Frequency Domain Theory of Nonlinear Volterra Systems based on Parametric Characteristic Analysis

By

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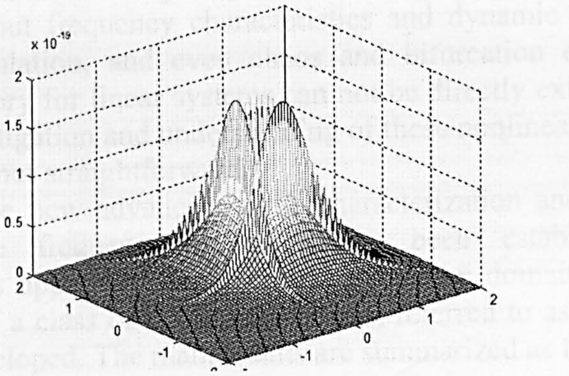
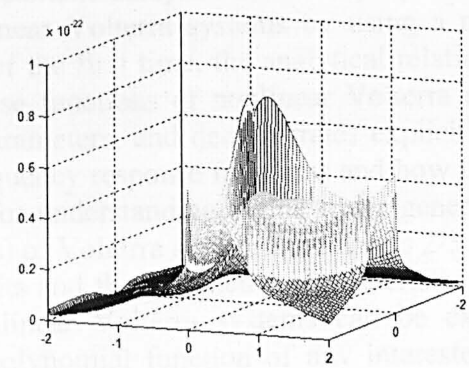
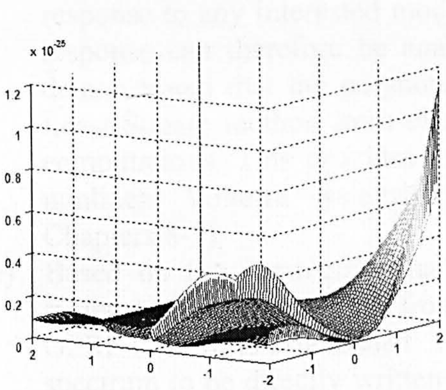
In pursuit of the nature of THE NATURE

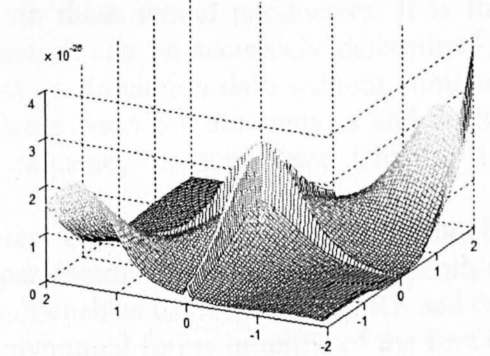
Take with the love of THE LOVE

-------To my family









ABSTRACT

The frequency domain methods for linear systems are well accepted by engineers and have been widely applied in engineering practice because the transfer function of linear systems can always provide a coordinate-free and equivalent description for system characteristics and are convenient to be used for the system analysis and design. Although the analysis and design of linear systems in the frequency domain have been well established and the frequency domain methods for nonlinear systems have already been investigated for many years, the frequency domain analysis for nonlinear systems is far from being fully developed. Nonlinear systems usually have very complicated output frequency characteristics and dynamic behaviour such as harmonics, inter-modulation, and even chaos and bifurcation etc. Therefore, the frequency domain theory for linear systems can not be directly extended to nonlinear systems, and the investigation and understanding of these nonlinear phenomena in the frequency domain are not straightforward.

In this study, some new advances in the characterization and understanding of nonlinearities in the frequency domain have been established, based on Volterra/Wiener series approach. A systematic frequency domain approach for the analysis and design of a class of nonlinear systems, referred to as nonlinear Volterra systems, has been developed. The main results are summarized as follows:

(a) A parametric characteristic analysis method is proposed for the frequency domain analysis of nonlinear Volterra systems by using a novel operator. The result clearly reveals, for the first time, the analytical relationship between high order frequency response functions of nonlinear Volterra systems and system time-domain model parameters, and demonstrates explicitly what model parameters affect system frequency response functions and how they do. This also provides a novel method for understanding higher order generalized frequency response functions (GFRFs) of Volterra systems. (Chapters 2-3 and Chapter 8)

(b) Based on the results and the parametric characteristic analysis in (a), the output spectrum of nonlinear Volterra systems can be explicitly expressed into a straightforward polynomial function of any interested model parameters with detailed parametric structure, which can directly relate system output frequency response to any interested model parameters such that system output frequency response can therefore be analyzed via these model parameters. It is further demonstrated that the polynomial function can be accurately determined by a Least Square method from experiment or simulation data without complicated computations. This provides a significant basis for the analysis and design of nonlinear Volterra systems in the frequency domain. (See Chapter 4 and Chapters 8-9)

(c) Based on the parametric characteristics of the nth-order GFRF of nonlinear systems, a novel mapping from the parametric characteristics of the nth-order GFRF to itself is established. This result enables the nth-order GFRF and output spectrum to be directly written as a polynomial forms in terms of the first order GFRF and model nonlinear parameters with a straightforward parametric relationship (Chapter 5). Based on this new mapping function, it is theoretically shown for the first time that under certain conditions, the output spectrum of a class of nonlinear systems can be expressed into an alternating series with

respect to some model nonlinear parameters. The result is of considerable practical significance for vibration suppressions (Chapter 6).

(d) Based on the parametric characteristic analysis in (a), the effects of different orders' system nonlinearity on the system output frequencies are also studied. This provides a novel insight into this issue and reveals many significant phenomena such as the counteraction between different nonlinearities at some specific frequencies, periodicity property of output frequencies, etc. These results can facilitate the structure selection and parameter determination for system modelling, identification, filtering and controller design (Chapter 7).

(e) Based on the new advances in the frequency domain theory of nonlinear systems achieved in the present study, a novel vibration control approach is proposed. This is a systematic frequency domain analysis based approach, which exploits the potential advantage of nonlinearities to achieve the purpose of vibration suppression (Chapter 9).

A series of systematic frequency domain analysis and design theories and methods for nonlinear Volterra systems have been established in the present study, The significances of these results are: (1) it can directly relate the nonlinear model parameters of interest to system frequency response functions, and therefore the nonlinear controller parameters or structural parameters can be analysed and designed in the frequency domain in a way which can relatively be easy to be implemented in engineering practice; (2) the method can be used not only to design a nonlinear feedback controller for a system by exploiting the potential advantages of nonlinearities, but also to analyse and design structural nonlinear characteristics which can be realized in a passive/active manner to achieve a desired passive structural physical characteristics; (3) it provides a novel approach to understanding the nature of a considerably large class of practical nonlinear systems.

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When one is pursuing something in the time and space, what can he take with? Is there anything external that his hope can always rest on, except the true love? For this reason, the author would like to give his sincere thanks with all heart and soul to his parents, wife, brothers and sisters for their pure love and invaluable support at all the time. Their love is just like an everlasting source of energy and strength, encouraging the author to go on. Especially, the author would like to very gratefully acknowledge the great effort that his wife (Ms. Xu GE) has always made in providing a warm, happy, healthy and active environment for the author. The author would also like to thank all his friends for their many help during studying in Sheffield.

What is life? What is nature? When understanding more about the nature, one will understand more about the life; and vice versa……

STATEMENT OF ORIGINALITY

Unless otherwise stated in the text, the work described in this dissertation was carried out solely by the candidate. None of this work has already been accepted for any degree, nor is it concurrently submitted in candidature for any degree.

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Date: 29/05/08

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LIST OF ACRONYMS

GFRF	Generalized Frequency Response Function
CE	Coefficient Extractor
PCA	Parametric Characteristic Analysis
OFRF	Output Frequency Response Function
NDE	Nonlinear Differential Equation
NARX	Nonlinear Auto-Regressive model with eXogenous input
SISO	Single Input Single Output
MIMO	Multiple Input Multiple Output
FFT	Fast Fourier Transform

Chapter 1
INTRODUCTION

1.1 Frequency domain methods for nonlinear systems

Frequency domain methods can usually provide some intuitive insights into system underlying mechanisms or characteristics of interest which are in most cases easier for engineers to understand. For example, the transfer function of a linear system is always coordinate-free and equivalent description whatever the model of the studied system is transformed by any linear transformations; the instability of a linear system is usually associated with at least one right-half-plane pole of the system; the peak of system output vibration often happens at the natural resonance frequency of the system, and so on. Therefore, frequency domain analysis and design of engineering systems are always one of the most favourite methodologies in practices and attract extensive research both in theory and application.

It is known that the analysis and synthesis of linear systems in the frequency domain have been well established. There are many methods and techniques that have been developed to cope with the analysis and design of linear systems in practice such as Bode diagram, root locus, Nyquist plot and so on (Ogota 1996). However, the frequency domain analysis for nonlinear systems is not straightforward. Nonlinear systems usually have very complicated output frequency characteristics and dynamic behaviour such as harmonics, inter-modulation, chaos and bifurcation, which can transfer signal energy between different frequencies to produce outputs at the frequency components of which may be quite different from the frequency components of the input. These phenomena complicate the study of nonlinear systems in the frequency domain, and the frequency domain theory for linear systems can not directly be extended to the nonlinear case. Therefore, the investigation and understanding of nonlinear phenomena in the frequency domain are far from being fully developed.

Frequency domain analysis of nonlinear systems has been studied since the fifties of last century. A traditional method was initiated by investigation of global stability of the stationary point within the frames of absolute stability theory, and then frequency domain methods for the analysis of stability of stationary sets and existence of cycles and homo-clinical orbits, as well as for the estimation of dimension of attractors etc were developed thereafter (Leonov et al 1996). Practically, the nonlinear behaviour or characteristics of a specific nonlinear part or nonlinear unite in a system can usually be analyzed by using describing functions or harmonic balance in the frequency domain. The describing function method represents a very powerful mathematical approach for the analysis and design of the behaviour of nonlinear systems with a single nonlinear component (Atherton 1975). It can be effectively applied to the analysis of limit cycle and oscillation for nonlinear systems in which the nonlinearity does not depend on frequency and produces no sub-harmonics etc. Applications for controller design based on describing function analysis have extensively been reported (Gelb and Vander Velde 1968, Taylor and Strobel 1985). However, limitations of the describing function methods are noticeable. For example, Engelberg (2002) provides a set of nonlinear systems for which the prediction of limit cycle by using describing functions is erroneous. Simultaneously, some improved methods were also developed (Sanders 1993, Elizalde and Imregun 2006, Nuij et al 2006). Another elegant method for the frequency domain analysis of nonlinear systems in practice is referred to as the harmonic balance (Solomou et al 2002, Peyton Jones 2003). This method provides an approximation of the amplitude of the steady state periodic response of a nonlinear system under the assumption that a Fourier series can represent the steady state solution. It can deal with more general problems of nonlinear systems such as the sub-harmonics and jump behaviour etc for both the time domain and the frequency domain responses. Except these well-established and noticeable methods, there are also some other results for the nonlinear system analysis in the frequency domain reported in literature. For example, based on the frequency domain methods for linear systems such as Bode diagrams, singular value decomposition, and the idea of varying eigenvalues or varying natural frequencies, the frequency domain methods for the analysis and synthesis of uncertain systems or time-varying systems were studied in Orlowski (2007), Glass and Franchek (1999), Shah and Franchek (1999) and Logemann and Townley (1997); and a frequency response function for convergent systems subject to a harmonic input was recently proposed in Pavlov (2007), etc.

For a class of nonlinear systems, which have a convergent Volterra series expansion, frequency domain analysis can be conducted based on the concept of generalized frequency response function (George 1959, Schetzen 1980, Rugh 1981). As studied in Boyd and Chua (1985), nonlinear systems, which are time invariant, causal and have fading memory, can be approximated by a Volterra series of a sufficiently high order. The results in Sandberg (1982, 1983) show that even nonlinear time varying systems have such a locally convergent Volterra series expansion under certain conditions. Therefore, this kind of frequency domain analysis methods can deal with a considerably large class of nonlinear systems which can be driven by any input signals and do not necessarily restrict to a specific nonlinear term, and thus is a more general methodology. Although the study on Volterra systems and the corresponding frequency domain methods has been carried out for several decades since the middle of last century, many problems still remain unsolved, relating to the application issues of this method both in theory and practices. The study in this dissertation is focused on this methodology and dedicated to the corresponding problems in applications.

