Axiomatisability problems for S-acts and S-posets

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Abstract

A non-empty set A is a left S-*act* if there exists a morphism from monoid S to the monoid of tranformations of A, denoted by $\mathcal{T}_A$. We note that $1 \mapsto I_A$ where I_A is the identity element of $\mathcal{T}_A$. Thus a left S-act A is a non-empty set on which S acts unitarily on the left. Therefore S-acts can be considered as a representation of monoids by transformations of sets. They also can be taken as a generalisation of modules over rings, therefore many questions of model theory existing for modules, such as the axiomatisability of certain classes, can also be asked in the S-act case. We are mainly interested in the theories of some categorically defined classses of S-acts, such as the classes of free, projective, and flat S-acts.

It is known that associated to a class of algebras $\mathcal{A}$ there is a first order language L. It is natural to ask, is there a set of sentences say $\sum$ in L such that a member $A \in \mathcal{A}$ has a property P if and only if $A \models \sum$. If such a set of sentences exists, we say that the subclass $\mathcal{B}$ of $\mathcal{A}$ whose members have the property P is *axiomatisable.*

We are interested in the literature of S-acts concentrating on the axiomatisability questions of free, projective, flat and weakly flat S-acts. In a sequence of articles, by Bulman Fleming, Gould and Stepanova [25, 29, 30, 50, 6], monoids are investigated such that these classes of S-acts are axiomatisable.

There are several more classes of S-acts, i.e. injective, weakly injective and α-injective, where α is a cardinal number, such that the questions of axiomatisability, and related notions of model theory have been discussed. We refer the interested reader to [20, 24, 26, 27, 28, 42, 43, 51], for further details.

Introductory work on the axiomatisability problems for S-acts was started by Gould in 1985, where the axiomatisability of strongly flat S-acts is described. She proved that the class of strongly flat S-acts is axiomatisable if and only if

$$\mathbf{R}(s,t) = \{(u,v) : su = tv\} \text{ and } \mathbf{r}(s,t) = \{u \in S : su = tu\}$$

are finitely generated for all $s, t \in S$. She also gave a partial answer to the question of axiomatisability of projective S-acts by showing that, if a monoid S is such that every ultrapower of S is projective, then the class of strongly flat S-acts is axiomatisable. Moreover S satisfies (M_R) (the descending chain condition on principal right ideals of S),

and if S satisfies (M^L) (the ascending chain condition on principal left ideals of S), then S has (A) (the ascending chain condition on cyclic S-subacts).

Gould was motivated by the corresponding questions of the axiomatisability of projective and flat R-module over a ring R, which had been solved by Eklof and Sabbagh [16]. We note that the definitions of strongly flat, flat and weakly flat coincide for R-modules over a ring R.

Left perfect rings were introduced in 1960 by Bass [1], and shown to be precisely those rings satisfying Condition (M_R). However, this is not enough for perfection in the monoid case. In 1971, Isbell [35] was the first who took the initiative and studied left perfect monoids. A monoid is *left perfect* if every left S-act has a projective cover.

The results of Isbell together with those of Fountain [19] proved that a monoid is left perfect if and only if it is satisfies (M_R) and an additional condition called (A).

A submonoid T of a monoid S is *right unitary* if $a, ba \in T$ implies that $b \in T$. By a result of [38], a submonoid T of S is right unitary if and only if T is the ρ-class of the identity, for some left congruence ρ on S. We note the following condition called (D), which says that every right unitary submonoid of S contains a minimal left ideal generated by an idempotent.

There is another condition called (K) due to Kilp [37], where he showed that, a monoid is left perfect if and only if it satisfies (A) and (K).

They have succeeded in showing the following:

Theorem 0.0.0.1. *[35, 19, 37] The following conditions are equivalent for a monoid S:*

(i) S is left perfect;

(ii) S satisfies (A) and (D);

(iii) S satisfies (A) and (M_R);

(iv) every strongly flat left S-act is projective;

(v) S satisfies (A) and (K).

In 1991/1992 Stepanova [50] gave a full answer to the axiomatisability of projective S-acts, which was partially answered by Gould in 1985, by using left perefct monoids. She proved that the class of projective left S-acts is axiomatisable if and only if S is left perfect and the class of strongly flat left S-acts is axiomatisable. Moreover she proved that for a monoid S, completeness, model completeness, and categoricity of projective S-acts are equivalent to S being a group. The same is true for the case of strongly flat

S-acts, where S is a commutative monoid. She also argued that if the class of free S-acts is axiomatisable, then it is complete, model complete and categorical.

Much later in 2002, Bulman-Fleming and Gould [6] gave an alternative proof of Stepanova's result of axiomatisability of projective S-acts. Using one trick of Stepanova's, they showed that, for a monoid S, if every ultraproduct of projective S-acts is projective then S satisfies (M^L). This provided the missing link to the argument of Gould in [24]. They also gave necessary and sufficient conditions on S such that the classes of flat and weakly flat left S-acts are axiomatisable.

In [29], Gould characterised those monoids such that the class of free S-acts is axiomatisable. Moreover she described connection between the conditions which arise on S, when considering the axiomatisability of different classes of S-acts, such as, free, projective, strongly flat, flat and weakly flat.

Recently, in 2007, there is a survey article [30] namely "Model theoretic properties of free, projective and flat S-acts", where the axiomatisability of free, projective and flat S-acts is discussed in more detail, along with the previous literature on axiomatisability of these classes. It also considers the further model theoretic notions of completness, model completness and categoricity of free, projective, and strongly flat S-acts.

A pomonoid is a monoid equipped with a partial order such that the partial order is compatible with the monoid operation on both sides. In a similar way to that in which S-acts correspond to the representation of monoids by transformations of sets, S-posets can be considered as a representation of pomonoids by transformations of posets. But there are major differences, since S-acts are merely algebras, whereas S-posets are relational structures as well as algebras. Therefore one needs to be very careful when dealing with S-posets due to the partial order involved.

The first aim of my thesis is to expand on the existing work on axiomatisabilty for classes of S-acts.

Chapter 1 consists of preliminary material on S-acts and S-posets. In particular, we define the various classes of S-acts and S-posets we will be considering.

It is known that there are three methods to axiomatise a given class of S-acts. The first method, which is mostly used for those classes given in terms of an 'interpolation type condition', we call the "elements" method. The other two methods involve replacement tossings and so we call these "replacement tossings" methods. In Chapter 2 we describe two general results, to axiomatise a given class of S-acts, by putting the two "replacement tossings" methods into an abstract general context. This method can then can be applied to obtain both known and new results. We also axiomatise some classes of S-acts defined by interpolation conditions which were not considered previously, e.g. classes of S-acts satisfying Conditions (EP),(PWP) and (W).

The next aim of my thesis is to introduce the notion of axiomatisabilty for classes of S-posets.

In Chapter 3 we focus on the axiomatisability of different classes of S-posets. We have succeeded in determining when the classes of, free, projective, strongly flat, flat, weakly flat, principally weakly flat, po-flat, weakly po-flat and principally weakly po-flat S-acts are axiomatisable. We also axiomatised some classes of S-posets satisfying conditions such as Condition (W), $(\mathrm{EP})^{\leq}$, (P_w), (PWP) and Condition (PWP_w).

Again in this chapter, along similar lines as for S-act case, we generalise the two methods of axiomatisability called "replacement tossing" methods for S-posets. We then apply these methods to axiomatise some classes satisfying flatness conditions, such as flat, weakly flat and principally weakly flat S-posets. Most of this work is along similar lines to the S-act case, except those differences which are due to the partial order. In view of this, some of the proofs are relegated to the Appendix.

As we mention above, the class of projective left S-acts is axiomatisable if and only if the class of strongly flat left S-acts is axiomatisable and S is left perfect monoid. Hence left perfect monoids play an important role in the questions of axiomatisability of projective S-acts. We investigate axiomatisability problems for projective S-posets over a pomonoid S, anticipating an analogous situation to the monoid setting. We therefore needed to begin an investigation of perfect pomonoids, which is topic of Chapter 4.

In Chapter 4 we initiate the investigation of poperfect pomonoids, concurrent with an article [52] by Pervukhin and Stepanova. We prove that a pomonoid is left poperfect if and only if it is satisfies (M_R) and "ordered" version Condition (A^{O}) of Condition (A). Moreover, we argue via direct limits of free S-posets, that (A) and (A^{O}) are equivalent, so that a pomonoid S is left perfect if and only if it is left poperfect. Some of our results coincide with those of [52]. We also investigate *right po-unitary* subpomonoids, where a subpomonoid is right po-unitary if it is the ρ-class of the identity, for some left po-congruence ρ. We show that a pomonoid S is left poperfect if and only if it satisfies $(\mathrm{A}^{\mathrm{O}})$ and $(\mathrm{D}^{\mathrm{O}})$, the ordered version of (D), which is given for a pomonoid S as follows:

every right po-unitary subpomonoid of S contains a minimal left ideal generated by an idempotent. We note that in the following theorem Condition (K^O) is the ordered version of (K), that is, every right collapsible subpomonoid of S contains a right zero.

We prove the ordered analogue of Theorem 0.0.0.1. Some of the techniques used are taken from those used in the monoid case but one needs to deal with care due to the ordering involved; for some steps we develop new strategies.

Theorem 0.0.0.2. *For a pomonoid S, the following are equivalent:*

(i) every strongly flat S-poset is projective;

(ii) S satisfies Conditions (A^O) and (M_R);

(iii) S satisfies Conditions (A^O) and (D^O);

(iv) S is left poperfect;

(v) S satisfies Conditions (A^O) and (K^O).

(i) every strongly flat S-act is projective;

(ii) S satisfies Conditions (A) and (M_R);

(iii) S satisfies Conditions (A) and (D);

(iv) S is left perfect;

(v) S satisfies Conditions (A) and (K).

In Chapter 5 we change direction again. We investigate the finitary conditions arising from questions of axiomatisability of classes of S-acts. For example, the class of S-acts satisfying Condition (PWP) is axiomatisable if and only if $\mathbf{R}(s, s)$ is finitely generated. We consider the question, what does this tell us about the structure of S? We concentrate on Conditions (P),(E) and (PWP).

We focus on the case when S is a Clifford monoid, that is, an inverse monoid with central idempotents. My results fall into two categories. In the first, we assume that the Clifford monoid has a least idempotent; in the second, we drop this assumption. Results in the first case are somewhat more pleasing. We further split our work into cases by making restrictions on the connecting homomorphisms: that they are trivial or, at the other extreme, one-one.

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Author's declaration

Some of the material in this Thesis appears in two joint papers [31, 32].

Chapter 1

An Introduction to S-acts and S-posets

Throughout this thesis, S will denote a monoid, or an ordered monoid, whichever the context dictates. We assume the reader to have a basic knowledge of the theory of semigroups and monoids, as may be found in [34]. Monoids have been widely studied via their representations as mappings of sets; that is, via the concept of S-act, which we explain below. Over the past three decades, an extensive theory of properties of S-acts has been developed (involving free, projective, and flat acts of various kinds). A fresh and comprehensive survey of this area was published in 2000 by M. Kilp, U. Knauer and A. Mikhalev in the monograph *Monoids, Acts and Categories* [39].

More recently, ordered monoids, known as pomonoids, have also been studied via their representations, this time as order-preserving maps of partially ordered sets, that is, by S-posets. To date there is no suitable text on S-posets, and only few articles attempting to generalise material from S-acts to S-posets.

In this chapter we will give a brief survey of S-acts and S-posets over monoids and pomonoids, respectively, concentrating on flatness properties, since the latter will be the topic of much of this thesis. Many definitions for S-posets are analogous to those for S-acts, but there are some crucial differences. We begin by reminding the reader of some basic concepts concerning ordered sets, and end with a discussion of first order languages and axiomatisability. The material in this Chapter can be found in [39], standard text-books on ordered structures such as [4], and articles on S-posets, to which we will refer explicitly.

1.1 Sets and relations

We begin with some details concerning sets, equivalence relations and ordered sets. We assume the reader to be familiar with the basic definitions associated with sets and categories. We remark that all our functions and category morphisms are written on the right of their arguments and are therefore composed left-to-right, unless otherwise stated.

Let X be a set and let ρ be a (binary) relation on X. We write ρ^{-1} for the set of pairs $\{(x,y) : (y,x) \in \rho\}$ and let ρ^t denote the transitive closure of ρ. Thus $x\,\rho^t\,y$ if and only if there exist $x_0 = x, x_1, \ldots, x_n = y$ such that

$$x_0\,\rho\,x_1\,\rho\,\ldots\,\rho\,x_n.$$

We remind the reader that a relation ρ on X is:

(1) *reflexive* if $a\,\rho\,a$ for all $a \in X$;

(2) *symmetric* if $a\,\rho\,b$ implies that $b\,\rho\,a$ for all $a,\ b \in X$;

(3) *anti-symmetric* if $a\,\rho\,b$ and $b\,\rho\,a$ implies that $a = b$ for all $a, b \in X$;

(4) *transitive* if $a\,\rho\,b$ and $b\,\rho\,c$ implies $a\,\rho\,c$ for all $a,\ b\,, c \in X$.

We recall that in a category, a morphism f is an *epimorphism* if from any equation of morphisms $fg = fh$ we deduce that $g = h$, that is, f is left cancellable. Dually, $f : A \to B$ is a *monomorphism* if f is right cancellable.

1.1.1 Equivalence relations and kernels

A relation $\equiv$ on a set X is called an *equivalence* relation if it is reflexive, symmetric and transitive. An equivalence relation $\equiv$ partitions X into equivalence classes. We will denote the equivalence class of $a \in X$ by $[a]$ (or $[a]_{\equiv}$), so that

$$[a] = \{b \in X : a \equiv b\}.$$

Note that $a \equiv b$ if and only if $[a] = [b]$. We denote by $X/\equiv$ the set $\{[a] : a \in X\}$ of equivalence classes; $X/\equiv$ is called the *quotient set* of X by $\equiv$.

If $f : X \to Y$ is a function between sets X and Y, then

$$\ker f = \{(x, x') : X \times X : xf = x'f\}$$

is the *kernel* of f. It is clear that $\ker f$ is an equivalence. Conversely, if $\equiv$ is an equivalence relation on a set X, then $\nu : X \to X/\equiv$ given by $x\nu = [x]$ is such that $\ker \nu = \equiv$. We say that ν is the *natural map*. The following result is standard.

Lemma 1.1.1.1. *Let $f : X \to Y, g : X \to Z$ be functions with f onto and $\ker f \subseteq \ker g$. Then there exists a unique function $h : Y \to Z$ such that $fh = g$.*

Proof. Put $(af)h = ag$ for all $af \in Y$. As $\ker f \subseteq \ker g$, h is well defined and clearly is the unique function with $fh = g$. $\square$

Corollary 1.1.1.2. *Let $f : X \to Y$ be an onto function. Then $\overline{f} : X/\ker f \to Y$ given by $[x]\overline{f} = xf$ is a bijection.*

1.1.2 Order relations and ordered kernels

A relation $\leq$ is a *quasiorder* (also known as a *quasi order, quasi-order, preorder* or *pre-order*) on X if it is reflexive and transitive, and a *partial order* if in addition it is anti-symmetric. If $\leq$ is a quasiorder (respectively, partial order), then we say that $(X, \leq)$, or, more briefly, X, is a *quasiordered set* (respectively, *partially ordered set* or *poset*).

We begin with a standard construction, which shows how to construct a partial order from any quasiorder.

Lemma 1.1.2.1. *Let $(X, \leq)$ be a quasiordered set. Define $\equiv$ on X by the rule that for any $a, b \in X$,*

$$a \equiv b \text{ if and only if } a \leq b \leq a.$$

Then $\equiv$ is an equivalence relation on X. Further, we may define a relation $\preceq$ on $X/\equiv$ by

$$[a] \preceq [b] \text{ if and only if } a \leq b,$$

under which $(X/\equiv, \preceq)$ is a partially ordered set.

Proof. The relation $\equiv$ is reflexive as $\leq$ is a reflexive relation. Let $a \equiv b$ so that $a \leq b \leq a$, then we will have $b \leq a \leq b$ which implies that $b \equiv a$.

If $a \equiv b \equiv c$ then $a \leq b \leq a$ and $b \leq c \leq b$. As $\leq$ is transitive relation so we will have $a \leq c \leq a$ which shows that $a \equiv c$. Thus $\equiv$ is an equivalence relation.

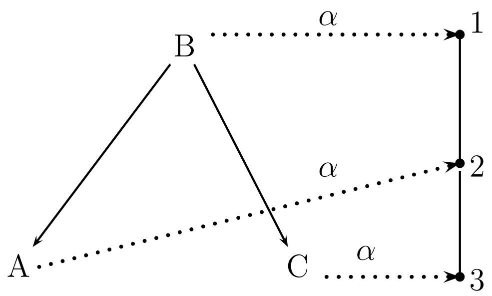
Moreover we can define a partial order relation $\preceq$ on $X/\equiv$ by $[a] \preceq [b]$ if and only if $a \leq b$. Note that $\preceq$ is well defined since if $a' \equiv a \leq b \equiv b'$ then we must have $a' \leq a \leq b \leq b'$ so that $a' \leq b'$ also. Now $\preceq$ is reflexive as $\leq$ is reflexive relation; now let

$[a] \preceq [b]$ and $[b] \preceq [a]$ then $a \leq b \leq a$ which implies that $a \equiv b$ and hence $[a] = [b]$ which shows that $\preceq$ is a anti-symmetric relation.

Let $[a] \preceq [b]$ and $[b] \preceq [c]$ then by definition $a \leq b \leq c$ as $\leq$ is transitive so we will have $a \leq c$ which is possible only if $[a] \preceq [c]$, and hence $\preceq$ is a partial order relation on $X/\equiv$.

□

We say that a function $f : A \to B$ between two posets A and B is a *poset morphism* or *pomorphism* if it is order preserving, i.e. if $a \leq a'$ in A then $af \leq a'f$ in B, for all $a, a' \in A$. A bijective pomorphism ν may have inverse that is not a pomorphism as the standard example illustrates:



An injective pomorphism $f : A \to B$ such that for all $a, a' \in A$, $a \leq a'$ if and only if $af \leq a'f$ is an *embedding*. A bijection f such that both f and f^{-1} are pomorphisms is a *po-isomorphism*.

It is slightly awkward to define the abstract notion corresponding to a 'congruence' for a poset, a difficulty which comes to the fore when we consider S-posets. We will say that an equivalence relation $\equiv$ on a poset A is an *order congruence* or *pocongruence* if $A/\equiv$ can be partially ordered in such a way that the natural map $\nu : A \to A/\equiv$ is a pomorphism. Thus, if $\equiv$ is a pocongruence, then $\equiv$ is $\ker \nu$ for a pomorphism ν.

Let $f : A \to B$ be a pomorphism where A and B are posets. The *ordered kernel* of f is

$$\{(a, a') : af \leq a'f, a\, , a' \in A\}$$

and is denoted by $\overrightarrow{\ker} f$. Note that $\overrightarrow{\ker} f$ is reflexive, transitive (so, a quasiorder) and contains $\leq$. We say that any relation ρ on a poset A is a *pseudo-order* if it is a quasiorder containing $\leq$. Thus, $\overrightarrow{\ker} f$ is a pseudo-order. Further, for any $a, a' \in A$, we have that

$$\ker f = \overrightarrow{\ker} f \cap (\overrightarrow{\ker} f)^{-1},$$

so that $\ker f$ is the equivalence relation associated with $\overrightarrow{\ker} f$. From Lemma 1.1.2.1, we know that the quotient $A/\ker f$ can be partially ordered by $[a] \preceq [b]$ if and only if $(a, b) \in \overrightarrow{\ker} f$. Since $\overrightarrow{\ker} f$ contains $\leq$, it is clear that the natural map $\nu : A \to A/\ker f$ is a pomorphism, so that $\ker f$ is a pocongruence. Further:

Lemma 1.1.2.2. *An equivalence relation $\equiv$ on a poset A is a pocongruence if and only if $\equiv$ is the equivalence relation associated to θ for a pseudo-order θ.*

Proof. Let $\equiv$ be a pocongruence. Then $\equiv$ is $\ker \nu$ for a pomorphism ν and $\ker \nu$ is the equivalence relation associated with $\overrightarrow{\ker} \nu$, where we have observed that $\overrightarrow{\ker} \nu$ is a pseudo-order.

Conversely, let $\equiv$ be the equivalence relation associated to θ for a pseudo-order θ. From Lemma 1.1.2.1, $A/\equiv$ is partially ordered by $\preceq$, where $[a] \preceq [b]$ if and only if $a \, \theta \, b$. Let $\nu : A \to A/\equiv$ be the natural map. If $a \leq b$, then $a \, \theta \, b$ as θ is a pseudo-order. Hence $a\nu = [a] \preceq [b] = b\nu$, so that ν is a pomorphism and consequently, $\equiv$ is a pocongruence. $\square$

We remark that θ in the above lemma may not be uniquely determined by $\equiv$ (see [11]).

Lemma 1.1.2.3. *Let A, B and C be posets and let $f : A \to B, g : A \to C$ be pomorphisms with f onto such that $\overrightarrow{\ker} f \subseteq \overrightarrow{\ker} g$. Then there exists a unique pomorphism $h : B \to C$ such that $fh = g$.*

Proof. We have that $\ker f \subseteq \ker g$ so by Lemma 1.1.1.1 there is a unique function $h : B \to C$ such that $fh = g$. If $b \leq b'$ where $b, b' \in B$, then if $b = af, b' = a'f$, we have that $(a, a') \in \overrightarrow{\ker} f$, so that $(a, a') \in \overrightarrow{\ker} g$ and so $bh = afh = ag \leq a'g = a'fh = b'h$, so that h is a pomorphism as required. $\square$

Corollary 1.1.2.4. *Let $f : A \to B$ be an onto pomorphism where A, B are posets. Then $\overline{f} : A/\ker f \to B$, (where $A/\ker f$ is partially ordered by $[a] \preceq [b]$ if and only if $(a, b) \in \overrightarrow{\ker} f$) given by $[x]\overline{f} = xf$ is a po-isomorphism.*

As a prelude to considering congruences on S-posets later in this chapter, we will now describe how to get a smallest pocongruence θ_γ, containing a given reflexive relation γ on a poset A.

We start by defining $\leq_\gamma$ such that $a \leq_\gamma b$ if there exists $n \in \mathbb{N} = \{1, 2, \ldots\}$ and $a_i, b_i \in A, 1 \leq i \leq n$, such that

$$a \leq a_1 \, \gamma \, b_1 \leq a_2 \, \gamma \, b_2 \leq \cdots a_n \, \gamma \, b_n \leq b.$$

Note that $\leq_\gamma$ is a pseudo-order containing γ and hence also the transitive closure γ^t of γ. By Lemma 1.1.2.1 we can define a relation $\equiv_\gamma$ on A such that

$$a \equiv_\gamma a' \text{ if and only if } a \leq_\gamma a' \leq_\gamma a$$

for all $a, a' \in A$.

Then $\equiv_\gamma$ is an equivalence relation on A. Further, the relation $\preceq_\gamma$ on $A/\equiv_\gamma$ given by

$$[a] \preceq_\gamma [b] \text{ if and only if } a \leq_\gamma b,$$

is a partial order on $A/\equiv_\gamma$, and as in Lemma 1.1.2.2 we may check that the natural map $\nu : A \to A/\equiv_\gamma$ given by $a\nu = [a]$ is a pomorphism such that if $a\,\gamma\,b$ then $a\nu \preceq_\gamma b\nu$, indeed $\overrightarrow{\ker}\nu = \leq_\gamma$.

Suppose now that B is any poset and $\phi : A \to B$ is a pomorphism such that if $a\,\gamma\,b$, then $a\phi \leq b\phi$. It is easy to see from the definition of $\leq_\gamma$ that $\overrightarrow{\ker}\nu = \leq_\gamma \subseteq \overrightarrow{\ker}\phi$ and consequently, $\equiv_\gamma \subseteq \ker\phi$. Moreover, from Lemma 1.1.2.3, there is a pomorphism $\psi : A/\equiv_\gamma \to B$ such that $\nu\psi = \phi$.

1.1.3 Direct limits in categories

We will occasionally need the notion of direct limit, which for the convenience of the reader we now recall.

Definition 1.1.3.1. A non-empty set A with a quasi-order $\leq$ with additional property that any two elements have upper bound is called a *directed set.*

Definition 1.1.3.2. Let $\{A_i : i \in I\}$ be a family of objects in a category $\mathbf{C}$ where I is a directed set.

Suppose we have a collection of morphisms in $\phi_{(i,j)} : A_i \to A_j$ in $\mathbf{C}$ where $i \leq j$, which satisfy the following two properties:

(i) $\phi_{(i,i)}$ is the identity map on A_i;

(ii) $\phi_{(i,k)} = \phi_{(i,j)}\,\phi_{(j,k)}$ where $i \leq j \leq k$.

We will say that $\mathcal{D} = \big(I, \{A_i\}_{i\in I}, \{\phi_{(i,j)}\}_{i\leq j}\big)$ is a *direct system* in $\mathbf{C}$ over the indexed set I.

Definition 1.1.3.3. Let $\mathcal{D} = \big(I, \{A_i\}_{i\in I}, \{\phi_{(i,j)}\}_{i\leq j}\big)$ be a direct system in a category $\mathbf{C}$. A *direct limit* of $\mathcal{D}$ is $\mathscr{L} = (B, \{\theta_i\}_{i\in I})$ where B is an object in $\mathbf{C}$ and for all $i \in I$, $\theta_i : A_i \to B$ in $\mathbf{C}$, such that $\theta_i = \phi_{(i,j)}\theta_j$ for all $i \leq j$, and having the property that if there exists any $D \in \mathbf{C}$ and collection of morphisms $\psi_i : A_i \to D$, $i \in I$, such that $\psi_i = \phi_{(i,j)}\psi_j$ for all $i \leq j$, then there exists a unique morphism $\phi : B \to D$ such that $\psi_i = \theta_i\phi$ for all $i \in I$.

Proposition 1.1.3.4. *[9] The direct limit of a directed system is unique.*

1.1.4 The category of sets

We will denote the *category of sets* by **Set** so that the objects of **Set** are sets and the morphisms between two objects are simply functions between the corresponding sets.

In **Set** epimorphisms coincide with surjective (onto) functions and monomorphisms with injective (one-one) functions [39].

1.1.5 The category of posets

The category of posets will be denoted by **Pos**; the objects are posets and morphisms are order preserving maps. In **Pos** epimorphisms coincide with onto pomorphisms and monomorphisms with one-one pomorphisms [9].

1.2 S-acts and S-posets

We give a brief introduction to S-acts over monoids and S-posets over pomonoids, more details will follow in later sections.

1.2.1 S-acts

Let A be a non-empty set and let S be a monoid, and suppose there is a function $S \times A \to A$, where $(s, a) \mapsto sa$ with the following properties:

(i) $s(t(a)) = (st)a$ for all $s, t \in S$ and $a \in A$;

(ii) $1a = a$ for all $a \in A$;

then A is said to be a *left S-act*. The notion of *right S-act* is defined dually. To emphasise that A is a left (right) S-act we may write ${}_SA$ (A_S). The class of all left and right S-acts are denoted by S-*Act* and *Act*-S, respectively. Notice that S may be regarded as both a left and a right S-act, with actions given by the binary operation in S. Indeed any left (right) ideal of S is a left (right) S-act.

A non-empty subset B of a left S-act A is an *S-subact* if B is closed under the action of S. Any left ideal of S is a subact of ${}_SS$ and dually, any right ideal is a subact of S_S.

Let A be a left S-act and ρ a relation on A. Then ρ is a *(left) S-act congruence* if ρ is an equivalence relation such that for any $a, b \in A$ and $s \in S$, if $a \,\rho\, b$, then $sa \,\rho\, sb$. An

S-act congruence on ${}_SS$ is called a left congruence on S. (Right) S-act congruences on right S-acts, and right congruences on S, are defined dually.

A function $\theta : A \to B$ from a left S-act A to a left S-act B is called an *S-morphism* if $(sa)\theta = s(a\theta)$ for all $s \in S$ and $a \in A$. A bijective S-morphism is called an *S-isomorphism*; if there exists an S-isomorphism from A to B, then we say that A and B are *isomorphic* and write $A \cong B$. The inverse of an S-isomorphism is, of course, itself an S-isomorphism. We will denote by **S-Act** the category with objects all left S-acts and morphisms the S-morphisms between them.

Dually, we can define S-morphisms between right S-acts. The category with objects all right S-acts and morphisms the S-morphisms between them will be denoted by **Act-S**. Indeed, any definition or result for left S-acts has its dual for right S-acts.

Let A be a left S-act and let $H \subseteq A \times A$. An *H-sequence* from a to a', where $a, a' \in A$, is a sequence of the form

$$a = s_1c_1,\ s_1d_1 = s_2c_2,\ \ldots,\ s_nd_n = a',$$

where $s_1, \ldots, s_n \in S$ and $(c_i, d_i) \in H, 1 \leq i \leq n$. The relation α_H on A is defined by the rule that $a\, \alpha_H\, b$ if and only if $a = b$ or there exists an H-sequence from a to b. We denote by $\rho(H)$ the smallest congruence on A containing H, that is, the *congruence generated by* H.

Lemma 1.2.1.1. *Let A be an S-act and let $H \subseteq A \times A$. Then*

$$\rho(H) = \alpha_{H \cup H^{-1}}.$$

Proof. It is very easy to prove that $\alpha_{H\cup H^{-1}}$ is reflexive, symmetric, and transitive. Moreover it preserves S-action, hence is an S-act congruence containing H, so that $\rho(H) \subseteq \alpha_{H\cup H^{-1}}$.

Let ν be an S-act congruence such that $H \subseteq \nu$. As ν is an S-act congruence so $H^{-1} \subseteq \nu$. Now let $a\, \alpha_{H\cup H^{-1}}\, a'$ then we have $a = a'$ or there exists $H \cup H^{-1}$-sequence such that

$$a = s_1c_1, s_1d_1 = s_2c_2, \ldots, s_nd_n = a'$$

where $s_1, \ldots, s_n \in S$ and $(c_i, d_i) \in H \cup H^{-1}$. Since ν is an S-act congruence containing $H \cup H^{-1}$, we will have $[a]_\nu = [s_1c_1]_\nu = [s_1d_1]_\nu = [s_2c_2]_\nu = \ldots = [s_nd_n]_\nu = [a']_\nu$. Hence $\alpha_{H\cup H^{-1}} \subseteq \nu$.

In particular $\alpha_{H\cup H^{-1}} \subseteq \rho(H)$ and so $\rho(H) = \alpha_{H\cup H^{-1}}$ as required. □

Let ρ be an S-act congruence on a left S-act A; Then A/ρ becomes an S-act under $s[a] = [sa]$ and $\nu_\rho : A \to A/\rho$ given by $a\nu_\rho = [a]$ is an S-morphism with ker $\nu_\rho = \rho$.

As in Corollary 1.1.1.2, it follows from standard arguments of universal algebra that if $\theta : A \to B$ is a left S-act morphism, then

$$\ker\,\theta = \{(a, a') \in A \times A : a\theta = a'\theta\}$$

is an S-act congruence on A such that

$$A/\ker\,\theta \cong \text{ im } \theta.$$

In **S-Act** epimorphisms coincide with onto S-morphisms and monomorphisms with one-one S-morphisms [39].

In any category of algebras, so, in particular, in **S-Act**, the direct limit of any directed system exists. The general recipe yields the following.

Proposition 1.2.1.2. *A direct limit $\mathscr{B}$ of a direct system $\mathcal{C} = \big(I, \{A_i\}_{i\in I}, \{\psi_{(i,j)}\}_{i\leq j}\big)$ of left S-acts exists, and can be considered as $\mathscr{B} = (A/\nu, \{\mu_i\}_{i\in I})$ where $A = \bigcup_i A_i$ (disjoint union) and ν and μ_i are given as below:*
(i) for $a_i \in A_i$ and $a_j \in A_j$,

$$a_i\,\nu\,a_j \text{ if and only if } a_i\psi_{(i,k)} = a_j\psi_{(j,k)} \text{ for some } k \geq i, j;$$

(ii) for each $i \in I$ and $a_i \in A_i$, $a_i\mu_i = [a_i]$.

The theory of S-acts is well established, what we want to do is introduce the reader to the notion of S-poset over an ordered monoid S.

1.2.2 S-posets

A *partially ordered monoid* or *pomonoid* is a monoid S with a partial order (usually written $\leq$) which is compatible with the binary operation on both sides. The canonical example of a pomonoid is an inverse monoid under the natural partial order, where $a \leq b$ if and only if $a = aa^{-1}b$, for all $a, b \in S$ [41].

Let A be a partially ordered set and let S be a pomonoid. We call A a *left S-poset* if A is a left S-act and if in addition:

i) if $s \leq t$ then $sa \leq ta$ for all $s, t \in S$ and $a \in A$;

ii) if $a \leq a'$ then $sa \leq sa'$ for all $a\,, a' \in A$ and $s \in S$.

Right S-posets are defined dually. The class of left (right) S-posets is denoted by *S-poset* (*poset-S*). An S-subposet of a left (right) S-poset A is a subposet B of A that is also an S-subact. Like S-acts, S can be regarded as a left or a right S-poset, and any left (right) ideal can be regarded as a left (right) S-subposet of S.

Let $S = \{1\}$ be a trivial monoid and let P be a poset. Then S-acts trivially on P, making P into a (left) S-poset.

An S-morphism $\phi : A \to B$ from a left S-poset A to a left S-poset B is called an *S-poset morphism* or more briefly, *S-pomorphism*, if it is order-preserving, that is, if $a \leq b$ in A implies that $a\phi \leq b\phi$ in B for all $a, b \in A$. It is an *embedding* if, in addition, it is an embedding of the underlying posets. An S-pomorphism from A to B is an *S-po-isomorphism* if it is also a po-isomorphism of the underlying posets. We then say that A and B are *isomorphic* and write $A \cong B$. Of course, we may define S-pomorphisms, embeddings and S-po-isomorphims for right S-posets in a dual manner. Indeed, any definition or result for left S-posets has its dual for right S-posets, which we may not explicitly state.

We will denote the category of left S-posets and S-pomorphisms by **S-Pos**. Note that in **S-Pos** epimorphisms are onto and monomorphisms are one-one [9]. Similarly, we denote the category of right S-posets and S-pomorphisms by **Pos-S**.

There is a fundamental difference between S-acts and S-posets: namely, the first are algebras and the second are relational structures. First concerns arise from the fact that a bijective order preserving S-pomorphism may not be an isomorphism and secondly, one has to be very careful about defining the notion of an ordered congruence (as we have seen for posets).

It is slightly complicated to define the notion of congruence on a relational structure, but is certainly possible. A general approach is given in [15], a specific description of congruences for S-posets may be found in [56]; we now give an account of the latter.

Let A be an S-poset and let $H \subseteq A \times A$. We carefully define a number of relations on A that can be obtained from H.

First, we consider the case for *pseudo-orders*. We extend the definition from posets to S-posets as follows.

Definition 1.2.2.1. Let A be an S-poset. A relation μ on A is a *pseudo-order* if μ contains $\leq$ and is reflexive, transitive and compatible with the action of S.

Let μ be a pseudo-order on an S-poset A. Then in particular μ is a quasiorder, so as in Lemma 1.1.2.1, we may define an equivalence relation, denoted here by $\equiv_\mu$, by the rule that for any $a, a' \in A$,

$$a \equiv_\mu a' \text{ if and only if } a \,\mu\, a' \,\mu\, a.$$

Further, $A/\equiv_\mu$ is partially ordered by $\preceq_\mu$, where

$$[a] \preceq_\mu [a'] \text{ if and only if } a \,\mu\, a'.$$

Since μ is compatible with the action of S, clearly so is $\equiv_\mu$, whence $A/\equiv_\mu$ is an S-act. Since $\leq \subseteq \mu$ it is easy to see that $A/\equiv_\mu$ is an S-poset. The natural map $\nu : A \to A/\equiv_\mu$ is an S-pomorphism such that $\overrightarrow{\ker}\nu = \mu$ and $\ker \nu = \equiv_\mu$.

Now we will define what we mean by a 'congruence' relation on an S-poset A. An S-act congruence ρ on A is called an *S-poset congruence* or *S-pocongruence* on A if A/ρ can be partially ordered such that it becomes an S-poset and the natural map $\nu : A \to A/\rho$ is a S-pomorphism [11].

Lemma 1.2.2.2. *Let μ be a pseudo-order on an S-poset A. Then $\equiv_\mu$ defined as above is an S-pocongruence such that if $a \,\mu\, a'$, then $[a]_{\equiv_\mu} \preceq [b]_{\equiv_\mu}$.*

Moreover, if ν is any S-pocongruence with the property that if $a \,\mu\, a'$, then $[a]_\nu \leq [a']_\nu$, then $\equiv_\mu \subseteq \nu$.

Proof. The first part of the lemma follows from earlier remarks.

Suppose now that ν is an S-pocongruence with the property that if $a \,\mu\, a'$, then $[a]_\nu \leq [a']_\nu$. If $a \equiv_\mu a'$, then $a \,\mu\, a' \,\mu\, a$, so that in A/ν we have

$$[a]_\nu \leq [a']_\nu \leq [a]_\nu$$

whence $[a]_\nu = [a']_\nu$, so that $a \,\nu\, a'$ and $\equiv_\mu \subseteq \nu$ as required. □

Corollary 1.2.2.3. *Let μ be a pseudo-order on an S-poset A. If $\mu \subseteq \nu$ for any S-pocongruence ν, then $\equiv_\mu \subseteq \nu$.*

If α and β are reflexive relations on A that are compatible with the S-action, then it is clear that the transitive closure $\gamma = (\alpha \cup \beta)^t$ of $\alpha \cup \beta$ is a quasi-order compatible with the S-action. Using reflexivity we see that for any $a, a' \in A$, we have $a \, \gamma \, a'$ if and only if there exists a sequence

$$a \, \alpha \, a_1 \, \beta \, b_1 \, \alpha \, a_2 \, \beta \, b_2 \, \ldots \, \beta \, b_n \, \alpha \, a',$$

for some $a_1, b_1, \ldots, a_n, b_n \in A$.

In particular, defining $\leq_\alpha$ to be $(\leq \cup \alpha)^t$, as in Section 1.1, we have that $\leq_\alpha$ is a pseudo-order and for any $a, a' \in A$, $a \leq_\alpha a'$ if and only if there is a sequence

$$a \leq a_1 \, \alpha \, b_1 \leq \, \ldots \leq a_n \, \alpha \, b_n \leq a'$$

for some $a_1, b_1, \ldots, a_n, b_n \in A$. We now let $\equiv_\alpha = \equiv_{\leq_\alpha}$ be the equivalence relation associated with $\leq_\alpha$, that is, for $a, a' \in A$,

$$a \equiv_\alpha a' \text{ if and only if } a \leq_\alpha a' \leq_\alpha a.$$

As above we have that $A/ \equiv_\alpha$ is partially ordered by $\preceq_\alpha$, where

$$[a] \preceq_\alpha [a'] \text{ if and only if } a \leq_\alpha a'$$

and further, $\equiv_\alpha$ is an S-pocongruence.

Lemma 1.2.2.4. *Let α be a reflexive relation on A that is compatible with the S-action. Then $\equiv_\alpha$ constructed as above is an S-pocongruence and is such that if $a \, \alpha \, a'$, then $[a] \preceq_\alpha [a']$. Further, if ν is any S-pocongruence with the property that $a \, \alpha \, a'$ implies that $[a]_\nu \leq [a']_\nu$, then $\equiv_\alpha \subseteq \nu$.*

Proof. We have argued that $\equiv_\alpha$ is an S-pocongruence.

Since $\alpha \subseteq \leq_\alpha$, we certainly have that if $a \, \alpha \, a'$, then $[a] \preceq_\alpha [a']$.

Suppose now that ν is an S-pocongruence with the property that if $a \, \alpha \, a'$, then $[a]_\nu \leq [a']_\nu$. If $a \leq_\alpha a'$ then we have a sequence

$$a \leq a_1 \, \alpha \, b_1 \leq \, \ldots \, \leq a_n \, \alpha \, b_n \leq a'$$

for some $a_1, b_1, \ldots, a_n, b_n \in A$. Since ν is a S-pocongruence, we have that

$$[a]_\nu \leq [a_1]_\nu \leq [b_1]_\nu \leq \ldots \leq [a']_\nu.$$

It follows that if $a \equiv_\alpha a'$, then $a \, \nu \, a'$ as required. □

Corollary 1.2.2.5. *Let α be a reflexive relation on an S-poset A that is compatible with the S-action. If $\alpha \subseteq \nu$ for any S-pocongruence ν, then $\equiv_\alpha \subseteq \nu$.*

We now consider the most general case of an arbitrary subset H of $A \times A$, where A is an S-poset. Recall that we denote by α_H the relation given by the rule that $a \, \alpha_H \, a'$ if and only if $a = a'$ or there is an H-sequence from a to a'. Certainly α_H is a compatible quasiorder. We abbreviate $\leq_{\alpha_H}$ by $\leq_H$, $\equiv_{\alpha_H}$ by $\equiv_H$ and $\preceq_{\alpha_H}$ by $\preceq_H$.

Lemma 1.2.2.6. *Let A be an S-poset and let $H \subseteq A \times A$. Then $\equiv_H$ is an S-pocongruence such that for any $(a, a') \in H$, we have that $[a] \preceq_H [a']$. Further, if ν is any S-pocongruence on A such that $[a]_\nu \leq [a']_\nu$ for all $(a, a') \in H$, then $\equiv_H \subseteq \nu$.*

Proof. From Lemma 1.2.2.4, we know that $\equiv_H$ is an S-poset congruence. Further, since $H \subseteq \alpha_H$, we certainly have that $[a] \preceq_H [a']$ for all $(a, a') \in H$.

Suppose now that ν is any S-poset congruence with the property that $[a]_\nu \leq [a']_\nu$ for all $(a, a') \in H$. If $a, a' \in A$ and $a \, \alpha_H \, a'$, then $a = a'$ (so that certainly $[a]_\nu \leq [a']_\nu$), or there exists an H-sequence

$$a = s_1 c_1, \; s_1 d_1 = s_2 c_2, \; \ldots, \; s_n d_n = a',$$

where $s_1, \ldots, s_n \in S$ and $(c_i, d_i) \in H$. Since ν is an S-poset congruence we have that

$$[a]_\nu = [s_1 c_1]_\nu = s_1 [c_1]_\nu \leq s_1 [d_1]_\nu = [s_1 d_1]_\nu$$

$$= [s_2 c_2]_\nu = \ldots = [s_n c_n]_\nu = s_n [c_n]_\nu \leq s_n [d_n]_\nu = [s_n d_n]_\nu = [a']_\nu.$$

From Lemma 1.2.2.4 we now have that $\equiv_H \subseteq \nu$. □

Definition 1.2.2.7. [10] The relation $\equiv_H$ that appears in Lemma 1.2.2.6 is the *S-poset congruence induced by H.*

It is sometime more convenient to use an alternative description of $\preceq_H$.

Let $(c_1, d_1), \cdots, (c_n, d_n) \in H$ and $s_1, \cdots s_n \in S$, $n \in \mathbb{N}^0 = \{0, 1, 2, \ldots\}$, be such that $(*)$ holds:

$$a \leq s_1 c_1, \; s_1 d_1 \leq s_2 c_2 \;, \ldots, s_n d_n \leq b.$$

Then either $n = 0$ and $a \leq b$, so that $a \preceq_H b$, or

$$a \leq s_1 c_1 \, \alpha_H \, s_1 d_1 \leq s_2 c_2 \, \alpha_H \cdots \alpha_H \, s_n d_n \leq b$$

so that again, $a \preceq_H b$.

Conversely, if $a \preceq_H b$ then

$$a \leq a_1 \, \alpha_H \, b_1 \leq \ldots \leq a_n \, \alpha_H \, b_n \leq b$$

for some $n \in \mathbb{N}$ and $a_1, b_1, \ldots, a_n, b_n \in A$. Where $a_i = b_i$ we just remove a_i and b_i from our sequence to obtain $b_{i-1} \leq a_{i+1}$, where $b_0 = a$ and $a_{n+1} = b$. Therefore, without loss of generality, we may assume that for each i, $1 \leq i \leq n$, there exists

$$(u_{i_1}, v_{i_1}), \cdots, (u_{i_{k(i)}}, v_{i_{k(i)}}) \in H$$

and $t_{i_1}, \cdots, t_{i_{k(i)}} \in S$ with

$$a_i = t_{i_1} u_{i_1}, \; t_{i_1} v_{i_1} = t_{i_2} u_{i_2}, \ldots, t_{i_{k(i)}} v_{i_{k(i)}} = b_i.$$

We now have

$$a \leq a_1 = t_{1_1} u_{1_1}, t_{1_1} v_{1_1} \leq t_{1_2} u_{1_2}, \ldots, t_{1_{k(1)}} v_{1_{k(1)}} = b_1 \leq a_2 = t_{2_1} u_{2_1},$$

$$t_{2_1} v_{2_1} \leq t_{2_1} u_{2_1}, \ldots, t_{n_{k(n)}} v_{n_{k(n)}} = b_n \leq b.$$

We conclude that $a \preceq_H b$ if and only if there exists a sequence as in $(*)$.

We again consider an arbitrary subset H of $A \times A$. We denote $\alpha_{H \cup H^{-1}}$ by β_H, so that β_H is $\rho(H)$, the S-act congruence generated by H. Accordingly we denote $\leq_{\beta_H}$ by $\leq_{H \cup H^{-1}}$, $\equiv_{\beta_H}$ by $\equiv_{H \cup H^{-1}}$ and $\preceq_{\beta_H}$ by $\preceq_{H \cup H^{-1}}$.

Lemma 1.2.2.8. *Let A be an S-poset and let $H \subseteq A \times A$. Then $\equiv_{H \cup H^{-1}}$ constructed as above is an S-pocongruence such that $H \subseteq \equiv_{H \cup H^{-1}}$. Further, if ν is any S-poset congruence with $H \subseteq \nu$, then $\equiv_{H \cup H^{-1}} \subseteq \nu$.*

Proof. We know from Lemma 1.2.2.6 that $\equiv_{H \cup H^{-1}}$ is an S-po-congruence. Further, if $(a, a') \in H$, then since $(a', a) \in H^{-1}$, the same result gives that

$$[a] \preceq_{H \cup H^{-1}} [a'] \preceq_{H \cup H^{-1}} [a]$$

whence $a \equiv_{H \cup H^{-1}} a'$.

Suppose now ν is any S-pocongruence on A with $H \subseteq \nu$. Then $H \cup H^{-1} \subseteq \nu$, so that for any $(a, a') \in H \cup H^{-1}$ we certainly have that $[a]_\nu = [a']_\nu$. Again from Lemma 1.2.2.6, we have that $\equiv_{H \cup H^{-1}} \subseteq \nu$. □

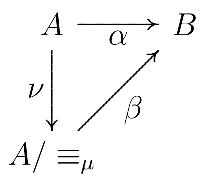
Definition 1.2.2.9. [10] Let A be an S-poset and let $H \subseteq A \times A$. Then $\equiv_{H \cup H^{-1}}$ defined as above is the *S-pocongruence generated by H*.

Definition 1.2.2.10. [10] An S-pocongruence ρ on an S-poset A is said to be *finitely generated* if $\rho = \equiv_H$ for some finite subset H of $A \times A$.

An S-poset A is *finitely presented* if $A \cong F/\rho$ for some finitely generated free S-poset F (see Section 1.3) and some $\rho = \equiv_H$ where H is a finite subset of $F \times F$.

Let $f : A \to B$ be an S-pomorphism where A and B are left S-posets. We know that $\overrightarrow{\ker} f$ is a poset pseudo-order, and, as it is compatible with the S-action, it is an S-poset pseudo-order.

Theorem 1.2.2.11. *(Fundamental Theorem of Morphisms for S-posets) [56] Let A and B be two left S-posets and $\alpha : A \to B$ an S-pomorphism. Then if μ is a pseudo-order on A such that $\mu \subseteq \overrightarrow{\ker}\alpha$, then there exists a unique S-pomorphism $\beta : A/\equiv_\mu \to B$ such that the diagram*



commutes, where $\nu : A \to A/\equiv_\mu$ is the natural map. Further, $\operatorname{im}\alpha = \operatorname{im}\beta$.

Conversely if μ is a pseudo-order on A, for which there exists a S-pomorphism $\beta : A/\equiv_\mu \to B$ such that the above diagram is commutative, then $\mu \subseteq \overrightarrow{\ker}\alpha$.

Proof. The natural map $\nu : A \to A/\equiv_\mu$ is an onto S-pomorphism with $\overrightarrow{\ker}\nu = \mu \subseteq \overrightarrow{\ker}\alpha$. From Lemma 1.1.2.3, there exists a unique pomorphism $\beta : A/\equiv_\mu \to B$ such that $\nu\beta = \alpha$. It is easy to check that β is an S-pomorphism.

For the converse, if $(a, b) \in \mu$, then $[a] \preceq_\mu [b]$ in $A/\equiv_\mu$, so that

$$a\alpha = a\nu\beta = [a]\beta \leq [b]\beta = b\nu\beta = b\alpha$$

in B, hence $\nu \subseteq \overrightarrow{\ker}\alpha$. □

Corollary 1.2.2.12. *[56] Let A and B be left S-posets, and let $\phi : A \to B$ be an S-pomorphism. Then $A/\ker\phi \cong \operatorname{im}\phi$, where for $[a], [b] \in A/\ker\phi, [a] \preceq [b]$ if and only if $a\phi \leq b\phi$.*

Corollary 1.2.2.13. *Suppose $\theta : A \to B$ is an S-pomorphism, and $H \subseteq \overrightarrow{\ker}\theta$. Then $\leq_H \subseteq \overrightarrow{\ker}\theta$. From Theorem 1.2.2.11, $\phi : A/{\equiv_H} \to B$ given by $[a]_{\equiv_H}\phi = a\,\theta$ is a well-defined S-pomorphism.*

Proof. Suppose that $a, b \in A$ and $a \leq_H b$. Then

$$a \leq s_1c_1, s_1d_1 \leq s_2c_2, \ldots, s_nd_n \leq b$$

for some $n \geq 0$, $(c_1, d_1), \ldots, (c_n, d_n) \in H$ and $s_1, \ldots, s_n \in S$. Then as θ is an S-pomorphism and $c_i\theta \leq d_i\theta$ for all $i \in \{1, \ldots, n\}$,

$$a\theta \leq (s_1c_1)\theta = s_1(c_1\theta) \leq s_1(d_1\theta) = (s_1d_1)\theta \leq \ldots = (s_nd_n)\theta \leq b\theta$$

and so $\leq_H \subseteq \overrightarrow{\ker}\theta$. Now use Theorem 1.2.2.11. □

Unlike the category **S-Act** of algebras, the category **S-Pos** is one of relational structures. Nevertheless, direct limits exist as expected.

Proposition 1.2.2.14. *[10] A direct limit* $\mathscr{L}$ *of a direct system* $\mathcal{D} = \big(I, \{A_i\}_{i\in I}, \{\phi_{(i,j)}\}_{i\leq j}\big)$ *of left* S*-posets exists, and can be represented as* $\mathscr{L} = (A/\gamma, \{\theta_i\}_{i\in I})$ *where* γ *is a congruence on* $A = \bigcup_i A_i$ *(disjoint union) with following properties:*

(i) γ *is the equivalence relation associated with the pseudo-order* μ*, where for* $a_i \in A_i, a_j \in A_j$*,* $a_i \,\mu\, a_j$ *if and only if* $a_i\phi_{(i,k)} \leq a_j\phi_{(j,k)}$ *for some* $k \geq i, j$*;*

(ii) for each $i \in I$ *and* $a_i \in A_i$*,* $a_i\theta_i = [a_i]$*;*

consequently

(a) $a_i \,\gamma\, a_j$*,* $a_i \in A_i\,, a_j \in A_j$*, if and only if* $a_i\phi_{(i,k)} = a_j\phi_{(j,k)}$ *for some* $k \geq i, j$*;*

(b) $[a_i] \leq [a_j]$*,* $a_i \in A_i\,, a_j \in A_j$*, if and only if* $a_i\phi_{(i,k)} \leq a_j\phi_{(j,k)}$ *for some* $k \geq i, j$*;*

(c) $a_i \,\gamma\, a_i\phi_{(i,j)}$ *for any* $a_i \in A_i, i \leq j$*.*

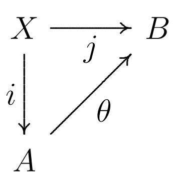
It is clear from Propositions 1.2.1.2 and Proposition 1.2.2.14 that the direct limit of $\mathcal{D}$ in **S-Pos** is essentially the direct limit in **S-Act** equipped with an ordering. We revisit this at the end of Chapter 4.

1.3 Free, projective and flat S-acts

We will be interested here in free, projective and flat S-acts.

Freeness and projectivity are given by the standard categorical definitions.

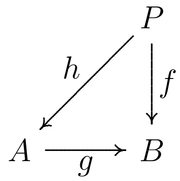
An S-act A is *free* on $X \subseteq A$ if for any S-act B and map $j : X \to B$ there is a unique S-morphism $\theta : A \to B$ such that $i\,\theta = j$, where $i : X \to A$ is inclusion, i.e. the diagram



commutes.

The class of free S-acts is denoted by $\mathcal{F}r$.

An S-act P is *projective* if for any onto S-morphism $g : A \to B$ and for any S-homomorphism $f : P \to B$ there exists a S-homomorphism $h : P \to A$ such that the following diagram



commutes. We will denote the class of projective S-acts by $\mathcal{P}r$.

Classical results determine completely the structure of free and projective S-acts. First, for a symbol x we let $Sx = \{sx \mid s \in S\}$ be a set of formal expressions in one-one correspondence with S; Sx becomes a left S-act (isomorphic to ${}_SS$) if we define $s(tx) = (st)x$ for all $s, t \in S$. In S-act, coproduct is the disjoint union, with S-action componentwise.

Proposition 1.3.0.15. *[39] A left S-act A is free on X if and only if $A \cong \bigsqcup_{x \in X} Sx$.*

Theorem 1.3.0.16. *[38] A left S-act A is called projective if and only if $A = \bigsqcup_{i \in I} P_i$ where $P_i \cong Se_i$ for idempotents $e_i \in S$, $i \in I$.*

The approach to concepts of flatness is rather more complicated, and involves the notion of tensor product, which we now describe.

Let A be a right S-act and B be a left S-act, take $A \times B$ be the Cartesian product of A and B. The *tensor product* of A and B is obtained by taking the quotient of $A \times B$ by the equivalence relation generated by the set $\{\big((as, b), (a, sb)\big) \mid a \in A, b \in B, s \in S\}$. We will use $A \otimes B$ to denote the tensor product of S-acts A and B. The equivalence class of $(a, b) \in A \times B$ will be denoted by $a \otimes b \in A \otimes B$.

We will need to look carefully at equalities of the form $a \otimes b = a' \otimes b'$.

Lemma 1.3.0.17. *[39] Let A be a right S-act and B a left S-act. Then for $a, a' \in A$ and $b, b' \in B$, $a \otimes b = a' \otimes b'$ if and only if there exist $s_1, t_1, s_2, t_2, \ldots, s_m, t_m \in S$, $a_2, \ldots, a_m \in A$ and $b_1, \ldots, b_m \in B$ such that*

$$\begin{array}{rclrcl} & & & b & = & s_1 b_1 \\ a s_1 & = & a_2 t_1 & t_1 b_1 & = & s_2 b_2 \\ a_2 s_2 & = & a_3 t_2 & t_2 b_2 & = & s_3 b_3 \\ & \vdots & & & \vdots & \\ a_{m-1} s_{m-1} & = & a_m t_{m-1} & t_{m-1} b_{m-1} & = & s_m b_m \\ a_m s_m & = & a' t_m & t_m b_m & = & b' \end{array}$$

The sequence presented in Lemma 1.3.0.17 will be called a *tossing* (or scheme) $\mathcal{T}$ of length m over A and B connecting (a, b) to (a', b'). The *skeleton* $\mathcal{S} = \mathcal{S}(\mathcal{T})$ of $\mathcal{T}$, is the sequence

$$\mathcal{S} = (s_1, t_1, \ldots, s_m, t_m) \in S^{2m}.$$

The set of all skeletons is denoted by $\mathbb{S}$. By considering trivial acts it is easy to see that $\mathbb{S}$ consists of all even length sequences of elements of S.

We know therefore that if $a, a' \in A$ and $b, b' \in B$, where A is a right S-act and B a left S-act, then $a \otimes b = a' \otimes b'$ in $A \otimes B$ if and only if there exists a tossing $\mathcal{T}$ from (a, b) to (a', b') over A and B, with skeleton $\mathcal{S}$, say. If the equality $a \otimes b = a' \otimes b'$ holds also in $C \otimes B$, where C is an S-subset of A containing a and a' (certainly in the case B is flat: see below) and is determined by some tossing $\mathcal{T}'$ from (a, b) to (a', b') over C and B with skeleton $\mathcal{S}' = \mathcal{S}(\mathcal{T}')$ then we say that $\mathcal{T}'$ is a *replacement tossing* for $\mathcal{T}$ and $\mathcal{S}'$ is a *replacement skeleton for* $\mathcal{S}$.

Now we define for an S-act B the functor $- \otimes B$: **Act-S** $\rightarrow$ **Set**, by

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & A' \\ \Big\downarrow {\scriptstyle - \otimes B} & & \Big\downarrow {\scriptstyle - \otimes B} \\ A \otimes B & \xrightarrow{\; f \otimes I_B \;} & A' \otimes B \end{array}$$

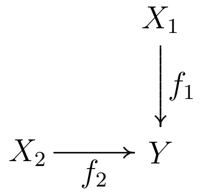
where $f \otimes I_B : A \otimes B \rightarrow A' \otimes B$, and for $a \otimes b \in A \otimes B$,

$$(a \otimes b)(f \otimes I_B) = af \otimes b$$

where we have $f : A \rightarrow A'$ an S-morphism in **Act-S**.

Now we will see that various notions of flatness can be drawn from this functor and involve it preserving monomorphisms, or related concepts such as pullbacks and equalisers.

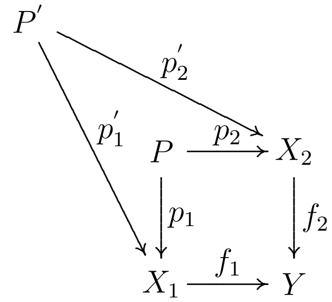
Consider the following diagram in **S-Act**,



The pair $(P, (p_1, p_2))$ where $p_i : P \to X_i$, $i = 1, 2$ are S-morphisms is called a *pullback* of the pair (f_1, f_2) if

(i) $p_1 f_1 = p_2 f_2$ and

(ii) if

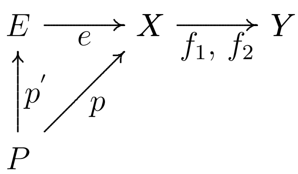


is a diagram in **S-Act** such that $p_1^{'} f_1 = p_2^{'} f_2$ then there exists a unique S-morphism say $\gamma : P^{'} \to P$ such that $\gamma p_1 = p_1^{'}$ and $\gamma p_2 = p_2^{'}$.

An *equalizer* diagram for f_1 and f_2, where $f_1, f_2 : X \to Y$ in **S-Act** is a pair (E, e) where $e : E \to X$ is an S-morphism, if

(i) $ef_1 = ef_2$ and

(ii) for any S-morphism $p : P \to X$ with $pf_1 = pf_2$ there exists a unique S-morphism $p^{'} : P \to E$ such that $p = p'e$.



A left S-act B is called *strongly flat* if the functor $- \otimes B$ preserves pullbacks and equalizers. There are well-known alternative descriptions, due to Stenström.

Theorem 1.3.0.18. *[49] Let S be a monoid. The following conditions are equivalent for a left S-act B:*

(i) B is strongly flat;

(ii) B is the direct limit of finitely generated free left S-acts;

(iii) B satisfies Conditions (P) and (E) which are defined as follows:

(P): for all $b, b' \in B$ and $s, s' \in S$ if $sb = s'b'$ then there exists $b'' \in B$ and $u, u' \in S$ such that $b = ub'', b' = u'b''$ and $su = s'u'$;

(E): for all $b \in B$ and $s, s' \in S$ if $sb = s'b$ then there exists $b'' \in B$ and $u \in S$ such that $b = ub''$ and $su = s'u$.

Conditions (P) and (E) have come to be known as examples of 'interpolation' conditions - flatness conditions not involving explicit mention of $\otimes$.

A left S-act B is *flat* if it preserves embeddings of right S-acts, which is easily seen to be equivalent to the following: if we have $a \otimes b = a' \otimes b'$ in $A \otimes B$ then the equality also holds in $aS \cup a'S \otimes B$ for all $a, a' \in A$ and $b, b' \in B$. The S-act B is called (*principally*) *weakly flat* if it preserves embeddings of (principal) right ideals of S into S; or, in other words, for any $m, m' \in K$, where K is a (principal) right ideal of S, and for any $b, b' \in B$, if $m \otimes b = m' \otimes b'$ in $S_S \otimes B$ then $m \otimes b = m' \otimes b'$ in $K \otimes B$. It is easy to see that B is principally weakly flat if and only if whenever $m \otimes b = m \otimes b'$ in $S_S \otimes B$, for $m \in S$ and $b, b' \in B$, then $m \otimes b = m \otimes b'$ in $mS \otimes B$.

We will denote the classes of strongly flat, flat, weakly flat and principally weakly flat left S-acts by $\mathcal{SF}$, $\mathcal{F}$, $\mathcal{WF}$, $\mathcal{PWF}$ respectively.

Unlike the case for strongly flat there are no simple conditions such as (P) and (E) in the flat or weakly flat or principally weakly flat cases. This makes the question of axiomatisability rather harder; we are forced to consider tossings explicitly.

In [7] Bulman-Fleming and McDowell prove that a left S-act B is weakly flat if and only if it is principally weakly flat and satisfies Condition (W):

(W) If $sa = ta'$ for $a, a' \in A$, $s, t \in S$ then there exists $a'' \in A$ $u \in sS \cap tS$ such that $sa = ta' = ua''$, where we can visualize u as $u = ss' = tt'$ for some $s', t' \in S$.

