LENGTH FUNCTIONS ON MODULES

by

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CONTENTS

P R E F A C E

In commutative algebra, real valued functions (which may attain infinity) such as length, rank, multiplicity etc. are frequently used. D.G.Northcott and M.Reufel were the first who observed their underlying common properties and started the study of functions satisfying certain properties [3].

Let R be a ring with identity element and assume that there is a function L which associates with each R-module a non-negative real number or plus infinity. We call L a length function on the category of R-modules if it satisfies the following conditions:

(i) $L(0) = 0$;

(ii) $L(A) = L(A') + L(A'')$ whenever $0 \to A' \to A \to A'' \to 0$ is an exact sequence of R-modules.

The object of the present thesis is to study length functions on modules. We will be particularly interested in the problem of the characterization of all the length functions on the category of R-modules.

In Chapter 1 we have collected the necessary prerequisites from general module theory. It could be said with a mild axaggeration that for the main body of this work one has only to know the Jordan-Hölder-Schreier theorem.

Very often it is more convenient to obtain a description of length functions on a certain subcategory of the R-modules than on the whole category. It is therefore necessary to develop a technique which enables us to 'ascend' from a subcategory to the full category of R-modules, i.e. to extend length functions. Chapter 2 is devoted to this problem. It is shown that this extension is always possible and there is a 'minimal' extension. This leads to the notion of continuity. Roughly speaking a length function L is continuous on a subcategory $\mathfrak{A}$ if L is uniquely determined by its values on $\mathfrak{A}$. In [3] the authors considered only those length functions whose values are determined on the finitely generated modules. We call these functions 'upper continuous'. Most of this chapter's material is contained in [7].

Chapter 3 contains the main results of the thesis. A length function L is called 'irreducible' if $L = L_1 + L_2$ implies that $L_1 = cL$ or $L_2 = cL$, $c > 0$. The main decomposition theorem (Theorem 3.12) states that if $\mathfrak{A}$ is a category with Krull-dimension then every length function on $\mathfrak{A}$ is a unique sum of irreducible functions.

The concept of the Krull-dimension of an Abelian category was introduced by P.Gabriel in his thesis [4]. His definition, however, was designed for the whole category and made use of the concept of quotient categories. To meet our different requirements we have had to modify his definition. It was felt at the same time that the employment of quotient categories would need a good working knowledge of Abelian categories. Accordingly, we will use an entirely elementary technique.

After the decomposition theorem we describe the irreducible length functions of the category of Noetherian modules for a given ring. It is found that the irreducible length functions are associated with indecomposable injective modules. In the case of a commutative ring the indecomposable injectives can be replaced by prime ideals.

Special questions of the general theory are discussed in Chapter 4. In Section 4.1 we describe the length functions on the category of Artinian modules over a commutative Noetherian ring. In Section 4.2 we prove that for Artinian rings and for (commutative) semi-local Dedekind rings every length function is determined by its values on Artinian and Noetherian modules. An example shows that the condition 'semi-local' cannot be dropped. D.G.Northcott and M.Reufel found that for a rank-one valuation ring the valuation induces a length function. We prove this result in Section 4.3. The last section of this chapter deals with rings R for which a length function L exists such that L(R) = 1.

After reading the first chapters the reader will probably find that the theory could easily be accommodated in Abelian categories. This is indeed true for the whole thesis, with the possible exception of Chapter 4. It was felt, however, that the possible gains in generality were not sufficient to warrant the use of abstract categories. On the other hand we made a determined effort to exploit aspects of duality. Thus finitely embedded modules are introduced in Section 1.3 and play a complementary rôle to that of finitely generated modules. Gabriel's definition of Krull-dimension has been modified and now has a self-dual nature. The result is that the decomposition theorem is applicable to the category of Artinian modules as well.

As an application of the general theory we present multiplicities in Chapter 5 as 'operators' on length functions. Again, Artinian modules are placed on an equal footing with Noetherian ones. In Section 5.2 we see the decomposition theorem at work in establishing the associative law for multiplicities without any restriction on the ring.

The ordered Grothendieck group is introduced in Chapter 6. We show that there is a one-to-one correspondence between the length functions and the order-preserving homomorphisms from this group into the real numbers.

In the present thesis we have attempted to present a general theory of length functions. On many problems only the first groping steps have been taken towards a solution. Of the numerous problems we mention only two: the characterization of length functions on categories without Krull-dimension and the problem of rank-rings. It seems that for both of these questions the ordered Grothendieck group holds the key.

I owe a considerable debt to the authorities of the University of Sheffield who helped to make it possible for me to stay in Great Britain. This debt I gladly acknowledge here. I also record my deep appreciation of the encouragement given to me by Professor D.G. Northcott, who stimulated me to embark on the investigation of length functions on modules.

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CHAPTER 1

CATEGORIES OF MODULES.

1.1 Introductory remarks.

It is assumed that the reader is familiar with the basic ideas of module theory, say chapter 1 of [1] and [2]. We make a few remarks, however, to clarify our terminology.

Throughout this thesis a ring R will mean a ring with identity element and an R-module is understood to be a unitary left R-module. The sign $\subset$ always stands for strict inclusion and inclusion in the wider sense is indicated by $\subseteq$. When we speak of a category we mean a non-empty full subcategory of the category of modules over a ring R i.e. a class of R-modules and all the homomorphisms between them. In addition, we assume that if a category contains a module M, then it contains every module isomorphic to M as well. The category of R-modules itself is denoted by $\mathfrak{M}(R)$.

The Jordan-Hölder-Schreier theorem plays a central rôle in our investigations. It might be helpful to recall this result here. Let A be an R-module. By a chain σ of A we mean a finite sequence of submodules of A of the form

$$\sigma : 0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A.$$

The modules $C_i = A_i/A_{i-1}$ $(1 \leqslant i \leqslant n)$ are called the chain factors of σ. Suppose that a second chain

$$\tau : 0 = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_m = A$$

of submodules of A is given with chain factors $D_i = B_i/B_{i-1}$ $(1 \leqslant i \leqslant m)$.

We say that τ is a refinement of σ if it is obtained from σ by inserting new submodules between those already present. The chains σ and τ are called equivalent if $n = m$ and the chain factors of σ and τ are isomorphic up to order i.e. there is a permutation φ such that $C_i \approx D_{\varphi(i)}$ $(1 \leq i \leq n)$. The Jordan-Hölder-Schreier theorem states that any two chains (and consequently a finite number of chains) of A have equivalent refinements.

1.2 Serre-categories

We wish to define two special types of categories which will frequently occur in the sequel. Let R be a ring and

$$0 \to A' \to A \to A'' \to 0$$

an exact sequence of R-modules.

Definition. A subcategory $\mathfrak{U}$ of $\mathfrak{M}(R)$ is called 'semi-closed' if it has the property that

$$A \in \mathfrak{U} \text{ implies } A', A'' \in \mathfrak{U}.$$

The category $\mathfrak{U}$ is said to be a 'Serre-category' if it has the property that

$$A \in \mathfrak{U} \text{ if and only if } A', A'' \in \mathfrak{U}.$$

It is clear that a Serre-category is semi-closed. Further, if σ is a chain of the module A in a semi-closed category $\mathfrak{U}$ then every chain factor belongs to $\mathfrak{U}$. If $\mathfrak{U}$ is a Serre-category then the converse holds as well.

An intersection of Serre (resp. semi-closed) categories is obviously a Serre (resp. semi-closed) category again. If $\mathfrak{U}$ is a category then the smallest Serre-category containing $\mathfrak{U}$ is called the

'<u>Serre-category generated by $\mathfrak{U}$</u>'. If set theory allowed us we could speak of this category as the intersection of all Serre-categories containing $\mathfrak{U}$. Instead, we proceed by construction.

If $B \subseteq C$ are submodules of a module A then the factor module C/B is called a '<u>segment</u>' of A. Suppose now that $\mathfrak{U}$ is a category. Set

$$\tilde{\mathfrak{U}} = \left\{ A \in \mathfrak{M}(R) : \begin{array}{l} \text{A has a chain } \sigma \text{ such that every chain factor of } \sigma \\ \text{is isomorphic to a segment of an element of } \mathfrak{U}. \end{array} \right\}$$

<u>Proposition 1.</u> <u>Let the situation be as described above. The category</u> $\tilde{\mathfrak{U}}$ <u>is the Serre-category generated by</u> $\mathfrak{U}$.

<u>Proof</u>. It is clear that $\mathfrak{U} \subseteq \tilde{\mathfrak{U}}$ and that every Serre-category containing $\mathfrak{U}$ contains $\tilde{\mathfrak{U}}$. It remains to be shown that $\tilde{\mathfrak{U}}$ is a Serre-category. Let

$$0 \to A' \to A \to A'' \to 0$$

be an exact sequence in $\mathfrak{M}(R)$ and suppose that $A', A'' \in \tilde{\mathfrak{U}}$. We may assume that A' is a submodule of A and $A'' = A/A'$. By piecing together any two chains of A' and A'' respectively we can obtain a chain for A. This shows that $A \in \tilde{\mathfrak{U}}$. Conversely, assume that $A \in \tilde{\mathfrak{U}}$. Then there is a chain σ of A such that the chain factors of σ are isomorphic to segments of modules of $\mathfrak{U}$. Obviously, any refinement of σ will have the same property since a segment of a segment is a segment again. Let σ' and τ' be equivalent refinements of σ and the chain $0 \subseteq A' \subseteq A$. Then τ' yields chains for A' and A/A' having the required property.

<u>Corollary</u>. <u>Let</u> $\mathfrak{U}$ <u>be a semi-closed category and</u> $\tilde{\mathfrak{U}}$ <u>the Serre-category generated by</u> $\mathfrak{U}$. <u>A module A belongs to</u> $\tilde{\mathfrak{U}}$ <u>if and only if it has a chain</u> σ <u>such that every chain factor of</u> σ <u>belongs to</u> $\mathfrak{U}$.

Note that the category containing the zero module alone is a Serre-category and contained in every semi-closed category. There can be no confusion in denoting this category by 0.

Lemma 2. *Let* A_1, A_2 *be submodules of* $A \in \mathfrak{M}(R)$ *and* $\mathfrak{U}$ *a Serre-category. Then*

(i) $A_1, A_2 \in \mathfrak{U}$ *if and only if* $A_1 + A_2 \in \mathfrak{U}$;

(ii) $A/A_1, A/A_2 \in \mathfrak{U}$ *if and only if* $A/A_1 \cap A_2 \in \mathfrak{U}$.

Proof. Consider the following exact sequences:

$$0 \to A_1 \to A_1 + A_2 \to A_2/A_1 \cap A_2 \to 0,$$

$$0 \to (A_1+A_2)/A_1 \to A/A_1 \cap A_2 \to A/A_2 \to 0,$$

and take into account that if $\mathfrak{U}$ contains a module then it contains all of its submodules and factor modules.

1.3 Finitely generated and finitely embedded modules.

In accordance with our programme we now define a class of modules, in a certain sense complementary to the class of finitely generated modules. First, however, we shall introduce injective modules.

Definition. *An R-module M is said to be 'injective' if for any diagram of R-modules*

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B \\ & & \beta\downarrow & \swarrow \gamma & \\ & & M & & \end{array}$$

with an exact row (i.e. α *is a monomorphism) a homomorphism* γ *can be found so that* $\gamma\alpha = \beta$.

We now briefly summarize the basic properties of injective modules. For the proofs the reader is referred to [2, chap.III,5-7].

For an R-module M the following are equivalent:

(i) M is injective;

(ii) M is a direct summand in every module containing it;

(iii) The functor $\mathrm{Hom}_R(-,M)$ from $\mathfrak{M}(R)$ into the category of Abelian groups is exact.

Definition. Let M be a submodule of the R-module N. We say that N is an 'essential extension' of M, if for a submodule P of N, $P \cap M = 0$ implies $P = 0$. For every R-module M there exists a module E(M) satisfying the following equivalent conditions:

(i) E(M) is a maximal essential extension of M;

(ii) E(M) is a minimal injective extension of M.

Moreover, if E′ is another module satisfying (i) - (ii) then there is an isomorphism between E(M) and E′ which is the identity on M. The module E(M) is called the 'injective envelope' of M and will always be denoted by E( ).

E.Matlis has shown the following [5, Proposition 2.1]: If M is a finite direct sum of modules $M = M_1 \oplus \ldots \oplus M_k$ then there is an isomorphism

$$E(M) \approx E(M_1) \oplus \ldots \oplus E(M_k).$$

Definition. Let M be an R-module. The 'socle' of M, denoted by S(M), is the sum of all simple submodules of M.†

Thus S(M) is the unique maximal semi-simple submodule of A and can be written as a direct sum of simple modules. It is easily seen that the socle commutes with direct sums.

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† In case M contains no simple modules $S(M) = 0$.

We now now ready to introduce the concept from which the heading of this section is derived.

Definition. An R-module M is said to be 'finitely embedded' if

$$E(M) \approx E(S_1) \oplus \dots \oplus E(S_k),$$

where each S_i is a simple R-module.†

Lemma 3. The module M is finitely embedded if and only if

(a) M is an essential extension of $S(M)$ and

(b) $S(M)$ is finitely generated.

Note that $S(M)$ is finitely generated if and only if it is both Noetherian and Artinian.

Proof. Suppose that (a) and (b) hold. Then $S(M) = S_1 \oplus \dots \oplus S_k$ where $S_1, \dots, S_k$ are simple modules and

$$E(M) = E(S(M)) = E(S_1 \oplus \dots \oplus S_k) \approx E(S_1) \oplus \dots \oplus E(S_k).$$

Conversely, if M is finitely embedded and $M \neq 0$ then

$$E = E(M) = E(S_1) \oplus \dots \oplus E(S_n) = E(S_1 \oplus \dots \oplus S_n)$$

for simple modules $S_1, \dots, S_n$ $(n \geq 1)$. Since $S_i \cap M \neq 0$ for each S_i, $S = S_1 \oplus \dots \oplus S_n \subseteq S(M)$. On the other hand

$$S(M) \subseteq S(E(M)) = S(E(S_1) \oplus \dots \oplus E(S_n)) = S.$$

We have $S = S(M)$ and M is an essential extension of $S(M)$ since $E(M) = E(S)$.

The propositions which follow are stated in dual forms for both finitely generated and finitely embedded modules. Since these are standard results for finitely generated modules, proofs will only be provided for the finitely embedded case.

A family $\{M_i\}_{i \in I}$ of submodules of M is said to be 'direct'

† The zero module as a vacuous sum is finitely embedded.

(resp. '<u>inverse</u>') if for any finite number $i_1,\ldots,i_k$ of elements of I there is an $i_0 \in I$ such that

$$M_{i_0} \supseteq M_{i_1} + \ldots + M_{i_k} \quad (\text{resp. } M_{i_0} \subseteq M_{i_1} \cap \ldots \cap M_{i_k}).$$

<u>Proposition 4.</u> <u>A module</u> M <u>is finitely generated if and only if every direct system of proper submodules of</u> M <u>is bounded above by a proper submodule of</u> M.

<u>Proposition 4*.</u> <u>A module</u> M <u>is finitely embedded if and only if every inverse system of non-zero submodules of</u> M <u>is bounded below by a non-zero submodule of</u> M.

<u>Proof.</u> Assume $M \neq 0$ is finitely embedded and $\{M_i\}_{i \in I}$ is an inverse system of non-zero submodules of M. Since M is an essential extension of its (Artinian) socle, $S(M_i) \neq 0$ and has minimal condition for all $i \in I$. We can choose an $i_0 \in I$ such that $S(M_{i_0})$ is minimal. For every $i \in I$ there exists a $j \in I$ such that $M_j \subseteq M_i \cap M_{i_0}$. It follows that $S(M_j) \subseteq S(M_i)$, $S(M_{i_0}) = S(M_j)$. Hence $S(M_{i_0}) \subseteq M_i$ for all $i \in I$.

Conversely, assume that any inverse system of non-zero submodules of M is bounded below. Let $N \subseteq M$ be a non-zero submodule. Then, by assumption, and Zorn's Lemma, there is a minimal submodule S of N which is, consequently, simple. Thus $0 \neq S(N) \subseteq N \cap S(M)$ and M is an essential extension of its socle which is clearly finitely generated.

<u>Proposition 5.</u> <u>A module</u> M <u>is Noetherian if and only if every submodule of</u> M <u>is finitely generated.</u>

<u>Proposition 5*.</u> <u>A module</u> M <u>is Artinian if and only if every factor module of</u> M <u>is finitely embedded.</u>

Proof. It will suffice to prove the "if" part. Let $A_1 \supseteq A_2 \supseteq \dots \supseteq A_n \supseteq \dots$ be a descending sequence of submodules of M and $A = \cap A_n$. Since M/A is finitely embedded we may assume that $A = 0$. If the sequence did not terminate it would be bounded by a non-zero submodule which is impossible.

Let $0 \to A' \xrightarrow{\varphi} A \to A'' \xrightarrow{\psi} 0$ be an exact sequence. Then we have:

Proposition 6. If A is finitely generated then so is A″. If A′ and A″ are finitely generated modules then A is a finitely generated module too.

Proposition 6*. If A is finitely embedded then so is A′. If A′ and A″ are finitely embedded modules then A is a finitely embedded module too.

Proof. By Proposition 4*, A′ is clearly finitely embedded whenever A is finitely embedded. For the second part let $\alpha : A' \to E(A')$, $\beta : A'' \to E(A'')$ be the embeddings of A′ and A″ into their injective envelopes. By definition $E(A') \oplus E(A'')$ is finitely embedded. Since $E(A')$ is injective, the mapping α can be extended to A, i.e. there is a $\varphi' : A \to E(A')$ such that $\varphi' \varphi = \alpha$. Then we have a monomorphism $\varphi' + \beta\psi : A \to E(A') \oplus E(A'')$ and the first part of the proposition implies the result.

Corollary. Let A_1, A_2 be submodules of a module M. If A_1, A_2 are finitely generated then $A_1 + A_2$ is finitely generated. If $M/A_1, M/A_2$ are finitely embedded then $M/A_1 \cap A_2$ is finitely embedded.

Proof. Observe that $M/A_1 \cap A_2$ can be mapped monomorphically into $M/A_1 \oplus M/A_2$ and use Proposition 6*.

Proposition 7. Let $\{M_i\}_{i\in I}$ be the family of finitely generated submodules of a module M. Then $\{M_i\}$ is a direct system and $\Sigma_{i\in I}M_i = M$.

Proposition 7*. Let $\{M_i\}_{i\in I}$ be the family of those submodules of a module M for which M/M_i is finitely embedded. Then $\{M_i\}$ is an inverse system and $\cap_{i\in I}M_i = 0$.

Proof. It is clear from the previous corollary that $\{M_i\}_{i\in I}$ is an inverse system. Let $x \in M$, $x \neq 0$. Then there is a simple module S and an epimorphism $Rx \to S$. Also, the mapping $Rx \to S \to E(S)$ can be extended to M since E(S) is injective. Therefore we have a homomorphism $\varphi : M \to E(S)$ such that $\varphi(x) \neq 0$. It follows that $M/\operatorname{Ker}\varphi$ is finitely embedded and $x \notin \operatorname{Ker}\varphi$. This proves that $\cap_{i\in I}M_i = 0$.

Remark. The duality here is very deceptive. Proposition 7 states, in effect, that every module is the direct limit of its finitely generated submodules. It is not true, however, that every module is the inverse limit of its finitely embedded factor modules. The reasons for this can be found in category theory, see e.g. [4, Proposition 6, chap.I.]

Definition. The R-module M is said to be 'singly embedded' if $E(M) = E(S)$ where S is a simple module.

Proposition 8. Every finitely generated module has a chain of submodules with singly generated chain factors.

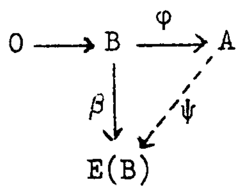
Proposition 8*. Every finitely embedded module has a chain of submodules with singly embedded chain factors.

Proof. Let M be a finitely embedded module and $E(M) = E(S_1) \oplus \dots \oplus E(S_k)$, $S_1, \dots, S_k$ simple modules. If $\pi_i : E(M) \to E(S_i)$ is the natural projection and $\alpha : M \to E(M)$ the injection of M into E(M), then M/N_i is singly embedded for $N_i = \operatorname{Ker} \pi_i \alpha$, $1 \leq i \leq k$. Also, $N_1 \cap \dots \cap N_k = 0$. A typical chain factor of the chain $0 = N_1 \cap \dots \cap N_k \subseteq N_1 \cap \dots \cap N_{k-1} \subseteq \dots \subseteq N_1 \subseteq M$ is of the form $N_1 \cap \dots \cap N_i / N_1 \cap \dots \cap N_{i+1} \approx N_1 \cap \dots \cap N_i + N_{i+1} / N_{i+1} \subseteq M/N_{i+1}$. The proposition now follows.

Proposition 9. *Let* $\varphi : A \to B$ *be an epimorphism of R-modules,* B *finitely generated. Then there is a monomorphism* $\alpha : A' \to A$ *such that* A' *is finitely generated and* $\operatorname{Im} \varphi\alpha = B$.

Proposition 9*. *Let* $\varphi : B \to A$ *be a monomorphism of R-modules,* B *finitely embedded. Then there is an epimorphism* $\alpha : A \to A'$ *such that* A' *is finitely embedded and* $\operatorname{Coim} \alpha\varphi = B$.

Proof. Consider the diagram



where β is the injection of B into E(B). Since E(B) is injective there is ψ such that $\psi\varphi = \beta$. Set $A' = \psi(A)$ and let α be the epimorphism $A \to A'$ induced by ψ. Then A' is finitely embedded and $\operatorname{Coim} \alpha\varphi = B$.

For a further discussion on finitely embedded modules and related problems the reader is referred to [8].

CHAPTER 2

LENGTH FUNCTIONS

2.1 Preliminaries.

Throughout section 1 and 2, the ring R will be kept fixed and all the categories are understood to be subcategories of $\mathfrak{M}(R)$.

Let $\mathfrak{A}$ be a semi-closed category and L a function from $\mathfrak{A}$ into the set of non-negative real numbers and plus infinity.

Definition. The function L on $\mathfrak{A}$ is called a 'length function' on $\mathfrak{A}$ if it satisfies the following conditions:

(i) $L(0) = 0$

(ii) $L(A) = L(A') + L(A'')$ whenever

$$0 \to A' \to A \to A'' \to 0$$

is an exact sequence in $\mathfrak{A}$.

Remark. Condition (i) is almost superfluous. In fact if $L(A) < +\infty$ for some $A \in \mathfrak{A}$, then (ii) implies (i). Condition (ii) will sometimes be referred to as the additivity property.

It follows immediately from the definition that if A,B are isomorphic modules in $\mathfrak{A}$ then $L(A) = L(B)$. Also, if A' is a submodule of $A \in \mathfrak{A}$ then $L(A') \leq L(A)$ and $L(A/A') \leq L(A)$. It is useful to note a few easy consequences of the axioms.

Let L be a length function on a semi-closed category $\mathfrak{A}$. If A is a module in $\mathfrak{A}$ and a chain

$$\sigma : 0 = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = A$$

of submodules of A is given then a straightforward induction argument yields the formula

$$L(A) = \Sigma_{i=1}^{n} L(A_i/A_{i-1}).$$

In particular, a finite direct sum $A = A_1 \oplus \ldots \oplus A_n$ gives $L(A) = L(A_1) + \ldots + L(A_n)$.

Consider now a long exact sequence

$$0 \to A_1 \to A_2 \to \ldots \to A_n \to 0 \qquad (1)$$

of modules in $\mathfrak{A}$. If $n = 1$ then $A_1 = 0$ and $L(A_1) = 0$. If $n = 2$ then $A_1 \approx A_2$ and $L(A_1) = L(A_2)$. Next we have $L(A_1) + L(A_3) = L(A_2)$ in the case $n = 3$. We contend that, in general

$$\sum_{i \text{ odd}} L(A_i) = \sum_{j \text{ even}} L(A_j) \quad (1 \leq i,j \leq n). \qquad (2)$$

The cases $n = 1,2,3$ have already been established. Assume that (2) holds for $n-1 > 2$. We may assume, without loss of generality, that n is odd, say. Put $B = \mathrm{Im}(A_{n-2} \to A_{n-1}) \approx \mathrm{Ker}(A_{n-1} \to A_n)$. The sequence (1) gives rise to exact sequences:

$$\begin{aligned} &0 \to A_1 \to \ldots \to A_{n-2} \to B \to 0 \text{ and} \\ &0 \to B \to A_{n-1} \to A_n \to 0. \end{aligned} \qquad (3)$$

Hence $L(A_1) + \ldots + L(A_{n-2}) = L(A_2) + \ldots + L(A_{n-3}) + L(B)$ by the induction hypothesis and $L(A_n) + L(B) = L(A_{n-1})$. Then $L(A_1) + \ldots + L(A_n) + L(B) = L(A_2) + \ldots + L(A_{n-1}) + L(B)$. If $L(B) < \infty$ then (2) follows. If $L(B) = \infty$ then $L(A_{n-1}) \geq L(B) = \infty$ and $L(A_{n-2}) \geq L(B) = \infty$ from (3). Thus (2) is valid again since both sides are equal to ∞.

We summarize these facts in the following:

Proposition 1. Let L be a length function on a semi-closed category $\mathfrak{A}$ and A_i, B_i, C_i $1 \leq i \leq n$ modules in $\mathfrak{A}$.

(i) *If* $0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A$ *is a chain of submodules of* A *then* $L(A) = \Sigma_{i=1}^{n} L(A_i/A_{i-1})$.

(ii) *If* $B = B_1 \oplus \cdots \oplus B_n$ *then* $L(B) = L(B_1) + \cdots + L(B_n)$.