1.2 Frequency domain analysis based on Volterra series expansion

As mentioned above, the input output relationship of nonlinear systems under certain conditions can be approximated by a Volterra series of a sufficiently high order (Boyd and Chua 1985, Sandberg 1982, 1983), which can be written as

$$y(t)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i \tag{1.1}$$

where N is the maximum order of the series, and $h_n(\tau_1,\cdots,\tau_n)$ is a scalar real valued function of $\tau_1,\cdots,\tau_n$, referred to as the nth order Volterra kernel. Generally, $y(t)$ is a scalar output and $u(t)$ is a scalar bounded input in (1.1). The nth order generalized frequency response function (GFRF) of nonlinear system (1.1) is defined as the multivariate Fourier transformation of $h_n(\tau_1,\cdots,\tau_n)$ (George 1959)

$$H_n(j\omega_1,\cdots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))d\tau_1\cdots d\tau_n \tag{1.2}$$

Clearly, (1.1) is a generalization of the traditional convolution description of linear systems to nonlinear systems. This fundamental property enables the Volterra series to have an extensive usefulness in modelling and analysis of a very wide class of nonlinear systems both in deterministic and stochastic (Volterra 1959, Van De Wouw et al 2002, Rugh 1981). This has been vindicated by a large number of applications of the Volterra series reported in modelling, identification, control and signal processing for different systems and engineering practices, which include electrical systems, biological systems, mechanical systems, communication systems, nonlinear filters, image processing, materials engineering, chemical engineering and so on (Fard et al 2005, Doyle et al 2002, French 1976, Boutabba et al 2003, Friston 2000, Yang and Tan 2006, Raz and Veen 1998, Bussgang et al 1974). Technically, most of these results are related to direct estimation or identification of the kernel $h_n(\tau_1,\cdots,\tau_n)$ or the GFRF $H_n(j\omega_1,\cdots,j\omega_n)$ from input output data (Brilliant 1958, Kim and Powers 1988, Bendat 1990, Nam and Powers 1994, Schetzen 1980, Schoukens 2003, Ljung 1999, Pintelon and Schoukens 2001).

Based on the existence of Volterra series expansion, the study of nonlinear systems in the frequency domain was initiated by the introduction of the concept of the generalized frequency response functions (GFRFs) as defined in (1.2). This provides a powerful technique for the study of nonlinear systems, which is similar to those based on the transfer function of linear systems. Thereafter, a fundamental method, referred to as Probing method (Rugh 1981), greatly promoted the development of this frequency domain method for nonlinear systems. By using the probing method, the GFRFs for a nonlinear system described by nonlinear differential equations (NDE) or nonlinear auto-regressive model with exogenous input (NARX) can directly be obtained from its model parameters. These results were further developed by Peyton-Jones and Billings (1989) and Billings and Peyton-Jones (1990), respectively. Based on these techniques, many results have been achieved for the frequency domain analysis of nonlinear systems which have a convergent Volterra series expansion. Swain and Billings (2001) extended the computation of GFRFs for SISO models to the case of MIMO nonlinear systems. The derivation of the GFRFs of nonlinear systems with mean level or DC terms was discussed in Zhang *et al.* (1995). The system output spectrum and output frequencies were studied in Lang and Billings (1996, 1997). Some preliminary results for the bound characteristics of the frequency response functions were given in Zhang and Billings (1996) and Billings and Lang (1996). These bound results were greatly generalized in Jing et al (2007) where the bound expressions are described into an elegant and concise form which is a polynomial of the first order GFRF with model nonlinear parameters as coefficients. The energy transfer characteristics of nonlinear systems were studied in Billings and Lang (2002) and Lang and Billings (2005) recently, and some diagram based techniques for the understanding of higher order GFRFs were discussed in Peyton-Jones and Billings (1990) and Yue et al (2005). Furthermore, the concept of Output Frequency Response Function of nonlinear systems was proposed in Lang et al (2006, 2007). These results form a fundamental basis for the development of frequency domain method for nonlinear systems studied in this dissertation.

1.3 Problems to be studied

As mentioned before, the frequency domain analysis of nonlinear systems is much more complicated than that for linear systems, because nonlinear systems usually have very complicated nonlinear behaviours such as super-harmonics, sub-harmonics, inter-modulation, and even bifurcation and chaos. These phenomena complicate the study of nonlinear systems in the frequency domain, and the frequency domain theory for linear systems can not directly be extended to the nonlinear case. Although there are some remarkable results having been developed as mentioned before, a systematic and more practical approach to the analysis and design for a much wider class of nonlinear systems in the frequency domain still remains to be developed.

In this dissertation, our study focuses on the frequency domain methods for the class of nonlinear systems which have a convergent Volterra series expansion for its input output relationship in the time domain as described in (1.1) (Sandberg 1982ab, 1983ab, Boyd and Chua 1985), and which are referred to as nonlinear Volterra systems in what follows. As discussed, the computation of the GFRFs and output spectrum is a key step in the frequency domain method based on Volterra series theory. To obtain the GFRFs for Volterra systems described by NDE models or NARX models, the probing method can be used (Rugh 1981). Once the GRFRs are obtained for a practical system, system output spectrum can then be evaluated (Lang 1996). These form a general procedure for this methodology. The advantages of this method, as mentioned, may lie in at least the following three points:

(a1) it is a mathematically elegant method for a considerably large class of nonlinear systems frequently encountered in practices of different fields, not restrict to a specific nonlinear unite or single nonlinear component;

(a2) it holds for any bounded input signals whatever the input is deterministic or stochastic, not restrict to some specific input signals such as harmonic or triangle or step inputs;

(a3) it provides very similar techniques to these for linear systems. For example the GFRFs for Volterra systems are similar to the FRF for linear systems, which are familiar to most engineers.

However, from previous research results, it can be seen that, the high order GFRF is actually a sequence of multivariable functions defined in a high dimensional frequency space. The evaluation of the values of the GFRFs higher than fourth or fifth order can become rather hard due to the large amount of algebra or symbolic manipulations that are involved (Yue *et al.* 2005). The situation may go worse in the computation of the system output spectrum of higher orders, since this involves a series of repetitive computations of the GFRFs from the first to the highest order that are involved. Moreover, the existing recursive algorithms for the computation of the GFRFs and output spectrum can not explicitly and simply reveal the analytical relationship between system time domain model parameters and system frequency response functions in a clear and straightforward manner. These inhibit the practical application of the existing theoretical results to such an extent that many problems remain unsolved regarding the nonlinear characteristics of the GFRFs and system output spectrum. For example, how these frequency response functions are influenced by the parameters of the underlying system model, what the connection to complex nonlinear behaviours is from the frequency response functions, and so on. From the viewpoint of practical applications, it can be seen that a straightforward analytical expression for the relationship between system time-domain model parameters and system frequency response functions (including the GFRFs and output spectrum) can considerably facilitate the analysis and design of nonlinear Volterra systems in the frequency domain.

1.4 Objective of this dissertation

In this dissertation, a novel systematic frequency domain method is developed for the class of nonlinear systems which have a convergent Volterra series expansion described by (1.1) for its input-output relationship in the time domain. Consider the input of (1.1) is any continuous and bounded input function $u(t)$ in $t \geq 0$ which has Fourier transform $U(j\omega)$ with input domain denoted by V, *i.e.*, $\omega \in V$. $u(t)$ may also be a multi-tone function in the following study, which is obviously a special case and can be described by

$$u(t) = \sum_{i=1}^{\bar{K}} |F_i| \cos(\omega_i t + \angle F_i) \tag{1.3}$$

where F_i is a complex number, $\angle F_i$ is the argument, $|F_i|$ is the modulus, $\bar{K} \in Z_+$, and Z_+ denotes all the positive integers. The class of input can be written formally as

$$K_u = \{u(t) \in C(\mathrm{R}) | \sup_{t \in R} |u(t)| \leq U_1, |u(t-\tau) - u(t)| \leq U_2 \tau, \text{for } \tau \geq 0\} \tag{1.4}$$

where $C(\mathrm{R})$ stands for the space of bounded continuous function on R which represents the set of all the real numbers, $|u(t)|$ denotes the absolute value of $u(t)$.

In the following studies, the Volterra systems of interest may be described by a NDE model as follows

$$\sum_{m=1}^{M} \sum_{p=0}^{m} \sum_{l_1, l_{p+q}=0}^{K} c_{p,q}(l_1, \cdots, l_{p+q}) \prod_{i=1}^{p} \frac{d^{l_i} y(t)}{dt^{l_i}} \prod_{i=p+1}^{p+q} \frac{d^{l_i} u(t)}{dt^{l_i}} = 0 \tag{1.5}$$

where $\left.\frac{d^l x(t)}{dt^l}\right|_{l=0} = x(t)$, $p+q=m$, $\sum_{l_1, l_{p+q}=0}^{K} (\cdot) = \sum_{l_1=0}^{K} \cdots \sum_{l_{p+q}=0}^{K} (\cdot)$, $M \in Z_+$ is the maximum degree of nonlinearity in terms of y(t) and u(t), $K \in Z_{0+}$ is the maximum order of the derivative, and Z_{0+} denotes all the non-negative integers. In this model, the parameters such as $c_{0,1}(.)$ and $c_{1,0}(.)$ are referred to as linear parameters, which correspond to the linear terms in the model, *i.e.*, $\frac{d^l y(t)}{dt^l}$ and $\frac{d^l u(t)}{dt^l}$ for k=0,1,...,K, and $c_{p,q}(\cdot)$ for $p+q>1$ are referred to as nonlinear parameters corresponding to nonlinear terms in the model of the form $\prod_{i=1}^{p} \frac{d^{l_i} y(t)}{dt^{l_i}} \prod_{i=p+1}^{p+q} \frac{d^{l_i} u(t)}{dt^{l_i}}$, *e.g.*, $y(t)^p u(t)^q$. $p+q$ is called the nonlinear degree of the nonlinear parameter $c_{p,q}(\cdot)$. A similar discrete nonlinear model known as NARX model is often used for practical nonlinear system identification from experimental data, which is given by

$$y(t) = \sum_{m=1}^{M} \sum_{p=0}^{m} \sum_{k_1, k_{p+q}=1}^{K} c_{p,q}(k_1, \cdots, k_{p+q}) \prod_{i=1}^{p} y(t-k_i) \prod_{i=p+1}^{p+q} u(t-k_i) \tag{1.6}$$

Note that (1.6) is normalized for the coefficient $y(t)$. For clarity and consistence with the discrete model (1.6), assume that $c_{1,0}(0)=1$ in (1.5).

Considering system (1.1) and all the systems which can be approximated by (1.1) and described by (1.5) or (1.6), for the problems discussed in the last section, the study in this dissertation is dedicated to develop some effective methods to understand, analyse, and characterize nonlinearities in the frequency domain, and therefore to establish a systematic frequency domain approach to the analysis and design of nonlinear Volterra systems in practices. Potential applications of the theoretical results will be validated by some detailed techniques and practical methods developed for some practical engineering problems.

1.5 Outline of the dissertation

The following chapters are organized as follows.

In Chapter 2, a novel and powerful operator is introduced and the concept of parametric characteristic analysis (PCA) for the nonlinear frequency response functions is defined and demonstrated, which is the fundamental basis of the whole study of this dissertation. It is shown that the PCA method is not only effective for the analysis of the frequency response functions of interest in this study, but may also be applicable for a class of parameterized polynomial systems. This part is mainly based on the published paper [2] and research reports [1] and [5] as listed in the Appendix.

Chapter 3 provides the fundamental results obtained by applying the PCA to the GFRFs of nonliear Volterra systems, which show an explicit relationship between the GFRFs and the system time-domain model parameters. Moreover, a correction for the recursive computation of the GFRFs given in Peyton-Jones and Billings (1989) and Billings and Peyton-Jones (1990) is discussed, and examples are provided to demonstrate the results. This part is mainly based on the published paper [2] and research reports [1] and [5] as listed in Appendix.

Based on the parametric characteristics of the GFRFs obtained in Chapter 3, the parametric characteristic of system output spectrum for Volterra systems is studied in Chapter 4. A novel frequency domain method for nonlinear Volterra systems based on the PCA method, referred to as the parametric characteristics based output spectrum analysis, is proposed with some fundamental techniques developed for practical applications. Some advantages and disadvantages of this new frequency domain method are demonstrated and compared with other methods. Simulation studies are conducted to demonstrate these results. This part is mainly based on the published papers [8] and [12] and research reports [1] and [5] as listed in Appendix.

