i+1)}((A - hc)^T)^i Q_g (A - hc)^i\Big)(A - hc)w \\ &= w^T\left(\sum_{i=0}^{\infty} r^{-2(i+1)}((A - hc)^T)^{(i+1)} Q_g (A - hc)^{(i+1)}\right) w \\ &\geqslant \lambda_{\min}(Q_g) w^T\left(\sum_{i=0}^{\infty} r^{-2(i+1)}((A - hc)^T)^{(i+1)}(A - hc)^{(i+1)}\right) w \\ &= \lambda_{\min}(Q_g) w^T(r^2 P_{g_I} - I)w \end{aligned} \tag{6.76}$$

Since for all real vectors w,

$$w^T(r^2 P_{g_Q} - Q_g)w \geqslant \lambda_{\min}(Q_g) w^T(r^2 P_{g_I} - I)w$$

then $r^2 P_{g_Q} - Q_g \geqslant \lambda_{\min}(Q_g)(r^2 P_{g_I} - I)$. □

Note that Lemma 6.8.1 and Corollary 6.8.1 are direct results of Theorem 6.8.2. Theorem 6.8.2 yields $\lambda_{\min}(Q_g)I - Q_g \geqslant r^2(P_{g_I} - P_{g_Q})$. Since $\lambda_{\min}(Q_g)I - Q_g$ is a nonpositive matrix, $r^2(\lambda_{\min}(Q_g)P_{g_I} - P_{g_Q})$ is also a nonpositive matrix. On the other hand, $r \neq 0$ yields $P_{g_Q} \geqslant \lambda_{\min}(Q_g)P_{g_I}$, i.e. Lemma 6.8.1 is obtained. Therefore, Theorem 6.8.2 clarifies the relationship between the solutions of DLE (6.67) which are obtained for $Q_g = I$ and an arbitrary p.d.s. matrix Q_g.

6.9 Examples

The following examples illustrate the design procedure for an asymptotically stable observer.

Example 6.9.1 Consider the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.60 & 0.04 \\ -0.01 & 0.90 & 0.12 \\ -0.16 & -1.25 & 0.57 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0.02 \\ 0.09 \\ 0.99 \end{bmatrix} u(k) + \begin{bmatrix} 0.002 \\ 0.009 \\ 0.099 \end{bmatrix} \xi$$

where ξ is a bounded random noise signal satisfying $|\xi| \leqslant 0.0106$ [110]. Taking

$$c = \begin{bmatrix} 0.44 & 1.7 & 0.85 \end{bmatrix}$$

$Q = I_3$ and $\alpha = 1$, the observer gain vector h is given by

$$h = \begin{bmatrix} 0.0246 & 0.2623 & 0.1225 \end{bmatrix}^T$$

Let η_0 be a real number and $\lambda = \eta_0 b$. For all η_0 the reconstruction error system in the sliding mode is independent of the perturbation signal because $\lambda = 10\eta_0\gamma = \eta\gamma$. Taking $\eta = 0.5$ yields

$$\lambda = \begin{bmatrix} 0.0010 & 0.0045 & 0.0495 \end{bmatrix}^T$$

The eigenvalues of $A - hc$ are 0.1388, $0.8852 \pm 0.1276i$. Hence the spectral radius $A - hc$ is 0.8943. Since all the eigenvalues of $A - hc$ are in the unit circle, the observer (6.55) is asymptotically stable.

Choosing $Q_g = I$ and $r = 0.9852$, the u.p.d.s. solution of the DRE (6.67) is P_g

$$P_g = \begin{bmatrix} 17.8877 & -1.3712 & -7.4347 \\ -1.3712 & 2.1476 & -2.3686 \\ -7.4347 & -2.3686 & 13.6161 \end{bmatrix}$$

with $\lambda_{\min}(Pg) = 1.0788$ and $\lambda_{\max}(Pg) = 23.4925$. On the other hand, $\xi_0 = 0.0113$ and $|\xi(k)| \leqslant 0.0113\|e(k)\|$. Since $\lambda_{\min}(Q_g) = 1$, $c = 1.9509$ and $\gamma = 0.0994$, then $\mu_0 = 0.0225$ and the right-hand side inequality (6.70) is

$$\frac{1}{\|c\|.\|\lambda\|}(\mu_0 - \xi_0\|\gamma\|) = 0.2208$$

Choose $W = 0.2198\sin(0.1k\pi)$. The feedforward injection input signal is defined as

$$v(k) = -0.2198\sin(0.1k\pi)\ |s(k)|, \quad \text{if} \quad s(k) \neq 0$$

where $s(k) = 0.44e_1(k) + 1.7e_2(k) + 0.85e_3(k)$ and $e(k) = [e_1(k)\ e_2(k)\ e_3(k)]^T$. Since the condition (6.72) is satisfied, the error system is asymptotically stable. Simulation results are shown in Fig. 6.7.

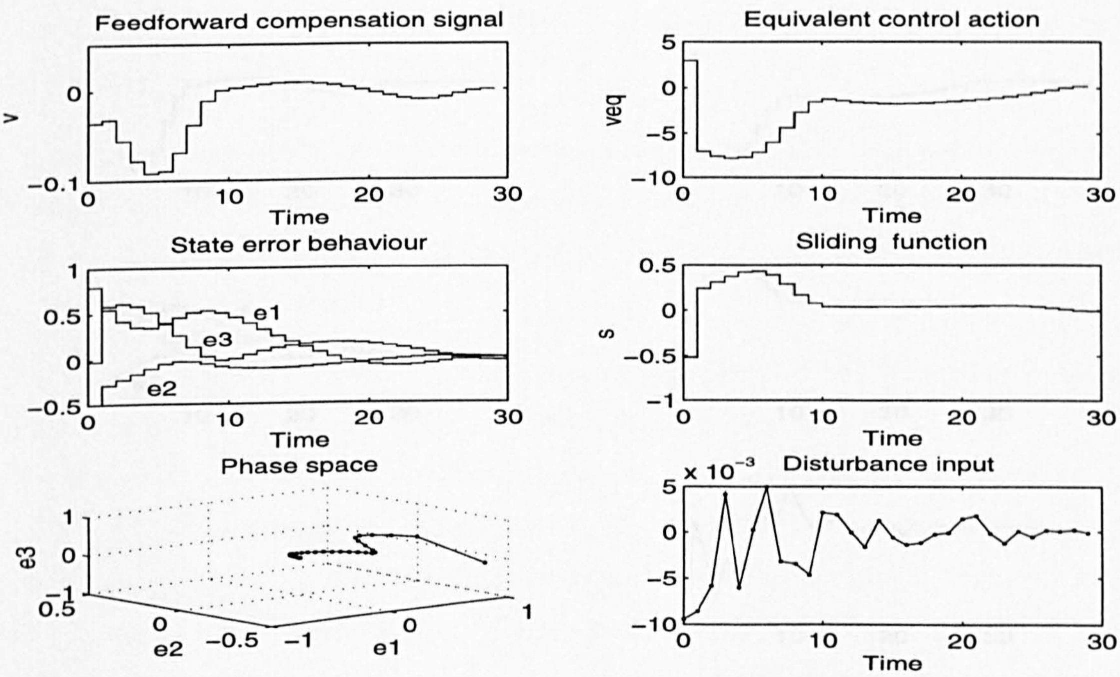

Figure 6.7: The responses of Example 6.9.1

Example 6.9.2 Consider the system as in Example 6.9.1. Let β be a real number and $Q_g = I_3$. Then

$$F = \begin{bmatrix} -0.3543 & 0.0545 & 0.9123 \end{bmatrix}^T$$

The eigenvalues of $A - Fc$ are 0.8950, $0.4314 \pm 0.1011i$ and lie inside the unit circle, but the condition $cF = 1$ is untrue. Consider $\hat{F} = F/(cF)$,

$$\hat{F} = \begin{bmatrix} -0.4973 & 0.0766 & 1.2808 \end{bmatrix}^T$$

with the eigenvalues of $A - \hat{F}M$ being 0.8908, −0.0371, 0.6163 which lie inside the unit circle. Lemma 6.7.1 is applied so that there exists a λ such that all the eigenvalues of $(I - \lambda(c\lambda)^{-1}c)A$ lie within the unit circle. Vector λ must be found so that

$$c \otimes \hat{F} - I_3 = \begin{bmatrix} -1.2188 & -0.8455 & -0.4227 \\ 0.0337 & -0.8699 & 0.0651 \\ 0.5636 & 2.1774 & 0.0887 \end{bmatrix}$$

Then $\lambda = \begin{bmatrix} 0.0186 & -0.0023 & 0.0495 \end{bmatrix}^T$, and h is obtained as in Example 6.9.1. So the eigenvalues of $A - hc$, and also r and P_g are the same as in Example 6.9.1. The responses are shown in Fig. 6.8.

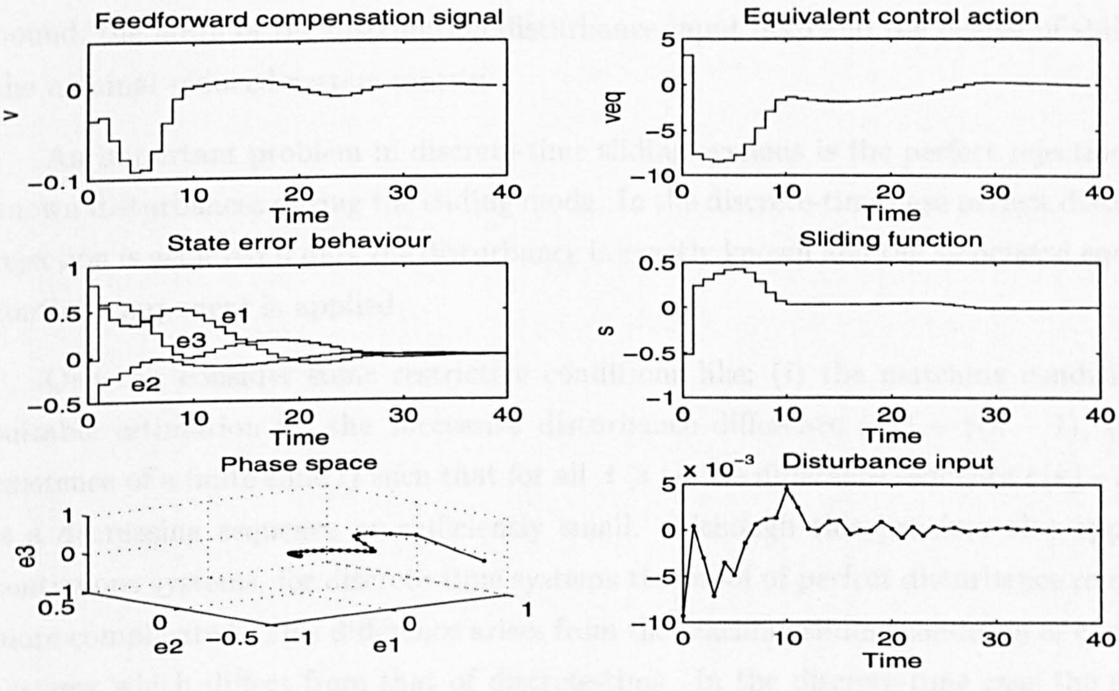

Figure 6.8: The responses of Example 6.9.2

6.10 Summary and Discussion

In this chapter the concept of the discrete-time sliding mode has been clarified and sufficient conditions for the existence of the discrete-time sliding mode have been presented. The sliding surface for the discrete-time systems is a lattice which is called the sliding latticewise surface or more concisely the sliding lattice. A method of control design using the properties of the discrete sliding mode has been proposed. This control guarantees the stability of the sliding mode and the stability of the system. This discrete-time control does not have the same structure as sliding mode continuous control. The behaviour of the system in the sliding mode, and stability conditions have been studied. It is concluded that if the nominal reduced order matrix is stable, then the state in the sliding mode is bounded. In this case the width of the boundary layer depends upon the disturbance bound, the norm of the distribution disturbance input map and the degree of stability of the nominal reduced system matrix.

An important problem in discrete-time sliding systems is the perfect rejection of unknown disturbances during the sliding mode. In the discrete-time case perfect disturbance rejection is achieved if only the disturbance is exactly known and the associated equivalent control component is applied.

One can consider some restrictive conditions like; (*i*) the matching condition, (*ii*) suitable estimation for the successive disturbance difference $\xi(k) - \xi(k-1)$, (*iii*) the existence of a finite time t_f such that for all $t \geqslant t_f$, the difference sequence $\xi(k) - \xi(k-1)$ is a decreasing sequence or sufficiently small. Although this problem also appears in continuous systems, for discrete-time systems the proof of perfect disturbance rejection is more complicated. This difference arises from the reaching sliding condition of continuous systems which differs from that of discrete-time. In the discrete-time case the stability of the system and the reaching sliding condition are established by considering a discrete Lyapunov function, and difference equations arises. In the case of continuous system the continuous Lyapunov function is utilized which results in derivatives.

The control design method as stated in this chapter needs an estimate for disturbance input. One may utilize the estimation as in [112]. The equivalent control with zero disturbance can be considered. In this way it is assumed that in the average sense, the disturbance does not affect the equivalent control. More precisely, since the equivalent control can be considered as the average of the control input and if the average of disturbance is zero, the equivalent control may be assumed independent of the disturbance input.

There are two ways for designing discrete-time sliding mode control: (*i*) in the first instance a dynamical sliding mode (or a control for the system) is specified, and then it is necessary to find the conventional control (sliding mode dynamics) [46], [21]; (*ii*) a control is determined by using the properties of the sliding mode such that the stability of the nominal systems in the sliding mode is conserved. In this chapter methods (*i*) and (*ii*) have been applied successfully for linear systems. In some previous work [126] there are errors [88] and also some restrictions [21].

The system stability has been studied and the design of the optimal sliding mode matrix is also extended to DSMC.

A method for discrete-time sliding mode observer design including external feedforward compensation and feedforward injection map has been presented, as have results for the stability of reconstruction error systems of linear systems. The problem of disturbance rejection has been studied. The cone condition for the error system is a limiting condition and satisfying this condition may be difficult in some practical problems. However, the stability of the system is guaranteed if one of the following conditions is satisfied: (*i*) the cone condition for the disturbance input with respect to the state; (*ii*) there exists a finite time instant such that after this time, the disturbance input sequence ξ is a decreasing sequence. So a simple condition on the disturbance should be found such that the stability of the error system is achieved.

A useful theorem, a corollary and a lemma have been stated in Section 6.8. Theorem 6.8.2 determines the relationship between the solution of the DLE (6.67) for arbitrary weighting function Q_g and the solution obtained for $Q = I$. An open problem is to find a bound for the ratio $\lambda_{\min}(P_g)/\lambda_{\min}(P_g)$ which appears in some stability conditions, e.g. (6.69).

Examples have been presented to demonstrate the techniques of the controller and observer design methods. These examples show that the results of this chapter can be successfully applied to many systems.

Chapter 7

Sliding Mode in Time-Delay Systems

7.1 Introduction

In recent years many methods have been reported for designing control for time-delay systems, for instance [81] and [100], and criteria for the stability of time-delay systems [23], [87] have been developed. Time-delays may appear in many ways; delays in measurement of system variables including physical properties of equipment used in the system or signal transmission; delays in control which arise in many chemical processes and radiation problems in physics. Time-delay systems are also used to model several different mechanisms in the dynamics of epidemics. Many problems such as incubation periods, maturation times, age structure, seasonal or diurnal variations, interactions across spatial distances or through complicated paths have been modelled by time-delay systems [16].

The work on the stability of time-delay linear systems has been reported by many authors and can be found in [22], [23], [54] and [87] amongst others. Work on the stabilization problem for a class of uncertain linear systems with delay on the state has been studied in [81]. The proof of stability of closed-loop time-delay systems with discontinuous control is more complicated than for the continuous case.

Stability criteria for time-delay systems can be classified into two categories: (i) there is no information about the delays, i.e. delay-independent criteria; (ii) there is some information about the delays, i.e. delay-dependent criteria. The delay-independent criteria are strong conditions to test the stability of the system. However, if the delays are small, these criteria may be useful. Delay-dependent criteria for a closed-loop system are dependent upon the kind of control which is applied to the system. To prove the stability of both open- and closed-loop time-delay systems, an appropriate Lyapunov function can be selected. The magnitude of the delay may not necessarily be important when establishing system stability. The stability may hold for certain sufficient conditions [54].

Sliding mode time-delay systems may employ the proportional-integral sliding mode (PISM) [100] and the traditional sliding mode. In the case of PISM the sliding surface may depend on delays and it is difficult to specify its dynamic performance; for the traditional sliding mode the sliding surface is independent of delay. In this chapter a method of designing the sliding surface and appropriate discontinuous control are presented to yield the stability of the sliding mode and the system.

7.2 Sliding Time-Delay Systems

Consider the following uncertain system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_0x(t-\tau) + Bu(t) + f(t,x,\tau) \\ x(t) &= \phi(t), \qquad t \in [-\tau, 0] \end{aligned} \tag{7.1}$$

where $x \in \mathbb{R}^n$ is the state variable, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ is full rank, $u \in \mathbb{R}^m$ is the input control, $C \in \mathbb{R}^{m\times n}$ such that CB is nonsingular, τ is a positive real number and $\phi(t)$ is a continuous vector-value initial function with $\|\phi\| = \sup_{-\tau\leqslant t\leqslant 0} \|\phi(t)\|$. Assume that (A, B) is a completely controllable pair, $m < n$ and the function $f(t,x,\tau) \in \mathbb{R}^n$ is a bounded disturbance or uncertain input signal. The sliding surface is defined as $s = Cx(t) = 0$. The ideal sliding mode exists if there is a finite time t_s such that

$$Cx = 0, \quad C\dot{x} = 0 \qquad t \geqslant t_s$$

where $C \in \mathbb{R}^{m\times n}$ is the sliding mode matrix. Then the virtual equivalent control is given by

$$u_{eq}(t) = -(CB)^{-1}\left(CAx(t) + CA_0x(t-\tau) + Cf(t,x,\tau)\right)$$

and the system in the sliding mode is

$$\dot{x}(t) = A_{eq}x(t) + \hat{A}_{eq}x(t-\tau) + B_{eq}f(t,x,\tau) \tag{7.2}$$

where $A_{eq} = (I - B(CB)^{-1}C)A$, $\hat{A}_{eq} = (I - B(CB)^{-1}C)A_0$, and $B_{eq} = (I - B(CB)^{-1}C)$.

Assumption: *Matching condition.* Assume that $f(t,x,\tau) = Bg(t,x,\tau)$. Then the system in the sliding mode

$$\dot{x}(t) = A_{eq}x(t) + \hat{A}_{eq}x(t-\tau) \tag{7.3}$$

is independent of the external input f. The matching condition is a suitable condition for the system in the sliding mode to be independent of the external uncertain input.

7.2.1 Sliding Control

Consider the control

$$u(t) = -(CB)^{-1}\left(CAx(t) + CA_0x(t-\tau) + \rho \text{ sgn } s(t)\right) \tag{7.4}$$

where $\rho = \rho(t, x(t), \tau) = \text{diag}(\rho_1, \rho_2, \ldots, \rho_m)$ with positive real functions $\rho_i = \rho_i(t, x(t), \tau)$ $(1 \leqslant i \leqslant m)$ and $\text{sgn } s = \begin{bmatrix} \text{sgn } s_1 & \text{sgn } s_2 & \ldots & \text{sgn } s_m \end{bmatrix}^T$. The system is now given by

$$\dot{x}(t) \;=\; A_{eq}x(t) + \hat{A}_{eq}x(t-\tau) + f(t,x,\tau) - B(CB)^{-1}\rho(t,x(t),\tau) \text{ sgn } s \tag{7.5}$$

Hence the sliding dynamics is governed by

$$\dot{s} = Cf(t, x(t), \tau) - \rho(t, x(t), \tau) \text{ sgn } s \tag{7.6}$$

and for all $0 \leqslant t \leqslant t_s$

$$s(t) = C\left(\int_{t_s}^{t} f(t, x(t), \tau)\, dt\right) - \rho(t, x(t), \tau)(t - t_s) \text{ sgn } s$$

The reaching sliding mode condition is

$$\dot{s}_i \text{ sgn } s_i < 0, \quad \forall i \quad 1 \leqslant i \leqslant m \tag{7.7}$$

in the neighbourhood of $s_i = 0$ [29], [121]. Multiplying the i-th $(1 \leqslant i \leqslant m)$ row of (7.6) by sgn s_i gives

$$\dot{s}_i \text{ sgn } s_i = C_i f(t, x(t), \tau) \text{ sgn } s_i - \rho_i(t, x(t), \tau) \tag{7.8}$$

Hence, a sufficient condition for the existence of the sliding mode is

$$C_i f(t, x(t), \tau)\text{sgn } s_i < \rho_i(t, x(t), \tau)$$

and a sufficient condition is that

$$\|Cf(t, x(t), \tau)\| < \min_{1 \leqslant i \leqslant m} \rho_i \tag{7.9}$$

7.3 System in the Sliding Mode

The behaviour of the system in the sliding mode is considered in this section. The system in the sliding mode is a subsystem of (7.1) of order $n - m$. Assume T is an orthogonal matrix (2.13)

$$TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

where B_2 is a nonsingular matrix. Let $z = Tx$, then

$$\dot{z}(t) = TAT^T z(t) + TA_0T^T z(t-\tau) + TBu(t) + TBg(t, x(t), \tau) \tag{7.10}$$

Now assume $z^T = [z_1,\ z_2]^T,\quad z_1 \in \mathbb{R}^{n-m},\quad z_2 \in \mathbb{R}^m$. Then

$$\dot{z}_1(t) = A_{11}z_1(t) + A_{12}z_2(t) + \hat{A}_{11}z_1(t-\tau) + \hat{A}_{12}z_2(t-\tau) \tag{7.11}$$

$$\dot{z}_2(t) = A_{21}z_1(t) + A_{22}z_2(t) + \hat{A}_{21}z_1(t-\tau) + \hat{A}_{22}z_2(t-\tau) + B_2u + B_2g(x(t), \tau, t) \tag{7.12}$$

where

$$TAT^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\quad TA_0T^T = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}$$

Subsystem (7.11) is the system in the sliding mode. So the sliding surface is

$$C_1z_1(t) + C_2z_2(t) = 0$$

where $CT^T = [C_1 \quad C_2]$. In the sliding mode $z_2(t) = -Kz_1(t)$ with $K = {C_2}^{-1}C_1$. Therefore the reduced order system (7.11) is converted to

$$\dot{z}_1(t) = (A_{11} - A_{12}K)z_1(t) + (\hat{A}_{11} - \hat{A}_{12}K)z_1(t-\tau) \tag{7.13}$$

For any $x(0)$ and any function $\phi \in C([-\tau, 0], \mathbb{R}^n)$ there exists a function $z_1(t)$ satisfying the differential equation (7.13) almost everywhere [25]. In this case, for all integers k the function $z_1(t)$ will be C^k on $((k-1)\tau, \infty)$, i.e. for all $1 \leqslant j \leqslant k$ the j-th derivative of function $z_1(t)$ is continuously differentiable on $((j-1)\tau, \infty)$. With these conditions the solution of (7.1) is

$$z_1(t) = e^{(A_{11}-A_{12}K)t}\phi_1(0) + \int_0^t e^{(A_{11}-A_{12}K)(t-w)} \times \left\{(\tilde{A}_{11} - \tilde{A}_{12}K)z_1(w-\tau)\right\} dw, \quad t \geqslant 0 \tag{7.14}$$

where $T\phi = [\phi_1 \quad \phi_2]^T$. Let $\|\phi_1\| = \sup_{-\tau \leqslant t \leqslant 0} \|\phi_1(t)\|$. There exist positive real numbers η and M such that $\|z_1(t)\| \leqslant (M\|\phi_1(0)\| + M^2\|\phi_1\|\tau)\, e^{(M^2+\eta)t}$ [25]. It is desired to find K such that $A_{11} - A_{12}K$ is stable. Consider the Riccati equation

$$A_{11}P + PA_{11}^T - PA_{12}R^{-1}A_{12}^TP = -Q \tag{7.15}$$

with Q and R are arbitrary semi-p.d.s. and p.d.s matrices respectively, having the u.p.d.s. matrix solution P. For $K = R^{-1}A_{12}^TP$, the matrix $A_{11} - A_{12}K$ is stable.

7.3.1 System Stability without Delay Information

For this case, system (7.1) is asymptotically stable if all the roots of the characteristic polynomial of the system (7.11) $p(s) = \det(sI - A_{11} + A_{12}K - (\hat{A}_{11} - \hat{A}_{12}K)e^{-\tau s})$ have negative real parts. If all the roots of $p(s) = 0$ lie in the open left-half complex plane, this condition is equivalent to $p(s) \neq 0$ for all $s \in \bar{\mathbb{C}}^+$ where $\bar{\mathbb{C}}^+$ is the set of all complex numbers with nonnegative real parts. Note that for $\tau > 0$ there may be infinitely many solutions, while for $\tau = 0$ there are finitely many solutions. So when $\tau > 0$, it is very difficult to find all the infinity roots of $p(s) = 0$ and check if the roots are in the open left-hand half-plane. This motivates one to avoid the delay-dependent condition $p(s) = 0$ for stability of the system and use the condition $p(s) \neq 0$ for all $s \in \bar{\mathbb{C}}^+$ to prove the asymptotic stability independent of delay. However, since $p(s)$ is an entire function, there are only a finite number of roots of $p(s) = 0$ in any compact set, in particular in a vertical strip of the complex plane. Furthermore, there exists a real number α such that all the roots of $p(s)$ lie to the left of the vertical line $x = \alpha$ [52].

The function $p(s)$ is an analytic function on $\bar{\mathbb{C}}^+$. Therefore according to the Maximum Principle Theorem [3, page 134] $p(s)$ takes the maximum value on its boundary, i.e. there exists a ω such that $\max p(s) = p(i\omega)$. Hence

$$\det(i\omega I - A_{11} + A_{12}K - (\hat{A}_{11} - \hat{A}_{12}K)e^{-i\omega\tau}) \neq 0, \quad \forall\omega \geqslant 0 \tag{7.16}$$

For simplicity, set $\tilde{A} = A_{11} - A_{12}K$ and $\tilde{A}_0 = \hat{A}_{11} - \hat{A}_{12}K$. So from [22] the following theorem is obtained.

Theorem 7.3.1: *Assume $\rho\left((\tilde{A})^{-1}\tilde{A}_0\right) < 1$. The system (7.11) is asymptotically stable independent of delay τ if and only if*

$$\rho\left((i\omega I - \tilde{A})^{-1}\tilde{A}_0\right) < 1, \qquad \forall\omega \geqslant 0 \tag{7.17}$$

Proof: Since $\tilde{A}$ is a stable matrix, $\rho\left((sI - \tilde{A})^{-1}\tilde{A}_0\right)$ is an analytic function in the right half-plane and according to the Maximum Principal Theorem [3, page 134] it is assumed that it takes its maximum value on the imaginary axis. Thus

$$\begin{aligned}\sup_{\omega\geqslant 0} \rho\left((i\omega I - \tilde{A})^{-1}\tilde{A}_0\right) &= \sup_{\omega\geqslant 0} \rho\left((i\omega I - \tilde{A})^{-1}\tilde{A}_0 e^{-i\omega\tau}\right) \\ &= \sup_{s\in\bar{\mathbb{C}}^+} \rho\left((sI - \tilde{A})^{-1}\tilde{A}_0 e^{-s\tau}\right)\end{aligned}$$

Necessary. Assume that the system (7.13) in the sliding mode is asymptotically stable independent of delay. Since $\tilde{A}$ is a stable matrix, for all $s \in \bar{\mathbb{C}}^+$, $\det(sI - \tilde{A}) \neq 0$.

