Stability properties of stochastic differential equations driven by Lévy noise

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Abstract

The main aim of this thesis is to examine stability properties of the solutions to stochastic differential equations (SDEs) driven by Lévy noise.

Using key tools such as Itô's formula for general semimartingales, Kunita's moment estimates for Lévy-type stochastic integrals, and the exponential martingale inequality, we find conditions under which the solutions to the SDEs under consideration are stable in probability, almost surely and moment exponentially stable. In addition, stability properties of stochastic functional differential equations (SFDEs) driven by Lévy noise are examined using Razumikhin type theorems.

In the existing literature the problem of stochastic stabilization and destabilization of first order non-linear deterministic systems has been investigated when the system is perturbed with Brownian motion. These results are extended in this thesis to the case where the deterministic system is perturbed with Lévy noise. We mainly focus on the stabilizing effects of the Lévy noise in the system, prove the existence of sample Lyapunov exponents of the trivial solution of the stochastically perturbed system, and provide sufficient criteria under which the system is almost surely exponentially stable. From the results that are established the Lévy noise plays a similar role to the Brownian motion in stabilizing dynamical systems.

We also establish the variation of constants formula for linear SDEs driven by Lévy noise. This is applied to study stochastic stabilization of ordinary functional differential equation systems perturbed with Lévy noise.

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"True wisdom comes to each of us when we realize how little we understand about life, ourselves, and the world around us. Wisdom begins in wonder". (attributed to Socrates)

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Contents

Notation and Terminology

$\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{R}^+ = [0, \infty)$.

$\mathbb{R}^d$ is d-dimensional Euclidean space, where $d \in \mathbb{N}$. Its elements $x = (x_1, x_2, \ldots, x_d)$ are vectors, with $x_i \in \mathbb{R}$ for each $1 \le i \le d$. The Euclidean inner-product in $\mathbb{R}^d$ between two vectors $x, y \in \mathbb{R}^d$ is denoted by $\langle x, y\rangle$ and is defined as

$$\langle x, y\rangle = \sum_{i=1}^{d} x_i y_i,$$

or as $x^T y$ where x^T denotes the transpose of the vector x, and the associated norm is

$$|x| = \langle x, x\rangle^{\frac{1}{2}} = \left(\sum_{i=1}^{d} {x_i}^2\right)^{\frac{1}{2}}.$$

$\mathcal{M}_{d,m}(\mathbb{R})$ will denote the space of all real-valued $d \times m$ matrices. We let $\mathcal{M}_d(\mathbb{R})$ denote $\mathcal{M}_{d,d}(\mathbb{R})$.
The trace of $A \in \mathcal{M}_d(\mathbb{R})$ is shortened to tr$A$.
We denote by $\|D\| = \left(\sum_{i=1}^{d}\sum_{j=1}^{m} D_{ij}^2\right)^{\frac{1}{2}}$ where $D \in \mathcal{M}_{d,m}(\mathbb{R})$. Note that $\|D\|$ is the Hilbert-Schmidt norm of a $d \times m$ matrix D i.e. $\|D\| = \sqrt{\mathrm{tr}(D^T D)}$ and if D is $d \times d$ symmetric matrix, it is well known that $\|D\| = \max_{i=1,2,\ldots,d} |\lambda_i|$ where $\{\lambda_i, i = 1, 2, \ldots, d\}$ are the eigenvalues of the matrix D.
Also denote by $\|D\|_1$ the matrix norm defined by $\|D\|_1 = \sup\{\|Dy\| : \|y\| = 1\}$ where D is a $d \times d$ matrix.
The kernel of a matrix $A \in \mathcal{M}_{d,m}(\mathbb{R})$ is denoted throughout the thesis as Ker(A) and by I we denote the identity matrix.
For a matrix $A \in \mathcal{M}_d(\mathbb{R})$ we say that is positive definite if $y^T A y > 0$ for all non-zero $y \in \mathbb{R}^d$.

In this thesis the Einstein summation convention is used, wherein summation is understood with respect to repeated upper and lower indices e.g. if $x, y \in \mathbb{R}^d$ and

$A = (A^i_j) \in \mathcal{M}_d(\mathbb{R})$ then

$$A^i_j x_i y^j = \sum_{i,j=1}^{d} A^i_j x_i y^j = (x, Ay).$$

$\#S$ means the number of elements in the set S.

S^c denotes the complement of a set S and $\overline{S}$ denotes the closure in some topology.

'Almost surely' is shorthanded to a.s. and 'almost everywhere' to a.e.
Also we denote $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$ for $a, b \in \mathbb{R}$.

We also need to define $C^n(\mathbb{R}^d)$, the space of n-times differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}$, all of whose derivatives are continuous. If $V \in C^2(\mathbb{R}^d; \mathbb{R})$ then we denote at $x \in \mathbb{R}^d$, $(\partial V(x)) = (\partial_1 V(x), \ldots, \partial_d V(x))^T$ where $\partial_i V(x)$ is the i-th first order partial derivative and $(\partial_i \partial_j V)(x)$ denotes $\frac{\partial^2 V}{\partial x_i \partial x_j}(x)$.

Throughout this thesis for each $x \in \mathbb{R}^d$, $c > 0$ we denote $B_c(x) = \{y \in \mathbb{R}^d : |y - x| < c\}$, $\hat{B}_c(x) = B_c(x) \setminus \{0\}$ and $B_c = B_c(0)$, $\hat{B}_c = \hat{B}_c(0)$.

Let $(S, \mathcal{F}, \mu)$ be a measure space and let $1 \leq p < \infty$. We denote by $L^p(S, \mathcal{F}, \mu; \mathbb{R}^d)$ the Banach space of all equivalence classes of mappings $f : S \to \mathbb{R}^d$ which agree almost everywhere (with respect to the measure μ) and for which $\|f\|_p < \infty$ where $\|\cdot\|_p$ is the following norm:

$$\|f\|_p = \left[\int_S |f(x)|^p \mu(dx)\right]^{1/p}.$$

$L^p(S; \mathbb{R}^d)$ is a shorthand for $L^p(S, \mathcal{F}, \mu; \mathbb{R}^d)$.

The Borel σ-algebra of a Borel set $A \subseteq \mathbb{R}^d$ is denoted by $\mathcal{B}(A)$.

Let $f : \mathbb{R}^d \to \mathbb{R}$. Then $\lim_{s \uparrow t} f(s) = k$ means $\lim_{s \to t, s < t} f(s) = k$, while $\lim_{s \downarrow t} f(s) = k$ implies $\lim_{s \to t, s > t} f(s) = k$.

We say that $f : \mathbb{R}^+ \to \mathbb{R}^d$ is càdlàg (*continue à droite avec des limites à gauche*) if it is right continuous with left limits and we denote the left limit at each point $t > 0$ by $f(t-) = \lim_{s \uparrow t} f(s)$.

Let $\tau \geq 0$. $\mathcal{D}([-\tau, 0]; \mathbb{R}^d)$ denotes the space of càdlàg mappings from $[-\tau, 0]$ to $\mathbb{R}^d$.

We denote by e_j the unit column vector, i.e. $e_j = (0, 0, \ldots, 0, 1^{(j)}, 0, \ldots, 0)^T$.

We define by $\delta_x(A)$ the Dirac mass concentrated at $x \in \mathbb{R}^d$

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A, \end{cases}$$

where $A \in \mathcal{B}(\mathbb{R}^d)$.

1_A is the indicator function of the Borel set A so

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Chapter 1

Introduction to Lévy Processes, Stochastic Differential Equations (SDEs) and Stability Theory

1.1 Introduction

This chapter falls into two parts. The first, is devoted to stochastic analysis of Lévy processes, while the second concerns stability theory. The former provides the appropriate mathematical tools that are needed to understand the concepts that will be developed in this thesis for the study of the stability of stochastic differential equations (SDEs) driven by Lévy processes. The latter gives some historical background and the appropriate mathematical mechanisms needed to develop stability theory.

1.2 Lévy processes: basic definitions and properties

In this section we provide a brief introduction to the fundamentals of Lévy processes theory.

Let $\mathcal{F}$ be a σ-algebra of a given set Ω. A family $(\mathcal{F}_t, t \geq 0)$ of sub σ-algebras of $\mathcal{F}$ is called a *filtration* if

$$\mathcal{F}_s \subseteq \mathcal{F}_t \qquad \text{whenever} \qquad s \leq t.$$

Assume that we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$ that satisfies the "usual hypotheses":
(a) Completeness: $\mathcal{F}_0$ contains all sets of P-measure zero.

(b) The filtration is right continuous, $\mathcal{F}_t = \mathcal{F}_{t+}$ where $\mathcal{F}_{t+} = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}$.

All random variables that are used in this thesis are considered to be defined on the same probability space.

Let $X = (X(t), t \geq 0)$ on $\mathbb{R}^d$ be a stochastic process defined on $(\Omega, \mathcal{F}, P)$.
Then X is adapted to the filtration $(\mathcal{F}_t, t \geq 0)$ (or $\mathcal{F}_t$-*adapted*) if

$$X(t) \text{ is } \mathcal{F}_t\text{-measurable for each } t \geq 0.$$

If X is $\mathcal{F}_t$-adapted and also satisfies for each $t \geq 0$ the integrability requirement $E(|X(t)|) < \infty$ then it is a *martingale* if, for all $0 \leq s < t < \infty$,

$$E(X(t)|\mathcal{F}_s) = X(s) \quad a.s.$$

Lévy processes, named in honor of the French mathematician Paul Lévy who first studied them in the 1930s, are stochastic processes with stationary and independent increments. They are the simplest class of processes having continuous paths that are interrupted with jump discontinuities of random size that appear at random times. Very important examples of Lévy processes are Brownian motion which has continuous sample paths, the Poisson process, the compound Poisson process, stable processes, subordinators and interlacing processes. Also a mixture of a Brownian motion and an independent compound Poisson process can give rise to a Lévy process and they constitute an important class of jump-diffusion processes. In addition Lévy processes form an important class of semimartingales (see Applebaum [1] pp. 115).

A formal definition follows.

Definition 1.2.1 The process $X = (X(t), t \geq 0)$ is a *Lévy process* if it is adapted to the filtration $(\mathcal{F}_t, t \geq 0)$ and the following conditions are satisfied.

(i) $X(0) = 0$ almost surely,
(ii) each $X(t) - X(s)$ is independent of $\mathcal{F}_s$, $\forall\ 0 \leq s < t < \infty$,
(iii) X has stationary increments i.e. $X(t) - X(s)$ has the same distribution as $X(t-s)$, for each $0 \leq s < t < \infty$,
(iv) X is stochastically continuous: for all $\alpha > 0$ and for all $s \geq 0$

$$\lim_{t \to s} P[|X(t) - X(s)| > \alpha] = 0.$$

Under these conditions, every Lévy process X has a càdlàg modification and is itself a Lévy process (see e.g. Applebaum [1] Theorem 2.1.7 pp. 74). For the rest of the thesis we will always consider this modification.

Historically the first investigations into the structure of Lévy processes can be tracked back to the late 1920s with the study of infinitely divisible distributions. The general structure of these was gradually discovered by de Finetti, Kolmogorov, Lévy and Itô and an important landmark is the celebrated Lévy-Khintchine formula. This formula gives a characterization of the infinite divisible probability measures through their characteristic functions. In the following we will present the Lévy-Khintchine formula but first we need some definitions quoted by Applebaum [1] Chapter 1.

Here $\mathcal{M}_1(\mathbb{R}^d)$ denotes the set of all Borel probability measures on $\mathbb{R}^d$.

The *convolution* of two probability measures $\mu_i \in \mathcal{M}_1(\mathbb{R}^d)$, for $i = 1, 2$ is defined as

$$(\mu_1 * \mu_2)(A) = \int_{\mathbb{R}^d} 1_A(x+y)\mu_1(dx)\mu_2(dy),$$

where $A \in \mathcal{B}(\mathbb{R}^d)$. Also define $(\mu)^{*^n} = \mu * ... * \mu$ (n times). Then $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ is *infinitely divisible* if μ has a convolution nth root i.e. if there exists a measure $\mu^{1/n} \in \mathcal{M}_1(\mathbb{R}^d)$ for which $(\mu^{1/n})^{*^n} = \mu$.

The *characteristic function* of $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ is defined as

$$\phi_\mu(u) = \int_{\mathbb{R}^d} e^{i\langle u,y\rangle}\mu(dy),$$

where $u \in \mathbb{R}^d$.

Now, let ν be a Borel measure defined on $\mathbb{R}^d \setminus \{0\} = \{x \in \mathbb{R}^d, x \neq 0\}$.
Then ν is a *Lévy measure* if

$$\int_{\mathbb{R}^d\setminus\{0\}} (|y|^2 \wedge 1)\nu(dy) < \infty. \tag{1.1}$$

The Lévy-Khintchine formula can now be introduced.

Theorem 1.2.2 *(Lévy-Khintchine formula)*
A Borel probability measure μ on $\mathbb{R}^d$ is infinitely divisible if there exists a vector $b \in \mathbb{R}^d$, a non-negative definite symmetric $d \times d$ matrix A and a Lévy measure ν on $\mathbb{R}^d \setminus \{0\}$ such that, for all $u \in \mathbb{R}^d$, the characteristic function of μ admits the representation

$$\phi_\mu(u) = \exp\left\{ i\langle b,u\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{\mathbb{R}^d\setminus\{0\}} \left[e^{i\langle u,y\rangle} - 1 - i\langle u,y\rangle 1_{\hat{B}}(y)\right]\nu(dy)\right\} \tag{1.2}$$

where $\hat{B} = \{y \in \mathbb{R}^d : 0 < |y| < 1\}$.
Conversely any mapping of the form (1.2) is the characteristic function of an infinitely divisible probability measure on $\mathbb{R}^d$.

For a proof see Sato [42] Theorem 8.1 pp. 37-45.

The triple (b, A, ν) is called the *characteristics* of μ.

Remark 1.2.3 The integral in the Lévy-Khintchine formula (1.2) converges. The integrand is $\mathcal{O}(y^2)$ as $|y| \to 0$ and is $\mathcal{O}(1)$ as $|y| \to \infty$. Hence, condition (1.1) is a sufficient condition for the integral to converge.

Now, the connection between the Lévy-Khintchine formula and Lévy processes is that each of the random variables comprising a Lévy process is infinitely divisible due to the stationarity and independence of the increments. Hence, the distribution of a Lévy process is determined by the form that the Lévy-Khintchine formula can take. Each $X(t)$ has characteristics $(tb, tA, t\nu)$ where the (b, A, ν) are the characteristics of $X(1)$. Also the triple (b, A, ν) is sufficient to determine some of the sample path properties of a Lévy process e.g. if $\nu = 0$ then X has a continuous sample paths (a.s.). For further details we refer to Sato [42], Section 21 and Kyprianou [27], Chapter 2 pp. 54-55.

1.2.1 Some examples of Lévy processes

(1) **Brownian motion:** A standard Brownian motion $B = (B(t), t \geq 0)$ is a Lévy process on $\mathbb{R}^d$ such that

(B1) $B(t) \sim N(0, tI)$ for each $t \geq 0$,
(B2) B has continuous sample paths.

The characteristic function of a standard Brownian motion is given by

$$\phi_{B(t)}(u) = \exp\left\{-\frac{1}{2}t|u|^2\right\},$$

for each $u \in \mathbb{R}^d$, $t \geq 0$.

(2) **Poisson process:** Let $N = (N(t), t \geq 0)$ be a Poisson process with intensity $\lambda > 0$. It is a Lévy process with non-negative integer values. For each $t > 0$, $N(t)$ follows a Poisson distribution with parameter λt so that

$$P\left(N(t) = n\right) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

for each $n = 0, 1, 2, \ldots$.
A Poisson process has piecewise constant paths on each finite time interval and at the random times

$$\tau_n = \inf\left\{t \geq 0; N(t) = n\right\}$$

it has jumps of size 1. Its characteristic function is given by

$$\phi_{N(t)}(u) = \exp\left[\lambda t\left(e^{iu} - 1\right)\right],$$

for each $u \in \mathbb{R}^d$, $t \geq 0$.

(3) **Compound Poisson process:** Let $(N(t), t \geq 0)$ be a Poisson random process with intensity $\lambda > 0$ and let $(U(m), m \in \mathbb{N})$ be a sequence of independent, identically distributed random variables on $\mathbb{R}^d$, independent of N and defined on the probability space $(\Omega, \mathcal{F}, P)$ with common law μ_U. The compound Poisson process $Z = (Z(t), t \geq 0)$ is a Lévy process, where for each $t \geq 0$

$$Z(t) = \sum_{k=1}^{N(t)} U(k).$$

The characteristic function of a compound Poisson process is

$$\phi_{Z(t)}(u) = \exp\left\{t\left[\int_{\mathbb{R}^d}(e^{i\langle u,y\rangle} - 1)\lambda\mu_U(dy)\right]\right\}$$

for each $u \in \mathbb{R}^d$, $t \geq 0$ and we can deduce that the Lévy measure for Z is $\lambda\mu_U$.

For further examples of processes that belong to the class of Lévy processes see e.g. Applebaum [1].

1.2.2 Lévy-Itô decomposition

Before we introduce another important tool in the theory of Lévy processes, the Lévy-Itô decomposition (see Ikeda-Watanabe [18] pp. 65 and Applebaum [1] pp. 108), we need to give some definitions.

We say that $A \in \mathcal{B}(\mathbb{R}^d\backslash\{0\})$ is *bounded below* if $0 \notin \overline{A}$.
Let S be a subset of $\mathbb{R}^d\backslash\{0\}$ and N be an integer-valued random measure on $\mathcal{U} = \mathbb{R}^+ \times S$ (for the definition of random measures see Applebaum [1] pp. 89). Then N is a *Poisson random measure* if

(1) For each $t \geq 0$ and $A \in \mathcal{B}(S)$ that is bounded below, $N(t, A)$ is Poisson distributed,
(2) For each $t_1, \ldots, t_n \in \mathbb{R}^+$ and each disjoint family $A_1, A_2, \ldots, A_n \in \mathcal{B}(S)$ that are bounded below, then the random variables $N(t_1, A_1), \ldots, N(t_n, A_n)$ are independent.

Its *intensity measure* is defined as $\lambda(A) = E(N(1, A))$ for all $A \in \mathcal{B}(S)$ that are bounded below.

Now let $S = \mathbb{R}^d\backslash\{0\}$.
The *jump process* associated to a Lévy process X is defined by $\Delta X = (\Delta X(t), t \geq 0)$

where

$$\Delta X(t) = X(t) - X(t-)$$

for each $t \geq 0$.
To count the jumps of the Lévy process X we define for each $E \in \mathcal{B}(\mathbb{R}^d \backslash \{0\})$

$$N(t,E) = \#\{0 \leq s \leq t; \Delta X(s) \in E\} = \sum_{0 \leq s \leq t} 1_E(\Delta X(s)).$$

For each $\omega \in \Omega$, $t \geq 0$ the set function $E \longrightarrow N(t,E)(\omega)$ is a counting measure on $\mathcal{B}(\mathbb{R}^d \backslash \{0\})$. Define an associated Borel measure $\nu(\cdot) = E(N(1,\cdot))$ (as in Applebaum [1] pp. 87). N is a Poisson random measure, and its intensity measure is $Leb \otimes \nu$, where *Leb* denotes Lebesgue measure on $\mathbb{R}^+$. We also define the *compensated Poisson random measure* by

$$\tilde{N}(t,E) = N(t,E) - t\nu(E).$$

Note that $(\tilde{N}(t,E), t \geq 0)$ is a martingale whenever $\nu(E) < \infty$.
For further details we refer to Applebaum [1] pp. 85-91.

For the rest of this subsection we denote by $B_A = (B_A(t), t \geq 0)$ in $\mathbb{R}^d$, the Gaussian process defined for each $t \geq 0$ by

$$B_A(t) = \left(B_A^1(t), \ldots, B_A^d(t)\right),$$

where for each $i = 1, 2, \ldots, d$

$$B_A^i(t) = \sum_{j=1}^{m} \sigma_j^i B^j(t)$$

and for each $j = 1, 2, \ldots, m$, B^j is a one-dimensional Brownian motion, $A \in \mathcal{M}_d(\mathbb{R})$ is a non-negative definite symmetric matrix, $\sigma \in \mathcal{M}_{d,m}(\mathbb{R})$ is a real-valued matrix, the square root of A such that $\sigma\sigma^T = A$. The process $B_A = (B_A(t), t \geq 0)$ is a Lévy process where for each $t \geq 0$, $B_A(t) \sim N(0, tA)$. We call the process $B_A = (B_A(t), t \geq 0)$ *Brownian motion with convariance A*.

Now the Lévy-Itô decomposition describes the sample paths of a Lévy process as a sum of a continuous part and jump parts. It effectively tells us that it can be decomposed into a Brownian motion with drift which is the continuous part, a compensated Poisson integral which represents the "small jumps" and a Poisson integral that describes the "large jumps" and is a compound Poisson process.

Theorem 1.2.4 *(Lévy-Itô decomposition)*
Every Lévy process $(X(t), t \geq 0)$ has the sample path decomposition

$$X(t) = bt + B_A(t) + \int_{|y|<1} y\tilde{N}(t, dy) + \int_{|y|\geq 1} yN(t, dy) \tag{1.3}$$

for each $t \geq 0$, where $b \in \mathbb{R}^d$, $(B_A, t \geq 0)$ is a Brownian motion with covariance matrix A and N is an independent Poisson random measure defined on $\mathbb{R}^+ \times (\mathbb{R}^d \setminus \{0\})$ with intensity measure $Leb \otimes \nu$.

Proof: See e.g. Applebaum [1] pp. 104-108.

Note that the three processes in (1.3) are all independent and the decomposition is unique (see Kunita [25] Theorem 2.7 pp. 327).

In general a Lévy process may not be a martingale, or even integrable but there are some important examples of Lévy processes that possess this property; for instance if a Lévy process has a zero mean then it is a martingale e.g. consider standard Brownian motion or the compensated Poisson process with intensity $\lambda > 0$.

For a more detailed treatment of the properties of Lévy processes we refer to Applebaum [1], Sato [42], Protter [40] and Kyprianou [27].

1.3 Stochastic Calculus - Itô's formula

In the following some basic tools from stochastic analysis theory and stochastic calculus will be introduced. These are needed for the purposes of this work. We will follow the definitions and notation of Applebaum [1].

Throughout this thesis, unless otherwise specified we assume that we are given an m-dimensional standard $\mathcal{F}_t$-adapted Brownian motion $B = (B(t), t \geq 0)$ with each $B(t) = (B^1(t), \ldots, B^m(t))$ and an independent $\mathcal{F}_t$-adapted Poisson random measure N defined on $\mathbb{R}^+ \times (\mathbb{R}^d \setminus \{0\})$ with compensator $\tilde{N}$ and intensity measure ν, where we will assume that ν is a Lévy measure.

Let $E \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ and $0 < T < \infty$. In order to assure the existence of stochastic integrals for which the integrator is a Lévy process, we need to define two spaces $\mathcal{P}_2(T, E)$ and $\mathcal{P}_2(T)$. The space $\mathcal{P}_2(T, E)$ is defined as the linear space of all predictable mappings $H : [0, T] \times E \times \Omega \to \mathbb{R}^d$ (for the definition of predictability see Applebaum [1] pp. 192) which satisfy

$$P\left[\int_0^T \int_E |H(s, y)|^2 \nu(dy) ds < \infty\right] = 1.$$

The space $\mathcal{P}_2(T)$ is the linear space of all predictable mappings $F : [0,T] \times \Omega \to \mathbb{R}^d$ for which $P\left[\int_0^T |F(t)|^2 dt < \infty\right] = 1$. For further details about stochastic integration we refer to Applebaum [1] Chapter 4.

For the rest of the thesis we take $E = \hat{B}_c$ where $0 < c \leq \infty$ plays the role of maximum allowable jump size. This parameter indicates what we mean by small and large jumps, and in the case where we want to consider small and large jumps under the same footing we take $c = \infty$.

A result that will play a vital role in this thesis is the celebrated Itô's formula. As will be revealed in the following chapters, it is a key technique applied to SDEs in order to establish stability.

Here we will give Itô's formula for a *Lévy-type stochastic integral*, with stochastic differential

$$dY(t) = G(t)dt + F(t)dB(t) + \int_{|y|<c} H(t,y)\tilde{N}(dt,dy) + \int_{|y|\geq c} K(t,y)N(dt,dy) \quad (1.4)$$

where for all $t \geq 0, 1 \leq i \leq d$, $1 \leq j \leq m$, $(G^i)^{1/2}, F^i_j \in \mathcal{P}_2(t)$, $H^i \in \mathcal{P}_2(t,E)$ and K is predictable. The construction of these integrals can be found in Applebaum [1] Chapter 4, pp. 205-212.

It is easily verified that Lévy-type stochastic integrals are semimartingales. Hence, the general Itô's formula for semimartingales can be applied (see e.g. Protter [40], pp. 271). Using algebra it is straightforward to prove the equivalence between Itô's formula for general semimartingales and Itô's formula in the form that follows. In this form it can be found in Ikeda-Watanabe [18] and Applebaum [1] pp. 226. Although it first appeared many years ago it is only in the last ten years that the importance of writing Itô's formula in this way has been appreciated.

First we need to define the *quadratic variation process* $[Y,Y] = ([Y,Y](t), t \geq 0)$ for $Y = (Y(t), t \geq 0)$ where Y as in (1.4). It is a $d \times d$ matrix-valued adapted process and its $(i,j)th$ entry, where $1 \leq i,j \leq d$, is defined by

$$[Y^i, Y^j](t) = [Y^i_c, Y^j_c](t) + \sum_{0 \leq s \leq t} \Delta Y^i(s) \Delta Y^j(s), \quad (1.5)$$

where Y_c is the continuous part of Y defined by $Y^i_c(t) = \int_0^t F^i_k(s)dB^k(s) + \int_0^t G^i(s)ds$ for each $t \geq 0$, $1 \leq i \leq d$, $1 \leq k \leq m$.
By Applebaum [1] corollary 4.4.9 pp. 228, for each $t \geq 0$ $[Y^i, Y^j](t)$ is almost surely

finite, and we have

$$[Y^i, Y^j](t) = \sum_{k=1}^{m} \int_0^t F_k^i(s) F_k^j(s) ds + \int_0^t \int_{|y|<c} H^i(s,y) H^j(s,y) N(ds,dy) + \int_0^t \int_{|y|\geq c} K^i(s,y) K^j(s,y) N(ds,dy).$$

For the validity of Itô's formula we need to impose the following local boundedness constraint on the jumps.

Assumption 1.3.1 For all $t \geq 0$,

$$\sup_{0\leq s\leq t} \sup_{0<|y|<c} |H(s,y)| < \infty \qquad a.s.$$

Theorem 1.3.2 *(Itô's formula)*
Let Y be a Lévy-type stochastic integral of the form (1.4). Then, for each $f \in C^2(\mathbb{R}^d)$, $t \geq 0$, with probability 1 we have

$$\begin{aligned} & f(Y(t)) - f(Y(0)) \\ &= \int_0^t \partial_i f(Y(s-)) dY_c^i(s) + \frac{1}{2} \int_0^t \partial_i \partial_j f(Y(s-)) d[Y_c^i, Y_c^j](s) \\ &+ \int_0^t \int_{|y|\geq c} [f(Y(s-) + K(s,y)) - f(Y(s-))] N(ds,dy) \\ &+ \int_0^t \int_{|y|<c} [f(Y(s-) + H(s,y)) - f(Y(s-))] \tilde{N}(ds,dy) \\ &+ \int_0^t \int_{|y|<c} [f(Y(s-) + H(s,y)) - f(Y(s-)) - H^i(s,y) \partial_i f(Y(s-))] \nu(dy) ds. \end{aligned}$$

For a proof see Applebaum [1] pp. 226.

For future reference we also state the following result.

Theorem 1.3.3 *(Itô's product formula)*
If Y^1 and Y^2 are real-valued Lévy-type stochastic integrals of the form (1.4) then, for all $t \geq 0$, with probability 1 we have that

$$Y^1(t)Y^2(t) = Y^1(0)Y^2(0) + \int_0^t Y^1(s-) dY^2(s) + \int_0^t Y^2(s-) dY^1(s) + \left[Y^1, Y^2\right](t). \quad (1.6)$$

For a proof see Applebaum [1], Theorem 4.4.13 pp. 231.

In differential form (1.6) is written as

$$d\left(Y^1(t)Y^2(t)\right) = Y^1(t-)dY^2(t) + Y^2(t-)dY^1(t) + d\left[Y^1, Y^2\right](t). \tag{1.7}$$

Note that we can derive the term $d\left[Y^1, Y^2\right](t)$, that is called an *Itô correction*, from the result of the following formal product relations between differentials:

$$dB^i(t)dB^j(t) = \delta^{ij}dt\ ; \qquad N(dt, dx)N(dt, dy) = N(dt, dx)\delta(x - y)$$

for $1 \leq i, j \leq m$.

For the last part of this section, we present three useful results that will be heavily exploited in the main part of the thesis.

(1) *Doob's martingale inequalities*
Let $(Z(t), t \geq 0)$ be an $\mathbb{R}^d$ valued martingale. Let $[a, b]$ be a closed interval in $\mathbb{R}$.
(i) If $p \geq 1$ and $Z(t) \in L^p(\Omega; \mathbb{R}^d)$ for all $t \geq 0$ then

$$P\left(\sup_{a \leq t \leq b} |Z(t)| \geq c\right) \leq \frac{E\left(|Z(b)|^p\right)}{c^p}$$

holds for all $c > 0$.
(ii) If $p > 1$ and $Z(t) \in L^p(\Omega; \mathbb{R}^d)$ for all $t \geq 0$ then

$$E\left(\sup_{a \leq t \leq b} |Z(t)|^p\right) \leq \left(\frac{p}{p-1}\right)^p E\left(|Z(b)|^p\right).$$

For further details see Mao [33] pp. 14.

The following well-known result is also a very useful tool in Chapter 3 and 4.
(2) *Gronwall's inequality*
Let $[a, b]$ be a closed interval in $\mathbb{R}$ and $\alpha, \beta : [a, b] \to \mathbb{R}$ be non-negative with α locally bounded and β integrable on $[a, b]$. If there exists $C \geq 0$ such that, for all $t \in [a, b]$,

$$\alpha(t) \leq C + \int_a^t \alpha(s)\beta(s)\, ds,$$

then we have

$$\alpha(t) \leq C \exp\left[\int_a^t \beta(s)\, ds\right]$$

for all $t \in [a, b]$.

Proof: See Applebaum [1], proposition 6.1.4 pp. 295-296.

Another helpful tool that is widely used in Chapter 4 is the strong law of large numbers for local martingales. But first we need to define the Meyer angle bracket process. By the Doob-Meyer decomposition (see Applebaum [1] pp. 81) if Y is a submartingale (see Applebaum [1] pp. 73) then there exists a unique predictable, increasing process $(A(t), t \geq 0)$ with $A(0) = 0$ (a.s.) such that $Y(t) - Y(0) - A(t)$ for each $t \geq 0$ is a uniformly integrable martingale. In the case that $Y(t) = M(t)^2$ where M is a square-integrable martingale then it is usual to write $\langle M, M\rangle(t) = A(t)$ for each $t \geq 0$ and we call the process $\langle M, M\rangle$ the *Meyer's angle-bracket process.* Sometimes $\langle M, M\rangle(t)$ is shortened to $\langle M\rangle(t)$ for each $t \geq 0$ in order to ease the notation. For example if $M = B$ where B is a one-dimensional standard Brownian motion then $\langle M\rangle(t) = t$ and if $M = \tilde{N}$ where $\tilde{N}$ is the compensated Poisson process with intensity λ then $\langle M\rangle(t) = \lambda t$ for each $t \geq 0$.

(3) *Strong law of large numbers*
Let $M = (M(t), t \geq t_0)$ be a local martingale. Define

$$\rho_M(t) = \int_{t_0}^{t} \frac{d\langle M\rangle(s)}{(1+s)^2}. \tag{1.8}$$

If the following condition holds

$$P\left(\lim_{t\to\infty} \rho_M(t) < \infty\right) = 1$$

then

$$P\left(\lim_{t\to\infty} \frac{M(t)}{t} \to 0\right) = 1. \tag{1.9}$$

Proof: See Liptser [29].

We also take the opportunity to mention some simple inequalities which we will use extensively throughout this thesis.

The logarithmic inequality:

$$\log(1+x) \leq x \quad \text{for} \quad x > -1, \tag{1.10}$$

from which it follows trivially that

$$\log(x) \leq x - 1 \quad \text{for} \quad x > 0. \tag{1.11}$$

The following inequality will also play a major role in the analysis of stability of SDEs driven by Lévy processes.
For all $a_1, \ldots, a_n \in \mathbb{R}^d$ it holds that

$$\left| \sum_{j=1}^{n} a_j \right|^2 \leq n \sum_{j=1}^{n} |a_j|^2. \tag{1.12}$$

This is established by using induction and the Cauchy-Schwarz inequality. □

1.4 Stochastic Differential Equations - SDEs

We can consider SDEs driven by Lévy processes of the form

$$dx^i(t) = l^i\,(x(t-))\,dX^i(t) \tag{1.13}$$

for each $1 \leq i \leq d$, $t \geq t_0$ where $X = (X(t), t \geq 0)$ is a Lévy process and $l_1, \ldots, l_d$ are suitable coefficients. Now if we replace X by its Lévy-Itô decomposition (see (1.3)) we see that we can generalize (1.13) and consider SDEs where time, the Brownian motion part, small and large jumps are all separately coupled to the system. This is the more general point of view that can be found in Applebaum [1] and is the one that we will use in this thesis.

Let $0 \leq t_0 \leq T \leq \infty$. Assume that the mappings $f : \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}^d \to \mathcal{M}_{d,m}(\mathbb{R})$, $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are Borel measurable.
We define a d-dimensional stochastic differential equation (SDE) by,

$$dx(t) = f(x(t-))dt + g(x(t-))dB(t) + \int_{|y|<c} H(x(t-), y)\tilde{N}(dt, dy) + \int_{|y|\geq c} K(x(t-), y)N(dt, dy) \tag{1.14}$$

on $t_0 \leq t \leq T$ with initial value $x(t_0) = x_0$, such that $x_0 \in \mathbb{R}^d$. The functions f, g are called *drift* and *diffusion coefficients* respectively and H and K are the *jump coefficients*.

To give (1.14) a rigorous mathematical meaning we rewrite it in an integral form, for each $t \geq t_0$, as

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^{t} f(x(s-))ds + \int_{t_0}^{t} g(x(s-))dB(s) + \int_{t_0}^{t}\int_{|y|<c} H(x(s-), y)\tilde{N}(ds, dy) \\ &\quad + \int_{t_0}^{t}\int_{|y|\geq c} K(x(s-), y)N(ds, dy) \quad \text{on } t_0 \leq t \leq T. \end{aligned}$$

As was mentioned in section 1.3, $c \in (0, \infty]$. In the case that $c = \infty$, the final integral in (1.14) is missing.

The uniqueness and existence of the solution of the SDE (1.14) is guaranteed from the theorem that follows, which will play a vital role for the rest of the thesis. By applying the Picard iteration technique, which is used by analogy to the ordinary differential equations-ODE case (see e.g. Boyce and Diprima [7] pp. 72-78) the existence and uniqueness of the solution for SDEs driven by a Lévy process can be proved if the following conditions are satisfied:

(C1)Lipschitz conditions: There exists a positive constant L such that, for all $x_1, x_2 \in \mathbb{R}^d$,

$$|f(x_1) - f(x_2)|^2 \leq L|x_1 - x_2|^2, \qquad \|g(x_1) - g(x_2)\|^2 \leq L|x_1 - x_2|^2, \quad (1.15)$$

$$\int_{|y|<c} |H(x_1, y) - H(x_2, y)|^2 \nu(dy) \leq L|x_1 - x_2|^2. \quad (1.16)$$

(C2)Growth conditions: There exists a positive constant K such that, for all $x \in \mathbb{R}^d$,

$$|f(x)|^2 \leq K(1 + |x|^2), \qquad \|g(x)\|^2 \leq K(1 + |x|^2), \quad (1.17)$$

$$\int_{|y|<c} |H(x, y)|^2 \nu(dy) \leq K(1 + |x|^2). \quad (1.18)$$

(C3)Big jumps condition: We require that the mapping $x \to K(x, y)$ is continuous for all $|y| \geq c$.

Note that Applebaum in [1] pp. 304 comments that (1.15) implies (1.17). However in later chapters of the thesis we will want to consider SDEs for which (1.17) and (1.18) hold, while (1.15) and (1.16) may not hold and so we list the conditions separately.

In the case of SDEs driven by a Lévy process with time-dependent coefficients, the conditions on the existence of a unique solution are analogous to (C1)-(C3), with $L > 0$ and $K > 0$ being replaced by the mappings $t \to L(t)$ and $t \to K(t)$ that are bounded and measurable (see Applebaum [1] pp. 312).

Theorem 1.4.1 *(Existence and Uniqueness)*
(i) Assume that (C1)-(C3) are satisfied. Then there exist a unique solution $x = (x(t), t \geq t_0)$ *to (1.14) with the initial condition* $x(t_0) = x_0$, *where* $x_0 \in \mathbb{R}^d$. *The process is adapted and càdlàg.*
(ii) For SDEs driven by a Lévy process with time-dependent coefficients, then a unique adapted and càdlàg solution exists under conditions analogous to (C1)-(C3).

For a proof see Applebaum [1] Theorem 6.2.3 pp. 304 and Theorem 6.2.9 pp. 311.

Solutions of SDEs driven by Lévy processes have many nice properties and we will mention some here for the sake of completeness. They form a class of Markov processes and in the case that there no jumps, diffusion processes. Their solutions form stochastic flows and under additional assumptions on the coefficients we can construct Feller processes. Full details can be found in Applebaum [1] Chapter 6.

Remark 1.4.2 In the proof of existence and uniqueness in Applebaum [1] Theorem 6.2.3 pp. 304 and Theorem 6.2.9 pp. 311 the author considers the initial condition to be an $\mathcal{F}_{t_0}$-measurable random variable i.e. $Y(0) = Y_0$ (a.s.). As we will discuss in due course for the purpose of this thesis it is sufficient for the initial condition to be constant.

We will also require the linear operator $\mathcal{L} : C^2(\mathbb{R}^d) \to C(\mathbb{R}^d)$ that has the following form (see Applebaum [1] Theorem 6.7.4 pp. 340):

$$\begin{aligned}(\mathcal{L}V)(x) &= f^i(x)(\partial_i V)(x) + \frac{1}{2}[g(x)g(x)^T]^{ik}(\partial_i\partial_k V)(x) \\ &+ \int_{|y|<c} \left[V(x+H(x,y)) - V(x) - H^i(x,y)(\partial_i V)(x)\right]\nu(dy) \\ &+ \int_{|y|\geq c} \left[V(x+K(x,y)) - V(x)\right]\nu(dy) \end{aligned} \tag{1.19}$$

where $V \in C^2(\mathbb{R}^d)$, $x \in \mathbb{R}^d$.

Remark 1.4.3 Note that if the coefficients of the SDE (1.14) satisfy additional assumptions besides the Lipschitz and growth conditions, then $(x(t), t \geq t_0)$ is a Feller process and $\mathcal{L}$ is the infinitesimal generator of the associated Feller semigroup. For further details we refer to Applebaum [1] pp. 331-342.

1.5 Stability Theory of SDEs

In this section we will include some of the historical background of stability theory in order to make our discussion more precise and clear. We will describe the celebrated Lyapunov theorem and we will give an introduction to the stochastic stability theory of SDEs.

In many applications where dynamical system models are used to represent the real world behavior, random processes are used in the model. They can capture the uncertainty of the environment in which the model is operating. The analysis and control of such systems involves evaluating stability properties, which is one of the qualitative properties of the random dynamical system being studied. Before we can consider the design of a system we need to make sure that the system is stable from input to output. Hence the study of stability properties of stochastic dynamical systems are considerable important.

The origin of stability theory for ordinary differential equations (ODEs) is due to A.M. Lyapunov. In [43] La Selle and Lefschetz refer to Lyapunov's paper (French translation) *"Probleme general de la stabilite du mouvement"* which proposed two different methods for determining the stability of deterministic systems. The first is known as the "*First method*" which requires the existence of a known explicit solution. Unfortunately this method is restrictive since for most differential equations (deterministic or stochastic) an explicit solution can be determined for only a very few cases e.g. linear SDEs driven by Brownian motion. For the vast majority of them, this is not possible. On the other hand, the" *Second method or direct Lyapunov method*" for determining the stability of a system, is more applicable since it does not require the knowledge of the explicit solution and that's why in recent years it has exhibited great power in applications specifically in engineering sciences and to mechanical and structural systems that have non-linear behavior (see Ariaratnam and Xie [4]). With the direct Lyapunov method we can get a lot of useful qualitative information about the behavior of the solution, without solving the equation. This includes asymptotic behavior and sensitivity of the solutions to small changes in the initial conditions. This information can be found from the coefficients in the differential equation.

In the following we will explain the direct Lyapunov method but first we need some definitions.

Assume that we are given a d-dimensional system of non-linear ordinary differential equations (ODEs) i.e.

$$\frac{dx(t)}{dt} = f(x(t)) \qquad \text{on } \ t \geq t_0 \tag{1.20}$$

with initial value $x(t_0) = x_0 \in \mathbb{R}^d$ and where $f : \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz continuous function. Assume that $f(0) = 0$ for all $t \geq t_0$. Hence, (1.20) has a solution $x(t) = 0$ corresponding to the initial value $x(t_0) = 0$, which is called the *trivial solution*. The trivial solution is said to be *stable* if, for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$|x(t)| < \varepsilon \qquad \text{for all } t \geq t_0$$

whenever $|x_0| < \delta$. The solution is called *asymptotically stable* if it is stable and there exists a $\delta_0(t_0) > 0$ such that

$$\lim_{t \to \infty} x(t) = 0$$

whenever $|x_0| < \delta_0$.

In the literature, authors also discuss about the stability of an *equilibrium point* or *steady state* $x = c$, which has the property that $f(c) = 0$ for all $t \geq t_0$. In the case where $c = 0$ we get $x(t) = 0$, the trivial solution.

Now define $\mathcal{K}$ to be the class of continuous non-decreasing functions $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ with $\mu(0) = 0$.

Then a continuous function V defined on B_h is said to be *positive-definite* if $V(0) = 0$ and, for some $\mu \in \mathcal{K}$, we have

$$V(x) \geq \mu(|x|) \quad \text{for all} \quad x \in B_h.$$

It is *negative-definite* if $-V$ is positive definite. It is said to be *decrescent* if for some $\lambda \in \mathcal{K}$, $V(x) \leq \lambda(|x|)$ for all $x \in B_h$.

The strength and the importance of the direct Lyapunov method lies in the fact that we have a test for the stability of an equilibrium point in terms of a function V, which is called the Lyapunov function. From a physical point of view the Lyapunov function can describe the total energy of a mechanical system. The idea is that if $\dot{V} \leq 0$, then V will not increase so the "distance" of $x(t)$ from the equilibrium point measured by V does not increase. In the case that $\dot{V} < 0$, this means that V will decrease to zero and as a result $x(t) \to 0$. These are the basic intuitive ideas behind the direct Lyapunov method that is contained in the celebrated Lyapunov theorem which is given below.

Theorem 1.5.1 *(Lyapunov Theorem)*
(i) If there exists a positive definite function $V \in C^1(B_h; \mathbb{R}^+)$, such that

$$\dot{V}(x(t)) = \frac{\partial V(x(t))}{\partial x} f(x(t)) \leq 0$$

for all $x \in B_h$, then the trivial solution of (1.20) is stable.

(ii) If there is a positive definite decrescent function $V \in C^1(B_h; \mathbb{R}^+)$ such that $\dot{V}(x(t))$ is negative definite, then the trivial solution of (1.20) is asymptotically stable.

If there is a function that satisfies the conditions of Theorem 1.5.1, then it is called a *Lyapunov function.*

Just as in the deterministic case, the main purpose of stochastic stability analysis is to determine the subsets in the space of the parameters of the SDE (such as the initial conditions or the time parameter) for which the solution is stable. In 1968 the Lyapunov theory was generalized for stochastic systems mainly through the contributions of Khasminski [22] and later on of Kushner [26]. Most of the recent literature, concerning the stability analysis of SDEs perturbed with noise, is dedicated to the case where the noise is Gaussian. But this is a fairly large restriction since many stochastic disturbances are not continuous. Specifically, SDEs driven by a Lévy process are now being intensively studied, particularly in mathematical finance (see e.g. Cont and Tankov [12], Applebaum [1] Chapter 5 and references within).

The main aim of this thesis is to examine stability of SDEs driven by Lévy noise. The general form of the SDEs is as defined in (1.14).

For the rest of the thesis we assume that conditions (C1)-(C3) are satisfied unless otherwise specified. Given the initial condition $x(t_0) = x_0 \in \mathbb{R}^d$ then the unique global solution of (1.14) is denoted by $x(t)$ for each $t \geq t_0$.