Based on the parametric characteristics of the GFRFs for nonlinear Volterra systems, a novel mapping function from the parametric characteristics of the nth-order GFRF to itself is established in Chapter 5. The GFRFs and output spectrum can therefore be directly written into a polynomial function in terms of model nonlinear parameters and the first order GFRF. This result can facilitate the analytical computation of the GFRFs and output spectrum and the analytical analysis of system nonlinear characteristics in the frequency domain. This part is mainly based on the published papers [7] and [9] and research reports [6] and [9] as listed in Appendix.

In Chapter 6, the effect of system nonlinearity on system output spectrum is studied by using the results in Chapter 5. Based on the novel mapping function established in Chapter 5, it is theoretically shown for the first time that under certain conditions the system output spectrum can be expressed into an alternating series with respect to some model nonlinear parameters. The results are verified by simulation studies. These results provide a novel investigation for the effect of nonlinearities on system output behaviours in the frequency domain. This part is mainly based on research reports [7] and [10] as listed in Appendix.

Chapter 7 investigates the nonlinear effect on the system output spectrum from another perspective. The output frequencies contributed by different system nonlinearities are studied and some significant properties, e.g. periodicity, of nonlinear system output frequencies are unveiled. Examples are given to demonstrate these results. This part is mainly based on research report [8] as listed in Appendix.

An extension of the results in Chapters 3 and 4 is provided in Chapter 8, which generalizes the results established for the SISO input-output models described by (1.5) and (1.6) to system models with a state space equation and a general nonlinear output function. This part is mainly based on the published papers [3-6, 10] and research report [4] as listed in Appendix.

Chapter 9 provides a practical application of the parametric characteristics-based output spectrum analysis method established in Chapter 4 and Chapter 8 for the analysis and design of a vibration suppression system. This part is mainly based on the published papers [1], [8] and [11] and research reports [2] and [3] as listed in Appendix.

A summary and overview for the research studies in the thesis is given in Chapter 10.

A publication list of the author during his studying for PhD degree is provided in Appendix, and all references of the thesis are then listed.

Chapter 2
PARAMETRIC CHARACTERISTIC ANALYSIS (PCA)

In this Chapter, the concept of parametric characteristic analysis (PCA) for a class of polynomial functions with parameterized coefficients is introduced, and based on this concept, a novel and powerful operator, which is referred to as Coefficient Extractor (CE), is defined and demonstrated, which plays a fundamental role for the purpose of parametric characteristic analysis for a class of parameterized polynomial functions with separable property.

2.1 Separable functions

Definition 2.1. A function $h(s; x)$ is said to be separable with respect to parameter x if it can be written as $h(s;x) = g(x)\cdot f_1(s) + f_0(s)$, where $f_i(.)$ for i=0,1 are functions of variable s but independent of the parameter x. □

A function $h(s; x)$ satisfying Definition 1 is referred to as x-separable function or simply separable function, where x is referred to as the parameter of interest which may be a parameter to be designed for a system, and s represents other parameters or variables, which may be a reference variable (or independent variable) of a system such as time or frequency.

Remark 2.1. In the definition of an x-separable function $h(s; x)$, x may be a vector including all the separable parameters of interest, and s denotes not only the independent variables of $h(.)$, but also may include all the other un-separable and uninterested parameters in $h(.)$. The parameter x and s are real or complex valued, but the detailed properties of the function $h(.)$ and its parameters are not necessarily considered here. Note also that in Definition 1, $f_0(s)$ and $f_1(s)$ are invariant with respect to x and $g(x)$. Thus $h(s; x)$ can be regarded as a pure function of x for any specific s. In this case, if $g(x)$ is known, and additionally the values of $h(s; x)$ and $g(x)$ under some different values of x, for example x_1 and x_2, can be obtained by certain methods (simulations or experimental tests), then the values of $f_0(s)$ and $f_1(s)$ can be achieved by the Least Square method, i.e.,

$$\begin{cases} h(s;x_1) = g(x_1)\cdot f_1(s) + f_0(s) \\ h(s;x_2) = g(x_2)\cdot f_1(s) + f_0(s) \end{cases} \Rightarrow \begin{bmatrix} f_0(s) \\ f_1(s) \end{bmatrix} = \begin{bmatrix} 1 & g(x_1) \\ 1 & g(x_2) \end{bmatrix}^{-1} \begin{bmatrix} h(s;x_1) \\ h(s;x_2) \end{bmatrix} \tag{2.1}$$

Thus the function $h(s; x)$ at a given s can be obtained which is an analytical function of the parameter x. This provides a numerical method to determine the relationship between the parameters of interest and the corresponding function. □

An x-separable function $h(s; x)$ at a given point s is denoted as $h(x)|_s$, or simply as $h(x)_s$.

Consider a parameterized function series

$$H(s;x) = g_1(x)f_1(s) + g_2(x)f_2(s) + \cdots + g_n(x)f_n(s) = G\cdot F^T \tag{2.2}$$

where n>1, $f_i(s)$ and $g_i(x)$ for i=1,…,n are all scalar functions, let F=[$f_1(s), f_2(s), \cdots, f_n(s)$] and G= [$g_1(x), g_2(x), \cdots, g_n(x)$], x and s are both parameterized vectors including the interested parameters and the other parameters, respectively. The series is obviously x-separable, thus $H(x)_s$ is completely determined by the parameters in x or the values of $g_1(x), g_2(x), \cdots, g_n(x)$. Note that at a given point s, the characteristics of the series $H(s;\, x)$ is completely determined by G, and how the parameters in x are included in $H(s;\, x)$ is completely demonstrated in G, too. Therefore, the parametric characteristics of the series $H(s;\, x)$ can be totally revealed by the function vector G. The vector G is referred to as the parametric characteristic vector of the series. If the characteristic vector G is determined, then following the method mentioned in Remark 2.1, the function $H(x)_s$ which shows the analytical relationship between the concerned parameter x and the series is achieved, and consequently the effects on the series from each parameter in x can be studied. The function $H(x)_s$ is referred to as parametric characteristic function of the series $H(s;\, x)$. Based on the discussions above, the following result can be concluded.

Lemma 2.1. If $H(s;\, x)$ is a separable function with respect to the parameter x, then there must exist a parametric characteristic vector G and an appropriate function vector F, such that $H(s;\, x) = G \cdot F^T$, where the elements of G are functions of x and independent of s, and the elements of F are functions of s but independent of x. □

According to the definition and discussion above, it will be seen that the nth-order GFRF of the NDE model in (1.5) and NARX model in (1.6) is separable with respect to any nonlinear parameters of the corresponding models. As mentioned, in order to study the relationship between an interested function $H(s;\, x)$ and its separable parameters x, the parametric characteristic vector G should be obtained. For a simple parameterized function, it may be easy to obtain parameterized vector G. But for a complicated function series with recursive computations, this is not straightforward. To this aim, and more importantly for the purpose of the parametric characteristic analysis for the nth-order GFRF and output spectrum of Volterra systems described by (1.5) or (1.6), a novel operator is introduced in the following section for the extraction of any parameters of interest involved in a separable parameterized polynomial function series.

2.2 Coefficient Extractor

Let C_s be a set of parameters which takes values in $\mathbb{C}$, let P_c be a monomial function set defined in C_s, *i.e.*, $P_c = \{c_1^{r_1} c_2^{r_2} \cdots c_I^{r_I} \,|\, c_i \in C_s, r_i \in Z_0, I = |C_s|\}$, where $|C_s|$ is the number of the parameters in C_s, Z+ denotes all the positive integers. Let W_s be another parameter set similar to C_s but $W_s \cap C_s = \phi$, and let P_f be a function set defined in W_s, *i.e.*, $P_f = \{f(w_1, \cdots, w_I) \,|\, w_i \in W_s, I = |W_s|\}$. Let Ξ denote all the finite order function series with coefficients in P_c timing some functions in P_f. A series in Ξ can be written as

$$H_{CF} = s_1 f_1 + s_2 f_2 + \cdots + s_\sigma f_\sigma \in \Xi \tag{2.3}$$

where $s_i \in P_c, f_i \in P_f$ for i=1,..., $\sigma \in Z_+$, $C = [s_1, s_2, \ldots, s_\sigma]$, and $F = [f_1, f_2, \ldots, f_\sigma]^T$. Obviously, this series is separable with respect to the parameters in C_s and W_s. Define a **Coefficient Extraction** operator $CE : \Xi \to P_c{}^\sigma$, such that

$$CE(H_{CF}) = [s_1, s_2, \cdots, s_\sigma] = C \in P_c{}^\sigma \tag{2.4}$$

where $P_c^\sigma = \{[s_1,s_2,\cdots,s_\sigma]|s_1,\cdots,s_\sigma \in P_c\}$. This operator has the following properties:

(1) Reduced vectorized sum "$\oplus$".
$CE(H_{C_1F_1}+H_{C_2F_2}) = CE(H_{C_1F_1})\oplus CE(H_{C_2F_2}) = C_1 \oplus C_2 = [C_1, C_2']$
and $C_2' = VEC(\overline{C}_2 - \overline{C}_1 \cap \overline{C}_2)$, where $\overline{C}_1 = \{C_1(i)|1 \le i \le |C_1|\}$, $\overline{C}_2 = \{C_2(i)|1 \le i \le |C_2|\}$, $VEC(.)$ is a vector consisting of all the elements in set (.). C_2' is a vector including all the elements in C_2 except the same elements as those in C_1.

(2) Reduced Kronecker product "$\otimes$".

$$\begin{aligned}&CE(H_{C_1F_1}\cdot H_{C_2F_2})\\ &= CE(H_{C_1F_1})\otimes CE(H_{C_2F_2})\\ &= C_1 \otimes C_2 \overset{\Delta}{=} VEC\left\{c \left| \begin{array}{l} C_3 = [C_1(1)C_2,\cdots,C_1(|C_1|)C_2] \\ c = C_3(i), 1 \le i \le |C_3| \end{array}\right.\right\}\end{aligned}$$

which implies that there are no repetitive elements in $C_1 \otimes C_2$.

(3) Invariant.
(i) $CE(\alpha \cdot H_{CF}) = CE(H_{CF}), \forall \alpha \notin C_s$; (ii) $CE(H_{CF_1}+H_{CF_2}) = CE(H_{C(F_1+F_2)}) = C$.

(4) Unitary. (i) If $\frac{\partial H_{CF}}{\partial c} = 0$ for $\forall c \in C_s$, then $CE(H_{CF}) = 1$; (ii) if H_{CF}=0 for $\forall c \in C_s$, then CE(H_{CF})=0. When there is a unitary 1 in CE(H_{CF}), there is a nonzero constant term in the corresponding series H_{CF} which has no relation with the parameters in C_s.

(5) Inverse. $CE^{-1}(C)=H_{CF}$. This implies any a vector C consisting of the elements in P_c should correspond to at least one series in Ξ.

(6) $CE(H_{C_1F_1}) \approx CE(H_{C_2F_2})$ if the elements of C_1 are the same as those of C_2, where "$\approx$" means equivalence. That is, both series are in fact the same result considering the order of s_if_i in the series has no effect on the value of a series H_{CF}. This further implies that the CE operator is also commutative and associative, for instance, $CE(H_{C_1F_1}+H_{C_2F_2}) = C_1 \oplus C_2 \approx CE(H_{C_2F_2}+H_{C_1F_1}) = C_2 \oplus C_1$. Hence, the results by the CE operator may be different but all may correspond to the same function series and are thus equivalent.

(7) Separable and parameters of interest only. A parameter in a series can only be extracted if the parameter is of interest and the series is separable with respect to this parameter. Thus the operation result is different for different purposes.
□

Note that from the definition of the CE operator above, all the operations are in terms of the parameters in C_s, and the CE operator sets up a mapping from Ξ to $P_c{}^\sigma$. For convenience, let $\underset{(*)}{\otimes}(\cdot)$ and $\underset{(*)}{\oplus}(\cdot)$ denote the multiplication and addition by the reduced Kronecker product "$\otimes$" and vectorized sum "$\oplus$" of the terms in (.) satisfying

(*), respectively; and $\overset{k}{\underset{i=1}{\otimes}} C_{p,q} = C_{p,q} \otimes \cdots \otimes C_{p,q}$ can be simply written as $C_{p,q}^k$. For model (1.5), define the $(p+q)$th degree nonlinear parameter vector as

$$C_{p,q} = [c_{p,q}(0,\cdots,0), c_{p,q}(0,\cdots,1),\cdots, c_{p,q}(\underbrace{K,\cdots,K}_{p+q=m})] \tag{2.5}$$

which includes all the nonlinear parameters of the form $c_{p,q}(.)$ in model (1.5). A similar definition for model (1.6) as

$$C_{p,q} = [c_{p,q}(1,\cdots,1), c_{p,q}(1,\cdots,2),\cdots, c_{p,q}(\underbrace{K,\cdots,K}_{p+q=m})] \tag{2.6}$$

Note that $C_{p,q}$ can also be regarded as a set of the $(p+q)$th degree nonlinear parameters of the form $c_{p,q}(.)$. Moreover, if all the elements of CE(H_{CF}) are zero, *i.e.*, CE(H_{CF})=0, then CE(H_{CF}) is also regarded as empty.