(iii) *If* $0 \to C_1 \to C_2 \to \cdots \to C_n \to 0$ *is an exact sequence then*

$$\sum_{\substack{i \text{ odd} \\ 1 \leq i \leq n}} L(C_i) = \sum_{\substack{j \text{ even} \\ 1 \leq j \leq n}} L(C_j).$$

Remark. If we have $L(C_i) < \infty$ $1 \leq i \leq n$ in (iii) then the result takes the form $\Sigma_{i=1}^{n}(-1)^i L(C_i) = 0$.

Proposition 2. *Let* L *be a length function on a Serre-category* $\mathfrak{U}$ *and* A_1, A_2, B_1, B_2 *submodules of the R-module* M. *Then we have*

(i) $L(A_1+A_2) + L(A_1 \cap A_2) = L(A_1) + L(A_2)$ *whenever* $A_1 + A_2 \in \mathfrak{U}$,

(ii) $L(M/B_1+B_2) + L(M/B_1 \cap B_2) = L(M/B_1) + L(M/B_2)$ *whenever* $M/B_1 \cap B_2 \in \mathfrak{U}$.

Proof. Note that all the modules which occur belong to $\mathfrak{U}$ by Lemma 1.2.

Consider the following exact sequences:

$$0 \to A_1 \to A_1+A_2 \to A_2/A_1 \cap A_2 \to 0 \tag{4}$$

$$0 \to A_1 \cap A_2 \to A_2 \to A_2/A_1 \cap A_2 \to 0. \tag{5}$$

Again, all the modules are in $\mathfrak{U}$. From (4) we obtain

$$L(A_1+A_2) = L(A_1) + L(A_2/A_1 \cap A_2).$$

Hence $L(A_1+A_2) + L(A_1 \cap A_2) = L(A_1) + L(A_2/A_1 \cap A_2) + L(A_1 \cap A_2)$ $= L(A_1) + L(A_2)$ since $L(A_2) = L(A_1 \cap A_2) + L(A_2/A_1 \cap A_2)$ from (5). Part (ii) is established in a similar way.

Let $L, \{L_i\}_{i \in I}$ be length functions on a semi-closed category $\mathfrak{U}$ and $c \geq 0$ a real number. The functions cL, $\Sigma_{i \in I} L_i$ defined by $(cL)(A) = cL(A)$; $(\Sigma_{i \in I} L_i)(A) = \sup_J \Sigma_{i \in J} L_i(A)$ where J ranges over the finite subsets of I, $A \in \mathfrak{A}$. It is easily seen that both cL and $\Sigma_{i\in I} L_i$ are length functions on $\mathfrak{A}$. Also, if L, L' are length functions on $\mathfrak{A}$ and $L \geq L'$ (i.e. $L(A) \geq L'(A)$ for all $A \in \mathfrak{A}$) then set

$$(L-L')(A) = \begin{cases} L(A) - L'(A) & \text{if } L(A) < \infty \\ \infty & \text{otherwise.} \end{cases}$$

We see at once that $L - L'$ is a length function and $L = L - L' + L'$.

For a Serre-category $\mathfrak{A}$ and a length function L on $\mathfrak{A}$ the <u>'kernel of L, Ker L</u> and the <u>'domain of finiteness of L' Fin L</u> are defined as follows:

$$\text{Ker } L = \{A \in \mathfrak{A} : L(A) = 0\},$$

$$\text{Fin } L = \{A \in \mathfrak{A} : L(A) < \infty\}.$$

It follows immediately that Ker L and Fin L are Serre-categories and $0 \subseteq \text{Ker} \subseteq L \text{ Fin } L \subseteq \mathfrak{A}$.

2.2 Extensions and continuity

Let $\mathfrak{A} \subseteq \mathfrak{B}$ be categories and L, L' length functions on $\mathfrak{A}$ and $\mathfrak{B}$ respectively. If $L'(A) = L(A)$ for all $A \in \mathfrak{A}$ then we say that <u>L' is an extension of L to $\mathfrak{B}$</u> and <u>L is the restriction of L' to $\mathfrak{A}$</u>.

If a length function L is defined on a Serre-category $\mathfrak{A}$ then we can easily extend L to $\mathfrak{M}(R)$ by simply setting $L(A) = \infty$ for $A \notin \mathfrak{A}$. The additivity of L - now defined on $\mathfrak{M}(R)$ - is easily checked. This extension is called the <u>trivial extension</u> of L. Before discussing a more satisfactory method of extensions we wish to introduce a convenient notational device. Assume that L is a function on a category $\mathfrak{A}$ and define

$$L_{\mathfrak{U}}(A) = \begin{cases} L(A) \text{ if } A \in \mathfrak{U} \\ 0 \quad \text{otherwise.} \end{cases} \quad (A \in \mathfrak{M}(R)).$$

For a given chain

$$\sigma : 0 = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n = A \tag{6}$$

of submodules of A write

$$L(\sigma, A, \mathfrak{U}) = \sum_{i=1}^{n} L_{\mathfrak{U}}(A_i/A_{i-1}).$$

We simply write $L(\sigma)$ or $L(\sigma,A)$ when A or $\mathfrak{U}$ or both are fixed, and this notation will be adopted throughout this section. Assume now that $\mathfrak{U}$ is a semi-closed category, L is a length function on $\mathfrak{U}$ and the chain τ of A is a refinement of the chain σ in (6). If $L_{\mathfrak{U}}(A_j/A_{j-1})$ $1 \leq j \leq n$ is not zero, i.e. $A_j/A_{j-1} \in \mathfrak{U}$, then all the chain factors of τ which arise by the refinement of the $A_{j-1} \subseteq A_j$ part of σ will again belong to $\mathfrak{U}$ since $\mathfrak{U}$ is semi-closed. Consequently $L(\sigma,A) \leq L(\tau,A)$. In other words, if σ is refined then $L(\sigma,A)$ is not decreased. Set

$$\hat{L}(A) = \sup_{\sigma} L(\sigma,A), \quad A \in \mathfrak{M}(R)$$

where the supremum is taken over all chains σ of the module A. We claim that $\hat{L}$ is a length function on $\mathfrak{M}(R)$ and extends L. For if A' is a submodule of A and σ is any chain of A then there are equivalent chains σ' and τ of A such that σ' is a refinement of σ and τ is a refinement of $0 \subseteq A' \subseteq A$. The chain τ induces chains τ' and τ'' of A' and A/A' respectively. Moreover,

$$L(\tau', A') + L(\tau'', A/A') = L(\tau, A) = L(\sigma', A) \geq L(\sigma, A).$$

Since σ was arbitrary, we deduce that

$$\hat{L}(A') + \hat{L}(A/A') \geq \hat{L}(A).$$

Conversely, let ρ,τ be chains of A' and A/A' respectively. If σ denotes the chain of A obtained by sticking ρ and τ together then $L(\rho,A') + L(\tau,A/A') = L(\sigma,A)$. Hence $\hat{L}(A') + \hat{L}(A/A') \leq \hat{L}(A)$.

Definition. The function $\hat{L}$ constructed above is called the 'continuous extension' of L with respect to the semi-closed category $\mathfrak{A}$.

We now characterize $\hat{L}$ as a minimal extension of L to $\mathfrak{M}(R)$. More precisely we prove:

Theorem 3. Let $\mathfrak{A}$ be a semi-closed category, L a length function on $\mathfrak{A}$, $\hat{L}$ the continuous extension of L and L' another extension of L to $\mathfrak{M}(R)$. Then $\hat{L}(A) \leq L'(A)$ for all $A \in \mathfrak{M}(R)$.

Proof. Let $A \in \mathfrak{M}(R)$ and σ a chain of A. We see from the definition of $L(\sigma,A)$ that $L(\sigma,A) \leq L'(A)$ since L and L' agree on $\mathfrak{A}$. Hence

$$\hat{L}(A) = \sup_{\sigma} L(\sigma,A) \leq L'(A).$$

Corollary. Let $\mathfrak{A}$ be a semi-closed category, L a length function on $\mathfrak{A}$ and $\tilde{\mathfrak{A}}$ the Serre-category generated by $\mathfrak{A}$. Then L has a unique extension to $\tilde{\mathfrak{A}}$.

Proof. Since L can be extended to $\mathfrak{M}(R)$ by Theorem 3 we have only to prove that this extension is unique. Assume that L_1, L_2 are length functions on $\tilde{\mathfrak{A}}$ and they both extend L. For each $A \in \tilde{\mathfrak{A}}$ there is a chain

$$\sigma : 0 = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n = A$$

such that the chain factors $C_i = A_i/A_{i-1}$ $1 \leq i \leq n$ belong to $\mathfrak{A}$ by Proposition 1.1 Corollary. Therefore

$$L_1(A) = \sum_i L_1(C_i) = \sum_i L_2(C_i) = L_2(A)$$

since L_1 and L_2 agree on $\mathfrak{A}$.

The corollary shows that when dealing with length functions on a semi-closed category we may assume, without loss of generality, that the category in question is a Serre-category.

While the continuous extension of a length function L on a semi-closed category $\mathfrak{A}$ turned out to be the 'minimal' among the extensions of L, the 'maximal' extension can be obtained as follows. First extend L to $\tilde{\mathfrak{A}}$, the Serre-category generated by $\mathfrak{A}$, and then take the trivial extension to $\mathfrak{M}(R)$.

Let $\mathfrak{A} \subseteq \mathfrak{B}$ be semi-closed categories and L a length function on $\mathfrak{B}$. The continuous extension of L with respect to $\mathfrak{A}$ is called the '<u>continuous component of L</u>' with respect to $\mathfrak{A}$. In other words this is the continuous extension of the restriction of L to $\mathfrak{A}$. For any length function L whose domain † contains $\mathfrak{A}$, the continuous component of L with respect to $\mathfrak{A}$ is denoted by $\hat{L}$. Thus $\hat{L}$ is a length function on $\mathfrak{M}(R)$. Since $\hat{L}$ and L agree on $\mathfrak{A}$, $\hat{L} \leqslant L$ on the domain of L, by Theorem 3. Clearly $\hat{\hat{L}} = \hat{L}$ and $L \to \hat{L}$ is a closure operation.

<u>Proposition 4</u>. <u>Let $\mathfrak{A}$ be a semi-closed category and $K, K_1, \{L_i\}_{i\in I}$ length functions whose domains contain $\mathfrak{A}$. If $\hat{L}$ denotes the continuous component of L with respect to $\mathfrak{A}$ then</u>

(i) $\widehat{(cK)} = c\hat{K}$ for a real number $c \geqslant 0$;

(ii) $K \leqslant K_1$ implies $\hat{K} \leqslant \hat{K}_1$;

(iii) $\widehat{(\Sigma_{i\in I} L_i)} = \Sigma_{i\in I} \hat{L}_i$.

<u>Proof</u>. Statements (i) amd (ii) follow immediately from the definition.

In order to prove (iii) assume, first, that $I = \{1,\ldots,n\}$ is a

† The semi-closed category on which L is defined.

finite set and let $A \in \mathfrak{M}(R)$. For any $\varepsilon > 0$ and $i \in I$ there is a chain σ_i of A such that $L_i(\sigma_i) = L_i(\sigma_i, A) > \hat{L}_i(A) - \varepsilon/n$. Let σ be a common refinement of the chains σ_i. then

$$\left(\sum_{i\in I} L_i\right)(\sigma) > \left(\sum_{i\in I} \hat{L}_i\right)(A) - \varepsilon.$$

Hence

$$\left(\widehat{\sum_{i\in I} L_i}\right)(A) \geq \left(\sum_{i\in I} \hat{L}_i\right)(A).$$

On the other hand

$$\left(\widehat{\sum_{i\in I} L_i}\right)(A) = \sup_{\sigma}\left(\sum_{i\in I} L_i\right)(\sigma) = \sup_{\sigma} \sum_{i\in I} L_i(\sigma) \leq$$

$$\sum_{i\in I} \sup_{\sigma_i} L_i(\sigma_i) = \sum_{i\in I} \hat{L}_i(A) = \left(\sum_{i\in I} \hat{L}_i\right)(A).$$

This shows that $(\widehat{\Sigma_{i\in I} L_i}) = \Sigma_{i\in I} \hat{L}_i$ when I is finite. Next, let I be an arbitrary index set. Using the above result for the finite case we obtain

$$\left(\widehat{\sum_{i\in I} L_i}\right)(A) = \sup_{\sigma}\left(\sum L_i\right)(\sigma) = \sup_{\sigma}\sup_{J}\left(\sum_{i\in J} L_i\right)(\sigma) =$$

$$\sup_{J}\sup_{\sigma}\left(\sum_{i\in J} L_i\right)(\sigma) = \sup_{J}\left(\sum_{i\in J} \hat{L}_i\right)(A) = \left(\sum_{i\in I} \hat{L}_i\right)(A),$$

where J and σ range over all the finite subsets of I and chains of submodules of A respectively.

<u>Definition.</u> <u>Let $\mathfrak{A}, \mathfrak{B}$ be semi-closed categories</u>, $\mathfrak{A} \subseteq \mathfrak{B}$. <u>A length function L on $\mathfrak{B}$ is said to be continuous on $\mathfrak{A}$ if $L = \hat{L}$ holds on $\mathfrak{B}$, where $\hat{L}$ is the continuous component of L with respect to $\mathfrak{A}$.</u>

Note that if a function is continuous on $\mathfrak{A}$ then it is uniquely determined by its values on $\mathfrak{A}$. Also $L = \hat{L} + L - \hat{L}$ on $\mathfrak{B}$ since $\hat{L} \leq L$, and $L - \hat{L}$ is 0 or ∞ on $\mathfrak{A}$. Thus every function admits a (unique) decomposition into 'continuous' and 'singular' parts.

<u>Proposition 5. Let the category $\mathfrak{U}$ be the intersection of the semi-closed categories $\mathfrak{U}_1,\dots,\mathfrak{U}_s$ and L a length function on $\mathfrak{M}(\mathbf{R})$. Then L is continuous on $\mathfrak{U}$ if and only if it is continuous on all the $\mathfrak{U}_i$'s.</u>

<u>Remark</u>. There is no loss of generality in assuming that L is on $\mathfrak{M}(R)$ and not on a category whose domain contains $\mathfrak{U}$. Indeed, one can always extend L to $\mathfrak{M}(R)$. As for continuity, see the corollaries after the proposition.

<u>Proof</u>. Let $\hat{L}^i$ and $\hat{L}$ be the continuous components of L with respect to $\mathfrak{U}_i$ and $\mathfrak{U}$ respectively. By Theorem 3 $\hat{L} \leq \hat{L}^i \leq L \quad 1 \leq i \leq s$. Therefore $L = \hat{L}^i \quad 1 \leq i \leq s$ if $L = \hat{L}$.

The second part is proved by induction on s. For $s = 1$ there is nothing to prove. Suppose that $s = 2$ and let $A \in \mathfrak{M}(R)$ and $\varepsilon > 0$ be given. Since L is continuous on $\mathfrak{U}_1$ we can find a chain

$$\sigma : 0 = A_0 \subsetneq A_1 \subsetneq \dots \subsetneq A_n = A, \quad C_i = A_i/A_{i-1} \quad 1 \leq i \leq n$$

such that $L(\sigma,A,\mathfrak{U}_1) > L(A) - \varepsilon$. Suppose that $C_{i_1},\dots,C_{i_k}$ are the chain factors lying in $\mathfrak{U}_1$, in other words those which contribute to $L(\sigma,A,\mathfrak{U}_1)$. Again, L is continuous on $\mathfrak{U}_2$ and we have chains $\tau_1,\dots,\tau_k$ of the modules $C_{i_1},\dots,C_{i_k}$ such that $L(\tau_j,C_{i_j},\mathfrak{U}_2) > L(C_{i_j}) - \varepsilon$ $1 \leq j \leq k$. But every chain factor of τ_j belongs to $\mathfrak{U}_1$, because $C_{i_j} \in \mathfrak{U}_1$ and $\mathfrak{U}_1$ is semi-closed. Accordingly $L_{\mathfrak{U}_2}$ and $L_{\mathfrak{U}}$ agree on $\mathfrak{U}_1$ since $\mathfrak{U} = \mathfrak{U}_1 \cap \mathfrak{U}_2$. Hence $L(\tau_j,C_{i_j},\mathfrak{U}_2) = L(\tau_j,C_{i_j},\mathfrak{U}) \quad 1 \leq j \leq k$. Thus, if σ' is the chain obtained from σ by inserting $\tau_1,\dots,\tau_k$ then

$$L(\sigma',A,\mathfrak{U}) \geq \sum_{j=1}^{k} L(\tau_j,C_{i_j},\mathfrak{U}_2) > \sum_{j=1}^{k} L(C_{i_j}) - k\varepsilon > L(A) - (k+1)\varepsilon.$$

It now follows from the definition of $\hat{L}$ that $\hat{L}(A) \geq L(A)$, whence $\hat{L} = L$.

Finally, assume $s > 2$ and the proposition is proved for $s - 1$.

Put $\mathfrak{B} = \mathfrak{U}_1 \cap \ldots \cap \mathfrak{U}_{s-1}$. Then L is continuous on $\mathfrak{B}$ by the induction hypothesis and L is continuous on $\mathfrak{U}_s$ by assumption. Since the case s = 2 has already been proved we can conclude that L is continuous on $\mathfrak{B} \cap \mathfrak{U}_s = \mathfrak{U}$.

Corollary 1. Suppose that two semi-closed categories $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are given and $\mathfrak{U}_1 \subseteq \mathfrak{U}_2$. If the length function L is continuous on $\mathfrak{U}_1$ then it is continuous on $\mathfrak{U}_2$.

Proof. Observe that $\mathfrak{U}_1 = \mathfrak{U}_1 \cap \mathfrak{U}_2$.

Corollary 2. Let $\mathfrak{U}$ be a semi-closed category, $\widetilde{\mathfrak{U}}$ the Serre-category generated by $\mathfrak{U}$ and L a length function. Then L is continuous on $\mathfrak{U}$ if and only if it is continuous on $\widetilde{\mathfrak{U}}$.

Proof. It will suffice to prove that L is continuous on $\widetilde{\mathfrak{U}}$ implies that L is continuous on $\mathfrak{U}$. Assume that L is continuous on $\widetilde{\mathfrak{U}}$ and let $\hat{L}$ be the continuous component of L with respect to $\mathfrak{U}$. Then $\hat{L} \leq L$. But L and $\hat{L}$ agree on $\widetilde{\mathfrak{U}}$ by the corollary to Theorem 3. Thus $\hat{L}$ extends L from $\widetilde{\mathfrak{U}}$. Therefore $L \leq \hat{L}$, this time by Theorem 1 itself.

Definition. A length function L on a semi-closed category $\mathfrak{U}$ is said to be 'upper continuous' if for each $A \in \mathfrak{U}$, $L(A) = \sup_X L(X)$, where X ranges over all the finitely generated submodules of A. Dually, L is called 'lower continuous' if for each $A \in \mathfrak{U}$, $L(A) = \sup_X L(X)$, where X ranges over all the finitely embedded factor modules of A.

In the work of Northcott and Reufel [3] upper continuity is incorporated into the definition of a length function. The term 'upper (lower) continuous' was used in [7] with a slightly different meaning.

If L is an upper (resp.lower) continuous length function on $\mathfrak{M}(R)$

then it is uniquely determined by its values on singly generated (resp. single embedded) modules by virtue of Proposition 1.8.

Proposition 6. Let L be an upper continuous length function on a semi-closed category $\mathfrak{A}$ and $\hat{L}$ the continuous extension of L to $\mathfrak{M}(R)$ (with respect to $\mathfrak{A}$). Then $\hat{L}$ is upper continuous.

Proposition 6*. If L is a lower continuous length function on a semi-closed category $\mathfrak{A}$ and $\hat{L}$ is the continuous extension of L to $\mathfrak{M}(R)$ then $\hat{L}$ is lower continuous.

Proof (of Proposition 6). We prove the proposition by showing that for any R-module A, $\varepsilon > 0$ and chain

$$\sigma : 0 = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n = A \tag{7}$$

of A, there is a finitely generated submodule X of A such that

$$\hat{L}(X) \geq L(\sigma, A) - \varepsilon. \tag{8}$$

We proceed by induction on n. If n = 1 then either $L(\sigma, A) = 0$ and there is nothing to prove, or $A \in \mathfrak{A}$ in which case $L(\sigma, A) = L(A)$. By assumption, L is upper continuous on $\mathfrak{A}$. Hence $L(X) > L(A) - \varepsilon$ for some finitely generated submodule X of A, $X \in \mathfrak{A}$. Therefore $\hat{L}(X) = L(X) > L(\sigma, A) - \varepsilon$. Assume now that (8) has been proved for all pairs σ, A, where σ has at most n-1 inclusions. Then we have a finitely generated X' of A_{n-1} in (7) such that $\hat{L}(X') > L(\sigma', A_{n-1}) - \varepsilon/2$ where σ' denotes the chain $0 = A_0 \subseteq \ldots \subseteq A_{n-2} \subseteq A_{n-1}$. If $A/A_{n-1} \notin \mathfrak{A}$ then $L(\sigma', A_{n-1}) = L(\sigma, A)$. We may, therefore, assume that $A/A_{n-1} \in \mathfrak{A}$. Using Proposition 1.3 and the fact that L is upper continuous on $\mathfrak{A}$, we find a finitely generated submodule Y of A such that $\hat{L}(Y+A_{n-1}/A_{n-1}) = L(Y+A_{n-1}/A_{n-1}) > L(A/A_{n-1}) - \varepsilon/2$. Put $X = X' + Y$.

Then

$$\hat{L}(X) = \hat{L}(X'+Y) = \hat{L}(X') + \hat{L}(Y/X'\cap Y) \geq \hat{L}(X') + \hat{L}(Y/A_{n-1}\cap Y) =$$
$$\hat{L}(X') + \hat{L}(Y + A_{n-1}/A_{n-1}) > L(\sigma', A_{n-1}) - \varepsilon/2 + L(A/A_{n-1}) - \varepsilon/2 =$$
$$L(\sigma, A) - \varepsilon.$$

This establishes (8). Taking supremums on both sides of (8) we obtain

$$\sup_X \hat{L}(X) \geq \hat{L}(A) - \varepsilon,$$

X ranges over all the finitely generated submodules of A. Since this holds for all $\varepsilon > 0$, $\sup_X \hat{L}(X) \geq \hat{L}(A)$. The proposition now follows.

The proof of Proposition 6* is analogous to Proposition 6 and omitted.

Proposition 7. Let $\mathfrak{F}, \mathfrak{H}$ denote the category of finitely generated and finitely embedded modules respectively.† If $\mathfrak{A}, \mathfrak{B}$ are semi-closed categories and L is a length function on $\mathfrak{M}(R)$ then:

(i) if $\mathfrak{A} \subseteq \mathfrak{F}$ (resp. $\mathfrak{A} \subseteq \mathfrak{H}$) and L is continuous on $\mathfrak{A}$ then L is upper (resp. lower) continuous;

(ii) if $\mathfrak{F} \subseteq \mathfrak{B}$ (resp. $\mathfrak{H} \subseteq \mathfrak{B}$) and L is upper (resp. lower) continuous then L is continuous on $\mathfrak{B}$.

Proof. If L is continuous on $\mathfrak{A}$ and $\mathfrak{A} \subseteq \mathfrak{F}$ then L is upper continuous on $\mathfrak{A}$ trivially. Now Proposition 6 establishes (i).

Suppose that L is upper continuous, $\mathfrak{F} \subseteq \mathfrak{B}$, and let $\hat{L}$ be the continuous component of L with respect to $\mathfrak{B}$. Then $\hat{L} \leq L$. If X is finitely generated then $X \in \mathfrak{B}$ and $L(X) = \hat{L}(X)$. Hence $L(A) = \sup_X L(X) = \sup_X \hat{L}(X) \leq \hat{L}(A)$ where $A \in \mathfrak{M}(R)$ and X ranges over the finitely generated submodules of A. Thus $L = \hat{L}$ and L is continuous on $\mathfrak{B}$.

† $\mathfrak{F}, \mathfrak{H}$ are not semi-closed. $\mathfrak{F}$ semi-closed <=> R is left-Noetherian. As for $\mathfrak{H}$, see [8].

Corollary. Let $\mathfrak{A}$ and $\mathfrak{B}$ be the category of Noetherian and Artinian modules respectively. Then $\mathfrak{A} \subseteq \mathfrak{F}$ and $\mathfrak{B} \subseteq \mathfrak{H}$. If L is continuous on $\mathfrak{A}$ (resp. $\mathfrak{B}$) then L is upper (resp. lower) continuous. If $\mathfrak{A} = \mathfrak{F}$ (resp. $\mathfrak{B} = \mathfrak{H}$) then the two concepts are equivalent.

Thus, if R is left Noetherian there is no need to distinguish between upper continuous functions and functions continuous on Noetherian modules. A different characterization is presented in the next theorem.

Theorem 8. For a length function L on $\mathfrak{M}(R)$ the following are equivalent:

(i) L is upper continuous;

(ii) for any module A and direct systems $\{A_i\}_{i\in I}$ of submodules of A such that $\Sigma_i A_i = A$, $L(A) = \sup_i L(A_i)$;

(iii) for any module A and totally ordered set $\{A_i\}_{i\in I}$ of submodules of A such that $\Sigma_i A_i = A$, $L(A) = \sup_i L(A_i)$.

Proof. (i) $\Rightarrow$ (ii). By definition $L(A) = \sup_X L(X)$ where the supremum is taken over the finitely generated submodules of A. For a typical finitely generated submodule X of A an index $j \in I$ can be found so that $X \subseteq A_j$. Consequently,

$$L(A) \geq \sup_i L(A_i) \geq \sup_X L(X) = L(A).$$

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) Suppose $L(A) > \sup_X L(X) = c$, where X varies over the finitely generated submodules of A. We can construct an ascending sequence $X_1, X_2, \ldots, X_n, \ldots$ of finitely generated submodules of A so that $L(X_n) > c - 1/n$. Hence for $B = \bigcup_{n=1}^{\infty} X_n$, $L(B) = c$ and $B \neq A$. By our assumption and Zorn's Lemma there is a submodule M of A which is maximal with respect to the property of containing B and $L(M) = c$.