Assume that $f(0) = 0$, $g(0) = 0$, $H(0, y) = 0$ for all $|y| < c$ and $K(0, y) = 0$ for all $|y| \geq c$ then (1.14) has a unique solution $x(t) = 0$ for all $t \geq t_0$ corresponding to the initial value $x(t_0) = 0$, which is called the *trivial solution.*

Stability in the deterministic or stochastic case means insensitivity of the system to changes i.e. whether small changes in the initial condition lead to small changes (stability) or to large changes (instability) in the solution of the system. Starting from a small vicinity of the trivial solution of the SDE we will investigate constraints on the parameters which ensures that the solution of the SDE converges to the trivial solution. In a stable system, trajectories which start very close to the origin remain close to the origin after a long time has passed and for unstable systems they may have moved a large distance away.

Stability theory for SDEs appears to be much richer than for ODEs. We will mainly focus on the three most important types of stochastic stability, these being the following:

- stability in probability

- moment stability

- almost sure stability

There also exists other types of stochastic stability but these may be too weak from the point of view of practical significance (e.g. asymptotic convergence in probability see Kozin [24] Example 2.1 pp. 96).

We quote the following definitions from Mao [33].

Definition 1.5.2 The trivial solution of (1.14) is said to be *stochastically stable* or *stable in probability* if for every pair of $\varepsilon \in (0,1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r, t_0)$ such that

$$P\left\{|x(t)| < r \text{ for all } t \geq t_0\right\} \geq 1 - \varepsilon \tag{1.21}$$

whenever $|x_0| < \delta$. Otherwise it is said to be *stochastically unstable.*

Definition 1.5.3 The trivial solution of (1.14) is said to be *almost surely exponentially stable* if

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t)| < 0 \qquad a.s. \tag{1.22}$$

for all $x_0 \in \mathbb{R}^d$. The quantity in the left hand side of (1.22) is called the *sample Lyapunov exponent.*

Definition 1.5.4 Assume that $p > 0$. The trivial solution of (1.14) is said to be *pth moment exponentially stable* if there is a pair of constants $\lambda > 0$ and $C > 0$ such that

$$E[|x(t)|^p] \leq C|x_0|^p \exp(-\lambda(t - t_0)) \qquad \text{for all } t \geq t_0 \tag{1.23}$$

for all $x_0 \in \mathbb{R}^d$. When $p = 2$, it is usually said to be *exponentially stable in mean square.* It follows from (1.23) that

$$\frac{1}{t}\log\left(E[|x(t)|^p]\right) \leq \frac{1}{t}\left(\log C|x_0|^p\right) + \frac{1}{t}\log\left(\exp(-\lambda(t - t_0))\right) \qquad \text{for all } t \geq t_0.$$

Hence,

$$\limsup_{t\to\infty} \frac{1}{t}\log\left(E[|x(t)|^p]\right) \leq -\lambda.$$

So the trivial solution of (1.14) is pth moment exponentially stable if

$$\limsup_{t\to\infty} \frac{1}{t}\log(E[|x(t)|]^p) < 0 \qquad a.s. \tag{1.24}$$

for all $x_0 \in \mathbb{R}^d$. The quantity in the left hand side of (1.24) is called *the pth moment Lyapunov exponent of the solution.*

The difference between almost sure exponential stability and pth moment exponential stability is that almost sure exponential stability implies that almost all sample paths of the solution will tend to the the equilibrium position exponentially fast, while the pth moment exponential stability implies that the pth moment of the solution will tend to zero exponentially fast.

Note that for the purposes of this thesis, is not necessary to consider the initial condition to be an $\mathcal{F}_{t_0}$-measurable random variable in order to establish stability of SDEs. It is enough to require it to be an $\mathbb{R}^d$-vector. Mao in [33] pp. 111, 120, 128 discusses this for SDEs driven by a Brownian motion. The same reasoning applies for the case of SDEs driven by a Lévy process.

In the existing literature there is little work done concerning stability of SDEs driven by discontinuous noise and that was a motivation for this thesis. A key paper is by Mao and Rodkina [35] where they investigate exponential stability of SDEs driven by discontinuous semimartingales. Although SDEs driven by Lévy processes belong to this class, the results that Mao and Rodkina have obtained do not seem to include the case that we consider. They examine the following SDE:

$$x(t) = x(0) + \int_0^t h(x(s-))dA(s) + \int_0^t w(x(s-))dM(s) \quad \text{on} \quad t \geq 0 \tag{1.25}$$

with initial condition $x(0) = x_0 \in \mathbb{R}^d$, where the process $A = (A(t), t \geq 0)$ is a process of bounded variation and the process $M = (M(t), t \geq 0)$ is a local martingale. They require the following hypothesis:

(H1) There exists a predictable process K of bounded variation, a predictable process α and an $m \times m$ matrix of predictable processes μ such that

$$A(t) = \int_0^t \alpha(s)dK(s) \quad \text{and} \quad \langle M, M\rangle(t) = \int_0^t \mu(s)dK(s).$$

Now for a generic Lévy process we can find from its Lévy-Itô decomposition that (see Applebaum [1] pp. 115)

$$A(t) = bt + \int_{|y|\geq c} yN(t, dy) \quad \text{and} \quad M(t) = B_A(t) + \int_{|y|<c} y\tilde{N}(t, dy).$$

We see that (H1) does not work for such a Lévy process since A is expressed in terms of the Poisson measure, while the process $\langle M, M\rangle$ involves only the Lévy measure. Hence, in this case we cannot express both of the processes A and $\langle M, M\rangle$ in terms of a single predictable process K as is required.

Mao and Rodkina in [35] also impose an additional hypothesis: For all $t \geq 0$

$$\int_0^\infty \mathcal{E}^{-1}(-\Gamma)\eta(t)dK(t) < \infty \quad a.s.$$

where η is a predictable process, $\eta(t) \geq 0$ for each $t \geq 0$, $\mathcal{E}$ is the Doléans-Dade exponential (see e.g. Applebaum [1] pp. 247) and $\Gamma(t) = \int_0^t \gamma(s)dK(s)$ where γ is a predictable process satisfying a number of properties that can be found on pp. 215 of Mao and Rodkina [35]. This condition may be extremely difficult to verify in practise.

In addition their equation (1.25) is less general than (1.14) since the driving coefficients h and w in (1.25) have no dependence on the jumps of the process.

We will aim to establish stability for solutions of (1.14) making more simple assumptions directly on the driving coefficients in a similar fashion to the results obtained in [33] by Mao for SDEs driven by Brownian motion.

1.6 Overview

The work that follows is mainly a partial generalization of Mao's theory [31, 32, 33], who has carried out extensive research in stochastic stability theory. Although Mao deals with non-homogenous SDEs driven by Brownian motion, for simplicity we extend his results in the homogenous case for SDEs and stochastic functional differential equations (SFDEs) driven by a Lévy process. We point out that the extension from homogeneous to inhomogeneous equations of the type considered by Mao [31, 32, 33] requires very little additional work.

This thesis is organized as follow:

Chapter 2 gives the variation of constants formula for linear inhomogenous SDEs driven by Lévy processes. This is based on Mao's work (see [33] pp. 92-98) where he derives the variation of constants formula for SDEs driven by Brownian motion. In the last part of this chapter we give the variation of constants formula for a special case, for an SDE driven by a Poisson process.

In Chapter 3 we focus on stability of the solution of SDEs driven by continuous noise interlaced by jumps. We provide an L^p estimate for the solution of the SDE under consideration. One of the key results of this chapter is the exponential martingale inequality which plays an important role in the work of this chapter. We continue our study by examining stability in probability and provide a theorem analogous to the Lyapunov theorem for the deterministic case. Mao in [33] extended the well-known Lyapunov theorem for the case of SDEs driven by Brownian motion. We imitate his proof and in a similar fashion we prove a Lyapunov theorem for SDEs driven by Lévy noise. Then we study almost sure asymptotic stability. We give an estimate for the sample Lyapunov exponent of the solution and criteria for almost sure exponential stability of the trivial solution for the SDE under consideration. In the last section of Chapter 3 we introduce moment exponential stability and we establish a relationship between moment exponential stability and almost sure exponential stability.

The aim of Chapter 4 is to provide conditions under which a non-linear deterministic system is almost surely exponentially stable when it is perturbed by random noise. As we will see the Lévy noise plays a similar role to the Brownian motion noise (as described in Mao [31, 33]) in stabilizing dynamical systems. To make our discussion simpler we examine separately the perturbation of the non-linear deterministic system first by small jumps and then by large jumps. Conditions for almost sure exponential stability for the trivial solution of the stochastically perturbed system driven by Lévy noise are obtained and some examples are given. Furthermore, we examine the stabilization of a one-dimensional linear deterministic system perturbed by a Brownian motion and a single Poisson process and then we focus on the stabilization of a non-linear system perturbed by Brownian motion and a compensated Poisson process, where additional insight can be gained. We have also established that a stable SDE driven by Poisson noise can be destabilized by Brownian motion provided that the dimension of the system is at least 2 and some additional conditions are satisfied.

In Chapter 5 we turn our attention to stochastic functional differential equations (SFDEs) driven by Lévy noise. We give a brief introduction to SFDEs and we establish existence and uniqueness for solutions of SFDEs by using the methodology of Applebaum in the case of SDEs driven by Lévy noise (see Applebaum [1] pp. 305-310). Then we extend Mao's approach, who studied stability properties of SFDEs driven by Brownian motion using Razumikhin type theorems (see Mao [32, 33]), for SFDEs and stochastic delay equations (SDDEs) where the driving noise is a Lévy process. Finally we apply the variation of constants formula that was developed in Chapter 2 to establish stabilization of an ordinary functional differential equation system when is perturbed by a compensated Poisson process.

Chapter 2

Properties of linear SDEs

2.1 Introduction

The variation of constants formula for linear ordinary differential equations (ODEs) has been used extensively for the study of stability and asymptotic behavior of solutions of ODEs (see Brauer [8, 9]). In this chapter we will derive the variation of constants formula for an SDE driven by a generic Lévy process and, as a special case, a Poisson process. These results will be applied in Chapter 5.

2.2 Variation of constants formula for an SDE driven by a Lévy process

In this section we will establish the variation of constants formula for an SDE driven by a Lévy process. Let the SDE be defined for each $t \geq 0$ by

$$\begin{aligned} dz(t) &= (F(t)z(t-) + f(t))\,dt + \sum_{k=1}^{m} (C_k(t)z(t-) + g_k(t))\,dB_k(t) \\ &+ \int_{|y|<c} \Big(H(t,y)z(t-) + h(t,y)\Big)\tilde{N}(dt,dy) + \int_{|y|\geq c} \Big(K(t,y)z(t-) + r(t,y)\Big)N(dt,dy) \end{aligned} \tag{2.1}$$

with initial condition $z(0) = z_0 \in \mathbb{R}^d$ where $F, C_k : [0,\infty) \to \mathcal{M}_d(\mathbb{R})$, $H, K : [0,\infty)\times\mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$, $f, g_k : [0,\infty) \to \mathbb{R}^d$, $h, r : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ and $B = (B(t), t \geq 0)$ and N are defined as in Chapter 1, section 1.3. Assume that $F, C_k, H, K, f, g_k, h, r$ are Borel measurable and F, C_k, f, g_k are bounded on $[0,\infty)$, for $1 \leq k \leq m$.

Assumption 2.2.1 For each $t \geq 0$, there exists $K_1(t) > 0$ and $K_2(t) > 0$ such that

$$(i) \int_{|y|<c} \left(\|H(t,y)\|^2 \vee \|H(t,y)\|^4 \right) \nu(dy) = K_1(t) < \infty$$

$$(ii) \int_{|y|<c} |h(t,y)|^2 \nu(dy) = K_2(t) < \infty$$

where we require that the mappings $t \to K_i(t)$ (i=1,2) are bounded and measurable.

For the remainder of this section, assumption 2.2.1 will always be satisfied.

Definition 2.2.2 If for all $t \geq 0$ $f(t) = 0$, $g_k(t) = 0$ for all $1 \leq k \leq m$, $h(t,y) = 0$ for all $|y| < c$ and $r(t,y) = 0$ for all $|y| \geq c$, then (2.1) is said to be *homogenous.*

For the existence and uniqueness of solutions to (2.1) we have to check if the Lipschitz and growth conditions for time-dependent coefficients hold (see Applebaum [1] pp. 312).

It is trivial to prove that the Lipschitz conditions on the drift and diffusion coefficients are satisfied. For the jump coefficient for all $t \geq 0$, $z_1, z_2 \in \mathbb{R}^d$ and $|y| < c$ we have that

$$\begin{aligned} \int_{|y|<c} |(H(t,y)z_1 + h(t,y)) - (H(t,y)z_2 + h(t,y))|^2 \nu(dy) &\leq \int_{|y|<c} \|H(t,y)\|^2 \nu(dy) |z_1 - z_2|^2 \\ &= K_3(t)|z_1 - z_2|^2 \end{aligned} \qquad (2.2)$$

where $K_3(t) = \int_{|y|<c} \|H(t,y)\|^2 \nu(dy)$ and by assumption 2.2.1 (i) is finite and the mapping $t \to K_3(t)$ is bounded and measurable. Hence, the Lipschitz conditions are satisfied.

As in the case of time independent coefficients, it can be easily proved that the Lipschitz conditions on the drift and diffusion coefficients imply the growth conditions (see Applebaum [1] pp. 304).

For the jump coefficient for all $t \geq 0$, $z \in \mathbb{R}^d$ and $|y| < c$, we have that

$$\begin{aligned} \int_{|y|<c} |H(t,y)z + h(t,y)|^2 \nu(dy) &\leq \int_{|y|<c} 2 \left(\|H(t,y)\|^2 |z|^2 + |h(t,y)|^2 \right) \nu(dy) \\ &\leq L(t)(1 + |z|^2) \end{aligned}$$

where $L(t) = 2 \max \left\{ \int_{|y|<c} \|H(t,y)\|^2 \nu(dy), \int_{|y|<c} |h(t,y)|^2 \nu(dy) \right\}$. By assumption 2.2.1 (i) and (ii) we deduce that $L(t)$ for each $t \geq 0$ is finite and the mapping $t \to L(t)$ is bounded and measurable.
Since the mapping

$$z \to K(t,y)z + r(t,y)$$

is linear, then is continuous for all $t \geq 0$, $|y| \geq c$, and all $z \in \mathbb{R}^d$. This implies that the required condition on the large jump time dependent coefficient (analogous to (C3), see Chapter 1, section 1.4) for existence and uniqueness is satisfied.
Hence, by Theorem 1.4.1 (ii), the SDE (2.1) has a unique solution.

Now, define the following homogeneous SDE for each $t \geq 0$

$$\begin{aligned} dx(t) &= F(t)x(t-)dt + \sum_{k=1}^{m} C_k(t)x(t-)dB_k(t) \\ &\quad + \int_{|y|<c} H(t,y)x(t-)\tilde{N}(dt,dy) + \int_{|y|\geq c} K(t,y)x(t-)N(dt,dy) \end{aligned} \tag{2.3}$$

with F, C_k $(1 \leq k \leq m), H, K$ as specified previously.

Following Mao [33] pp. 92, for each $1 \leq j \leq d$, let $\Phi_j = (\Phi_j(t), t \geq 0)$ be the solution to (2.3) with initial condition $x(0) = e_j$, where for each $t \geq 0$ $\Phi_j(t) = (\Phi_{1j}(t), \Phi_{2j}(t), \ldots, \Phi_{dj}(t))^T$. Hence, we can define for each $t \geq 0$, the matrix $\Phi(t) = (\Phi_{ij}(t)) \in \mathcal{M}_d(\mathbb{R})$ where for $1 \leq i, j \leq d$,

$$\begin{aligned} d\Phi_{ij}(t) &= \sum_{p=1}^{d} F_{ip}(t)\Phi_{pj}(t-)dt + \left(\sum_{k=1}^{m}\sum_{p=1}^{d} C_{ip}^{k}(t)\Phi_{pj}(t-)dB_k(t)\right) \\ &+ \sum_{p=1}^{d}\left(\int_{|y|<c} H_{ip}(t,y)\Phi_{pj}(t-)\tilde{N}(dt,dy)\right) + \sum_{p=1}^{d}\left(\int_{|y|\geq c} K_{ip}(t,y)\Phi_{pj}(t-)N(dt,dy)\right). \end{aligned} \tag{2.4}$$

Writing this equation in matrix form we have that

$$\begin{aligned} d\Phi(t) &= F(t)\Phi(t-)dt + \sum_{k=1}^{m} C_k(t)\Phi(t-)dB_k(t) + \int_{|y|<c} H(t,y)\Phi(t-)\tilde{N}(dt,dy) \\ &\quad + \int_{|y|\geq c} K(t,y)\Phi(t-)N(dt,dy) \end{aligned} \tag{2.5}$$

with initial condition $\Phi(0) = I$.

Note that the solution of (2.5), when it exists, will be an $\mathcal{M}_d(\mathbb{R})$-valued stochastic process $(\Phi(t), t \geq 0)$.

The next theorem shows that a unique solution of (2.3) exists and can be expressed in terms of the random matrix-valued process Φ.

Theorem 2.2.3 *The unique solution of (2.3) is*

$$x(t) = \Phi(t)x_0$$

given that $x(0) = x_0 \in \mathbb{R}^d$ *is the initial condition.*

The proof is omitted as it is exactly the same as in Mao [33] Theorem 2.1 pp. 93, except that there it is only done for an SDE driven by a Brownian motion while we use an SDE driven by a Lévy process.

Now that we have established existence and uniqueness of the solution to (2.1) we will derive the variation of constants formula for an SDE driven by a generic Lévy process of the form (2.1). We need the following tools.

Let $a : [0, \infty) \to \mathbb{R}$, $b_k : [0, \infty) \to \mathbb{R}$ for $1 \le k \le m$ be bounded Borel measurable functions. Assume that $w : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$, $\delta : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ and $q : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ are Borel measurable, such that $\inf_{y \in \hat{B}_c} (\delta(t, y) + w(t, y)) > -1$ and $\inf_{y \in \mathbb{R}^d \setminus B_c} q(t, y) > -1$. Define the following SDE

$$\begin{aligned} dz(t) &= \left(a(t) + \int_{|y|<c} w(t, y)\nu(dy) \right) z(t-)dt + \left(\sum_{k=1}^{m} b_k(t)z(t-)dB_k(t) \right) \\ &+ \int_{|y|\ge c} q(t, y)z(t-)N(dt, dy) + \int_{|y|<c} (\delta(t, y) + w(t, y))\, z(t-)\tilde{N}(dt, dy) \end{aligned} \tag{2.6}$$

with initial condition $z(0) = z_0 \in \mathbb{R}^d$.

Assumption 2.2.4 For each $t \ge 0$, there exists $K_4(t) > 0$ such that

$$\int_{|y|<c} \left(|w(t, y)| \vee |w(t, y)|^2 \vee |\delta(t, y)|^2 \right) \nu(dy) = K_4(t) < \infty,$$

where the map $t \to K_4(t)$ is bounded and measurable.

Taking into account assumption 2.2.4, since a, b_k for $1 \le k \le m$ are bounded, the mapping $z \to q(t, y)z$ is linear hence continuous for all $|y| \ge c$ and $z \in \mathbb{R}^d$, and as (2.6) is of linear form, it can be easily proved that the conditions on the drift, diffusion and jump coefficients for existence and uniqueness of the solutions to (2.6) (see Chapter 1, section 1.4) are satisfied.

Lemma 2.2.5 The SDE defined in (2.6) has a unique solution which is given by

$$\begin{aligned} z(t) &= z_0 \exp\Bigg[\int_0^t \left(a(s) - \frac{1}{2}\sum_{k=1}^m b_k{}^2(s) + \int_{|y|<c} [\log(w(s,y)+\delta(s,y)+1) - \delta(s,y)]\,\nu(dy)\right) ds \\ &\quad + \sum_{k=1}^m \int_0^t b_k(s)dB_k(s) + \int_0^t \int_{|y|<c} \log(w(s,y)+\delta(s,y)+1)\tilde{N}(ds,dy) \\ &\quad + \int_0^t \int_{|y|\geq c} \log(q(s,y)+1)N(ds,dy)\Bigg]. \end{aligned}$$

Remark 2.2.6 Note that we can also express (2.6) as the stochastic exponential (see Applebaum [1] pp. 247) of

$$dz(t) = z(t-)dY(t)$$

where $Y = (Y(t), t \geq 0)$ is the Lévy-type stochastic integral that has the form

$$\begin{aligned} dY(t) &= \left(a(t) + \int_{|y|<c} w(t,y)\right) dt + \sum_{k=1}^m b_k(t)dB_k(t) + \int_{|y|<c} (w(t,y)+\delta(t,y))\,\tilde{N}(dt,dy) \\ &\quad + \int_{|y|\geq c} q(t,y)N(dt,dy). \end{aligned}$$

The result in Lemma 2.2.5 is a special case of equation (5.2) in Applebaum [1] pp. 248 where the author deals with more general stochastic exponentials. We include a short proof here for completeness, of the specific result that we need.

Proof: Define an adapted process $(x(t), t \geq 0)$ whose stochastic differential is

$$\begin{aligned} dx(t) &= \Bigg[\left(a(t) - \frac{1}{2}\sum_{k=1}^m b_k{}^2(t) + \int_{|y|<c} [\log(w(t,y)+\delta(t,y)+1) - \delta(t,y)]\,\nu(dy)\right) dt \\ &\quad + \sum_{k=1}^m b_k(t)dB_k(t) + \int_{|y|<c} \log(w(t,y)+\delta(t,y)+1)\tilde{N}(dt,dy) \\ &\quad + \int_{|y|\geq c} \log(q(t,y)+1)N(dt,dy)\Bigg]. \end{aligned}$$

Apply Itô's formula to $z(t) = z_0 e^{x(t)}$. Then, for each $t \geq 0$,

$$\begin{aligned}
z(t) &= z_0 + \int_0^t z_0 e^{x(s-)} \Bigg(a(s) - \frac{1}{2}\sum_{k=1}^m {b_k}^2(s) \\
&\qquad + \int_{|y|<c} \left[\log\left(w(s,y) + \delta(s,y) + 1\right) - \delta(s,y)\right] \nu(dy) \Bigg) ds \\
&+ \frac{1}{2}\int_0^t \sum_{k=1}^m z_0 e^{x(s-)} {b_k}^2(s) ds + \int_0^t \sum_{k=1}^m z_0 e^{x(s-)} b_k(s) dB_k(s) \\
&+ \int_0^t \int_{|y|\geq c} z_0 e^{x(s-)} \left[e^{\log(q(s,y)+1)} - 1 \right] N(ds,dy) \\
&+ \int_0^t \int_{|y|<c} z_0 e^{x(s-)} \left[e^{\log(w(s,y)+\delta(s,y)+1)} - 1 \right] \tilde{N}(ds,dy) \\
&+ \int_0^t \int_{|y|<c} z_0 e^{x(s-)} \left[e^{\log(w(s,y)+\delta(s,y)+1)} - 1 - \log(w(s,y) + \delta(s,y) + 1) \right] \nu(dy) ds \\
&= z_0 + \int_0^t \left(a(s) + \int_{|y|<c} w(s,y)\nu(dy) \right) z(s-) ds + \int_0^t \sum_{k=1}^m b_k(s) z(s-) dB_k(s) \\
&\quad + \int_0^t \int_{|y|<c} \left(w(s,y) + \delta(s,y)\right) z(s-) \tilde{N}(ds,dy) + \int_0^t \int_{|y|\geq c} q(s,y) z(s-) N(ds,dy).
\end{aligned}$$

Hence, z is a solution to (2.6) and by the existence and uniqueness theorem (see Theorem 1.4.1 (ii)) it is the unique one. □

Now consider the one-dimensional SDE

$$\begin{aligned}
dQ(t) &= \sum_{k=1}^m \sum_{i=1}^d C_{ii}^k(t) Q(t-) dB_k(t) \\
&+ \left(\sum_{i=1}^d F_{ii}(t) + \sum_{k=1}^m \sum_{1\leq i<j\leq d} \left(C_{ii}^k(t) C_{jj}^k(t) - C_{ij}^k(t) C_{ji}^k(t) \right) \right) Q(t-) dt \\
&+ \int_{|y|\geq c} \left(\sum_{i=1}^d K_{ii}(t,y) + \sum_{1\leq i<j\leq d} \left(K_{ii}(t,y) K_{jj}(t,y) - K_{ij}(t,y) K_{ji}(t,y) \right) \right) Q(t-) N(dt,dy) \\
&+ \int_{|y|<c} \left(\sum_{i=1}^d H_{ii}(t,y) + \sum_{1\leq i<j\leq d} \left(H_{ii}(t,y) H_{jj}(t,y) - H_{ij}(t,y) H_{ji}(t,y) \right) \right) Q(t-) \tilde{N}(dt,dy) \\
&+ \int_{|y|<c} \left(\sum_{1\leq i<j\leq d} \left(H_{ii}(t,y) H_{jj}(t,y) - H_{ij}(t,y) H_{ji}(t,y) \right) \right) Q(t-) \nu(dy) dt
\end{aligned} \tag{2.7}$$

with initial condition $Q(0) = 1$.

Before we establish existence and uniqueness to solutions of (2.7), we need the following results.

Proposition 2.2.7 If $A \in \mathcal{M}_d(\mathbb{R})$ then

$$\mathrm{tr}\left(A^2\right) \leq \|A\|^2.$$

Proof: Applying the Cauchy-Schwarz inequality twice then

$$\begin{aligned}\mathrm{tr}\left(A^2\right) = \sum_{i=1}^{d}\left(\sum_{j=1}^{d} A_{ij}A_{ji}\right) &\leq \sum_{i=1}^{d}\left(\sum_{j=1}^{d} A_{ij}^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{d} A_{ji}^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{i=1}^{d}\sum_{j=1}^{d} A_{ij}^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{d}\sum_{j=1}^{d} A_{ji}^2\right)^{\frac{1}{2}} = \|A\|^2.\end{aligned}$$

□

Lemma 2.2.8 For each $t \geq 0$, there exists $K_5(t) > 0$ such that

$$(i)\ \int_{|y|<c}\left|\left(\sum_{1\leq i<j\leq d}(H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y))\right)\right|\nu(dy) \leq \beta_1 K_3(t) < \infty, \quad (2.8)$$

$$(ii)\ \int_{|y|<c}\left[\left|\sum_{i=1}^{d} H_{ii}(t,y)\right|^2 + \left|\sum_{1\leq i<j\leq d}(H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y))\right|^2\right]\nu(dy) \leq \beta_2 K_5(t) < \infty, (2.9)$$

$$(iii)\ \int_{|y|<c}\left|\sum_{i=1}^{d} H_{ii}(t,y) + \sum_{1\leq i<j\leq d}(H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y))\right|^2\nu(dy) \leq 2\beta_2 K_5(t) < \infty, (2.10)$$

where $\beta_1 = \left(\frac{d+1}{2}\right)$, $\beta_2 = \left(\frac{2d+d^2+1}{2}\right)$, and the mapping $t \to K_5(t)$ is bounded and measurable.

Proof: We first note the easily verified fact that for every $a_1, \ldots, a_d \in \mathbb{R}$,

$$\left(\sum_{i=1}^{d} a_i\right)^2 = \sum_{i=1}^{d} a_i^2 + 2\sum_{1\leq i<j\leq d} a_i a_j. \quad (2.11)$$

Using the identity (2.11) then

$$\sum_{1\le i<j\le d} (H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y))$$
$$= \frac{1}{2}\left[\left(\sum_{i=1}^{d} H_{ii}(t,y)\right)^2 - \sum_{i=1}^{d}(H_{ii}(t,y))^2\right] - \sum_{1\le i<j\le d} H_{ij}(t,y)H_{ji}(t,y)$$
$$= \frac{1}{2}\left(\sum_{i=1}^{d} H_{ii}(t,y)\right)^2 - \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} H_{ij}(t,y)H_{ji}(t,y). \tag{2.12}$$

(i) To prove (2.8), by (2.12), the Cauchy-Schwarz inequality, (1.12), proposition 2.2.7 and assumption 2.2.1 we deduce that

$$\int_{|y|<c}\left|\left(\sum_{1\le i<j\le d} (H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y))\right)\right|\nu(dy)$$
$$\le \int_{|y|<c} \frac{1}{2}\left|\left(\sum_{i=1}^{d} H_{ii}(t,y)\right)^2 - \sum_{i=1}^{d}\sum_{j=1}^{d} H_{ij}(t,y)H_{ji}(t,y)\right|\nu(dy)$$
$$\le \int_{|y|<c}\left(\frac{1}{2}\left|d\sum_{i=1}^{d}|H_{ii}(t,y)|^2\right| + \frac{1}{2}\left|\sum_{i=1}^{d}\sum_{j=1}^{d} H_{ij}(t,y)H_{ji}(t,y)\right|\right)\nu(dy)$$
$$\le \int_{|y|<c}\left(\frac{d}{2}\|H(t,y)\|^2 + \frac{1}{2}\mathrm{tr}\left(H(t,y)^2\right)\right)\nu(dy)$$
$$\le \int_{|y|<c}\left(\frac{d}{2}\|H(t,y)\|^2 + \frac{1}{2}\|H(t,y)\|^2\right)\nu(dy) = \left(\frac{d+1}{2}\right)K_3(t) < \infty, \tag{2.13}$$

where for each $t \ge 0$, $K_3(t)$ is defined as in (2.2).

(ii) Using the same arguments that were used to obtain (i), we have

$$\int_{|y|<c}\left[\left|\sum_{i=1}^{d} H_{ii}(t,y)\right|^2 + \left|\sum_{1\le i<j\le d} (H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y))\right|^2\right]\nu(dy)$$
$$\le \int_{|y|<c}\left[d\sum_{i=1}^{d}|H_{ii}(t,y)|^2 + \left(\frac{d}{2}\|H(t,y)\|^2 + \frac{1}{2}\mathrm{tr}\left(H(t,y)^2\right)\right)^2\right]\nu(dy)$$
$$\le \int_{|y|<c}\left[d\sum_{i=1}^{d}|H_{ii}(t,y)|^2 + \frac{d^2}{2}\|H(t,y)\|^4 + \frac{1}{2}\left|\mathrm{tr}\left(H(t,y)^2\right)\right|^2\right]\nu(dy)$$
$$\le \int_{|y|<c}\left[d\|H(t,y)\|^2 + \frac{d^2}{2}\|H(t,y)\|^4 + \frac{1}{2}\|H(t,y)\|^4\right]\nu(dy) \le \left(\frac{2d+d^2+1}{2}\right)K_5(t) < \infty,$$

where $K_5(t) = \int_{|y|<c}\left(\|H(t,y)\|^2 \vee \|H(t,y)\|^4\right)\nu(dy)$.

(iii) This follows easily from the result of (ii) where we use the inequality $|a+b|^2 \leq 2\left(|a|^2+|b|^2\right)$ for $a, b \in \mathbb{R}$. $\square$

Theorem 2.2.9 *There exists a unique adapted and càdlàg solution $Q = (Q(t), t \geq 0)$ to (2.7) with initial condition $Q(0) = 1$.*

Proof: For the existence of a unique solution to (2.7) we have to show that the Lipschitz, growth and the large jump conditions for time-dependent coefficients are satisfied (see section 1.4).

Since for each $1 \leq k \leq m$, C_k and F are bounded then for each $1 \leq k \leq m$

$$\|C_k(t)\|^2 = \sum_{i=1}^{d}\sum_{j=1}^{d}\left(C_{ij}^k(t)\right)^2 \leq d^2 \max_{1\leq i,j\leq d}\sup_{t\geq 0}\left|C_{ij}^k(t)\right|^2 < \infty. \quad (2.14)$$

The same argument applies to the matrix F.

For each $t \geq 0$ and $1 \leq k \leq m$ define

$$L(t) = d^2 \max_{1\leq i\leq d}\sup_{t\geq 0}|F_{ii}(t)|^2 < \infty \text{ and } M^k(t) = d^2 \max_{1\leq i,j\leq d}\sup_{t\geq 0}\left|C_{ij}^k(t)\right|^2 < \infty.$$

We also define, for each $t \geq 0$,

$$D(t) = \left|\int_{|y|<c}\left(\sum_{1\leq i<j\leq d}(H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y))\right)\nu(dy)\right| \text{ and}$$

$$P(t) = \int_{|y|<c}\left|\left(\sum_{i=1}^{d}H_{ii}(t,y) + \sum_{1\leq i<j\leq d}(H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y))\right)\right|^2 \nu(dy)$$

which by Lemma 2.2.8 (i) and (iii) respectively are finite and the mappings $t \rightarrow D(t)$ and $t \rightarrow P(t)$ are bounded and measurable.

Now for each $t \geq 0$ and for all $Q_1, Q_2 \in \mathbb{R}$ by (1.12) and (2.14) it holds that

$$\begin{aligned}
&\Bigg|\Bigg(\sum_{i=1}^{d} F_{ii}(t) + \sum_{k=1}^{m}\sum_{1\leq i<j\leq d}\left(C_{ii}^k(t)C_{jj}^k(t) - C_{ij}^k(t)C_{ji}^k(t)\right) \\
&\quad + \int_{|y|<c}\Big(\sum_{1\leq i<j\leq d}(H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y))\Big)\nu(dy)\Bigg)(Q_2 - Q_1)\Bigg|^2 \\
= &\Bigg|\sum_{i=1}^{d} F_{ii}(t) + \sum_{k=1}^{m}\sum_{1\leq i<j\leq d}\left(C_{ii}^k(t)C_{jj}^k(t) - C_{ij}^k(t)C_{ji}^k(t)\right) \\
&+ \int_{|y|<c}\left(\sum_{1\leq i<j\leq d}(H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y))\right)\nu(dy)\Bigg|^2 |Q_2 - Q_1|^2 \\
\leq\ & 3\Bigg[\left|\sum_{i=1}^{d} F_{ii}(t)\right|^2 + \left|\sum_{k=1}^{m}\sum_{1\leq i<j\leq d}\left(C_{ii}^k(t)C_{jj}^k(t) - C_{ij}^k(t)C_{ji}^k(t)\right)\right|^2 \\
&+ \left|\int_{|y|<c}\left(\sum_{1\leq i<j\leq d}(H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y))\right)\nu(dy)\right|^2\Bigg]|Q_2 - Q_1|^2 \\
\leq\ & 3\left[d\sum_{i=1}^{d}|F_{ii}(t)|^2 + \left|\sum_{k=1}^{m}\sum_{1\leq i<j\leq d}\left(C_{ii}^k(t)C_{jj}^k(t) - C_{ij}^k(t)C_{ji}^k(t)\right)\right|^2 + D(t)^2\right]|Q_2 - Q_1|^2 \\
\leq\ & 3\left[d^2\max_{1\leq i\leq d}\sup_{t\geq 0}|F_{ii}(t)|^2 + d^4 m\sum_{k=1}^{m}\left|\max_{1\leq i,j\leq d}\sup_{t\geq 0}\left|C_{ij}^k(t)\right|^2\right|^2 + D(t)^2\right]|Q_2 - Q_1|^2 \\
\leq\ & 9\max\{L(t), m\sum_{k=1}^{m}M^k(t)^2, D(t)^2\}\,|Q_2 - Q_1|^2,
\end{aligned}$$

where $L(t), M^k(t)$ and $D(t)$ are defined as above. Hence, the drift coefficient in (2.7) is Lipschitz continuous with respect to Q.

For the diffusion coefficient for each $t \geq 0$ and all $Q_1, Q_2 \in \mathbb{R}$ by (1.12) we find that

$$\begin{aligned}
\left|\sum_{k=1}^{m}\sum_{i=1}^{d}C_{ii}^k(t)Q_2 - \sum_{k=1}^{m}\sum_{i=1}^{d}C_{ii}^k(t)Q_1\right|^2 &= \left|\sum_{k=1}^{m}\sum_{i=1}^{d}C_{ii}^k(t)\right|^2 |Q_2 - Q_1|^2 \\
&\leq dm\sum_{k=1}^{m}\sum_{i=1}^{d}\left|C_{ii}^k(t)\right|^2 |Q_2 - Q_1|^2 \\
&\leq d^2 m\sum_{k=1}^{m}\max_{1\leq i\leq d}\sup_{t\geq 0}\left|C_{ii}^k(t)\right|^2 |Q_2 - Q_1|^2 \\
&\leq m\sum_{k=1}^{m}M^k(t)\,|Q_2 - Q_1|^2.
\end{aligned}$$

The jump coefficient satisfies the Lipschitz condition since for each $t \geq 0$ and all $Q_1, Q_2 \in \mathbb{R}$ we have that

$$\begin{aligned}
&\int_{|y|<c} \left| \left(\sum_{i=1}^{d} H_{ii}(t,y) + \sum_{1 \leq i < j \leq d} (H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y)) \right) (Q_2 - Q_1) \right|^2 \nu(dy) \\
&= \int_{|y|<c} \left| \left(\sum_{i=1}^{d} H_{ii}(t,y) + \sum_{1 \leq i < j \leq d} (H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y)) \right) \right|^2 \nu(dy) \, |Q_2 - Q_1|^2 \\
&= P(t) \, |Q_2 - Q_1|^2 ,
\end{aligned}$$

with $P(t)$ defined as above.

As was mentioned previously the growth conditions for time-independent drift and diffusion coefficients are a consequence of the Lipschitz conditions (see Applebaum [1] pp. 304). The same will apply for time-dependent coefficients. For the "small" jump coefficient for each $t \geq 0$ and all $Q \in \mathbb{R}$

$$\begin{aligned}
&\int_{|y|<c} \left| \left(\sum_{i=1}^{d} H_{ii}(t,y) + \sum_{1 \leq i < j \leq d} (H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y)) \right) Q \right|^2 \nu(dy) \\
&= \int_{|y|<c} \left| \left(\sum_{i=1}^{d} H_{ii}(t,y) + \sum_{1 \leq i < j \leq d} (H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y)) \right) \right|^2 \nu(dy) \, |Q|^2 \\
&= P(t) \, |Q|^2 \leq P(t) \left(1 + |Q|^2\right).
\end{aligned}$$

Now for all $t \geq 0$, the mapping

$$Q \to \left(\sum_{i=1}^{d} K_{ii}(t,y) + \sum_{1 \leq i < j \leq d} (K_{ii}(t,y)K_{jj}(t,y) - K_{ij}(t,y)K_{ji}(t,y)) \right) Q \qquad (2.15)$$

is continuous for all $|y| \geq c$ and $Q \in \mathbb{R}$, as it is a linear mapping.

Hence, by Theorem 1.4.1 (ii), (2.7) has a unique solution. □

Now let $W(t)$ be the determinant of the matrix $\Phi(t)$ for each $t \geq 0$ and note that $W(0) = 1$.

Assumption 2.2.10 For all $t \geq 0$,

$$\inf_{y \in \hat{B}_c} \operatorname{tr}(H(t,y)) + \frac{1}{2}(\operatorname{tr}(H(t,y)))^2 - \frac{1}{2}\operatorname{tr}\left(H(t,y)^2\right) > -1$$

$$\inf_{y \in \mathbb{R}^d \setminus B_c} \operatorname{tr}(K(t,y)) + \frac{1}{2}(\operatorname{tr}(K(t,y)))^2 - \frac{1}{2}\operatorname{tr}\left(K(t,y)^2\right) > -1$$

We require that assumption 2.2.10 holds for the rest of this section.

Remark 2.2.11 The conditions in assumption 2.2.10 may appear rather strange. However, we can simplify the form of the left hand sides if we express them in terms of eigenvalues. Suppose that $T \in \mathcal{M}_d(\mathbb{R})$ has eigenvalues $\lambda_1, \ldots, \lambda_d$. Then

$$\begin{aligned}
\operatorname{tr}(T) + \frac{1}{2}(\operatorname{tr}(T))^2 - \frac{1}{2}\operatorname{tr}(T^2) &= \sum_{i=1}^{d} \lambda_i + \frac{1}{2}\left(\sum_{i=1}^{d} \lambda_i\right)^2 - \frac{1}{2}\sum_{i=1}^{d} \lambda_i{}^2 \\
&= \sum_{i=1}^{d} \lambda_i + \sum_{1 \leq i < j \leq d} \lambda_i \lambda_j .
\end{aligned}$$

Thanks to the previous results now we are able to prove the following important theorem.

Theorem 2.2.12 *For each $t \geq 0$, the matrix $\Phi(t)$ is invertible almost surely and its determinant has the following form*

$$\begin{aligned}
W(t) &= \exp\left[\int_0^t \left(tr(F(s)) - \frac{1}{2}\sum_{k=1}^{m} tr\left(C_k(s)^2\right)\right) ds + \sum_{k=1}^{m} tr(C_k(s))\, dB_k(s)\right. \\
&+ \int_0^t \int_{|y| \geq c} \log\left(tr(K(s,y)) + \frac{1}{2}(tr(K(s,y)))^2 - \frac{1}{2}tr\left(K(s,y)^2\right) + 1\right) N(ds,dy) \\
&+ \int_0^t \int_{|y| < c} \log\left(tr(H(s,y)) + \frac{1}{2}(tr(H(s,y)))^2 - \frac{1}{2}tr\left(H(s,y)^2\right) + 1\right) \tilde{N}(ds,dy) \\
&+ \int_0^t \int_{|y| < c} \left(\log\left(tr(H(s,y)) + \frac{1}{2}(tr(H(s,y)))^2 - \frac{1}{2}tr\left(H(s,y)^2\right) + 1\right)\right. \\
&\qquad \left.\left. - \, tr(H(s,y))\right)\nu(dy)ds\right]
\end{aligned}$$

almost surely.

Proof: Following Mao [33] pp. 94, by Itô's product formula (see Theorem 1.3.3) we obtain

$$dW(t) = \sum_{i=1}^{d} dc_i(t) + \sum_{1\le i<j\le d} dh_{ij}(t) \tag{2.16}$$

where $dc_i(t) = \begin{vmatrix} \Phi_{11}(t) & \dots & \Phi_{1d}(t) \\ \vdots & & \vdots \\ d\Phi_{i1}(t) & \cdots & d\Phi_{id}(t) \\ \vdots & & \vdots \\ \Phi_{d1}(t) & \cdots & \Phi_{dd}(t) \end{vmatrix}$ and $dh_{ij}(t) = \begin{vmatrix} \Phi_{11}(t) & \dots & \Phi_{1d}(t) \\ \vdots & & \vdots \\ d\Phi_{i1}(t) & \dots & d\Phi_{id}(t) \\ \vdots & & \vdots \\ d\Phi_{j1}(t) & \cdots & d\Phi_{jd}(t) \\ \vdots & & \vdots \\ \Phi_{d1}(t) & \cdots & \Phi_{dd}(t) \end{vmatrix}$.

Hence, using (2.4) for $1 \le i \le d$

$$\begin{aligned} dc_i(t) &= F_{ii}(t)W(t-)dt + \sum_{k=1}^{m} C_{ii}^k(t)W(t-)dB_k(t) + \int_{|y|<c} H_{ii}(t,y)W(t-)\tilde{N}(dt,dy) \\ &\quad + \int_{|y|\ge c} K_{ii}(t,y)W(t-)N(dt,dy) \end{aligned}$$

and for $1 \le i < j \le d$

$$\begin{aligned} dh_{ij}(t) &= \sum_{k=1}^{m} \left(C_{ii}^k(t)C_{jj}^k(t) - C_{ij}^k(t)C_{ji}^k(t) \right) W(t-)dt \\ &\quad + \int_{|y|\ge c} \left(K_{ii}(t,y)K_{jj}(t,y) - K_{ij}(t,y)K_{ji}(t,y) \right) W(t-)N(dt,dy) \\ &\quad + \int_{|y|<c} \left(H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y) \right) W(t-)\tilde{N}(dt,dy) \\ &\quad + \int_{|y|<c} \left(H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y) \right) W(t-)\nu(dy)dt. \end{aligned}$$

Then (2.16) becomes

$$
\begin{aligned}
dW(t) &= \sum_{k=1}^{m}\sum_{i=1}^{d} C_{ii}^{k}(t)W(t-)dB_k(t) \\
&+ \left[\sum_{i=1}^{d} F_{ii}(t) + \sum_{k=1}^{m}\sum_{1\le i<j\le d}\left(C_{ii}^{k}(t)C_{jj}^{k}(t) - C_{ij}^{k}(t)C_{ji}^{k}(t)\right)\right] W(t-)dt \\
&+ \int_{|y|\ge c}\left(\sum_{i=1}^{d} K_{ii}(t,y) + \sum_{1\le i<j\le d}(K_{ii}(t,y)K_{jj}(t,y) - K_{ij}(t,y)K_{ji}(t,y))\right) W(t-)N(dt,dy) \\
&+ \int_{|y|<c}\left(\sum_{i=1}^{d} H_{ii}(t,y) + \sum_{1\le i<j\le d}(H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y))\right) W(t-)\tilde{N}(dt,dy) \\
&+ \int_{|y|<c}\left(\sum_{1\le i<j\le d}(H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y))\right) W(t-)\nu(dy)dt. \qquad (2.17)
\end{aligned}
$$

Hence, $W(t)$ for each $t \ge 0$ satisfies (2.7) and by Theorem 2.2.9 it follows that it is the unique solution to this SDE.