The CE operator provides a useful tool for the analysis of the parametric characteristics of separable functions. It can be shown that the nonlinear parametric characteristics of the GFRFs for (1.5) or (1.6) can be obtained by directly substituting the operations "+" and "." by "$\oplus$" and "$\otimes$" in the corresponding recursive algorithms, respectively, and neglecting the corresponding multiplied frequency functions. This is demonstrated by the following example.

Example 2.1. Computation of the parametric characteristics of the 2[nd] order GFRF of model (1.5). The 2[nd] order GFRF from Billings and Peyton-Jones (1990) is

$$L(n)\cdot H_n(j\omega_1,\cdots,j\omega_n) = \sum_{k_1,k_n=1}^{K} c_{0,n}(k_1,\cdots,k_n)(j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n}$$
$$+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K} c_{p,q}(k_1,\cdots,k_{p+q})(j\omega_{n-q+1})^{k_{p+1}}\cdots(j\omega_n)^{k_{p+q}} H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}) \tag{2.7}$$
$$+\sum_{p=2}^{n}\sum_{k_1,k_p=0}^{K} c_{p,0}(k_1,\cdots,k_p) H_{n,p}(j\omega_1,\cdots,j\omega_n)$$

for n=2, where $L(2)= -\sum_{k_1=0}^{K} c_{1,0}(k_1)(j\omega_1 + j\omega_2)^{k_1}$, $H_{1,1}(j\omega_1) = H_1(j\omega_1)(j\omega_1)^{k_1}$, $H_{2,2}(\cdot) = H_1(j\omega_1)H_{1,1}(j\omega_2)(j\omega_1)^{k_2}$.

Applying the CE operator to (2.7) for nonlinear parameters and using the notation in (2.5), it can be obtained that

$$CE(H_2(\cdot)) = CE(L(2)\cdot H_2(\cdot)) = C_{0,2} \oplus \left(\overset{2-1}{\underset{q=1}{\oplus}} \overset{2-q}{\underset{p=1}{\oplus}} (C_{p,q} \otimes CE(H_{2-q,p}(\cdot))) \right) \oplus \left(\overset{2}{\underset{p=2}{\oplus}} C_{p,0} \otimes CE(H_{2,p}(\cdot)) \right)$$
$$= C_{0,2} \oplus \left(C_{1,1} \otimes CE(H_{1,1}(\cdot))\right) \oplus \left(C_{2,0} \otimes CE(H_{2,2}(\cdot))\right)$$

Note that $H_1(.)$ has no relationship with nonlinear parameters, from the definition of CE operator, it can be obtained that CE($H_1(.)$)=1. Similarly, it can be obtained that CE($H_{2,2}(.)$)=1. Therefore, the parametric characteristic vector of the second order GFRF is

$$CE(H_2(\cdot)) = C_{0,2} \oplus C_{1,1} \oplus C_{2,0} \tag{2.8}$$

(2.8) shows clearly that nonlinear parameters in $C_{0,2}$, $C_{1,1}$ and $C_{2,0}$ have independent effects on the 2[nd] order GFRF without interference, and no any other nonlinear parameters have any influence on the 2[nd] order GFRF. This provides an explicit insight into the relationship between the 2[nd] order GFRF and nonlinear parameters. For example, if $H_2(.)$ is required to be a special spectrum or magnitude, only the parameters in $C_{0,2}$, $C_{1,1}$ and $C_{2,0}$ may need to be designed purposely. □

Example 2.1 shows that the CE operator is very effective for the derivation of the parametric characteristic vector of a separable function series about the parameters of interest. It provides a fundamental technique for the study of parametric effects on the involved parameter-separable function series for any systems. In the present study, in most cases, the CE operator will be applied for all the nonlinear parameters in model (1.5) or model (1.6). When the CE operator is applied for a specific nonlinear parameter c, the parametric characteristic of the nth-order GFRF will be denoted by $\mathrm{CE}(H_n(.))_c$.

2.3 Summary

The purpose of the parametric characteristic analysis proposed in this chapter is to reveal how the parameters of interest in a separable parameterized function series or polynomial affect the function series or polynomial and what the possible effects are. Obviously, the CE operator provides an important and fundamental technique for this analysis. The following chapters will demonstrate the usefulness and significance of these results.

Chapter 3
PARAMETRIC CHARACTERISTIC ANALYSIS FOR THE GFRFS OF NDE AND NARX MODELS

In this chapter, the GFRFs for nonlinear Volterra systems described by (1.5) and (1.6) are discussed firstly. Then by using the novel operator defined in Chapter 2, the parametric characteristic analysis is conducted for the GFRFs of nonlinear Volterra systems described by model (1.5) and some fundamental and theoretical results are obtained for the parametric characteristics of the GFRFs. The results explicitly show what model nonlinear parameters affect the nth-order GFRF and how the effect is. Consequently, the analytical polynomial relationship between the GFRFs and model nonlinear parameters is clearly revealed. These provide a significant insight into the effect of system nonlinear parameters on the GFRFs. Similar results also hold for the NARX model described in (1.6).

3.1 The GFRFs

As discussed before, the concept of the GFRFs provides a basis for the study of nonlinear Volterra systems in the frequency domain. By using the probing method (Rugh 1981), an algorithm to compute the *n*th-order GFRF for nonlinear Volterra systems described by the NDE model (1.5) was provided in Billings and Peyton-Jone (1990):

$$L_n(j\omega_1+\cdots+j\omega_n)\cdot H_n(j\omega_1,\cdots,j\omega_n)=\sum_{k_1,k_n=1}^{K}c_{0,n}(k_1,\cdots,k_n)(j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n}$$
$$+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K}c_{p,q}(k_1,\cdots,k_{p+q})(j\omega_{n-q+1})^{k_{n-q+1}}\cdots(j\omega_{p+q})^{k_{p+q}}H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}) \quad (3.1)$$
$$+\sum_{p=2}^{n}\sum_{k_1,k_p=0}^{K}c_{p,0}(k_1,\cdots,k_p)H_{n,p}(j\omega_1,\cdots,j\omega_n)$$

$$H_{n,p}(\cdot)=\sum_{i=1}^{n-p+1}H_i(j\omega_1,\cdots,j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_i)^{k_p} \quad (3.2)$$

$$H_{n,1}(j\omega_1,\cdots,j\omega_n)=H_n(j\omega_1,\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_n)^{k_1} \quad (3.3)$$

where

$$L_n(j\omega_1+\cdots+j\omega_n)=-\sum_{k_1=0}^{K}c_{1,0}(k_1)(j\omega_1+\cdots+j\omega_n)^{k_1} \quad (3.4)$$

Moreover, $H_{n,p}(j\omega_1,\cdots,j\omega_n)$ in (3.2) can also be written as

$$H_{n,p}(j\omega_1,\cdots,j\omega_n)=\sum_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p+1}\prod_{i=1}^{p}H_{r_i}(j\omega_{X+1},\cdots,j\omega_{X+r_i})(j\omega_{X+1}+\cdots+j\omega_{X+r_i})^{k_i} \quad (3.5)$$

where

$$X=\sum_{x=1}^{i-1}r_x \quad (3.6)$$

3.1.1 A correction and revision for the computation of the *n*th-order GFRF

In the recursive algorithm for the computation of the GFRFs above, the second term in the right side of Equation (3.1), *i.e.*,

$$\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K} c_{p,q}(k_1,\cdots,k_{p+q})(j\omega_{n-q+1})^{k_{n-q+1}}\cdots(j\omega_{p+q})^{k_{p+q}} H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})$$

should be

$$\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K} c_{p,q}(k_1,\cdots,k_{p+q})(\prod_{i=1}^{q}(j\omega_{n-q+i})^{k_{p+i}})H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}) \tag{3.7}$$

The correction is in the superscripts for $(j\omega_{n-q+1})^{k_{n-q+1}}\cdots(j\omega_{p+q})^{k_{p+q}}$. That is, Equation (3.1) should be corrected as

$$\begin{aligned} L_n(j\omega_1+\cdots+j\omega_n)\cdot H_n(j\omega_1,\cdots,j\omega_n) &= \sum_{k_1,k_n=1}^{K} c_{0,n}(k_1,\cdots,k_n)(j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n} \\ &+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K} c_{p,q}(k_1,\cdots,k_{p+q})(\prod_{i=1}^{q}(j\omega_{n-q+i})^{k_{p+i}})H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}) \\ &+\sum_{p=2}^{n}\sum_{k_1,k_p=0}^{K} c_{p,0}(k_1,\cdots,k_p)H_{n,p}(j\omega_1,\cdots,j\omega_n) \end{aligned} \tag{3.8}$$

This result can be shown by directly applying the probing method for the cross input-output nonlinear terms labelled by nonlinear parameter $c_{pq}(.)$ for $p\ge 1, q\ge 1$ in model (1.5) as demonstrated in Billings and Peyton Jones (1990).

For clarity, consider a simple cross nonlinear term $c_{1,2}(k_1,k_2,k_3)\frac{d^{k_1}y(t)}{dt^{k_1}}\frac{d^{k_2}u(t)}{dt^{k_2}}\frac{d^{k_3}u(t)}{dt^{k_3}}$. The contribution to the asymmetric *n*th-order GFRF from this specific term is

$$\begin{aligned} &C_n\left[\sum_{r_1=1}^{n} H_{r_1}(j\omega_1\cdots j\omega_{r_1})(j\omega_1+\cdots+j\omega_{r_1})^{k_1}e^{j(\omega_1+\cdots+\omega_{r_1})t}\cdot\sum_{r=1}^{n}(j\omega_r)^{k_2}e^{j\omega_r t}\cdot\sum_{r=1}^{n}(j\omega_r)^{k_3}e^{j\omega_r t}\right] \\ &= H_{n-2}(j\omega_1\cdots j\omega_{n-2})(j\omega_1+\cdots+j\omega_{n-2})^{k_1}e^{j(\omega_1+\cdots+\omega_{n-2})t}\cdot(j\omega_{n-1})^{k_2}e^{j\omega_{n-1}t}\cdot(j\omega_n)^{k_3}e^{j\omega_n t} \\ &= H_{n-2}(j\omega_1\cdots j\omega_{n-2})(j\omega_1+\cdots+j\omega_{n-2})^{k_1}\cdot(j\omega_{n-1})^{k_2}\cdot(j\omega_n)^{k_3}\cdot e^{j(\omega_1+\cdots+\omega_n)t} \end{aligned} \tag{3.9}$$

where $C_n[.]$ denote the operation of extracting the coefficient of $e^{j(\omega_1+\cdots+\omega_n)t}$ (Billings and Peyton Jones 1990). By using (3.2) and (3.5), (3.9) is equal to

$$(\prod_{i=1}^{2}(j\omega_{n-2+i})^{k_{1+i}})H_{n-2,1}(j\omega_1,\cdots,j\omega_{n-2})$$

This result is consistent with (3.7). Following the same method and extending to the general case, (3.7) and (3.8) can be achieved. Moreover, for convenience in further derivation, let

$$H_{0,0}(\cdot)=1,\ H_{n,0}(\cdot)=0 \text{ for } n>0,\ H_{n,p}(\cdot)=0 \text{ for } n<p, \text{ and } \prod_{i=1}^{q}(\cdot)=\begin{cases}1 & q=0, p>1\\ 0 & q=0, p\le 1\end{cases} \tag{3.10}$$

Then (3.8) can be written in a more concise form as

$$H_n(j\omega_1,\cdots,j\omega_n)=\frac{1}{L_n(j\sum_{i=1}^{n}\omega_i)}\sum_{q=0}^{n}\sum_{p=0}^{n-q}\sum_{k_1,k_{p+q}=0}^{K} c_{p,q}(k_1,\cdots,k_{p+q})(\prod_{i=1}^{q}(j\omega_{n-q+i})^{k_{p+i}})H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}) \tag{3.11}$$

Therefore, the corrected recursive algorithm for the computation of the GFRFs is (3.8 or 3.11, 3.10, 3.2-3.5). Note that the GFRFs here are asymmetric and the symmetric GFRFs can be obtained as

$$H_n^{sym}(j\omega_1,\cdots,j\omega_n)=\frac{1}{n!}\sum_{\substack{\text{all the permutations}\\ \text{of }\{1,2,\ldots,n\}}}H_n(j\omega_1,\cdots,j\omega_n) \tag{3.12}$$

From the recursive algorithm for the computation of the GFRFs in ((3.8 or 3.11, 3.10, 3.2-3.5)) for model (1.5), it can be seen that the nth-order GFRF is a parameter-separable polynomial function with respect to the nonlinear parameters in model (1.5). For convenience, let

$$C(n,K)=\left(c_{p,q}(k_1,\cdots,k_{p+q})\left|\begin{array}{c}p=0\cdots m,\ p+q=m,\\ 2\le m\le n\\ k_i=0\cdots K,\ i=1\cdots p+q\end{array}\right.\right) \tag{3.13}$$

which includes all the nonlinear parameters from degree 2 to n. Obviously, $C(M,K)$ include all the nonlinear parameters involved in model (1.5).