Thus $M \neq A$ and there is a singly generated submodule $S \subseteq A$, $S \nsubseteq M$. We have

$$d = L(S/S \cap M) = L(S+M/M) = L(S+M) - L(M) > 0 \qquad (9)$$

by the maximality of M. (Note that $L(M) < \infty$ since $L(M) = c < L(A)$.) Choose the positive integer k such that $L(X_k) > c-d$, Then $X_k + S$ is finitely generated and

$$L(X_k+S) = L(X_k) + L(S/X_k \cap S) \geq L(X_k) + L(S/M \cap S) > c$$

by (9). On the other hand $L(X_k+S) \leq c$, a contradiction.

Remark. This theorem does not dualize, not, at least, without modification. The reason for this is, that we had to use the fact that every module is a direct limit of finitely generated submodules. This phenomenon has no counterpart as it was pointed out in section 3 of chapter 1.

The essence of Theorem 8 is, roughly speaking, that upper continuous functions commute with direct limits. Before giving the exact meaning of this it might be helpful to give the definition of direct and inverse limits.

The partially ordered set I is called directed if any two (and consequently finite number) elements of I have an upper bound in I. Let I be a directed set and $\{A_i\}_{i \in I}$ a family of R-modules with a collection of homomorphisms $\alpha_{i,j} : A_i \to A_j$, $i \leq j$, satisfying $\alpha_{j,k}\alpha_{i,j} = \alpha_{i,k}$ whenever $i \leq j \leq k$. Let D be an R-module and $\varphi_i : A_i \to D$ homomorphisms for each $i \in I$ satisfying

$$\varphi_j \alpha_{i,j} = \varphi_i, \text{ for all pairs } i \leq j. \qquad (10)$$

The pair (D, φ_i) is called the direct limit of the system $(A_i, \alpha_{i,j})$ if, for any other module D' the homomorphisms $\varphi'_i : A_i \to D'$ satisfying

(10) there is a unique homomorphism $\psi : D \to D'$ such that $\psi\varphi_i = \varphi'_i$ for all $i \in I$. Inverse limits are defined analogously. It is well known that direct and inverse limits always exist in $\mathfrak{M}(R)$ and they are unique up to isomorphism. If (D,φ_i) is the direct limit of the directed system $(A_i,\alpha_{i,j})$ (notation $D = \varinjlim A_i$) then the submodules $\{\mathrm{Im}\,\varphi_i\}_{i\in I}$ of D form a direct system and $\Sigma_i \mathrm{Im}\,\varphi_i = D$. Consequently we can state

Corollary. The length function L on $\mathfrak{M}(R)$ is upper continuous if and only if for each direct limit of modules $(D,\varphi_i) = \varinjlim A_i$ we have $L(D) = \sup_i L(A_i/\mathrm{Ker}\,\varphi_i)$.

The easy part of Theorem 8 can be carried over to the lower continuous case:

Proposition 9. Let L be a length function on $\mathfrak{M}(R)$ and assume that for each module A and inverse system of submodules $\{A_i\}_{i\in I}$ of A such that $\cap_i A_i = 0$, $L(A) = \sup_i L(A/A_i)$. Then L is lower continuous.

Proof. According to Proposition 1.7 the family of submodules of X of A for which A/X is finitely embedded form an inverse system with zero intersection. Hence L is lower continuous.

2.3 Change of rings.

Let R,S be rings and T an exact† functor from $\mathfrak{M}(R)$ to $\mathfrak{M}(S)$. Let L be a length function on $\mathfrak{M}(S)$ and define L_T on $\mathfrak{M}(R)$ by

$$L_T(A) = L(T(A)), \qquad A \in \mathfrak{M}(R) \tag{11}$$

Evidently, L_T is a length function on $\mathfrak{M}(R)$. In most cases in the present work T is induced by the functors Hom and $\otimes$. It is assumed that the reader is familiar with the definitions and the elementary properties of these functors. For an introduction to the tensor

† It is immaterial, at this point, whether T is covariant or contravariant.

product $\otimes$, the reader is referred to [2 chap.V. 1-5]. We now briefly illustrate how these functors will arise.

Let ${}_S M_R$ be a right R and left S-bimodule, i.e. M is a left S, a right R-module and $s(ar) = (sa)r$ for all $r \in R, s \in S, a \in M$. Write

$$T(A) = {}_S M_R \otimes_R A, \quad A \in \mathfrak{M}(R).$$

Then T is a covariant functor and commutes with direct limits. Also, M is called <u>right R-flat</u> if T is exact. If T is exact and $\{A_i\}_{i\in I}$ is a direct family of submodules of A and $\Sigma_i A_i = A$ then $\{T(A_i)\}_{i\in I}$ form a direct family of submodules of T(A) and $\Sigma_{i\in I} T(A_i) = T(A)$. Thus L_T in (11) is upper continuous whenever L is upper continuous.

Next, consider a left R-S-bimodule ${}_{R,S}M$, i.e. a module M which is a left R, left S-module and $r(s(a)) = s(r(a))$ for all $r \in R, s \in S, a \in M$. Set

$$U(A) = \mathrm{Hom}_R(A,M), \quad A \in \mathfrak{M}(R).$$

Then U is a contravariant functor from $\mathfrak{M}(R)$ to $\mathfrak{M}(S)$ and turns direct limits into inverse limits. Also, U is exact if and only if M, as an R-module, is injective (see chapter I, section 3). If L is a length function on $\mathfrak{M}(S)$ such that $L(M) = \sup_i L(M/M_i)$ whenever $\{M_i\}_{i\in I}$ is an inverse system of submodules of M, $\cap_i M_i = 0$ and if U is exact then L_T in (11) is upper continuous. Summing up we obtain

<u>Proposition 10.</u> <u>Let</u> R,S <u>be rings,</u> T <u>an exact functor from</u> $\mathfrak{M}(R)$ <u>to</u> $\mathfrak{M}(S)$ <u>and for each length function</u> L <u>on</u> $\mathfrak{M}(S)$ <u>define</u>

$$L_T(A) = L(T(A)) \quad A \in \mathfrak{M}(R).$$

<u>Then</u> L_T <u>is a length function on</u> $\mathfrak{M}(R)$ <u>and</u> L_T <u>is upper continuous if</u>

either of the following conditions is satisfied:

(i) $T(A) = {}_S M_R \otimes_R A$ where the S,R-bimodule ${}_S M_R$ is R-flat and L is upper continuous;

(ii) $T(A) = \mathrm{Hom}_R(A, {}_{R-S}M)$ where the R-S-bimodule ${}_{R-S}M$ is R-injective and $L(B) = \sup_i L(B/B_i)$ for all S-modules B and inverse system of submodules $\{B_i\}_{i \in I}$, $\cap_i B_i = 0$.

The next problem we propose to consider is the relation between length functions on $\mathfrak{M}(R)$ and on $\mathfrak{M}(R/I)$, I a two-sided ideal of R. If we denote by $\mathfrak{M}_I$ the category of R-modules annihilated by I then $\mathfrak{M}_I$ and $\mathfrak{M}(R/I)$ can be identified in the obvious way. Note that $\mathfrak{M}_I$ is a semi-closed subcategory of $\mathfrak{M}(R)$. It is clear that every length function on $\mathfrak{M}(R)$ induces a length function - its restriction to $\mathfrak{M}_I$ - on $\mathfrak{M}(R/I)$. Conversely, a length function on $\mathfrak{M}(R/I)$ yields a length function on $\mathfrak{M}_I$ and this, in turn, can be extended to $\mathfrak{M}(R)$ since $\mathfrak{M}_I$ is semi-closed. Further, Proposition 6 and 6* tell us that this extension preserves upper (resp. lower) continuity. Thus we have obtained

Proposition 11. Let R be a ring and I a two-sided ideal of R. There is a one-to-one correspondence between the length functions on $\mathfrak{M}(R/I)$ and the length functions on $\mathfrak{M}(R)$ which are continuous on $\mathfrak{M}_I$. Moreover, this correspondence preserves upper (lower) continuity.

Finally, we briefly mention two special cases. We refer the reader to [9] for a full description of the functors which will occur.

If the ring R is the finite direct sum of the rings $R_1, \ldots, R_n$ then $1_R = 1_{R_1} + \ldots + 1_{R_n}$ uniquely. If $A \in \mathfrak{M}(R)$ then

$A = 1_{R_1} A \oplus \ldots \oplus 1_{R_n} A$. The Serre-category $\mathfrak{M}_i = \{1_{R_i} A : A \in \mathfrak{M}(R)\}$ can be identified with $\mathfrak{M}(R_i)$ and the functors $D_i : \mathfrak{M}(R) \to \mathfrak{M}(R_i)$, $D_i(A) = 1_{R_i} A$, are exact, $1 \leq i \leq n$. It now follows that any length function on $\mathfrak{M}(R)$ is a unique sum of length functions on $\mathfrak{M}(R_i)$, $1 \leq i \leq n$.

Consider now R^n the ring of $n \times n$ matrices over a ring R, $n > 0$. It is known (e.g. [9]) that the categories $\mathfrak{M}(R^n)$ and $\mathfrak{M}(R)$ are equivalent, i.e. there are exact functors $S : \mathfrak{M}(R^n) \to \mathfrak{M}(R)$, $T : \mathfrak{M}(R) \to \mathfrak{M}(R^n)$ such that ST and TS are naturally equivalent to the identity functors of $\mathfrak{M}(R)$ and $\mathfrak{M}(R^n)$ respectively. This equivalence of categories induces a one-to-one correspondence between their respective length functions.

2.4 Examples.

I. The classical length function

Let R be a ring and $\mathfrak{S}$ the category of simple R-modules and 0. Then $\mathfrak{S}$ is semi-closed and if

$$0 \to A' \to A \to A'' \to 0$$

is an exact sequence in $\mathfrak{S}$ then either $A' \approx A$ and $A'' = 0$ or $A \approx A''$ and $A' = 0$. Define the function L on $\mathfrak{S}$ by setting $L(0) = 0$ and $L(S) = 1$, $S \in \mathfrak{S}$, $S \neq 0$. Then L is a length function on $\mathfrak{S}$. Let ℓ be the continuous extension of L to $\mathfrak{M}(R)$. The length function ℓ on $\mathfrak{M}(R)$ is called the <u>classical length function</u> and will always be denoted by ℓ. If $\mathfrak{A}$ and $\mathfrak{B}$ denotes the category of Noetherian and Artinian modules and $\tilde{\mathfrak{S}}$ the Serre-category generated by $\mathfrak{S}$ then $\mathfrak{S} \subseteq \tilde{\mathfrak{S}} = \mathfrak{A} \cap \mathfrak{B}$. Therefore ℓ is both upper and lower continuous by Proposition 7.

Assume now that L is a length function on $\mathfrak{M}(R)$ satisfying $L(S) = 1$ for all simple modules S in $\mathfrak{M}(R)$. Then L and ℓ agree on $\mathfrak{S}$ and hence on $\tilde{\mathfrak{S}}$. (Theorem 3 Corollary). It is easily seen that any non-zero module $A \in \mathfrak{M}(R)$ has a segment which is a simple module. Hence $L(A) < \infty$ if and only if $A \in \widetilde{\mathfrak{S}} = \mathfrak{A} \cap \mathfrak{B}$. In other words, $\operatorname{Ker} L = 0$, $\operatorname{Fin} L = \mathfrak{A} \cap \mathfrak{B}$. Thus we have obtained the following characterisation of the classical length function.

<u>Proposition 12</u>. <u>If</u> L <u>is a length function on</u> $\mathfrak{M}(R)$ <u>and</u> $L(S) = 1$ <u>for every simple module</u> S <u>in</u> $\mathfrak{M}(R)$ <u>then</u> $L = \ell$.

Note the following additional property of ℓ. If $\{A_i\}_{i \in I}$ is an inverse family of submodules of A and $\cap_i A_i = 0$ then $\ell(A) = \sup_i \ell(A/A_i)$. It will suffice to consider the case $\sup_i \ell(A/A_i) = n < \infty$. Then there is a $j \in I$ such that $\ell(A/A_j) = n$. If there were a j' with $A_{j'} \subset A_j$ then $\ell(A_j/A_{j'}) > 0$ which is impossible since $\ell(A/A_{j'}) = n$. Thus A_j is minimal in $\{A_i\}_{i \in I}$ which shows that $A_j = 0$ and $\ell(A) = n$.

<u>II. The rank</u>

The rank of a module is usually defined for modules over a commutative domain by means of 'dependence'. The non-commutative case is studied in [10] and [11]. The reader is advised to consult these papers if interested in the relationship between rank and dependence in modules.

A (not necessarily commutative) ring R is called a <u>domain</u> if $rs = 0$ implies $r = 0$ or $s = 0$, $r, s \in R$. Let R be a domain and Q a ring containing R as a subring. Then Q is called the <u>(left) quotient field</u> of R if the following conditions are satisfied:

(i) if $r \neq 0$, $r \in R$ then r^{-1} exists in Q;

(ii) every element of Q can be written in the form $r^{-1}s$, $r, s \in R$, $r \neq 0$.

A domain is called an <u>Ore-domain</u> if it has a quotient field. Quotient fields are unique up to isomorphism. Every commutative domain is an Ore-domain. Let R be an Ore-domain and Q its (left) quotient field. As a right R-module Q is flat (see e.g. [12]). Also, $\mathfrak{M}(Q)$ is the category of (left) vector spaces over Q and the classical length function ℓ on $\mathfrak{M}(R)$ assigns to each vector space V its dimension. The function L_r on $\mathfrak{M}(R)$, defined by

$$L_r(A) = \dim_Q(Q \otimes_R A)$$

is called the <u>rank-function</u> of $\mathfrak{M}(R)$. By Proposition 11, L_r is an upper continuous length function on $\mathfrak{M}(R)$. Also, $L_r(R) = 1$ since $Q \otimes_R R \approx Q$ as Q-modules.

<u>Proposition 13. Let R be a (left) Ore-domain and L an upper continuous length function on $\mathfrak{M}(R)$ such that L(R) = 1. Then L is the rank-function on $\mathfrak{M}(R)$.</u>

The commutative version of Proposition 12 was proved in [3, Theorem 2].

<u>Proof</u>. It is enough to show that for each left ideal I of R, $L(R/I) = L_r(R/I)$ because both functions are upper continuous. We may assume that $I \neq 0$. Choose a non-zero element $a \in I$. Then $L(R/I) \leq L(R/Ra) = L(R) - L(Ra) = 0$ since $R \approx Ra$. Hence $L(R/I) = 0$ for $I \neq 0$. Similarly, $L_r(R/I) = 0$ if $I \neq 0$. This establishes the proposition.

<u>III. Trivial functions</u>

The length function L on $\mathfrak{M}(R)$ is said to be '<u>trivial</u>' if it has values only 0 or ∞. Let $\mathfrak{A}$ be a Serre-category and L defined by $L(A) = 0$ if $A \in \mathfrak{A}$ and ∞ otherwise. Then L is a trivial length function and $\operatorname{Ker} L = \mathfrak{U}$. Conversely, if L is a trivial function then $\operatorname{Ker} L$ is a Serre-category. Thus there is a one-to-one correspondence between the Serre-categories of $\mathfrak{M}(R)$ and the trivial functions on $\mathfrak{M}(R)$.

If L is a length function on $\mathfrak{M}(R)$ and $\hat{L}$ is the continuous component of L with respect to $\operatorname{Fin} L$ then $L = \hat{L} + L - \hat{L}$ and $L - \hat{L}$ is a trivial function whose kernel is $\operatorname{Fin} L$.

CHAPTER 3

CHARACTERIZATION OF LENGTH FUNCTIONS ON CATEGORIES WITH KRULL-DIMENSION

In his fundamental work [4], P. Gabriel defined the Krull-dimension of an Abelian category. The terminology is justified by the fact that for a commutative Noetherian ring R, the Krull-dimension of $\mathfrak{M}(R)$ is equal to the Krull-dimension of the ring R. For our purpose Gabriel's definition is too restrictive. Accordingly, we give a modified definition which is more applicable to the problem of characterizing length functions. The terminology, however, is kept; partly because of the similarities and partly because they are equivalent on the category of Noetherian modules.

3.1. Quasi-simple modules

Let $\mathfrak{U},\mathfrak{B}$ be Serre-categories of $\mathfrak{M}(R)$, $\mathfrak{U} \subseteq \mathfrak{B}$.

Definition. A module S in $\mathfrak{B}$ is called '$\mathfrak{U}$-simple' if $S \notin \mathfrak{U}$ and for any submodule S' of S either $S' \in \mathfrak{U}$ or $S/S' \in \mathfrak{U}$.

Taking $\mathfrak{U} = 0$ and $\mathfrak{B} = \mathfrak{M}(R)$ we obtain the usual definition of a simple module. We also notice that the category $\mathfrak{B}$ plays a purely restrictive role and its presence in favour to $\mathfrak{M}(R)$ is only a technical convenience. The module S in $\mathfrak{B}$ is said to be '*quasi-simple*' if it is $\mathfrak{U}$-simple for some Serre-category $\mathfrak{U} \subseteq \mathfrak{B}$.

We now fix two Serre-categories $\mathfrak{U} \subseteq \mathfrak{B}$ in $\mathfrak{M}(R)$ and consider the $\mathfrak{U}$-simple modules in $\mathfrak{B}$. The following lemma is an immediate consequence of the definition.

<u>Lemma 1.</u> <u>Let</u> S <u>be a $\mathfrak{A}$-simple module and</u> σ <u>a chain of submodules of</u> S. <u>Then there is exactly one chain factor of</u> σ <u>which does not belong to</u> $\mathfrak{A}$.

<u>Lemma 2.</u> <u>Let</u> S' <u>be a segment of a $\mathfrak{A}$-simple module</u> S. <u>If</u> S' <u>does not belong to</u> $\mathfrak{A}$ <u>then</u> S' <u>is $\mathfrak{A}$-simple.</u>

<u>Proof.</u> Let N be a submodule of S' and suppose that $S' \notin \mathfrak{A}$. We can find a chain σ of submodules of S such that N and S'/N are chain factors of σ because S' is a segment of S. Using Lemma 1 we find that either $N \in \mathfrak{A}$ or $S'/N \in \mathfrak{A}$. Thus S' is $\mathfrak{A}$-simple.

<u>Definition.</u> <u>Let</u> S,P <u>be $\mathfrak{A}$-simple modules. We say that</u> S <u>is equivalent to</u> P, <u>write</u> $S \sim P$, <u>if</u> S <u>and</u> P <u>have isomorphic segments which in turn are $\mathfrak{A}$-simple modules.</u>

It is evident from the definition that the relation $\sim$ is symmetric and reflexive.

<u>Lemma 3.</u> <u>If</u> S,P,Q <u>are $\mathfrak{A}$-simple modules and</u> $S \sim P$, $P \sim Q$ <u>then</u> $S \sim Q$.

<u>Proof.</u> Let A,B be segments of P and M,N segments of S and Q respectively such that A,B,M,N are $\mathfrak{A}$-simple modules and $M \approx A, B \approx N$. There are chains σ,τ of P such that A is a factor of σ and B is a factor of τ. Let σ',τ' be equivalent refinements of σ and τ. By Lemma 1 there is exactly one chain factor, say A' of σ' and B' in τ' such that $A' \notin \mathfrak{A}, B' \notin \mathfrak{A}$. Then A' and B' must necessarily be segments of A and B respectively, and $A' \approx B'$. Moreover, A' and B' are $\mathfrak{A}$-simple by Lemma 2. The isomorphisms $M \approx A, B \approx N$ yield segments M' of M and N' of N such that $M' \approx A', B' \approx N$. Thus M' and N' are $\mathfrak{A}$-simple segments of S and Q and $M' \approx N'$. Thus $S \sim Q$.

Proposition 4. Let S be a Noetherian $\mathfrak{A}$-simple module. Then there is a left ideal I of R satisfying the following conditions:

(i) R/I is a Noetherian $\mathfrak{A}$-simple module;

(ii) Every proper factor module of R/I is in $\mathfrak{A}$;

(iii) $R/I \sim S$.

Proof. Let M be the maximal submodule of S which belongs to $\mathfrak{A}$. Then S/M is Noetherian and does not belong to $\mathfrak{A}$. Hence S/M is $\mathfrak{A}$-simple and $S/M \sim S$. If $x \neq 0$ $x \in S/M$ then Rx is Noetherian and the only submodule of Rx which lies in $\mathfrak{A}$ is the zero module. Again, Rx is $\mathfrak{A}$-simple and $Rx \sim S$. Finally $Rx \approx R/I$ where $I = 0 :_R x$.†

Dually, we have

Proposition 4*. Let S be an Artinian $\mathfrak{A}$-simple module. Then there is a simple module M and a submodule I of E(M) satisfying the following conditions:

(i) I is an Artinian $\mathfrak{A}$-simple module;

(ii) Every proper submodule of I is in $\mathfrak{A}$;

(iii) $S \sim I$.

As before, let two Serre-categories $\mathfrak{A} \subseteq \mathfrak{B}$ be given. Throughout this chapter, $\mathfrak{A}'$ will denote the Serre-category generated by $\mathfrak{A}$ and the $\mathfrak{A}$-simple modules in $\mathfrak{B}$. Thus strictly speaking $\mathfrak{A}'$ depends on $\mathfrak{B}$ as well. It may happen, of course, that there are no $\mathfrak{A}$-simple modules, i.e. $\mathfrak{A} = \mathfrak{A}'$. Clearly $\mathfrak{A} \subseteq \mathfrak{A}' \subseteq \mathfrak{B}$.

Proposition 5. Let the situation be as described above. In order that a module M should belong to $\mathfrak{A}'$ it is necessary and sufficient that a chain σ of submodules of M exists with the property that each chain factor of σ is either a $\mathfrak{A}$-simple module or belongs to $\mathfrak{A}$.

† $0 :_R x = \{r \in R : rx = 0\}$.

Roughly speaking $\mathfrak{A}'$ consists of modules with a '$\mathfrak{A}$-composition series'.

Proof. Combine Proposition 1.1 and Lemma 2.

3.2. The Krull dimension

Let $\mathfrak{A} \subseteq \mathfrak{B}$ be Serre-categories. For each ordinal number α a Serre-category will be defined such that $\mathfrak{A} \subseteq \mathfrak{A}_\alpha \subseteq \mathfrak{A}_\beta \subseteq \mathfrak{B}$ whenever $\alpha \leq \beta$. This sequence of Serre-categories will be referred to as the Krull sequence between $\mathfrak{A}$ and $\mathfrak{B}$. We start with the number -1 and set:[†]

$$\mathfrak{A}_{-1} = \mathfrak{A}.$$

Assume that $\mathfrak{A}_\beta$ has already been defined for ordinals $\beta < \alpha$ then

$$\mathfrak{A}_\alpha = (\mathfrak{A}_\beta)' \text{ if } \alpha = \beta + 1 \text{ and}$$

$$\mathfrak{A}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta \text{ if } \alpha \text{ is a limit ordinal.}$$

The Krull dimension of $\mathfrak{B}$ over $\mathfrak{A}$, dim $\mathfrak{B}/\mathfrak{A}$, is the smallest ordinal α for which $\mathfrak{A}_\alpha = \mathfrak{B}$. If there is no ordinal α such that $\mathfrak{A}_\alpha = \mathfrak{B}$ then we write dim $\mathfrak{B}/\mathfrak{A} = \infty$. Note that when we write dim $\mathfrak{B}/\mathfrak{A} < \infty$ we actually mean that dim $\mathfrak{B}/\mathfrak{A} \neq \infty$ and not that dim $\mathfrak{B}/\mathfrak{A}$ is an integer. The Krull dimension of a module M over $\mathfrak{A}$ is defined similarly. Thus if $M \in \mathfrak{B}$ then dim $M/\mathfrak{A}$ stands for the smallest ordinal α for which $M \in \mathfrak{A}_\alpha$. If no such ordinal exists then put dim $M/\mathfrak{A} = \infty$. It is clear from the definition that the Krull dimension of a module in $\mathfrak{B}$ is never a limit ordinal. (∞ is only a symbol and not an ordinal number).

Proposition 6. Let $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{B}_1, \mathfrak{B}_2$ be Serre-categories and $\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \mathfrak{B}_2 \subseteq \mathfrak{B}_1$. Then dim $\mathfrak{B}_2/\mathfrak{A}_2 \leq$ dim $\mathfrak{B}_1/\mathfrak{A}_1$.

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† The reason is purely aesthetic , we want our dimension concept to coincide with the usual Krull dimension of a ring when the latter is defined.

Proof. We may assume that dim $\mathfrak{B}_1/\mathfrak{A}_1 = \alpha < \infty$. Let $\{\mathfrak{D}_\beta\}$ be the Krull sequence between $\mathfrak{A}_1$ and $\mathfrak{B}_1$, $\mathfrak{D}_{-1} = \mathfrak{A}_1$, $\mathfrak{D}_\alpha = \mathfrak{B}_1$ and let $\mathfrak{H}_\beta$ be a typical element of the Krull sequence between $\mathfrak{A}_2$ and $\mathfrak{B}_2$. The Serre-category generated by $\mathfrak{A}_2$ and $\mathfrak{D}_\beta \cap \mathfrak{B}_2$ is denoted by $\mathfrak{S}_\beta$. We claim that $\mathfrak{S}_\beta \subseteq \mathfrak{H}_\beta$ for all $\beta \leqslant \alpha$. Since $\mathfrak{S}_{-1} = \mathfrak{A}_2 = \mathfrak{H}_{-1}$ the statement is true for $\beta = -1$. Assume that it has already been proved for all ordinals γ, $\gamma < \beta \leqslant \alpha$. If β is a limi ordinal then $\mathfrak{S}_\beta = \bigcup_{\gamma<\beta}\mathfrak{S}_\gamma \subseteq \bigcup_{\gamma<\beta}\mathfrak{H}_\gamma = \mathfrak{H}_\beta$. Assume now that $\beta = \delta + 1$. By the induction hypothesis $\mathfrak{S}_\delta \subseteq \mathfrak{H}_\delta$. If M is a $\mathfrak{S}_\delta$-simple module in $\mathfrak{B}_2$ and $M \notin \mathfrak{H}_\delta$ then M is obviously a $\mathfrak{H}_\delta$-simple module. Consequently $(\mathfrak{S}_\delta)' \subseteq \mathfrak{H}_{\delta+1}$. Suppose that M is a $\mathfrak{D}_\delta$-simple module and $M \in \mathfrak{B}_2$. Then either $M \in \mathfrak{S}_\delta$ or M is a $\mathfrak{S}_\delta$-simple module in $\mathfrak{B}_2$. It follows that $\mathfrak{S}_{\delta+1} \subseteq (\mathfrak{S}_\delta)' \subseteq \mathfrak{H}_{\delta+1}$. Thus $\mathfrak{S}_\beta \subseteq \mathfrak{H}_\beta$ for all $\beta \leqslant \alpha$. In partioular $\mathfrak{B}_2 = \mathfrak{S}_\alpha \subseteq \mathfrak{H}_\alpha$ shows that dim $\mathfrak{B}_2/\mathfrak{A}_2 \leqslant \alpha$ as required.