Using (2.12) then the "small" jump coefficient in (2.17) for $t \ge 0$ and $|y| < c$ becomes

$$
\begin{aligned}
&\sum_{i=1}^{d} H_{ii}(t,y) + \sum_{1\le i<j\le d}(H_{ii}(t,y)H_{jj}(t,y) - H_{ij}(t,y)H_{ji}(t,y)) \\
&= \sum_{i=1}^{d} H_{ii}(t,y) + \frac{1}{2}\left(\sum_{i=1}^{d} H_{ii}(t,y)\right)^2 - \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} H_{ij}(t,y)H_{ji}(t,y) \\
&= \operatorname{tr}(H(t,y)) + \frac{1}{2}(\operatorname{tr}(H(t,y)))^2 - \frac{1}{2}\operatorname{tr}\left(H(t,y)^2\right).
\end{aligned}
$$

The same reasoning applies for the "large" jump coefficient.

Now under assumption 2.2.10 and (2.14) we can see that (2.17) is of the same form as (2.6) and by (2.8) and (2.9) assumption 2.2.4 is satisfied. Hence, due to Lemma 2.2.5

we have that

$$
\begin{aligned}
W(t) &= \exp\Bigg[\int_0^t \left(\sum_{i=1}^d F_{ii}(s) + \sum_{k=1}^m \sum_{1\le i<j\le d} \left(C_{ii}^k(s)C_{jj}^k(s) - C_{ij}^k(s)C_{ji}^k(s)\right)\right) ds \\
&\quad -\frac{1}{2}\sum_{k=1}^m \int_0^t \left(\sum_{i=1}^d C_{ii}^k(s)\right)^2 ds + \sum_{k=1}^m \int_0^t \sum_{i=1}^d C_{ii}^k(s) dB_k(s) \\
&\quad + \int_0^t \int_{|y|\ge c} \log\left(\mathrm{tr}\,(K(s,y)) + \frac{1}{2}(\mathrm{tr}\,(K(s,y)))^2 - \frac{1}{2}\mathrm{tr}\left(K(s,y)^2\right) + 1\right) N(ds,dy) \\
&\quad + \int_0^t \int_{|y|<c} \log\left(\mathrm{tr}\,(H(s,y)) + \frac{1}{2}(\mathrm{tr}\,(H(s,y)))^2 - \frac{1}{2}\mathrm{tr}\left(H(s,y)^2\right) + 1\right) \tilde{N}(ds,dy) \\
&\quad + \int_0^t \int_{|y|<c} \Bigg(\log\left(\mathrm{tr}\,(H(s,y)) + \frac{1}{2}(\mathrm{tr}\,(H(s,y)))^2 - \frac{1}{2}\mathrm{tr}\left(H(s,y)^2\right) + 1\right) \\
&\qquad\qquad - \mathrm{tr}\,(H(s,y))\Bigg)\nu(dy)ds\Bigg],
\end{aligned}
$$

and after some algebra this yields that

$$
\begin{aligned}
W(t) &= \exp\Bigg[\int_0^t \left(\mathrm{tr}\,(F(s)) - \frac{1}{2}\sum_{k=1}^m \mathrm{tr}\left(C_k(s)^2\right)\right) ds + \sum_{k=1}^m \mathrm{tr}\,(C_k(s))\, dB_k(s) \\
&\quad + \int_0^t \int_{|y|\ge c} \log\left(\mathrm{tr}\,(K(s,y)) + \frac{1}{2}(\mathrm{tr}\,(K(s,y)))^2 - \frac{1}{2}\mathrm{tr}\left(K(s,y)^2\right) + 1\right) N(ds,dy) \\
&\quad + \int_0^t \int_{|y|<c} \log\left(\mathrm{tr}\,(H(s,y)) + \frac{1}{2}(\mathrm{tr}\,(H(s,y)))^2 - \frac{1}{2}\mathrm{tr}\left(H(s,y)^2\right) + 1\right) \tilde{N}(ds,dy) \\
&\quad + \int_0^t \int_{|y|<c} \Bigg(\log\left(\mathrm{tr}\,(H(s,y)) + \frac{1}{2}(\mathrm{tr}\,(H(s,y)))^2 - \frac{1}{2}\mathrm{tr}\left(H(s,y)^2\right) + 1\right) \\
&\qquad\qquad - \mathrm{tr}\,(H(s,y))\Bigg)\nu(dy)ds\Bigg].
\end{aligned}
$$

This implies that $W(t) > 0$ (a.s.) and therefore the matrix $\Phi(t)$ is (a.s.) invertible for each $t \ge 0$. □

Now we are ready to prove the variation of constants formula for SDEs driven by a Lévy process.

Theorem 2.2.13 *(Variation of constants formula)*
Assume that for each $t \ge 0$ $I + H(t,y)$, $I + K(t,y)$ are invertible for all $y \in \hat{B}_c$ and $y \in \mathbb{R}^d \setminus B_c$ respectively and that $\int_{|y|<c} H(t,y)\,(I + H(t,y))^{-1}\, h(t,y)\nu(dy) < \infty$. Then

the unique solution of (2.1) is the following

$$
\begin{aligned}
z(t) &= \Phi(t)\Bigg[z_0 + \int_0^t \Phi(s-)^{-1}\Bigg(f(s) - \sum_{k=1}^m C_k(s) g_k(s) \\
&\qquad - \int_{|y|<c} H(s,y)\,(I + H(s,y))^{-1}\, h(s,y)\nu(dy) \Bigg) ds \\
&\quad + \int_0^t \sum_{k=1}^m \Phi(s-)^{-1} g_k(s) dB_k(s) + \int_0^t \int_{|y|<c} \Phi(s-)^{-1}\,(I + H(s,y))^{-1}\, h(s,y)\tilde{N}(ds,dy) \\
&\quad + \int_0^t \int_{|y|\geq c} \Phi(s-)^{-1}\,(I + K(s,y))^{-1}\, r(s,y) N(ds,dy) \Bigg]
\end{aligned}
$$

where for each $t \geq 0$, $\Phi(t)$ is defined as in (2.5).

Proof: We follow the arguments given by Mao [33] pp. 96 for the Brownian motion case. Define for each $t \geq 0$

$$
\begin{aligned}
d\xi(t) &= \Phi(t-)^{-1}\Bigg(f(t) - \sum_{k=1}^m C_k(t) g_k(t) - \int_{|y|<c} H(t,y)\,(I + H(t,y))^{-1}\, h(t,y)\nu(dy) \Bigg) dt \\
&\quad + \sum_{k=1}^m \Phi(t-)^{-1} g_k(t) dB_k(t) + \int_{|y|<c} \Phi(t-)^{-1}\,(I + H(t,y))^{-1}\, h(t,y)\tilde{N}(dt,dy) \\
&\quad + \int_{|y|\geq c} \Phi(t-)^{-1}\,(I + K(t,y))^{-1}\, r(t,y) N(dt,dy) \qquad (2.18)
\end{aligned}
$$

and further define $v(t) = \Phi(t)\xi(t)$ with initial condition $v(0) = z_0$. Applying Itô's product formula (see Theorem 1.3.3), then

$$
dv(t) = d\Phi(t)\xi(t-) + \Phi(t-)d\xi(t) + d[\Phi, \xi](t).
$$

Hence by (2.5) and (2.18)

$$
\begin{aligned}
dv(t) &= F(t)\Phi(t-)\xi(t-)dt + \sum_{k=1}^{m} C_k(t)\Phi(t-)\xi(t-)dB_k(t) + \int_{|y|<c} H(t,y)\Phi(t-)\xi(t-)\tilde{N}(dt,dy) \\
&+ \int_{|y|\geq c} K(t,y)\Phi(t-)\xi(t-)N(dt,dy) \\
&+ \Phi(t-)\Bigg[\Phi(t-)^{-1}\left(f(t) - \sum_{k=1}^{m} C_k(t)g_k(t) - \int_{|y|<c} H(t,y)\left(I+H(t,y)\right)^{-1} h(t,y)\nu(dy)\right) dt \\
&\qquad + \sum_{k=1}^{m} \Phi(t-)^{-1} g_k(t)dB_k(t) + \int_{|y|<c} \Phi(t-)^{-1}\left(I+H(t,y)\right)^{-1} h(t,y)\tilde{N}(dt,dy) \\
&\qquad + \int_{|y|\geq c} \Phi(t-)^{-1}\left(I+K(t,y)\right)^{-1} r(t,y)N(dt,dy)\Bigg] \\
&+ \Bigg[\sum_{k=1}^{m} C_k(t)g_k(t)dt + \int_{|y|<c} H(t,y)\left(I+H(t,y)\right)^{-1} h(t,y)N(dt,dy) \\
&\qquad + \int_{|y|\geq c} K(t,y)\left(I+K(t,y)\right)^{-1} r(t,y)N(dt,dy)\Bigg] \\
&= \left(F(t)v(t-) + f(t)\right) dt + \sum_{k=1}^{m} \left(C_k(t)v(t-) + g_k(t)\right) dB_k(t) \\
&+ \int_{|y|<c} H(t,y)v(t-)\tilde{N}(dt,dy) + \int_{|y|<c} \left(I+H(t,y)\right)\left(I+H(t,y)\right)^{-1} h(t,y)\tilde{N}(dt,dy) \\
&+ \int_{|y|\geq c} K(t,y)v(t-)N(dt,dy) + \int_{|y|\geq c} \left(I+K(t,y)\right)\left(I+K(t,y)\right)^{-1} r(t,y)N(dt,dy) \\
&= \left(F(t)v(t-) + f(t)\right) dt + \sum_{k=1}^{m} \left(C_k(t)v(t-) + g_k(t)\right) dB_k(t) \\
&+ \int_{|y|<c} \left(H(t,y)v(t-) + h(t,y)\right)\tilde{N}(dt,dy) + \int_{|y|\geq c} \left(K(t,y)v(t-) + r(t,y)\right) N(dt,dy).
\end{aligned}
$$

Then v is a solution to (2.1) with initial condition $v(0) = z_0$ and by the existence and uniqueness theorem (see Theorem 1.4.1 (ii)) it is the unique one. The required result follows. □

2.3 Special case: Variation of constants formula for an SDE driven by a Poisson process

In this section we will establish the variation of constants formula, for a special case of (2.1) which is an SDE driven by a Poisson process. This will be useful for us later on in Chapter 5.

Let $(N(t), t \geq 0)$ be a Poisson process with intensity $\lambda > 0$. We can trivially associate a Poisson random measure to this Poisson process if we define for each $t \geq 0$, $A \in \mathcal{B}(\mathbb{R}^d)$ $N(t, A) = N(t)\delta_1(A)$ where $\delta_1(A)$ is a Dirac mass concentrated at 1. Then the Lévy measure is $E(N(1, A)) = E(N(1)\delta_1(A)) = \lambda\delta_1(A)$.

Now set in (2.1) $C_k = g_k = 0$ for each $1 \leq k \leq m$, $H = h = 0$. Then (2.1) becomes

$$dz(t) = (F(t)z(t-) + f(t))\, dt + (K(t)z(t-) + r(t))\, dN(t) \quad \text{for} \quad t \geq 0 \tag{2.19}$$

with initial condition $z(0) = z_0 \in \mathbb{R}^d$.
Hence, we get the variation of constants formula for (2.19) from the results of section 2.2.

Theorem 2.3.1 *(Variation of constants formula)*
Assume that $I + K(t)$ is invertible for all $t \geq 0$. Then the unique solution of (2.19) is

$$z(t) = \Phi(t) \left(z_0 + \int_0^t \Phi(s)^{-1} f(s)ds + \int_0^t \Phi(s)^{-1} (I + K(s))^{-1} r(s)dN(s) \right)$$

where Φ is defined as in (2.5).

Now for the purposes of work to be done in Chapter 5, suppose that in (2.19) $z = (z(t), t \geq 0)$ is a one-dimensional process and $F(\cdot)$,$K(\cdot)$, $f(\cdot)$ and $r(\cdot)$ are real Borel measurable functions. Combining the results of Theorem 2.2.3 and Lemma 2.2.5 for an SDE driven by a Poisson process then we have an explicit expression for Φ which is given, for each $t \geq 0$, by

$$\Phi(t) \quad = \quad \exp\left[\int_0^t F(s)ds + \int_0^t \log(K(s) + 1)dN(s) \right]. \tag{2.20}$$

Chapter 3

Stability of SDEs

3.1 Introduction

In this chapter we will examine SDEs of the form

$$dx(t) = f(x(t-))dt + g(x(t-))dB(t) + \int_{|y|<c} H(x(t-), y)\tilde{N}(dt, dy) \qquad \text{on } t \geq t_0 \quad (3.1)$$

with initial value $x(t_0) = x_0$, such that $x_0 \in \mathbb{R}^d$, where $c \in (0, \infty]$ and f, g, H and $B = (B(t), t \geq 0)$ and $\tilde{N}$ are defined as in Chapter 1, section 1.3 and section 1.4.
In stochastic integral form this equation can be written as

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s-))ds + \int_{t_0}^{t} g(x(s-))dB(s) + \int_{t_0}^{t}\int_{|y|<c} H(x(s-), y)\tilde{N}(ds, dy). \tag{3.2}$$

We will focus in finding L^p estimates for the solution of (3.1) and examining sufficient criteria for stability of the solution in various senses e.g. almost sure exponential stability, moment stability and stability in probability (in terms of Lyapunov functions).

3.2 L^p Estimates

In this section we are interested in finding L^p estimates (for $p \geq 2$) for the solution of a stochastic differential equation that is defined as in (3.1).

In the following we will present Kunita's estimates for $p \geq 2$ (see Kunita [25]) for the solution of an SDE of the form (3.1), which will be very useful throughout this thesis.

Kunita's estimates are an extension of the well-known Burkholder's inequality, which gives estimates for moments of stochastic integrals driven by Brownian motions (see Applebaum [1] pp. 234 and second edition pp. 387).

Theorem 3.2.1 *(Kunita) For all $p \geq 2$, there exists $C_p > 0$ such that for each $t > t_0 \geq 0$,*

$$\begin{aligned} E\left[\sup_{t_0 \leq s \leq t} |x(s)|^p\right] \leq\; & C_p\Bigg\{|x_0|^p + E\left[\int_{t_0}^{t} |f(x(r-))|^p dr\right] + E\left[\int_{t_0}^{t} \|g(x(r-))\|^p dr\right] \\ & + E\left[\int_{t_0}^{t}\left(\int_{|y|<c} |H(x(r-),y)|^2 \nu(dy)\right)^{\frac{p}{2}} dr\right] \\ & + E\left[\int_{t_0}^{t}\int_{|y|<c} |H(x(r-),y)|^p \nu(dy) dr\right]\Bigg\} \end{aligned} \tag{3.3}$$

where $x(t_0) = x_0 \in \mathbb{R}^d$ is the initial condition.

Proof: See Kunita [25] pp. 332-335.

Assumption 3.2.2 For all $2 \leq q \leq p$ and $K > 0$

$$\int_{|y|<c} |H(x,y)|^q \nu(dy) \leq K|x|^q.$$

We require that assumption 3.2.2 holds for the remainder of this chapter.

The following result is also due to Kunita [25] pp. 341. Kunita does not provide a proof and so we give one below.

Corollary 3.2.3 For all $p \geq 2$ and $x_0 \in \mathbb{R}^d$, there exists $C_p'' > 0$ such that for each $t > t_0 \geq 0$,

$$E\left[\sup_{t_0 \leq s \leq t}\Big(1 + |x(s)|\Big)^p\right] \leq C_p''\left\{\Big(1 + |x_0|\Big)^p + \int_{t_0}^{t} E\left[\sup_{t_0 \leq s \leq r}\Big(1 + |x(s)|\Big)^p\right] dr\right\}. \tag{3.4}$$

Proof: Applying Jensen's inequality in (3.2) and then taking expectations, we obtain

$$\begin{aligned} E\Big[(1 + |x(t)|)^p\Big] \leq 4^{p-1}\Bigg\{ & (1 + |x_0|)^p + E\left[\left|\int_{t_0}^{t} f(x(s-))ds\right|^p\right] + E\left[\left|\int_{t_0}^{t} g(x(s-))dB(s)\right|^p\right] \\ & + E\left[\left|\int_{t_0}^{t}\int_{|y|<c} H(x(s-),y)\tilde{N}(ds,dy)\right|^p\right]\Bigg\}. \end{aligned}$$

By the same arguments as were used to deduce (3.3), (see Kunita [25] pp. 332-335) we then get

$$
\begin{aligned}
E\Big[\sup_{t_0\le s\le t}\Big(1+|x(s)|\Big)^p\Big] \;&\le\; C_p'\Bigg\{(1+|x_0|)^p+E\left[\int_{t_0}^{t}|f(x(s-))|^p ds\right]\\
&+\; E\left[\int_{t_0}^{t}\|g(x(s-))\|^p ds\right]\\
&+\; E\left[\int_{t_0}^{t}\left(\int_{|y|<c}\Big|H(x(s-),y)\Big|^2\nu(dy)\right)^{\frac{p}{2}} ds\right]\\
&+\; E\left[\int_{t_0}^{t}\int_{|y|<c}|H(x(s-),y)|^p\,\nu(dy)ds\right]\Bigg\} \qquad (3.5)
\end{aligned}
$$

where $C_p' = 4^{p-1}C_p > 0$ and C_p is a positive constant. Applying the growth condition (C2) we deduce that

$$
\begin{aligned}
E\left[\int_{t_0}^{t}|f(x(s-))|^p ds\right] \;&=\; E\left[\int_{t_0}^{t}\left(|f(x(s-))|^2\right)^{\frac{p}{2}} ds\right]\\
&\le\; E\left[\int_{t_0}^{t}\left(K\left(1+|x(s)|^2\right)\right)^{\frac{p}{2}} ds\right]\\
&=\; K^{\frac{p}{2}}\int_{t_0}^{t}E\left[\left(1+|x(s)|^2\right)^{\frac{p}{2}}\right] ds\\
&\le\; K^{\frac{p}{2}}\int_{t_0}^{t}E\left[(1+|x(s)|)^p\right] ds, \qquad (3.6)
\end{aligned}
$$

using Fubini's theorem (see e.g. Applebaum [1] Theorem 1.1.7 pp. 12) and the elementary inequality, that for $a \ge 0$, $(1+a^2) \le (1+a)^2$.
The same arguments apply for the diffusion term and for the penultimate term in (3.5). For the last term of (3.5) we will apply assumption 3.2.2. Hence, by Fubini's theorem

$$
\begin{aligned}
E\left[\int_{t_0}^{t}\int_{|y|<c}|H(x(s-),y)|^p\nu(dy)ds\right] \;&\le\; K\int_{t_0}^{t}E\left[|x(s)|^p\right] ds\\
&\le\; K\int_{t_0}^{t}E\left[\Big(1+|x(s)|\Big)^p\right] ds, \qquad (3.7)
\end{aligned}
$$

since for $a \ge 0$, $a^p \le (1+a)^p$.
Substituting (3.6) and (3.7) into (3.5) then

$$
\begin{aligned}
E\left[\sup_{t_0\le s\le t}\Big(1+|x(s)|\Big)^p\right] \;&\le\; C_p'\left\{(1+|x_0|)^p+Z_p\int_{t_0}^{t}E\left[\Big(1+|x(s)|\Big)^p\right] ds\right\}\\
&\le\; C_p'\left\{(1+|x_0|)^p+Z_p\int_{t_0}^{t}E\left[\sup_{t_0\le s\le r}\Big(1+|x(s)|\Big)^p\right] dr\right\}. \qquad (3.8)
\end{aligned}
$$

where $Z_p = K + 3K^{\frac{p}{2}}$. Set $C_p'' = \max\{C_p', C_p' Z_p\}$ and the result follows. □

Proposition 3.2.4 Let $p \geq 2$. Then there exist constants $C_p' > 0$ and $K_p' > 0$ such that for all $x_0 \in \mathbb{R}^d$ and each $t \geq t_0$

$$(i)\quad E\left[\sup_{t_0 \leq s \leq t}\left(1 + |x(s)|\right)^p\right] \leq C_p'(1 + |x_0|)^p \exp\left\{K_p'(t - t_0)\right\},$$

$$(ii)\quad E\left[|x(t)|^p\right] \leq C_p'(1 + |x_0|)^p \exp\left\{K_p'(t - t_0)\right\}.$$

Proof: (i) The result follows on applying Gronwall's inequality (see Chapter 1, section 1.3) to (3.8) with

$$\alpha(t) = E\left[\sup_{t_0 \leq s \leq t}(1 + |x(s)|)^p\right], \quad C = C_p'(1 + |x_0|)^p \text{ and } \beta = Z_p C_p' = K_p'.$$

(ii) This follows easily from (i), since each

$$|x(t)|^p \leq \sup_{t_0 \leq s \leq t}(1 + |x(s)|)^p.$$

□

3.2.1 Exponential martingale inequality

The classical exponential martingale inequality for a continuous martingale $M = (M(t), t \geq 0)$ is

$$P\left(\sup_{0 \leq t \leq T}\left[M(t) - \frac{\alpha}{2}\langle M, M\rangle(t)\right] > \beta\right) \leq \exp(-\alpha\beta)$$

where T, α, β are positive constants and $\langle M, M\rangle$ is the quadratic variation of M, as for continuous semimartingales it holds that $\langle M, M\rangle = [M, M]$ (see Mao [33] pp. 12).

Mao in [33] applies this to the special case where $M(t) = \int_0^t g(s)dB(s)$ for each $t \geq 0$ and he has obtained the following result

$$P\left[\sup_{0 \leq t \leq T}\left\{\int_0^t g(s)\,dB(s) - \frac{\alpha}{2}\int_0^t |g(s)|^2\,ds\right\} > \beta\right] \leq \exp(-\alpha\beta)$$

where $g = (g_1, \ldots, g_m) \in P_2(T)$, $B = (B(t), t \geq 0)$ is an m-dimensional Brownian motion and T, α, β are positive constants. This result plays an important role in Mao's work on stability.

Since we are dealing with processes of the type

$$M(t) = \int_0^t g(s)dB(s) + \int_0^t \int_{|y|<c} H(s,y)\,\tilde{N}(ds,dy)$$

for each $t \geq 0$, where $H : \mathbb{R}^+ \times E \to \mathbb{R} \in \mathcal{P}_2(T,E)$ and $\tilde{N}$ is the compensated Poisson measure, we need to prove the exponential martingale inequality for this type of processes. This will be an essential tool for the work later on.

The following is a generalization of Mao [33] Chapter 1, Theorem 7.4 pp. 44.

Theorem 3.2.5 *(Exponential Martingale Inequality)*
Let T, α, β be any positive numbers. Assume that $g \in \mathcal{P}_2(T)$ and the mapping $H : \mathbb{R}^+ \times E \to \mathbb{R} \in \mathcal{P}_2(T,E)$. Then

$$P\Big[\sup_{0\leq t\leq T}\Big\{\int_0^t g(s)\,dB(s) - \frac{\alpha}{2}\int_0^t |g(s)|^2\,ds + \int_0^t \int_{|y|<c} H(s,y)\,\tilde{N}(ds,dy) \\ -\frac{1}{\alpha}\int_0^t \int_{|y|<c}\Big[\exp\big(\alpha H(s,y)\big) - 1 - \alpha H(s,y)\Big]\nu(dy)ds\Big\} > \beta\Big] \leq \exp\big(-\alpha\beta\big). \quad (3.9)$$

Proof: Define a sequence of stopping times $(\tau_n, n \geq 1)$ as follows. For each $n \geq 1$

$$\tau_n = \inf\Big\{t \geq 0 : \Big|\int_0^t g(s)\,dB(s)\Big| + \frac{\alpha}{2}\int_0^t |g(s)|^2\,ds + \Big|\int_0^t \int_{|y|<c} H(s,y)\,\tilde{N}(ds,dy)\Big| \\ +\frac{1}{\alpha}\Big|\int_0^t \int_{|y|<c}\Big[\exp\big(\alpha H(s,y)\big) - 1 - \alpha H(s,y)\Big]\,\nu(dy)ds\Big| \geq n\Big\},$$

and note that $\tau_n \uparrow \infty$ almost surely.

Define the Itô process

$$x_n(t) = \alpha\int_0^t g(s)I_{[0,\tau_n]}(s)\,dB(s) - \frac{\alpha^2}{2}\int_0^t |g(s)|^2 I_{[0,\tau_n]}(s)\,ds \\ + \int_0^t \int_{|y|<c} \alpha H(s,y)I_{[0,\tau_n]}(s)\,\tilde{N}(ds,dy) \\ - \int_0^t \int_{|y|<c}\Big[\exp\big(\alpha H(s,y)\big) - 1 - \alpha H(s,y)\Big]I_{[0,\tau_n]}(s)\,\nu(dy)ds,$$

for each $t \geq 0$. Then for all $0 \leq t \leq T$

$$\begin{aligned}
|x_n(t)| &\leq \alpha \int_0^t |g(s)| I_{[0,\tau_n]}(s)\, dB(s) + \frac{\alpha^2}{2} \int_0^t |g(s)|^2 I_{[0,\tau_n]}(s)\, ds \\
&\quad + \alpha \left| \int_0^t \int_{|y|<c} H(s,y) I_{[0,\tau_n]}(s)\, \tilde{N}(ds,dy) \right| \\
&\quad + \left| \int_0^t \int_{|y|<c} \Big[\exp\big(\alpha H(s,y)\big) - 1 - \alpha H(s,y) \Big] I_{[0,\tau_n]}(s) \nu(dy) ds \right| \leq \alpha n
\end{aligned}$$

which means that the mapping $t \to x_n(t)$ is bounded from $[0,T] \to \mathbb{R}$ (a.s.).

Let $Z(t) = \exp x_n(t)$. We apply Itô's formula with jumps (see Chapter 1, section 1.3), to obtain

$$\begin{aligned}
&\exp x_n(t) \\
&= 1 + \alpha \int_0^t \exp x_n(s) g(s) I_{[0,\tau_n]}(s)\, dB(s) - \frac{\alpha^2}{2} \int_0^t \exp x_n(s) |g(s)|^2 I_{[0,\tau_n]}(s)\, ds \\
&- \int_0^t \int_{|y|<c} \Big[\exp x_n(s) \left(\exp\big(\alpha H(s,y)\big) - 1 - \alpha H(s,y) \right) I_{[0,\tau_n]}(s) \Big] \nu(dy) ds \\
&+ \frac{\alpha^2}{2} \int_0^t \exp x_n(s) |g(s)|^2 I_{[0,\tau_n]}(s) ds \\
&+ \int_0^t \int_{|y|<c} \exp x_n(s) \big[\exp\big(\alpha H(s,y)\big) - 1 \big] I_{[0,\tau_n]}(s)\, \tilde{N}(ds,dy) \\
&+ \int_0^t \int_{|y|<c} \Big[\exp x_n(s) \big(\exp\big(\alpha H(s,y)\big) - 1 \big) - \alpha H(s,y) \exp x_n(s) \Big] I_{[0,\tau_n]}(s) \nu(dy) ds,
\end{aligned}$$

and it follows immediately that

$$\begin{aligned}
\exp x_n(t) &= 1 + \alpha \int_0^t \exp x_n(s) g(s) I_{[0,\tau_n]}\, dB(s) \\
&\quad + \int_0^t \int_{|y|<c} \exp x_n(s) \big[\exp\big(\alpha H(s,y)\big) - 1 \big] I_{[0,\tau_n]}(s)\, \tilde{N}(ds,dy).
\end{aligned}$$

Now each process $(\exp x_n(t), 0 \leq t \leq T)$ is a local martingale (see e.g. Applebaum [1] Chapter 5, Corollary 5.2.2 pp. 253).
Since we also have

$$\sup_{t \in [0,T]} \exp x_n(t) \leq \exp \alpha n, \qquad a.s.$$

then there exists a sequence of stopping times $(T_m, m \in \mathbb{N})$ with $(T_m \to \infty)$ (a.s.) as

$m \to \infty$ such that for all $0 \le s \le t \le T$

$$E\left[\exp(x_n(t \wedge T_m))|\mathcal{F}_s\right] = \exp(x_n(s \wedge T_m)) \le \exp \alpha n \qquad a.s.$$

An application of the dominated convergence theorem (conditional version) gives

$$\begin{aligned} E[\exp x_n(t)|\mathcal{F}_s] &= \lim_{m\to\infty} E[\exp\left(x_n(t \wedge T_m)\right)|\mathcal{F}_s] \\ &= \lim_{m\to\infty} \exp\left(x_n(s \wedge T_m)\right) \\ &= \exp x_n(s), \end{aligned}$$

i.e. $Z(t) = \exp x_n(t)$ is a martingale for all $0 \le t \le T$. By using Applebaum [1] Chapter 5, Theorem 5.2.4 pp. 254, it follows that $E[\exp x_n(t)] = 1$ for all $0 \le t \le T$.

Now, applying Doob's martingale inequality (see Chapter 1, section 1.3), we obtain

$$P\left[\sup_{0\le t\le T} \exp\left(x_n(t)\right) \ge \exp\left(\alpha\beta\right)\right] \le \exp(-\alpha\beta) E\left[\exp\left(x_n(T)\right)\right] = \exp(-\alpha\beta).$$

Hence,

$$\begin{aligned} &P\Bigg[\sup_{0\le t\le T}\Bigg\{\int_0^t g(s) I_{[0,\tau_n]} s\, dB(s) - \frac{\alpha}{2}\int_0^t |g(s)|^2 I_{[0,\tau_n]}(s)\, ds \\ &+ \int_0^t \int_{|y|<c} H(s,y) I_{[0,\tau_n]}(s)\, \tilde{N}(ds,dy) \\ &-\frac{1}{\alpha}\int_0^t \int_{|y|<c} \left[\exp\left(\alpha H(s,y)\right) - 1 - \alpha H(s,y)\right] I_{[0,\tau_n]}(s)\, \nu(dy) ds\Bigg\} > \beta\Bigg] \le \exp\left(-\alpha\beta\right). \end{aligned} \tag{3.10}$$

Define

$$\begin{aligned} A_n = \Bigg\{\omega \in \Omega : \sup_{0\le t\le T} \Bigg[&\int_0^t g(s) I_{[0,\tau_n]}(s)\, dB(s) - \frac{\alpha}{2}\int_0^t |g(s)|^2 I_{[0,\tau_n]}(s)\, ds \\ &+ \int_0^t \int_{|y|<c} H(s,y) I_{[0,\tau_n]}(s)\, \tilde{N}(ds,dy) \\ &-\frac{1}{\alpha}\int_0^t \int_{|y|<c} \left[\exp\left(\alpha H(s,y)\right) - 1 - \alpha H(s,y)\right] I_{[0,\tau_n]}(s)\, \nu(dy) ds\Bigg] > \beta\Bigg\}. \end{aligned}$$

It is well known that (see e.g. Applebaum [1] pp. 112)

$$P\left[\liminf_{n\to\infty} A_n\right] \le \liminf_{n\to\infty} P\left[A_n\right] \le \limsup_{n\to\infty} P\left[A_n\right] \le P\left[\limsup_{n\to\infty} A_n\right]. \tag{3.11}$$

By (3.10)

$$\limsup_{n\to\infty} P[A_n] \le \exp(-\alpha\beta). \tag{3.12}$$

Since,

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\left(\bigcap_{m=n}^{\infty} A_m\right) \qquad \text{and} \qquad \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\left(\bigcup_{m=n}^{\infty} A_m\right)$$

then

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = A \tag{3.13}$$

where

$$A = \Big\{\omega \in \Omega : \sup_{0\le t\le T}\Big[\int_0^t g(s)dB(s) - \frac{\alpha}{2}\int_0^t |g(s)|^2 ds + \int_0^t\int_{|y|<c} H(s,y)\tilde{N}(ds,dy)$$
$$-\frac{1}{\alpha}\int_0^t\int_{|y|<c}\Big[\exp\big(\alpha H(s,y)\big) - 1 - \alpha H(s,y)\Big]\nu(dy)ds\Big] > \beta\Big\}.$$

Combining (3.11), (3.12) and (3.13) then $P[A] \le \exp(-\alpha\beta)$ as required. □

In the sequel we will apply the exponential martingale inequality (3.9) for a special case. This is done in order to simplify the calculations of a theorem that will follow later.

Assume that $H : \mathbb{R}^+ \times E \to \mathbb{R}$ and set $F(t,y) = e^{H(t,y)} - 1$.
Then, $H(t,y) = \log(1 + F(t,y))$ where $0 \le t \le T$, $y \in \mathbb{R}^d$.
Hence, for $\alpha > 0$

$$\begin{aligned}\exp\big(\alpha H(t,y)\big) - 1 - \alpha H(t,y) &= \exp\big(\alpha\log(1+F(t,y))\big) - 1 - \alpha\log\big(1+F(t,y)\big)\\ &= \big(1+F(t,y)\big)^{\alpha} - 1 - \log\big(1+F(t,y)\big)^{\alpha}.\end{aligned}$$

As a result the exponential martingale inequality (3.9) now takes the form,

$$P\Big[\sup_{0\le t\le T}\Big\{\int_0^t g(s)\,dB(s) - \frac{\alpha}{2}\int_0^t |g(s)|^2\,ds + \int_0^t\int_{|y|<c}\log(1+F(s,y))\,\tilde{N}(ds,dy)$$
$$-\frac{1}{\alpha}\int_0^t\int_{|y|<c}\Big[\big(1+F(s,y)\big)^{\alpha} - 1 - \log\big(1+F(s,y)\big)^{\alpha}\Big]\nu(dy)ds\Big\} > \beta\Big] \le \exp(-\alpha\beta).$$
(3.14)

In particular for $\alpha = 1$,

$$P\left[\sup_{0\le t\le T}\left\{\int_0^t g(s)\,dB(s) - \frac{1}{2}\int_0^t |g(s)|^2\,ds + \int_0^t\int_{|y|<c} \log(1+F(s,y))\,\tilde{N}(ds,dy) + \int_0^t\int_{|y|<c}\Big[\log\left(1+F(s,y)\right) - F(s,y)\Big]\nu(dy)ds\right\} > \beta\right] \le \exp\left(-\beta\right). \tag{3.15}$$

3.3 Stability in probability

The aim of this section is to study stability in probability.

The theorem that follows is analogous to the Lyapunov theorem for the deterministic case (see Chapter 1, Theorem 1.5.1). Mao in [33], Chapter 4, Theorem 2.2 pp. 111 extended the well-known Lyapunov theorem for the case of SDEs driven by a Brownian motion. We imitate his proof and in a similar fashion we prove the Lyapunov theorem for SDEs driven by a Lévy process.

In the following we will require $\mathcal{L}$, the linear operator associated to the SDE for the system under consideration (3.1) when $c \in (0,\infty)$. It is a special case of a more general formula in Chapter 1, section 1.4, (1.19), as the Lévy process has bounded jumps. For the convenience of the reader we give the precise expression for $\mathcal{L}$ in this case.

$$\begin{aligned}(\mathcal{L}V)(x) &= f^i(x)(\partial_i V)(x) + \frac{1}{2}[g(x)g(x)^T]^{ik}(\partial_i\partial_k V)(x) \\ &\quad + \int_{|y|<c}\left[V(x+H(x,y)) - V(x) - H^i(x,y)(\partial_i V)(x)\right]\nu(dy)\end{aligned} \tag{3.16}$$

where $V \in C^2(\mathbb{R}^d)$, $x \in \mathbb{R}^d$.

For the next theorem fix $h > 0$ such that $h \ge 2c$, where h is the radius of a ball B_h in $\mathbb{R}^d$, and c, the maximum allowable jump size, as defined in Chapter 1, section 1.3, is finite.

Remark 3.3.1 Although the next proof goes along similar lines to Mao's proof in [33] pp. 111, there is a variation due to the fact that solutions to SDEs driven by Brownian motion have continuous paths, while the solution to (3.1) has càdlàg paths.

Theorem 3.3.2 *Assume that there exists a positive definite function $V \in C^2(B_h;\mathbb{R}^+)$ such that*

$$\mathcal{L}V(x) \le 0$$

for all $x \in B_h$. Then the trivial solution of (3.1) is stable in probability.

Proof: Since V is positive definite it holds that $V(0) = 0$ and there exist a function $\mu \in \mathcal{K}$ such that

$$V(x) \geq \mu(|x|) \text{ for all } x \in B_h \qquad \text{(see section 1.5).}$$

Let $\varepsilon \in (0,1)$ and $r > 0$. By the continuity of V, for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, r) > 0$ such that

$$\sup_{x \in B_\delta} V(x) \leq \mu(r)\varepsilon. \tag{3.17}$$

Now fix $x_0 \in B_\delta$. Without loss of generality let $0 < r < \frac{h}{2}$ and define the stopping time

$$\tau = \inf\{t \geq t_0 : |x(t)| \geq r\}.$$

Applying Itô's formula to V for any $t \geq t_0$,

$$\begin{aligned} V(x(t\wedge\tau)) &= V(x_0) + \int_{t_0}^{t\wedge\tau} \partial_i V(x(s-)) \left[f^i(x(s-))ds + g^{ij}(x(s-))dB_j(s)\right] \\ &+\frac{1}{2}\int_{t_0}^{t\wedge\tau} \partial_i\partial_k V(x(s-))[g(x(s-))g(x(s-))^T]^{ik}ds \\ &+ \int_{t_0}^{t\wedge\tau}\int_{|y|<c} \left[V\big(x(s-)+H(x(s-),y)\big) - V(x(s-))\right]\tilde{N}(ds,dy) \\ &+ \int_{t_0}^{t\wedge\tau}\int_{|y|<c} \left[V\big(x(s-)+H(x(s-),y)\big) - V(x(s-))\right. \\ &\qquad\qquad \left. - H^i(x(s-),y)\partial_i V(x(s-))\right]\nu(dy)ds. \end{aligned}$$

Hence,

$$\begin{aligned} V(x(t\wedge\tau)) &= V(x_0) + \int_{t_0}^{t\wedge\tau} \mathcal{L}V(x(s-))ds + \int_{t_0}^{t\wedge\tau} \partial_i V(x(s-))g^{ij}(x(s-))dB_j(s) \\ &+ \int_{t_0}^{t\wedge\tau}\int_{|y|<c} \left[V\big(x(s-)+H(x(s-),y)\big) - V(x(s-))\right]\tilde{N}(ds,dy). \end{aligned}$$

Using the fact that $\mathcal{L}V \leq 0$ and taking expectations, it follows that

$$E[V(x(t\wedge\tau))] \leq V(x_0) \qquad a.s.$$

Now, $|x(t\wedge\tau)| < r$ (a.s.) for $t < \tau$. So it follows that for all $\omega \in \{\tau < \infty\}$ $|x(\tau)(\omega)| \leq r + c$. Hence, $|x(\tau)(\omega)| < \frac{h}{2} + \frac{h}{2} = h$. Also since $V(x) \geq \mu(|x|)$ for all $x \in B_h$ we have that for all $\omega \in \{\tau < \infty\}$,

$$V(x(\tau)(\omega)) \geq \mu(|x(\tau)(\omega)|) \geq \mu(r) \tag{3.18}$$

as μ is increasing (see section 1.5).

From the previous relations we deduce that

$$V(x_0) \geq E\left[V(x(t \wedge \tau))\right] \geq E\left[1_{\{\tau \leq t\}} V(x(\tau))\right] \geq P(\tau \leq t)\mu(r).$$

Then, by (3.17),

$$P(\tau \leq t)\mu(r) \leq V(x_0) \leq \sup_{x \in B_\delta} V(x) \leq \mu(r)\varepsilon.$$

Hence,

$$P(\tau \leq t) \leq \varepsilon.$$

Letting $t \to \infty$, we get $P(\tau < \infty) \leq \varepsilon$ which is the required result i.e.

$$P\big(|x(t)| < r \quad \text{for all } t \geq t_0\big) \geq 1 - \varepsilon.$$

□

Remark 3.3.3 The function V that appears in Theorem 3.3.2, is the *stochastic Lyapunov function.*

3.4 Almost surely asymptotic estimates

This section includes an almost surely asymptotic estimate for the sample Lyapunov exponent and criteria for almost sure exponential stability of the solution of the stochastic differential equation driven by continuous noise interspersed by jumps. We will also provide conditions under which the solution of (3.1) (with probability 1) will never reach zero provided that the solution of the stochastic system starts from a non-zero point.

Pioneering work was carried out by Khasminski in [22] who gave a necessary and sufficient condition for almost sure exponential stability of the linear Itô equation and opened new chapters in stochastic stability theory. Most of the researchers continuing Khasminski's work have devoted their research examining almost sure exponential stability for SDEs driven by Brownian motion. However, Grigoriu in [13] studies almost sure stability for non-Gaussian noise. He investigates stability of non-linear SDEs driven by a compound Poisson process by a linearization technique applied to one-dimensional non-linear system. The author uses Lyapunov exponents to identify subsets in the space of the parameters for which the solution of the SDE is stable.

Also Grigoriu and Samorodnitsky in [14] propose two methods for determining asymptotic stability of the trivial solution of linear stochastic differential equations driven by Poisson noise. The first method extends a result obtained by Khasminski in [22] for diffusion processes to the case of SDEs driven by a Poisson process which is based on Itô's formula and the use of Lyapunov exponents. Applying this technique Grigoriu and Samorodnitsky have managed to obtain results for one-dimension SDEs but for $d > 1$ their method was unsuccessful. The second method is based on a geometric ergodic theorem for Markov chains. They consider a certain Markov chain $(x_n, n \in \mathbb{N})$ associated with the solution x of a linear SDE system driven by Poisson noise. In their paper they have shown that if $(x_n, n \in \mathbb{N})$ is ergodic so is x and they have established conditions under which $(x_n, n \in \mathbb{N})$ is ergodic or not. They acknowledge that the first method has limitations compared with the second and the second provides a general criterion for assessing the long term behavior of x_n and x. Nevertheless by using Mao's points of view (see Mao [33]) for almost sure exponential stability we have managed to use Lyapunov exponents and Itô's formula to obtain almost sure exponential stability for $\mathbb{R}^d$-valued processes determined by non-linear SDEs driven by a general Poisson random measure.

In the next theorem we will give an estimate of the sample Lyapunov exponent of the solution to (3.1). In the event that we know a unique solution exists without requiring (C1)-(C2) to hold, then we can still obtain an estimate for the sample Lyapunov exponent of the solution. But first we need to impose a condition on the coefficients of (3.1), the *monotone condition.* This is as follows.

There is a positive constant a such that for all $x \in \mathbb{R}^d$

$$x^T f(x) + \frac{1}{2}\|g(x)\|^2 + \frac{1}{2}\int_{|y|<c} |H(x,y)|^2 \nu(dy) \leq a\left(1 + |x|^2\right). \tag{3.19}$$

In the case that we have to impose (C1)-(C2), for the existence and uniqueness of a solution to (3.1), then by the growth condition (C2), (3.19) is satisfied with $a = (\sqrt{K} + K)$.

The following is a generalization of Mao's work [33] Chapter 2, Theorem 5.1 pp. 63.

Theorem 3.4.1 *Under the monotone condition (3.19), the sample Lyapunov exponent of the solution of (3.1) exists and satisfies the following*

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| \leq a \qquad a.s.$$

where a is as in (3.19).