Remark 1.3.0.19. [39] In **S-Act** we have

$$\mathcal{F}r \Rightarrow \mathcal{P}r \Rightarrow \mathcal{SF} \Rightarrow \mathcal{F} \Rightarrow \mathcal{WF} \Rightarrow \mathcal{PWF}$$

1.4 Free, projective and flat S-posets

Free and projective S-posets have the standard categorical definitions, as for S-acts. We can see that study of projectives in **S-Pos** closely parallels that for **S-Act**.

In **S-Pos** *coproducts* of left S-posets are disjoint unions where the components of the coproduct are incomparable and the S-action is componentwise.

We show how to construct a free S-poset over a pomonoid S. First, for a symbol x we let $Sx = \{sx \mid s \in S\}$ be a set of formal expressions in one-one correspondence with S as before. Then Sx becomes a left S-poset (isomorphic to ${}_SS$) if we define $s(tx) = (st)x$ for all $s, t \in S$ and $sx \leq tx$ if and only if $s \leq t$ in S.

Theorem 1.4.0.20. *[10] An S-poset A is free on a set X if and only if $A \cong \bigsqcup_{x \in X} Sx$.*

Theorem 1.4.0.21. *[10] Every S-poset is isomorphic to the quotient of a free S-poset.*

As in the unordered case, every free S-poset is projective and the converse is not true.

Proposition 1.4.0.22. *[48] Let S be a pomonoid. Then*
(i) Se is a projective S-poset for any idempotent $e \in S$;
(ii) a disjoint union of S-posets P_i is projective if and only if each P_i is projective for every $i \in I$;
(iii) an S-poset is projective if and only if it is isomorphic to a coproduct of S-posets of the form Se, where e is idempotent.

We now consider tensor products for S-posets. Let A be a right S-poset and B a left S-poset, then the tensor product, which is denoted by $A \otimes B$, is the quotient of the poset $A \times B$ by the pocongruence relation θ on $A \times B$ generated by

$$\{(as, b), (a, sb) : s \in S, a \in A, b \in B\}.$$

We will denote the equivalence class of $(a, b) \in A \times B$ with respect to congruence θ by $a \otimes b$. The following lemma explains the ordering relation in $A \otimes B$.

Lemma 1.4.0.23. *[48] Let A be a right S-poset, B a left S-poset, $a, a' \in A$, $b, b' \in B$. Then $a \otimes b \leq a' \otimes b'$ in $A \otimes B$ if and only if there exists $a_2, a_3, \cdots, a_m \in A$, $b_1, b_2, \cdots, b_m \in B$ and $s_1, t_1, \cdots, s_m, t_m \in S$ such that*

$$\begin{array}{rclrcl} & & & b & \leq & s_1 b_1 \\ as_1 & \leq & a_2 t_1 & t_1 b_1 & \leq & s_2 b_2 \\ a_2 s_2 & \leq & a_3 t_2 & t_2 b_2 & \leq & s_3 b_3 \\ & \vdots & & & \vdots & \\ a_{m-1} s_{m-1} & \leq & a_m t_{m-1} & t_{m-1} b_{m-1} & \leq & s_m b_m \\ a_m s_m & \leq & a' t_m & t_m b_m & \leq & b' \end{array} \qquad (*)$$

It follows that $a' \otimes b' \leq a \otimes b$ *if and only if there exists* $c_2, \cdots c_n \in A$ *and* $d_1, \cdots, d_n \in B$ *and* $u_1, v_1, \cdots, u_n, v_n \in S$ *such that*

$$\begin{array}{rclrcl} & & & b' & \leq & u_1 d_1 \\ a'u_1 & \leq & c_2 v_1 & v_1 d_1 & \leq & u_2 d_2 \\ c_2 u_2 & \leq & c_3 v_2 & v_2 d_2 & \leq & u_3 d_3 \\ & \vdots & & & \vdots & \\ c_{n-1} u_{n-1} & \leq & c_n v_{n-1} & v_{n-1} d_{n-1} & \leq & u_n d_n \\ c_n u_n & \leq & a v_n & v_n d_n & \leq & b \end{array} \qquad (**)$$

Thus $a \otimes b = a' \otimes b'$ *in* $A \otimes B$ *if and only if* $(*)$ *and* $(**)$ *exist.*

Definition 1.4.0.24. The sequences $(*)$ and $(**)$ in the above theorem are called a *double ordered tossing* $\mathcal{DT}$ of length $m+n$, from (a, b) to (a', b') over A and B. The *skeleton* of $\mathcal{DT}$ is

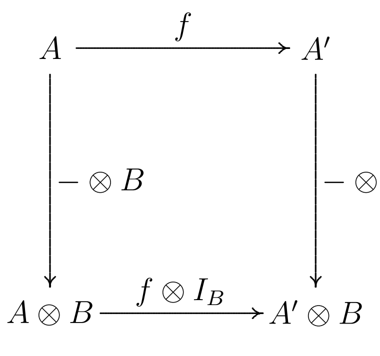
$$\mathcal{S}(\mathcal{DT}) = (s_1, t_1, \cdots, s_m, t_m, u_1, v_1, \cdots, u_n, v_n).$$

The sequence $(*)$ is called an *ordered tossing* $\mathcal{T}$ of length m from (a, b) to (a', b') over A and B. The *skeleton* of $\mathcal{T}$ is

$$\mathcal{S}(\mathcal{T}) = (s_1, t_1, \cdots, s_m, t_m).$$

If $\mathcal{S}_1 = (s_1, t_1, \ldots, s_m, t_m)$ and $\mathcal{S}_2 = (u_1, v_1, \ldots, u_n, v_n)$ are skeletons of ordered tossings, then we may (abusing notation) write $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ as shorthand for the skeleton $\mathcal{S} = (s_1, t_1, \ldots, s_m, t_m, u_1, v_1, \ldots, u_n, v_n)$ of the corresponding double ordered tossing.

As in the case of S-acts different notions of flatness are drawn from the tensor functor $- \otimes B$, where $- \otimes B$: **Pos-S**$\rightarrow$ **Pos** is a functor from the category of right S-posets to the category of posets, given by



where $f \otimes I_B : A \otimes B \rightarrow A' \otimes B$ is defined by

$$(a \otimes b)(f \otimes I_B) = af \otimes b$$

where we have $f : A \rightarrow A'$ an S-pomorphism in **Pos-S**.

Consider the following diagram in **S-Pos**:

$$\begin{array}{ccc} & & X_1 \\ & & \downarrow{\scriptstyle f_1} \\ X_2 & \xrightarrow[f_2]{} & Y \end{array}$$

The pair $\big(P, (p_1, p_2)\big)$ where $p_i : P \to X_i$, $i = 1, 2$ are S-pomorphisms is called a *subpullback* of the pair (f_1, f_2) if

(i) $p_1 f_1 \leq p_2 f_2$ and

(ii) if there exists $(p_1^{'}, p_2^{'})$ in **S-Pos** such that $p_1^{'} f_1 \leq p_2^{'} f_2$ then there exists a unique S-pomorphism say $\gamma : P^{'} \to P$ such that $\gamma p_1 = p_1^{'}$ and $\gamma p_2 = p_2^{'}$.

A *subequalizer* diagram for f_1 and f_2, where $f_1, f_2 : X \to Y$ in **S-Pos** is a pair (E, e) where $e : E \to X$ is an S-pomorphism, if

(i) $e f_1 \leq e f_2$ and

(ii) for any S-pomorphism $p : P \to X$ with $p f_1 \leq p f_2$ there exists a unique S-pomorphism $p^{'} : P \to E$ such that $p = p' e$.

A left S-poset B is called *flat* if the functor $- \otimes B$ takes embeddings in the category of **Pos-S** to one-one maps in the category **Pos** of posets. It is called *weakly flat* if the functor $- \otimes B$ takes embeddings of right ideals of S into S to one-one maps in the category **Pos**. Similarly for a *principally weakly flat* S-poset B, the functor $- \otimes B$ takes embeddings of principal right ideals in S into S to one-one maps in the category **Pos**. As for S-acts, a left S-poset B is principally weakly flat if and only if for any $m \in S$ and $b, b' \in B$, if $m \otimes b = m \otimes b'$ in $S \otimes B$, then $m \otimes b = m \otimes b'$ in $mS \otimes B$.

A left S-poset B is called *strongly flat* if the functor $- \otimes B$ preserves subpullbacks and subequalizers. The notion of strong flatness has several alternative characterisations. We now describe the two we will use. For the first, we need Conditions (P) and (E), defined for left S-posets as follows:

(P): for all $b, b' \in B$ and $s, s' \in S$ if $s\, b \leq s'\, b'$ then there exists $b'' \in B$ and $u, u' \in S$ such that $b = u\, b''$, $b' = u'\, b''$ and $s\, u \leq s'\, u'$;

(E): for all $b \in B$ and $s, s' \in S$ if $s\, b \leq s'\, b$ then there exists $b'' \in B$ and $u \in S$ such that $b = u\, b''$ and $s\, u \leq s'\, u$.

Such flatness conditions, i.e. using elements of S and S-posets rather than tossings explictly, we call *interpolation conditions.* Weaker than either (P) or (E) we have Condition (EP): for all $b \in B$ and $s, s' \in S$, if $sb \leq s'b$ then there exists $b'' \in B$ and $u, u' \in S$ such that $b = ub'' = u'b''$ and $su \leq s'u'$. The unordered version of this condition was introduced for M-acts in [22].

For the second alternative approach to strong flatness, we use the notion of direct limit.

Theorem 1.4.0.25. *[10] The following are equivalent for a left S-poset B:*
(i) B satisfies Condition (P) and Condition (E);
(ii) B is isomorphic to a direct limit of a family of finitely generated free S-posets;
(iii) B is subpullback flat and subequalizer flat.

Proposition 1.4.0.26. *Let $\mathcal{D} = \big(I, \{A_i\}_{i\in I}, \{\phi_{(i,j)}\}_{i\leq j}\big)$ be a directed system of strongly flat left S-posets, and let $\mathscr{L} = (A/\gamma, \{\theta_i\}_{i\in I})$ be the direct limit constructed as above. Then A/γ is strongly flat.*

Proof. We show that A/γ satisfies conditions (P) and (E), so that by the Theorem 1.4.0.25, it is strongly flat.

Suppose that $a \in A_i, a' \in A_j, s, s' \in S$ and $s[a] \leq s'[a']$. Then $[sa] \leq [s'a']$ so there exists $k \geq i, j$ such that $(sa)\phi_{(i,k)} \leq (s'a')\phi_{(j,k)}$. Thus in A_k we have that $s(a\,\phi_{(i,k)}) \leq s'(a'\,\phi_{(j,k)})$ so that as A_k satisfies condition (P) we have $a'' \in A_k$ and $u, u' \in S$ such that $a\phi_{(i,k)} = ua'', a'\phi_{(j,k)} = u'a''$ and $su \leq s'u'$.

From (c) of Proposition 1.2.2.14 we have that $[a] = [ua''] = u[a'']$ and $[a'] = [u'a''] = u'[a'']$ and A/γ satisfies (P) as required. Similarly, A/γ has Condition (E).

□

The notion of strong flatness simplifies for cyclic left S-posets.

Definition 1.4.0.27. A left S-poset A is called *cyclic* if $A = Sa$ for some $a \in A$.

Lemma 1.4.0.28. *A left S-poset A is cyclic if and only if there exists an S-pocongruence ρ on ${}_SS$ such that $A \cong S/\rho$.*

Proof. Let $A = Sa$ for some $a \in A$, then we define $\rho_a : S \to A$ by $s\rho_a = sa$ and whenever $s \leq t$, we will have $sa \leq ta$ which shows that ρ_a is a surjective S-pomorphism. Let ρ be the equivalence relation associated with $\overrightarrow{\ker}\rho_a$. Then ρ is an S-pocongruence and by Corollary 1.2.2.12 $S/\rho \cong \operatorname{im}\rho_a = A$.

Conversely, $S/\rho = S[1]_\rho$ is cyclic. □

The following is easy to check.

Lemma 1.4.0.29. *[46] A cyclic left S-poset $A = Sa$ is strongly flat, if and only if for $x, y \in S$, if $xa \leq ya$ then there exists $u \in S$ such that $a = ua$, $xu \leq yu$.*

Corollary 1.4.0.30. *[46] Let ρ be a left S-pocongruence on a pomonoid S. The following conditions are equivalent:*
(i) S/ρ is strongly flat;
(ii) if $[s] \leq [t]$ where $s, t \in S$ then there exists $u \in S$ such that $su \leq tu$ and $1 \rho\, u$.

In **Pos-S**, one-one S-pomorphisms do not coincide with embeddings; to see this consider the case for posets regarded as S-posets over a trivial pomonoid S. In [46] Shi defined notions of po-flat, weakly po-flat, principally weakly po-flat S-posets, as follows: An S-poset B is called *po-flat* if the functor $- \otimes B$ takes embeddings in the category of **Pos-S** to embeddings in **Pos**. It is *weakly po-flat* (*principally weakly po-flat*) if the functor $- \otimes B$ preserves the embeddings of (principal) right ideals of S into S.

In [46] Shi has shown that analogues of some properties in **S-Act** are not true in **S-Pos**. For example all S-acts satisfy Condition (P) (for acts) if and only if S is a group, but even if we take an ordered group we can find an S-poset which does not satisfy Condition (P) (for S-posets). Shi then defined another notion similar to Condition (P), called Condition (P_w).

We say that a left S-poset B satisfies *Condition (P_w)* whenever for all $b, b' \in B$ and $s, s' \in S$ if $s\,b \leq s'\,b'$ then there exists $b'' \in B$, $u, u' \in S$ such that $s\,u \leq s'\,u'$, $b \leq u\,b''$, $u'\,b'' \leq b'$.

Further, let G be an ordered group, then all G-posets satisfy Condition (P_w)[46] and Condition $(\mathrm{P}_w) \Rightarrow$ po-flat $\Rightarrow$ flat, hence for an ordered group G every left G-poset is po-flat, that is every left G-poset is flat.

Shi [46] has shown that a left S-poset B is weakly po-flat if and only if it is principally weakly po-flat and satisfies:
Condition (W): for any $b, b' \in B$ and $s, s' \in S$, if $sb \leq s'b'$ then implies that there exists $b'' \in B$, $p \in sS, p' \in s'S$ such that $p \leq p'$, $sb \leq pb''$, $p'b'' \leq s'b'$. Shi's proof is along the same lines as that for S-acts by Syd Bulman-Fleming and McDowell in [39], who have proved that a left S-act A is weakly flat if and only if it is principally weakly flat and satisfies a condition analogous to Condition (W) for S-acts. A proof analogous to those in [39, 46] gives the following.

Lemma 1.4.0.31. *Let S be a pomonoid. A left S-poset B is weakly flat if and only if it is principally weakly flat and satisfies:*
Condition (U): for all $b, b' \in B$ and $s, s' \in S$, if $sb = sb'$ then there exists $b'' \in B$, $p \in sS, p' \in s'S$, with $p \leq p'$ and $sb = pb'' = p'b'' = s'b'$.

We will denote the classes of free, projective, strongly flat, flat, weakly flat, principally weakly flat, po-flat, weakly po-flat, principally weakly po-flat left S-posets by

$$\mathcal{F}r,\ \mathcal{P}r,\ \mathcal{SF},\ \mathcal{F},\ \mathcal{WF},\ \mathcal{PWF},\ \mathcal{PF},\ \mathcal{WPF},\ \mathcal{PWPF}$$

respectively. The classes of left S-posets satisfying Condition (P), Condition (E) Condition (EP) or Condition (P_w) will be denoted by $\mathcal{P}, \mathcal{E}, \mathcal{EP}$ and $\mathcal{P}_w$, respectively. We note that many classes and conditions for S-posets have the same notation as for S-acts, but the meaning should always be clear from the context.

Finally in our list of flatness properties we turn out attention to those introduced in [23] by Golchin and Rezaei. They define Conditions (WP),(WP_w),(PWP) and (PWP_w) for S-posets, which are derived from the concepts of subpullback diagrams in **S-Pos**. For our purposes here it is enough to define (PWP) and (PWP_w) for a left S-poset B:
Condition (PWP): for all $b, b' \in B$ and $s \in S$, if $sb \leq sb'$ then there exits $u, u' \in S$ and $b'' \in B$ such that $b = ub''$, $b' = u'b''$ and $su \leq su'$;
Condition (PWP_w): for all $b, b' \in B$ and $s \in S$, if $sb \leq sb'$ then there exist $u, u' \in S$ and $b'' \in B$ such that $b \leq ub''$, $u'b'' \leq b'$ and $su \leq su'$.

We denote by

$$\mathcal{WP}, \mathcal{WP}_w, \mathcal{PWP} \text{ and } \mathcal{PWP}_w$$

the classes of left S-posets satisfying Conditions (WP),(WP_w),(PWP) and (PWP_w), respectively.
Remark 1.4.0.32. [23][48] In **S-Pos** we have the following implications, all of which are known to be strict except for Condition (P_w) implies po-flat:

$$\begin{array}{ccccccccccc}
\mathcal{F}r & \Rightarrow & \mathcal{P}r & \Rightarrow & \mathcal{SF} & \Rightarrow & \mathcal{P} & \Rightarrow & \mathcal{WP} & \Rightarrow & \mathcal{PWP} \\
 & & & & & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & & & & & \mathcal{P}_w & \Rightarrow & \mathcal{WP}_w & \Rightarrow & \mathcal{PWP}_w \\
 & & & & & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & & & & & \mathcal{PF} & \Rightarrow & \mathcal{WPF} & \Rightarrow & \mathcal{PWPF} \\
 & & & & & & \Downarrow & & \Downarrow & & \Downarrow \\
 & & & & & & \mathcal{F} & \Rightarrow & \mathcal{WF} & \Rightarrow & \mathcal{PWF}
\end{array}$$

We are interested in finding for which pomonoids these classes are axiomatisable.

1.5 Conditions

We now give a number of finitary conditions for monoids and pomonoids which will be used in this thesis, particularly when characterising left perfect pomonoids in Chapter 4.

Let S be a monoid. Conditions (A), (M_R), (M_L), (M^R) and (M^L) are defined as follows:

(M_R)/(M_L): S satisfies the descending chain condition for principal right/left ideals;

(M^R)/(M^L): S satisfies the ascending chain condition for principal right/left ideals;

(A): every left S-act satisfies the ascending chain condition on cyclic S-subacts;

Now let S be a pomonoid. In addition to the above conditions we define Condition (A^{O}):

(A^{O}): every left S-poset satisfies the ascending chain condition for cyclic S-subposets.

It is clear that Condition (A) implies Condition (A^{O}); we show later that, in fact, these two conditions are equivalent.

1.6 First order languages and axiomatisability

Relating to any class $\mathcal{C}$ of universal algebras or relational structures of the same type (where we are not properly defining type here) there exists a first order language. One can then ask, which of the properties defined for $A \in \mathcal{C}$ can be captured by first order sentences of the language, that is, which of the properties are axiomatisable? For the convenience of the reader we will describe here a short account of first order logic and the notion of axiomatisability. Subsequently we will consider the first order languages associated to M-acts over a monoid M and S-posets over a pomonoid S, and give some illustrations of axiomatisable properties.

1.6.1 First order languages

To define a first order language L we must first specify its *alphabet*; this consists of variable, constant, function and predicate (relational) symbols, together with punctuation, quantifiers and logical connectives.

(1) *Variable Symbols*

An infinite set of variables $\{x_i : i \in I\}$, normally assumed to be countable.

(2) *Constant Symbols*

A (possibly empty) set $\{c_j : j \in J\}$ of constant symbols.

(3) *Function Symbols*

A (possibly empty) set

$$\{f_j^i : i \in \mathbb{N}, j \in J_i\},$$

where the superscript i represents that the function is i-place e.g. f_1^3 indicates a function symbol to be interpreted by a ternary function.

(4) *Predicate Symbols*

A set of predicate letters

$$\{P_k^i : i \in \mathbb{N}, k \in M_i\}.$$

Here the superscript indicates the arity of the predicate, e.g. P_1^3 denotes a predicate symbol to be interpreted by a ternary relation. We make the convention that $=$ (or, more accurately, a symbol that will be interpreted as equality) is a binary predicate.

(5) *Punctuation*

We allow commas ',' and parentheses '(' and ')' as punctuation.

(6) *Quantifier Symbol*

A single symbol $\forall$.

(7) *Logical connectives* The symbols $\neg$ and $\rightarrow$.

We use additional symbols as shorthand, for example, $\exists x P_1^1(x)$ is shorthand for $\neg\forall x \neg P_1^1(x)$ and $P_1^1(x) \vee P_2^1(x)$ is shorthand for $\neg P_1^1(x) \rightarrow P_2^1(x)$.

Given an alphabet as above for L, we must now define terms and well formed formulae.

The set of *terms* of L is defined inductively. Any constant or variable is a term, if $t_1, \ldots, t_n$ are terms then so is any expression $f_j^n(t_1, \ldots, t_n)$.

The set of *well formed formulae* of L is again defined inductively:

(i) $P_i^n(x_1, x_2, \cdots, x_n)$ where P_i^n is a predicate and $x_1, x_2, \cdots, x_n$ are terms, is a well formed formula, usually called an *atomic formula*;

(ii) if θ is a well formed formula then so is $\neg\theta$;

(iii) if θ and ϕ are well formed formulae then $(\theta \to \phi)$ is a well formed formula;

(iv) if θ is a well formed formula then $\forall\, x\, \theta$ is a well formed formula, for any variable x.

For clarity we may insert extra parentheses, denoting for example $\forall x \exists y\, x = f_1^1(y)$ by $(\forall x)(\exists y)\big(x = f_1^1(y)\big)$. We note here that *terms* and *formulae* are series of symbols; as yet they have no intrinsic meanings. Formally, we should say that L is a *first order language with equality.* For brevity we may refer to well formed formulae simply as *formulae.*

We now give some examples of first order languages.

Groups There is more than one choice for a first order language associated with the class $\mathcal{G}$ of all groups. One possibility is $L_{\mathcal{G}}$, where $L_{\mathcal{G}}$ has a constant symbol 1, a unary function f_1^1 and a binary function f_1^2. We usually write $f_1^1(x)$ as x^{-1} and $f_1^2(x, y)$ as xy, as we will be 'interpreting' f_1^1 and f_1^2 as inversion and product, respectively.

S-**acts** Let S be a monoid. For left S-acts the first order language has no constant or relational symbols (other than $=$) and consists of a unary function symbol say λ_s (dropping the superscript 1) for each $s \in S$. We denote the first order language relating to left S-acts by L_S. We may write $\lambda_s(x)$ as sx.

S-**posets** Similarly if S is a pomonoid we can define a language $L_S^{\leq}$ having no constant symbols, equipped with a binary relation symbol $\leq$, and having a unary function symbol λ_s for each $s \in S$. Again, we may write $\lambda_s(x)$ as sx.

Sentences A *sentence* of L is a (well formed) formula with no free variables, that is, if an x appears in the formula, it is governed by a $\forall x$. Of the two well formed formulae

$(\forall x)(sx = y)$ and $(\forall x)\big(sx = x \to (\exists y)(x = ty)\big)$ in L_S, the first is not a sentence but the second is. We now comment on the notion of interpretation, without going into full detail.

1.6.2 Interpretations

Let L be a first order language, an *interpretation* (or *L-structure*) I for L consists of a set D_I called the *domain* of the interpretation, a subset $\{\overline{c}_i : i \in I\}$ of elements of D_I called *constants*, a set $\{\overline{f}^i_j : i \in \mathbb{N}, j \in J_i\}$ of functions where $\overline{f}^i_j : D^i_I \to D_I$ and a set of relations $\{\overline{P}^i_k : i \in \mathbb{N}, k \in M_i\}$ on D_I where each $\overline{P}^i_k$ is i-ary. The constants, functions and relations 'interpret' those of L; we insist that the equality symbol in L is interpreted by the relation of equality in D_I.

It is rather involved to say exactly what we mean by 'interpret', but the reader will not go wrong by relying on intuition, as we only consider simple languages such as L_S and $L^{\leq}_S$. It is also complicated, and does not add to clarity here, to explain exactly what it means for an interpretation to *satisfy* a formula of L. Essentially, given a formula of L, and translating it into a statement about D_I, by interpreting the constants, functions and relations it contains by the corresponding ones of I, it is satisfied by I if it is a true statement about D_I. We normally say simply that the formula of L is 'true in I' or even just 'true in D_I'. If φ is a formula that is satisfied by I we denote this by $I \models \varphi$ or $D_I \models \varphi$.

Let S be a monoid and let A be a left S-act. Then A gives us an interpretation of the language L_S, where λ_s is the function $a \mapsto sa$, for all $s \in S$. The sentence $(\forall x)(\lambda_s(\lambda_t(x)) = \lambda_{st}(x))$ is satisfied by the interpretation, as it is a true statement about A. The sentence

$$(\forall x)(\forall y)(\lambda_s(x) = \lambda_s(y) \to x = y)$$

would be true in some S-acts (for example, if S is a group) but not others.

Now let S be a pomonoid and let A be a left S-poset. The formal symbol $\leq$ in $L^{\leq}_S$ is interpreted by the partial order $\leq$ in S. If $s, t \in S$ with $s \leq t$, then, for example,

$$(\forall x)(\lambda_s(x) \leq \lambda_t(x))$$

is true in A.

Models Let L be a first order language and let T be a set of sentences of L. We say that an interpretation I of L is a *model* of T if every sentence of T is true in I and we denote it by $I \models T$ or $D_I \models T$.

1.6.3 Axiomatisability

We now come to one of the central concepts of this thesis. Let T be a set of sentences of a first order language L. Then the collection $\{I : I \text{ inteprets L}, I \models T\}$ is said to be *axiomatised* by T. Let $\mathcal{C}$ be a class of interpretations of L. Then $\mathcal{C}$ is *axiomatisable* if there is a set T of sentences axiomatising $\mathcal{C}$; let us stress that this means for an interpretation I, I lies in $\mathcal{C}$ if and only if $I \models T$.

As an illustration pertinent to this thesis, we consider S-acts and S-posets.

Let S be a monoid. Then the class S-Act is axiomatised by

$$\sum\nolimits_S = \{(\forall x)(\lambda_1(x) = x)\} \bigcup \{\varphi_{s,t} : s, t \in S\}$$

where

$$\varphi_{s,t} := (\forall x)(\lambda_s(\lambda_t(x)) = \lambda_{st}(x)).$$

Let S be a pomonoid. Then the class S-poset is axiomatised by

$$\sum\nolimits_S^{\leq} = \sum\nolimits_S \bigcup \{\psi_{s,t} : s \leq t\} \bigcup \{\theta_s : s \in S\} \bigcup \{\ \pi\ \}$$

where for $s \leq t$

$$\psi_{s,t} := (\forall x)(\lambda_s(x) \leq \lambda_t(x))$$

and for any s

$$\theta_s := (\forall x)(\forall y)(x \leq y \to \lambda_s(x) \leq \lambda_s(y)).$$

We include the following set of sentences in $\sum_S^{\leq}$ for the partial order relation involve:

$\pi := \big\{(\forall x)\big(x \leq x\big) \bigwedge (\forall x,\, y)\big((x \leq y \,\wedge\, y \leq x) \rightarrow (x = y)\big) \bigwedge (\forall x,\, y,\, z)\big((\, x \leq y \,\wedge\, y \leq z) \rightarrow (x \leq z)\big)\big\}$.

We point out the obvious here that in L_S (S a monoid) and $L_S^{\leq}$ (S a pomonoid), we ***cannot quantify over elements of*** S.

Ultraproducts Ultraproducts play an important role in axiomatisability problems. We define first ultrafilters on a non-empty set I followed by the definition of ultraproducts.

Let I be a non-empty set; we denote the set of subsets of I by $\mathcal{P}(I)$. We say that $\Phi \subseteq \mathcal{P}(I)$ is a *filter* over I if (i) $I \in \Phi$, (ii) for any $X,\, Y \in \mathcal{P}(I)$ implies that $X \cap Y \in \Phi$, and (iii) if $X \in \Phi$ and $X \subseteq Y \subseteq I$ then $Y \in \Phi$. A filter Φ is called an *ultrafilter* if for all $X \in \mathcal{P}(I)$, $X \in \Phi$ if and only if $I \setminus X \notin \Phi$.

Theorem 1.6.3.1. *[13] A filter D is maximal if and only if it is an ultrafilter.*

Definition 1.6.3.2. Let F be a subset of $\mathcal{P}(I)$, we say that F has the *finite intersection property* if and only if the intersection of finitely many elements of F is non-empty.

The next result is Proposition 4.1.3 of [13].

Theorem 1.6.3.3. (Ultrafilter theorem)*[13] For any proper subset Θ of $P(I)$, such that Θ has the finite intersection property, an ultrafilter Φ can be constructed such that $\Theta \subseteq \Phi$.*

Every proper filter over I satisfies the finite intersection property, hence can be extended to an ultrafilter over I.

Let L be a first order language and let $\{A_i : i \in I\}$ be a set of interpretations of L. Let Φ be an ultrafilter on I and let $A = \prod_{i \in I} A_i$ be the Cartesian product of the A_is. With a view to controlling the notation, we denote by (a_i) the element f of A such that $if = a_i$, for any $i \in I$. For a constant c, a function f and a relation P of L, we denote for each $j \in I$ the interpretation of c, f and P in A_j by c_j, f_j and P_j, respectively.

We define a relation $\equiv_\Phi$ by the rule that

$$(a_i) \equiv_\Phi (b_i) \text{ if and only if } \{i : a_i = b_i\} \in \Phi.$$

It is easy to see from the definition of filter that $\equiv_\Phi$ is an equivalence. We denote by $(a_i)_\Phi$ the equivalence class of an element (a_i) of A under $\equiv_\Phi$ and put $\mathcal{U} = A/\equiv_\Phi$, often abbreviated by $\mathcal{U} = A/\Phi$.

We now make $\mathcal{U}$ into an interpretation of L. A constant c is interpreted by

$$c_{\mathcal{U}} = (c_i)_\Phi$$

an n-place function f by $f_{\mathcal{U}}$ where

$$f_{\mathcal{U}}((a_i^1)_\Phi, \dots, (a_i^n)_\Phi) = (f_i(a_i^1, \dots, a_i^n))_\Phi$$

and an n-ary relation P by

$$((a_i^1), \dots, (a_i^n)) \in P \text{ if and only if } \{i : (a_i^1, \dots, a_i^n) \in P_i\} \in \Phi.$$

Of course, some work is required to show that the interpretation of f and P is well defined. The interested reader may find this in Proposition 4.1.7 of [13]. We call the interpretation $\mathcal{U}$ constructed in this way an *ultraproduct* of the interpretations A_i.

The next result is crucial.

Theorem 1.6.3.4. (Łos's Theorem)*[13] Let L be a first order language, and let $\mathcal{C}$ be a class of interpretations of L. If $\mathcal{C}$ is axiomatisable, then $\mathcal{C}$ is closed under ultraproducts.*

We now consider ultraproducts of S-acts and S-posets. Let S be a monoid and let $\{A_i : i \in I\}$ be a family of left S-acts and put $A = \Pi\{A_i : i \in I\}$. Let Φ be an ultrafilter on I and put $\mathcal{U} = A/\Phi$. Then, for any $s \in S$ and $(a_i)_\Phi \in \mathcal{U}$ we have that

$$s(a_i)_\Phi = (sa_i)_\Phi.$$

Similarly we can define ultraproduct $\mathcal{U}$ of a family $\{A_i : i \in I\}$ of S-posets over a pomonoid S, with respect to an ultrafilter Φ. The action of S is as above, but now we must specify the interpretation of $\leq$. From the general recipe we have that

$$(a_i)_\Phi \leq (b_i)_\Phi \text{ if and only if } \{i : a_i \leq b_i\} \in \Phi.$$

Chapter 2

Axiomatisability problems of S-acts

In this chapter, we will be considering axiomatisability problems for classes of (left) S-acts. To simplify notation in the first order language L_S, relating to left S-acts, we will replace expressions of the form '$\lambda_s(x)$' by sx in formulae. Then the class of all S-acts which we denote by S-*Act* is axiomatised by

$$\sum_S = \{(\forall x)(1x = x)\} \bigcup \{\varphi_{s,t} : s, t \in S\}$$

where

$$\varphi_{s,\,t} := (\forall x)(s(t(x)) = st(x)).$$

For more details we refer reader to Section 1.6 of Chapter 1.

It can be noted that there are certain classes of S-acts that are axiomatisable for all monoids S, e.g. S-*Acts*. Less trivially, we denote by $\mathcal{T}'$ the class of torsion free left S-acts. A left S-act A is *torsion free* if

$$sa = sb \text{ implies that } a = b$$

for all $s \in \mathcal{LC}$, where $\mathcal{LC}$ denotes the set of left cancellable elements of S. Clearly $\mathcal{T}'$ is axiomatised by

$$\Sigma_S \cup \big\{(\forall x, y)(sx = sy \to x = y) : s \in \mathcal{LC}\big\}.$$

For brevity, here and elsewhere, we drop explicit mention of Σ_S and say more simply that $\mathcal{T}'$ is axiomatised within S-*Act* by

$$\big\{(\forall x, y)(sx = sy \to x = y) : s \in \mathcal{LC}\big\}.$$

However, there are some classes of S-acts which are axiomatisable for some monoids and not for others e.g. the classes $\mathcal{SF}$ and $\mathcal{P}r$ of strongly flat and projective S-acts are axiomatisable if S is finite or a group, but for the monoid C where $C = \{1 = e_0, e_1, e_2, \cdots\}$ and $e_i e_j = e_{max\{i,j\}}$, that is, C is an inverse ω-chain, the class $\mathcal{SF}$ is axiomatisable but $\mathcal{P}r$ is not [25].

Introductory work on axiomatisability problems for S-acts was done by Gould [25]. She considered the following questions: for which monoids S are the classes of $\mathcal{SF}$ and $\mathcal{P}r$ axiomatisable? She described necessary and sufficient conditions on S such that $\mathcal{SF}$ is axiomatisable and obtained partial results for $\mathcal{P}r$. The full answer for $\mathcal{P}r$ was provided by Stepanova [50]. The kind of conditions that arise, here as for other questions, are finitary in nature.

Later Bulman-Fleming and Gould [6] gave an alternative proof of Stepanova's result of axiomatisability of projective S-acts. They also characterised those monoids such that the classes $\mathcal{F}$(flat) and $\mathcal{WF}$(weakly flat) of S-acts are axiomatisable. Subsequently, Gould [29] characterised those monoids S such that the class $\mathcal{F}r$(free) S-acts were axiomatisable. In [29] there is a discussion of the relations between the conditions on a monoid S that arise while axiomatising certain classes of S-acts such as $\mathcal{F}r$, $\mathcal{P}r$, $\mathcal{SF}$, $\mathcal{F}$ or $\mathcal{WF}$. Recently Gould, Stepanova, Mikhalev and Palyutin [30] gave a comrehensive survey named "Model Theoretic Properties of Free, Projective and Flat S-acts" which includes much additional model theoretic material.

The aim of this chapter is to add to the theory of axiomatisability of classes of S-acts over a monoid S. We put some of the techniques of earlier articles into a general setting. In Chapter 3 we use these methods to develop the theory of axiomatisability of S-posets over a pomonoid S.

It is known that there are three familiar methods to axiomatise classes of S-acts. The first of them is the simplest making use of interpolation conditions on S-acts to produce finitary conditions on S. This method has been used by Gould for $\mathcal{SF}$ [25]; we will refer to this as the "elements" method. We have used this in the context of, for example, Condition (EP),(W), and (PWP), for S-acts.

The second two methods both involve "replacement tossings" and have been developed by Bulman-Fleming and Gould in [6] for $\mathcal{F}$ and $\mathcal{WF}$; we will refer to these as "replacement tossings" methods; we have used these in the perspective of, Condition (E),(P),(EP),(W) and (PWP), for S-acts.

First, we consider the axiomatisability of some classes of S-acts related to flatness, such as $\mathcal{F}, \mathcal{WF}$ and $\mathcal{PWF}$(principally weakly flat), where the first two are previously discussed by Bulman-Fleming and Gould [6]. In Section 2.1 we demonstrate a more general way to axiomatise these classes, putting the two of the "replacement tossings" methods into an abstract context. These can then be specialised to prove both new and known results.

In Section 2.2, we investigate the axiomatisability of the classes $\mathcal{EP}$, $\mathcal{W}$. For the definitions of these classes see Chapter 1. We determine when these classes of S-acts are axiomatisable by using both the "elements" method and by using "replacement tossings".

In Section 2.3 we attempt some examples of axiomatisability. We develop the connection between axiomatisability conditions of different classes. We know that if $\mathcal{P}$ is axiomatisable then so is $\mathcal{W}$. We give an example of a monoid such that $\mathcal{W}$ is axiomatisable but $\mathcal{P}$ is not. It is known that $\mathcal{E}$ implies $\mathcal{EP}$, we would like to know whether $\mathcal{EP}$ is axiomatisable if $\mathcal{E}$ is axiomatisable but this is still unknown.

We note that if Condition A implies Condition B, where A and B are conditions on left S-acts, then we usually expect that if the class $\mathcal{A}$ of left S-acts satisfying Condition A is axiomatisable, then so is the class $\mathcal{B}$ of the left S-acts satisfying Condition B.

Lemma 2.0.3.5. *Let S be a monoid, and let $\mathcal{U}, \mathcal{V}$ be classes of left S-acts such that $S \in \mathcal{U}$, and $\mathcal{U} \subseteq \mathcal{V}$. Suppose that $\mathcal{V}$ is axiomatisable if and only if every ultrapower of S lies in $\mathcal{V}$. Now if $\mathcal{U}$ is axiomatisable then so is $\mathcal{V}$.*

Surprisingly, we have managed to show without using Lemma 2.0.3.5, which has been extensively used throughout this Chapter, that if $\mathcal{P}$ is axiomatisable then so is $\mathcal{EP}$.

2.1 General results on axiomatisability

Let $\mathcal{C}$ be a class of embeddings of right S-acts, for example, all embeddings, or all inclusions of right ideals into S. A left S-act B is called *$\mathcal{C}$-flat* if the functor $-\otimes B$ maps embeddings in $\mathcal{C}$ to one-one maps in **Set**, that is, if $\tau : A \to A'$ is in $\mathcal{C}$, then $\tau \otimes I_B$ is one-one. In terms of elements this says that if $a, a' \in A$ and $b, b' \in B$ and $a\tau \otimes b = a'\tau \otimes b'$ in $A' \otimes B$, then $a \otimes b = a' \otimes b'$ in $A \otimes B$. We denote the class of $\mathcal{C}$-flat left S-acts by $\mathcal{CF}$. Note: for S-posets there will be two variations of the notion of $\mathcal{C}$-flat, as we explain in Chapter 3.

We introduce Condition (Free) on $\mathcal{C}$ below. In Subsection 2.1.1 we find necessary and sufficient conditions for the class $\mathcal{CF}$ of $\mathcal{C}$-flat left S-acts to be axiomatisable if $\mathcal{C}$ satisfies Condition (Free). The result of Bulman-Fleming and Gould axiomatising $\mathcal{F}$ becomes a special case. In Subsection 2.1.2 we drop the assumption of Condition (Free). We have a general result to determine for which monoids S is $\mathcal{CF}$ axiomatisable. The result of Bulman-Fleming and Gould axiomatising $\mathcal{WF}$ then becomes a special case. We can also deduce the axiomatisability result for $\mathcal{PWF}$ using this method.

The two general results in this Section involve "replacement tossings". Some of the arguments are rather intricate. The reader wanting an easier introduction to axiomatisability problems could look at Section 2.2 of this chapter first.

2.1.1 Axiomatisability of $\mathcal{CF}$ with Condition (Free)

In this subsection we find necessary and sufficient conditions on S such that a class $\mathcal{CF}$ is axiomatisable, where $\mathcal{C}$ is a class of embeddings of left S-acts satisfying Condition (Free). We first describe this condition.

It is convenient to introduce some notation. Let

$$\mathcal{S} = (s_1, t_1, \cdots, s_n, t_n) \in \mathbb{S}$$

be a skeleton. Let R_S be the first order language relating to right S-acts.

We define a formula $\epsilon_{\mathcal{S}} \in R_S$, as follows:

$$\epsilon_{\mathcal{S}}(x, x_2, \cdots, x_n, x') := xs_1 = x_2t_1 \wedge x_2s_2 = x_3t_2 \wedge \cdots \wedge x_ns_n = x't_n$$

and put

$$\delta_{\mathcal{S}}(x,x') := (\exists x_2 \cdots \exists x_n)\epsilon_{\mathcal{S}}(x,x_2,\cdots,x_n,x').$$

On the other hand we define the formula

$$\theta_{\mathcal{S}}(x,x_1,\cdots,x_n,x') := x = s_1x_1 \wedge t_1x_1 = s_2t_2 \wedge \cdots \wedge t_nb_n = x'$$

of L_S and put

$$\gamma_{\mathcal{S}}(x,x') := (\exists x_1 \cdots \exists x_n)\theta_{\mathcal{S}}(x,x_1,\cdots,x_n,x').$$

Remark 2.1.1.1. Let A, B be right and left S-acts, respectively, let $a, a' \in A$ and $b, b' \in B$.

(i) The pair (a, b) is connected to the pair (a', b') via a tossing with skeleton $\mathcal{S}$ if and only if $\delta_{\mathcal{S}}(a, a')$ is true in A and $\gamma_{\mathcal{S}}(b, b')$ is true in B.

(ii) If $\delta_{\mathcal{S}}(a, a')$ is true in A and $\psi : A \to A'$ is a (right) S-morphism, then $\delta_{\mathcal{S}}(a\psi, a'\psi)$ is true in A'.

(iii) If $\gamma_{\mathcal{S}}(b, b')$ is true in B and $\tau : B \to B'$ is (left) S-morphism, then $\gamma_{\mathcal{S}}(b\tau, b'\tau)$ is true in $B\tau$.

Definition 2.1.1.2. We say that $\mathcal{C}$ satisfies *Condition (Free)* if for each $\mathcal{S} \in \mathbb{S}$ there is an embedding $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}}$ in $\mathcal{C}$ and $u_{\mathcal{S}}, u'_{\mathcal{S}} \in W_{\mathcal{S}}$ such that $\delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$ and further, for any embedding $\mu : A \to A'$ in $\mathcal{C}$ and any $a, a' \in A$ such that $\delta_{\mathcal{S}}(a\mu, a'\mu)$ is true in A', there is a morphism $\nu : W'_{\mathcal{S}} \to A'$ such that $u_{\mathcal{S}}\tau_{\mathcal{S}}\nu = a\mu$, $u'_{\mathcal{S}}\tau_{\mathcal{S}}\nu = a'\mu$ and $W_{\mathcal{S}}\tau_{\mathcal{S}}\nu \subseteq A\mu$.

Lemma 2.1.1.3. *Let $\mathcal{C}$ be a class of embeddings of right S-acts satisfying* Condition (Free). *Then the following conditions are equivalent for a left S-act B:*

(i) B is $\mathcal{C}$-flat;

(ii) $-\otimes B$ preserves all embeddings $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}}$;

(iii) if $(u_{\mathcal{S}}\tau_{\mathcal{S}}, b)$ and $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, b')$ are connected by a tossing over $W'_{\mathcal{S}}$ and B with skeleton $\mathcal{S}$, then $(u_{\mathcal{S}}, b)$ and $(u'_{\mathcal{S}}, b')$ are connected by a tossing over $W_{\mathcal{S}}$ and B.

Proof. Clearly we need only show that (iii) implies (i). Suppose that (iii) holds, let $\mu : A \to A'$ lie in $\mathcal{C}$ and suppose that

$$(a\mu, b), (a'\mu, b') \in A' \times B$$

are connected via a tossing over A' and B with skeleton $\mathcal{S}$. From considering the left hand side of the tossing, we have that $\delta_{\mathcal{S}}(a\mu, a'\mu)$ is true in A' and from considering the right hand side, $\gamma_{\mathcal{S}}(b, b')$ is true in B. By assumption there is an embedding $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}}$ in $\mathcal{C}$ and $u_{\mathcal{S}}, u'_{\mathcal{S}} \in W_{\mathcal{S}}$ such that $\delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$, and a morphism $\nu : W'_{\mathcal{S}} \to A'$ such that $u_{\mathcal{S}}\,\tau_{\mathcal{S}}\nu = a\mu$, $u'_{\mathcal{S}}\,\tau_{\mathcal{S}}\nu = a'\mu$ and $W_{\mathcal{S}}\tau_{\mathcal{S}}\nu \subseteq A\mu$. Since $\delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$, there is a tossing from $(u_{\mathcal{S}}\tau_{\mathcal{S}}, b)$ to $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, b')$ over $W'_{\mathcal{S}}$ and B with skeleton $\mathcal{S}$. From (iii), it follows that $(u_{\mathcal{S}}, b)$ and $(u'_{\mathcal{S}}, b')$ are connected via a tossing over $W_{\mathcal{S}}$ and B with skeleton $\mathcal{T}$ say. It follows that $\delta_{\mathcal{T}}(u_{\mathcal{S}}, u'_{\mathcal{S}})$ is true in $W_{\mathcal{S}}$ and so $\delta_{\mathcal{T}}(u_{\mathcal{S}}\tau_{\mathcal{S}}\nu, u'_{\mathcal{S}}\tau_{\mathcal{S}}\nu)$, that is, $\delta_{\mathcal{T}}(a\mu, a'\mu)$ is true in $A\mu$. Since μ is an embedding we deduce that $\delta_{\mathcal{T}}(a, a')$ is true in A and consequently, (a, b) and (a', b') are connected via a tossing with skeleton $\mathcal{T}$ over A and B. Hence B is $\mathcal{C}$-flat as required. □

We use "The Finitely Presented Flatness Lemma" [6] for S-acts to construct an example of the use of Condition (Free). Specifically, we show that the class of all right S-acts has Condition (Free).

Let $\mathcal{S} = (s_1, t_1, \cdots, s_m, t_m)$ be a skeleton and let F^{m+1} be the free right S-act

$$xS \,\dot{\cup}\, x_2S \,\dot{\cup} \cdots \dot{\cup}\, x_mS \,\dot{\cup}\, x'S.$$

Let $\rho_{\mathcal{S}}$ be the S-act congruence on F^{m+1} generated by the relation $R_{\mathcal{S}}$

$$\{(xs_1, x_2t_1), (x_2s_2, x_3t_2), \cdots, (x_{m-1}s_{m-1}, x_mt_{m-1}), (x_ms_m, x't_m)\}.$$

We denote the $\rho_{\mathcal{S}}$-class of $w \in F^{m+1}$ by $[w]$.

If B is a left S-act and $b, b_1, \cdots, b_m, b' \in B$ are such that

$$b = s_1b_1,\ t_1b_1 = s_2b_2, \cdots, t_mb_m = b'$$

that is, $\theta_{\mathcal{S}}(b, b_1, \cdots, b_m, b')$ is true, then the tossing

$$\begin{array}{rclrcl}
 & & & b & = & s_1b_1 \\
{[x]s_1} & = & [x_2]t_1 & t_1b_1 & = & s_2b_2 \\
{[x_2]s_2} & = & [x_3]t_2 & t_2b_2 & = & s_3b_3 \\
 & \vdots & & & \vdots & \\
{[x_{m-1}]s_{m-1}} & = & [x_m]t_{m-1} & t_{m-1}b_{m-1} & = & s_mb_m \\
{[x_m]s_m} & = & [x']t_m & t_mb_m & = & b'
\end{array}$$

over $F^{m+1}/\rho_{\mathcal{S}}$ and B is called the *standard tossing* with skeleton

$$\mathcal{S} = (s_1, t_1, \cdots, s_m, t_m)$$

of length m connecting $([x], b)$ to $([x'], b')$.

Lemma 2.1.1.4. *[6] The following conditions are equivalent for a left S-act B:*

(i) B is flat;

(ii) $- \otimes B$ maps the embeddings of $[x]S \cup [x']S$ into $F^{m+1}/\rho_{\mathcal{S}}$ in the category ***Act*-S** *to monomorphisms in the category of* **Set**, *for every skeleton $\mathcal{S}$;*

(iii) if $\big([x], b\big)$ and $\big([x'], b'\big)$ are connected by a standard tossing over $F^{m+1}/\rho_{\mathcal{S}}$ and B with skeleton $\mathcal{S}$, then they are connected by a tossing over $[x]S \cup [x']S$ and B.

We therefore are able to show that:

Lemma 2.1.1.5. *The class Act-S of all right S-acts has Condition (Free).*

Proof. Let $\mathcal{S}$ be a skeleton of length m, and let $W'_{\mathcal{S}} = F^{m+1}/\rho_{\mathcal{S}}$, $W_{\mathcal{S}} = [x]S \cup [x']S$, and let $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}}$ denote inclusion. Then for $[x], [x'] \in W_{\mathcal{S}}$, put $u_{\mathcal{S}} = [x]$ and $u'_{\mathcal{S}} = [x']$. Clearly, $\delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$.

Suppose that $\mu : A \to A'$ is any right S-act embedding and $\delta_{\mathcal{S}}(a\mu, a'\mu)$ is true in A'

$$\begin{aligned} a\mu s_1 &= a_2 t_1 \\ a_2 s_2 &= a_3 t_2 \\ &\vdots \\ a_m s_m &= a'\mu t_m. \end{aligned}$$

Define $\psi : F^{m+1} \to A'$ by $x\psi = a\mu,\ x_i\psi = a_i,\ \ 2 \le i \le m,\ \ x'\psi = a'\mu.$

Then $\rho_{\mathcal{S}} \subseteq \ker\psi$ so there exists $\nu = \overline{\psi} : F^{m+1}/\rho_{\mathcal{S}} \to A'$, given by $[k]\overline{\psi} = k\psi$. We have

$$u_{\mathcal{S}}\tau_{\mathcal{S}}\nu = [x]\overline{\psi} = x\psi = a\mu,\ \ u'_{\mathcal{S}}\tau_{\mathcal{S}}\nu = [x']\overline{\psi} = x'\psi = a'\mu$$

so that

$$W_{\mathcal{S}}\tau_{\mathcal{S}}\nu = ([x]S \cup [x']S)\overline{\psi} = a\mu S \cup a'\mu S \subseteq A\mu.$$

Thus we can see that Condition (Free) holds.

□

Let $\mathcal{C}$ be a class of embeddings of right S-acts, and let $\overline{\mathcal{C}}$ be the set of products of morphisms in $\mathcal{C}$ (with the obvious definition).

Lemma 2.1.1.6. *Let $\mathcal{C}$ be a class of embeddings of right S-acts, satisfying Condition (Free). If a left S-act B is $\mathcal{C}$-flat, then it is $\overline{\mathcal{C}}$-flat.*

Proof. Let I be an indexing set and let $\gamma_i : A_i \to A'_i \in \mathcal{C}$ for all $i \in I$. Let $A = \prod_{i\in I} A_i,\ A' = \prod_{i \in I} A'_i$ and let $\gamma : A \to A'$ be the canonical embedding.

Suppose B is a $\mathcal{C}$-flat left S-act, then for any $a_i,\ a'_i \in A_i,\ b, b' \in B$ if $a_i\gamma_i \otimes b = a'_i\gamma_i \otimes b'$ over $A'_i \otimes B$ then the equality $a_i \otimes b = a'_i \otimes b'$ also holds in $A_i \otimes B$.

Suppose that $\underline{a}\,,\ \underline{a}' \in A$ are such that $\underline{a}\gamma \otimes b = \underline{a}'\gamma \otimes b$ in $A' \otimes B$, where $\underline{a}\gamma = (a_i\gamma_i)$ and $\underline{a}'\gamma = (a'_i\gamma_i)$. Then

$$\begin{array}{rclrcl} & & & b &=& s_1 b_1 \\ \underline{a}\gamma s_1 &=& \underline{c_2} t_1 & t_1 b_1 &=& s_2 b_2 \\ \vdots & & & & \vdots & \\ \underline{c_n} s_n &=& \underline{a}'\gamma t_n & t_n b_n &=& b' \end{array}$$

where $\mathcal{S} = (s_1, t_1, \cdots, s_n, t_n) \in \mathbb{S}$ is a skeleton of length n with $\underline{c_2}, \cdots, \underline{c_n} \in A', b_1, \cdots, b_n \in B$. By assumption we will have $\tau = \tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}} \in \mathcal{C}$ and $u = u_{\mathcal{S}}, u' = u'_{\mathcal{S}} \in W_{\mathcal{S}}$ such that $\delta_{\mathcal{S}}(u\tau, u'\tau)$ is true in $W'_{\mathcal{S}}$. Notice that for each $i \in I$, as $\gamma_i : A_i \to A'_i \in \mathcal{C}$ and $\delta_{\mathcal{S}}(a_i\gamma_i, a'_i\gamma_i)$ is true in A'_i, there exists a morphism say $\nu_i : W'_{\mathcal{S}} \to A'_i$ such that $u\,\tau\nu_i = a_i\gamma_i,\ u'\tau\nu_i = a'_i\gamma_i$ and $W_{\mathcal{S}}\tau\nu_i \subseteq A_i\gamma_i$ for all i.

We have $\delta_{\mathcal{S}}(u\tau, u'\tau)$ is true in $W'_{\mathcal{S}}$ and $\gamma_{\mathcal{S}}(b, b')$ is true in B, so that $u\tau \otimes b = u'\tau \otimes b'$ in $W'_{\mathcal{S}} \otimes B$. As B is a $\mathcal{C}$-flat left S-act, and $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}} \in \mathcal{C}$, there exists a replacement tossing say

$$\begin{array}{rclrcl} & & & b & = & u_1w_1 \\ uu_1 & = & d_2v_1 & v_1w_1 & = & u_2w_2 \\ & \vdots & & & \vdots & \\ d_mu_m & = & u'v_m & v_mw_m & = & b' \end{array}$$

over $W_{\mathcal{S}}$ and B with replacement skeleton $\mathcal{S} = (u_1, v_1, \cdots, u_m, v_m)$ where each $d_i \in W_{\mathcal{S}}$ and each $w_j \in B$. So we will have for each $i \in I$,

$$\begin{array}{rclrcl} & & & b & = & u_1w_1 \\ u\tau\nu_iu_1 & = & d_2\tau\nu_iv_1 & v_1w_1 & = & u_2w_2 \\ & \vdots & & & \vdots & \\ d_m\tau\nu_iu_m & = & u'\tau\nu_iv_m & v_mw_m & = & b' \end{array}$$

and so as $W_{\mathcal{S}}\tau\nu_i \subseteq A_i\gamma_i$,

$$\begin{array}{rclrcl} & & & b & = & u_1w_1 \\ a_i\gamma_iu_1 & = & g_{2,i}\gamma_iv_1 & v_1w_1 & = & u_2w_2 \\ & \vdots & & & \vdots & \\ g_{m,i}\gamma_iu_m & = & a'_i\gamma_iv_m & v_mw_m & = & b'. \end{array}$$

where $g_{2,i}, \ldots, g_{m,i} \in A_i$ for $i \in I$. As each γ_i is an embedding we will have

$$\begin{array}{rclrcl} & & & b & = & u_1w_1 \\ a_iu_1 & = & g_{2,i}v_1 & v_1w_1 & = & u_2w_2 \\ & \vdots & & & \vdots & \\ g_{m,i}u_m & = & a'_iv_m & v_mw_m & = & b' \end{array}$$

so with $\underline{c_2} = (g_{2,i}), \cdots, \underline{c_m} = (g_{m,i})$ we will finally have

$$\begin{array}{rclrcl} & & & b & = & u_1w_1 \\ \underline{a}u_1 & = & \underline{c_2}v_1 & v_1w_1 & = & u_2w_2 \\ & \vdots & & & \vdots & \\ \underline{c_m}u_m & = & \underline{a}'v_m & v_mw_m & = & b' \end{array}$$

and hence $\underline{a} \otimes b = \underline{a}' \otimes b'$ over $A \otimes B$. $\square$

We now come to our first main result. The technique used is that of [6], but we are working in a more general context.

Theorem 2.1.1.7. *Let $\mathcal{C}$ be a class of embeddings of right S-acts satisfying Condition (Free). Then the following conditions are equivalent for a monoid S:*

(i) the class $\mathcal{CF}$ is axiomatisable;

(ii) the class $\mathcal{CF}$ is closed under formation of ultraproducts;

(iii) for every skeleton $\mathcal{S} \in \mathbb{S}$ there exist finitely many replacement skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(\mathcal{S})}$ such that, for any embedding $\gamma : A \to A'$ in $\mathcal{C}$ and any $\mathcal{C}$-flat left S-act B, if $(a\gamma, b)$, $(a'\gamma, b') \in A' \times B$ are connected by a tossing $\mathcal{T}$ over A' and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a', b') are connected by a tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(\mathcal{S})\}$;

(iv) for every skeleton $\mathcal{S} \in \mathbb{S}$ there exist finitely many replacement skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\beta(\mathcal{S})}$ such that, for any $\mathcal{C}$-flat left S-act B, if $(u_{\mathcal{S}}\tau_{\mathcal{S}}, b)$ and $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, b')$ are connected by a tossing $\mathcal{T}$ over $W'_{\mathcal{S}}$ and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $(u_{\mathcal{S}}, b)$, and $(u'_{\mathcal{S}}, b')$ are connected by a tossing $\mathcal{T}'$ over $W_{\mathcal{S}}$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \beta(\mathcal{S})\}$.

Proof. The implication (i) implies (ii) is clear from Łos's Theorem. To prove (ii) implies (iii), we suppose that $\mathcal{CF}$, the class of $\mathcal{C}$-flat left S-acts, is closed under formation of ultraproducts. We also assume that (iii) is false. Let J be the family of finite subsets of $\mathbb{S}$. We suppose that there exists a skeleton $\mathcal{S}_0 = (s_1, t_1, \cdots, s_m, t_m) \in \mathbb{S}$ such that for every subset f of J, there exists an embedding $\gamma_f : A_f \to A'_f \in \mathcal{C}$, a $\mathcal{C}$-flat left S-act B_f, and pairs $(a_f\gamma_f, b_f)$, $(a'_f\gamma_f, b'_f) \in A'_f \times B_f$ such that $(a_f\gamma_f, b_f)$ and $(a'_f\gamma_f, b'_f)$ are connected over A'_f and B_f by a tossing $\mathcal{T}_f$ with skeleton $\mathcal{S}_0$, but no replacement tossing over A_f and B_f connecting (a_f, b_f) and (a'_f, b'_f) has a skeleton belonging to the set f.

Let $J_{\mathcal{S}} = \{f \in J : \mathcal{S} \in f\}$ for each $\mathcal{S} \in \mathbb{S}$. Now we are able to define an ultrafilter Φ on J containing each $J_{\mathcal{S}}$ for all $\mathcal{S} \in \mathbb{S}$, as each intersection of finitely many of the sets $J_{\mathcal{S}}$ is non-empty.

We now define $A' = \prod_{f\in J} A'_f$, $A = \prod_{f\in J} A_f$ and $B = \prod_{f\in J} B_f$. Let $\gamma : A \to A'$ be the product embedding which is given by $(a_f)\gamma = (a_f\gamma_f)$. We note here that $\underline{a}\gamma \otimes \underline{b} = \underline{a}'\gamma \otimes \underline{b}'$ in $A' \otimes B$, where $\underline{a} = (a_f)$, $\underline{a}' = (a'_f)$, $\underline{b} = (b_f)$ and $\underline{b}' = (b'_f)$ and that this equality is determined by a tossing over A' and B (the " product" of the tossings $\mathcal{T}_f$'s) having skeleton $\mathcal{S}_0$. It follows that the equality for $\underline{a}\gamma \otimes \underline{b}_\Phi = \underline{a}'\gamma \otimes \underline{b}'_\Phi$ holds also in $A' \otimes \mathcal{U}$ where $\mathcal{U} = (\prod_{f\in J} B_f)/\Phi$, and can be determined by a tossing over A' and $\mathcal{U}$ with skeleton $\mathcal{S}_0$.

By assumption, $\mathcal{U}$ is $\mathcal{C}$-flat, and by Lemma 2.1.1.6 above $\underline{a} \otimes \underline{b}_\Phi = \underline{a}' \otimes \underline{b}'_\Phi$ in $A \otimes \mathcal{U}$. So there exists a replacement tossing $\mathcal{T}'$ over A and $\mathcal{U}$ with replacement skeleton $\mathcal{S}(\mathcal{T}') = \mathcal{S}' = (u_1, v_1, \cdots, u_n, v_n)$ connecting $\underline{a}\otimes\underline{b}_\Phi$ and $\underline{a}'\otimes\underline{b}'_\Phi$ and hence there exists $\underline{c_2}, \cdots, \underline{c_n} \in A$ and $\underline{b_1}_\Phi, \cdots, \underline{b_n}_\Phi \in \mathcal{U}$ and $u_1, v_1, \cdots, u_n, v_n \in S$ such that for

$$\begin{array}{rclrcl} & & & \underline{b}_\Phi & = & u_1\underline{b_1}_\Phi \\ \underline{a}u_1 & = & \underline{c_2}v_1 & v_1\underline{b_1}_\Phi & = & u_2\underline{b_2}_\Phi \\ \vdots & & & & \vdots & \\ \underline{c_n}u_n & = & \underline{a}'v_n & v_n\underline{b_n}_\Phi & = & \underline{b}'_\Phi. \end{array}$$

For $f \in J$ and $2 \le i \le n$, suppose that $\underline{c_i}(f) = c_{i,f} \in A_f$ and for $1 \le i \le n$ suppose that $\underline{b_i}(f) = b_{i,f} \in B_f$.

As Φ is closed under finite intersections, there exists $D \in \Phi$ such that

$$\begin{array}{rclrcl} & & & b_f & = & u_1b_{1,f} \\ a_fu_1 & = & c_{2,f}v_1 & v_1b_{1,f} & = & u_2b_{2,f} \\ c_{2,f}u_2 & = & c_{3,f}v_2 & v_2b_{2,f} & = & u_3b_{3,f} \\ \vdots & & & & \vdots & \\ c_{n,f}u_n & = & a'_fv_n & v_nb_{n,f} & = & b'_f \end{array}$$

whenever $f \in D$.