If $s = 0$ then $\rho\left((\tilde{A})^{-1}\tilde{A}_0\right) < 1$ and there is nothing to prove. Suppose that $s \neq 0$ then $\det\left(I - (sI - \tilde{A})^{-1}\tilde{A}_0 e^{-s\tau}\right) \neq 0$, i.e. for all $s \in \bar{\mathbb{C}}^+$, 1 is not an eigenvalue of $(sI - \tilde{A})^{-1}\tilde{A}_0 e^{s\tau}$, so

$$\rho\left((i\omega I - \tilde{A})^{-1}\tilde{A}_0\right) \neq 1, \qquad \forall \omega \geqslant 0 \tag{7.18}$$

Assume there exists a ω such that $\rho\left((i\omega I - \tilde{A})^{-1}\tilde{A}_0\right) > 1$. Since $\rho\left((i\omega I - \tilde{A})^{-1}\tilde{A}_0\right)$ is a continuous function, there is a ω_0 such that $\rho\left((i\omega_0 I - \tilde{A})^{-1}\tilde{A}_0\right) = 1$ which contradicts (7.18). Therefore $\rho\left((\omega I - \tilde{A})^{-1}\tilde{A}_0\right) < 1 \quad \forall s \in \bar{\mathbb{C}}^+$.

Sufficiency. Assume that (7.17) is satisfied. Hence

$$\sup_{\omega \geqslant 0} \ \rho\left((i\omega I - \tilde{A})^{-1}\tilde{A}_0\right) < 1$$

Thus $p(s) = \det\left((sI - \tilde{A})^{-1}\tilde{A}_0 e^{-s\tau}\right) \neq 0, \quad \forall s \in \bar{\mathbb{C}}^+$. □

Since

$$\rho\left((i\omega I - (A_{11} - A_{12}K))^{-1}(\hat{A}_{11} - \hat{A}_{12}K)\right) \leqslant \|(i\omega I - (A_{11} - A_{12}K))^{-1}(\hat{A}_{11} - \hat{A}_{12}K)\| \qquad \forall \omega \geqslant 0$$

for (7.16) to be satisfied it is sufficient that

$$\|(i\omega I - (A_{11} - A_{12}K))^{-1}(\hat{A}_{11} - \hat{A}_{12}K)\| < 1, \qquad \forall \omega \geqslant 0$$

To assess the stability by direct applying Theorem 7.3.1 is difficult. So for practical problems, it is necessary to find a criterion to test the stability. The following theorem gives such a condition free of frequency. Its proof is similar to that in [23].

Theorem 7.3.2: *Let P be the u.p.d.s. solution of the Riccati equation*

$$A_{11}P + PA_{11}^T - 2PA_{12}R^{-1}A_{12}^T P = -Q \tag{7.19}$$

where Q and R are arbitrary semi-p.d.s. and p.d.s. matrices respectively. Then if

$$\left\|\left(\hat{A}_{11} - \hat{A}_{12}K\right)P\right\| < \lambda_{\min}(Q)/2, \qquad K = R^{-1}{A_{12}}^T P \tag{7.20}$$

the system in the sliding mode is asymptotically stable independent of delay.

Proof: Let P be the u.p.d.s. matrix solution of the Riccati equation (7.19). Adding and subtracting $i\omega P$ on the left-hand side of (7.19) gives

$$(-A_{11} + A_{12}K + i\omega I)^* P + P(-A_{11} + A_{12}K + i\omega I) = Q \tag{7.21}$$

and by pre- and post-multiplication by

$$(\hat{A}_{11} - \hat{A}_{12}K)^T((-A_{11} + A_{12}K + i\omega I)^*)^{-1}$$

and

$$(-A_{11} + A_{12}K + i\omega I)^{-1}(\hat{A}_{11} - \hat{A}_{12}K)$$

yields

$$\begin{aligned}(\hat{A}_{11} - \hat{A}_{12}K)^T & ((-A_{11} + A_{12}K + i\omega I)^*)^{-1} P(\hat{A}_{11} - \hat{A}_{12}K) \\ & +(\hat{A}_{11} - \hat{A}_{12}K)^T P(-A_{11} + A_{12}K + i\omega I)^{-1}(\hat{A}_{11} - \hat{A}_{12}K) \\ = & \ (\hat{A}_{11} - \hat{A}_{12}K)^T(-A_{11} + A_{12}K + i\omega I)^{-1}Q \\ & (-A_{11} + A_{12}K + i\omega I)^{-1}(\hat{A}_{11} - \hat{A}_{12}K) \qquad (7.22)\end{aligned}$$

So

$$\begin{aligned}& \|(-A_{11} + A_{12}K + i\omega I)^{-1}(\hat{A}_{11} - \hat{A}_{12}K)\|^2\lambda_{\min}(Q) < \\ & \quad 2\|P(\hat{A}_{11} - \hat{A}_{12}K)\|.\|(-A_{11} + A_{12}K + i\omega I)^{-1}(\hat{A}_{11} - \hat{A}_{12}K)\| \\ & \quad < \lambda_{\min}(Q)\ \|(-A_{11} + A_{12}K + i\omega I)^{-1}(\hat{A}_{11} - \hat{A}_{12}K)\|\end{aligned}$$

This complete the proof. □

7.3.2 A Criterion for Sliding System Stability

If the delay-independent criteria fail, then the stability of the sliding system should be tested by delay-dependent criteria. In this case, some information regarding the delay is necessary. However, when there is no information about the delay, for establishing the time-delay system stability, the use of delay-independent criteria is a useful and powerful method. Also, it is straightforward to check the stability condition. Now, the stability of the sliding system (7.11) is studied. A suitable Lyapunov function is

$$V(t, z_1(t), z_1(t-\tau)) = z_1^T(t)Pz_1(t) + \int_{t-\tau}^{t} z_1^T(\theta)Qz_1(\theta)d\theta$$

where P is the p.d.s. solution of the Riccati equation (7.15). Let $K = R^{-1}A_{12}^T P$, then

$$\begin{aligned}\dot{V} = & \ z_1^T(t)\left\{(A_{11} - A_{12}K)^T P + P(A_{11} - A_{12}K)\right\} z_1(t) \\ & + z_1^T(t-\tau)(\hat{A}_{11} - \hat{A}_{12}K)^T P z_1(t) \\ & + z_1^T(t)P(\hat{A}_{11} - \hat{A}_{12}K)z_1(t-\tau) + z_1^T(t)Qz_1(t) - z_2^T(t-\tau)Qz_1(t-\tau)\end{aligned}$$

$$= -z_1^T(t)PA_{11}R^{-1}A_{12}^TPz_1(t) + z_1^T(t-\tau)(\hat{A}_{11} - \hat{A}_{12}K)^TPz_1(t) + z_1^T(t)P(\hat{A}_{11} - \hat{A}_{12}K)z_1(t-\tau) - z_1^T(t-\tau)Qz_1(t-\tau)$$

$$= -\begin{bmatrix} z_1(t) \\ z_1(t-\tau) \end{bmatrix}^T \begin{bmatrix} PA_{12}R^{-1}A_{12}^TP & -P(\hat{A}_{11} - \hat{A}_{12}K) \\ -(\hat{A}_{11} - \hat{A}_{12}K)^TP & Q \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_1(t-\tau) \end{bmatrix} \tag{7.23}$$

If $\|A_{12}\|_{R^{-1}} > \|\hat{A}_{11} - \hat{A}_{12}K\|_{Q^{-1}}$, i.e.

$$A_{12}R^{-1}A_{12}^T - (\hat{A}_{11} - \hat{A}_{12}K)Q^{-1}(\hat{A}_{11} - \hat{A}_{12}K)^T > 0$$

then the matrix

$$\begin{bmatrix} PA_{12}R^{-1}A_{12}^TP & -P(\hat{A}_{11} - \hat{A}_{12}K) \\ -(\hat{A}_{11} - \hat{A}_{12}K)^TP & Q \end{bmatrix}$$

is p.d. and $\dot{V} < 0$; and the system is stable. Note that if

$$\sigma_M(\hat{A}_{11} - \hat{A}_{12}K) < \sqrt{\frac{\lambda_{\min}(Q)}{\lambda_{\max}(R)}}\sigma_m(A_{12})$$

then

$$A_{12}R^{-1}A_{12}^T - (\hat{A}_{11} - \hat{A}_{12}K)Q^{-1}(\hat{A}_{11} - \hat{A}_{12}K) > 0$$

and the system is stable.

7.4 Global Stability of the System

If the ideal sliding mode occurs, the system (7.1) is converted to the reduced order system (7.13). Otherwise, the states lie in a boundary layer of the sliding surface $s = 0$ and the dynamical motion is no longer governed by a reduced order system, i.e. there is no finite time instant such that after this time instant the states lie on the sliding surface. The system is now given by (7.5). The stability of the sliding system with $f = 0$ is studied in this section. Consider system (7.1) with control (7.4). Let $\epsilon > \max \Re(\lambda(A_{eq}))$ where $\Re(\cdot)$ denotes the real part of complex number $(\cdot)$. Since one eigenvalue of A_{eq} is 0, $\epsilon > 0$. Assume that $h(s) = C\left(sI - (A_{eq} - \epsilon I)\right)^{-1}B$ is strictly positive real. Then matrix $A_{eq} - \epsilon I$ is stable and the Lyapunov equation

$$(A_{eq} - \epsilon I)^TP + P(A_{eq} - \epsilon I) = -Q \tag{7.24}$$

where Q is an arbitrary p.d.s. matrix, has a u.p.d. solution P. Consider the Lyapunov function

$$V(t, x(t), x(t-\tau)) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(\theta)Rx(\theta)d\theta$$

where P is the p.d.s. solution of the Lyapunov equation (7.24) and R is an arbitrary p.d.s. matrix. For simplicity, consider $R = rI$ where r is an arbitrary positive real number. Then

$$\begin{aligned}
\dot{V} &= x^T(t)[(A_{eq} - \epsilon I)^T P + P(A_{eq} - \epsilon I) + R]x(t) + \\
&\quad x^T(t-\tau)\hat{A}_{eq}^T Px(t) + x^T(t)P\hat{A}_{eq}x(t-\tau) - \\
&\quad x^T(t-\tau)Rx(t-\tau) + 2\epsilon x^T(t)Px(t) - \\
&\quad (x^T(t)PB + B^T Px(t))\rho(t, x(t), \tau)\text{sgn}(Cx(t)) \\
&= x^T(t)(-Q + 2\epsilon P + rI + r^{-1}P\hat{A}_{eq}\hat{A}_{eq}^T P)x(t) - \\
&\quad r\left\{x(t-\tau) - r^{-1}\hat{A}_{eq}^T Px(t)\right\}^T \left\{x(t-\tau) - r^{-1}\hat{A}_{eq}^T Px(t)\right\} \\
&\quad -(x^T(t)PB + B^T Px(t))\rho(t, x(t), \tau)\text{sgn}(Cx(t))
\end{aligned} \tag{7.25}$$

Since $B^T P = \alpha C$ for some $\alpha > 0$. Then $\dot{V} < 0$ if

$$-Q + 2\epsilon P + rI + r^{-1}P\hat{A}_{eq}\hat{A}_{eq}^T P$$

is a negative definite matrix. So if

$$-\lambda_{\min}(Q) + 2\epsilon\lambda_{\max}(P) + r + r^{-1}||P\hat{A}_{eq}||^2 < 0$$

then $\dot{V} < 0$. Assume that

$$\epsilon < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$$

and

$$r = \frac{\lambda_{\min}(Q) - 2\epsilon\lambda_{\max}(P)}{2}$$

Then $\dot{V} < 0$ if $r \geqslant ||P\hat{A}_{eq}||$. The stability of the system with a strictly positive real condition has been proved. However, the stability of the system without this condition can be proved by choosing $P = C^T C$ and a Q matrix satisfying (7.24).

7.5 Example

Example 7.5.1: Consider the system

$$\dot{x}(t) = \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0.4 \\ 0 & 1 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g(t, x, \tau)$$

where $g(t, x, \tau)$ is an external input signal with $|g(t, x, \tau)| < 1$. Choose $x(t) = \phi(t) = [0, -1]$ for $-\tau \leqslant t \leqslant 0$. The u.p.d. solution of (7.15) is $P = 2.4142$. So $K = -2.4142$ and $C = [-2.4142 \quad 1]$. Choose the control

$$u(t) = \begin{bmatrix} 2.4142 & -0.4142 \end{bmatrix} x(t) - \begin{bmatrix} 2.4142 & 0.0343 \end{bmatrix} x(t-\tau) - \text{sgn } s(t)$$

where $s(t) = Cx(t)$. The sliding surface is $s = -2.4142x_1(t) + x_2(t) = 0$. Since the condition (7.9) is satisfied, the sliding mode occurs. The system in the sliding mode is given by $\dot{x}_1 = x_1 - x_2 - x_1(t-\tau) + 0.4x_2(t-\tau)$. Let $R = Q = 1$. The system in the sliding mode is given by $\dot{x}_1 = -1.4142x_1 - 0.0343x_1(t-\tau)$. Since

$$\left\| \left(\hat{A}_{11} - \hat{A}_{12}K \right) P \right\| = 0.1268 < \lambda_{\min}(Q)/2 = 0.5$$

the system is stable independent of delay because Theorem 7.3.2 is satisfied. Simulation results for $\tau = 0.4$ are shown in Fig. 7.1.

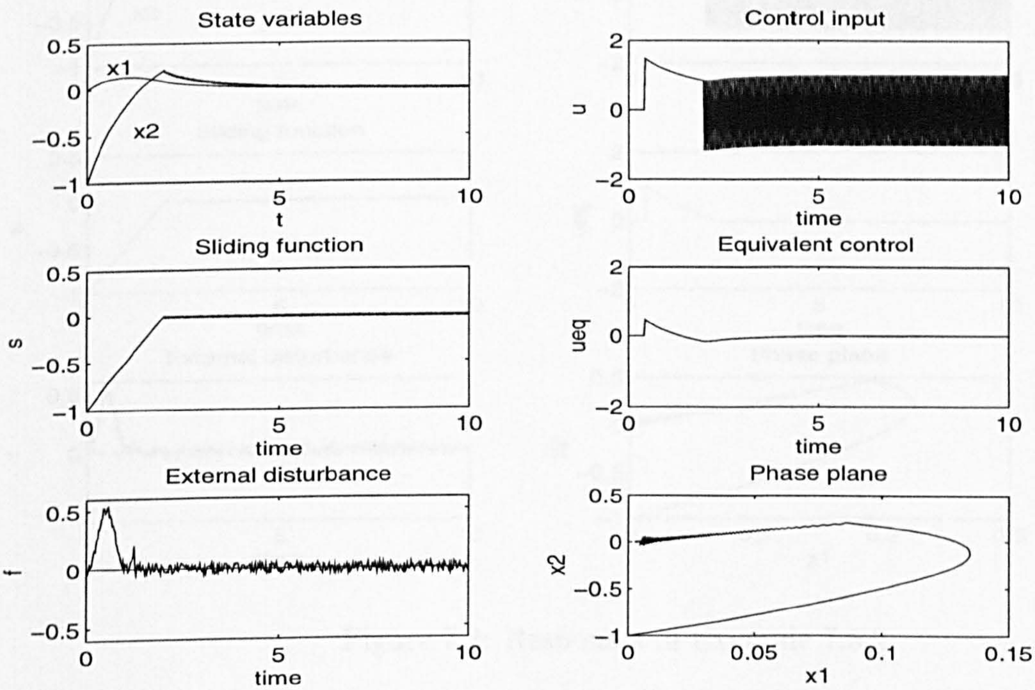

Figure 7.1: Responses of Example 7.5.1.

Example 7.5.2: Consider the system

$$\dot{x}(t) = \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0.2 \\ 0 & 1 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g(t, x, \tau)$$

where $g(t, x, \tau)$ is an external input signal with $|g(t, x, \tau)| < 1$. So the u.p.d. solution of (7.15) is $P = 2.4142$. Consider the sliding surface and ϕ as in Example 7.5.1. Select the discontinuous control

$$u(t) \;=\; \begin{bmatrix} 2.4142 & -0.4142 \end{bmatrix} x(t) + \begin{bmatrix} 0 & -0.5172 \end{bmatrix} x(t-\tau) - \text{sgn } s(t)$$

where $s(t) = Cx(t)$. The condition (7.9) is satisfied, so the sliding mode occurs. The system in the sliding mode is given by $\dot{x}_1 = x_1 - x_2 + 0.2x_2(t-\tau)$. Let $R = Q = 1$. The system in the sliding mode is given by $\dot{x}_1 = -1.4142x_1 + 0.4828x_1(t-\tau)$. On the other hand $\left\| \left(\hat{A}_{11} - \hat{A}_{12}K \right) P \right\| = 0.7464 > \lambda_{\min}(Q)/2 = 0.5$. So the condition (7.20) is not satisfied. But the system is stable independent of delay. Therefore the condition (7.20) is only a sufficient condition. Simulation results for $\tau = 0.4$ are shown in Fig. 7.2.

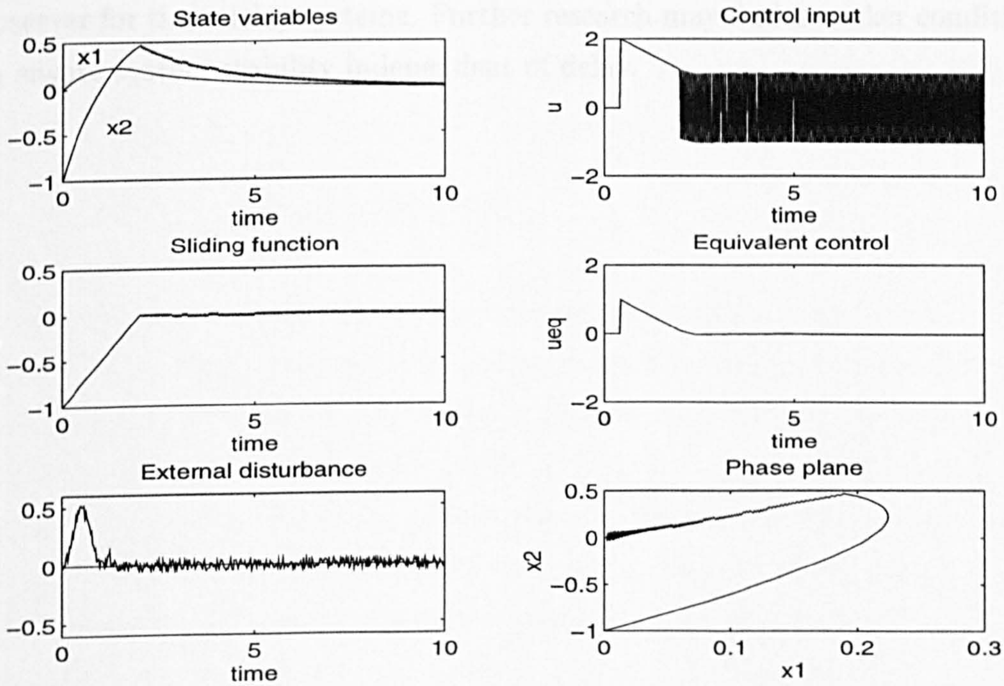

Figure 7.2: Responses of Example 7.5.2

7.6 Summary and Discussion

Time-delay systems appear in many practical problems. The sliding mode on a specified surface is achieved if the state converges to the surface. Two kinds of sliding surface can be designed: (*i*) when the sliding surface is independent of the delays; or (*ii*) the sliding surface depends on the delays. In the second case the delays should be constant, otherwise the sliding surface is not a simple hyperplane. In this chapter the stability of the sliding mode control of a system with a delay on the state has been considered.

The delay is assumed known. If the delay is unknown, the problem then is how to define the sliding control such that the state lies on a certain sliding surface. The extension of the results of this chapter for systems with finite and varying delays is straightforward. The results can also be extended to systems with delay in control and to the sliding mode observer for time-delay systems. Further research may find a weaker condition than (7.20) to ensure system stability independent of delay.

Chapter 8

Pole Assignment of Linear Systems and Sliding Hyperplanes

8.1 Pole Assignment Techniques for Control and Sliding Hyperplane Design

Pole assignment is applied not only to ensure system stability but is carried out also to achieve other aims. The choice of the eigenvalues influences the stability and response of the system. Therefore to get desired responses, it is required to locate poles in desired positions. A popular method for designing the linear feedback gain matrix or the sliding hyperplane requires the exact specification of the desired closed-loop eigenvalues. This method is often too rigid a design requirement, since in many practical problems exact eigenvalue specification may not be required. In general, a set of exact eigenvalues for the closed-loop system may not be known, so it is useful to be able to specify a region of the left-hand half-plane within which the eigenvalues should lie. The fundamental basis of the design method is based on linear quadratic optimal control (LQ) methods, as well as pole assignment methods.

A method has been studied for yielding closed-loop poles at desired locations which is capable of shifting both real and imaginary parts to any location (Rousan and Sawan [91]). Some work has been done on the placing of closed-loop eigenvalues within particular regions by linear state feedback methods. Another technique is the placing of the eigenvalues within a hyperbola with major and minor axes at 45° to the x and y axes by employing two connected Riccati equations (Kawasaki and Shimemura [64]). In this method the controller design is achieved by utilizing an iterative algorithm for finding the solutions of the appropriate Riccati equations and then the feedback gain matrix. Some work has been done on necessary and sufficient conditions for the eigenvalues of both real and complex matrices to lie in a specified region (Gutman [49], Gutman and Vaisberg [51]). Root clustering using Lyapunov type approaches is another way of designing a controller (Gutman and Jury [50], Abdul Wahab [1], [2], Woodham [129], Woodham and Zinober [130], [131]). More recently work has been published on the pole placement problem using H_∞ methods (Saeki [96]). The problem of pole placement in a vertical strip has been studied by Shieh at el [101] using LQ methods. Their method has some inaccuracies and errors which will be corrected in Section 8.6. A recursive method using the LQ approach in order to shift the open-loop poles inside a vertical strip has been considered by Arar et al [7]. The most interesting regions are: a disc, a hyperbola, an infinite vertical strip and a sector in the left-hand half-plane.

Most research work based on the Lyapunov approach has proposed sufficient conditions to guarantee that all the eigenvalues of the system, with parameter uncertainty in the state matrix of the system, lie inside the specified region (Juang [60], Jaung et al [61], Abdul Wahab [1], [2] and Horng et al [58]. In their studies all the eigenvalues of the state matrix A are assumed to be in the specified region. Then some conditions on a matrix E are found to ensure that all the eigenvalues of $A + E$ lie in the same region. These methods differ from the methods for the case where there is no such information about matrix A, or all the eigenvalues of A are not in the left-hand half plane. Some of these approaches to obtain robust stability benefit from the matrix measure properties and an appropriate Lyapunov equation (Juang [60]). Woodham [129] and Woodham and Zinober [130], [131] have proposed a method for designing a feedback gain matrix and sliding hyperplane. They used a complex Riccati equation and found a feedback gain matrix such that all the eigenvalues lie in the specified sector S bounded by a line at an angle θ to the imaginary axis and the reflection of this line in the real axis. Their method has some limitations, i.e. it cannot be applied to all sectors. Their method fails even for $\theta = 0$, but may work for a range of θ for some systems. However, it is very difficult to specify precisely such a range or type of system. The technique usually holds for sectors with an angle to the imaginary axis $\theta \in (0, 40°)$. This estimate of the range of θ was obtained by testing about 100000 random controllable canonical form linear systems. There is no exact mathematical method available to predict this range. These inaccuracies arise from (i): using the absolute value of elements of a conventional p.d. Hermitian matrix solution of appropriate Riccati equation; (ii) applying some properties of real matrices which may not be valid for complex matrices. In Section 8.4 some examples illustrate these results.

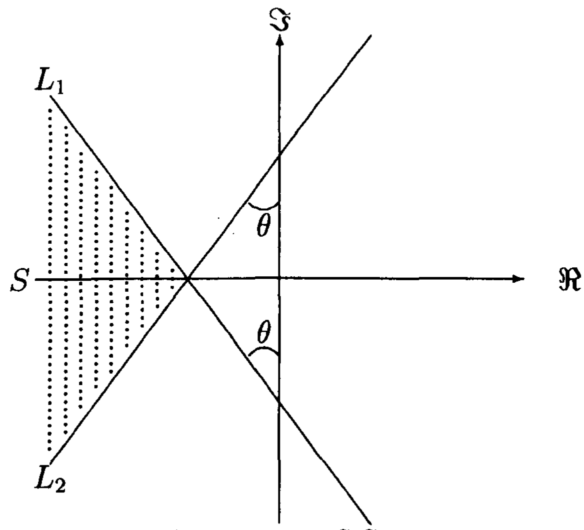

Figure 8.1: S Sector

In this chapter the pole placement problem within a sector by using suitable Lyapunov and Riccati equations, is first considered. Then errors in the statement and proof of Theorem 1 in [101] are presented for an infinite vertical strip in the left-hand complex plane. These inaccuracies will be corrected by proposing a modified theorem with new conditions and proof.

In Section 8.2 the complex Lyapunov equation and the problem of system stability are discussed; and in Section 8.2 and 8.3 the conditions for all the eigenvalues of $A+BF$ to lie in a specified sector are presented. Some of these conditions are necessary and sufficient, others only sufficient. However, these methods are new and work for all sectors. These results are based on the solutions of appropriate Riccati equations.

The design of the feedback gain matrix or the sliding hyperplane by using the complex Riccati equation is complicated as shown in Section 8.2. Straightforward methods are proposed in Sections 8.5 and 8.6 and Tables 8.1-8.3 summarise the results.

In this chapter the control law is $u = Fx$, S is a sector with the vertex α and its edges form angles of θ, $0 \leqslant \theta < \pi/2$, with the coordinate axes as in Fig. 8.1. When $\theta = \pi/2$, the region S is the null set. The matrix F refers to the sliding mode gain matrix as stated in Chapter 2.

New methods of pole placement in the sector S			
Tools	Validity of θ	Section	Theorem
CARE	$0 \leqslant \theta < 90°$	8.2	Theorems 8.2.2 and 8.2.3
ARE, Shifted and Rotation	$0 \leqslant \theta \leqslant 45°$	8.3	
ARE with zero right-hand side	$0 \leqslant \theta \leqslant 90°$	8.5	Proposition 8.5.1

Table 8.1: Pole placement methods for the eigenvalues to lie in the sector S

New methods of pole placement in a specified region		
Region	Restriction	Section
Hyperbola	See page 172	8.5.2
Between two sectors	See page 171	8.5.1

Table 8.2: Pole placement methods for the eigenvalues to lie in the specified region symmetric with respect to x-axis

Errors in the previous work				
Method	Pole placement	Problem description	Page	Section
Shieh et al [101]	in a vertical strip	Errors and inaccuracies	174	8.6
		Restrictions	175	8.6
Woodham [129]	in a vertical strip and a sector	Solution of the ARE with zero right-hand side	187, 188 191	8.7.2-8.7.3
Woodham and Zinober [131]	Pole placement in a sector	Error in using CARE	165	8.4
		Weighting matrices	163, 197	8.4
		Their suggested feedback	163-166	8.4

Table 8.3: Pole placement methods for the eigenvalues to lie in the specified region

8.2 Pole Placement in a Sector using Complex Riccati Equation

The Lyapunov equation is important for the proof of the stability of systems. A matrix is stable if and only if the appropriate Lyapunov equation has a p.d.s. solution matrix. The stability approach can be studied in a number of ways; for instance, (i) all the closed-loop eigenvalues lying in the specified regions in the left-hand half-plane, (ii) h-stability with $h > 0$ means all the eigenvalues lie to the left of the line $x = -h$, (iii) $\mathscr{S}_{(\alpha,\theta)}$-stability ($\alpha < 0$ and $0 \leqslant \theta < \pi/2$) denotes all the eigenvalues are in a symmetric sector with respect to the real axis, with angle θ with respect to the imaginary axis and with the vertex α in the left-hand half-plane. One way to prove the $\mathscr{S}_{(\alpha,\theta)}$-stability of a system is that to show an appropriate generalized complex algebraic Riccati equation (CARE) has a u.p.d. Hermitian solution. Another is to use a real algebraic Riccati equation (ARE) with zero right-hand side. This problem is considered in Theorems 8.2.2 and 8.2.3 for the complex case, and Proposition 8.5.1 for the real case. In this section the complex Lyapunov equation and the system stability are studied. A method for obtaining a solution of the CARE and further discussion are presented in Chapter 9.