Proof: An application of Itô's formula with jumps to $Z(t) = \log(1+|x(t)|^2)$ yields that, for each $t \geq t_0$,

$$
\begin{aligned}
&\log\left(1+|x(t)|^2\right) \\
&= \log\left(1+|x_0|^2\right) + \int_{t_0}^{t} \frac{2x(s-)^T}{1+|x(s-)|^2} f(x(s-))\,ds + \int_{t_0}^{t} \frac{2x(s-)^T}{1+|x(s-)|^2} g(x(s-))\,dB(s) \\
&+ \int_{t_0}^{t} \left(\frac{\|g(x(s-))\|^2}{1+|x(s-)|^2} - \frac{2|x(s-)^T g(x(s-))|^2}{(1+|x(s-)|^2)^2} \right) ds \\
&+ \int_{t_0}^{t}\int_{|y|<c} \left[\log\left(1+\left|x(s-)+H(x(s-),y)\right|^2\right) - \log\left(1+|x(s-)|^2\right)\right] \tilde{N}(ds,dy) \\
&+ \int_{t_0}^{t}\int_{|y|<c} \left[\log\left(1+\left|x(s-)+H(x(s-),y)\right|^2\right) - \log\left(1+|x(s-)|^2\right) \right. \\
&\qquad\qquad\qquad\qquad\qquad \left. - \frac{2x(s-)^T H(x(s-),y)}{1+|x(s-)|^2} \right] \nu(dy)ds.
\end{aligned}
$$

We can rewrite
$\log\left(1+|x(s-)+H(x(s-),y)|^2\right) - \log\left(1+|x(s-)|^2\right)$

$$
\begin{aligned}
&= \log\left(\frac{1+|x(s-)|^2 + 2x(s-)^T H(x(s-),y) + |H(x(s-),y)|^2}{1+|x(s-)|^2} \right) \\
&= \log\left(1 + \frac{2x(s-)^T H(x(s-),y) + |H(x(s-),y)|^2}{1+|x(s-)|^2} \right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\log\left(1+|x(t)|^2\right) &= \log\left(1+|x_0|^2\right) + \int_{t_0}^{t} \frac{1}{1+|x(s-)|^2} \left[2x(s-)^T f(x(s-)) + \left\|g(x(s-))\right\|^2 \right] ds \\
&\quad +2\int_{t_0}^{t} \frac{x(s-)^T g(x(s-))}{1+|x(s-)|^2} dB(s) - 2\int_{t_0}^{t} \frac{\left|x(s-)^T g(x(s-))\right|^2}{\left(1+|x(s-)|^2\right)^2} ds \\
&\quad + \int_{t_0}^{t}\int_{|y|<c} \log\left(1+\frac{2x(s-)^T H(x(s-),y) + |H(x(s-),y)|^2}{1+|x(s-)|^2}\right) \tilde{N}(ds,dy) \\
&\quad + \int_{t_0}^{t}\int_{|y|<c} \left[\log\left(1+\frac{2x(s-)^T H(x(s-),y) + |H(x(s-),y)|^2}{1+|x(s-)|^2}\right) \right. \\
&\qquad\qquad\qquad\qquad \left. - \frac{2x(s-)^T H(x(s-),y)}{1+|x(s-)|^2} \right] \nu(dy)ds. \qquad (3.20)
\end{aligned}
$$

Define

$$
\begin{aligned}
M(t) &= 2\int_{t_0}^{t} \frac{x(s-)^T g(x(s-))}{1+|x(s-)|^2} dB(s) \\
&\quad + \int_{t_0}^{t}\int_{|y|<c} \log\left(1+\frac{2x(s-)^T H(x(s-),y) + |H(x(s-),y)|^2}{1+|x(s-)|^2}\right) \tilde{N}(ds,dy).
\end{aligned}
$$

Using the exponential martingale inequality (3.15), for $\alpha = 1$, $\beta = \log n^2$ and $T = n$ where $n \in \mathbb{N}$, one sees that

$$P\Big[\sup_{t_0\le t\le n}\Big(M(t) - 2\int_{t_0}^{t}\frac{|x(s-)^T g(x(s-))|^2}{(1+|x(s-)|^2)^2}ds + \int_{t_0}^{t}\int_{|y|<c}\Big[\log\Big(1+\frac{2x(s-)^T H(x(s-),y) + |H(x(s-),y)|^2}{1+|x(s-)|^2}\Big) - \frac{2x(s-)^T H(x(s-),y) + |H(x(s-),y)|^2}{1+|x(s-)|^2}\Big]\nu(dy)ds\Big) > 2\log n\Big] \le \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$, an application of the Borel-Cantelli lemma, yields that

$$P\Big[\limsup_{n\to\infty}\Big\{\sup_{t_0\le t\le n}\Big(M(t) - 2\int_{t_0}^{t}\frac{|x(s-)^T g(x(s-))|^2}{(1+|x(s-)|^2)^2}ds + \int_{t_0}^{t}\int_{|y|<c}\Big[\log\Big(1+\frac{2x(s-)^T H(x(s-),y) + |H(x(s-),y)|^2}{1+|x(s-)|^2}\Big) - \frac{2(x(s-)^T H(x(s-),y)) + |H(x(s-),y)|^2}{1+|x(s-)|^2}\Big]\nu(dy)ds\Big) > 2\log n\Big\}\Big] = 0.$$

Using elementary probability calculations, we then have that

$$P\Big[\liminf_{n\to\infty}\Big\{\sup_{t_0\le t\le n}\Big(M(t) - 2\int_{t_0}^{t}\frac{|x(s-)^T g(x(s-))|^2}{(1+|x(s-)|^2)^2}ds + \int_{t_0}^{t}\int_{|y|<c}\Big[\log\Big(1+\frac{2x(s-)^T H(x(s-),y) + |H(x(s-),y)|^2}{1+|x(s-)|^2}\Big) - \frac{2x(s-)^T H(x(s-),y) + |H(x(s-),y)|^2}{1+|x(s-)|^2}\Big]\nu(dy)ds\Big) \le 2\log n\Big\}\Big] = 1.$$

So, for $n \ge n_0(\omega)$, $t_0 \le t \le n$,

$$\begin{aligned} M(t) \quad + \quad & \int_{t_0}^{t}\int_{|y|<c}\Big\{\log\Big(1+\frac{2x(s-)^T H(x(s-),y) + |H(x(s-),y)|^2}{1+|x(s-)|^2}\Big) - \frac{2x(s-)^T H(x(s-),y)}{1+|x(s-)|^2}\Big\}\nu(dy)ds \\ & \le 2\int_{t_0}^{t}\frac{|x(s-)^T g(x(s-))|^2}{(1+|x(s-)|^2)^2}ds + \int_{t_0}^{t}\int_{|y|<c}\frac{|H(x(s-),y)|^2}{1+|x(s-)|^2}\nu(dy)ds + 2\log n \end{aligned}$$

almost surely. As a result substituting the previous relation into (3.20) we obtain, almost surely

$$\begin{aligned} \log(1+|x(t)|^2) \quad \le \quad & \log(1+|x_0|^2) + \int_{t_0}^{t}\frac{1}{1+|x(s-)|^2}[2x(s-)^T f(x(s-)) + \|g(x(s-))\|^2]\,ds \\ & + \int_{t_0}^{t}\int_{|y|<c}\frac{|H(x(s-),y)|^2}{1+|x(s-)|^2}\nu(dy)ds + 2\log n. \end{aligned}$$

and by the monotone condition (3.19), we deduce that

$$\log(1+|x(t)|^2) \leq \log(1+|x_0|^2) + 2a(t-t_0) + 2\log n$$

for all $n \geq n_0(\omega)$, $t_0 \leq t \leq n$, almost surely. Now, for almost all $\omega \in \Omega$ if $n \geq n_0(\omega)$, $n-1 \leq t \leq n$,

$$\frac{1}{t}\log(1+|x(t)|^2) \quad \leq \quad \frac{1}{n-1}\log(1+|x(t)|^2).$$

Then,

$$\begin{aligned}
\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| &\leq \limsup_{t\to\infty}\frac{1}{2t}\log(1+|x(t)|^2) \\
&\leq \limsup_{n\to\infty}\frac{1}{2(n-1)}\left[\log\left(1+|x_0|^2\right)+2a(n-t_0)+2\log n\right] \\
&= a \qquad\qquad\qquad a.s.
\end{aligned}$$

as required. □

In order to be able to develop the theory in this section we need the following technical inequality.

Lemma 3.4.2 If $x, y \in \mathbb{R}^d$, $x, y, x+y \neq 0$ then

$$\frac{1}{|x+y|} - \frac{1}{|x|} + \frac{\langle x,y\rangle}{|x|^3} \leq \frac{2|y|}{|x|^2}\left(\frac{|y|+|x|}{|x+y|}\right).$$

Proof: Using the elementary inequality $|a|-|b| \leq |a+b| \leq |a|+|b|$ for $a, b \in \mathbb{R}^d$ and the Cauchy-Schwarz inequality $|\langle y,z\rangle| \leq |y|.|z|$ for $y, z \in \mathbb{R}^d$ we have that

$$\begin{aligned}
\frac{1}{|x+y|} - \frac{1}{|x|} + \frac{\langle x,y\rangle}{|x|^3} &= \frac{|x|^3 - |x|^2.|x+y| + |x+y|.\langle x,y\rangle}{|x|^3.|x+y|} \\
&\leq \frac{|x|^3 - |x|^2.|x+y| + (|x+y|).|x|.|y|}{|x|^3.|x+y|} \\
&= \frac{|x|^2 - |x|.|x+y| + (|x+y|).|y|}{|x|^2.|x+y|} \\
&\leq \frac{|x|^2 - |x|.(|x|-|y|) + |y|.(|x|+|y|)}{|x|^2.|x+y|} \\
&= \frac{|y|^2 + 2|x|.|y|}{|x|^2.|x+y|} \leq \frac{2|y|}{|x|^2}.\left(\frac{|y|+|x|}{|x+y|}\right).
\end{aligned}$$

□

The following lemma is a generalization of Mao's work [31, 33] pp. 280-281 and pp. 120-121 respectively, that refers to an SDE driven by a Brownian motion, and the main results on the sections that follow depend critically on the result of the lemma below. We will prove that under some conditions the solution of (3.1) can never reach the origin provided that $x_0 \neq 0$.

Assumption 3.4.3 We suppose that H is always such that

$$\nu\left\{y \in (-c,c), \text{there exists } x \neq 0 \text{ such that } x + H(x,y) = 0\right\} = 0.$$

We require that assumption 3.4.3 holds for the rest of this section.

Lemma 3.4.4 Assume that for any $\theta > 0$ there exists $K_\theta > 0$, such that

$$|f(x)| + \|g(x)\| + 2\int_{|y|<c} |H(x,y)| \left(\frac{|x| + |H(x,y)|}{|x + H(x,y)|}\right) \nu(dy) \leq K_\theta |x| \quad \text{if } |x| \leq \theta. \tag{3.21}$$

If $x_0 \neq 0$ then

$$P\left(x(t) \neq 0 \ \text{ for all } t \geq t_0\right) = 1. \tag{3.22}$$

Proof: Assume that (3.22) is false. This implies that for some $x_0 \neq 0$ there will be a stopping time τ with $P(\tau < \infty) > 0$ when the solution will be zero for the first time:

$$\tau = \inf\{t \geq t_0 : |x(t)| = 0\}.$$

Since the paths of x are almost surely càdlàg there exists $T > t_0$ and $\theta > 1$ such that $P(B) > 0$ where

$$B = \left\{\omega \in \Omega : \tau(\omega) \leq T \text{ and } |x(t)(\omega)| \leq \theta - 1 \ \text{ for all } t_0 \leq t \leq \tau(\omega)\right\}.$$

Let $V(x) = |x|^{-1}$. If $0 < |x| \leq \theta$ it follows that

$$\begin{aligned}
\mathcal{L}V(x) &= -\frac{x^T f(x)}{|x|^3} + \frac{1}{2}\left(-\frac{\|g(x)\|^2}{|x|^3} + \frac{3|x^T g(x)|^2}{|x|^5}\right) \\
&\qquad + \int_{|y|<c} \left[\frac{1}{|x + H(x,y)|} - \frac{1}{|x|} + \frac{\langle x, H(x,y)\rangle}{|x|^3}\right] \nu(dy) \\
&\leq \frac{|f(x)|}{|x|^2} + \frac{\|g(x)\|^2}{|x|^3} + 2\int_{|y|<c} \left[\frac{|H(x,y)|}{|x|^2}\left(\frac{|H(x,y)| + |x|}{|x + H(x,y)|}\right)\right] \nu(dy),
\end{aligned} \tag{3.23}$$

where we have used the result of Lemma 3.4.2.
Applying (3.21) to (3.23) then

$$\mathcal{L}V(x) \leq \alpha V(x) \qquad \text{if} \quad 0 < |x| \leq \theta$$

where α is a positive constant that depends on K_θ.

Now define the following family of stopping times

$$\tau_\varepsilon = \inf\{t \geq t_0 : |x(t)| \leq \varepsilon \text{ or } \ |x(t)| \geq \theta\}$$

for each $0 < \varepsilon < |x_0|$. By Itô's formula with jumps (see Theorem 1.3.2, Chapter 1) applied to $Z(t) = e^{-\alpha(t-t_0)}V(x(t))$ we obtain

$$\begin{aligned}
&e^{-\alpha(\tau_\varepsilon \wedge T - t_0)}V(x(\tau_\varepsilon \wedge T)) \\
&\quad = V(x_0) + \int_{t_0}^{\tau_\varepsilon \wedge T} -\alpha e^{-\alpha(s-t_0)}V(x(s-))ds \\
&\quad + \int_{t_0}^{\tau_\varepsilon \wedge T} e^{-\alpha(s-t_0)}\partial_i V(x(s-))\left[f^i(x(s-))ds + g^{ij}(x(s-))dB_j(s)\right] \\
&\quad + \frac{1}{2}\int_{t_0}^{\tau_\varepsilon \wedge T} e^{-\alpha(s-t_0)}\partial_i\partial_k V(x(s-))[g(x(s-))g(x(s-))^T]^{ik}ds \\
&\quad + \int_{t_0}^{\tau_\varepsilon \wedge T}\int_{|y|<c} e^{-\alpha(s-t_0)}\Big[V\big(x(s-) + H(x(s-),y)\big) - V(x(s-))\Big]\tilde{N}(ds,dy) \\
&\quad + \int_{t_0}^{\tau_\varepsilon \wedge T}\int_{|y|<c} e^{-\alpha(s-t_0)}\Big[V\big(x(s-) + H(x(s-),y)\big) - V(x(s-)) \\
&\qquad\qquad - H^i(x(s-),y)\partial_i V(x(s-))\Big]\nu(dy)ds.
\end{aligned}$$

Hence,

$$E\left[e^{-\alpha(\tau_\varepsilon \wedge T - t_0)}V(x(\tau_\varepsilon \wedge T))\right] = V(x_0) + E\left[\int_{t_0}^{\tau_\varepsilon \wedge T} e^{-\alpha(s-t_0)}\big[-\alpha V(x(s-)) + \mathcal{L}V(x(s-))\big]ds\right].$$

Since $\mathcal{L}V(x) \leq \alpha V(x)$ it follows that

$$E\left[e^{-\alpha(\tau_\varepsilon \wedge T - t_0)}V(x(\tau_\varepsilon \wedge T))\right] \ \leq \ V(x_0).$$

If $\omega \in B$, then $\tau_\varepsilon(\omega) \leq T$ and $|x(\tau_\varepsilon(\omega))| \leq \varepsilon$. Then,

$$\begin{aligned}
E\left[e^{-\alpha(T-t_0)}\varepsilon^{-1}1_B\right] \ &\leq \ E\left[e^{-\alpha(\tau_\varepsilon - t_0)}|x(\tau_\varepsilon(\omega))|^{-1}1_B\right] = E\left[e^{-\alpha(\tau_\varepsilon \wedge T - t_0)}V(x(\tau_\varepsilon \wedge T))1_B\right] \\
&\leq \ E\left[e^{-\alpha(\tau_\varepsilon \wedge T - t_0)}V(x(\tau_\varepsilon \wedge T))\right] \leq V(x_0).
\end{aligned}$$

Hence,

$$P(B) \leq \varepsilon\, e^{\alpha(T-t_0)}|x_0|^{-1}, \quad \text{for all } \varepsilon \geq 0.$$

Now let $\varepsilon \to 0$. Then it follows that $P(B) = 0$ which contradicts the definition of the set B and the required result follows. □

Remark 3.4.5 Condition (3.21) in Lemma 3.4.4 seems quite complicated. We will now show that there is a natural class of mappings H for which this is satisfied, at least in the case $d = 1$. To begin suppose that we can find a mapping H_1 for which

$$\int_{|y|<c} |H_1(x,y)|\nu(dy) < K_\theta|x|, \text{ for all } x \in \mathbb{R}.$$

Now let $A = \{(x,y) \in \mathbb{R}^2 : x \geq 0, H_1(x,y) \geq 0\} \cup \{(x,y) \in \mathbb{R}^2 : x \leq 0, H_1(x,y) \leq 0\}$ and so $A^c = \{(x,y) \in \mathbb{R}^2 : x \geq 0, H_1(x,y) < 0\} \cup \{(x,y) \in \mathbb{R}^2 : x \leq 0, H_1(x,y) > 0\}$.

Define $H(x,y) = \big(1_A(x,y) - 1_{A^c}(x,y)\big)H_1(x,y)$.
Hence,

$$|H_1(x,y)| = |H(x,y)| \text{ and } |x + H(x,y)| = |x| + |H_1(x,y)|.$$

Then we find that

$$\int_{|y|<c} |H(x,y)|.\left(\frac{|x| + |H(x,y)|}{|x + H(x,y)|}\right)\nu(dy) = \int_{|y|<c} |H_1(x,y)|\nu(dy) < K_\theta|x|, \text{ for all } x \in \mathbb{R}.$$

To construct specific examples of mappings of the form H_1 we can take e.g. $H_1(x,y) = H_2(x)y^2$ where $\frac{H_2(x)}{x}$ is bounded.

For the next two results, we require that the following local boundedness constraint on the jumps holds:

Assumption 3.4.6 For all bounded sets M in $\mathbb{R}^d$,

$$\sup_{x\in M}\sup_{0<|y|<c} |H(x,y)| < \infty.$$

Assumption 3.4.6 is related to assumption 1.3.1, that is a requirement for applying Itô's formula. Also note that assumption 3.4.6 is always satisfied if H is jointly continuous.

In the sequel conditions for almost sure exponential stability of the trivial solution of (3.1) will be obtained. First we need a useful technical result.

Let $V \in C^2(\mathbb{R}^d; \mathbb{R}^+)$ be such that $V(x) \neq 0$ for every $x \in \mathbb{R}^d$. Define the following processes $I_1 = (I_1(t), t \geq t_0)$, $I_2 = (I_2(t), t \geq t_0)$ and $I = (I(t), t \geq t_0)$ where for each

$t \geq t_0$

$$I_1(t) = \int_{t_0}^{t} \int_{|y|<c} \left(\frac{V\big(x(s-) + H(x(s-),y) - V(x(s-))\big)}{V(x(s-))} - \frac{H^i(x(s-),y)}{V(x(s-))} \partial_i V(x(s-)) \right) \nu(dy)ds, \tag{3.24}$$

$$I_2(t) = \int_{t_0}^{t} \int_{|y|<c} \left(\log\left(\frac{V\big(x(s-) + H(x(s-),y)\big)}{V(x(s-))} \right) + 1 - \frac{V\big(x(s-) + H(x(s-),y)\big)}{V(x(s-))} \right) \nu(dy)ds, \tag{3.25}$$

$$I(t) = \int_{t_0}^{t} \int_{|y|<c} \left(\log\left(\frac{V\big(x(s-) + H(x(s-),y)\big)}{V(x(s-))} \right) - \frac{H^i(x(s-),y)}{V(x(s-))} \partial_i V(x(s-)) \right) \nu(dy)ds. \tag{3.26}$$

Note that for each $t \geq t_0$, $I(t) = I_1(t) + I_2(t)$.

Lemma 3.4.7 Let $I_1 = (I_1(t), t \geq t_0)$, $I_2 = (I_2(t), t \geq t_0)$ and $I = (I(t), t \geq t_0)$ be defined for each $t \geq t_0$ as in (3.24), (3.25), (3.26) respectively. Then for each $t \geq t_0$, it holds that

$$(i)\ |I_1(t)| < \infty, \quad (ii)\ |I(t)| < \infty, \quad \text{and} \quad (iii)\ |I_2(t)| < \infty \quad a.s.$$

Proof: (i) Following Kunita's arguments in [25] pp. 317, by using a Taylor's series expansion with integral remainder term (see Burkill [10], Theorem 7.7) we obtain for each $y \in \hat{B}_c$ and $x \in \mathbb{R}^d$

$$V\big(x + H(x,y)\big) - V(x) - H^i(x,y)\partial_i V(x) = \int_0^1 \partial_i \partial_j V\big(x + \theta H(x,y)\big)(1-\theta)d\theta H^i(x,y)H^j(x,y).$$

Hence,

$$\begin{aligned} &|I_1(t)| \\ &\leq \int_{t_0}^{t} \int_{|y|<c} \left| \frac{V\big(x(s-) + H(x(s-),y)\big) - V(x(s-)) - H^i(x(s-),y)\partial_i V(x(s-))}{V(x(s-))} \right| \nu(dy)ds \\ &\leq \frac{1}{2} \int_{t_0}^{t} \int_{|y|<c} \left| \sup_{0 \leq \theta \leq 1} \frac{\partial_i \partial_j V\big(x(s-) + \theta H(x(s-),y)\big)}{V(x(s-))} \right| \left| H^i(x(s-),y) H^j(x(s-),y) \right| \nu(dy)ds, \end{aligned} \tag{3.27}$$

since $\int_0^1 (1-\theta)d\theta = \frac{1}{2}$.

For each $z \in \mathbb{R}^d$, $y \in \hat{B}_c$, $1 \le i, j \le d$, define

$$f_{ij}^V(z,y) = \sup_{0\le\theta\le1} \frac{\partial_i\partial_j V\big(z+\theta H(z,y)\big)}{V(z)}.$$

By assumption 3.4.6 it follows that

$$\sup_{t_0\le s\le t}\sup_{0<|y|<c} |f_{ij}^V(x(s-),y)| < \infty \qquad a.s.$$

Using the Cauchy-Schwarz inequality it follows from (3.27) that

$$\begin{aligned} |I_1(t)| &\le \frac{1}{2}\sup_{t_0\le s\le t}\sup_{0<|y|<c} |f_{ij}^V(x(s-),y)| \int_{t_0}^t\int_{|y|<c} \left|H^i(x(s-),y)H^j(x(s-),y)\right| \nu(dy)ds \\ &\le \frac{1}{2}\left(\sum_{i,j=1}^d \sup_{t_0\le s\le t}\sup_{0<|y|<c} |f_{ij}^V(x(s-),y)|^2\right)^{\frac{1}{2}} \int_{t_0}^t\int_{|y|<c} |H(x(s-),y)|^2\nu(dy)ds < \infty, \end{aligned} \tag{3.28}$$

almost surely.

(ii) Using the same arguments as in (i), we obtain

$$\begin{aligned} |I(t)| &\le \int_{t_0}^t\int_{|y|<c}\bigg|\log\big(V\big(x(s-)+H(x(s-),y)\big)\big) - \log\big(V(x(s-))\big) \\ &\qquad\qquad - \frac{H^i(x(s-),y)}{V(x(s-))}\partial_i V(x(s-))\bigg|\nu(dy)ds \\ &= \int_{t_0}^t\int_{|y|<c}\bigg|\bigg(\frac{\partial_i\partial_j V\big(x(s-)+\theta H(x(s-),y)\big)}{V\big(x(s-)+\theta H(x(s-),y)\big)} \\ &\qquad - \frac{\partial_i V\big(x(s-)+\theta H(x(s-),y)\big)\partial_j V\big(x(s-)+\theta H(x(s-),y)\big)}{\big(V\big(x(s-)+\theta H(x(s-),y)\big)\big)^2}\bigg) \\ &\qquad \times \int_0^1 (1-\theta)d\theta H^i(x(s-),y)H^j(x(s-),y)\bigg|\nu(dy)ds. \end{aligned}$$

Define for each $z \in \mathbb{R}^d$, $y \in \hat{B}_c$, $1 \le i, j \le d$,

$$h_{ij}^V(z,y) = \sup_{0\le\theta\le1}\left(\frac{\partial_i\partial_j V\big(z+\theta H(z,y)\big)}{V\big(z+\theta H(z,y)\big)} - \frac{\partial_i V\big(z+\theta H(z,y)\big)\partial_j V\big(z+\theta H(z,y)\big)}{\big(V\big(z+\theta H(z,y)\big)\big)^2}\right).$$

Then we have

$$\sup_{t_0\le s\le t}\sup_{0<|y|<c} |h_{ij}^V(x(s-),y)| < \infty \qquad a.s.$$

By using the Cauchy-Schwarz inequality as in (3.28) the required result follows.

(iii) Recall that for each $t \geq t_0$ $I_2(t) = I(t) - I_1(t)$. Then by (i) and (ii) it follows that $|I_2(t)| \leq |I(t)| + |I_1(t)| < \infty$ almost surely for each $t \geq t_0$. □

The following is a generalization of Mao's work [33] Chapter 4, Theorem 3.3 pp. 121.

Theorem 3.4.8 *Assume that (3.21) holds. Let* $V \in C^2(\mathbb{R}^d; \mathbb{R}^+)$ *and let* $p > 0, c_1 > 0, c_2 \in \mathbb{R}, c_3 \geq 0$ *and* $c_4 > 0$ *be such that for all* $x \neq 0$

$$\begin{aligned}
&(i) \quad c_1|x|^p \leq V(x),\\
&(ii) \quad \mathcal{L}V(x) \leq c_2V(x),\\
&(iii) \quad \left|(\partial V(x))^T g(x)\right|^2 \geq c_3\big(V(x)\big)^2,\\
&(iv) \quad \int_{|y|<c}\left[\log\left(\frac{V(x+H(x,y))}{V(x)}\right) - \frac{V(x+H(x,y)) - V(x)}{V(x)}\right]\nu(dy) \leq -c_4.
\end{aligned}$$

Then

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \leq -\frac{c_3 + 2c_4 - 2c_2}{2p} \qquad a.s. \tag{3.29}$$

and furthermore if $c_3 > 2c_2 - 2c_4$, *then the trivial solution of (3.1) is almost surely exponentially stable for all* $x_0 \in \mathbb{R}^d$.

Remark 3.4.9 Using the logarithmic inequality $\log(x) \leq x - 1$ for $x > 0$ then

$$\int_{|y|<c}\left[\log\left(\frac{V(x+H(x,y))}{V(x)}\right) - \frac{V(x+H(x,y)) - V(x)}{V(x)}\right]\nu(dy) \leq 0.$$

Hence condition (iv) in Theorem 3.4.8 is a reasonable constraint to require.

Proof: For $x_0 = 0$, then $x = 0$ hence (3.29) holds trivially. For the rest of the proof we assume that $x_0 \neq 0$. Due to Lemma 3.4.4, then $x(t) \neq 0$ for all $t \geq t_0$ almost surely.

Apply Itô's formula to $Z(t) = \log(V(x(t)))$. Then for each $t \geq t_0$,

$$\begin{aligned}
\log(V(x(t))) &= \log(V(x_0)) + \int_{t_0}^{t} \frac{1}{V(x(s-))}\partial_i V(x(s-)) \left[f^i(x(s-))ds + g^{ij}(x(s-))dB_j(s)\right] \\
&+\frac{1}{2}\int_{t_0}^{t} \Big[\frac{1}{V(x(s-))}\partial_i\partial_k V(x(s-))[g(x(s-))g(x(s-))^T]^{ik} \\
&\qquad - \frac{1}{(V(x(s-)))^2}\left|(\partial V(x(s-)))^T g(x(s-))\right|^2\Big] ds \\
&+ \int_{t_0}^{t}\int_{|y|<c} \left[\log\left(V\big(x(s-) + H(x(s-),y)\big)\right) - \log\left(V(x(s-))\right)\right] \tilde{N}(ds,dy) \\
&+ \int_{t_0}^{t}\int_{|y|<c} \Big[\log\left(V\big(x(s-) + H(x(s-),y)\big)\right) - \log\left(V(x(s-))\right) \\
&\qquad - \frac{1}{V(x(s-))}\partial_i V(x(s-))H^i(x(s-),y)\Big]\nu(dy)ds.
\end{aligned} \tag{3.30}$$

Note that the last integral in (3.30), for each $t \geq t_0$ can be written as $I(t) = I_1(t)+I_2(t)$ where $I_1(t), I_2(t), I(t)$ are as defined in (3.24), (3.25), (3.26). By Lemma 3.4.7 it follows that $I_1(t)$ and $I_2(t)$ are finite almost surely. Now using the linear operator $\mathcal{L}$ defined in (1.19) we obtain

$$\begin{aligned}
\log(V(x(t))) &= \log(V(x_0)) + \int_{t_0}^{t} \frac{\mathcal{L}V(x(s-))}{V(x(s-))}ds + \int_{t_0}^{t} \frac{1}{V(x(s-))}\partial_i V(x(s-))g^{ij}(x(s-))dB_j(s) \\
&-\frac{1}{2}\int_{t_0}^{t} \frac{1}{(V(x(s-)))^2}\left|(\partial V(x(s-)))^T g(x(s-))\right|^2 ds \\
&+ \int_{t_0}^{t}\int_{|y|<c} \log\left(\frac{V\big(x(s-) + H(x(s-),y)\big)}{V(x(s-))}\right)\tilde{N}(ds,dy) + I_2(t).
\end{aligned}$$

Define for each $t \geq t_0$

$$\begin{aligned}
M(t) &= \int_{t_0}^{t} \frac{1}{V(x(s-))}\partial_i V(x(s-))g^{ij}(x(s-))dB_j(s) \\
&\qquad + \int_{t_0}^{t}\int_{|y|<c} \log\left(\frac{V\big(x(s-) + H(x(s-),y)\big)}{V(x(s-))}\right)\tilde{N}(ds,dy)
\end{aligned}$$

where the first integral is a continuous martingale and the second is a local martingale. We may then write

$$\begin{aligned}
\log(V(x(t))) &\leq \log(V(x_0)) + \int_{t_0}^{t} \frac{\mathcal{L}V(x(s-))}{V(x(s-))}ds + M(t) \\
&- \frac{1}{2}\int_{t_0}^{t} \frac{1}{(V(x(s-)))^2}\left|(\partial V(x(s-)))^T g(x(s-))\right|^2 ds + I_2(t).
\end{aligned} \tag{3.31}$$

We now use the exponential martingale inequality (3.9) for $T = n$, $\alpha = \varepsilon$ and $\beta = \varepsilon n$ where $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$. Then for every integer $n \geq t_0$, we find that

$$P\Bigg[\sup_{t_0\le t\le n}\Bigg\{M(t) - \frac{\varepsilon}{2}\int_{t_0}^{t}\frac{1}{(V(x(s-)))^2}\left|(\partial V(x(s-)))^T g(x(s-))\right|^2 ds - \frac{1}{\varepsilon}\int_0^t\int_{|x|<c}\Bigg[e^{\log\left(\frac{V\left(x(s-)+H(x(s-),y)\right)}{V(x(s-))}\right)^{\varepsilon}} - 1 - \varepsilon\log\left(\frac{V\big(x(s-)+H(x(s-),y)\big)}{V(x(s-))}\right)\Bigg]\nu(dy)ds\Bigg\} > \varepsilon n\Bigg] \le e^{-\varepsilon^2 n}.$$

Since $\sum_{n=1}^{\infty} e^{-\varepsilon^2 n} < \infty$ then an application of the Borel-Cantelli lemma and elementary probability calculations, as in the proof of Theorem 3.4.1, yields that

$$P\Bigg[\liminf_{n\to\infty}\Bigg\{\sup_{t_0\le t\le n}\Bigg(M(t) - \frac{\varepsilon}{2}\int_{t_0}^{t}\frac{1}{(V(x(s-)))^2}\left|(\partial V(x(s-)))^T g(x(s-))\right|^2 ds - \frac{1}{\varepsilon}\int_0^t\int_{|x|<c}\Bigg(\frac{V\big(x(s-)+H(x(s-),y)\big)}{V(x(s-))}\Bigg)^{\varepsilon} - 1 - \varepsilon\log\left(\frac{V\big(x(s-)+H(x(s-),y)\big)}{V(x(s-))}\right)\nu(dy)ds\Bigg) \le \varepsilon n\Bigg\}\Bigg] = 1.$$

Hence for almost all $\omega \in \Omega$ there is a random integer $n_0 = n_0(\omega)$ such that for $n \geq n_0$, $t_0 \leq t \leq n$,

$$\begin{aligned} M(t) \le\ & \frac{\varepsilon}{2}\int_{t_0}^{t}\frac{1}{(V(x(s-)))^2}\left|(\partial V(x(s-)))^T g(x(s-))\right|^2 ds + \varepsilon n \\ & + \frac{1}{\varepsilon}\int_{t_0}^t\int_{|y|<c}\Bigg[\Bigg(\frac{V\big(x(s-)+H(x(s-),y)\big)}{V(x(s-))}\Bigg)^{\varepsilon} - 1 \\ & - \varepsilon\log\left(\frac{V\big(x(s-)+H(x(s-),y)\big)}{V(x(s-))}\right)\Bigg]\nu(dy)ds. \end{aligned} \tag{3.32}$$

Substituting (3.32) into (3.31) and using conditions (ii) and (iii) it follows immediately that

$$
\begin{aligned}
&\log(V(x(t))) \\
&\leq \log(V(x_0)) - \frac{1}{2}[(1-\varepsilon)c_3 - 2c_2](t-t_0) + \varepsilon n \\
&+ \int_{t_0}^{t}\int_{|y|<c}\left[\log\left(\frac{V(x(s-)+H(x(s-),y))}{V(x(s-))}\right) + 1 - \frac{V(x(s-)+H(x(s-),y)))}{V(x(s-))}\right]\nu(dy)ds \\
&+\frac{1}{\varepsilon}\int_{t_0}^{t}\int_{|y|<c}\left[\left(\frac{V(x(s-)+H(x(s-),y))}{V(x(s-))}\right)^{\varepsilon} - 1\right. \\
&\qquad\qquad \left. - \varepsilon\log\left(\frac{V(x(s-)+H(x(s-),y))}{V(x(s-))}\right)\right]\nu(dy)ds \qquad (3.33)
\end{aligned}
$$

for $n \geq n_0$, $t_0 \leq t \leq n$.

Fix $x \in \mathbb{R}^d$ and define for $y \in \hat{B}_c$, $h_\varepsilon(y) = \frac{1}{\varepsilon}\left|\left(\frac{V(x+H(x,y))}{V(x)}\right)^{\varepsilon} - 1 - \varepsilon\log\left(\frac{V(x+H(x,y))}{V(x)}\right)\right|$. We easily deduce that $\left(\frac{V(x+H(x,y))}{V(x)}\right)^{\varepsilon} - 1 - \varepsilon\log\left(\frac{V(x+H(x,y))}{V(x)}\right) \geq 0$ for all $y \in \hat{B}_c$, by using the elementary inequality $e^b - 1 - b \geq 0$ for $b \in \mathbb{R}$. Since $\varepsilon \in (0,1)$ then we can use the inequality $b^c < 1 + c(b-1)$ for $0 < c < 1$ and $b > 0$ (see Hardy, Littlewood and Pólya [17] pp. 40) to deduce that for all $y \in \hat{B}_c$,

$$
\begin{aligned}
h_\varepsilon(y) &\leq \frac{1}{\varepsilon}\left[1 + \varepsilon\left(\frac{V(x+H(x,y))}{V(x)} - 1\right) - 1 - \varepsilon\log\left(\frac{V(x+H(x,y))}{V(x)}\right)\right] \\
&\leq \frac{V(x+H(x,y))}{V(x)} - 1 - \log\left(\frac{V(x+H(x,y))}{V(x)}\right). \qquad (3.34)
\end{aligned}
$$

Note that by Lemma 3.4.7 (iii) the right hand side of (3.34) is Lebesgue integrable and is ν-integrable on $\hat{B}_c$. It follows that the same applies for $h_\varepsilon(y)$.

Now let $\varepsilon \to 0$.

The dominated convergence theorem yields that for all $t \geq t_0$

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{t_0}^{t}\int_{|y|<c}\frac{1}{\varepsilon}\left[\left(\frac{V(x(s-)+H(x(s-),y))}{V(x(s-))}\right)^{\varepsilon} - 1\right. \\
&\qquad\qquad \left. - \varepsilon\log\left(\frac{V(x(s-)+H(x(s-),y))}{V(x(s-))}\right)\right]\nu(dy)ds \\
&= \int_{t_0}^{t}\int_{|y|<c}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\left(\frac{V(x(s-)+H(x(s-),y))}{V(x(s-))}\right)^{\varepsilon} - 1\right] \\
&\qquad\qquad - \log\left(\frac{V(x(s-)+H(x(s-),y))}{V(x(s-))}\right)\nu(dy)ds \\
&= 0. \qquad (3.35)
\end{aligned}
$$

Hence by (3.35) for $n \geq n_0$, $t_0 \leq t \leq n$, (3.33) becomes

$$\begin{aligned}\log(V(x(t))) \leq & \log(V(x_0)) - \frac{1}{2}(c_3 - 2c_2)(t - t_0) \\ & + \int_{t_0}^{t} \int_{|y|<c} \left[\log\left(\frac{V(x(s-) + H(x(s-), y))}{V(x(s-))} \right) + 1 \right. \\ & \left. - \frac{V(x(s-) + H(x(s-), y))}{V(x(s-))} \right] \nu(dy)ds. \qquad (3.36)\end{aligned}$$

Now substituting condition (iv) into (3.36), we see that for almost all $\omega \in \Omega$, $t_0 + n - 1 \leq t \leq t_0 + n$, $n \geq n_0$

$$\frac{1}{t}\log(V(x(t))) \leq -\frac{t - t_0}{2t}(c_3 - 2c_2) + \frac{\log(V(x(t_0)))}{t_0 + n - 1} - \frac{t - t_0}{t}c_4.$$

Now applying condition (i) we obtain that

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \leq -\frac{c_3 - 2c_2 + 2c_4}{2p} \qquad a.s.$$

□

3.5 Moment exponential stability

The main aim of this section is to introduce criteria for the solution of an SDE driven by a Lévy process to be moment exponentially stable and to derive a relation between moment and almost sure exponential stability.

The following result is an extension of Mao's work [33] Chapter 4, Theorem 4.4 pp. 130.

Theorem 3.5.1 *Let $p, \alpha_1, \alpha_2, \alpha_3$ be positive constants. If $V \in C^2(\mathbb{R}^d; \mathbb{R}^+)$ satisfies*

(i) $\alpha_1|x|^p \leq V(x) \leq \alpha_2|x|^p$,

(ii) $\mathcal{L}V(x) \leq -\alpha_3 V(x)$,

for all $x \in \mathbb{R}^d$, then

$$E\left[|x(t)|^p\right] \leq \frac{\alpha_2}{\alpha_1}|x_0|^p \exp(-\alpha_3(t - t_0)) \qquad \text{for all } t \geq t_0 \qquad (3.37)$$

for all $x_0 \in \mathbb{R}^d$. As a result the trivial solution of (3.1) is pth moment exponentially stable under conditions (i) and (ii) and the pth moment Lyapunov exponent should not be greater than $-\alpha_3$.

Proof: Fix any $x_0 \neq 0$ on $\mathbb{R}^d$. For each $n \geq |x_0|$ define the stopping time

$$\tau_n = \inf\{t \geq t_0 : |x(t)| \geq n\}$$

so that, $\tau_n \uparrow \infty$ almost surely.
Apply Itô's formula to $Z(t) = \exp(\alpha_3(t - t_0))V(x(t))$. Then, for each $t \geq t_0$,

$$\begin{aligned}
&\exp(\alpha_3(t \wedge \tau_n - t_0))V(x(t \wedge \tau_n)) \\
&\quad = V(x_0) + \int_{t_0}^{t\wedge\tau_n} \alpha_3 \exp(\alpha_3(s - t_0))V(x(s-))ds \\
&\quad + \int_{t_0}^{t\wedge\tau_n} \exp(\alpha_3(s - t_0))\partial_i V(x(s-)) \left[f^i(x(s-))ds + g^{ij}(x(s-))dB_j(s)\right] \\
&\quad + \frac{1}{2}\int_{t_0}^{t\wedge\tau_n} \exp(\alpha_3(s - t_0))\partial_i\partial_k V(x(s-))[g(x(s-))g(x(s-))^T]^{ik}ds \\
&\quad + \int_{t_0}^{t\wedge\tau_n}\int_{|y|<c} \exp(\alpha_3(s - t_0))\Big[V\big(x(s-) + H(x(s-), y)\big) - V(x(s-))\Big]\tilde{N}(ds, dy) \\
&\quad + \int_{t_0}^{t\wedge\tau_n}\int_{|y|<c} \exp(\alpha_3(s - t_0))\Big[V\big(x(s-) + H(x(s-), y)\big) - V(x(s-)) \\
&\qquad\qquad - \partial_i V(x(s-))H^i(x(s-), y)\Big]\nu(dy)ds.
\end{aligned}$$

Hence

$$\begin{aligned}
E\big[\exp(\alpha_3(t \wedge \tau_n - t_0))V(x(t \wedge \tau_n))\big] &= V(x_0) + E\Big[\int_{t_0}^{t\wedge\tau_n} \exp(\alpha_3(s - t_0))\big[\alpha_3 V(x(s-)) \\
&\qquad + \mathcal{L}V(x(s-))\big]ds\Big]. \qquad (3.38)
\end{aligned}$$

Now, apply condition (ii) within (3.38) to obtain

$$E\big[\exp(\alpha_3(t \wedge \tau_n - t_0))V(x(t \wedge \tau_n))\big] \leq V(x_0). \qquad (3.39)$$

From condition (i) (left hand inequality), we have

$$\alpha_1|x(t \wedge \tau_n)|^p \exp(\alpha_3(t \wedge \tau_n - t_0)) \leq V(x(t \wedge \tau_n)) \exp(\alpha_3(t \wedge \tau_n - t_0)).$$

Taking expectations and using (3.39) and then applying condition (i) again (right-hand inequality), it follows that

$$\begin{aligned}
\alpha_1 E\big[|x(t \wedge \tau_n)|^p \exp(\alpha_3(t \wedge \tau_n - t_0))\big] &\leq E\big[V(x(t \wedge \tau_n)) \exp(\alpha_3(t \wedge \tau_n - t_0))\big] \\
&\leq V(x_0) \leq \alpha_2|x_0|^p. \qquad (3.40)
\end{aligned}$$

Hence,

$$E\Big[|x(t\wedge\tau_n)|^p\exp(\alpha_3(t\wedge\tau_n-t_0))\Big] \;\le\; \frac{\alpha_2}{\alpha_1}|x_0|^p$$

which implies that

$$E\Big[1_{[0,\tau_n]}\exp(\alpha_3(t-t_0))|x(t)|^p\Big] \;\le\; \frac{\alpha_2}{\alpha_1}|x_0|^p.$$

Using the fact that for each $t\ge t_0$, $1_{[0,\tau_n]}\exp\big(\alpha_3(t-t_0)\big)|x(t)|^p$ forms a monotonic increasing sequence of random variables then by the monotone convergence theorem it can be deduced that

$$E\Big[\exp(\alpha_3(t-t_0))|x(t)|^p\Big] \;=\; \lim_{n\to\infty}E\Big[1_{[0,\tau_n]}\exp(\alpha_3(t-t_0))|x(t)|^p\Big]\le\frac{\alpha_2}{\alpha_1}|x_0|^p.$$

And the result follows. □

Proposition 3.5.2 If the trivial solution of (3.1) is pth moment exponentially stable then the trivial solution is qth moment exponentially stable for every $q<p$.

Proof: Let $q<p$. An application of Holder's inequality yields that

$$E\left[|x(t)|^q\right]\le E\left[|x(t)|^p\right]^{\frac{q}{p}}. \tag{3.41}$$

Since the trivial solution of (3.1) is pth moment exponentially stable using (1.23) and (3.41) then,

$$E\left[|x(t)|^q\right] \;\le\; \Big[C|x_0|^p\exp\left(-\lambda(t-t_0)\right)\Big]^{\frac{q}{p}}=C^{\frac{q}{p}}|x_0|^q\exp\left(-\left(\frac{\lambda q}{p}\right)(t-t_0)\right).$$

Hence,

$$E[|x(t)|^q]\le D|x_0|^q\exp(-\lambda'(t-t_0))$$

where $D=C^{\frac{q}{p}}>0$ and $\lambda'=\frac{\lambda q}{p}>0$ which means that the solution is qth moment exponentially stable. □

It is very interesting to mention that in general there is no obvious relation between exponential and almost sure stability. Kozin in [24] pp. 107 refers to an example of an SDE driven by a Brownian motion where the system is almost surely stable but is not moment exponentially stable. However it is possible when moment stability holds to deduce almost sure stability under some additional conditions. Mao in [33] Theorem

4.2, Chapter 4 pp. 128 establishes such a relation. In the following we will generalize Mao's result and give the relationship between the pth moment exponential stability and almost sure exponential stability for the trivial solution of (3.1).

We will need to use among others, two inequalities: the Burkholder-Davis-Gundy (BDG) inequality and Kunita's estimate for a compensated Poisson integral. For the sake of completeness we state the BDG inequality here.

Theorem 3.5.3 *(Burkholder-Davis-Gundy inequality)*
Let $b \in P_2(T)$. Define for $t \geq t_0$

$$x(t) = \int_{t_0}^{t} b(s)dB(s) \qquad \text{and} \qquad A(t) = \int_{t_0}^{t} \|b(s)\|^2 ds.$$

Then for every $k > 0$ there exist universal positive constants c_k, C_k, such that

$$c_k E\left[|A(t)|^{\frac{k}{2}}\right] \leq E\left[\sup_{t_0 \leq s \leq t} |x(t)|^k\right] \leq C_k E\left[|A(t)|^{\frac{k}{2}}\right]$$

for all $t \geq t_0$. In particular we may take the following choices

$$\begin{aligned} c_k &= \left(\frac{k}{2}\right)^k & C_k &= \left(\frac{32}{k}\right)^{\frac{k}{2}} & &\text{if } 0 < k < 2, \\ c_k &= 1 & C_k &= 4 & &\text{if } k = 2, \\ c_k &= (2k)^{-\frac{k}{2}} & C_k &= \left(\frac{k^{k+1}}{2(k-1)^{k-1}}\right)^{\frac{k}{2}} & &\text{if } k > 2. \end{aligned}$$

Proof: See Mao [33] pp. 40. □

Now we need to discuss Kunita's estimate in a little more detail for reasons that will become obvious below. For the purposes of the following proof let $Y(t) = \int_{t_0}^{t} \int_{|y|<c} H(x(s-), y)\tilde{N}(ds, dy)$. Applying Ito's formula to the process $|Y(t)|^p$ for each $t \geq t_0$, we have

$$\begin{aligned} |Y(t)|^p &= \int_{t_0}^{t} \int_{|y|<c} \Big\{|Y(s-) + H(x(s-), y)|^p - |Y(s-)|^p\Big\}\tilde{N}(ds, dy) \\ &\quad + \int_{t_0}^{t} \int_{|y|<c} \Big\{|Y(s-) + H(x(s-), y)|^p - |Y(s-)|^p \\ &\qquad - p|Y(s-)|^{p-2}Y(s-)^T H(x(s-), y)\Big\}\nu(dy)ds, \end{aligned} \tag{3.42}$$

where the first term in (3.42) is a local martingale.