3.2 Parametric characteristics of the GFRFs

A fundamental result can be obtained firstly for the parametric characteristic of the nth-order GFRF of model (1.5), which provides an important basis for the parametric characteristic analysis of the frequency response functions in the following studies.

Proposition 3.1. Consider the GFRFs for model (1.5). There exists a complex valued function vector with appropriate dimension $f_n(j\omega_1,\cdots,j\omega_n)$ which is a function of $j\omega_1,\cdots,j\omega_n$ and the linear parameters in model (1.5), such that

$$H_n(j\omega_1,\cdots,j\omega_n)=CE(H_n(j\omega_1,\cdots,j\omega_n))\cdot f_n(j\omega_1,\cdots,j\omega_n) \tag{3.14}$$

where $CE(H_n(j\omega_1,\cdots,j\omega_n))$ is the parametric characteristic vector of the nth-order GFRF for model (1.5) whose elements include and only include all the nonlinear parameters in $C_{0,n}$ and all the parameter monomials in $C_{pq}\otimes C_{p_1q_1}\otimes C_{p_2q_2}\otimes\cdots\otimes C_{p_kq_k}$ for $0\le k\le n-2$, whose subscripts satisfy

$$p+q+\sum_{i=1}^{k}(p_i+q_i)=n+k,\ 2\le p_i+q_i\le n-k,\ 2\le p+q\le n-k \text{ and } 1\le p\le n-k \tag{3.15}$$

Proof. Equation (3.14) is directly followed from Lemma 2.1 and the corresponding discussions in Chapter 2. It can be derived by applying the CE operator to Equations (3.2-3.5, 3.8) that

$$CE(H_n(j\omega_1,\cdots,j\omega_n))=C_{0,n}\oplus\left(\mathop{\oplus}_{q=1}^{n-1}\mathop{\oplus}_{p=1}^{n-q}C_{p,q}\otimes CE(H_{n-q,p}(\cdot))\right)\oplus\left(\mathop{\oplus}_{p=2}^{n}C_{p,0}\otimes CE(H_{n,p}(\cdot))\right) \tag{3.16a}$$

$$CE(H_{n,p}(\cdot))=\mathop{\oplus}_{i=1}^{n-p+1}CE(H_i(\cdot))\otimes CE(H_{n-i,p-1}(\cdot)) \text{ or } CE(H_{n,p}(\cdot))=\mathop{\oplus}_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p+1}\mathop{\otimes}_{i=1}^{p}CE(H_{r_i}(\cdot)) \tag{3.16b}$$

$$CE(H_{n,1}(\cdot))=CE(H_n(\cdot))) \tag{3.16c}$$

Obviously, $C_{0,n}$ is the first term in equation (3.16a). For clarity, consider a simpler case that there is only output nonlinearities in (3.16a), then (3.16a) is reduced to the last term of equation (3.16a), *i.e.*, $\mathop{\oplus}_{p=2}^{n}C_{p,0}\otimes CE(H_{n,p}(\cdot))=\mathop{\oplus}_{p=2}^{n}C_{p,0}\otimes\mathop{\oplus}_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p+1}\mathop{\otimes}_{i=1}^{p}CE(H_{r_i}(\cdot))$.

Note that $\bigoplus_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p+1} \bigotimes_{i=1}^{p} CE(H_{r_i}(\cdot))$ includes all the combinations of $(r_1, r_2, ..., r_p)$ satisfying $\sum_{i=1}^{p} r_i = n$, $1 \le r_i \le n-p+1$, and $2 \le p \le n$. Moreover, CE($H_1(\cdot)$)=1 since there are no nonlinear parameters in it, and any repetitive combinations have no contribution. Hence, $\bigoplus_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p+1} \bigotimes_{i=1}^{p} CE(H_{r_i}(\cdot))$ must include all the possible non-repetitive combinations of $(r_1, r_2, ..., r_k)$ satisfying $\sum_{i=1}^{k} r_i = n-p+k$, $2 \le r_i \le n-p+1$ and $1 \le k \le p$. So does CE($H_n(j\omega_1,\cdots,j\omega_n)$). Each of the subscript combinations corresponds to a monomial of the involved nonlinear parameters. Thus, by including the term $C_{p,0}$ and considering the range of each variable (*i.e.*, r_i, p, and k), CE($H_n(j\omega_1,\cdots,j\omega_n)$) must include all the possible non-repetitive monomial functions of the nonlinear parameters of the form $C_{p0} \otimes C_{r_1 0} \otimes C_{r_2 0} \otimes \cdots \otimes C_{r_k 0}$ satisfying $p + \sum_{i=1}^{k} r_i = n+k$, $2 \le r_i \le n-k$, $0 \le k \le n-2$ and $2 \le p \le n-k$.

When the other types of nonlinearities are considered, by extending the results above to a more general case such that the nonlinear parameters appear in the form $C_{pq} \otimes C_{p_1 q_1} \otimes C_{p_2 q_2} \otimes \cdots \otimes C_{p_k q_k}$ and the subscripts satisfy $p+q+\sum_{i=1}^{k}(p_i+q_i) = n+k$, $2 \le p_i + q_i \le n-k$, $0 \le k \le n-2$, $2 \le p+q \le n-k$ and $1 \le p \le n-k$, the same conclusion can be reached. Hence, the proposition is proved. □

Remark 3.1. The result in Proposition 3.1 also holds for the NARX model (Jing et al 2006). In Proposition 3.1, $f_n(j\omega_1,\cdots,j\omega_n)$ is not a function of $CE(H_n(j\omega_1,\cdots,j\omega_n))$ and is invariant at a specific point $(\omega_1,\cdots,\omega_n)$ if the linear parameters of model (1.5) are fixed. Proposition 3.1 provides for the first time an explicit analytical expression for the *n*th-order GFRF which reveals a straightforward relationship between the nonlinear parameters of model (1.5) and the system GFRFs, and is an explicit function of the nonlinear parameters at any specific frequency point $(\omega_1,\cdots,\omega_n)$. Equation (3.14) is referred to as the parametric characteristic function of the *n*th-order GFRF, which is denoted by $H_n(C(n,K))_{(\omega_1,\cdots,\omega_n)}$.□

Remark 3.2. As mentioned above, the CE operator sets up a mapping from Ξ to $P_c{}^{\sigma}$. When applying the CE operator to the GFRFs of model (1), C_s=$C(M,K)$, W_s= $\{\omega_1,\cdots,\omega_N\} \cup \{c_{1,0}(k_1), c_{0,1}(k_1) | 0 \le k_1 \le K\}$, $P_c = \{c_1^{r_1} c_2^{r_2} \cdots c_I^{r_I} | c_i \in C(M,K), r_i \in \mathrm{Z}_0, I = |C(M,K)|\}$ and $\Xi = \{H_n(\cdot) | 1 \le n \le N\}$. The condition described by (3.15) in Proposition 3.1 provides a sufficient and necessary condition on what nonlinear parameters of model (1.5) can appear in the *n*th-order GFRF, and also how parameters determine the GFRF.

For a better understanding of the parametric characteristic $CE(H_n(j\omega_1,\cdots,j\omega_n))$, the following properties of $CE(H_n(j\omega_1,\cdots,j\omega_n))$ for the NDE model (1.5) can be obtained, based on Proposition 3.1.

Definition 3.1. If a nonlinear parameter monomial $\prod_{i=1}^{k} c_{p_i,q_i}^{j_i}(\cdot)$ (k>0, $j_i \geq 0$) is an element of $CE(H_n(j\omega_1,\cdots,j\omega_n))$, then it has an independent contribution to $H_n(j\omega_1,\cdots,j\omega_n)$, and is referred to as a complete monomial of order *n* (simply as *n*-order complete); otherwise, if it is part of an *n*-order complete monomial, then it is referred to as *n*-order incomplete.

Obviously, all the elements in $CE(H_n(j\omega_1,\cdots,j\omega_n))$ are *n*-order complete.

Property 3.1. The largest nonlinear degree of the nonlinear parameters appearing in $CE(H_n(j\omega_1,\cdots,j\omega_n))$ is *n* corresponding to nonlinear parameters $c_{p,q}(.)$ with $p+q=n$, and the *n*-degree nonlinear parameters of form $c_{p,q}(.)$ ($p+q=n$) are all *n*-order complete.

Proof. In (3.15) when $p+q=n$, then $p+q+\sum_{i=1}^{k}(p_i+q_i)=n+\sum_{i=1}^{k}(p_i+q_i)=n+k$, which further yields $\sum_{i=1}^{k}(p_i+q_i)=k$. Note that $2 \leq p_i+q_i \leq n-k$ and $0 \leq k \leq n-2$, thus $k=p_i=q_i=0$. Therefore, the property is proved. □

Property 3.2. $c_{p,q}(.)$ is *j*-order incomplete for $j>p+q$. That is, for a nonlinear parameter $c_{p,q}(.)$, it will appear in all the GFRFs of order larger than $p+q$.

Proof. This property can be seen from the recursive equations (3.16a-c) and can also be proved from Proposition 3.1. Suppose $c_{p,q}(.)$ does not appear in $H_n(j\omega_1,\cdots,j\omega_n)$, where n>p+q. Consider a monomial $c_{p,q}(.)c_{2,0}^{k}(.)$ with $k=n-p-q$. It can be verified from Proposition 3.1 that $c_{p,q}(.)c_{2,0}^{n-p-q}(.)$ is n-order complete. This results in a contradiction. □

Properties 3.1-3.2 show that only the nonlinear parameters of degree from 2 to *n* have contribution to $CE(H_n(j\omega_1,\cdots,j\omega_n))$, and the *n*-degree nonlinear parameters contribute to all the GFRFs of order $\geq n$.

Property 3.3. If $2 \leq p_i+q_i$, $1 \leq k$ and there is at least one p_i satisfying $1 \leq p_i$ except for $k=1$, then $c_{p_1q_1}(\cdot)c_{p_2q_2}(\cdot)\cdots c_{p_kq_k}(\cdot)$ is *Z*-order complete, where $Z=\sum_{i=1}^{k}(p_i+q_i)-k+1$. Moreover, $\prod_{i=1}^{k} c_{p_i,q_i}(\cdot)$ are *j*-order incomplete for $j>Z$, and have no effect on the GFRFs of order less than *Z*. □

The proof of Property 3.3 is given in Section 3.4. Given any monomial $c_{p_1q_1}(\cdot)c_{p_2q_2}(\cdot)\cdots c_{p_kq_k}(\cdot)$, it can be easily determined from Property 3.3 that, to which order GFRF the monomial contributes independently. For instance, consider a nonlinear parameter $c_{3,2}(.)$, which corresponds to the nonlinear term $\prod_{i=1}^{3}\frac{d^{k_i}y(t)}{dt^{k_i}}\prod_{i=4}^{5}\frac{d^{k_i}u(t)}{dt^{k_i}}$. It follows from Property 3.3 that Z=(3+2)-1+1=5. Thus this nonlinear term has an independent contribution to the 5th order GFRF $H_5(.)$ and affects all the GFRFs of order larger than 5. Moreover, it has no effect on the GFRFs less than the 5th order.