Now suppose that $f \in D \cap J_{\mathcal{S}'}$, then from the tossing just considered, we see that $\mathcal{S}'$ is the skeleton of a tossing over A_f and B_f connecting the pairs (a_f, b_f) and (a'_f, b'_f); that is, $\mathcal{S}'$ a replacement skeleton for skeleton $\mathcal{S}_0$ of the tossing $\mathcal{T}_f$. But $\mathcal{S}'$ belongs to f, a contradiction. This completes the proof that (ii) implies that (iii).

It is clear that (iii) implies that (iv).

Now we want to prove (iv) implies (i). We assume that (iv) holds. We aim to use this condition to construct a set of axioms for $\mathcal{CF}$.

Let $\mathbb{S}_1$ denote the set of all elements of $\mathbb{S}$ such that if $\mathcal{S} \in \mathbb{S}_1$, then there is no $\mathcal{C}$-flat left S-act B such that $\gamma_{\mathcal{S}}(b, b')$ is true for any $b, b' \in B$. For $\mathcal{S} \in \mathbb{S}_1$ we put

$$\psi_{\mathcal{S}} := (\forall x)(\forall x')\neg\gamma_{\mathcal{S}}(x, x')$$

For $\mathcal{S} \in \mathbb{S}_2 = \mathbb{S} \setminus \mathbb{S}_1$, there must be a $B \in \mathcal{CF}$ and $b, b' \in B$ such that $\gamma_{\mathcal{S}}(b, b')$ is true in B. As $\mathcal{C}$ satisfies Condition (Free), there exists an embedding $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}} \in \mathcal{C}$ such that $\delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$, whence there is a tossing from $(u_{\mathcal{S}}\tau_{\mathcal{S}}, b)$ to $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, b')$ over $W'_{\mathcal{S}}$ and B with skeleton $\mathcal{S}$.

Let $\mathcal{S}_1, \cdots, \mathcal{S}_{\beta(\mathcal{S})}$ be a minimum set of replacement skeletons for tossings with skeleton $\mathcal{S}$ connecting pairs of the form $(u_{\mathcal{S}}\tau_{\mathcal{S}}, c)$ to $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, c')$ where $c, c' \in C$ and C ranges over $\mathcal{CF}$. Hence for each $k \in \{1, \cdots, \beta(\mathcal{S})\}$ if

$$\mathcal{S}_k = (u_1, v_1, \cdots, u_{h_k}, v_{h_k})$$

there exists a $\mathcal{C}$-flat left S-act C_k, elements $c_k, c'_k \in C_k$ such that $\gamma_{\mathcal{S}_k}(c_k, c'_k)$ is true in C_k and $\delta_{\mathcal{S}_k}(u_{\mathcal{S}_k}, u'_{\mathcal{S}_k})$ is true in $W_{\mathcal{S}}$.

We define $\phi_{\mathcal{S}}$ to be the sentence

$$\phi_{\mathcal{S}} := (\forall y)(\forall y')\big(\gamma_{\mathcal{S}}(y, y') \to \gamma_{\mathcal{S}_1}(y, y') \vee \ldots \vee \gamma_{\mathcal{S}_{\beta(\mathcal{S})}}(y, y')\big).$$

Let

$$\sum\nolimits_{\mathcal{CF}} = \{\psi_{\mathcal{S}} : \mathcal{S} \in \mathbb{S}_1\} \cup \{\phi_{\mathcal{S}} : \mathcal{S} \in \mathbb{S}_2\}.$$

We claim that $\sum_{\mathcal{CF}}$ axiomatises $\mathcal{CF}$.

Suppose first that D is any $\mathcal{C}$-flat left S-act. By choice of $\mathbb{S}_1$, it is clear that $D \models \psi_{\mathcal{S}}$ for any $\mathcal{S} \in \mathbb{S}_1$.

Now take any $\mathcal{S} \in \mathbb{S}_2$, and suppose that $d, d' \in D$ are such that D satisfies $\gamma_{\mathcal{S}}(d, d')$. Then, as noted earlier $(u_{\mathcal{S}}\tau_{\mathcal{S}}, d)$ and $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, d')$ are joined over $W'_{\mathcal{S}}$ and D by a tossing with skeleton $\mathcal{S}$, and therefore, by assumption, there is a tossing over $W_{\mathcal{S}}$ and D joining $(u_{\mathcal{S}}, d)$ and $(u'_{\mathcal{S}}, d')$ with skeleton $\mathcal{S}_k$ for some $k \in \{1, \cdots, \beta(\mathcal{S})\}$. It is now clear that $\gamma_{\mathcal{S}_k}(d, d')$ holds in D, as required. We have now shown that $D \models \Sigma_{\mathcal{CF}}$.

Finally we show that a left S-act C that satisfies $\Sigma_{\mathcal{CF}}$ must be a $\mathcal{C}$-flat. We need to show that condition (iii) of Lemma 2.1.1.3 holds for C. Let

$$\mathcal{S} = (s_1, t_1, \ldots, s_m, t_m) \in \mathbb{S}$$

and suppose we have a tossing

$$\begin{array}{rclrcl} & & & c & = & s_1 c_1 \\ u_{\mathcal{S}}\tau_{\mathcal{S}} s_1 & = & w_2 t_1 & t_1 c_1 & = & s_2 c_2 \\ \vdots & & & & \vdots & \\ w_m s_m & = & u'_{\mathcal{S}}\tau_{\mathcal{S}} t_m & t_m c_m & = & c' \end{array}$$

over $W'_{\mathcal{S}}$ and C.

If $\mathcal{S}$ belonged to $\mathbb{S}_1$, then C will satisfy the sentence $(\forall y)(\forall y')\neg\gamma_{\mathcal{S}}(y, y')$ and so $\neg\gamma_{\mathcal{S}}(c, c')$ would hold, which is a contradiction as we have sequence of equalities in the right-hand side of the above tossing. Therefore we conclude that $\mathcal{S}$ belongs to $\mathbb{S}_2$. Because C satisfies $\phi_{\mathcal{S}}$ and because $\gamma_{\mathcal{S}}(c, c')$ holds, it follows that $\gamma_{\mathcal{S}_k}(c, c')$ holds for some $k \in \{1, 2, \cdots, \beta(\mathcal{S})\}$. Since $W_{\mathcal{S}}$ satisfies $\delta_{\mathcal{S}_k}(u_{\mathcal{S}}, u'_{\mathcal{S}})$, we have a tossing over $W_{\mathcal{S}}$ and C connecting $(u_{\mathcal{S}}, c)$ and $(u'_{\mathcal{S}}, c')$, showing that C is $\mathcal{C}$-flat.

□

We recall that the definition of a flat S-act is that it is $\mathcal{C}$-flat where $\mathcal{C}$ is the class of *all* embeddings of right S-acts. The class of all flat left S-acts is denoted by $\mathcal{F}$.

By Lemma 2.1.1.5, the class of all right S-acts has Condition (Free), so from Theorem 2.1.1.7, we immediately have the following corollary. Note the extra equivalent condition, to bring it into line with [6, Theorem 12].

Corollary 2.1.1.8. *[6] The following conditions are equivalent for a monoid S:*
(i) the class $\mathcal{F}$ is axiomatisable;
(ii) the class $\mathcal{F}$ is closed under formation of ultraproducts;
(iii) for every skeleton $\mathcal{S} \in \mathbb{S}$ there exist finitely many replacement skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(\mathcal{S})}$ such that, for any right S-act embedding $\gamma : A \to A'$, and any flat left S-act B, if $(a\gamma, b)$, $(a'\gamma, b') \in A' \times B$ are connected by a tossing $\mathcal{T}$ over A' and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a', b') are connected by a tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(\mathcal{S})\}$;
(iv) for every skeleton $\mathcal{S} \in \mathbb{S}$ there exist finitely many replacement skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(\mathcal{S})}$ such that, for any right S-act A and any flat left S-act B, if (a, b), $(a', b') \in A \times B$ are connected by a tossing $\mathcal{T}$ over A and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a', b') are connected by a tossing $\mathcal{T}'$ over $aS \cup a'S$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(\mathcal{S})\}$;
(v) for every skeleton $\mathcal{S} \in \mathbb{S}$ there exists finitely many replacement skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\beta(\mathcal{S})}$ such that, for any flat left S-act B, if $([x], b)$ and $([x'], b')$ are connected by a tossing $\mathcal{T}$ over $F^{m+1}/\rho_{\mathcal{S}}$ and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $([x], b)$, and $([x'], b')$ are connected by a tossing $\mathcal{T}'$ over $[x]S \cup [x']S$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \beta(\mathcal{S})\}$.

2.1.2 Axiomatisability of $\mathcal{CF}$ in general case

We continue to consider a class $\mathcal{C}$ of embeddings of right S-acts, but now drop our assumptions that Condition (Free) holds. The results and proofs of this section are analogous to those for weakly flat S-acts in [6]. Note that the conditions in (iii) below appear weaker than those in Theorem 2.1.1.7, as we are only asking that for specific elements a, a' and skeleton $\mathcal{S}$, there are finitely many replacement skeletons, in the sense made specific below.

Theorem 2.1.2.1. *Let $\mathcal{C}$ be a class of embeddings of right S-acts.*
The following conditions are equivalent:
(i) the class $\mathcal{CF}$ is axiomatisable;
(ii) the class $\mathcal{CF}$ is closed under ultraproducts;

(iii) for every skeleton $\mathcal{S}$ over S and $a, a' \in A$, where $\mu : A \to A'$ is in $\mathcal{C}$, there exist finitely many skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(a,\mathcal{S},a',\mu)}$, such that for any $\mathcal{C}$-flat left S-act B, if $(a\mu, b)$, $(a'\mu, b')$ are connected by a tossing $\mathcal{T}$ over A' and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a', b') are connected by a tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(a, \mathcal{S}, a', \mu)\}$.

Proof. The implication (i) implies (ii) is clear from Łos's Theorem.

To prove (ii) implies (iii), we suppose that $\mathcal{CF}$, the class of $\mathcal{C}$-flat left S-acts, is closed under formation of ultraproducts. We also assume that (iii) is false. Let J be the family of finite subsets of $\mathbb{S}$. We suppose that for some skeleton $\mathcal{S}_0 = (s_1, t_1, \cdots, s_m, t_m) \in \mathbb{S}$, for some embedding $\nu : A \to A' \in \mathcal{C}$, and $a, a' \in A$, then for every $f \in J$ there is a $\mathcal{C}$-flat left S-act B_f, and $b_f, b'_f \in B_f$ with the pairs $(a\mu, b_f)$, $(a'\mu, b'_f) \in A' \times B_f$ such that $(a\mu, b_f)$ and $(a'\mu, b'_f)$ are connected over A' and B_f by a tossing $\mathcal{T}_f$ with skeleton $\mathcal{S}_0$, but such that no replacement tossing over A and B_f connecting (a, b_f) and (a', b'_f) has a skeleton belonging to the set f.

Let $J_{\mathcal{S}} = \{f \in J : \mathcal{S} \in f\}$ for each $\mathcal{S} \in \mathbb{S}$. Now we are able to define an ultrafilter Φ on J containing each $J_{\mathcal{S}}$ for all $\mathcal{S} \in \mathbb{S}$, as each intersection of finitely many of the sets $J_{\mathcal{S}}$ is non-empty.

We note here that $a\mu \otimes \underline{b} = a^{'}\mu \otimes \underline{b}^{'}$ in $A' \otimes B$, where $B = \prod_{f \in J} B_f$ and $\underline{b} = (b_f)$ and $\underline{b}^{'} = (b^{'}_f)$ and that this equality is determined by a tossing over A' and B (the "product" of the tossings $\mathcal{T}_f$'s) having skeleton $\mathcal{S}_0$. It follows that the equality for $a\mu \otimes \underline{b}_\Phi = a^{'}\mu \otimes \underline{b}^{'}_\Phi$ holds also in $A' \otimes \mathcal{U}$ where $\mathcal{U} = (\prod_{f \in J} B_f)/\Phi$, and can be determined by a tossing over A' and $\mathcal{U}$ with skeleton $\mathcal{S}_0$.

By assumption $\mathcal{U}$ is $\mathcal{C}$-flat, so that $(a, \underline{b}_\Phi)$ and $(a', \underline{b}^{'}_\Phi)$ are connected via a replacement tossing $\mathcal{T}^{'}$ over A and $\mathcal{U}$, say

$$
\begin{array}{rclrcl}
 & & & \underline{b}_\Phi & = & u_1 \underline{d_1}_\Phi \\
au_1 & = & c_2 v_1 & v_1 \underline{d_1}_\Phi & = & u_2 \underline{d_2}_\Phi \\
c_2 u_2 & = & c_3 v_2 & v_2 \underline{d_2}_\Phi & = & u_3 \underline{d_3}_\Phi \\
 & \vdots & & & \vdots & \\
c_n u_n & = & a' v_n & v_n \underline{d_n}_\Phi & = & \underline{b}^{'}_\Phi
\end{array}
$$

where $\underline{d_i}(f) = d_{i,f}$ for any $f \in J$ and $i \in \{1, \cdots, n\}$. We put $\mathcal{S}^{'} = \mathcal{S}(\mathcal{T}')$.

As Φ is closed under finite intersections, there exists $D \in \Phi$ such that

$$
\begin{array}{rclrcl}
 & & & b_f & = & u_1 d_{1,f} \\
au_1 & = & c_2 v_1 & v_1 d_{1,f} & = & u_2 d_{2,f} \\
c_2 u_2 & = & c_3 v_2 & v_2 d_{2,f} & = & u_3 d_{3,f} \\
 & \vdots & & & \vdots & \\
c_n u_n & = & a' v_n & v_n d_{n,f} & = & b'_f
\end{array}
$$

whenever $f \in D$.

Now suppose that $f \in D \cap J_{\mathcal{S}'}$. Then from the tossing, just considered, we see that $\mathcal{S}^{'}$ is the skeleton of a tossing over A and B_f connecting the pairs (a, b_f) and (a', b'_f); that is, $\mathcal{S}^{'}$ is a replacement skeleton for skeleton $\mathcal{S}_0$ of the tossing $\mathcal{T}_f$. But $\mathcal{S}^{'}$ belongs to f, a contradiction. This completes the proof that (ii) implies that (iii).

Finally, suppose that (iii) holds. Let

$$\mathbb{T}' = \{(a, \mathcal{S}, a', \mu) : \mathcal{S} \in \mathbb{S}, \mu : A \to A' \in \mathcal{C}, a, a' \in A', \delta_{\mathcal{S}}(a\mu, a'\mu) \text{ holds}\}.$$

We introduce a sentence corresponding to elements of $\mathbb{T}'$ in such a way that the resulting set of sentences axiomatises the class $\mathcal{CF}$.

We let $\mathbb{T}_1$ be the set of $(a, \mathcal{S}, a', \mu) \in \mathbb{T}'$ such that $\gamma_{\mathcal{S}}(b, b')$ does not hold for any b, b' in any $\mathcal{C}$-flat left S-act B, and put $\mathbb{T}_2 = \mathbb{T}' \setminus \mathbb{T}_1$. For $T = (a, \mathcal{S}, a', \mu) \in \mathbb{T}_1$ we let

$$\psi_T = \psi_{\mathcal{S}} := (\forall x)(\forall x')\neg\gamma_{\mathcal{S}}(x, x').$$

If $T = (a, \mathcal{S}, a', \mu) \in \mathbb{T}_2$, then $\mathcal{S}$ is the skeleton of some scheme joining $(a\mu, b)$ to $(a'\mu, b')$ over A' and some $\mathcal{C}$-flat left S-act B. By our assumption (iii), there is a finite list of replacement skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(T)}$. Then, for each $k \in \{1, \cdots, \alpha(T)\}$, if $\mathcal{S}_k = (u_1, v_1, \cdots, u_{h_k}, v_{h_k})$, there exist a $\mathcal{C}$-flat left S-act C_k and elements $c_k, c_k^{'} \in C_k$ such that $\delta_{\mathcal{S}_k}(a, a')$ is true in A and $\gamma_{\mathcal{S}_k}(c_k, c_k^{'})$ is true in C_k. We let ϕ_T be the sentence

$$\phi_T := (\forall y)(\forall y')(\gamma_{\mathcal{S}}(y, y') \to \gamma_{\mathcal{S}_1}(y, y') \vee \cdots \vee \gamma_{\mathcal{S}_{\alpha(T)}}(y, y'))$$

Let

$$\sum\nolimits_{\mathcal{CF}} = \{\psi_T : T \in \mathbb{T}_1\} \cup \{\phi_T : T \in \mathbb{T}_2\}.$$

We claim that $\sum_{\mathcal{CF}}$ axiomatises $\mathcal{CF}$.

Suppose first that D is any $\mathcal{C}$-flat left S-act. Let $T = (a, \mathcal{S}, a', \mu) \in \mathbb{T}_1$. Then $\gamma_{\mathcal{S}}(b, b')$ is not true for any $b, b' \in B$, for any $\mathcal{C}$-flat left S-act B, so certainly $D \models \psi_T$.

On the other hand, for $T = (a, \mathcal{S}, a', \mu) \in \mathbb{T}_2$, if $d, d' \in D$ are such that $\gamma_{\mathcal{S}}(d, d')$ is true, together with the fact $\delta_{\mathcal{S}}(a\mu, a'\mu)$ holds, gives that $(a\mu, d)$ is connected to $(a'\mu, d')$ over A' and D via a tossing with skeleton $\mathcal{S}$. Because D is $\mathcal{C}$-flat, (a, d) and (a', d') are connected over A and D, and by assumption (iii), we can take the tossing to have skeleton one of $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(T)}$, say $\mathcal{S}_k$. Thus $D \models \gamma_{\mathcal{S}_k}(d, d^{'})$ and it follows that $D \models \phi_T$. Hence D is a model of $\sum_{\mathcal{CF}}$.

Conversely, we show that every model of $\sum_{\mathcal{CF}}$ is $\mathcal{C}$-flat. Let $C \models \sum_{\mathcal{CF}}$ and suppose that $\mu : A \to A' \in \mathcal{C}$, $a, a' \in A$, $c, c' \in C$ and we have a tossing

$$\begin{array}{rclrcl} & & & c & = & s_1c_1 \\ a\mu s_1 & = & w_2t_1 & t_1c_1 & = & s_2c_2 \\ w_2s_2 & = & w_3t_2 & t_2c_2 & = & s_3c_3 \\ & \vdots & & & \vdots & \\ w_ms_m & = & a'\mu t_m & t_mc_m & = & c' \end{array}$$

with skeleton $\mathcal{S} = (s_1, t_1, \cdots, s_mt_m)$ over A' and C. Then the quadruple $T = (a, \mathcal{S}, a', \mu) \in \mathbb{T}'$. Since $\gamma_{\mathcal{S}}(c, c')$ holds, C cannot be a model of ψ_T. Since $C \models \sum_{\mathcal{CF}}$ it follows that $T \in \mathbb{T}_2$. But then ϕ_T holds in C so that for some $k \in \{1, \cdots, \alpha(T)\}$ we have that $\gamma_{\mathcal{S}_k}(c, c')$ is true. As $\delta_{\mathcal{S}_k}(a, a')$ is true in A, we have tossing over A and C connecting (a, c) to (a', c'). Thus C is $\mathcal{C}$-flat.

□

We now explain why the axiomatisability of weakly flat S-acts as given in [6] then becomes a special case. We recall that a left S-act B is called *weakly flat* if the functor $- \otimes B$ maps inclusions of right ideals in the category of **S-Act** to one-one maps in the category of **Set**. So, B is weakly flat if it is $\mathcal{C}$-flat where $\mathcal{C}$ is the class of all inclusions of right ideals of S into S. The class of weakly flat left S-acts is denoted by $\mathcal{WF}$. In our Corollary, we do not need to mention the embeddings μ, since they are all inclusion maps of right ideals into S.

Corollary 2.1.2.2. *[6, Theorem 13] The following are equivalent for a monoid S:*
(i) the class $\mathcal{WF}$ is axiomatisable;
(ii) the class $\mathcal{WF}$ is closed under ultraproducts;
(iii) for every skeleton $\mathcal{S}$ over S and $a, a' \in S$ there exists finitely many skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\beta(a,\mathcal{S},a')}$ over S, such that for any weakly flat left S-act B, if (a,b), $(a',b') \in S \times B$ are connected by a tossing $\mathcal{T}$ over S and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a,b) and (a',b') are connected by a tossing $\mathcal{T}'$ over $aS \cup a'S$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$ for some $k \in \{1, \cdots, \beta(a,\mathcal{S},a')\}$.

We say that a left S-act B is *principally weakly flat* if it is $\mathcal{C}$-flat where $\mathcal{C}$ is the set of all inclusions of principal right ideals of S into S. We end this section by considering the axiomatisability of principally weakly flat S-acts. We first remark that if aS is a principal right ideal of S and B is a left S-act, then

$$au \otimes b = av \otimes b' \text{ in } aS \otimes B \text{ if and only if } a \otimes ub = a \otimes vb' \text{ in } aS \otimes B$$

with a similar statement for $S \otimes B$. Thus B is principally weakly flat if and only if for all $a \in S$, if $a \otimes b = a \otimes b'$ in $S \otimes B$, then $a \otimes b = a \otimes b'$ in $aS \otimes B$.

Our next result follows from Theorem 2.1.2.1 and its proof.

Corollary 2.1.2.3. *The following conditions are equivalent for a monoid $\mathcal{S}$:*
(i) the class $\mathcal{PWF}$ is axiomatisable;
(ii) the class $\mathcal{PWF}$ is closed under ultraproducts;
(iii) for every skeleton $\mathcal{S}$ over S and $a \in S$ there exists finitely many skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\tau(a,\mathcal{S})}$ over S, such that for any principally weakly flat left S-act B, if (a,b), $(a,b') \in S \otimes B$ are connected by a tossing $\mathcal{T}$ over S and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a,b) and (a,b') are connected by a tossing $\mathcal{T}'$ over aS and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$ for some $k \in \{1, \cdots, \tau(a,\mathcal{S})\}$.

2.2 Axiomatisability of specific classes of S-acts

We now examine specific classes of S-acts which can be axiomatisable by various techniques. Axiomatisability of classes $\mathcal{E}$ and $\mathcal{P}$ using the "elements" method are given in [25], we will be discussing axiomatisability of these classes by using "replacement tossings" methods here.

To axiomatise classes such as $\mathcal{EP}$, $\mathcal{W}$, $\mathcal{PWP}$ we use both methods of proof, i.e. "elements" and "replacement tossings" methods.

2.2.1 Axiomatisability of Condition (P) for S-acts

We recall the definition of Condition (P), as follows:

Definition 2.2.1.1. A left S-act A satisfies *Condition (P)* if for any $s, s' \in S$ and $a, a' \in A$, if $s\,a = s'\,a'$ then there exists $a'' \in A$, u, $u' \in S$ such that $a = u\,a''$, $a' = u'\,a''$ and $s\,u = s'\,u'$.

Let S be a monoid. For any $s, t \in S$ we put

$$\mathbf{R}(s,t) = \{(u,v) \in S \times S : su = tv\}$$

and notice that $\mathbf{R} = \emptyset$ or is an S-subact of $S \times S$.

The following result is implicit in [25] and made explicit in [30].

Theorem 2.2.1.2. *[25, 30] The following conditions are equivalent for a monoid S:*
(i) the class $\mathcal{P}$ is axiomatisable;
(ii) the class $\mathcal{P}$ is closed under ultraproducts;
(iii) the class $\mathcal{P}$ is closed under ultrapowers;
(iv) every ultrapower of S lies in $\mathcal{P}$;
(v) for any $s, t \in S$, $\mathbf{R}(s,t) = \emptyset$ or is finitely generated.

We now rephrase the above in terms of replacement tossings.

Remark 2.2.1.3. Observe that if $sa = tb$ for some $s, t \in S, a, b \in B$, then

$$\begin{array}{rcl rcl} & & & a & = & 1\,a \\ s\,1 & = & 1\,s & s\,a & = & t\,b \\ 1\,t & = & t\,1 & 1\,b & = & b \end{array}$$

so (s, a), (t, b) are connected via a tossing of length 2 over S and B with skeleton $(1,\, s,\, t,\, 1)$.

Conversely if (s, a), (t, b) are connected with skeleton $(1,\, s,\, t,\, 1)$ in the way

$$\begin{array}{rcl rcl} & & & a & = & 1\,b_1 \\ s\,1 & = & a_2 s & s\,b_1 & = & t\,b_2 \\ a_2\,t & = & t\,1 & 1\,b_2 & = & b \end{array}$$

then $sa = tb$.

Remark 2.2.1.4. Suppose $su = tv$, $a = uc, b = vc$ then

$$\begin{array}{rcl rcl} & & & a & = & u\,c \\ su & = & tv & vc & = & b \end{array}$$

is a length 1 tossing connecting (s,a) to (t,b) over S and B or over $sS \cup tS$ and B with skeleton (u,v).

Conversely if there exists a length 1 tossing connecting (s,a) to (t,b) over S and B with skeleton (u,v) it must look like

$$\begin{array}{rclrcl} & & & a & = & u\,b_1 \\ su & = & tv & vb_1 & = & b. \end{array}$$

so $(u,v) \in \mathbf{R}(s,t)$.

Corollary 2.2.1.5. *The following conditions are equivalent for a monoid S:*

(i) the class $\mathcal{P}$ is axiomatisable;

(ii) for every skeleton $\mathcal{S} = (1,\, s,\, t,\, 1)$ over S, there exists finitely many replacement skeletons $\mathcal{S}_1 = (u_1, v_1), \cdots, \mathcal{S}_{n(\mathcal{S})} = (u_{n(\mathcal{S})}, v_{n(\mathcal{S})})$ of length one such that for any $a, b \in B \in \mathcal{P}$ and $sa = tb$ (equivalently, (s,a) is connected to (t,b) via a tossing with skeleton $\mathcal{S}$), then (s,a) is connected to (t,b) via a replacement tossing with skeleton $\mathcal{S}_i$, for some $1 \leq i \leq n(\mathcal{S})$.

Proof. Suppose that (i) holds. Let $\mathcal{S} = (1,s,t,1)$ be a skeleton. From Theorem 2.2.1.2, $\mathbf{R}(s,t) = \emptyset$ or $\mathbf{R}(s,t)$ is finitely generated. In the first case, set $n(\mathcal{S}) = 0$ and in the second, suppose that

$$\mathbf{R}(s,t) = \bigcup_{i=1}^{i=n} (u_i, v_i)S.$$

Put $n(\mathcal{S}) = n$ and let $\mathcal{S}_i = (u_i, v_i)$ for $1 \leq i \leq n$.

Let $B \in \mathcal{P}$ and suppose that $sa = tb$ for some $s,t \in S$ and $a,b \in B$. Then $ss' = tt'$ and $a = s'c$, $b = t'c$ for some $s', t' \in S$ and $c \in B$. But then $(s',t') = (u_i, v_i)r$ for some $i \in \{1, \ldots, n\}$ and $r \in S$, so that $a = u_i d, b = v_i d$ for some $d = rc \in B$ and (u_i, v_i) is the skeleton of a replacement tossing. Hence (ii) holds.

Conversely, suppose that (ii) holds. If $\mathbf{R}(s,t) \neq \emptyset$, let $(u,v) \in \mathbf{R}(s,t)$. Then $su = tv$ and as $S \in \mathcal{P}$ we have that there is a replacement tossing with skeleton (u_i, v_i) connecting (s,u) to (t,v). Perforce we have that $(u_i, v_i) \in \mathbf{R}(s,t)$, $u = u_i c, v = v_i c$ so that $(u,v) = (u_i, v_i)c$ for some $c \in S$. It follows that $\mathbf{R}(s,t)$ is finitely generated. By Theorem 2.2.1.2, $\mathcal{P}$ is axiomatisable. □

2.2.2 Axiomatisability of Condition (E) for S-acts

We recall the definition of Condition (E) as follows:

Definition 2.2.2.1. A left S-act A satisfies *Condition (E)* if $s\,a = s'\,a$ then there exists $a' \in A$, $u \in S$ such that $a = u\,a'$ with $s\,u = s'\,u$.

Let S be a monoid. For any $s,t \in S$ we put

$$\mathbf{r}(s,t) = \{u \in S : su = tu\}$$

and notice that $\mathbf{r}(s,t) = \emptyset$ or is a right ideal of S.

The following result is given in [25, 30].

Theorem 2.2.2.2. *[25, 30] The following conditions are equivalent for a monoid S:*
(i) the class $\mathcal{E}$ is axiomatisable;
(ii) the class $\mathcal{E}$ is closed under ultraproducts;
(iii) the class $\mathcal{E}$ is closed under ultrapowers;
(iv) every ultrapower of S lies in $\mathcal{E}$;
(v) for any $s, t \in S$, $\mathbf{r}(s,t) = \emptyset$ or is finitely generated.

We now rephrase the above in terms of replacement tossings.

Remark 2.2.2.3. Suppose $su = tu$, $a = uc$ then

$$\begin{array}{ccccc} & & a & = & u\,c \\ su & = & tu \quad uc & = & a \end{array}$$

is a length 1 tossing connecting (s,a) to (t,a) over S and B with skeleton (u,u). We will say a skeleton of the form (u,u) a *trivial* skeleton.

Conversely if there exists length 1 tossing connecting (s,a) to (t,a) over S and B by a trivial skeleton (s_1, s_1) it must look like

$$\begin{array}{ccccc} & & a & = & s_1\,b_1 \\ ss_1 & = & ts_1 \quad s_1b_1 & = & a \end{array}$$

notice that $s_1 \in \mathbf{r}(s,t)$.

Corollary 2.2.2.4. *The following conditions are equivalent for a monoid S:*
(i) the class $\mathcal{E}$ is axiomatisable;
(ii) for every skeleton $\mathcal{S} = (1,\ s,\ t,\ 1)$ over S, there exists finitely many trivial replacement skeletons $\mathcal{S}_1 = (u_1, u_1), \cdots, \mathcal{S}_{m(\mathcal{S})} = (u_{m(\mathcal{S})}, u_{m(\mathcal{S})})$ such that for any $a \in B \in \mathcal{E}$ and $sa = ta$ (equivalently, (s,a) is connected to (t,a) via a tossing with skeleton $\mathcal{S}$), then (s,a) is connected to (t,a) via a replacement tossing with trivial skeleton $\mathcal{S}_i$, $1 \leq i \leq m(\mathcal{S})$.

Proof. Suppose that (i) holds. Let $\mathcal{S} = (1, s, t, 1)$ be a skeleton. From Theorem 2.2.2.2, $\mathbf{r}(s,t) = \emptyset$ or $\mathbf{r}(s,t)$ is finitely generated right ideal of S. In the first case, set $m(\mathcal{S}) = 0$ and in the second, suppose that

$$r(s,t) = \bigcup_{i=1}^{i=m} u_i S.$$

Put $m(\mathcal{S}) = m$ and let $\mathcal{S}_i = (u_i, u_i)S$ for $1 \leq i \leq m$.

Let $A \in \mathcal{E}$ and suppose that $sa = ta$ for some $s, t \in S$ and $a \in A$. Then $ss' = ts'$ and $a = s'c$ for some $s' \in S$ and $c \in A$. But then $s' = u_i r$ for some $i \in \{1, \ldots, m\}$ and $r \in S$, so that $a = u_i d$ for some $d = rc \in A$, and (u_i, u_i) is the trivial skeleton of a replacement tossing connecting (s,a) to (t,a). Hence (ii) holds.

Conversely, suppose that (ii) holds. If $\mathbf{r}(s,t) \neq \emptyset$, let $u \in \mathbf{r}(s,t)$. Then $su = tu$ and as $S \in \mathcal{E}$ we have that there is a replacement tossing with trivial skeleton (u_i, u_i) connecting (s,u) to (t,u). We have that $u_i \in \mathbf{r}(s,t)$ and $u = u_i c$ which gives that $\mathbf{r}(s,t)$ is finitely generated. By Theorem 2.2.2.2, $\mathcal{E}$ is axiomatisable. $\square$

2.2.3 Axiomatisability of Condition (EP)

In [22] Golchin and Mohammadzadeh defined a new flatness property of acts over monoids which is an extended version of Conditions (E) and (P).

Definition 2.2.3.1. A left S-act A satisfies Condition (EP) if whenever $s\,a = t\,a$ for some $s, t \in S$ and $a \in A$, then there exists $a'' \in A$, $u, v \in S$ such that $a = u\,a'' = v\,a''$ with $s\,u = t\,v$. We will denote the class of left S-acts satisfying Condition (EP), by $\mathcal{EP}$.

Remark 2.2.3.2. [22] Condition (E) implies Condition (EP) and Condition (P) implies Condition (EP) but neither converse is true.

Theorem 2.2.3.3. *The following conditions are equivalent for a monoid S:*
(i) the class $\mathcal{EP}$ is axiomatisable;
(ii) $\mathcal{EP}$ is closed under ultraproducts;
(iii) for any $s, t \in S$ either $sa \neq ta$ for all $a \in A \in \mathcal{EP}$ or there exists $f \subseteq \mathbf{R}(s,t)$, f is finite, such that if

$$sa = ta,\ a \in A \in \mathcal{EP} \text{ then } (a, a) = (u, v)b$$

for some $(u, v) \in f$ and $b \in A$.

Proof. (i) implies (ii): this follows from Łos's Theorem.

(ii) implies (iii): suppose for each finite subset f of $\mathbf{R}(s,t)$, there exists $A_f \in \mathcal{EP}$, $a_f \in A_f$ with $sa_f = ta_f$ and $(a_f, a_f) \notin f\,A_f$. Let J be the set of finite subsets of $\mathbf{R}(s,t)$. For each $(u, v) \in \mathbf{R}(s,t)$ we define

$$J_{(u,v)} = \{f \in J : (u, v) \in f\}.$$

As each intersection of finitely many of the sets $J_{(u,v)}$ is non-empty, we are able to define an ultrafilter Φ on J, such that each $J_{(u,v)} \in \Phi$ for all $(u, v) \in \mathbf{R}(s,t)$.

Now $s\underline{a} = t\underline{a}$ in A where $A = \prod_{f \in J} A_f$ and $\underline{a} = (a_f)$; this equality is determined by a product of the elements $sa_f = ta_f$ with $a_f \in A_f$, for each $f \in J$. It follows that this equality $s\underline{a}_\Phi = t\underline{a}_\Phi$ also holds in $\mathcal{U}$ where $\mathcal{U} = \prod_{f \in J} A_f / \Phi$; by assumption $\mathcal{U}$ has Condition (EP), so there exists $u, v \in S$, and $\underline{r}_\Phi = (r_f)_\Phi \in \mathcal{U}$ such that

$$\underline{a}_\Phi = u\underline{r}_\Phi = v\underline{r}_\Phi,\ su = tv.$$

As Φ is closed under finite intersections, there must exist $T \in \Phi$ such that $a_f = ur_f = vr_f$ for all $f \in T$.

Now suppose that $f \in T \cap J_{(u,v)}$, then $(u, v) \in f$ and

$$(a_f, a_f) = (u, v)r_f \in fA_f$$

a contradiction to our assumption, hence (ii) implies (iii).

(iii) implies (i): we will show that the class of left S-acts satisfying Condition (EP) is axiomatisable by giving explicitly a set of sentences axiomatising it.

For any element $\rho = (s,t) \in S \times S$ with $sa = ta$, for some $a \in A$ where $A \in \mathcal{EP}$, we choose and fix a finite set of elements $\{(u_{\rho,1}, v_{\rho,1}) \cdots (u_{\rho,n(\rho)}, v_{\rho,n(\rho)})\}$ of $\mathbf{R}(\rho)$ as guaranteed by (iii). We define sentences ϕ_ρ of L_s as follows:

If $sa \neq ta$ for all $a \in A \in \mathcal{EP}$, let

$$\phi_\rho = (\forall x)(sx \neq tx);$$

otherwise,

$$\phi_\rho = (\forall x)\big(sx = tx \to (\exists z)(\bigvee_{i=1}^{n(\rho)} (x = u_{\rho,i} z \ = v_{\rho,i} z))\big).$$

Let

$$\sum\nolimits_{\mathcal{EP}} = \{\phi_\rho : \rho \in S \times S\}.$$

We claim that $\sum_{\mathcal{EP}}$ axiomatises the class $\mathcal{EP}$.

Suppose that A is an S-act satisfying Condition (EP) and $\rho \in S \times S$, where $\rho = (s,t)$. If $sb \neq tb$, for all $b \in B \in \mathcal{EP}$, then certainly this is true for A, so that $A \models \phi_\rho$.

On the other hand if $sb = tb$ for some $b \in B \in \mathcal{EP}$ we have

$$\phi_\rho = (\forall x)\big(sx = tx \to (\exists z)(\bigvee_{i=1}^{n(\rho)} (x = u_{\rho,i} z \ = v_{\rho,i} z))\big).$$

Suppose $sa = ta$ where $a \in A$ then we must have

$$\phi_\rho = (\forall x)\big(sx = tx \to (\exists z)(\bigvee_{i=1}^{n(\rho)} (x = u_{\rho,i} z \ = v_{\rho,i} z))\big).$$

By (iii), $(a,a) = (u_{\rho,i}, v_{\rho,i})c$ for some $i \in \{1, 2, \ldots, n(\rho)\}$ and $c \in A$. Hence $A \models \phi_\rho$.

Conversely if A is a model of $\sum_{\mathcal{EP}}$ and if $s\,a = t\,a$ where $s,t \in S$ and $a \in A$, we cannot have that ϕ_ρ is $(\forall x)(sx \neq tx)$. It follows that for some $b \in B \in \mathcal{EP}$ we have $sb = tb$ and $f = \{(u_{\rho,1}, v_{\rho,1}), \cdots, (u_{\rho,n(\rho)}, v_{\rho,n(\rho)})\}$ exists as in (iii) and ϕ_ρ is

$$(\forall x)\big(sx \ = \ tx \ \to \ (\exists\, z)(\bigvee_{i=1}^{n(\rho)} (\, x \ = \ u_{\rho,i} z \ = \ v_{\rho,i}\, z))\big).$$

Hence there exists an element $c \in A$ with $a = u_{\rho,i}\, c = v_{\rho,i}\, c$ for some $i \in \{1, 2, \ldots, n(\rho)\}$. By definition of $u_{\rho,i}, v_{\rho,i}$ we have $su_{\rho,i} \ = tv_{\rho,i}$. Thus A satisfies Condition (EP) and so $\sum_{\mathcal{EP}}$ axiomatises $\mathcal{EP}$. □

Remark 2.2.3.4. Note that for any $a \in A \in \mathcal{EP}$, if $sa = sa$ then certainly $(a,a) = (1,1)a$ and $(1,1) \in R(s,s)$. So that to check the condition (iii) of Theorem 2.2.3.3 holds, it is enough to consider the cases where $s \neq t$.

If S is a monoid such that $\mathbf{R}(s,t)$ is finitely generated for all $s,t \in S$ with $s \neq t$, then $\mathcal{EP}$ is axiomatisable. To see this let S be a monoid such that $\mathbf{R}(s,t)$ is finitely generated, let $sa = ta$ for some $a \in A \in \mathcal{EP}$, then $a = ua' = va'$ for some $u, v \in S$ and $a' \in A$ with $su = tv$, so that $(u,v) = (u_i, v_i)t$ for some $t \in S$ and $i \in \{1, \cdots, n\}$. Now $a = u_i a'' = v_i a''$

where $a'' = ta'$. We can therefore choose $f = \{(u_1, v_1), \cdots, (u_n, v_n)\}$, and Condition (iii) of Theorem 2.2.3.3 satisfied.

We can conclude that if $\mathcal{P}$ is axiomatisable, so is $\mathcal{EP}$.

Remark 2.2.3.5. Suppose $su = tv$, $a = ua'' = va''$ then

$$\begin{array}{ccccc} & & a & = & u\,a'' \\ su & = & tv & va'' = & a \end{array}$$

is a length 1 tossing connecting (s, a) to (t, a) over S and B with skeleton (u, v).

Conversely if there exists length 1 tossing connecting (s, a) to (t, a) over S and B by a skeleton $\mathcal{S} = (s_1, t_1)$ it must look like

$$\begin{array}{ccccc} & & a & = & s_1\,a_1 \\ ss_1 & = & tt_1 & t_1a_1 = & a \end{array}$$

so that $(s_1, t_1) \in \mathbf{R}(s, t)$.

Remark 2.2.3.6. From Remark 2.2.1.3 it is obvious that $sa = ta$ if and only if (s, a) connected to (t, a) over S and B via a tossing of length 2 with skeleton $(1, s, t, 1)$.

Corollary 2.2.3.7. *The following conditions are equivalent for a monoid S:*

(i) the class $\mathcal{EP}$ is axiomatisable;

(ii) for every skeleton $\mathcal{S} = (1, s, t, 1)$ over S, there exists finitely many replacement skeletons $\mathcal{S}_1 = (u_1, v_1), \cdots, \mathcal{S}_{p(\mathcal{S})} = (u_{p(\mathcal{S})}, v_{p(\mathcal{S})})$ such that for any $a \in B \in \mathcal{EP}$ and $sa = ta$ (equivalently, (s, a) is connected to (t, a) via a tossing with skeleton $\mathcal{S}$), then (s, a) is connected to (t, a) via a replacement tossing of skeleton $\mathcal{S}_i$, $1 \leq i \leq p(\mathcal{S})$.

Proof. We follow the similar arguments as given in the proof of Corollaries 2.2.1.5 and 2.2.2.4. □

2.2.4 Axiomatisability of Condition (W) for S-acts

In [39] Bulman-Fleming and McDowell introduced an interpolation type condition called Condition (W). We will describe the condition on a monoid S such that $\mathcal{W}$ is axiomatisable.

We remind the reader of the following definition:

Definition 2.2.4.1. A left S-act A satisfies *Condition (W)*, if whenever $sa = ta'$ for $a, a' \in A$, $s, t \in S$, then there exists $a'' \in A$, $u \in sS \cap tS$, such that $sa = ta' = ua''$. We will denote the class of left S-acts satisfying Condition (W) as $\mathcal{W}$.

Remark 2.2.4.2. The monoid S satisfies Condition (W) as an S-act.

Theorem 2.2.4.3. *The following conditions are equivalent for a monoid S:*

(i) the class $\mathcal{W}$ is axiomatisable;

(ii) $\mathcal{W}$ is closed under ultraproducts;

(iii) $\mathcal{W}$ is closed under ultrapowers;

(iv) every ultrapower of S lies in $\mathcal{W}$;

(v) for any $s, t \in S$, $sS \cap tS = \emptyset$ or $sS \cap tS$ is finitely generated as a right ideal of S.

Proof. (i) implies (ii): this follows from Łos's Theorem; (ii) implies (iii) is clear and (iii) implies (iv) is obvious as S satisfies Condition (W) as an S-act by Remark 2.2.4.2.

(iv) implies (v): let $s, t \in S$ and suppose that $sS \cap tS \neq \emptyset$. Clearly $sS \cap tS$ is a right ideal of S and so in particular is a right S-act. We suppose that $sS \cap tS$ is not finitely generated. Let $\{u_\beta : \beta < \gamma\}$ be a generating subset of $sS \cap tS$ of cardinality γ, where $u_\beta = sx_\beta = ty_\beta$ for some x_β, y_β in S.

By assumption γ is a limit ordinal. We may suppose that for any $\beta < \gamma$, u_β is not in the right ideal generated by the preceding elements u_τ that is $u_\beta \notin \bigcup_{\tau<\beta} u_\tau S$.

Let Φ be a uniform ultrafilter on γ, that is, Φ is an ultrafilter on γ such that all sets in Φ have the same cardinality γ. Let $\mathcal{U} = S^\gamma/\Phi$. By assumption, $\mathcal{U}$ satisfies Condition (W) as an S-act.

Define elements $\underline{a} = (x_\beta)$ and $\underline{b} = (y_\beta)$ and consider $\underline{a}_\Phi$, $\underline{b}_\Phi \in \mathcal{U}$. Since $sx_\beta = u_\beta = ty_\beta$ for all $\beta < \gamma$, clearly $s\underline{a}_\Phi = t\underline{b}_\Phi$. By assumption $\mathcal{U}$ satisfies Condition (W) so there exists $\underline{c}_\Phi \in \mathcal{U}$ and $u \in sS \cap tS$ such that $s\underline{a}_\Phi = t\underline{b}_\Phi = u\,\underline{c}_\Phi$. Let $\underline{c}_\Phi = (z_\beta)_\Phi$ so there exists sets $T_1, T_2 \in \Phi$ such that $s\,x_\beta = u\,z_\beta$ for all $\beta \in T_1$ and $t\,y_\beta = u\,z_\beta$ for all $\beta \in T_2$. Since $u \in sS \cap tS$ there exists $\sigma < \gamma$ and $h \in S$ with $u = u_\sigma\,h$. Using the fact that $T_1 \cap T_2 \in \Phi$ and Φ is a uniform ultrafilter, $T_1 \cap T_2$ contains an ordinal say $\alpha \geq \sigma + 1$. Then

$$u_\alpha = s\,x_\alpha = t\,y_\alpha = u\,z_\alpha = u_\sigma\,h\,z_\alpha$$

and so $u_\alpha \in u_\sigma S$, a contradiction. Thus $sS \cap tS$ is finitely generated.

(v) implies (i): we show that the class of S-acts satisfying Condition (W) is axiomatisable by giving explicitly a set of sentences that axiomatises $\mathcal{W}$.

For any element $\rho = (s, t)$ of $S \times S$ with $sS \cap tS \neq \emptyset$, we choose and fix a finite set of generators $\{u_{\rho,1}, \cdots, u_{\rho,n(\rho)}\}$ of $sS \cap tS$. For $s, t \in S$ we define sentences Υ_ρ , as follows:

If $sS \cap tS = \emptyset$ then

$$\Upsilon_\rho := (\forall x)(\forall y)(sx \neq ty);$$

if $sS \cap tS \neq \emptyset$ then

$$\Upsilon_\rho := (\forall x)(\forall y)\Big(sx = ty \to (\exists z)(\bigvee_{i=1}^{n(\rho)} sx = ty = u_{\rho,i}\,z)\Big).$$

Let

$$\sum\nolimits_{\mathcal{W}} = \{\Upsilon_\rho : \rho \in S \times S\}.$$

We claim that $\Sigma_\mathcal{W}$ axiomatises $\mathcal{W}$.

Suppose that A is a S-act satisfying Condition (W) and $\rho \in S \times S$, where $\rho = (s, t)$. If $sS \cap tS = \emptyset$ and there exists $a, b \in A$ such that $sa = tb$, then since A satisfies Condition (W), there exists $u \in sS \cap tS$ (such that $sa = tb = uc$ for some $c \in A$), a contradiction. Thus $A \models \Upsilon_\rho$.

If $sS \cap tS \neq \emptyset$ and $sa = tb$ where $a, b \in A$ then again using the fact that A satisfies Condition (W) there are elements $u \in sS \cap tS$ and $a' \in A$ such that $sa = tb = ua'$. Now $u \in sS \cap tS$ and so $u = u_{\rho,i}h$ for some $i \in \{1, 2, \cdots, n(\rho)\}$ and $h \in S$. Thus $sa = tb = u_{\rho,i}ha'$, where $ha' \in A$. Hence $A \models \Upsilon_\rho$.

Conversely if A is a model of $\Sigma_\mathcal{W}$ and if $sa = tb$ where $s, t \in S$ and $a, b \in A$, then since $A \models \Upsilon_\rho$, where $\rho = (s, t)$ it follows that $sS \cap tS$ cannot be empty and Υ_ρ is

$$(\forall x)(\forall y)\big(sx = ty \to (\exists z)(\bigvee_{i=1}^{n(\rho)} sx = ty = u_{\rho,i}z)\big).$$

Hence there exists an element $c \in A$ such that $sa = tb = u_{\rho,i}c$ for some $i \in \{1, 2, \cdots, n(\rho)\}$. By definition of $u_{\rho,i}$ we have $u_{\rho,i} \in sS \cap tS$. Thus A satisfies Condition (W) and so $\Sigma_{\mathcal{W}}$ axiomatises $\mathcal{W}$.

$\square$

We now explain the axiomatisability of Condition (W) in terms of replacement skeletons.

Remark 2.2.4.4. Observe that if $sa = tb = ua'$ for some $s, t, u \in S, a, b, a' \in B$, then

$$\begin{array}{rclrcl} & & & a & = & 1\,a \\ s\,1 & = & 1\,s & sa & = & ua' \\ 1\,u & = & 1\,u & ua' & = & tb \\ 1\,t & = & t\,1 & 1\,b & = & b \end{array}$$

so (s, a), (t, b) are connected via a tossing of length 3 over S and B with skeleton $(1, s, u, u, t, 1)$.

Conversely if (s, a) and (t, b) are connected via a tossing with skeleton $(1, s, u, u, t, 1)$, we have

$$\begin{array}{rclrcl} & & & a & = & 1\,b_1 \\ s\,1 & = & a_2 s & sb_1 & = & ub_2 \\ a_2 u & = & a_3 u & ub_2 & = & tb_3 \\ a_3 t & = & t\,1 & 1\,b_3 & = & b, \end{array}$$

then $sa = tb = ub_2$ for some $b_2 \in B$.

Corollary 2.2.4.5. *The following conditions are equivalent for a monoid S:*

(i) the class $\mathcal{W}$ is axiomatisable;

(ii) for every skeleton $\mathcal{S} = (1,\ s,\ t,\ 1)$ over S, there exists finitely many replacement skeletons

$$\mathcal{S}_1 = (1, s, u_1, u_1, t, 1), \ldots, \mathcal{S}_{n(\mathcal{S})} = (1, s, u_{n(\mathcal{S})}, u_{n(\mathcal{S})}, t, 1)$$

where $u_i \in sS \cap tS$, such that for any $a, b \in B \in \mathcal{W}$ and $sa = tb$ (equivalently, (s, a) connected with (t, b) via a tossing over S and B with skeleton $\mathcal{S}$), then (s, a) is connected to (t, b) via a tossing over S and B with skeleton $\mathcal{S}_i$(equivalently, $sa = tb = u_i d$ for some $d \in B$), for some $1 \le i \le n$.

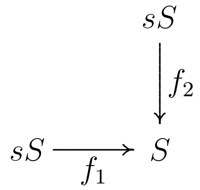
Proof. Suppose that $\mathcal{W}$ is axiomatisable. Let $\mathcal{S} = (1, s, t, 1)$ be a skeleton. If $sS \cap tS = \emptyset$, we put $n = 0$. Otherwise, we know from Theorem 2.2.4.3 that $sS \cap tS$ is finitely generated, say by $u_1, \ldots, u_n$. Let $\mathcal{S}_i = (1, s, u_i, u_i, t, 1)$. If $B \in \mathcal{W}$ and $sa = tb$, then $sa = tb = vc$ for some $v \in sS \cap tS$ and $c \in B$. But then $v = u_i r$ for some $i \in \{1, \ldots, n\}$, giving $sa = tb = u_i d$ where $d = rc$. Thus (ii) holds by Remark 2.2.4.4.

Conversely, suppose that (ii) holds and let $s, t \in S$. Suppose that $sS \cap tS \neq \emptyset$ and let $r = sa = tb$ where $r, a, b \in S$. Certainly S has Condition (W), so that there is a replacement tossing $\mathcal{S}_i = (1, s, u_i, u_i, t, 1)$ for some $u_i \in sS \cap tS$ and $i \in \{1, \ldots, n(\mathcal{S})\}$. Hence $r = sa = tb = u_i d$ for some $d \in S$ so that $r \in u_i S$ and we deduce $sS \cap tS = \bigcup_{1 \le i \le n(\mathcal{S})} u_i S$. By Theorem 2.2.4.3, $\mathcal{W}$ is axiomatisable. $\square$

Remark 2.2.4.6. We have replaced a smaller tossing with longer one. This is concerned with having a common tossing $(s,a) \to (u_i,d)$ and $(t,b) \to (u_i,d)$.

2.2.5 Axiomatisability of Condition (PWP)

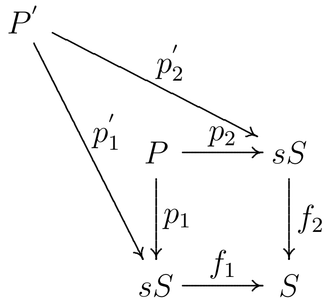
We recall the definition of pullback diagram as given in Chapter 1 Section 1.3. Consider the following diagram in **Act-S**.



The pair $(P,(p_1,p_2))$ where $p_i : P \to sS$, $i = 1,2$ are S-morphisms is called a *pullback* of the pair (f_1,f_2) if

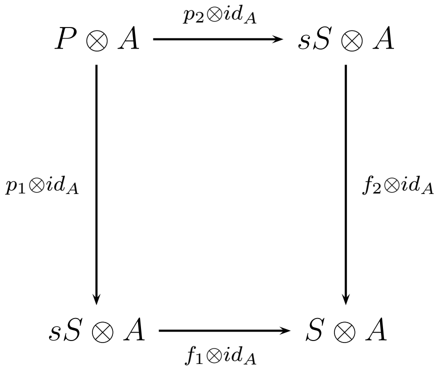
(i) $p_1 f_1 = p_2 f_2$ and,

(ii) if



is a diagram in **Act-S** such that $p_1^{'} f_1 = p_2^{'} f_2$ then there exists a unique S-morphism say $\gamma : P^{'} \to P$ such that $\gamma p_1 = p_1^{'}$ and $\gamma p_2 = p_2^{'}$. We can assume that $P = \{(x,y) \in sS \times sS : xf_1 = yf_2\}$, and p_1 and p_2 are projections onto the first and second co-ordinates.

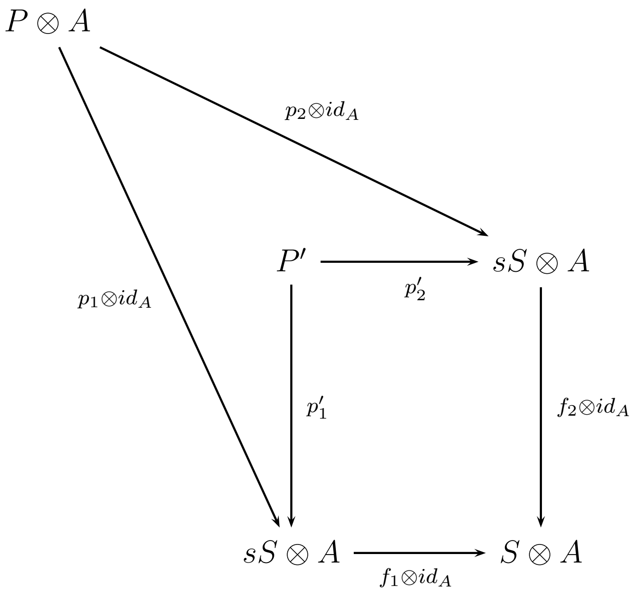
After tensoring the pullback diagram $(P,(p_1,p_2))$ of the pair (f_1,f_2) with a left S-act A we get a commutative diagram



in the category **Set** of sets and maps. Notice that

$$P \otimes A = \{(x, y) \otimes a : (x, y) \in sS \times sS \,\text{and}\, xf_1 = yf_2\}.$$

Also it is given in [21], if we have a pullback of the pair (f_1, f_2) then we have a pullback of the pair $(f_1 \otimes id_A, f_2 \otimes id_A)$. This is of the form $(P^{'}, (p_1^{'}, p_2^{'}))$



where

$$P^{'} = \{(su \otimes a, sv \otimes a') \in (sS \otimes A_S) \times (sS \otimes A_S) : (su)f_1 \otimes a = (sv)f_2 \otimes a'\}.$$

Consider the diagram given above, by definition of pullback there exists a unique map $\gamma : P \otimes A \to P^{'}$ such that $\gamma p_1^{'} = p_1 \otimes id_A$ and $\gamma p_2^{'} = p_2 \otimes id_A$. It follows that γ is defined by $((su, sv) \otimes a)\gamma = (su \otimes a, sv \otimes a)$, for all $s, u, v \in S$ and $a \in A$, see [5]. We call the map γ the *corresponding map.*

A left S-act satisfies condition (PWP) if for every pullback diagram $(P, (p_1, p_2))$ of the pair (f, f) where $f : sS \to S$, the corresponding map γ is surjective.
Equivalently [40], a left S-act satisfies condition (PWP) if

$\forall\, a,\, a' \in A, \forall\, t \in S, ta = ta' \;\Rightarrow\; \exists\, a'' \in A,\; u, v \in S$ such that

$$a = ua'' \wedge a' = va'' \wedge tu = tv.$$

We note that $\mathbf{R}(t, t)$ is as follows:

$$\mathbf{R}(t, t) = \{(u, v) \in S \times S : tu = tv\}.$$

Remark 2.2.5.1. Note that S satisfies Condition (PWP).

We will denote the class of S-acts satisfying Condition (PWP) by $\mathcal{PWP}$.

Theorem 2.2.5.2. *The following conditions are equivalent for a monoid S:*
(i) the class $\mathcal{PWP}$ is axiomatisable;
(ii) $\mathcal{PWP}$ is closed under ultraproducts;
(iii) $\mathcal{PWP}$ is closed under ultrapowers;
(iv) every ultrapower of S has $\mathcal{PWP}$;
(v) for any $t \in S$, $\mathbf{R}(t, t)$ is finitely generated as an S-subact of $S \times S$.

Proof. (i) implies (ii): this follows from Łos's Theorem, (ii) implies (iii) is clear; (iii) implies (iv) is obvious as S satisfies Condition (PWP) as an S-act. .

(iv) implies (v): suppose $\mathbf{R}(t, t)$ is not finitely generated. Suppose for each finite subset f of $\mathbf{R}(t, t)$, there exists $a_f, a_f^{'} \in S$ with $ta_f = ta_f^{'}$ and $(a_f, a_f^{'}) \notin f\,S$.

Let J be the set of finite subsets of $\mathbf{R}(t, t)$. For each $(u, v) \in \mathbf{R}(t, t)$ we define

$$J_{(u,v)} = \{f \in J : (u, v) \in f\}.$$

As each intersection of finitely many of the sets $J_{(u,v)}$ is non-empty, so we are able to define an ultrafilter Φ on J, such that each $J_{(u,v)} \in \Phi$ for all $(u, v) \in \mathbf{R}(t, t)$.

Let $\underline{a} = (a_f)$ and $\underline{a}^{'} = (a_f^{'})$ then $t\underline{a} = t\underline{a}^{'}$ in $\prod_{f \in J} S^f$, where each S^f is a copy of S, as $ta_f = ta_f^{'}$, for each $f \in J$. It follows that this equality $t\underline{a}_\Phi = t\underline{a}_\Phi^{'}$ also holds in $\mathcal{U}$ where $\mathcal{U} = \prod_{f \in J} S^f/\Phi$; by assumption $\mathcal{U}$ has $\mathcal{PWP}$, so there exists $u, v \in S$, and $\underline{r}_\Phi = (r_f)_\Phi \in \mathcal{U}$ such that

$$\underline{a}_\Phi = u\underline{r}_\Phi,\;\; \underline{a}_\Phi^{'} = v\underline{r}_\Phi,\;\; tu = tv.$$

As Φ is closed under finite intersections, there must exists $T \in \Phi$ such that

$$a_f = ur_f, \;\; a_f^{'} = vr_f$$

for all $f \in T$.

Now suppose that $f \in T \cap J_{(u,v)}$, then $(u, v) \in f$ so

$$(a_f, a_f^{'}) = (u, v)r_f \; \in fS$$

a contradiction to our assumption that is $(a_f, a_f^{'}) \notin fS$. Hence (iv) implies (v).

(v) implies (i) : we will show that the class of left S-acts satisfying Condition (PWP) is axiomatisable by giving explicitly a set of sentences that axiomatises this class. For any element $t \in S$, we choose and fix a finite set of elements

$$\{(u_{t,1}, v_{t,1}) \cdots (u_{t,n(t)}, v_{t,n(t)})\}$$

of $\mathbf{R}(t, t)$. We define sentences ϕ_t of L_s as follows:

$$\phi_t := (\forall x)(\forall x')\big(tx = tx' \to (\exists z)(\bigvee_{i=1}^{n(t)} (x = u_{t,i}z \wedge x' = v_{t,i}z))\big).$$

Let

$$\sum\nolimits_{\mathcal{PWP}} = \{\phi_t : t \in S\}.$$

We claim that $\Sigma_{\mathcal{PWP}}$ axiomatises the class $\mathcal{PWP}$.

Let A be an S-act satisfying Condition (PWP). Given that $\mathbf{R}(t, t) \neq \emptyset$, suppose $ta = ta'$ where $a, a' \in A$. Then using the fact that A satisfies Condition (PWP) there are elements $s', t' \in S$ and $c \in A$ such that $ts' = tt'$, $a = s'\,c$, $a' = t'\,c$. Now $(s', t') \in \mathbf{R}(t, t)$ so that ϕ_t is

$$(\forall x)(\forall x')\big(tx = tx' \to (\exists z)(\bigvee_{i=1}^{n(t)} (x = u_{t,i}z \wedge x' = v_{t,i}z))\big)$$

then $(s', t') = (u_{t,i}, v_{t,i})s$ for some $i \in \{1, 2, \ldots, n(t)\}$ and $s \in S$. Thus $a = u_{t,i}sc$, $a' = v_{t,i}sc$. Hence $A \models \phi_t$.