For $p \geq 2$ and $Y(t) = \int_{t_0}^{t} \int_{|y|<c} H(x(s-), y)\tilde{N}(ds, dy)$ and letting $f = 0$ and $g = 0$ then

Kunita's estimate (Theorem 3.2.1) becomes

$$E\left[\sup_{t_0\le s\le t}|Y(s)|^p\right] \le c_7(p)E\left[\left(\int_{t_0}^t\int_{|y|<c}|H(x(s-),y)|^2\nu(dy)ds\right)^{\frac{p}{2}}\right] + c_8(p)E\left[\left(\int_{t_0}^t\int_{|y|<c}|H(x(s-),y)|^p\nu(dy)ds\right)\right] \quad (3.43)$$

where $c_7(p)$ and $c_8(p)$ are positive constants that depend only on p.

For a proof see Kunita [25] pp. 334.

Remark 3.5.4 Recall that in the context of stability theory we are always assuming that $f(0) = 0$ and $g(0) = 0$, hence from the Lipschitz conditions on the drift and diffusion coefficients we deduce that for all $x \in \mathbb{R}^d$ there exists $L > 0$ such that

$$|f(x)| \le \sqrt{L}|x| \qquad \text{and} \qquad \|g(x)\|^2 \le L|x|^2.$$

Hence,

$$x^T f(x) \vee \|g(x)\|^2 \le |x^T f(x)| \vee \|g(x)\|^2 \le |x|.|f(x)| \vee \|g(x)\|^2 \le L'|x|^2 \quad (3.44)$$

where $L' = \max\{\sqrt{L}, L\}$ and this will be used in the proof of the theorem below.

Theorem 3.5.5 *Assume that assumption 3.2.2 holds. For $p \ge 2$, pth moment exponential stability, of the trivial solution to (3.1), implies almost sure exponential stability.*

Proof: Fix any $x_0 \neq 0$ on $\mathbb{R}^d$. If the solution is pth moment exponentially stable then

$$E[|x(t)|^p] \le C'|x_0|^p \exp(-\lambda(t-t_0)) \qquad \text{on } t \ge t_0 \quad (3.45)$$

with λ and C' positive constants.

Let $n \in \mathbb{N}$. Apply Ito's formula to $Z(t) = |x(t)|^p$. For $t_0 + n - 1 \le t \le t_0 + n$ the following holds almost surely

$$\begin{aligned}
|x(t)|^p &= |x(t_0+n-1)|^p + \int_{t_0+n-1}^{t} p|x(s-)|^{p-2}x(s-)^T f(x(s-))ds \\
&+ \frac{1}{2}\int_{t_0+n-1}^{t} \Big(p|x(s-)|^{p-2}\|g(x(s-))\|^2 + p(p-2)|x(s-)|^{p-4}|x(s-)^T g(x(s-))|^2\Big) ds \\
&+ \int_{t_0+n-1}^{t} p|x(s-)|^{p-2}x(s-)^T g(x(s-))dB(s) \\
&+ \int_{t_0+n-1}^{t}\int_{|y|<c} \Big(|x(s-)+H(x(s-),y)|^p - |x(s-)|^p\Big)\tilde{N}(ds,dy) \\
&+ \int_{t_0+n-1}^{t}\int_{|y|<c} \Big(|x(s-)+H(x(s-),y)|^p - |x(s-)|^p \\
&\qquad - p|x(s-)|^{p-2}x(s-)^T H(x(s-),y)\Big)\nu(dy)ds.
\end{aligned}$$

Applying (3.44) then,

$$\begin{aligned}
|x(t)|^p &\leq |x(t_0+n-1)|^p + c_1\int_{t_0+n-1}^{t} |x(s-)|^p ds \\
&+ \int_{t_0+n-1}^{t} p|x(s-)|^{p-2}x(s-)^T g(x(s-))dB(s) \\
&+ \int_{t_0+n-1}^{t}\int_{|y|<c} \Big(|x(s-)+H(x(s-),y)|^p - |x(s-)|^p\Big)\tilde{N}(ds,dy) \\
&+ \int_{t_0+n-1}^{t}\int_{|y|<c} \Big(|x(s-)+H(x(s-),y)|^p - |x(s-)|^p \\
&\qquad - p|x(s-)|^{p-2}x(s-)^T H(x(s-),y)\Big)\nu(dy)ds
\end{aligned}$$

where $c_1 = pL' + \frac{pL'}{2}[1+(p-2)]$. Hence,

$$\begin{aligned}
&E\left[\sup_{t_0+n-1\leq t\leq t_0+n} |x(t)|^p\right] \\
&\leq E\left[|x(t_0+n-1)|^p\right] + c_1\int_{t_0+n-1}^{t_0+n} E\left[|x(s-)|^p\right] ds \\
&\quad + E\left[\sup_{t_0+n-1\leq t\leq t_0+n}\int_{t_0+n-1}^{t} p|x(s-)|^{p-2}x(s-)^T g(x(s-))dB(s)\right] \\
&\quad + E\Bigg[\sup_{t_0+n-1\leq t\leq t_0+n}\Bigg\{\int_{t_0+n-1}^{t}\int_{|y|<c}\Big(|x(s-)+H(x(s-),y)|^p - |x(s-)|^p\Big)\tilde{N}(ds,dy) \\
&\qquad + \int_{t_0+n-1}^{t}\int_{|y|<c}\Big(|x(s-)+H(x(s-),y)|^p - |x(s-)|^p \\
&\qquad\qquad - p|x(s-)|^{p-2}x(s-)^T H(x(s-),y)\Big)\nu(dy)ds\Bigg\}\Bigg]. \qquad (3.46)
\end{aligned}$$

For the Brownian motion integral, by the Burkholder-Davis-Gundy inequality with

$b(t) = p|x(t-)|^{p-2}x(t-)^T g(x(t-))$ and $k = 1$ then (see Mao [33] pp. 129)

$$E\left[\sup_{t_0+n-1\leq t\leq t_0+n} \int_{t_0+n-1}^{t} p|x(s-)|^{p-2}x(s-)^T g(x(s-))dB(s)\right]$$
$$\leq \frac{1}{2}E\left[\sup_{t_0+n-1\leq t\leq t_0+n} |x(t-)|^p\right] + 16p^2 L' \int_{t_0+n-1}^{t_0+n} E[|x(s-)|^p]ds. \quad (3.47)$$

Define

$$\begin{aligned} I_1 &= E\Bigg[\sup_{t_0+n-1\leq t\leq t_0+n} \Bigg\{ \int_{t_0+n-1}^{t}\int_{|y|<c} \Big(|x(s-)+H(x(s-),y)|^p - |x(s-)|^p\Big)\tilde{N}(ds,dy) \\ &+ \int_{t_0+n-1}^{t}\int_{|y|<c} \Big(|x(s-)+H(x(s-),y)|^p - |x(s-)|^p \\ &\qquad - p|x(s-)|^{p-2}x(s-)^T H(x(s-),y)\Big)\nu(dy)ds\Bigg\}\Bigg]. \end{aligned}$$

Since the integrals have the same form as in (3.42), applying Kunita's estimate (3.43) it follows that

$$\begin{aligned} I_1 &\leq c_7(p)E\left[\left(\int_{t_0+n-1}^{t_0+n}\int_{|y|<c} |H(x(s-),y)|^2\nu(dy)ds\right)^{\frac{p}{2}}\right] \\ &\qquad + c_8(p)E\left[\left(\int_{t_0+n-1}^{t_0+n}\int_{|y|<c} |H(x(s-),y)|^p\nu(dy)ds\right)\right]. \quad (3.48) \end{aligned}$$

An application of Holder's inequality to the first term on the right-hand side of (3.48) yields

$$\begin{aligned} I_1 &\leq c_7(p,t)E\left[\int_{t_0+n-1}^{t_0+n}\left(\int_{|y|<c} |H(x(s-),y)|^2\nu(dy)\right)^{\frac{p}{2}} ds\right] \\ &\qquad + c_8(p)E\left[\left(\int_{t_0+n-1}^{t_0+n}\int_{|y|<c} |H(x(s-),y)|^p\nu(dy)ds\right)\right]. \quad (3.49) \end{aligned}$$

Using assumption 3.2.2 within (3.49), we obtain

$$\begin{aligned} I_1 &\leq c_7(p,t)E\left[\int_{t_0+n-1}^{t_0+n} (K|x(s-)|^2)^{\frac{p}{2}}\, ds\right] + c_8(p)E\left[\int_{t_0+n-1}^{t_0+n} K|x(s-)|^p ds\right] \\ &= C_p(t)E\left[\int_{t_0+n-1}^{t_0+n} |x(s-)|^p ds\right] \quad (3.50) \end{aligned}$$

where $C_p(t) = c_7(p,t)K^{\frac{p}{2}} + c_8(p)K$. By (3.47) and (3.50) then (3.46) becomes

$$E\left[\sup_{t_0+n-1\le t\le t_0+n}|x(t)|^p\right] \le E[|x(t_0+n-1)|^p] + \frac{1}{2}E\left[\sup_{t_0+n-1\le t\le t_0+n}|x(t-)|^p\right]$$
$$+(c_1 + 16p^2L' + C_p(t))\left(\int_{t_0+n-1}^{t_0+n}E[|x(s)|^p]ds\right). \quad (3.51)$$

Rearranging, for $p \ge 2$

$$E\left[\sup_{t_0+n-1\le t\le t_0+n}|x(t)|^p\right] \le 2E[|x(t_0+n-1)|^p] + C\int_{t_0+n-1}^{t_0+n}E[|x(s)|^p]ds \quad (3.52)$$

where C is a positive constant depending on p, t, K and L'.
Define $L_1 = E[|x(t_0+n-1)|^p]$. Since the solution is pth moment exponentially stable, then by (3.45)

$$L_1 = E[|x(t_0+n-1)|]^p \le C'|x_0|^p e^{-\lambda(n-1)}.$$

Define

$$L_2 = \int_{t_0+n-1}^{t_0+n}E[|x(r)|^p]dr.$$

Again using (3.45) and integrating, we see that

$$\begin{aligned} L_2 = \int_{t_0+n-1}^{t_0+n}E[|x(r)|^p]dr &\le C'\frac{|x_0|^p}{\lambda}\left(e^{-\lambda(n-1)} - e^{-\lambda n}\right) \\ &\le C'\frac{|x_0|^p}{\lambda}e^{-\lambda(n-1)}. \end{aligned}$$

Hence,

$$E\left[\sup_{t_0+n-1\le s\le t_0+n}|x(s)|^p\right] \le \left\{2L_1 + CL_2\right\} = C_3e^{-\lambda(n-1)} \quad (3.53)$$

where $C_3 = |x_0|^p(2C' + \frac{CC'}{\lambda})$.
Now we argue as in Mao [33] pp. 129-130 and the required result follows. □

In applications Lyapunov functions are often taken to be *quadratic functions* of the form

$$V(x) = x^TQx \quad (3.54)$$

where Q is a symmetric positive definite matrix.

In the following corollary we will give some explicit conditions on the coefficients of (3.1) using a Lyapunov function based on (3.54), in order to prove that the trivial solution of (3.1) is pth moment exponentially stable for $p > 0$. This is a generalization of Corollary 4.6, Chapter 4 pp. 131-132 in Mao [33].

For completeness, we first provide a proof of the following well-known result, that will be used in the corollary below and in other parts of this thesis.

Proposition 3.5.6 If D is a $d \times d$ real-valued symmetric matrix then for all $x \neq 0$

$$\min\{\mu_i\} \leq \frac{x^T D x}{|x|^2} \leq \max\{\mu_i\} \tag{3.55}$$

where μ_i, $1 \leq i \leq d$ are the eigenvalues of the matrix D.

Proof: If D is a $d \times d$ symmetric matrix then it has real eigenvalues $\mu_1, \ldots, \mu_d$ and there exists an orthogonal matrix O such that $O^T D O = \text{diag}(\mu_1, \ldots, \mu_d)$. Hence,

$$\begin{aligned} \frac{x^T D x}{|x|^2} &= \frac{x^T D x}{x^T x} = \frac{x^T O O^T D O O^T x}{x^T O O^T x} = \frac{x_1^T O^T D O x_1}{x_1^T x_1} \quad \text{where} \quad x_1 = O^T x \\ &= \frac{\sum_{i=1}^d \mu_i (x_1^i)^2}{\sum_{i=1}^d (x_1^i)^2}. \end{aligned}$$

Hence,

$$\min\{\mu_i\} \leq \frac{\sum_{i=1}^d \mu_i (x_1^i)^2}{\sum_{i=1}^d (x_1^i)^2} \leq \max\{\mu_i\}$$

and the required result follows. □

Corollary 3.5.7 Assume that for all $x \in \mathbb{R}^d$ $\int_{|y|<c} |x^T Q H(x,y)| \nu(dy) < \infty$ and the following assumptions hold:

$$\begin{aligned} &(i) \quad x^T Q f(x) + \frac{1}{2}\text{tr}\left(g^T(x) Q g(x)\right) - \int_{|y|<c} x^T Q H(x,y) \nu(dy) \leq b_1 x^T Q x, \\ &(ii) \quad b_2 x^T Q x \leq |x^T Q g(x)| \leq b_3 x^T Q x, \\ &(iii) \quad \int_{|y|<c} [((x + H(x,y))^T Q (x + H(x,y))^{\frac{p}{2}} - (x^T Q x)^{\frac{p}{2}} \nu(dy)] \leq b_4 p (x^T Q x)^{\frac{p}{2}}, \end{aligned}$$

where Q is a symmetric positive-definite $d \times d$ matrix and b_1 to b_4 are constants.
(a) If $b_1 < 0$ and $b_4 < 0$ then the solution to (3.1) is pth moment exponentially stable if it holds that $p < 2 + 2|b_1|/{b_3}^2 + 2|b_4|/b_3^2$.

(b) If $b_1 \geq 0$, $b_4 \geq 0$ such that $0 \leq b_1 + b_4 < b_2^2$ then the trivial solution of (3.1) is pth moment exponentially stable if it holds that $p < 2 - 2b_1/{b_2}^2 - 2b_4/{b_2}^2$.

Proof: Since Q is a symmetric positive-definite matrix it has real positive eigenvalues $\lambda_1, \ldots, \lambda_d$. Define λ_{min} and λ_{max} to be the smallest and largest eigenvalues of Q respectively. Then by proposition 3.5.6 we see that for $x \neq 0$

$$\lambda_{min}|x|^2 \leq x^T Q x \leq \lambda_{max}|x|^2.$$

and for $p \geq 2$, we have that

$$\lambda_{min}{}^{\frac{p}{2}}|x|^p \leq (x^T Q x)^{\frac{p}{2}} \leq \lambda_{max}{}^{\frac{p}{2}}|x|^p.$$

Define $V(x) = (x^T Q x)^{\frac{p}{2}}$. Hence,

$$b_1|x|^p \leq V(x) \leq b_2|x|^p$$

where b_1, b_2 are positive constants and we can see that condition (i) from Theorem 3.5.1 is satisfied.

Now the linear operator $\mathcal{L}$ as given by (1.19) applied to the function V yields

$$\begin{aligned} \mathcal{L}V(x) &= p(x^TQx)^{(\frac{p}{2}-1)}(x^TQf(x) + \frac{1}{2}\text{tr}(g^T(x)Qg(x)) + p\left(\frac{p}{2}-1\right)(x^TQx)^{(\frac{p}{2}-2)}|x^TQg(x)|^2 \\ &\quad + \int_{|y|<c}\Bigg[((x+H(x,y))^TQ(x+H(x,y)))^{\frac{p}{2}} - (x^TQx)^{\frac{p}{2}} \\ &\qquad - p(x^TQx)^{(\frac{p}{2}-1)}x^TQH(x,y)\Bigg]\nu(dy). \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{L}V(x) &= p(x^TQx)^{(\frac{p}{2}-1)}\left(x^TQf(x) + \frac{1}{2}\text{tr}(g^T(x)Qg(x)) - \int_{|y|<c} x^TQH(x,y)\nu(dy)\right) \\ &+ p\left(\frac{p}{2}-1\right)(x^TQx)^{(\frac{p}{2}-2)}|x^TQg(x)|^2 \\ &+ \int_{|y|<c}\left[((x+H(x,y))^TQ(x+H(x,y)))^{\frac{p}{2}} - (x^TQx)^{\frac{p}{2}}\right]\nu(dy). \end{aligned}$$

Case 1: $b_1 < 0$, $b_4 < 0$ and $p < 2 + 2|b_1|/b_3^2 + 2|b_4|/b_3^2$
This implies that $|b_1| - \left(\frac{p}{2}-1\right){b_3}^2 + |b_4| > 0$. Applying conditions (i) to (iii) then,

$$\begin{aligned} \mathcal{L}V(x) &\leq -p\left[|b_1| - \left(\frac{p}{2}-1\right){b_3}^2 + |b_4|\right]V(x) \\ \text{i.e.} \quad \mathcal{L}V(x) &\leq -c_3V(x) \end{aligned}$$

where $c_3 = p\left[|b_1| - \left(\frac{p}{2}-1\right){b_3}^2 + |b_4|\right]$ is a positive constant.

We see that under these assumptions condition (ii) of Theorem 3.5.1 is satisfied, and it follows that the trivial solution of (3.1) is pth moment exponentially stable.

Case 2: $b_1 \geq 0$, $b_4 \geq 0$ such that $0 \leq b_1 + b_4 < b_2^2$ and $p < 2 - 2b_1/b_2^2 - 2b_4/b_2^2$.
Again applying conditions (i) to (iii) then,

$$\begin{aligned} \mathcal{L}V(x) &\leq -p\left[-b_1 - \left(\frac{p}{2} - 1\right) {b_2}^2 - b_4\right] V(x) \\ \text{i.e.} \quad \mathcal{L}V(x) &\leq -c_4 V(x) \end{aligned}$$

where $c_4 = p\left[-b_1 - \left(\frac{p}{2} - 1\right) {b_2}^2 - b_4\right]$ is a positive constant.
Since the conditions of Theorem 3.5.1 are satisfied then again the trivial solution of (3.1) is pth moment exponentially stable. □

Chapter 4

Stochastic Stabilization and Destabilization

4.1 Introduction

Assume that we have the system of non-linear ordinary differential equations

$$\frac{dx(t)}{dt} = f(x(t)) \qquad \text{on } t \geq t_0 \tag{4.1}$$

with $x(t_0) = x_0 \in \mathbb{R}^d$, where $f : \mathbb{R}^d \to \mathbb{R}^d$.

Assumption 4.1.1 We assume that $f : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz continuous function and furthermore, for some $K > 0$

$$|f(x)| \leq K|x| \quad \text{for all } x \in \mathbb{R}^d, \tag{4.2}$$

e.g. $f(x) = xe^{-x^2}$ for $x \in \mathbb{R}^d$ or $f(x) = 1 - e^{-x}$ for $x > 0$.

We require that this assumption holds for the rest of this chapter.

Now suppose that the non-linear system (4.1) is perturbed by random noise and therefore a stochastic system is created. It is possible when a system is perturbed by noise that it can be stabilized if it is unstable, in the sense that by adding noise we can force the solution of the stochastic differential equation to converge to the trivial solution as time increases indefinitely. This is the goal of stochastic stabilization. On the other hand, a system can be destabilized when perturbed by noise, in the case that the unperturbed system is stable. In this instance, by adding noise the sample paths of the process escape to infinity (a.s.) instead of converging to the trivial solution as time increases indefinitely.

These ideas of stabilization have evolved through the work of Bellman [6] and Merkov [36] who used deterministic periodic noise to stabilize a system. Pioneering work on stochastic stabilization has been carried out by Khasminski and Arnold in [22] and [5] respectively. Khasminski has used two independent white noises to stabilize a system and Arnold has shown that a linear ODE system of the form $dx(t) = Ax(t)dt$ can be stabilized by a zero mean stationary noise if and only if $trA < 0$. Mao in [31] and [33] pp. 135-141 uses an m-dimensional Brownian motion process $B = (B(t), t \geq 0)$, where for each $t \geq 0$ $B(t) = (B^1(t), \ldots, B^m(t))$, as the source of noise to perturb the given system (4.1) and he establishes a general theory of stochastic stabilization for the given non-linear system. The perturbed system has the form:

$$dx(t) = f(x(t))dt + \sum_{k=1}^{m} G_k x(t) dB_k(t) \qquad \text{on} \quad t \geq 0 \tag{4.3}$$

where $G_k \in \mathcal{M}_d(\mathbb{R})$, for $1 \leq k \leq m$, and initial condition $x(0) = x_0 \in \mathbb{R}^d$.

Motivated by Mao's theory (see Mao [31] and [33] pp. 135-141) we would like to extend his results and investigate conditions under which the given non-linear system (4.1) can be stabilized or destabilized, if it is perturbed by a more general noise. The main purpose of this chapter is to examine sufficient criteria for stability of non-linear deterministic systems when they are perturbed by Lévy-type stochastic integral terms. In fact we will examine almost sure exponential stability.

Although moment exponential stability is stronger than almost sure exponential stability (see Chapter 3, Theorem 3.5.5); as shown by Kozin [24] pp. 107 and Mao [33] it is possible for the stochastically perturbed system to have unbounded moments but to be almost surely exponentially stable. Hence for examining stabilization is better to avoid moment properties and aim to stabilize (4.1) in an almost sure exponential way.

In the following we will give the proper mathematical set up for this chapter.
Assume that we are given an m-dimensional standard $\mathcal{F}_t$-adapted Brownian motion process $B = (B(t), t \geq 0)$ with $B(t) = (B^1(t), \ldots, B^m(t))$ for each $t \geq 0$ and an independent $\mathcal{F}_t$-adapted Poisson random measure N defined on $\mathbb{R}^+ \times (\mathbb{R}^m \setminus \{0\})$ with compensator $\tilde{N}$ of the form $\tilde{N}(dt, dy) = N(dt, dy) - \nu(dy)dt$ where ν is a Lévy measure.

In this chapter the stochastically perturbed system corresponding to (4.1) will have the following form

$$\begin{aligned} dx(t) &= f(x(t-))dt + \sum_{k=1}^{m} G_k x(t-) dB_k(t) + \int_{|y|<c} D(y)x(t-)\tilde{N}(dt, dy) \\ &\quad + \int_{|y|\geq c} E(y)x(t-)N(dt, dy) \end{aligned} \tag{4.4}$$

for all $t \geq t_0$, where $G_k \in \mathcal{M}_d(\mathbb{R})$, for $1 \leq k \leq m$, and D and E are suitable matrix-valued functions.

Given the initial condition $x(t_0) = x_0 \in \mathbb{R}^d$, the uniqueness and existence of the solution of the SDE (4.4) when $f : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz continuous function, is guaranteed from Theorem 4.2.1 that will be proved in the sequel (see section 4.2). The unique solution to (4.4) is denoted by $x(t)$ for each $t \geq t_0$. For all $x \in \mathbb{R}^d$ define $g(x) = \sum_{k=1}^{m} G_k x$, $H(x, y) = D(y)x$ for $|y| < c$ and $K(x, y) = E(y)x$ for $|y| \geq c$. Assume that $f(0) = 0$, then (4.4) has a solution $x(t) = 0$ for all $t \geq t_0$ corresponding to the initial value $x(t_0) = 0$, which is the trivial solution.

This chapter will be organized as follows:
In section 4.2 we will give the proof for the existence and uniqueness of solutions for Lévy driven SDEs with locally Lipschitz coefficients under some additional conditions. In section 4.3 we will describe some of the mathematical tools needed to develop the analysis in this chapter. In the other sections we turn our attention to the stabilizing effects of the noise when it is added to the deterministic system (4.1). Our aim in the later sections is to find conditions under which the Lyapunov exponent is negative so that the solution of the stochastically perturbed system of the non-linear deterministic system is almost sure exponentially stable. In order to make our discussion simple we will examine separately the perturbation of (4.1) first by small jumps (by a compensated Poisson integral) and then by large jumps (by a Poisson integral), in sections 4.4 and 4.5 respectively. In section 4.6 we will combine the results of sections 4.4 and 4.5 in order to have conditions for almost sure exponential stability for the trivial solution of the stochastically perturbed system driven by a Lévy process. In sections 4.7 and 4.8 we will consider some special cases of the theory that was developed in the previous sections. In section 4.7 we will examine the stabilization of a one-dimensional linear deterministic system perturbed by a Brownian motion and a single Poisson process. Finally, in section 4.8 we focus on the stabilization of a non-linear system perturbed with Brownian motion and a compensated Poisson process and on the destabilization of a stable SDE driven by Poisson noise perturbed with Brownian motion.

4.2 Existence and uniqueness theorem for Lévy driven SDEs with locally Lipschitz coefficients

Since $f : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz continuous function and not necessarily Lipschitz continuous it is not possible to apply Theorem 1.4.1 (i). In the following we will prove the existence of a unique solution to SDEs driven by a Lévy process when the coefficients satisfy local Lipschitz conditions and the growth conditions (C2).

Consider the following general d-dimensional SDE

$$dx(t) = f(x(t-))dt + g(x(t-))dB(t) + \int_{|y|<c} H(x(t-), y)\tilde{N}(dt, dy) \tag{4.5}$$

on $t_0 \leq t \leq T$ with initial value $x(t_0) = x_0$, such that $x_0 \in \mathbb{R}^d$.

For the rest of this section we impose the following conditions on the coefficients of (4.5).

Assume that the growth conditions (C2) as defined in Chapter 1 hold and

Local Lipschitz conditions: For every $n \geq 1$ there exists a positive integer K_n such that for all $x_1, x_2 \in \mathbb{R}^d$ with $|x_1| \vee |x_2| \leq n$

$$|f(x_1) - f(x_2)|^2 \leq K_n|x_1 - x_2|^2, \qquad \|g(x_1) - g(x_2)\|^2 \leq K_n|x_1 - x_2|^2 \quad \text{and} \tag{4.6}$$

$$\int_{|y|<c} |H(x_1, y) - H(x_2, y)|^2\nu(dy) \leq K_n|x_1 - x_2|^2. \tag{4.7}$$

In Applebaum [1] Theorem 6.2.11 pp. 313 it has been proved that under local Lipschitz and local growth conditions on the coefficients of (4.5) there exists a unique local solution. As we will show in the sequel under the local Lipschitz conditions and the global growth conditions (C2) we can prove that their exists a global solution to (4.5).

The following is a generalization of Mao's work [33] pp. 57, that deals with the case $H \equiv 0$.

Theorem 4.2.1 *Assume that the local Lipschitz conditions (4.6), (4.7) and linear growth conditions (C2) hold. Under these conditions then (4.5) has a unique càdlàg, adapted solution for all* $t \in [t_0, T]$.

Before we prove Theorem 4.2.1 we need to prepare a lemma.

For $n \geq 1$ define

$$f_n(x) = \begin{cases} f(x) & \text{if } |x| \leq n \\ f\left(\frac{nx}{|x|}\right) & \text{if } |x| > n \end{cases} \tag{4.8}$$

$$g_n(x) = \begin{cases} g(x) & \text{if } |x| \leq n \\ g\left(\frac{nx}{|x|}\right) & \text{if } |x| > n \end{cases} \tag{4.9}$$

$$H_n(x, y) = \begin{cases} H(x, y) & \text{if } |x| \leq n, |y| < c \\ H\left(\frac{nx}{|x|}, y\right) & \text{if } |x| > n, |y| < c \end{cases} \tag{4.10}$$

where f, g and H satisfy the local Lipschitz conditions (4.6) and (4.7), and the growth conditions (C2).

Lemma 4.2.2 Let f_n, g_n, H_n be defined as in (4.8), (4.9) and (4.10) respectively. Then f_n, g_n and H_n satisfy the global Lipschitz conditions (C1).

Proof: As $\left|\frac{nx_i}{|x_i|}\right| = n$ for all $x_i \in \mathbb{R}^d$, $i = 1, 2$, when $|x_1|, |x_2| > n$ we have

$$\begin{aligned} |f_n(x_1) - f_n(x_2)|^2 &= \left|f\left(\frac{nx_1}{|x_1|}\right) - f\left(\frac{nx_2}{|x_2|}\right)\right|^2 \leq K_n \left|\frac{nx_1}{|x_1|} - \frac{nx_2}{|x_2|}\right|^2 = n^2 K_n \left(2 - \frac{2x_1.x_2}{|x_1|.|x_2|}\right) \\ &= \frac{n^2 K_n}{|x_1|.|x_2|} \left(2|x_1|.|x_2| - 2x_1.x_2\right), \end{aligned} \tag{4.11}$$

with $K_n > 0$. Then by (4.11) we have that

$$\begin{aligned} &|f_n(x_1) - f_n(x_2)|^2 - \frac{n^2 K_n}{|x_1|.|x_2|} |x_1 - x_2|^2 \\ &\quad \leq \frac{n^2 K_n}{|x_1|.|x_2|}\Big(2|x_1|.|x_2| - 2x_1.x_2\Big) - \frac{n^2 K_n}{|x_1|.|x_2|}\Big(|x_1|^2 + |x_2|^2 - 2x_1.x_2\Big) \\ &\quad = \frac{n^2 K_n}{|x_1|.|x_2|} \left(2|x_1|.|x_2| - |x_1|^2 - |x_2|^2\right) = -\frac{n^2 K_n}{|x_1|.|x_2|} \left(|x_1| - |x_2|\right)^2 \leq 0. \end{aligned}$$

Hence,

$$|f_n(x_1) - f_n(x_2)|^2 \leq \frac{n^2 K_n}{|x_1|.|x_2|} |x_1 - x_2|^2 \leq K_n |x_1 - x_2|^2.$$

Now for $|x_1|, |x_2| \leq n$ the local Lipschitz constant is K_n.

For $|x_1| > n$ and $|x_2| \leq n$ we have

$$\begin{aligned} |f_n(x_1) - f_n(x_2)|^2 &= \left|f\left(\frac{nx_1}{|x_1|}\right) - f(x_2)\right|^2 \leq K_n \left|\frac{nx_1}{|x_1|} - x_2\right|^2 \\ &= K_n \left(n^2 + |x_2|^2 - \frac{2nx_1.x_2}{|x_1|}\right). \end{aligned} \tag{4.12}$$

By (4.12) it follows that

$$\begin{aligned} &|f_n(x_1) - f_n(x_2)|^2 - \frac{nK_n}{|x_1|} |x_1 - x_2|^2 \\ &\leq K_n \left(n^2 + |x_2|^2 - \frac{2nx_1.x_2}{|x_1|}\right) - \frac{nK_n}{|x_1|} \left(|x_1|^2| + |x_2|^2 - 2x_1.x_2\right) \\ &= K_n \left(|x_2|^2 \left(1 - \frac{n}{|x_1|}\right) + n(n - |x_1|)\right) = K_n \left(\frac{|x_2|^2}{|x_1|} (|x_1| - n) + n(n - |x_1|)\right) \\ &\leq K_n \left(n\left(|x_1| - n\right) - n\left(|x_1| - n\right)\right) = 0, \end{aligned}$$

and this implies that

$$|f_n(x_1) - f_n(x_2)|^2 \leq \frac{nK_n}{|x_1|} |x_1 - x_2|^2 \leq K_n |x_1 - x_2|^2 .$$

The same arguments applies for $|x_2| > n$ and $|x_1| \leq n$.

The arguments for g_n and H_n are similar in all cases.

Hence with this truncation procedure we have shown that for all $x_1, x_2 \in \mathbb{R}^d$, f_n, g_n, H_n satisfy the global Lipschitz conditions (C1) with Lipschitz constant $L = K_n$. □

Now we are ready to proceed with the proof of Theorem 4.2.1.

Proof of existence: Define for $n \geq 1$

$$x_n(t) = x_0 + \int_{t_0}^{t} f_n(x_n(s-))ds + \int_{t_0}^{t} g_n(x_n(s-))dB(s) + \int_{t_0}^{t} \int_{|y|<c} H_n(x_n(s-), y)\tilde{N}(ds, dy) \tag{4.13}$$

where f_n, g_n, H_n satisfy (4.8), (4.9) and (4.10) respectively. Then by Lemma 4.2.2 f_n, g_n and H_n satisfy the global Lipschitz conditions (C1) with Lipschitz constants $L = K_n$, and the growth conditions (C2). Hence, by Theorem 1.4.1 (i) for all $t \in [t_0, T]$ there exists a unique solution to (4.13).

Now define the stopping times

$$\sigma_n = \inf \{t \in [t_0, T] : |x_n(t)| \geq n\} \quad \text{and} \quad \tau_n = T \wedge \sigma_n. \tag{4.14}$$

We claim that for $n \geq 1$

$$x_n(t) = x_{n+1}(t) \;\; (a.s.) \;\; \text{if} \;\; t_0 \leq t \leq \tau_n. \tag{4.15}$$

For $t_0 \leq t \leq \tau_n$, we have $|x_n(t)| \leq n$ (a.s.). By (4.8) it follows that $f_n(x_n(t)) = f(x_n(t))$ (a.s.). Also since $|x_n(t)| \leq n < n+1$, then $f_{n+1}(x_n(t)) = f(x_n(t))$ (a.s.). Hence,

$$f_n(x_n(t)) = f_{n+1}(x_n(t)) \;\; (a.s) \quad \text{for } t_0 \leq t \leq \tau_n. \tag{4.16}$$

The same applies for the diffusion and jump coefficient.

Hence for $t_0 \leq t \leq \tau_n$ by (4.16) we have that with probability 1

$$
\begin{aligned}
&x_{n+1}(t) - x_n(t) \\
&= \int_{t_0}^{t} [f_{n+1}(x_{n+1}(s-)) - f_n(x_n(s-))]\, ds + \int_{t_0}^{t} [g_{n+1}(x_{n+1}(s-)) - g_n(x_n(s-))]\, dB(s) \\
&\quad + \int_{t_0}^{t}\int_{|y|<c} [H_{n+1}(x_{n+1}(s-), y) - H_n(x_n(s-), y)]\, \tilde{N}(ds, dy) \\
&= \int_{t_0}^{t} [f_{n+1}(x_{n+1}(s-)) - f_{n+1}(x_n(s-))]\, ds + \int_{t_0}^{t} [g_{n+1}(x_{n+1}(s-)) - g_{n+1}(x_n(s-))]\, dB(s) \\
&\quad + \int_{t_0}^{t}\int_{|y|<c} [H_{n+1}(x_{n+1}(s-), y) - H_{n+1}(x_n(s-), y)]\, \tilde{N}(ds, dy). \qquad (4.17)
\end{aligned}
$$

Using (1.12), with $n = 3$, taking expectations, then using standard arguments (see e.g. Applebaum [1] pp. 306-307) and applying the global Lipschitz conditions (C1) with $L = K_n$ we see that

$$
E\left[\sup_{t_0 \le s \le t} |x_{n+1}(s \wedge \tau_n) - x_n(s \wedge \tau_n)|^2\right] \le C_n \int_{t_0}^{t} E\left[\sup_{t_0 \le r \le s} |x_{n+1}(r \wedge \tau_n) - x_n(r \wedge \tau_n)|^2\right] ds,
$$

where $C_n > 0$.

Now applying Gronwall's inequality we get

$$
E\left[\sup_{t_0 \le s \le t} |x_{n+1}(s \wedge \tau_n) - x_n(s \wedge \tau_n)|^2\right] \le 0.
$$

Hence the claim (4.15) is verified.

Now, since (4.15) holds for $t_0 \le t \le \tau_n$, we have that

$$
\begin{aligned}
\tau_n &= T \wedge \inf\{t \in [t_0, T] : |x_{n+1}(t)| \ge n\} \\
&\le T \wedge \inf\{t \in [t_0, T] : |x_{n+1}(t)| \ge n+1\} \\
&= \tau_{n+1}. \qquad (4.18)
\end{aligned}
$$

Hence, $(\tau_n, n \in \mathbb{N})$ is an increasing sequence.

For $t \in [t_0, T]$, then

$$
\begin{aligned}
x_n(t \wedge \tau_n) &= x_0 + \int_{t_0}^{t} f_n(x_n(s-)) 1_{[t_0, \tau_n]} ds + \int_{t_0}^{t} g_n(x_n(s-)) 1_{[t_0, \tau_n]} dB(s) \\
&\quad + \int_{t_0}^{t}\int_{|y|<c} H_n(x_n(s-), y) 1_{[t_0, \tau_n]} \tilde{N}(ds, dy). \qquad (4.19)
\end{aligned}
$$

Using (1.12), with $n = 4$, and following standard arguments (see e.g. Applebaum [1] pp. 306) we can see that

$$E\left[\sup_{t_0\le s\le t}\left(1+|x_n(s\wedge\tau_n)|^2\right)\right] \le 4\left(1+|x_0|^2\right)+4(T-t_0)\int_{t_0}^{t}E\Big[\big|f_n(x_n(s-))1_{[t_0,\tau_n]}\big|^2\Big]ds$$
$$+16\int_{t_0}^{t}E\Big[\big|g_n(x_n(s-))1_{[t_0,\tau_n]}\big|^2\Big]ds$$
$$+16\int_{t_0}^{t}\int_{|y|<c}E\Big[\big|H_n(x_n(s-),y)1_{[t_0,\tau_n]}\big|^2\Big]\nu(dy)ds. \tag{4.20}$$

Now applying the growth conditions (C2) and Gronwall's inequality, there exists $K'>0$ such that

$$E\left[\sup_{t_0\le s\le t}|x_n(s\wedge\tau_n)|^2\right]\le E\left[\sup_{t_0\le s\le t}\left(1+|x_n(s\wedge\tau_n)|^2\right)\right]\le 4(1+|x_0|^2)e^{4K'(T-t_0)} \tag{4.21}$$

By Chebyshev's inequality and (4.21), for $n\ge 1$ we obtain that

$$P\left(\sup_{t_0\le s\le t}|x_n(s\wedge\tau_n)|\ge n\right)\le\frac{E\left[\sup_{t_0\le s\le t}|x_n(s\wedge\tau_n)|^2\right]}{n^2}\le\frac{4(1+|x_0|)^2e^{4K'(T-t_0)}}{n^2}.$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty$ we can apply the Borel-Cantelli Lemma to deduce that for almost all $\omega\in\Omega$ there is a random integer $n_0=n_0(\omega)\in\mathbb{N}$ such that for $n\ge n_0(\omega)$

$$P\left(\liminf_{n\to\infty}\sup_{t_0\le s\le t}|x_n(s\wedge\tau_n)|<n\right)=1.$$

Hence for $n\ge n_0(\omega)$, $|x_n(t\wedge\tau_n)|<n$ a.s.

Assume that $\sigma_n\to T_0<T$ as $n\to\infty$ (a.s.). Then it follows that there exists an integer $m_0\in\mathbb{N}$ such that $|x_m(t\wedge\tau_n)|\ge m$ for all $m\ge m_0$ and this yields a contradiction. Hence we can deduce that for $\sigma_n\to\infty$ (a.s.) and $\tau_n\to T$ (a.s.) as $n\to\infty$.

Now define $x'(t)=x_{n_0}(t)$ for $t\in[t_0,T]$.
For $n\ge n_0$, applying (4.15) repeatedly yields that $x_{n_0}(t\wedge\tau_n)=x_n(t\wedge\tau_n)$. Hence, $x'(t\wedge\tau_n)=x_n(t\wedge\tau_n)$. Then (4.13) becomes

$$x'(t\wedge\tau_n) = x_0+\int_{t_0}^{t\wedge\tau_n}f_n(x'(s-))ds+\int_{t_0}^{t\wedge\tau_n}g_n(x'(s-))dB(s)$$
$$+\int_{t_0}^{t\wedge\tau_n}\int_{|y|<c}H_n(x'(s-),y)\tilde{N}(ds,dy). \tag{4.22}$$

Since for $n\ge n_0$, $|x'(t\wedge\tau_n)|\le n$ then by (4.8), (4.9) and (4.10), (4.22) can be written as

$$x'(t \wedge \tau_n) = x_0 + \int_{t_0}^{t\wedge\tau_n} f(x'(s-))ds + \int_{t_0}^{t\wedge\tau_n} g(x'(s-))dB(s) + \int_{t_0}^{t\wedge\tau_n}\int_{|y|<c} H(x'(s-),y)\tilde{N}(ds,dy).$$

Let $n \to \infty$. Since $\lim_{n\to\infty}\tau_n = T$ (a.s.) then

$$x'(t) = x_0 + \int_{t_0}^{t} f(x'(s-))ds + \int_{t_0}^{t} g(x'(s-))dB(s) + \int_{t_0}^{t}\int_{|y|<c} H(x'(s-),y)\tilde{N}(ds,dy).$$

Hence, we see that $x'(t)$ is a solution to (4.5).

Proof of uniqueness: Let $(x(t), t \in [t_0,T])$ and $(y(t), t \in [t_0,T])$ be two distinct global solutions of (4.5). Let $(\tau_n, n \in \mathbb{N})$ be as defined in (4.14). Since x and y are both local solutions for all $t_0 \le t \le \tau_n$ then by uniqueness of local solutions (see Applebaum [1] Theorem 6.2.11 pp. 313) we have that $x(t) = y(t)$ (a.s.) for all $t_0 \le t \le \tau_n$. Then by the continuity of probability we obtain

$$P\Big(x(t) = y(t) \text{ for all } t \in [t_0,T]\Big) = P\left(\bigcap_{n\in\mathbb{N}} (x(t) = y(t) \text{ for all } t_0 \le t \le \tau_n)\right) = 1,$$

as required. □

Remark 4.2.3 If the assumptions (4.6) and (4.7) hold on every finite subinterval $[t_0,T]$ of $[t_0,\infty)$, along with the growth conditions (C2), then we will have a unique global solution on $[t_0,\infty)$ (see e.g. Mao [33] pp. 58).

We also remark that (4.2) implies the growth condition (C2) on the drift coefficient.

4.3 Some useful results

In order to provide a complete presentation of the arguments in this chapter we first need some ideas from linear algebra.

Lemma 4.3.1 If $x \in \mathbb{R}^d$, with $x \neq 0$ and F is a $d \times d$ real-valued matrix then

$$\log(|(I+F)x|^2) - \log(|x|^2) \le \log\left((1+\|F\|_1)^2\right). \tag{4.23}$$

Proof: $\log(|x+Fx|^2) - \log(|x|^2)$

$$\begin{aligned}
&= \log\left(\frac{|x|^2 + 2x^T Fx + |Fx|^2}{|x|^2}\right) \leq \log\left(\frac{|x|^2 + 2\|F\|_1.|x|^2 + \|F\|_1^2.|x|^2}{|x|^2}\right) \\
&= \log\left(\frac{|x|^2.(1+\|F\|_1)^2}{|x|^2}\right) = \log\left(1+\|F\|_1\right)^2. \qquad (4.24)
\end{aligned}$$

□

Now consider the following general d-dimensional SDE given by

$$dx(t) = f(x(t-))dt + g(x(t-))dB(t) + \int_{|y|<c} H(x(t-),y)\tilde{N}(dt,dy) \qquad \text{on } t \geq t_0 \quad (4.25)$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}^d \to \mathcal{M}_{d,m}(\mathbb{R})$, $H : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ and we have the initial value $x(t_0) = x_0$ where $x_0 \in \mathbb{R}^d$.

We require that assumption 3.4.3 holds for the rest of this chapter.

Assume that (3.21) holds. Then by Lemma 3.4.4 the solution of (4.25) can never reach the origin provided that $x_0 \neq 0$.

Now let $z = (z(t), t \geq t_0)$ be the solution of the following SDE that includes all the noise components of a generic Lévy process.

$$dz(t) = f(z(t-))dt + g(z(t-))dB(t) + \int_{|y|<c} H(z(t-),y)\tilde{N}(dt,dy) + \int_{|y|\geq c} E(y)z(t-)N(dt,dy) \quad (4.26)$$

on $t \geq t_0$ with initial value $z(t_0) = z_0$, such that $z_0 \in \mathbb{R}^d$.

The next result generalizes Lemma 3.4.4 to the case where large jumps are included as in (4.26). We remark that Li, Dong and Situ [28] pp. 123 have obtained the same result for more general jump-diffusion processes but using more complicated techniques.