Lemma 8.2.1: *If $A \in \mathbb{R}^{n\times n}$ is a constant matrix then all the eigenvalues of A lie in the region S (see Fig. 8.1) if and only if the matrix $e^{-i\theta}(A - \alpha I)$ is stable.*

Proof: If all the eigenvalues of A lie in the region S, then all the eigenvalues of $e^{-i\theta}(A-\alpha I)$ lie inside the left-hand half-plane. Conversely, if all the eigenvalues of $e^{-i\theta}(A - \alpha I)$ lie inside the left complex plane, then all the eigenvalues of $e^{i\theta}(e^{-i\theta}(A - \alpha I)) = A - \alpha I$ lie within a sector with the origin as vertex and the edges parallel to the sector S. So all the eigenvalues of A are located in the sector S. □

Remark 8.2.1: If $A \in \mathbb{C}^{n\times n}$ is a constant complex matrix and all the eigenvalues of A lie in the region S then $e^{i\theta}(A - \alpha I)$ is a stable matrix but the converse is not always true as in Example 8.2.1.

Example 8.2.1 (Counterexample): Let

$$A = \begin{bmatrix} -0.5412 - 1.3066i & 0 & 0 \\ 0 & -1.1945 - 1.0360i & 0 \\ 0 & 0 & 0.1121 - 1.5772i \end{bmatrix}$$

and $\alpha = 0$ and $\theta = \pi/12$. The eigenvalues of A are $-0.5412 - 1.3066i$, $-1.1945 - 1.0360i$, $0.1121 - 1.5772i$ and the argument of the eigenvalues of A are

$$-112.5000°, -139.0651°, -85.9349°$$

Assume $A_1 = Ae^{-i\pi/12}$, i.e.

$$A_1 = \begin{bmatrix} -0.8609 - 1.1220i & 0 & 0 \\ 0 & -1.4219 - 0.6915i & 0 \\ 0 & 0 & -0.2999 - 1.5524i \end{bmatrix}$$

The eigenvalues of A_1 are $-0.8609 - 1.1220i, -1.4219 - 0.6915i, -0.2999 - 1.5524i$ and the argument of eigenvalues A_1 are

$$-127.5000°, -154.0651°, -100.9349°.$$

Therefore the eigenvalues of A lie in the right-hand half-plane but the eigenvalues of A_1 are not in the sector. So Lemma 8.2.1 is not satisfied if A is a complex matrix.

To prove all the eigenvalues of a real matrix A lie in the sector S, one can study a $2n \times 2n$ conventional real matrix. The general case of this corresponding matrix will be discussed in detail in Chapter 9. The following Lemma is equivalent to Lemma 8.2.1.

Lemma 8.2.2: *If $A \in \mathbb{R}^{n\times n}$ is a constant matrix, then all the eigenvalues of A lie in the region S if and only if*

$$A_s = (A - \alpha I) \otimes \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} (A-\alpha I)\cos\theta & -(A-\alpha I)\sin\theta \\ (A-\alpha I)\sin\theta & (A-\alpha I)\cos\theta \end{bmatrix}$$

is a stable matrix.

Proof: The eigenvalues of the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

are $e^{\pm i\theta}$. For any eigenvalue λ of A, $\lambda_s = e^{\pm i\theta}\lambda$ are the eigenvalues A_s (see Appendix B). So λ is in sector S if and only if λ_s is in the left-hand half-plane. □

Lemma 8.2.3: *If $A \in \mathbb{C}^{n\times n}$ is a constant matrix, then all the eigenvalues of A lie in the region S if and only if both the matrices $e^{-i\theta}(A - \alpha I)$ and $e^{i\theta}(A - \alpha I)$ are stable.*

Proof: If $e^{-i\theta}(A-\alpha I)$ and $e^{i\theta}(A-\alpha I)$ are stable then all the eigenvalues of A are located under the line L_1 and above of the line L_2, respectively (See Fig. 8.1). □

Theorem 8.2.1 (Lyapunov) [11]: *Assume $A \in \mathbb{C}^{n\times n}$ is a matrix. The matrix A is stable if and only if the Lyapunov equation*

$$A^*P + PA = -Q \tag{8.1}$$

where Q is an arbitrary p.d. Hermitian matrix, has a u.p.d. Hermitian matrix solution P.

Corollary 8.2.1: *If $A \in \mathbb{R}^{n\times n}$ is a constant matrix, then all the eigenvalues of A lie in the region S if and only if the Lyapunov equation*

$$e^{i\theta}(A-\alpha I)^T P + e^{-i\theta}P(A-\alpha I) = -Q \tag{8.2}$$

where Q is an arbitrary p.d.s. matrix, has a unique p.d.s. Hermitian matrix solution P.

Proof: The proof follows from Theorem 8.2.1 and Lemma 8.2.1. □

Remark 8.2.2: Note that if $A \in \mathbb{C}^{n\times n}$, according to Remark 8.2.1, when the Lyapunov equation (8.2) has a u.p.d. Hermitian matrix solution, then $e^{-i\theta}(A-\alpha I)$ is a stable matrix but all the eigenvalues of A may not be located in the region S. However, if $e^{-i\theta}(A-\alpha I)$ is a stable matrix, the arguments of all the eigenvalues of $e^{-i\theta}(A-\alpha I)$ are less than $3\pi/2 - 2\theta$ and the real parts of all the eigenvalues of A are less than α. Then all the eigenvalues of $e^{-i\theta}(A-\alpha I)$ are located in the region S. In fact the following corollary is satisfied.

Corollary 8.2.2: *If $A \in \mathbb{C}^{n\times n}$ is a constant matrix, all the eigenvalues of A lie in the region S if and only if both Lyapunov equations*

$$e^{i\theta}(A-\alpha I)^* P_+ + P_+ e^{-i\theta}(A-\alpha I) = -Q \tag{8.3}$$

$$e^{-i\theta}(A-\alpha I)^* P_- + P_- e^{i\theta}(A-\alpha I) = -Q \tag{8.4}$$

where Q is an arbitrary p.d. Hermitian matrix, have u.p.d. Hermitian matrix solutions.

Proof: By using Lemma 8.2.3 similarly as in Corollary 8.2.1 the proof is obtained. □

Note that when $A \in \mathbb{R}^{n\times n}$ Corollaries 8.2.1 and 8.2.2 are the same, because in this case, when P_+ is the solution of equation (8.3) then $\bar{P_+}$ is the solution of (8.4). So $P_- = \bar{P_+}$.

Theorem 8.2.2: *Assume F is a real feedback matrix such that $u = Fx$, then all the eigenvalues of $A + BF$ lie in the region S if and only if the generalized Riccati equation*

$$e^{i\theta}(A-\alpha I)^T P + Pe^{-i\theta}(A-\alpha I) + e^{i\theta}(BF)^T P + e^{-i\theta}P(BF) = -Q \tag{8.5}$$

where Q is an arbitrary p.d.s. matrix, has a u.p.d. Hermitian matrix solution.

Proof: According to Corollary 8.2.1 all the eigenvalues of $A + BF$ lie in the region S if and only if

$$e^{i\theta}(A + BF - \alpha I)^T P + Pe^{-i\theta}(A + BF - \alpha I) = -Q \tag{8.6}$$

has a u.p.d. Hermitian matrix solution P. Then (8.6) is readily obtained. □

Note that for $\theta = \pi/2$, the Hermitian solution (8.5) is a skew-symmetric matrix which is obviously not a p.d. matrix. Therefore, it is impossible to find a p.d. Hermitian solution for (8.5). In fact, in this case S is the null set. When θ tends to $\pi/2$, the eigenvalues of $A+BF$ are approximately real. If the matrix P is an Hermitian matrix, then $P = P_1 + iP_2$ where P_1 is p.d.s. and P_2 is a skew-symmetric matrix. Then (8.5) implies

$$\begin{aligned}(\cos\theta + i\sin\theta)(A-\alpha I)^T(P_1 + iP_2) + (\cos\theta - i\sin\theta)(P_1 + iP_2)(A-\alpha I) +\\ (\cos\theta + i\sin\theta)(BF)^T(P_1 + iP_2) + (\cos\theta - i\sin\theta)(P_1 + iP_2)(BF) = -Q\end{aligned} \tag{8.7}$$

Then

$$\begin{aligned}(A-\alpha I)^T(\cos\theta P_1 - \sin\theta P_2) + (\cos\theta P_1 + \sin\theta P_2)(A-\alpha I) + (BF)^T\\ (\cos\theta P_1 - \sin\theta P_2) + (\cos\theta P_1 + \sin\theta P_2)(BF) = -Q\end{aligned} \tag{8.8}$$

and

$$\begin{aligned}(A-\alpha I)^T(\sin\theta P_1 + \cos\theta P_2) + (\cos\theta P_2 - \sin\theta P_1)(A-\alpha I) +\\ (BF)^T(\cos\theta P_2 + \sin\theta P_1) + (\cos\theta P_2 - \sin\theta P_1)(BF) = 0\end{aligned} \tag{8.9}$$

The gain matrix F can be found by using optimization methods similar to the method in [8] such that both generalized real Riccati equations (8.8) and (8.9) are satisfied. Similarly to the real case, i.e. as when $\theta = 0$, let $F = -R^{-1}B^T\tilde{P}$ where R is an arbitrary p.d.s. matrix and $\tilde{P}$ is a p.d.s. matrix to be determined. Then

$$\begin{aligned}(A-\alpha I)^T(\cos\theta P_1 - \sin\theta P_2) + (\cos\theta P_1 + \sin\theta P_2)(A-\alpha I) - \tilde{P}BR^{-1}B^T\\ \times(\cos\theta P_1 - \sin\theta P_2) - (\cos\theta P_1 + \sin\theta P_2)BR^{-1}B^T\tilde{P} = -Q\end{aligned} \tag{8.10}$$

and

$$(A-\alpha I)^T(\sin\theta P_1+\cos\theta P_2)+(\cos\theta P_2-\sin\theta P_1)(A-\alpha I)-\tilde{P}BR^{-1}B^T \times(\cos\theta P_2+\sin\theta P_1)-(\sin\theta P_1-\cos\theta P_2)BR^{-1}B^T\tilde{P}=0 \qquad (8.11)$$

This analysis leads to the following theorem.

Theorem 8.2.3: *All the eigenvalues of $A+BF$ with $F=-R^{-1}B^T\tilde{P}$ lie in the sector S if and only if the dual Riccati equations (8.10) and (8.11) have the p.d.s., skew symmetric and p.d.s. solutions P_1, P_2 and $\tilde{P}$, respectively.* □

As already proved, equations (8.8) and (8.9) are immediately results of (8.5). So, to test for a given real F, whether all the eigenvalues of the closed-loop $A+BF$ lie in the sector S, it is sufficient that (8.5) has a p.d. Hermitian solution P. Conversely, if all the eigenvalues of $A+BF$ lie in the sector S, F and P must satisfy (8.6). On the other hand, the p.d.s. matrix P_1 and the skew symmetric matrix P_2 are the solutions of the dual ARE (8.8) and (8.9) if and only if $P=P_1+iP_2$ is a solution of (8.5). So, all the eigenvalues of $A+BF$ lie in the sector S if and only if the dual ARE (8.8) and (8.9) have p.d.s. and skew symmetric matrices P_1 and P_2. However, if the feedback gain matrix F is in the form of the optimal gain, there is a p.d.s. matrix $\tilde{P}$ such that $F=-R^{-1}B^T\tilde{P}$. Theorem 8.2.3 implies that for a given p.d.s. matrix $\tilde{P}$, all the eigenvalues of $A+BF$ with $F=-R^{-1}B^T\tilde{P}$ lie in the sector S if and only if (8.10) and (8.11) are satisfied by p.d.s. and skew symmetric matrices P_1 and P_2. Therefore, Theorems 8.2.2 and 8.2.3 do not yield a direct way to find the solutions of these equations. These theorems give only criteria for establishing whether for given real F and p.d.s. matrices $\tilde{P}$, the eigenvalues of $A+BF$ lie in the sector S.

To obtain such a real F, a straightforward method is given in Proposition 8.5.1. Further research is needed to find a method for obtaining the solutions of the dual ARE (8.10) and (8.11). The following example illustrates the above results.

Example 8.2.2 [129]: Consider the system (2.4) with

$$A=\begin{bmatrix}-1 & 1 & 0 & 0 & 0\\ 0 & -2 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}, \quad B=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 1\\ 0 & 1\\ 1 & 0\end{bmatrix}$$

The orthogonal transformation matrix T (2.13) is

$$T = \begin{bmatrix} -1.0000 & 0 & 0 & 0 & 0 \\ 0 & -0.7071 & -0.5000 & 0.5000 & 0 \\ 0 & -0.7071 & 0.5000 & -0.5000 & 0 \\ 0 & 0 & 0 & 0 & -1.0000 \\ 0 & 0 & -0.7071 & -0.7071 & 0 \end{bmatrix}$$

Then the matrices A_{11} and A_{12} (2.16) are

$$A_{11} = \begin{bmatrix} -1.0000 & 0.7070 & 0.7070 \\ 0 & -0.8965 & -1.1035 \\ 0 & -0.3965 & -1.6035 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 \\ -0.5000 & 0.8535 \\ 0.5000 & 0.1465 \end{bmatrix}$$

Let $Q = I_3$, $\alpha = -2$ and $\theta = 60°$. Consider

$$F = \begin{bmatrix} 0.7390 & -0.3290 & 3.8290 \\ 3.4859 & 3.2673 & 3.8277 \end{bmatrix}$$

The solution of the dual equations (8.10) and (8.11) are

$$P_1 = 10^3 \times \begin{bmatrix} 5.7785 & 3.0104 & -1.6647 \\ 3.0104 & 1.5843 & -0.8902 \\ -1.6647 & -0.8902 & 0.5135 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0 & -260.4701 & 377.9352 \\ 260.4701 & 0 & 122.2723 \\ -377.9352 & -122.2723 & 0 \end{bmatrix}$$

In fact

$$P = 10^3 \times \begin{bmatrix} 5.7785 - 0.0000i & 3.0104 - 0.2605i & -1.6647 + 0.3779i \\ 3.0104 + 0.2605i & 1.5843 - 0.0000i & -0.8902 + 0.1223i \\ -1.6647 - 0.3779i & -0.8902 - 0.1223i & 0.5135 - 0.0000i \end{bmatrix}$$

is the u.p.d. Hermitian solution of (8.5). Therefore all the eigenvalues of $A_{11} - A_{12}F$ are in the sector S. The eigenvalues of $A_{11} - A_{12}F$ are $-2.1148 \pm 0.0542i$ and -4.6988. Note that in this case, $F = R^{-1}A_{12}^T\tilde{P}$ where

$$\tilde{P} = \begin{bmatrix} 9.6268 & 3.2694 & 4.7475 \\ 3.2694 & 3.3637 & 2.7058 \\ 4.7475 & 2.7058 & 10.3639 \end{bmatrix}$$

$\tilde{P}$ is p.d.s. matrix which satisfies the dual equations (8.10) and (8.11).

Remark 8.2.3: Note that when F is not a real matrix, Remark 8.2.2 implies that, if all the eigenvalues of $A + BF$ are in region S, the generalized Riccati equation (8.5) has a u.p.d. Hermitian matrix solution but the converse is not true unless the argument of all the eigenvalues of $A + BF - \alpha I$ are less than $3\pi/2 - 2\theta$ and the real parts of all the eigenvalues A+BF are less than α.

Remark 8.2.4: Suppose $F = -\tilde{R}^{-1}B^TP$ where $\tilde{R}$ is an arbitrary p.d. Hermitian matrix and P is the u.p.d. matrix solution of the generalized Riccati equation (8.5). Then (8.5) is converted to

$$e^{i\theta}(A - \alpha I)^T P + e^{-i\theta}P(A - \alpha I) - 2\cos\theta PB\tilde{R}^{-1}B^TP = -Q \tag{8.12}$$

and if consider $R = \tilde{R}/2\cos\theta$, then (8.12) is given by

$$e^{i\theta}(A - \alpha I)^T P + e^{-i\theta}P(A - \alpha I) - PBR^{-1}B^TP = -Q \tag{8.13}$$

When $\theta = 0$, the left-hand of (8.11) is zero, because in this case, $P_2 = 0$. So (8.11) is converted to $0 = 0$. Therefore, the equations (8.10) and (8.13) are the same and $\tilde{P} = P$ is a p.d.s. real matrix. □

When F is complex, the converse of Theorem 8.2.2 is not true. In this case for all the eigenvalues of $A + BF$ to lie in the sector S, *two* generalized Riccati equations

$$e^{i\theta}(A - \alpha I)^* P_+ + P_+ e^{-i\theta}(A - \alpha I) + e^{i\theta}(BF)^* P_+ + e^{-i\theta}P_+(BF) = -Q \tag{8.14}$$
$$e^{-i\theta}(A - \alpha I)^* P_- + P_- e^{i\theta}(A - \alpha I) + e^{-i\theta}(BF)^* P_- + e^{i\theta}P_-(BF) = -Q \tag{8.15}$$

should have p.d. Hermitian solutions. □

It is clear that, when $A+BF$ is a real matrix, the equations (8.14) and (8.15) are equivalent because if P_+ is the u.p.d. Hermitian matrix solution of (8.14), $\bar{P}_+$ the complex conjugate of P_+, is the u.p.d. matrix solution of (8.15). Assume $F = -\hat{R}^{-1}B^TP_+$ where $\hat{R}$ is a p.d.s. matrix and P_+ is the u.p.d. matrix solution of (8.14). Then

$$e^{i\theta}(A - \alpha I)^* P + e^{-i\theta}P(A - \alpha I) - 2\cos\theta PB\hat{R}^{-1}B^TP = -Q. \tag{8.16}$$

Assume $R = \hat{R}/2\cos\theta$. Then the equation (8.16) will be

$$e^{i\theta}(A - \alpha I)^* P + e^{-i\theta}P(A - \alpha I) - PBR^{-1}B^TP = -Q \tag{8.17}$$

If P is the u.p.d. Hermitian matrix solution of the Riccati equation (8.17), then all the eigenvalues of $A - BR^{-1}B^TP$ may not be in the region S unless $\theta = 0$. This shows that for $\theta \neq 0$ the feedback gain matrix is not in the form of $-R^{-1}B^TP$ or $-R^{-1}B^T\bar{P}$ where $\bar{P}$, the complex conjugate of P, is the solution of the equation

$$e^{-i\theta}(A-\alpha I)^*\bar{P} + e^{-i\theta}\bar{P}(A-\alpha I) - \bar{P}BR^{-1}B^T\bar{P} = -Q \tag{8.18}$$

Remark 8.2.5: The Riccati equation (8.17) is equivalent to

$$\tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} - \tilde{P}\tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{P} = -\tilde{Q} \tag{8.19}$$

where

$$\tilde{A} = \begin{bmatrix} (A-\alpha I)\cos\theta & -(A-\alpha I)\sin\theta \\ (A-\alpha I)\sin\theta & (A-\alpha I)\cos\theta \end{bmatrix}, \quad \tilde{P} = \begin{bmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{bmatrix}$$

$$\tilde{Q} = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \quad \tilde{R} = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$$

The Riccati equation (8.19) has a u.p.d.s. real matrix solution $\tilde{P}$. The feedback gain matrix is given by

$$\tilde{F} = -\tilde{R}^{-1}\tilde{B}^T\tilde{P} = -\begin{bmatrix} R^{-1}B^TP_1 & -R^{-1}B^TP_2 \\ R^{-1}B^TP_2 & R^{-1}B^TP_1 \end{bmatrix} = -R^{-1}B^T \otimes \tilde{P} \tag{8.20}$$

The feedback gain matrix (8.20) relates to the $2n \times 2n$ system

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}\tilde{u} \\ \tilde{w} &= \tilde{C}\tilde{x} \end{aligned} \tag{8.21}$$

But the original system is an $n \times n$ system. The problem is how can a feedback gain matrix be obtained from the $2n \times 2n$ feedback gain matrix for the $n \times n$ system. This problem is discussed in Section 9.4.

8.3 Shifted and Rotation Method

For $\theta \leqslant 45°$, another method exists for designing a feedback gain matrix such that all the eigenvalues lie in the sector S. Let P be the u.p.d. solution of the Riccati equation

$$A^TP + PA - PBR^{-1}B^TP = -Q \tag{8.22}$$

All the eigenvalues of $A - BR^{-1}B^TP$ lie in the left-hand complex half-plane. Assume the eigenvalues of $A - BR^{-1}B^TP$ are $\lambda_1, \lambda_2, \ldots, \lambda_m, \bar{\lambda}_1, \bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_m$ and the remaining

$\lambda_{2m+1}, \lambda_{2m+2}, \ldots, \lambda_n$ are real, where $\bar{\lambda}_k$ stands for the complex conjugate of λ_k, $k = 1, \ldots, m$. Without loss of generality assume the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ have positive imaginary parts, then $e^{i\theta}(\lambda_1) + \alpha$, $e^{i\theta}(\lambda_2) + \alpha$, $\ldots, e^{i\theta}(\lambda_m) + \alpha$, $e^{-i\theta}(\bar{\lambda}_1) + \alpha$, $e^{-i\theta}(\bar{\lambda}_2) + \alpha, \ldots, e^{-i\theta}(\bar{\lambda}_m) + \alpha$, $\lambda_{2m+1} + \alpha, \lambda_{2m+2} + \alpha, \ldots, \lambda_n + \alpha$ lie in the region S if $\theta \leqslant 45°$. It is desired to find the real feedback gain matrix such that the eigenvalues η_i of the closed-loop system are $\eta_1 = e^{i\theta}(\lambda_1)+\alpha$, $\eta_2 = e^{i\theta}(\lambda_2) + \alpha, \ldots,$ $\eta_m = e^{i\theta}(\lambda_m)+\alpha$, $\eta_{m+1} = e^{-i\theta}(\bar{\lambda}_1)+\alpha$, $\eta_{m+2} = e^{-i\theta}(\bar{\lambda}_2) + \alpha), \ldots, \eta_{2m} = e^{-i\theta}(\bar{\lambda}_m) + \alpha), \eta_{2m+1} = \lambda_{2m+1} + \alpha, \eta_{2m+2} = \lambda_{2m+2} + \alpha$, $\ldots, \eta_n = \lambda_n + \alpha$. To determine the feedback gain matrix K pole assignment methods can now be applied. For instance, using the method in [79, page 43], [11, page 268] gives the appropriate feedback gain. The controller matrix K is decomposed as $K = EM^T$ where E and M are m- and n-vectors. The vector E is chosen such that the auxiliary single input plant

$$\dot{x} = Ax + BE\tilde{u} \tag{8.23}$$

is completely controllable via $\tilde{u}$. This is possible whenever the system (2.4) is completely controllable and the matrix A is cyclic, i.e. its characteristic polynomial is a minimal polynomial. Assume the eigenvalues of A^T are $\xi_1, \xi_2, \ldots, \xi_n$ with $w_1, w_2, \ldots, w_n$ the corresponding eigenvectors. Let

$$M = \sum_{k=1}^{k=n} \ell_k w_k \tag{8.24}$$

where

$$\ell_k = \frac{\prod_{j=1}^{j=n} (\xi_k - \eta_j)}{w_k^T BE \prod_{\substack{j=1 \\ j \neq k}}^{j=n} (\xi_k - \xi_j)}$$

So all the eigenvalues of $A + BK$ are $\eta_1, \eta_2, \ldots, \eta_n$ and lie in the specified sector. The following example illustrates the results of the Shifted and Rotation method 8.3 .

Example 8.3.1: Consider the system (2.4) with $\alpha = -2$, $\theta = 45°$, $Q = I_3$ and $R = I_2$. The u.p.d.s. solution of the ARE (8.22) with A_{11} replacing A, and A_{12} replacing B is

$$P = \begin{bmatrix} 0.4919 & 0.1316 & 0.0853 \\ 0.1316 & 0.5564 & -0.1901 \\ 0.0853 & -0.1901 & 0.4458 \end{bmatrix}$$

The eigenvalues of $A_{11} - A_{12}R^{-1}A_{12}^T P$ are $-1.0771 \pm 0.1721i$ and -2.0589. So

$$\eta_1 = -2.8833 + 0.6399i, \; \eta_2 = -2.8833 - 0.6399i, \; \eta_3 = -4.0589$$

Using the pole assignment method the feedback gain is obtained as

$$F = \begin{bmatrix} 4.3768 & -1.0225 & 5.0192 \\ 5.5974 & 3.5255 & 2.0172 \end{bmatrix}$$

The eigenvalues of $A_{11} - A_{12}F$ are η_1, η_2, η_3 which obviously lie in the sector.

8.4 Examples

Now the work of Woodham [129] and Woodham et al [131] is discussed and some errors and inaccuracies in their method are shown. Their method is known to fail for some cases, i.e. for θ less than 60° and also $\theta = 0$. Also it has been shown by Cao and Sun in [20] that the Woodham et al [131] method does not work for all sectors. A counterexample for the case $\theta = 40°$ is presented [20] and it is shown that the weighting matrices in [131] are not Hermitian matrices.

The following example shows that even if $\theta = 0$ and P is the u.p.d. Hermitian solution of (8.13), all the eigenvalues of $A_{11} - A_{12}R^{-1}A_{12}^T\hat{P}$ may not lie in the sector S where $\hat{P}$ is defined such that its elements are the absolute value of the elements of P. If p_{ij} is the (i,j)-th element of P, $\hat{p}_{ij} = \sqrt{p_{ij}\bar{p}_{ij}}$ is the (i,j)-th element of $\hat{P}$.

Example 8.4.1: Consider the following example of a remotely piloted vehicle (RPV) (Safonov et al [94]; Safonov and Chiang [95]).

$$A = \begin{bmatrix} -0.0257 & -36.6170 & -18.8970 & -32.0900 & 3.2509 & -0.7626 \\ 0.0001 & -1.8997 & 0.9831 & -0.0007 & -0.1708 & -0.0050 \\ 0.0123 & 11.7200 & -2.6316 & 0.0009 & -31.6040 & 22.3960 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -30.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & -30.0000 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 30 & 0 \\ 0 & 30 \end{bmatrix}$$

Then the orthogonal matrix (2.13) is given by

$$T = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

and from (2.16)

$$A_{11} = \begin{bmatrix} -0.0257 & 32.0900 & 18.8970 & -36.6170 \\ 0 & 0 & 1.0000 & 0 \\ -0.0123 & 0.0009 & -2.6316 & -11.7200 \\ 0.0001 & 0.0007 & -0.9831 & -1.8997 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 3.2509 & -0.7626 \\ 0 & 0 \\ 31.6040 & -22.3960 \\ -0.1708 & -0.0050 \end{bmatrix}$$

Assume $R = I_2$, $Q = I_4$, $\alpha = -2.1896$, $\theta = 0$. The u.p.d.s of the Riccati equation (8.13) is

$$P = \begin{bmatrix} 0.2633 & -8.3023 & -0.0082 & -5.0598 \\ -8.3023 & 381.9231 & 1.4851 & 226.8472 \\ -0.0082 & 1.4851 & 0.0406 & 0.7875 \\ -5.0598 & 226.8472 & 0.7875 & 136.8960 \end{bmatrix}$$

Assume

$$F_1 = R^{-1} A_{12}^T \hat{P}$$

then

$$\hat{P} = \begin{bmatrix} 0.2633 & 8.3023 & 0.0082 & 5.0598 \\ 8.3023 & 381.9231 & 1.4851 & 226.8472 \\ 0.0082 & 1.4851 & 0.0406 & 0.7875 \\ 5.0598 & 226.8472 & 0.7875 & 136.8960 \end{bmatrix}$$

and therefore

$$F_1 = \begin{bmatrix} 0.2494 & 35.1795 & 1.1740 & 17.9540 \\ -0.4087 & -40.7259 & -0.9186 & -22.1792 \end{bmatrix}$$

The eigenvalues $A_{11} - A_{12}F_1$ are $-27.7918 \pm 19.7733i$, -4.8295, 0.0119. Since 0.0119 is not in the left-hand half-plane, the Woodham et al [131] method fails for $\theta = 0$. However, it is very difficult to specify a value of θ for which their method works. Since $F_2 = R^{-1} A_{12}^T P$,

$$F_2 = \begin{bmatrix} 1.4624 & -18.8002 & 1.1210 & -14.9435 \\ 0.0072 & -28.0633 & -0.9062 & -14.4620 \end{bmatrix}$$

the eigenvalues $A_{11} - A_{12}F_2$ are $-33.8104, \ -25.3151, \ -4.8263, \ -3.7015$ which lie within the specified region.