We write dim $\mathfrak{A}$ and dim A for dim $\mathfrak{A}/0$ and dim A/0 respectively.

Theorem 7. If $\mathfrak{A}$ and $\mathfrak{B}$ denote the categories of Noetherian and Artinian modules respectively then dim $\mathfrak{A} < \infty$ and dim $\mathfrak{B} < \infty$. In particular, dim $A < \infty$ for a module A in either $\mathfrak{A}$ or $\mathfrak{B}$.

Proof. For a change consider the Artinian case. Let A be an Artinian module and $\mathfrak{B}_\alpha$ the α-th element of the Krull sequence of $\mathfrak{B}$. Suppose that dim A = ∞ and choose the submodule M of A to be minimal with respect to this property, i.e. dim M = ∞ but dim N < ∞ for any proper submodule N of M. Note that $M \neq 0$. Put $\gamma = \sup\{\dim N : N \subset M\}$. Then $M \notin \mathfrak{B}_\gamma$ by assumption. But every proper submodule of M belongs to $\mathfrak{B}_\gamma$. Hence M is a $\mathfrak{B}_\gamma$-simple module and $M \in \mathfrak{B}_{\gamma+1}$ which contradicts our assumption that dim M = ∞. Thus dim A < ∞ for all $A \in \mathfrak{B}$. The Noetherian case can be proved similarly.

In order to complete the proof we need the following property of $\mathfrak{A}$ and $\mathfrak{B}$: there is a set of modules generating $\mathfrak{A}$ (resp. $\mathfrak{B}$). For $\mathfrak{A}$, the set of modules of the form R/L, L a left ideal of R, R/L is Noetherian do generate $\mathfrak{A}$ in view of Proposition 1.8. Similarly, using Proposition 1.8* we find that Artinian singly embedded modules generate $\mathfrak{B}$. Let $\{L_i\}_{i\in I}$ be the set of maximal left ideals and choose an injective envelope E_i for each R/L_i. Then the Artinian submodules of the modules E_i form a set of representatives for Artinian single embedded modules.

We now complete the proof of Theorem 7. Let $\{N_i\}_{i\in I}$ and $\{A_j\}_{j\in J}$ be sets of modules generating $\mathfrak{A}$ and $\mathfrak{B}$ respectively. As we have already seen $\dim N_i < \infty$ and $\dim A_j < \infty$ for all $i \in I, j \in J$. Put $\alpha = \sup_i \dim N_i$, $\beta = \sup_j \dim A_j$. If $\mathfrak{A}_\gamma, \mathfrak{B}_\gamma$ denote the typical elements of the Krull sequences of $\mathfrak{A}$ and $\mathfrak{B}$ respectively then $N_i \in \mathfrak{A}_\alpha$ and $A_j \in \mathfrak{B}_\beta$ for all $i \in I, j \in J$. Therefore $\mathfrak{A} \subseteq \mathfrak{A}_\alpha$ $\mathfrak{B} \subseteq \mathfrak{B}_\beta$ whence $\dim \mathfrak{A} \leq \alpha$ and $\dim \mathfrak{B} \leq \beta$.

3.3 The main decomposition theorem

We wish to characterize length functions by representing them as linear combinations of others. The natural building blocks for such a representation theory are length functions which cannot be decomposed any further. Thus we make the following

Definition. A non-trivial length function L on a category $\mathfrak{A}$ is called irreducible if for length functions L_1 and L_2 on $\mathfrak{A}$, $L = L_1 + L_2$ implies that either $L_1 = cL$ or $L_2 = cL$ for some real number $c > 0$.

Proposition 8. <u>If the length function L is irreducible on a semi-closed category $\mathfrak{A}$ then its continuous extension to $\mathfrak{M}(R)$ is irreducible on $\mathfrak{M}(R)$.</u>

Proof. Assume that $\hat{L} = L_1 + L_2$ where $\hat{L}$ is the continuous extension of L to $\mathfrak{M}(R)$. Since L is irreducible on $\mathfrak{A}$ we may assume that $L_1 = cL$, $c > 0$ on $\mathfrak{A}$. Using Theorem 2.3 and Proposition 2.4 we obtain

$$c\hat{L} = \hat{L}_1 \leq L_1, \ \hat{L} = \hat{L}_1 + \hat{L}_2, \ \hat{L}_2 \leq L_2 \text{ on } \mathfrak{M}(R). \qquad (1)$$

We contend that $L_1 = c\hat{L}$. Let $A \in \mathfrak{M}(R)$ and assume that $\hat{L}(A) = \infty$. Then $L_1(A) \geq \hat{L}_1(A) = c\hat{L}(A) = \infty$. Next, if $\hat{L}(A) < \infty$ then $(L_1(A) - \hat{L}_1(A)) + (L_2(A) - \hat{L}_2(A)) = 0$. Hence $L_1(A) = \hat{L}_1(A)$.

A length function L is said to be '<u>finite on $\mathfrak{A}$</u>', $\mathfrak{A}$ a semi-closed category, if $\mathfrak{A} \subseteq$ Fin L. Let L be a length function on $\mathfrak{M}(R)$ (or on a semi-closed category) and let $\hat{L}$ be the continuous component of L with respect to Fin L. Then $L = \hat{L} + (L-\hat{L})$ and the latter function is trivial. Further, if L admits a representation as a sum of irreducible functions on Fin L, then this representation can be extended by continuity to $\hat{L}$. This is the reason why we can focus our attention on decomposing a function over its domain of finiteness. If we obtain a decomposition for L over Fin L, then L will be expressed as a sum of irreducible and, possibly, trivial functions over $\mathfrak{M}(R)$.

To avoid set-theoretical difficulties we assume that the following condition is always satisfied in this section.

(Δ) If $\mathfrak{A}$ and $\mathfrak{B}$ are Serre-categories, $\mathfrak{A} \subseteq \mathfrak{B}$, then there is a set Π and a family of $\mathfrak{A}$-simple modules $\{S_\pi\}_{\pi\in\Pi}$ in $\mathfrak{B}$ such that for every $\mathfrak{A}$-simple module S in $\mathfrak{B}$, $S \sim S_\pi$ for some $\pi \in \Pi$.

In other words the equivalence classes of $\mathfrak{A}$-simple modules form a set.

The results in the previous section show that (Δ) is satisfied if $\mathfrak{B}$ is the category of Noetherian or Artinian modules.

Let $\mathfrak{U} \subsetneq \mathfrak{B}$ Serre-categories, $\dim \mathfrak{B}/\mathfrak{U} = 0$ and $\{S_\pi\}_{\pi\in\Pi}$ a set of representatives of $\mathfrak{U}$-simple modules. If $\widetilde{\mathfrak{U}}$ is the semi-closed category comprising $\mathfrak{U}$ and the $\mathfrak{U}$-simple modules and

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence in $\widetilde{\mathfrak{U}}$ then we have either of the following three possibilities:

(a) $M, M', M'' \in \mathfrak{U}$;

(b) $M \sim M'$ and $M'' \in \mathfrak{U}$;

(c) $M \sim M''$ and $M' \in \mathfrak{U}$.

Let L be a length function on $\mathfrak{B}$ such that $\mathfrak{U} \subseteq \operatorname{Ker} L$. Then $L(S) = L(S')$ whenever S and S' are equivalent $\mathfrak{U}$-simple modules in $\mathfrak{B}$ (Lemma 1). It now follows that the functions $\{L_\pi\}_{\pi\in\Pi}$ on $\widetilde{\mathfrak{U}}$ defined by

$$L_\pi(M) = \begin{cases} 1 \text{ if } M \text{ is } \mathfrak{U}\text{-simple and } M \sim S_\pi \\ 0 \text{ otherwise,} \end{cases}$$

are length functions on $\widetilde{\mathfrak{U}}$. The continuous extension of L_π to $\mathfrak{M}(R)$ (still denoted by L_π) is called the <u>length function associated to the quasi-simple module</u> S_π. Note that L_π has integer values.

<u>Lemma 9.</u> <u>Let the situation be as described above and let</u> L_1, L_2 <u>be length functions on</u> $\mathfrak{B}$ <u>such that</u> $\mathfrak{U} \subseteq \operatorname{Ker} L_1 \cap \operatorname{Ker} L_2$. <u>Then</u> $L_1 = L_2$ <u>if and only if</u> $L_1(S_\pi) = L_2(S_\pi)$ <u>for all</u> $\pi \in \Pi$.

<u>Proof</u>. Suppose that $L_1(S_\pi)=L_2(S_\pi)$ for all $\pi \in \Pi$. Then if S is a $\mathfrak{U}$-simple module, $S \sim S_\mu$ for some $\mu \in \Pi$ and $L_1(S) = L_1(S_\mu) = L_2(S_\mu) = L_2(S)$. Hence $L_1 = L_2$ on $\widetilde{\mathfrak{U}}$ and consequently on $\mathfrak{U}$. (Theorem 2.3 Cor.)

<u>Proposition 10.</u> <u>Let</u> $\mathfrak{U} \subsetneq \mathfrak{B}$ <u>be Serre-categories</u>, $\dim \mathfrak{B}/\mathfrak{U} = 0$ <u>and</u> $(S_\pi)_{\pi\in\Pi}$ <u>a minimal set of representatives</u>† <u>of</u> $\mathfrak{U}$-<u>simple modules. Let</u>

† one from each equivalent class,

L *be a finite function on* $\mathfrak{B}$ *such that* $\mathfrak{U} \subseteq \text{Ker } L$. *Then*

(i) L *is irreducible if and only if* $L = cL_\pi$, $c > 0$ *for some* $\pi \in \Pi$;

(ii) $L = \Sigma_{\pi\in\Pi} c_\pi L_\pi$, $c_\pi = L(S_\pi)$ *is the unique representation of* L *on* $\mathfrak{B}$ *as a linear combination of the* L_π's.

Proof. Write $L' = \Sigma_{\pi\in\Pi} c_\pi L_\pi$ with $c_\pi = L(S_\pi)$. From Lemma 9 and the definitions of the L_π's we see that $L = L'$. If $L = \Sigma_{\pi\in\Pi} d_\pi L_\pi$ then $d_\pi = L(S_\pi) = c_\pi$ for all $\pi \in \Pi$. The proof will be completed by the verification of (i). Assume that $L_\pi = L_1 + L_2$ on $\mathfrak{B}$. We may assume that $c = L_1(S_\pi) \neq 0$. Then $L_1 = cL$ by Lemma 9. Conversely, if L is irreducible on $\mathfrak{B}$, ($\mathfrak{U} \subseteq \text{Ker } L$, $\mathfrak{B} \subseteq \text{Fin } L$) then $L \neq 0$ since L is not trivial. Hence there is a $\mu \in \Pi$ such that $L(S_\mu) \neq 0$. Put $L' = \Sigma_{\pi\neq\mu} L(S_\pi) L_\pi$. Then $L = L' + L(S_\mu)L_\mu$. Since $L'(S_\mu) = 0$, L' cannot be a multiple of L. Hence $L_\mu = cL$ and $L = c^{-1}L_\mu$.

Corollary 1. The functions $L_\pi, \pi\in\Pi$ *are irreducible on* $\mathfrak{M}(R)$.

Proof. Apply Proposition 8.

Corollary 2. If $M \neq 0$ *is a Noetherian (resp. Artinian) module then there is a length function* L *on* $\mathfrak{M}(R)$ *such that* $0 < L(M) < \infty$.

Proof. By Theorem 7, $\dim M = \alpha + 1$ in the category of Noetherian (resp. Artinian) modules. Let $\{L_\pi\}_{\pi\in\Pi}$ be the set of length functions associated to $\mathfrak{U}_\alpha$, the term in the Krull sequence corresponding to the ordinal α. Set $L = \Sigma_{\pi\in\Pi} L_\pi$. Then $\mathfrak{U}_\alpha = \text{Ker } L \subset \mathfrak{U}_{\alpha+1} \subseteq \text{Fin } L$. Hence $0 < L(M) < \infty$. Also, $L(M)$ is an integer.

One may utilize the above result and define the 'length of an ideal' in arbitrary rings. Observe that if I is an ideal of a commutative Noetherian ring then the number $L(R/I)$ obtained in the proof is the same as the length of I as it is defined in the classical theory.

Now we turn to the general case. Let $\mathfrak{A} \subseteq \mathfrak{B}$ be Serre-categories and $\dim \mathfrak{B}/\mathfrak{A} = \gamma < \infty$. The elements of the Krull sequence are denoted by $\mathfrak{A}_\alpha$, $-1 \leq \alpha \leq \gamma$. Let L be a finite function on $\mathfrak{B}$ and $\mathfrak{A} \subseteq \operatorname{Ker} L$. For each α, $-1 \leq \alpha \leq \gamma$, L^α denotes the continuous component of L with respect to $\mathfrak{A}_\alpha$. By Theorem 2.3 $L^\alpha \leq L^{\alpha+1}$. Let L_α be the continuous component of $L^{\alpha+1} - L^\alpha$ with respect to $\mathfrak{A}_{\alpha+1}$, $-1 \leq \alpha \leq \gamma$. The functions L_α, $-1 \leq \alpha \leq \gamma$ are called the <u>Krull components</u> of L.

<u>Lemma 11. Let the situation be as described above. Then</u>

(i) $\mathfrak{A}_\alpha \subseteq \operatorname{Ker} L_\alpha$ <u>and</u> L_α <u>is continuous on</u> $\mathfrak{A}_{\alpha+1}$ $(\alpha \neq \gamma)$, <u>and</u>

(ii) $L^\alpha = \sum_{-1 \leq \beta < \alpha} L_\beta$ <u>for all</u> α, $-1 \leq \alpha \leq \gamma$.

<u>Moreover, the Krull components of</u> L <u>are completely characterized by</u> (i) - (ii) <u>and</u> $L = \sum_{-1 \leq \alpha < \gamma} L_\alpha$ <u>on</u> Fin L.

<u>Proof</u>. It is evident that L^α and $L^{\alpha+1}$ agree on $\mathfrak{A}_\alpha$. Hence (i). Next, $L^{-1} = 0$ and $L^0 = L_{-1}$. Assume that (ii) has already been proved for all ordinals $\beta < \alpha$ and consider $L' = \sum_{-1 \leq \beta < \alpha} L_\beta$. It follows from this assumption and (i) that L' and L agree on $\mathfrak{A}_\beta$, $-1 = \beta < \alpha$. Hence $L^\alpha = L'$ if α is a limit ordinal. If $\alpha = \delta + 1$ then

$$L' = \sum_{-1 \leq \beta < \delta} L_\beta + L_\delta = L^\delta + L_\delta = L^{\delta+1} = L^\alpha \text{ from the definition.}$$

Let K_α be a second sequence satisfying (i) and (ii) and let α be an ordinal, $-1 \leq \alpha < \gamma$. Then $L_\alpha = L^{\alpha+1} - L^\alpha = K_\alpha$ on $\mathfrak{A}_{\alpha+1}$. But L_α, K_α are continuous on $\mathfrak{A}_{\alpha+1}$. Hence $L_\alpha = K_\alpha$.

Theorem 12. Let $\mathfrak{U} \subseteq \mathfrak{B}$ be Serre-categories, $\dim \mathfrak{B}/\mathfrak{U} = \gamma < \infty$, and let $\{\mathfrak{U}_\alpha\}$ $-1 \leq \alpha \leq \gamma$ be the Krull sequence between $\mathfrak{U}$ and $\mathfrak{B}$. For each α, $-1 \leq \alpha < \gamma$, $\{S_\pi\}_{\pi \in \Pi_\alpha}$ is a set of representatives of $\mathfrak{U}_\alpha$-simple modules, one from each equivalent class and Π is the disjoint union of the sets Π_α. If L is a finite function on $\mathfrak{B}$ such that $\mathfrak{U} \subseteq \operatorname{Ker} L$ then

(i) L is irreducible on $\mathfrak{B}$ if and only if $L = cL_\pi$, $c > 0$, for some $\pi \in \Pi$

(ii) $L = \sum_{\pi \in \Pi} c_\pi L_\pi$ on $\mathfrak{B}$, $c_\pi = L_\alpha(S_\pi)$ if $\pi \in \Pi_\alpha$ and L_α is the α-Krull component of L. Moreover, this is the unique representation of L as a linear combination of the L_π's.

Proof. Let L_α, $-1 \leq \alpha < \gamma$ be the Krull components of L. Then by Lemma 11, $L = \sum_{-1 \leq \alpha < \gamma} L_\alpha$ on $\mathfrak{B}$. For each α, L_α is a finite function on $\mathfrak{U}_{\alpha+1}$ and $\mathfrak{U}_\alpha \subseteq \operatorname{Ker} L_\alpha$. Hence $L_\alpha = \sum_{\pi \in \Pi_\alpha} c_\pi L_\pi$, $c_\pi = L_\alpha(S_\pi)$ on $\mathfrak{U}_{\alpha+1}$ by Proposition 10. Since all the functions involved are continuous on $\mathfrak{U}_{\alpha+1}$ we deduce that $L_\alpha = \sum_{\pi \in \Pi_\alpha} c_\pi L_\pi$ on $\mathfrak{M}(R)$. Thus $L = \sum_{\pi \in \Pi} c_\pi L_\pi$ on $\mathfrak{B}$ with $c_\pi = L_\alpha(S_\pi)$ for $\pi \in \Pi_\alpha$. Let $L = \sum_{\pi \in \Sigma} d_\pi L_\pi$ be a second representation of L as a linear combination of the functions L_π. Put $K_\alpha = \sum_{\pi \in \Pi_\alpha} d_\pi L_\pi$, $-1 \leq \alpha < j$. Then the K_α's clearly satisfy conditions (i) - (ii) in Lemma 11. Hence $K_\alpha = L_\alpha$ and Proposition 10 implies the identities $d_\pi = c_\pi$ for all $\pi \in \Pi$. To complete the proof we have to verify (i). By Proposition 10, Cor.1, L_π is irreducible on $\mathfrak{B}$ for all $\pi \in \Pi$. Conversely, if L is an irreducible length function on $\mathfrak{B}$ such that $\mathfrak{U} \subseteq \operatorname{Ker} L$, $\mathfrak{B} \subseteq \operatorname{Fin} L$

then $L_\alpha \neq 0$ for at least one α, $-1 \leq \alpha < \gamma$. (By definition, an irreducible function is not trivial.) Let α be the smallest ordinal for which $L_\alpha \neq 0$ and put $L' = \sum_{\alpha<\beta<\gamma} L_\beta$. Then $L = L_\alpha + L'$. But L is irreducible. Hence $L_\alpha = cL$ or $L' = cL$, $c > 0$. Since $\operatorname{Ker} L \nsubseteq \mathfrak{U}_{\alpha+1} \subseteq \operatorname{Ker} L'$ we must have $L_\alpha = cL$. Now L_α is a finite irreducible function on $\mathfrak{U}_{\alpha+1}$ and $\mathfrak{U}_\alpha \subseteq \operatorname{Ker} L_\alpha$. Hence Proposition 10 can be applied to obtain $L = c^{-1} L_\alpha = dL_\pi$ on $\mathfrak{U}_{\alpha+1}$, $d > 0$, $\pi \in \Pi_\alpha$. But both L_α and L_π are continuous on $\mathfrak{U}_{\alpha+1}$. Thus $L = dL_\pi$ on $\mathfrak{B}$ as required.

Corollary 1. Let L be a length function on $\mathfrak{M}(R)$ and $\dim \operatorname{Fin} L/\operatorname{Ker} L < \infty$. Then L can be expressed on $\operatorname{Fin} L$ as a sum of irreducible length functions. If L is continuous on $\operatorname{Fin} L$ then this representation holds true on $\mathfrak{M}(R)$.

Proof. Immediate.

Corollary 2. If L is a length function on the category of Noetherian (resp. Artinian) modules then the function L can be expressed on $\operatorname{Fin} L$ as a sum of irreducible length functions.

Proof. Combine Proposition 6 and Theorem 7.

Remark. The representation in Theorem 12 is unique in the following sense: Suppose that $L = \sum_{i\in I} L_i = \sum_{j\in J} L_j$ are representations of L, L_i, L_j are irreducible functions. Then by Theorem 12, L_i, L_j, $i\in I, j\in J$ are scalar multiples of the functions $\{L_\pi\}_{\pi\in\Pi}$. It follows that there is a bijection $\varphi : I \to J$ and real numbers $c_i > 0$ such that

$L_i = c_i L_{\varphi(i)}$, for all $i \in I$.

The length function L on a Serre-category $\mathfrak{U}$ is called 'locally discrete' if for every module $A \in \mathfrak{U}$, $\inf\{L(S) : L(S) > 0, S \text{ is a segment of } A\} > 0$.

A sum of length functions $L = \Sigma_i L_i$ on a category $\mathfrak{U}$ is called '<u>discrete</u>' if for every module $A \in \mathfrak{U}$, $L_i(A) = 0$ for all but a finite number of i.

<u>Theorem 13. Let L be a length function on $\mathfrak{M}(R)$ and dimFin L/Ker L $< \infty$. Then the following are equivalent:</u>

(i) dimFin L/Ker L ≤ 0;

(ii) L <u>is a discrete sum of irreducible functions on</u> Fin L;

(iii) L <u>is locally discrete</u>.

<u>Proof</u>. (i) $\Rightarrow$ (ii) If dimFin L/Ker L = -1 there is nothing to prove. Assume dim Fin L/Ker L = 0. By Proposition 10, $L = \Sigma_\pi c_\pi L_\pi$ where the L_π's are the irreducible functions associated to the Ker L-simple modules. If $A \in$ Fin L then A has a chain σ such that the chain factors are either Ker L-simple modules or elements of Ker L. (Proposition 5). Let $L_{\pi_1}, \dots, L_{\pi_k}$ be the functions associated to the Ker L-simple chain factors of σ. Then $c_\pi L_\pi(A) = 0$ whenever $\pi \neq \pi_1, \dots, \pi \neq \pi_k$.

(ii) $\Rightarrow$ (iii) By Theorem 10 we may assume $L = \sum_{\pi \in \Pi} c_\pi L_\pi$ where the functions L_π have only integer values and this sum is discrete.
In order to prove that L is locally discrete we may confine ourselves to modules $A \in$ Fin L, $A \notin$ Ker L. Let A be such a module and S a segment of A. If $c_\pi L_\pi(A) = 0$ then $c_\pi L_\pi(S) = 0$ as well. It now follows that there are indices $\pi_1, \dots, \pi_k$ such that $c_{\pi_1} > 0, \dots, c_{\pi_k} > 0$ and $L(S) = c_{\pi_1} L_{\pi_1}(S) + \dots + c_{\pi_k} L_{\pi_k}(S)$ for any segment S of A. Thus $L(S) \geq \min(c_{\pi_1}, \dots, c_{\pi_k})$ whenever $L(S) > 0$.

(iii) $\Rightarrow$ (i) Let Ker L $= \mathfrak{U}_{-1} \subsetneq \mathfrak{U}_0 \subsetneq \mathfrak{U}_1 \subsetneq \dots$ be the Krull sequence between Ker L and Fin L and assume that $\mathfrak{U}_0 \neq$ Fin L. Since dimFin $L/\text{Ker}\, L < \infty$ we have $\mathfrak{U}_{-1} \subset \mathfrak{U}_0 \subset \mathfrak{U}_1$. Choose a $\mathfrak{U}_0$-simple module S in Fin L. If A is a submodule of S then either $A \in \mathfrak{U}_0$ or $S/A \in \mathfrak{U}_0$. Since S is $\mathfrak{U}_0$-simple and not $\mathfrak{U}_{-1}$-simple there must be a submodule A of S such that either $A \in \mathfrak{U}_0$ but $A \notin \mathfrak{U}_{-1}$ or $S/A \in \mathfrak{U}_0$ but $S/A \notin \mathfrak{U}_{-1}$. Let A_1 be the one which belongs to $\mathfrak{U}_0$ but not to $\mathfrak{U}_{-1}$ and B_1 the other. Then $L(S) = L(A_1) + L(B_1)$, $L(A_1) \neq 0$, $L(B_1) \neq 0$ and B_1 is $\mathfrak{U}_0$-simple. Hence the procedure can be repeated by B_1 etc. In this way two sequences of modules $A_1, A_2, \ldots, A_n, .$. and $B_1, B_2, \ldots, B_n ..$ are generated such that $L(A_n) \neq 0, L(B_n) \neq 0$ and $L(S) = L(A_1) + \ldots + L(A_n) + L(B_n)$ for all $n > 0$. Consequently $L(A_n) \to 0$ and A_n is a segment of S. But this contradicts the fact that L is locally discrete.

<u>Theorem 14.</u> <u>If the finite values of a length function</u> L <u>on</u> $\mathfrak{M}(R)$ <u>are integers then</u> dimFin $L/\text{Ker}\, L \leq 0$.