Conversely, let A be a model of $\Sigma_{\mathcal{PWP}}$. Let $t\,a = t\,a'$ where $t \in S$ and $a, a' \in A$. It follows that

$$f = \{(u_{t,1}, v_{t,1}), \cdots, (u_{t,n(t)}, v_{t,n(t)})\}$$

exists as in (v), and ϕ_t is

$$(\forall x)(\forall x')\big(tx = tx' \to (\exists\, z)(\bigvee_{i=1}^{n(t)} (x = u_{t,i}z \wedge x' = v_{t,i}\, z))\big).$$

Hence there exists an element $c \in A$ with $a = u_{t,i}\,c = v_{t,i}\,c$ for some $i \in \{1, 2, \ldots, n(t)\}$. By definition of $u_{t,i}, v_{t,i}$ we have $s\,u_{t,i} = tv_{t,i}$. Thus A satisfies Condition (PWP) and so $\Sigma_{\mathcal{PWP}}$ axiomatises the class $\mathcal{PWP}$. $\square$

Corollary 2.2.5.3. *The following conditions are equivalent for a monoid S:*

(i) the class $\mathcal{PWP}$ is axiomatisable;

(ii) for every skeleton $\mathcal{S} = (1,\, t,\, t,\, 1)$ over S, there exists finitely many replacement skeletons $\mathcal{S}_1 = (u_1, v_1), \cdots, \mathcal{S}_{q(\mathcal{S})} = (u_{q(\mathcal{S})}, v_{q(\mathcal{S})})$ of length one such that for any $a, b \in B \in \mathcal{PWP}$ and $ta = tb$ (equivalently, (t, a) is connected to (t, b) via a tossing with skeleton $\mathcal{S}$), then (t, a) is connected to (t, b) via a replacement tossing with skeleton $\mathcal{S}_i$, for some $1 \leq i \leq q(\mathcal{S})$.

Proof. We follow the same argument given in Theorem 2.2.1.5, putting $s = t$ in Remarks 2.2.1.3 and 2.2.1.4 and using $R(t, t)$ rather than $R(s, t)$. □

2.3 Examples

Example (1): Let G be a group with identity ϵ, let $S_1 = G$. For the monoid S_1 we show the classes $\mathcal{E}$, $\mathcal{P}$, $\mathcal{EP}, \mathcal{W}$ and $\mathcal{PWP}$ of S-acts are axiomatisable.

First note that for any $s, t \in G, \mathbf{r}(s, t)$ is a right ideal, so that $\mathbf{r}(s, t) = G$ and so is finitely generated. Thus $\mathcal{E}$ is axiomatisable. Also $\mathbf{R}(s, t) = (s^{-1}t, \epsilon)G$ is finitely generated, so $\mathcal{P}$ is axiomatisable by Theorem 2.2.1.2 (see also [25]), and by Remark 2.2.3.4, $\mathcal{EP}$ is also axiomatisable. Since every right ideal of G is simply G, by Theorem 2.2.4.3, $\mathcal{W}$ is axiomatisable.

Note that S_1 being an inverse semigroup is absolutely flat by [8] and [18], so that $\mathcal{F}$, $\mathcal{WF}$ and indeed $\mathcal{PWF}$ are axiomatisable.

Example (2): Let T be an infinite null semigroup. Consider $S_2 = T \cup \epsilon$, where ϵ is an adjoined identity. For $s \neq t$, with $s, t \in T$, $\mathbf{r}(s, t) = T$ but T is a non-finitely generated (right) ideal of S_2.

Moreover $\mathbf{R}(s, t)$ is not always finitely generated. For $s, t \in T$ with $s \neq t$,

$$\mathbf{R}(s, t) = \{(u, v) : u, v \in T\}.$$

Suppose on the contrary, $\mathbf{R}(s, t)$ is finitely generated and let

$$\{(u_1, v_1), \cdots, (u_n, v_n)\}$$

be a finite set of generators.

For any $m \in T$ we have $(m, m) \in \mathbf{R}(s, t)$, so $(m, m) = (u_i, v_i)p$ for some $u_i, v_i \in T$ and $p \in S_2$. If $p = \epsilon$, then $m = u_i = v_i$, if $p \neq \epsilon$, then $m = 0$. It follows that T is finite, a contradiction. We therefore have $\mathcal{P}$ and $\mathcal{E}$ are not axiomatisable by Theorems 2.2.1.2 and 2.2.2.2.

Note that $\mathbf{R}(\epsilon,\epsilon)=(\epsilon,\epsilon)S_2$. However, for $s\in T$,

$$\mathbf{R}(s,s)=\{(\epsilon,\epsilon)\}\cup(T\times T).$$

If $\{(\epsilon,\epsilon),(p_1,q_1),\cdots,(p_n,q_n)\}$ is a finite set of generators of $\mathbf{R}(s,s)$, then for any $m\in T$ we have $(s,m)\in\mathbf{R}(s,s)$, so $(s,m)=(p_i,q_i)t$ for some $t\in S$; it follows as above that $m=q_i$ or $m=0$. Hence T is finite, a contradiction. Thus $\mathbf{R}(s,s)$ is not finitely generated. We therefore conclude from Theorem 2.2.5.2 that $\mathcal{PWP}$ is not axiomatisable.

We also note that $\epsilon S_2=S_2$, $sS_2=\{s,0\}$ and $tS_2=\{t,0\}$ for any $s,t\in T$ and $sS_2\cap tS_2=\{0\}$, $sS_2\cap\epsilon S_2=\{s,0\}$. Therefore by Theorem 2.2.4.3, $\mathcal{W}$ is axiomatisable.

Moreover $\mathcal{WF}$ and hence $\mathcal{F}$ are not axiomatisable, see Example 2 of [6] for detail.

Example (3): Let S_3 be a monoid which is a semilattice $\{0,1\}$ of groups G_1, G_0 with trivial connecting homomorphism. Let e,ϵ be the identities of G_1 and G_0 respectively. If G_1 is finite then for the monoid S_3 the classes $\mathcal{P}$, $\mathcal{E}$, $\mathcal{W}$ and $\mathcal{PWP}$ are axiomatisable, as we now show.

We are supposing that S_3 is the union of groups G_0 and G_1. Since each (right) ideal is a union of G_0 and G_1, it follows that S_3 has only finitely many ideals. Then every ideal of S_3 is finitely generated, so $\mathbf{r}(s,t)$ is finitely generated. Therefore $\mathcal{E}$ is axiomatisable. In fact, $S_3=eS_3$ and $G_0=\epsilon S_3$ are the only right ideals, so that $\mathcal{W}$ is also axiomatisable.

We will now check that $\mathbf{R}(s,t)$ is finitely generated for all $s,t\in S$, such that $s\neq t$.

Let $s,t\in G_0$. We claim that $\mathbf{R}(s,t)=\mathbf{R}$ where

$$\mathbf{R}=(e,t^{-1}s)S_3\cup(s^{-1}t,e)S_3.$$

First note that $se=s=\epsilon s=tt^{-1}s$ hence $(e,t^{-1}s)\in\mathbf{R}(s,t)$ and so $(e,t^{-1}s)S_3\subseteq\mathbf{R}(s,t)$. With the dual we have $(e,t^{-1}s)S_3\cup(s^{-1}t,e)S_3$ is contained in $\mathbf{R}(s,t)$, then clearly, $\mathbf{R}\subseteq\mathbf{R}(s,t)$.

Conversely, suppose that $(u,v)\in\mathbf{R}(s,t)$, so that $su=tv$. If $u,v\in G_1$, from $su=tv$ we have $s=t$ a contradiction. If $u\in G_0$ then we have that $u=\epsilon u=s^{-1}su=s^{-1}tv$

so that $(u, v) = (s^{-1}t, e)v$ and so $(u, v) \in \mathbf{R}$. Together with the dual this tells us that $\mathbf{R}(s, t) \subseteq \mathbf{R}$ and so $\mathbf{R}(s, t) = \mathbf{R}$ as required.

If $s \in G_0, t \in G_1$ we claim that $\mathbf{R}(s, t) = \mathbf{R}$ where

$$\mathbf{R} = (e, t^{-1}s)S_3.$$

To see this, notice that $se = s = es = tt^{-1}s$, so that $(e, t^{-1}s) \in \mathbf{R}(s, t)$. Consequently, $\mathbf{R} \subseteq \mathbf{R}(s, t)$.

Conversely, suppose that $(u, v) \in \mathbf{R}(s, t)$. If $u, v \in G_1$, then $su = tv \in G_0 \cap G_1$, a contradiction.

Let $u \in G_0$. We cannot have $v \in G_1$, else $su = tv \in G_0 \cap G_1$, a contradiction. If $v \in G_0$, then from $su = tv$ we have $v = t^{-1}su$, so $(u, v) = (e, t^{-1}s)u \in \mathbf{R}$.

On the other hand, if $u \in G_1$ and $v \in G_0$, then $t^{-1}su = t^{-1}tv = ev = v$ so that $(u, v) = (e, t^{-1}s)u \in \mathbf{R}$. This yields that $\mathbf{R}(s, t) \subseteq \mathbf{R}$ and so $\mathbf{R}(s, t) = \mathbf{R}$ as required.

Let $s, t \in G_1$, we claim that $\mathbf{R}(s, t) = \mathbf{R}$ where

$$\mathbf{R} = (\epsilon, \epsilon)S \cup (s^{-1}t, e)S.$$

Since the connecting homomorphism is trivial, $s\epsilon = \epsilon = t\epsilon$ so $\mathbf{R} \subseteq \mathbf{R}(s, t)$.

Suppose that $(u, v) \in \mathbf{R}(s, t)$. If $u, v \in G_0$ then $su = tv$ implies $u = v$, so that $(u, v) = (\epsilon, \epsilon)u$. The cases where $u \in G_0$, $v \in G_1$ or $u \in G_1, v \in G_0$ are not possible. Let $u, v \in G_1$ with $su = tv$ then $u = s^{-1}tv$ so that $(u, v) = (s^{-1}t, e)v$ where $(s^{-1}t, e) \in \mathbf{R}$. Thus $\mathbf{R}(s, t)$ is finitely generated as required.

It follows from Theorem 2.2.1.2 that $\mathcal{P}$ is axiomatisable, also by Remark 2.2.3.4, $\mathcal{EP}$ is axiomatisable.

We note from Theorems 2.2.4.3 and 2.2.5.2, the classes $\mathcal{W}$ and $\mathcal{PWP}$ are axiomatisable if and only if every ultrapower of S lies in $\mathcal{W}$ and $\mathcal{PWP}$ respectively. Also $\mathcal{P} \subseteq \mathcal{W}$, and $\mathcal{P} \subseteq \mathcal{PWP}$, so by using Lemma 2.0.3.5 we conclude that $\mathcal{W}$ and $\mathcal{PWP}$ are axiomatisable.

Example (4): Let $\mathbb{Z} \times \mathbb{Z}$ be the semigroup with binary operation given by

$$(a, b)(c, d) = (a - b + max\{b, c\}, d - c + max\{b, c\})$$

and let $S_4 = (\mathbb{Z} \times \mathbb{Z}) \cup \{\epsilon\}$ where ϵ is an adjoined identity. For more details we refer the reader to [29] and [54].

Not every right ideal of S_4 is finitely generated, but $\mathbf{r}(\alpha, \beta)$ is finitely generated for all $\alpha, \beta \in S_4$ [29]. It follows from Theorem 2.2.2.2 that $\mathcal{E}$ is axiomatisable. Again from [29], $\mathbf{R}(\alpha, \beta)$ is not necessarily finitely generated, so $\mathcal{P}$ is not axiomatisable.

Moreover as S_4 is an inverse semigroup, and hence absolutely flat see [8] and [18], it follows that $\mathcal{F}$, $\mathcal{WF}$ and $\mathcal{PWF}$ are axiomatisable.

We know that $\mathcal{P} \subseteq \mathcal{W}$, and by using Lemma 2.0.3.5, if $\mathcal{P}$ is axiomatisable for a monoid S, then so is $\mathcal{W}$. The converse of this statement is not true in general. However S_4 satisfies Condition (iv) of Theorem 2.2.4.3 as the principal right ideals are linearly ordered, so the intersection of two such ideals is again principal, see [29]. Thus $\mathcal{W}$ is axiomatisable.

Example (5): Consider $S_5 = (\mathbb{N}^{\epsilon}, \min)$ where ϵ is the adjoined identity element.

Let $s, t \in S_5$ with $s \neq t$. Without loss of generality, take $s < t$. Then $\mathbf{R}(s, t) = \{(1,1), \cdots, (s,s), (s+1, s), \cdots\} = (s,s)S_5 \cup (\epsilon, s)S_5$ so that $\mathbf{R}(s,t)$ is finitely generated. It follows from Remark 2.2.3.4 that $\mathcal{EP}$ is axiomatisable. However, we can check that $\mathbf{R}(s,s)$ is not finitely generated for every s.

Suppose on contrary, $\mathbf{R}(1,1) = S_5 \times S_5$ is finitely generated. If $(u_1, v_1), \cdots, (u_n, v_n)$ is a finite set of generators of $\mathbf{R}(1,1)$, let $(\epsilon, m) = (u_i, v_i)t$ for some $i \in \mathbb{N}$ and for some $t \in S_5$; therefore $u_i = \epsilon, t = \epsilon$ and so $m = v_i$, hence S_5 is finite, a contradiction. It follows from Theorems 2.2.1.2 and 2.2.5.2 that $\mathcal{P}$ and $\mathcal{PWP}$ are not axiomatisable.

Since the principal ideals of S_5 are linearly ordered, $\mathcal{W}$ is axiomatisable by 2.2.4.3.

For $s < t$ we have that $su = tu$ if and only if $u \leq s$, so that $\mathbf{r}(s,t) = uS_5$ and is finitely generated. Hence $\mathcal{E}$ is axiomatisable by Theorem 2.2.2.2.

Remark 2.3.0.4. We make the following connections between the axiomatisability conditions of the following classes of S-acts, pointing out some of them which are still unknown.

$\mathcal{P}$	$\Rightarrow$	$\mathcal{EP}$	*Remark* 2.2.3.4
$\mathcal{P}$	$\not\Leftarrow$	$\mathcal{EP}$	*Example* 5
$\mathcal{E}$	$\Rightarrow$	$\mathcal{EP}$	*Unknown*
$\mathcal{E}$	$\not\Leftarrow$	$\mathcal{EP}$	*Unknown*
$\mathcal{P}$	$\Rightarrow$	$\mathcal{W}$	*Lemma* 2.0.3.5
$\mathcal{P}$	$\not\Leftarrow$	$\mathcal{W}$	*Example* 4
$\mathcal{P}$	$\Rightarrow$	$\mathcal{PWP}$	*Lemma* 2.0.3.5
$\mathcal{P}$	$\not\Leftarrow$	$\mathcal{PWP}$	*Unknown*
$\mathcal{E}$	$\not\Rightarrow$	$\mathcal{P}$	*Example* 4
$\mathcal{P}$	$\not\Rightarrow$	$\mathcal{E}$	*Unknown*

2.4 Some Open Problems

In addition to deciding the unknown implications in the above diagram, there are also the questions of deciding the connections between the axiomatisability of $\mathcal{PWF}$ and $\mathcal{WF}$, and (harder) $\mathcal{WF}$ and $\mathcal{F}$. Indeed it is an open problem to determine for which monoids S is $\mathcal{WF} = \mathcal{F}$?

There are also further classes of " flat " S-acts such as those satisfying Condition (WP) introduced in [40] that we have not yet considered from the point of view of axiomatisability.

A left S-act satisfies *Condition* (WP) if for every pullback diagram $(P, (p_1, p_2))$ of the pair (f, f) where I is a right ideal, and $f : I \to S$ is a S-morphism, and the corresponding map γ is surjective.

Or equivalently [40], a left S-act A satisfies condition (WP) if and only if for every S-morphism $f : (sS \cup tS)_S \to S_S$ where $s, t \in S$ and all $a, a' \in A$ if $(s)fa = (t)fa'$ then there exists $a'' \in A$, $u, v \in S$, $s', t' \in \{s, t\}$ such that $(s'u)f = (t'v)f$, $s \otimes a = s'u \otimes a''$, and $t \otimes a' = t'v \otimes a''$ in $(sS \cup tS)_S \otimes_S A$.

We aim to axiomatise the class of left S-acts satisfying Condition (WP).

Chapter 3

Axiomatisability problems for S-posets

In Chapter 2 we added to the theory of axiomatisability problems of S-acts. In this chapter we initiate the investigation of axiomatisability problems for S-posets over a pomonoid S. We have succeeded in determining when the classes

$$\mathcal{SF},\ \mathcal{F},\ \mathcal{WF}, \mathcal{PWF}, \mathcal{PF}, \mathcal{WPF}, \mathcal{PWPF}$$

of strongly flat, flat, weakly flat, principally weakly flat, po-flat, weakly po-flat and principally weakly po-flat S-posets, respectively, are axiomatisable. Most of the proofs are along the same lines as those for S-acts. In addition we have axiomatised some conditions such as Condition (P), Condition (E) (which together give us $\mathcal{SF}$), (EP), (W), (P_w), (PWP) and Condition $(\mathrm{PWP})_w$ for S-posets. These conditions and classes are all defined in Chapter 1.

We recall that, associated with the class S-Pos for a pomonoid S, we have a first order language $L_S^{\leq}$, which has no constant symbols, a unary function symbol λ_s for each $s \in S$, and (other than $=$), a single relational symbol $\leq$, with $\leq$ being binary. An S-poset provides an interpretation of $L_S^{\leq}$ in the obvious way, indeed in $L_S^{\leq}$ we write sx for $\lambda_s(x)$. We note that S-Pos itself is axiomatisable amongst all interpretations of $L_S^{\leq}$. For any $s, t \in S$ and $u, v \in S$ with $u \leq v$ we define sentences

$$\varphi_{s,t} := (\forall x)\big(s(t(x)) = (st)x\big),\ \theta_s := (\forall x, y)\big(x \leq y \to sx \leq sy\big)$$

$$\psi_{u,v} := (\forall x)(ux \leq vx).$$

and

$\pi := \big\{(\forall x)\big(x \leq x\big) \bigwedge (\forall x,\, y)\big((x \leq y \,\wedge\, y \leq x) \to (x = y)\big) \bigwedge (\forall x,\, y,\, z)\big((\, x \leq y \,\wedge\, y \leq z) \to (x \leq z)\big)\big\}.$

Then for any $s, t, u, v \in S$, Π_S axiomatises S-*Pos*; where

$$\Pi_S \;=\; \{(\forall x)(1\,x = x)\} \cup \big\{\varphi_{s,t} : s, t \in S\big\} \cup \big\{\theta_s : s \in S\big\} \cup \big\{\psi_{u,v} : u \leq v : u, v \in S\big\} \cup \big\{\pi\big\}.$$

Some classes of left S-posets are axiomatisable for *any* monoid S. For example, the class $\mathcal{T}$ of left S-posets with the trivial partial order is axiomatised by

$$\Pi_S \cup \{(\forall x, y)\big(x \leq y \to x = y\big)\}.$$

To save repetition, we will assume from now on that when axiomatising a class of left S-posets, Π_S is understood, so that we would say $\{(\forall x, y)\big(x \leq y \to x = y\big)\}$ axiomatises $\mathcal{T}$. Other natural classes of left S-posets are axiomatisable for some pomonoids and not for others and it is our aim here to investigate the pomonoids that arise.

Corresponding questions for classes of M-acts over a monoid M have been answered in [25, 50, 6, 29, 30] and revisited in Chapter 2. In Chapter 2 we developed two general methods of axiomatisability which led to axiomatisability results for some of classes such as flat, weakly flat and principally weakly flat M-*acts* as special cases. The classes of projective (strongly flat, po-flat, weakly po-flat) left S-posets $\mathcal{P}r(\mathcal{SF}, \mathcal{PF}, \mathcal{WPF})$ have recently been considered in [52] (which uses slightly different terminology) as has the class $\mathcal{F}r$ of free left S-posets in the case where S has only finitely many right ideals. We note that many of the techniques of [52] follow those in the M-act case and, for this reason, we aim here to produce two general strategies, as in Chapter 2, that will deal with a number of axiomatisability questions for classes of S-posets. In particular they may be applied to $\mathcal{PF}$ and $\mathcal{WPF}$. Just as many concepts of flatness that are equivalent for R-modules over a unital ring R are different for M-acts, so many concepts that coincide for M-acts split for S-posets. Thus [52] left a number of classes open; we address many of them here, with both our general techniques and ad hoc methods.

The structure of this chapter is as follows. We present in Section 3.1 and Section 3.2 our *general* axiomatisability results, which apply to various classes defined by flatness properties, as in Section 2.1 of Chapter 2. In the case of S-posets we have two variations i.e. when the functor $-\otimes B$ maps embeddings to one-one maps and when $-\otimes B$ preserves embeddings. There are two kinds of results, for each variation, all phrased in terms of 'replacement tossings'. In Section 3.1 we consider the case when $-\otimes B$ maps embeddings to one-one maps; we then apply our results to determine when $\mathcal{F}, \mathcal{WF}$ and $\mathcal{PWF}$ are axiomatisable. Next, in Section 3.2, we consider the case when $-\otimes B$ preserves embeddings. We show how our results may be applied to reproduce the results of [52] determining for which pomonoids $\mathcal{PF}$ or $\mathcal{WPF}$ are axiomatisable, together with a further application to $\mathcal{PWPF}$. For the sake of clarity, where proofs are similar to those for M-acts over a monoid M, we relegate them to the Appendix.

In Section 3.3 we then consider classes defined by flatness conditions that translate into so called 'interpolation conditions'. In these cases we can give rather more direct arguments, avoiding the concept of replacement tossing. Section 3.4 briefly visits the question of axiomatisability of $\mathcal{F}r$ and $\mathcal{P}r$; the results here are easily deducible from the corresponding ones for M-acts. Finally in Section 3.5 we present some open problems.

We need to give some definitions in the context of S-posets which will be used later:

Let $\mathcal{C}$ be a class of ordered embeddings of right S-posets. We recall that an S-pomorphism $\alpha : A \to B$ between two left S-posets A and B is called an *embedding* if it satisfies the condition

$$a \leq a' \Leftrightarrow a\alpha \leq a'\alpha.$$

Thus if $\alpha : A \to B$ is an embedding, then A is isomorphic as an S-poset to $A\alpha$.

A left S-poset B is called $\mathcal{C}$*-flat* if the functor $-\otimes B$ takes embeddings in $\mathcal{C}$ to one-one maps in the category of **Pos**, that is, if $\alpha : A \to A'$ is in $\mathcal{C}$ then $\alpha \otimes I_B$ is one-one, or in terms of elements, if for some $a, a' \in A$ and $b, b' \in B$ we have $a\alpha \otimes b = a'\alpha \otimes b'$ in $A' \otimes B$, then $a \otimes b = a' \otimes b'$ in $A \otimes B$. We will denote the class of $\mathcal{C}$-flat left S-posets by $\mathcal{CF}$.

A left S-poset B is called a *$\mathcal{C}$-po-flat* if the functor $- \otimes B$ takes embeddings in $\mathcal{C}$ to embeddings in the category of **Pos** that is, if $\beta : A \to A'$ is in $\mathcal{C}$, then $\beta \otimes I_B$ is an embedding, or in terms of elements, if for some $a, a' \in A$ and $b, b' \in B$ we have $a\,\beta \otimes b \leq a'\,\beta \otimes b'$ in $A' \otimes B$, then $a \otimes b \leq a' \otimes b'$ in $A \otimes B$. We will denote the class of $\mathcal{C}$-po-flat left S-posets by $\mathcal{C}$-$\mathcal{PF}$.

We follow the pattern of presentation of Chapter 2. As for S-acts we introduce conditions on $\mathcal{C}$ called (Free) and (Free)$^{\leq}$.

If $\mathcal{C}$ satisfies condition (Free) (respectively, (Free$^{\leq}$)), then we can find necessary and sufficient conditions for the classes $\mathcal{CF}$ (respectively, $\mathcal{C}$-$\mathcal{PF}$) to be axiomatisable. Then our results for axiomatising $\mathcal{F}$ (the class of flat left S-posets) and $\mathcal{PF}$ (the class of po-flat S-posets) become special cases.

As in the unordered case our next step is to drop the assumption of Condition (Free) or (Free$^{\leq}$). Along similar lines as for S-acts we then have general results giving conditions that determine when $\mathcal{CF}$, and $\mathcal{C}$-$\mathcal{PF}$ are axiomatisable. The results for axiomatising $\mathcal{WF}$ (class of weakly flat left S-posets) and $\mathcal{WPF}$ (class of weakly po-flat left S-posets) then become special cases.

3.1 Axiomatisability of $\mathcal{CF}$

In this section we describe our two general results involving 'replacement tossings' for $\mathcal{CF}$. We first consider the case when $\mathcal{C}$ satisfies Condition (Free). This will enable us to specialise to the case where $\mathcal{C}$ is the class of all right S-poset embeddings. For the second we consider an arbitary class $\mathcal{C}$; we then specialise to the cases where $\mathcal{C}$ consists of all inclusions of (principal) right ideals into S. We remark that similar methods have been applied to axiomatisability problems for S-acts over a monoid S, as shown in Chapter 2.

3.1.1 Axiomatisability of $\mathcal{C}$-flat S-posets with Condition (Free)

It is convenient to introduce some notation.

Let

$$\mathcal{S} = (s_1, t_1, \ldots, s_m, t_m)$$

be an ordered skeleton of length m.

We define a formula $\epsilon_{\mathcal{S}}$ of $R_S^{\leq}$, where $R_S^{\leq}$ is the first order language associated with right S-posets, as follows:

$$\epsilon_{\mathcal{S}}(x, x_2, \cdots, x_m, x') := \big(xs_1 \leq x_2t_1 \wedge x_2s_2 \leq x_3t_2 \wedge \ldots \wedge x_ms_m \leq x't_m\big)$$

and a formula $\theta_{\mathcal{S}}$ of $L_S^{\leq}$ by

$$\theta_{\mathcal{S}}(x, x_1, \cdots, x_m, x') := \big(x \leq s_1x_1 \wedge t_1x_1 \leq s_2x_2 \wedge \ldots \wedge t_mx_m \leq x'\big).$$

Suppose now that

$$\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2) = (s_1, t_1, \ldots, s_m, t_m, u_1, v_1, \ldots, u_n, v_n)$$

is a double ordered skeleton of length $m+n$. We put

$$\delta_{\mathcal{S}}(x, x') := (\exists x_2 \ldots \exists x_m \exists y_2 \ldots \exists y_n)\epsilon_{\mathcal{S}_1}(x, x_2, \ldots, x_m, x') \wedge \epsilon_{\mathcal{S}_2}(x', y_2, \ldots, y_n, x).$$

On the other hand we define the formula

$$\gamma_{\mathcal{S}}(x, x') := (\exists x_1 \cdots \exists x_m \exists y_1 \cdots \exists y_n)\theta_{\mathcal{S}_1}(x, x_1, \ldots, x_m, x') \wedge \theta_{\mathcal{S}_2}(x', y_1, \ldots, y_n, x').$$

Remark 3.1.1.1. Let A, B be right and left S-posets, respectively, let $a, a' \in A$ and $b, b' \in B$.

(i) The pair (a, b) is connected to the pair (a', b') via a double ordered tossing with double ordered skeleton $\mathcal{S}$ if and only if $\delta_{\mathcal{S}}(a, a')$ is true in A and $\gamma_{\mathcal{S}}(b, b')$ is true in B.

(ii) If $\delta_{\mathcal{S}}(a, a')$ is true in A and $\psi : A \to A'$ is a (right) S-pomorphism, then $\delta_{\mathcal{S}}(a\psi, a'\psi)$ is true in A'.

(iii) If $\gamma_{\mathcal{S}}(b, b')$ is true in B and $\tau : B \to B'$ is an S-pomorphism, then $\gamma_{\mathcal{S}}(b\tau, b'\tau)$ is true in $B\tau$.

Definition 3.1.1.2. We say that $\mathcal{C}$ *satisfies Condition (Free)* if for each double ordered skeleton $\mathcal{S}$ there is an embedding $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}}$ in $\mathcal{C}$ and $u_{\mathcal{S}}, u'_{\mathcal{S}} \in W_{\mathcal{S}}$ such that $\delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$ and further, for any embedding $\mu : A \to A' \in \mathcal{C}$ and any $a, a' \in A$ such that $\delta_{\mathcal{S}}(a\mu, a'\mu)$ is true in A' there is a morphism $\nu : W'_{\mathcal{S}} \to A'$ such that $u_{\mathcal{S}}\tau_{\mathcal{S}}\nu = a\mu$, $u'_{\mathcal{S}}\tau_{\mathcal{S}}\nu = a'\mu$ and $W_{\mathcal{S}}\tau_{\mathcal{S}}\nu \subseteq A\mu$.

Lemma 3.1.1.3. *Let $\mathcal{C}$ be a class of embeddings of right S-posets satisfying Condition (Free). Then the following are equivalent for a left S-poset B:*

(i) B is $\mathcal{C}$-flat;

*(ii) $-\otimes B$ maps the embeddings $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}}$ in the category **Pos-S** to monomorphisms in the category of* **Pos**, *for every double ordered skeleton $\mathcal{S}$;*

(iii) if $(\mu_{\mathcal{S}}\tau_{\mathcal{S}}, b)$ and $(\mu'_{\mathcal{S}}\tau_{\mathcal{S}}, b')$ are connected by a double ordered tossing over $W'_{\mathcal{S}}$ and B with double ordered skeleton $\mathcal{S}$, then $(u_{\mathcal{S}}, b)$ and $(u'_{\mathcal{S}}, b')$ are connected by a double ordered tossing over $W_{\mathcal{S}}$ and B.

Proof. See Appendix. □

Our next aim is to show that the class of all embeddings of right S-posets has Condition (Free). To this end we present a 'Finitely Presented Flatness Lemma' for S-posets.

For a double ordered skeleton $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ where

$$\mathcal{S}_1 = (s_1, t_1, \cdots, s_m, t_m) \text{ and } \mathcal{S}_2 = (u_1, v_1, \cdots, u_n, v_n),$$

we let F^{m+n} be the free right S-poset

$$xS \,\dot\cup\, x_2S \,\dot\cup\, \ldots \dot\cup\, x_mS \,\dot\cup\, y_2S \,\dot\cup\, y_3S \,\dot\cup \ldots \dot\cup\, y_nS \,\dot\cup\, x'S$$

and let $R_{\mathcal{S}}$ be the set

$$\big\{(xs_1, x_2t_1), (x_2s_2, x_3t_2), \ldots, (x_{m-1}s_{m-1}, x_mt_{m-1}), (x_ms_m, x't_m),$$

$$(x'u_1, y_2v_1), (y_2u_2, y_3v_2), \ldots, (y_nu_n, xv_n)\big\}.$$

Let us abbreviate by $\equiv_{\mathcal{S}}$ the S-poset congruence $\equiv_{R_{\mathcal{S}}}$ induced by $R_{\mathcal{S}}$. We abbreviate the order $\preceq_{R_{\mathcal{S}}}$ on $F^{m+n}/\equiv_{\mathcal{S}}$ by $\preceq_{\mathcal{S}}$.

If B is a left S-poset and b, b_1, $,\cdots, b_m$, d_1, d_2, $,\cdots, d_n$, $b' \in B$ are such that

$$\theta_{\mathcal{S}_1}(b, b_1, \ldots, b_m, b') \text{ and } \theta_{\mathcal{S}_2}(b', d_1, \ldots, d_n, b)$$

hold, then the double ordered tossing

$$\begin{array}{rclrcl} & & & b & \leq & s_1b_1 \\ [x]s_1 & \leq & [x_2]t_1 & t_1b_1 & \leq & s_2b_2 \\ [x_2]s_2 & \leq & [x_3]t_2 & t_2b_2 & \leq & s_3b_3 \\ & \vdots & & & \vdots & \\ [x_m]s_m & \leq & [x']t_m & t_mb_m & \leq & b' \end{array}$$

$$\begin{array}{rclrcl} & & & b' & \leq & u_1d_1 \\ [x']u_1 & \leq & [y_2]v_1 & v_1d_1 & \leq & u_2d_2 \\ [y_2]u_2 & \leq & [y_3]v_2 & v_2d_2 & \leq & u_3d_3 \\ & \vdots & & & \vdots & \\ [y_n]u_n & \leq & [x]v_n & v_nd_n & \leq & b \end{array}$$

over $F^{m+n}/\equiv_{\mathcal{S}}$ and B is called a *double ordered standard tossing*; clearly it has double ordered skeleton $\mathcal{S}$.

It is clear that (by considering a trivial left S-poset B), the set of all double ordered skeletons $\mathbb{DOS}$ is the set of all finite even length sequences of elements of S, of length at least 4.

Lemma 3.1.1.4. *The following conditions are equivalent for a left S-poset B:*

(i) B is flat;

(ii) $-\otimes B$ maps embeddings of $[x]S \cup [x']S$ into $F^{m+n}/\equiv_{\mathcal{S}}$ in the category ***Pos*-S** *to monomorphisms in the category of* **Pos***, for every double ordered skeleton $\mathcal{S}$;*

(iii) if $\big([x], b\big)$ and $\big([x'], b'\big)$ are connected by a double ordered standard tossing over $F^{m+n}/\equiv_{\mathcal{S}}$ and B (with double ordered skeleton $\mathcal{S}$), then they are connected by a double ordered tossing over $[x]S \cup [x']S$ and B.

Proof. We will prove here only $(iii) \Rightarrow (i)$. Suppose that B satisfies condition (iii), let a, a' belongs to any right S-poset A, let $b, b' \in B$, and suppose that $a\otimes b = a'\otimes b'$ in $A\otimes B$ via a double ordered tossing with double ordered skeleton $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$, where $\mathcal{S}_1, \mathcal{S}_2$ have lengths m and n, respectively. By Remark 3.1.1.1, $\delta_{\mathcal{S}}(a, a')$ is true in A and $\gamma_{\mathcal{S}}(b, b')$ is true in B. Since $\delta_{\mathcal{S}}([x], [x'])$ holds in $F^{m+n}/\equiv_{\mathcal{S}}$, we have that $\big([x], b\big)$ and $\big([x'], b'\big)$ are connected by a double ordered standard tossing over $F^{m+n}/\equiv_{\mathcal{S}}$ and B. By the given hypothesis we have that $\big([x], b\big)$ and $\big([x'], b'\big)$ connected via a double ordered tossing in $\big([x]S \cup [x']S\big) \otimes B$, say with double ordered skeleton $\mathcal{U}$.

Since $\delta_{\mathcal{S}}(a, a')$ is true in A, there are elements $a_2, \ldots, a_m, c_2, \ldots, c_n \in A$ such that

$$\epsilon_{\mathcal{S}_1}(a, a_2, \ldots, a_m, a') \text{ and } \epsilon_{\mathcal{S}_2}(a', c_2, \ldots, c_n, a)$$

hold in A. Let $\phi : F^{m+n} \to A$ be the S-pomorphism which is defined by $x\phi = a$, $x_i\phi = a_i$ $(2 \leq i \leq m)$, $x'\phi = a'$ and $y_j\phi = c_j$ $(2 \leq j \leq n)$. Since $u\phi \leq u'\phi$ for all $(u, u') \in R_{\mathcal{S}}$, we have that $\overline{\phi} : F^{m+n}/\equiv_{\mathcal{S}} \to A$ given by $[z]\overline{\phi} = z\phi$ is a well defined S-pomorphism. We have that $\delta_{\mathcal{U}}([x],[x'])$ holds in $[x]S \cup [x']S$, so that by Remark 3.1.1.1, $\delta_{\mathcal{U}}(a,a')$ holds in $aS \cup a'S$. Since also $\gamma_{\mathcal{U}}(b,b')$ holds in B, we have that (a,a') and (b,b') are connected by a double ordered tossing over $aS \cup a'S$ and B, so that $a \otimes b = a' \otimes b'$ in $(aS \cup a'S) \otimes B$. Thus B is flat, as required. □

With a similar argument, we prove the following.

Lemma 3.1.1.5. *The class Pos-S of all right S-posets has Condition (Free).*

Proof. Let $\mathcal{S}$ be a double ordered skeleton of length $m+n$, let $W'_{\mathcal{S}} = F^{m+n}/\equiv_{\mathcal{S}}$, $W_{\mathcal{S}} = [x]S \cup [x']S$ and let $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}}$ denote inclusion. Then $[x],[x'] \in W_{\mathcal{S}}$ and $\delta_{\mathcal{S}}([x]\tau_{\mathcal{S}},[x']\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$.

Let $\mu : A \to A'$ be any right S-poset embedding such that $\delta_{\mathcal{S}}(a\mu, a'\mu)$ holds in A', for some $a,a' \in A$. As in Lemma 3.1.1.4, there is as a consequence an S-pomorphism $\nu : W'_{\mathcal{S}} \to A'$ such that $[x]\tau_{\mathcal{S}}\nu = a\mu$ and $[x']\tau_{\mathcal{S}}\nu = a'\mu$. Clearly

$$W_{\mathcal{S}}\tau_{\mathcal{S}}\nu = ([x]S \cup [x']S)\tau_{\mathcal{S}}\nu = [x]\tau_{\mathcal{S}}\nu S \cup [x']\tau_{\mathcal{S}}\nu S = a\mu S \cup a'\mu S = (aS \cup a'S)\mu \subseteq A\mu.$$

Thus, with $u_{\mathcal{S}} = [x]$ and $u'_{\mathcal{S}} = [x']$, we see that Condition (Free) holds.

□

Lemma 3.1.1.6. *Let $\mathcal{C}$ be a class of embeddings of right S-posets, satisfying Condition (Free). Let $\overline{\mathcal{C}}$ be the set of products of morphisms in $\mathcal{C}$. If a left S-poset B is $\mathcal{C}$-flat, then it is $\overline{\mathcal{C}}$-flat.*

Proof. See Appendix. □

We now come to our first main result. The technique used is inspired by that of Chapter 2, but there are some differences due to the fact that we are dealing with orderings.

Theorem 3.1.1.7. *Let $\mathcal{C}$ be a class of embeddings of right S-posets satisfying Condition (Free). Then the following conditions are equivalent for a pomonoid S:*

(i) the class $\mathcal{CF}$ is axiomatisable;

(ii) the class $\mathcal{CF}$ is closed under formation of ultraproducts;

(iii) for every double ordered skeleton $\mathcal{S} \in \mathbb{DOS}$ there exist finitely many double ordered replacement skeletons $\mathcal{S}_1, \ldots, \mathcal{S}_{\alpha(\mathcal{S})}$ such that, for any embedding $\gamma : A \to A'$ in $\mathcal{C}$ and any $\mathcal{C}$-flat left S-poset B, if $(a\gamma, b)$, $(a'\gamma, b') \in A' \times B$ are connected by a double ordered tossing $\mathcal{T}$ over A' and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a,b) and (a',b') are connected by a double ordered tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(\mathcal{S})\}$;

(iv) for every double ordered skeleton $\mathcal{S} \in \mathbb{DOS}$ there exists finitely many double ordered replacement skeletons $\mathcal{S}_1, \ldots, \mathcal{S}_{\beta(\mathcal{S})}$ such that, for any $\mathcal{C}$-flat left S-poset B, if $(u_{\mathcal{S}}\tau_{\mathcal{S}}, b)$ and $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, b')$ are connected by the double ordered tossing $\mathcal{T}$ over $W'_{\mathcal{S}}$ and B (with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$), then $(u_{\mathcal{S}}, b)$, and $(u'_{\mathcal{S}}, b')$ are connected by a double ordered tossing $\mathcal{T}'$ over $W_{\mathcal{S}}$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \beta(\mathcal{S})\}$.

Proof. See Appendix. □

The axiomatisability of the class $\mathcal{F}$ of flat S-posets follows the first pattern which is explained as follows.

Clearly, an S-poset is flat if and only if it is $\mathcal{C}$-flat where $\mathcal{C}$ is the class of *all* embeddings of right S-posets. By Lemma 3.1.1.5, the class of all right S-posets has Condition (Free), so from Theorem 3.1.1.7, we immediately have the following corollary.

Corollary 3.1.1.8. *The following conditions are equivalent for a pomonoid S:*

(i) the class $\mathcal{F}$ is axiomatisable;

(ii) the class $\mathcal{F}$ is closed under formation of ultraproducts;

(iii) for every double ordered skeleton $\mathcal{S} \in \mathbb{DOS}$ there exist finitely many double ordered replacement skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(\mathcal{S})}$ such that, for any right S-poset embedding $\gamma : A \to A'$, and any flat left S-poset B, if $(a\gamma, b)$, $(a'\gamma, b') \in A' \times B$ are connected by a double ordered tossing $\mathcal{T}$ over A' and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a', b') are connected by a double ordered tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(\mathcal{S})\}$;

(iv) for every double ordered skeleton $\mathcal{S} \in \mathbb{DOS}$ there exist finitely many double ordered replacement skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(\mathcal{S})}$ such that, for any right S-poset A and any flat left S-poset B, if (a, b), $(a', b') \in A \times B$ are connected by a double ordered tossing $\mathcal{T}$ over A and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a', b') are connected by a double ordered tossing $\mathcal{T}'$ over $aS \cup a'S$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(\mathcal{S})\}$;

(v) for every double ordered skeleton $\mathcal{S} \in \mathbb{DOS}$ there exists finitely many double ordered replacement skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\beta(\mathcal{S})}$ such that, for any flat left S-poset B, if $([x], b)$ and $([x'], b')$ are connected by a double ordered tossing $\mathcal{T}$ over $F^{m+n}/\equiv_{\mathcal{S}}$ and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $([x], b)$ and $([x'], b')$ are connected by a double ordered tossing $\mathcal{T}'$ over $[x]S \cup [x']S$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \beta(\mathcal{S})\}$.

3.1.2 Axiomatisability of $\mathcal{C}$-flat S-posets without Condition (Free)

We continue to consider a class $\mathcal{C}$ of embeddings of right S-posets, but now drop our assumption that Condition (Free) holds. The results and proofs of this section are analogous to those for weakly flat S-acts in [6] and in Chapter 2, Subsection 2.1.2. Note that the conditions in (iii) below appear weaker than those in Theorem 3.1.1.7, as we are only asking that for specific elements a, a' and double ordered skeleton $\mathcal{S}$, there are finitely many double ordered replacement skeletons, in the sense made specific below.

Theorem 3.1.2.1. *Let $\mathcal{C}$ be a class of embeddings of right S-posets.*

The following conditions are equivalent:

(i) the class $\mathcal{CF}$ is axiomatisable;

(ii) the class $\mathcal{CF}$ is closed under ultraproducts;

(iii) for every double ordered skeleton $\mathcal{S} \in \mathbb{DOS}$ and $a, a' \in A$, where $\mu : A \to A'$ is in $\mathcal{C}$, there exist finitely many double ordered skeleton $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(a,\mathcal{S},a',\mu)}$, such that for any $\mathcal{C}$-flat left S-poset B, if $(a\mu, b)$, $(a'\mu, b')$ are connected by a double ordered tossing $\mathcal{T}$ over

A' and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a', b') are connected by a double ordered tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(a, \mathcal{S}, a', \mu)\}$.

Proof. See Appendix. $\square$

We now give two applications of Theorem 3.1.2.1.

We recall definition of weakly flat S-poset. A left S-poset B is called *weakly flat* if the functor $- \otimes B$ maps inclusions of right ideals in the category of **S-Pos** to one-one order preserving maps in the category of **Pos**. Note that if the left S-poset B is weakly flat then the equality $a \otimes b = a' \otimes b'$ also holds in $(aS \cup a'S) \otimes B$. Recall that the class of weakly flat left S-posets is denoted by $\mathcal{WF}$.

In the following two corollaries we do not need to mention the embeddings μ, since they are all inclusion maps of right ideals into S.

Corollary 3.1.2.2. *The following are equivalent for a pomonoid S:*

(i) the class $\mathcal{WF}$ is axiomatisable;

(ii) the class $\mathcal{WF}$ is closed under ultraproducts;

(iii) for every double ordered skeleton $\mathcal{S}$ and $a, a' \in S$ there exists finitely many double ordered skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\beta(a,\mathcal{S},a')}$ such that for any weakly flat left S-poset B, if (a, b), $(a', b') \in S \times B$ are connected by a double ordered tossing $\mathcal{T}$ over S and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$ then (a, b) and (a', b') are connected by a double ordered tossing $\mathcal{T}'$ over $aS \cup a'S$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$ for some $k \in \{1, \cdots, \beta(a, \mathcal{S}, a')\}$.

We end this section by considering the axiomatisability of principally weakly flat left S-posets. A left S-poset B is called *principally weakly flat* if the functor $- \otimes B$ maps embeddings of principal right ideals in the category of **S-Pos** to one-one order preserving maps in the category of **Pos**. The class of all principally weakly flat left S-posets is denoted by $\mathcal{PWFS}$.

We first remark that if aS is a principal right ideal of S and B is a left S-poset, then

$$au \otimes b = av \otimes b' \text{ in } aS \otimes B \text{ if and only if } a \otimes ub = a \otimes vb' \text{ in } aS \otimes B$$

with a similar statement for $S \otimes B$. Thus B is principally weakly flat if and only if for all $a \in S$, if $a \otimes b = a \otimes b'$ in $S \otimes B$, then $a \otimes b = a \otimes b'$ in $aS \otimes B$. From Theorem 3.1.2.1 and its proof we have the following result for $\mathcal{PWF}$.

Corollary 3.1.2.3. *The following conditions are equivalent for a pomonoid $\mathcal{S}$:*

(i) the class $\mathcal{PWF}$ is axiomatisable;

(ii) the class $\mathcal{PWF}$ is closed under ultraproducts;

(iii) for every double ordered skeleton $\mathcal{S}$ over S and $a \in S$ there exists finitely many double ordered skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\gamma(a,\mathcal{S})}$ over S, such that for any principally weakly flat left S-poset B, if (a,b), $(a,b') \in S \otimes B$ are connected by a double ordered tossing $\mathcal{T}$ over S and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$ then (a,b) and (a,b') are connected by a double ordered tossing $\mathcal{T}'$ over aS and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$ for some $k \in \{1, \cdots, \gamma(a,\mathcal{S})\}$.

3.2 Axiomatisability of $\mathcal{C}$-$\mathcal{PF}$

In this section we explain how the methods and results of Section 3.1 may be adapted to the case when $-\otimes B$ preserves embeddings, rather than merely taking embeddings to monomorphisms. We omit proofs, as they follow now established patterns. Further details may be found in the appendix.

We introduce a condition on a class $\mathcal{C}$ of embeddings of right S-posets called Condition (Free)$^{\leq}$.

Let

$$\mathcal{S} = (s_1, t_1, \ldots, s_m, t_m)$$

be an ordered skeleton of length m. We put

$$\delta^{\leq}_{\mathcal{S}}(x, x') := (\exists x_2 \ldots \exists x_m)\epsilon_{\mathcal{S}}(x, x_2, \ldots, x_m, x')$$

and

$$\gamma^{\leq}_{\mathcal{S}}(x, x') := (\exists x_1 \ldots \exists x_m)\theta_{\mathcal{S}}(x, x_1, \ldots, x_m, x')$$

where ϵ and θ are defined as in Section 3.1. Notice that similar comments to those in Remark 3.1.1.1 hold, in particular, if A is a right and B a left S-poset, then the pair $(a,b) \in A \times B$ is connected to the pair $(a',b') \in A \times B$ via an ordered tossing with ordered skeleton $\mathcal{S}$ if and only if $\delta^{\leq}_{\mathcal{S}}(a,a')$ is true in A and $\gamma^{\leq}_{\mathcal{S}}(b,b')$ is true in B.

3.2.1 Axiomatisability of $\mathcal{C}$-$\mathcal{PF}$ S-posets with Condition (Free$^{\leq}$)

We define the condition corresponding to (Free) in this case as follows;

Definition 3.2.1.1. We say that $\mathcal{C}$ satisfies Condition (Free)$^{\leq}$ if for each ordered skeleton $\mathcal{S}$ there is an embedding $\kappa_{\mathcal{S}} : V_{\mathcal{S}} \to V'_{\mathcal{S}}$ in $\mathcal{C}$ and $v_{\mathcal{S}}, v'_{\mathcal{S}} \in V_{\mathcal{S}}$ such that $\delta^{\leq}_{\mathcal{S}}(v_{\mathcal{S}}\kappa_{\mathcal{S}}, v'_{\mathcal{S}}\kappa_{\mathcal{S}})$ is true in $V'_{\mathcal{S}}$ and further for any embedding $\mu : A \to A' \in \mathcal{C}$ and any $a, a' \in A$ such that $\delta^{\leq}_{\mathcal{S}}(a\mu, a'\mu)$ is true in A' there is a morphism $\nu : V'_{\mathcal{S}} \to A'$ such that $u_{\mathcal{S}}\kappa_{\mathcal{S}}\nu = a\mu$, $u'_{\mathcal{S}}\kappa_{\mathcal{S}}\nu = a'\mu$ and $V_{\mathcal{S}}\kappa_{\mathcal{S}}\nu \subseteq A\mu$.

Now we show that if $\mathcal{C}$ is a class of embeddings of right S-posets satisfying Condition (Free)$^{\leq}$, then to show that a left S-poset B is in $\mathcal{C}$-$\mathcal{PF}$, that is, B is $\mathcal{C}$-po-flat, it is enough to show that for any ordered skeleton $\mathcal{S}$, if $(v_{\mathcal{S}}\kappa_{\mathcal{S}}, b)$ and $(v'_{\mathcal{S}}\kappa_{\mathcal{S}}, b')$ are connected by an ordered tossing over $V'_{\mathcal{S}}$ and B with ordered skeleton $\mathcal{S}$, then $(v_{\mathcal{S}}, b)$ and $(v'_{\mathcal{S}}, b')$ are connected by an ordered tossing over $V_{\mathcal{S}}$ and B.

Lemma 3.2.1.2. *Let $\mathcal{C}$ be a class of embeddings of right S-posets satisfying Condition (Free)$^{\leq}$. Then the following are equivalent for a left S-poset B:*

(i) B is $\mathcal{C}$-po-flat;

*(ii) $- \otimes B$ maps the embeddings $\kappa_{\mathcal{S}} : V_{\mathcal{S}} \to V'_{\mathcal{S}}$ in the category **Pos-S** to embeddings in the category of **Pos**, for every ordered skeleton $\mathcal{S}$;*

(iii) if $v_{\mathcal{S}}\kappa_{\mathcal{S}} \otimes b \leq v'_{\mathcal{S}}\kappa_{\mathcal{S}} \otimes b'$ as the inequality is given by an ordered tossing over $V'_{\mathcal{S}}$ and B with ordered skeleton $\mathcal{S}$, then $v_{\mathcal{S}} \otimes b \leq v'_{\mathcal{S}} \otimes b'$ in $V_{\mathcal{S}} \otimes B$.

Proof. See Appendix. □

We recall Lemma 3.1.1.6 in terms of po-flatness.

Lemma 3.2.1.3. *Let $\mathcal{C}$ be a class of embeddings of right S-posets, satisfying Condition (Free)$^{\leq}$. Let $\overline{\mathcal{C}}$ be the set of products of morphisms in $\mathcal{C}$. If a left S-poset B is $\mathcal{C}$-po-flat, then it is $\overline{\mathcal{C}}$-po-flat.*

Proof. See Appendix. □

Theorem 3.2.1.4. *Let $\mathcal{C}$ be a class of embeddings of right S-posets satisfying Condition (Free)$^{\leq}$. Then the following conditions are equivalent for a pomonoid S;*

(i) the class $\mathcal{C}$-$\mathcal{PF}$ is axiomatisable;

(ii) the class $\mathcal{C}$-$\mathcal{PF}$ is closed under formation of ultraproducts;

(iii) for every ordered skeleton $\mathcal{S}$ there exist finitely many replacement ordered skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(\mathcal{S})}$ such that, for any embedding $\gamma : A \to A'$ in $\mathcal{C}$ and any $\mathcal{C}$-po-flat left S-poset B, if $a\gamma \otimes b \leq a'\gamma \otimes b' \in A' \otimes B$ via an ordered tossing $\mathcal{T}$ with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $a \otimes b \leq a' \otimes b'$ via an ordered tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(\mathcal{S})\}$;

(iv) for every ordered skeleton $\mathcal{S}$ there exists finitely many replacement ordered skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\beta(\mathcal{S})}$ such that, for any $\mathcal{C}$-po-flat left S-poset B, if $(v_{\mathcal{S}}\kappa_{\mathcal{S}}, b)$ and $(v'_{\mathcal{S}}\kappa_{\mathcal{S}}, b')$ are such that $v_{\mathcal{S}}\kappa_{\mathcal{S}} \otimes b \leq v'_{\mathcal{S}}\kappa_{\mathcal{S}} \otimes b'$ by an ordered tossing $\mathcal{T}$ over $V'_{\mathcal{S}}$ and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $v_{\mathcal{S}} \otimes b \leq v'_{\mathcal{S}} \otimes b'$ are connected by an ordered tossing $\mathcal{T}'$ over $V_{\mathcal{S}}$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \beta(\mathcal{S})\}$.

Proof. See Appendix. □

To show that the class of all embeddings of right S-posets has Condition (Free)$^{\leq}$, for an ordered skeleton

$$\mathcal{S} = (s_1, t_1, \ldots, s_m, t_m)$$

we let F^m be the free right S-poset

$$xS \,\dot{\cup}\, x_2 S \,\dot{\cup}\, \ldots \dot{\cup}\, x_m S \,\dot{\cup}\, x' S$$

and put

$$T_{\mathcal{S}} = \big\{(xs_1, x_2t_1), (x_2s_2, x_3t_2), \ldots, (x_ms_m, x't_m)\big\}.$$

Let $\equiv_{\mathcal{S}}$ be $\equiv_{T_{\mathcal{S}}}$, the S-poset congruence which is induced by $T_{\mathcal{S}}$. Abbreviate the order $\preceq_{T_{\mathcal{S}}}$ by $\leq_{\mathcal{S}}$ so that $[a] \leq_{\mathcal{S}} [b]$ for all $(a, b) \in T_{\mathcal{S}}$. We defined an *ordered standard tossing* from $([x], b)$ to $([x'], b')$ where $b, b' \in B$ for a left S-poset B in the analogous way to a double ordered standard tossing.

If B is a left S-poset and

$$b,\ b_1,\ , \cdots, b_m,\ b' \in B$$

are such that the right hand side of the following inequalities exists, then the ordered tossing

$$\begin{array}{rclcrcl}
 & & & & b & \leq & s_1b_1 \\
[x]s_1 & \leq & [x_2]t_1 & & t_1b_1 & \leq & s_2b_2 \\
[x_2]s_2 & \leq & [x_3]t_2 & & t_2b_2 & \leq & s_3b_3 \\
 & \vdots & & & & \vdots & \\
[x_{m-1}]s_{m-1} & \leq & [x_m]t_{m-1} & & t_{m-1}b_{m-1} & \leq & s_mb_m \\
[x_m]s_m & \leq & [x']t_m & & t_mb_m & \leq & b'
\end{array}$$

over $F^m/\equiv_{\mathcal{S}}$ and B is called a *ordered standard tossing* with ordered skeleton $\mathcal{S}$.

The proof of the next lemma follows that of Lemma 3.1.1.4.

Lemma 3.2.1.5. *The following conditions are equivalent for a left S-poset B:*

(i) B is po-flat;

(ii) $-\otimes B$ maps the embeddings of $[x]S \cup [x']S$ into $F^m/\equiv_{\mathcal{S}}$ in the category ***Pos*-S** *to embeddings in the category of* **Pos**, *for every ordered skeleton $\mathcal{S}$;*

(iii) if the inequality $[x] \otimes b \leq [x'] \otimes b'$ holds by an ordered standard tossing over $F^m/\equiv_{\mathcal{S}}$ and B with ordered skeleton $\mathcal{S}$, then $[x] \otimes b \leq [x'] \otimes b'$ holds by an ordered tossing over $[x]S \cup [x']S$ and B.

Proof. See Appendix. ☐

As in Lemma 3.1.1.5 we then have:

Lemma 3.2.1.6. *The class Pos-S of all right S-posets has Condition (Free)$^{\leq}$.*

Proof. See Appendix. ☐

We can now deduce the following corollary, which appears without proof in [52]. The reader should note that in [52], (weakly) po-flat S-posets are referred to as being (weakly) flat. Recall that we denote the class of po-flat left S-poset by $\mathcal{PF}$.

Corollary 3.2.1.7. *[52] The following conditions are equivalent for a pomonoid S:*
(i) the class $\mathcal{PF}$ is axiomatisable;
(ii) the class $\mathcal{PF}$ is closed under formation of ultraproducts;
(iii) for every ordered skeleton $\mathcal{S} \in \mathbb{OS}$ there exist finitely many ordered replacement skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(\mathcal{S})}$ such that, for any right S-poset embedding $\gamma : A \to A'$, and any po-flat left S-poset B, if $a\gamma \otimes b, \leq a'\gamma \otimes b' \in A' \times B$ via an ordered tossing $\mathcal{T}$ over A' and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $a \otimes b \leq a' \otimes b'$ via an ordered tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(\mathcal{S})\}$;
(iv) for every ordered skeleton $\mathcal{S}$ there exist finitely many replacement ordered skeletons $\mathcal{S}_1, \ldots, \mathcal{S}_{\alpha(\mathcal{S})}$ such that, for any right S-poset A and any po-flat left S-poset B, if $a \otimes b \leq a' \otimes b'$ exists in $A \otimes B$ via an ordered tossing $\mathcal{T}$ with ordered skeleton $\mathcal{S}$, then $a \otimes b \leq a' \otimes b'$ also exists in $(aS \cup a'S) \otimes B$ by a replacement ordered tossing $\mathcal{T}'$ such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(\mathcal{S})\}$.
(v) for every ordered skeleton $\mathcal{S} \in \mathbb{OS}$ there exists finitely many ordered replacement skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\beta(\mathcal{S})}$ such that, for any po-flat left S-poset B, if $[x] \otimes b \leq [x'] \otimes b'$ via an ordered tossing $\mathcal{T}$ over $F^m / \equiv_{\mathcal{S}}$ and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $[x] \otimes b \leq [x'] \otimes b'$ via an ordered tossing $\mathcal{T}'$ over $[x]S \cup [x']S$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \beta(\mathcal{S})\}$.

3.2.2 Axiomatisability of $\mathcal{C}$-$\mathcal{PF}$ without Condition (Free$^{\leq}$)

We now drop our assumption that Condition (Free)$^{\leq}$ holds. The proof of the next result follows that of Theorem 3.1.2.1.

Theorem 3.2.2.1. *The following conditions are equivalent for a monoid S:*
(i) the class $\mathcal{C}$-$\mathcal{PF}$ is axiomatisable;
(ii) the class $\mathcal{C}$-$\mathcal{PF}$ is closed under ultraproducts;
(iii) for every ordered skeleton $\mathcal{S}$ over S and $a, a' \in A$, where $\mu : A \to A'$ is in $\mathcal{C}$, there exist finitely many ordered skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(a,\mathcal{S},a',\mu)}$, such that for any $\mathcal{C}$-po-flat left S-act B, if $a\mu \otimes b \leq a'\mu \otimes b'$ by an ordered tossing $\mathcal{T}$ over A' and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $a \otimes b \leq a' \otimes b'$ by an ordered tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(a, \mathcal{S}, a', \mu)\}$.

Proof. See Appendix. ☐

As for $\mathcal{WF}$ and $\mathcal{PWF}$, Theorem 3.2.2.1 can be specialised to the cases where $\mathcal{C}$ consists of all inclusions of (principal) right ideals of S into S, thus giving necessary and sufficient conditions on S such that $\mathcal{WPF}$ (a result also found in [52]) ($\mathcal{PWPF}$) is axiomatisable. The statements of these results are obtained from those of Corollaries 3.1.2.2 and 3.1.2.3, with the word 'double' omitted and 'flat' replaced by 'po-flat'.

As in Section 3.1, we need not make specific mention of the embeddings of right ideals in S, since they are always inclusion.

Corollary 3.2.2.2. *The following are equivalent for an ordered monoid S:*
(i) the class $\mathcal{WPF}$ is axiomatisable;
(ii) the class $\mathcal{WPF}$ is closed under ultraproducts;
(iii) for every ordered skeleton $\mathcal{S}$ over S and $a, a' \in S$ there exists finitely many ordered skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\beta(a,\mathcal{S},a')}$ over S, such that for any weakly po-flat left S-poset B, if $a \otimes b \leq a' \otimes b'$ in $S \otimes B$ by an ordered tossing $\mathcal{T}$ over S and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $a \otimes b \leq a' \otimes b'$ by an ordered tossing $\mathcal{T}'$ over $aS \cup a'S$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$ for some $k \in \{1, \cdots, \beta(a, \mathcal{S}, a')\}$.

We follow same line as for axiomatisability of weakly flat S-acts considering $a \otimes b \leq a' \otimes b'$ instead of the equality $a \otimes b = a' \otimes b'$.

We recall that we will denote the class of all principally weakly po-flat S-posets by $\mathcal{PWPF}$.

Corollary 3.2.2.3. *The following conditions are equivalent for a pomonoid $\mathcal{S}$:*
(i) the class $\mathcal{PWPF}$ is axiomatisable;
(ii) the class $\mathcal{PWPF}$ is closed under ultraproducts;
(iii) for every ordered skeleton $\mathcal{S}$ over S and $a \in S$ there exists finitely many ordered skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\gamma(a,\mathcal{S})}$ over S, such that for any principally weakly po-flat left S-poset B, if $a \otimes b \leq a \otimes b'$ in $S \otimes B$ by an ordered tossing $\mathcal{T}$ over S and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $a \otimes b \leq a \otimes b'$ by an ordered tossing $\mathcal{T}'$ over aS and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$ for some $k \in \{1, \cdots, \gamma(a, \mathcal{S})\}$.

3.3 Axiomatisability of specific classes of S-posets

We now concentrate on axiomatisability problems for certain classes of S-posets, in the cases that we can avoid the 'replacement tossings' arguments of the Sections 3.1 and 3.2. We consider the classes of S-posets satisfying Condition (P) and (E) (which together give us the class of strongly flat S-posets), and the classes of S-posets satisfying Condition (EP), (W), (P_w), (PWP) and (PWP_w).

Let S be a pomonoid and let $(s,t) \in S \times S$. We define

$$\mathbf{R}^{\leq}(s,t) = \{(u,v) \in S \times S : su \leq tv\} \text{ and } \mathbf{r}^{\leq}(s,t) = \{u \in S : su \leq tu\}$$

so that $\mathbf{R}^{\leq}(s,t)$ is either empty or is an S-subposet of the right S-poset $S \times S$, and $\mathbf{r}^{\leq}(s,t)$ is either empty or is a right ideal of S. Note that in [52], $\mathbf{R}^{\leq}(s,t)$ and $\mathbf{r}^{\leq}(s,t)$ are written as $\mathbf{R}^{<}(s,t)$ and $\mathbf{r}^{<}(s,t)$.