Property 3.4. If $1 \le r_i$ and $1 \le k$, then the elements of $CE(\prod_{i=1}^{k} H_{r_i}(\cdot))$ are all Z-order complete, where $Z = \sum_{i=1}^{k} r_i - k + 1$, and are all j-order incomplete for $j>Z$, and have no effect on the GFRFs of order less than Z. Similarly, the elements of $\prod_{i=1}^{k_1} c_{p_i q_i}(\cdot) \otimes CE(\prod_{i=1}^{k_2} H_{r_i}(\cdot))$ are all Z-order complete, where $Z = \sum_{i=1}^{k_1}(p_i + q_i) + \sum_{i=1}^{k_2} r_i - k_1 - k_2 + 1$, and are all j-order incomplete for $j>Z$, and have no effect on the GFRFs of order less than Z. □

The proof of Property 3.4 is given in Section 3.4. Obviously, this property is an extension of Property 3.3, which shows that some computation by "$\otimes$" between some parameters and the parametric characteristics of some different order GFRFs may result in the same parametric characteristic.

Property 3.5. $CE(H_{n,p}(\cdot)) = CE(H_{n-p+1}(\cdot))$. □

The proof of Property 3.5 is given in Section 3.4. This property, together with Property 3.4, provides a simplified approach to the recursive computation of the parametric characteristic of the nth-order GFRF in Equations (3.16a-c), which is summarized in Corollary 3.1 as follows.

Corollary 3.1. The parametric characteristic of the nth-order GFRF for model (1.5) can be recursively determined as

$$CE(H_n(j\omega_1,\cdots,j\omega_n)) = C_{0,n} \oplus \bigoplus_{q=1}^{n-1}\left\{C_{n-q,q} \oplus \left(\bigoplus_{p=1}^{n-q-1} C_{p,q} \otimes \chi_C(n,p,q,\lfloor\tfrac{n-q}{2}\rfloor)\right)\right\} \oplus \left\{C_{n,0} \oplus \left(\bigoplus_{p=2}^{n-1} C_{p,0} \otimes \chi_C(n,p,0,\lfloor\tfrac{n+1}{2}\rfloor)\right)\right\} \tag{3.17}$$

where $\lfloor\cdot\rfloor$ is to take the integer part, $\chi_C(n,p,q,\aleph) = \begin{cases} CE(H_{n-p-q+1}(\cdot)) & p \le \aleph \\ C_{0,n-p-q+1} & p > \aleph \end{cases}$, and $\aleph$ is a positive integer.

Proof. Using Property 3.5, (3.16a) can be written as (n>1)

$$CE(H_n(j\omega_1,\cdots,j\omega_n)) = C_{0,n} \oplus \left(\bigoplus_{q=1}^{n-1}\bigoplus_{p=1}^{n-q} C_{p,q} \otimes CE(H_{n-q-p+1}(\cdot))\right) \oplus \left(\bigoplus_{p=2}^{n} C_{p,0} \otimes CE(H_{n-p+1}(\cdot))\right) \tag{3.18}$$

Note from Property 3.4 that some computations in the second and third part of the last equation are repetitive. For example, the monomials in $C_{n-2,1} \otimes CE(H_{n-n+2-1+1}(\cdot)) = C_{n-2,1} \otimes CE(H_2(\cdot))$ (n>2) are included in $C_{1,1} \otimes CE(H_{n-1}(\cdot)) \cup C_{2,0} \otimes CE(H_{n-1}(\cdot))$, except the monomials in $C_{n-2,1} \cdot C_{0,2}$. For this reason, (3.18) can be further written as

$$CE(H_n(j\omega_1,\cdots,j\omega_n))$$
$$= C_{0,n} \oplus \mathop{\oplus}_{q=1}^{n-1}\left\{C_{n-q,q} \oplus \left(\mathop{\oplus}_{p=1}^{\lfloor n-q/2 \rfloor} C_{p,q} \otimes CE(H_{n-q-p+1}(\cdot))\right) \oplus \left(\mathop{\oplus}_{p=\lfloor n-q/2 \rfloor+1}^{n-q-1} C_{p,q} \otimes C_{0,n-q-p+1})\right)\right\}$$
$$\oplus \left\{C_{n,0} \oplus \left(\mathop{\oplus}_{p=2}^{\lfloor n+1/2 \rfloor} C_{p,0} \otimes CE(H_{n-p+1}(\cdot))\right) \oplus \left(\mathop{\oplus}_{p=\lfloor n+1/2 \rfloor+1}^{n-1} C_{p,0} \otimes C_{0,n-p+1}\right)\right\}$$

This produces Equation (3.17). The proof is completed. □

Remark 3.3. Corollary 3.1 provides an alternative recursive way to determine the parametric characteristic of the *n*th-order GFRF. If there are only some nonlinear parameters in (3.13) of interest, then Equation (3.17) and all the results above can still be used by taking other parameters as 1 if they are nonzero, or as zero if they are zero. Therefore, whatever nonlinear parameters (for instance *x*) are concerned, the parametric characteristic function with respect to *x* denoted by $H_n(x)_{(\omega_1,\cdots,\omega_n;C(n,K)\backslash x)}$ and the parametric characteristic $CE(H_n(j\omega_1,\cdots,j\omega_n))$ can all be derived by following the same method established above. □

The parametric characteristic analysis of this section can not only provide guidance to the computation and analysis of the GFRFs, but also demonstrate how the parameters of interest affect the GFRFs and consequently provide useful information for the system analysis. The following example provides an illustration for this.

Example 3.1. Consider the parametric characteristics of the following two cases:

Case 1: Suppose there is only one input nonlinear term $C_{0,3} \neq 0$, and all the other nonlinear parameters are zero in model (1.5). Then the parametric characteristics of the nth-order GFRF can be computed as

If n<3, it follows from Property 3.1 that $CE(H_n(j\omega_1,\cdots,j\omega_n))$ =0.

If n=3, it also follows from Property 3.1 that the parameters in $C_{0,3}$ are all 3-order complete. Thus $CE(H_3(j\omega_1,\cdots,j\omega_3))=C_{0,3}$.

If n>3, it follows from Property 3.2, $C_{0,3}$ should be *n*-order incomplete in this case. However, from the Definition 3.1, a complete monomial should have at least one $p \geq 1$. Since there are no other nonzero nonlinear parameters, $CE(H_n(j\omega_1,\cdots,j\omega_n))$ =0 for this case.

Therefore, $CE(H_n(j\omega_1,\cdots,j\omega_n))$ =0 for n $\neq$ 1 and n $\neq$ 3 in Case 1. That is, only $H_1(j\omega)$ and $H_3(j\omega_1,\cdots,j\omega_3)$ are nonzero in this case. Obviously, the computation of the parametric characteristics can provide guidance to the computation and analysis of the GFRFs from this case study.

Case 2: Suppose only $C_{0,3} \neq 0$ and $C_{2,0} \neq 0$, and all the other nonlinear parameters are zero. Then the parametric characteristics of the GFRFs can be simply determined as

$$CE(H_1(j\omega_1)) = 1,\ CE(H_2(j\omega_1,j\omega_2)) = C_{2,0},\ CE(H_3(j\omega_1,\cdots,j\omega_3)) = C_{2,0}^2 \oplus C_{0,3}$$
$$CE(H_4(j\omega_1,\cdots,j\omega_4)) = C_{2,0}^3 \oplus C_{0,3} \otimes C_{2,0},\ CE(H_5(j\omega_1,\cdots,j\omega_5)) = C_{2,0}^4 \oplus C_{0,3} \otimes C_{2,0}^2$$
$$CE(H_6(j\omega_1,\cdots,j\omega_6)) = C_{2,0}^6 \oplus C_{0,3} \otimes C_{2,0}^3 \oplus C_{0,3}^2 \otimes C_{2,0}$$

Especially, if only $c_{0,3}$ is of interest for analysis, then $c_{2,0}$ can be regarded as constant 1. In this case, the parametric characteristics of the GFRFs can be obtained as

$$CE(H_1(j\omega_1)) = CE(H_2(j\omega_1, j\omega_2)) = 1,\ CE(H_3(j\omega_1,\cdots,j\omega_3)) = c_{0,3}$$

$$CE(H_4(j\omega_1,\cdots,j\omega_4)) = c_{0,3},\ CE(H_5(j\omega_1,\cdots,j\omega_5)) = c_{0,3},\ CE(H_6(j\omega_1,\cdots,j\omega_6)) = c_{0,3} \oplus c_{0,3}^2$$

Note that different parametric characteristics of the GFRFs correspond to different polynomial functions with respect to the parameters of interest, which can demonstrate how the parameters of interest affect the GFRFs and thus provide some useful information for the system analysis. For example, from the parametric characteristics in Case 2, it can be seen that the sensitivity of the GFRFs for n<6 with respect to $c_{0,3}$ is a constant when $c_{2,0}$ and the linear parameters are constant. This may imply that in order to make the system less sensitive to the input nonlinearity with coefficient $c_{0,3}$, it needs only to adjust the parameters in $c_{2,0}$ and the linear parameters of model (1.5) to reduce the corresponding constants in Case 2 under certain conditions. □

The parametric characteristic and its properties developed in this section for the nth-order GFRF demonstrate what the parametric characteristics of the GFRFs are, and how the nonlinear parameters in $C(n,K)$ make contributions to the nth-order GFRF. As demonstrated in Example 3.1, these fundamental results can be used to reveal how the nonlinear parameters affect the GFRFs and how the frequency response functions of model (1.5) are constructed and thus dominated by the model parameters which define system nonlinearities. Based on these results, useful results can be developed and will be discussed in more details in the following sections and chapters.

Moreover, it should be noted that all the results above developed for the NDE model (1.5) also hold for the NARX model (1.6) (Jing et al 2006).

3.3 Parametric characteristics based analysis

Based on the parametric characteristics of the GFRFs established in the last section, many significant results can be obtained. The parametric characteristic analysis can provide an important insight into at least the following aspects:

(a) The system nonlinear effects on the frequency response functions (including the GFRFs and output spectrum) ------ mainly discussed in this section, Chapter 4 and Chapter 7;
(b) The detailed polynomial structure of the frequency response functions ------ mainly discussed in this section and Chapter 4;
(c) Computations of the GFRFs and output spectrum ------ mainly discussed in Chapter 4 and Chapter 5;
(d) Understanding of nonlinear behaviour in the frequency domain ------ mainly discussed in Chapter 6;
(e) Analysis and design of system output behaviour by using nonlinearities ------ mainly discussed in Chapter 9.

In this section, some of these results are given, and more detailed results will be discussed later in the following chapters.

3.3.1 Nonlinear effect on the GFRFs from different nonlinear parameters

The nonlinearities in model (1.5) or model (1.6) can be classified into three categories as follows:

(a) Pure input nonlinearities. This refers to the nonlinear parameters $c_{0,n}(.)$, which are the first term in the parametric characteristics in equation (3.17);
(b) Pure output nonlinearities. This refers to the nonlinear parameters $c_{n,0}(.)$, which are the last term in equation (3.17);
(c) Input-output cross nonlinearities, This refers to the nonlinear parameters $c_{p,q}(.)$, which are the second term in (3.17).

It is known that different nonlinearity has a different effect on system dynamics. Different nonlinear parameters correspond to different degree and category of nonlinearities. Hence, the frequency characteristics of frequency response functions and the effects of different nonlinear parameters on system output behaviour can be revealed by the parametric characteristic analysis of the corresponding frequency response functions. Since the GFRFs represent system frequency characteristics, the study on the nonlinear effect on the GFRFs from different categories of nonlinearities can provide an important insight into the relationship between the system frequency characteristics and physical model parameters. In this section, the parametric characteristics based analysis is investigated and discussed for the GFRFs in order to reveal how different model parameters have their effect on the frequency response functions for model (1.5), and therefore affect the system frequency characteristics. In what follows, the k+1 in monomial $C_{pq} \otimes C_{p_1q_1} \otimes C_{p_2q_2} \otimes \cdots \otimes C_{p_kq_k}$ is referred to as the power of the monomial.