Lemma 4.3.2 Assume that the hypothesis of Lemma 3.4.4 hold and $I + E(y)$ is invertible for all $y \in B_c(0)^c$. Then

$$P\left(z(t) \neq 0 \text{ for all } t \geq t_0\right) = 1.$$

Proof: Let $(P(t), t \geq 0)$ be the compound Poisson process where for each $t \geq 0$, $P(t) = \int_{|y|\geq c} yN(dt,dy)$ and let $(\tau_n, n \in \mathbb{N})$ be the arrival times for the jumps of the process $(P(t) - P(t_0), t \geq t_0)$. Let $x = (x(t), t \geq t_0)$ be the solution of (4.25) and

$z = (z(t), t \geq t_0)$ be the solution of (4.26) with initial condition $x(t_0) = z(t_0) = y \neq 0$. Using the interlacing technique (see Applebaum [1] Theorem 6.2.9 pp. 311) we will construct a solution to the SDE (4.26) in the following way:

$$\begin{aligned}
z(t) &= x(t) && \text{for } t_0 \leq t < \tau_1, \\
z(\tau_1) &= x(\tau_1-) + E(\Delta P(\tau_1))x(\tau_1-) && \text{for } t = \tau_1, \\
&= (I + E(\Delta P(\tau_1)))x(\tau_1-) && \\
z(t) &= z(\tau_1) + x(t) - x(\tau_1) && \text{for } \tau_1 < t < \tau_2, \\
z(\tau_2) &= (I + E(\Delta P(\tau_2)))z(\tau_2-) && \text{for } t = \tau_2,
\end{aligned}$$

and so on recursively. For $t_0 \leq t < \tau_1$, $z(t) \neq 0$ as in Lemma 3.4.4. Assume that $z(\tau_1) = 0$ (a.s.). Since $I + E(y)$ is invertible, $\text{Ker}(I + E(y)) = \{0\}$ and this implies that $x(\tau_1-) = 0$. But by Lemma 3.4.4 this can never happen so we can conclude that $z(\tau_1) \neq 0$. Now for $\tau_1 < t < \tau_2$, $x(t) - x(\tau_1)$ is a solution to (4.25). Using Lemma 3.4.4 then $x(t) - x(\tau_1) \neq 0$ and since $z(\tau_1) \neq 0$ then this implies that for $\tau_1 < t < \tau_2$, $z(t) \neq 0$. For $t = \tau_2$, the same argument applies here as for $t = \tau_1$. Hence $z(\tau_2) \neq 0$. Now define

$$A_n = \{\omega \in \Omega : z(t)(\omega) \neq 0 \text{ for all } t_0 \leq t \leq \tau_n\}.$$

By iterating the above argument we have that $P(A_n) = 1$ for all $n \in \mathbb{N}$. Since $\tau_n \uparrow \infty$ (a.s.) as $n \to \infty$ then by elementary probability arguments (see e.g. Itô [19] pp. 42) we have that

$$P\left(\bigcap_{n \in \mathbb{N}} A_n\right) = 1$$

and the required result follows. □

4.4 Stabilization of a system by a compensated Poisson integral

In this section we will mainly be interested in the case where $H(x, y) = D(y)x$ in (4.25). So to satisfy assumption 3.4.3 we want $x + D(y)x = 0$ to have no non-trivial solutions (except on a set of possible ν-measure zero) which is true if $D(y)$ is positive definite or more generally $D(y)$ does not have an eigenvalue of -1 (ν almost everywhere).

Hence, we will examine stability conditions for the non-linear deterministic system (4.1) which is perturbed by noise as follows.

$$dx(t) = f(x(t-))dt + \sum_{k=1}^{m} G_k x(t-)dB_k(t) + \int_{|y|<c} D(y)x(t-)\tilde{N}(dt, dy) \tag{4.27}$$

on $t \geq t_0$, $x(t_0) = x_0 \in \mathbb{R}^d$ where $G_k \in \mathcal{M}_d(\mathbb{R})$ for $1 \leq k \leq m$, and $y \to D(y)$ is a measurable map from $\mathbb{R}^m \to \mathcal{M}_d(\mathbb{R})$. We emphasize that the compensated Poisson random measure in (4.27) is defined on $\mathbb{R}^+ \times \mathbb{R}^m$.

Assumption 4.4.1 We require that

$$Z_c = \int_{|y|<c} \left(\|D(y)\|_1 \vee \|D(y)\|_1{}^2 \right) \nu(dy) < \infty,$$

and that $D(y)$ does not have any eigenvalue equal to -1.

By Theorem 4.2.1 we can assert the existence and uniqueness of the solution to (4.27). The diffusion and jump coefficients are linear and taking into account assumption 4.4.1 it can easily be shown that they satisfy global Lipschitz and global growth conditions. Now the drift coefficient satisfies assumption 4.1.1 and hence by Theorem 4.2.1 and remark 4.2.3 we deduce that (4.27) has a unique solution.

In the following theorem we will establish conditions on the coefficients of (4.27) for the trivial solution of the perturbed system (4.27) to be almost surely exponentially stable. In particular this indicates that the compensated Poisson noise can have a similar role to the Brownian motion (as in Mao [31] and [33] pp. 135-141) in stabilizing dynamical systems.

Theorem 4.4.2 *Let assumption 3.4.3 and (3.21) hold. Suppose that the following conditions are satisfied where* $\xi > 0, \gamma \geq 0,\ \delta \geq 0$

$$(i) \sum_{k=1}^{m} |G_k x|^2 \leq \xi |x|^2 \qquad \text{and} \qquad (ii) \sum_{k=1}^{m} |x^T G_k x|^2 \geq \gamma |x|^4$$

$$(iii) \int_{|y|<c} x^T D(y) x \, \nu(dy) \geq \delta |x|^2$$

for all $x \in \mathbb{R}^d$*. Then the sample Lyapunov exponent of the solution of (4.27) exists and satisfies*

$$\limsup_{t \to \infty} \frac{1}{t} \log |x(t)| \leq -\left(\gamma - K - \frac{\xi}{2} - \int_{|y|<c} \log \left(1 + \|D(y)\|_1\right) \nu(dy) + \delta \right) \quad a.s.$$

for any $x_0 \neq 0$*. If* $\gamma > K + \frac{\xi}{2} - \delta + \int_{|y|<c} \log \left(1 + \|D(y)\|_1\right) \nu(dy)$ *then the trivial solution of the system (4.27) is almost surely exponentially stable.*

Proof: Fix $x_0 \neq 0$. Due to Lemma 3.4.4, then $x(t) \neq 0$ for all $t \geq t_0$ almost surely. Apply Itô's formula with jumps (see Theorem 1.3.2) to $\log(|x(t)|^2)$. Hence for each $t \geq t_0$

$$\begin{aligned}\log(|x(t)|^2) &= \log(|x_0|^2) + \int_{t_0}^{t} \frac{2x(s-)^T}{|x(s-)|^2} f(x(s-))\,ds + \sum_{k=1}^{m} \int_{t_0}^{t} \frac{2x(s-)^T}{|x(s-)|^2} G_k x(s-)\,dB_k(s) \\ &\quad + \sum_{k=1}^{m} \int_{t_0}^{t} \left(\frac{|G_k x(s-)|^2}{|x(s-)|^2} - \frac{2|x(s-)^T G_k x(s-)|^2}{|x(s-)|^4} \right) ds \\ &\quad + \int_{t_0}^{t} \int_{|y|<c} \left[\log\left(|x(s-) + D(y)x(s-)|^2\right) - \log\left(|x(s-)|^2\right) \right] \tilde{N}(ds, dy) \\ &\quad + \int_{t_0}^{t} \int_{|y|<c} \left[\log\left(|x(s-) + D(y)x(s-)|^2\right) - \log\left(|x(s-)|^2\right) \right. \\ &\qquad \left. - \frac{2x(s-)^T D(y) x(s-)}{|x(s-)|^2} \right] \nu(dy)ds. \end{aligned} \tag{4.28}$$

Applying Lemma 4.3.1 and condition (iii) we obtain an estimate for the last integral in (4.28):

$$\begin{aligned} &\int_{t_0}^{t} \int_{|y|<c} \left[\log\left(|x(s-) + D(y)x(s-)|^2\right) - \log\left(|x(s-)|^2\right) - \frac{2x(s-)^T D(y) x(s-)}{|x(s-)|^2} \right] \nu(dy)ds \\ &\leq \int_{t_0}^{t} \int_{|y|<c} \left[\log\left(1 + \|D(y)\|_1\right)^2 \nu(dy) - 2\delta \right] ds \\ &= 2(t - t_0) \left[\int_{|y|<c} \log\left(1 + \|D(y)\|_1\right) \nu(dy) - \delta \right] \end{aligned} \tag{4.29}$$

where we note that $\int_{|y|<c} \log\left(1 + \|D(y)\|_1\right) \nu(dy) < \infty$, by the logarithmic inequality (1.10) and assumption 4.4.1.

Define $M_1 = (M_1(t), t \geq t_0)$ where for each $t \geq t_0$

$$M_1(t) = 2 \sum_{k=1}^{m} \int_{t_0}^{t} \frac{x(s-)^T G_k x(s-)}{|x(s-)|^2}\, dB_k(s).$$

This is a continuous local martingale. We also define the process $M_2 = (M_2(t), t \geq t_0)$ where for each $t \geq t_0$

$$M_2(t) = \int_{t_0}^{t} \int_{|y|<c} \log\left(\frac{|x(s-) + D(y)x(s-)|^2}{|x(s-)|^2} \right) \tilde{N}(ds, dy)$$

and this is a càdlàg local martingale.

Substituting (4.29) into (4.28) we obtain

$$
\begin{aligned}
\log(|x(t)|^2) \;\leq\; & \log(|x_0|^2) + \int_{t_0}^{t} \frac{2x(s-)^T}{|x(s-)|^2} f(x(s-))ds \\
& + M_1(t) + \sum_{k=1}^{m} \int_{t_0}^{t} \left(\frac{|G_k x(s-)|^2}{|x(s-)|^2} - \frac{2|x(s-)^T G_k x(s-)|^2}{|x(s-)|^4} \right) ds \\
& + M_2(t) + 2(t-t_0) \left[\int_{|y|<c} \log\left(1 + \|D(y)\|_1\right) \nu(dy) - \delta \right]. \qquad (4.30)
\end{aligned}
$$

For the Brownian motion integral we have as in Mao [31] pp. 282 that

$$
\langle M_1 \rangle(t) = 4 \sum_{k=1}^{m} \int_{t_0}^{t} \frac{\left|x(s-)^T G_k x(s-)\right|^2}{|x(s-)|^4} ds \leq 4(t-t_0) \sum_{k=1}^{m} \|G_k\|_1{}^2
$$

and

$$
\begin{aligned}
\lim_{t\to\infty} \rho_{M_1}(t) \;&\leq\; 4 \sum_{k=1}^{m} \|G_k\|_1{}^2 \lim_{t\to\infty} \int_{t_0}^{t} \frac{ds}{(1+s)^2} \\
&\leq\; \frac{4}{1+t_0} \sum_{k=1}^{m} \|G_k\|_1{}^2 < \infty
\end{aligned}
$$

where ρ_{M_1} is as defined in (1.8).
Hence, by the strong law of large numbers (see (1.9)) it holds that

$$
\lim_{t\to\infty} \frac{M_1(t)}{t} \to 0 \qquad a.s. \qquad (4.31)
$$

Now due to Kunita [25] pp. 323 we have that

$$
\langle M_2 \rangle(t) = \int_{t_0}^{t} \int_{|y|<c} \left[\log\left(\frac{|x(s-) + D(y)x(s-)|^2}{|x(s-)|^2} \right) \right]^2 \nu(dy)ds.
$$

Using Lemma 4.3.1 and the elementary inequality (1.10), we obtain

$$
\begin{aligned}
\langle M_2 \rangle(t) \;&\leq\; \int_{t_0}^{t} \int_{|y|<c} \left[\log\left((1 + \|D(y)\|_1)^2 \right) \right]^2 \nu(dy)ds \\
&=\; 4 \int_{t_0}^{t} \int_{|y|<c} \left[\log\left((1 + \|D(y)\|_1) \right) \right]^2 \nu(dy)ds \\
&\leq\; 4(t-t_0) \int_{|y|<c} \|D(y)\|_1{}^2 \nu(dy) < \infty.
\end{aligned}
$$

Hence,

$$\begin{aligned}\lim_{t\to\infty}\rho_{M_2}(t) &\le \lim_{t\to\infty}\int_{t_0}^{t}\int_{|y|<c}\frac{4}{(1+s)^2}\|D(y)\|_1{}^2\nu(dy)ds\\ &= 4\int_{|y|<c}\|D(y)\|_1{}^2\nu(dy).\lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{(1+s)^2}\\ &= \frac{4}{1+t_0}\int_{|y|<c}\|D(y)\|_1{}^2\nu(dy)<\infty.\end{aligned}$$

Consequently, by the strong law of large numbers

$$\lim_{t\to\infty}\frac{M_2(t)}{t}\to 0 \qquad a.s. \tag{4.32}$$

Then by (4.30), (4.31), (4.32) and applying conditions $(i)-(ii)$ we deduce that

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)|\le-\left(\gamma-K-\frac{\xi}{2}-\int_{|y|<c}\log\left(1+\|D(y)\|_1\right)\nu(dy)+\delta\right) \qquad a.s.$$

□

4.5 Stabilization by Poisson noise

In this section we will perturb the non-linear system (4.1) by a Poisson random measure which represents the "large jumps" of a Lévy process i.e.

$$dx(t)=f(x(t-))dt+\int_{|y|\ge c}E(y)x(t-)N(dt,dy) \quad \text{on} \quad t\ge t_0 \tag{4.33}$$

with initial condition $x(t_0)=x_0\in\mathbb{R}^d$ and where $y\to E(y)$ is a measurable map from $\mathbb{R}^m\to\mathcal{M}_d(\mathbb{R})$.

For the rest of this section we impose the following constraint:

Assumption 4.5.1 We require that

$$\int_{|y|\ge c}\|E(y)\|_1{}^2\nu(dy)<\infty,$$

and that $E(y)$ does not have any eigenvalue equal to -1.

Recall that f is locally Lipschitz and satisfies (4.2) which implies the growth condition (C2). Hence the unperturbed system (4.1) has a global solution by Theorem 4.2.1, and so the usual construction of the solution $(x(t), t \geq t_0)$ of (4.33) by interlacing gives a global solution (see Applebaum [1] Theorem 6.2.9 pp. 311).

Under the assumption $\int_{|y|\geq c} \|E(y)\|_1{}^2 \nu(dy) < \infty$, by the Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned}\int_{|y|\geq c} \|E(y)\|_1 \, \nu(dy) &\leq \left(\int_{|y|\geq c} \|E(y)\|_1{}^2 \nu(dy)\right)^{\frac{1}{2}} \left(\int_{|y|\geq c} 1^2 \nu(dy)\right)^{\frac{1}{2}} \\ &= \left(\nu\left(|y| \geq c\right)\right)^{\frac{1}{2}} \left(\int_{|y|\geq c} \|E(y)\|_1{}^2 \nu(dy)\right)^{\frac{1}{2}} < \infty, \quad (4.34)\end{aligned}$$

and hence we can write (4.33) as

$$dx(t) = \left(f(x(t-)) + \int_{|y|\geq c} E(y)x(t-)\nu(dy)\right) dt + \int_{|y|\geq c} E(y)x(t-)\tilde{N}(dt, dy).$$

We could now investigate stabilization by using Theorem 4.4.2, but we will instead develop the theory again as the structure of (4.33) enables new insights to be obtained.

Lemma 4.5.2 If $\int_{|y|\geq c} \|E(y)\|_1{}^2 \nu(dy) < \infty$, then

$$\sup_{x\in\mathbb{R}^d\setminus\{0\}} \int_{|y|\geq c} \left[\log\left(|x + E(y)x|\right) - \log\left(|x|\right)\right] \nu(dy) < \infty. \quad (4.35)$$

Proof: Using the result of Lemma 4.3.1, (1.10) and (4.34)

$$\begin{aligned}\int_{|y|\geq c} \left[\log\left(\left|x + E(y)x\right|\right) - \log\left(|x|\right)\right] \nu(dy) &\leq \int_{|y|\geq c} \log(1 + \|E(y)\|_1)\nu(dy) \\ &\leq \int_{|y|\geq c} \|E(y)\|_1 \nu(dy) < \infty,\end{aligned}$$

and the required result follows. □

Next we will prove that the system (4.1) can be stabilized almost surely if is perturbed by the "large jumps" part of the Lévy noise.

Theorem 4.5.3 *If*

$$\sup_{z\in\mathbb{R}^d\setminus\{0\}} \int_{|y|\geq c} \left[\log\left(\left|z + E(y)z\right|\right) - \log\left(|z|\right)\right]\nu(dy) \leq -K, \quad (4.36)$$

then the sample Lyapunov exponent of the solution of (4.33) exists and satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| \le K + \sup_{z\in\mathbb{R}^d\setminus\{0\}} \int_{|y|\ge c} \left[\log\left(|z+E(y)z|\right) - \log\left(|z|\right) \right] \nu(dy).$$

Then the trivial solution of the system (4.33) is almost surely exponentially stable.

Proof: Fix $x_0 \neq 0$. On account of Lemma 4.3.2 then $x(t) \neq 0$ for all $t \ge t_0$ almost surely. Apply Itô's formula (see Theorem 1.3.2) to $\log(|x(t)|^2)$. Then for each $t \ge t_0$

$$\begin{aligned}\log(|x(t)|^2) &= \log(|x_0|^2) + \int_{t_0}^t \frac{2x(s-)^T}{|x(s-)|^2} f(x(s-))\,ds \\ &+ \int_{t_0}^t \int_{|y|\ge c} \left[\log\left(|x(s-)+E(y)x(s-)|^2\right) - \log\left(|x(s-)|^2\right)\right] N(ds,dy).\end{aligned}$$

Hence, by (4.35) we deduce that

$$\begin{aligned}\log(|x(t)|^2) &= \log(|x_0|^2) + \int_{t_0}^t \frac{2x(s-)^T}{|x(s-)|^2} f(x(s-))\,ds \\ &+ \int_{t_0}^t \int_{|y|\ge c} \left[\log\left(|x(s-)+E(y)x(s-)|^2\right) - \log\left(|x(s-)|^2\right)\right] \tilde{N}(ds,dy) \\ &+ \int_{t_0}^t \int_{|y|\ge c} \left[\log\left(|x(s-)+E(y)x(s-)|^2\right) - \log\left(|x(s-)|^2\right)\right] \nu(dy)ds.\end{aligned} \tag{4.37}$$

Define the process $M_3 = (M_3(t), t \ge t_0)$ where for each $t \ge t_0$

$$M_3(t) = \int_{t_0}^t \int_{|y|\ge c} \log\left(\frac{|x(s-)+E(y)x(s-)|^2}{|x(s-)|^2}\right) \tilde{N}(ds,dy)$$

and this is a càdlàg local martingale.
Applying the law of large numbers for the process $M_3 = (M_3(t), t \ge t_0)$, as in Theorem 4.4.2 for $M_2 = (M_2(t), t \ge t_0)$, we see that

$$\lim_{t\to\infty} \frac{M_3(t)}{t} \to 0 \qquad a.s. \tag{4.38}$$

By (4.2) then

$$\begin{aligned}\frac{1}{t}\log(|x(t)|^2) &\le \frac{1}{t}\log(|x_0|^2) + 2K\frac{(t-t_0)}{t} + \frac{M_3(t)}{t} \\ &+ \frac{(t-t_0)}{t} \sup_{t_0\le s\le t} \int_{|y|\ge c} \left[\log\left(|x(s-)+E(y)x(s-)|^2\right) - \log\left(|x(s-)|^2\right)\right]\nu(dy).\end{aligned}$$

Letting $t \to \infty$ and applying the law of large numbers then the sample Lyapunov exponent exists and satisfies

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \;\leq\; K+\sup_{t_0\leq s\leq\infty}\int_{|y|\geq c}\Big[\log\left(|x(s-)+E(y)x(s-)|\right)-\log\left(|x(s-)|\right)\Big]\nu(dy).$$

To obtain that the trivial solution of the system (4.33) is almost surely exponentially stable $\limsup_{t\to\infty}\frac{1}{t}\log|x(t)|<0$ must hold. By (4.36) the required result follows. □

Condition (4.36) is the most general that we can find but it is not immediately clear how to find examples that satisfy this condition. In the following part we will investigate some explicit conditions.

Before we study our problem further we need a useful lemma.

Lemma 4.5.4 If $z\in\mathbb{R}^d$ with $z\neq 0$ and F is a $d\times d$ real valued symmetric matrix then

$$-\infty<\log\left(|z+Fz|\right)<\log\left(|z|\right) \tag{4.39}$$

if and only if the eigenvalues of F belong to the set $(-2,-1)\cup(-1,0)$.

Proof: In order for the left hand-side of the inequality (4.39) to be satisfied F cannot have an eigenvalue $\lambda=-1$. For if F has such an eigenvalue and if z is the corresponding eigenvector then $\log|z+Fz|=\log 0=-\infty$.

Now by the monotonicity of the logarithmic function the right-hand side of the inequality holds if and only if for every $z\neq 0$

$$|z|>|z+Fz|,\quad \text{i.e.}\quad |z|^2>|z+Fz|^2=|z|^2+2\langle z,Fz\rangle+|Fz|^2.$$

This holds if and only if

$$\begin{aligned}2\langle z,Fz\rangle+|Fz|^2<0 \quad &<=> \quad 2\langle z,Fz\rangle+\langle Fz,Fz\rangle<0 <=> \langle z,(F^TF+2F)z\rangle<0\\ &<=> \quad \langle z,(F^2+2F)z\rangle<0 \end{aligned}\tag{4.40}$$

(since F is symmetric) and this is true if and only if each of the eigenvalues satisfies the following relation

$$\lambda^2+2\lambda<0 \qquad \text{i.e.} \qquad -2<\lambda<0, \quad \text{and the result follows.}$$

□

By Lemma 4.5.4 we can ensure that the left-hand side of the inequality (4.36) is negative if $E(y)$ is symmetric and all the eigenvalues of $E(y)$ belong to the set $(-2,-1)\cup(-1,0)$.

In the following we will give some examples illustrating the theory developed above.

Example 4.5.5 Fix $c = d = m = 1$ and let $E(y) = b$ for all $y \in \mathbb{R}$, where $b \in (-2,-1)\cup(-1,0)$. Now for any Lévy measure ν we have that

$$\int_{|y|\geq 1} \log(|1+b|)\nu(dy) = \log(|1+b|)\nu(|y|\geq 1) = R\log(|1+b|) < 0$$

where $R = \nu(|y|\geq 1) < \infty$.

We require that (4.36) is satisfied. Hence, if $0 < K < -R\log(|1+b|)$ then the trivial solution of

$$dx(t) = f(x(t-))dt + \int_{|y|\geq 1} bx(t-)N(dt,dy)$$

is almost surely exponentially stable since (4.36) clearly holds.

Example 4.5.6 Again take $c = d = m = 1$ and define $E(y) = y\mathbf{1}_A(y)$ where $A = (-2,-1)$. Note that any finite measure on $\mathbb{R}\backslash\{0\}$ can be a Lévy measure. Define ν to be the Lebesgue measure on the set $(-2,-1)$ and 0 everywhere else. Hence by (1.11)

$$\begin{aligned}\int_{|y|\geq 1} \log\left(|1+y\mathbf{1}_A(y)|^2\right)\nu(dy) &\leq \int_{|y|\geq 1}\left[|1+y\mathbf{1}_A(y)|^2-1\right]\nu(dy)\\ &= \int_{-2}^{-1}(2y+y^2)dy = -\frac{2}{3}.\end{aligned}$$

In order for the trivial solution of (4.33) to be almost surely exponentially stable we require that $0 < K < \frac{1}{3}$. Then with these values of $E(\cdot)$ and K, the stochastically perturbed system is stable as (4.36) again holds.

4.6 Stabilization of SDEs driven by Lévy noise

In this section we are interested in finding stability conditions for SDEs driven by generic Lévy noise i.e.

$$\begin{aligned}dx(t) &= f(x(t-))dt + \sum_{k=1}^{m} G_k x(t-)dB_k(t) + \int_{|y|<c} D(y)x(t-)\tilde{N}(dt,dy)\\ &\quad + \int_{|y|\geq c} E(y)x(t-)N(dt,dy) \qquad (4.41)\end{aligned}$$

on $t \geq t_0$ and initial condition $x(t_0) = x_0 \in \mathbb{R}^d$.

Combining the results of Theorem 4.4.2 and Theorem 4.5.3 we obtain the following.

Theorem 4.6.1 *Let assumption 3.4.3 and (3.21) hold. Assume that the following conditions are satisfied where* $\xi > 0$, $\gamma \geq 0$, $\delta \geq 0$

$$(i) \sum_{k=1}^{m} |G_k x|^2 \leq \xi |x|^2 \qquad and \qquad (ii) \sum_{k=1}^{m} |x^T G_k x|^2 \geq \gamma |x|^4$$

$$(iii) \int_{|y|<c} x^T D(y) x \, \nu(dy) \geq \delta |x|^2$$

for all $x \in \mathbb{R}^d$,

$$Z_c = \int_{|y|<c} \left(\|D(y)\|_1 \vee \|D(y)\|_1{}^2 \right) \nu(dy) < \infty \qquad and \qquad K_c = \int_{|y|\geq c} \|E(y)\|_1{}^2 \nu(dy) < \infty$$

where neither of D *or* $E : \mathbb{R}^m \to \mathcal{M}_d(\mathbb{R})$ *have an eigenvalue equal to -1.*
Then the sample Lyapunov exponent of the solution of (4.41) exists and satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| \leq -\left(\gamma - K - \frac{\xi}{2} - \int_{|y|<c} \log\left(1 + \|D(y)\|_1\right) \nu(dy) + \delta - M(c) \right)$$

where $M(c) = \sup_{z\in\mathbb{R}^d\setminus\{0\}} \int_{|y|\geq c} \left[\log\left(|z + E(y)z| \right) - \log\left(|z| \right) \right] \nu(dy) < \infty$.
If $\gamma > K + \frac{\xi}{2} - \delta + \int_{|y|<c} \log(1 + \|D(y)\|_1)\nu(dy) + M(c)$ *then the trivial solution of (4.41) is almost surely exponentially stable.*

Proof: Immediate from the proofs of Theorem 4.4.2 and Theorem 4.5.3.

Depending on the noise that perturbs the non-linear deterministic system (4.1) the trivial solution of (4.41) is almost surely exponentially stable if the following conditions are satisfied in the stated cases below, where γ, ξ, δ, $M(c)$ are as defined previously.

Case 1: Only the Brownian motion perturbs the non-linear system (4.1) (as in Mao [31] pp. 282 and [33] pp. 137)

$$\gamma > K + \frac{\xi}{2}.$$

Case 2: The system (4.1) is perturbed only by small jumps. It must hold that

$$-K - \int_{|y|<c} \log\left(1 + \|D(y)\|_1\right) \nu(dy) + \delta > 0.$$

Case 3: The system is perturbed by a mixture of Brownian motion, infinitely small jumps with maximum allowable size c and a finite number of large jumps of minimum allowable jump size c. The stochastic system is stable after adding the small and large jumps if the following condition holds

$$\gamma > K + \frac{\xi}{2} - \delta + \int_{|y|<c} \log\left(1 + \|D(y)\|_1\right) \nu(dy) + M(c).$$

4.6.1 A special case of an SDE driven by Lévy noise

Consider the following stochastically perturbed system

$$dx(t) = h(x(t-))dt + \sum_{i=1}^{m} G_i x(t-) dY^i(t) \tag{4.42}$$

where $Y = (Y(t), t \geq t_0)$ is a Lévy process taking values in $\mathbb{R}^m$. Let $h : \mathbb{R}^d \to \mathbb{R}^d$ be Borel measurable and for $1 \leq i \leq m$, $G_i \in \mathcal{M}_d(\mathbb{R})$. The initial condition is $x(t_0) = x_0 \in \mathbb{R}^d$.

We denote the Lévy-Itô decomposition of $Y = (Y(t), t \geq t_0)$ as

$$Y^i(t) = b^i t + B_A^i(t) + \int_{|y|<1} y^i \tilde{N}(t, dy) + \int_{|y|\geq 1} y^i N(t, dy)$$

for each $1 \leq i \leq m$, $t \geq t_0$ where $b \in \mathbb{R}^m$, $(B_A(t), t \geq t_0)$ is an m-dimensional Brownian motion with covariance matrix A and for each $1 \leq i \leq m$, $t \geq t_0$ $B_A^i(t) = \sum_{k=1}^{p} \sigma^{ik} B_k(t)$ where $B_1, \ldots, B_p$ are standard one-dimensional Brownian motion and σ is an $m \times p$ real-valued matrix for which $\sigma\sigma^T = A$ (see Applebaum [1] pp. 109).

Therefore the Brownian motion component of (4.42) will have the following form

$$\sum_{i=1}^{m} G_i x(t-) dB_A^i(t) = \sum_{i=1}^{m} \sum_{k=1}^{p} G_i x(t-) \sigma^{ik} dB_k(t). \tag{4.43}$$

Now, let $G'_k = \sum_{i=1}^{m} G_i \sigma^{ik}$. It follows that (4.43) can be written as

$$\sum_{i=1}^{m} G_i x(t-) dB_A^i(t) = \sum_{k=1}^{p} G'_k x(t-) dB_k(t).$$

Assumption 4.6.2

$$(i) \int_{\mathbb{R}^m \setminus \{0\}} |y|^2 \nu(dy) < \infty \quad \text{and} \quad (ii) \int_{|y|<1} |y| \nu(dy) < \infty.$$

We require that assumption 4.6.2 holds for the rest of this subsection.

Remark 4.6.3 Note that (i) in assumption 4.6.2 is a necessary and sufficient condition for a Lévy process to have a finite second moment (see Sato [42] pp. 159-163) and (ii) is a necessary and sufficient condition to have finite variation if $A = 0$ holds (see e.g. Kyprianou [27] pp. 54).

By assumption 4.6.2 we can write the Lévy-Itô decomposition of $Y = (Y(t), t \geq t_0)$ for each $1 \leq i \leq m$, $t \geq t_0$ as

$$Y^i(t) = \tilde{b}^i t + B_A^i(t) + \int_{\mathbb{R}^m \setminus \{0\}} y^i \tilde{N}(t, dy)$$

where $\tilde{b}^i = b^i + \int_{|y| \geq 1} y^i \nu(dy)$.

Hence (4.42) can be written as

$$\begin{aligned} dx(t) &= \left(h(x(t-)) + \sum_{i=1}^{m} G_i x(t-) \tilde{b}^i \right) dt + \sum_{k=1}^{p} G'_k x(t-) dB_k(t) \\ &\quad + \sum_{i=1}^{m} \int_{\mathbb{R}^m \setminus \{0\}} G_i x(t-) y^i \tilde{N}(dt, dy). \end{aligned} \tag{4.44}$$

Now, we have reduced our problem to an SDE driven by a Brownian motion and a compensated Poisson integral. If we define $f(\cdot)$, $g(\cdot)$ and $H(\cdot, y)$ the drift, diffusion and jump coefficients of (4.27) in terms of the coefficients in (4.44), it follows that (4.44) is of the same form as (4.27).

As it will be shown below the assumptions of section 4.4 are satisfied.
Since h is locally Lipschitz continuous, and for each $1 \leq i \leq m$ $\tilde{b}^i \in \mathbb{R}$ and G_i is a constant matrix, then f is a locally Lipschitz continuous function. Also for all $x \in \mathbb{R}^d$ we have

$$|f(x)| = \left| h(x) + \sum_{i=1}^{m} G_i x \tilde{b}^i \right| \leq |h(x)| + \sum_{i=1}^{m} \left| G_i x \tilde{b}^i \right| \leq L|x| + \sum_{i=1}^{m} ||G_i||_1 |\tilde{b}^i| |x| \leq (L + K')|x|$$

where L and K' are positive constants. Hence assumption 4.1.1 is satisfied. By assumption 4.6.2 it follows that the requirements on the coefficient of the compensated Poisson integral (assumption 4.4.1) are fulfilled. Hence, fitting the coefficients of (4.44) to the conditions of Theorem 4.4.2 we have the following.

Corollary 4.6.4 Let an SDE be of the form (4.42) with G_i, for $1 \leq i \leq m$ fixed. Let assumption 3.4.3 and (3.21) hold. If the following conditions are satisfied where $\xi > 0, \gamma \geq 0, \delta \geq 0$

$$(i) \sum_{k=1}^{p} |G'_k x|^2 \leq \xi |x|^2 \qquad \text{and} \qquad (ii) \sum_{k=1}^{p} |x^T G'_k x|^2 \geq \gamma |x|^4$$

$$(iii) \sum_{i=1}^{m} \int_{\mathbb{R}^m\setminus\{0\}} y^i x^T G_i x \, \nu(dy) \geq \delta|x|^2$$

for all $x \in \mathbb{R}^d$ then the sample Lyapunov exponent of the solution of (4.42) exists and satisfies

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \leq -\left(\gamma - K - \frac{\xi}{2} - \sum_{i=1}^{m}\int_{\mathbb{R}^m\setminus\{0\}} \log\left(1 + \|G_i\|_1|y^i|\right)\nu(dy) + \delta\right) \qquad a.s.$$

for any $x_0 \neq 0$. If $\gamma \geq K + \frac{\xi}{2} + \sum_{i=1}^{m}\int_{\mathbb{R}^m\setminus\{0\}} \log\left(1 + \|G_i\|_1|y^i|\right)\nu(dy) - \delta$ then the trivial solution of the system (4.42) is almost surely exponentially stable.

In the following we will demonstrate the theory that we have developed for SDEs of the form (4.42), by giving an example.

Example 4.6.5 (CGMY process)
We will use as a Lévy process the CGMY process. The CGMY process is a pure jump process with Lévy measure

$$\nu(dy) = \begin{cases} C'\frac{\exp(-Q|y|)}{|y|^{1+\alpha}} & \text{for } y < 0 \\ \\ C\frac{\exp(-M|y|)}{|y|^{1+\alpha}} & \text{for } y > 0 \end{cases} \tag{4.45}$$

where $C > 0$, $C' > 0$, $Q > 0$, $M > 0$ and $0 \leq \alpha < 2$. For $0 \leq \alpha < 1$ the process has finite variation and for $1 < \alpha < 2$ has infinite variation (for further details see Carr, Geman, Madan and Yor [11]).

For simplicity we assume that the CGMY process has only positive jumps and $0 \leq \alpha < 1$. Consider that $d = m = 1$ and $G > 0$.
Since the CGMY process is a pure jump process there is no Brownian motion component and therefore we will only need to check if assumption 4.6.2, and condition (iii) of Corollary 4.6.4 are satisfied.
Now we will check if assumption 4.6.2 (i) holds.

$$\begin{aligned} I &= \int_{\mathbb{R}\setminus\{0\}} |y|^2\nu(dy) = \int_0^\infty y^2 C\frac{e^{-My}}{y^{1+\alpha}}dy \\ &= \int_0^\infty y^{1-\alpha}Ce^{-My}dy = CM^{\alpha-2}\Gamma(2-\alpha) < \infty. \end{aligned}$$

In order to find a value for δ we compute

$$\int_{\mathbb{R}\setminus\{0\}} |y|\nu(dy) = \int_0^\infty y^{-\alpha}Ce^{-My}dy = CM^{\alpha-1}\Gamma(1-\alpha) < \infty, \tag{4.46}$$

and we see that condition (iii) is satisfied with $\delta = GCM^{\alpha-1}\Gamma(1-\alpha) > 0$.

From (4.46) we can also deduce that $\int_{|y|<1} |y|\nu(dy) < \infty$, and hence assumption 4.6.2 (ii) is satisfied.

Then the sample Lyapunov exponent of the solution of (4.42) exists and satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t)| \leq K + \int_0^\infty \log(1 + \|G\|_1 y)\, \nu(dy) - GCM^{\alpha-1}\Gamma(1-\alpha) \qquad a.s.$$

If $K + \int_0^\infty \log(1+Gy)\,\nu(dy) - GCM^{\alpha-1}\Gamma(1-\alpha) < 0$ then the trivial solution of (4.42) is almost surely exponentially stable.

A special case of the CGMY process is the Variance Gamma process (V.G) where $\alpha = 0$, $C = C' = \frac{1}{k}$, $Q = M = \frac{1}{\sigma}\sqrt{\frac{2}{k}}$ where k and σ are positive constants. Hence the Lévy measure is now

$$\nu(dy) = \frac{1}{k|y|} \exp\left(-\frac{|y|}{\sigma}\sqrt{\frac{2}{k}}\right) dy \quad y \neq 0$$

(see Madan and Seneta [30] pp. 519).

Then by the above calculations the sample Lyapunov exponent of the solution of (4.42) exists and satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t)| \leq K + \int_0^\infty \log(1 + \|G\|_1 y)\, \frac{1}{ky}.e^{\left(\frac{-y}{\sigma}.\sqrt{\frac{2}{k}}\right)} dy - \frac{\sigma G}{\sqrt{2k}} \qquad a.s.$$

If $K + \int_0^\infty \log(1 + \|G\|_1 y)\, \frac{1}{ky}.e^{\left(\frac{-y}{\sigma}.\sqrt{\frac{2}{k}}\right)} dy - \frac{\sigma G}{\sqrt{2k}} < 0$, then the trivial solution of (4.42) is almost surely exponentially stable.

4.7 Stochastic stabilization of linear systems

In this section we will consider a special case of (4.1), a linear unstable one-dimensional deterministic system. We will investigate if it's possible to stabilize this system by adding a mixture of Brownian motion and Poisson noise. Of course this case is already covered by Theorem 4.4.2 but we will find that using a single Poisson process allows us to have greater flexibility.

For simplicity in this section we take $t_0 = 0$.
Consider the following one-dimensional system

$$\frac{dy(t)}{dt} = ay(t) \qquad \text{on } t \geq 0$$

with initial condition $y(0) = x_0 \in \mathbb{R}$ and $a > 0$. The solution of the system is

$y(t) = x_0 \exp(at)$ for each $t \geq 0$, which yields immediately that the system is unstable since

$$\lim_{t \to \infty} y(t) = \infty.$$

Suppose that we perturb the system with noise and the system now has the following form:

$$dx(t) = ax(t-)dt + bx(t-)dB(t) + cx(t-)d\tilde{N}(t) \tag{4.47}$$

where $b > 0$, $c > -1$, $B(t)$ is a one-dimensional Brownian motion and $(\tilde{N}(t), t \geq 0)$ is the compensated Poisson process with $\tilde{N}(t) = N(t) - \lambda t$ where $\lambda > 0$ is the intensity of the Poisson process $(N(t), t \geq 0)$. Assume that the processes $(B(t), t \geq 0)$ and $(N(t), t \geq 0)$ are independent.

Hence,

$$\begin{aligned} dx(t) &= (a - \lambda c)x(t-)dt + bx(t-)dB(t) + cx(t-)dN(t) \\ &= x(t-)dZ(t) \end{aligned} \tag{4.48}$$

where $Z(t) = (a - \lambda c)t + bB(t) + cN(t)$, for each $t \geq 0$.

We see immediately that $x(t) = x_0 \mathcal{E}_Z(t)$, where $\mathcal{E}_Z$ is the stochastic exponential of the semimartingale Z (see Applebaum [1] pp. 249), with $x(0) = x_0 \in \mathbb{R}$. For simplicity we assume that $x_0 = 1$ from now on.

Define the processes $X = (X(t), t \geq 0)$ with $X(t) = bB(t) + (a - \lambda c)t$ for each $t \geq 0$ and $Y = (Y(t), t \geq 0)$ with $Y(t) = cN(t)$ for each $t \geq 0$, where X and Y are semimartingales. The stochastic exponential of two semimartingales has the following property (see Applebaum [1] pp. 249)

$$\mathcal{E}_X(t)\mathcal{E}_Y(t) = \mathcal{E}_{X+Y+[X+Y]}(t)$$

where $[X, Y]$ is the quadratic variation of the processes X and Y (for the definition see section 1.3, (1.5)). In this case $[X, Y] = 0$ hence,

$$\mathcal{E}_X(t)\mathcal{E}_Y(t) = \mathcal{E}_{X+Y}(t). \tag{4.49}$$

The process Y can be written as a compound Poisson process where for each $t \geq 0$

$$Y(t) = \underbrace{c + c + \ldots + c}_{N(t) \text{ times}}.$$

By Applebaum [1] pp. 250 the stochastic exponential of this compound Poisson process is

$$\mathcal{E}_Y(t) = (1+c)^{N(t)}.$$

The stochastic exponential of the process X is a geometric Brownian motion

$$\mathcal{E}_X(t) = \exp\left[\left(a - \lambda c - \frac{1}{2}b^2\right)t + bB(t)\right].$$

Hence by (4.49) the stochastic exponential of the process x for each $t \geq 0$ is

$$x(t) = \mathcal{E}_Z(t) = e^{(a-\lambda c-\frac{1}{2}b^2)t}e^{bB(t)}(1+c)^{N(t)}.$$

So,

$$\frac{1}{t}\log|x(t)| = \left(a - \lambda c - \frac{1}{2}b^2\right) + \frac{N(t)}{t}\log(1+c) + b\frac{B(t)}{t}.$$

By the law of large numbers for a Brownian motion and Poisson process respectively (see Sato [42] pp. 246) we have

$$\lim_{t\to\infty}\frac{B(t)}{t} = 0 \qquad a.s. \quad \text{and} \quad \lim_{t\to\infty}\frac{N(t)}{t} = \lambda \qquad a.s.$$

Hence,

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| = \left(a - \lambda c - \frac{1}{2}b^2\right) + \lambda\log(1+c). \qquad a.s. \tag{4.50}$$

If $\left(a - \lambda c - \frac{1}{2}b^2\right) + \lambda\log(1+c) < 0$, then the perturbed system (4.48) becomes almost surely stable.

In the following we will give conditions that should be examined in order to see if the perturbed system is stable or not.

Case 1: *Stabilization only from the Brownian motion* - $c = 0$.
The Brownian motion can stabilize the unstable system if

$$b > \sqrt{2a} \qquad \text{or} \qquad b < -\sqrt{2a}.$$

Otherwise the system remains unstable.

Case 2: *Stabilization only from the Poisson noise* - $b = 0$, $-1 < c < 0$ or $c > 0$.
There is only Poisson noise. If the following condition is satisfied then the system can

be stabilized. Otherwise the system remains unstable.

$$c - \log(1+c) > \frac{a}{\lambda}.$$

Case 3: Stabilization from the Brownian motion in presence of the Poisson noise.
If $b > \sqrt{2a}$ or $b < -\sqrt{2a}$, then $a - \frac{1}{2}b^2 < 0$. The unstable system is stabilized by the Brownian motion process, no matter what the value of c is, since $-\lambda c + \lambda \log(1+c) < 0$ always hold, and hence in this case $(a - \frac{1}{2}b^2 - \lambda c) + \lambda \log(1+c) < 0$ will always be satisfied. It follows that the Poisson noise cannot destabilize the system that it can be stabilized by the Brownian motion.

Case 4: Destabilization from the Brownian motion but stabilization from the Poisson noise.
If $-\sqrt{2a} \leq b \leq \sqrt{2a}$, then $a - \frac{1}{2}b^2 \geq 0$, and the system remains unstable by the Brownian motion noise. It can be stabilized by the Poisson noise, for $c > 0$ or $-1 < c < 0$, if the following is satisfied

$$c - \log(1+c) > \frac{a - \frac{1}{2}b^2}{\lambda}.$$

In the case that the previous condition is not satisfied then the system remains unstable.

4.8 Perturbation of non-linear deterministic systems

Let the non-linear system (4.1) be perturbed by Brownian motion and Poisson noise as is shown below.

$$dx(t) = f(x(t-))dt + \sum_{k=1}^{m} G_k x(t-)dB_k(t) + Dx(t-)d\tilde{N}(t) \qquad \text{on} \qquad t \geq t_0$$

i.e.

$$dx(t) = [f(x(t-)) - \lambda Dx(t-)]dt + \sum_{k=1}^{m} G_k x(t-)dB_k(t) + Dx(t-)dN(t) \tag{4.51}$$

where $D \in \mathcal{M}_d(\mathbb{R})$.

4.8.1 Stabilization

Let $x = (x(t), t \geq t_0)$ be the solution of

$$dx(t) = h(x(t-))dt + g(x(t-))dB(t) + Dx(t-)dN(t) \qquad \text{on } t \geq t_0 \tag{4.52}$$

where $h(x) = f(x) - \lambda Dx$. From now on we assume that D is symmetric and that -1 is not an eigenvalue of D. It follows that $I + D$ is invertible.

Since (4.52) is a special case of (4.26) and $I + D$ is invertible then by Lemma 4.3.2 the solution of the SDE (4.52) will never be zero provided that the initial condition $x(t_0) = x_0 \neq 0$.

The result that follows is a special case of Theorem 4.4.2. However we present a separate proof below. The fact that D does not depend on the jumps, allows us to get more direct and simple results.

Then for the stochastically perturbed system (4.51) the following applies.