Remark 3.3.0.4. We would like to mention here, that, with the same techniques as in Chapter 2, the "elements methods" can be related to 'replacement tossings methods" for the classes of $\mathcal{P}, \mathcal{E}, \mathcal{EP}, \mathcal{PWP}, \mathcal{PWP}_w$ and $\mathcal{W}$.

3.3.1 Axiomatisability of $\mathcal{P}$, $\mathcal{E}$ and $\mathcal{SF}$

For completeness we give the following results from [52]. The proofs follow closely those of the unordered case in [25],[29] and [30] and may be found in the Appendix.

Theorem 3.3.1.1. *[52] The following conditions are equivalent for an ordered monoid S:*
(i) the class of S-posets satisfying Condition (P) is axiomatisable;
(ii) every ultraproduct of S-posets satisfying Condition (P) also satisfies Condition (P);
(iii) every ultrapower of S satisfies Condition (P);
(iv) for any $s,t \in S$, $\mathbf{R}^{\leq}(s,t) = \emptyset$ or $\mathbf{R}^{\leq}(s,t)$ is finitely generated as a right S-subact of $S \times S$.

Proof. See Appendix. □

Theorem 3.3.1.2. *[52] The following conditions are equivalent for an ordered monoid S:*
(i) the class of S-posets satisfying Condition (E) is axiomatisable;
(ii) every ultraproduct of S-posets satisfying Condition (E) also satisfies Condition (E);
(iii) every ultrapower of S satisfies Condition (E);
(iv) for any $s,t \in S$, $\mathbf{r}^{\leq}(s,t) = \emptyset$ or $\mathbf{r}^{\leq}(s,t)$ is finitely generated as a right ideal of S.

Proof. See Appendix. □

Theorem 3.3.1.3. *The following conditions are equivalent for an ordered monoid S:*
(i) the class $\mathcal{SF}$ is axiomatisable;
(ii) the class $\mathcal{SF}$ is closed under ultraproducts;
(iii) every ultrapower of S is in $\mathcal{SF}$;
(iv) for every $s,t \in S$, $\mathbf{R}^{\leq}(s,t)$ is empty or is finitely generated and $\mathbf{r}^{\leq}(s,t)$ is empty or is finitely generated.

Proof. Proof follows from Theorem 3.3.1.1 and Theorem 3.3.1.2. □

3.3.2 Axiomatisability of $\mathcal{EP}$

We recall that, in the terminology introduced above, a left S-poset A satisfies Condition (EP) if, given $sa \leq ta$ for any $s, t \in S$ and $a \in A$, we have that

$$a = ua' = va' \text{ for some } (u, v) \in \mathbf{R}^{\leq}(s, t) \text{ and } a' \in A.$$

We will denote the class of left S-posets satisfying Condition (EP) by $\mathcal{EP}$.

Theorem 3.3.2.1. *The following conditions are equivalent for a pomonoid S:*
(i) the class $\mathcal{EP}$ is axiomatisable;
(ii) the class $\mathcal{EP}$ is closed under ultraproducts;
(iii) for any $s, t \in S$, either $sa \not\leq ta$ for all $a \in A \in \mathcal{EP}$ or there exists a finite subset f of $\mathbf{R}^{\leq}(s, t)$, such that for any $a \in A \in \mathcal{EP}$

$$sa \leq ta \Rightarrow (a, a) = (u, v)b \text{ for some } (u, v) \in f \text{ and } b \in A.$$

Proof. (i) implies (ii): this follows from Łos's Theorem.
(ii) implies (iii): suppose $sa \leq ta$ for some $a \in A \in \mathcal{EP}$ and for each finite subset f of $\mathbf{R}^{\leq}(s, t)$, there exists $A_f \in \mathcal{EP}$ and $a_f \in A_f$ with $sa_f \leq ta_f$ and $(a_f, a_f) \notin fA_f$.
Let J be the set of finite subsets of $\mathbf{R}^{\leq}(s, t)$. For each $(u, v) \in \mathbf{R}^{\leq}(s, t)$ we define

$$J_{(u,v)} = \{f \in J : (u, v) \in f\}.$$

As each intersection of finitely many of the sets $J_{(u,v)}$ is non-empty, we are able to define an ultrafilter Φ on J, such that each $J_{(u,v)} \in \Phi$ for all $(u, v) \in \mathbf{R}^{\leq}(s, t)$.
Now $s(a_f) \leq t(a_f)$ in A where $A = \prod_{f \in J} A_f$, and it follows that the inequality $s(a_f)_\Phi \leq t(a_f)_\Phi$ holds in $\mathcal{U}$ where $\mathcal{U} = \prod_{f \in J} A_f/\Phi$. By assumption $\mathcal{U}$ lies in $\mathcal{EP}$, so there exists $(u, v) \in \mathbf{R}^{\leq}(s, t)$, and $r_f \in A_f$ such that

$$(a_f)_\Phi = u(r_f)_\Phi = v(r_f)_\Phi.$$

As Φ is closed under finite intersections, there must exist $T \in \Phi$ such that $a_f = ur_f = vr_f$ for all $f \in T$.
Now suppose that $f \in T \cap J_{(u,v)}$, then $(u, v) \in f$ and

$$(a_f, a_f) = (u, v)r_f \in fA_f$$

a contradiction to our assumption, hence (ii) implies (iii).
(iii) implies (i): given that (iii) holds, we give an explicit set of sentences that axiomatises $\mathcal{EP}$.
For any element $\rho = (s, t) \in S \times S$ with $sa \leq ta$, for some $a \in A$ where $A \in \mathcal{EP}$, we choose and fix a finite set of elements $\{(u_{\rho 1}, v_{\rho 1}) \cdots (u_{\rho\, n(\rho)}, v_{\rho\, n(\rho)})\}$ of $\mathbf{R}^{\leq}(\rho)$ as guaranteed by (iii). We define sentences ϕ_ρ of $L_S^{\leq}$ as follows:
If $sa \not\leq ta$ for all $a \in A \in \mathcal{EP}$, let

$$\phi_\rho := (\forall x)(sx \not\leq tx);$$

otherwise,

$$\phi_\rho := (\forall x)\big(sx \leq tx \to (\exists z)(\bigvee_{i=1}^{n(\rho)} (x = u_{\rho i}z \ = v_{\rho i}z))\big).$$

Let

$$\sum\nolimits_{\mathcal{EP}} = \{\phi_\rho : \rho \in S \times S\}.$$

We claim that $\sum_{\mathcal{EP}}$ axiomatises the class $\mathcal{EP}$.

Suppose that $A \in \mathcal{EP}$ and $\rho = (s,t) \in S \times S$. If $sb \not\leq tb$, for all $b \in B \in \mathcal{EP}$, then certainly this is true for A, so that $A \models \phi_\rho$.

Suppose on the other hand that $sb \leq tb$, for some $b \in B \in \mathcal{EP}$; then

$$\phi_\rho := (\forall x)\big(sx \leq tx \to (\exists z)(\bigvee_{i=1}^{n(\rho)} (x = u_{\rho i}z \ = v_{\rho i}z))\big).$$

Suppose $sa \leq ta$ where $a \in A$. As $A \in \mathcal{EP}$, (iii) tells us that there is an element $b \in A$ and $(u_{\rho i}, v_{\rho i})$ for some $i \in \{1, \dots, n(\rho)\}$ with $a = u_{\rho i}b = v_{\rho i}b$. Hence $A \models \phi_\rho$.

Conversely suppose that A is a model of $\sum_{\mathcal{EP}}$ and $sa \leq ta$ where $s,t \in S$ and $a \in A$. We cannot have that ϕ_ρ is $(\forall x)(sx \not\leq tx)$. It follows that for some $b \in B \in \mathcal{EP}$ we have $sb \leq tb$, and

$$f = \{(u_{\rho\, 1}, v_{\rho\, 1}), \cdots, (u_{\rho\, n(\rho)}, v_{\rho\, n(\rho)})\}$$

exists as in (iii) and ϕ_ρ is

$$(\forall x)\big(sx \ \leq \ tx \ \to \ (\exists\, z)(\bigvee_{i=1}^{n(\rho)} (\, x \ = \ u_{\rho i}z \ = \ v_{\rho i}\, z))\big).$$

Hence there exists an element $c \in A$ with $a = u_{\rho i}c = v_{\rho i}c$ for some $i \in \{1, 2, \dots, n(\rho)\}$. By definition of $u_{\rho i}, v_{\rho i}$ we have $su_{\rho i} \ \leq tv_{\rho i}$. Thus A satisfies Condition (EP) and so $\sum_{\mathcal{EP}}$ axiomatises $\mathcal{EP}$. □

3.3.3 Axiomatisability of $\mathcal{PWP}$

We recall Condition (PWP) for a left S-poset B: for all $b, b' \in B$ and $s \in S$, if $sb \leq sb'$ then there exits $u, u' \in S$ and $b'' \in B$ such that $b = ub'', b' = u'b''$ and $su \leq su'$. The class of left S-posets satisfying Condition (PWP) is denoted by $\mathcal{PWP}$. We solve the axiomatisability problem for $\mathcal{PWP}$ by following similar lines to those for $\mathcal{EP}$. For any $s \in S$ we have that $\mathbf{R}^{\leq}(s,s) \neq \emptyset$ and this enables us to concentrate on ultrapowers of S, resulting in a slight simplification as compared to the final condition in Theorem 3.3.2.1.

Theorem 3.3.3.1. *The following conditions are equivalent for a pomonoid S:*

(i) the class $\mathcal{PWP}$ is axiomatisable;

(ii) the class $\mathcal{PWP}$ is closed under ultraproducts;

(iii) every ultrapower of S lies in $\mathcal{PWP}$;

(iv) $\mathbf{R}^{\leq}(s,s)$ is finitely generated for any $s \in S$.

Proof. See Appendix. □

3.3.4 Axiomatisability of $\mathcal{P}_w$

We recall that a left S-poset A satisfies Condition (P_w) if for any $a, a' \in A$ and $s, t \in S$, if $sa \leq ta'$, then there exists $a'' \in A$ and $u, u' \in S$ with $(u, u') \in \mathbf{R}^{\leq}(s,t)$, $a \leq ua''$ and $u'a'' \leq a'$. The class of left S-posets satisfying Condition (P_w) is denoted by $\mathcal{P}_w$.

Theorem 3.3.4.1. *The following conditions are equivalent for a pomonoid S:*
(i) the class $\mathcal{P}_w$ is axiomatisable;
(ii) the class $\mathcal{P}_w$ is closed under ultraproducts;
(iii) every ultrapower of S satisfies Condition (P_w);
(iv) for any $\rho = (s,t) \in S \times S$, either $\mathbf{R}^{\leq}(s,t) = \emptyset$ or there exist finitely many

$$(u_{\rho 1}, v_{\rho 1}), \ldots, (u_{\rho n(\rho)}, v_{\rho n(\rho)}) \in \mathbf{R}^{\leq}(s,t)$$

such that for any $(x,y) \in \mathbf{R}^{\leq}(s,t)$,

$$x \leq u_{\rho i}h \text{ and } v_{\rho i}h \leq y$$

for some $i \in \{1, \ldots n(\rho)\}$ and $h \in S$.

Proof. (iii) implies (iv): suppose that every ultrapower of S satisfies Condition (P_w) but (iv) does not hold. Then there exists $\rho = (s,t) \in \mathbf{R}^{\leq}(s,t)$ with $\mathbf{R}^{\leq}(s,t) \neq \emptyset$ but such that no finite subset of $\mathbf{R}^{\leq}(s,t)$ exists as in (iv).

Let $\{(u_\beta, v_\beta) : \beta < \gamma\}$ be a set of minimal (infinite) cardinality γ contained in $\mathbf{R}^{\leq}(s,t)$ such that if $(x,y) \in \mathbf{R}^{\leq}(s,t)$, then

$$x \leq u_\beta h \text{ and } v_\beta h \leq y$$

for some $\beta < \gamma$ and $h \in S$. From the minimality of γ we may assume that for any $\alpha < \beta < \gamma$, it is not true that both

$$u_\beta \leq u_\alpha h \text{ and } v_\alpha h \leq v_\beta$$

for any $h \in S$.

Let Φ be a uniform ultrafilter on γ, that is Φ is an ultrafilter on γ such that all sets in Φ have cardinality γ. Let $\mathcal{U} = S^\gamma/\Phi$, by assumption $\mathcal{U}$ satisfies Condition (P_w).

Since $su_\beta \leq tv_\beta$ for all $\beta < \gamma$, $s(u_\beta)_\Phi \leq t(v_\beta)_\Phi$. As $\mathcal{U}$ satisfies Condition (P_w), there exists $(u,v) \in \mathbf{R}^{\leq}(s,t)$ and $(w_\beta)_\Phi \in \mathcal{U}$ such that

$$(u_\beta)_\Phi \leq u(w_\beta)_\Phi \text{ and } v(w_\beta)_\Phi \leq (v_\beta)_\Phi.$$

Let $D \in \Phi$ be such that

$$u_\beta \leq uw_\beta \text{ and } vw_\beta \leq v_\beta$$

for all $\beta \in D$. Now $(u,v) \in \mathbf{R}^{\leq}(s,t)$ so that

$$u \leq u_\sigma h \text{ and } v_\sigma h \leq v$$

for some $\sigma < \gamma$. Choose $\beta \in D$ with $\beta > \sigma$. Then

$$u_\beta \leq uw_\beta \leq u_\sigma hw_\beta \text{ and } v_\sigma hw_\beta \leq vw_\beta \leq v_\beta,$$

a contradiction. Thus (iv) holds. (iv) implies (i): suppose that (iv) holds.

Let $\rho = (s,t) \in S \times S$. If $\mathbf{R}^{\leq}(s,t) = \emptyset$ we put

$$\Omega_\rho := (\forall x)(\forall y)(sx \not\leq ty).$$

If $\mathbf{R}^{\leq}(s,t) \neq \emptyset$, let

$$(u_{\rho 1}, v_{\rho 1}), \ldots, (u_{\rho\, n(\rho)}, v_{\rho\, n(\rho)}) \in \mathbf{R}^{\leq}(s,t)$$

be the finite set given by our hypothesis, and put

$$\Omega_\rho := (\forall x)(\forall y)\big(sx \leq ty \to (\exists z)(\bigvee_{i=1}^{n(\rho)} (x \leq u_{\rho i}\, z \wedge v_{\rho i}\, z \leq y))\big).$$

Let

$$\sum\nolimits_{\mathcal{P}_w} = \{\Omega_\rho : \rho \in S \times S\}.$$

We claim that $\sum_{\mathcal{P}_w}$ axiomatises $\mathcal{P}_w$.

Let $A \in \mathcal{P}_w$ and let $\rho = (s,t) \in S \times S$. Suppose first that $\mathbf{R}^{\leq}(s,t) = \emptyset$. If $sa \leq tb$ for some $a, b \in S$, then as A satisfies (P_w) we have, in particular, that $\mathbf{R}^{\leq}(s,t) \neq \emptyset$, a contradiction. Hence $A \models \Omega_\rho$.

On the other hand, if $\mathbf{R}^{\leq}(s,t) \neq \emptyset$, then

$$\Omega_\rho := (\forall x)(\forall y)\big(sx \leq ty \to (\exists z)(\bigvee_{i=1}^{n(\rho)} (x \leq u_{\rho i}\, z \wedge v_{\rho i}\, z \leq y))\big).$$

If $sa \leq tb$ where $a, b \in A$, then there exists $(u,v) \in \mathbf{R}^{\leq}(s,t)$ and $c \in A$ with

$$a \leq uc \text{ and } vc \leq b.$$

By hypothesis we have that

$$u \leq u_{\rho i} h \text{ and } v_{\rho i} h \leq v$$

for some $h \in S$ and $i \in \{1, \ldots, n(\rho)\}$. Now

$$a \leq uc \leq u_{\rho i} hc \text{ and } v_{\rho i} hc \leq vc \leq b$$

so that (with $z = hc$), $A \models \Omega_\rho$. Hence $A \models \sum_{\mathcal{P}_w}$.

Conversely, suppose that $A \models \sum_{\mathcal{P}_w}$ and $sa \leq tb$ for some $\rho = (s,t) \in S \times S$ and $a, b \in S$. We must therefore have that $\mathbf{R}^{\leq}(s,t) \neq \emptyset$ and consequently, Ω_ρ is

$$(\forall x)(\forall y)\big(sx \leq ty \to (\exists z)(\bigvee_{i=1}^{n(\rho)} (x \leq u_{\rho i}\, z \wedge v_{\rho i}\, z \leq y))\big).$$

Hence $a \leq u_{\rho i} c$ and $v_{\rho i} c \leq b$ for some $c \in A$. By definition, $(u_{\rho i}, v_{\rho i}) \in \mathbf{R}^{\leq}(s,t)$, so that A lies in $\mathcal{P}_w$. □

3.3.5 Axiomatisability of $\mathcal{PWP}_w$

We recall Condition (PWP_w) for a left S-poset B: for all $b, b' \in B$ and $s \in S$, if $sb \leq sb'$ then there exist $u, u' \in S$ and $b'' \in B$ such that $b \leq ub'', u'b'' \leq b'$ and $su \leq su'$. The class of left S-posets satisfying Condition (PWP_w) is denoted by $\mathcal{PWP}_w$.

We solve the axiomatisability problem for Condition (PWP_w) by following similar lines to those for Condition (P_w). Of course in this case $\mathbf{R}^{\leq}(s,s) \neq \emptyset$ for any $s \in S$ and so our result is as follows.

Theorem 3.3.5.1. *The following conditions are equivalent for a pomonoid S:*
(i) the class $\mathcal{PWP}_w$ is axiomatisable;
(ii) the class $\mathcal{PWP}_w$ is closed under ultraproducts;
(iii) every ultrapower of S satisfies Condition (PWP_w);
(iv) for any $s \in S$ there exists finitely many

$$(u_{\rho 1}, v_{\rho 1}), \ldots, (u_{\rho\, n(\rho)}, v_{\rho\, n(\rho)}) \in \mathbf{R}^{\leq}(s,s)$$

such that for any $(x,y) \in \mathbf{R}^{\leq}(s,s)$,

$$x \leq u_{\rho i} h \text{ and } v_{\rho i} h \leq y$$

for some $i \in \{1, \ldots n(\rho)\}$ and $h \in S$.

Proof. See Appendix. □

3.3.6 Axiomatisability of $\mathcal{W}$

For our final class defined by an interpolation condition, we consider $\mathcal{W}$. We recall Condition (W) for a left S-poset A: for all $a, a' \in A$ and $s, t \in S$, if $s\,a \leq t\,a'$, then there exists $a'' \in A$, $p \in sS$, $q \in tS$ such that $p \leq q$, $s\,a \leq p\,a''$ and $q\,a'' \leq t\,a'$. The class of left S-posets satisfying Condition (W) is denoted by $\mathcal{W}$.

Theorem 3.3.6.1. *The following conditions are equivalent for a pomonoid S:*
(i) the class $\mathcal{W}$ is axiomatisable;
(ii) the class $\mathcal{W}$ is closed under ultraproducts;
(iii) every ultrapower of S lies in $\mathcal{W}$;
(iv) for any $s, t \in S$ there exists an integer $n \geq 0$,

$$p_1, \cdots, p_n \in sS \text{ and } q_1, \cdots, q_n \in tS$$

such that for all $i \in \{1, \ldots, n\}$ we have $p_i \leq q_i$, and if $su \leq tv$ then there exists $i \in \{1, \cdots, n\}$ and $z \in S$ with

$$su \leq p_i z \text{ and } q_i z \leq tv.$$

Proof. (iii) implies (iv): suppose that every ultrapower of S satisfies Condition (W) but that (iv) fails. Then there exist $s, t \in S$ such that there does not exist any finite list $p_1, \ldots, p_n, q_1, \ldots, q_n$ satisfying the conditions of (iv).

Let γ be a cardinal minimal with respect to the existence of a set $\{(u_\beta, v_\beta) : \beta < \gamma\}$ such that $u_\beta \in sS$, $v_\beta \in tS$, $u_\beta \leq v_\beta$ and if $su \leq tv$ then there exists $\beta < \gamma$ and $z \in S$ with $su \leq u_\beta z$, $v_\beta z \leq tv$.

Certainly γ exists since we could consider $\{(sx, ty) : x, y \in S, sx \leq ty\}$. We are assuming that γ is infinite. By the minimality of γ we can assume that it is not true that for any $\gamma > \beta > \sigma$, we have both $u_\beta \leq u_\sigma k$ and $v_\sigma k \leq v_\beta$.

Let Φ be a uniform ultrafilter on γ and let $\mathcal{U} = S^\gamma/\Phi$; by assumption $\mathcal{U}$ satisfies Condition (W).

Since each $u_\beta \in sS, u_\beta = sx_\beta$ for some $x_\beta \in S$; similarly, $v_\beta = ty_\beta$ for some $y_\beta \in S$. Now $u_\beta \leq v_\beta$ for all $\beta < \gamma$, so that $s(x_\beta)_\Phi \leq t(y_\beta)_\Phi$ and as $\mathcal{U}$ satisfies Condition (W), there exists $(w_\beta)_\Phi \in \mathcal{U}$, $p \in sS$ and $q \in tS$ with

$$p \leq q, s(x_\beta)_\Phi \leq p(w_\beta)_\Phi \text{ and } q(w_\beta)_\Phi \leq t(y_\beta)_\Phi.$$

Let $D \in \Phi$ be such that

$$sx_\beta \leq pw_\beta \text{ and } qw_\beta \leq ty_\beta$$

for all $\beta \in D$. As $p \leq q$ there exists $\sigma < \gamma$ and $z \in S$ with

$$p \leq u_\sigma z \text{ and } v_\sigma z \leq q.$$

Hence, choosing $\beta \in D$ with $\beta > \sigma$,

$$u_\beta = sx_\beta \leq pw_\beta \leq u_\sigma z w_\beta \text{ and } v_\sigma z w_\beta \leq qw_\beta \leq ty_\beta = v_\beta,$$

a contradiction. Hence (iv) holds.

(iv) implies (i): suppose now that (iv) holds. For each $\rho = (s, t) \in S \times S$ let

$$p_{\rho 1}, \ldots, p_{\rho\, n(\rho)}, q_{\rho 1}, \ldots, q_{\rho\, n(\rho)}$$

be the list of elements of S guaranteed by (iv). If $n(\rho) = 0$, let

$$\Omega_\rho := (\forall x)(\forall y)(sx \not\leq ty).$$

If $n(\rho) \geq 1$, let

$$\Omega_\rho := (\forall x)(\forall y)\big(sx \leq ty \rightarrow (\exists z)(\bigvee_{i=1}^{n(\rho)} (sx \leq p_{\rho i} z \wedge q_{\rho i} z \leq ty))\big)$$

and let

$$\sum\nolimits_{\mathcal{W}} = \{\Omega_\rho : \rho \in S \times S\}.$$

We claim that $\sum_{\mathcal{W}}$ axiomatises $\mathcal{W}$.

Let $A \in \mathcal{W}$ and $\rho = (s, t) \in S \times S$. If $n(\rho) = 0$ and $sa \leq tb$, for some $a, b \in A$, then, in particular, $su \leq tv$ for some $u, v \in S$. By as (iv) holds this gives that $n(\rho) \geq 1$, a contradiction. Hence $A \models \Omega_\rho$.

Suppose now that $n(\rho) \geq 1$, so that

$$\Omega_\rho := (\forall x)(\forall y)\big(sx \leq ty \to (\exists z)(\bigvee_{i=1}^{n(\rho)} (sx \leq p_{\rho i}z \wedge q_{\rho i}z \leq ty))\big).$$

If $sa \leq tb$ for some $a, b \in A$, then there exists $p \in sS, q \in tS$ and $c \in A$ such that

$$p \leq q, sa \leq pc \text{ and } qc \leq tb.$$

By (iv),

$$p \leq p_{\rho i}z \text{ and } q_{\rho i}z \leq q$$

for some $i \in \{1, \ldots, n(\rho)\}$ and $z \in S$. Hence

$$sa \leq p_{\rho i}zc \text{ and } q_{\rho i}zc \leq tb$$

so that $A \models \Omega_\rho$. Hence $A \models \sum_{\mathcal{W}}$.

Conversely, if $A \models \sum_{\mathcal{W}}$ and $sa \leq tb$ for some $\rho = (s, t) \in S \times S$ and $a, b \in A$, then we must have $n(\rho) \geq 1$ and

$$\Omega_\rho = (\forall x)(\forall y)\big(sx \leq ty \to (\exists z)(\bigvee_{i=1}^{n(\rho)} (sx \leq p_{\rho i}z \wedge q_{\rho i}z \leq ty))\big).$$

Then

$$sa \leq p_{\rho i}c \text{ and } q_{\rho i}c \leq tb$$

for some $i \in \{1, \ldots, n(\rho)\}$ and $c \in A$. By choice of $p_{\rho i}, q_{\rho i}$, we see that $A \in \mathcal{W}$. Hence $\sum_{\mathcal{W}}$ axiomatises $\mathcal{W}$ as required.

□

3.4 Axiomatisability of $\mathcal{P}r$ and $\mathcal{F}r$

The question of the axiomatisability of $\mathcal{P}r$ was addressed in [52]. Without giving much detail, Pervukhin and Stepanova indicate that if every ultrapower of a pomonoid S is projective as a left S-poset, then it can be argued, following the corresponding proofs for S-acts, that S is poperfect, which here can be taken to mean $\mathcal{SF} = \mathcal{P}r$ in the class of left S-posets. In [52] this is then utilised to show that $\mathcal{P}r$ is axiomatisable if and only if $\mathcal{SF}$ is axiomatisable and $\mathcal{SF} = \mathcal{P}r$. Notice that in [52], the classes $\mathcal{SF}$ and $\mathcal{P}r$ are denoted by $\mathcal{SF}^<$ and $\mathcal{P}^<$, to distinguish them from the classes of strongly flat and projective left S-*acts*, a convention we have not followed here.

In Chapter 4 we show that a pomonoid S is left perfect as a *monoid* if and only if it is left perfect as a *pomonoid*. With this in mind we can give a short and direct proof of the following.

Theorem 3.4.0.2. *The following are equivalent for a pomonoid S:*

(i) the class $\mathcal{P}r$ is axiomatisable;

(ii) every ultrapower of S is projective as a left S-poset;

(iii) the class $\mathcal{SF}$ is axiomatisable and $\mathcal{SF} = \mathcal{P}r$.

Proof. Clearly we need only prove that (ii) implies (iii); suppose that (ii) holds.

Let $\mathcal{U} = S^{\gamma}/\Phi$ be an ultrapower of S *as a left S-act*, then

$$\mathcal{U} = \prod_{\gamma \in \wedge} S^{\gamma}/\equiv$$

where

$$(a_i) \equiv (b_i) \Leftrightarrow \{i : a_i = b_i\} \in \Phi$$

and

$$s(a_i)_{\Phi} = (sa_i)_{\Phi} \text{ is a well-defined } S\text{-action.}$$

Consider the corresponding ultrapower of S as a left S-poset, that is, $\mathcal{U}' = S^{\gamma}/\Phi$. Here $\equiv$ and the S-action are defined as before and

$$(a_i)_{\Phi} \leq (b_i)_{\Phi} \Leftrightarrow \{i : a_i \leq b_i\} \in \Phi \qquad (*).$$

In other words $\mathcal{U}'$ is $\mathcal{U}$ equipped with the partial order defined as in $(*)$.

We are supposing $\mathcal{U}'$ is projective as a left S-poset, that is, there exists a disjoint union $\bigcup_{i \in I} Se_i$ where e_is are idempotents, and an S-po-isomorphism $\theta : \mathcal{U}' \to \bigcup_{i \in I} Se_i$. Regarding $\bigcup_{i \in I} Se_i$ as an S-act, $\theta : \mathcal{U} \to \bigcup_{i \in I} Se_i$ is certainly an S-act isomorphism. We can conclude that every ultrapower of S as a left S-act is projective. From [30, Theorem 8.6], S is left perfect, so from Chapter 4 or [31, Theorem 6.3], S is left poperfect. Hence $\mathcal{SF} = \mathcal{P}r$. From [52, Theorem 4.8], we also have that $\mathcal{SF}$ is axiomatisable.

□

3.4.1 Axiomatisability of $\mathcal{F}r$.

To explain our result we need to recall the following definition from [29]. Let $e \in E(S)$, where $E(S)$ is the set of idempotents of a monoid S, and let $a \in S$. We say that $a = xy$ is an *e-good factorisation of a through x* if $y \neq wz$ for any w, z with $e = xw$ and $e \mathcal{L} w$ (see [29]).

Theorem 3.4.1.1. *The following conditions are equivalent for a pomonoid S:*

(i) every ultrapower of the left S-poset S is free;

(ii) $\mathcal{P}r$ is axiomatisable and S satisfies $(*)$: for all $e \in E(S) \setminus 1$, there exists a finite set $f \subseteq S$ such that any $a \in S$ has an e-good factorization through x, for some $x \in f$;

(iii) the class $\mathcal{F}r$ is axiomatisable.

Proof. $(i) \Rightarrow (ii)$: since every ultrapower of S is free as a left S-poset, it is free as a left S-act with the same argument as in Theorem 3.4.0.2. By [29, Theorem 5.3], S satisfies $(*)$. Also by Theorem 3.4.0.2, $\mathcal{P}r$ is axiomatisable.

(ii) implies (iii): if $\mathcal{P}r$ is axiomatisable, then every ultrapower of copies of S is projective as a left S-poset, and hence as a left S-act. From [30, Lemma 8.4], it follows that for any $e \in E(S)$ and $u \in S$, there are only finitely many $x \in S$ such that $e = ux$. This permits us to define the sentences φ_e as in [30]. Let $\sum_{\mathcal{P}r}$ be the set of sentences axiomatising the projective left S-posets. Then, as in [30, Theorem 9.1],

$$\sum_{\mathcal{P}r} \cup \big\{\varphi_e : e \in E(S) \setminus \{1\}\big\}.$$

axiomatises $\mathcal{F}r$.

$\square$

3.5 Some Open Problems

We aim to axiomatise the class of left S-posets satisfying Condition (WP), $(\mathrm{WP})_w$ and Condition (U). The first two conditions appear in [23]. The final one we define here. It may easily be seen that a principally weakly po-flat S-poset is weakly po-flat if and only if it satisfies Condition (U).

A left S-poset A satisfies Condition (WP) if for any S-pomorphism $\beta : (sS \cup tS) \to S$ where $s, t \in S$, and $a, a' \in A$, if $(s)\beta a \leq (t)\beta a'$ then there exist $a'' \in A$, $u, v \in S$ and $s', t' \in \{s, t\}$ such that $(s'u)\beta \leq (t'v)\beta$, $s \otimes a = s'u \otimes a''$ and $t \otimes a' = t'v \otimes a''$ in $(sS \cup tS) \otimes A$.

A left S-poset A satisfies Condition (WP_w) if for any S-pomorphism $\beta : (sS \cup tS) \to S$ where $s, t \in S$, and $a, a' \in A$, if $(s)\beta a \leq (t)\beta a'$ then there exist $a'' \in A$, $u, v \in S$ and $s', t' \in \{s, t\}$ such that $(s'u)\beta \leq (t'v)\beta$, $s \otimes a \leq s'u \otimes a''$ and $t'v \otimes a'' \leq t \otimes a'$ in $(sS \cup tS) \otimes A$.

Condition (U): for all $b, b' \in B$ and $s, s' \in S$, if $sb = sb'$ then there exists $b'' \in B$, $p \in sS, p' \in s'S$, with $p \leq p'$ and $sb = pb'' = p'b'' = s'b'$.

Chapter 4

Perfect Pomonoids

Following standard terminology from the theories of R-modules over a unital ring R, and S-acts over a monoid S, we say that a pomonoid S is *left poperfect* if every left S-poset has a projective cover.

Left perfect rings were introduced in 1960 in a seminal paper of Bass [1] and shown to be precisely those rings satisfying M_R, the descending chain condition on principal right ideals. In 1971, inspired by the results of Bass and Chase [12], Isbell was the first to study left perfect monoids [35]. The results of [35], together with those of Fountain [19], show that a monoid is left perfect if and only if it satisfies M_R and, in addition, a finitary condition dubbed Condition (A).

A further characterisation of left perfect rings was given in [12], where Chase proved that a ring is left perfect if and only if every flat left module is projective; the corresponding result for M-acts was demonstrated in [19].

In this Chapter we initiate the study of left poperfect pomonoids, concurrent with the recently appeared article [52] of Pervukhin and Stepanova. We introduce the terminology *po*perfect in order to distinguish the two possible definitions of left perfection of a pomonoid S, that is, as a monoid and as a pomonoid. In fact, they transpire to be equivalent. We show, as in [52] that a pomonoid S is left poperfect if and only if it satisfies (M_R) and the 'ordered' version Condition (A^{O}) of Condition (A) and that further, these conditions are equivalent to every strongly flat left S-poset being projective. On the other hand, we argue via an analysis of direct limits that Conditions (A) and (A^{O})

are equivalent, so that a pomonoid S is left perfect if and only if it is left poperfect. Our results and many of our techniques certainly correspond to those for monoids, but we must take careful account of the partial ordering on S, and in places introduce alternative strategies to those found in [35] and [19].

A left S-poset A over a pomonoid S is called a *cover* for a left S-poset B if there exists an S-poset epimorphism (an *S-po-epimorphism*) $\beta : A \to B$, such that any restriction of β to a proper S-subposet of A is not an S-po-epimorphism. Such a map β is called a *coessential S-po-epimorphism*. The pomonoid S is said to be *left poperfect* if every left S-poset has a projective cover. In this chapter as we deal exclusively with left S-acts and S-posets, 'S-acts' and 'S-posets' will always be *left*, without us explicitly saying so.

Perfection for Monoids

Left perfect monoids were introduced by Isbell in [35]. Characterisations of left perfect monoids were given in [35] and subsequently by Fountain [19] and Kilp [37]. Since their results inform ours, we now pause to explain them.

A *projective cover* of an S-act A is an S-act epimorphism $f : P \to A$ where P is a projective S-act, such that the restriction of f to any proper S-subact of P is not an S-act epimorphism.

We say that a monoid S is *left perfect* if every S-act has a projective cover.

A submonoid T of a monoid S is *right unitary* if $a, ba \in T$ implies that $b \in T$.

Lemma 4.0.0.2. *[39, Corollary 1.4.9] A submonoid T of a monoid S is right unitary if and only if T is the ρ-class of the identity, for some left congruence ρ on S.*

Let S be a monoid. A submonoid T of S is *right collapsible* if for any $a, b \in T$ we can find $c \in T$ with $ac = bc$. We recall from Chapter 1 the following conditions that we need below:

Condition (A): every S-act satisfies the ascending chain condition for cyclic subacts;

Condition (D): every right unitary submonoid of S contains a minimal left ideal generated by an idempotent;

Condition (K): every right collapsible submonoid of S contains a right zero;

Condition ($\mathbf{M_R}$): S satisfies the descending chain condition on principal right ideals.

Theorem 4.0.0.3. *[35, 19, 37] The following conditions are equivalent for a monoid S:*
(i) *S is left perfect;*
(ii) *S satisfies (A) and (D);*
(iii) *S satisfies (A) and (M_R);*
(iv) *every strongly flat S-act is projective;*
(v) *S satisfies (A) and (K).*

Perfection for Pomonoids

In a series of steps we prove the ordered analogue of Theorem 4.0.0.3. Some of our techniques are taken from those used in the monoid case, but these need careful adjustment to deal with the orderings involved; for some steps we develop new strategies. After giving the requisite background results in Section 4.1, we concentrate in Section 4.2 on characterising those pomonoids S such that every strongly flat S-poset is projective, and show that these are precisely those that satisfy Conditions (M_R) and ($\mathrm{A^O}$), the ordered version of Condition (A), defined as follows:

Condition ($\mathbf{A^O}$): every S-poset satisfies the ascending chain condition on cyclic S-subposets.

Conditions (A) and ($\mathrm{A^O}$) are intimately related to the behaviour of direct limits of sequences of copies of S. Careful analysis of these direct limits enables us to show that (A) and ($\mathrm{A^O}$) are equivalent for a pomonoid.

In Section 4.3 we turn our attention explicitly to poperfect pomonoids. We investigate conditions under which a subpomonoid is the ρ-class of the identity, for some left po-congruence ρ: we call such subpomonoids *right po-unitary subpomonoids*. We show that a pomonoid S is left poperfect if and only if it satisfies ($\mathrm{A^O}$) and ($\mathrm{D^O}$), the ordered version of (D), defined as for a pomonoid S follows:

Condition ($\mathbf{D^O}$): every right po-unitary subpomonoid of S contains a minimal left ideal generated by an idempotent.

We observe that if ρ is a left po-congruence on S such that S/ρ is strongly flat, then S/ρ is strongly flat as an S-act: it follows from [36] that ρ-class of the identity is right collapsible. In Section 4.4 we show that all strongly flat cyclic S-posets are projective if and only if S satisfies (K).

In Section 4.5 we show that in the presence of Condition (A^{O}), Conditions (M_R) and (D^{O}) are equivalent. One way is relatively straightforward, but to show that (D^{O}) follows from (M_R) we require a mixture of the techniques of [35] and a classic semigroup theoretic argument. This completes the proof of the following theorem.

Theorem 4.0.0.4. *The following conditions are equivalent for a pomonoid S:*
(i) *S is left poperfect;*
(ii) *S satisfies (A^O) and (D^O);*
(iii) *S satisfies (A^O) and (M_R);*
(iv) *every strongly flat S-poset is projective;*
(v) *S satisfies (A^O) and (K).*

Since (A) and (A^{O}) are interchangeable, the conditions of the above result are also equivalent to those in Theorem 4.0.0.3.

Note that our results are more extensive than those that appeared in [52]. They have not checked that (A) and (A^{O}) are equivalent, or considered Condition (K) or (D^{O}), and so they have not shown that perfection and po-perfection are equivalent for a pomonoid.

Some of the minor results in this Chapter have recently been announced, without proof, in [53]. We note, however, that the author of [53] does not distinguish between congruence classes of S-poset congruences, and congruence classes of S-act congruences, a distinction we feel to be necessary. For completeness we provide proofs, whilst making reference to [53].

4.1 Preliminaries

In this section we outline the concepts related to pomonoids and S-posets needed for the rest of the chapter; for definitions relating to acts over monoids, we refer the reader to Chapter 1 and the monograph [39]. Throughout this chapter, S will denote a pomonoid.

We need to recall the notion of congruence for S-posets. Let A be a S-poset. For the purposes of this chapter we give one description of the S-poset congruence generated by $H \subseteq A \times A$. This follows from comments in Chapter 1 but we would like to make it explicit. We let $\mathbf{H} = H \cup H^{-1}$. Note that in [31], we denote $\leq_{\mathbf{H}}$ by $\leq_H$. First we note that $a \leq_{\mathbf{H}} b$ if and only if there exists $n \geq 0$ and

$$(c_1, d_1), \ldots, (c_n, d_n) \in H \cup H^{-1} \text{ and } s_1, \ldots, s_n \in S$$

such that

$$a \leq s_1 c_1, s_1 d_1 \leq s_2 c_2, \ldots, s_n d_n \leq b.$$

We know that $\leq_{\mathbf{H}}$ is reflexive, transitive, contains the relation $\leq$ and is compatible with the action of S. It follows that the relation $\equiv_{\mathbf{H}}$ given by $a \equiv_{\mathbf{H}} b$ if and only if $a \leq_{\mathbf{H}} b \leq_{\mathbf{H}} a$ is an S-act congruence. Moreover, $A/\equiv_{\mathbf{H}}$ may be partially ordered by

$$[a] \preceq_{\mathbf{H}} [b] \text{ if and only if } a \leq_{\mathbf{H}} b,$$

and the natural map $A \to A/\equiv_{\mathbf{H}}$ is an S-pomorphism. That is, $\equiv_{\mathbf{H}}$ is an S-poset congruence, the *S-poset congruence generated by* H. Notice that for any $(a, b) \in H$, $[a] = [b]$. We refer the reader to Lemma 1.2.2.8 of Chapter 1.

Free, projective and strongly flat S-posets are defined in Chapter 1 Section 1.4. We will denote the class of free, projective and strongly flat S-posets by $\mathcal{F}r$, $\mathcal{P}r$ and $\mathcal{SF}$ respectively.

It is clear that A is cyclic if and only if A is isomorphic to S/ρ for some left po-congruence on S. We remark that, from Proposition 1.4.0.22, of Chapter 1, an indecomposable projective S-poset A is cyclic and therefore of the form Sa, where there is an S-po-isomorphism $\phi : Sa \to Se$ for some idempotent $e \in E(S)$, with $a\phi = e$. Consequently, for any $s, t \in S$ we have that $sa \leq ta$ if and only if $se \leq te$; we say that a is *ordered right e-cancellative*. In fact the following is true.

Lemma 4.1.0.5. *[47] Let λ be a left po-congruence on S then S/λ is projective if and only if there exists an idempotent $e \in S$ such that $1 \,\lambda\, e$ and $[s] \leq [t]$ implies $se \leq te$.*

We recall that a S-poset A is *strongly flat* if the functor $- \otimes A$ from **Pos-S** to the category **Pos** of partially ordered sets, preserves subpullbacks and subequalisers or equivalently if A satisfies Condition (P) and Condition (E) which are defined as follows:

(P): for all $a, a' \in A$ and $s, s' \in S$ if $s\,a \leq s'\,a'$ then there exists $a'' \in A$ and $u, u' \in S$ such that $a = u\,a''$, $a' = u'\,a''$ and $s\,u \leq s'\,u'$;

(E): for all $a \in A$ and $s, s' \in S$ if $s\,a \leq s'\,a$ then there exists $a'' \in A$ and $u \in S$ such that $a = u\,a''$ and $s\,u \leq s'\,u$.

Before stating our next result, we remark that in **S-Pos**, direct limits of directed systems of S-posets exist, as observed in [10], where they are referred to as *directed colimits.*

Theorem 4.1.0.6. *[10] The following conditions are equivalent for a S-poset A:*
(i) A is strongly flat;
(ii) A is isomorphic to a direct limit of finitely generated free S-posets;
(iii) A satisfies (P) and (E):
(iv) A is subpullback flat and subequalizer flat.

The next observation is straightforward, and also appears in [52]. We will employ it from time to time to simplify our approach to strongly flat S-posets. It follows from an analysis of direct limits of free S-acts and S-posets. We prefer to argue directly from interpolation conditions.

Lemma 4.1.0.7. *Let A be a strongly flat S-poset. Then A is strongly flat as an S-act.*

Proof. Let $sa = tb$ where $s, t \in S$ and $a, b \in A$. Then certainly $sa \leq tb$ so that by Condition (P), $su \leq tv$, $a = uc$ and $b = vc$ for some $u, v \in S$ and $c \in A$. We then have that $tvc \leq suc$ and so by Condition (E) $tvw \leq suw$ for some $w \in S$ with $c = wd$. Now

$$suw \leq tvw \leq suw$$

so that $suw = tvw$, $a = uc = uwd$ and $b = vc = vwd$. If $a = b$ then by (E), we can take $u = v$, so that $uw = vw$. By Theorem 1.3.0.18 of Chapter 1, due to Stenström, A is strongly flat as an S-act. □

4.2 Pomonoids for which $\mathcal{SF} = \mathcal{P}r$

Just as for R-modules over a unital ring R, and M-acts over a monoid M, any projective S-poset is strongly flat [10], that is $\mathcal{P}r \subseteq \mathcal{SF}$. A natural question, which we address in this section, asks under what conditions on S do we have that $\mathcal{SF} = \mathcal{P}r$?

We have two strategies to answer this question. Both involve a careful study of direct limits of free S-acts versus free S-posets over a pomonoid S. One approach is to then consider under which conditions S-morphisms automatically become S-pomorphisms, and call upon the result of [35, 19]. We prefer to take first a more direct strategy, on the way making clear a number of arguments sketched in [35].

The construction in the next result is crucial, particularly in understanding the connections between perfection and poperfection for a pomonoid S. It is implicit in [35] in the unordered case, taken up and made rather more explicit in [19]. Here we aim for an even more direct presentation for S-posets, noting that we have difficulties to overcome due to the partial orders involved.

In [50] Stepanova proved, in the S-act case, that $\mathcal{P}$ is axiomatisable if and only if $\mathcal{SF}$ is axiomatisable and S is left perfect monoid. In Theorem 3.4.0.2, Chapter 3 we showed the ordered analogue of this result, by assuming that a pomonoid is left perfect as a monoid if and only if it is left poperfect as a pomonoid. We showed that, for pomonoids the class $\mathcal{P}r$ is axiomatisable if and only if the class $\mathcal{SF}$ is axiomatisable and S is left poperfect. We aim to prove our assumption here.

Lemma 4.2.0.8. *Let S be a pomonoid and let $\underline{a} = (a_1, a_2, \ldots)$ be a sequence of elements of S. Let*

$$F = Sx_1 \cup Sx_2 \cup \ldots$$

be the free S-poset on $\{x_i : i \in \mathbb{N}\}$ and let

$$H = \{(x_i, a_i x_{i+1}) : i \in \mathbb{N}\} \subseteq F \times F.$$

(i) For any $sx_m, tx_n \in F$,

$$sx_m \leq_{\mathbf{H}} tx_n \textit{ if and only if } sa_m a_{m+1} \ldots a_w \leq ta_n a_{n+1} \ldots a_w$$

for some $w \geq \max\{m, n\}$. Further,

$$sx_m \equiv_{\mathbf{H}} tx_n \textit{ if and only if } sa_m a_{m+1} \ldots a_v = ta_n a_{n+1} \ldots a_v$$

for some $v \geq \max\{m, n\}$.

(ii) The S-poset $F(\underline{a}) = F/\equiv_{\mathbf{H}}$ is the direct limit of the directed sequence

$$Sx_1 \to Sx_2 \to \ldots$$

where $\alpha_i : Sx_i \to Sx_{i+1}$ is given by $x_i\alpha_i = a_i x_{i+1}$.

(iii) The S-poset $F(\underline{a})$ is strongly flat.

Proof. (i) Suppose that

$$sx_m \leq_{\mathbf{H}} tx_n;$$

then there exist $h \in \mathbb{N}^0$ and $s_i \in S$ and $(y_i, z_i) \in H \cup H^{-1}$, $1 \leq i \leq h$ such that

$$sx_m \leq s_1 y_1, s_1 z_1 \leq s_2 y_2, \ldots, s_h z_h \leq tx_n.$$

We proceed by induction on h. If $h = 0$, then

$$sx_m \leq tx_n \text{ in } F$$

so that $m = n$ and $s \leq t$ in S. Certainly

$$sa_m \leq ta_m = ta_n.$$

Suppose inductively that from

$$ux_i = s_1 z_1 \leq s_2 y_2, \ldots, s_h z_h \leq tx_n$$

we can deduce that

$$ua_i \ldots a_o \leq ta_n \ldots a_o$$

for some $o \geq \max\{i, n\}$.

Case (I): $(y_1, z_1) = (x_j, a_j x_{j+1})$.
From $sx_m \leq s_1 y_1 = s_1 x_j$ we have that $m = j$ and $s \leq s_1$; from $ux_i = s_1 z_1 = s_1 a_j x_{j+1}$ we deduce that $i = j + 1$ and $u = s_1 a_j$. Hence

$$\begin{aligned} sa_m \ldots a_o &= sa_j a_{j+1} \ldots a_o \\ &\leq s_1 a_j a_{j+1} \ldots a_o \\ &= ua_{j+1} \ldots a_o \\ &= ua_i \ldots a_o \\ &\leq ta_n \ldots a_o \end{aligned}$$

and $o \geq \max\{i, n\} \geq \max\{m, n\}$.

Case (II): $(y_1, z_1) = (a_j x_{j+1}, x_j)$. From $sx_m \leq s_1 y_1 = s_1 a_j x_{j+1}$ we have that

$$m = j + 1, s \leq s_1 a_j$$

and from $ux_i = s_1 z_1 = s_1 x_j$ we have that

$$i = j \text{ and } u = s_1.$$

Hence $s \leq ua_i$, so that if $i = o$,

$$s \leq ua_i \leq ta_n \ldots a_o$$

giving that

$$sa_m \leq ta_n \ldots a_o a_m$$

where $m > i = o \geq n$. On the other hand, if $i < o$, so that $o \geq m$,

$$\begin{aligned} sa_m \ldots a_o &\leq ua_i a_m \ldots a_o \\ &= ua_i a_{i+1} \ldots a_o \\ &\leq ta_n \ldots a_o \end{aligned}$$

where $o \geq \max\{m, n\}$.

Conversely, suppose that $sa_m \ldots a_w \leq ta_n \ldots a_w$ where $w \geq \max\{m, n\}$. Then

$$sx_m \leq_{\mathbf{H}} sa_m x_{m+1} \leq_{\mathbf{H}} \ldots \leq_{\mathbf{H}} sa_m \ldots a_w x_{w+1}$$

$$\leq ta_n \ldots a_w x_{w+1} \leq_{\mathbf{H}} ta_n \ldots a_{w-1} x_w \leq_{\mathbf{H}} \ldots \leq_{\mathbf{H}} tx_n$$

so that $sx_m \leq_{\mathbf{H}} tx_n$ as required.

Clearly if $sa_m \ldots a_w = ta_n \ldots a_w$ for some $w \geq \max\{m, n\}$, then $sx_m \leq_{\mathbf{H}} tx_n \leq_{\mathbf{H}} sx_m$, so that $sx_m \equiv_{\mathbf{H}} tx_n$.

On the other hand, if $sx_m \equiv_{\mathbf{H}} tx_n$, then from $sx_m \leq_{\mathbf{H}} tx_n \leq_{\mathbf{H}} sx_m$ we have that

$$sa_m \ldots a_w \leq ta_n \ldots a_w, ta_n \ldots a_v \leq sa_m \ldots a_v$$

for some $v, w \geq \max\{m, n\}$. Without loss of generality assume that $v \geq w$. Then

$$sa_m \ldots a_w a_{w+1} \ldots a_v \leq ta_n \ldots a_w a_{w+1} \ldots a_v \leq sa_m \ldots a_v$$

so that $sa_m \ldots a_v = ta_n \ldots a_v$ as required.

(ii) Define $\beta_i : Sx_i \to F(\underline{a})$ by $x_i\beta_i = [x_i]$. Notice that if $i < j$ then

$$x_i\alpha_i \ldots \alpha_{j-1}\beta_j = (a_i a_{i+1} \ldots a_{j-1} x_j)\beta_j = [a_i a_{i+1} \ldots a_{j-1} x_j] = [x_i] = x_i\beta_i.$$

Now let P be an S-poset, and $\gamma_i : Sx_i \to P$, $i \in \mathbb{N}$, a set of S-pomorphisms such that for any $i < j$ we have that $\gamma_i = \alpha_i \ldots \alpha_{j-1}\gamma_j$.

Define $[ux_i]\delta$ to be $(ux_i)\gamma_i$. If $[ux_i] \leq [vx_j]$, then from (i) we know that there exists $k \geq \max\{i, j\}$ such that

$$ua_i \ldots a_k \leq va_j \ldots a_k.$$

It follows that

$$\begin{aligned} [ux_i]\delta &= (ux_i)\gamma_i \\ &= ux_i\alpha_i \ldots \alpha_{k-1}\gamma_k \\ &= (ua_i \ldots a_{k-1}x_k)\gamma_k \\ &\leq (va_j \ldots a_{k-1}x_k)\gamma_k \\ &= (vx_j\alpha_i \ldots \alpha_{k-1})\gamma_k \\ &= (vx_j)\gamma_j \\ &= [vx_j]\delta \end{aligned}$$

so that δ is well-defined, order preserving and clearly compatible with the action of S. We also have that for each $i \in \mathbb{N}$, $\beta_i\delta = \gamma_i$, and δ is unique with respect to the latter property. Hence $F(\underline{a})$ is indeed the direct limit of the given system.

(iii) This follows from Theorem 4.1.0.6. □

We remark that the above is a special case of a more general result concerning direct limits of free S-acts and S-posets; for the details, see Section 4.6, where we give an alternative approach to direct limits of free S-posets.

The equivalence of (i) and (iv) in the next lemma is implicit in [35]. We note that in (i) and (ii) it is clear that

$$Sb_1 \subseteq Sb_2 \subseteq \ldots.$$

Proposition 4.2.0.9. *Let S be a pomonoid, let $\underline{a} = (a_1, a_2, \ldots)$ be a sequence of elements of S and let $n \in \mathbb{N}$. The following conditions are equivalent:*

(i) for every S-act A and for every sequence of elements $b_1, b_2, \ldots$ of A such that $b_i = a_i b_{i+1}$ for all $i \in \mathbb{N}$,

$$Sb_n = Sb_{n+1} = \ldots;$$

(ii) for every S-poset A and for every sequence of elements $b_1, b_2, \ldots$ of A such that $b_i = a_i b_{i+1}$ for all $i \in \mathbb{N}$,

$$Sb_n = Sb_{n+1} = \ldots;$$

(iii) in $F(\underline{a})$ we have that

$$S[x_n] = S[x_{n+1}] = \ldots;$$

(iv) for all $i \geq n$ there exists $j_i \geq i + 1$ such that

$$Sa_i a_{i+1} \ldots a_{j_i} = Sa_{i+1} \ldots a_{j_i}.$$

Proof. It is clear that (i) implies (ii) and that (ii) implies (iii).

We suppose now that (iii) holds. Let $i \geq n$, so that $S[x_i] = S[x_{i+1}]$. Then $[x_{i+1}] = u[x_i]$ for some $u \in S$, so that by Lemma 4.2.0.8 there exists $j_i \geq i + 1$ such that

$$a_{i+1} a_{i+2} \ldots a_{j_i} = u a_i a_{i+1} a_{i+2} \ldots a_{j_i}.$$

Then

$$\begin{aligned} Sa_i \ldots a_{j_i} &\subseteq Sa_{i+1} \ldots a_{j_i} \\ &= Sua_i a_{i+1} \ldots a_{j_i} \\ &\subseteq Sa_i \ldots a_{j_i}, \end{aligned}$$

so that $Sa_i \ldots a_{j_i} = Sa_{i+1} \ldots a_{j_i}$ as required.

Finally, assume that (iv) is true, let A be an S-act and let $b_i \in A$ be such that $b_i = a_i b_{i+1}$ for $i \in \mathbb{N}$. Then for any $i \geq n$ we have that

$$\begin{aligned} Sb_i &\subseteq Sb_{i+1} \\ &= Sa_{i+1} \ldots a_{j_i} b_{j_i+1} \\ &= Sa_i a_{i+1} \ldots a_{j_i} b_{j_i+1} \\ &= Sb_i, \end{aligned}$$

so that $Sb_n = Sb_{n+1} = \ldots$ as claimed. □

Our next corollary is now immediate.

Corollary 4.2.0.10. *A pomonoid S has Condition (A) if and only if it has Condition (A^O).*

Proof. Let S be a pomonoid with Condition (A^{O}). Let A be a S-act and let $b_1, b_2, \cdots$ be a sequence of elements of A such that

$$Sb_1 \subseteq Sb_2 \subseteq \cdots$$

Then $b_i = a_i b_{i+1}$ for some $a_1, a_2, \cdots \in S$. Let

$$a = (a_1, a_2, \cdots)$$

In the S-poset $F(\underline{a})$ we have

$$S[x_1] \subseteq S[x_2] \subseteq \cdots$$

so that as S has (A^O),

$$S[x_n] = S[x_n + 1] = \cdots$$

By (iii) implies (i) of Proposition 4.2.0.9

$$Sb_n = Sb_{n+1} = \cdots$$

We deduce that A has the ascending chain condition on cyclic S-subacts, and so as A was an arbitrary S-act, Condition (A) holds for S. Conversely, if S has Condition (A), then certainly every S-poset has the ascending chain condition on cyclic S-subposets, as every S-(sub)poset is an S-(sub)act.

□

We say that an S-poset A over a pomonoid S is *locally cyclic* if every finitely generated S-subposet of A is contained in a cyclic S-poset.

Lemma 4.2.0.11. *(c.f [35, Result 1.2]) The following are equivalent for a pomonoid S;*
(i) *for any sequence $\underline{a} = (a_1, a_2, \ldots)$ of elements of S, $F(\underline{a})$ is cyclic;*
(ii) *any direct limit of a sequence of copies of the S-poset S is cyclic;*
(iii) *S satisfies Condition (A^O) (or equivalently, Condition (A));*
(iv) *any locally cyclic S-poset is cyclic.*

Proof. The equivalence of (i) and (ii) is clear, since any direct limit of a sequence of copies of S must be constructed in the manner of $F(\underline{a})$.

Suppose now that (i) holds. Let $\underline{a} = (a_1, a_2, \ldots)$ be a sequence of elements of S; it is clear that in $F(\underline{a})$ we have that

$$S[x_1] \subseteq S[x_2] \subseteq \ldots$$

so that as $F(\underline{a})$ is cyclic,

$$S[x_1] \subseteq S[x_2] \subseteq \ldots \subseteq Su[x_n]$$

for some $u \in S$ and $n \in \mathbb{N}$. It now follows that for any $j \geq n$,

$$S[x_j] \subseteq Su[x_n] \subseteq S[x_n] \subseteq S[x_j]$$

so that

$$S[x_n] = S[x_{n+1}] = \ldots$$

and (iii) holds from Proposition 4.2.0.9.

To show that (iii) implies (iv), let S have Condition (A^O) and let B be a locally cyclic S-poset. Let $b_1 \in B$; if B is not cyclic then $Sb_1 \subset B$, so there exists $b_1' \notin Sb_1$. Now B is locally cyclic, so that $Sb_1 \cup Sb_1' \subseteq Sb_2$ for some $b_2 \in B$, and clearly, $Sb_1 \subset Sb_2$. Continuing in this manner we obtain an infinite ascending chain of cyclic S-subposets of B, contradicting the existence of Condition (A^O). Hence B is cyclic.

Finally, assume that (iv) is true. Since $F(\underline{a})$ is the union of an ascending chain of cyclic S-subposets, it is clear that $F(\underline{a})$ is locally cyclic, hence cyclic by assumption.

□

We now focus on the question of when $\mathcal{SF} = \mathcal{P}r$.

Lemma 4.2.0.12. *Let S be a pomonoid such that every S-poset $F(\underline{a})$ is projective. Then S satisfies Condition (A^O) (or equivalently, Condition (A)).*

Proof. As $F(\underline{a})$ is a union of an ascending chain of cyclic S-subposets, if projective it must therefore be cyclic. The result now follows from Lemma 4.2.0.11. □

Every S-poset $F(\underline{a})$ is strongly flat from Lemma 4.2.0.8.

Corollary 4.2.0.13. *Let S be a pomonoid such that every strongly flat S-poset is projective. Then S satisfies Condition (A^O) (or equivalently, Condition (A)).*

The following argument is essentially that of [19]; we include it here for completeness, since all the preliminaries are set up.

Lemma 4.2.0.14. *Let S be a pomonoid such that every S-poset $F(\underline{a})$ is projective. Then S satisfies (M_R).*

Proof. Let

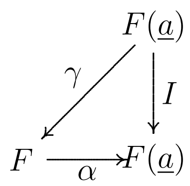
$$a_1S \supseteq b_1S \supseteq b_2S \supseteq \cdots$$

be a decreasing sequence of principal right ideals of S. Then there are elements $a_i, i \geq 2$ such that $b_i = b_{i-1}a_{i+1}$ (where $b_0 = a_1$). Then

$$b_1 = a_1a_2, b_2 = b_1a_3 = a_1a_2a_3, \ldots.$$

Let $\underline{a} = (a_1, a_2, \ldots)$ and let $F(\underline{a})$ be defined as in Lemma 4.2.0.8.

Let I be the identity map in $F(\underline{a})$ and let $\alpha : F \to F(\underline{a})$ be the canonical S-pomorphism. Since $F(\underline{a})$ is projective, there exists an S-pomorphism $\gamma : F(\underline{a}) \to F$ such that



commutes.

Suppose that for each $i \in \mathbb{N}$ we have that

$$[x_i]\gamma = c_ix_{j(i)}.$$

Then for any $i \geq 2$,

$$c_1x_{j(1)} = [x_1]\gamma = (a_1 \ldots a_{i-1}[x_i])\gamma = a_1 \ldots a_{i-1}c_ix_{j(i)},$$

so that $j(i) = j(1) = j$ say, and moreover

$$c_1 = a_1 \ldots a_{i-1}c_i$$

for all i. It follows that $c_1S \subseteq a_1 \dots a_{i+1}S$, that is, $c_1S \subseteq b_iS$ for all $i \in \mathbb{N}$. Now

$$[x_1] = [x_1]I = [x_1]\gamma\alpha = c_1x_j\alpha = [c_1x_j],$$

so by Lemma 4.2.0.8,

$$a_1 \dots a_n = c_1a_j \dots a_n$$

for some $n \geq j$. Hence

$$b_{n-1}S = a_1 \dots a_nS \subseteq c_1S$$

so that $b_{n-1}S = b_nS = \dots$ and our descending chain terminates as required.

□

Corollary 4.2.0.15. *Let S be a pomonoid such that every strongly flat S-poset is projective. Then S satisfies (M_R).*

Our next technical lemma has two significant uses. The strategy for the proof is again taken from the unordered case in [19], but note that that article omits the proof that c is idempotent.

Lemma 4.2.0.16. *Let S be a pomonoid and let ρ be a strongly flat left po-congruence on S such that the set $\{dS : d \in B\}$ has a minimal element, where $B = [1]$. Then S/ρ is projective.*

Proof. From Lemma 4.1.0.7, S/ρ is strongly flat as a S-act. Let $c \in B$ be such that cS is minimal in $\mathcal{I} = \{dS : d \in B\}$. We will now show that c is idempotent. Since $c\,\rho\,c^2$, by the Corollary to Result 4 of [19] we have $cu = c^2u$ for some $u \in S$ with $u\,\rho\,1$. Then $c^2uS \subseteq cS$ but cS is minimal in $\mathcal{I}$, hence $c\,\mathcal{R}\,c^2u$. Hence $c = c^2ux$ for some $x \in S$ and so

$$c^2 = c^3ux = c^2ux = c.$$

Let $d \in B$, so that $d\,\rho\,c$. Exactly as in [19] we have that $dv = cv$ for some $v \in B$ and then

$$cS = cvS = dvS \subseteq dS.$$

Thus cS is *minimum* in $\mathcal{I}$.