Woodham et al [131] claimed that their method works for $\theta < 60°$. But the following example shows that this statement is not always true, i.e. for a real feedback matrix all the eigenvalues may not be located in the sector S for some $\theta < 60°$.

Example 8.4.2: Consider A_{11} and A_{12} as in Example 8.4.1. Assume $\alpha = -5.8898$, $\theta = 50°$, $Q = I_4$, $R = I_2$. Then the u.p.d. Hermitian matrix solution of the Riccati equation (8.13) is

$$P = \begin{bmatrix} 0.61 & -18.44 - 1.50i & -0.07 - 0.04i & -9.07 - 0.31i \\ -18.44 + 1.50i & 856.01 & 3.58 & 503.09 + 10.48i \\ -0.07 + 0.04i & 3.58 & 0.05 & 1.85 + 0.14i \\ -9.07 + 0.31i & 503.09 - 10.48i & 1.85 - 0.14 & 329.86 \end{bmatrix}$$

The eigenvalues of $A_{11} - A_{12}F_1$ are $-43.1168 \pm 59.7971i, \ -5.1977, \ -1.4200$, where $F_1 = R^{-1}A_{12}^T\hat{P}$. Obviously, one of eigenvalues -1.42 is not in the sector S. Let $F_2 = R^{-1}A_{12}^TP$. The eigenvalues of $A_{11} - A_{12}F_2$ are $-57.8464 + 17.0566i, \ -5.8683 - 15.4634i, \ -5.8287 - 1.5450i, \ -7.3276 - 0.0481i$. So all the eigenvalues of $A_{11} - A_{12}R^{-1}{A_{12}}^TP$ are not in the sector S. This example shows that even if the Hermitian matrix solution of the CARE (8.13) is utilized to design a complex feedback, all the eigenvalues of $A_{11} - A_{12}R^{-1}A_{12}^TP$ may not lie in the sector S.

For $\alpha = -9.1032, \ \theta = 40°, \ Q = I_4, \ R = I_2$, the u.p.d. Hermitian matrix solution of the Riccati equation (8.13) is

$$P = \begin{bmatrix} 1.6 & -36.8 - 3.1i & -0.1 - 0.1i & -10.9 + 0.4i \\ -36.8 + 3.1i & 1656.2 & 6.8 + 0.1i & 958.4 + 25.8i \\ -0.1 + 0.1i & 6.8 - 0.1i & 0.1 & 3.5 + 0.7i \\ -10.9 - 0.4i & 958.4 - 25.8i & 3.5 - 0.7i & 725.5 \end{bmatrix}$$

Therefore

$$F_1 = R^{-1}A_{12}^T\hat{P} = \begin{bmatrix} 8.6378 & 171.1769 & 2.2386 & 24.1044 \\ -4.9753 & -185.2597 & -1.7824 & -91.6911 \end{bmatrix}$$

The eigenvalues of $A_{11} - A_{12}F_1$ are $-66.0690 \pm 97.6081i, \ -2.3101, \ -8.9935$. Hence two eigenvalues of the closed-loop matrix $A_{11} - A_{12}F_1$ are not in the sector S. On the other hand

$$\begin{aligned} F_2 &= R^{-1}A_{12}^TP \\ &= \begin{bmatrix} 2.83 + 2.87i & -68.33 - 8.76i & 1.27 - 0.17i & -49.00 + 23.00i \\ 1.89 - 1.99i & -129.05 + 4.68i & -1.55 + 0.07i & -73.53 - 15.74i \end{bmatrix} \end{aligned}$$

and the eigenvalues of $A_{11} - A_{12}F_2$ are $-63.7920 + 21.0003i$, $-8.5553 - 16.9893i$, $-9.9447 - 3.9711i$, $-13.6067 - 0.0398i$. Clearly, the eigenvalue $-8.5553 - 16.9893i$ of $A_{11} - A_{12}F_2$ is not in the specified sector, i.e. in this case the complex feedback F_2 also does not work.

Now let us alter α. Suppose $\alpha = -2.6614$, $\theta = 40°$ and P is the u.p.d. Hermitian matrix solution of the Riccati equation (8.13)

$$P = \begin{bmatrix} 0.27 & -8.31 - 0.37i & -0.02 - 0.02i & -4.95 - 0.04i \\ -8.31 + 0.37i & 384.90 & 1.59 + 0.04i & 227.62 - 0.24i \\ -0.02 + 0.02i & 1.59 - 0.04i & 0.04 & 0.83 - 0.03i \\ -4.95 + 0.04i & 227.62 + 0.24i & 0.83 + 0.03i & 137.91 \end{bmatrix}$$

The eigenvalues of $A_{11} - A_{12}R^{-1}A_{12}^T\hat{P}$ are $-31.1910 \pm 29.3426i$, -4.7796, -0.6119. Obviously, -0.6119 is not in the sector S.

These examples establish that the Woodham et al [131] method for pole placement in a specified sector fails for certain values of α and θ. The interval of values θ for which their method works, is unknown and the above examples are counterexamples. The main reason for the failure is that if the u.p.d. Hermitian matrix solution P of the ARE (8.13) is used, all the eigenvalues of the associated closed-loop system may not be inside the sector S as stated in Remarks 8.2.2 and 8.2.3.

8.5 Eigenvalue Assignment Method by using Real ARE

As stated, the pole placement method utilizing the *complex* Riccati equation is complicated. Most effort has been focused on the real case. In this section a method is proposed to construct a feedback gain matrix such that all the eigenvalues of the closed-loop system lie in the specified sector. First three lemmas are stated which are used in the proof of a proposition and theorem.

Definition 8.5.1 Let P_1 and P_2 be two matrices. P_1 is said to be greater than P_2, $P_1 \geqslant P_2$, if and only if $P_1 - P_2$ is a semi-p.d. matrix. P_1 is the *maximum* solution of an ARE if, for any solution P_2, $P_1 \geqslant P_2$.

The so-called mirror-image shift lemma (Molinari [85]) is now stated. This lemma presents the basic idea about the solution of the Riccati equation with zero right-hand side.

Lemma 8.5.1 [64], [101]: *Let λ_i $(1 \leqslant i \leqslant n)$ be the eigenvalues of A, and P the maximum solution of the algebraic Riccati equation*

$$A^T P + PA - PBR^{-1}B^T P = 0 \tag{8.25}$$

Assume s_i $(1 \leqslant i \leqslant n)$ are the eigenvalues of $A - BR^{-1}B^T P$. Then

$$s_i = \begin{cases} \lambda_i & \text{if } \Re(\lambda_i) \leqslant 0 \\ -\lambda_i & \text{if } \Re(\lambda_i) > 0 \end{cases}$$

□

The maximum solution P of (8.25) is a semi-p.d.s. matrix because a solution of (8.25) is $P_0 = 0$ and if P_m is the maximum solution of (8.25), then $P_m - P_0 = P_m \geqslant 0$. The following lemma shows the relationship between the Riccati equation (8.25) and a certain Lyapunov equation. Note that A is a completely unstable matrix if $-A$ is a stable matrix.

Lemma 8.5.2: *Assume A is a completely unstable matrix, (A, B) is a completely controllable pair and $m \times m$ matrix R is an arbitrary p.d.s. matrix solution. Then P is a nonsingular solution of ARE (8.25) if and only if $S = P^{-1}$ is the solution of the Lyapunov equation*

$$S(-A)^T + (-A)S = -BR^{-1}B^T \tag{8.26}$$

Moreover, if P is a p.d.s. matrix, so is S.

Proof: Since (A, B) is controllable, $(A, BR^{-\frac{1}{2}})$ is controllable (Anderson and Moore [6]). Assume P is the solution of ARE (8.25) so

$$A^T P + PA = PBR^{-1}B^T P$$

Pre- and post-multiplying by P^{-1}

$$P^{-1}A^T + AP^{-1} = BR^{-1}B^T$$

So $S = P^{-1}$ is the solution of (8.26). Moreover, S is symmetric if and only if P is symmetric, and S is p.d. if and only if P is p.d.. □

Fact 8.5.1: Equation (8.25) has no p.d. solution unless A is a completely unstable matrix (see Theorem A.1.1). This is now proved in detail. Since (A, B) is a completely controllable pair, if A is a stable matrix, the Lyapunov equation

$$SA^T + AS = -BR^{-1}B^T \tag{8.27}$$

has a u.p.d.s. solution S [47], [127]. Hence $-S^{-1}$ is the solution of (8.25) which is negative definite. On the other hand, if P is a p.d.s. solution of (8.25) then $-P^{-1}$ is a solution of (8.27), which yields $-A$ stable. This contradicts the stability of A. Therefore when A is stable, (8.25) has no nonzero semi-p.d.s. and the maximum solution is the trivial solution $P = 0$. Except for this case, the maximum solution of (8.25) is a nonzero matrix. In particular, if A is a completely unstable matrix, the Riccati equation (8.25) has a p.d.s. solution matrix.

Now the solution of the Riccati equation (8.25) is studied. Let $\lambda_1^-, \lambda_2^-, \ldots, \lambda_{n^-}^-$ be the eigenvalues of A which are in the left-hand half-plane and $\xi_1^-, \xi_2^-, \ldots, \xi_{n^-}^-$ be the corresponding eigenvectors. The maximum solution of (8.25) satisfies

$$\mathcal{N}(P) = \text{span}\{\xi_1^-, \xi_2^-, \ldots, \xi_{n^-}^-\}$$

where $\mathcal{N}(P)$ and $\text{span}\{\xi_1^-, \xi_2^-, \ldots, \xi_{n^-}^-\}$ denote the null space of P and the linear subspace spanned by vectors $\xi_1^-, \xi_2^-, \ldots, \xi_{n^-}^-$ [64]. Therefore the nullity of P is n^-. On the other hand, the dimension of range P (rank P) plus the nullity of P equals n, the dimension of the space. Therefore the dimension of range P is $n - n^- = n^+$. Hence n^- eigenvalues of P are zero and the remaining n^+ eigenvalues lie in the right-hand half-plane. Moreover, if A has no eigenvalues in the right-hand half-plane, the range of P is zero, i.e. in this case $P = 0$. If A has no eigenvalues in the left-hand half-plane, the dimension of the range of P is n. So P is the p.d.s. solution of the Riccati equation (8.25).

The following lemma allocates the eigenvalues of the shifted closed-loop matrix $A - BK$.

Lemma 8.5.3: *Let λ_i $(1 \leqslant i \leqslant n)$ be the eigenvalues of A, α a nonnegative real number and P the maximum solution of the ARE*

$$(A^T + \alpha I)P + P(A + \alpha I) - PBR^{-1}B^TP = 0 \tag{8.28}$$

Assume s_i $(1 \leqslant i \leqslant n)$ are the eigenvalues of $A - BR^{-1}B^TP$. Then

$$s_i = \begin{cases} \lambda_i & \text{if } \Re(\lambda_i) \leqslant -\alpha \\ -2\alpha - \lambda_i & \text{if } \Re(\lambda_i) > -\alpha \end{cases}$$

Proof: Let $\hat{A} = A + \alpha I$ and $\hat{\lambda}_i$ $(1 \leqslant i \leqslant n)$ be the eigenvalues of $\hat{A}$. Assume $\hat{s}_i$ $(1 \leqslant i \leqslant n)$ are the eigenvalues of $\hat{A} - BR^{-1}B^TP$, then Lemma 8.5.1 gives

$$\hat{s}_i = \begin{cases} \hat{\lambda}_i & \text{if } \Re(\hat{\lambda}_i) \leqslant 0 \\ -\hat{\lambda}_i & \text{if } \Re(\hat{\lambda}_i) > 0 \end{cases} \tag{8.29}$$

Substituting $\hat{s}_i = s_i + \alpha$ and $\hat{\lambda}_i = \lambda_i + \alpha$ in (8.29) gives

$$s_i = \begin{cases} \lambda_i & \text{if } \Re(\lambda_i) \leqslant -\alpha \\ -\lambda_i - 2\alpha & \text{if } \Re(\lambda_i) > -\alpha \end{cases}$$

□

Proposition 8.5.1: *Assume λ_i $(1 \leqslant i \leqslant n)$ are the eigenvalues of A which are not necessary distinct, and S is a sector with vertex at nonpositive real α and the angle with the imaginary axis θ (see Fig.8.1). Suppose*

$$\eta = \tan\theta \max_{1 \leqslant i \leqslant n} |\Im(\lambda_i)| \tag{8.30}$$

and P is the maximum solution of ARE

$$(A^T - (\alpha - \eta)I)P + P(A - (\alpha - \eta)I) - PBR^{-1}B^T P = 0 \tag{8.31}$$

Then all the eigenvalues of $A - BR^{-1}B^T P$ are in the sector S if $\Re(\lambda_i) \neq \alpha - \eta$ $(1 \leqslant i \leqslant n)$. If there is a λ_i such that $\Re(\lambda_i) = \alpha - \eta$, P is the p.d.s. solution of ARE

$$(A^T - (\alpha - \epsilon - \eta)I)P + P(A - (\alpha - \epsilon - \eta)I) - PBR^{-1}B^T P = 0 \tag{8.32}$$

where ϵ is small positive real number such that for all i, $\lambda_i \neq \alpha - \eta - \epsilon$.

Proof: Using Lemma 8.5.3 the eigenvalues of $A - BR^{-1}B^T P$, s_i $(1 \leqslant i \leqslant n)$ are

$$s_i = \begin{cases} \lambda_i & \text{if } \Re(\lambda_i) \leqslant -(\eta - \alpha) \\ -2(\eta - \alpha) - \lambda_i & \text{if } \Re(\lambda_i) > -(\eta - \alpha) \end{cases}$$

Assume $\Re(\lambda_i) < -(\eta - \alpha)$ then $\Re(\lambda_i) \leqslant \alpha - \Im(\lambda_i)\tan\theta$, i.e. $\lambda_i = \Re(\lambda_i) + i\Im(\lambda_i)$ is inside the sector with boundary lines

$$\begin{aligned} L_1: \quad x &= \alpha - y\tan\theta \\ L_2: \quad x &= \alpha + y\tan\theta \end{aligned}$$

as shown in Fig.8.1. Suppose $\Re(\lambda_i) > -(\eta - \alpha)$ then

$$\begin{aligned} \Re(s_i) &= -2(\eta - \alpha) - \Re(\lambda_i) \\ &< -2(\eta - \alpha) + (\eta - \alpha) \\ &< \alpha - \eta \end{aligned}$$

Since

$$\Im(s_i) = \Im(\lambda_i), \qquad \forall i \quad 1 \leqslant i \leqslant n$$

therefore

$$\Re(s_i) \leqslant \alpha - \Im(s_i)\tan\theta$$

i.e. s_i lies in the sector S. If there is an i such that $\Re(\lambda_i) = -(\eta - \alpha)$, λ_i is a point on the boundary of the sector. Let ϵ be a real positive number such that for all i, $\lambda_i \neq \alpha - \eta - \epsilon$, and P be the p.d.s. solution of ARE (8.32). Converting α to $\alpha - \epsilon$ in the proof of the first part of the proposition, the desired result is obtained. □

Note that when all the eigenvalues of A are real, $\eta = 0$ and for all values of θ the solution P of ARE is invariant. Therefore in this case the closed-loop eigenvalues are also invariant for all values of θ.

It is possible to find various feedback gain matrices such that all the eigenvalues of the closed-loop system lie in the sector S. Here a method is presented which enables one to find feedback gain matrices such that all the closed-loop eigenvalues lie within the sector without any restriction on placement in the sector. This method can be applied as an iterative method. Horng et al [58] presented conditions for all the eigenvalues of the summation of two matrices A and G, i.e. matrix $A + G$, to lie in a specified region when all the eigenvalues of matrix A are inside that region. Now a similar method is applied and a new feedback gain is found.

Let $F = -BR^{-1}B^TP$ where P is the maximum solution of the ARE (8.31). According to Proposition 8.5.1 all the eigenvalues of $A + BF$ are in the sector.

Assume $\tilde{A} = A + BF$ and consider the new system

$$\dot{x} = \tilde{A}x + Bu \tag{8.33}$$

The control law is $u = \tilde{F}x$. Since all the eigenvalues of $\tilde{A}$ are in the sector S, then the Lyapunov equation

$$e^{i\theta}(\tilde{A} - \alpha I)\tilde{P} + e^{-i\theta}\tilde{P}(\tilde{A} - \alpha I) = -Q \tag{8.34}$$

where Q is an arbitrary p.d.s. matrix, has the u.p.d. Hermitian matrix solution $\tilde{P}$. Then if

$$e^{-i\theta}(\tilde{A} - \alpha I)^T\tilde{P} + e^{i\theta}\tilde{P}(\tilde{A} - \alpha I) + e^{-i\theta}(B\tilde{F})^T\tilde{P} + e^{i\theta}\tilde{P}(B\tilde{F}) < 0 \tag{8.35}$$

all the eigenvalues of $A + B\tilde{F}$ lie in the region S. Using (8.34) in (8.35) yields

$$-Q + e^{-i\theta}(B\tilde{F})^T\tilde{P} + e^{i\theta}\tilde{P}(B\tilde{F}) < 0 \tag{8.36}$$

A sufficient condition for (8.36) to hold is that

$$\sigma_M(e^{-i\theta}(B\tilde{F})^T\tilde{P} + e^{i\theta}\tilde{P}(B\tilde{F})) < \sigma_m(Q) \tag{8.37}$$

Let $\tilde{F} = -R^{-1}B^T\hat{P}$ where $\hat{P}$ is a real p.d.s. matrix. Now

$$\begin{aligned}
&\sigma_M(e^{-i\theta}\hat{P}BR^{-1}B^T\tilde{P} + e^{i\theta}\tilde{P}BR^{-1}B^T\hat{P}) \\
&\qquad \leqslant \sigma_M(e^{-i\theta}\hat{P}BR^{-1}B^T\tilde{P}) + \sigma_M(e^{i\theta}PBR^{-1}B^T\hat{P}) \\
&\qquad \leqslant \sigma_M(\hat{P})\sigma_M(BR^{-1}B^T\tilde{P}) + \sigma_M(\tilde{P}BR^{-1}B^T)\sigma_M(\hat{P}) \\
&\qquad \leqslant 2\sigma_M(\tilde{P})\sigma_M(B\tilde{R}^{-1}B^T\hat{P})
\end{aligned} \tag{8.38}$$

From (8.37) and (8.38) one can imply if

$$2\sigma_M(\tilde{P}BR^{-1}B^T)\sigma_M(\hat{P}) < \sigma_m(Q) \tag{8.39}$$

then (8.37) is satisfied. But (8.39) can be replaced by

$$\sigma_M(\hat{P}) < \frac{\sigma_m(Q)}{2\sigma_M(\tilde{P}BR^{-1}B^T)} = c_s \tag{8.40}$$

The closed-loop matrix is

$$\begin{aligned}
\tilde{A} + B\tilde{F} &= A + BF + B\tilde{F} \\
&= A - BR^{-1}B^TP - R^{-1}B^T\hat{P} \\
&= A - BR^{-1}B^T(P + \hat{P})
\end{aligned} \tag{8.41}$$

To find $\hat{P}$, noting (8.40) consider an $n \times n$ arbitrary nonsingular matrix E and $\hat{P} = (d_s/\sigma_M{}^2(E))\ EE^T$ where d_s is a positive real number such that $d_s < c_s$. Therefore, for any arbitrary nonsingular matrix E, a matrix $\hat{P}$ is obtained which gives a new feedback gain matrix $\hat{F} = -R^{-1}B^T(P + \hat{P})$.

8.5.1 Extension Technique for a Region Bounded by Two Sectors

A technique for placing all the closed-loop poles in a region bounded by the intersection of two sectors, will now be considered. Suppose the first sector S_1 has boundary lines crossing the real axis at α in the left-hand half-plane and with angle to the imaginary axis θ. The second sector S_2 has boundary lines crossing the real axis at β in the left-hand half-plane and with angle to the imaginary axis ϕ. Consider a new sector S_3 with vertex $\eta = \min\{\alpha, \beta\}$ and the angle with the imaginary axis $\hat{\theta} = \max\{\phi, \theta\}$ (see Fig.

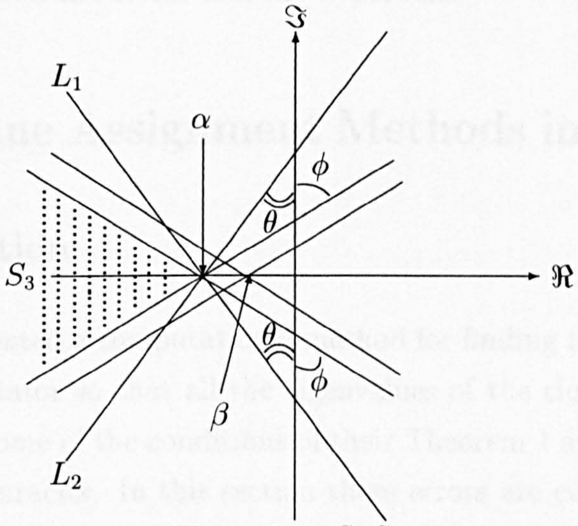

Figure 8.2: S_3 Sector

8.2). By applying Proposition 8.5.1, the gain feedback F can be found such that all the eigenvalues of $A + BF$ lie within the sector S_3. It is clear in this case that all the eigenvalues of $A + BF$ lie in the intersection of the two sectors S_1 and S_2. The only restriction of this method is that the eigenvalues of $A+BF$ cannot lie in the small common region of two sectors S_1 and S_2, and outside the sector S_3.

8.5.2 Pole Placement in a Hyperbola

The extension of the technique for placing all the closed-loop system eigenvalues within a hyperbola in the left-hand half-plane is now considered. The pole placement within the hyperbola with asymptotic lines $y = \pm x$ has been considered by Kawasaki and Shimemura [64]. They obtained two associated Riccati equations with conventional weighting matrices and an iterative computational algorithm.

A new method is presented which guarantees that all the closed-loop eigenvalues lie in a hyperbola. Consider the hyperbola

$$\frac{(\Re(\lambda) + \alpha)^2}{a^2} - \frac{(\Im(\lambda))^2}{b^2} = 1 \tag{8.42}$$

Let S be a sector with boundary lines crossing $\beta = \alpha - a$ on the real axis and parallel to the asymptotic lines of the hyperbola, i.e. the boundary lines of the sector are

$$\frac{\Re(\lambda) + \alpha}{a} \pm \frac{\Im(\lambda)}{b} = 1 \tag{8.43}$$

Obviously, when all the eigenvalues of the closed-loop system are in sector S, they also lie inside the hyperbola. The only restriction of this method is that the eigenvalues cannot lie in the region between the sector and the hyperbola.

8.6 Eigenvalue Assignment Methods in a Vertical Strip

8.6.1 Introduction

Shieh et al [101] presented a computational method for finding the feedback control of the linear quadratic regulator so that all the eigenvalues of the closed-loop system lie in an open vertical strip. Some of the conditions of their Theorem 1 are incorrect and also their proof has some inaccuracies. In this section these errors are corrected and the modified theorem with a new conventional proof is proved. Illustrative examples are presented in Section 8.7.

$\Re(\cdot)$ and $\Im(\cdot)$ indicate the real and imaginary parts of complex number $(\cdot)$, and $\mathrm{Tr}(\cdot)$ the trace of matrix $(\cdot)$. To avoid any misunderstanding the notation below is the same as in paper [101], i.e. the system is given by

$$\dot{x} = Ax + Bu \tag{8.44}$$

where A and B are $n \times n$ and $n \times m$ real matrices respectively, and (A, B) is a completely controllable pair. Let h_1 and h_2 be two nonnegative real numbers. The closed vertical strip is specified by the closed interval $[-h_2, -h_1]$ with $h_2 \geqslant h_1$ and the open vertical strip is given by $(-h_2, -h_1)$ with $h_2 > h_1$. Assume λ_1^-, $\lambda_2^-, \dots, \lambda_{n^-}^-$ are the eigenvalues of A which are in the closed left-hand half-plane, and λ_1^+, $\lambda_2^+, \dots, \lambda_{n^+}^+$ are the open right-hand half-plane eigenvalues of A. The shifted system matrix is $\hat{A} = A + h_1 I_n$. Let $\hat{\lambda}_1^-$, $\hat{\lambda}_2^-, \dots, \hat{\lambda}_{\hat{n}^-}^-$ be the closed left-hand half-plane eigenvalues of $\hat{A}$, and $\hat{\lambda}_1^+$, $\hat{\lambda}_2^+, \dots, \hat{\lambda}_{\hat{n}^+}^+$ be the open right-hand half-plane eigenvalues of $\hat{A}$. The feedback control is $u = -rKx$ where r is a real number and K is an $m \times n$ matrix to be defined later. Let P be the maximum solution [64] of the Riccati equation, their (6c),

$$PBR^{-1}B^TP - \hat{A}^TP - P\hat{A} = 0$$

where R is an arbitrary positive definite symmetric matrix. Then the eigenvalues of the closed-loop system

$$\dot{x} = (\hat{A} - rBK)x, \qquad r > 0.5$$

where $K = R^{-1}B^TP$, are $\hat{\lambda}_1^-$, $\hat{\lambda}_2^-, \dots, \hat{\lambda}_{\hat{n}^-}^-$, $\tilde{\lambda}_1$, $\tilde{\lambda}_2, \dots, \tilde{\lambda}_{\hat{n}^+}$. The $\tilde{\lambda}_i$ $(1 \leqslant i \leqslant \hat{n}^+)$ with $\Re(\tilde{\lambda}_i) = -\tilde{\alpha}_i < 0$ are the newly placed left-hand open half-plane eigenvalues in the shifted coordinates [101].

Now the errors in [101] are stated. These errors, inaccuracies and restrictions are divided into two parts as follows:

Error and inaccuracies in their theory:

1. Error in the proof of their Theorem 1 [101]: The main part of Theorem 1 gives conditions for the eigenvalues of closed-loop system to lie in the open vertical strip $(-h_2, -h_1)$. In their proof Shieh et al obtained the result, their $(9h)$,

$$\sum_{i=1}^{\hat{n}^+} \tilde{\alpha}_i = h_2 - h_1$$

   and stated that "Since each $\tilde{\alpha}_i$ is a positive real value and

$$\sum_{i=1}^{\hat{n}^+} \tilde{\alpha}_i = h_2 - h_1 > 0$$

   therefore, each $\tilde{\alpha}_i < h_2 - h_1$, and the newly placed eigenvalues of $(\hat{A} - rBK)$ for $r > 0.5$ lie inside the vertical strip, $\{-(h_2 - h_1), 0\}$ in the shifted coordinates."

   This is wrong because if $\hat{A}$ has *only* one eigenvalue in the right-hand half-plane, i.e. $\hat{n}^+ = 1$, then

$$\sum_{i=1}^{\hat{n}^+} \tilde{\alpha}_i = \tilde{\alpha}_1$$

   and their $(9h)$ gives

$$\tilde{\alpha}_1 = h_2 - h_1$$

   which contradicts $\tilde{\alpha}_1 < h_2 - h_1$ implied in the first paragraph after $(9h)$ in their paper, i.e. in this case $h_2 - h_1 < h_2 - h_1$ which is wrong.