A. Pure input nonlinearities

As mentioned, this category of nonlinearities correspond to the nonlinear parameters of the form $c_{0,q}(.)$ with q>1. If n=q, then from Property 3.1 the parametric characteristic of the nth-order GFRF with respect to the parameters in $C_{0,q}$ is

$$CE(H_n(j\omega_1,\cdots,j\omega_n))_{C_{0,q}} = C_{0,q} \tag{3.19a}$$

and if n<q,

$$CE(H_n(j\omega_1,\cdots,j\omega_n))_{C_{0,q}} = 1 \tag{3.19b}$$

For n>q, since there is at least one parameter $c_{p,q}(.)$ with p>0 for any complete monomials (except $c_{0,n}(.)$) in $CE(H_n(j\omega_1,\cdots,j\omega_n))_{c_{0,q}(\cdot)}$ from Proposition 3.1, thus $c_{0,q}(.)^{\rho}$ for any ρ >0 can not be an independent entry in $CE(H_n(j\omega_1,\cdots,j\omega_n))_{c_{0,q}(\cdot)}$. The largest power ρ can only appear in the monomial $c_{0,q}(.)^{\rho} c_{p',q'}(.)$, where $c_{p',q'}(.)$ is nonzero, satisfies $p' \ge 1$ and $p'+q' \ge 2$ and has the smallest $p'+q'$. In this case, ρ can be computed from Property 3.3 as

$$\rho(n,0,q) = \tfrac{n-p'-q'}{q-1}$$

For example, if $p'+q'=2$, then

$$\rho(n,0,q) = \begin{cases} \left\lfloor \frac{n-1}{q-1} \right\rfloor & \text{if } \frac{n-1}{q-1} \text{ is not an integer} \\ \frac{n-q}{q-1} & \text{else} \end{cases}$$

Therefore, for n>q,

$$CE(H_n(j\omega_1,\cdots,j\omega_n))_{C_{0,q}} = [1 \quad C_{0,q} \quad C_{0,q}^{\ 2} \quad \cdots \quad C_{0,q}^{\ \rho(n,0,q)}] \tag{3.19c}$$

In particular, when all the other nonlinear parameters are zero except for $C_{0,q}$, then (n>1)

$$CE(H_n(j\omega_1,\cdots,j\omega_n))_{C_{0,q}} = \begin{cases} C_{0,q} & \text{if q = n} \\ 0 & \text{else} \end{cases} \tag{3.19d}$$

It can further be verified that the parametric characteristic $CE(H_n(j\omega_1,\cdots,j\omega_n))_{C_{0,q}}$ is the same as (3.19d) even when only all the other categories of nonlinear parameters are zero except for the input nonlinearity.

From the parametric characteristic analysis of the *n*th-order GFRF for the input nonlinearity, it can be concluded that,

(A1) The parametric characteristic function with respect to the input nonlinearity for the nth-order GFRF is a polynomial of the largest degree $\rho(n,0,q)$, i.e.,

$$H_n(C_{0,q})_{(j\omega_1,\cdots,j\omega_n;C(n,K)\backslash C_{0,q})} = [1 \quad C_{0,q} \quad C_{0,q}^{\ 2} \quad \cdots \quad C_{0,q}^{\ \rho(n,0,q)}]\cdot f_n(j\omega_1,\cdots,j\omega_n;C(n,K)\backslash C_{0,q})$$

where $f_n(j\omega_1,\cdots,j\omega_n;C(n,K)\backslash C_{0,q})$ is an appropriate function vector.

(A2) The largest power for the input nonlinearity of an independent contribution in $CE(H_n(j\omega_1,\cdots,j\omega_n))$ is 1, which corresponds to the nonlinear parameters in $C_{0,n}$.

(A3) For comparison with the other categories of nonlinearities, considering the individual effect of pure input nonlinearity when there are no other categories of nonlinearities, *i.e.*, output nonlinearity and input-output cross nonlinearity, it can be seen from (3.19d) that the input nonlinearities have no auto-crossing effects on system dynamics. That is, each degree of the input nonlinearities has an independent contribution to the corresponding order GFRF and the largest power of a complete monomial from input nonlinearities is 1, i.e., the nth-order GFRF is simply $H_n(j\omega_1,\cdots,j\omega_n) = C_{0,q}\cdot f_n(j\omega_1,\cdots,j\omega_n)$ from Proposition 3.1. Obviously, if $C_{0,n}$=0, there will be no contribution from the input nonlinearities in the nth-order GFRF. It will be seen that these demonstrate a quite different property for the input nonlinearity from other categories of nonlinearities.

It is known that a difficulty in the analysis of Volterra systems is that the Volterra kernel functions in the time domain usually interact with each order due to the crossing nonlinear effects from different nonlinearities, and so are the GFRFs in the frequency domain. From the discussions above, this difficulty does not hold for the case that there are only input nonlinearities, *e.g.*, for the class of Volterra systems studied in Kotsios (1997). The parametric characteristic analysis for the input nonlinearities can also make light on the selection of different parameters for the energy transfer filter design in Billings and Lang (2002).

B. Pure output nonlinearities

This category of nonlinearities correspond to the nonlinear parameters of the form $c_{p,0}(.)$ with p>1. If n=p, then from Property 3.1

$$CE(H_n(j\omega_1,\cdots,j\omega_n))_{C_{p,0}} = C_{p,0} \tag{3.20a}$$

If n<p, also from Property 3.1

$$CE(H_n(j\omega_1,\cdots,j\omega_n))_{C_{p,0}}=1 \tag{3.20b}$$

These are similar to the input nonlinearity. If $n>p$, then from Properties 3.1-3.3 $C_{p,0}$ will contribute to all the GFRFs of order lager than p. From Property 3, $c_{p,0}(.)^{\rho}$ for $\rho>0$ is a complete monomial for the Zth-order GFRFs where $Z=(p-1)\rho+1$. For the nth-order GFRF with $n>p$, the largest power ρ can be computed from Property 3.3 as

$$\rho(n,p,0)=\left\lfloor \frac{n-1}{p-1} \right\rfloor$$

Thus, for $n>p$,

$$CE(H_n(j\omega_1,\cdots,j\omega_n))_{C_{p,0}}=[1 \quad C_{p,0} \quad C_{p,0}{}^2 \quad \cdots \quad C_{p,0}{}^{\rho(n,p,0)}] \tag{3.20c}$$

Consider the particular case where all nonlinear parameters are zero except the parameters in $C_{p,0}$, then for $n>1$

$$CE(H_n(j\omega_1,\cdots,j\omega_n))_{C_{p,0}}=\begin{cases}0 & \text{if } p>n \text{ or } \frac{n-1}{p-1} \text{ is not an integer}\\ C_{p,0}{}^{\rho(n,p,0)} & \text{else}\end{cases} \tag{3.20d}$$

However, when all other nonlinear parameters are zero except output nonlinear parameters, the parametric characteristic $CE(H_n(j\omega_1,\cdots,j\omega_n))_{C_{p,0}}$ for $n>p$ is the same as (3.20c).

From the parametric characteristic analysis of the nth-order GFRF for the pure output nonlinearity, it can be concluded that,

(B1) The parametric characteristic function with respect to the output nonlinearity for the nth-order GFRF is a polynomial of the largest degree $\rho(n,p,0)$, i.e.,

$$H_n(C_{p,0})_{(j\omega_1,\cdots,j\omega_n;C(n,K)\backslash C_{p,0})}=[1 \quad C_{p,0} \quad C_{p,0}{}^2 \quad \cdots \quad C_{p,0}{}^{\rho(n,p,0)}]\cdot f_n(j\omega_1,\cdots,j\omega_n;C(n,K)\backslash C_{p,0})$$

where $f_n(j\omega_1,\cdots,j\omega_n;C(n,K)\backslash C_{p,0})$ is an appropriate function vector. Note that $\rho(n,p,0)\geq\rho(n,0,q)$, which may imply that for the same nonlinear degree, output nonlinearity has a larger effect on the system than input nonlinearity.

(B2) The largest power for the output nonlinear parameter $C_{p,0}$ of an independent contribution in $CE(H_n(j\omega_1,\cdots,j\omega_n))$ is $\rho(n,p,0)$, which corresponds to the n-order complete monomial $C_{p,0}{}^{\rho(n,p,0)}$. However, the largest power for the output nonlinearity of a complete monomial in $CE(H_n(j\omega_1,\cdots,j\omega_n))$ is k, corresponding to the monomial $C_{p_1,0}\otimes C_{p_2,0}\otimes\cdots\otimes C_{p_k,0}$, where $k=p_1+\cdots+p_k+1-n$. This is quite different with the input nonlinearity.

(B3) Considering the individual effect of pure output nonlinearity when there are no other categories of nonlinearities, *i.e.*, input nonlinearity and input-output cross nonlinearity, it can be seen from (3.20c) that the output nonlinearities have auto-crossing nonlinear effects on system dynamics. That is, different degree of output nonlinearities can form a complete monomial in the nth-order GFRF and the largest power of this kind of complete monomials from output nonlinearities is k as mentioned in (B2). Obviously, if the degree-n nonlinear parameter $C_{n,0}=0$, there are still contributions from the output nonlinearities in the nth-order GFRF if there are other nonzero output nonlinear parameters of degree less than n. These may imply that output nonlinearity has more complicated and larger effect on the system than input nonlinearity of the same order, which shows a property different from that of the input nonlinearity as mentioned in (A3).

(B4) It can be seen from (3.20c-d) that $C_{p,0}$ will contribute independently to the GFRFs whose orders are $(p-1)i+1$ for i=1,2,3,.... It is known that for a Volterra system, the system nonlinear dynamics is usually dominated by the first several order GFRFs (Taylor 1999, Boyd and Chua 1985). This implies that the nonlinear terms with coefficient $C_{p,0}$ of smaller nonlinear degree, *e.g.*, 2 and 3, take much greater roles in the GFRFs than other pure output nonlinearities. This property is significant for the design of nonlinear feedback controller design, where a desired degree of nonlinearity should be determined for control objectives (Jing et al 2006a, Van Moer et al 2001). This will be further discussed in Chapter 9.

C. Input-output cross nonlinearities

This category of nonlinearities corresponds to the nonlinear parameters of the form $c_{p,q}(.)$ with $p \ge 1$ and $q \ge 1$. It can be verified that the parametric characteristics of the GFRFs with respect to such nonlinearities are very similar to those for the pure output nonlinearities as shown in B, and the conclusions held for the output nonlinearity still hold for the input-output cross nonlinearity. Thus the detailed discussions are omitted here. For a summary, the following parametric characteristics hold for both of these two categories of nonlinearities

$$CE(H_n(j\omega_1,\cdots,j\omega_n))_{C_{p,q}} = \begin{cases} 1 & \text{if } n < p+q \\ \begin{bmatrix} 1 & C_{p,q} & C_{p,q}{}^2 & \cdots & C_{p,q}{}^{\rho(n,p,q)} \end{bmatrix} & \text{else} \end{cases} \quad (3.21)$$

where, n>1, $\rho(n,p,q) = \lfloor \frac{n-1}{p+q-1} \rfloor$, $p \ge 1$ and $p+q \ge 2$.

A difference between the input-output cross nonlinearity and the pure output nonlinearity may be that the output nonlinearity can be relatively easily realized by a nonlinear state or output feedback control in practice.

Remark 3.4. Based on the parametric characteristic of the *n*th-order GFRF with respect to nonlinear parameters in $C_{p,q}$, the sensitivity of the GFRFs with respect to these nonlinear parameters can also be studied. From Proposition 3.1, the sensitivity of $H_n(j\omega_1,\cdots,j\omega_n)$ with respect to a specific nonlinear parameter c can be computed as

$$\frac{\partial H_n(c)_{(\omega_1,\cdots,\omega_n;C(M,K,n)\backslash c)}}{\partial c} = \frac{\partial H_n(j\omega_1,\cdots,j\omega_n)}{\partial c} = \frac{\partial CE(H_n(j\omega_1,\cdots,j\omega_n))}{\partial c}\cdot f_n(j\omega_1,\cdots,j\omega_n) \quad (3.22)$$

Thus, the sensitivity of the *n*th-order GFRF with respect to any nonlinear parameter $c=c_{p,q}(.)$ with $p \ge 1$ and $p+q \ge 2$ can be obtained from (3.21) as:

$$\frac{\partial H_n(c)_{(\omega_1,\cdots,\omega_n;C(K,n)\backslash c)}}{\partial c} = \begin{bmatrix} 0 & 1 & 2c & \cdots & \rho(n,p,q)c^{\rho(n,p,q)-1} \end{bmatrix} \cdot \bar{f}_n(j\omega_1,\cdots,j\omega_n;C(K,n)\backslash c) \quad (3.23)$$

where $\bar{f}_n(j\omega_1,\cdots,j\omega_n;C(K,n)\backslash c)$ is an appropriate function vector defined in Proposition 3.1. Obviously, the sensitivity to a specific parameter is still an analytical polynomial function of the nonlinear parameter. From the parametric characteristics in (3.19-3.21), it can be concluded that the sensitivity of the *n*th-order GFRF with respect to an input nonlinear parameter must be zero or constant when there are no other category of nonlinearities. However, this can only happen to the output nonlinear parameters and input-output cross nonlinear parameters if the nonlinear degree of the parameter of interest is *n*. Otherwise, the sensitivity function with respect to an output or an input-output cross nonlinear parameter is still an analytical polynomial function of the parameter of interest and some other nonzero parameters.