Now let $\theta : S/\rho \to Sc$ be defined by $[u]\theta = uc$. Then $[u] \leq [v]$ implies that there exists $w\,\rho\,1$ such that $uw \leq vw$. Since $w \in B$ we have that $cS \subseteq wS$, so that $c = wt$ for some $t \in S$. Therefore $uwt \leq vwt$ implies that $uc \leq vc$ hence θ is well-defined and order-preserving. To check that θ preserves the S-action,

$$(s[u])\theta = [su]\theta = (su)c = s(uc) = s[u]\theta.$$

To check the injectivity let $sc \leq tc$; then

$$[s] = s[1] = s[c] = [sc] \leq [tc] = t[c] = t[1] = [t]$$

as ρ is a po-congruence. Thus θ is injective and clearly θ is a surjective S-pomorphism; moreover, we have also shown that the inverse of θ preserves order, so that θ is an S-poset isomorphism. As c is an idempotent, by Proposition 1.4.0.22 of Chapter 1, Sc and hence S/ρ are projective. □

Theorem 4.2.0.17. *If S satisfies (M_R), then every strongly flat cyclic S-poset is projective.*

Proof. Let C be a strongly flat cyclic S-poset. By Corollary 1.4.0.30 of section 4.1, $C \cong S/\rho$ where ρ is a strongly flat left po-congruence. Let $B = [1]$. Since S has (M_R), there is an element $c \in B$ such that cS is minimal in $\{dS : d \in B\}$. The result now follows from Lemma 4.2.0.16. □

We will call a generating set X of an S-poset A *independent* if for any $x, x' \in X$ such that $x \in Sx'$ we have $x = x'$.

Lemma 4.2.0.18. *Let A be an S-poset which satisfies the ascending chain condition for cyclic S-subposets. If X is a set of generators for A, then X contains an independent set of generators for A.*

Proof. Regarded as an S-act, A satisfies the ascending chain condition for cyclic S-subacts (since these coincide with the cyclic S-subposets). The result now follows from that in the S-act case (Lemma 2 of [19]). □

Lemma 4.2.0.19. *Let A be a strongly flat S-poset which satisfies the ascending chain condition for cyclic S-subposets. If A is indecomposable, then A is cyclic.*

Proof. This follows immediately from Lemma 4.1.0.7 and Lemma 3 of [19]. □

Corollary 4.2.0.20. *If S satisfies Condition (A^O), then every strongly flat S-poset is a disjoint union of cyclic strongly flat S-posets.*

Proof. It is clear that if A is a strongly flat S-poset, then so are its indecomposable components. It is then immediate from Lemma 4.2.0.19 that the indecomposable components are cyclic. □

We now come to the main theorem of this section. We remark that the equivalence of (i) and (iii) is given in [52].

Theorem 4.2.0.21. *Let S be a pomonoid. Then the following conditions are equivalent:*
(i) *every strongly flat S-poset is projective;*
(ii) *every S-poset of the form $F(\underline{a})$ is projective;*
(iii) *S satisfies Condition (A^O) and (M_R);*
(iv) *S satisfies Condition (A) and (M_R);*
(v) *every strongly flat S-act is projective.*

Proof. Since every $F(\underline{a})$ is strongly flat, clearly (i) implies (ii). If every S-poset $F(\underline{a})$ is projective, then S has (M_R) from Lemma 4.2.0.14 and (A^O) from Lemma 4.2.0.12, so that (ii) implies (iii).

Now suppose that (iii) holds. As S satisfies Condition (A^O), from Corollary 4.2.0.20, every strongly flat S-poset A is a disjoint union of strongly flat cyclic S-posets; as in addition S has (M_R), then in view of Theorem 4.2.0.17, these are all projective, and it follows that A is projective and (iii) implies (i).

The remainder of the result follows from Theorem 4.0.0.3 and Corollary 4.2.0.10. □

4.3 Poperfect Pomonoids

We recall that a pomonoid is *left poperfect* if every S-poset has a *projective cover*, that is, a cover that is projective.

Lemma 4.3.0.22. *(cf. [53]) A cover of a cyclic S-poset is cyclic.*

Proof. Suppose that $A = Sa$ is a cyclic S-poset and suppose that $\beta : B \to A$ is a coessential S-po-epimorphism. Let $b \in B$ be such that $b\beta = a$; then $\beta' = \beta|_{Sb} : Sb \to A$ is an S-po-epimorphism. Since β is coessential we must have that $B = Sb$ and B is cyclic as required. □

We now wish to identify those subpomonoids of S that are the po-congruence classes of the identity, for any left po-congruence. This will enable us to find conditions under which cyclic S-posets have projective covers.

Definition 4.3.0.23. A subpomonoid P of a pomonoid S is *right po-unitary* if for any

$$p, a_1, b_1, \cdots, a_n, b_n, q \in P, s_1, \ldots, s_n \in S,$$

if

$$p \leq s_1 a_1,\ s_1 b_1 \leq s_2 a_2,\ \cdots, s_n b_n \leq q,$$

then

$$s_1, s_2, \cdots, s_n \in P.$$

Lemma 4.3.0.24. *Let S be a pomonoid. If U is a right po-unitary subpomonoid, then U is right unitary.*

Proof. Suppose that $a, ba \in U$. Then as

$$ba \leq b \cdot a,\ b \cdot a \leq ba$$

the definition of po-unitary gives us that $b \in U$. □

The following fact concerning right unitary submonoids is useful.

Lemma 4.3.0.25. *Let U be a right unitary submonoid of a monoid S. Then for $a, b \in U$,*

$$Ua \subseteq Ub \text{ if and only if } Sa \subseteq Sb.$$

Proof. If $Ua \subseteq Ub$, then certainly $Sa \subseteq Sb$.

Conversely, if $Sa \subseteq Sb$, then $a = ub$ for some $u \in S$, but as U is right unitary, $u \in U$ so that $Ua \subseteq Ub$ as required. □

Notice that a right unitary submonoid need not be right po-unitary. For an example, take that of $\mathbb{N}^0 = \{0, 1, 2, \ldots\}$ under $+$, with the usual ordering. Then $\mathbb{E} = \{2n : n \in \mathbb{N}^0\}$ is (right) unitary. Notice that

$$0 \leq 1 + 0, 1 + 0 \leq 2$$

but $1 \notin \mathbb{E}$.

Lemma 4.3.0.26. *Let S be a pomonoid and let $P \subseteq S$. Then $P = [1]$ for a left po-congruence on S if and only if P is a right po-unitary subpomonoid of S.*

Proof. Let ρ be a left po-congruence on S and let $P = [1]$. Then P is a subpomonoid of S, as if $p_1, p_2 \in P$, then

$$p_1 p_2 \,\rho\, p_1 1 \,\rho\, p_1 \,\rho\, 1.$$

Suppose now that $p, a_1, b_1, \cdots, a_n, b_n, q \in P$ and $s_1, \ldots s_n \in S$ are such that

$$p \leq s_1 a_1,\ s_1 b_1 \leq s_2 a_2, \cdots, s_n b_n \leq q.$$

As ρ is a left po-congruence, we have in S/ρ that

$$[1] = [p] \leq [s_1 a_1] = s_1[a_1] = [s_1] = s_1[b_1] = [s_1 b_1]$$

$$\leq [s_2 a_2] \ldots = [s_n b_n] \leq [q] = [1]$$

which implies that

$$[1] \leq [s_1] \leq [s_2] \ldots [s_n] \leq [1]$$

so that

$$[1] = [s_1] = \ldots = [s_n] = [1]$$

as required.

Conversely, let P be a left po-unitary subpomonoid of S. Let ρ be $\equiv_{P \times P}$, the S-po-congruence generated by $P \times P$ (note that $P \times P = (P \times P) \cup (P \times P)^{-1}$). From the construction of $\equiv_{P \times P}$, we have that $P \times P \subseteq \rho$ so that as $1 \in P$ we have $P \subseteq [1]$.

Let $w \in [1]$. Then there are elements

$$s_1, \ldots, s_n, t_1, \ldots, t_m \in S$$

and

$$(u_1, v_1), \ldots, (u_n, v_n), (x_1, y_1), \ldots, (x_m, y_m) \in P \times P$$

such that

$$1 \leq s_1 u_1, s_1 v_1 \leq s_2 v_2, \ldots, s_n v_n \leq w = w\,1,$$

$$w\,1 \leq t_1 x_1, t_1 y_1 \leq t_2 y_2, \ldots, t_m y_m \leq 1$$

so that as P is left po-unitary we have that $w \in P$ and $P = [1]$ as required.

□

We note that the result below also appears without proof in [53], but the preceding lemma in that article, characterising po-congruence classes of identities, is incorrect if applied to S-poset po-congruences.

Proposition 4.3.0.27. *(cf. [53] and [39, Proposition III 17.22]) Let ρ be a left po-congruence on a pomonoid S. The cyclic S-poset S/ρ has a projective cover if and only if the subpomonoid $R = [1]$ contains a minimal left ideal generated by an idempotent.*

Proof. Suppose that the cyclic S-poset S/ρ has a projective cover; from Lemma 4.3.0.22 this must be cyclic. Without loss of generality, let $\alpha : Se \to S/\rho$ be a coessential S-po-epimorphism. Then for some $u \in S$,

$$(ue)\alpha = [1] = u(e\alpha).$$

Since α is coessential, $Sue = Se$ so that $e = que$ for some $q \in S$; we can assume that $q = eq$. Calculating, we have that

$$(uq)^2 = (uq)(uq) = u(qu)eq = u(que)q = ueq = uq,$$

so that $uq \in E(S)$. Moreover,

$$[1] = (ue)\alpha = (uque)\alpha = uq(ue)\alpha = (uq)[1] = [uq]$$

so that $uq \in R = [1]$.

Suppose now that $w \in R$ and $Rw \subseteq Ruq$. Then $w = wuq$ and

$$(wue)\alpha = w(ue)\alpha = w[1] = [w],$$

so that

$$wue \, \mathcal{L} \, e \text{ in } S.$$

We then have that

$$w = wuq = wueq \, \mathcal{L} \, eq = q \text{ in } S$$

and so

$$Sq = Sw = Swuq \subseteq Suq \subseteq Sq.$$

By Lemma 4.3.0.24 and 4.3.0.25, $Rw = Ruq$ so that Ruq is a minimal left ideal in R.

Conversely, suppose that $R = [1]$ contains a minimal left ideal Re, where $e \in E(R)$. Define $\theta : Se \to S/\rho$ by $(se)\theta = [s]$. If $se \leq te$ then as ρ is a po-congruence, we have that

$$[s] = s[1] = s[e] = [se] \leq [te] = t[e] = t[1] = [t],$$

so that θ is well defined and order preserving. It is now easy to see that θ is an onto S-pomorphism. Notice that $e\theta = [1]$.

If $Spe \subseteq Se$ and $\theta|_{Spe} : Spe \to S/\rho$ is onto, then we must have that $(rpe)\theta = [1]$ for some $r \in S$. It follows that $rp \in R$ so that $Rrpe = Re$ and consequently, $Srpe = Se$. We then have that $Spe = Se$ so that θ is coessential as required.

□

Our next corollary follows immediately from Proposition 4.3.0.27 and the comment following Definition 1.4.0.27 in Chapter 1.

Corollary 4.3.0.28. *A pomonoid S satisfies Condition (D^O) if and only if every cyclic S-poset has a projective cover.*

Lemma 4.3.0.29. *(cf. [53]) If an S-poset A is the union of an infinite strictly ascending chain of cyclic S-subposets then A does not have a projective cover.*

Proof. Suppose $A = \cup_{n \in \mathbb{N}} Sa_n$ and

$$Sa_1 \subset Sa_2 \subset \cdots Sa_n \subset \cdots$$

where all inclusions are strict, is an ascending chain of cyclic S-subposets of A and assume that A has a projective cover P with coessential S-po-epimorphism $\alpha : P \to A$.

Now $P = \cup_{i \in I} P_i$ and we can assume that each $P_i = Se_i$ for some idempotent e_i in S. If $|I| > 1$, take $i \in I$; then if $e_i\alpha \in Sa_n$ for some $n \in \mathbb{N}$, we have that $P_i\alpha \subseteq Sa_n$. Then $\alpha|_{P \setminus P_i}$ is still an S-po-epimorphism and thus P cannot be a cover for A. Finally if $|I| = 1$, say $I = \{1\}$, then if $e_1\alpha \in Sa_m$, the image of α is contained in Sa_m, a contradiction. □

Theorem 4.3.0.30. *Let S be a pomonoid. Then S is left poperfect if and only if S satisfies Conditions (A^O) and (D^O).*

Proof. Suppose S is left poperfect. Then Condition ($\mathrm{A^O}$) and Condition ($\mathrm{D^O}$) follow from Lemma 4.3.0.29 and Corollary 4.3.0.28, respectively.

Conversely, suppose that S satisfies Conditions ($\mathrm{A^O}$) and ($\mathrm{D^O}$). By Corollary 4.3.0.28, every cyclic S-poset has a projective cover.

Let A be an arbitrary S-poset. From Lemma 4.2.0.18, A has an independent set X of generators. For each $x \in X$, let $\alpha_x : Se_x \to Sx$ be a coessential S-po-epimorphism, where $e_x \in E(S)$. Let $G = \bigcup_{\bar{x} \in X} Se_x\bar{x}$ be the S-subposet of the free left S-poset on $\overline{X} = \{\bar{x} : x \in X\}$ and define $\alpha : G \to A$ by $(se_x\bar{x})\alpha = (se_x)\alpha_x$. Clearly, α is an S-po-epimorphism.

Suppose that α is not coessential. Then there exists $y \in X$ and a strict left ideal I of Se_y such that

$$\alpha : \bigcup_{x \in X \setminus \{y\}} Se_x\bar{x} \ \cup I\bar{y} \to A$$

is onto. Consequently, $y = (ue_x\bar{x})\alpha_x \in Sx$ for some $x \neq y$, a contradiction, or $y = (pe_y\bar{y})\alpha$ for some $pe_y \in I$ and so $\alpha_y : I \to Sy$ is onto, contradicting the coessentiality of α_y. Hence α is coessential. □

4.4 Right collapsible subpomonoids

In this section we consider Condition (K) for a pomonoid S, introduced by Kilp for monoids in [36]. In [37], it is proved that a monoid is left perfect if and only if it satisfies Condition (A) and (K). Similar techniques are employed in the article of Renshaw [44]. Our aim here is to show the ordered analogue.

Our first result follows immediately from Lemma 4.1.0.7 and Lemma 2.2 of [36].

Lemma 4.4.0.31. *Let ρ be a left po-congruence on S such that S/ρ is strongly flat and let $P = [1]$. Then P is a right collapsible subpomonoid.*

Lemma 4.4.0.32. *Let $P \subseteq S$ be a right collapsible subpomonoid and let ρ be the relation $\equiv_{P\times P}$ on S. Then*
(i) ρ is a left po-congruence;
(ii) $P \subseteq [1]$
and
(iii) S/ρ is strongly flat.

Proof. (i) and (ii) are clear from the definition of $\equiv_{P\times P}$.
(iii) Suppose now that $[s] \leq [t]$ in S/ρ. Then

$$s \leq u_1p_1,\; u_1q_1 \leq u_2p_2, \ldots, u_nq_n \leq t$$

for some $p_1, q_1, \ldots, p_n, q_n \in P$ and $u_1, \ldots, u_n \in S$. Since P is right collapsible, we can find $z_1 \in P$ with $p_1z_1 = q_1z_1$. Then

$$sz_1 \leq u_1p_1z_1 = u_1q_1z_1.$$

If $n = 1$, we then have that $sz_1 \leq tz_1$. Otherwise, $sz_1 \leq u_2p_2z_1$ and we pick $z_2 \in P$ with $p_2z_1z_2 = q_2z_1z_2$. Then

$$sz_1z_2 \leq u_2p_2z_1z_2 = u_2q_2z_1z_2.$$

If $n = 2$ we obtain that $sz_1z_2 \leq tz_1z_2$, if not we continue in this manner, until we obtain that $sz_1 \ldots z_n \leq tz_1 \ldots z_n$. As $z_1 \ldots z_n \in P$, and $P \subseteq [1]$, we have that S/ρ is strongly flat by Corollary 1.4.0.30. □

We can now verify the ordered analogue of Theorem 2.3 of [36].

Lemma 4.4.0.33. *Let S be a pomonoid. All strongly flat cyclic S-posets are projective if and only if S satisfies Condition (K).*

Proof. Suppose that all strongly flat cyclic S-posets are projective. Let $P \subseteq S$ be a right collapsible subpomonoid. By the above lemma we can construct a left po-congruence ρ on S such that S/ρ is strongly flat and $P \subseteq [1]$. By assumption, S/ρ is projective, and so there exists an idempotent $e \in S$ with $e\,\rho\,1$ and such that for all $s, t \in S$, if $[s] \leq [t]$ then $se \leq te$.

As in Lemma 4.4.0.32, we know that if $s\,\rho\,t$, then there exists $y \in P$ with $sy \leq ty$. We have that $1\,\rho\,e$ and so $z \leq ez$ for some $z \in P$. Now $ez\,\rho\,z$, and so there exists $w \in P$ with $ezw \leq zw$. We therefore have

$$ezw \leq zw \leq ezw$$

and so $ezw = zw$. Let $x \in P$; since $1\,\rho\,x$ for all $x \in P$, we will have $e = xe$ from Lemma 4.1.0.5.

Now let $x \in P$ be an arbitrary element and let $l = zw$. Then

$$xl = xel = el = l,$$

so that l is a right zero for P.

Conversely, suppose that (K) holds. Let ρ be a left po-congruence on S such that S/ρ is strongly flat; we must show that S/ρ is isomorphic to some Se, where $e \in E(S)$, *as an S-poset*. Let $P = [1]$; then P is a right collapsible subpomonoid of S by Lemma 4.4.0.31. By assumption there exists a right zero say $e \in P$. Then e is an idempotent and $1 \,\rho\, e$.

Suppose $[s] \leq [t]$ for some $s, t \in S$. As S/ρ is strongly flat, there exists $u \in S$ such that $u \,\rho\, 1$ and $su \leq tu$. Note that

$$se = s(ue) \leq t(ue) = te,$$

hence S/ρ is projective by Lemma 4.1.0.5. □

4.5 Left poperfect pomonoids and $\mathcal{SF} = \mathcal{P}r$

The aim of this section is to show that a pomonoid is left poperfect if and only if $\mathcal{SF} = \mathcal{P}r$. In view of Corollary 4.2.0.20 and Lemma 4.5.0.34 this amounts to showing that in the presence of Condition ($\mathrm{A^O}$), Condition ($\mathrm{D^O}$) is equivalent to (M_R). It will then follow immediately that a pomonoid is left poperfect if and only if it is left perfect.

Lemma 4.5.0.34. *If S satisfies Condition (D^O), then every strongly flat cyclic S-poset is projective.*

Proof. As in Theorem 4.2.0.17 a strongly flat cyclic S-poset is isomorphic to S/ρ where ρ is some strongly flat left po-congruence and $B = [1]$ is a left po-unitary subpomonoid of S. Condition ($\mathrm{D^O}$) gives that B has a minimal left ideal say Be generated by an idempotent e. By Lemma (8.12) in [14], eB is a minimal right ideal of B.

Suppose now that $d \in B$ and $dS \subseteq eS$. Then $d = ed$, so that $dB \subseteq eB$ and so the minimality of eB gives that $dB = eB$. Consequently, $eS = dS$, so that eS is minimal in $\mathcal{I} = \{dS : d \in B\}$. The result now follows from Lemma 4.2.0.16. □

Let S be a pomonoid. Given that we proved in Section 4.2 that Conditions (A) and ($\mathrm{A^O}$) are equivalent, the proof of the next result could essentially be taken from [35]. However, a significant part of the proof of Result 1.7 of that article relies on categorical techniques that we have avoided below. Our argument is in some sense a clarification of that in [35].

Theorem 4.5.0.35. *Let S be a pomonoid such that S satisfies Condition (M_R) and Condition (A^O). Then S has Condition (D^O).*

Proof. If S has (M_R) and ($\mathrm{A^O}$), then as every strongly flat S-poset is projective, it follows from Theorem 4.1.0.6, that every direct limit of copies of S, regarded as a left S-poset, is projective.

Let S/ρ be a cyclic S-poset; to avoid ambiguity in this proof we denote the ρ-class of $a \in S$ by $[a]_\rho$. Let $B = [1]_\rho$.

Suppose $v \in E(S) \cap B$ and $t \in B$ with $St \subseteq Sv$. As $t \in B$ and B is a submonoid it is clear that $t^n \in B$.

Let

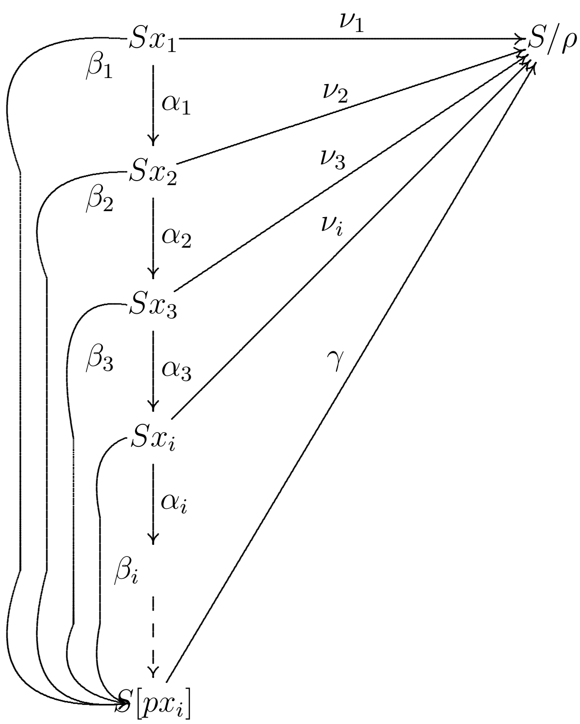
$$Sx_1 \to Sx_2 \to \dots$$

be a direct sequence of copies of S, where $x_i\alpha_i = tx_{i+1}$ for all $i \in \mathbb{N}$. Put

$$\underline{t} = (t, t, \dots)$$

so that by Lemma 4.2.0.8, the direct limit is $F(\underline{t})$. By assumption, $F(\underline{t})$ is projective, so as it is indecomposable, $F(\underline{t}) = S[px_i]$ for some px_i where $[px_i]$ is ordered right e-cancellable for some $e \in E(S)$.

Let $\nu_i : Sx_i \to S/\rho$ be defined by $x_i\nu_i = [1]_\rho$.



We note that

$$x_i\,\alpha_i\,\nu_{i+1} = (tx_{i+1})\nu_{i+1} = t[1]_\rho = [t]_\rho = [1]_\rho = x_i\nu_i$$

which implies that $\alpha_i\nu_{i+1} = \nu_i$. By definition of direct limit, there exists an S-pomorphism $\gamma : S[px_i] \to S/\rho$ such that $\beta_i\gamma = \nu_i$ for all $i \in \mathbb{N}$.

Define $\tau : S[px_i] \to Sx_i$ by $(u[px_i])\tau = uepx_i$. As $[px_i]$ is ordered right e-cancellative, it follows that τ is well defined. It is easy to see τ is an S-pomorphism.

Now

$$[px_i]\tau\beta_i = (epx_i)\beta_i = [epx_i] = e[px_i] = [px_i],$$

so that

$$\tau\,\beta_i = I_{S[px_i]}.$$

Put $\psi = \beta_{i+1}\tau\alpha_i : Sx_{i+1} \to Sx_{i+1}$; then

$$\psi^2 = (\beta_{i+1}\tau\alpha_i)(\beta_{i+1}\tau\alpha_i) = \beta_{i+1}\tau(\alpha_i\beta_{i+1})\tau\alpha_i = \beta_{i+1}\tau\beta_i\tau\alpha_i = \beta_{i+1}I_{S[px_i]}\tau\alpha_i = \beta_{i+1}\tau\alpha_i = \psi.$$

It is then easy to see that

$$x_{i+1}\psi = hx_{i+1}$$

for some $h \in E(S)$.

Calculating,

$$hx_{i+1} = x_{i+1}\psi = x_{i+1}\beta_{i+1}\tau\alpha_i = (wx_i)\alpha_i = wtx_{i+1}$$

for some $w \in S$ and therefore $h = wt$, giving that $Sh \subseteq St$.

We check that

$$\beta_{i+1}\,\tau\,\alpha_i\,\nu_{i+1} = \beta_{i+1}\,\tau\,\alpha_i\,\beta_{i+1}\,\gamma = \beta_{i+1}\,\gamma = \nu_{i+1}$$

and

$$[h]_\rho = (h\,x_{i+1})\nu_{i+1} = x_{i+1}\psi\nu_{i+1} = x_{i+1}\nu_{i+1} = [1]_\rho$$

thus $h \in B$.

Suppose now that

$$Se_1 \supseteq Se_2 \supseteq Se_3 \cdots$$

is a desending chain of principal left ideals generated by idempotents $e_i \in S$. From Lemma 1.2.10 of [33], there are idempotents $g_1, g_2, \ldots$ such that for all $i \in \mathbb{N}$, we have that $Sg_i = Se_i$ and

$$g_1 \geq g_2 \geq \ldots$$

under the natural partial order on $E(S)$. Higgins remarks on [33, page 28] that if S is regular and satisfies M_R, then it also satisfies M_L. Here we do not know that S is regular, but certainly

$$g_1S \supseteq g_2S \supseteq \ldots$$

and as S has M_R we deduce that for some $n \in \mathbb{N}$,

$$g_nS = g_{n+1}S = \ldots$$

and hence $g_n = g_{n+1} = \ldots$. Consequently,

$$Se_n = Se_{n+1} = \ldots.$$

Certainly $1 \in B$ and we have shown that every principal left ideal St where $t \in B$ contains a principal left ideal Sh where $h \in E(S) \cap B$. It follows from the above that there is an idempotent $e' \in B$ such that Se' is minimal with respect to being generated by an element of B. By Lemma 4.3.0.25, Be' is a minimal left ideal of B. Hence S satisfies Condition (D^{O}). □

We can now give the final result of this section.

Theorem 4.5.0.36. *For a pomonoid S, the following are equivalent:*

(i) every strongly flat S-poset is projective;
(ii) S satisfies Conditions (A^O) and (M_R);
(iii) S satisfies Conditions (A^O) and (D^O);
(iv) S is left poperfect;
(v) S satifies Conditions (A^O) and (K^O).
(i) every strongly flat S-act is projective;
(ii) S satisfies Conditions (A) and (M_R);
(iii) S satisfies Conditions (A) and (D);
(iv) S is left perfect;
(v) S satifies Conditions (A) and (K).

Proof. In view of Theorems 4.0.0.3, 4.2.0.21, 4.3.0.30, 4.5.0.35 and Corollary 4.2.0.10, we need only to show that (*ii*) and (*iii*) are equivalent.

If (*iii*) holds, by Corollary 4.2.0.20, every strongly flat S-poset can be written as a disjoint union of cyclic strongly flat S-posets which are projective by Lemma 4.5.0.34 as S satisfies Condition $(\mathrm{D^O})$, hence every strongly flat S-poset is projective. By Theorem 4.2.0.21, S satisfies (*ii*).

Conversely, suppose that (*ii*) holds, then (*iii*) follows from Theorem 4.5.0.35.

□

We remark that it is clear that Condition (D) implies $(\mathrm{D^O})$, and in view of Lemma 4.5.0.34, $(\mathrm{D^O})$ implies (K). It is known [37] that (K) does not imply (D), and the same example (of the free monogenic monoid) with length as partial order, shows that (K) does not imply $(\mathrm{D^O})$. It remains to show whether (D) and $(\mathrm{D^O})$ are equivalent.

4.6 Direct limits of free S-posets revisited

In this section we briefly analyse the connection between direct limits of (free) S-acts, and direct limits of (free) S-posets, over a pomonoid S.

Lemma 4.6.0.37. *Let S be a pomonoid. Let*

$$\mathcal{D} = (I, \{A_i\}_{i\in I}, \{\phi_{(i,j)}\}_{i\leq j})$$

be a direct system in **S**-***Pos***. *Note that $\mathcal{D}$ may also be regarded as a direct system in* **S**-***Act***. *Let $\mathcal{V} = (V, \{\theta_i\}_{i\in I})$ be the direct limit of $\mathcal{D}$ in* **S**-***Act***, *constructed as in Proposition 1.2.1.2. Then V is an S-poset under the ordering given by, for any $a_i \in A_i, a_j \in A_j$,*

$$[a_i] \leq [a_j] \text{ if and only if } a_i\phi_{(i,k)} \leq a_j\phi_{(j,k)} \text{ for some } k \geq i, j,$$

the natural maps θ_i are S-pomorphisms, and with this ordering, $\mathcal{V} = (V, \{\theta_i\}_{i\in I})$ is the direct limit of $\mathcal{D}$ in **S**-***Pos***.

Proof. This follows from Proposition 1.2.2.14. □

We can say even more if we focus on free S-posets and S-acts.

Lemma 4.6.0.38. *Let S be a pomonoid and let $F = \cup_{i \in I} Sx_i$ be a free S-poset (so that F may also be regarded as a free S-act). Let A be an S-poset and let $\phi : F \to A$ be an S-morphism. Then ϕ is an S-pomorphism.*

Proof. For each $i \in I$, let $x_i\phi = a_i$. Let $sx_i \leq tx_j$ in F, then $i = j$ and $s \leq t$ in S. We have $sa_i \leq ta_i$ and

$$(sx_i)\phi = s(x_i\phi) = sa_i \leq ta_i = ta_j = t(x_j\phi) = (tx_j)\phi.$$

□

We know from Theorem 2.6 of [52], which was argued using interpolation conditions, that a strongly flat S-poset is strongly flat as an S-act. We can say rather more. First, a straightforward observation, the proof of which we leave to the reader.

Lemma 4.6.0.39. *Let B be an S-act and C an S-poset over a pomonoid S, and suppose that $\phi : B \to C$ is an S-isomorphism. Then, defining $\leq$ on B by $a \leq b$ if and only if $a\phi \leq b\phi$ in C, we have that B is an S-poset and ϕ is an S-po-isomorphism.*

Corollary 4.6.0.40. *Let S be a pomonoid and let A be a strongly flat S-poset. Then A is a strongly flat S-act. Conversely, if B is a strongly flat S-act, then there exists a partial order on B such that B is a strongly flat S-poset.*

Proof. Let A be a strongly flat S-poset. Then by Theorem 1.4.0.25, A is isomorphic as an S-poset to a direct limit V of a directed system

$$\mathcal{D} = (I, \{A_i\}_{i \in I}, \{\phi_{(i,j)}\}_{i \leq j})$$

in **S-Pos**, where each A_i is a finitely generated free S-poset. Clearly then A is isomorphic to V as an S-act, and, if we forget the partial ordering in the A_i and in V, we have that $\mathcal{D}$ is a directed system of finitely generated free S-acts in **S-Act**, and V as an S-act is the direct limit in **S-Act** of $\mathcal{D}$. Thus by Theorem 1.3.0.18, A is strongly flat as an S-act.

Conversely, if B is strongly flat as an S-act, then by Theorem 1.3.0.18 of Chapter 1, there is an S-act isomorphism $\theta : B \to C$, where C is a direct limit in **S-Act** of a directed system

$$\mathcal{D} = (I, \{A_i\}_{i \in I}, \{\phi_{(i,j)}\}_{i \leq j})$$

of finitely generated free S-acts. Clearly, each A_i can be ordered so that it becomes a free S-poset, and by Lemma 4.6.0.38, the $\phi_{(i,j)}$ become S-pomorphisms. In this way, $\mathcal{D}$ is a directed system in **S-Pos**. Now by Lemma 4.6.0.37, C can be ordered so that it is the direct limit of $\mathcal{D}$ in **S-Pos** and so is strongly flat as an S-poset. By Lemma 4.6.0.39, B can be ordered in such a way that it is isomorphic to C in **S-Pos**.

□

Chapter 5

Clifford Monoids

The second aim of my thesis is to investigate the finitary conditions arising from questions of axiomatisability of classes of S-acts and S-posets. For example, from [25] we know that for a monoid S, $\mathcal{SF}$ is axiomatisable if and only if $\mathbf{r}(s,t)$ and $\mathbf{R}(s,t)$ are finitely generated for all $s,t \in S$. What does this tell us about the structure of S?

We focus on the case where S is a Clifford monoid. We recall from [34] that a Clifford monoid is an inverse monoid with central idempotents, or equivalently, a monoid that is a (strong) semilattice Y of groups $G_\alpha, \alpha \in Y$. We denote the identity of G_α by $e_\alpha, \alpha \in Y$. Since S is a monoid, Y has a maximum element μ where e_μ is the identity of S. We note that for such a monoid, $\mathcal{R} = \mathcal{L} = \mathcal{H}$ so that left, right and (two-sided) ideals coincide, and the $\mathcal{H}$-classes are the groups $G_\alpha, \alpha \in Y$. Throughout this chapter we denote the identity element of Y by μ and the zero element of Y, where it exists, by 0.

For our own convenience we introduce the following notations in this chapter. Where there is possibility of ambiguity over the monoid in question, we use a superscript in the notation $\mathbf{R}^S(s,t)$ and $\mathbf{r}^S(s,t)$, that is, for elements s,t in a monoid S

$$\mathbf{R}^S(s,t) = \{(u,v) \in S \times S : su = tv\},$$

and

$$\mathbf{r}^S(s,t) = \{u \in S : su = tu\}.$$

Our motivation for doing so is to relate some conditions for a semilattice Y of groups $G_\alpha, \alpha \in Y$, in terms of the corresponding conditions on Y.

In Section 5.1, we consider Clifford monoids with least idempotents. After some general results we then concentrate on the case where the connecting homomorphisms are trivial. We investigate conditions on such a monoid S, such that $\mathbf{R}(s,t)$ and $\mathbf{r}(s,t)$ are finitely generated.

In Section 5.2, we investigate Clifford monoids (not necessarily with least idempotents), while considering the cases when connecting homomorphisms are general, trivial and one-one. We find necessary and sufficient conditions on S, such that $\mathbf{R}(s,t)$ and $\mathbf{r}(s,t)$ are finitely generated.

5.1 Clifford monoids with least idempotents

In this section we consider a Clifford monoid S with a least idempotent.

5.1.1 Clifford monoids with least idempotent(general case)

In this subsection we deal with Clifford monoids with a least idempotent and no restriction on the connecting homomorphisms. Our aim here to find necessary and sufficient conditions on S, such that $\mathbf{R}(s,t)$ and $\mathbf{r}(s,t)$ are finitely generated.

Proposition 5.1.1.1. *Let S be a monoid which is a semilattice Y of groups G_β, $\beta \in Y$ and suppose in addition that Y has a zero element 0. If $S \setminus G_0$ is finite, then the strongly flat left S-acts are axiomatisable.*

Proof. We are supposing that $S \setminus G_0$ is finite. Clearly then Y is finite, so that S is the union of finitely many groups. Since each (right) ideal is a union of G_αs, and there are only finitely many such, it follows that S has only finitely many ideals. Certainly then every ideal of S is finitely generated, so that each ideal of the form $\mathbf{r}(s,t)$ is finitely generated. It remains to prove that $\mathbf{R}(s,t)$ is finitely generated for all $s,t \in S$.

Case (I) Let $s,t \in G_0$. We claim that $\mathbf{R}(s,t) = \mathbf{R}$ where

$$\mathbf{R} = (e_\mu, t^{-1}s)S \cup (s^{-1}t, e_\mu)S \cup \bigcup_{\substack{u,v \in S\setminus G_0 \\ su=tv}} (u,v)S.$$

To prove our claim first note that,

$$se_\mu = s = e_0 s = tt^{-1}s$$

hence $(e_\mu, t^{-1}s) \in \mathbf{R}(s,t)$ and so $(e_\mu, t^{-1}s)S \subseteq \mathbf{R}(s,t)$. With the dual we deduce that

$$(e_\mu, t^{-1}s)S \cup (s^{-1}t, e_\mu)S \subseteq \mathbf{R}(s,t)$$

and clearly then $\mathbf{R} \subseteq \mathbf{R}(s,t)$.

Conversely, suppose that $(u,v) \in \mathbf{R}(s,t)$, so that $su = tv$. If $u, v \in S \setminus G_0$, then clearly $(u,v) \in \mathbf{R}$. If $u \in G_0$ then we have that

$$u = e_0 u = s^{-1} su = s^{-1} tv$$

so that $(u,v) = (s^{-1}t, e_\mu)v$ and so $(u,v) \in \mathbf{R}$. Together with the dual this yields that $\mathbf{R}(s,t) \subseteq \mathbf{R}$ and so $\mathbf{R}(s,t) = \mathbf{R}$ as required.

Case (II) If $s \in G_0, t \in G_\beta, \beta > 0$ we claim that $\mathbf{R}(s,t) = \mathbf{R}$ where

$$\mathbf{R} = \bigcup_{\substack{\delta \in Y \\ \beta\delta = 0}} (s^{-1}t, e_\delta)S \cup (e_\mu, t^{-1}s)S \cup \bigcup_{\substack{u,v \in S \setminus G_0 \\ su = tv}} (u,v)S.$$

To see this, notice that if $\beta\delta = 0$, then

$$s(s^{-1}t) = (ss^{-1})t = e_0 t = te_0 = te_\beta e_\delta = te_\delta$$

so that $(s^{-1}t, e_\delta) \in \mathbf{R}(s,t)$. Also,

$$se_\mu = s = e_\beta s = tt^{-1}s,$$

so that $(e_\mu, t^{-1}s) \in \mathbf{R}(s,t)$. Consequently, $\mathbf{R} \subseteq \mathbf{R}(s,t)$.

Conversely, suppose that $(u,v) \in \mathbf{R}(s,t)$. If $u, v \in S \setminus G_0$, then clearly $(u,v) \in \mathbf{R}$. If $u \in G_0$ and $v \in G_\delta$, then from $su = tv$ we have that $\beta\delta = 0$ and

$$u = s^{-1}su = s^{-1}tv$$

so that $(u,v) = (s^{-1}t, e_\delta)v \in \mathbf{R}$. On the other hand, if $u \in G_\nu$ and $v \in G_0$, then

$$t^{-1}su = t^{-1}tv = e_\beta v = v$$

so that $(u,v) = (e_\mu, t^{-1}s)u \in \mathbf{R}$. Hence $\mathbf{R} = \mathbf{R}(s,t)$ as required.

Case (III) Suppose now that if $s \in G_\beta$ and $t \in G_\gamma$ where $\beta, \gamma > 0$. We claim that $\mathbf{R}(s,t) = \mathbf{R}$ where

$$\mathbf{R} = \bigcup_{\delta\gamma = 0} (s^{-1}te_0, e_\delta)S \cup \bigcup_{\delta\beta = 0} (e_\delta, t^{-1}se_0)S \cup \bigcup_{\substack{u,v \in S \setminus G_0 \\ su = tv}} (u,v)S.$$

First, if $\delta\gamma = 0$, then

$$ss^{-1}te_0 = te_0 = te_\gamma e_\delta = te_\delta$$

so that $(s^{-1}te_0, e_\delta) \in \mathbf{R}(s,t)$. Together with the dual we see that $\mathbf{R} \subseteq \mathbf{R}(s,t)$.

Conversely, suppose that $(u,v) \in \mathbf{R}(s,t)$ and at least one of $u, v \in G_0$; without loss of generality suppose that $u \in G_0$ and $v \in G_\delta$. Then from $su = tv$ we have that $\gamma\delta = 0$ and

$$u = s^{-1}su = s^{-1}tv.$$

But $u = ue_0$ so that $u = s^{-1}te_0 v$ and $(u,v) = (s^{-1}te_0, e_\delta)v \in \mathbf{R}$. It follows that $\mathbf{R}(s,t) \subseteq \mathbf{R}$ and so $\mathbf{R} = \mathbf{R}(s,t)$ as required.

□

Remark 5.1.1.2. Note that for any monoid S, if $a\,\mathcal{L}\,b$ where $\mathcal{L}$ is *the left Green's relation* defined by

$$a\,\mathcal{L}\,b \Leftrightarrow a = xb \text{ and } b = ya$$

for some $x, y \in S^1$, then $\mathbf{R}(a,a) = \mathbf{R}(b,b)$.

In particular, if S is a semilattice Y of groups G_α, $\alpha \in Y$, and e_α is the identity of each G_α for each $\alpha \in Y$, then for any $s \in G_\alpha$, $\mathbf{R}(s,s) = \mathbf{R}(e_\alpha, e_\alpha)$.

The following result is now straightforward.

Lemma 5.1.1.3. *Let $s \in G_\alpha$ for some $\alpha \in Y$, then $\mathbf{R}(s,s)$ is finitely generated if and only if $\mathbf{R}(e_\alpha, e_\alpha)$ is finitely generated.*

We aim towards a converse to Proposition 5.1.1.1.

Lemma 5.1.1.4. *Let S be a monoid which is a semilattice Y of groups G_α, $\alpha \in Y$ and suppose in addition that Y has a zero element 0. If $\mathbf{R}(e_0, e_0)$ is finitely generated, then Y is finite.*

Proof. As $\mathbf{R}(e_0, e_0)$ is finitely generated, there exists a set of generators $\{(u_1, v_1), \ldots, (u_n, v_n)\}$ of $\mathbf{R}(e_0, e_0)$. Take $0 < \beta < \mu$, as $e_0 e_\mu = e_0 e_\beta$ so $(e_\mu, e_\beta) \in \mathbf{R}(e_0, e_0)$ implies that $(e_\mu, e_\beta) = (u_i, v_i)t$ for some $i \in \{1, \ldots, n\}$ and $t \in S$. We must have $t \in G_\mu$ which shows that $v_i \in G_\beta$ and hence Y is finite, as there are only finitely many v_i. □

Definition 5.1.1.5. A *chain*, or *linearly ordered set*, is a partially ordered set $(A, \leq)$ such that any two elements of A are comparable.

Definition 5.1.1.6. A semilattice Y is *finite above* if for any $\alpha \in Y$, $|\{\beta : \beta \geq \alpha\}| < \infty$.

In the proof of following theorem, even though we are calling upon later results, the arguments are not circular.

Theorem 5.1.1.7. *Let S be a monoid which is a semilattice Y of groups G_α, $\alpha \in Y$. Suppose Y has a zero element 0 and in addition that Y is a chain. Then the following conditions are equivalent;*

(i) Y is finite and for each $\beta \geq \gamma$, $\ker \phi_{\beta,\gamma}$ is finite;
(ii) $\mathcal{PWP}$ is axiomatisable;
(iii) $\mathbf{R}(s,s)$ is finitely generated, for all $s \in S$;
(iv) $\mathbf{R}(e_\alpha, e_\alpha)$ is finitely generated, for all $\alpha \in Y$.

Proof. $(iv) \Rightarrow (i)$ follows from Lemmas 5.1.1.4 and 5.2.0.11, $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ are clear.

(i) implies (ii) holds from Corollary 5.2.0.12. □

Example: Let S be a monoid which is a semilattice Y of groups $G_\beta, \beta \in Y$ such that Y has a zero element. Then axiomatisability of $\mathcal{PWP}$ implies axiomatisability of $\mathcal{E}$.

Proof. Let S be a monoid which is a semilattice Y of groups G_α, $\alpha \in Y$. Let 0 be the zero element of Y. Let $\mathbf{R}(e_0, e_0)$ be finitely generated, then by Lemma 5.1.1.4, Y is finite. As in Proposition 5.1.1.1, $\mathbf{r}(s,t)$ is finitely generated for all $s, t \in S$. Therefore $\mathcal{E}$ is axiomatisable.

□

5.1.2 Clifford monoids with least idempotents and trivial connecting homomorphisms

We now concentrate on Clifford monoids having a least idempotent and such that the connecting homomorphisms are trivial. We manage to show that in presence of a least idempotent, for a Clifford monoid S, strongly flat S-acts are axiomatisable if and only if $\mathbf{R}(e_0, e_0)$ is finitely generated, or equivalently $\mathbf{R}(e_\beta, e_\beta)$ is finitely generated, for all $\beta \in Y$.

We note that the following result also follows from Lemma 5.2.0.11, but we wish to show it directly.

Lemma 5.1.2.1. *Let S be a monoid which is a semilattice Y of groups G_α, $\alpha \in Y$ and suppose in addition that Y has a zero element 0 and the connecting homomorphisms of S are trivial. If $\mathbf{R}(e_0, e_0)$ is finitely generated, then for each $\beta > 0$, G_β is finite.*

Proof. As $\mathbf{R}(e_0, e_0)$ is finitely generated, there exists a set of generators $\{(u_1, v_1), \ldots, (u_n, v_n)\}$ of $\mathbf{R}(e_0, e_0)$. Let $u \in G_\beta$ where β is lying above 0. Then $(u, e_\beta) \in \mathbf{R}(e_0, e_0)$ as $e_0 u = e_0 = e_0 e_\beta$ and hence $(u, e_\beta) = (u_i, v_i)t$ for some $i \in \{1, 2, \ldots, n\}$ and $t \in G_\zeta$ for some $\zeta \in Y$. This implies that

$$u = ue_\beta^{-1} = u_i t t^{-1} v_i^{-1} = u_i v_i^{-1} e_\zeta.$$

But Y is finite by Lemma 5.1.1.4 and it follows that G_β is finite.

□

From Proposition 5.1.1.1, Lemma 5.1.1.4, and Lemma 5.1.2.1 we immediately have the next result.

Theorem 5.1.2.2. *Let S be a monoid which is a semilattice Y of groups G_α, $\alpha \in Y$ and suppose in addition that Y has a zero 0 and the connecting homomorphisms of S are trivial. Then the following are equivalent.*

(i) $S \setminus G_0$ is finite;

(ii) $\mathcal{SF}$ is axiomatisable;

(iii) $\mathbf{R}(s, t)$ is finitely generated;

(iv) $\mathbf{R}(e_\beta, e_\beta)$ is finitely generated $\forall \beta \in Y$;

(v) $\mathbf{R}(e_0, e_0)$ is finitely generated.

The next step will be to investigate Clifford monoids without the restriction that Y has a least element.

5.2 Clifford monoids

In this section we investigate Clifford monoids, without having any restrictions on connecting homomorphisms. We first note the following definition.

Definition 5.2.0.3. (i) A semigroup S has ascending chain condition (a.c.c.) on (principal) ideals if there exists no infinite chain

$$I_1 \subset I_2 \subset I_3 \subset \cdots$$

of (principal) ideals of S.

(ii) A poset Y has the ascending chain condition (a.c.c.) if there is no infinite chain

$$a_1 < a_2 < \cdots$$

of elements of Y.

Lemma 5.2.0.4. *Let S be a monoid which is a semilattice Y of groups G_α, $\alpha \in Y$. Then*

(i) if Z is an ideal of Y, $I = \cup_{j \in Z} G_j$ is an ideal of S;

(ii) if I is an ideal of S then $I = \cup_{j \in Z} G_j$ for some ideal Z in Y.

If $I = \cup_{j \in Z} G_j$ is an ideal of S, where Z is an ideal of Y, then I is finitely generated if and only if Z is finitely generated.

Proof. (i) If $g_j \in I$ where $j \in Z$ and $g_k \in G_k$ where $k \in Y$, then $g_j g_k \in G_{jk}$ and $jk \in Z$ as Z is an ideal. Therefore $g_j g_k \in I$.

(ii) As an ideal is a union of $\mathcal{H}$- classes, therefore $I = \cup_{j \in Z} G_j$ for some $Z \subseteq Y$. If $j \in Z$ and $k \in Y$, $e_j \in I$, and $e_k \in S$, therefore $e_j e_k = e_{jk} \in I$, then $jk \in Z$, therefore Z is an ideal of Y.

For the final part, let $I = \cup_{j \in Z} G_j$ be an ideal of S, where Z is an ideal of Y. It is easy to see that $\{s_j : j \in J\}$ is a generating set for I, where $s_j \in G_j$, if and only if $\{e_j : j \in J\}$ is a generating set for I. Moreover, $\{e_j : j \in J\}$ is a generating set for I if and only if J is a generating set for Z. The result follows.

□

We can now deduce the following.

Lemma 5.2.0.5. *Let S be a monoid, which is a semilattice Y of groups G_α, $\alpha \in Y$.*

(i) S has ascending chain condition on ideals if and only if Y has the ascending chain condition on ideals.

(ii) S has ascending chain condition on principal ideals if and only if Y has the ascending chain condition on principal ideals or equivalently, Y has the ascending chain condition as a poset.

Proof. (i) Let S have the a.c.c. on ideals and let Z be an ideal of Y. By Theorem 5.2.0.4, $I = \cup_{j \in Z} G_j$ is an ideal of S. By assumption, I is finitely generated, so that by Theorem 5.2.0.4, Z is finitely generated.

Conversely, suppose that Y has the a.c.c. on ideals and I is an ideal of S. Again by Theorem 5.2.0.4, $I = \cup_{j \in Z} G_j$ for some ideal Z of Y. By assumption, Z is finitely generated and it follows that I is finitely generated.

(ii) For the second part, it is clear that

$$\alpha_1 Y \subset \alpha_2 Y \subset \ldots$$

if and only if

$$\alpha_1 < \alpha_2 < \ldots$$

so that Y has a.c.c. on principal ideals if and only if Y has the a.c.c. as a poset.

For the first claim, it is easy to see that

$$e_\alpha S \subset e_\beta S \text{ if and only if } \alpha < \beta$$

from which the result follows.

□

Lemma 5.2.0.6. *If Y is a semilattice with identity such that for all $\alpha \in Y$ there exists only finitely many β with $\beta \geq \alpha$, then Y is a lattice.*

Proof. Let $\alpha, \beta \in Y$. Let $\mathcal{C} = \{\gamma : \gamma \geq \alpha, \beta\}$, then $\mathcal{C} \neq \emptyset$ as $\mu \in \mathcal{C}$. Let $\mathcal{C} = \{\gamma_1, \cdots, \gamma_n\}$ then $\gamma_1 \wedge \cdots \wedge \gamma_n$ exist. We claim that $\gamma_1 \wedge \cdots \wedge \gamma_n$ is the join (least upper bound) of α and β.

As $\alpha \leq \gamma_1, \cdots, \gamma_n$, therefore $\alpha \leq \gamma_1 \wedge \cdots \wedge \gamma_n$, similarly $\beta \leq \gamma_1 \wedge \cdots \wedge \gamma_n$. If $\delta \geq \alpha, \beta$ then $\delta \in \mathcal{C}$, so $\delta = \gamma_i$ say. Then $\delta \geq \gamma_1 \wedge \cdots \wedge \gamma_n$. Therefore $\gamma_1 \wedge \cdots \wedge \gamma_n$ is the join of $\{\alpha, \beta\}$. □

The following lemma will be useful.

Lemma 5.2.0.7. *Let Y be a semilattice. Then for any $\alpha \in Y$,*

$$D_\alpha = \{\tau \in Y : \tau \ngeq \alpha\}$$

is either empty or is an ideal of Y.

Proof. If $\tau \ngeq \alpha$, then for any $\eta \in Y$, $\eta\tau \ngeq \alpha$, as $\tau \geq \eta\tau$. Therefore D_α is an ideal. □

The following Theorem gives necessary condition on a Clifford monoid S, with Y is a chain, such that the class $\mathcal{P}$ is axiomatisable.

Theorem 5.2.0.8. *Let S be a monoid, which is a semilattice Y of groups G_α, $\alpha \in Y$. Suppose that Y is a chain and for each $\beta \in Y$*

$$|\cup_{\gamma \geq \beta} \ker \phi_{\gamma,\beta}| < \infty.$$

Then $\mathbf{R}(s,t)$ is finitely generated for all $s, t \in S$.

Proof. Notice first that the hypothesis guarantees that the semilattice Y is finite above.

Case (I): Let $s, t \in G_\alpha$. Let $\gamma \geq \alpha$. If $s^{-1}t \notin \text{ Im } \phi_{\gamma,\alpha}$, let $\mathbf{T}_\gamma = \emptyset$. If $s^{-1}t \in \text{ Im } \phi_{\gamma,\alpha}$, let

$$(s^{-1}t)\phi^{-1}_{\gamma,\alpha} = \{w_1^\gamma, \cdots, w_n^\gamma\}.$$

Then for all $\beta \geq \alpha$,

$$sw_i^\gamma = s(w_i^\gamma)\phi_{\gamma,\alpha} = s(s^{-1}t) = t = te_\beta$$

and

$$t(w_i^\gamma)^{-1} = t((w_i^\gamma)^{-1}\phi_{\gamma,\alpha}) = t(w_i^\gamma\phi_{\gamma,\alpha})^{-1} = t(s^{-1}t)^{-1} = s = se_\beta$$

so that $(w_i^\gamma, e_\beta), (e_\beta, (w_i^\gamma)^{-1}) \in \mathbf{R}(s,t)$. Let

$$\mathbf{T}_\gamma = \bigcup_{\beta \geq \alpha} \{(w_i^\gamma, e_\beta), (e_\beta, (w_i^\gamma)^{-1}) : 1 \leq i \leq n\}.$$

Put $\mathbf{T} = \bigcup_{\gamma \geq \alpha} \mathbf{T}_\gamma$. Note that $\mathbf{T}$ is finite and $\mathbf{T} \subseteq \mathbf{R}(s,t)$. Also, $ss^{-1}t = t = te_\alpha$, so $(s^{-1}t, e_\alpha) \in \mathbf{R}(s,t)$. Let

$$\mathbf{R} = \mathbf{T} \cup (s^{-1}t, e_\alpha)S$$

so that $\mathbf{R} \subseteq \mathbf{R}(s,t)$.

Conversely, let $(u,v) \in \mathbf{R}(s,t)$. Let $u \in G_\tau, v \in G_\eta$. Notice that $\tau \geq \alpha$ if and only if $\eta \geq \alpha$.

If $u \in G_\tau, v \in G_\eta$ with $\tau, \eta < \alpha$, then $\tau = \eta$ and $su = tv$ implies $u = s^{-1}tv$ so

$$(u,v) = (s^{-1}t, e_\alpha)v.$$

Otherwise $u \in G_\tau, v \in G_\eta$, where $\tau, \eta \geq \alpha$. Suppose that $\tau \geq \eta$ then from $su = tv$ we have

$$e_\alpha u = s^{-1}tv$$

$$e_\alpha uv^{-1} = s^{-1}t$$

$$\Rightarrow \; uv^{-1} \in G_\eta, \; (uv^{-1})\phi_{\eta,\alpha} = s^{-1}t.$$

It follows that $\mathbf{T}_\eta \neq \emptyset$, let $\{w_1^\eta, \cdots, w_n^\eta\} = (s^{-1}t)\phi_{\eta,\alpha}^{-1}$. Then

$$uv^{-1} = w_i^\eta \text{ for some } i, \, 1 \leq i \leq n.$$

Now $vu^{-1} = (w_i^\eta)^{-1}$ therefore $v = (w_i^\eta)^{-1}u$, giving

$$(u,v) = (e_\tau, (w_i^\eta)^{-1})u.$$

Similarly, if $\eta \geq \tau$ then $(u,v) = (w_i^\tau, e_\eta)v$. Therefore $\mathbf{R}(s,t) \subseteq \mathbf{R}$ and hence $\mathbf{R}(s,t) = \mathbf{R}$.

Case (II): Let $s \in G_\alpha, t \in G_\beta$, $\alpha > \beta$. Let

$$\mathbf{R} = \bigcup_{\beta \leq \delta} (s^{-1}t, e_\delta)S.$$

We note that $ss^{-1}t = e_\alpha t = t = te_\beta$ and $t = te_\delta$ for any $\delta \geq \beta$. Therefore $\mathbf{R} \subseteq \mathbf{R}(s,t)$.

Conversely, if $su = tv$ and if $u \in G_\gamma, v \in G_\delta$ then $\alpha\gamma = \beta\delta$ so $\alpha\gamma \leq \beta$, therefore $\alpha\gamma = \gamma \leq \beta < \alpha$. Since Y is a chain, γ must be equal to β or δ.

Case (i): If $\gamma = \beta$ then $(u,v) = (s^{-1}t, e_\delta)v$ and $\beta \leq \delta$.
Case(ii): If $\gamma = \delta$ then $(u,v) = (s^{-1}t, e_\beta)v$ as $\delta \leq \beta$.

Therefore $\mathbf{R} = \mathbf{R}(s,t)$.

□

The proof of the next lemma follows that of Lemma 5.1.1.4.

Lemma 5.2.0.9. *Let S be a monoid which is a semilattice Y of groups $G_\alpha, \alpha \in Y$. Suppose that $\mathbf{R}(e_\alpha, e_\alpha)$ is finitely generated for some $\alpha \in Y$. Then $\{\beta \in Y : \beta > \alpha\}$ is finite.*

Remark 5.2.0.10. In view of Lemma 5.2.0.9 we observe, if S is a monoid which is a semilattice Y of groups G_α for each $\alpha \in Y$, such that $\mathbf{R}(e_\alpha, e_\alpha)$ is finitely generated, then S is a lattice.

Lemma 5.2.0.11. *Let S be a monoid which is a semilattice Y of groups G_α, $\alpha \in Y$. Suppose $\mathbf{R}(e_\gamma, e_\gamma)$ is finitely generated for $\gamma \in Y$. Then for all $\beta \geq \gamma$,* $\ker\phi_{\beta,\gamma}$ *is finite.*

Proof. We have $u \in \ker\phi_{\beta,\gamma} \Leftrightarrow u\,\phi_{\beta,\gamma} = e_\gamma \Leftrightarrow e_\gamma\, u = e_\gamma e_\beta \Leftrightarrow (u, e_\beta) \in \mathbf{R}(e_\gamma, e_\gamma)$. Let $(u_1, v_1), \cdots, (u_n, v_n)$ be a finite set of generators for $\mathbf{R}(e_\gamma, e_\gamma)$.

Let $u \in \ker\phi_{\beta,\gamma}$. Then $(u, e_\beta) = (u_i, v_i)t$ for some i and $t \in G_\tau$ where $\tau \geq \beta$. Now,

$$\begin{aligned} u = ue_\beta^{-1} &= u_i t t^{-1} v_i^{-1} \\ &= u_i v_i^{-1} e_\tau \\ &= u_i v_i^{-1}. \end{aligned}$$

Therefore $\ker\phi_{\beta,\gamma}$ is finite.

□

The next result follows from Theorem 5.2.0.8, Lemma 5.2.0.11 and Lemma 5.2.0.9.

Corollary 5.2.0.12. *Let S be a monoid which is a chain Y of groups $G_\alpha, \alpha \in Y$. Then $\mathbf{R}^S(s, s)$ is finitely generated for all $s \in S$ if and only if $\bigcup_{\gamma > \alpha} \ker\phi_{\gamma,\alpha}$ is finite, if and only if $\mathbf{R}^S(s, t)$ is finitely generated for all $s, t \in S$.*

Lemma 5.2.0.13. *Let S be a monoid which is a semilattice Y of groups $G_\alpha, \alpha \in Y$. If $\mathbf{R}^S(e_\alpha, e_\alpha)$ is finitely generated for all $\alpha \in Y$, then $\mathbf{R}^Y(\alpha, \alpha)$ is finitely generated for all $\alpha \in Y$.*

Proof. Let $\alpha \in Y$ and put

$$\mathbf{R}^S(e_\alpha, e_\alpha) = \bigcup_{i \in X} (u_i, v_i)S$$

where X is finite. Suppose that for each $i \in X$ we have that $u_i \in G_{\mu_i}$ and $v_i \in G_{\nu_i}$. Then $(\mu_i, \nu_i) \in \mathbf{R}^Y(\alpha, \alpha)$.

Conversely if $(\beta, \gamma) \in \mathbf{R}^Y(\alpha, \alpha)$ then $\alpha\,\beta = \alpha\,\gamma$ so we will have

$$(e_\beta, e_\gamma) = (u_i, v_i)s$$

for some $i \in X$ and $s \in S$, say $s \in G_\delta$. But then $(\beta, \gamma) = (\mu_i, \nu_i)\delta$. It follows that

$$\mathbf{R}^Y(\alpha, \alpha) = \bigcup_{i \in X} (\mu_i, \nu_i)Y.$$

□

We aim now to characterise those conditions on a Clifford monoid S such that $\mathbf{R}^S(s,s)$ is finitely generated. In the following, if $\alpha, \beta \in Y$ where Y is a semilattice, then $\alpha \perp \beta$ means that α and β are incomparable.

Proposition 5.2.0.14. *Let S be a monoid which is a semilattice Y of groups $G_\alpha, \alpha \in Y$. Suppose that:*
(i) $\mathbf{R}^Y(\alpha,\alpha)$ is finitely generated for all $\alpha \in Y$;
(ii) $\mid \cup_{\beta>\alpha} G_\beta \mid < \infty$ for all $\alpha \in Y$;
(iii) for any $\alpha \in Y$, there exist only finitely many β with $\alpha \perp \beta$ and $\mid G_\beta \mid > 1$.
Then $\mathbf{R}^S(s,s)$ is finitely generated for all $s \in S$.

Proof. Let $s \in G_\alpha$. By Lemma 5.1.1.3 , it is enough to show that $\mathbf{R}^S(e_\alpha, e_\alpha)$ is finitely generated.

Let $K = \bigcup_{\substack{\beta \perp \alpha \\ |G_\beta|>1}} G_\beta$. Notice that if K is not empty, there exists $\beta \perp \alpha$ with G_β not trivial, then as $\alpha\beta < \beta$ we have that G_β is finite. It follows that $|K| < \infty$.

Let $v \in K$ be such that $v \in G_\tau$ where $\tau \perp \alpha$. Then if $e_\alpha v = e_\alpha t$ we have that $t \in G_\gamma$ for $\gamma \geq \alpha\tau$. We define

$$W_v = \{t : e_\alpha v = e_\alpha t, t \in G_\gamma, \gamma > \alpha\tau\}.$$

Note that W_v is finite.