<u>Proof.</u> Assume that the theorem is not true and $(\text{Ker}\, L)' \subset \text{Fin}\, L$. Choose a module A in Fin L, $A \notin (\text{Ker}\, L)'$ so that $L(A)$ is minimal. For a submodule B of A we have $L(A) = L(B) + L(A/B)$. If $L(B) < L(A)$ and $L(A/B) < L(A)$ then $A, A/B \in (\text{Ker}\, L)'$ by the choice of A and $A \in (\text{Ker}\, L)'$ since $(\text{Ker}\, L)'$ is a Serre-category. Since $A \notin (\text{Ker}\, L)'$, either $L(B) = L(A)$ or $L(A/B) = L(A)$. Therefore either B or A/B belongs to Ker L for any choice of B. But then **A** is a Ker L-simple module and, again, $A \in (\text{Ker}\, L)'$, contradicting our assumption. Thus $(\text{Ker}\, L)' = \text{Fin}\, L$ and dimFin $L/\text{Ker}\, L \leq 0$.

<u>3.4 The category of Noetherian modules.</u>

We have seen in the previous section how the irreducible length functions can be used as building blocks in the representation problem. Apart from their existence, however, our theory provided little information. In the present section we set out to show that the irreducible length functions of the category of Noetherian modules can be realised by means of injective modules.

Definition. An R-module M is said to be 'indecomposable' if its only direct summands are 0 and M.

Proposition 15 [5, Proposition 2.2]. For an R-module M the following are equivalent:

(i) E(M) is an injective envelope of every non-zero submodule of itself;

(ii) M contains no non-zero submodules S and T such that $S \cap T = 0$,

(iii) E(M) is indecomposable.

Proof. (i) ⇒ (ii) ⇒ (iii) Clear.

(iii) ⇒ (i). If E(M) is indecomposable then it contains no non-zero injective submodules. Thus E(M) is the minimal injective extension of every non-zero submodule of itself.

Let E be an injective R-module and let S be the ring of endomorphisms of E. Then E becomes a left R, left S-bimodule. The functor $T(M) = \mathrm{Hom}_R(M,E)$ is an exact contravariant functor from $\mathfrak{M}(R)$ to $\mathfrak{M}(S)$. The S-module structure of T(M) is given by

$$f(a) = f \cdot a, \quad a : M \to E, \; f : E \to E.$$

For each R-module M, put $L_E(M) = \ell(T(M))$ where ℓ is the classical length function on $\mathfrak{M}(S)$. Then L_E is an upper continuous length function on $\mathfrak{M}(S)$. (Proposition 2.10). Suppose that for a module

$M \in \mathfrak{M}(R)$, $T(M) \neq 0$ and every non-zero element of $T(M)$ is a monomorphism $M \to E$. The next lemma tells us that $L_E(M) = 1$.

Lemma 16. Let the situation be as described above. Then $L_E(M) = 1$.

Proof. It will suffice to prove that $T(M)$, as an S-module, is simple. Since $T(M) \neq 0$, the lemma will follow if we show that every non-zero element of $T(M)$ generates $T(M)$. Suppose $f, g \in T(M)$, $f \neq 0$. Then f is a monomorphism and the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{f} & E \\ & & {\scriptstyle g}\downarrow & \swarrow{\scriptstyle h} & \\ & & E & & \end{array}$$

can be completed by an $h : E \to E$ such that $g = hf$, $h \in S$.

For the rest of this chapter $\mathfrak{F}$ denotes the category of Noetherian modules in $\mathfrak{M}(R)$. Suppose L is an irreducible length function on $\mathfrak{F}$. Then L is called 'normalized' if $\{L(A) : A \in \mathrm{Fin}\,L\} = \{0,1,2,\ldots,n,\ldots\}$. By Theorem 12, $L = cL'$, $c > 0$ on $\mathrm{Fin}\,L$ and L' is normalized since it is an irreducible length function associated to a quasi-simple module. But an irreducible length function is continuous on its domain of finiteness. (For $L = \hat{L} + L - \hat{L}$, $\hat{L}$ is the continuous component of L with respect to $\mathrm{Fin}\,L$. But then $\hat{L} = dL$, $d > 0$ and $L = \hat{L}$ on $\mathrm{Fin}\,L$. Hence $L = \hat{L}$). Therefore $L = cL'$ on $\mathfrak{M}(R)$. Thus every irreducible length function is a scalar multiple of a normalized length function.

We say that a module $E \in \mathfrak{M}(R)$ is an '$\mathfrak{F}$-injective' if it is an indecomposable injective module and contains a non-zero Noetherian module.

<u>Lemma 17. Let E be an $\mathfrak{F}$-injective module. Then there is a submodule P of E, $P \neq 0$, $P \in \mathfrak{F}$ such that every non-zero homomorphism from P to E is a monomorphism.</u>

<u>Proof</u>. Let N be a non-zero Noetherian submodule of E. Choose a submodule M of N so that $\mathrm{Hom}_R(N/M,E) \neq 0$ and M is maximal with respect to this property. Then there is a non-zero homomorphism $f : N/M \to E$ and $P = f(N/M)$ has the required property.

Proposition 15 shows that an indecomposable injective module is the injective envelope of a cyclic module. Therefore one may speak of the set of isomorphism classes of indecomposable injective modules.

<u>Theorem 18. There is a one-to-one correspondence between the normalized irreducible length functions of $\mathfrak{F}$ and the set of isomorphism classes of $\mathfrak{F}$-injective modules, given by $E \Leftrightarrow L_E$.</u>

<u>Proof</u>. We recall that if E is injective then $L_E(M) = \ell(\mathrm{Hom}_R(M,E))$, $M \in \mathfrak{M}(R)$. Here ℓ is the classical length function over the endomorphism ring of E.

Let $\mathfrak{U} \subseteq \mathfrak{F}$ be a Serre-category, P a $\mathfrak{U}$-simple module in $\mathfrak{F}$ and L_P the associated length function. We are going to prove that

(a) we may assume that E(P) is $\mathfrak{F}$-injective and P satisfies the conditions of Lemma 17, and in this case

(b) $L_P = L_{E(P)}$.

Indeed, by Proposition 4, we may assume that every proper factor module of P (i.e. not P itself) belongs to $\mathfrak{U}$. Suppose that S and T are non-zero submodules of P and $S \cap T = 0$. Then P/S and P/T belong to $\mathfrak{U}$ and there is a monomorphism $P \to P/S \oplus P/T$.

This shows that $P \in \mathfrak{U}$ which is impossible. We can then conclude that $E = E(P)$ is an $\mathfrak{F}$-injective module. (Proposition 15). If $f : P \to E$ is a homomorphism and $f(P) \neq 0$ then $f(P) \cap P \neq 0$ since E is indecomposable. Accordingly $f(P) \cap P \notin \mathfrak{U}$. At the same time f(P) is a factor module of P and belongs to $\mathfrak{U}$ if it is proper, i.e. if $\operatorname{Ker} f \neq 0$. Thus $\operatorname{Ker} f = 0$. We now prove (b).

First let $M \in \mathfrak{M}(R)$ and $L_E(M) > 0$. We show that M has a segment isomorphic to a proper submodule of P or in other words M has a $\mathfrak{U}$-simple segment equivalent to P. For if $L_E(M) > 0$ then there is a non-zero homomorphism $f : M \to E$ and $f(M) \cap P \neq 0$.

It now follows that $L_E(M) > 0 \Rightarrow L_P(M) > 0$, i.e. $\mathfrak{U} \subseteq \operatorname{Ker} L_P \subseteq \operatorname{Ker} L_E$ Also, if Q is an $\mathfrak{U}$-simple module and $L_E(Q) > 0$ then $Q \sim P$. Moreover, $L_E(P) = 1$ by Lemma 16 and (a). Using Lemma 9 we find that L_P and L_E agree on $\mathfrak{U}' \cap \mathfrak{F}$. Further, L_P is continuous on $\mathfrak{U}'$ by definition. Thus $L_P \leq L_E$ (on $\mathfrak{M}(R)$). If $L_P(M) > 0$ then M has a $\mathfrak{U}$-simple segment equivalent to P. Hence $L_P(M) > 0$ implies $L_E(M) > 0$ which shows that $\operatorname{Ker} L_P = \operatorname{Ker} L_E$. Also, by Theorem 14 $\operatorname{dimFin} L_P/\operatorname{Ker} L_P = 0$. In order to show that $L_P = L_E$ on $\operatorname{Fin} L_P$ we use Lemma 9 again. Evidently, P and every non-zero submodule of P, is a $\operatorname{Ker} L_P$-simple module. Let Q be a $\operatorname{Ker} L_P$-simple module. Since L_P is irreducible, $L_P(Q) > 0$ implies that $Q \sim P$ by Proposition 10. If $L_E(Q) > 0$ then Q has a segment isomorphic to a non-zero submodule of P whence $Q \sim P$. Thus $L_P = L_E$ on $\operatorname{Fin} L_P$. But $L_P \leq L_E$ on $\mathfrak{M}(R)$. Hence $L_P = L_E$ on $\mathfrak{M}(R)$.†

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† In Lemma 9 and Proposition 10 the condition (Δ) was implicit. The following is easily verified. For an upper continuous length function L the equivalence classes of Ker L-simple modules form a set. Indeed, if A is Ker L-simple then it has a cyclic submodule $B \notin \operatorname{Ker} L$ for otherwise $L(A) = 0$. Thus every Ker L-simple module is equivalent to a module of the form R/I, I a left ideal. In our case both L_E and L_P are upper continuous.

We now prove Theorem 18 proper. Let E be an $\mathfrak{F}$-injective module and set $\mathfrak{U} = \text{Ker } L_E \cap \mathfrak{F}$. By Lemma 17 there is a submodule P of E such that $P \neq 0$, $P \in \mathfrak{F}$ and the non-zero homomorphisms from P to E are monomorphisms. Obviously, P is $\mathfrak{U}$-simple and (a) - (b) imply that $L_E = L_P$. Hence L_E is a normalized irreducible length function on $\mathfrak{F}$ and $E \to L_E$ is well defined.

If L is a normalized irreducible length function on $\mathfrak{F}$ then $L = L_P$ for some quasi-simple module P in $\mathfrak{F}$. Applying (a) - (b) we see that $L = L_E$ for some $\mathfrak{F}$-injective E. Thus $E \to L_E$ is onto.

Assume now that E and E′ are $\mathfrak{F}$-injective modules and $L_E = L_{E'}$. Choose a non-zero Noetherian submodule P of E such that the non-zero homomorphisms in $\text{Hom}_R(P,E)$ are monomorphisms. (Lemma 17). Then $1 = L_E(P) = L_{E'}(P)$ and there must be a non-zero homomorphism $f : P \to E'$. Therefore $L_{E'}(f(P)) \neq 0$ and $L_E(f(P)) \neq 0$. Again, we have a non-zero homomorphism $g : f(P) \to E$. Hence gf is not zero and f must be a monomorphism. Consequently $E' = E(f(P)) \approx E(P) = E$. This completes the proof of Theorem 17.

<u>Corollary</u>. <u>Let</u> $\{E_\pi\}_{\pi\in\Pi}$ <u>be a set of representatives of</u> $\mathfrak{F}$-<u>injective modules, one from each isomorphism class, and put</u> $L_\pi = L_{E_\pi}$. <u>If</u> L <u>is a length function on</u> $\mathfrak{F}$ <u>then</u> L <u>can be uniquely written as a linear combination of the</u> L_π's. <u>This representation is valid on</u> Fin L, <u>or on</u> $\mathfrak{F}$ <u>if</u> L <u>is continuous on</u> Fin L.

The corollary is an immediate consequence of the theorem and Theorem 12 Cor.2.

3.5 The commutative case.

Throughout this section R will be a fixed commutative ring and $\mathfrak{F}$ denotes the category of Noetherian R-modules. We will demonstrate that the commutativity of the ring R enables one to carry out further simplifications. Our first result characterizes the $\mathfrak{F}$-injective modules.

An ideal P of R is called a 'prime ideal' if $P \neq R$ and $rs \in P$ implies $r \in P$ or $s \in P$ for all $r,s \in R$. We see at once that if I and J are ideals, $I \supset P$, $J \supset P$, then $I \cap J \supset P$. Thus $E(R/P)$ is indecomposable for a prime ideal P. By an '$\mathfrak{F}$-prime' ideal P we mean a prime ideal P of R such that $R/P \in \mathfrak{F}$, i.e. R/P is Noetherian.

Proposition 19. There is a one-to-one correspondence between the set of $\mathfrak{F}$-prime ideals of R and the set of isomorphism classes of $\mathfrak{F}$-injective modules, given by $P \leftrightarrow E(R/P)$.

This result was proved in [5] under the assumption that R is Noetherian.

Proof. If P is an $\mathfrak{F}$-prime then $E(R/P)$ is $\mathfrak{F}$-injective. Conversely, if E is an $\mathfrak{F}$-injective module then by Lemma 17 we can find a sub-module N of E such that $N \neq 0$, $N \in \mathfrak{F}$ and the non-zero homomorphisms from N to E are monomorphisms. Choose an element $e \in N$, $e \neq 0$. Then Re has the same properties as N since homomorphisms from Re can be extended to E. Let $P = 0 : e$ and $rs \in P$, $s \notin P$ for elements $r,s \in R$. By assumption $x \to sx$, $x \in Re$, is a monomorphism since $se \neq 0$. Hence $0 = s(re) \Rightarrow re = 0$ and $r \in P$. Thus P is a prime ideal. Finally $Re \approx R/P$ implies that $E \approx E(R/P)$.

Now suppose that P, P' are prime ideals of R such that

$E(R/P) \approx E(R/P')$. Then $E(R/P)$ has a submodule N isomorphic to R/P', and $N \cap (R/P) \neq 0$. Consider a non-zero element x of $N \cap (R/P)$, and suppose that it corresponds to $y \in R/P'$. Then $P = 0 : x = 0 : y = P'$. This completes the proof.

According to this last result, the irreducible length functions on $\mathfrak{F}$ take the form $L_{E(R/P)}$ where P runs through the $\mathfrak{F}$-prime ideals of R. It is not these functions, however, but the principle of localization which is widely used in commutative algebra. It will be presently shown that the numerical outcome is the same whichever technique one uses.

Let P be a prime ideal of R. A '<u>localization of R with respect to P</u>' is a ring R_P with a ring homomorphism $\varphi : R \to R_P$ satisfying the following conditions:

(i) $\text{Ker } \varphi = \{r \in R : rt = 0 \text{ for some } t \in R - P\}$;

(ii) $\varphi(t)$ is a unit in R_P for all $t \in R - P$;

(iii) Every element of R_P can be expressed in the form

$$\varphi(r)\varphi(t)^{-1} \quad (r \in R,\ t \in R - P).$$

Then R_P is unique up to isomorphism and one can simply refer to R_P as the localization with respect to P, without reference to φ. The localization of R with respect to a prime ideal always exists. We need only a few well-known facts concerning localizations. The ring R_P is flat when regarded as an R-module. Accordingly $A \to R_P \otimes_R A$ is an exact functor from $\mathfrak{M}(R)$ to $\mathfrak{M}(R_P)$. If I is an ideal of R then $R/I \otimes_R R_P = 0$ if and only if $I \not\subseteq P$. There is a unique maximal ideal of R_P isomorphic to $P \otimes_R R_P$, and $R/P \otimes_R R_P$ is isomorphic to the simple R_P-module, necessarily unique up to isomorphism.

Let P be a prime ideal of R, ℓ the classical length function on $\mathfrak{M}(R_P)$ and set

$$L_P(A) = \ell(A \otimes_R R_P),\ A \in \mathfrak{M}(R).$$

<u>Proposition 20.</u> <u>Let</u> P <u>be a prime ideal of</u> R <u>and</u> E = E(R/P). <u>The functions</u> L_E <u>and</u> L_P <u>are identical on</u> $\mathfrak{M}(R)$.

<u>Proof</u>. The functions L_E and L_P are upper continuous by Proposition 2.10, and all of their finite values are integers. If $L_E \neq L_P$ then there is an integer $n \geq 0$ such that the statement $L_E(A) = n \iff L_P(A) = n$ for all $A \in \mathfrak{M}(R)$, is not true. Let n be minimal with respect to this property.

For an ideal I of R, $L_P(R/I) = 0$ if and only if $I \not\subseteq P$. If $I \subseteq P$ then $L_E(R/I) \geq L_E(R/P) = 1$. Conversely, assume that $L_E(R/I) > 0$ for an ideal $I \subseteq R$. Then there is a non-zero homomorphism $f : R/I \to E$ and $I \subseteq 0 : f(R/I) \subseteq 0 : f(R/I) \cap R/P = P$ since $f(R/I) \cap R/P \neq 0$. Thus for any ideal I of R, $L_P(R/I) = 0 \iff L_E(R/I) = 0$. It follows that $L_P(A) = 0 \iff L_E(A) = 0$ for all $A \in \mathfrak{M}(R)$ since the functions are upper continuous. Hence $n > 0$.

Assume now that $L_P(R/I) = n$ for an ideal $I \subseteq R$. Then $I \subseteq P$ and $L_P(R/I) = L_P(R/P) + L_P(P/I) = 1 + L_E(P/I) = L_E(R/P)$ since $L_P(P/I) = n - 1$. Similarly, $L_E(R/I) = n \Rightarrow L_P(R/I) = n$. Using upper continuity we see that $L_P(A) = n \iff L_E(A) = n$ for all $A \in \mathfrak{M}(R)$. This, however, contradicts our assumption on n. Thus L_P and L_E agree on $\mathfrak{M}(R)$.

The most striking difference between the commutative and non-commutative case is, that if L is a length function on $\mathfrak{F}$ (R is commutative) then dimFin L/Ker L < 0. The result depends largely on the following simple lemma which is of some interest in its own right.

Lemma 21. Let L be a length function on $\mathfrak{M}(R)$ and P a prime ideal of R such that $L(R/P) < \infty$. If the ideal I strictly contains P then $L(R/I) = 0$.

Proof. The exact sequence

$$0 \to I/P \to R/P \to R/I \to 0$$

yields $L(R/I) = L(R/P) - L(I/P)$. Choose an element $r \in I$, $r \notin P$. Then the natural map $R \to rR$ induces a monomorphism $R/P \to I/P$. Thus $L(R/P) \leq L(R/I)$ and $L(R/I) = 0$.

Lemma 22. The family of modules R/P, P is an $\mathfrak{F}$-prime, form a set of representatives for quasi-simple modules in $\mathfrak{F}$, one for each equivalent class.

This is implicit in Proposition 19 but we give a direct proof.

Proof. Let $\mathfrak{U} \subseteq \mathfrak{F}$ be a Serre-category and S a $\mathfrak{U}$-simple module in $\mathfrak{F}$. By Proposition 4, there is an ideal P in R such that $S \sim R/P$ and every proper factor module of R/P is in $\mathfrak{U}$. If $r,s \in R$, $r \notin P$, then the homomorphism $f : R/P \to r \cdot R/P \subseteq R/P$ is not zero. Hence $f(R/P)$ is a non-zero submodule of R/P and $f(R/P) \notin \mathfrak{U}$. Accordingly, $\operatorname{Ker} f = 0$. If $rs \in P$ then $r(sR/P) = 0$ implies $sR/P = 0$ and $s \in P$. Thus P is a prime ideal. It is clear that if P is an $\mathfrak{F}$-prime then R/P is Ker L_P-simple. Finally, different primes give rise to non-equivalent quasi-simple modules since the induced length functions are different (Proposition 19).

Theorem 23. If L is a length function on $\mathfrak{F}$ then dimFin $L/\operatorname{Ker} L \leq 0$.

Proof. Assume that dimFin $L/\operatorname{Ker} L > 0$. Then we must have

Ker L $\subset$ (Ker L)$' \subset$ Fin L since dimFin L/Ker L $< \infty$. We can choose a (Ker L)$'$-simple module S in Fin L of the form $S = R/P$, P an $\mathfrak{F}$-prime (Lemma 22). Now Lemma 21 shows that every proper factor module of S is in Ker L. Hence S is Ker L-simple contrary to our assumption.

Let $\mathfrak{U} \subseteq \mathfrak{F}$ be a Serre-category. A prime ideal P of R is said to be a '<u>minimal prime ideal of $\mathfrak{U}$</u>' if $R/P \in \mathfrak{U}$ and for any prime ideal P' of R, $R/P' \in \mathfrak{U}$ and $P' \subseteq P$ implies that $P' = P$.

<u>Proposition 24</u>. <u>If</u> L <u>is a length function on</u> $\mathfrak{F}$ <u>then</u> L <u>admits a unique decomposition on</u> Fin L <u>of the term</u> $L = \Sigma_P c_P L_P$, <u>where</u> P <u>ranges over the minimal prime ideals of</u> Fin L <u>and</u> $c_P = L(R/P)$. <u>Moreover</u>, L <u>is the discrete sum of the</u> L_P's <u>and the decomposition holds on</u> $\mathfrak{F}$ (<u>or on</u> $\mathfrak{M}(R)$ <u>if</u> L <u>is continuous on</u> Fin L).

<u>Proof</u>. By Theorem 23 dimFin L/Ker L ≤ 0. If Ker L = Fin L, i.e. $L = 0$ there is nothing to prove. Assume now that dimFin L/Ker L $= 0$ and let $\{P_i\}_{i \in I}$ be a family of $\mathfrak{F}$-prime ideals such that $\{R/P_i\}_{i \in I}$ is a set of representatives of Ker L-simple modules in Fin L, $P_i \neq P_j$ for $i \neq j$. Then by Proposition 10, $L = \Sigma_{i \in I} c_{P_i} L_{P_i}$ on Fin L, $c_{P_i} = L(R/P_i)$ and this representation of L as a linear combination of the L_{P_i}'s is unique. For each $i \in I$ we have $0 < L(R/P_i) < \infty$. It follows from Lemma 21 that P_i is a minimal prime ideal of Fin L. If P is a minimal prime ideal of Fin L then either $0 < L(R/P)$, whence R/P is Ker L-simple and $P = P_i$ for some $i \in I$, or $L(R/P) = 0$. Thus we can let P run through the minimal primes of Fin L. The sum is discrete by Theorem 13.

3.6 A counter-example

The aim of this section is to show that Theorem 23 is no longer valid if the commutativity of the ring is dropped.

Proposition 25. There exists a ring R and a length function L on the category of Noetherian R-modules such that $\dim \mathrm{Fin}\, L/\mathrm{Ker}\, L > 0$.

Proof. Let F be a field and V a countable dimensional vector space over F. Let $\{X_i\}_{i=1}^{\infty}$ be a base of V and define endomorphisms $\{e_j\}_{j=1}^{\infty}$ for V by

$$e_j(X_i) = \begin{cases} X_i & i < j \\ X_{i+1} & i = j \\ 0 & j < i \end{cases} \qquad 1 \leq i,j < \infty.$$

Let R be the subring of the ring of endomorphisms of V generated by F and $\{e_j\}_{j=1}^{\infty}$. Then V is a left R-module in a natural way. It is easily seen from the construction that

$$V = RX_1 \supset RX_2 \supset \ldots \supset RX_n \supset \ldots$$

are the only R-submodules of V. Hence V is Noetherian. It is clear that $S_i = RX_i/RX_{i+1}$ is a simple R-module for every $i \geq 1$. If $j > i$ then $e_j S_j = 0$ but $e_j S_i \neq 0$. Therefore $S_i \approx S_j$ if and only if $i = j$. Let $\mathfrak{U}$ be the category of R-modules of finite length, (i.e. the category of modules which are both Noetherian and Artinian) and $\mathfrak{F}$ the category of Noetherian R-modules. Then $V \in \mathfrak{F}$ but $V \notin \mathfrak{U}$. Also, V is $\mathfrak{U}$-simple since every proper factor module of V belongs to $\mathfrak{U}$. Let L_i be the length function associated to the simple module S_i and set $L = \Sigma_{i=1}^{\infty} 2^{-i} L_i$. Then $L(V) = 1 \Rightarrow V \in \mathrm{Fin}\, L$ but $V \notin (\mathrm{Ker}\, L)'$. Thus $\dim \mathrm{Fin}\, L/\mathrm{Ker}\, L > 0$. Since we considered L as a function over $\mathfrak{F}$, $\mathrm{Fin}\, L \subseteq \mathfrak{F}$ and $\dim \mathrm{Fin}\, L/\mathrm{Ker}\, L < \infty$.

CHAPTER 4

SPECIAL CATEGORIES

4.1 Artinian modules over commutative Noetherian rings

The main decomposition theorem (Theorem 12) applies to the category of Artinian modules as well as to the Noetherian one. In the general case, however, we do not have such a transparent description of the irreducible functions as given for the category of Noetherian modules in section 3.4.

Throughout this section R denotes a fixed commutative Noetherian ring and $\mathfrak{H}$ stands for the category of Artinian R-modules. For each maximal ideal M of R the local ring R_M has a natural topology induced by the powers of the maximal ideal of R_M. The completion of R_M in this topology is denoted by $\hat{R}_M$. Details of this construction can be found in [1, section 9.11], together with the result that $\hat{R}_M$ is a commutative Noetherian ring. Let $\mathfrak{I}_M$ be the category of Noetherian modules over $\hat{R}_M$. E.Matlis established a perfect duality between the Noetherian and Artinian modules of a complete local ring [5, Cor. 4.3].

We first show that this result can be extended to a perfect duality between $\mathfrak{H}$ and the 'direct sum' of the $\mathfrak{I}_M$'s. Our aim in this section is to describe the length functions on $\mathfrak{H}$ and this duality will enable us to pass to the study of length functions on the $\mathfrak{I}_M$'s. We shall rely heavily upon the methods and results of E.Matlis [5] and [6].

Let $A \in \mathfrak{H}$, M a maximal ideal of R and define

$$T_M(A) = \{x \in A : M^n x = 0 \text{ for some } n > 0\}.$$

It is easily seen that $T_M(A)$ is a submodule of A. Let B be a second element of $\mathfrak{H}$ and $f : A \to B$ an R-homomorphism. Let $T_M(f) : T_M(A) \to T_M(B)$ be the restriction of f to $T_M(A)$. We see at once that T_M is a functor from $\mathfrak{H}$ to $\mathfrak{H}$. For each maximal ideal M of R, E_M denotes the injective envelope of R/M. From [5] and [6] we need the following results:

(a) The functors $T_M : \mathfrak{H} \to \mathfrak{H}$ are left exact and commute with the taking of injective envelopes [5, Proposition 1].