2. Their Riccati equation $(6c)$ has no p.d.s. matrix solution unless the matrix $-\hat{A}$ is a stable matrix, i.e. all the eigenvalues of A lie to the right of the vertical line $-h_1$. This has been proved in Section (8.5).

3. The conditions $h_2 > \max |\Re(\hat{\lambda}_i^-)| + h_1$ and $\max \Re(\hat{\lambda}_i^-) \neq 0$ are necessary conditions for all the invariant eigenvalues to lie in the open vertical strip while the condition $h_2 \geqslant \max |\Re(\hat{\lambda}_i^-)| + h_1$ is only a necessary condition for all the invariant eigenvalues to lie in the closed vertical strip. Condition $\max \Re(\hat{\lambda}_i^-) \neq 0$ ensures that for all $1 \leqslant i \leqslant \hat{n}^-$, $\Re(\hat{\lambda}_i^-) \neq 0$, i.e. the matrix A has no eigenvalue on the vertical line $x = -h_1$. If $h_2 < \max |\Re(\hat{\lambda}_i^-)| + h_1$, A has an eigenvalue to the left of the vertical line $-h_2$. The eigenvalue of A with the real part $-\max |\Re(\lambda_i^-)|$ satisfies this

condition. Thus the matrix A has an eigenvalue which is an invariant eigenvalue of $A - rBk$ and is not in the vertical strip. So without these conditions it is impossible for the invariant eigenvalues of closed-loop matrix $A - rBK$ to lie in the desired vertical strip. These conditions are required for the theorem to be satisfied.

4. If λ_i^- satisfies $\Re(\lambda_i^-) = -h_1$, all the conditions are satisfied, but λ_i^- is not in the specified open vertical strip $(-h_2, -h_1)$. Moreover, the condition
$$r = \frac{1}{2} + \frac{h_2 - h_1}{2\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+}$$
is not always true. For example when $\hat{A}$ has just one eigenvalue in the right-hand half-plane, their $(9h)$ yields $-(h_2 - h_1)$ as the real part of the eigenvalue of $\hat{A} - rBK$ which is not in the open infinite vertical strip $(-(h_2 - h_1), 0)$, i.e. the real part of one eigenvalue of $A - rBK$ is $-h_2$ which is not in the open vertical strip $(-h_2, -h_1)$.

Restrictions in [101]: There are some restrictions in their Theorem 1 [101]:

(*i*) When $r = 1/2$ then $h_2 = h_1$ and vice versa. So if $h_2 \neq h_1$, r cannot take the value $1/2$. Thus when $r = 1/2$ all the eigenvalues of $A - rBk$ should lie in the vertical line $h_1 = h_2$. This excludes a large class of matrices A with eigenvalues to the left of the vertical line $x = -h_1$, when there is no choice in selecting h_2 except $h_2 = h_1$.

(*ii*) Also by choosing $h_1 > 0$, the dominant eigenvalues of $A - rBk$ cannot lie in the vertical strip $(-h_1, 0)$. Even when h_1 is a small positive real number, this selection of h_1 is such that the eigenvalues of $A - rBk$ cannot lie sufficiently near the imaginary axis, i.e. inside the vertical strip $(-h_1, 0)$. Note that choosing $h_1 = 0$ gives $r = \frac{1}{2}(1 + h_2 / \sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+)$ which differs from [101] and the matrix $A - rBK$ is different.

These inaccuracies are illustrated by an example.

Example 8.6.1 (Counterexample): Consider the system (8.44) with
$$A = \begin{bmatrix} 1 & 1 \\ 0 & -2.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The eigenvalues of A are 1 and -2.5. First choose $h_1 = 1$ and $h_2 = 3$. So $\hat{A} = A + h_1 I_2$ with $\hat{n}^+ = 1$, $\hat{\lambda}_i^+ = 2$. Then $\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ = 2$ and $r = 1$. The Riccati equation $(6c)$ has four symmetric solutions
$$P_0 = 0, \quad P_1 = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 49 & 14 \\ 14 & 4 \end{bmatrix}$$

Since the eigenvalues of $P_3 - P_2$ are 51 and 0, $P_3 - P_2 \geqslant 0$. Also $P_3 - P_1 \geqslant 0$ because the eigenvalues of $P_3 - P_1$ are 53.2389 and 2.7611. Hence the maximum solution of the Riccati equation (6c) is P_3. Matrix P_3 is semi-positive definite symmetric because the eigenvalues of P_3 are 53 and 0. Therefore the Riccati equation (6c) may not generally have positive definite symmetric solution. Only when all the eigenvalues of $\hat{A}$ are in the right-hand half-plane, does the Riccati equation (6c) have a p.d.s. matrix solution. Except for this case, the solution is semi-positive definite symmetric. Hence $K = [14 \quad 4]$ and the eigenvalues of $\hat{A} - BK$ are -2 and -1.5. So, the eigenvalues of $A - rBK$ are -3 and -2.5. Here $h_2 - h_1 = 2 = \tilde{\alpha}$, so according to the proof of Theorem 1 in [101], $\tilde{\alpha} < h_2 - h_1$, i.e. $2 = h_2 - h_1 < h_2 - h_1 = 2$, which not true. Moreover, for $h_2 < 2.5 = \max|\Re(\hat{\lambda}_i^-)| + h_1$, -3 is not inside the vertical strip $[-h_2, -h_1]$. Therefore the condition $h_2 \geqslant \max|\Re(\hat{\lambda}_i^+)| + h_1$ is a necessary condition for A to have an eigenvalue to the left line of the vertical line $x = -h_1$.

Note that $\mathcal{N}(P_3)$ has one element $\xi_1^- = [-0.2747 \quad 0.9615]^T$ which is the eigenvector corresponding to the eigenvalue -1.5 of $\hat{A}$ and the range of $\hat{A}$ has one element because rank $P_3 = 1$. As already stated, one eigenvalue of P is zero.

Next consider $h_1 = 2.5$ and $h_2 = 3$.

$$h_2 > \max\left\{|\Re(\lambda_i^-)|\right\} = \max\{1, |-2.5|\} = 2.5$$

All the conditions of their Theorem 1 are satisfied, but -2.5, which is an eigenvalue of $A - rBK$ with $r = 0.5713$, is not inside the *open* vertical strip $(-3, -2.5)$. Matrix A should have no eigenvalues on the vertical line $x = -h_1$. Moreover, for $h_1 \neq h_2$, r could take values greater than or equal to 1/2. Particularly, taking $r = 1/2$ yields all the eigenvalues of $A - rBk$ in the closed vertical strip $[-h_2, -h_1]$ which is not obtained from the method in [101]. Take $h_1 = 1$ and $h_2 = 3$, then with the method in [101] r cannot be 1/2. However, for $r = 1/2$ the eigenvalues of $A - rBK$ are -2.5 and 0, i.e. in the vertical strip $[-h_2, -h_1] = [-3, -1]$ with $h_1 \neq h_2$.

Now consider $h_1 = 0$, $h_2 = 3$; then $r = 2$ and the maximum solution of their Riccati equation (6c) is

$$P = \begin{bmatrix} 24.5 & 7 \\ 7 & 2 \end{bmatrix}$$

with eigenvalues 26.5 and 0. The eigenvalues of $A - rBK$ are -2.5, -3 in the vertical strip $[-3, 0]$. This result is not obtained from the method stated in [101].

Summary:

Hence, for the eigenvalues of $A - rBK$ to lie in the *open* vertical strip $(-h_2, -h_1)$, the correct necessary conditions should be

$$h_2 > \max\left\{|\Re(\hat{\lambda}_i^-)|\right\} + h_1 = -\min\left\{\Re(\hat{\lambda}_i^-)\right\} + h_1$$

$$h_1 \neq \Re(\lambda_i^-), \quad \text{for all } i = 1, 2, \ldots, n^-$$

and

$$\frac{1}{2} < r < \frac{1}{2} + \frac{h_2 - h_1}{2\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+}$$

For the eigenvalues of the closed-loop system to lie in the *closed* vertical strip, the correct conditions are

$$h_2 \geqslant \max\left\{|\Re(\hat{\lambda}_i^-)|\right\} + h_1 = -\min\left\{\Re(\hat{\lambda}_i^-)\right\} + h_1$$

and

$$\frac{1}{2} \leqslant r \leqslant \frac{1}{2} + \frac{h_2 - h_1}{2\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+}$$

Note that when r varies from $\frac{1}{2}$ to $\frac{1}{2}\left(1 + (h_2 - h_1)/\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+\right)$, some of the eigenvalues of $A - rBK$ can move from one place to another location in the open vertical strip. So, if the accumulation of the dominant eigenvalues near the vertical line $-h_1$ is required, r should be chosen sufficiently near $1/2$; and if they should be near the vertical line $-h_2$, r should be selected sufficiently near

$$\frac{1}{2} + \frac{h_2 - h_1}{2\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+}$$

The choice

$$r = \frac{1}{4}\left(2 + \frac{h_2 - h_1}{\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+}\right)$$

guarantees the dominant eigenvalues to be concentrated towards the the centre of the vertical strip, i.e. near the vertical line $x = -(h_2 + h_1)/2$.

Now Theorem 1 in [101] is modified such that with a new conventional proof, the above obstacles in the statement, proof and restrictions are removed. This proof is completely different from that of Shieh et al. Note that the theorem gives only a sufficient range for variation of r, i.e. there may be a value of r, for which the eigenvalues of $A - rBK$ lie in the vertical strip, greater than $\frac{1}{2}\left(1 + (h_2 - h_1)/\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+\right)$.

Theorem 8.6.1: *Let h_1 and h_2 be nonnegative real numbers with $h_2 \geqslant h_1$, $A \in \mathbb{R}^{n\times n}$ and $\hat{A} = A + h_1 I_n$. Assume $\lambda_1^-, \lambda_2^-, \ldots, \lambda_{n^-}^-$ are the eigenvalues of A which are in* *the closed left-hand half-plane and* $\lambda_1^+,\ \lambda_2^+, \ldots, \lambda_{n^+}^+$ *are the open right-hand half-plane eigenvalues of* A. *Let* $\hat{\lambda}_1^-,\ \hat{\lambda}_2^-, \ldots, \hat{\lambda}_{\hat{n}^-}^-$ *be the closed left-hand half-plane eigenvalues of* $\hat{A}$ *and* $\hat{\lambda}_1^+,\ \hat{\lambda}_2^+, \ldots, \hat{\lambda}_{\hat{n}^+}^+$ *be the open right-hand half-plane eigenvalues of* $\hat{A}$ *and*

$$\eta = \frac{1}{2} + \frac{h_2 - h_1}{2\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+}$$

Let P *be the maximum solution of the algebraic Riccati equation*

$$\hat{A}^T P + P\hat{A} - PBR^{-1}B^T P = 0 \tag{8.45}$$

where R *is an arbitrary positive definite symmetric matrix. Suppose* r *is an arbitrary real number. The intervals* $[-h_2, -h_1]$ *with* $h_2 \geqslant h_1$ *and* $(-h_2, -h_1)$ *with* $h_2 > h_1$ *specify the closed and open vertical strip, respectively. Then*

(i) *if* $\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ \neq 0$, *the eigenvalues of* $A - rBK$ *are inside*

(a) *the open vertical strip* $(-h_2, -h_1)$ *if* $1/2 < r < \eta$, $\max\left\{\Re(\hat{\lambda}_i^-)\right\} \neq 0$ *and*

$$h_2 > \max\left\{|\Re(\hat{\lambda}_i^-)|\right\} + h_1 = -\min\left\{\Re(\hat{\lambda}_i^-)\right\} + h_1$$

(b) *the closed vertical strip* $[-h_2, -h_1]$ *if* $1/2 \leqslant r \leqslant \eta$ *and*

$$h_2 \geqslant \max\left\{|\Re(\hat{\lambda}_i^-)|\right\} + h_1 = -\min\left\{\Re(\hat{\lambda}_i^-)\right\} + h_1$$

.

(ii) *if* $\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ = 0$, *the eigenvalues of* $A - rBK$ *lie in*

(a) *the open vertical strip* $(-h_2, -h_1)$ *if,*

$$h_2 > \max\left\{|\Re(\hat{\lambda}_i^-)|\right\} + h_1 = -\min\left\{\Re(\hat{\lambda}_i^-)\right\} + h_1$$

and $\max\left\{\Re(\hat{\lambda}_i^-)\right\} \neq 0$.

(b) *the closed vertical strip* $[-h_2, -h_1]$ *if*

$$h_2 \geqslant \max\left\{|\Re(\hat{\lambda}_i^-)|\right\} + h_1 = -\min\left\{\Re(\hat{\lambda}_i^-)\right\} + h_1$$

In Case (ii) the maximum solution of (8.25) is $P = 0$ *and all the eigenvalues of* $A - rBK$ *are the same as the eigenvalues of* A.

Proof: Using Lemma 8.5.1 the spectrum of $\hat{A} - BK$ is given by

$$\sigma(\hat{A} - BK) = \left\{\hat{\lambda}_1^-,\ \hat{\lambda}_2^-, \dots, \hat{\lambda}_{\hat{n}^-}^-, -\hat{\lambda}_1^+,\ -\hat{\lambda}_2^+, \dots, -\hat{\lambda}_{\hat{n}^+}^+\right\}$$

Therefore

$$\mathrm{Tr}(\hat{A} - BK) = \sum_{i=1}^{\hat{n}^-} \hat{\lambda}_i^- - \sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ \tag{8.46}$$

On the other hand

$$\begin{aligned} \mathrm{Tr}(\hat{A} - BK) &= \mathrm{Tr}(\hat{A}) - \mathrm{Tr}(BK) \\ &= \sum_{i=1}^{\hat{n}^-} \hat{\lambda}_i^- + \sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ - \mathrm{Tr}(BK) \end{aligned} \tag{8.47}$$

From (8.46) and (8.47)

$$\mathrm{Tr}(BK) = 2\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ \tag{8.48}$$

as in [101]. Using (8.48)

$$\begin{aligned} \mathrm{Tr}(\hat{A} - rBK) &= \sum_{i=1}^{\hat{n}^-} \hat{\lambda}_i^- + \sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ - r\mathrm{Tr}(BK) \\ &= \sum_{i=1}^{\hat{n}^-} \hat{\lambda}_i^- + \sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ - 2r\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ \\ &= \sum_{i=1}^{\hat{n}^-} \hat{\lambda}_i^- - (2r-1)\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ \end{aligned} \tag{8.49}$$

Since $\left\{\hat{\lambda}_1^-,\ \hat{\lambda}_2^-, \dots, \hat{\lambda}_{\hat{n}^-}^-\right\}$ is an invariant set for all $r \geqslant \frac{1}{2}$, therefore $-(2r-1)\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+$ is the summation of the remaining eigenvalues of $\hat{A} - rBK$ corresponding to

$$\left\{-\hat{\lambda}_1^+,\ -\hat{\lambda}_2^+, \dots, -\hat{\lambda}_{\hat{n}^+}^+\right\}$$

A sufficient condition for the eigenvalues of $\hat{A} - rBK$ to lie in the closed (open) left-hand half-plane is that $2r - 1 \geqslant 0$ $(2r - 1 > 0)$.

1. Assume $\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ \neq 0$.

   (*i*) If $h_2 > \max\left\{|\Re(\hat{\lambda}_i^-)|\right\} + h_1$ and $\max \Re(\hat{\lambda}_i^-) \neq 0$ are satisfied and

$$0 < (2r-1)\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ < h_2 - h_1 \tag{8.50}$$

then all the eigenvalues of $\hat{A}-rBK$ are in the open vertical strip $(-(h_2-h_1), 0)$, and the eigenvalues of $A-rBK$ are in the open vertical strip $(-h_2, -h_1)$. Equation (8.50) is equivalent to $1/2 < r < \eta$.

(*ii*) If $h_2 \geqslant \max\left\{|\Re(\hat{\lambda}_i^-)|\right\} + h_1$ is satisfied and

$$0 \leqslant (2r-1)\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ \leqslant h_2 - h_1 \tag{8.51}$$

then all the eigenvalues of $\hat{A}-rBK$ are in the closed vertical strip $[-(h_2-h_1), 0]$. Thus the eigenvalues of $A-rBK$ are in the closed vertical strip $[-h_2, -h_1]$. Equation (8.51) is true if and only if $1/2 \leqslant r \leqslant \eta$.

2. If $\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ = 0$, from (8.49) $\mathrm{Tr}(\hat{A}-rBK) = \sum_{i=1}^{\hat{n}^-} \hat{\lambda}_i^-$.

(*i*) If the conditions $h_2 > \max\left\{|\Re(\hat{\lambda}_i^-)|\right\} + h_1$ and $\max \Re(\hat{\lambda}_i^-) \neq 0$ are satisfied, then all the eigenvalues of $A-rBK$ are in the vertical strip $(-h_2, -h_1)$.

(*ii*) For all the eigenvalues $A-rBK$ to lie in the closed vertical strip $[-h_2, -h_1]$ the condition $h_2 > \max\left\{|\Re(\hat{\lambda}_i^-)|\right\} + h_1$ is sufficient.

For $r = \frac{1}{2}$, (8.49) implies that all the eigenvalues of $\hat{A} - \frac{1}{2}BK$ are the eigenvalues of $\hat{A}$ which are in the left-hand half-plane or on the imaginary axis, while for $r = \eta$ and $\hat{n}^+ = 1$, all the corresponding eigenvalues of A to the right of the vertical line $x = -h_1$ lie on the vertical line h_2. □

If $h_1 = 0$ and A has an eigenvalue on the imaginary axes, the condition for the eigenvalues of the closed-loop system to lie in the open vertical strip cannot be derived from Theorem 8.6.1. To ensure the stability of the closed-loop system it is required that all the eigenvalues $A-rBK$ lie in the left-hand half-plane. This problem is a particular case of the following.

Remark 8.6.1 Suppose one of the eigenvalues of A is $-h_1$. Consider $\hat{A} = A + (h_1+\epsilon)I_n$ where ϵ is a small positive real number such that $h_1+\epsilon$ is not the real part of an eigenvalue of A, and $-(h_1+\epsilon)$ is not an eigenvalue of A. Assume all the conditions of Theorem 8.6.1 are satisfied, then all the eigenvalues of $A-rBK$ lie within the vertical strip $(-h_2, -h_1)$.

Theorem 8.6.1 yields only a range of r satisfying the desired condition. However, the greatest upper value of r with the desired property can be found by trial and error.

The real number r_0 is said to be an upper bound satisfying (*u.b.s.*) the desired condition if for any $0.5 < r \leqslant r_0$, all the eigenvalues of $A-rBK$ lie in the vertical strip. An u.b.s.

r_0 is said to be the greatest upper bound satisfying (*g.u.b.s.*) the desired condition if for any $r > r_0$, there exists one eigenvalue of $A - rBK$ outside the region. Abbreviations *u.b.s.* and *g.u.b.s.* denote these properties.

8.7 Examples

Examples 8.7.1-8.7.3 below illustrate the numerous results about pole placement in a vertical strip stated in Section 8.6. The remaining examples 8.7.4 and 8.7.5 illustrate the results of Section 8.5 about pole placement in a sector and address the errors in [129] for obtaining the solution of the ARE (8.45).

Example 8.7.1 [101]: Consider the system (8.44) with

$$A = \begin{bmatrix} -0.5000 & 0.1100 & -0.6600 & -0.2200 \\ 0 & -1.0300 & -0.4100 & 2.0700 \\ 0 & -1.3200 & -0.3300 & 2.6400 \\ 0 & -0.0300 & 0.0300 & 0.0600 \end{bmatrix}, \quad B = \begin{bmatrix} 0.80 & 0.20 \\ 0 & 0.52 \\ -0.20 & 0.36 \\ -0.04 & 0.10 \end{bmatrix}$$

The spectrum of A is

$$\sigma(A) = \{\lambda_1 = -0.5000,\ \lambda_2 = -1.5002,\ \lambda_3 = 0.0017,\ \lambda_4 = 0.1985\}$$

Let $h_1 = 1$ and $h_2 = 2$, then

$$\sigma(\hat{A}) = \left\{\hat{\lambda}_1 = 0.5000,\ \hat{\lambda}_2 = -0.5002,\ \hat{\lambda}_3 = 1.0017,\ \hat{\lambda}_4 = 1.1985\right\}$$

Then $\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ = \hat{\lambda}_1 + \hat{\lambda}_3 + \hat{\lambda}_4 = 2.7002$ and an u.b.s. of r is

$$0.5(1 + (h_2 - h_1)/\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ = 0.6852$$

i.e. r can take all values between 0.5 and 0.6852. However, the g.u.b.s. r is not 0.6852. As already stated in Remark 8.6.1, the g.u.b.s. r_0 of r with desired property can be found by trial and error. When r tends to r_0 some of the eigenvalues of the closed-loop $A - rBK$ move to the vertical line $x = -h_2$.

The semi-p.d.s. matrix solution of ARE (8.45) is

$$P = \begin{bmatrix} 16.49831928793 & -74.18522113962 & 56.48932042071 & 149.06146039136 \\ -74.18522113962 & 374.55921933460 & -290.26838686087 & -721.59550305191 \\ 56.48932042071 & -290.26838686087 & 225.50261148977 & 556.19493305580 \\ 149.06146039136 & -721.59550305191 & 556.19493305580 & 1584.72921533561 \end{bmatrix}$$

and

$$K = \begin{bmatrix} -4.0617 & 27.5693 & -22.1569 & -55.3790 \\ -0.0343 & 3.2776 & -2.8413 & 13.2857 \end{bmatrix}$$

Let $r = 0.5$, then

$$A - rBK = \begin{bmatrix} 1.1281 & -11.2455 & 8.4869 & 20.6030 \\ 0.0089 & -1.8822 & 0.3287 & -1.3843 \\ -0.4000 & 0.8470 & -2.0343 & -5.2893 \\ -0.0795 & 0.3575 & -0.2711 & -1.7119 \end{bmatrix}$$

and the eigenvalues of $A - rBK$ are $-1 \pm 0.7742i$, -1.5002, -1 are not all in the open vertical strip $(-h_2, -h_1)$. In fact, three eigenvalues of $A - rBK$ are on the vertical line $x = -h_1$. Let $r = 0.51$, then

$$A - rBK = \begin{bmatrix} 1.1607 & -11.4726 & 8.6698 & 21.0195 \\ 0.0091 & -1.8992 & 0.3435 & -1.4534 \\ -0.4080 & 0.8903 & -2.0683 & -5.4479 \\ -0.0811 & 0.3653 & -0.2771 & -1.7473 \end{bmatrix}$$

and the eigenvalues of $A - rBK$ are $-1.0170 \pm 0.7740i$, -1.5002, -1.0200. Taking $r = 0.6852$

$$A - rBK = \begin{bmatrix} 1.7312 & -15.4516 & 11.8749 & 28.3159 \\ 0.0122 & -2.1978 & 0.6024 & -2.6638 \\ -0.5481 & 1.6496 & -2.6655 & -8.2264 \\ -0.1090 & 0.5010 & -0.3826 & -2.3682 \end{bmatrix}$$

and the eigenvalues of $A - rBK$ are $-1.3146 \pm 0.7074i$, -1.5002, -1.3710. Theorem 8.6.1 guarantees that all the eigenvalues of $A - rBK$ are inside the vertical strip if $0.5 < r < 0.6852$.

In the following some values of r are chosen to find the influence of value r on the dominant eigenvalues of $A - rBk$. Let $r = 0.55$, then

$$A - rBK = \begin{bmatrix} 1.2909 & -12.3810 & 9.4016 & 22.6853 \\ 0.0098 & -1.9674 & 0.4026 & -1.7297 \\ -0.4400 & 1.0637 & -2.2047 & -6.0823 \\ -0.0875 & 0.3963 & -0.3012 & -1.8891 \end{bmatrix}$$

with the eigenvalues $-1.0849 \pm 0.7695i$, -1.5002, -1.1002. Choosing $r = 0.57$

$$A - rBK = \begin{bmatrix} 1.3560 & -12.8353 & 9.7674 & 23.5182 \\ 0.0102 & -2.0015 & 0.4322 & -1.8679 \\ -0.4560 & 1.1503 & -2.2729 & -6.3994 \\ -0.0906 & 0.4118 & -0.3132 & -1.9599 \end{bmatrix}$$

the eigenvalues are $-1.1189 \pm 0.7650i$, -1.5002, -1.1402. For $r = 0.6700$

$$A - rBK = \begin{bmatrix} 1.6817 & -15.1064 & 11.5968 & 27.6828 \\ 0.0120 & -2.1719 & 0.5799 & -2.5587 \\ -0.5360 & 1.5837 & -2.6137 & -7.9853 \\ -0.1066 & 0.4893 & -0.3734 & -2.3143 \end{bmatrix}$$

and the eigenvalues are $-1.2888 \pm 0.7183i$, -1.5002, -1.3405.

Therefore, when r increases from 0.5, the eigenvalues of $A - rBK$ corresponding to the eigenvalues of A which lie in the right-hand half-plane, move to the vertical line $x = -2$ and there is an r_0 such that for all $r > r_0$ all the eigenvalues are not inside the strip. This r_0 is the maximum (g.u.b.s.) of all the values r which all the eigenvalues of $A - rBK$ lie inside the vertical strip. However, the g.u.b.s. r with this property is 0.9708, i.e. for all $0.5 < r < 0.9709$, all the eigenvalues of $A - rBK$ are inside the vertical strip. For all values r outside this interval, the eigenvalues are not inside the vertical strip. The g.u.b.s. $r = 0.9708$ can be obtained by a trial and error method. Take $r = 0.9708$. Then

$$A - rBK = \begin{bmatrix} 2.6611 & -21.9378 & 17.0996 & 40.2100 \\ 0.0173 & -2.6846 & 1.0243 & -4.6368 \\ -0.7766 & 2.8874 & -3.6390 & -12.75559 \\ -0.15449 & 0.7224 & -0.5546 & -3.3803 \end{bmatrix}$$

The eigenvalues of $A - rBK$ are -1.5991, -1.9998, -1.9435, -1.5002 which are in the vertical strip. But for $r = 0.9709$

$$A - rBK = \begin{bmatrix} 2.6614 & -21.9401 & 17.1014 & 40.2141 \\ 0.0173 & -2.6847 & 1.0245 & -4.6375 \\ -0.7767 & 2.8878 & -3.6393 & -12.7572 \\ -0.1544 & 0.7225 & -0.5546 & -3.3806 \end{bmatrix}$$

and the eigenvalues of $A - rBK$ are -1.5986, -2.0007, -1.9437, -1.5002. Some of these eigenvalues lie outside the strip.

The eigenvalues of $A - rBK$ for various r are shown in Table 8.4. When r increases from 0.5, the three eigenvalues of $A - rBK$ move from the vertical line $x = -1$ to the vertical line $x = -2$, and when $r > 0.9708$ then at least one eigenvalues lies outside the open vertical strip, i.e. the real part of this eigenvalue is equal to or less than -2. Therefore, the range of variation of r depends upon the value h_2.