Suppose $\mathbf{R}^Y(\alpha,\alpha) = \bigcup_{j\in X}(\mu_j,\nu_j)Y$ where X is finite. Then $(e_{\mu_j}, e_{\nu_j}) \in \mathbf{R}^S(e_\alpha, e_\alpha)$.

We claim that $\mathbf{R}^S(e_\alpha, e_\alpha) = \mathbf{R}$ where

$$\mathbf{R} = \bigcup_{j\in X}(e_{\mu_j}, e_{\nu_j})S \cup \bigcup_{\substack{\alpha<\eta, \alpha<\delta, u\in G_\eta, \\ v\in G_\delta, e_\alpha u = e_\alpha v}} (u,v)S \cup \bigcup_{v\in K}\bigcup_{t\in W_v}\big((v,t)S \cup (t,v)S\big).$$

Clearly $\mathbf{R} \subseteq \mathbf{R}^S(e_\alpha, e_\alpha)$.

Notice that for any $\beta, \gamma \geq \alpha$, we have that $(e_\beta, e_\gamma) = (e_{\mu_i}, e_{\nu_i})e_\delta$ for some $\delta \in Y$.

Conversely, suppose $(u,v) \in \mathbf{R}^S(e_\alpha, e_\alpha)$ then $e_\alpha u = e_\alpha v$. Certainly by the remark above, $(e_\alpha, e_\alpha) = (e_{\mu_i}, e_{\nu_i})e_\gamma$ for some $i \in X$ and $\gamma \in Y$. Consequently, if u , $v \in G_\alpha$, then we have that $u = v$ so that

$$(u,v) = (e_\alpha, e_\alpha)u = (e_{\mu_i}, e_{\nu_i})e_\gamma u.$$

If we have $u \in G_\gamma$, $v \in G_\alpha$ where $\gamma > \alpha$ then $e_\alpha u = e_\alpha v$ which shows that $v = e_\alpha u$ which implies that

$$(u,v) = (e_\gamma, e_\alpha)u$$

and so is in $\mathbf{R}$, with the same reasoning as above. Dually if $u \in G_\alpha, v \in G_\gamma$ with $\gamma > \alpha$.

If $u \in G_\eta$, $v \in G_\delta$ with $\eta, \delta > \alpha$, then clearly $(u,v) \in \mathbf{R}$. We have now exhausted the cases where $u \in G_\beta, v \in G_\gamma$ and $\beta, \gamma \geq \alpha$. Notice that if $u \in G_\beta$ where $\beta \geq \alpha$ then we must have $v \in G_\gamma$ where $\gamma \geq \alpha$ also.

If $u \in G_\gamma$ where $\gamma < \alpha$, then $v \in G_\delta$ where $\delta \ngeq \alpha$ and $\gamma = \alpha\delta$. Hence $(\gamma, \delta) \in \mathbf{R}^Y(\alpha,\alpha)$ so there exists (μ_i, ν_i), $i \in X$ such that $(e_\gamma, e_\delta) = (e_{\mu_i}, e_{\nu_i})e_\omega$ for some $\omega \in Y$. We have

$$u = e_\alpha u = e_\alpha v = e_\alpha e_\delta v = e_\gamma v$$

so that

$$(u,v) = (e_\gamma, e_\delta)v \in \mathbf{R}.$$

Dually if $v \in G_\gamma$ where $\gamma < \alpha$.

We are left with the case where $u \in G_\beta, v \in G_\gamma$, $\beta \perp \alpha \perp \gamma$.

Suppose first that $|G_\beta| > 1$. Then $u \in K$ and as $e_\alpha u = e_\alpha v$ we have that $\alpha\beta = \alpha\gamma$ so that $\gamma \geq \alpha\beta$. If $\gamma = \alpha\beta$ then $\gamma \leq \alpha$, a contradiction. On the other hand if $\gamma > \alpha\beta$ then $v \in W_u$ so that $(u,v) \in \mathbf{R}$. Dually if $|G_\gamma| > 1$.

Finally, suppose that $|G_\beta| = |G_\gamma| = 1$. Then $u = e_\beta$, $v = e_\gamma$ and so $e_\alpha e_\beta = e_\alpha e_\gamma$ and it follows that $(u,v) \in \mathbf{R}$.

Thus we can conclude that $\mathbf{R}^S(e_\alpha, e_\alpha) = \mathbf{R}$ and so is finitely generated. □

5.2.1 Clifford monoids with trivial connecting homomorphisms

In this subsection we investigate conditions on a Clifford monoid S, with trivial connecting homomorphisms, such that $\mathcal{SF}$ is axiomatisable. We first consider the axiomatisability of the class $\mathcal{E}$ and then move onto the case of $\mathcal{P}$.

Lemma 5.2.1.1. *Let S be a monoid which is a semilattice Y of groups $G_\alpha, \alpha \in Y$. Then for any $\alpha, \beta \in Y$,*

$$\mathbf{r}^S(e_\alpha, e_\beta) = \bigcup_{\gamma \in \mathbf{r}^Y(\alpha,\beta)} G_\gamma.$$

Proof. Let $s \in \mathbf{r}^S(e_\alpha, e_\beta)$, with $s \in G_\gamma$. Then $e_\alpha s = e_\beta s$ so that $\alpha\gamma = \beta\gamma$ and $\gamma \in \mathbf{r}^Y(\alpha, \beta)$. Thus

$$\mathbf{r}^S(e_\alpha, e_\beta) \subseteq \bigcup_{\gamma \in \mathbf{r}^Y(\alpha,\beta)} G_\gamma.$$

On the other hand, if $t \in G_\gamma$ where $\gamma \in \mathbf{r}^Y(\alpha, \beta)$, then $e_\alpha e_\gamma = e_\beta e_\gamma$ gives us $e_\alpha t = e_\beta t$ and so $t \in \mathbf{r}^S(e_\alpha, e_\beta)$. Hence

$$\mathbf{r}^S(e_\alpha, e_\beta) \supseteq \bigcup_{\gamma \in \mathbf{r}^Y(\alpha,\beta)} G_\gamma.$$

□

Lemma 5.2.1.2. *Let S be a monoid which is a semilattice Y of groups $G_\alpha, \alpha \in Y$, with trivial connecting homomorphisms. Then if $\alpha > \beta$, $s \in G_\alpha$, and $t \in G_\beta$, then $\mathbf{r}(s,t) = \mathbf{r}(e_\alpha, t)$.*

Proof. Let $u \in S$, then

$$\begin{aligned} su = tu &\Rightarrow s^{-1}su = s^{-1}tu \\ &\Rightarrow e_\alpha u = tu \\ &\Rightarrow se_\alpha u = stu. \\ &\Rightarrow su = tu \end{aligned}$$

□

Lemma 5.2.1.3. *Let S be a monoid which is a semilattice Y of groups G_α, $\alpha \in Y$, such that the connecting homomorphisms are trivial. Let $I_\alpha = \bigcup_{\tau \in D_\alpha} G_\tau$, where $D_\alpha = \{\tau \in Y : \tau \not\geq \alpha\}$ as given in Lemma 5.2.0.7. Then $I_\alpha = \mathbf{r}(s,t)$ for any $s, t \in G_\alpha$ with $s \neq t$.*

Proof. Let $s, t \in G_\alpha$, $s \neq t$. Let $I_\alpha = \cup_{\tau \in D_\alpha} G_\tau$, since D_α is an ideal of Y, so I_α is an ideal of S, by Theorem 5.2.0.4.

Let $e_\tau \in I_\alpha$ so that $\tau \in D_\alpha$. Then $\tau \not\geq \alpha$ so that $\alpha\tau < \alpha$. We have

$$se_\tau = se_\alpha e_\tau = se_{\alpha\tau} = e_{\alpha\tau} = te_\tau.$$

Therefore $e_\tau \in \mathbf{r}(s,t)$, and so $G_\tau \subseteq \mathbf{r}(s,t)$ and hence $I_\alpha \subseteq \mathbf{r}(s,t)$.

Conversely, let $G_\kappa \subseteq \mathbf{r}(s,t)$ so that $se_\kappa = te_\kappa$. If $\kappa \geq \alpha$, then this would give $s = t$, a contradiction, therefore $\kappa \not\geq \alpha$, and so $\kappa \in D_\alpha$, then $G_\kappa \subseteq I_\alpha$. We therefore have $\mathbf{r}(s,t) \subseteq I_\alpha$. □

Lemma 5.2.1.4. *Let S be a monoid which is a semilattice Y of groups $G_\alpha, \alpha \in Y$, such that the connecting homomorphisms are trivial. If $\alpha \perp \beta$ then for any $s \in G_\alpha, t \in G_\beta$, $\mathbf{r}(s,t) = \cup_{\gamma \in \mathbf{r}(\alpha,\beta)} G_\gamma$, where $\mathbf{r}(s,t) = \mathbf{r}^S(s,t)$ and $\mathbf{r}(\alpha,\beta) = \mathbf{r}^Y(\alpha,\beta)$.*

Proof. If $\alpha\gamma = \beta\gamma$, then $\alpha\gamma < \alpha$ (else $\alpha = \beta\gamma \leq \beta$), so $se_\gamma = e_{\alpha\gamma} = e_{\beta\gamma} = te_\gamma$ implies that $G_\gamma \subseteq \mathbf{r}(s,t)$ so that $\cup_{\gamma \in \mathbf{r}(\alpha,\beta)} G_\gamma \subseteq \mathbf{r}(s,t)$.

If $g_\gamma \in \mathbf{r}(s,t)$ then $sg_\gamma = tg_\gamma$ so $\alpha\gamma = \beta\gamma$ and $g_\gamma \in \cup_{\gamma \in \mathbf{r}(\alpha,\beta)} G_\gamma$. Therefore the claim is proved.

□

Lemma 5.2.1.5. *Let S be a monoid which is a semilattice Y of groups G_α, $\alpha \in Y$, such that the connecting homomorphisms are trivial. Suppose $G_\beta \neq \{e_\beta\}$ and $\beta < \alpha$. Let $U_{\alpha,\beta} = \{\gamma : \alpha\gamma < \beta\}$. Then $U_{\alpha,\beta}$ is an ideal of Y. Let $t \in G_\beta \setminus \{e_\beta\}$ then $\mathbf{r}(e_\alpha, t) = \bigcup_{\tau \in U_{\alpha,\beta}} G_\tau$.*

Proof. Let $u \in G_\gamma$ such that $u \in \mathbf{r}(e_\alpha, t)$, $e_\alpha u = tu$ implies $e_\alpha e_\gamma = te_\gamma$ so that $\alpha\gamma = \beta\gamma \leq \beta$.

If $\alpha\gamma = \beta$, $\beta \leq \gamma$ then $e_\beta = te_\gamma = t$, a contradiction.

Therefore $\alpha\gamma < \beta$ so that $\gamma \in U_{\alpha,\beta}$ and $u \in \bigcup_{\tau \in U_{\alpha,\beta}} G_\tau$, hence $\mathbf{r}(e_\alpha, t) \subseteq \cup_{\tau \in U_{\alpha,\beta}} G_\tau$.

Conversely, let $u \in \bigcup_{\tau \in U_{\alpha,\beta}} G_\tau$. Then $u \in G_\tau$ for some τ with $\alpha\tau < \beta$.

Also

$$\alpha\tau = \alpha\tau\beta = \alpha\beta\tau = \beta\tau < \beta$$

and

$$\begin{aligned} e_\alpha e_\tau = e_\beta e_\tau &\Rightarrow e_\alpha e_\tau = e_\beta e_{\alpha\tau} = te_{\alpha\tau} \\ &\Rightarrow e_\alpha e_\tau u = te_\alpha e_\tau u \\ &\Rightarrow e_\alpha u = tu \\ &\Rightarrow u \in \mathbf{r}(e_\alpha, t) \end{aligned}$$

□

Theorem 5.2.1.6. *Let S be a monoid which is a semilattice Y of groups G_α, $\alpha \in Y$, such that the connecting homomorphisms are trivial. Then $\mathcal{E}$ is axiomatisable if and only if*

(i) D_α is finitely generated for any α with $|G_\alpha| > 1$.

(ii) for any α, β, $\mathbf{r}(\alpha, \beta)$ is finitely generated.

(iii) for any β with $G_\beta \neq \{e_\beta\}$ and $\alpha > \beta$, $U_{\alpha,\beta}$ is finitely generated.

Proof. We know that $\mathcal{E}$ is axiomatisable if and only if $\mathbf{r}(a, b)$ is finitely generated for all $a, b \in S$.

Let $a \in G_\alpha$ and $b \in G_\beta$. If $\alpha \perp \beta$, then by Proposition 5.2.1.4, $\mathbf{r}(a, b) = \bigcup_{\gamma \in \mathbf{r}(\alpha,\beta)} G_\gamma$. If $\alpha > \beta$, then $\mathbf{r}(a, b) = r(e_\alpha, b)$ by Proposition 5.2.1.2. If $b = e_\beta$, then $\mathbf{r}(a, b) = \mathbf{r}(e_\alpha, e_\beta) = \bigcup_{\gamma \in \mathbf{r}(\alpha,\beta)} G_\gamma$ by Lemma 5.2.1.1. If $b \neq e_\beta$, then $\mathbf{r}(a, b) = \bigcup_{\gamma \in U_{\alpha,\beta}} G_\gamma$ by Proposition 5.2.1.5. Finally, if $\alpha = \beta$, then either $a = b$ so that $\mathbf{r}(a, b) = S$, or if $a \neq b$, then $\mathbf{r}(a, b) = \bigcup_{\gamma \in D_\alpha} G_\gamma$ by Proposition 5.2.1.3.

The result now follows from the final part of Lemma 5.2.0.4. □

Remark 5.2.1.7. If Y has the ascending chain condition on ideals, clearly all these properties hold.

Consider the case where Y is a chain. Then for $\alpha < \beta$ we have that $\mathbf{r}^Y(\alpha, \beta) = \alpha Y$ and also $D_\alpha = \alpha Y$, so are finitely generated. Thus $\mathcal{E}$ is axiomatisable if and only if every $U_{\alpha,\beta}$ is finitely generated for every $\alpha > \beta$ with G_β non-trivial. But, that is equivalent to every element β of Y with G_β non-trivial having a greatest predecessor.

We recall from Lemma 5.2.0.9 that if S is a monoid which is a semilattice Y of groups $G_\alpha, \alpha \in Y$, then if $\mathbf{R}(e_\alpha, e_\alpha)$ is finitely generated, there are only finitely many elements of Y above α. The next lemma follows from Lemma 5.1.2.1, but we give a direct argument.

Lemma 5.2.1.8. *Let S be a monoid which is a semilattice Y of groups $G_\alpha, \alpha \in Y$, with trivial connecting homomorphisms. Suppose that $\mathbf{R}(e_\alpha, e_\alpha)$ is finitely generated for some $\alpha \in Y$. Then each G_β is finite for $\beta > \alpha$.*

Proof. Since $\mathbf{R}(e_\alpha, e_\alpha)$ is finitely generated, it has a finite set of generators say

$$\{(u_1, v_1), \cdots, (u_n, v_n)\}.$$

Let $u \in G_\beta$ where $\beta > \alpha$. Then $(u, e_\beta) \in \mathbf{R}(e_\alpha, e_\alpha)$ as the connecting homomorphisms are trivial. Hence $(u, e_\beta) = (u_i, v_i)\, t$ for some $i \in \{1, 2, \cdots, n\}$ and $t \in G_\gamma$ with $\gamma \geq \beta$. This implies that

$$u = ue_\beta^{-1} = u_i t t^{-1} v_i^{-1} = u_i v_i^{-1} e_\gamma.$$

Thus G_β is finite. □

The following is immediate from Lemma 5.2.1.8 and Lemma 5.2.0.9.

Corollary 5.2.1.9. *Let S be a monoid which is a semilattice Y of groups $G_\alpha, \alpha \in Y$, with trivial connecting homomorphisms. Suppose that $\mathbf{R}(e_\alpha, e_\alpha)$ is finitely generated for some $\alpha \in Y$. Then*

$$| \cup_{\beta > \alpha} G_\beta | < \infty.$$

Lemma 5.2.1.10. *Let S be a monoid which is a semilattice Y of groups $G_\alpha, \alpha \in Y$ such that the connecting homomorphisms are trivial. Suppose that $\mathbf{R}^S(e_\alpha, e_\alpha)$ is finitely generated for all $\alpha \in Y$. Then for any $\alpha \in Y$, there exist only finitely many β with $\alpha \perp \beta$ and $\mid G_\beta \mid > 1$.*

Proof. Let
$$\{(u_1, v_1), \dots, (u_n, v_n)\}$$
be a set of generators for $\mathbf{R}^S(e_\alpha, e_\alpha)$.

Suppose that $\{\beta_i : i \in Z\}$ is the set of elements of Y such that $\alpha \perp \beta_i$ for all $i \in Z$. Let
$$Z' = \{i \in Z : \{u_i, v_i : 1 \leq i \leq n\} \cap G_{\beta_i} \neq \emptyset\}$$
and observe that Z' is finite.

For any $u, v \in G_{\beta_i}$ where $i \notin Z'$, we have $e_\alpha u = e_\alpha v = e_{\alpha\beta_i}$, so that
$$\begin{aligned}(u, v) &= (u_k, v_k)\, t = (t, t) \quad if\, t \in G_{\beta_i} \\ &= (e_{\beta_i}, e_{\beta_i}) \quad if\, t \notin G_{\beta_i}\end{aligned}$$
which implies that $u = v$ so there exists only finitely many G_{β_i} such that G_{β_i} are not trivial.

□

We can now put together in order to show the following.

Theorem 5.2.1.11. *Let S be a monoid which is a semilattice Y of groups $G_\alpha, \alpha \in Y$ such that all connecting morphisms are trivial. Then $\mathbf{R}^S(s, s)$ is finitely generated for all $s \in S$ if and only if*

(i) $\mathbf{R}^Y(\alpha, \alpha)$ is finitely generated for all $\alpha \in Y$;

(ii) $\mid \cup_{\beta > \alpha} G_\beta \mid < \infty$ for all $\alpha \in Y$;

(iii) for any $\alpha \in Y$, there exist only finitely many β with $\alpha \perp \beta$ and $\mid G_\beta \mid > 1$.

Proof. If Conditions (i), (ii), (iii) hold, then $\mathbf{R}^S(s, s)$ is finitely generated for all $s \in S$, by Proposition 5.2.0.14.

Conversely let $\mathbf{R}^S(s, s)$ be finitely generated then as $\mathbf{R}^S(s, s) = \mathbf{R}^Y(e_\alpha, e_\alpha)$ if $s \in G_\alpha$, we have $\mathbf{R}^S(e_\alpha, e_\alpha)$ is finitely generated and hence Conditions $(i), (ii)$ and (iii) hold by Lemma 5.2.0.13, Lemma 5.2.0.11 and Lemma 5.2.1.10 respectively. □

Lemma 5.2.1.12. *Let S be a monoid which is a semilattice Y of groups $G_\alpha, \alpha \in Y$ such that the connecting homomorphisms are trivial. Suppose that $\mathbf{R}(s, t)$ is finitely generated for all $s, t \in S$.*

Let $s \in G_\alpha$, $t \in G_\beta$ and let $\mathbf{R}(s, t) = \bigcup_{i \in I}(u_i, v_i)S$ where $u_i \in G_{\mu_i}$, $v_i \in G_{\nu_i}$. Suppose that $\delta \perp \beta$. Then there exists only finitely many γ with $\alpha\,\gamma = \beta\,\delta$, and $\alpha \perp \gamma$.

Proof. Let $v \in G_\delta$, $v \neq e_\delta$. Let $\gamma \in Y$ with $\alpha \perp \gamma$ and $\alpha\gamma = \beta\delta$. Then
$$se_\gamma = e_{\alpha\gamma} = e_{\beta\delta} = tv$$
so $(e_\gamma, v) \in \mathbf{R}(s, t)$ which implies that $(e_\gamma, v) = (u_i, v_i)u$ say where $u \in G_\mu$.

Let
$$T = \{\mu_i \tau : i \in I, \tau \geq \delta\}$$

so that T is finite.

Now $v = v_i u \in G_{\nu_i \mu = \delta}$ therefore $\mu \geq \delta$. Now $\gamma = \mu_i \mu \in T$.

□

Lemma 5.2.1.13. *Let S be a monoid which is a semilattice Y of groups $G_\alpha, \alpha \in Y$. Suppose that the class of S-acts satisfying Condition (P) is axiomatisable. Then $\mathbf{R}^Y(\alpha, \beta)$ is finitely generated.*

Proof. Let $\alpha, \beta \in Y$ and let

$$(u_1, v_1)S \cup \cdots \cup (u_n, v_n)S = \mathbf{R}^S(e_\alpha, e_\beta).$$

Then $e_\alpha u_i = e_\beta v_i$ for all i where $u_i \in G_{\mu_i}$, $v_i \in G_{\nu_i}$ therefore $\alpha\, \mu_i = \beta\, \nu_i$ so $(\mu_i, \nu_i) \in \mathbf{R}^Y(\alpha, \beta)$.

If $\alpha\, \gamma = \beta\, \delta$ then $e_\alpha e_\gamma = e_\beta e_\delta$ so

$$(e_\gamma, e_\delta) = (u_i, v_i)t \;\text{ where } t \in G_\mu$$

$$\Rightarrow \gamma = \mu_i \mu, \;\; \delta = \nu_i \mu$$

$$\Rightarrow (\gamma, \delta) = (\mu_i, \nu_i)\mu$$

so $\mathbf{R}^Y(\alpha, \beta) = \bigcup_{1 \leq i \leq n} (\mu_i, \nu_i)Y$. □

We restate Corollary 5.2.0.12 in the current context.

Theorem 5.2.1.14. *Let S be a monoid which is a chain Y of groups $G_\alpha, \alpha \in Y$ such that the connecting homomorphisms are trivial. Then the following conditions are equivalent:*
(i) $|\bigcup_{\gamma > \alpha} G_\gamma| < \infty$;
(ii) $\mathbf{R}(s, s)$ is finitely generated for all $s \in S$;
(iii) $\mathbf{R}(s, t)$ is finitely generated for all $s, t \in S$;
(iv) $\mathcal{PWP}$ is axiomatisable;
(v) $\mathcal{P}$ is axiomatisable;
(vi) $\mathcal{SF}$ is axiomatisable.

In the case where Y has a zero and the connecting homomorphisms are trivial, $\mathbf{R}(e_\beta, e_\beta)$ is finitely generated, for all $e_\beta \in S$ if and only if $\mathcal{SF}$ is axiomatisable. We have to work much harder where Y does not have a zero, as we now see.

Theorem 5.2.1.15. *Let S be a monoid which is a semilattice Y of groups G_α, $\alpha \in Y$, such that the connecting homomorphisms are trivial. Then $\mathbf{R}(s, t)$ is finitely generated for all $s, t \in S$ with $s\, \mathcal{H}\, t$ if and only if*
(i) $\mathbf{R}^Y(\alpha, \alpha)$ is finitely generated for all $\alpha \in Y$;
(ii) $|\bigcup_{\beta > \alpha} G_\beta| < \infty$ for all $\alpha \in Y$;
(iii) for any $\alpha \in Y$ there exists only finitely many β with $\alpha \perp \beta$ and $|G_\beta| > 1$;
(iv) let $\alpha \in Y$ and suppose that $\delta \perp \alpha$; then there exists only finitely many γ with $\gamma \perp \alpha$ and $\alpha\gamma = \alpha\delta$;

(v) let $G_\alpha \neq \{e_\alpha\}$ then $\mathbf{R}^Y(\alpha, \alpha)$ has a finite set of generators $\{(\mu_i, \nu_i) : 1 \leq i \leq n\}$ such that if $\gamma \perp \alpha \perp \delta$ and $\alpha\,\gamma = \alpha\,\delta$, then

$$(\gamma, \delta) = (\mu_i, \nu_i)\mu$$

for some i with $\mu_i \not\geq \alpha,\ \nu_i \not\geq \alpha$.

Proof. Suppose that $\mathbf{R}(s, t)$ is finitely generated for all $s, t \in G_\alpha$, for some $\alpha \in Y$. Then (i), (ii) and (iii) hold by Theorem 5.2.1.11.

(iv) Holds by Proposition 5.2.1.12, adjusted to the case where $\alpha = \beta$.

(v) Let $s \neq t \in G_\alpha$ and let

$$\mathbf{R}(s, t) = \bigcup_{1 \leq i \leq m} (u_i, v_i)S$$

where $u_i \in G_{\mu_i},\ v_i \in G_{\nu_i}$. Then $su_i = tv_i$ for each $i \in I$ so that $\alpha\mu_i = \alpha\nu_i$ and

$$(\mu_i, \nu_i) \in \mathbf{R}^Y(\alpha, \alpha).$$

Now $\mu_i \geq \alpha \Leftrightarrow \mu_i\alpha = \alpha \Leftrightarrow \nu_i\alpha = \alpha \Leftrightarrow \nu_i \geq \alpha$.

If $\mu_i,\ \nu_i > \alpha$ then $s = t$ which gives a contradiction.

Suppose $\gamma \perp \alpha \perp \delta$ and $\alpha\gamma = \alpha\delta$ then

$$se_\gamma = e_{\alpha\gamma} = e_{\alpha\delta} = te_\delta$$

so $(e_\gamma, e_\delta) = (u_i, v_i)r$ for some $r \in G_\mu$ therefore $\gamma = \mu_i\mu,\ \ \delta = \nu_i\mu$. If $\mu_i \geq \alpha$ and $\nu_i \geq \alpha$ then we must have $\mu_i = \alpha$ and $\gamma \leq \alpha$, or $\nu_i = \alpha$ and $\delta \leq \alpha$ a contradiction, therefore $\mu_i,\ \nu_i \not\geq \alpha$.

Let H be a finite set of generators for $\mathbf{R}^Y(\alpha, \alpha)$ guaranteed by (i). If H does not have the described property we can augment H with suitable generators (μ_i, ν_i) as above.

Conversely, suppose conditions (i) to (v) are true. From Theorem 5.2.1.11, $\mathbf{R}(s, s)$ is finitely generated for all $s \in S$. We show that $\mathbf{R}(s, t)$ is finitely generated for all $s, t \in G_\alpha$ and for some $\alpha \in Y$.

If G_α is trivial, then $s = t$ and we know $\mathbf{R}(s, s)$ is finitely generated. Therefore we suppose that $G_\alpha \neq \{e_\alpha\}$ and $s \neq t$.

Let

$$T_1 = \{(u, v) : u \in G_\gamma, v \in G_\beta, \gamma > \alpha,\ \beta > \alpha, su = tv\}$$

and let

$$\mathbf{R}_1 = T_1 S,$$

so that clearly

$$\mathbf{R}_1 \subseteq \mathbf{R}(s, t).$$

Let

$$\{(\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)\}$$

be the finite set of generators of $\mathbf{R}^Y(\alpha, \alpha)$ guaranteed by (v). Let

$$T_2 = \{(e_{\mu_i} s^{-1} t, e_{\nu_i}), (e_{\mu_i}, t^{-1} s e_{\nu_i}) : 1 \leq i \leq n\}$$

and let $\mathbf{R}_2 = T_2 S$. Notice that

$$se_{\mu_i}s^{-1}t = e_\alpha e_{\mu_i}t = e_\alpha e_{\nu_i}t = e_{\nu_i}t = te_{\nu_i}$$

so that we see $\mathbf{R}_2 \subseteq \mathbf{R}(s,t)$.

Suppose $\delta \perp \alpha$ and G_δ is not trivial. Let $\gamma_1, \cdots, \gamma_n$ be such that $\gamma_i \perp \alpha$ and $\alpha\gamma_i = \alpha\delta$. Notice that by (iii), G_δ and each G_{γ_i} is finite. Let

$$T_\delta = \{(u,v) : u \in G_{\gamma_i}, 1 \le i \le n, v \in G_\delta\}$$

and notice that $T_\delta \subseteq \mathbf{R}(s,t)$. Let

$$T_3 = \bigcup_{\substack{\delta\perp\alpha \\ |G_\delta|>1}} T_\delta \text{ and } \mathbf{R}_3 = T_3 S.$$

By (iii), T_3 is finite and we have $\mathbf{R}_3 \subseteq \mathbf{R}(s,t)$.

Let $J \subseteq \{1, \ldots, n\}$ be such that $j \in J$ if and only if $\mu_j \not\geq \alpha$ and $\nu_j \not\geq \alpha$. Notice that

$$se_{\mu_j} = e_{\alpha\mu_j} = e_{\alpha\nu_j} = te_{\nu_j}$$

so that with

$$T_4 = \{(e_{\mu_j}, e_{\nu_j}) : j \in J\}$$

we have that

$$\mathbf{R}_4 = T_4 S \subseteq \mathbf{R}(s,t).$$

Let

$$\mathbf{R} = \mathbf{R}_1 \cup \mathbf{R}_2 \cup \mathbf{R}_3 \cup \mathbf{R}_4,$$

so that $\mathbf{R} \subseteq \mathbf{R}(s,t)$.

Conversely, let $(u,v) \in \mathbf{R}(s,t)$ and $u \in G_\gamma$, $v \in G_\delta$ so that $\alpha\gamma = \alpha\delta$ and $(\gamma,\delta) \in \mathbf{R}^Y(\alpha,\alpha)$. Then $\gamma \ge \alpha$ if and only if $\delta \ge \alpha$. If $\gamma > \alpha$ and $\delta > \alpha$ then $(u,v) \in \mathbf{R}_1$.

If $\gamma \le \alpha$ then $su \in G_\gamma = G_{\alpha\,\delta}$ and $u = s^{-1}tv$, also

$$(e_\gamma, e_\delta) = (e_{\mu_i}, e_{\nu_i})e_\mu$$

for some $i \in \{1, \ldots, n\}$. We have

$$(u,v) = (e_{\mu_i}e_\mu s^{-1}tv, e_{\nu_i}e_\mu v) = (e_{\mu_i}s^{-1}t, e_{\nu_i})e_\mu v \in \mathbf{R}_2.$$

Dually, if $\delta \le \alpha$.

Suppose $\gamma \perp \alpha \;\perp \delta$. If G_δ is not trivial, then as $\delta \perp \alpha$, $\gamma \perp \alpha$, and $\alpha\gamma = \alpha\delta$, we have $\gamma = \gamma_i$ and $(u,v) \in \mathbf{R}_3$.

Dually if G_γ is not trivial.

Finally suppose G_γ, G_δ are trivial, so $u = e_\gamma$ and $v = e_\delta$. By assumption $(\gamma,\delta) = (\mu_i,\nu_i)\mu$ for some μ_i, $\nu_i \in Y$ with $\mu_i, \nu_i \not\geq \alpha$. We have $(\mu_i,\nu_i) \in T_4$ and

$$(u,v) = (e_\gamma, e_\delta) = (e_{\mu_i}, e_{\nu_i})e_\mu \in \mathbf{R}_4.$$

Thus $\mathbf{R}(s,t) \subseteq \mathbf{R}$ and $\mathbf{R}(s,t)$ is finitely generated. □

We conjecture that with similar (but more complicated versions) of (v) in the above result, we can find necessary and sufficient conditions such that $\mathbf{R}(s,t)$ is finitely generated for all $s,t \in S$. Of course, these are in terms of the corresponding conditions on Y, so we would like to further investigate such conditions on semilattices.

5.2.2 Clifford monoids with one-one connecting homomorphism

In this subsection we specialise Corollary 5.2.0.12 to the case where the connecting homomorphisms are one-one.

Proposition 5.2.2.1. *Let S be a monoid which is a chain Y of groups G_α, $\alpha \in Y$ with connecting homomorphisms are one-one. Then the following conditions are equivalent:*
(i) Y is finite above;
(ii) the strongly flat S-acts are axiomatisable.

Proof. Suppose that Y is finite above. Then $|\bigcup_{\beta>\gamma} \ker \phi_{\beta,\gamma}| < \infty$, so that by Theorem 5.2.0.8, $\mathbf{R}(s,t)$ is finitely generated. Clearly any ideal of S is finitely generated, so that any ideal of the form $\mathbf{r}(s,t)$ is finitely generated. Therefore $\mathcal{SF}$ is axiomatisable.

Conversely, if $\mathcal{SF}$ is axiomatisable, then by Corollary 5.2.0.12,

$$|\bigcup_{\beta>\gamma} \ker \phi_{\beta,\gamma}| < \infty$$

so that Y is finite above. $\square$

We are able to axiomatise certain classes of S-posets for a pomonoid S in Chapter 4, the answers being in terms of finitary conditions on S. Therefore, many questions which have been asked for Clifford monoids can be asked for Clifford pomonoids. If we take the ordering to be the natural partial order, that is,

$$a \leq b \text{ if and only if } a = aa^{-1}b,$$

then we can obtain similar results to those of this chapter.

Chapter 6

Appendix

In this appendix we provide the proofs that we have omitted in the main body of the text, since they follow closely others already provided. For the convenience of the reader, we restate results before giving the proofs.

Lemma 3.1.1.3. *Let $\mathcal{C}$ be a class of embeddings of right S-posets satisfying Condition (Free). Then the following are equivalent for a left S-poset B:*

(i) B is $\mathcal{C}$-flat;

*(ii) $-\otimes B$ maps the embeddings $\tau_{\mathcal{S}}: W_{\mathcal{S}} \to W'_{\mathcal{S}}$ in the category **Pos-S** to monomorphisms in the category of **Pos**, for every double ordered skeleton $\mathcal{S}$;*

(iii) if $(\mu_{\mathcal{S}}\tau_{\mathcal{S}}, b)$ and $(\mu'_{\mathcal{S}}\tau_{\mathcal{S}}, b')$ are connected by a double ordered tossing over $W'_{\mathcal{S}}$ and B with double ordered skeleton $\mathcal{S}$, then $(u_{\mathcal{S}}, b)$ and $(u'_{\mathcal{S}}, b')$ are connected by a double ordered tossing over $W_{\mathcal{S}}$ and B.

Proof. Clearly we need only show that (iii) implies (i). Suppose that (iii) holds, let $\mu : A \to A'$ lie in $\mathcal{C}$ and suppose that

$$(a\mu, b), (a'\mu, b') \in A' \times B$$

are connected via a double ordered tossing with double ordered skeleton $\mathcal{S}$, so that $\gamma_{\mathcal{S}}(b, b')$ holds. From considering the left hand side of the double ordered tossing, we have that $\delta_{\mathcal{S}}(a\mu, a'\mu)$ is true in A'. By assumption there is an embedding $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}}$ in $\mathcal{C}$ and $u_{\mathcal{S}}, u'_{\mathcal{S}} \in W_{\mathcal{S}}$ such that $\delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$, and a morphism $\nu : W'_{\mathcal{S}} \to A'$ such that $u_{\mathcal{S}}\,\tau_{\mathcal{S}}\nu = a\mu$, $u'_{\mathcal{S}}\,\tau_{\mathcal{S}}\nu = a'\mu$ and $W_{\mathcal{S}}\tau_{\mathcal{S}}\nu \subseteq A\mu$. Since $\delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$, there is an double ordered tossing from $(u_{\mathcal{S}}\tau_{\mathcal{S}}, b)$ to $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, b')$ over $W'_{\mathcal{S}}$ and B, with double ordered skeleton $\mathcal{S}$. From (iii), it follows that $(u_{\mathcal{S}}, b)$ and $(u'_{\mathcal{S}}, b')$ are connected via a double ordered tossing over $W_{\mathcal{S}}$ and B with double ordered skeleton $\mathcal{T}$ say. It follows that $\delta_{\mathcal{T}}(u_{\mathcal{S}}, u'_{\mathcal{S}})$ is true in $W_{\mathcal{S}}$ and so $\delta_{\mathcal{T}}(u_{\mathcal{S}}\tau_{\mathcal{S}}\nu, u'_{\mathcal{S}}\tau_{\mathcal{S}}\nu)$, that is, $\delta_{\mathcal{T}}(a\mu, a'\mu)$ is true in $A\mu$. Since μ is an ordered embedding we deduce that $\delta_{\mathcal{T}}(a, a')$ is true in A and consequently, (a, b) and (a', b') are connected via a double ordered tossing with double ordered skeleton $\mathcal{T}$ over A and B. Hence B is $\mathcal{C}$-flat as required. $\square$

Lemma 3.1.1.6. *Let $\mathcal{C}$ be a class of embeddings of right S-posets, satisfying Condition (Free). Let $\overline{\mathcal{C}}$ be the set of products of morphisms in $\mathcal{C}$. If a left S-poset B is $\mathcal{C}$-flat, then it is $\overline{\mathcal{C}}$-flat.*

Proof. Let I be an indexing set and let $\gamma_i : A_i \to A'_i \in \mathcal{C}$ for all $i \in I$. Let $A = \prod_{i \in I} A_i$, $A' = \prod_{i \in I} A'_i$ and let $\gamma : A \to A'$ be the canonical embedding, so that $(a_i)\gamma = (a_i \gamma_i)$.

Suppose B is a $\mathcal{C}$-flat left S-poset. Let $\underline{a} = (a_i), \underline{a}' = (a'_i) \in A$ and $b, b' \in B$ be such that $\underline{a}\gamma \otimes b = \underline{a}'\gamma \otimes b$ in $A' \otimes B$. Then for some double ordered skeleton $\mathcal{S}$,

$$A' \models \delta_{\mathcal{S}}(\underline{a}\gamma, \underline{a}'\gamma) \text{ and } B \models \gamma_{\mathcal{S}}(b, b').$$

It follows that for each $i \in I$,

$$A_i^{'} \models \delta_{\mathcal{S}}(a_i\gamma_i, a'_i\gamma_i).$$

By assumption that $\mathcal{C}$ has Condition (Free), there exist $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}} \in \mathcal{C}$ and $u_{\mathcal{S}}, u'_{\mathcal{S}} \in W_{\mathcal{S}}$ such that $\delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$. Further, for each $i \in I$, as $\delta_{\mathcal{S}}(a_i\gamma_i, a'_i\gamma_i)$ is true in A'_i, there exists an S-pomorphism $\nu_i : W'_{\mathcal{S}} \to A'_i$ such that $u_{\mathcal{S}}\,\tau_{\mathcal{S}}\nu_i = a_i\gamma_i$, $u'_{\mathcal{S}}\tau_{\mathcal{S}}\nu_i = a'_i\gamma_i$ and $W_{\mathcal{S}}\tau_{\mathcal{S}}\nu_i \subseteq A_i\gamma_i$.

We have $\delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}})$ is true in $W'_{\mathcal{S}}$ and $\gamma_{\mathcal{S}}(b, b')$ is true in B, giving that $u_{\mathcal{S}}\tau_{\mathcal{S}} \otimes b = u'_{\mathcal{S}}\tau_{\mathcal{S}} \otimes b'$ in $W'_{\mathcal{S}} \otimes B$. As B is a $\mathcal{C}$-flat left S-poset and $\tau_{\mathcal{S}} : W_{\mathcal{S}} \to W'_{\mathcal{S}} \in \mathcal{C}$, we have that $u_{\mathcal{S}} \otimes b = u'_{\mathcal{S}} \otimes b'$ in $W_{\mathcal{S}} \otimes B$, say via a double ordered tossing with double ordered skeleton $\mathcal{U}$. It follows that

$$W_{\mathcal{S}} \models \delta_{\mathcal{U}}(u_{\mathcal{S}}, u'_{\mathcal{S}}) \text{ and } B \models \gamma_{\mathcal{U}}(b, b').$$

By *(ii)* of Remark 3.1.1.1, we have that

$$A_i\gamma_i \models \delta_{\mathcal{U}}(u_{\mathcal{S}}\tau_{\mathcal{S}}\nu_i, u'_{\mathcal{S}}\tau_{\mathcal{S}}\nu_i),$$

that is,

$$A_i\gamma_i \models \delta_{\mathcal{U}}(a_i\gamma_i, a'_i\gamma_i).$$

Writing $\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2)$ where $\mathcal{U}_1$ has length h and $\mathcal{U}_2$ has length k, we have that there are elements $w_{i,2}, \dots, w_{i,h}, z_{i,2}, \dots, z_{i,k} \in A_i$ such that

$$\epsilon_{\mathcal{U}_1}(a_i\gamma_i, w_{i,2}\gamma_i, \dots, w_{i,h}\gamma_i, a'_i\gamma_i) \text{ and } \epsilon_{\mathcal{U}_2}(a'_i\gamma_i, z_{i,2}\gamma_i, \dots, z_{i,k}\gamma_i, a_i\gamma_i)$$

are true. But γ_i is an *embedding*, so that

$$\epsilon_{\mathcal{U}_1}(a_i, w_{i,2}, \dots, w_{i,h}, a'_i) \text{ and } \epsilon_{\mathcal{U}_2}(a'_i, z_{i,2}, \dots, z_{i,k}, a_i)$$

hold in A_i. Hence $\delta_{\mathcal{U}}(a_i, a'_i)$ is true in each A_i and so $\delta_{\mathcal{U}}(\underline{a}, \underline{a}')$ holds in A. Together with $\gamma_{\mathcal{U}}(b, b')$ being true in B, we deduce that $\underline{a} \otimes b = \underline{a}' \otimes b'$ in $A \otimes B$, as required. □

Theorem 3.1.1.7. *Let $\mathcal{C}$ be a class of embeddings of right S-posets satisfying Condition (Free). Then the following conditions are equivalent for a pomonoid S:*

(i) the class $\mathcal{CF}$ is axiomatisable;

(ii) the class $\mathcal{CF}$ is closed under formation of ultraproducts;

(iii) for every double ordered skeleton $\mathcal{S} \in \mathbb{DOS}$ there exist finitely many double ordered replacement skeletons $\mathcal{S}_1, \dots, \mathcal{S}_{\alpha(\mathcal{S})}$ such that, for any embedding $\gamma : A \to A'$ in $\mathcal{C}$ and any $\mathcal{C}$-flat left S-poset B, if $(a\gamma, b)$, $(a'\gamma, b') \in A' \times B$ are connected by a double ordered tossing $\mathcal{T}$ over A' and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a', b') are connected by a double ordered tossing $\mathcal{T}^{'}$ over A and B such that $\mathcal{S}(\mathcal{T}^{'}) = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(\mathcal{S})\}$;

(iv) for every double ordered skeleton $\mathcal{S} \in \mathbb{DOS}$ there exists finitely many double ordered replacement skeletons $\mathcal{S}_1, \dots, \mathcal{S}_{\beta(\mathcal{S})}$ such that, for any $\mathcal{C}$-flat left S-poset B, if $(u_{\mathcal{S}}\tau_{\mathcal{S}}, b)$ and $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, b')$ are connected by the double ordered tossing $\mathcal{T}$ over $W'_{\mathcal{S}}$ and B (with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$), then $(u_{\mathcal{S}}, b)$, and $(u'_{\mathcal{S}}, b')$ are connected by a double ordered tossing $\mathcal{T}^{'}$ over $W_{\mathcal{S}}$ and B such that $\mathcal{S}(\mathcal{T}^{'}) = \mathcal{S}_k$, for some $k \in \{1, \cdots, \beta(\mathcal{S})\}$.

Proof. The implication (i) implies (ii) is clear from Łos's Theorem.

To prove (ii) implies (iii), we suppose that $\mathcal{CF}$, the class of $\mathcal{C}$-flat left S-posets is closed under formation of ultraproducts and that (iii) is false. Let J be the family of finite subsets of $\mathbb{DOS}$. We suppose that there exists a double ordered skeleton $\mathcal{S}_0 \in \mathbb{DOS}$ such that for every subset f of J, there exists an embedding $\gamma_f : A_f \to A'_f \in \mathcal{C}$, a $\mathcal{C}$-flat left S-poset B_f, and pairs $(a_f\gamma_f, b_f)$, $(a'_f\gamma_f, b'_f) \in A'_f \times B_f$ such that $(a_f\gamma_f, b_f)$ and $(a'_f\gamma_f, b'_f)$ are connected over A'_f and B_f by a double ordered tossing $\mathcal{T}_f$ with double ordered skeleton $\mathcal{S}_0$, but such that no double ordered replacement tossing over A_f and B_f connecting (a_f, b_f) and (a'_f, b'_f) has a double ordered skeleton belonging to the set f.

Let $J_{\mathcal{S}} = \{f \in J : \mathcal{S} \in f\}$ for each $\mathcal{S} \in \mathbb{DOS}$. Then there exists an ultrafilter Φ on J containing each $J_{\mathcal{S}}$, as each intersection of finitely many of the sets $J_{\mathcal{S}}$ is non-empty.

We now define $A' = \prod_{f\in J} A'_f$, $A = \prod_{f\in J} A_f$ and $B = \prod_{f\in J} B_f$. Let $\gamma : A \to A'$ be the embedding given by $(a_f)\gamma = (a_f\gamma_f)$. We note here that $\underline{a}\gamma \otimes \underline{b} = \underline{a}'\gamma \otimes \underline{b}'$ in $A' \otimes B$, where $\underline{a} = (a_f)$, $\underline{a}' = (a'_f)$, $\underline{b} = (b_f)$ and $\underline{b}' = (b'_f)$ and that this equality is determined by a double ordered tossing over A' and B (the "product" of the double ordered tossings $\mathcal{T}_f$'s) having double ordered skeleton $\mathcal{S}_0$. It follows that the equality for $\underline{a}\gamma \otimes \underline{b}_\Phi = \underline{a}'\gamma \otimes \underline{b}'_\Phi$ holds also in $A' \otimes \mathcal{U}$ where $\mathcal{U} = (\prod_{f\in J} B_f)/\Phi$, and can be determined by a double ordered tossing over A' and $\mathcal{U}$ with double ordered skeleton $\mathcal{S}_0$.

By assumption, $\mathcal{U}$ is $\mathcal{C}$-flat, and by Lemma 3.1.1.6, $\underline{a} \otimes \underline{b}_\Phi = \underline{a}' \otimes b'_\Phi$ in $A \otimes \mathcal{U}$, say via a double ordered tossing with double ordered skeleton $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2)$ of length $h + k$, say

$$\mathcal{V}_1 = (d_1, e_1, \dots, d_h, e_h) \text{ and } \mathcal{V}_2 = (g_1, \ell_1, \dots, g_k, \ell_k).$$

Hence

$$A \models \delta_{\mathcal{V}}(\underline{a}, \underline{a}') \text{ and } \mathcal{U} \models \gamma_{\mathcal{V}}(\underline{b}_\Phi, \underline{b}'_\Phi).$$

Certainly $A_f \models \delta_{\mathcal{V}}(a_f, a'_f)$ for every f. Considering now the truth of $\gamma_{\mathcal{V}}(\underline{b}_\Phi, \underline{b}'_\Phi)$, there exist

$$(b_{1,f})_\Phi, \dots, (b_{h,f})_\Phi, (c_{1,f})_\Phi, \dots, (c_{k,f})_\Phi \in \mathcal{U}$$

such that

$$\begin{array}{rclrcl} \underline{b}_\Phi & \leq & d_1(b_{1,f})_\Phi & \underline{b}'_\Phi & \leq & g_1(c_{1,f})_\Phi \\ e_1(b_{1,f})_\Phi & \leq & d_2(b_{2,f})_\Phi & \ell_1(c_{1,f})_\Phi & \leq & g_2(c_{2,f})_\Phi \\ & \vdots & & & \vdots & \\ e_h(b_{h,f})_\Phi & \leq & \underline{b}'_\Phi & \ell_k(c_{k,f})_\Phi & \leq & \underline{b}_\Phi. \end{array}$$

As Φ is closed under finite intersections, there exists $D \in \Phi$ such that

$$\begin{array}{rclrcl} b_f & \leq & d_1 b_{1,f} & b'_f & \leq & g_1 c_{1,f} \\ e_1 b_{1,f} & \leq & d_2 b_{2,f} & \ell_1 c_{1,f} & \leq & g_2 c_{2,f} \\ & \vdots & & & \vdots & \\ e_h b_{h,f} & \leq & b'_f & \ell_k c_{k,f} & \leq & b_f \end{array}$$

for all $f \in D$.

Now suppose that $f \in D \cap J_{\mathcal{V}}$, then from the double ordered tossing just considered, we see that $\mathcal{V}$ is the double ordered skeleton of a double ordered tossing over A_f and B_f connecting the pairs (a_f, b_f) and (a'_f, b'_f); that is, $\mathcal{V}$ a double ordered replacement skeleton for the double ordered skeleton $\mathcal{S}_0$ of the double ordered tossing $\mathcal{T}_f$. But $\mathcal{V}$ belongs to f, a contradiction. This completes the proof that (ii) implies (iii).

It is clear that (iii) implies (iv).

Now we want to prove that (iv) implies (i). We assume that (iv) holds. We aim to use this condition to construct a set of axioms for $\mathcal{CF}$.

Let $\mathbb{S}_1$ denote the set of all elements of $\mathbb{DOS}$ such that if $\mathcal{S} \in \mathbb{S}_1$, then there is no $\mathcal{C}$-flat left S-poset B such that $\gamma_{\mathcal{S}}(b, b') \in B$ for any $b, b' \in B$. For $\mathcal{S} \in \mathbb{S}_1$ we put

$$\psi_{\mathcal{S}} := (\forall x)(\forall x')\neg\gamma_{\mathcal{S}}(x, x')$$

For $\mathcal{S} \in \mathbb{S}_2 = \mathbb{DOS} \setminus \mathbb{S}_1$, there must be a $B \in \mathcal{CF}$ and $b, b' \in B$ such that $\gamma_{\mathcal{S}}(b, b')$ is true in B, whence there is a double ordered tossing from $(u_{\mathcal{S}}\tau_{\mathcal{S}}, b)$ to $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, b')$ over $W'_{\mathcal{S}}$ and B with double ordered skeleton $\mathcal{S}$.

Let $\mathcal{S}_1, \cdots, \mathcal{S}_{\beta(\mathcal{S})}$ be a minimum set of double ordered replacement skeletons for double ordered tossings with double ordered skeleton $\mathcal{S}$ connecting pairs of the form $(u_{\mathcal{S}}\tau_{\mathcal{S}}, c)$ to $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, c')$ where $c, c' \in C$ and C ranges over $\mathcal{CF}$. Hence for each k in $\{1, \cdots, \beta(\mathcal{S})\}$, there exists a $\mathcal{C}$-flat left S-poset C_k, elements $c_k, c'_k \in C_k$ such that

$$W_{\mathcal{S}} \models \delta_{\mathcal{S}_k}(u_{\mathcal{S}}, u'_{\mathcal{S}}) \text{ and } C_k \models \gamma_{\mathcal{S}_k}(c_k, c'_k).$$

We define $\phi_{\mathcal{S}}$ to be the sentence

$$\phi_{\mathcal{S}} := (\forall y)(\forall y')\big(\gamma_{\mathcal{S}}(y, y') \to \gamma_{\mathcal{S}_1}(y, y') \vee \ldots \vee \gamma_{\mathcal{S}_{\beta(\mathcal{S})}}(y, y')\big).$$

Let

$$\sum\nolimits_{\mathcal{CF}} = \{\psi_{\mathcal{S}} : \mathcal{S} \in \mathbb{S}_1\} \cup \{\phi_{\mathcal{S}} : \mathcal{S} \in \mathbb{S}_2\}.$$

We claim that $\sum_{\mathcal{CF}}$ axiomatises $\mathcal{CF}$.

Suppose first that D is any $\mathcal{C}$-flat left S-poset. By choice of $\mathbb{S}_1$, it is clear that $D \models \psi_{\mathcal{S}}$ for any $\mathcal{S} \in \mathbb{S}_1$.

Now take any $\mathcal{S} \in \mathbb{S}_2$, and suppose that $d, d' \in D$ are such that D satisfies $\gamma_{\mathcal{S}}(d, d')$. Then, as noted earlier $(u_{\mathcal{S}}\tau_{\mathcal{S}}, d)$ and $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, d')$ are joined over $W'_{\mathcal{S}}$ and D by a double ordered tossing with double ordered skeleton $\mathcal{S}$, and therefore, by assumption, there is a double ordered tossing over $W_{\mathcal{S}}$ and D joining $(u_{\mathcal{S}}, d)$ and $(u'_{\mathcal{S}}, d')$ with double ordered skeleton $\mathcal{S}_k$ for some $k \in \{1, \cdots, \beta(\mathcal{S})\}$. It is now clear that $\gamma_{\mathcal{S}_k}(d, d')$ holds in D, as required. We have shown that $D \models \sum_{\mathcal{CF}}$.

Finally, we show that a left S-poset C that satisfies $\Sigma_{\mathcal{CF}}$ must be a $\mathcal{C}$-flat. We need to show that Condition (iii) of Lemma 3.1.1.3 holds for C. Let $\mathcal{S} \in \mathbb{DOS}$ and suppose we have a double ordered tossing with double ordered skeleton $\mathcal{S}$ connecting $(u_{\mathcal{S}}\tau_{\mathcal{S}}, c)$ and $(u'_{\mathcal{S}}\tau_{\mathcal{S}}, c')$ over $W'_{\mathcal{S}}$ and C. Then

$$W'_{\mathcal{S}} \models \delta_{\mathcal{S}}(u_{\mathcal{S}}\tau_{\mathcal{S}}, u'_{\mathcal{S}}\tau_{\mathcal{S}}) \text{ and } C \models \gamma_{\mathcal{S}}(c, c').$$

If $\mathcal{S}$ belonged to $\mathbb{S}_1$, then C would satisfy the sentence

$$(\forall y)(\forall y')\neg\gamma_{\mathcal{S}}(y, y')$$

and so $\neg\gamma_{\mathcal{S}}(c, c')$ hold, which is a contradiction. Therefore we conclude that $\mathcal{S}$ belongs to $\mathbb{S}_2$. Because C satisfies $\phi_{\mathcal{S}}$ and because $\gamma_{\mathcal{S}}(c, c')$ holds, it follows that $\gamma_{\mathcal{S}_k}(c, c')$ holds for some $k \in \{1, 2, \cdots, \beta(\mathcal{S})\}$. But $W_{\mathcal{S}} \models \delta_{\mathcal{S}_k}(u_{\mathcal{S}}, u'_{\mathcal{S}})$, whence $(u_{\mathcal{S}}, c)$ and $(u'_{\mathcal{S}}, c')$ are connected via a double ordered tossing over $W_{\mathcal{S}}$ and C with double ordered skeleton $\mathcal{S}_k$, showing that C is $\mathcal{C}$-flat. □

Theorem 3.1.2.1. *Let $\mathcal{C}$ be a class of embeddings of right S-posets.*

The following conditions are equivalent:

(i) the class $\mathcal{CF}$ is axiomatisable;

(ii) the class $\mathcal{CF}$ is closed under ultraproducts;

(iii) for every double ordered skeleton $\mathcal{S} \in \mathbb{DOS}$ and $a, a' \in A$, where $\mu : A \to A'$ is in $\mathcal{C}$, there exist finitely many double ordered skeleton $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(a,\mathcal{S},a',\mu)}$, such that for any $\mathcal{C}$-flat left S-poset B, if $(a\mu, b)$, $(a'\mu, b')$ are connected by a double ordered tossing $\mathcal{T}$ over A' and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then (a, b) and (a', b') are connected by a double ordered tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(a, \mathcal{S}, a', \mu)\}$.

Proof. The implication (i) implies (ii) is clear from Łos's Theorem.

To prove (ii) implies (iii), we suppose that $\mathcal{CF}$, the class of $\mathcal{C}$-flat left S-posets, is closed under formation of ultraproducts, and assume that (iii) is false. Let J be the family of finite subsets of $\mathbb{DOS}$. We suppose that for some double ordered skeleton $\mathcal{S}_0 \in \mathbb{DOS}$, for some embedding $\mu : A \to A' \in \mathcal{C}$, and for some $a, a' \in A$, for every $f \in J$ there is a $\mathcal{C}$-flat left S-poset B_f, and $b_f, b'_f \in B_f$ such that $(a\mu, b_f)$ and $(a'\mu, b'_f)$ are connected over A' and B_f by a double ordered tossing $\mathcal{T}_f$ with double ordered skeleton $\mathcal{S}_0$, but such that no double ordered replacement tossing over A and B_f connecting (a, b_f) and (a', b'_f) has a double ordered skeleton belonging to the set f.

Let $J_{\mathcal{S}} = \{f \in J : \mathcal{S} \in f\}$ for each $\mathcal{S} \in \mathbb{DOS}$. Now we are able to define an ultrafilter Φ on J containing each $J_{\mathcal{S}}$ for all $\mathcal{S} \in \mathbb{DOS}$, as each intersection of finitely many of the sets $J_{\mathcal{S}}$ is non-empty.

We note here that $a\mu \otimes \underline{b} = a^{'}\mu \otimes \underline{b}^{'}$ in $A' \otimes B$, where $B = \prod_{f \in J} B_f$, $\underline{b} = (b_f)$ and $\underline{b}' = (b'_f)$, and that this equality is determined by a double ordered tossing over A' and B (the "product" of the double ordered tossings $\mathcal{T}_f$) having double ordered skeleton $\mathcal{S}_0$. It follows that the equality for $a\mu \otimes \underline{b}_\Phi = a^{'}\mu \otimes \underline{b}^{'}_\Phi$ holds also in $A' \otimes \mathcal{U}$ where $\mathcal{U} = (\prod_{f \in J} B_f)/\Phi$, and can be determined by a double ordered tossing over A' and $\mathcal{U}$ with double ordered skeleton $\mathcal{S}_0$.

By assumption $\mathcal{U}$ is $\mathcal{C}$-flat, so that $(a, \underline{b}_\Phi)$ and $(a', \underline{b}^{'}_\Phi)$ are connected via a double ordered replacement tossing over A and $\mathcal{U}$, with double ordered skeleton $\mathcal{V}$ say. Hence

$$A \models \delta_{\mathcal{V}}(a, a') \text{ and } \mathcal{U} \models \gamma_{\mathcal{V}}(\underline{b}_\Phi, \underline{b}_\Phi).$$

As in Theorem 3.1.1.7, there exists $D \in \Phi$ such that $B_f \models \gamma_{\mathcal{V}}(b_f, b'_f)$ for all $f \in D$.

Now suppose that $f \in D \cap J_{\mathcal{V}}$. Then $\mathcal{V}$ is the double ordered skeleton of a double ordered tossing over A and B_f connecting the pairs (a, b_f) and (a', b'_f); that is, $\mathcal{V}$ is a double ordered replacement skeleton for the double ordered skeleton $\mathcal{S}_0$ of the double ordered tossing $\mathcal{T}_f$. But $\mathcal{S}$ belongs to f, a contradiction. This completes the proof that (ii) implies (iii).

Finally, suppose that (iii) holds. Let

$$\mathbb{T}' = \{(a, \mathcal{S}, a', \mu) : \mathcal{S} \in \mathbb{DOS}, \mu : A \to A' \in \mathcal{C}, a, a' \in A', \delta_{\mathcal{S}}(a\mu, a'\mu) \text{ holds}\}.$$

We introduce sentences corresponding to elements of $\mathbb{T}'$ in such a way that the resulting set of sentences axiomatises the class $\mathcal{CF}$.

We let $\mathbb{T}_1$ be the set of $(a, \mathcal{S}, a', \mu) \in \mathbb{T}'$ such that $\gamma_{\mathcal{S}}(b, b')$ does not hold for any b, b' in any $\mathcal{C}$-flat left S-poset B, and put $\mathbb{T}_2 = \mathbb{T}' \setminus \mathbb{T}_1$. For $T = (a, \mathcal{S}, a', \mu) \in \mathbb{T}_1$ we let

$$\psi_T = \psi_{\mathcal{S}} := (\forall x)(\forall x')\neg\gamma_{\mathcal{S}}(x, x').$$

If $T = (a, \mathcal{S}, a', \mu) \in \mathbb{T}_2$, then $\mathcal{S}$ is the double ordered skeleton of some double ordered tossing joining $(a\mu, b)$ to $(a'\mu, b')$ over A' and some $\mathcal{C}$-flat left S-poset B. By our assumption (*iii*), there is a finite list of double ordered replacement skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(T)}$. Choosing $\alpha(T)$ to be minimal, for each $k \in \{1, \cdots, \alpha(T)\}$, there exist a $\mathcal{C}$-flat left S-poset C_k and elements $c_k, c_k' \in C_k$, such that

$$A \models \delta_{\mathcal{S}_k}(a, a') \text{ and } C_k \models \gamma_{\mathcal{S}_k}(c_k, c_k').$$

We let ϕ_T be the sentence

$$\phi_T := (\forall y)(\forall y')(\gamma_{\mathcal{S}}(y, y') \to \gamma_{\mathcal{S}_1}(y, y') \vee \cdots \vee \gamma_{\mathcal{S}_{\alpha(T)}}(y, y')).$$

Let

$$\sum\nolimits_{\mathcal{CF}} = \{\psi_T : T \in \mathbb{T}_1\} \cup \{\phi_T : T \in \mathbb{T}_2\}.$$

We claim that $\sum_{\mathcal{CF}}$ axiomatises $\mathcal{CF}$.

Suppose first that D is any $\mathcal{C}$-flat left S-poset. Let $T = (a, \mathcal{S}, a', \mu) \in \mathbb{T}_1$. Then $\gamma_{\mathcal{S}}(b, b')$ is not true for any $b, b' \in B$, for any $\mathcal{C}$-flat left S-poset B, so certainly $D \models \psi_T$.

On the other hand, let $T = (a, \mathcal{S}, a', \mu) \in \mathbb{T}_2$, and let $d, d' \in D$ be such that $\gamma_{\mathcal{S}}(d, d')$ is true. Together with the fact $\delta_{\mathcal{S}}(a\mu, a'\mu)$ holds, we have that $(a\mu, d)$ is connected to $(a'\mu, d')$ over A' and D via a double ordered tossing with double ordered skeleton $\mathcal{S}$. Because D is $\mathcal{C}$-flat, (a, d) and (a', d') are connected over A and D, and by assumption (*iii*), we can take the double ordered replacement tossing to have double ordered skeleton one of $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(T)}$, say $\mathcal{S}_k$. Thus $D \models \gamma_{\mathcal{S}_k}(d, d')$ and it follows that $D \models \phi_T$. Hence D is a model of $\sum_{\mathcal{CF}}$.