(b) For maximal ideals M, M' of R, $E_M \in \mathfrak{H}$† and $T_{M'}(E_M) = E_M$ if $M = M'$ and 0 if $M \neq M'$. [6, Proposition 3] and [5, Theorem 3.4]

(c) For each $A \in \mathfrak{H}$, $A = \oplus_M T_M(A)$ where M ranges over all the maximal ideals of R, and $T_M(A) = 0$ for all but a finite number of maximal ideals [6, Theorem 1].

(d) There is a ring isomorphism $\mathrm{Hom}_R(E_M, E_M) \approx \hat{R}_M$ and an $\hat{R}_M$-isomorphism $E_M \approx E(S)$ where S is the only simple $\hat{R}_M$-module. [5, Theorem 3.7].

For each maximal ideal M of R set

$$\mathfrak{H}_M = \{A \in \mathfrak{H} : T_M(A) = A\}.$$

Lemma 1. *Let M be a maximal ideal of R. Then $\mathfrak{H}_M$ is a Serre-category and $T_M : \mathfrak{H} \to \mathfrak{H}_M$ is an exact functor.*

Proof. Suppose $A, B \in \mathfrak{H}$ and $f : A \to B$ is a homomorphism. Clearly

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† $E_M \in \mathfrak{H}$ for each maximal ideal amounts to saying that an R-module is finitely embedded if and only if it is Artinian.

$f(T_M(A)) \subseteq T_M(B)$ for each maximal ideal M of R. It follows that if f is an epimorphism then $T_M(f)$ is an epimorphism by (c). Hence T_M is exact. Moreover, $A \in \mathfrak{H}_M$ if and only if $T_{M'}(A) = 0$ for maximal ideals $M' \neq M$. Since the functors T_M are exact, $\mathfrak{H}_M$ is a Serre-category for all maximal ideals M of R.

Proposition 2. The categories $\mathfrak{H}_M$ and $\mathfrak{T}_M$ are equivalent, i.e. there are exact functors $F : \mathfrak{H}_M \to \mathfrak{T}_M$ and $G : \mathfrak{T}_M \to \mathfrak{H}_M$ such that FG and GF are naturally equivalent to the identity functors.

Proof. Let F and G be defined by

$$F(A) = \operatorname{Hom}_R(A, E_M), \quad A \in \mathfrak{H}_M \text{ and}$$

$$G(B) = \operatorname{Hom}_{\hat{R}_M}(B, E_M) \quad B \in \mathfrak{T}_M.$$

Both F and G are exact since E_M is injective even if regarded as an $\hat{R}_M$-module by (d). The isomorphisms $F(E_M) \approx \hat{R}_M$ and $G(\hat{R}_M) \approx E_M$ imply that $F(A) \in \mathfrak{T}_M$ and $G(B) \in \mathfrak{H}_M$ whenever $A \in \mathfrak{H}_M$ and $B \in \mathfrak{T}_M$ respectively. (Here the first isomorphism is an $\hat{R}_M$-isomorphism while the second is an R isomorphism.) Let $A \in \mathfrak{H}_M$ and consider the mapping

$$f_A : A \to \operatorname{Hom}_{\hat{R}_M}(\operatorname{Hom}(A, E_M), E_M) = GF(A),$$

defined by $f_A(a)(x) = x(a)$, $a \in A$, $x \in \operatorname{Hom}_R(A, E_M)$. Then f is natural and a monomorphism. Indeed, for any $A \in \mathfrak{H}_M$, $E(A) = E(T_M(A)) = T_M(E(A)) = E_M^k$ for some integer $k > 0$ by (a) and (c), where E_M^k denotes the direct sum of k copies of E_M. Hence $F(A) \neq 0$ if $A \neq 0$ and for any element $a \in A$, $a \neq 0$ there is a homomorphism $x : A \to E_M$ such that $x(a) \neq 0$. This shows that f_A is a monomorphism.

Let $B = E(A)/A = E_M^k/A$. Then the following diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \to & E_M^k & \to & B & \to & 0 \\ & & \downarrow f & & \downarrow f & & \downarrow f & & \\ 0 & \to & GF(A) & \to & GF(E_M^k) & \to & GF(B) & \to & 0 \end{array}$$

is commutative with exact rows and monomorphisms between them. But the middle f is an isomorphism, and so are all f's. This proves that f is a natural isomorphism. A similar argument gives the required equivalence for the functor FG.

Let L be a length function on $\mathfrak{S}$ and for each maximal ideal M of R set

$$L_M(A) = L(T_M(A)), \quad A \in \mathfrak{S}.$$

Since T_M is exact, L_M is a length function on $\mathfrak{S}$.

Proposition 3. For maximal ideals $M \neq M'$ of R, $L_M(A) = 0$ if $A \in \mathfrak{S}_{M'}$. Further, $L = \Sigma_M L_M$ where M ranges over all the maximal ideals of R and this sum is discrete.

Proof. If $A \in \mathfrak{S}_{M'}$ and $M \neq M'$ then $T_M(A) = 0$ and $L_M(A) = 0$ as well. Let $A \in \mathfrak{S}$. Then by (c), $A = \oplus_M T_M(A)$, where the summation is taken over all the maximal ideals of R. But $T_M(A) = 0$ for all but a finite number of maximal ideals since A is Artinian. Accordingly, $L(A) = \Sigma_M L(T_M(A)) = \Sigma_M L_M(A)$.

In view of the above proposition, we obtain a full description of the length functions on $\mathfrak{S}$ once we determine the length functions on the $\mathfrak{S}_M$'s.

Let M be a maximal ideal of R and F,G the functors between $\mathfrak{S}_M$ and $\mathfrak{T}_M$ described in Proposition 2. If L, L' are length functions on $\mathfrak{S}_M$ and $\mathfrak{T}_M$ respectively then the functions L_G, L'_F defined by

$$L_G(B) = L(G(B)),\ B \in \mathfrak{F}_M$$

$$L'_F(A) = L'(F(A)),\ A \in \mathfrak{H}_M$$

are length functions on $\mathfrak{F}_M$ and $\mathfrak{H}_M$ respectively.

Theorem 4. There is a one-to-one correspondence between the length functions on $\mathfrak{H}_M$ and the length functions on $\mathfrak{F}_M$, given by

$$L \leftrightarrow L_G \quad (L' \leftrightarrow L'_F).$$

Proof. Let L be a length function on $\mathfrak{H}_M$ and L' a length function on $\mathfrak{F}_M$. Then by Proposition 3, $L_G = L'$ if and only if $L = L'_F$. The theorem now follows.

4.2 Artinian and Dedekind rings

As we have seen in the earlier sections, we can gain useful information about length functions by considering them on categories with Krull dimension, in particular on the categories of Noetherian and Artinian modules. The question arises as to what extent is a length function determined by its behaviour on the categories of Noetherian and Artinian modules. In order to put the question in a more precise form, suppose that R is a ring and $\mathfrak{F}$ and $\mathfrak{H}$ denote the categories of Noetherian and Artinian modules respectively. Let L be a length function on $\mathfrak{M}(R)$ and L_1 the continuous component of L with respect to $\mathfrak{F}$. Then $L = L_1 + (L-L_1)$ where $L - L_1$ has values 0 or ∞ on $\mathfrak{F}$. If we repeat the process with $L - L_1$ and $\mathfrak{H}$ then we obtain $L = L_1 + L_2 + L'$ where L_2 is the continuous component of $L - L_1$ with respect to $\mathfrak{H}$ and L' has values 0 or ∞ on $\mathfrak{F}$ and $\mathfrak{H}$. Thus our original question takes the following form. Is a length function trivial on $\mathfrak{M}(R)$ if it is trivial on $\mathfrak{F}$ and $\mathfrak{H}$? If the answer is 'yes' then every length function on $\mathfrak{M}(R)$ is a sum of trivial functions and length functions which are continuous on $\mathfrak{F}$ or on $\mathfrak{H}$. In this section we decide the question in the affirmative for two classes of rings, and show that in general the answer is 'no'. The following simple lemma will be useful in the sequel.

Lemma 5. Let R be a ring, L a length function on $\mathfrak{M}(R)$ and $M \in \mathfrak{M}_{(R)}$. Set $M^I = \oplus_{i \in I} M$. Then $L(M^I) = 0$ or ∞ in either of the following cases.

(i) I is infinite,

(ii) $L(M) = 0$ or ∞.

Proof. If the index set I is infinite then $M^I \approx M^I \oplus M^I$ whence $L(M^I) = L(M^I) + L(M^I)$. Thus $L(M^I) = 0$ or ∞. If I is finite then the result follows immediately from (ii).

The 'Jacobson radical' of a ring R is the intersection of the annihilators of the simple R-modules. Let R be a ring and J its Jacobson radical. It is well-known that if R is (left) Artinian then J is nilpotent. If R/J is Artinian then every R-module annihilated by J is semi-simple, i.e. a direct sum of simple modules. Moreover, there are only a finite number of non-isomorphic simple R-modules. Our first result concerns the class of those rings R which satisfy:

(a) R/J is (left) Artinian;

(b) J is nilpotent.

For example, every Artinian ring satisfies (a) and (b). Also, if R satisfies (a) and (b) then $\mathfrak{F} = \mathfrak{H}$ = modules with finite (classical) length.

Theorem 6. Let R satisfy conditions (a) - (b) and let L be a length function on $\mathfrak{M}(R)$ such that $L(S) = 0$ or ∞ for every simple R-module S. Then L is trivial on $\mathfrak{M}(R)$.

Proof. Let $M \in \mathfrak{M}(R)$ and suppose that M is semi-simple. We can arrange that $M = M_1 \oplus \ldots \oplus M_k$ where each of the M_i's is a direct sum of mutually isomorphic simple modules. By Lemma 5, $L(M_i) = 0$ or ∞ for all $i = 1,\ldots,k$. Hence $L(M) = 0$ or ∞. Assume now that M is an arbitrary element of $\mathfrak{M}(R)$. Since the Jacobson radical J of R is nilpotent, $J^n = 0$ for some $n > 0$. Then $M = J^0M \supseteq JM \supseteq \ldots \supseteq J^{n-1}M \supseteq J^nM = 0$ is a chain of submodules of M and $L(M) = \Sigma_{i=0}^{n-1} L(J^iM/J^{i+1}M)$. But $J^iM/J^{i+1}M$ is annihilated by J whence semi-simple. It follows from the first part of the proof that $L(J^iM/J^{i+1}M) = 0$ or ∞ for all $i = 0,\ldots,n-1$. Thus $L(M) = 0$ or ∞ and L is trivial on $\mathfrak{M}(R)$.

A commutative domain whose ideals are totally ordered by inclusion is called a '<u>valuation ring</u>'. If R is a valuation ring then every finitely generated ideal is principal. If, in addition, R is Noetherian then there is an element $p \in R$ such that every proper ideal is of the form Rp^k, $k > 0$. A commutative Noetherian domain R is said to be a '<u>Dedekind domain</u>' if, for each maximal ideal $\mathcal{M}$ of R, $R_{\mathcal{M}}$ is a valuation ring. The ring R is called '<u>semi-local</u>' if it has only a finite number of maximal ideals.

Theorem 7. *Let R be a semi-local Dedekind domain and let L be a length function on $\mathfrak{M}(R)$ such that L has values only 0 or ∞ on Noetherian and Artinian modules. Then L is trivial on $\mathfrak{M}(R)$.*

Proof. We call an R-module A '<u>torsion</u>' if every element of A has a non-zero annihilator ideal. The proof will be afforded in a number of steps, the first of which shows that it is sufficient to consider torsion modules over a valuation ring.

We assume that there is a module $A \in \mathfrak{M}(R)$ such that $0 < L(A) < \infty$ and we want to derive a contradiction.

(a) We may assume that A is a torsion module and R is a valuation ring. For if T is the torsion submodule of A (the maximal torsion module in A) then A/T is torsion free and $L(A) = L(T) + L(A/T)$. Further, we can find a maximal free submodule F of A/T and $T' = (A/T)/F$ is a torsion module. Thus $L(A) = L(T) + L(T') + L(F)$. By Lemma 5, $L(F) = 0$ or ∞, in our case $L(F) = 0$ since $L(F) \leq L(A) < \infty$. Hence $L(A) = L(T) + L(T')$ and either $L(T)$ or $L(T')$ is finite and non-zero. Let $P_1, \ldots, P_k$ be the maximal ideals of R. If A is a torsion R-module then $A = A_1 \oplus \ldots \oplus A_k$ where every element of A_i is annihilated by a power of the ideal P_i, $1 \leq i \leq k$. (c.f. [6, Theorem 1].) If $0 < L(A) < \infty$ then the same holds true for at least one A_i, $1 \leq i \leq k$. It is easily seen that the elements of A_i are (uniquely) divisible by the elements in $R - P_i$. Thus each of the A_i's can be regarded, in a natural way, as an R_{P_i}-module.

(b) Suppose that R is a Noetherian valuation ring and A is a torsion R-module such that $0 < L(A) < \infty$. There exists a 'basic' submodule B of A such that B is the direct sum of cyclic modules and A/B is injective. [14, Section 29, p.97-98]. Then $L(A) = L(B) + L(A/B)$. But by [5, Theorem 2.5] and Proposition 3.19, A/B is a direct sum of copies of $E(R/P)$ where P is the maximal ideal of R. Also, $E(R/P)$ is Artinian by [6, Proposition 3]. Using Lemma 5 we see that $L(A/B) = 0$. Hence $L(A) = L(B)$.

(c) Assume now that R is a Noetherian valuation ring and

P its maximal ideal. Let A be a direct sum of cyclic torsion R-modules such that $0 < L(A) < \infty$. Then $A = \oplus_{k=1}^{\infty} A_k$, where A_k is the direct sum of copies of R/P^k. Set

$$B_k = \begin{cases} A_k & \text{if } A_k \text{ is a finite direct sum,} \\ 0 & \text{otherwise;} \end{cases}$$

$$C_k = \begin{cases} A_k & \text{if } A_k \text{ is an infinite direct sum,} \\ 0 & \text{otherwise,} \end{cases}$$

$B = \oplus_{k=1}^{\infty} B_k$, $C = \oplus_{k=1}^{\infty} C_k$. Then $A_k = B_k \oplus C_k$ $(1 \leq k < \infty)$ and $A = B \oplus C$. But $C \approx C \oplus C$ and therefore $L(C) = 0$, $L(A) = L(B)$. Since L is 0 or ∞ on Noetherian modules, B must be an infinite sum. Consequently $B = \oplus_{i=1}^{\infty} Rb_i$ and $0 : b_i \supsetneq 0 : b_{i+1}$ for all i. Set $B_1 = \oplus_{i=1}^{\infty} Rb_{2i-1}$ and $B_2 = \oplus_{i=1}^{\infty} Rb_{2i}$. Then $B = B_1 \oplus B_2$ and $L(B) = L(B_1) + L(B_2)$. For $i < j$ we have monomorphisms $f_{i,j} : Rb_i \to Rb_j$ which yield monomorphisms $g : B \to B_1$, $h : B \to B_2$ defined by $g(b_i) = f_{i,2i-1}(b_i)$ and $h(b_i) = f_{i,2i}(b_i)$ $1 \leq i < \infty$. Thus $L(B) \leq L(B_1)$, $L(B) \leq L(B_2)$ and these imply

$$2L(B) \leq L(B_1) + L(B_2) = L(B).$$

It follows that $L(B) = 0$, the required contradiction.

We close this section with an example which shows that the theorem does not generalize to Dedekind domains having an infinite number of prime ideals. For the sake of simplicity we consider the ring of integers, Z.

Let p be a prime number and $Z(p^\infty)$ the injective envelope of $Z/(p)$. The completion of $Z_{(p)}$ (called the ring of p-adic integers), $\hat{Z}_p$ is a commutative domain. Let L_r be the rank function over this ring. As we have seen $\mathrm{Hom}_Z(Z(p^\infty), Z(p^\infty)) \approx \hat{Z}_p$. The function L_p

defined on $\mathfrak{M}(Z)$ by

$$L_p(A) = L_r(\mathrm{Hom}(A, Z(p^\infty))), \; A \in \mathfrak{M}(Z)$$

is a length function. Since $L_p(Z) = 0$, L_p is zero on the Noetherian Abelian groups. Let $p_1, \ldots, p_n, \ldots$ be the sequence of rational primes and $L_i = L_{p_i}$ the functions defined above. The subcategory $\mathfrak{A}$ of $\mathfrak{M}(Z)$, consisting of all the Abelian groups A for which the sequence $\{L_i(A)\}_{i=1}^\infty$ is bounded, is a Serre-category and contains the Noetherian and Artinian Abelian groups. Let S be the partially ordered Abelian group of bounded sequences of integers, addition and order defined componentwise. The subgroup D of S consisting of sequences with finitely many non-zero terms is convex in S. There is an order preserving homomorphism φ from S into the real numbers which vanishes on D, yet $\varphi(1,1,\ldots,1,\ldots) = 1$. This may be seen as follows. The group S/D is torsion-free, whence the partial order can be extended to a full order [15, Cor 13, p.39]. Since every convex subgroup of S containing $(1,\ldots,1,\ldots)$ is equal to S, we can find a maximal convex subgroup M/D of the fully ordered group S/D by Zorn's lemma. Then S/M is a fully ordered rank-one group, whence it is order-isomorphic to a subgroup of the reals.

The function L on $\mathfrak{M}(Z)$ defined by

$$L(A) = \varphi(L_1(A), L_2(A), \ldots, L_n(A), \ldots), \; A \in \mathfrak{A}$$

is a length function on $\mathfrak{A}$ which vanishes on Noetherian groups since each L_i does so. If A is Artinian then $(L_1(A), \ldots, L_n(A), \ldots) \in D$ and $\varphi(D) = 0$. Hence L vanishes on Artinian modules too. But $L(A) = 1$ for $A = \oplus_{i=i}^\infty Z(p_i^\infty)$. The extension of L to $\mathfrak{M}(Z)$ is the required example.

4.3 Valuation rings

In [3] the authors determined all the upper continuous length functions over a valuation ring R. In case R is Noetherian our section 3.5 provides the (otherwise trivial) answer. The interesting case is, of course, the one where R is not Noetherian. In this case a completely new type of length function makes its appearance. We feel that no account of length functions can be complete without the presentation of this 'truly' upper continuous function.

Let R be a valuation ring and M its maximal ideal. If there is no prime ideal between M and 0, ($M \neq 0$) then R is said to be of rank one. In this case there is a function v from R into the real numbers and infinity satisfying:

(i) $v(a) \geq 0$ for all $a \in R$ and $v(a) = \infty$ if and only if $a = 0$;

(ii) $v(ab) = v(a) + v(b)$ for all $a,b \in R$;

(iii) $v(a+b) \geq \min(v(a),v(b))$.

Such a function is called a valuation of R. A proof of the existence of a valuation can be found in [13].

Let R be a rank-one valuation ring and v a valuation on R. For an ideal I of R set $v(I) = \inf_{a \in I} v(a)$. Then $v(0) = \infty$ and $v(R) = 0$ since $v(1) = 0$. Also, if M is the maximal ideal of R then $v(M) > 0$ implies that R is Noetherian.

Lemma 7. *Let R be a rank-one valuation ring and v a valuation on R. If I,J are ideals of R, where $I \subseteq J$ then $v(I:J) + v(J) = v(I)$.*

Proof. If R is Noetherian then every ideal is a principal ideal and the lemma follows immediately. Also, if $J = 0$ there is nothing to prove. Thus we may assume that $J \supset 0$ and $v(M) = 0$ where M is the maximal ideal of R.

If $b \in I:J$ and $c \in J$ then $bc \in I$ and $v(bc) = v(b) + v(c)$. Accordingly, $v(I:J) + v(J) \geq v(I)$. Conversely, let $a \in IM$. Then $v(a) - v(J) > v(I) - v(J)$ and we can find an element $b \in J$ such that $v(a) - v(b) > v(I) - v(J)$. Hence $a = bc$ for some $c \in R$. Further, if $d \in J$ then $v(cd) = v(c) + v(d) = v(a) - v(b) + v(d) > v(I) - v(J) + v(d) \geq v(I)$. It follows that $cd \in I$ and $c \in I:J$ since d was an arbitrary element of J. Thus $v(a) = v(b) + v(c)$ where $b \in J$, $c \in I:J$. Consequently, for every element $a \in IM$, $v(a) \geq v(J) + v(I:J)$. This implies that $v(IM) \geq v(J) + v(I:J)$. But $v(IM) = v(I)$ since $v(M) = 0$.

Let the situation be as described in Lemma 7. We wish to show that the valuation v on R induces an upper-continuous length function L on $\mathfrak{M}(R)$ such that $L(R/I) = v(I)$ for an ideal I of R. If R is Noetherian then L is just a positive multiple of the classical length function. If R is not Noetherian then $L(R/M) = 0$ and so L vanishes on Noetherian R-modules. (It is easy to see that Noetherian R-modules have finite classical length.) Yet L is upper-continuous.

Set $\mathfrak{U} = \{A \in \mathfrak{M}(R) : A$ is isomorphic to a segment of $R\}$. Thus if $A \in \mathfrak{U}$ then $A \approx I/J$ where $J \subsetneq I$ are ideals of R. Obviously, $\mathfrak{U}$ is semi-closed. Define L on $\mathfrak{U}$ by writing $L(0) = 0$ and $L(A) = v(J) - v(I)$ if $A \approx I/J$ and $I \supset J$ are ideals of R. If $I \supset J$, $I' \supset J'$ are ideals of R such that $I/J \approx I'/J'$ then taking annihilators on both sides: $I : J = I' : J'$. Applying Lemma 7 we see that $v(J) - v(I) = v(J') - v(I')$. Thus L is well defined and $L(A) = L(B)$ if $A \approx B, A, B \in \mathfrak{U}$. It now follows that L is a length function on $\mathfrak{U}$. Let $I \supset J$ be ideals of R. Then $L(I/J) = v(J) - v(I) = v(J) - \inf_{a \in I-J} v(a) = \sup_{a \in I-J} (v(J) - v(a)) = \sup_{a \in I-J} L(Ra/J)$.

This shows that L is upper continuous on $\mathfrak{A}$. Using Proposition 2.6 we find that the continuous extension of L to $\mathfrak{M}(R)$ is upper continuous. Thus we have obtained

Theorem 8. [3, Theorem 12]. Let R be a rank-one valuation ring and v a valuation on R. Then there is an upper-continuous length function L on $\mathfrak{M}(R)$ such that $L(R/I) = v(I)$ for every ideal I of R.

It was shown in [3] that the general situation, when the valuation ring R is arbitrary, can be reduced to the rank-one case. We note also that L in Theorem 8 is irreducible but not associated to a quasi-simple module when R is not Noetherian.

4.4 Rank-rings

We call a ring R a 'rank-ring' if there exists a length function L on $\mathfrak{M}(R)$ such that $L(R) = 1$. Equivalently, if $0 < L(R) < \infty$ is satisfied by a length function L on $\mathfrak{M}(R)$, then R is a rank-ring. The characterization of the class of rank-rings seems to be an interesting, though difficult, problem. The present section records the little information on rank rings which can be deducted from our investigations so far. The problem will be taken up again in chapter 6. (Theorem 6.4)

A left-Noetherian ring is a rank-ring by Proposition 3.10, Cor.2.

Let S,R be rings and assume that S is a right R, left S bi-module and S is flat as a right R-module. If S is a rank-ring then R is a rank-ring too. For if L is a length function on $\mathfrak{M}(S)$ such that $L(S) = 1$ then L' defined by $L'(A) = L(S \otimes_R A)$, $A \in \mathfrak{M}(R)$ is a length function on $\mathfrak{M}(R)$ and $L'(R) = 1$. Thus an Ore-domain is a rank-ring as we have seen in example II of section 2.4. (Or more generally, if the classical left-quotient ring of R exists and it is a rank-ring then R is a rank-ring too.) If R is a commutative ring and P is a prime ideal of R such that R_P is a rank-ring then R is a rank-ring as well. This is the case, in particular, when R_P is Noetherian for a minimal prime ideal P of R.

The class of rank-rings is closed under finite direct **sums** and the forming of full matrix rings. (See section 2.3.)

Let R^k denote the direct sum of k copies of the ring R, k > 0. The ring R is said to have IBN (invariant basis number) if $R^k \approx R^n$ implies k = n. We see at once that if R is a rank-ring then it has IBN. It is known that every commutative ring has IBN but there are rings without IBN (c.f. [16]). Therefore not every ring is a rank-ring. In fact we can say more.

Theorem 9. There is a commutative ring R, having exactly one prime ideal which is not a rank-ring.

Proof. Assume that R is a commutative ring and M is the only prime ideal of R. Let M be nilpotent and L a length function on $\mathfrak{M}(R)$. If $0 < L(R/M) < \infty$ then L is a positive multiple of the classical length function on $\mathfrak{M}(R)$ by Proposition 2.12. On the other hand, $L(R/M) = 0$ implies that L is trivial on $\mathfrak{M}(R)$ by Theorem 6. If, therefore, $L(R) = 1$ then L is a multiple of the classical length function and R is Artinian. Hence R is not a rank-ring if it is not Artinian. Such a ring can be constructed as follows. Let F be a field, $\{X_n\}_{n=1}^{\infty}$ a countable number of indeterminates and $S = F[X_1,\dots,X_n,\dots]$ the polynomial ring over F. Let A be the ideal in S generated by $\{X_iX_j\}_{i,j}$ and consider the ring $R = S/A$. It is easily seen that if M is the ideal in S generated by the indeterminates $\{X_i\}_i$ then M/A is the only prime ideal of R. Further, $(M/A)^2 = 0$ and R is not Artinian. Thus the ring R is not a rank-ring.