Now let $h_1 = 0.5$, $h_2 = 2$ and $\hat{A} = A + h_1$. It can be shown that 0.5 is an eigenvalue of

$h_1 = 1$ and $h_2 = 2$				
r	Eigenvalues of $A - rBK$			
0.5	$-1.0000 + 0.7742i$	$-1.0000 - 0.7742i$	-1.5002	-1.0000
0.54	$-1.0679 + 0.7712i$	$-1.0679 - 0.7712i$	-1.5002	-1.0801
0.58	$-1.1359 + 0.7621i$	$-1.1359 - 0.7621i$	-1.5002	-1.1602
0.62	$-1.2038 + 0.7469i$	$-1.2038 - 0.7469i$	-1.5002	-1.2404
0.66	$-1.2718 + 0.7249i$	$-1.2718 - 0.7249i$	-1.5002	-1.3205
0.7	$-1.3397 + 0.6956i$	$-1.3397 - 0.6956i$	-1.5002	-1.4006
0,74	$-1.4077 + 0.6581i$	$-1.4077 - 0.6581i$	-1.5002	-1.4807
0.78	$-1.4756 + 0.6108i$	$-1.4756 - 0.6108i$	-1.5002	-1.5609
0.82	$-1.5436 + 0.5512i$	$-1.5436 - 0.5512i$	-1.5002	-1.6410
0.86	$-1.6115 + 0.4748i$	$-1.6115 - 0.4748i$	-1.5002	-1.7211
0.9	$-1.6795 + 0.3710i$	$-1.6795 - 0.3710i$	-1.5002	-1.8012
0.94	$-1.7475 + 0.2018i$	$-1.7475 - 0.2018i$	-1.5002	-1.8812
0.9708	-1.5991	-1.9998	-1.5002	-1.9435
0.9709	-1.5986	-2.0007	-1.5002	-1.9437
0.98	-1.5595	-2.0710	-1.5002	-1.9617

Table 8.4: The eigenvalues of $A - rBK$ for various values of r

$A - rBK$. The semi-p.d.s. solution of ARE (8.45) is

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 23.501070148989 & -20.799096827320 & -31.645246165015 \\ 0 & -20.899096827320 & 18.408231026207 & 28.207531424950 \\ 0 & -31.645246165015 & 28.207531424951 & 131.007541236564 \end{bmatrix}$$

Therefore

$$K = \begin{bmatrix} 0 & 5.4256 & -4.8099 & -10.8818 \\ 0 & 1.5684 & -1.3678 & 6.7999 \end{bmatrix}$$

and the eigenvalues of $A - rBK$ are -0.5000, -1.1985, -1.0017, -1.5002 which are not all in the open vertical strip, i.e. Theorem 1 in [101] is wrong. But our Theorem 8.6.1 gives suitable feedback gain matrices. Let $\epsilon = 0.005$, $\hat{h}_1 = h_1 + \epsilon = 0.505$ and $\hat{A} = A + \hat{h}_1$. Then the semi-p.d.s. solution of ARE (8.45) is

$$P = \begin{bmatrix} 0.028714648294 & -0.167700002142 & 0.132519208919 & 0.335294603867 \\ -0.1677700002142 & 24.655229461686 & -21.727613589579 & -33.800431560986 \\ 0.1322519208919 & -21.727613589579 & 19.156542679854 & 29.931096893038 \\ 0.3355294603867 & -33.801431560986 & 29.931096893038 & 135.992881929296 \end{bmatrix}$$

and

$$K = \begin{bmatrix} -0.0169 & 5.5634 & -4.9225 & -11.1577 \\ -0.0002 & 1.5852 & -1.3824 & 6.8653 \end{bmatrix}$$

Let $h_2 = 1.7002$. Since the eigenvalues of $\hat{A}$ are 0.0050, −0.9952, 0.5067, 0.7035, $\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ = 1.2152$ and an u.b.s. η of r is

$$\eta = 0.5 \left(1 + (h_2 - h_1) / \sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ \right) = 0.9938$$

For $r = 0.5$

$$A - rBK = \begin{bmatrix} -0.4932 & -2.2739 & 1.4473 & 3.5566 \\ 0.0001 & -1.4422 & -0.0506 & 0.2850 \\ -0.0017 & -1.0490 & -0.5734 & 0.2885 \\ -0.0003 & 0.0020 & 0.0007 & -0.5064 \end{bmatrix}$$

with eigenvalues $-0.5050 \pm 0.0595i$, -0.5050 and -1.5002. For $r = \eta = 0.9938$

$$A - rBK = \begin{bmatrix} -0.4865 & -4.6283 & 3.5285 & 7.2865 \\ 0.0001 & -1.8492 & 0.3044 & -1.4779 \\ -0.0033 & -0.7813 & -0.8138 & -2.0340 \\ -0.0007 & 0.0336 & -0.0283 & -1.0659 \end{bmatrix}$$

with eigenvalues −0.5101, −1.1997, −1.5002 and −1.0054. The value $\eta = 0.9938$ is obtained from Theorem 8.6.1. In fact the g.u.b.s. of r is 1.3454 which is obtained by trial and error, since

$$A - 1.3454BK = \begin{bmatrix} -0.4817 & -6.3046 & 5.0102 & 9.9420 \\ 0.0002 & -2.1390 & 0.5571 & -2.7330 \\ -0.0045 & -0.5908 & -0.9850 & -3.6875 \\ -0.0009 & 0.0561 & -0.0489 & -1.4641 \end{bmatrix}$$

with eigenvalues −0.5079, −1.70015, −1.5002, −1.3616. For $r = 1.3455$

$$A - 1.3455BK = \begin{bmatrix} -0.4817 & -6.3050 & 5.0107 & 9.9428 \\ 0.0002 & -2.1391 & 0.5572 & -2.7334 \\ -0.0045 & -0.5907 & -0.9851 & -3.6880 \\ -0.0009 & 0.0561 & -0.0489 & -1.4642 \end{bmatrix}$$

with eigenvalues −0.5079, −1.7003, −1.5002 and −1.3617.

So when r increases from $x = 0.5$ to 1.3454, the dominant eigenvalues move from the vertical line $x = h_1$ to the vertical line $x = h_2$.

Example 8.7.2 [129]: Consider the system (2.4) with

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The orthogonal transformation matrix T (2.13) is taken as

$$T = \begin{bmatrix} -1.0000 & 0 & 0 & 0 & 0 \\ 0 & -0.7071 & -0.5000 & 0.5000 & 0 \\ 0 & -0.7071 & 0.5000 & -0.5000 & 0 \\ 0 & 0 & 0 & 0 & -1.0000 \\ 0 & 0 & -0.7071 & -0.7071 & 0 \end{bmatrix}$$

Then the matrices A_{11} and A_{12} (2.16) are

$$A_{11} = \begin{bmatrix} -1.0000 & 0.7071 & 0.7071 \\ 0 & -0.8964 & -1.1036 \\ 0 & -0.3964 & -1.6036 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 0 \\ -0.5000 & 0.8536 \\ 0.5000 & 0.1464 \end{bmatrix}$$

The eigenvalues of A_{11} are -2, -1, -0.5. Consider the strip $(-h_2, -h_1)$ with $h_1 = 1.5$ and $h_2 = 2.5$. Let $\hat{A}_{11} = A_{11} + h_1 I_3$. The eigenvalues of $\hat{A}_{11}$ are 0.5, 1, -0.5 and $\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ = 1.5$. Thus an u.b.s. for r is

$$\eta = 0.5(1 + (h_2 - h_1)/\sum_{i=1}^{\hat{n}^+} \hat{\lambda}_i^+ = 0.8333$$

By considering $\hat{A}_{11}$ as A, and A_{12} in place of B, the semi-p.d.s. solution P of ARE (8.45) is

$$P = \begin{bmatrix} 2.45457213932604 & 1.46296027139282 & 2.00829564804722 \\ 1.46296027244758 & 2.20516150011468 & -0.13624308281623 \\ 2.00829564697450 & -0.13624308431857 & 2.97637478827414 \end{bmatrix} \qquad (8.52)$$

The eigenvalues of $\hat{A}_{11} - A_{12}R^{-1}A_{12}^T P$ are -0.5, -1. So Lemma 8.5.1 is satisfied and P (8.52) is an acceptable solution of the ARE (8.45). However, the exact solution P of the ARE (8.45) is semi-p.d.s. with a zero eigenvalue. So P (8.52) is an approximation to the correct solution. In fact, some entries of the exact P have an infinite decimal expansion; so some the elements of P are only an approximation to the correct solution. Therefore the gain matrix F obtained from this P is valid. But P in [129, page 52] is incorrect, it does not satisfy Lemma 8.5.1. This error arises form their incorrect theory about finding the solution of the ARE with zero right-hand side. So their matrices F and C are not correct despite the fact that the eigenvalues of $\hat{A}_{11} - A_{12}R^{-1}A_{12}^TP$ lie inside the strip $(-1.5, -2.5)$.

For $r = \eta = 0.8333$, F is given by

$$F = \begin{bmatrix} 0.2272 & -0.9755 & 1.2969 \\ 1.2856 & 1.5519 & 0.2662 \end{bmatrix}$$

and

$$C = \begin{bmatrix} -0.2272 & -0.2272 & 1.1363 & -1.1363 & -1.0000 \\ -1.2857 & -1.2857 & -1.3500 & -0.0642 & 0 \end{bmatrix}$$

which is correct. Let

$$F_1 = R^{-1}A_{12}^TP = \begin{bmatrix} 0.2727 & -1.1707 & 1.5563 \\ 1.5428 & 1.8624 & 0.3194 \end{bmatrix}$$

The eigenvalues of $A_{11} - A_{12}F$ are $-2.0000 \pm 0.1508i$, -2. These eigenvalues are within the vertical strip $(-2.5, -1.5)$. Since two eigenvalues of $\hat{A}_{11}$ are in the right-hand half-plane, the proof of Theorem 8.6.1 implies that the two eigenvalues of $A_{11} - 0.5A_{12}F$ take the value h_1. In fact, the eigenvalues of $A_{11} - rA_{12}F$ are $-1.5000 \pm 0.3016i$, -2. By testing various values of r, the g.u.b.s. r with the desired property can be found. The g.u.b.s. r is 0.99998, because the eigenvalues of $A_{11} - 0.99998A_{12}F$ are $-2, -2.4999$ and the eigenvalues of $A_{11} - 0.99999A_{12}F$ are -2, -2.5. For $r = 0.99998$

$$C = \begin{bmatrix} -0.2727 & -0.2727 & 1.3635 & -1.3635 & -1.0000 \\ -1.5428 & -1.5428 & -1.4786 & 0.0643 & 0 \end{bmatrix}$$

The eigenvalues of $A_{11} - rA_{12}F$ for various r are shown in Table 8.5. When r increases from 0.5, the three eigenvalues of $A_{11} - rA_{12}F$ move from the vertical line $x = -h_1 = -1.5$ to the vertical line $x = h_2 = -2.5$. If $r > 0.99998$ then at least one eigenvalue lies outside the open vertical strip, i.e. the real part is equal to or greater than -2.5. Therefore, the range of variation of r depends upon the value h_2. Now consider $h_1 = 1$ and $h_2 = 2.5$. Since -1 is an eigenvalue of A_{11}, for all values of r one of the eigenvalues of $\hat{A}_{11} - rA_{12}$ is zero. Therefore, the vertical line $x = -h_1$ should be shifted to $x = h_1 + \epsilon$ where ϵ is a small positive real number, say 0.0005. Suppose $\hat{A}_{11} = A_{11} + h_1\epsilon$. The eigenvalues of $\hat{A}_{11}$ are then 0.0005, 0.5005, -0.9995 and $\sum_{i=1}^{\hat{n}+} \hat{\lambda}_i^+ = 0.0505$. So an u.b.s. r of is 1.9970.

$h_1 = 1.5$ and $h_2 = 2.5$			
r	Eigenvalues of $A_{11} - rA_{12}F$		
0.5000	$-1.5000 + 0.3016i$	$-1.5000 - 0.3016i$	-2
0.5400	$-1.5600 + 0.2999i$	$-1.5600 - 0.2999i$	-2
0.5800	$-1.6200 + 0.2950i$	$-1.6200 - 0.2950i$	-2
0.6200	$-1.6800 + 0.2865i$	$-1.6800 - 0.2865i$	-2
0.6600	$-1.7400 + 0.2743i$	$-1.7400 - 0.2743i$	-2
0.7000	$-1.8000 + 0.2577i$	$-1.8000 - 0.2577i$	-2
0.7400	$-1.8600 + 0.2358i$	$-1.8600 - 0.2358i$	-2
0.7800	$-1.9200 + 0.2069i$	$-1.9200 - 0.2069i$	-2
0.8200	$-1.9800 + 0.1676i$	$-1.9800 - 0.1676i$	-2
0.8600	$-2.0400 + 0.1068i$	$-2.0400 - 0.1068i$	-2
0.9000	-2.0148	-2.1852	-2
0.9400	-1.9930	-2.3270	-2
0.9800	-1.9953	-2.4447	-2
0.99998	-2.0000	-2.4999	-2
0.99999	-2.0000	-2.5000	-2
1.0200	-2.0061	-2.5539	-2
1.0600	-2.0213	-2.6587	-2
1.1000	-2.0394	-2.7606	-2
1.1400	-2.0594	-2.8606	-2

Table 8.5: The eigenvalues of $A_{11} - rA_{12}F$ for various values of r

The semi-p.d.s. solution of ARE (8.45) is

$$P = \begin{bmatrix} 0.001002421 12344 & 0.00104171782023 & 0.00037590613253 \\ 0.00104171782023 & 0.66835723935475 & -0.66688404201337 \\ 0.00037590613253 & -0.66688404201337 & 0.66741564846600 \end{bmatrix}$$

Since the eigenvalues of $\hat{A}_{11} - A_{12}R^{-1}A_{12}^TP$ are -0.0005, -0.5005, -0.9995 and the eigenvalues of $\hat{A}_{11}$ are 0.0005, 0.5005, -0.9995, respectively, (8.45) gives the correct solution P and the corresponding F. But P in [129, page 52] is incorrect. The author also mentioned that it is not clear why this solution is not satisfied in the ARE and desired properties. In fact, P in [129] is not satisfied by Lemma 8.5.1 and F obtained from P is wrong.

An u.b.s. of r is 1.9970 because $\sum_{i=1}^{\hat{n}+} \hat{\lambda}_i^+ = 0.5010$. Consider $F = 1.997F_1$ where

$$F_1 = R^{-1}A_{12}^T P = \begin{bmatrix} -0.0003 & -0.6676 & 0.6671 \\ 0.0009 & 0.4729 & -0.4715 \end{bmatrix}$$

Then

$$C = \begin{bmatrix} 0.0007 & 0.0007 & 1.3328 & -1.3328 & -1.0000 \\ -0.0019 & -0.0019 & -1.6501 & 0.2359 & 0 \end{bmatrix}$$

The eigenvalues of $\hat{A}_{11} - 1.997A_{12}R^{-1}A_{12}^T P$ and the eigenvalues of $\hat{A}_{11} - 1.998A_{12}R^{-1}A_{12}^T P$ are -1.0011, -2, -2.4999 and -1.001, -2, -2.5, respectively. Therefore the g.u.v. r is 1.997. Let

$$F_1 = R^{-1}A_{12}^T P = \begin{bmatrix} -0.0003 & -0.6676 & 0.6672 \\ 0.0009 & 0.4729 & -0.4715 \end{bmatrix}$$

The eigenvalues of $A_{11} - rA_{12}F_1$ for various r are shown in Table 8.6. As r increases from 0.5, the eigenvalues $A_{11} - rA_{12}F$ move from the vertical line $x = -h_1 = -1$ to the vertical line $x = h_2 = -2.5$, and if $r > 1.997$ then at least one of them is located outside the open vertical strip, i.e. the real part of this eigenvalue equals or is greater than -2.5. When the ϵ method is used, $r = 0.5$ is also an acceptable value because h_1 is one of the eigenvalues of A_{11} and $-(h_1 + \epsilon)$ is an eigenvalue of $A_{11} - A_{12}F$ which is clearly in the vertical strip $(-h_2, h_2)$. As the value of r increases the eigenvalues corresponding to the eigenvalues of $\hat{A}_1$ which are in the right-hand half-plane, move to different places. These results are shown in Table 8.6.

Example 8.7.3 [129]: Consider the system (2.4) with

$$A = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0 & -0.3320 & 0 & 0.0187 \\ 0 & 0 & 0 & 1.0000 \\ 0 & 0.7830 & 0 & -0.1914 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 130.8 & -308.3 \\ 0 & 0 \\ -308.3 & 3155.4 \end{bmatrix}$$

Assume T is the transformation matrix given by (2.13)

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -0.3906 & 0 & 0.9206 \\ 0 & -0.9206 & 0 & -0.3906 \end{bmatrix}$$

$h_1 = 1$ and $h_2 = 2.5$			
r	The eigenvalues of $A_{11} - rA_{12}F_1$		
0.500	$-1.0005 + 0.0129i$	$-1.0005 - 0.0129i$	−2
0.650	−1.0017	−1.1496	−2
0.800	−1.0012	−1.3004	−2
0.950	−1.0010	−1.4509	−2
1.100	−1.0010	−1.6012	−2
1.250	−1.0010	−1.7515	−2
1.400	−1.0010	−1.9018	−2
1.550	−1.0010	−2.0521	−2
1.700	−1.0010	−2.2024	−2
1.850	−1.0011	−2.3526	−2
1.997	−1.0011	−2.4999	−2
1.998	−1.0011	−2.5009	−2
2.000	−1.0011	−2.5029	−2
2.150	−1.0012	−2.6532	−2
2.300	−1.0012	−2.8034	−2
2.450	−1.0012	−2.9537	−2
2.600	−1.0013	−3.1039	−2
2.750	−1.0013	−3.2542	−2
2.900	−1.0014	−3.4044	−2
3.050	−1.0014	−3.5547	−2
3.200	−1.0015	−3.7050	−2

Table 8.6: The eigenvalues e of $A_{11} - rA_{12}F_1$ for various values of r

Then from (2.16)

$$A_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -0.3906 & -0.9206 \\ 0.9206 & -0.3906 \end{bmatrix}$$

The eigenvalues of A_{11} are a double root at 0. Selecting $h_1 = 2$ and $h_2 = 3$ gives the matrix $\hat{A}_{11} = A_{11} + 2I_2$. The eigenvalues of $\hat{A}_{11}$ are double repeated eigenvalues 2 and $Tr(\hat{A}_{11}^{+}) = 4$. Therefore an u.b.s. for r is

$$\eta = 0.5(1 + (h_2 - h_1)/\sum_{i=1}^{\hat{n}^{+}} \hat{\lambda}_i^{+} = 0.625$$

The p.d.s. solution P of ARE (8.45) with $\hat{A}_{11}$ as $\hat{A}$ and A_{12} in place of B is

$$P = \begin{bmatrix} \frac{13751}{3438} & 0 \\ 0 & \frac{13751}{3438} \end{bmatrix}$$

$$F_1 = R^{-1}A_{12}^T P = \begin{bmatrix} -1.5623 & 3.6821 \\ -3.6821 & -1.5623 \end{bmatrix}$$

It is straightforward to show that this P is an exact solution of (8.45) and satisfies Lemma 8.5.3. The sliding matrix for $r = 0.625$ is

$$C_1 = \begin{bmatrix} 0.625F_1 & I_2 \end{bmatrix} T = \begin{bmatrix} -0.9764 & -0.3906 & 2.3013 & 0.9206 \\ -2.3013 & -0.9206 & -0.9764 & -0.3906 \end{bmatrix}$$

The p.d.s. solution P (8.45) in [129, page 59] is incorrect, because this P does not satisfy Lemma 8.5.3, in spite of the eigenvalues of the closed-loop reduced system lying in the strip. The eigenvalues λ of $A_{11} - rA_{12}F_1$ for various values of r are shown in Table 8.7. The g.u.b.s. r for all the eigenvalues of $A_{11} - rA_{12}F_1$ to lie in the strip is 0.7499. Note that for all r, $A_{11} - rA_{12}F_1$ has only a double eigenvalue.

$h_1 = 2$ and $h_2 = 3$										
r	0.50	0.53	0.56	0.59	0.62	0.625	0.65	0.68	0.71	0.74
λ	−2.00	−2.12	−2.24	−2.36	−2.48	−2.50	−2.60	−2.72	2.84	−2.96
r	0.7499	0.75	0.77	0.80	0.83	0.86	0.89	0.92	0.95	
λ	−2.9996	−3.00	−3.08	−3.20	−3.320	−3.44	−3.56	−3.68	−3.80	

Table 8.7: The eigenvalues λ of $A_{11} - rA_{12}F$ for various values of r

Example 8.7.4: Consider Example 8.4.1. Assume $\alpha = -2$ and $\theta = 30°$. The solution of the Riccati equation (8.31) is

$$P = \begin{bmatrix} 0.0130 & -0.4584 & 0.0039 & -0.2922 \\ -0.4584 & 16.8859 & -0.0896 & 10.6139 \\ 0.0039 & -0.0896 & 0.0084 & -0.0832 \\ -0.2922 & 10.6139 & -0.0832 & 6.7645 \end{bmatrix}$$

and

$$F = \begin{bmatrix} 0.2142 & -6.1348 & 0.2926 & -4.7346 \\ -0.0949 & 2.3032 & -0.1909 & 2.0523 \end{bmatrix}$$

The eigenvalues of $A_{11} - A_{12}F$ are -5.6757, $-4.9732 \pm 0.2462i$, -4.0251 and the sliding matrix is

$$C = \begin{bmatrix} -0.2142 & 4.7346 & 0.2926 & -6.1348 & -1.0000 & 0 \\ 0.0949 & -2.0523 & -0.1909 & 2.3032 & 0 & -1.0000 \end{bmatrix}$$

The method of Section 8.2 is now applied. Let $Q = I_4$. Then $d_s = 0.37211 \times 10^{-6}$. Taking $\hat{P} = d_s I_4$ the eigenvalues of $A_{11} - A_{12}R^{-1}A_{12}^T(P + \hat{P})$ are -5.7336, $-4.9506 \pm 0.3301i$, -4.0129 while the eigenvalues of $A_{11} - A_{12}R^{-1}A_{12}^T P$ are -5.6757, $-4.9732 \pm 0.2462i$, -4.0251. So by using this method various gain matrices can be found such that all eigenvalues of the closed-loop system lie in the specified sector.The eigenvalues of the closed-loop system for $\alpha = -2$ and various θ are shown in Table 8.8.

$\alpha = -2$			
θ	Eigenvalues of $A_{11} - A_{12}F$		
0°	−5.6757	$-4.6889 \pm 0.2462i$	−3.7408
10°	−5.6757	$-4.7757 \pm 0.2462i$	−3.8276
20°	−5.6757	$-4.8681 \pm 0.2462i$	−3.9200
30°	−5.6757	$-4.9732 \pm 0.2462i$	−4.0251
40°	−5.6757	$-5.1021 \pm 0.2462i$	−4.1540
50°	−5.6757	$-5.2757 \pm 0.2462i$	−4.3276
60°	−5.6757	$-5.5417 \pm 0.2462i$	−4.5936
70°	−5.6757	$-6.0417 \pm 0.2462i$	−5.0936
80°	−5.6757	$-7.4813 \pm 0.2462i$	−6.5332

Table 8.8: The eigenvalues of $A_{11} - A_{12}F$

When α decreases, the real parts of the eigenvalues of $A_{11} - A_{12}F$ also decrease progressively at a regular rate. However, for certain θ the imaginary part of the eigenvalues are invariant with respect to variations in α. This result is obtained form the mobilization of Proposition 8.5.1. The variation of the eigenvalues of $A_{11} - A_{12}F$ for $\theta = 80°$ and various values of α are shown in Tables 8.9. The eigenvalues of $A_{11} - A_{12}F$ are far from the vertex α; in fact the real parts of the eigenvalues $A_{11} - A_{12}F$ lie within the range $2\alpha \pm 6$.

Example 8.7.5: Consider Example 8.7.3 again. Choose $\theta = 30°$ and $\alpha = -2.0001$. The p.d.s. solution of ARE (8.31) is

$$P = \begin{bmatrix} 5.55861265813242 & 2.81835724537085 & 5.04106085141026 \\ 2.81835724536768 & 3.42901981419127 & 0.55622119703102 \\ 5.04106085141342 & 0.55622119703550 & 6.57230440878359 \end{bmatrix}$$

which gives

$$F = \begin{bmatrix} 1.1114 & -1.4364 & 3.0080 \\ 3.1438 & 3.0084 & 1.4370 \end{bmatrix}$$

The eigenvalues of $A_{11} - A_{12}F$ are -2.0002, -3.0002, -3.5002 and the sliding matrix is

$$C = \begin{bmatrix} -1.1114 & -1.1113 & 2.2222 & -2.2222 & -1.0000 \\ -3.1438 & -3.1434 & -1.4928 & 0.0786 & 0 \end{bmatrix}$$

Since all the eigenvalues of A_{11} are real, $\eta = 0$, i.e. for all values $0° \leqslant \theta < 90°$ the solution P of the ARE (8.31) is invariant, which causes the gain matrix F and the sliding matrix C to be invariant for all values of θ. However, all the eigenvalues lie inside the sector.

Assume $\alpha = -2$. Since -2 is an eigenvalue of $\hat{A}_{11}$, it is also an eigenvalue of the closed-loop matrix $A_{11} - A_{12}F$ which is on the boundary of the sector. More precisely it is the vertex of the sector. In this case let ϵ be a small positive real number and consider $\hat{A}_{11} = A_{11} + (\alpha + \epsilon)I_2$. In fact, by using this method the sector is shifted to a new sector with vertex $\alpha + \epsilon$ and boundary lines parallel with the boundary lines of the previous sector. For example, take $\epsilon = 0.0001$ and $\alpha = -2$.