Theorem 4.8.1 *Suppose that the following conditions are satisfied for all $x \in \mathbb{R}^d$, where $\xi > 0, \gamma \geq 0$*

$$(i) \sum_{k=1}^{m} |G_k x|^2 \leq \xi |x|^2 \qquad and \qquad (ii) \sum_{k=1}^{m} |x^T G_k x|^2 \geq \gamma |x|^4 \tag{4.53}$$

and D is a $d \times d$ symmetric positive definite matrix. Then the sample Lyapunov exponent of the solution of (4.51) exists and satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| \leq -\left(\gamma + \lambda\mu_{min} - K - \frac{\xi}{2} - \lambda \log\left(1 + \mu_{max}\right)\right) \qquad a.s.$$

for any $x_0 \neq 0$, where μ_{min} and μ_{max} are the minimum and maximum eigenvalues of the matrix D respectively and λ is the intensity of the Poisson process. In particular the trivial solution of (4.51) is almost surely exponentially stable if the following relationship is satisfied

$$\gamma + \lambda\mu_{min} - K - \tfrac{\xi}{2} - \lambda \log\left(1 + \mu_{max}\right) > 0.$$

Proof: Fix any $x_0 \neq 0$. Apply Itô's formula (see Theorem 1.3.2) to $\log(|x(t)|^2)$. Then, for each $t \geq t_0$,

$$\begin{aligned} \log(|x(t)|^2) \;=\;& \log(|x_0|^2) + \int_{t_0}^{t} \frac{2x(s-)^T}{|x(s-)|^2} f(x(s-))\, ds - \lambda \int_{t_0}^{t} \frac{2x(s-)^T D x(s-)}{|x(s-)|^2}\, ds \\ &+ \sum_{k=1}^{m} \int_{t_0}^{t} \frac{2x(s-)^T}{|x(s-)|^2} G_k x(s-)\, dB_k(s) \\ &+ \sum_{k=1}^{m} \int_{t_0}^{t} \left(\frac{|G_k x(s-)|^2}{|x(s-)|^2} - \frac{2|x(s-)^T G_k x(s-)|^2}{|x(s-)|^4} \right) ds \\ &+ \int_{t_0}^{t} \left[\log\left(\left| x(s-) + Dx(s-) \right|^2 \right) - \log\left(|x(s-)|^2 \right) \right] dN(s). \end{aligned} \tag{4.54}$$

Due to Proposition 3.5.6 we have that

$\int_{t_0}^{t} \left[\log(|x(s-) + Dx(s-)|^2) - \log(|x(s-)|^2)\right] dN(s)$

$$
\begin{aligned}
&= \int_{t_0}^{t} \left[\log\left(1 + \frac{2x(s-)^T Dx(s-)}{|x(s-)|^2} + \frac{|Dx(s-)|^2}{|x(s-)|^2}\right)\right] dN(s) \\
&\leq \log\left((1 + 2\mu_{max} + \mu_{max}^2)\right)(N(t) - N(t_0)) \\
&= \log\left((1 + \mu_{max})^2\right)(N(t) - N(t_0)), \qquad (4.55)
\end{aligned}
$$

where μ_{max} is the maximum eigenvalue of matrix D.

Then by (3.55) and (4.55), we have the following estimate for (4.54)

$$
\begin{aligned}
\log(|x(t)|^2) \leq\; & \log(|x_0|^2) + \int_{t_0}^{t} \frac{2x(s-)^T}{|x(s-)|^2} f(x(s-))\, ds - 2\lambda\mu_{min}(t - t_0) \\
& + \sum_{k=1}^{m} \int_{t_0}^{t} \frac{2x(s-)^T}{|x(s-)|^2} G_k x(s-) dB_k(s) \\
& + \sum_{k=1}^{m} \int_{t_0}^{t} \left(\frac{|G_k x(s-)|^2}{|x(s-)|^2} - \frac{2|x(s-)^T G_k x(s-)|^2}{|x(s-)|^4}\right) ds \\
& + \log(1 + \mu_{max})^2 (N(t) - N(t_0)).
\end{aligned}
$$

Applying conditions (4.2) and (4.53), we obtain

$$
\begin{aligned}
\frac{1}{t}\log(|x(t)|^2) \leq\; & \frac{1}{t}\log(|x_0|^2) + \frac{(t - t_0)}{t}(2K - 2\lambda\mu_{min} + \xi - 2\gamma) \\
& + \frac{1}{t}\sum_{k=1}^{m} \int_{t_0}^{t} \frac{2x(s-)^T}{|x(s-)|^2} G_k x(s-)\, dB_k(s) \\
& + 2\log(1 + \mu_{max}) \frac{N(t) - N(t_0)}{t}.
\end{aligned}
$$

Define

$$
M_1(t) = \sum_{k=1}^{m} \int_{t_0}^{t} \frac{2x(s-)^T}{|x(s-)|^2} G_k x(s-) dB_k(s).
$$

Then by the law of large numbers for $(M_1(t), t \geq t_0)$ as in the proof of Theorem 4.4.2 and the well known law of large numbers for Poisson processes (see Sato [42] pp. 246) we have respectively that

$$
\lim_{t\to\infty} \frac{M_1(t)}{t} \to 0 \qquad a.s. \quad \text{and} \quad \lim_{t\to\infty} \frac{N(t)}{t} = \lambda \qquad a.s.
$$

where λ is the intensity of the Poisson process. Hence,

$$
\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|^2) \leq (2K - 2\lambda\mu_{min} + \xi - 2\gamma) + 2\lambda\log\left(1 + \mu_{max}\right) \qquad a.s,
$$

and the required result follows. □

Depending on the noise that perturbs the non-linear deterministic system (4.1) such as, a Brownian motion only, or a Poisson process only, or a mixture of Brownian motion and Poisson process, then the trivial solution of (4.51) is almost surely exponentially stable if the following conditions are satisfied in the stated cases below.

Case 1: The system is perturbed by the Brownian motion only, $D = 0$. The condition that must be satisfied is (as in Mao [31] pp. 282 and [33] pp. 137)

$$\gamma > K + \frac{\xi}{2}.$$

Case 2: There is no Brownian motion part, so $\gamma = 0$, $\xi = 0$. Only the Poisson noise perturbs the non-linear system (4.1). If it holds that

$$\mu_{min} - \log(1 + \mu_{max}) > \frac{K}{\lambda},$$

then the Poisson noise can stabilize the deterministic system.

Case 3: The system is perturbed by a mixture of the Brownian motion and Poisson noise. Assume that $\gamma > K + \frac{\xi}{2}$. In order for the trivial solution of (4.51) to remain almost surely exponentially stable the following condition must be satisfied

$$\gamma > K + \frac{\xi}{2} - \lambda\mu_{min} + \lambda \log(1 + \mu_{max}).$$

Case 4: If $\gamma < K + \frac{\xi}{2}$ it is not clear if the given non-linear deterministic system can be stabilized by the Brownian motion or not. By adding the Poisson noise if

$$\mu_{min} - \log(1 + \mu_{max}) > \frac{K + \frac{\xi}{2} - \gamma}{\lambda}$$

holds, then the trivial solution of (4.51) is almost surely exponentially stable.

4.8.2 Stochastic destabilization of non-linear systems.

Assume that (4.1) is unstable. We have seen in section 4.8.1 that we can stabilize (4.1) by adding Poisson noise. To be precise consider the following stochastically perturbed system

$$dx(t) = f(x(t-))dt + Dx(t-)d\tilde{N}(t) \qquad \text{on} \qquad t \geq t_0 \tag{4.56}$$

which is a special case of (4.51). Consider that D is a $d \times d$ symmetric positive definite matrix and

$$\lambda\mu_{min} - K - \lambda \log\left(1 + \mu_{max}\right) > 0, \tag{4.57}$$

holds, where μ_{min} and μ_{max} are the minimum and maximum eigenvalues of the matrix D respectively and λ is the intensity of the Poisson process. Then by Theorem 4.8.1, the trivial solution of (4.56) is almost surely exponentially stable.

In the following we will investigate the conditions under which (4.56) is destabilized when it is perturbed by Brownian motion. So we consider the following SDE

$$dx(t) = f(x(t-))dt + Dx(t-)d\tilde{N}(t) + \sum_{k=1}^{m} G_k x(t-)dB_k(t) \qquad \text{on} \qquad t \geq t_0. \tag{4.58}$$

Theorem 4.8.2 *Assume that D is a $d\times d$ symmetric positive definite matrix with μ_{min} and μ_{max} its minimum and maximum eigenvalues respectively and (4.57) holds. If the following conditions are satisfied for all $x \in \mathbb{R}^d$ where $\xi > 0, \gamma \geq 0$*

$$(i) \sum_{k=1}^{m} |G_k x|^2 \geq \xi |x|^2 \qquad\qquad (ii) \sum_{k=1}^{m} |x^T G_k x|^2 \leq \gamma |x|^4 \tag{4.59}$$

then

$$\liminf_{t\to\infty} \frac{1}{t} \log|x(t)| \geq \left(-K - \lambda\mu_{max} + \frac{\xi}{2} - \gamma + \lambda \log\left(1 + \mu_{min}\right)\right) \qquad a.s.$$

for any $x_0 \neq 0$. In particular if $\xi > 2K + 2\lambda\mu_{max} + 2\gamma - 2\lambda\log\left(1 + \mu_{min}\right)$ then the trivial solution of (4.58) tends to infinity almost surely exponentially fast.

Proof: Fix $x_0 \neq 0$. Due to Lemma 4.3.2 then $x(t) \neq 0$ for all $t \geq t_0$. If we apply Itô's formula (see Theorem 1.3.2) to $\log(|x(t)|^2)$ we get (4.54).

Now by Proposition 3.5.6

$\int_{t_0}^{t} \log(|x(s-) + Dx(s-)|^2) - \log(|x(s-)|^2) dN(s)$

$$\begin{aligned} &= \log\left(\frac{|x(s-)|^2 + 2x(s-)^T Dx(s-) + |Dx(s-)|^2}{|x(s-)|^2}\right)(N(t) - N(t_0)) \\ &\geq \log\left(\left(1 + 2\mu_{min} + \mu_{min}^2\right)\right)(N(t) - N(t_0)) = \log\left(\left(1 + \mu_{min}\right)^2\right)(N(t) - N(t_0)). \end{aligned} \tag{4.60}$$

Applying conditions (4.2) and (4.59) and using the result of (4.60) we get an estimate of (4.54) where $M_1(t)$ is defined as in the proof of Theorem 4.8.1.

$$\begin{aligned}\log(|x(t)|^2) &\geq \log(|x_0|^2) + M_1(t) + (-2K - 2\lambda\mu_{max} + \xi - 2\gamma)(t - t_0) \\ &\quad + \log\left((1+\mu_{min})^2\right)(N(t) - N(t_0)).\end{aligned}$$

Using the same arguments as in Theorem 4.8.1, then

$$\lim_{t\to\infty} \frac{M_1(t)}{t} = 0 \qquad a.s. \quad \text{and} \quad \lim_{t\to\infty} \frac{N(t)}{t} = \lambda \qquad a.s.$$

Hence, we deduce that

$$\liminf_{t\to\infty} \frac{1}{t} \log|x(t)|^2 \geq \Big(-2K - 2\lambda\mu_{max} + \xi - 2\gamma + 2\lambda\log\left(1+\mu_{min}\right)\Big)$$

i.e.

$$\liminf_{t\to\infty} \frac{1}{t} \log|x(t)| \geq \Big(-K - \lambda\mu_{max} + \frac{\xi}{2} - \gamma + \lambda\log\left(1+\mu_{min}\right)\Big) \qquad a.s.$$

□

Remark 4.8.3 We have examined the case where (4.57) is not assumed and investigated if a stable ODE system can be destabilized by Poisson noise. Our results were inconclusive. However, it is known that the Brownian motion can destabilize a stable ODE system (see Mao [31] pp. 285-286 and [33] pp. 139).

The following examples are considered by Mao (see [31] pp. 287 and [33] pp. 140-141) for the case that (4.1) is perturbed by Brownian motion. We will perturb (4.56) by continuous noise where the continuous noise has the same form as in Mao's work.

Example 4.8.4 Consider the following stochastic perturbation of the system (4.56).

$$dx(t) = f(x(t-))dt + \sum_{i=1}^{p} D_i x(t-)d\tilde{N}_i(t) + \sum_{k=1}^{m} G_k x(t-)dB_k(t) \tag{4.61}$$

with initial condition $x(t_0) = x_0 \in \mathbb{R}^d$ and $d = 2p$ ($p \geq 1$). Let $(B(t), t \geq t_0)$ be an m-dimensional Brownian motion where for each $t \geq t_0$ $B(t) = (B_1(t), B_2(t), \ldots, B_m(t))$ with $B_1, B_2, \ldots, B_m$ independent one-dimensional Brownian motions and $(N(t), t \geq t_0)$ a p-dimensional Poisson process with $N_1, \ldots, N_p$ being one-dimensional independent Poisson processes which are all independent of the Brownian motion. Let G_k for $1 \leq k \leq m$ be a $d \times d$ constant matrix with $G_k = 0$ for $2 \leq k \leq m$ and

$$G_1 = \begin{pmatrix} 0 & \sigma & \dots & 0 \\ -\sigma & 0 & \dots & \\ \dots & & & \\ \dots & 0 & 0 & \sigma \\ 0 & \dots & -\sigma & 0 \end{pmatrix}$$

where σ is a constant and D_1 be a $d \times d$ diagonal matrix such that $D_1 = \text{diag}(a, \cdots, a)$, where $a > 0$ and $D_i = 0$ for $2 \leq i \leq p$. As a result the system has the form of

$$dx(t) = f(x(t-))dt + D_1 x(t-)d\tilde{N}_1(t) + G_1 x(t-)dB_1(t)$$

and it becomes

$$dx(t) = f(x(t-)) + ax(t-)d\tilde{N}_1(t) + \sigma y(t-)dB_1(t) \tag{4.62}$$

where $y(t)^T = (x_2(t), -x_1(t), \dots, x_{2p}(t), -x_{2p-1}(t))$ for each $t \geq t_0$. The hypotheses (i) and (ii) of Theorem 4.8.2 are satisfied since

$$(i) \sum_{k=1}^{m} |G_k x|^2 = \sigma^2 |x|^2 \qquad \text{and} \qquad (ii) \sum_{k=1}^{m} |x^T G_k x|^2 = 0.$$

as in Mao [31] pp. 287 and [33] pp. 140-141. Hence,

$$\liminf_{t \to \infty} \frac{1}{t} \log |x(t)| \geq \left(\frac{1}{2}\sigma^2 - K - \lambda a + \lambda \log(1+a) \right) \qquad a.s. \tag{4.63}$$

If $\sigma^2 > 2K + 2\lambda a - 2\lambda \log(1+a)$ then the trivial solution of (4.62) is almost surely exponentially unstable.

In example 4.8.4 we were restricted to even cases. In the following example we drop this constraint and examine the case where the dimension of the system is odd.

Example 4.8.5 Consider the stochastic perturbation of the system (4.56) as in (4.61) and suppose that $d \geq 3$ and $m = p = d$. Assume that $\sigma \in \mathbb{R}$ and define for each $k = 1, 2, \dots, d-1$, $G_k = (G_k^{ij})$ where $G_k^{ij} = \sigma$ if $i = k$ and $j = k+1$, and $G_k^{ij} = 0$ otherwise. For $k = d$ let $G_d = (G_d^{ij})$ where $G_d^{ij} = \sigma$ if $i = d$ and $j = 1$ and $G_d^{ij} = 0$ otherwise. Let D for all $1 \leq i \leq p$ be defined as in example 4.8.4.
Now the system has the form of

$$dx(t) = f(x(t-))dt + D_1 x(t-)d\tilde{N}_1(t) + \sigma \begin{pmatrix} x_2(t)dB_1(t) \\ \vdots \\ x_d(t)dB_{d-1}(t) \\ x_1(t)dB_d(t) \end{pmatrix} \tag{4.64}$$

As was shown in Mao [31] pp. 286 and [33] pp. 140, the following holds

$$\sum_{i=1}^{d} |G_k x|^2 = \sigma^2 |x|^2 \qquad \text{and} \qquad \sum_{i=1}^{d} |x^T G_k x|^2 \le \frac{1}{3}\sigma^2 |x|^4.$$

By Theorem 4.8.2 then

$$\liminf_{t\to\infty} \frac{1}{t} \log |x(t)| \ge \left(\frac{1}{6}\sigma^2 - K - \lambda a + \lambda \log\left(1+a\right) \right) \qquad a.s.$$

If $\sigma^2 > 6\left(K + \lambda a - \lambda \log\left(1+a\right)\right)$ then the trivial solution of (4.64) tends to infinity almost surely exponentially fast.

Remark 4.8.6 From the above calculations, we see that any stable system of the form $dy(t) = f(y(t-))dt + Dy(t-)d\tilde{N}(t)$ can be destabilized by Brownian motion provided that the dimension of the system is $d \ge 2$ and (4.2) and (4.57) are satisfied.

Concerning one-dimensional systems, Mao in [31, 33] pp. 287 and pp. 141 respectively, has shown that the exponentially stable one-dimensional system

$$\frac{dx(t)}{dt} = ax(t) \qquad \text{on } t \ge t_0 \tag{4.65}$$

with initial condition $x(t_0) = x_0 \in \mathbb{R}$ and $a < 0$ cannot be destabilized by Brownian motion if we restrict the stochastic perturbation to the linear form $\sum_{k=1}^{m} \sigma_k x(t) dB_k(t)$. In this case the stochastically perturbed system is

$$dx(t) = ax(t)dt + \sum_{k=1}^{m} \sigma_k x(t) dB_k(t) \tag{4.66}$$

and he shows that

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| = \left(a - \frac{1}{2} \sum_{k=1}^{m} \sigma_k^2 \right) < 0.$$

In the following we will perturb (4.65) by the Poisson noise i.e.

$$dx(t) = ax(t-)dt + cx(t-)d\tilde{N}(t), \tag{4.67}$$

where $c > -1$. Then, by (4.50)

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| = (a - \lambda c + \lambda \log(1+c)) < 0.$$

Hence the trivial solution of (4.67) is almost surely exponentially stable. Now we will examine if the stochastically stable system (4.67) can be destabilized by Brownian motion. The new stochastic perturbed system is given by

$$dx(t) = ax(t-)dt + cx(t-)d\tilde{N}(t) + \sum_{k=1}^{m} \sigma_k x(t-)dB_k(t) \tag{4.68}$$

where for each $1 \leq k \leq m$, $\sigma_k > 0$ and $c > -1$. Using (4.50), then

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t)| = \left(a - \lambda c - \frac{1}{2}\sum_{k=1}^{d} \sigma_k^2\right) + \lambda \log(1+c).$$

Since $\left(a - \frac{1}{2}\sum_{k=1}^{m} \sigma_k^2\right) < 0$ and for $c > -1$, $\lambda \log(1+c) - \lambda c < 0$ then the system will remain stable i.e. $\left(a - \lambda c - \frac{1}{2}\sum_{k=1}^{d} \sigma_k^2 + \lambda \log(1+c)\right) < 0$. Hence, from this example we can see that the exponentially stable one-dimensional system (4.67) cannot be destabilized by the Brownian motion with the form of perturbation as given.

Hence, we can deduce that one-dimensional exponentially stable ODE systems and one-dimensional exponentially stable SDEs driven by Poisson noise cannot be destabilized by Brownian motion noise.

Chapter 5

Stochastic Functional Differential Equations (SFDEs) driven by Lévy noise

5.1 Introduction

In this chapter we will concentrate on stochastic functional differential equations (SFDEs) driven by Lévy noise. In SFDEs the drift, diffusion and jump coefficients depend on the past of the solution, while in SDEs the coefficients just depend on the present. Obviously an SDE belongs to the class of SFDEs with zero time delay but stability theory of SFDEs and SDEs has significant differences. Also SFDEs contain an important class of stochastic delay equations (SDDEs), which (in the Brownian motion case) have found a number of applications to e.g. mathematical finance (see e.g. Kazmerchuka, Swishchukb, and Wu [21]). Stochastic functional differential equations appear in many different contexts including the study of materials with memory (viscoelastic materials), mathematical demography and population dynamics (see Mohammed [37]).

This chapter is self-contained and its structure will be the following:
Firstly a brief introduction to SFDEs is given. Then we will turn our attention to stability properties of SFDEs and stochastic differential delay equations (SDDEs) driven by Lévy noise using Razumikhin type theorems. And finally, the central point of this chapter, conditions for the stabilization of an unstable functional ODE system using noise associated with a Lévy process will be established.

5.2 Existence and uniqueness of solutions

Assume that we are given an m-dimensional standard $\mathcal{F}_t$-adapted Brownian motion $B = (B(t), t \geq 0)$ with each $B(t) = (B^1(t), \ldots, B^m(t))$ and an independent $\mathcal{F}_t$-adapted Poisson random measure N defined on $\mathbb{R}^+ \times (\mathbb{R}^d \backslash \{0\})$ with compensator $\tilde{N}$ and intensity measure ν, where we assume that ν is a Lévy measure.

Let $\tau > 0$. Consider that $0 < c \leq \infty$ and $f : \mathcal{D}([-\tau, 0]; \mathbb{R}^d) \to \mathbb{R}^d$, $g : \mathcal{D}([-\tau, 0]; \mathbb{R}^d) \to \mathcal{M}_{d,m}(\mathbb{R})$ and $H : \mathcal{D}([-\tau, 0]; \mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ are Borel measurable. A stochastic functional differential equation driven by Lévy noise is defined as

$$dx(t) = f(x_{t-})dt + g(x_{t-})dB(t) + \int_{|y|<c} H(x_{t-}, y)\, \tilde{N}(dt, dy) \quad \text{on } t \geq 0 \tag{5.1}$$

and to give (5.1) a rigorous meaning we rewrite it in an integral form, for each $t \geq 0$, as

$$x(t) = x_0(0) + \int_0^t f(x_{s-})ds + \int_0^t g(x_{s-})dB(s) + \int_0^t \int_{|y|<c} H(x_{s-}, y)\, \tilde{N}(ds, dy) \tag{5.2}$$

where x_0 is the initial condition and $\big(x_t, t \geq 0\big)$ is the process taking values in $\mathcal{D}([-\tau, 0]; \mathbb{R}^d)$, which is defined as follows.
For each $t \geq 0$, $x_t : \Omega \to \mathcal{D}([-\tau, 0]; \mathbb{R}^d)$ where for each $\omega \in \Omega$, $\theta \in [-\tau, 0]$,

$$x_t(\omega)(\theta) = x(t + \theta)(\omega).$$

It then follows that

$$x_{t-}(\omega)(\theta) = \lim_{s \uparrow t} x_s(\omega)(\theta) = \lim_{s \uparrow t} x(s + \theta)(\omega) = x(t\text{-} + \theta)(\omega).$$

By $\mathcal{D}_0\left([-\tau, 0]; \mathbb{R}^d\right)$ we denote the family of all bounded $\mathcal{F}_0$-measurable $\mathcal{D}\left([-\tau, 0]; \mathbb{R}^d\right)$-valued random variables. For $0 < p < \infty$ and $t \geq 0$ we denote by $L_t^p\left([-\tau, 0]; \mathbb{R}^d\right)$ the family of all $\mathcal{F}_t$-measurable $\mathcal{D}\left([-\tau, 0]; \mathbb{R}^d\right)$-valued random variables ϕ such that $\sup_{-\tau \leq \theta \leq 0} E|\phi(\theta)|^p < \infty$. Also $\|\cdot\|$ here denotes the supremum norm in $\mathcal{D}\left([-\tau, 0]; \mathbb{R}^d\right)$. To avoid any confusions, we denote the norm of a matrix $A \in \mathcal{M}_{d,m}(\mathbb{R})$ as $\|A\|_* = \left(\sum_{i=1}^d \sum_{j=1}^m A_{ij}^2\right)^{\frac{1}{2}}$.

For the purposes of this chapter we assume that the initial condition x_0 is a fixed $\mathcal{D}_0\left([-\tau, 0]; \mathbb{R}^d\right)$-valued random variable.

Definition 5.2.1 (c.f. Mao [33], definition 5.2.1 pp. 149) A stochastic process $x = (x(t), t \geq -\tau)$ is called a solution to (5.1) with initial condition x_0 if it has the following properties

(i) It has a càdlàg modification and $(x_t, t \geq 0)$ is $\mathcal{F}_t$-adapted,
(ii) For each $\mathcal{F}_t$-adapted process $(x_t, t \geq 0)$, the processes $(f(x_t), t \geq 0)$, $(g(x_t), t \geq 0)$ and $(H(x_t, y), t \geq 0)$ for all $y \in \hat{B}_c$ are also $\mathcal{F}_t$-adapted and for each $t \geq 0$, $\int_0^t |f(x_{s-})|ds < \infty$ (a.s.), $\int_0^t \|g(x_{s-})\|_*^2 ds < \infty$ (a.s.) and $\int_0^t \int_{|y|<c} |H(x_{s-}, y)|^2 \nu(dy)ds < \infty$ (a.s.),
(iii) For each $t \geq 0$, $x(t)$ satisfies (5.2) almost surely.
A solution x is said to be unique if any other solution x' is indistinguishable from x i.e.

$$P\left(x(t) = x'(t) \ \text{ for all } \ t \geq -\tau\right) = 1.$$

Now for the existence of a unique solution to (5.1) we need to impose the following conditions:
(F1)Lipschitz conditions: There exists a positive constant L such that, for all $\varphi, \phi \in \mathcal{D}([-\tau, 0]; \mathbb{R}^d)$,

$$|f(\varphi) - f(\phi)|^2 \leq L\|\varphi - \phi\|^2, \qquad \|g(\varphi) - g(\phi)\|_*^2 \leq L\|\varphi - \phi\|^2, \tag{5.3}$$

$$\int_{|y|<c} |H(\varphi, y) - H(\phi, y)|^2 \nu(dy) \leq L\|\varphi - \phi\|^2. \tag{5.4}$$

(F2)Growth conditions: There exists a positive constant K such that, for all $\varphi \in \mathcal{D}([-\tau, 0]; \mathbb{R}^d)$,

$$|f(\varphi)|^2 \leq K(1 + \|\varphi\|^2), \qquad \|g(\varphi)\|_*^2 \leq K(1 + \|\varphi\|^2), \tag{5.5}$$

$$\int_{|y|<c} |H(\varphi, y)|^2 \nu(dy) \leq K(1 + \|\varphi\|^2). \tag{5.6}$$

Mao in [33] pp. 150-152 proves existence and uniqueness of solutions of SFDEs which are driven by Brownian motion under the Lipschitz and growth conditions, using the Picard iteration technique. Now, the existence and uniqueness of the solutions of SFDEs driven by Lévy processes can be proved quite easily, by a mixture of Mao's approach and Applebaum's arguments that were used for establishing existence and uniqueness of solutions of SDEs driven by Lévy processes (see Applebaum [1] pp. 305-309).

Theorem 5.2.2 *Assume that the Lipschitz and growth conditions (F1) and (F2) hold. Then there exists a unique solution $x = (x(t), t \geq -\tau)$ to (5.1) with initial data x_0. The process x is adapted and càdlàg.*

Proof: As the proof is a straightforward extension of arguments in the literature, we do not include it here. However it is presented in full as Appendix A.

For the rest of this chapter we require that (F1) and (F2) hold.

Assume that $f(0) = 0$, $g(0) = 0$ and $H(0, y) = 0$ for all $|y| < c$. Then (5.1) admits a solution $x(t) = 0$ (a.s.) for all $t \geq 0$ corresponding to initial value $x_0 = 0$, which is called the *trivial solution.*

The definitions of stability for the trivial solution of SFDEs are obvious modification of those given for SDEs as in Chapter 1, section 1.5.

Again we are considering SFDEs of homogeneous form to avoid complications in the notation. As was mentioned in the case of SDEs, the extension from homogeneous to inhomogeneous form (see Mao [32] and Chapter 5 [33] for the Brownian motion case) does not require very much additional work.

5.3 Razumikhin type theorems for stochastic functional differential equations

In the literature there are many different points of view for examining exponential stability of SFDEs. One method is the use of Lyapunov functionals which is a generalization of the direct Lyapunov method that we have used in the previous chapters to examine stability of SDEs (see Kolmanovskii and Nosov [23]). Another method is application of the Razumikhin type theorems based on the use of functions in $\mathbb{R}^d$ similar to the Razumikhin theory that was developed for the stability of deterministic functional differential equations (see Hale and Lunel [16] pp. 151-161). Novel work in stability of SFDEs driven by Brownian motion using the Razumikhin technique, has been carried out by Mao in [32]. For further techniques that can be used for the study of exponential stability of SFDEs (e.g. LaSalle-type theorems) we refer to Mao's review [34].

The aim of this section is to generalize Mao's paper [32] and use the Razumikhin type approach to examine stability of SFDEs driven by Lévy noise.

In this section we denote by $\mathcal{L}V : \mathcal{D}([-\tau, 0]; \mathbb{R}^d) \to \mathbb{R}$ the linear functional associated to (5.1) which is defined as:

$$\begin{aligned}(\mathcal{L}V)(\phi) &= f^i(\phi)(\partial_i V)(\phi(0)) + \frac{1}{2}\left[g(\phi)g(\phi)^T\right]^{ik}(\partial_i\partial_k V)(\phi(0)) \\ &+ \int_{|y|<c}\left[V(\phi(0) + H(\phi, y)) - V(\phi(0)) - H^i(\phi, y)(\partial_i V)(\phi(0))\right]\nu(dy)\end{aligned} \tag{5.7}$$

where $V \in C^2(\mathbb{R}^d;\mathbb{R}^+)$, and $\phi \in \mathcal{D}([-\tau,0];\mathbb{R}^d)$.

Now to give some insight into the Razumikhin technique, let $x = (x(t), t \geq -\tau)$ be the solution of (5.1) and $V \in C^2(\mathbb{R}^d;\mathbb{R}^+)$. By Itô's formula we have that

$$V(x(t)) - V(x(0)) - \int_0^t \mathcal{L}V(x_{s-})ds$$

is a local martingale. This implies that

$$E\left(V(x(t))\right) - E\left(V(x(0))\right) = \int_0^t E\left(\mathcal{L}V(x_s)\right)ds,$$

hence

$$\frac{dE\left(V(x(t))\right)}{dt} = E\left(\mathcal{L}V(x_t)\right).$$

Now to establish stability of the trivial solution, traditional stability techniques technically require $E\left(\mathcal{L}V(\phi)\right) < 0$ for all $\phi \in L_t^p([-\tau,0];\mathbb{R}^d)$. Razumikhin in 1956 proposed for the case of ordinary differential delay equations (ODDEs), that is not necessary to impose conditions on V for all $\phi \in \mathcal{C}([-\tau,0];\mathbb{R}^d)$. This was extended in the stochastic case as well, and results that are based on negativeness of $E\left(\mathcal{L}V(\phi)\right)$ for certain $\phi \in L_t^p([-\tau,0];\mathbb{R}^d)$ are called Razumikhin type theorems. For further explanations see Hale and Lunel [16] pp. 151-161 and Myshkis [39].

Assumption 5.3.1 For all $2 \leq q \leq p$, there exists a constant $K > 0$ such that, for all $\phi \in \mathcal{D}([-\tau,0];\mathbb{R}^d)$,

$$\int_{|y|<c} \left|H(\phi,y)\right|^q \nu(dy) \leq K\left(1 + \|\phi\|^q\right).$$

We require assumption 5.3.1 to hold for the remainder of this chapter.

The following is a generalization of Mao's work [32] Theorem 2.1 where he obtains pth moment exponential stability for SFDEs driven by a Brownian motion process. As will be shown below if we further add the jumps of a Lévy process then this will not change the form of Mao's results or the conditions that have been imposed on the SFDE driven by a Brownian motion. Hence some details are omitted where arguments are exactly the same as in Mao [32].

For the proof below we need the following lemma.

Lemma 5.3.2 Let $q : [-\tau, 0] \to \mathbb{R}^+$ be a continuous function and define $\max_{-\tau \leq \theta \leq 0} q(\theta) = L$. Given $0 < \varepsilon < L - q(0)$, there exists $\delta > 0$ such that $|k| < \delta$, implies that $q(k) \leq L$.

Proof: Assume that q is a non-constant function. Let θ' be the largest $\theta_k \in [-\tau, 0]$ such that $q(\theta_k) = L$. If $\theta' \neq 0$, then $q(\theta) < L$ for all $\theta \in (\theta', 0]$ and in particular $q(0) < L$. Now as q is continuous, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|k| < \delta$ implies $|q(k) - q(0)| < \varepsilon$ and hence $q(k) < q(0) + \varepsilon$. Provided that we take $\varepsilon < L - q(0)$ then $q(k) < L$. □

Theorem 5.3.3 *Let $V \in C^2(\mathbb{R}^d; \mathbb{R}^+)$, $h > 1$ and p, q_1, q_2, ρ be positive constants. Suppose that*

$$(i)\ q_1|x|^p \leq V(x) \leq q_2|x|^p \quad \text{for all } x \in \mathbb{R}^d.$$

(ii) For all $\phi \in L_t^p([-\tau, 0]; \mathbb{R}^d)$ such that

$$E\left(V(\phi(\theta))\right) < hE\left(V(\phi(0))\right) \quad \text{for all} \quad -\tau \leq \theta \leq 0,$$

we require

$$E\left(\mathcal{L}V(\phi)\right) \leq -\rho E\left(V(\phi(0))\right).$$

Then for all initial conditions the solution of (5.1) is pth moment exponentially stable and satisfies

$$E\left(|x(t)|^p\right) \leq \frac{q_2}{q_1} E\left(\|x_0\|^p\right) e^{-\beta t} \quad \text{for all} \quad t \geq 0, \tag{5.8}$$

where $\beta = \min\{\rho, \log(h)/\tau\}$ and the pth moment sample Lyapunov exponent is not greater that $-\beta$.

Proof: Let $\alpha \in (0, \beta)$ be arbitrary and $\bar{\beta} = \beta - \alpha$. Define

$$U(t) = \max_{-\tau \leq \theta \leq 0} \left[e^{\bar{\beta}(t+\theta)} E\left(V(x(t+\theta))\right)\right] \quad \text{for all} \quad t \geq 0.$$

Since V is a continuous function and x is càdlàg then $V \circ x$ is càdlàg as the composition of a continuous and a càdlàg function is càdlàg (see Applebaum [1] pp. 118). By applying Itô's formula with jumps in (5.1), then taking expectations, a standard argument shows that the map $t \to E\left(V(x(t))\right)$ is continuous. Hence, U is continuous with respect to time.

Define

$$D_+U(t) = \limsup_{k\to 0^+} \frac{U(t+k)-U(t)}{k} \quad \text{for all } t \geq 0. \tag{5.9}$$

Our aim is to prove that $D_+U(t) \leq 0$ for all $t \geq 0$.

Fix $t \geq 0$ and define $\theta' \in [-\tau, 0]$ to be the largest $\theta \in [-\tau, 0]$ such that $U(t) = e^{\bar{\beta}(t+\theta)}E\left(V(x(t+\theta))\right)$. Hence $U(t) = e^{\bar{\beta}(t+\theta')}E\left(V(x(t+\theta'))\right)$.
Since $\theta' \in [-\tau, 0]$ then either $\theta' < 0$ or $\theta' = 0$.

Case I: $\theta' < 0$
Fix $t \geq 0$. If $\theta' < 0$, then

$$e^{\bar{\beta}(t+\theta)}E\left(V(x(t+\theta))\right) < e^{\bar{\beta}(t+\theta')}E\left(V(x(t+\theta'))\right) \quad \text{for all } \theta' < \theta \leq 0.$$

Define $q(\theta) = e^{\bar{\beta}(t+\theta)}E\left(V(x(t+\theta))\right)$ where $\theta \in [-\tau, 0]$ and choose L as in Lemma 5.3.2. Then by Lemma 5.3.2, given sufficiently small $\varepsilon > 0$, we can find $\delta > 0$ such that for $0 < k < \delta$

$$e^{\bar{\beta}(t+k)}E\left(V(x(t+k))\right) \leq e^{\bar{\beta}(t+\theta')}E\left(V(x(t+\theta'))\right).$$

Hence $U(t+k) \leq U(t)$, and so $D_+U(t) \leq 0$.

Case II: $\theta' = 0$
Fix $t \geq 0$. For $\theta' = 0$, we will only give an outline of the proof. Following the same line of arguments as in Mao [32], Theorem 2.1 pp. 237 (to which we refer for further details) we obtain

$$E\left(V(x(t+\theta))\right) \leq e^{\bar{\beta}\tau}E\left(V(x(t))\right) \quad \text{for all} \quad -\tau \leq \theta \leq 0, \tag{5.10}$$

and as $V \in C^2(\mathbb{R}^d; \mathbb{R}^+)$ then either $E\left(V(x(t))\right) = 0$ or $E\left(V(x(t))\right) > 0$.
If $E\left(V(x(t))\right) = 0$, then by (5.10) and condition (i), $x(t+\theta) = 0$ (a.s.) for all $-\tau \leq \theta \leq 0$. From the assumptions it holds that $f(0) = 0$, $g(0) = 0$ and $H(0, y) = 0$ for all $|y| < c$, hence $x(t) = 0$ (a.s.) for all $t \geq 0$. It follows that $x(t+k) = 0$ (a.s.) for all $k > 0$, hence $U(t+k) = 0$, and so $D_+U(t) = 0$. Now assume that $E\left(V(x(t))\right) > 0$. By the definition of β it follows that $e^{\bar{\beta}\tau} < h$ with $h > 1$, and then by (5.10)

$$E\left(V(x(t+\theta))\right) \leq e^{\bar{\beta}\tau}E\left(V(x(t))\right) < hE\left(V(x(t))\right) \quad \text{for all} \quad -\tau \leq \theta \leq 0,$$

and hence by condition (ii) we have

$$E(\mathcal{L}V(x_{t-})) \leq -\rho E\left(V(x(t))\right).$$

By applying Itô's formula to $Z(t) = e^{\bar{\beta}(t+k)}V(x(t+k))$ for $k > 0$ we have the following

$$\begin{aligned} e^{\bar{\beta}(t+k)}V(x(t+k)) &= e^{\bar{\beta}t}V(x(t)) + \int_t^{t+k} \bar{\beta}e^{\bar{\beta}s}V(x(s-))ds \\ &+ \int_t^{t+k} e^{\bar{\beta}s}\partial_i V(x(s-)) \left[f^i(x_{s-})ds + g^{ij}(x_{s-})dB_j(s)\right] \\ &+ \frac{1}{2}\int_t^{t+k} e^{\bar{\beta}s}\,\partial_i\partial_k V(x(s-))[g(x_{s-})g(x_{s-})^T]^{ik}ds \\ &+ \int_t^{t+k}\int_{|y|<c} e^{\bar{\beta}s}\left[V(x(s-)+H(x_{s-},y)) - V(x(s-))\right]\tilde{N}(ds,dy) \\ &+ \int_t^{t+k}\int_{|y|<c} e^{\bar{\beta}s}\Big[V\left(x(s-)+H(x_{s-},y)\right) - V(x(s-)) \\ &\qquad - H^i(x_{s-},y)\partial_i V(x(s-))\Big]\nu(dy)ds. \end{aligned}$$

Hence by (5.7)

$$e^{\bar{\beta}(t+k)}E\left(V(x(t+k))\right) - e^{\bar{\beta}t}E\left(V(x(t))\right) = \int_t^{t+k} e^{\bar{\beta}s}\left[\bar{\beta}E\left(V(x(s-))\right) + E(\mathcal{L}V(x_{s-}))\right]ds. \tag{5.11}$$

Now apply condition (ii) to see that

$$\bar{\beta}E\left(V(x(t))\right) + E(\mathcal{L}V(x_{t-})) \leq -(\rho-\bar{\beta})E\left(V(x(t))\right) < 0 \tag{5.12}$$

since $\rho > \bar{\beta}$. From this point we argue as in Mao [32] pp. 237 and the required result follows. □

The theorem below shows that under some additional assumptions, the pth moment exponential stability, of the trivial solution of (5.1), implies the almost sure exponential stability, generalizing Theorem 2.2 pp. 238 in Mao [32].

Theorem 5.3.4 *Let $p \geq 2$. Assume that the hypotheses of Theorem 5.3.3 hold and there exists constants $K_1 > 0$ and $K_2 > 0$ such that, for all $\phi \in L^p_t([-\tau,0];\mathbb{R}^d)$, the following conditions are satisfied.*

$$(i)\ E\left(|f(\phi)|^p\right) + E\left(tr\left[g(\phi)^T g(\phi)\right]\right)^{\frac{p}{2}} + E\left(\int_{|y|<c}|H(\phi,y)|^2\,\nu(dy)\right)^{\frac{p}{2}} \leq K_1 \sup_{-\tau\leq\theta\leq 0} E\left(|\phi(\theta)|^p\right). \tag{5.13}$$

$$(ii)\ E\left(\int_{|y|<c}|H(\phi,y)|^p\,\nu(dy)\right) \leq K_2 \sup_{-\tau\leq\theta\leq 0} E\left(|\phi(\theta)|^p\right). \tag{5.14}$$

Then

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \le -\frac{\beta}{p} \qquad a.s.$$

where $\beta = \min\{\rho, \log(h)/\tau\}$.

Proof: For $t \ge \tau$, by Jensen's inequality we can deduce that

$$\begin{aligned} E\left(\|x_{t+\tau}\|^p\right) = E\left(\sup_{0\le h\le\tau}|x(t+h)|^p\right) &\le 4^{p-1}\Bigg(E\left(|x(t)|^p\right) + E\left[\int_t^{t+\tau}|f(x_{s-})|\,ds\right]^p \\ &+ E\left[\sup_{0\le h\le\tau}\left|\int_t^{t+h} g(x_{s-})dB(s)\right|^p\right] \\ &+ E\left[\sup_{0\le h\le\tau}\left|\int_t^{t+h}\int_{|y|<c} H\left(x_{s-},y\right)\tilde{N}(ds,dy)\right|^p\right]\Bigg). \end{aligned}$$

Using Holder's inequality, condition (5.13) and the results of Theorem 5.3.3, then

$$E\left[\int_t^{t+\tau}|f(x_{s-})|\,ds\right]^p \le \frac{K_1 q_2 \tau^p}{q_1}E\left(\|x_0\|^p\right)e^{-\beta(t-\tau)}. \tag{5.15}$$

By the Burkholder-Davis-Gundy inequality (see Chapter 3, Theorem 3.5.3) and following the same line of arguments that were used to deduce (5.15), we have

$$E\left[\sup_{0\le h\le\tau}\left|\int_t^{t+h} g(x_{s-})dB(s)\right|^p\right] \le \frac{C_p K_1 q_2 \tau^{\frac{p}{2}}}{q_1}E\left(\|x_0\|^p\right)e^{-\beta(t-\tau)} \tag{5.16}$$

where C_p is a positive constant depending only on p.
The estimates in (5.15) and (5.16) are exactly the same estimates obtained in the Brownian motion case, as in Mao [32] pp. 238-239.

Now for the compensated Poisson integral we will use Kunita's estimates (see Chapter 3, (3.43)). Applying Holder's inequality to the first term of the right-hand side of the inequality below and then conditions (i) and (ii) we deduce that

$$
\begin{aligned}
E&\left[\sup_{0\le h\le\tau}\left|\int_t^{t+h}\int_{|y|<c} H\left(x_{s-},y\right)\tilde{N}(ds,dy)\right|^p\right] \\
&\le c_3(p)E\left[\left(\int_t^{t+\tau}\int_{|y|<c}\left|H\left(x_{s-},y\right)\right|^2\nu(dy)ds\right)^{\frac{p}{2}}\right] \\
&\quad + c_4(p)E\left[\int_t^{t+\tau}\int_{|y|<c}\left|H\left(x_{s-},y\right)\right|^p\nu(dy)ds\right] \\
&\le c_3(p)\tau^{\frac{p}{2}-1}E\left[\int_t^{t+\tau}\left(\int_{|y|<c}\left|H\left(x_{s-},y\right)\right|^2\nu(dy)\right)^{\frac{p}{2}}ds\right] \\
&\quad + c_4(p)E\left[\int_t^{t+\tau}\int_{|y|<c}\left|H\left(x_{s-},y\right)\right|^p\nu(dy)ds\right] \\
&\le c_3(p)\tau^{\frac{p}{2}-1}\left(\int_t^{t+\tau}K_1\sup_{-\tau\le\theta\le 0}E\big(|x(s+\theta)|^p\big)ds\right) \\
&\quad + c_4(p)\left(\int_t^{t+\tau}K_2\sup_{-\tau\le\theta\le 0}E\big(|x(s+\theta)|^p\big)ds\right),
\end{aligned}
\tag{5.17}
$$

where $c_3(p)$ and $c_4(p)$ positive constants that depend on p.

Hence by (5.8) and (5.17) we obtain

$$
E\left[\sup_{0\le h\le\tau}\left|\int_t^{t+h}\int_{|y|<c} H\left(x_{s-},y\right)\tilde{N}(ds,dy)\right|^p\right] \le \left(c_3(p)\tau^{\frac{p}{2}}K_1 + c_4(p)\tau K_2\right)\frac{q_2}{q_1}E\big(\|x_0\|^p\big)e^{-\beta(t-\tau)}. \tag{5.18}
$$

Combining (5.15), (5.16) and (5.18) we find that

$$
E\left(\|x_{t+\tau}\|^p\right) \le Le^{-\beta t} \qquad \text{for all } t\ge\tau
$$

where

$$
L = \frac{4^{p-1}q_2}{q_1}E\big(\|x_0\|^p\big)\left[1+e^{\beta\tau}\left(K_1\left(\tau^p + C_p\tau^{\frac{p}{2}} + c_3(p)\tau^{\frac{p}{2}}\right) + c_4(p)K_2\tau\right)\right].
$$

From this point we follow Mao's exact arguments in [32] pp. 239 and the required result follows. □

5.4 Stochastic differential delay equations

In this section we consider stochastic differential delay equations (SDDE) driven by Lévy noise and examine their stability properties.

The first article on stochastic delay equations driven by Brownian motion was by Itô and Nisio in [20]. In [15] Gushchin and Küchler give necessary and sufficient conditions for the existence of a stationary solution to an SDDE driven by a Lévy process. Stationary and Feller properties of SDDEs driven by Lévy processes can be found in Reiß, Riedle and Gaans [41], while Mohammed and Scheutzow in [38] discuss flow and stability properties. For a detailed treatment of SDDEs we refer to these.