Using Theorem 8 one can obtain a rank-ring which is not related to Noetherian rings in any of the earlier described ways. Indeed, let R be a non-Noetherian rank-one valuation ring, v a valuation on R and M the maximal ideal of R. If I is an ideal of R so that $0 \subset I \subset M$ then $0 < v(I) < \infty$. If L is the length function in Theorem 8 then L induces a length function on $\mathfrak{M}(R/I)$ which makes R/I a rank-ring. But R/I is not Noetherian and does not seem to have a flat over ring which is Noetherian. Also M/I is the only prime ideal of R/I.

CHAPTER 5

MULTIPLICITY THEORY

5.1 The multplicity operator

Throughout this chapter $\mathfrak{F}$ and $\mathfrak{H}$ denote the category of Noetherian and Artinian R-modules respectively for a ring R. The ring R will be kept fixed in the first section. Let $\mathfrak{A}$ be a subcategory of $\mathfrak{M}(R)$. If for each length function L on $\mathfrak{A}$ there is associated a length function eL on $\mathfrak{A}$ then we say that e is an operator on the class of length functions on $\mathfrak{A}$. The product and sum of the operators is defined in the obvious way, i.e. if e_1 and e_2 are operators on length functions on $\mathfrak{A}$ then $(e_1+e_2)L = e_1L + e_2L$ and $(e_1e_2)L = e_1(e_2L)$ for every length function L on $\mathfrak{A}$. We let Γ denote the centre of the ring R.

For each central element $\gamma \in \Gamma$ the multiplicity symbol $e[\gamma]$ will be defined as an operator acting on length functions on $\mathfrak{F}$ and $\mathfrak{H}$. Our aim in this chapter is twofold. First we introduce the multiplicity operators and prove their elementary properties restricting ourselves to a bare minimum. At this stage the emphasis is laid upon the parallel development, i.e. we wish to demonstrate that multplicity theory can be developed on Artinian modules just as well as on Noetherian ones. In section two the associative law will be established without any restriction on the ring R.

This work owes much to the treatment of multiplicities in [1, chapter 7] and the reader should consult this account for further information.

Let $\gamma_1,\dots,\gamma_n$ be central elements and L, L^* length functions on $\mathfrak{F}$ and $\mathfrak{H}$ respectively. Define the categories $\mathfrak{S}(\gamma_1,\dots,\gamma_n,L)$ and $\mathfrak{S}(\gamma_1,\dots,\gamma_n,L^*)$ as follows:

$$\mathfrak{S}(\gamma_1,\dots,\gamma_n,L) = \{A \in \mathfrak{F} : A/\gamma_1 A + \dots + \gamma_n A \in \operatorname{Fin} L\};$$

$$\mathfrak{S}(\gamma_1,\dots,\gamma_n,L^*) = \{A \in \mathfrak{H} : 0 :_A \gamma_1 \cap \dots \cap 0 :_A \gamma_n \in \operatorname{Fin} L^*\}.$$

Lemma 1. *Let* $0 \to A' \to A \to A'' \to 0$ *be an exact sequence in* $\mathfrak{M}(R)$ *and* $\gamma \in \Gamma$. *Then there is an exact sequence of the form:*

$$0 \to 0 :_{A'} \gamma \to 0 :_A \gamma \to 0 :_{A''} \gamma \to A'/\gamma A' \to A/\gamma A \to A''/\gamma A'' \to 0. \qquad (1)$$

This is a special case of the so-called 'Ker-Coker sequence'. For a direct proof see [1, Lemma 3, p.301].

Proposition 2. *Let* $\gamma \in \Gamma$ *and* L, L^* *be length functions on* $\mathfrak{F}$ *and* $\mathfrak{H}$ *respectively. Then* $\mathfrak{S}(\gamma,L)$ *and* $\mathfrak{S}(\gamma,L^*)$ *are Serre-categories.*

Proof. Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence in $\mathfrak{M}(R)$. Suppose that $A', A'' \in \mathfrak{S}(\gamma,L)$. From the exact sequence (1) we see that

$$A'/\gamma A' \to A/\gamma A \to A''/\gamma A'' \to 0$$

is exact. Put $B = \operatorname{Im}(A'/\gamma A' \to A/\gamma A)$. Then $B \in \operatorname{Fin} L$, $A''/\gamma A'' \in \operatorname{Fin} L$ and we have an exact sequence

$$0 \to B \to A/\gamma A \to A''/\gamma A'' \to 0.$$

Hence $A/\gamma A \in \operatorname{Fin} L$ since $\operatorname{Fin} L$ is a Serre-category. Thus $A \in \mathfrak{S}(\gamma,L)$. (Note that $\operatorname{Fin} L \subseteq \mathfrak{F}$ since $\mathfrak{F}$ is the domain of L.) If $A \in \mathfrak{S}(\gamma,L)$ then $A'' \in \mathfrak{S}(\gamma,L)$ since $A/\gamma A \to A''/\gamma A'' \to 0$ is exact.

One may establish in a similar manner, (using the first part of (1)), that $A', A'' \in \mathfrak{S}(\gamma,L^*) \Rightarrow A \in \mathfrak{S}(\gamma,L^*)$ and $A \in \mathfrak{S}(\gamma,L^*) \Rightarrow A' \in \mathfrak{S}(\gamma,L^*)$.

Assume now that $A \in \mathfrak{S}(\gamma, L)$. In order to show that $A' \in \mathfrak{S}(\gamma, L)$ we may assume that A' is a submodule of A. Suppose that $A'/\gamma A' \notin \mathrm{Fin}\, L$. Then we can find a submodule B of A which is maximal with respect to the property that $B/\gamma B \notin \mathrm{Fin}\, L$. Put $B' = B :_A \gamma$. Then we have $\gamma^2 B' \subseteq \gamma B \subseteq \gamma B' \subseteq B \subseteq B'$. If $B \subset B'$ then $L(B'/\gamma B') < \infty$ by the maximality of B. Further, $L(B/\gamma B) \leq L(B'/\gamma^2 B') = L(B'/\gamma B') + L(\gamma B'/\gamma^2 B') < \infty$ since $\gamma B'/\gamma^2 B' \approx B'/\gamma B' + 0 :_{B'} \gamma$. If, on the other hand, $B = B'$ then $\gamma A \cap B = \gamma B$ and $L(B/\gamma B) = L(B/\gamma A \cap B) = L(B + \gamma A/\gamma A) \leq L(A/\gamma A) < \infty$. In either way a contradiction is obtained. Thus, if $L(A/\gamma A) < \infty$ for $A \in \mathfrak{F}$ then $L(A'/\gamma A') < \infty$ for every submodule of A.

Now we have only to consider the Artinian case. To this end, suppose that $A \in \mathfrak{S}(\gamma, L^*)$, i.e. $A \in \mathfrak{H}$ and $L^*(0 :_A \gamma) < \infty$. If there is a submodule $B \subseteq A$ such that $A/B \notin \mathfrak{S}(\gamma, L^*)$ then choose B to be minimal with respect to this property. Note that for any submodule C of A, $0 :_{A/C} \gamma \approx C :_A \gamma/C$. Put $B' = B :_A \gamma$. Then $\gamma B \subseteq B \subseteq B' \subseteq \gamma B :_A \gamma^2$. If $\gamma B \subset B$ then, by the minimality of B, $L^*(\gamma B :_A \gamma/\gamma B) < \infty$ and $L^*(\gamma B :_A \gamma^2/\gamma B) = L^*(\gamma B :_A \gamma^2/\gamma B :_A \gamma) + L^*(\gamma B :_A \gamma/\gamma B) < \infty$ since $\gamma B :_A \gamma^2/\gamma B :_A \gamma \approx (\gamma B :_A \gamma) \cap \gamma A/\gamma B \subseteq \gamma B :_A \gamma/\gamma B$. Thus $L^*(B'/B) \leq L^*(\gamma B :_A \gamma^2/\gamma B) < \infty$. If, on the other hand, $\gamma B = B$ then it is easy to see that $B' = 0 :_A \gamma + B$. Hence $L^*(B'/B) = L^*(0 :_A \gamma/0 :_B \gamma) \leq L^*(0 :_A \gamma) < \infty$. In either way, we obtain $L^*(B'/B) < \infty$. But $B'/B \approx 0 :_{A/B} \gamma$ which contradicts our assumption on B. Thus, for every submodule B of A, $A/B \in \mathfrak{S}(\gamma, L^*)$ and the proof of Proposition 2 is concluded.

We are now ready to define the multiplicity operator. Let $\gamma \in \Gamma$ and L, L^* be length functions on $\mathfrak{F}$ and $\mathfrak{H}$ respectively. Since $\mathfrak{S}(\gamma, L)$ and $\mathfrak{S}(\gamma, L^*)$ are Serre-categories by Proposition 2, we find that $0 :_A \gamma \in \mathfrak{S}(\gamma, L)$ (resp. $A/\gamma A \in \mathfrak{S}(\gamma, L^*)$) whenever $A \in \mathfrak{S}(\gamma, L)$ (resp. $A \in \mathfrak{S}(\gamma . L^*)$). Hence $0 :_A \gamma \in \text{Fin}\, L$ (resp. $A/\gamma A \in \text{Fin}\, L^*$). Now define the operator $e[\gamma]$ by

$$e[\gamma]L(A) = \begin{cases} L(A/\gamma A) - L(0 :_A \gamma) & \text{if } A \in \mathfrak{S}(\gamma . L), \\ \infty & \text{otherwise,} \end{cases} \qquad (A \in \mathfrak{F})$$

$$e[\gamma]L^*(A) = \begin{cases} L(0 :_A \gamma) - L(A/\gamma A) & \text{if } A \in \mathfrak{S}(\gamma, L^*) \\ \infty & \text{otherwise.} \end{cases} \qquad (A \in \mathfrak{H})$$

We cannot say, as yet, that $e[\gamma]L$, $e[\gamma]L^*$ are length functions. So we prove

<u>Proposition 3.</u> <u>The functions</u> $e[\gamma]L$ <u>and</u> $e[\gamma]L^*$ <u>are additive.</u>

<u>Proof.</u> Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence in $\mathfrak{F}$. Since $\mathfrak{S}(\gamma, L)$ is a Serre-category, $A \notin \mathfrak{S}(\gamma, L)$ implies that either $A' \notin \mathfrak{S}(\gamma, L)$ or $A'' \notin \mathfrak{S}(\gamma, L)$. Hence $e[\gamma]L(A) = e[\gamma]L(A') + e[\gamma]L(A'')$ if $A \notin \mathfrak{S}(\gamma, L)$. Assume now that $A \in \mathfrak{S}(\gamma, L)$. Then from the exact sequence (1) and Proposition 2.1, $L(0 :_{A'} \gamma) + L(0 :_{A''} \gamma) + L(A/\gamma A) = L(0 :_A \gamma) + L(A'/\gamma A') + L(A''/\gamma A'')$. But all the modules which occur belong to Fin L. Hence, by rearrangement, $e[\gamma]L(A) = e[\gamma]L(A') + e[\gamma]L(A'')$. The additivity of $e[\gamma]L^*$ is proved similarly.

It is clear that $e[\gamma]L(0) = e[\gamma]L^*(0) = 0$. We have only to show, therefore, that the functions $e[\gamma]L$ and $e[\gamma]L^*$ are non-negative. This will follow from the following.

Proposition 4. If $A \in \mathfrak{S}(\gamma, L)$ (resp. $A \in \mathfrak{S}(\gamma, L^*)$) and $\gamma^n A = 0$ for some $n > 0$ then $e[\gamma]L(A) = 0$ (resp. $e[\gamma]L^*(A) = 0$).

Proof. We proceed by induction on n. If $\gamma A = 0$ then $A/\gamma A = A$ and $0 :_A \gamma = A$. Also, $A \in \mathfrak{S}(\gamma, L)$ (resp. $A \in \mathfrak{S}(\gamma, L^*)$). Hence $e[\gamma]L(A) = L(A) - L(A) = 0$. (resp. $a[\gamma]L^*(A) = L^*(A) - L^*(A) = 0$).

Assume now that $n > 1$ and the proposition has been established for all values less than n. Consider the exact sequence

$$0 \to \gamma A \to A \to A/\gamma A \to 0.$$

Then $\gamma^{n-1}(\gamma A) = 0$ and $\gamma(A/\gamma A) = 0$. The result now follows from Proposition 3.

Corollary 1. The functions $e[\gamma]L$, $e[\gamma]L^*$ are non-negative. Therefore $e[\gamma]$ is an operator on the length functions of $\mathfrak{F}$ and on the length functions of $\mathfrak{H}$.

Proof. First consider $A \in \mathfrak{F}$. We may assume that $A \in \mathfrak{S}(\gamma, L)$. Let $0 :_A \gamma^k$ be maximal among the submodules of A of the form $0 :_A \gamma^n$, $n > 0$. By Proposition 4, $e[\gamma]L(0 :_A \gamma^k) = 0$ and $e[\gamma]L(A) = e[\gamma]L(A/0 :_A \gamma^k)$ since $e[\gamma]$ is additive. Put $B = A/0 :_A \gamma^k$. It follows from the maximality of $0 :_A \gamma^k$ that $0 :_B \gamma = 0$. Hence $e[\gamma]L(A) = e[\gamma]L(B) = L(B/\gamma B) \geq 0$. (If $A \notin \mathfrak{S}(\gamma, L)$, then $e[\gamma]L(A) = L(A/\gamma A) = \infty$.)

Assume now that $A \in \mathfrak{S}(\gamma, L^*)$ and let $\gamma^k A$ be minimal among the submodules of A of the form $\gamma^n A$, $n > 0$. Then $e[\gamma]L^*(A) = e[\gamma]L^*(\gamma^k A)$ since $\gamma^k(A/\gamma^k A) = 0$. But $\gamma(\gamma^k A) = \gamma^k A$ by the minimality of $\gamma^k A$. Hence $e[\gamma]L^*(\gamma^k A) = L^*(0 :_{\gamma^k A} \gamma) \geq 0$.

In view of the above corollary we can now rightly call $e[\gamma]$ an operator.

Corollary 2. *The operator* $e[\gamma]$ *preserves local discreteness.*

Proof. Recall that a length function L is called locally discrete (section 3.3) if for each module A in the domain of L, $\inf\{L(S) : S \text{ a segment of } A, L(S) > 0\} > 0$. Let $\gamma \in \Gamma$ and L be a length function on $\mathfrak{F}$. We saw in the proof of the previous corollary that for every $A \in \mathfrak{F}$, $e[\gamma]L(A) = L(S)$ for a factor module S of A. Hence $\inf\{e[\gamma]L(S) : S \text{ a segment of } A, e[\gamma]L(S) > 0\} \geq \inf\{L(S) : S \text{ a segment of } A, L(S) > 0\} > 0$. Analogously, if L^* is a locally discrete length function on $\mathfrak{H}$ then so is $e[\gamma]L^*$. Note also that $e[\gamma]L$ (resp. $e[\gamma]L^*$) will have integer values whenever L (resp. L^*) has.

Let $\gamma_1,\ldots,\gamma_n$ be central elements. For the product $e[\gamma_1]e[\gamma_2],\ldots,e[\gamma_n]$ of the operators $e[\gamma_1],\ldots,e[\gamma_n]$ we simply write $e[\gamma_1,\ldots,\gamma_n]$. Note that $e[\gamma_1,\gamma_2]$ and $e[\gamma_1\gamma_2]$ are different operators.

Proposition 5. *Let* $\gamma_1,\ldots,\gamma_n$ *be central elements and* L,L^* *length functions on* $\mathfrak{F}$ *and* $\mathfrak{H}$ *respectively. Then*

$$\text{Fin } e[\gamma_1,\ldots,\gamma_n]L = \mathfrak{S}(\gamma_1,\ldots,\gamma_n,L) \text{ and}$$

$$\text{Fin } e[\gamma_1,\ldots,\gamma_n]L^* = \mathfrak{S}(\gamma_1,\ldots,\gamma_n,L^*).$$

In particular, $\mathfrak{S}(\gamma_1,\ldots,\gamma_n,L)$ *and* $\mathfrak{S}(\gamma_1,\ldots,\gamma_n,L^*)$ *are Serre-categories.*

Proof. If $n = 1$ then the statement follows a fortiori from the definition. Assume that $n > 1$ and the proposition has been established for $n - 1$. Put $K = e[\gamma_2,\ldots,\gamma_n]L$. Then $e[\gamma_1,\ldots,\gamma_n]L = e[\gamma_1]K$. Let $A \in \mathfrak{F}$. Then we have:

$$A \in \mathrm{Fin}\ e[\gamma_1,\dots,\gamma_n]L \iff A \in \mathrm{Fin}\ e[\gamma_1]K \iff A/\gamma_1 A \in \mathrm{Fin}\,K \iff A/\gamma_1 A \in \mathfrak{S}(\gamma_2,\dots,\gamma_n,L)$$

by the induction hypothesis. Next, we have an isomorphism $B/\gamma_2 B + \dots + \gamma_n B \approx A/\gamma_1 A + \dots + \gamma_n A$ where $B = A/\gamma_1 A$. This shows that $A/\gamma_1 A \in \mathfrak{S}(\gamma_2,\dots,\gamma_n,L)$ if and only if $A \in \mathfrak{S}(\gamma_1,\dots,\gamma_n,L)$.

The formula $\mathrm{Fin}\ e[\gamma_1,\dots,\gamma_n]L^* = \mathfrak{S}(\gamma_1,\dots,\gamma_n,L^*)$ is proved similarly.

Theorem 6. *Let $\gamma_1,\dots,\gamma_k$ be central elements and L, L^* length functions on $\mathfrak{F}$ and $\mathfrak{H}$ respectively. If $A \in \mathfrak{F}$ and $B \in \mathfrak{H}$ then*

$$e[\gamma_1,\dots,\gamma_k]L(A) = \inf_{n_1,\dots,n_k} \frac{L(A/\gamma_1^{n_1}A + \dots + \gamma_k^{n_k}A)}{n_1\cdots n_k}, \qquad (2)$$

$$e[\gamma_1,\dots,\gamma_k]L^*(B) = \inf_{n_1,\dots,n_k} \frac{L^*(0 :_B \gamma_1^{n_1} \cap \dots \cap 0 :_B \gamma_k^{n_k})}{n_1\cdots n_k}. \qquad (3)$$

In the case $k = 1$, $\inf_{n_1}$ can be replaced by $\operatorname{Lim}_{n_1\to\infty}$ in the formulae.

Remark. In fact $\inf_{n_1,\dots,n_k}$ can always be replaced by $\operatorname{Lim}_{n\to\infty}$ where $n = \min(n_1,\dots,n_k)$. In this form the first formula is known as the limit formula of Lech. The replacement of inf by Lim seems to give a genuinely stronger result. For a proof of Lech's formula see [1, Theorem 10, p.314].

Proof. We will only prove (2) since (3) can be dealt with in a similar manner.

First, let $k = 1$ and put $\gamma = \gamma_1$. If $A \notin \mathfrak{S}(\gamma, L)$ then $L(A/\gamma A) = \infty$. Hence $L(A/\gamma^n A) = \infty$ for all n and there is nothing to prove.

Suppose that $A \in \mathfrak{S}(\gamma,L)$ and $0 :_A \gamma = 0$. Then for each $n > 0$ there is an isomorphism $\gamma^n A/\gamma^{n+1}A \approx A/\gamma A$. Consequently

$e[\gamma]L(A) = L(A/\gamma A) = n^{-1}L(A/\gamma^n A)$ and the result follows. Next consider a general element $A \in \mathfrak{S}(\gamma, L)$. Choose the integer m so that $0 :_A \gamma^m = 0 :_A \gamma^s$ for all $s \geq m$. Then it is easily seen that $0 :_A \gamma^m \cap \gamma^s A = 0$ for all $s \geq m$. Put $B = A/0 :_A \gamma^m$. Then $0 :_B \gamma = 0$. From the first part of the proof we find that $e[\gamma]L(B) = n^{-1}L(B/\gamma^n B)$. Next, from the isomorphism $B/\gamma^n B \approx A/(0 :_A \gamma^m + \gamma^n A)$ we deduce that $L(B/\gamma^n B) = L(A/\gamma^n A) - L(0 :_A \gamma^m/0 :_A \gamma^m \cap \gamma^n A)$. Put $C = L(0 :_A \gamma^m) < \infty$ $(A \in \mathfrak{S}(\gamma, L))$. Then for $n \geq m$, $L(B/\gamma^n B) = L(A/\gamma^n A) - C$. Finally $e[\gamma]L(A) = e[\gamma]L(B) = n^{-1}L(A/\gamma^n A) - n^{-1}C$ if $n \geq m$. This shows that $e[\gamma]L(A) = \lim_{n\to\infty} n^{-1}L(A/\gamma^n A) = \inf_n n^{-1}L(A/\gamma^n A)$.

Assume now that $k > 1$ and (2) has been established for the case when the multiplicity operator contains less than k central elements. Put $L' = e[\gamma_2, \ldots, \gamma_k]L$. Then $e[\gamma_1, \ldots, \gamma_k]L = e[\gamma_1]L'$ and the induction hypothesis yields

$$e[\gamma_1]L'(A) = \inf_{n_1} \frac{L'(A/\gamma_1^{n_1}A)}{n_1} = \inf_{n_1} \inf_{n_2,\ldots,n_k} \frac{L(A/\gamma_1^{n_1}A + \ldots + \gamma_k^{n_k}A)}{n_1 \ldots n_k}$$

$$= \inf_{n_1,\ldots,n_k} \frac{L(A/\gamma_1^{n_1}A + \ldots + \gamma_k^{n_k}A)}{n_1 \ldots n_k} .$$

Here we used the isomorphism $B/\gamma_2^{n_2}B + \ldots + \gamma_k^{n_k}B \approx A/\gamma_1^{n_1}A + \ldots + \gamma_k^{n_k}A$; B stands for $A/\gamma_1^{n_1}A$. The proof of the theorem is now complete.

Corollary 1. The operators $e[\gamma]$ commute with each other, i.e. for central elements γ_1, γ_2 $e[\gamma_1, \gamma_2] = e[\gamma_2, \gamma_1]$.†

Proof. Observe that the right hand sides of (2) and (3) are symmetric in the γ_i's. In general, $e[\gamma_1, \ldots, \gamma_n]$ does not depend on the order of the γ_i's.

† Equality of the operators means the obvious, i.e. $e_1 = e_2$ if $e_1L = e_2L$ for all length functions L on which e_1 and e_2 are acting.

Corollary 2. For central elements γ_1, γ_2 we have $e[\gamma_1\gamma_2] = e[\gamma_1] + e[\gamma_2]$.

Proof. We will only prove that for each length function L on $\mathfrak{F}$, $e[\gamma_1\gamma_2]L = e[\gamma_1]L + e[\gamma_2]L$. The corresponding statement for length functions on $\mathfrak{H}$ is analogous.

If $A \in \mathfrak{F}$ then by Theorem 6,

$$e[\gamma_1\gamma_2]L(A) = \lim_{n\to\infty} n^{-1}L(A/\gamma_1^n\gamma_2^n A) = \lim_{n\to\infty} n^{-1}\{L(A/\gamma_1^n A) + L(\gamma_1^n A/\gamma_1^n\gamma_2^n A)\}.$$

Accordingly, it is enough to show that $e[\gamma_2]L(A) = \lim_{n\to\infty} n^{-1}L(\gamma_1^n A/\gamma_1^n\gamma_2^n A)$. We may assume that $A \in \mathfrak{S}(\gamma_2, L)$. For each n we have an isomorphism $\gamma_1^n A/\gamma_1^n\gamma_2^n A \approx A/\gamma_2^n A + 0 :_A \gamma_1^n$. This gives

$$L(\gamma_1^n A/\gamma_1^n\gamma_2^n A) = L(A/\gamma_2^n A) - L(S_n) \text{ where } S_n = 0 :_A \gamma_1^n/0 :_A \gamma_1^n \cap \gamma_2^n A.$$

If, however, n is large enough then $0 :_A \gamma_1^n = 0 :_A \gamma_1^m$ for some fixed $m > 0$. There exists, therefore, a finite number c such that $0 \leq L(S_n) \leq c$. It now follows that $\lim_{n\to\infty} n^{-1}L(\gamma_1^n A/\gamma_1^n\gamma_2^n A) = \lim_{n\to\infty} n^{-1}L(A/\gamma_2^n A) = e[\gamma_2]L(A)$. This establishes the corollary.

5.2 The associative law

In this section we will only consider length function on $\mathfrak{F}$. There are two reasons for this. The first is that a formal associative law can be established in the Artinian case exactly the same way as in the Noetherian one. The special forms of the associative law, however, cannot be obtained in the general Artinian case because we do not have the concrete description of the irreducible functions. We note, however, that if the ring is commutative and Noetherian then the results of section 4.1 can be used to obtain an analogue of Proposition 9.

If mention is being made to several rings, then $\mathfrak{F}(R)$ is used to denote the category of Noetherian R-modules. As before, we write simply $\mathfrak{F}$ when there can be no confusion.

Proposition 7. *Let* L, $\{L_i\}_{i\in I}$ *be length functions on* $\mathfrak{F}$ *and assume that* L *is the discrete sum of the* L_i's *on* Fin L. *If* $\gamma_1,\ldots,\gamma_n$ *are central elements then* $e[\gamma_1,\ldots,\gamma_n]L$ *is the discrete sum of the* $e[\gamma_1,\ldots,\gamma_n]L_i$'s *on* Fin $e[\gamma_1,\ldots,\gamma_n]L$.