Conversely, we show that every model of $\sum_{\mathcal{CF}}$ is $\mathcal{C}$-flat. Let $C \models \sum_{\mathcal{CF}}$ and suppose that $\mu : A \to A' \in \mathcal{C}$, $a, a' \in A$, $c, c' \in C$ and $a\mu \otimes c = a'\mu \otimes c'$ in $A' \otimes C$, say with double ordered tossing having double ordered skeleton $\mathcal{S}$. Then the quadruple $T = (a, \mathcal{S}, a', \mu) \in \mathbb{T}'$. Since $\gamma_{\mathcal{S}}(c, c')$ holds, C cannot be a model of ψ_T. Since $C \models \sum_{\mathcal{CF}}$ it follows that $T \in \mathbb{T}_2$. But then ϕ_T holds in C so that for some $k \in \{1, \cdots, \alpha(T)\}$ we have that $\gamma_{\mathcal{S}_k}(c, c')$ is true. We also know that $A \models \delta_{\mathcal{S}_k}(a, a')$, so that we have double ordered tossing over A and C connecting (a, c) to (a', c'). Thus C is $\mathcal{C}$-flat.

$\square$

Lemma 3.2.1.2. *Let $\mathcal{C}$ be a class of embeddings of right S-posets satisfying Condition (Free)$^{\leq}$. Then the following are equivalent for a left S-poset B:*

(i) B is $\mathcal{C}$-po-flat;

(ii) $- \otimes B$ maps the embeddings $\kappa_{\mathcal{S}} : V_{\mathcal{S}} \to V_{\mathcal{S}}'$ in the category ***Pos*-S** *to embeddings in the category of* **Pos***, for every ordered skeleton $\mathcal{S}$;*

(iii) if $v_{\mathcal{S}}\kappa_{\mathcal{S}} \otimes b \leq v_{\mathcal{S}}'\kappa_{\mathcal{S}} \otimes b'$ as the inequality is given by an ordered tossing over $V_{\mathcal{S}}'$ and B with ordered skeleton $\mathcal{S}$, then $v_{\mathcal{S}} \otimes b \leq v_{\mathcal{S}}' \otimes b'$ in $V_{\mathcal{S}} \otimes B$.

Proof. Clearly we need only show that (*iii*) implies (*i*). Suppose that (*iii*) holds, let $\nu : A \to A'$ lie in $\mathcal{C}$ and suppose that

$$(a\nu, b), (a'\nu, b') \in A' \times B$$

are connected via an ordered tossing with ordered skeleton $\mathcal{S}$, so that $\gamma_{\mathcal{S}}^{\leq}(b, b')$ holds. From considering the left hand side of the ordered tossing, we have that $\delta_{\mathcal{S}}^{\leq}(a\nu, a'\nu)$ is true in A'. By assumption there is an embedding $\kappa_{\mathcal{S}} : V_{\mathcal{S}} \to V'_{\mathcal{S}}$ in $\mathcal{C}$ and $v_{\mathcal{S}}, v'_{\mathcal{S}} \in V_{\mathcal{S}}$ such that $\delta^{\leq}_{\mathcal{S}}(v_{\mathcal{S}}\kappa_{\mathcal{S}}, v'_{\mathcal{S}}\kappa_{\mathcal{S}})$ is true in $V'_{\mathcal{S}}$, and a morphism $\alpha : V'_{\mathcal{S}} \to A'$ such that $v_{\mathcal{S}}\,\kappa_{\mathcal{S}}\alpha = a\nu$, $v'_{\mathcal{S}}\,\kappa_{\mathcal{S}}\alpha = a'\nu$ and $V_{\mathcal{S}}\kappa_{\mathcal{S}}\alpha \subseteq A\nu$. Since $\delta^{\leq}_{\mathcal{S}}(v_{\mathcal{S}}\kappa_{\mathcal{S}}, v'_{\mathcal{S}}\kappa_{\mathcal{S}})$ is true in $V'_{\mathcal{S}}$, there is an ordered tossing from $(v_{\mathcal{S}}\kappa_{\mathcal{S}}, b)$ to $(v'_{\mathcal{S}}\kappa_{\mathcal{S}}, b')$ over $V'_{\mathcal{S}}$ and B, with ordered skeleton $\mathcal{S}$. From (iii), it follows that $(v_{\mathcal{S}}, b)$ and $(v'_{\mathcal{S}}, b')$ are connected via an ordered tossing over $V_{\mathcal{S}}$ and B with ordered skeleton $\mathcal{T}$ say. Consequently, $\delta^{\leq}_{\mathcal{T}}(v_{\mathcal{S}}, v'_{\mathcal{S}})$ is true in $V_{\mathcal{S}}$ and so $\delta^{\leq}_{\mathcal{T}}(v_{\mathcal{S}}\kappa_{\mathcal{S}}\alpha, v'_{\mathcal{S}}\kappa_{\mathcal{S}}\alpha)$, that is, $\delta^{\leq}_{\mathcal{T}}(a\nu, a'\nu)$ is true in $A\nu$. Since ν is an ordered embedding we deduce that $\delta^{\leq}_{\mathcal{T}}(a, a')$ is true in A. Also, $\gamma^{\leq}_{\mathcal{T}}(b, b')$ is true in B and so (a, b) and (a', b') are connected via an ordered tossing with ordered skeleton $\mathcal{T}$ over A and B. Hence B is $\mathcal{C}$-po-flat as required. □

Lemma 3.2.1.3. *Let $\mathcal{C}$ be a class of embeddings of right S-posets, satisfying Condition (Free)$^{\leq}$. Let $\overline{\mathcal{C}}$ be the set of products of morphisms in $\mathcal{C}$. If a left S-poset B is $\mathcal{C}$-po-flat, then it is $\overline{\mathcal{C}}$-po-flat.*

Proof. Let I be an indexing set and let $\gamma_i : A_i \to A'_i \in \mathcal{C}$ for all $i \in I$. Let $A = \prod_{i\in I} A_i$, $A' = \prod_{i\in I} A'_i$ and let $\gamma : A \to A'$ be the canonical embedding, so that $(a_i)\gamma = (a_i\gamma_i)$.

Suppose B is a $\mathcal{C}$-po-flat left S-poset. Let $\underline{a} = (a_i), \underline{a}' = (a'_i) \in A$ and $b, b' \in B$ be such that $\underline{a}\gamma \otimes b \leq \underline{a}'\gamma \otimes b$ in $A' \otimes B$. Then for some ordered skeleton $\mathcal{S}$,

$$A' \models \delta^{\leq}_{\mathcal{S}}(\underline{a}\gamma, \underline{a}'\gamma) \text{ and } B \models \gamma^{\leq}_{\mathcal{S}}(b, b').$$

It follows that for each $i \in I$,

$$A_i \models \delta^{\leq}_{\mathcal{S}}(a_i\gamma_i, a'_i\gamma_i).$$

By assumption that $\mathcal{C}$ has Condition (Free)$^{\leq}$, there exist $\kappa_{\mathcal{S}} : V_{\mathcal{S}} \to V'_{\mathcal{S}} \in \mathcal{C}$ and $v_{\mathcal{S}}, v'_{\mathcal{S}} \in V_{\mathcal{S}}$ such that $\delta^{\leq}_{\mathcal{S}}(v_{\mathcal{S}}\kappa_{\mathcal{S}}, v'_{\mathcal{S}}\kappa_{\mathcal{S}})$ is true in $V'_{\mathcal{S}}$. Further, for each $i \in I$, as $\delta^{\leq}_{\mathcal{S}}(a_i\gamma_i, a'_i\gamma_i)$ is true in A'_i, there exists an S-pomorphism $\alpha_i : V'_{\mathcal{S}} \to A'_i$ such that $v_{\mathcal{S}}\,\kappa_{\mathcal{S}}\alpha_i = a_i\gamma_i$, $v'_{\mathcal{S}}\kappa_{\mathcal{S}}\alpha_i = a'_i\gamma_i$ and $V_{\mathcal{S}}\kappa_{\mathcal{S}}\alpha_i \subseteq A_i\gamma_i$.

We have $\delta^{\leq}_{\mathcal{S}}(v_{\mathcal{S}}\kappa_{\mathcal{S}}, v'_{\mathcal{S}}\kappa_{\mathcal{S}})$ is true in $V'_{\mathcal{S}}$ and $\gamma^{\leq}_{\mathcal{S}}(b, b')$ is true in B, giving that $v_{\mathcal{S}}\kappa_{\mathcal{S}} \otimes b \leq v'_{\mathcal{S}}\kappa_{\mathcal{S}} \otimes b'$ in $V'_{\mathcal{S}} \otimes B$. As B is a $\mathcal{C}$-po-flat left S-poset and $\kappa_{\mathcal{S}} : V_{\mathcal{S}} \to V'_{\mathcal{S}} \in \mathcal{C}$, we have that $v_{\mathcal{S}} \otimes b \leq v'_{\mathcal{S}} \otimes b'$ in $V_{\mathcal{S}} \otimes B$, say via an ordered tossing with ordered skeleton $\mathcal{U}$. It follows that

$$V_{\mathcal{S}} \models \delta^{\leq}_{\mathcal{U}}(v_{\mathcal{S}}, v'_{\mathcal{S}}) \text{ and } B \models \gamma^{\leq}_{\mathcal{U}}(b, b').$$

Corresponding to (ii) of Remark 3.1.1.1, we have that

$$A_i\gamma_i \models \delta^{\leq}_{\mathcal{U}}(v_{\mathcal{S}}\kappa_{\mathcal{S}}\alpha_i, v'_{\mathcal{S}}\kappa_{\mathcal{S}}\alpha_i)$$

that is,

$$A_i\gamma_i \models \delta^{\leq}_{\mathcal{U}}(a_i\gamma_i, a'_i\gamma_i).$$

Let $\mathcal{U}$ has length h, we have that there are elements $w_{i,2}, \ldots, w_{i,h} \in A_i$ such that

$$\epsilon^{\leq}_{\mathcal{U}}(a_i\gamma_i, w_{i,2}\gamma_i, \ldots, w_{i,h}\gamma_i, a'_i\gamma_i)$$

is true. But γ_i is an *embedding*, so that

$$\epsilon^{\leq}_{\mathcal{U}}(a_i, w_{i,2}, \ldots, w_{i,h}, a'_i)$$

holds in A_i. Hence $\delta^{\leq}_{\mathcal{U}}(a_i, a'_i)$ is true in each A_i and so $\delta^{\leq}_{\mathcal{U}}(\underline{a}, \underline{a}')$ holds in A. Together with $\gamma^{\leq}_{\mathcal{U}}(b, b')$ being true in B, we deduce that $\underline{a} \otimes b \leq \underline{a}' \otimes b'$ in $A \otimes B$, as required. □

Theorem 3.2.1.4. *Let $\mathcal{C}$ be a class of embeddings of right S-posets satisfying Condition (Free)$^{\leq}$. Then the following conditions are equivalent for a pomonoid S;*

(i) the class $\mathcal{C}$-$\mathcal{PF}$ is axiomatisable;

(ii) the class $\mathcal{C}$-$\mathcal{PF}$ is closed under formation of ultraproducts;

(iii) for every ordered skeleton $\mathcal{S}$ there exist finitely many replacement ordered skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(\mathcal{S})}$ such that, for any embedding $\gamma : A \to A'$ in $\mathcal{C}$ and any $\mathcal{C}$-po-flat left S-poset B, if $a\gamma \otimes b \leq a'\gamma \otimes b' \in A' \otimes B$ via an ordered tossing $\mathcal{T}$ with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $a \otimes b \leq a' \otimes b'$ via an ordered tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(\mathcal{S})\}$;

(iv) for every ordered skeleton $\mathcal{S}$ there exists finitely many replacement ordered skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\beta(\mathcal{S})}$ such that, for any $\mathcal{C}$-po-flat left S-poset B, if $(v_{\mathcal{S}}\kappa_{\mathcal{S}}, b)$ and $(v'_{\mathcal{S}}\kappa_{\mathcal{S}}, b')$ are such that $v_{\mathcal{S}}\kappa_{\mathcal{S}} \otimes b \leq v'_{\mathcal{S}}\kappa_{\mathcal{S}} \otimes b'$ by an ordered tossing $\mathcal{T}$ over $V'_{\mathcal{S}}$ and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $v_{\mathcal{S}} \otimes b \leq v'_{\mathcal{S}} \otimes b'$ are connected by an ordered tossing $\mathcal{T}'$ over $V_{\mathcal{S}}$ and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \beta(\mathcal{S})\}$.

Proof. The implication (i) implies (ii) is clear from Łos's Theorem.

To prove (ii) implies (iii), we suppose that $\mathcal{C}$-$\mathcal{PF}$, the class of $\mathcal{C}$-po-flat left S-posets, is closed under formation of ultraproducts and that (iii) is false. Let J be the family of finite subsets of $\mathbb{OS}$, the set of *ordered skeletons*. We suppose that there exists an ordered skeleton $\mathcal{S}_0 \in \mathbb{OS}$ such that for every subset f of J, there exists an embedding $\gamma_f : A_f \to A'_f \in \mathcal{C}$, a $\mathcal{C}$-po-flat left S-poset B_f, and pairs $(a_f\gamma_f, b_f)$, $(a'_f\gamma_f, b'_f) \in A'_f \times B_f$ such that $a_f\gamma_f \otimes b_f \leq a'_f\gamma_f \otimes b'_f$ over A'_f and B_f by an ordered tossing $\mathcal{T}_f$ with ordered skeleton $\mathcal{S}_0$, but such that there is no ordered replacement tossing over A_f and B_f giving $a_f \otimes b_f \leq a'_f \otimes b'_f$ via an ordered skeleton belonging to the set f.

Let $J_{\mathcal{S}} = \{f \in J : \mathcal{S} \in f\}$ for each $\mathcal{S} \in \mathbb{OS}$. Then there exists an ultrafilter Φ on J containing each $J_{\mathcal{S}}$, as each intersection of finitely many of the sets $J_{\mathcal{S}}$ is non-empty.

We now define $A' = \prod_{f \in J} A'_f$, $A = \prod_{f \in J} A_f$ and $B = \prod_{f \in J} B_f$. Let $\gamma : A \to A'$ be the embedding given by $(a_f)\gamma = (a_f\gamma_f)$. We note here that $\underline{a}\gamma \otimes \underline{b} \leq \underline{a}'\gamma \otimes \underline{b}'$ in $A' \otimes B$, where $\underline{a} = (a_f)$, $\underline{a}' = (a'_f)$, $\underline{b} = (b_f)$ and $\underline{b}' = (b'_f)$ and that this inequality is determined by an ordered tossing over A' and B (the "product" of the ordered tossings $\mathcal{T}_f$'s) having ordered skeleton $\mathcal{S}_0$. It follows that the inequality for $\underline{a}\gamma \otimes \underline{b}_\Phi \leq \underline{a}'\gamma \otimes \underline{b}'_\Phi$ holds also in $A' \otimes \mathcal{U}$ where $\mathcal{U} = (\prod_{f \in J} B_f)/\Phi$, and can be determined by an ordered tossing over A' and $\mathcal{U}$ with an ordered skeleton $\mathcal{S}_0$.

By assumption, $\mathcal{U}$ is $\mathcal{C}$-po-flat, and by Lemma 3.2.1.3, $\underline{a} \otimes \underline{b}_\Phi \leq \underline{a}' \otimes \underline{b}'_\Phi$ in $A \otimes \mathcal{U}$, say via an ordered tossing with ordered skeleton $\mathcal{V}$ of length h, say

$$\mathcal{V} = (d_1, e_1, \ldots, d_h, e_h).$$

Hence

$$A \models \delta^{\leq}_{\mathcal{V}}(\underline{a}, \underline{a}') \text{ and } \mathcal{U} \models \gamma^{\leq}_{\mathcal{V}}(\underline{b}_\Phi, \underline{b}'_\Phi).$$

Certainly $A_f \models \delta^{\leq}_{\mathcal{V}}(a_f, a'_f)$ for every f. Considering now the truth of $\gamma^{\leq}_{\mathcal{V}}(\underline{b}_\Phi, \underline{b}'_\Phi)$, there exist

$$(b_{1,f})_\Phi, \ldots, (b_{h,f})_\Phi \in \mathcal{U}$$

such that

$$\begin{aligned} \underline{b}_\Phi &\leq d_1(b_{1,f})_\Phi \\ e_1(b_{1,f})_\Phi &\leq d_2(b_{2,f})_\Phi \\ &\vdots \\ e_h(b_{h,f})_\Phi &\leq \underline{b}'_\Phi \end{aligned}$$

As Φ is closed under finite intersections, there exists $D \in \Phi$ such that

$$\begin{array}{rcl} b_f & \leq & d_1 b_{1,f} \\ e_1 b_{1,f} & \leq & d_2 b_{2,f} \\ & \vdots & \\ e_h b_{h,f} & \leq & b'_f \end{array}$$

for all $f \in D$, that is,

$$B_f \models \gamma^{\leq}_{\mathcal{V}}(b_f, b'_f)$$

for all $f \in D$.

Now suppose that $f \in D \cap J_{\mathcal{V}}$, then from the ordered tossing just considered, we see that $\mathcal{V}$ is the ordered skeleton of an ordered tossing over A_f and B_f such that $a_f \otimes b_f \leq a'_f \otimes b'_f$; that is, $\mathcal{V}$ an ordered replacement skeleton for the ordered skeleton $\mathcal{S}_0$ of the ordered tossing $\mathcal{T}_f$. But $\mathcal{V}$ belongs to f, a contradiction. This completes the proof that (ii) implies (iii).

It is clear that (iii) implies (iv). Now we want to prove that (iv) implies (i). We assume that (iv) holds. We aim to use this condition to construct a set of axioms for $\mathcal{C}$-$\mathcal{PF}$.

Let $\mathbb{S}_1$ denote the set of all elements of $\mathbb{OS}$ such that if $\mathcal{S} \in \mathbb{S}_1$, then there is no $\mathcal{C}$-po-flat left S-poset B such that $\gamma^{\leq}_{\mathcal{S}}(b, b') \in B$ for any $b, b' \in B$. For $\mathcal{S} \in \mathbb{S}_1$ we put

$$\psi_{\mathcal{S}} := (\forall x)(\forall x')\neg\gamma^{\leq}_{\mathcal{S}}(x, x')$$

For $\mathcal{S} \in \mathbb{S}_2 = \mathbb{OS} \setminus \mathbb{S}_1$, there must be a $B \in \mathcal{C}$-$\mathcal{PF}$ and $b, b' \in B$ such that $\gamma^{\leq}_{\mathcal{S}}(b, b')$ is true in B, whence there is an ordered tossing from $(v_{\mathcal{S}}\kappa_{\mathcal{S}}, b)$ to $(v'_{\mathcal{S}}\kappa_{\mathcal{S}}, b')$ over $V'_{\mathcal{S}}$ and B with an ordered skeleton $\mathcal{S}$.

Let $\mathcal{S}_1, \cdots, \mathcal{S}_{\beta(\mathcal{S})}$ be a minimum set of ordered replacement skeletons for ordered tossings with ordered skeleton $\mathcal{S}$ such that $v_{\mathcal{S}}\kappa_{\mathcal{S}} \otimes c \leq v'_{\mathcal{S}}\kappa_{\mathcal{S}} \otimes c'$ where $c, c' \in C$ and C ranges over $\mathcal{C}$-$\mathcal{PF}$. Hence for each k in $\{1, \cdots, \beta(\mathcal{S})\}$, there exists a $\mathcal{C}$-po-flat left S-poset C_k, elements $c_k, c'_k \in C_k$ such that

$$V_{\mathcal{S}} \models \delta^{\leq}_{\mathcal{S}_k}(v_{\mathcal{S}}, v'_{\mathcal{S}}) \text{ and } C_k \models \gamma^{\leq}_{\mathcal{S}_k}(c_k, c'_k).$$

We define $\phi_{\mathcal{S}}$ to be the sentence

$$\phi_{\mathcal{S}} := (\forall y)(\forall y')\big(\gamma^{\leq}_{\mathcal{S}}(y, y') \to \gamma^{\leq}_{\mathcal{S}_1}(y, y') \vee \ldots \vee \gamma^{\leq}_{\mathcal{S}_{\beta(\mathcal{S})}}(y, y')\big).$$

Let

$$\sum\nolimits_{\mathcal{C}\text{-}\mathcal{PF}} = \{\psi_{\mathcal{S}} : \mathcal{S} \in \mathbb{S}_1\} \cup \{\phi_{\mathcal{S}} : \mathcal{S} \in \mathbb{S}_2\}.$$

We claim that $\sum_{\mathcal{C}\text{-}\mathcal{PF}}$ axiomatises $\mathcal{C}$-$\mathcal{PF}$.

Suppose first that D is any $\mathcal{C}$-po-flat left S-poset. By choice of $\mathbb{S}_1$, it is clear that $D \models \psi_{\mathcal{S}}$ for any $\mathcal{S} \in \mathbb{S}_1$.

Now take any $\mathcal{S} \in \mathbb{S}_2$, and suppose that $d, d' \in D$ are such that D satisfies $\gamma^{\leq}_{\mathcal{S}}(d, d')$. Then, as noted earlier, $v_{\mathcal{S}}\kappa_{\mathcal{S}} \otimes d \leq v'_{\mathcal{S}}\kappa_{\mathcal{S}} \otimes d'$ in $V'_{\mathcal{S}} \otimes D$ via an ordered tossing with ordered skeleton $\mathcal{S}$, and therefore, by assumption, there is an ordered tossing over $V_{\mathcal{S}}$ and D joining $(v_{\mathcal{S}}, d)$ and $(v'_{\mathcal{S}}, d')$ with ordered skeleton $\mathcal{S}_k$ for some $k \in \{1, \cdots, \beta(\mathcal{S})\}$. It is now clear that $\gamma^{\leq}_{\mathcal{S}_k}(d, d')$ holds in D, as required. We have now shown that $D \models \sum_{\mathcal{C}\text{-}\mathcal{PF}}$.

Finally we show that a left S-poset C that satisfies $\sum_{\mathcal{C}\text{-}\mathcal{PF}}$ must be a $\mathcal{C}$-po-flat. We need to show that Condition (iii) of Lemma 3.2.1.2 of Chapter 3 holds for C. Let $\mathcal{S} \in \mathbb{OS}$ and suppose we have an ordered tossing with ordered skeleton $\mathcal{S}$ connecting $(v_{\mathcal{S}}\kappa_{\mathcal{S}}, c)$ and $(v'_{\mathcal{S}}\kappa_{\mathcal{S}}, c')$ over $V'_{\mathcal{S}}$ and C. Then

$$V'_{\mathcal{S}} \models \delta^{\leq}_{\mathcal{S}}(v_{\mathcal{S}}\kappa_{\mathcal{S}}, v'_{\mathcal{S}}\kappa_{\mathcal{S}}) \text{ and } C \models \gamma^{\leq}_{\mathcal{S}}(c, c').$$

If $\mathcal{S}$ belonged to $\mathbb{S}_1$, then C would satisfy the sentence

$$(\forall y)(\forall y')\neg\gamma^{\leq}_{\mathcal{S}}(y, y')$$

and so $\neg\gamma^{\leq}_{\mathcal{S}}(c, c')$ would hold, which would be a contradiction. Therefore we conclude that $\mathcal{S}$ belongs to $\mathbb{S}_2$. Because C satisfies $\phi_{\mathcal{S}}$ and because $\gamma^{\leq}_{\mathcal{S}}(c, c')$ holds, it follows that $\gamma^{\leq}_{\mathcal{S}_k}(c, c')$ holds for some $k \in \{1, 2, \cdots, \beta(\mathcal{S})\}$. But $V_{\mathcal{S}} \models \delta^{\leq}_{\mathcal{S}_k}(v_{\mathcal{S}}, v'_{\mathcal{S}})$, whence $(v_{\mathcal{S}}, c)$ and $(v'_{\mathcal{S}}, c')$ are connected via an ordered tossing over $V_{\mathcal{S}}$ and C with an ordered skeleton $\mathcal{S}_k$, showing that C is $\mathcal{C}$-po-flat. □

Lemma 3.2.1.5. *The following conditions are equivalent for a left S-poset B:*

(i) B is po-flat;

(ii) $-\otimes B$ maps the embeddings of $[x]S \cup [x']S$ into $F^m/\equiv_{\mathcal{S}}$ in the category ***Pos*-S** *to embeddings in the category of* **Pos***, for every ordered skeleton $\mathcal{S}$;*

(iii) if the inequality $[x] \otimes b \leq [x'] \otimes b'$ holds by an ordered standard tossing over $F^m/\equiv_{\mathcal{S}}$ and B with ordered skeleton $\mathcal{S}$, then $[x] \otimes b \leq [x'] \otimes b'$ holds by an ordered tossing over $[x]S \cup [x']S$ and B.

Proof. We will prove here only (iii) implies (i). Suppose that B satisfies Condition (iii), let a, a' belongs to any right S-poset A, let $b, b' \in B$, and suppose that $a \otimes b \leq a' \otimes b'$ in $A \otimes B$ via an ordered tossing with ordered skeleton $\mathcal{S}$, of length m, so that $\delta^{\leq}_{\mathcal{S}}(a, a')$ is true in A and $\gamma^{\leq}_{\mathcal{S}}(b, b')$ is true in B. From the construction of $F^m/\equiv_{\mathcal{S}}$ we have that $[x] \otimes b \leq [x'] \otimes b'$ via an ordered standard tossing over $F^m/\equiv_{\mathcal{S}}$ and B. By the given hypothesis we have that $[x] \otimes b \leq [x'] \otimes b'$ via an ordered tossing in $\big([x]S \cup [x']S\big) \otimes B$, say with an ordered skeleton $\mathcal{U}$.

Since $\delta^{\leq}_{\mathcal{S}}(a, a')$ is true in A, there are elements $a_2, \ldots, a_m \in A$ such that

$$\epsilon^{\leq}_{\mathcal{S}}(a, a_2, \ldots, a_m, a')$$

hold in A. Let $\phi : F^m \to A$ be the S-pomorphism which is defined by $x\phi = a$, $x_i\phi = a_i$ $(2 \leq i \leq m)$, $x'\phi = a'$. Since $u\phi \leq u'\phi$ for all $(u, u') \in T_{\mathcal{S}}$, by Theorem 1.2.2.11 of Chapter 1, we have that $\overline{\phi} : F^m/\equiv_{\mathcal{S}} \to A$ given by $[z]\overline{\phi} = z\phi$ is a well defined S-pomorphism. We have that $\delta^{\leq}_{\mathcal{U}}([x], [x'])$ holds in $[x]S \cup [x']S$, so that $\delta^{\leq}_{\mathcal{U}}(a, a')$ holds in $aS \cup a'S$. Since also $\gamma^{\leq}_{\mathcal{U}}(b, b')$ holds in B, we have that (a, b) and (a', b') are connected by an ordered tossing over $aS \cup a'S$ and B, so that $a \otimes b \leq a' \otimes b'$ in $aS \cup a'S \otimes B$. Thus B is po-flat, as required. □

Lemma 3.2.1.6. *The class Pos-S of all right S-posets has Condition (Free)$^{\leq}$.*

Proof. Let $\mathcal{S}$ be an ordered skeleton of length n, let $V'_{\mathcal{S}} = F^m/\equiv_{\mathcal{S}}$, $V_{\mathcal{S}} = [x]S \cup [x']S$ and let $\kappa_{\mathcal{S}} : V_{\mathcal{S}} \to V'_{\mathcal{S}}$ denote inclusion. Then $[x], [x'] \in V_{\mathcal{S}}$ and $\delta^{\leq}_{\mathcal{S}}([x]\kappa_{\mathcal{S}}, [x']\kappa_{\mathcal{S}})$ is true in $V'_{\mathcal{S}}$.

Let $\nu : A \to A'$ be any right S-poset embedding such that $\delta^{\leq}_{\mathcal{S}}(a\nu, a'\nu)$ holds in A', for some $a, a' \in A$. As in Lemma 3.2.1.2, there is as a consequence an S-pomorphism $\alpha : V'_{\mathcal{S}} \to A'$ such that $[x]\kappa_{\mathcal{S}}\alpha = a\nu$ and $[x']\kappa_{\mathcal{S}}\alpha = a'\nu$. Clearly

$V_{\mathcal{S}}\kappa_{\mathcal{S}}\alpha = ([x]S \cup [x']S)\kappa_{\mathcal{S}}\alpha = [x]\kappa_{\mathcal{S}}\alpha S \cup [x']\kappa_{\mathcal{S}}\alpha S = a\nu S \cup a'\nu S = (aS \cup a'S)\nu \subseteq A\nu.$

Thus, with $v_{\mathcal{S}} = [x]$ and $v'_{\mathcal{S}} = [x']$, we see that Condition (Free)$^{\leq}$ holds.

□

Axiomatisability of $\mathcal{C}$-$\mathcal{PF}$ without Condition (Free)$^{\leq}$

Theorem 3.2.2.1. *The following conditions are equivalent for a monoid S:*

(i) the class $\mathcal{C}$-$\mathcal{PF}$ is axiomatisable;

(ii) the class $\mathcal{C}$-$\mathcal{PF}$ is closed under ultraproducts;

(iii) for every ordered skeleton $\mathcal{S}$ over S and $a, a' \in A$, where $\mu : A \to A'$ is in $\mathcal{C}$, there exist finitely many ordered skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(a,\mathcal{S},a',\mu)}$, such that for any $\mathcal{C}$-po-flat left S-act B, if $a\mu \otimes b \leq a'\mu \otimes b'$ by an ordered tossing $\mathcal{T}$ over A' and B with $\mathcal{S}(\mathcal{T}) = \mathcal{S}$, then $a \otimes b \leq a' \otimes b'$ by an ordered tossing $\mathcal{T}'$ over A and B such that $\mathcal{S}(\mathcal{T}') = \mathcal{S}_k$, for some $k \in \{1, \cdots, \alpha(a, \mathcal{S}, a', \mu)\}$.

Proof. The implication (i) implies (ii) is clear from Łos's Theorem.

To prove (ii) implies (iii), we suppose that $\mathcal{C}$-$\mathcal{PF}$, the class of $\mathcal{C}$-po-flat left S-posets, is closed under formation of ultraproducts, and assume that (iii) is false. Let J be the family of finite subsets of $\mathbb{OS}$. We suppose that for some ordered skeleton $\mathcal{S}_0 \in \mathbb{OS}$, for some ordered embedding $\mu : A \to A' \in \mathcal{C}$, and for some $a, a' \in A$, for every $f \in J$ there is a $\mathcal{C}$-po-flat left S-poset B_f, and $b_f, b'_f \in B_f$ such that $a\mu \otimes b_f \leq a'\mu \otimes b'_f$ are connected over A' and B_f by an ordered tossing $\mathcal{T}_f$ with ordered skeleton $\mathcal{S}_0$, but such that no ordered replacement tossing over A and B_f connecting (a, b_f) and (a', b'_f) has an ordered skeleton belonging to the set f.

Let $J_{\mathcal{S}} = \{f \in J : \mathcal{S} \in f\}$ for each $\mathcal{S} \in \mathbb{S}$. Now we are able to define an ultrafilter Φ on J containing each $J_{\mathcal{S}}$ for all $\mathcal{S} \in \mathbb{S}$, as each intersection of finitely many of the sets $J_{\mathcal{S}}$ is non-empty.

We note here that $a\mu \otimes \underline{b} \leq a'\mu \otimes \underline{b}'$ in $A' \otimes B$, where $B = \prod_{f \in J} B_f$, $\underline{b} = (b_f)$ and $\underline{b}' = (b'_f)$, and that this inequality is determined by an ordered tossing over A' and B (the "product" of the ordered tossings $\mathcal{T}_f$) having ordered skeleton $\mathcal{S}_0$. It follows that the inequality for $a\mu \otimes \underline{b}_\Phi \leq a'\mu \otimes \underline{b}'_\Phi$ holds also in $A' \otimes \mathcal{U}$ where $\mathcal{U} = (\prod_{f \in J} B_f)/\Phi$, and can be determined by an ordered tossing over A' and $\mathcal{U}$ with ordered skeleton $\mathcal{S}_0$.

By assumption $\mathcal{U}$ is $\mathcal{C}$-po-flat, so that $a \otimes \underline{b}_\Phi \leq a' \otimes \underline{b}'_\Phi$ via an ordered replacement tossing over A and $\mathcal{U}$, with ordered skeleton $\mathcal{V}$ say. Hence

$$A \models \delta^{\leq}_{\mathcal{V}}(a, a') \text{ and } \mathcal{U} \models \gamma^{\leq}_{\mathcal{V}}(\underline{b}_\Phi, \underline{b}_\Phi).$$

By a familiar argument there exists $D \in \Phi$ such that $B_f \models \gamma^{\leq}_{\mathcal{V}}(b_f, b'_f)$ for all $f \in D$.

Now suppose that $f \in D \cap J_{\mathcal{V}}$. Then $\mathcal{V}$ is the ordered skeleton of an ordered tossing over A and B_f such that $a \otimes b_f \leq a' \otimes b'_f$; that is, $\mathcal{V}$ is an ordered replacement skeleton for ordered skeleton $\mathcal{S}_0$ of the ordered tossing $\mathcal{T}_f$. But $\mathcal{V}$ belongs to f, a contradiction. This completes the proof that (ii) implies (iii).

Finally, suppose that (iii) holds. Let

$$\mathbb{T}' = \{(a, \mathcal{S}, a', \mu) : \mathcal{S} \in \mathbb{OS}, \mu : A \to A' \in \mathcal{C}, a, a' \in A', \delta^{\leq}_{\mathcal{S}}(a\mu, a'\mu) \text{ holds}\}.$$

We introduce sentences corresponding to elements of $\mathbb{T}'$ in such a way that the resulting set of sentences axiomatises the class $\mathcal{C}$-$\mathcal{PF}$.

We let $\mathbb{T}_1$ be the set of $(a, \mathcal{S}, a', \mu) \in \mathbb{T}'$ such that $\gamma_{\mathcal{S}}^{\leq}(b, b')$ does not hold for any b, b' in any $\mathcal{C}$-po-flat left S-poset B, and put $\mathbb{T}_2 = \mathbb{T}' \setminus \mathbb{T}_1$. For $T = (a, \mathcal{S}, a', \mu) \in \mathbb{T}_1$ we let

$$\psi_T = \psi_{\mathcal{S}} := (\forall x)(\forall x')\neg\gamma_{\mathcal{S}}^{\leq}(x, x').$$

If $T = (a, \mathcal{S}, a', \mu) \in \mathbb{T}_2$, then $\mathcal{S}$ is the ordered skeleton of some ordered tossing joining $a\mu \otimes b \leq a'\mu \otimes b'$ over A' and some $\mathcal{C}$-po-flat left S-poset B. By our assumption (iii), there is a finite list of ordered replacement skeletons $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(T)}$. Choosing $\alpha(T)$ to be minimal, for each $k \in \{1, \cdots, \alpha(T)\}$, there exist a $\mathcal{C}$-po-flat left S-poset C_k and elements $c_k, c_k' \in C_k$, such that

$$A \models \delta_{\mathcal{S}_k}^{\leq}(a, a') \text{ and } C_k \models \gamma_{\mathcal{S}_k}^{\leq}(c_k, c_k').$$

We let ϕ_T be the sentence

$$\phi_T := (\forall y)(\forall y')(\gamma_{\mathcal{S}}^{\leq}(y, y') \rightarrow \gamma_{\mathcal{S}_1}^{\leq}(y, y') \vee \cdots \vee \gamma_{\mathcal{S}_{\alpha(T)}}^{\leq}(y, y')).$$

Let

$$\sum\nolimits_{\mathcal{C}\text{-}\mathcal{PF}} = \{\psi_T : T \in \mathbb{T}_1\} \cup \{\phi_T : T \in \mathbb{T}_2\}.$$

We claim that $\sum_{\mathcal{C}\text{-}\mathcal{PF}}$ axiomatises $\mathcal{C}$-$\mathcal{PF}$.

Suppose first that D is any $\mathcal{C}$-po-flat left S-poset.

Let $T = (a, \mathcal{S}, a', \mu) \in \mathbb{T}_1$. Then $\gamma_{\mathcal{S}}^{\leq}(b, b')$ is not true for any $b, b' \in B$, for any $\mathcal{C}$-po-flat left S-poset B, so certainly $D \models \psi_T$.

On the other hand, let $T = (a, \mathcal{S}, a', \mu) \in \mathbb{T}_2$, and let $d, d' \in D$ be such that $\gamma_{\mathcal{S}}^{\leq}(d, d')$ is true. Together with the fact $\delta_{\mathcal{S}}^{\leq}(a\mu, a'\mu)$ holds, we have that $a\mu \otimes d \leq a'\mu \otimes d'$ over A' and D via an ordered tossing with ordered skeleton $\mathcal{S}$. Because D is $\mathcal{C}$-po-flat, $a \otimes d \leq a' \otimes d'$ over A and D, and by assumption (iii), we can take the ordered replacement tossing to have ordered skeleton one of $\mathcal{S}_1, \cdots, \mathcal{S}_{\alpha(T)}$, say $\mathcal{S}_k$. Thus $D \models \gamma_{\mathcal{S}_k}^{\leq}(d, d')$ and it follows that $D \models \phi_T$. Hence D is a model of $\sum_{\mathcal{C}\text{-}\mathcal{PF}}$.

Conversely, we show that every model of $\sum_{\mathcal{C}\text{-}\mathcal{PF}}$ is $\mathcal{C}$-po-flat. Let $C \models \sum_{\mathcal{C}\text{-}\mathcal{PF}}$ and suppose that $\mu : A \rightarrow A' \in \mathcal{C}$, $a, a' \in A$, $c, c' \in C$ and $a\mu \otimes c \leq a'\mu \otimes c'$ in $A' \otimes C$, say with ordered tossing having ordered skeleton $\mathcal{S}$. Then the ordered quadruple $T = (a, \mathcal{S}, a', \mu) \in \mathbb{T}'$. Since $\gamma_{\mathcal{S}}^{\leq}(c, c')$ holds, C cannot be a model of ψ_T. Since $C \models \sum_{\mathcal{C}\text{-}\mathcal{PF}}$ it follows that $T \in \mathbb{T}_2$. But then ϕ_T holds in C so that for some $k \in \{1, \cdots, \alpha(T)\}$ we have that $\gamma_{\mathcal{S}_k}^{\leq}(c, c')$ is true. We also know that $A \models \delta_{\mathcal{S}_k}^{\leq}(a, a')$, so that we have ordered tossing over A and C such that $a \otimes c \leq a' \otimes c'$. Thus C is $\mathcal{C}$-po-flat.

□

Theorem 3.3.1.1. *[52] The following conditions are equivalent for an ordered monoid S:*

(i) the class of S-posets satisfying Condition (P) is axiomatisable;

(ii) every ultraproduct of S-posets satisfying Condition (P) also satisfies Condition (P);

(iii) every ultrapower of S satisfies Condition (P);

(iv) for any $s, t \in S$, $\mathbf{R}^{\leq}(s, t) = \emptyset$ or $\mathbf{R}^{\leq}(s, t)$ is finitely generated as a right S-subact of $S \times S$.

Proof. (i) implies (ii): this follows from Łos's theorem.

(ii) implies (iii): this is obvious since S satisfies Condition (P) as a S-poset.

(iii) implies (iv): let $s, t \in S$ and suppose that $\mathbf{R}^{\leq}(s,t) \neq \emptyset$. We suppose that $\mathbf{R}^{\leq}(s,t)$ is not finitely generated.

Let $\{(u_\beta, v_\beta) : \beta < \gamma\}$ be a generating subset of $\mathbf{R}^{\leq}(s,t)$ of minimum cardinality γ. By assumption, γ is a limit ordinal. We may suppose that for any $\beta < \gamma$, (u_β, v_β) is not in the right S-subact generated by the preceding elements (u_τ, v_τ), that is, $(u_\beta, v_\beta) \notin \bigcup_{\tau<\beta} (u_\tau, v_\tau)S$.

Let Φ be a uniform ultrafilter on γ, that is Φ is an ultrafilter on γ such that all sets in Φ have cardinality γ.

Put $\mathcal{U} = S^\gamma/\Phi$. By assumption $\mathcal{U}$ satisfies Condition (P). Define elements $\underline{a}$ and $\underline{b}$ of $\mathcal{U}$ by $\underline{a} = (u_\beta)_\Phi$, $\underline{b} = (v_\beta)_\Phi$. Since $su_\beta \leq tv_\beta$ for all $\beta < \gamma$, clearly $s\,\underline{a} \leq t\,\underline{b}$.

By assumption $\mathcal{U}$ satisfies Condition (P) so there exists $s', t' \in S$ and $\underline{c} \in \mathcal{U}$ such that $\underline{a} = s'\underline{c}$, $\underline{b} = t'\underline{c}$ and $ss' \leq tt'$. Let $\underline{c} = (w_\beta)_\Phi$; from $ss' \leq tt'$ we have $(s', t') \in \mathbf{R}^{\leq}(s,t)$ and so we have $(s', t') = (u_\sigma, v_\sigma)h$ for some $\sigma < \gamma$ and $h \in S$. Since $\underline{a} = s'\underline{c}$ and $\underline{b} = t'\underline{c}$ there exists sets T_1 and T_2 in Φ such that $u_\beta = s'w_\beta$ for all $\beta \in T_1$ and $v_\beta = t'w_\beta$ for all $\beta \in T_2$. Using the fact that $T_1 \cap T_2 \in \Phi$ and Φ is uniform, $T_1 \cap T_2$ contains an ordinal $\alpha \geq \sigma + 1$. Then

$$(u_\alpha, v_\alpha) = (s'w_\alpha, t'w_\alpha) = (s', t')w_\alpha = (u_\sigma, v_\sigma)hw_\alpha$$

and so $(u_\alpha, v_\alpha) \in (u_\sigma, v_\sigma)S$, a contradiction. Thus $\mathbf{R}^{\leq}(s,t)$ is finitely generated.

(iv) implies (i): we show that the class of left S-posets satisfying Condition (P) is axiomatisable by giving explicitly a set of sentences that axiomatises, the class of S-posets which satisfies Condition (P). For any element $\rho \in S \times S$ with $\mathbf{R}^{\leq}(\rho) \neq \emptyset$, we choose and fix a finite set of generators $\{(u_{\rho 1}, v_{\rho 1}) \cdots (u_{\rho n}, v_{\rho n})\}$ of $\mathbf{R}^{\leq}(\rho)$. For ρ in $S \times S$ where $\rho = (s,t)$, define sentences ϕ_ρ of $L_s^{\leq}$ as follows:

If $\mathbf{R}^{\leq}(\rho) = \emptyset$ then

$$\phi_\rho := (\forall x)(\forall y)(sx \not\leq ty)$$

if $\mathbf{R}^{\leq}(\rho) \neq \emptyset$ then

$$\phi_\rho := (\forall x)(\forall y)\big(sx \leq ty \to (\exists z)(\bigvee_{i=1}^{n(\rho)} (x = u_{\rho i}z \wedge y = v_{\rho i}z))\big).$$

Let

$$\sum\nolimits_{\mathcal{P}} = \{\phi_\rho : \rho \in S \times S\}.$$

We claim that $\sum_{\mathcal{P}}$ axiomatises the class of S-posets satisfying Condition (P).

Suppose that A is an S-poset satisfying Condition (P) and $\rho \in S \times S$, where $\rho = (s,t)$. If $\mathbf{R}^{\leq}(\rho) = \emptyset$ and there exists $a, b \in A$ such that $sa \leq tb$, then since A satisfies Condition (P), $s\,s' \leq t\,t'$ for some $s', t' \in S$, a contradiction. Thus $A \models \phi_\rho$. If $\mathbf{R}^{\leq}(\rho) \neq \emptyset$ and $sa \leq tb$ where $a, b \in A$ then again using the fact that A satisfies Condition (P) there are elements $s', t' \in S$ and $c \in A$ such that $ss' \leq tt'$, $a = s'c$, $b = t'c$. Now $(s', t') \in \mathbf{R}^{\leq}(s,t)$ and so $(s', t') = (u_{\rho i}, v_{\rho i})h$ for some $i \in \{1, 2, \ldots, n(\rho)\}$ and $h \in S$. Thus $a = u_{\rho i}hc$, and $b = v_{\rho i}hc$ where $hc \in A$. Hence $A \models \phi_\rho$.

Conversely let A be a model of $\sum_{\mathcal{P}}$. If $s\,a \leq t\,b$ where $s, t \in S$ and $a, b \in A$, then since $A \models \phi_\rho$, where $\rho = (s, t)$ it follows that $\mathbf{R}^{\leq}(\rho)$ cannot be empty and ϕ_ρ is

$$(\forall x)(\forall y)\big(sx \leq ty \rightarrow (\exists\, z)(\bigvee_{i=1}^{n(\rho)} (\, x = u_{\rho i} z \wedge y = v_{\rho i}\, z))\big)$$

where $\{(u_{\rho\,1}, v_{\rho\,1}), \cdots, (u_{\rho\,n}, v_{\rho\,n})\}$ is a finite set of generators of $\mathbf{R}^{\leq}(\rho)$.

Hence there exists an element $c \in A$ with $a = u_{\rho i}\, c$, $b = v_{\rho i}\, c$ for some $i \in \{1, 2, \ldots, n(\rho)\}$. By definition of $u_{\rho i}, v_{\rho i}$ we have $s\, u_{\rho i} \leq t v_{\rho i}$. Thus A satisfies Condition (P) and so $\sum_{\mathcal{P}}$ axiomatises the class of S-posets satisfying Condition (P). □

Theorem 3.3.1.2. *[52] The following conditions are equivalent for an ordered monoid S:*

(i) the class of S-posets satisfying Condition (E) is axiomatisable;

(ii) every ultraproduct of S-posets satisfying Condition (E) also satisfies Condition (E);

(iii) every ultrapower of S satisfies Condition (E);

(iv) for any $s, t \in S$, $\mathbf{r}^{\leq}(s, t) = \emptyset$ or $\mathbf{r}^{\leq}(s, t)$ is finitely generated as a right ideal of S.

Proof. (i) implies (ii): this follows from Łos's theorem.

(ii) implies (iii): this is obvious since S satisfies Condition (E) as a left S-poset.

(iii) implies (iv): let $s, t \in S$ suppose that $\mathbf{r}^{\leq}(s, t) \neq \emptyset$ and $\mathbf{r}^{\leq}(s, t)$ is not finitely generated.

Let $\{u_\beta : \beta \leq \gamma\}$ be a generating subset of $\mathbf{r}^{\leq}(s, t)$ of minimum cardinality γ. By assumption γ is a limit ordinal. We may suppose that u_β is not in the right S-subposet generated by preceding elements, that is $u_\beta \notin \bigcup_{\tau<\beta} u_\tau S$.

Let Φ be a uniform ultrafilter on γ, that is Φ is an ultrafilter on γ such that all sets in D have cardinality γ. Put $\mathcal{T} = S^\gamma/\Phi$. By assumption, $\mathcal{T}$ satisfies Condition (E) as a S-poset.

Define an element $\underline{a} = (u_\beta)_\Phi$. Since $su_\beta \leq tu_\beta$ for all $\beta < \gamma$, clearly $s\,\underline{a} \leq t\,\underline{a}$. Since $\mathcal{T}$ satisfies Condition (E), there exists $s' \in S$ and $\underline{c} \in \mathcal{T}$ with $ss' \leq ts'$ and $\underline{a} = s'c$. Put $\underline{c} = (w_\beta)_\Phi$.

From $ss' \leq ts'$ we have $s' \in \mathbf{r}^{\leq}(s, t)$ and so $s' = (u_\sigma)h$ for some $\sigma < \gamma$ and $h \in S$. Since $\underline{a} = s'\underline{c}$ this implies that there exists T in Φ such that $u_\beta = s'w_\beta$ for all $\beta \in T$, but $s' = (u_\sigma)h$ so $u_\beta = u_\sigma h w_\beta$ implying that $u_\beta \in u_\sigma S$, a contradiction. Thus $\mathbf{r}^{\leq}(s, t)$ is finitely generated.

(iv) implies (i): we show that class of S-posets satisfying Condition (E) is axiomatisable by giving explicitly a set of sentences that axiomatises this class.

For any element ρ of $S \times S$ with $\mathbf{r}^{\leq}(\rho) \neq \emptyset$, choose and fix a set of generators $\{w_{\rho\,1} \cdots w_{\rho\, m(\rho)}\}$ of $\mathbf{r}^{\leq}(\rho)$. For $\rho \in S \times S$ where $\rho = (s, t)$ define sentences ϕ_ρ of $L_S^{\leq}$ as follows:

If $\mathbf{r}^{\leq}(\rho) = \emptyset$, then

$$\phi_\rho := (\forall\, x)(\, sx \not\leq\, ty).$$

If $\mathbf{r}^{\leq}(\rho) \neq \emptyset$ then

$$\phi_\rho \, := \, (\forall\, x)\big(\, sx \, \leq \, tx \rightarrow (\exists\, z)(\bigvee_{i=1}^{m(\rho)} x \, = \, w_{\rho i}\, z\,)\big).$$

Let

$$\sum\nolimits_{\mathcal{E}} = \{\phi_\rho \,:\, \rho \in S \times S\}.$$

We claim that $\sum_{\mathcal{E}}$ axiomatises the class of S-posets satisfying Condition (E). Suppose first that A satisfy Condition (E) and $\rho \in S \times S$, where $\rho = (s,t)$. If $\mathbf{r}^{\leq}(s,t) = \emptyset$ and there exists $a \in A$ such that $s\,a \leq t\,a$, then since A satisfy Condition (E), we have that $s\,s' \leq t\,s'$ for some $s' \in S$, a contradiction. Thus $A \models \phi_\rho$.

If $\mathbf{r}^{\leq}(\rho) \neq \emptyset$ and $s\,a \leq t\,a$ where $a \in A$, then again since A satisfies Condition (E), there exists $s' \in S$ and $c \in A$ such that $s\,s' \leq t\,s'$ and $a = s'c$. Now $s' \in \mathbf{r}^{\leq}(s,t)$ and so $s' = w_{\rho i}h$ for some $i \in \{1, 2, \cdots, m(\rho)\}$ and $h \in S$. Thus $a = w_{\rho i}\,h\,c$ and $h\,c \in A$, hence $A \models \phi_\rho$. Clearly A is a model of $\sum_{\mathcal{E}}$.

Conversely, let A be a model of $\sum_{\mathcal{E}}$. If $s\,a \leq t\,a$ where $s,t \in S$ and $a \in A$, then since $A \models \phi_\rho$, where $\rho = (s,t)$, it follows that $\mathbf{r}^{\leq}(\rho)$ cannot be empty and ϕ_ρ is

$$(\forall x)\big(sx \leq tx \rightarrow (\exists z)(\bigvee_{i=1}^{m(\rho)} (x = w_{\rho i}z))\big).$$

Hence there exists an element c of A with $a = w_{\rho i}c$ for some $i \in \{1, 2, \cdots, m(\rho)\}$. By definition of $w_{\rho i}$, $sw_{\rho i} \leq tw_{\rho i}$. Thus A satisfies Condition (E), and so $\sum_{\mathcal{E}}$ axiomatises class of S-posets satisfying Condition (E).

□

Theorem 3.3.3.1. *The following conditions are equivalent for a pomonoid S:*
(i) the class $\mathcal{PWP}$ is axiomatisable;
(ii) the class $\mathcal{PWP}$ is closed under ultraproducts;
(iii) every ultrapower of S lies in $\mathcal{PWP}$;
(iv) $\mathbf{R}^{\leq}(s,s)$ is finitely generated for any $s \in S$.

Proof. (i) implies (ii): this follows from Łos's theorem.

(ii) implies (iii): this is obvious since S satisfies Condition (PWP) as a S-poset.

(iii) implies (iv): let $s \in S$ and suppose that $\mathbf{R}^{\leq}(s,s)$ is not finitely generated.

Let $\{(u_\beta, v_\beta) : \beta < \gamma\}$ be a generating subset of $\mathbf{R}^{\leq}(s,s)$ of minimum cardinality γ. By assumption, γ is a limit ordinal. We may suppose that for any $\beta < \gamma$, (u_β, v_β) is not in the right S-subact generated by the preceding elements (u_τ, v_τ), that is, $(u_\beta, v_\beta) \notin \bigcup_{\tau<\beta}(u_\tau, v_\tau)S$.

Let Φ be a uniform ultrafilter on γ, that is Φ is an ultrafilter on γ such that all sets in Φ have cardinality γ.

Put $\mathcal{U} = S^\gamma/\Phi$. By assumption $\mathcal{U}$ satisfies Condition (PWP). Define elements $\underline{a}$ and $\underline{b}$ of $\mathcal{U}$ by $\underline{a} = (u_\beta)_\Phi$, $\underline{b} = (v_\beta)_\Phi$. Since $su_\beta \leq sv_\beta$ for all $\beta < \gamma$, clearly $s\,\underline{a} \leq s\,\underline{b}$.

By assumption $\mathcal{U}$ satisfies Condition (PWP) so there exists $s', t' \in S$ and $\underline{c} \in \mathcal{U}$ such that $\underline{a} = s'\underline{c}$, $\underline{b} = t'\underline{c}$ and $ss' \leq st'$. Let $\underline{c} = (w_\beta)_\Phi$; from $ss' \leq st'$ we have $(s',t') \in \mathbf{R}^{\leq}(s,s)$ and so we have $(s',t') = (u_\sigma, v_\sigma)h$ for some $\sigma < \gamma$ and $h \in S$. Since $\underline{a} = s'\underline{c}$ and $\underline{b} = t'\underline{c}$ there exists sets T_1 and T_2 in Φ such that $u_\beta = s'w_\beta$ for all $\beta \in T_1$ and $v_\beta = t'w_\beta$ for all

$\beta \in T_2$. Using the fact that $T_1 \cap T_2 \in \Phi$ and Φ is a uniform ultrafilter, $T_1 \cap T_2$ contains an ordinal $\alpha \geq \sigma + 1$. Then

$$(u_\alpha, v_\alpha) = (s'w_\alpha, t'w_\alpha) = (s', t')w_\alpha = (u_\sigma, v_\sigma)hw_\alpha$$

and so $(u_\alpha, v_\alpha) \in (u_\sigma, v_\sigma)S$, a contradiction. Thus $\mathbf{R}^{\leq}(s,s)$ is finitely generated.

(iv) implies (i): we show that the class $\mathcal{PWP}$ is axiomatisable by giving explicitly a set of sentences that axiomatises it. For any element $s \in S$, we choose and fix a finite set of generators

$$\{(u_{s\,1}, v_{s\,1}) \cdots (u_{s\,n(s)}, v_{s\,n(s)})\}$$

of $\mathbf{R}^{\leq}(s,s)$. For s in S, define sentences ϕ_s of $L_S^{\leq}$ as follows:

$$\phi_s = (\forall x)(\forall y)\big(sx \leq sy \to (\exists z)(\bigvee_{i=1}^{n(s)} (x = u_{s\,i}z \wedge y = v_{s\,i}z))\big).$$

Let

$$\sum\nolimits_{\mathcal{PWP}} = \{\phi_s : s \in S\}.$$

We claim that $\sum_{\mathcal{PWP}}$ axiomatises the class of S-posets satisfying Condition (PWP).

Suppose that A is an S-poset satisfying Condition (PWP). If $sa \leq sb$ where $a, b \in A$ then using the fact that A satisfies Condition (PWP) there are elements $s', t' \in S$ and $c \in A$ such that $ss' \leq st'$, $a = s'c$, $b = t'c$. Now $(s', t') \in \mathbf{R}^{\leq}(s,s)$ and so $(s', t') = (u_{s\,i}, v_{s\,i})h$ for some $i \in \{1, 2, \ldots, n(s)\}$ and $h \in S$. Thus $a = u_{s\,i}hc$ and $b = v_{s\,i}hc$ where $hc \in A$. Hence $A \models \phi_s$.

Conversely, let A be a model of $\sum_{\mathcal{PWP}}$. If $s\,a \leq s\,b$ where $s \in S$ and $a, b \in A$, then since $A \models \phi_s$, where ϕ_s is

$$(\forall x)(\forall y)\big(sx \leq sy \to (\exists\, z)(\bigvee_{i=1}^{n(s)} (\, x = u_{s\,i}z \wedge y = v_{s\,i}\, z))\big)$$

there exists an element $c \in A$ with $a = u_{s\,i}\, c$ and $b = v_{s\,i}\, c$ for some $i \in \{1, 2, \ldots, n(s)\}$. By definition of $u_{s\,i}, v_{s\,i}$ we have $s\, u_{s\,i} \leq sv_{s\,i}$. Thus A satisfies Condition (PWP) and so $\sum_{\mathcal{PWP}}$ axiomatises the class of S-posets satisfying Condition (PWP). □

Theorem 3.3.5.1. *The following conditions are equivalent for a pomonoid S:*
(i) the class $\mathcal{PWP}_w$ is axiomatisable;
(ii) the class $\mathcal{PWP}_w$ is closed under ultraproducts;
(iii) every ultrapower of S satisfies Condition (PWP_w);
(iv) for any $s \in S$ there exists finitely many

$$(u_{\rho 1}, v_{\rho 1}), \ldots, (u_{\rho\, n(\rho)}, v_{\rho\, n(\rho)}) \in \mathbf{R}^{\leq}(s,s)$$

such that for any $(x, y) \in \mathbf{R}^{\leq}(s,s)$,

$$x \leq u_{\rho i}h \text{ and } v_{\rho i}h \leq y$$

for some $i \in \{1, \ldots n(\rho)\}$ and $h \in S$.

Proof. (*i*) implies (*ii*): follows from Łos's theorem.

(*ii*) implies (*iii*) is obvious since S satisfies Condition (PWP_w) as a left S-poset.

(*iii*) implies (*iv*): suppose that every ultrapower of S has (PWP_w) but that (*iv*) does not hold.

Let $\{(u_\beta, v_\beta) : \beta < \gamma\}$ be a set of minimal (infinite) cardinality γ contained in $\mathbf{R}^{\leq}(s,s)$ such that if $(x,y) \in \mathbf{R}^{\leq}(s,s)$, then

$$x \leq u_\beta h \text{ and } v_\beta h \leq y$$

for some $\beta < \gamma$ and $h \in S$. From the minimality of γ we may assume that for any $\alpha < \beta < \gamma$, it is not true that both

$$u_\beta \leq u_\alpha h \text{ and } v_\alpha h \leq v_\beta$$

for any $h \in S$.

Let Φ be a uniform ultrafilter on γ, that is Φ is an ultrafilter on γ such that all sets in Φ have cardinality γ. Let $\mathcal{U} = S^\gamma/\Phi$, by assumption $\mathcal{U}$ satisfies Condition (PWP_w).

Since $su_\beta \leq sv_\beta$ for all $\beta < \gamma$, $s(u_\beta)_\Phi \leq s(v_\beta)_\Phi$. As $\mathcal{U}$ satisfies Condition (PWP_w), there exists $(u,v) \in \mathbf{R}^{\leq}(s,s)$ and $(w_\beta)_\Phi \in \mathcal{U}$ such that

$$(u_\beta)_\Phi \leq u(w_\beta)_\Phi \text{ and } v(w_\beta)_\Phi \leq (v_\beta)_\Phi.$$

Let $D \in \Phi$ be such that

$$u_\beta \leq uw_\beta \text{ and } vw_\beta \leq v_\beta$$

for all $\beta \in D$. Now $(u,v) \in \mathbf{R}^{\leq}(s,s)$ so that

$$u \leq u_\sigma h \text{ and } v_\sigma h \leq v$$

for some $\sigma < \gamma$. Choose $\beta \in D$ with $\beta > \sigma$. Then

$$u_\beta \leq uw_\beta \leq u_\sigma h w_\beta \text{ and } v_\sigma h w_\beta \leq vw_\beta \leq v_\beta$$

a contradiction. Thus (*iv*) holds.

(*iv*) implies (*i*): we will show that the class of left S-posets satisfying Condition (PWP_w) is axiomatisable by giving explicitly a set of sentences that axiomatises this class. For any element $s \in S$, we choose and fix a finite set of elements

$$\{(u_{s\,1}, v_{s\,1}) \cdots (u_{s\,n(s)}, v_{s\,n(s)})\}$$

of $\mathbf{R}^{\leq}(s,s)$. We define sentences of $L_S^{\leq}$ as follows:

$$\Omega_s := (\forall x)(\forall y)\big(sx \leq sy \rightarrow (\exists z)(\bigvee_{i=1}^{n(s)} (x \leq u_{s\,i}\, z \wedge v_{s\,i}\, z \leq y))\big).$$

Let

$$\sum\nolimits_{\mathcal{PWP}_w} = \{\Omega_s : s \in S\}.$$

We claim that $\sum_{\mathcal{PWP}_w}$ axiomatises $\mathcal{PWP}_w$.

Let A be an S-poset satisfying Condition (PWP_w).

Suppose $sa \leq sb$ where $a, b \in A$, then using the fact that A satisfies Condition (PWP$_w$) there are elements $s', t' \in S$ and $c \in A$ such that $ss' \leq st'$, $a \leq s'c$ and $t'c \leq a'$. We have $s' \leq u_{s\,i}t,\ v_{s\,i}t \leq t'$ for some $i \in \{1, \cdots, n(s)\}$ and $t \in S$. Hence $a \leq u_{s\,i}tc$ and $v_{s\,i}tc \leq a'$ so $A \models \Omega_s$.

Conversely, suppose that $A \models \sum_{\mathcal{PWP}_w}$ and $sa \leq sb$ for some $s \in S$ and $a, b \in S$. Since $A \models \Omega_s$ where Ω_s is

$$(\forall x)(\forall y)\big(sx \leq sy \rightarrow (\exists z)(\bigvee_{i=1}^{n(s)} (x \leq u_{s\,i}\, z \wedge v_{s\,i}\, z \leq y))\big)$$

we have $a \leq u_{\rho i}c$ and $v_{\rho i}c \leq b$ for some $c \in A$. By definition, $(u_{\rho i}, v_{\rho i}) \in \mathbf{R}^{\leq}(s, s)$, so that A lies in $\mathcal{PWP}_w$. $\square$

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