Now a general SDDE driven by Lévy noise has the following form

$$\begin{aligned} dx(t) &= F\big(x(t\text{-}), x(t\text{-} - \mu_1(t)), \ldots, x(t\text{-} - \mu_k(t))\big)dt \\ &\quad + G\big(x(t\text{-}), x(t\text{-} - \mu_1(t)), \ldots, x(t\text{-} - \mu_k(t))\big)dB(t) \\ &\quad + \int_{|y|<c} K\big(x(t\text{-}), x(t\text{-} - \mu_1(t)), \ldots, x(t\text{-} - \mu_k(t)), y\big)\tilde{N}(dt, dy) \end{aligned} \tag{5.19}$$

for $t \geq 0$, where for $1 \leq n \leq k$, $\mu_n : \mathbb{R}^+ \to [0, \tau]$ are continuous functions, $F : \mathbb{R}^d \times \mathcal{M}_{d,k}(\mathbb{R}) \to \mathbb{R}^d$, $G : \mathbb{R}^d \times \mathcal{M}_{d,k}(\mathbb{R}) \to \mathcal{M}_{d,m}(\mathbb{R})$ and $H : \mathbb{R}^d \times \mathcal{M}_{d,k}(\mathbb{R}) \times \mathbb{R}^d \to \mathbb{R}^d$ and the initial data x_0 is a fixed $\mathcal{D}([-\tau, 0]; \mathbb{R}^d)$-valued random variable.

We assume that $F(0, 0, \ldots, 0) = 0$, $G(0, 0, \ldots, 0) = 0$ and $K(0, 0, \ldots, 0, 0) = 0$ so that for $x_0 = 0$ we see that (5.19) admits a trivial solution.

Define for $t \geq 0$, $\phi \in \mathcal{D}([-\tau, 0]; \mathbb{R}^d)$

$$f(\phi) = F\left(\phi_t(0), \phi_t(-\mu_1(t)), \ldots, \phi_t(-\mu_k(t))\right), \quad g(\phi) = G\big(\phi_t(0), \phi_t(-\mu_1(t)), \ldots, \phi_t(-\mu_k(t))\big),$$

$$\text{and} \qquad H(\phi, y) = K\big(\phi_t(0), \phi_t(-\mu_1(t)), \ldots, \phi_t(-\mu_k(t)), y\big).$$

Assume that F, G, H satisfy (F1) and (F2). Then (5.19) is of the same form as (5.1), with coefficients given by f, g, H as defined previously. Hence, we can apply the results that have been obtained in section 5.3 for SFDEs to examine the stability properties of SDDEs.

In this case the functional $\mathcal{L}V : \mathcal{D}([-\tau, 0]; \mathbb{R}^d) \to \mathbb{R}$ will have the following form

$$
\begin{aligned}
(\mathcal{L}V)(\phi) &= F^i\Big(\phi(0), \phi(-\mu_1(t)), \ldots, \phi(-\mu_k(t))\Big)(\partial_i V)(\phi(0)) \\
&+\frac{1}{2}\left[G\Big(\phi(0), \phi(-\mu_1(t)), \ldots, \phi(-\mu_k(t))\Big)G\Big(\phi(0), \phi(-\mu_1(t)), \ldots, \phi(-\mu_k(t))\Big)^T\right]^{ir} (\partial_i\partial_r V)(\phi(0)) \\
&+\int_{|y|<c}\Big[V\Big(\phi(0) + K(\phi(0), \phi(-\mu_1(t)), \ldots, \phi(-\mu_k(t)), y\Big) - V(\phi(0)) \\
&\qquad - K^i\Big(\phi(0), \phi(-\mu_1(t)), \ldots, \phi(-\mu_k(t)), y\Big)(\partial_i V)(\phi(0))\Big]\nu(dy) \qquad (5.20)
\end{aligned}
$$

where $V \in C^2(\mathbb{R}^d; \mathbb{R}^+)$, and $\phi \in \mathcal{D}([-\tau, 0]; \mathbb{R}^d)$.

The following is a generalization of Mao's work [32], Theorem 3.1.

Theorem 5.4.1 *Let $V \in C^2(\mathbb{R}^d; \mathbb{R}^+)$, $h > 1$ and $p, q_1, q_2, \rho, \rho_1, \ldots, \rho_n$ be positive constants. If*

$$(i)\ q_1|x|^p \leq V(x) \leq q_2|x|^p \quad \text{for all } x \in \mathbb{R}^d,$$

and the functional $\mathcal{L}V : \mathcal{D}([-\tau, 0]; \mathbb{R}^d) \to \mathbb{R}$ as defined in (5.20) satisfies

$$(ii)\ (\mathcal{L}V)(\phi) \leq -\rho V(\phi(0)) + \sum_{n=1}^{k} \rho_n V(\phi(-\mu_n(t))) \qquad (5.21)$$

where $\phi \in \mathcal{D}([-\tau, 0]; \mathbb{R}^d)$, then the trivial solution of (5.19) is pth moment exponentially stable if $\rho > \sum_{n=1}^k \rho_n$ and moreover its pth moment Lyapunov exponent satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log(E(|x(t)|)^p) < -\left(\rho - h\sum_{n=1}^{k} \rho_n\right) \qquad a.s. \qquad (5.22)$$

where $h \in (1, \rho/\sum_{n=1}^k \rho_n)$ and is the unique solution of $\rho - h\sum_{n=1}^k \rho_n = \log(h)/\tau$.

If $p \geq 2$ and the following conditions are satisfied for all $(x, z_1, \ldots, z_k) \in \mathbb{R}^d \times \mathcal{M}_{d,k}(\mathbb{R})$ with $M_1 > 0$, $M_2 > 0$

$$|F(x, z_1, \ldots, z_k)|^2 + tr\left[g(x, z_1, \ldots, z_k)^T g(x, z_1, \ldots, z_k)\right] \leq M_1\left(|x|^2 + \sum_{n=1}^{k} |z_n|^2\right) (5.23)$$

and

$$\int_{|y|<c} |H(x, z_1, \ldots, z_k, y)|^p \nu(dy) \leq M_2\left(|x|^p + \sum_{n=1}^{k} |z_n|^p\right) \qquad . \qquad (5.24)$$

then the sample Lyapunov exponent

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| < -\left(\frac{\rho - h\sum_{n=1}^{k}\rho_n}{p}\right) \qquad a.s.$$

If $\rho - h\sum_{n=1}^{k}\rho_n > 0$ then the trivial solution of (5.19) is almost surely exponentially stable.

Remark 5.4.2 The uniqueness of solution to $\rho - h\sum_{n=1}^{k}\rho_n = \log(h)/\tau$ is proved in Appendix B.

Proof: For $t \geq 0$, choose $\phi_t \in L_t^p([-\tau,0];\mathbb{R}^d)$ which satisfies

$$E\left(V(\phi_t(\theta))\right) < hE\left(V(\phi_t(0))\right) \quad \text{for all} \quad -\tau \leq \theta \leq 0 \tag{5.25}$$

where $h > 1$. Then by (5.21) and (5.25), it holds that

$$\begin{aligned} E\left(\mathcal{L}V(\phi_t)\right) &\leq -\rho E(V(\phi_t(0))) + \sum_{n=1}^{k}\rho_n E(V(\phi_t(-\mu_n(t)))) \\ &\leq -\left(\rho - h\sum_{n=1}^{k}\rho_n\right)E(V(\phi_t(0))). \end{aligned} \tag{5.26}$$

Since $\rho > \sum_{n=1}^{k}\rho_n$ and $h \in (1, \rho/\sum_{n=1}^{k}\rho_n)$ then by (i), (5.25) and (5.26) the conditions of Theorem 5.3.3 are satisfied, with β defined as $\beta = \min\{\rho - h\sum_{n=1}^{k}\rho_n, \log(h)/\tau\}$. It follows immediately that the trivial solution is pth moment exponentially stable and the pth moment Lyapunov exponent satisfies

$$\limsup_{t\to\infty} \frac{1}{t}\log\left(E[|x(t)|]^p\right) < -\left(\rho - h\sum_{n=1}^{k}\rho_n\right) \qquad a.s.$$

Now consider that $p \geq 2$. For all $t \geq 0$ and $\phi \in L_t^p([-\tau,0];\mathbb{R}^d)$ then by (5.23), (5.24) and Jensen's inequality applied twice we can show that

$$\begin{aligned} &E\left(|f(\varphi)|^p\right) + E\left(\mathrm{tr}\left[g(\varphi)^T g(\varphi)\right]\right)^{\frac{p}{2}} + E\left(\int_{|y|<c}|H(\varphi,y)|^2\,\nu(dy)\right)^{\frac{p}{2}} \\ &\leq 3E\left(M\left[|\phi(0)|^2 + \sum_{n=1}^{k}|\phi(-\mu_n(t))|^2\right]\right)^{\frac{p}{2}} \\ &\leq 3M^{\frac{p}{2}}(1+k)^{\frac{p-2}{2}}E\left(|\phi(0)|^p + \sum_{n=1}^{k}|\phi(-\mu_n(t))|^p\right) \leq 3\left(M(1+k)\right)^{\frac{p}{2}}\sup_{-\tau\leq\theta\leq 0}E\left(|\phi(\theta)|^p\right), \end{aligned}$$

where $M = \max\{M_1, M_2\}$. Also,

$$E\left(\int_{|y|<c} |H(\varphi, y)|^p \nu(dy)\right) \leq E\left(M_2\left(|\phi(0)|^p + \sum_{n=1}^{k} |\phi(-\mu_n(t))|^p\right)\right)$$
$$\leq (k+1)M_2 \sup_{-\tau\leq\theta\leq 0} E(|\phi(\theta)|^p).$$

Hence, conditions (5.13) and (5.14) are satisfied and as the conditions of Theorem 5.3.3 hold, then by Theorem 5.3.4 the trivial solution of (5.19) is almost surely exponentially stable, and so

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t)| \leq -\left(\frac{\rho - h\sum_{n=1}^{k} \rho_n}{p}\right) \qquad a.s.$$

□

5.5 Stabilization of SFDEs by a Poisson process

Assume that we are given the following non-linear functional ODE system

$$\frac{dx(t)}{dt} = f(x_{t-}) \qquad \text{on} \qquad t \geq 0 \tag{5.27}$$

with $f : \mathcal{D}([-\tau, 0]; \mathbb{R}^d) \to \mathbb{R}^d$ and initial condition $x_0 \in \mathcal{D}([-\tau, 0]; \mathbb{R}^d)$.
Consider that (5.27) is unstable, in the sense that as time increases indefinitely the solutions of (5.27) escape to infinity.

Recent developments in the area of stochastic stabilization of SFDEs have been carried out by Appleby [2]. Appleby considers an unstable functional ODE system and he perturbs it with noise of the form $\sigma x(t)dB(t)$, where $\sigma \neq 0$ is a real number and B is a one-dimensional standard Brownian motion. The system in homogenous form, for each $t \geq 0$, is the following:

$$dx(t) = \left(f_1(x(t), x(t-\tau_1(t)), \ldots, x(t-\tau_n(t))) + \int_{t-\tau_0(t)}^{t} f_2(x(s))ds\right)dt + \sigma x(t)dB(t)$$

where $n \in \mathbb{N}$. Appleby proves that if f_1 and f_2 satisfy global Lipschitz conditions and global linear bounds and if the finite time delay $\tau' = \sup_{t\geq 0} \max_{i=0,1,\ldots,n} \tau_i(t)$ is sufficiently small, then we can choose σ sufficiently large to stabilize the unstable functional ODE system in an almost sure exponential way.

An improvement of Appleby's results has been made in [3] by Mao and Appleby. They consider a more general stochastic perturbation of the form $\Sigma x(t)dB(t)$ with $\Sigma \in \mathcal{M}_d(\mathbb{R})$ and for each $t \geq 0$, $B(t) = (B^1(t), \ldots, B^m(t))$ is an m-dimensional Brownian motion. Namely, the stochastically perturbed system is

$$dx(t) = f(x_t)dt + \Sigma x(t)dB(t) \tag{5.28}$$

with $f(x_t) = f_1(x(t), x(t-\tau_1(t)), \ldots, x(t-\tau_n(t))) + \int_{t-\tau_0(t)}^{t} f_2(x(s))ds$, now satisfying weaker conditions than the ones that Appleby imposed in [2]. To be precise, they only require locally Lipschitz and one-sided linear bound conditions. In their work [3] they have managed to stabilize (5.28) under certain conditions, and in particular, they have shown that for $\Sigma = \sigma M$, where $\sigma \neq 0$ and $M \in \mathcal{M}_d(\mathbb{R})$ satisfying $\min_{|x|=1}\left(2\langle x, Mx\rangle^2 - |Mx|^2\right) > 0$, the solution of (5.28) tends to zero in an almost sure exponential way when the noise intensity is large enough and the time delay is sufficiently small.

Now, following Mao's and Appleby's ideas in [3] we will try and establish conditions under which the unstable functional ODE system (5.27) can be stabilized by Lévy noise.

Remark 5.5.1 Note that from the Lipschitz condition on the drift coefficient and the fact that $f(0) = 0$, it follows that f satisfies a one-sided linear bound i.e.

$$\langle \phi(0), f(\phi)\rangle \leq |\phi(0)||f(\phi)| \leq \sqrt{L}|\phi(0)|.||\phi|| \tag{5.29}$$

for every $\phi \in \mathcal{D}([-\tau, 0]; \mathbb{R}^d)$, where $L > 0$ is as in (5.3), and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^d$.

Many of the results of this chapter extend to SFDEs satisfying only local Lipschitz conditions and a one-sided linear bound as in Mao and Appleby [3], where the Brownian motion case is treated.

Assume the system (5.27) is perturbed by a one-dimensional compensated Poisson process $(\tilde{N}(t), t \geq 0)$ of intensity $\lambda > 0$. Now, the new system forms a stochastic functional differential equation i.e.

$$dx(t) = f(x_{t-})dt + Dx(t-)d\tilde{N}(t) \quad \text{for} \quad t \geq 0 \tag{5.30}$$

with initial condition a fixed random variable $x_0 \in \mathcal{D}([-\tau, 0]; \mathbb{R}^d)$. We take $D \in \mathcal{M}_d(\mathbb{R})$ to be a symmetric positive definite matrix with ρ_{min} and ρ_{max} its smallest and largest eigenvalues respectively. Hence,

$$dx(t) = [f(x_{t-}) - \lambda Dx(t-)]\, dt + Dx(t-)dN(t). \tag{5.31}$$

Existence and uniqueness of solutions to (5.30) follows from Theorem 5.2.2. Since the drift coefficient in (5.30) satisfies (F1)-(F2) and the coefficient of the compensated Poisson process is linear, then (5.30) admits a unique càdlàg solution.

Remark 5.5.2 It was shown in Chapter 3 and Chapter 4, (Lemma 3.4.4 and Lemma 4.3.2) if the initial condition is non-zero, then the solution of an SDE driven by a Lévy process will be non-zero for all time. This property does not necessarily hold for SFDEs (see Appleby [2]). Hence, we are not able to apply the same method of proof that was used in Chapter 4 for the stabilization of SDEs to the case of SFDEs.

Now define for each $t \geq 0$ and $x_0 \neq 0$

$$\delta(t) = \begin{cases} \frac{|Dx(t)|}{|x(t)|} & \text{for } x(t) \neq 0, \\ \delta_0 & \text{for } x(t) = 0 \end{cases} \tag{5.32}$$

$$\varepsilon(t) = \begin{cases} \frac{2\langle x(t), Dx(t)\rangle}{|x(t)|^2} & \text{for } x(t) \neq 0, \\ \varepsilon_0 & \text{for } x(t) = 0 \end{cases} \tag{5.33}$$

where $\delta_0 \geq 0$ and $\varepsilon_0 \geq 0$ are such that $0 \leq \delta_0 \leq \|D\|_*$ and $\varepsilon_0 = 2\|D\|_*$.
Note that for $x(t) \neq 0$

$$|Dx(t)|^2 = \delta(t)^2|x(t)|^2 \quad \text{and} \quad 2\langle x(t), Dx(t)\rangle = \varepsilon(t)|x(t)|^2. \tag{5.34}$$

For $x(t) \neq 0$, $-\varepsilon(t) \leq -2\rho_{min}$ and

$$\begin{aligned} \log\left(\varepsilon(t) + \delta(t)^2 + 1\right) &= \log\left(\frac{2\langle x(t), Dx(t)\rangle}{|x(t)|^2} + \frac{|Dx(t)|^2}{|x(t)|^2} + 1\right) \\ &\leq \log\left(\frac{|x(t)|^2 + 2\|D\|_*|x(t)|^2 + \|D\|_*^2.|x(t)|^2}{|x(t)|^2}\right) = \log\left(1 + \|D\|_*\right)^2 \end{aligned}$$

and for $x(t) = 0$

$$\log\left(\varepsilon(t) + \delta(t)^2 + 1\right) = \log\left(\varepsilon_0 + \delta_0{}^2 + 1\right) \leq \log(1 + \|D\|_*)^2 \quad \text{and}$$

$$-\varepsilon(t) = -\varepsilon_0 = -2\|D\|_* \leq -2\rho_{min}.$$

Hence for all $t \geq 0$ we have that

$$\log\left(\varepsilon(t) + \delta(t)^2 + 1\right) \leq \log\left(1 + \|D\|_*\right)^2 \quad \text{and} \quad -\varepsilon(t) \leq -2\rho_{min}.$$

To establish stabilization of (5.30) we will follow the same line of arguments that were used by Appleby and Mao in [3], Theorem 2.6, where they have proved the stabilization of an unstable functional ODE system that is stochastically perturbed by Brownian motion.

Theorem 5.5.3 *The sample Lyapunov exponent of the solution of (5.30) satisfies*

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \le -\left[\lambda\left(\rho_{min}-\log(1+\rho_{max})\right)-\sqrt{L}\exp\left(\lambda\tau\rho_{max}\right)\right] \quad a.s. \tag{5.35}$$

In particular if $\lambda(\rho_{min}-\log(1+\rho_{max})) > \sqrt{L}\exp(\tau\lambda\rho_{max})$ *then the trivial solution of (5.30) is almost surely exponentially stable.*

Proof: For $x_0 = 0$ obviously (5.35) holds, since for each $t \ge 0$, $x(t) = 0$. Consider now that $x_0 \neq 0$. Applying Itô's formula to $Y(t) = |x(t)|^2$, we find, for each $t \ge 0$,

$$\begin{aligned}
|x(t)|^2 &= |x_0|^2 + \int_0^t \left[2\langle x(s-), f(x_{s-})\rangle - 2\lambda\langle x(s-), Dx(s-)\rangle\right] ds \\
&\quad + \int_0^t \left[|x(s-)+Dx(s-)|^2 - |x(s-)|^2\right] dN(s) \\
&= |x_0|^2 + \int_0^t \left[2\langle x(s-), f(x_{s-})\rangle - 2\lambda\langle x(s-), Dx(s-)\rangle\right] ds \\
&\quad + \int_0^t \left[2x(s-)^T Dx(s-) + |Dx(s-)|^2\right] dN(s).
\end{aligned} \tag{5.36}$$

Using (5.34) then (5.36) becomes

$$Y(t) = Y(0) + \int_0^t \left[2\langle x(s-), f(x_{s-})\rangle - \lambda\varepsilon(s-)Y(s-)\right] ds + \int_0^t \left(\varepsilon(s-)+\delta(s-)^2\right) Y(s-) dN(s).$$

By the variation of constants formula for an SDE driven by a Poisson process (see Chapter 2, Theorem 2.3.1), then, for each $t \ge 0$, we have

$$Y(t) = \Phi(t)\left[Y(0) + \int_0^t 2\Phi(s)^{-1}\langle x(s-), f(x_{s-})\rangle ds\right] \tag{5.37}$$

where by (2.20)

$$\Phi(t) = \exp\left[-\int_0^t \lambda\varepsilon(s-)ds + \int_0^t \log\left(\varepsilon(s-)+\delta(s-)^2+1\right) dN(s)\right]. \tag{5.38}$$

We extend the definition of Φ to the whole of $[-\tau, \infty)$ by

$$\Phi(t) = \begin{cases} \exp\left[-\int_0^t \lambda\varepsilon(s-)ds + \int_0^t \log\left(\varepsilon(s-)+\delta(s-)^2+1\right) dN(s)\right] & \text{for } t \ge 0 \\ 1 & \text{for } t \in [-\tau, 0] \end{cases} \tag{5.39}$$

and define $Z(t) = \Phi(t)^{-1}Y(t)$ for $t \geq -\tau$. Now by (5.37) and (5.29), for $t \geq 0$ we have

$$\begin{aligned} Z(t) &\leq Y(0) + 2\sqrt{L}\int_0^t \Phi(s)^{-1}|x(s-)|\,.\|x_{s-}\|ds \\ &= Y(0) + 2\sqrt{L}\int_0^t \Phi(s)^{-\frac{1}{2}}.Z(s-)^{\frac{1}{2}}\|x_{s-}\|\,ds. \end{aligned}$$

From this point we will follow exactly the same arguments that were used by Appleby and Mao in [3] pp. 1077 to obtain an estimate of the form

$$Z(t)^{\frac{1}{2}} \leq \|x_0\|\exp\left(\sqrt{L}\int_0^t g(s)ds\right) \quad \text{for } t \geq 0 \tag{5.40}$$

where for each $t \geq 0$, $g(t) = \Phi(t)^{-\frac{1}{2}}\sup_{t-\tau\leq u\leq t}\Phi(u)^{\frac{1}{2}}$.
Since $Z(t) = \Phi(t)^{-1}|x(t)|^2$ then (5.40) becomes

$$|x(t)| \leq \|x_0\|\Phi(t)^{\frac{1}{2}}\exp\left(\sqrt{L}\int_0^t g(s)ds\right) \quad \text{for } t \geq 0. \tag{5.41}$$

Hence combining (5.38) and (5.41), then

$$\begin{aligned} \limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \leq \limsup_{t\to\infty}\frac{1}{t}\Big[&-\frac{\lambda}{2}\int_0^t \varepsilon(s-)ds + \frac{1}{2}\int_0^t \log\left(\varepsilon(s-)+\delta(s-)^2+1\right)dN(s) \\ &+ \sqrt{L}\int_0^t g(s)ds\Big]. \end{aligned} \tag{5.42}$$

Now to obtain an estimate for the sample Lyapunov exponent of the solution of (5.30) we must obtain an almost sure estimate for each of the terms on the right-hand side of (5.42). From the way that ε is defined (see (5.33)) we see that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t -\varepsilon(s-)ds \leq -2\rho_{min}. \tag{5.43}$$

For the second term of the right-hand side of (5.42), by the strong law of large numbers for a Poisson process (see Sato [42] pp. 246), it holds that

$$\lim_{t\to\infty}\frac{N(t)}{t} = \lambda \qquad a.s.$$

and hence,

$$\begin{aligned} \limsup_{t\to\infty}\frac{1}{t}\left[\int_0^t \log\left(\varepsilon(s-)+\delta(s-)^2+1\right)dN(s)\right] &\leq \limsup_{t\to\infty}\frac{1}{t}\left[\log(1+\|D\|_*)^2N(t)\right] \\ &= 2\lambda\log(1+\rho_{max}). \end{aligned} \tag{5.44}$$

Recall that for each $t \geq 0$, $g(t) = \Phi(t)^{-\frac{1}{2}} \sup_{t-\tau \leq u \leq t} \Phi(u)^{\frac{1}{2}}$. Now for $t \geq \tau$

$$\begin{aligned} g(t) &= \exp\left(\frac{1}{2}\int_0^t \lambda\varepsilon(s)ds - \frac{1}{2}\int_0^t \log\left(\varepsilon(s-) + \delta(s-)^2 + 1\right)dN(s)\right) \\ &\quad \times \sup_{t-\tau \leq u \leq t} \exp\left(-\frac{1}{2}\int_0^u \lambda\varepsilon(s)ds + \frac{1}{2}\int_0^u \log\left(\varepsilon(s-) + \delta(s-)^2 + 1\right)dN(s)\right) \\ &= \sup_{t-\tau \leq u \leq t} \exp\left(\frac{1}{2}\int_u^t \lambda\varepsilon(s)ds\right) \times \sup_{t-\tau \leq u \leq t} \exp\left(-\frac{1}{2}\int_u^t \log\left(\varepsilon(s-) + \delta(s-)^2 + 1\right)dN(s)\right) \\ &\leq \exp\left(\lambda\|D\|_*\tau\right) \times \sup_{t-\tau \leq u \leq t} \exp\left(-\int_u^t \log(1 + \|D\|_*)dN(s)\right) \\ &\leq \exp\left(\lambda\|D\|_*\tau\right) = \exp\left(\tau\lambda\rho_{max}\right). \end{aligned}$$

Hence,

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t g(s)ds \leq \limsup_{t\to\infty} \frac{1}{t}\int_0^t \exp\left(\lambda\rho_{max}\tau\right)ds = \exp\left(\lambda\rho_{max}\tau\right). \tag{5.45}$$

Applying (5.43), (5.44), and (5.45) into (5.42) then

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \leq \lambda\left(\log(1 + \rho_{max}) - \rho_{min}\right) + \sqrt{L}\exp\left(\lambda\rho_{max}\tau\right) \qquad a.s.$$

□

Remark 5.5.4 To obtain almost sure exponential stability for the trivial solution of (5.30) we can see from (5.35) that the time delay τ has to be sufficiently small.

Appendix A

Existence and uniqueness of SFDEs

In this section we will prove the existence and uniqueness of solutions of SFDEs driven by Lévy processes under the Lipschitz and growth conditions (F1)-(F2). The proof is based on Picard iteration and a combination of arguments used by Applebaum [1] pp. 305-309 in the proof of existence and uniqueness of SDEs driven by a Lévy process, and by Mao for SFDEs driven by Brownian motion in [33] pp. 150-152. Note that in the proof below we are using $\|\cdot\|_2 = \left(E(|\cdot|^2)\right)^{\frac{1}{2}}$ in $L^2(\Omega, \mathcal{F}, P)$.

Theorem A.0.5 *Assume that the Lipschitz and growth conditions (F1) and (F2) hold. Then there exists a unique solution $x = (x(t), t \geq -\tau)$ to (5.1) with initial data x_0. The process x is adapted and càdlàg.*

Proof of existence: Define a sequence of processes $(x^n, n \in \mathbb{N} \cup \{0\})$ such that for each $t \geq 0$, $x^0(t) = x_0(0)$ and $x_t^0 = x_0$ and for each $n \in \mathbb{N} \cup \{0\}, t \geq 0$,

$$x^{n+1}(t) = x_0(0) + \int_0^t f(x_{s-}^n)ds + \int_0^t g(x_{s-}^n)dB(s) + \int_0^t \int_{|y|<c} H(x_{s-}^n, y)\tilde{N}(ds, dy). \tag{A.1}$$

Now each of the Itô integrals in (A.1) is a stochastic integral of a predictable process, which by Applebaum [1] Theorem 4.2.3 and Theorem 4.2.12 is $\mathcal{F}_t$-adapted and has a càdlàg modification. Using this property and a simple inductive argument we can show that each x^n, for $n \in \mathbb{N} \cup \{0\}$, is adapted and càdlàg.

Now for each $n \in \mathbb{N} \cup \{0\}, t \geq 0$, we have

$$\begin{aligned} x^{n+1}(t) - x^n(t) &= \int_0^t \left[f(x_{s-}^n) - f(x_{s-}^{n-1})\right] ds + \int_0^t \left[g(x_{s-}^n) - g(x_{s-}^{n-1})\right] dB(s) \\ &+ \int_0^t \int_{|y|<c} \left[H(x_{s-}^n, y) - H(x_{s-}^{n-1}, y)\right] \tilde{N}(ds, dy). \end{aligned} \tag{A.2}$$

We claim that for all $n \in \mathbb{N}$, $t \geq 0$,

$$E\left(\sup_{0 \leq s \leq t} \left|x^n(s) - x^{n-1}(s)\right|^2\right) \leq \frac{C_2(t)^n M^n}{n!} \tag{A.3}$$

where $C_2(t) = tC_1(t)$, $C_1(t) = 3t + 24$ and $M = \max\left\{L, K\left(1 + E\left(\|x_0\|^2\right)\right)\right\}$ with L and K the Lipschitz and growth constants in (F1)-(F2) respectively.
We will show this by induction.

For $n = 0$, by (1.12), we deduce that

$$\begin{aligned} \left|x^1(t) - x^0(t)\right|^2 &= \left[\int_0^t f(x_{s-}^0)ds + \int_0^t g(x_{s-}^0)dB(s) + \int_0^t \int_{|y|<c} H(x_{s-}^0, y)\tilde{N}(ds, dy)\right]^2 \\ &\leq 3\left\{\left[\int_0^t f(x_0)ds\right]^2 + \left[\int_0^t g(x_0)dB(s)\right]^2\right. \\ &\qquad \left. + \left[\int_0^t \int_{|y|<c} H(x_0, y)\tilde{N}(ds, dy)\right]^2\right\} \end{aligned} \tag{A.4}$$

for each $t \geq 0$ and $-\tau \leq \theta \leq 0$. Firstly we take expectations. Then, for the drift term we use the Cauchy-Schwarz inequality and for the other two terms apply Doobs's martingale inequality (see Chapter 1, section 1.3) and then Itô's isometry property (see Applebaum [1] Theorem 4.2.3). Finally an application of the linear growth condition (F2) yields that

$$\begin{aligned} E\left(\sup_{0 \leq s \leq t} \left|x^1(t) - x^0(t)\right|^2\right) &\leq 3t \int_0^t E\left(|f(x_0)|^2\right) ds + 12 \int_0^t E\left(\|g(x_0)\|_*^2\right) ds \\ &\quad +12 \int_0^t \int_{|y|<c} E\left(|H(x_0, y)|^2\right) \nu(dy)ds \\ &\leq 3Kt \int_0^t \left(1 + E\left(\|x_0\|^2\right)\right) ds + 12K \int_0^t \left(1 + E\left(\|x_0\|^2\right)\right) ds \\ &\quad +12K \int_0^t \left(1 + E\left(\|x_0\|^2\right)\right) ds \\ &= (3t + 24)\, Kt\left(1 + E\left(\|x_0\|^2\right)\right) = C_1(t)Kt\left(1 + E\left(\|x_0\|^2\right)\right) \end{aligned}$$

where $C_1(t) = 3t + 24$. Therefore

$$E\left(\sup_{0\le s\le t}\left|x^1(t) - x^0(t)\right|^2\right) \le C_2(t)M \tag{A.5}$$

with $C_2(t) = tC_1(t)$ and $M = KE\left(1 + \|x_0\|^2\right)$.

Now we assume that (A.3) is true for arbitrary $n \in \mathbb{N}$ and we will check that (A.3) holds for $n+1$. Hence,

$$\begin{aligned}
\left|x^{n+1}(t) - x^n(t)\right|^2 &= \left[\int_0^t \left[f(x^n_{s-}) - f(x^{n-1}_{s-})\right] ds + \int_0^t \left[g(x^n_{s-}) - g(x^{n-1}_{s-})\right] dB(s)\right. \\
&\quad \left. + \int_0^t \int_{|y|<c} \left[H(x^n_{s-}, y) - H(x^{n-1}_{s-}, y)\right] \tilde{N}(ds, dy)\right]^2 \\
&\le 3\left\{\left[\int_0^t \left[f(x^n_{s-}) - f(x^{n-1}_{s-})\right] ds\right]^2 + \left[\int_0^t \left[g(x^n_{s-}) - g(x^{n-1}_{s-})\right] dB(s)\right]^2\right. \\
&\quad \left. + \left[\int_0^t \int_{|y|<c} \left[H(x^n_{s-}, y) - H(x^{n-1}_{s-}, y)\right] \tilde{N}(ds, dy)\right]^2\right\}.
\end{aligned}$$

By the same arguments as was used in deducing (A.5),

$$\begin{aligned}
E\left(\sup_{0\le s\le t}\left|x^{n+1}(s) - x^n(s)\right|^2\right) &\le \left[3E\left(\sup_{0\le s\le t}\left\{\int_0^s \left[f(x^n_{u-}) - f(x^{n-1}_{u-})\right] du\right\}^2\right)\right. \\
&\quad + 12E\left(\left\{\int_0^t \left[g(x^n_{s-}) - g(x^{n-1}_{s-})\right] dB(s)\right\}^2\right) \\
&\quad \left. + 12E\left(\left\{\int_0^t \int_{|y|<c} \left[H(x^n_{s-}, y) - H(x^{n-1}_{s-}, y)\right] \tilde{N}(ds, dy)\right\}^2\right)\right] \\
&\le C_1(t)\left[\int_0^t E\left(\left|f(x^n_{s-}) - f(x^{n-1}_{s-})\right|^2\right) ds\right. \\
&\quad + \int_0^t E\left(\|g(x^n_{s-}) - g(x^{n-1}_{s-})\|_*^2\right) ds \\
&\quad \left. + \int_0^t \int_{|y|<c} E\left(\left|H(x^n_{s-}, y) - H(x^{n-1}_{s-}, y)\right|^2\right) \nu(dy) ds\right]
\end{aligned}$$

where $C_1(t) = 3t + 24$.
Now by the Lipschitz condition (F1) we have that

$$\begin{aligned} E\left(\sup_{0\leq s\leq t}\left|x^{n+1}(s)-x^{n}(s)\right|^{2}\right) &\leq C_1(t)L\int_0^t E\left(\left\|x_{s-}^{n}-x_{s-}^{n-1}\right\|^{2}\right)ds \\ &= C_1(t)L\int_0^t E\left(\sup_{0\leq u\leq s}\left|x^{n}(u)-x^{n-1}(u)\right|^{2}\right)ds, \end{aligned} \tag{A.6}$$

and by the induction assumption we obtain

$$\begin{aligned} E\left(\sup_{0\leq s\leq t}\left|x^{n+1}(s)-x^{n}(s)\right|^{2}\right) &\leq C_1(t)L\int_0^t \frac{C_2(s)^n M^n}{n!}ds \leq \frac{C_1(t)^{n+1}M^{n+1}t^{n+1}}{(n+1)!} \\ &= \frac{M^{n+1}C_2(t)^{(n+1)}}{(n+1)!}. \end{aligned} \tag{A.7}$$

Hence (A.3) is true for all $n \in \mathbb{N}$.

Next we shall proof that $(x^n(t), t \geq 0)$ is convergent in the L^2 sense for each $t \geq 0$. First note that from (A.3) for each $m, n \in \mathbb{N}$ and for each $0 \leq s \leq t$ we have that

$$\|x^n(s) - x^m(s)\|_2 \leq \sum_{r=m+1}^{n} \|x^r(s) - x^{r-1}(s)\|_2 \leq \sum_{r=m+1}^{n} \frac{C_2(t)^{r/2}M^{r/2}}{(r!)^{1/2}}.$$

Using a ratio test argument, we can see that the series on the right is a convergent series. Consequently each $(x^n(s), n \in \mathbb{N})$ is Cauchy and it follows immediately that it is convergent to some $x(s) \in L^2(\Omega, \mathcal{F}, P)$. Define x to be the process $(x(t), t \geq 0)$. Hence for each $n \in \mathbb{N} \cup \{0\}$, $0 \leq s \leq t$,

$$\|x(s) - x^n(s)\|_2 \leq \sum_{r=n+1}^{\infty} \frac{C_2(t)^{r/2}M^{r/2}}{(r!)^{1/2}}. \tag{A.8}$$

In the next lines we will prove the almost sure convergence of $(x^n, n \in \mathbb{N})$.
The Chebychev inequality and (A.3) yields that

$$\begin{aligned} P\left(\sup_{0\leq s\leq t}|x^n(s) - x^{n-1}(s)| \geq \frac{1}{2^n}\right) &\leq (2^n)^2 E\left(\sup_{0\leq s\leq t}\left|x^n(s) - x^{n-1}(s)\right|^2\right) \\ &\leq \frac{(4C_2(t)M)^n}{n!}. \end{aligned}$$

Hence, by the Borel-Cantelli lemma

$$P\left(\limsup_{n\to\infty}\sup_{0\leq s\leq t}|x^n(s) - x^{n-1}(s)| \geq \frac{1}{2^n}\right) = 0,$$

and it follows that

$$P\left(\liminf_{n\to\infty}\sup_{0\le s\le t}|x^n(s)-x^{n-1}(s)|<\frac{1}{2^n}\right)=1.$$

Now given any $\delta>0$ there exists $n_0\in\mathbb{N}$ such that for any $m,n>n_0$ we have that

$$\sup_{0\le s\le t}|x^n(s)-x^m(s)|\le\sum_{r=m}^{n-1}\sup_{0\le s\le t}|x^{r+1}(s)-x^r(s)|<\sum_{r=m}^{n-1}\frac{1}{2^{r+1}}<\delta$$

with probability 1. As a result $(x^n, n\in\mathbb{N})$ is almost surely uniformly Cauchy in finite intervals $[0,t]$ and hence almost surely uniformly convergent on finite intervals to the process $x=(x(t),t\ge 0)$. Since the uniform limit of a sequence of càdlàg functions in a finite interval is càdlàg, then x is càdlàg (see Applebaum [1] pp. 119). Also we can find a subsequence $(x^{n_k}, n_k\in\mathbb{N})$ for which the uniform convergence will still hold almost surely, and hence $x=(x(t),t\ge 0)$ is adapted (see Applebaum [1] pp. 71).

Now we need to show that $x(t)$, for each $t\ge 0$, is a solution to (A.1). Denote by $\tilde{x}=(\tilde{x}(t),t\ge 0)$ the stochastic process given by

$$\tilde{x}(t) = x_0(0)+\int_0^t f(x_{s-})ds+\int_0^t g(x_{s-})dB(s)+\int_0^t\int_{|y|<c}H(x_{s-},y)\tilde{N}(ds,dy).$$

Then, for each $n\in\mathbb{N}\cup\{0\}$, we obtain

$$\begin{aligned}\tilde{x}(t)-x^n(t) &= \int_0^t\left[f(x_{s-})-f(x^n_{s-})\right]ds+\int_0^t\left[g(x_{s-})-g(x^n_{s-})\right]dB(s)\\ &+ \int_0^t\int_{|y|<c}\left[H(x_{s-},y)-H(x^n_{s-},y)\right]\tilde{N}(ds,dy).\end{aligned}$$

Following the same arguments as used in deducing (A.6) i.e. the Cauchy-Schwarz inequality, the Itô isometry property, the Lipschitz conditions and the result of (A.8), then for all $0\le s\le t\le\infty$,

$$\begin{aligned}E\left(|\tilde{x}(s)-x^n(s)|^2\right) &\le C_1(t)L\int_0^t E\left(\|x_{s-}-x^n_{s-}\|^2\right)ds\\ &= C_1(t)L\int_0^t E\left(\sup_{0\le u\le s}|x(u)-x^n(u)|^2\right)ds\\ &\le C_2(t)L\sup_{0\le s\le t}E\left(\sup_{0\le u\le s}|x(u)-x^n(u)|^2\right)\\ &\le C_2(t)L\left(\sum_{r=n+1}^{\infty}\frac{C_2(t)^{r/2}M^{r/2}}{(r!)^{1/2}}\right)^2.\end{aligned}$$

The right hand side of the inequality converges to 0 as $n \to \infty$, hence $\tilde{x}(s) = L^2 - \lim_{n\to\infty} x^n(s)$. But from (A.8) each $x(s) = L^2 - \lim_{n\to\infty} x^n(s)$. Hence, by the uniqueness of limits $\tilde{x}(s) = x(s)$ almost surely for all $0 \le s \le t \le \infty$.

Proof of uniqueness: Denote by x and x' two distinct solutions of (5.1). For each $t \ge 0$,

$$\begin{aligned} x'(t) - x(t) &= \int_0^t \left[f(x'_{s-}) - f(x_{s-})\right] ds + \int_0^t \left[g(x'_{s-}) - g(x_{s-})\right] dB(s) \\ &\quad + \int_0^t \int_{|y|<c} \left[H(x'_{s-}, y) - H(x_{s-}, y)\right] \tilde{N}(ds, dy). \end{aligned}$$

Using the same arguments that were used to deduce (A.6), the Cauchy-Schwarz inequality, the Itô isometry property and the Lipschitz conditions then we have that

$$\begin{aligned} E\left(\sup_{0\le s\le t} |x'(s) - x(s)|^2\right) &\le C_1(t)L\int_0^t E\left(\|x'_{s-} - x_{s-}\|^2\right) ds \\ &= C_1(t)L\int_0^t E\left(\sup_{0\le u\le s} |x'(u) - x(u)|^2\right) ds. \end{aligned}$$

Applying the Gronwall's inequality then $E\left(\sup_{0\le s\le t} |x'(t) - x(t)|^2\right) = 0$. This yields that $x'(s) = x(s)$ for all $0 \le s \le t$ almost surely and hence for all $-\tau \le s \le t$, since both satisfy (5.1) with initial condition x_0. Now using the continuity of probability we deduce that

$$P\left(x'(t) = x(t) \text{ for all } \ t \ge -\tau\right) = P\left(\bigcap_{N\in\mathbb{N}} \left(x'(t) = x(t) \text{ for all } \ -\tau \le t \le N\right)\right) = 1$$

and the required result follows. □

In Chapter 5 we have considered "small" and "large" jumps under the same footing by allowing c to take values in $(0, \infty]$. However, this is unnecessarily restrictive. Hence for the sake of completeness we will show by interlacing that SFDEs that deal separately the "small" and "large" jumps of the Lévy process have a unique solution. In this case we can consider the whole class of Lévy processes as driving noise.

Let $z = (z(t), t \ge t_0)$ be the solution of

$$\begin{aligned} z(t) &= z_0(0) + \int_0^t f(z_{s-})ds + \int_0^t g(z_{s-})dB(s) + \int_0^t \int_{|y|<c} H\left(z_{s-}, y\right) \tilde{N}(ds, dy) \\ &\quad + \int_0^t \int_{|y|\ge c} K\left(z_{s-}, y\right) N(ds, dy) \quad \text{on } t \ge 0, \end{aligned} \tag{A.9}$$

where z_0 is the fixed initial condition, $K : \mathcal{D}([-\tau, 0]; \mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ is Borel measurable and the mappings f, g, H and the process B and the random measure N as defined in Chapter 5.

Assumption A.0.6 We require that

$$x \to K(x, y)$$

is a continuous mapping for all $|y| \geq c$.

Theorem A.0.7 *The solution of (A.9), $z = (z(t), t \geq 0)$ is a unique adapted càdlàg process.*

Proof: Let $(\tau_n, n \in \mathbb{N})$ be the arrival times for the jumps of the compound Poisson process $(P(t), t \geq 0)$ where $P(t) = \int_{|y|\geq c} yN(t, dy)$. Let $x = (x(t), t \geq 0)$ be the solution of (5.1) and $z = (z(t), t \geq 0)$ be the solution of (A.9) with $z_0(0) = x_0(0)$. Using the interlacing technique (see Applebaum [1] pp. 311) we will construct a solution to the SFDE (A.9) in the following way:

$$\begin{aligned}
z(t) &= x(t) && \text{for } 0 \leq t < \tau_1, \\
z(\tau_1) &= x(\tau_1-) + K\left(z_{\tau_1-}, \Delta P(\tau_1)\right) && \text{for } t = \tau_1, \\
z(t) &= z(\tau_1) + x_1(t) - x_1(\tau_1) && \text{for } \tau_1 < t < \tau_2, \\
z(\tau_2) &= z(\tau_2-) + K\left(z_{\tau_2-}, \Delta P(\tau_2)\right) && \text{for } t = \tau_2,
\end{aligned}$$

and so on recursively, where for each $i \in \mathbb{N}$, x_i is the solution of (5.1) with $x_i(0) = z(\tau_i)$ and $z_{\tau_i} : \Omega \to \mathcal{D}([-\tau, 0]; \mathbb{R}^d)$ is defined as

$$z_{\tau_i}(\omega)(\theta) = z(\tau_i + \theta)(\omega)$$

for each $\omega \in \Omega$ and $\theta \in [-\tau, 0]$. Now z is a solution to (A.9) and is easily seen to be an adapted and càdlàg process. Note that assumption A.0.6 ensures the predictability of the Poisson integrals. By the uniqueness of the solution of (5.1) (see Theorem A.0.5) and the form of the interlacing structure, z is unique. □

Appendix B

A useful lemma

The following lemma applies for Theorem 5.4.1, Chapter 5.

Lemma B.0.8 Let $h > 1$ and $\rho, \rho_1, \ldots, \rho_k, \tau$ be positive constants. Then

$$\rho - h \sum_{n=1}^{k} \rho_n = \log(h)/\tau \tag{B.1}$$

has a unique solution.

Proof: Let $h_1 > 1$ and $h_2 > 1$ be two distinct solutions to (B.1). Hence,

$$\rho - h_1 \sum_{n=1}^{k} \rho_n = \log(h_1)/\tau \quad \text{and} \quad \rho - h_2 \sum_{n=1}^{k} \rho_n = \log(h_2)/\tau.$$

This implies that

$$\tau(h_1 - h_2) \sum_{n=1}^{k} \rho_n + \log\left(\frac{h_1}{h_2}\right) = 0. \tag{B.2}$$

Now if $h_1 > h_2$ then for all $h_1, h_2 > 1$ we have that $\tau(h_1 - h_2) \sum_{n=1}^{k} \rho_n + \log\left(\frac{h_1}{h_2}\right) > 0$. If $h_1 < h_2$ then for all $h_1, h_2 > 1$ we obtain that $\tau(h_1 - h_2) \sum_{n=1}^{k} \rho_n + \log\left(\frac{h_1}{h_2}\right) < 0$. Hence, (B.2) is satisfied only if $h_1 = h_2$. □

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