Proof. We recall that if L is the discrete sum of the L_i's on Fin L then for every module $A \in$ Fin L, $L(A) = \Sigma_i L_i(A)$ and $L_i(A) = 0$ for all but a finite number of i in I. Also Fin $e[\gamma_1,\ldots,\gamma_n]L = \mathfrak{S}(\gamma_1,\ldots,\gamma_n,L)$ by Proposition 5. Let $A \in$ Fin $e[\gamma_1]L = \mathfrak{S}(\gamma_1,L)$. Then $A/\gamma_1 A$ and $0 :_A \gamma_1$ belong to Fin L and hence to Fin L_i, $i \in I$. Also, the sum of the L_i's is discrete on Fin L and therefore $L_i(A/\gamma_1 A) = L_i(0 :_A \gamma_1) = 0$ for all but a finite number of i. Accordingly,

$$L(A/\gamma_1 A) - L(0 :_A \gamma_1) = \Sigma_i\{L_i(A/\gamma_1 A) - L_i(0 :_A \gamma_1)\} = \Sigma_i e[\gamma_1]L_i(A)$$

and $e[\gamma]L_i(A) = 0$ for all but a finite number of i. Thus $e[\gamma_1]L$ is the discrete sum of the $e[\gamma_1]L_i$'s on Fin $e[\gamma_1]L$. We can now take the operators $e[\gamma_2],\ldots,e[\gamma_n]$ successively and the result is obtained. Note that the order of the operators is immaterial by virtue of Theorem 6, Cor.1.

Theorem 8. (Associative Law). *Let* L *be a locally discrete length function on* $\mathfrak{F}$, $\gamma_1,\ldots,\gamma_n$ *central elements and* i *an integer satisfying* $0 \leq i \leq n$. *If* $\{S_\pi\}_{\pi\in\Pi}$ *is a set of representatives of* Ker $e[\gamma_{i+1},\ldots,\gamma_n]L$*-simple modules in* $\mathfrak{S}(\gamma_{i+1},\ldots,\gamma_n,L)$, *one from each isomorphism class, then the decomposition*

$$e[\gamma_1,\ldots,\gamma_n]L = \Sigma_\pi e[\gamma_{i+1},\ldots,\gamma_n]L(S_\pi)\cdot e[\gamma_1,\ldots,\gamma_i]L_\pi \qquad (4)$$

<u>holds on $\mathfrak{S}(\gamma_1,\ldots,\gamma_n,L)$. Here L_π is the irreducible length function associated to S_π, and the sum is discrete on</u> $\mathfrak{S}(\gamma_1,\ldots,\gamma_n,L)$.

<u>Remarks</u>. We note that the theorem is true if $\mathfrak{F}$ is replaced by $\mathfrak{H}$. The proofs are identical.

When there are no central elements the operator $e[\cdot]$ is understood to be the identity operator. Thus $e[\cdot]L = L$. Also, we make the convention that $\mathfrak{S}(\cdot,L) = \text{Fin } L$.

The assumption that L is locally discrete is not too severe. It is certainly satisfied if L is taken to be the classical length function. Also, if the ring is commutative then every length function on $\mathfrak{F}$ is locally discrete by Theorem 3.23.

<u>Proof</u>. Put $L' = e[\gamma_{i+1},\ldots,\gamma_n]L$. Then L' is locally discrete by Proposition 4, Cor.2. Also, $\text{Fin } L' = \mathfrak{S}(\gamma_{i+1},\ldots,\gamma_n,L)$ by Proposition 5. Using Theorem 3.14 and Proposition 3.10, we find that the decomposition

$$L' = \Sigma_\pi L'(S_\pi)L_\pi$$

holds on Fin L' and this sum is discrete. Put $c_\pi = L'(S_\pi) = e[\gamma_{i+1},\ldots,\gamma_n]L(S_\pi)$. Applying Proposition 7 we obtain that

$$e[\gamma_1,\ldots,\gamma_n]L = e[\gamma_1,\ldots,\gamma_i]L' = \Sigma_\pi e[\gamma_1,\ldots,\gamma_i](c_\pi L_\pi)$$

holds on Fin $e[\gamma_1,\ldots,\gamma_n]L = \mathfrak{S}(\gamma_1,\ldots,\gamma_n,L)$. The theorem now follows from the following trivial fact: for any length function K on $\mathfrak{F}$, $\gamma \in \Gamma$ and real number $c \geq 0$, $e[\gamma](cK) = c\, e[\gamma]K$.

We can put (4) in a slightly different form. Using Proposition 3.4 and Theorem 3.14, a set of left ideals $\{I_\pi\}_{\pi\in\Pi}$ of R can be found such that $E_\pi = E(R/I_\pi)$ is indecomposable for all $\pi \in \Pi$ and R/I_π form a set of representatives of Ker $e[\gamma_{i+1},\ldots,\gamma_n]L$-simple modules in $\mathfrak{S}(\gamma_{i+1},\ldots,\gamma_n,L)$ one from each equivalence class. Then the L_{E_π} 's are the associated irreducible functions by Theorem 3.18.

<u>Corollary 1. Let the situation be as described above. Then (4) takes the form</u>

$$e[\gamma_1,\ldots,\gamma_n]L = \Sigma_\pi e[\gamma_{i+1},\ldots,\gamma_n]L(R/I_\pi)\cdot e[\gamma_1,\ldots,\gamma_i]L_{E_\pi}.$$

Assume now that the ring R is commutative. Then L is automatically locally discrete by Theorem 3.23. Also, with L′ as above,

$$L' = \Sigma_P L'(R/P)L_P \qquad (5)$$

where the summation is taken over all the minimal prime ideals of $\mathfrak{S} = \mathfrak{S}(\gamma_{i+1},\ldots,\gamma_n,L)$ by Theorem 3.24. As before, the decomposition (5) holds on $\mathfrak{S}$ and it is discrete there. In the commutative case, therefore, $e[\gamma_1,\ldots,\gamma_n]L$ admits the following decomposition on $\mathfrak{S}(\gamma_1,\ldots,\gamma_n,L)$:

$$e[\gamma_1,\ldots,\gamma_n]L = \Sigma_P e[\gamma_{i+1},\ldots,\gamma_n]L(R/P)\cdot e[\gamma_1,\ldots,\gamma_i]L_P. \qquad (6)$$

Here P ranges over all the minimal prime ideals of $\mathfrak{S}$ and L_P is the irreducible length function associated to P. By Proposition 3.20, for all $A \in \mathfrak{M}(R)$, $L_P(A) = \ell_{R_P}(A \otimes_R R_P)$ where ℓ_{R_P} is the classical length function on $\mathfrak{M}(R_P)$. We now show that in (6) we can let P run through only those minimal prime ideals of $\mathfrak{S}$ which contain the ideal $I = R\gamma_1 +\ldots+ R\gamma_i$. For suppose that P is a minimal prime of $\mathfrak{S}$ and P does not contain I. For a module $A \in \mathfrak{S}(\gamma_1,\ldots,\gamma_n,L)$ we have

$$0 \le e[\gamma_1,\ldots,\gamma_i]L_P(A) \le L_P(A/IA)$$

by Theorem 6. But $I(A/IA) = 0$ and P does not contain I. Hence

$t(A/IA) = 0$ for some $t \in R-P$. It now follows that $L_P(A/IA) = 0$ and so $e[\gamma_1,\dots,\gamma_i]L_P(A) = 0$. Thus the term corresponding to P in (6) is zero and we can leave it out.

In order to transform (6) into a more familiar form we make a number of observations concerning change of rings. For a prime ideal P of R, let $\varphi_P : R \to R_P$ and $\psi_P \to R/P$ be the canonical ring homomorphisms. Then $R/P \in \mathfrak{F}(R/P)$ whenever $R/P \in \mathfrak{F}(R)$ and L can be regarded, in a natural way, as a length function on $\mathfrak{F}(R/P)$. Further, for each R-module A annihilated by P, and $\gamma \in R$, $\gamma A = \psi_P(\gamma)A$. Accordingly, if $e_{R/P}[\psi_P(\gamma)]$ denotes the multiplicity operator acting on the length functions of $\mathfrak{F}(R/P)$ then $e_{R/P}[\psi_P(\gamma)]L(A) = e[\gamma]L(A)$. In particular, $e[\gamma_{i+1},\dots,\gamma_n]L(R/P) = e_{R/P}[\psi_P(\gamma_{i+1}),\dots,\psi_P(\gamma_n)]L(R/P)$ in (6).

Similarly, $A \in \mathfrak{F}(R)$ implies that $A \otimes_R R_P \in \mathfrak{F}(R_P)$. Also, we have an R_P-isomorphism

$$(A/\gamma_1 A+\dots+\gamma_i A) \otimes_R R_P \approx A \otimes_R R_P/\varphi(\gamma_1)(A \otimes_R R_P)+\dots+ \varphi(\gamma_i)(A \otimes_R R_P)$$

since the functor $- \otimes_R R_P$ is exact. In particular, for a module $A \in \mathfrak{F}(R)$ we have by Theorem 6

$$e[\gamma_1,\dots,\gamma_i]L_P(A) = \inf_{n_1,\dots,n_i} \frac{\ell_{R_P}\left\{(A/\gamma_1^{n_1}A +\dots+ \gamma_i^{n_i}A) \otimes_R R_P\right\}}{n_1\dots n_i}.$$

Consequently, $e[\gamma_1,\dots,\gamma_i]L_P(A) = e_{R_P}[\varphi(\gamma_1),\dots,\varphi(\gamma_n)]\ell_{R_P}(A \otimes_R R_P)$.

We record these results in the following:

<u>Proposition 9</u>. <u>Let</u> R <u>be a commutative ring</u>, L <u>a length function on</u> $\mathfrak{F}(R)$ <u>and</u> $\gamma_1,\dots,\gamma_n$ <u>elements of</u> R. <u>If now</u> i <u>is an integer satisfying</u> $0 \leqslant i \leqslant n$, <u>then for a module</u> $A \in \mathfrak{S}(\gamma_1,\dots,\gamma_n,L)$,

$$e[\gamma_1,\ldots,\gamma_n]_L(A) = \Sigma_P e_{R/P}[\psi_P(\gamma_{i+1}),\ldots,\psi_P(\gamma_n)]_{L(R/P)}$$

$$\cdot e_{R_P}[\varphi_P(\gamma_1),\ldots,\varphi_P(\gamma_i)]\ell_{R_P}(A \otimes_R R_P).$$

Here P ranges over all those minimal prime ideals of $\mathfrak{S}(\gamma_{i+1},\ldots,\gamma_n,L)$ which contain I, and $\varphi_P : R \to R_P$, $\psi_P : R \to R/P$ are the canonical ring homomorphisms.

CHAPTER 6

THE ORDERED GROTHENDIECK GROUP

6.1 The ordered Grothendieck group as a solution of a universal problem

We call a category $\mathfrak{U}$ '<u>small</u>' if there is a set of modules $\{A_i\}_{i\in I}$ in $\mathfrak{U}$ such that for every $M \in \mathfrak{U}, A_i \approx M$ for some $i \in I$. In other words, $\mathfrak{U}$ is small if the isomorphism classes of modules of $\mathfrak{U}$ form a set. Examples of small subcategories of $\mathfrak{M}(R)$ include the Serre-categories generated by the finitely generated and finitely embedded modules respectively. In particular, the categories of Noetherian and Artinian modules are small. The construction of the Grothendieck group of a small category $\mathfrak{U}$ of $\mathfrak{M}(R)$ is well-known. We wish to show that a 'natural' pre-order can be defined on the Grothendieck group. The ordered Grothendieck group is then obtained by factorizing through the equivalent classes with respect to this preorder and thus turning it into a proper partial order.

Our terminology on partially ordered groups will be that of [15], unless defined otherwise. It is assumed that the elementary properties of partially ordered groups are known. Details can be found in [15]. Throughout this chapter a '<u>p.o. group</u>' means a partially ordered Abelian group.

Let $\mathfrak{U}$ be a subcategory of $\mathfrak{M}(R)$ and G a p.o. group. A function $v : \mathfrak{U} \to G$ is called a '<u>valuation</u>' if it satisfies the following two conditions:

(i) $v(A) \geq 0$ for all $A \in \mathfrak{U}$;

(ii) $v(A) = v(A') + v(A'')$ whenever $0 \to A' \to A \to A'' \to 0$ is an exact sequence in $\mathfrak{U}$.

We see at once that $v(0) = 0$, $(0 \in \mathfrak{U})$. A valuation v of $\mathfrak{U}$ is a finite length function on $\mathfrak{U}$ if and only if the range of v is a subgroup of the real numbers.

Let $\mathfrak{U}$ be a subcategory of $\mathfrak{M}(R)$ and $v : \mathfrak{U} \to G(\mathfrak{U})$ a valuation from $\mathfrak{U}$ into a p.o. group $G(\mathfrak{U})$.

<u>Definition</u>. The p.o.group $G(\mathfrak{U})$ is called the 'ordered Grothendieck group of $\mathfrak{U}$' (and v the canonical valuation) if for any p.o.group G' and valuation $v' : \mathfrak{U} \to G'$ there is a unique order-preserving homomorphism $f : G(\mathfrak{U}) \to G'$ such that $v' = fv$.

It is easily seen that the ordered Grothendieck group is unique up to order-isomorphism. Our next task is to establish the existence.

<u>Proposition 1.</u> <u>For each small subcategory</u> $\mathfrak{U}$ <u>of</u> $\mathfrak{M}(R)$ <u>the ordered Grothendieck group of</u> $\mathfrak{U}$ <u>exists</u>.

<u>Proof</u>. For a module $A \in \mathfrak{U}$, $\bar{A}$ denotes the isomorphism class of A. Since $\mathfrak{U}$ is small, the $\bar{A}$'s form a set $\bar{\mathfrak{U}}$. For elements $\bar{A}, \bar{B} \in \bar{\mathfrak{U}}$ write $\bar{A} \leq \bar{B}$ if A is isomorphic to a segment of B. It is clear that this relation is well-defined and turns $\bar{\mathfrak{U}}$ into a preordered set, i.e. $\leq$ is reflexive and transitive. Let F be the free Abelian group on $\bar{\mathfrak{U}}$. Then F consists of elements of the form

$$a_1\bar{A}_1 + \dots + a_k\bar{A}_k,\ a_1,\dots,a_k \text{ integers},\ \bar{A}_1,\dots,\bar{A}_k \in \bar{\mathfrak{U}}$$

We turn F into a preordered group by defining the positive cone P of F as

$$P = \{a_1(\bar{A}_1-\bar{B}_1) + \dots + a_k(\bar{A}_k-\bar{B}_k) : a_1 \geq 0,\dots,a_k \geq 0; \bar{A}_1 \geq \bar{B}_1,\dots,\bar{A}_k \geq \bar{B}_k\}.$$

In particular, $\bar{A} = \bar{A} - \bar{0} \in P$ for all $\bar{A} \in \bar{\mathfrak{U}}$. Indeed, all one has to check is that $X, Y \in P \Rightarrow X + Y \in P$.[†] This, however, is trivially satisfied. If $X, Y \in F$ then we write $X \leq Y$ if $Y - X \in P$. Let D be the convex subgroup of F generated by elements of the form

$\bar{A} - \bar{B} - \bar{C}, \bar{A}, \bar{B}, \bar{C} \in \bar{\mathfrak{U}}$ and $0 \to B \to A \to C \to 0$ is exact in $\mathfrak{U}$.

(A set $S \subsetneq F$ is convex if $X \leq Y \leq Z$ and $X, Z \in S$ implies $Y \in S$.) Put $G(\mathfrak{U}) = F/D$ and for each $A \in \mathfrak{U}$ let $[A]$ denote the coset $\bar{A} + D$ in $G(\mathfrak{U})$. We claim that $G(\mathfrak{U})$ is the Grothendieck group of $\mathfrak{U}$ and $A \to [A]$ is the canonical valuation. The partial order on $G(\mathfrak{U})$ is the induced partial order, i.e. the order relation between the cosets is defined by the rule $X + D \leq Y + D$ if and only if, $X \leq Y + Z$ for some $Z \in D$. This is the standard method and one easily checks that this relation on $G(\mathfrak{U})$is not only a pre-order (since it is indeed anti-symmetric). It now follows that $A \to [A]$ is a valuation of $\mathfrak{U}$.

Assume now that G' is a p.o.group and $v : \mathfrak{U} \to G'$ is a valuation. Obviously, $v(A) = v(B)$ if $A \approx B$. Hence v induces a homomorphism g from F to G'. Further, $v(A) \leq v(B)$ whenever A is isomorphic to a segment of B. Hence $g : F \to G'$ is (pre)order-preserving. Consequently Ker g is convex in F. Since v is a valuation, $v(B) = v(A) + v(C)$ whenever $0 \to B \to A \to C \to 0$ is exact in $\mathfrak{U}$. Hence $g(\bar{A}-\bar{B}-\bar{C}) = 0$. Thus $D \subsetneq \text{Ker } g$ and there is a unique order-preserving homomorphism $f : G(\mathfrak{U}) \to G'$ such that $v(A) = f([A])$ for all $A \in G(\mathfrak{U})$. This establishes the proposition.

It should be noted that the ordered Grothendieck group of a

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† F is only a pre-ordered group.

(small) category $\mathfrak{A}$ is not isomorphic to the standard Grothendieck group as an Abelian group, but a factor of it. This can be seen immediately from the construction if one observes that the standard Grothendieck group is obtained from the free group F by factorising through a group contained in D.

6.2 Length functions and the ordered Grothendieck group.

Let $\mathfrak{A}$ be a small semi-closed subcategory of $\mathfrak{M}(R)$, $G(\mathfrak{A})$ the ordered Grothendieck group of $\mathfrak{A}$ and $v : \mathfrak{A} \to G(\mathfrak{A})$ the canonical valuation. The connection between $G(\mathfrak{A})$ and the length functions on $\mathfrak{A}$ is revealed in the next theorem.

Theorem 2. There is a one-to-one correspondence between the finite length functions on $\mathfrak{A}$ and the order preserving homomorphisms from $G(\mathfrak{A})$ into the real numbers.

Proof. The set of real numbers is a p.o.group under addition and natural order. Let H be the set of order-preserving homomorphisms from $G(\mathfrak{A})$ into the real numbers. If $f \in H$ then fv is clearly a finite length function on $\mathfrak{A}$. Conversely, given a finite length function L on $\mathfrak{A}$ there exists a unique order-preserving homomorphism f in H such that $L = fv$. Thus $f \leftrightarrow fv$ $(f \in H)$ is the required one-to-one correspondence.

Theorem 2 suggests that the study of (finite) length functions may be replaced by the study of ordered Grothendieck groups. The latter concept is probably a more natural invariant of a category than the family of length functions on it. At the moment, however, we can compute the ordered Grothendieck groups of certain categories with the aid of length functions and not the other way round.

The next result is a good illustration of this point.

Theorem 3. Let R be a commutative ring and $\mathfrak{U}$ a Serre subcategory of the category of Noetherian R-modules. Let $G(\mathfrak{U})$ be the ordered Grothendieck group of $\mathfrak{U}$ and $v : \mathfrak{U} \to G(\mathfrak{U})$ the canonical valuation. Then $G(\mathfrak{U})$ is a free group, order defined componentwise. The elements $v(R/P)$, P a minimal prime ideal of $\mathfrak{U}$, form a basis of $G(\mathfrak{U})$.

For the proof we need the following lemma.

Lemma 4. Let R be a commutative ring and A a Noetherian R-module, $A \neq 0$. There are prime ideals $P_0, \ldots, P_{n-1}$ of R and a chain of submodules

$$0 = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = A$$

such that $A_{i+1}/A_i \approx R/P_i$, $i = 0, \ldots, n-1$.

This is a well-known and interesting result. A proof can be found in [1, Prop.9, section 7.9, p.338].

We now prove Theorem 3. By Theorem 2, $G(\mathfrak{U})$ exists and the set $\{v(A) : A \in \mathfrak{U}\}$ generates $G(\mathfrak{U})$. Now Lemma 4 shows that $\{v(R/P) : P \text{ a prime ideal, } R/P \in \mathfrak{U}\}$ is a generating set for $G(\mathfrak{U})$. Next, if $P \subset P'$ are prime ideals and $R/P \in \mathfrak{U}$ then $R/P' \in \mathfrak{U}$ and $v(R/P) = v(R/P') + v(P'/P)$. But P'/P contains a submodule isomorphic to R/P (see Lemma 3.21). Accordingly, $v(R/P) \geq v(R/P') + v(R/P)$ and so $0 \leq v(R/P') \leq 0$. Hence $v(R/P') = 0$ whenever P' is not a minimal prime ideal of $\mathfrak{U}$. Thus the set $v(R/P)$, P a minimal prime of $\mathfrak{U}$, generates $G(\mathfrak{U})$. If P is a minimal prime ideal of $\mathfrak{U}$ then L_P, the associated irreducible length function, is finite on $\mathfrak{A}$ (Theorem 3.24). Consequently, for each minimal prime P of $\mathfrak{A}$ there is an order preserving homomorphism f_P such that $L_P = f_P v$.

Let $P_1, \ldots, P_k$ be distinct minimal prime ideals of $\mathfrak{A}$ and $a_1, \ldots, a_k, b_1, \ldots, b_k$ be integers. If $a_1 v(R/P_1) + \ldots + a_k v(R/P_k) = 0$ then applying f_{P_i} we obtain $a_i = 0$, $i = 1, \ldots, k$. Hence the elements $v(R/P)$, P is a minimal prime of $\mathfrak{A}$, form a basis. Further, if

$$a_1 v(R/P_1) + \ldots + a_k v(R/P_k) \geq b_1 v(R/P_1) + \ldots + b_k v(R/P_k)$$

then, using f_{P_i} again, $a_i \geq b_i$ is obtained. Thus order is defined componentwise.

If R is a commutative Noetherian ring then for a Serre-category $\mathfrak{A}$ of Artinian modules the ordered Grothendieck group $G(\mathfrak{A})$ of $\mathfrak{A}$ can be computed by using the duality in section 4.1 and Theorem 3 above. We find that $G(\mathfrak{A})$ is free and order is defined componentwise.

Several other ordered Grothendieck groups can be computed by means of length functions. As another example we note that if R is a rank-one valuation ring then Theorem 4.8 can be used to compute the ordered Grothendieck group of the Serre-category generated by the finitely generated torsion R-modules. This group is not free if R is not Noetherian.

Finally a theorem on rank-rings.

<u>Theorem 4.</u> <u>Let</u> R <u>be a ring and</u> G(R) <u>the ordered Grothendieck group of the Serre-category generated by the module</u> R. <u>Then</u> R <u>is a rank-ring if and only if</u> $G(R) \neq 0$.

Proof. If R is a rank-ring then there is a length function L on $\mathfrak{M}(R)$ such that $L(R) = 1$. Consequently, if $\mathfrak{F}$ denotes the Serre-category generated by R, then L is a finite non-zero function on $\mathfrak{F}$. Hence $G(R) \not\leq 0$ by Theorem 2. Conversely, assume that $G(R) \not\leq 0$ and let $v : \mathfrak{F} \to G(R)$ be the canonical valuation. Proposition 1.1 shows that if $A \in \mathfrak{F}$ then A has a chain of submodules σ such that every chain factor of σ is isomorphic to a segment of R. Accordingly, $v(A) \leq nv(R)$, $n > 0$ $A \in \mathfrak{F}$. Hence $v(R) \not\leq 0$ if $G(R) \not\leq 0$. Further, $nv(R) = 0$, $n > 0$, implies that $0 \leq v(R) \leq nv(R)$ and $v(R) = 0$. Thus $G(R) \not\leq 0$ implies that $v(R) \not\leq 0$ and $G(R)$ is not a torsion group. If T denotes the torsion subgroup of $G(R)$ then $G = G(R)/T$ is a torsion free p.o. group under the induced order. (T is completely unordered in $G(R)$.) Also, the partial order on G can be extended to a full order. [15, Corollary 5, p.36]. Let v' be the composite valuation $v' : \mathfrak{F} \overset{v}{\to} G(R) \to G$. If a convex subgroup of G contains $v'(R)$ then it contains $v'(A)$ for all $A \in \mathfrak{F}$ and so it is equal to G. There is, therefore, a maximal convex subgroup D of G, $v'(R) \notin D$. But G/D is fully ordered and of rank-one. This means that G/D is order isomorphic to a subgroup of the real numbers. This shows that there is a non-zero order preserving homomorphism f from $G(R)$ into the real numbers. Then fv is a finite length function on $\mathfrak{F}$. The extension L of fv to $\mathfrak{M}(R)$ is such that $0 < L(R) < \infty$. Hence R is a rank-ring.

REFERENCES

[1] D.G.Northcott, 'Lessons on rings, modules and multiplicities', Cambridge University Press, 1968.

[2] S.McLane, 'Homology', Springer, Berlin, 1963.

[3] D.G.Northcott and M.Reufel, 'A generalization of the concept of length', Quart.J.of Math. (Oxford) (2) 16 (1965) 297-321.

[4] P.Gabriel, 'Des categories abéliennes', Bull.Soc.Math., France, 90 (1962) 323-448.

[5] E.Matlis, 'Injective modules over Noetherian rings', Pacific J. Math. 8 (1958) 511-528.

[6] E.Matlis, 'Modules with descending chain condition', Trans. Amer.Math.Soc. 97 (1960) 495-508.

[7] P.Vámos, 'Additive functions and duality over Noetherian rings', Quart.J. of Maths. (Oxford) (2) 17 (1968) 43-55.

[8] P.Vámos, 'The dual of the notion of "finitely generated" ', J.London Math.Soc. 43 (1968).

[9] P.Vámos, 'On ring classes defined by modules', Publ.Math. Debrecen, 14 (1967) 1-8.

[10] L.Fuchs, 'Ranks of modules', Annales Univ.Sci., Budapest, Sectio Math., 6 (1963) 71-82.

[11] A.Kertész, 'On ranks of modules', ibid. 6 (1963) 83-98.

[12] V.P. Elizarov, 'Plane extension of rings', Soviet Math.Dokl. 8 (1967), 905-907.

[13] O.F.G.Schilling, 'The theory of valuations', Amer.Math.Soc., New York, 1950.

[14] L.Fuchs, 'Abelian groups', Pergamon Press, Oxford, 1960.

[15] L.Fuchs, 'Partially ordered algebraic systems', Pergamon Press, Oxford, 1963.

[16] P.Cohn, 'Some remarks on the invariant basis property', Topology, 5 (1966) 215-228.