Now consider the method for placing the poles of the closed-loop system within the intersection of two sectors as stated in Section 8.5.1. Take $\alpha = -2$, $\theta = 30°$, $\beta = -4$ and $\phi = 45°$. Consider a new sector with vertex $\eta = -4$ and angle with imaginary axis $\hat{\theta} = 45°$. This sector is inside the intersection of the two sectors. The semi-p.d.s. solution of ARE (8.31) is

$$P = \begin{bmatrix} 347.279364009904 & 43.769241278813 & 54.454529393184 \\ 43.769241278813 & 11.859265309981 & 6.863925898970 \\ 54.454529393184 & 6.863925898970 & 26.196398665093 \end{bmatrix}$$

and then

$$F = \begin{bmatrix} 5.3426 & -2.4977 & 9.6662 \\ 45.3336 & 11.1279 & 9.6942 \end{bmatrix}$$

The eigenvalues of $A_{11} - A_{12}F$ are -6, -7, -7.5 and the sliding matrix for the full state is found to be

$$C = \begin{bmatrix} -5.3426 & -5.0689 & 6.0820 & -6.0820 & -1.0000 \\ -45.3336 & -14.7235 & -1.4240 & 0.0098 & 0 \end{bmatrix}$$

Example 8.7.6 Now consider Example 8.7.3 again. Choose $\alpha = -2$. Since zero is a repeated eigenvalue of A, 2α is an eigenvalue of $A_{11} - A_{12}F$ and Proposition 8.5.1 gives a single invariant gain matrix for all θ. The solution of ARE (8.31) is

$$P = \begin{bmatrix} 3.999709 & 0 \\ 0 & 3.999709 \end{bmatrix}$$

and then

$$F = \begin{bmatrix} -1.5623 & 3.6821 \\ -3.6821 & -1.5623 \end{bmatrix}$$

The poles of the closed-loop reduced system are the eigenvalues of $A_{11} - A_{12}F$ which are a double root at -4. After transforming to the full state space, the sliding matrix is found to be

$$C = \begin{bmatrix} -1.5623 & -0.3906 & 3.6821 & 0.9206 \\ -3.6821 & -0.9206 & -1.5623 & -0.3906 \end{bmatrix}$$

Now consider the method for placing the poles of the closed-loop system within the intersection of two sectors. Take $\alpha = -2$, $\theta = 30°$, $\beta = -4$ and $\phi = 45°$. Consider a new sector with vertex $\eta = -4$ and angle $\hat{\theta} = 45°$. This sector is inside the intersection of two sectors. A_{11} has double eigenvalues 0. The double eigenvalue of $A_{11} - A_{12}F$ is -8. To find F and then C, the semi-p.d.s. solution of ARE (8.31) should first be obtained. So

$$P = \begin{bmatrix} 7.9994183 & 0 \\ 0 & 7.9994183 \end{bmatrix}$$

and then

$$F = \begin{bmatrix} -3.1246 & 7.3643 \\ -7.3643 & -3.1246 \end{bmatrix}$$

The eigenvalues of $A_{11} - A_{12}F$ are a double root at -8. After transforming to the full state space, the sliding matrix is found to be

$$C = \begin{bmatrix} -3.1246 & -0.3906 & 7.3643 & 0.9206 \\ -7.3643 & -0.9206 & -3.1246 & -0.3906 \end{bmatrix}$$

$\theta = 80°$			
α	Eigenvalues of $A_{11} - A_{12}F$		
0	−5.6757	$-3.4813 \pm 0.2462i$	−2.5332
−0.5	−5.6757	$-4.4813 \pm 0.2462i$	−3.5332
−1	−5.6757	$-5.4813 \pm 0.2462i$	−4.5332
−1.5	−5.6757	$-6.4813 \pm 0.2462i$	−5.5332
−2	−5.6757	$-7.4813 \pm 0.2462i$	−6.5332
−2.5	−5.6757	$-8.4813 \pm 0.2462i$	−7.5332
−3	−5.6757	$-9.4813 \pm 0.2462i$	−8.5332
−3.5	−5.6757	$-10.4813 \pm 0.2462i$	−9.5332
−4	−5.6757	$-11.4813 \pm 0.2462i$	−10.5332
−4.5	−6.1167	$-12.4813 \pm 0.2462i$	−11.5332
−5	−7.1167	$-13.4813 \pm 0.2462i$	−12.5332
−5.5	−8.1167	$-14.4813 \pm 0.2462i$	−13.5332
−6	−9.1167	$-15.4813 \pm 0.2462i$	−14.5332
−6.5	−10.1167	$-16.4813 \pm 0.2462i$	−15.5332
−7	−11.1167	$-17.4813 \pm 0.2462i$	−16.5332
−7.5	−12.1167	$-18.4813 \pm 0.2462i$	−17.5332
−8	−13.1167	$-19.4813 \pm 0.2462i$	−18.5332
−8.5	−14.1167	$-20.4813 \pm 0.2462i$	−19.5332
−9	−15.1167	$-21.4813 \pm 0.2462i$	−20.5332
−9.5	−16.1167	$-22.4813 \pm 0.2462i$	−21.5332
−10	−17.1167	$-23.4813 \pm 0.2462i$	−22.5332
−10.5	−18.1167	$-24.4813 \pm 0.2462i$	−23.5332
−11	−19.1167	$-25.4813 \pm 0.2462i$	−24.5332
−11.5	−20.1167	$-26.4813 \pm 0.2462i$	−25.5332
−12	−21.1167	$-27.4813 \pm 0.2462i$	−26.5332
−12.5	−22.1167	$-28.4813 \pm 0.2462i$	−27.5332
−13	−23.1167	$-29.4813 \pm 0.2462i$	−28.5332
−13.5	−24.1167	$-30.4813 \pm 0.2462i$	−29.5332
−14	−25.1167	$-31.4813 \pm 0.2462i$	−30.5332
−14.5	−26.1167	$-32.4813 \pm 0.2462i$	−31.5332
−15	−27.1167	$-33.4813 \pm 0.2462i$	−32.5332

Table 8.9: The eigenvalues of $A_{11} - A_{12}F$ for various values of α

$\theta = 80°$			
α	Eigenvalues of $A_{11} - A_{12}F$		
−15.5	−28.1167	$-34.4813 \pm 0.2462i$	−33.5332
−16	−29.1167	$-35.4813 \pm 0.2462i$	−34.5332
−16.5	−30.1167	$-36.4813 \pm 0.2462i$	−35.5332
−17	−31.1167	$-37.4813 \pm 0.2462i$	−36.5332
−17.5	−32.1167	$-38.4813 \pm 0.2462i$	−37.5332
−18	−33.1167	$-39.4813 \pm 0.2462i$	−38.5332
−18.5	−34.1167	$-40.4813 \pm 0.2462i$	−39.5332
−19	−35.1167	$-41.4813 \pm 0.2462i$	−40.5332
−19.5	−36.1167	$-42.4813 \pm 0.2462i$	−41.5332
−20	−37.1167	$-43.4813 \pm 0.2462i$	−42.5332
−20.5	−38.1167	$-44.4813 \pm 0.2462i$	−43.5332
−21	−39.1167	$-45.4813 \pm 0.2462i$	−44.5332
−21.5	−40.1167	$-46.4813 \pm 0.2462i$	−45.5332
−22.5	−42.1167	$-48.4813 \pm 0.2462i$	−47.5332
−23	−43.1167	$-49.4813 \pm 0.2462i$	−48.5332
−23.5	−44.1167	$-50.4813 \pm 0.2462i$	−49.5332
−24	−45.1167	$-51.4813 \pm 0.2462i$	−50.5332
−24.5	−46.1167	$-52.4813 \pm 0.2462i$	−51.5332
−25	−47.1167	$-53.4813 \pm 0.2462i$	−52.5332
−25.5	−48.1167	$-54.4813 \pm 0.2462i$	−53.5332
−26	−49.1167	$-55.4813 \pm 0.2462i$	−54.5332
−26.5	−50.1167	$-56.4813 \pm 0.2462i$	−55.5332
−27	−51.1167	$-57.4813 \pm 0.2462i$	−56.5332
−27.5	−52.1167	$-58.4813 \pm 0.2462i$	−57.5332
−28	−53.1167	$-59.4813 \pm 0.2462i$	−58.5332
−28.5	−54.1167	$-60.4813 \pm 0.2462i$	−59.5332
−29	−55.1167	$-61.4813 \pm 0.2462i$	−60.5332
−29.5	−56.1167	$-62.4813 \pm 0.2462i$	−61.5332
−30	−57.1167	$-63.4813 \pm 0.2462i$	−62.5332

Table 8.9 (contd) The eigenvalues of $A_{11} - A_{12}F$ for various values of α

8.8 Summary and Discussion

In this chapter the Lyapunov and Riccati equations, and stability properties of complex and real systems have been considered. Several new methods have been proposed to find feedback matrices such that all the eigenvalues of $A_{11} - A_{12}F$ lie in a specified sector or vertical strip. These techniques have been extended to the hyperbola and the region between two specified sectors. The errors and inaccuracies in [101] have been clarified, and a new theorem and its proof have been presented to yield all the eigenvalues of the closed-loop system to lie in a specified vertical strip in the left-hand half-plane.

All these methods are based on properties of the Riccati equation. The ARE with zero right-hand side has a semi-p.d.s. solution matrix, and only if the state matrix is completely unstable does this ARE have a p.d.s. solution. When the matrix is stable, the semi-p.d.s. solution is the zero matrix. All these facts have been proved and illustrated by examples. Further research is needed to obtain conditions for all the eigenvalues to lie in a specified region by using the CARE. More details of the CARE will be presented in Chapter 9.

The work by Woodham [129] also has some error and inaccuracies, which arise from applying the Shieh et al [101] method. Shieh et al [101] believed that the ARE with zero right-hand has a p.d.s. solution matrix but this is generally not true. The methods in [129] and [131] fail because, for some sectors, all the eigenvalues obtained by these algorithms may not lie in the specified sector. However, the associated CARE has a p.d. Hermitian solution. So, if all the eigenvalues of the closed-loop system are in the specified sector, the associated CARE has a p.d. Hermitian solution, but the converse is always not true. Another reason for the failure is that they considered pure complex weighting matrices (i.e. for all $\theta \neq 0$ the weighting matrix is not real) and finally apply *real* weighting matrices. They also consider the absolute values of a matrix while the absolute value of a p.d.s matrix is not necessarily a p.d.s. matrix. Note that the positivity definition of a matrix is not related to the positivity of all the elements of the matrix.

It should be emphasized that Woodham [129] stated that her feedback gain matrix seems to work in some cases. She did not specify for which systems the feedback matrix is valid and indicated that more work was needed. The definition of a p.d. matrix in the statistics literature is equivalent to the p.d. of the elements, but this definition is not valid in control theory.

Many examples have been presented to illustrate the results. Some conditions are necessary and sufficient, others only sufficient. Further work is needed to obtain conditions which are weaker than those stated in this chapter. Using the transfer function of the closed-loop system may prove fruitful.

Chapter 9

Matrix Complex Vector Space and Linear Complex Systems

The real algebraic Riccati equation is important because system stabilization can be achieved by using a suitable feedback gain corresponding to its solution. Also H_∞ theory extensively uses solutions arising from the Riccati equation. Many methods have been proposed for obtaining the solution of this equation. The Schur method using an associated Hamiltonian matrix has been proposed by Laub [77]. Another method, the so-called Macfarlane-Potter-Fath method (Kailath [63]), uses eigenvalue decomposition of an associated Hamiltonian matrix.

If the weighting matrix of the right hand-side of the ARE is zero as in (8.25), the Hamiltonian matrix may not have sufficient eigenvalues with negative real parts, because the eigenvalues of the Hamiltonian matrix are the eigenvalues of matrices A and $-A^T$. In this case, the eigenvalues of the closed-loop matrix are the eigenvalues of A and $-A$ which lie in the left-hand closed half-plane. The solution of this ARE cannot be obtained from the Hamiltonian matrix. In fact the matrix which can be used for obtaining the solution is not invertible (see Appendix C). This solution is semi-p.d. and the number of zero eigenvalues of the solution matrix is the same as the number of eigenvalues of A which are in the left-hand closed half-plane. The nature of the solution of this Riccati equation has been studied in Chapter 8. Byers's solution method [17] is only for SISO problems. The more general MIMO case still remains unsolved.

As seen in Chapter 8, the complex Riccati equation appears in many control problems. However, an algorithm for finding the solution of the complex Riccati equation has not been considered in the established literature. In this chapter the real vector space of complex square matrices is studied and the positivity concept of a matrix is extended to the complex case. In Section 9.1 the relationship between the real vector spaces of complex and real square matrices is clarified. This relationship yields links between the real and complex Riccati equations. In Section 9.2 the concept of the p.d. complex matrix is defined and the concept of positive definiteness is extended to the complex case. Some theorems yield the necessary and sufficient conditions for positivity of a complex matrix. In Section 9.3 the complex Riccati equation and some related aspects are considered. Complex systems and their application are considered in Section 9.4.

In this chapter $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ denote the real vector space of real $n \times n$ matrices and the real vector space of complex $n \times n$ matrices, respectively.

9.1 Complex Matrix Vector Space

Some basic concepts about the real vector spaces of real and complex square matrices are reviewed in this section. The relationship between real vector spaces of real and complex square matrices is also considered. In fact there is an isomorphism from $M_n(\mathbb{C})$ onto a subspace of $M_{2n}(\mathbb{R})$. Note that the dimensions of $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ over the field $\mathbb{R}$ are n^2 and $2n^2$, respectively.

Let $M \in M_n(\mathbb{C})$, then $M = A + iB$ where $A, B \in M_n(\mathbb{R})$. Then, for any $z = x + iy \in \mathbb{C}^n$

$$\begin{aligned} Mz &= (A + iB)(x + iy) \\ &= (Ax - By) + i(Ay + Bx) \end{aligned} \tag{9.1}$$

Assuming $Mz = \hat{x} + i\hat{y}$, then

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{9.2}$$

Equation (9.2) shows that the map $\Phi : \mathbb{C}^n \longrightarrow \mathbb{R}^{2n}$ defined by

$$\Phi(z_1, z_2, \dots, z_n) = (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n)$$

where $z_j = x_j + iy_j$ for $1 \leqslant j \leqslant n$, is an isomorphism of $\mathbb{R}$-vector spaces.

Define the operator $J : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ with

$$J(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n) = (-y_1, -y_2, \dots, -y_n, x_1, x_2, \dots, x_n)$$

Thus the operator J is

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \in M_{2n}(\mathbb{R})$$

Suppose

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A, B, C, D \in M_n(\mathbb{R})$. Then $MJ = JM$ if and only if $A = D$ and $B = -C$. Let $N : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ and $M : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be linear operators related by $\Phi M = N\Phi$. Then M is linear if and only if $MJ = JM$. On the other hand, for any $M \in M_{2n}(\mathbb{R})$, M is in the form of $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ if and only if $MJ = JM$. Define

$$\begin{aligned} \hat{M}(\mathbb{C}) &= \{M \in M_{2n}(\mathbb{R}) : MJ = JM\} \\ &= \left\{ \begin{bmatrix} A & -B \\ B & A \end{bmatrix} : A, B \in M_n(\mathbb{R}) \right\} \end{aligned} \tag{9.3}$$

Therefore, the following theorem is well established.

Theorem 9.1.1 *Let $\psi : M_n(\mathbb{C}) \longrightarrow \hat{M}_n(\mathbb{C})$ be a map which is defined by*

$$A + iB \longrightarrow \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

ψ is an isomorphism in the sense of rings and $\mathbb{R}$-vector spaces.

Proof: The proof is straightforwardly achieved by checking the properties of the isomorphism. □

Since the complex vector space $M_n(\mathbb{C})$ on the field $\mathbb{R}$ has properties corresponding to the $\mathbb{R}$-vector space $\hat{M}_n(\mathbb{C})$, this theorem is very important. For example a complex matrix is invertible if and only if its corresponding real matrix is invertible.

9.2 Positivity of a Complex Matrix

In this section the positivity concept of a complex matrix, necessary and sufficient conditions for positivity, and some related results are presented. The positivity concept is defined to include the traditional definition for a real matrix. In fact the real matrix is a particular case of a complex matrix with zero complex parts.

Definition 9.2.1 A complex matrix $A = A_1 + iA_2 \in M_n(\mathbb{C})$ is called p.d. if for all complex vectors $x \neq 0$, $\Re(x^*Ax) > 0$, i.e. $A = A_1 + iA_2$ is p.d. if for all $x = x_1 + ix_2 \in \mathbb{C}^n$

$$x_1^T A_1 x_1 - x_1^T A_2 x_2 + x_2^T A_2 x_1 + x_2^T A_1 x_2 > 0 \tag{9.4}$$

If $A = A_1 + iA_2$ is a complex matrix and $x \in \mathbb{R}^n$, then $\Re(x^*Ax) = x^T A_1 x$. When A is a real matrix and $x \in \mathbb{R}^n$, then $x^*Ax = x^TAx$ and the definitions coincide. Also, if A is an Hermitian matrix, then for all $x \in \mathbb{C}^n$, $(x^*Ax)^* = x^*Ax$ and $(x^*Ax)^* \in \mathbb{R}$. Since the eigenvalues of an Hermitian matrix are real, the eigenvalues of a p.d. Hermitian matrix are positive real [42, page 105]. It is necessary to deal with the general case including non-Hermitian matrices.

Lemma 9.2.1: *Let $A = A_1 + iA_2$ be a complex matrix and $x = x_1 + ix_2 \in \mathbb{C}^n$, then*

$$\Re(x^*Ax) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{9.5}$$

Proof:

$$\begin{aligned} x^*Ax &= ({x_1}^T - ix_2^T)(A_1 + iA_2)(x_1 + ix_2) \\ &= x_1^T A_1 x_1 - x_1^T A_2 x_2 + x_2^T A_2 x_1 + x_2^T A_1 x_2 + \\ &\qquad i(x_1^T A_2 x_1 + x_1^T A_1 x_2 - x_2^T A_1 x_1 + x_2^T A_2 x_2) \end{aligned} \tag{9.6}$$

Hence

$$\begin{aligned} \Re(x^*Ax) &= x_1^T A_1 x_1 - x_1^T A_2 x_2 + x_2^T A_2 x_1 + x_2^T A_1 x_2 \\ &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

□

Lemma 9.2.2: *Let $A = A_1 + iA_2 \in M_n(\mathbb{C})$ be a complex matrix. A necessary and sufficient condition that A is p.d. is that the matrix*

$$\begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix}$$

is a p.d. real matrix.

Proof: Let A be a p.d. complex matrix and

$$\tilde{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^{2n\times 1}, \quad x_1, x_2 \in \mathbb{R}^{n\times 1}$$

be an arbitrary vector. Let $x = x_1 + ix_2$. Then, by using Lemma 9.2.1, $\Re(x^*Ax) > 0$ if and only if

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0$$

Lemma 9.2.2 shows that the positivity of complex matrix is converted to the positivity of a corresponding real matrix.

Lemma 9.2.3: *Let $A = A_1 + iA_2 \in M_n(\mathbb{C})$ be a matrix. A necessary and sufficient condition that A is an Hermitian matrix is that the matrix*

$$\begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix}$$

is a symmetric real matrix.

Lemma 9.2.4: *Let $A = A_1 + iA_2 \in M_n(\mathbb{C})$ be a complex matrix. A necessary and sufficient condition that A is stable, is that the matrix*

$$\tilde{A} = \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix}$$

is stable.

Proof: If λ_k, $k = 1, 2, \ldots, n$, are the eigenvalues of the complex matrix A, then the eigenvalues of $\tilde{A}$ are λ_k and its conjugates $\bar{\lambda}_k$. □

Lemma 9.2.4 implies that all the eigenvalues of the complex matrix $A = A_1 + iA_2$ lie in a specified region symmetric with respect to the real axis, if and only if all the eigenvalues of its corresponding real matrix $\tilde{A}$ lie in this region.

Lemma 9.2.5: *Let $A_1, A_2 \in M_n(\mathbb{R})$ with $|A_1| \neq 0$ and $|A_1 + A_2 A_1{}^{-1} A_2| \neq 0$. Then*

$$(A_1 + A_2 A_1{}^{-1} A_2)^{-1} A_2 A_1{}^{-1} = A_1{}^{-1} A_2 (A_1 + A_2 A_1{}^{-1} A_2)^{-1}$$

Proof:

$$\begin{aligned}(A_1 + A_2 A_1{}^{-1} A_2) A_1{}^{-1} A_2 &= A_2 + A_2 A_1{}^{-1} A_2 A^{-1} A_2 \\ &= A_2 A_1{}^{-1} (A_1 + A_2 A_1{}^{-1} A_2)\end{aligned}$$

□

The following theorem is obtained directly from the above Lemma.

Theorem 9.2.1: *Assume $A = A_1 + iA_2 \in M_n(\mathbb{C})$ is an invertible complex matrix with $|A_1| \neq 0$ and $|A_1 + A_2 A_1{}^{-1} A_2| \neq 0$. Then*

$$A^{-1} = (I_n - iA_1{}^{-1} A_2)(A_1 + A_2 A_1{}^{-1} A_2)^{-1}$$

Corollary 9.2.1: *Assume $A = A_1 + iA_2 \in M_n(\mathbb{C})$. A sufficient condition for the invertibility of the matrix A is that $|A_1| \neq 0$ and $|A_1 + A_2 A_1{}^{-1} A_2| \neq 0$. This condition is not necessary.*

Proof: The proof results immediately from Theorem 9.2.1. This condition is not necessary. Note that $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ has an inverse but the conditions of Corollary 9.2.1 are not satisfied. □

If $A \in \hat{M}(\mathbb{C})$, $|A_1| \neq 0$ and $|A_1 + A_2 A_1{}^{-1} A_2| \neq 0$. Corollary 9.2.1 and Theorem 9.2.1 yield

$$\begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix}^{-1} = \begin{bmatrix} (A_1 + A_2 A_1{}^{-1} A_2)^{-1} & A_1{}^{-1} A_2 (A_1 + A_2 A_1{}^{-1} A_2)^{-1} \\ -A_1{}^{-1} A_2 (A_1 + A_2 A_1{}^{-1} A_2)^{-1} & (A_1 + A_2 A_1{}^{-1} A_2)^{-1} \end{bmatrix}$$

9.3 Complex Algebraic Riccati Equation (CARE)

In Section 9.1 some properties of complex matrices and relationships between the real vector spaces of real and complex square matrices have been stated. In this section an important problem, i.e. the complex Riccati equation is considered. The complex Riccati equation is related to its real counterpart. This correspondence characterizes a way to convert the complex problem to the real matrix case.

Consider the complex Riccati equation

$$A^* P + PA - PBR^{-1} B^T P = -Q \tag{9.7}$$

where A, B are $n \times n$ and $n \times m$ complex matrices. Also R, Q are arbitrary p.d. and semi-p.d. Hermitian $m \times m$ and $n \times n$ complex matrices, respectively. Therefore $A = A_1 + iA_2$, $B = B_1 + iB_2$, $R = R_1 + iR_2$ and $Q = Q_1 + iQ_2$ where R_1, Q_1 are symmetric matrices and R_2, Q_2 are skew-symmetric matrices. Now it is desired to to find a p.d. Hermitian matrix solution $P = P_1 + iP_2$ such that

$$(P_1 + iP_2)(A_1 + iA_2) + (A_1^T - iA_2^T)(P_1 + iP_2) - (P_1 + iP_2)(B_1 + iB_2)$$
$$(R_1 + iR_2)^{-1}(B_1 + iB_2)^T(P_1 + iP_2) = -(Q_1 + iQ_2) \tag{9.8}$$

Since R is p.d.s matrix, $R_1 + R_2{R_1}^{-1}R_2 > 0$ and $R_1 > 0$. Therefore, $|R_1| \neq 0$ and $|R_1 + R_2{R_1}^{-1}R_2| \neq 0$. Thus, from Theorem 9.2.1 the inverse of R exists and $R^{-1} = \Delta^{-1} + i\Lambda$, where $\Delta = R_1 + R_2{R_1}^{-1}R_2$ and $\Lambda = -{R_1}^{-1}R_2(R_1 + R_2{R_1}^{-1}R_2)^{-1}$. Hence

$$\{P_1A_1 - P_2A_2 + A_1^TP_1 + A_2^TP_2 - ((P_1B_1 - P_2B_2)\Delta^{-1} - (P_2B_1 + P_1B_2)\,\Lambda)$$
$$(B_1^TP_1 - B_2^TP_2) + ((P_2B_1 + P_1B_2)\Delta^{-1} + (P_1B_1 - P_2B_2)\Lambda)(B_2^TP_1 + B_1^TP_2)\}$$

$$+i\,\{P_2A_1 + P_1A_2 + A_1^TP_2 - A_2^TP_1 - ((P_2B_1 + P_1B_2)\Delta^{-1} + (P_1B_1 - P_2B_2)\Lambda)$$
$$(B_1^TP_1 + B_2^TP_2) - ((P_1B_1 - P_2B_2)\Delta^{-1} - (P_2B_1 + P_1B_2)\Lambda)\,(B_2^TP_1 + B_1^TP_2)\}$$
$$= -(Q_1 + iQ_2) \tag{9.9}$$

Therefore

$$P_1A_1 - P_2A_2 + A_1^TP_1 + A_2^TP_2 - ((P_1B_1 - P_2B_2)\Delta^{-1} - (P_2B_1 + P_1B_2)\Lambda)$$
$$(B_1^TP_1 + B_2^TP_2) + ((P_2B_1 + P_1B_2)\Delta^{-1} + (P_1B_1 - P_2B_2)\,\Lambda)$$
$$(B_2^TP_1 + B_1^TP_2) = -Q_1 \tag{9.10}$$

and

$$P_2A_1 + P_1A_2 + A_1^TP_2 - A_2^TP_1 - ((P_2B_1 + P_1B_2)\Delta^{-1} + (P_1B_1 - P_2B_2)\Lambda)$$
$$(B_1^TP_1 - B_2^TP_2) + ((P_1B_1 - P_2B_2)\Delta^{-1} - (P_2B_1 + P_1B_2)\Lambda)$$
$$(B_2^TP_1 + B_1^TP_2) = -Q_2 \tag{9.11}$$

Consider the Riccati equation

$$\tilde{P}\tilde{A} + \tilde{A}^T\tilde{P} - \tilde{P}\tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{P} = \tilde{Q} \tag{9.12}$$

where

$$\tilde{A} = \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1 & -B_2 \\ B_2 & B_1 \end{bmatrix}, \quad \tilde{P} = \begin{bmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} Q_1 & -Q_2 \\ Q_2 & Q_1 \end{bmatrix}$$

and

$$\tilde{R}^{-1} = \begin{bmatrix} \Delta^{-1} & -\Lambda \\ \Lambda & \Delta^{-1} \end{bmatrix}$$

If $R_2 = 0$, $R = R_1$ and $\Lambda = 0$

$$\tilde{R}^{-1} = \begin{bmatrix} R^{-1} & 0 \\ 0 & R^{-1} \end{bmatrix}$$

The equation (9.12) includes both the equations (9.10) and (9.11). This implies the following theorem:

Theorem 9.3.1: *The p.d. Hermitian matrix $P = P_1 + iP_2$ is a solution of the Riccati equation (9.7) if and only if the p.d.s. matrix*

$$\tilde{P} = \begin{bmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{bmatrix}$$

is a solution of (9.12). $\tilde{P}$ is unique if and only if the matrix P is unique.

Theorem 9.3.2: *Assume that P and $\tilde{P}$ are the p.d. matrix solution of equations (9.7) and (9.15), respectively. $A - BR^{-1}B^TP$ is stable if and only if $\tilde{A} - \tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{P}$ is stable.*

Proof: $\tilde{A} - \tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{P}$ is stable if and only if the Riccati equation (9.12) has an u.p.d.s. matrix solution. But Theorem 9.3.1 implies that $\tilde{P}$ is the u.p.d.s. matrix solution of (9.12) if and only if P is the u.p.d. Hermitian matrix solution of (9.7). This is equivalent to the stability of the matrix $A - BR^{-1}B^TP$. □

Corollary 9.3.1: *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A - BR^{-1}B^TP$. Then the eigenvalues of $\tilde{A} - \tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{P}$ are $\lambda_1, \lambda_2, \ldots, \lambda_n, \bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n$.*

Proof: The proof follows immediately from Lemma 9.2.4. □

From Corollary 9.3.1 the eigenvalues of $\tilde{A} - \tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{P}$ are $\lambda_1, \lambda_2, \ldots, \lambda_n, \bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n$. So if $\tilde{A} - \tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{P}$ has a real eigenvalue, this eigenvalue appears twice in the spectrum. The n eigenvalues of $A - BR^{-1}B^TP$ are contained in the eigenvalue set of $\tilde{A} - \tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{P}$ and if λ_i is an pure complex eigenvalue (i.e. $\lambda_i \notin \mathbb{R}$) of $A - BR^{-1}B^TP$, then $\bar{\lambda}_i$ may not be an eigenvalue of $A - BR^{-1}B^TP$. However, all the eigenvalues of $A - BR^{-1}B^TP$ and $\tilde{A} - \tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{P}$ lie in the same region.

9.4 Complex Systems

When the terms "complex system" or "system with complex state and control matrices" are used, it may seem that the systems under consideration are not directly real life problems. But as stated in Section 9.1, there is an isomorphism from the $\mathbb{R}$-linear vector space of complex matrices onto a subspace of the $\mathbb{R}$-linear vector space of real matrices which gives information about the corresponding system with complex states in the real world. In this section an example of this type of system model is presented. Complex systems (with complex state and control matrices) have been studied in a few papers (Martin [83], Hazewinkel and Martin [56], Byrnes et al [18]). Consider the system

$$\dot{x} = Ax + Bu \tag{9.13}$$

where $A = A_1 + iA_2 \in \mathbb{C}^{n\times n}$, $B = B_1 + iB_2 \in \mathbb{C}^{n\times m}$, $u \in \mathbb{C}^m$, $x \in \mathbb{C}^n$. Assume $x = x_1 + ix_2$ and $u = u_1 + iu_2$ where $x_1,\ x_2 \in \mathbb{R}^n$, $u_1,\ u_2 \in \mathbb{R}^m$. The matrices $\tilde{A},\ \tilde{B}$ are defined as in (9.12) and

$$\tilde{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \tilde{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Then the complex system (9.13) is equivalent to the real system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & -B_2 \\ B_2 & B_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{9.14}$$

Therefore, any complex system of order n is equivalent to a $2n$-th order real system. Theorem 9.3.2 gives the relationship between the stability of a complex system and appropriate real system. If $\tilde{\mathscr{F}} = \begin{bmatrix} \mathscr{F}_1 & -\mathscr{F}_2 \\ \mathscr{F}_2 & \mathscr{F}_1 \end{bmatrix}$ is a feedback for the system (9.14), then $\mathscr{F} = \mathscr{F}_1 + i\mathscr{F}_2$ is a feedback for the complex system (9.13). When $\tilde{A} + \tilde{B}\tilde{\mathscr{F}}$ is a stable matrix, $A + B\mathscr{F}$ is also stable. In fact the poles of the system (9.13) lie within the same region as the poles of the real system (9.14). Therefore an optimal feedback for the real system (9.14) is $\tilde{u} = \tilde{\mathscr{F}}\tilde{x}$ with

$$\tilde{\mathscr{F}} = \begin{bmatrix} \mathscr{F}_1 & -\mathscr{F}_2 \\ \mathscr{F}_2 & \mathscr{F}_1 \end{bmatrix}$$

where

$$\begin{aligned} \mathscr{F}_1 &= -\Delta^{-1}B_1^T P_1 + \Delta^{-1}B_2^T P_2 + \Lambda B_2^T P_1 + \Lambda B_1^T P_2 \\ \mathscr{F}_2 &= -\Delta^{-1}B_1^T P_2 - \Delta^{-1}B_2^T P_1 + \Lambda B_2^T P_2 - \Lambda B_1^T P_1 \end{aligned}$$

with the same Λ, Δ and P as in Section 9.3.

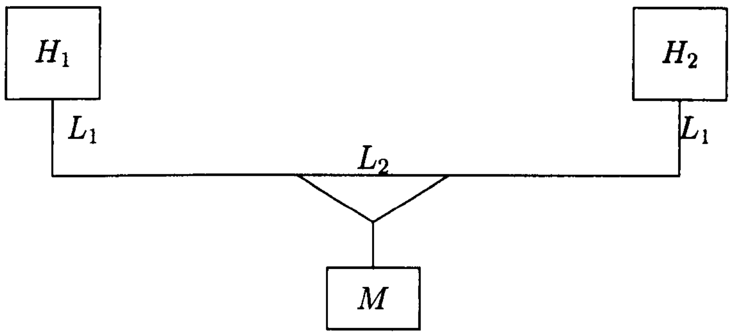

Figure 9.1: Two helicopters with a rigid bar

9.4.1 Example of a Complex System Model

The twin lift helicopter system (Martin [83], Hazewinkel and Martin [56], Byrnes et al [18]) is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{9.15}$$

It has been shown by Wang et al [124] that if (A_1, B) is controllable and

$$\tilde{A} = \begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$$

$(\tilde{A},\tilde{B})$ may not be controllable. Therefore, this system cannot be asymptotically stabilized by local state feedback. The system (9.15) is equivalent to the complex system

$$\dot{x} = (A_1 + iA_2)x + Bu \tag{9.16}$$

where $x = x_1 + ix_2$ and $u = u_1 + iu_2$. The system (9.16) (and also (9.15)) is controllable if and only if $(A_1 + iA_2, B)$ is controllable.

The moving of large loads with helicopters is important in commercial and military operations. The Sikorski CH−53E is a large helicopter able to carry a payload of approximately 40,000 lbs. Consider using two helicopters for carrying large loads; this is known as twin lift [83]. Assume two helicopters H_1, H_2 are carrying a rigid bar attached by cables such that the mass, M, is in the centre of the cable as is shown in Fig. 9.1. The system is modelled by linear dynamics of the form (9.15) where the helicopter dynamics are modelled by the matrix A_2, representing the coupling between the two systems and containing the effect of the parameters L_1 and L_2. The local feedback of x_1 by u_1, and x_2 by u_2 should have the same gain matrix K, i.e. $u_j = Kx_j,\ j = 1, 2$ and hence $u = u_1 + iu_2 = K(x_1 + ix_2)$. The control for a complex system can be represented as a complex vector. It is desired to obtain real feedback. Generally, a complex system is in the form

$$\dot{x} = Ax + Bu \tag{9.17}$$

where A, B, x and u are complex matrices, i.e. $A = A_1 + iA_2$ and $B = B_1 + iB_2$ where A_1, B_1 are symmetric and A_2, B_2 skew-symmetric matrices.

9.5 Summary and Discussion:

In this chapter the $\mathbb{R}$-vector spaces of complex and real matrices have been studied and the relationship between the spaces have been clarified. More precisely, there is an isomorphism from the $\mathbb{R}$-vector space of complex $n \times n$ matrices onto a subspace of the $\mathbb{R}$-vector space of $2n \times 2n$ real matrices in the sense of rings and vector spaces. This yields a suitable manner of finding the solution of the CARE. The existence conditions for the CARE u.p.d. solution have been presented. The relationship between the complex system of order n and its equivalent system which is a $2n$-th order real system has been given. Any problem in the $\mathbb{R}$-vector space of complex matrices can be converted to the $\mathbb{R}$-vector space of real matrices. Complex systems can be converted to real ones. Complex feedback should be considered and the associated real feedback applied.

Chapter 10

Conclusions and Suggestion for Future Work

10.1 Conclusions

Sliding mode control is a well-known approach to the problem of control of uncertain systems, since it is invariant to a class of parameter uncertainty. Well-established investigations have shown that the sliding mode controller/observer is a good approach from the point of view of robustness, implementation, numerical stability, applicability, ease of design tuning and overall evaluation. In the sliding mode control approach the controller and/or observer is designed so that the state trajectory converges to a surface named the sliding surface. It is desired to design the sliding surface so that the system stability is achieved.

The **continuous** and **discrete-time optimal** sliding mode and optimal control have been studied. The sliding mode in regulator and tracking problems, and also a class of servo-mechanism and reference signal systems have been considered. Using the linear quadratic cost functional guarantees the stability of the system. When the system has a reference signal input, the design of the sliding surface is different. In this case the reference input and its effect on the system must be considered. Therefore by considering output tracking and regulator problems the sliding surface can be obtained. The reference input has been considered as a dynamic system, and the sliding control and the sliding surface have been obtained. In this case the reference system is a dynamic system independent of the system and operates as a signal reference controller or generator.

Frequency shaping control design linked with linear quadratic optimal control and sliding mode control is a technique for controlling systems with uncertainties. A *new* method for designing the control and the sliding surface has been proposed when the LQ weighting functions are not constant for all frequencies. By using this method pre- and post-compensators have been designed. The resulting augmented system is a combination of the LQ system and compensators. The order of the augmented system depends upon the dimension of the LQ system state and the weighting functions. It has been concluded that the sliding mode can be expressed as a linear operator of states, i.e. a dynamic system. Additionally, conditions have been obtained to retain the spectrum of the LQ reduced system as a subset of the spectrum of the augmented system. This is important if compensators for the system are required such that the eigenvalues of the LQ system are the eigenvalues of the augmented system. Furthermore an iterative constructive procedure for the optimal sliding mode has been developed. This method enables one to find various sliding surfaces and by looking at the eigenvalue locations in the left-hand half-plane a sliding surface can then be selected to suit.

The aim of **observer design** is to find an estimate for the state and, if the input is unknown, estimate a suitable input. Using the sliding control input form observer a suitable estimated input can be obtained. In this thesis a discontinuous observer for full order systems with disturbance input has been designed. The system may not be ideally in the sliding mode and the uncertainty may not satisfy the matching condition. The proposed sliding observer design method yields an estimated state which nearly approaches the actual state. A sufficient condition ensures the asymptotic stability of the system with some limitations on the uncertainty input. Otherwise, this method ensures only that the estimated state tends approximately to the actual state. The bounds of this approximation have been expressed precisely. In this way, the state error trajectory enters a compact set at a finite time and thereafter remains there. The sliding mode also occurs after a finite time; so there exists a finite time such that the state trajectories enter the 'sliding region' and move to the origin along and in the vicinity of the sliding surface. An controller-observer design method has been presented by considering the sliding mode properties of linear systems.

The sliding dynamics for SISO and MIMO linear systems and conditions for the existence of the sliding mode in the presence of uncertainty, have been studied. The existence of the sliding mode guarantees that the state trajectories converge to a sliding surface at a finite time and move along the surface to the origin thereafter. However, the system may generally not be stable. For the system to be asymptotically stable, some further conditions may be needed.

A new straightforward technique for determining whether the system in the sliding mode is independent of the perturbation input has been presented. Also sufficient conditions for the sliding mode control design of systems with disturbance input and the sliding mode dynamics have been obtained.

The sliding surface for the **discrete-time systems** is a *lattice* called the sliding latticewise surface or more concisely the sliding lattice. A *new* control design technique using the properties of the discrete sliding mode has been proposed. This control guarantees the stability of the sliding mode and the stability of the system. This control does not have the same structure as continuous SMC. The behaviour of the system in the sliding mode and stability conditions have been studied. If the nominal reduced order matrix is stable, then the state in the sliding mode is bounded. In this case the width of the boundary layer depends upon the disturbance bound, the norm of the distribution disturbance input map and the degree of stability of the nominal reduced system matrix.

The main problem in discrete-time sliding systems is the perfect rejection of unknown disturbances during the sliding mode. In the discrete-time case perfect disturbance rejection is achieved if only the disturbance is exactly known and the associated equivalent control component is applied. One needs to consider various restrictive conditions like; (*i*) the matching condition, (*ii*) suitable estimation for the successive disturbance difference (the variation of the disturbance sequence) $\xi(k) - \xi(k-1)$, (*iii*) the difference sequence $\xi(k) - \xi(k-1)$ is a decreasing sequence or sufficiently small after a finite time. Although this problem also appears in continuous systems, in the case of discrete-time systems the proof of perfect disturbance rejection is more complicated. This difference arises because the reaching sliding condition of continuous systems differs from that of discrete-time systems. In discrete-time systems in the study of the stability of the system and also the reaching sliding condition, the discrete Lyapunov function is employed and difference equations appear. In the case of continuous systems the continuous Lyapunov function is utilized which results in derivatives. In the discrete-time control design method one needs to use an estimate for the disturbance input. One may achieve the estimation as in [112]. Another approach uses the equivalent control with zero disturbance and one assumes that the disturbance does not affect the equivalent control. More precisely, since the equivalent control can be considered as the average of the control input, if the mean of the disturbance is zero, then the equivalent control may be assumed to be "independent" of the disturbance input.

There are two approaches to the design of a discrete-time sliding mode controller: (*i*) in the first instance a dynamical sliding mode(or the control) is specified, and then it is necessary to find the conventional control (sliding mode dynamics) [46], [21]; (*ii*) a control is found by using the properties of the sliding mode so that the stability of the nominal systems in the sliding mode is conserved. In this thesis both methods (*i*) and (*ii*) have been applied successfully to linear systems. New discrete-time system stability conditions have been proposed and the design of the optimal sliding mode matrix has been extended to DSMC. Most previous DSMC research has considered only SISO systems.

New results for the stability of reconstruction error systems of linear discrete-time systems have been proposed. The same difficulties which occur in the disturbance rejection problem exist for the discrete-time sliding observer. The cone condition for the error system is a boundedness condition and satisfying this condition is difficult in practice. However, the stability of the system is guaranteed if one of the following conditions is satisfied: (*i*) the cone condition for the disturbance input with respect to the state; (*ii*) there exists a finite time instant such that after this time the disturbance input sequence ξ is a decreasing sequence. So a simple condition on the disturbance should yield the stability of the error system. Further research is needed to find this condition.

The sliding mode control of time-delay systems has been considered. Time-delay sliding system stability has been studied for the cases of having full information about the delay and also lack of information. The sliding surface is delay-independent as for the traditional sliding surface, and the reaching condition is achieved by applying a conventional discontinuous control. The sliding mode on a specified surface is achieved if the state converges to the surface. Two kinds of sliding surface can be designed: (*i*) the sliding surface is independent of the delays; (*ii*) the sliding surface depends on the delays. In the second case the delays should be constant, otherwise the sliding surface is not a simple hyperplane.

The Lyapunov and Riccati equations, and stability properties of complex and real systems have been considered. By using these equations the sliding surface and feedback gain matrix can be found such that all the eigenvalues of the closed-loop system lie in a specified region.

Several new methods have been proposed for all the eigenvalues of the closed-loop system to lie in a specified region. Eigenvalues can be specified in a region in the left-hand half-plane for the system and design the gain feedback matrix to yield these eigenvalues. This method can also be applied to the design of the sliding gain matrix. The following regions have been considered; a sector, an infinite vertical strip, a disc, a hyperbola and the intersection of two sectors. A modified new theorem with a new proof has been presented to place all the eigenvalues of the closed-loop system in a specified vertical strip in the left-hand half-plane. These methods are based on properties of the Riccati equation. The ARE with zero right-hand side has a semi-p.d.s. solution matrix, and only if the state matrix is completely unstable, does this ARE have a p.d.s. solution. When the matrix is stable, the semi-p.d.s. solution is the zero matrix. Illustrative examples have been presented. If all the eigenvalues of the closed-loop system are in the specified sector, the associated CARE has a p.d. Hermitian solution, but the converse is not always true. The definition of a p.d. matrix in the statistics literature is equivalent to the p.d. of the elements but this definition is not appropriate in control theory.

The *complex* equation, positivity of a complex matrix and the control of complex systems are significant problems which appear in many control theory problems. Generalized complex Riccati equations have been considered. The positivity concept of a matrix has been carefully defined. A method for finding the solution of the complex Riccati equation has been proposed.

There is an isomorphism from $M_n(\mathbb{C})$ into $M_{2n}(\mathbb{R})$, so the $\mathbb{R}$-vector space of complex matrices is isomorphic to the conventional $\mathbb{R}$-vector spaces of real matrices in the sense of rings and vector spaces. This yields a suitable way of finding the solution of CARE. Existence conditions for the u.p.d. solution of CARE have been presented. The relationship between a complex system of order n and its equivalent system, which is a $2n$-th order real system, has been given. Any problem in the $\mathbb{R}$-vector space of complex matrices can be converted to the $\mathbb{R}$-vector space of real matrices. For instance, complex systems can be converted to their real counterpart.

10.2 Suggestions for Further Research

In this section some problems requiring further research and some open questions are discussed. Possibly most results of this thesis could be extended to nonlinear affine systems

$$\dot{x} = A(t,x)x + B(t,x)u + f(t,x,u)$$

Also some results for continuous systems could be extended to discrete-time systems. Some of these problems are:

(i) The optimal sliding tracking problem for a nonlinear affine system with uncertainty needs to be investigated; also linear and nonlinear discrete-time systems. If the bounded reference input is a stepwise function, the stability of the system is achieved. But if the reference signal is not a stepwise function and the reference input is only bounded, the stability of the system needs be investigated.

Useful results may possibly be obtained by investigating of the sliding condition in the sense of the boundary layer and also in extending the theory to nonlinear systems.

(*ii*) Further research should address H_∞ and the sliding mode, and extend the work in [55]. Possibly a generalized system could be found and then H_∞ methods could be used. Similarly to [55], by using the H_∞ approach, the sliding gain matrix could be found for cases (*iii*) and (*iv*) of Section 4.1. Moreover, in [55] only one way has been presented for obtaining the sliding gain matrix. However, some H_∞ methods to obtain the feedback gain matrix, could be adapted to those augmented systems which have been discussed in Chapter 4.

(*iii*) Further research should investigate bounds tighter rather than those stated in Chapter 5. The results may be extended to nonlinear systems with nonlinearities only in the disturbance input term.

It is interesting to consider the system with unmatched uncertainty and obtain a relaxed sliding condition to impose asymptotic stability.

(*iv*) Theorem 6.8.2 determines the relationship between the solution of the DLE (6.67) for arbitrary weighting function Q_g and the solution for $Q = I$. An open problem is to find a relationship between the eigenvalues of the p.d.s. matrix solution of the discrete-time Lyapunov equation, particularly a bound for the ratio $\lambda_{\min}(P_g)/\lambda_{\min}(P_g)$ which appears in some stability conditions, such as (6.69).

(*v*) The sliding mode control of delay systems is a relatively new field which needs to be developed. The stability of the sliding mode control of a system with a delay on the state has been considered. It was assumed that the delay is constant, but if the delay is a function of time, how should sliding control be defined so that the state lies in a certain sliding surface. The extension of the results of this thesis to systems with finite delays is straightforward. The results should also be extended to systems with delay in the control and to the sliding mode observer for time-delay systems. Further research may yield a weaker condition than (7.20) to ensure system stability independent of the delay.

(*vi*) Further work is needed to obtain conditions weaker than those in this thesis, for the eigenvalues of a closed-loop continuous system to lie in the specified region in the left-hand half-plane. By using the CARE one may obtain appropriate conditions. Using the transfer function of the closed-loop system may also prove fruitful.

(*vii*) Complex feedback may be considered to find the associated real feedback so that the stability of the system and other desired properties are preserved. As stated in Chapter 8, the CARE gives a complex feedback gain matrix which has not the same properties as real case. However the CARE yields a associated real feedback for a real system with order two times higher than original system. The problem is whether one can drive a real feedback gain matrix by using the complex feedback gain matrix or more precisely with the solution of the CARE.

(*viii*) More general cases of the Riccati equation and some related problems are still open questions. A method is also required to obtain the semi-p.d.s. solution of the ARE when the weighting matrix of the right-hand side is zero (see equation (8.25)).

(*ix*) More research is required to obtain a dynamical well-behaved sliding mode so that the stability of the system in the presence of external unmatched disturbance input is achieved. Using the transfer function of the system may be a possible way to design a sliding mode. This method may need some information about the zeros of the system.

(*x*) There are many ways forward regarding the adaptive sliding surface design for continuous and discrete-time systems. Sliding surfaces could possibly be designed by using a method similar to the backstepping approach [128].

Appendix A

A.1 Stability

Consider the time-varying system

$$\dot{x} = f(x, t) \tag{A.1}$$

where $x \in \mathbb{R}^n$ and $f : D \times \mathbb{R}_+ \to \mathbb{R}^n$ where the domain $D \subset \mathbb{R}^n$, is a piecewise continuous function in t and locally Lipschitz in x.

Suppose $\bar{x} \in D$ is an equilibrium point with $f(\bar{x}, t) = 0$ for all $t \geqslant t_0$. Any equilibrium point can be shifted to the origin via a change of variables, $y = x - \bar{x}$. So without loss of generality, all the definitions are stated for the case when the equilibrium point is at the origin.

Definition A.1.1: The equilibrium point 0 of (A.1) is

(*i*) *stable*, if for each $\epsilon > 0$ there exists a $\delta(\epsilon, t_0) > 0$ such that for all $t \geqslant t_0$

$$||x(t_0; x_0, t_0)|| < \delta \Rightarrow ||x(t; x_0, t_0)|| < \epsilon \tag{A.2}$$

(*ii*) *unstable*, if it is not stable

(*iii*) *asymptotically stable*, if it is stable and there exists a δ such that

$$||x(t_0; x_0, t_0)|| < \delta \Rightarrow \lim_{t \to \infty} x(t; x_0, t_0) = 0 \tag{A.3}$$

Definition A.1.2: The equilibrium point 0 (A.1) is

(*i*) *exponentially stable*, if there exist two positive real numbers, α and β, such that for for all $t \geqslant t_0$,

$$||x(t; x_0, t_0)|| < \alpha ||x(t_0; x_0, t_0)|| e^{-\beta(t - t_0)} \tag{A.4}$$

(*ii*) *quadratically stable*, if there exist p.d.s. matrices P, $Q \in \mathbb{R}^{n \times n}$ such that for all $t \geqslant t_0$,

$$x^T P f(x,t) < -x^T Q x \tag{A.5}$$

Corollary A.1.1: *Consider the system (A.1) satisfying (A.5). If the equilibrium point* 0 *is quadratically stable, then it is exponentially stable.*

Proof: Consider the Lyapunov function $V = x^T P x$. From (A.5)

$$\frac{dV(x(t))}{dt} \leqslant -2x^T Q x \tag{A.6}$$

From (A.6) one can show that (A.4) holds with

$$\alpha = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}}, \quad \beta = \lambda_{\min}(P^{-1}Q)$$

□

Theorem A.1.1: *Consider the linear time-invariant system*

$$\dot{x}(t) = Ax(t) \tag{A.7}$$

The origin is

(*i*) asymptotically stable *if all the eigenvalues of A lie in the open left-hand half-plane*

(*ii*) stable, *but not asymptotically stable, if all the eigenvalues of A lie in the closed left-hand half-plane, i.e. A has one eigenvalue on the imaginary axis*

(*iii*) unstable *if at least one eigenvalue of A has a positive real part*

(*iv*) completely unstable *if the real part of all the eigenvalues of A are positive.*

□

Theorem A.1.2: *Consider the linear time-invariant discrete-time system*

$$x(k+1) = Ax(k) \tag{A.8}$$

The origin is

(*i*) asymptotically stable *if all the eigenvalues of A lie inside the unit circle.*

(*ii*) stable, *but not asymptotically stable, if A has at least one eigenvalue on the boundary of the unit circle.*

(*iii*) unstable *if at least one eigenvalue of A is outside the unit circle*

(*iv*) completely unstable *if all the eigenvalues of A are outside the unit circle.*

A.2 Boundedness

Definition A.2.1 The solution $x(t; x_0, t_0)$ of (A.1) is

(*i*) *bounded*, if there exists a constant $h(x_0, t_0) > 0$ such that

$$\|x(t; x_0, t_0)\| < h(x_0, t_0) \tag{A.9}$$

(*ii*) *uniformly bounded*, if there exists a constant $h_1(x_0) > 0$, possibly dependent on x_0 but not on t_0, such that

$$\|x(t; x_0, t_0)\| < h_1(x_0) \tag{A.10}$$

(*iii*) *ultimately bounded*, with respect to a compact set $X \subset \mathbb{R}^n$, if there exists a nonnegative time $T(t_0, x_0, X)$ such that for all $t \geqslant t_0 + T(t_0, x_0, X)$, $x(t) \in X$.

(*iv*) *uniformly ultimately bounded* with respect to a compact set $X \in \mathbb{R}^n$, if $T(x_0, X)$, possibly dependent on x_0 but not on t_0 defined as (*iii*) is independent of t_0, i.e. there exists a nonnegative time $T(x_0, X)$ such that for all $t \geqslant t_0 + T(x_0, X)$, $x(t) \in X$.

Definition (*iv*) (A.2.1) can be stated as follows [62, page 202]:

The solution $x(t; x_0, t_0)$ of (A.1) is *uniformly ultimately bounded*, if there exists constants a and c, and for every $\alpha \in (0, c)$ there is a constant $T = T(\alpha)$ such that, for all $t > t_0 + T$,

$$\|x(t_0; x_0, t_0)\| < \alpha \Rightarrow \|x(t; x_0, t_0)\| < b \tag{A.11}$$

Appendix B

B.1 Kronecker Product

Let $A = (a_{ij}) \in \mathbb{C}^{n\times m}$ and $B = (b_{ij}) \in \mathbb{C}^{p\times q}$. Then $A \otimes B$ is an $np \times mq$ matrix having the (i, j)-th block $a_{ij}B$. Basic properties are:

(B_1) $(A \otimes B)^* = A^* \otimes B^*$.

(B_2) Let C and D be $m \times r$ and $q \times s$, respectively; then $(A \otimes B)(C \otimes D) = AC \otimes BD$

(B_3) If A and B are nonsingular, $m = n$ and $p = q$, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

(B_4) If $m = n$ and $p = q$, and λ_i $(1 \leqslant i \leqslant n)$ and $\mu_j (1 \leqslant j \leqslant p)$ are the eigenvalues of A and B, respectively, then

(i) the eigenvalues of $A \otimes B$ are $\lambda_i \mu_j$

(ii) the eigenvalues of $A \otimes I_p + I_n \otimes B$ are $\lambda_i + \mu_j$.

Appendix C

C.1 Real Algebraic Riccati Equation (ARE)

Let $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{r\times n}$, $Q \in \mathbb{R}^{r\times r}$, $R \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{n\times m}$. Suppose R and Q are p.d. matrices, $r \leqslant n$ and $m \leqslant n$. Assume that (A, B) is a stabilizable pair and (A, C) is a detectable pair. The real continuous algebraic Riccati equation

$$A^T P + PA - PBR^{-1}B^T P = C^T QC \tag{C.1}$$

has a u.p.d.s. P matrix solution and all the eigenvalues of $A - BR^{-1}B^T P$ lie in the left-hand half-plane. The Hamiltonian matrix is defined as

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -C^T QC & -A^T \end{bmatrix} \tag{C.2}$$

If λ is an eigenvalue of H, then so is $-\lambda$ (with the same multiplicity). Let U be the matrix of eigenvectors of H, ordered so that the n left-most columns correspond to eigenvalues with negative real parts, and the n right-most columns correspond to eigenvalues with positive real parts. If $(A, B, Q^{1/2}C)$ is minimal, H has no eigenvalues on the imaginary axis [80, page 226]. Now partition U into $n \times n$ blocks

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \tag{C.3}$$

The solution to (C.1) is then given by $P = U_{21}U_{11}^{-1}$.

Theorem C.1.1: *For the matrix H in (C.2) let U (C.3) be any matrix which transforms H into upper Jordan form, $U^{-1}HU = J$. Then provided that U_{11} is nonsingular, the solution to (C.1) is given by $P = U_{21}U_{11}^{-1}$ and the eigenvalues of $A - BR^{-1}B^T P$ are the same as those of S_{11} where*

$$J = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}$$

Remark C.1.1: As already stated, it can be shown that provided (A, B) is completely controllable and $(A, Q^{1/2}C)$ completely observable, then H in (C.2) has no eigenvalues on the imaginary axis [80, page 50], [11, pages 282-284]. This ensures that there exist nonsingular matrices U_{11}. □

C.2 Positivity

Theorem C.2.1 *The real matrix*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is p.d. if and only if one of the following conditions is satisfied:

(*i*) P_{11} *and* $P_{21}P_{22}^{-1}P_{12}$ *are p.d. matrices*

(*ii*) P_{22} *and* $P_{12}P_{11}^{-1}P_{21}$ *are p.d. matrices.*

Lemma C.2.1 *Let* $b = [b_1, b_2, \ldots, b_n]^T$ *and* $c = [c_1, c_2, \ldots, c_n]$. *Then*

1. $\det(\lambda I - bc) = (-1)^n \lambda^{n-1}(\lambda - cb)$
2. $\det(\lambda I - (I - bc)) = (-1)^n(\lambda - 1)^{n-1}(\lambda - 1 + cb)$
3. $\|\frac{bc}{cb}\| = \frac{\|b\| \ \|c\|}{|cb|}$
4. $\|I - \frac{bc}{cb}\| = \frac{\|b\| \ \|c\|}{|cb|}$
5. $\det(\lambda I - (\frac{bc}{cb})^T(\frac{bc}{cb})) = (-1)^n(\lambda - 1)^{n-2}\left(\lambda - \frac{\|b\|^2\|c\|^2}{(cb)^2}\right)$

Proof:

$$\begin{aligned} \left(\frac{bc}{cb}\right)^T \left(\frac{bc}{cb}\right) &= \frac{c^T b^T bc}{(cb)^2} \\ &= \frac{c^T \|b\|^2 c}{(cb)^2} \\ &= \frac{\|b\|^2}{(cb)^2} c^T c \end{aligned}$$

The matrix $c^T c$ has just one nonzero eigenvalue which equals the trace of $c^T c$, i.e., $\|c\|^2$. Therefore, the maximum eigenvalue of $(\frac{bc}{cb})^T(\frac{bc}{cb})$ is $\frac{\|b\|^2\|c\|^2}{(cb)^2}$. □

For MIMO systems the following lemma holds:

Lemma C.2.2 *Let B and C be $n \times m$ and $m \times n$ matrices. Then*

1. *m eigenvalues of $B(CB)^{-1}C$ are 1 and the remaining $(n-m)$ are zero.*
2. *$n-m$ eigenvalues of $(I - B(CB)^{-1}C)$ are 1 and the remaining m are zero.*
3. *the matrix $B(CB)^{-1}C$ has m nonzero and $n-m$ zero singular values.*
4. *m singular values of $I - B(CB)^{-1}C$ are zero and the remaining $n-m$ are nonzero.*
5. *$\|B(CB)^{-1}C\| = \|I - B(CB)^{-1}C\|$.*

□

C.3 Barbalat Lemma

Theorem C.3.1 (Barbalat): *Consider the function $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}$. If ϕ is uniformly continuous and $\lim_{t\to\infty} \int_0^t \phi(\tau)d\tau = 0$ exists and is finite, then*

$$\lim_{t\to\infty} \phi(t) = 0$$

Proof: See [62, page 186]

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Index