3.4 Proofs

- **Proof of Property 3.3**

From Proposition 3.1, $CE(H_Z(\cdot))$ includes all non-repetitive monomial functions of the nonlinear parameters in model (1.5) of the form $C_{pq} \otimes C_{p_1q_1} \otimes C_{p_2q_2} \otimes \cdots \otimes C_{p_kq_k}$, where the subscripts satisfy $p+q+\sum_{i=1}^{k'}(p_i+q_i)=Z+k'$, $2 \le p_i+q_i \le Z-k'$, $0 \le k' \le Z-2$, $2 \le p+q \le Z-k'$, and noting $1 \le p \le Z-k'$, thus $\overset{k}{\underset{i=1}{\otimes}} C_{p_iq_i}$ is included in $CE(H_Z(\cdot))$. Moreover, substitute k by $k+x$, where x>0 is an integer, then $Z'=\sum_{i=1}^{k+x}(p_i+q_i)-k-x+1$, which further yields $Z'-Z=\sum_{i=1}^{x}(p_i+q_i)-x$. Note that $2 \le p_i+q_i$, thus $Z'-Z \ge \sum_{i=1}^{x}2-x=x$. Therefore, $\overset{k}{\underset{i=1}{\otimes}} C_{p_iq_i}$ must appear in $CE(H_j(j\omega_1,\cdots,j\omega_j))$ for j>Z and but must not appear in the GFRFs of order less than j. This completes the proof. □

- **Proof of Property 3.4**

From Proposition 3.1, any element $c_{p_1,q_1}(\cdot)c_{p_2,q_2}(\cdot)\cdots c_{p_{k_{r_i}},q_{k_{r_i}}}(\cdot)$ in CE($H_{r_i}(\cdot)$) with r_i>1 satisfy

$$r_i=\sum_{i=1}^{k_{r_i}}(p_i+q_i)-k_{r_i}+1$$

Note that if r_i=1, then CE($H_{r_i}(\cdot)$)=1. In this case, suppose $(p_i+q_i)=1$ for consistence. Therefore,

$$\sum_{i=1}^{k}r_i-k+1=(\sum_{i=1}^{k}\sum_{j=1}^{k_{r_i}}(p_j+q_j)-\sum_{i=1}^{k}k_{r_i}+k)-k+1=\sum_{i=1}^{k}\sum_{j=1}^{k_{r_i}}(p_j+q_j)-\sum_{i=1}^{k}k_{r_i}+1=Z.$$

This proves the first part of this property. The second part follows from the first part and Property 3.3. □

- **Proof of Property 3.5**

A different proof was given in Proposition 3 of Jing et al (2006), but here presents a more concise proof based on the properties developed in Section 3.2. Applying the CE operator to Equation (3.5), it can be obtained that

$$CE(H_{n,p}(j\omega_1,\cdots,j\omega_n))=\underset{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}{\overset{n-p+1}{\oplus}}\overset{p}{\underset{i=1}{\otimes}}CE(H_{r_i}(\cdot))=CE(H_{n-p+1}(\cdot))\oplus\left(\underset{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}{\overset{n-p}{\oplus}}\overset{p}{\underset{i=1}{\otimes}}CE(H_{r_i}(\cdot))\right)$$

From Property 3.4, it follows that all the elements in $\underset{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}{\overset{n-p}{\oplus}}\overset{p}{\underset{i=1}{\otimes}}CE(H_{r_i}(\cdot))$ should be Z-order complete, where $Z=\sum_{i=1}^{p}r_i-p+1=n-p+1$. This completes the proof. □

3.5 Summary

The parametric characteristic analysis proposed in Chapter 2 has been used in this Chapter for the study of the GFRFs of nonlinear Volterra systems described by model (1.5) or model (1.6). Fundamental and significant results have been established for the parametric characteristics of the GFRFs of the nonlinear systems. The method has been shownd to be of great significance in understanding the system's frequency response functions. As mentioned in Section 3.3, the significance has at least five aspects, some of which have been demonstrated in this chapter and more will be discussed later.

From the results of this Chapter, it can be seen that, the parametric characteristics of the GFRFs can explicitly reveal the relationship between the time domain model parameters and the GFRFs and therefore provide a useful insight into the analysis and design of nonlinear systems in the frequency domain. By using the parametric characteristic analysis, system nonlinear frequency domain characteristics can be studied in terms of the time domain model parameters which define system nonlinearities, and the dependence of the frequency response functions of nonlinear systems on model parameters can be revealed. As it has been shown, the analytical relationship between system output spectrum and model parameters can be determined explicitly, and the nonlinear effect on the system output frequency response from different nonlinearities can be unveiled. This will facilitate the study of nonlinear behaviours in the frequency domain and unveil the effects of different categories of system nonlinearities on the output frequency response. These will be further studied in the following chapters. It will be seen that, all these results provide a novel approach to the frequency domain analysis of nonlinear systems, which may be difficult to be addressed before.

Chapter 4
PARAMETRIC CHARACTERISTIC ANALYSIS FOR SYSTEM OUTPUT SPECTRUM

The parametric characteristics of system output spectrum of model (1.5) are studied firstly, especially with respect to specific nonlinear parameters of interest. Then, a systematic frequency domain method based on the parametric characteristic analysis results, referred to as the parametric characteristics based output spectrum analysis, is established and discussed in detail for nonlinear Volterra systems described by model (1.5) or model (1.6).

4.1 Parametric characteristics of system output spectrum

The system output spectrum of model (1.1) can be described as (Lang and Billings 1996):

$$Y(j\omega)=\sum_{n=1}^{N}Y_n(j\omega) \tag{4.1}$$

when subject to a general input $u(t)$, in (4.1)

$$Y_n(j\omega)=\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}H_n(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega \tag{4.2}$$

When the input is a specific multi-tone function described by (1.3), i.e.,

$$u(t)=\sum_{i=1}^{\bar{K}}|F_i|\cos(\omega_i t+\angle F_i)$$

in (4.1)

$$Y_n(j\omega)=\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}H_n(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n}) \tag{4.3}$$

where

$$F(\omega)=\begin{cases}|F_i|e^{j\angle F_i} & \text{if } \omega\in\{\omega_k, k=\pm1,\cdots,\pm K\}\\ 0 & \text{else}\end{cases} \tag{4.4}$$

Definition 4.1. A function $y(h;s)$ is homogeneous of degree d with respect to h if $y(ch;s)=c^d y(h;s)$, where c is a constant, s denotes the independent variables of $y(.)$, and h may be a parameter or a function of certain variables and parameters.

The detailed properties of the functions and variables in Definition 4.1 are not necessarily considered here. The definition of a homogeneous function can also be referred to Rugh (1981). From Definition 4.1, it can be verified that Equation (4.2) and Equation (4.3) are both 1-degree homogeneous with respect to the nth-order GFRF $H_n(\cdot)$. From this definition, the following lemma is obvious.

Lemma 4.1. If $y(h;s_1)$ is a homogeneous function of degree d, and $h(.)$ is a separable function with respect to parameter x whose parametric characteristic function can be written as $h(x)=g(x)f(s_2)$, then $y(h;s_1)$ is a separable function with respect to x and its parametric characteristic function can be written as $y(x)_s=g(x)\,)^{[d]}f_y(f(s_2);s_1)$, where s_1 denotes the un-separable or un-interested parameters or variables in $h(.)$, s_2 denotes some variables in $y(.)$, $f_y(f(s_2);s_1)$ is an appropriate function vector, and $g(x)^{[d]}$ is the d times reduced kronecker product of $g(x)$.

From Proposition 3.1, Lemma 4.1 and Equations (4.1-4.2), the following result can be obtained for a homogeneous function $Y(H_n(.); s)$ of degree d, where $H_n(.)$ is the nth-order GFRF of model (1.5).

Proposition 4.1. $Y_n(H_n(j\omega_1,...,j\omega_n);\omega_1,\cdots,\omega_n)$ is a homogeneous function of degree d with respect to the nth-order GFRF $H_n(j\omega_1,...,j\omega_n)$ of (1.5). Then $Y_n(H_n(j\omega_1,...,j\omega_n);\omega_1,\cdots,\omega_n)$ is a separable function with respect to the nonlinear parameters in (3.13), whose parametric characteristic function can be described by

$$Y_n(C(M,K))_{\omega_1,\cdots,\omega_n} = \text{CE}(H_n(j\omega_1,...,j\omega_n))^{[d]}\tilde{Y}_n(f_n(j\omega_1,\cdots,j\omega_n);\omega_1,\cdots,\omega_n) \tag{4.5}$$

The sensitivity of the homogeneous function with respect to a specific parameter c is

$$\frac{\partial Y_n(C(M,K))_{\omega_1,\cdots,\omega_n}}{\partial c} = \frac{\partial CE(H_n(j\omega_1,\cdots,j\omega_n))^{[d]}}{\partial c}\cdot Y_n(f_n(j\omega_1,\cdots,j\omega_n);\omega_1,\cdots,\omega_n) \tag{4.6}$$

where $\tilde{Y}_n(f_n(j\omega_1,\cdots,j\omega_n);\omega_1,\cdots,\omega_n)$ is an appropriate function vector, and when d=1

$$\tilde{Y}_n(f_n(j\omega_1,\cdots,j\omega_n);\omega_1,\cdots,\omega_n) = Y_n(f_n(j\omega_1,\cdots,j\omega_n);\omega_1,\cdots,\omega_n) \tag{4.7}$$

Proof. The results are straightforward from Proposition 3.1, Lemma 4.1 and Equations (4.1-4.2). □

The following result can be concluded directly from Proposition 4.1 for the output spectrum of model (1.5).

Corollary 4.1. The output frequency response function $Y(j\omega)$ in (4.1) for model (1.5) is separable with respect to the nonlinear parameters in (3.13), whose parametric characteristic function can be described by

$$Y(C(M,K))_\omega = \sum_{n=1}^{N} CE(H_n(\cdot))\cdot Y_n(f_n(\cdot);j\omega) \tag{4.8a}$$

and whose parametric characteristic is

$$CE(Y(j\omega)) = \bigoplus_{n=1}^{N} CE(H_n(\cdot)) \tag{4.8b}$$

The sensitivity of the output frequency response with respect to a specific parameter c is

$$\frac{\partial Y(j\omega)}{\partial c} = \sum_{n=1}^{N}\frac{\partial CE(H_n(\cdot))}{\partial c}\cdot Y_n(f_n(\cdot);j\omega) \tag{4.9}$$

where, if the input is a general function, then $\omega = \omega_1 + \cdots + \omega_n$,

$$Y_n(f_n(\cdot);j\omega) = Y_n(f_n(j\omega_1,\cdots,j\omega_n);j\omega) = \frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega} f_n(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega \tag{4.10}$$

if the input is the multi-tone function given in (1.3), then $\omega = \omega_{k_1} + \cdots + \omega_{k_n}$,

$$Y_n(f_n(\cdot);j\omega) = Y_n(f_n(j\omega_{k_1},\cdots,j\omega_{k_n});j\omega) = \frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} f_n(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n}) \tag{4.11}$$

□

From these results, it is noted that the system output spectrum can also be expressed by a polynomial function of the nonlinear parameters in $C(M,K)$ based on the parametric characteristics of the GFRFs, and the detailed structure of this polynomial function with respect to any parameters of interest is completely determined by its parametric characteristic. Therefore, how the nonlinear parameters affect the system output spectrum can be studied through the parametric characteristic analysis as discussed in Chapter 3.

Remark 4.1. Note that $CE(H_n(\cdot))$ can be derived from the system model parameters according to the results developed in Chapter 3. Given a specific system described by model (1.5) or model (1.6), $Y(C(M,K)_\omega$ can be obtained by the FFT of the time domain output data from simulations or experiments at frequency ω. Therefore, $Y_n(f_n(\cdot);j\omega)$ for n=1,...,N can be obtained by the Least Square method as mentioned in Remark 2.1. And then $Y_n(C(n,K))_{\omega=\omega_1+\cdots+\omega_n} = CE(H_n(\cdot))\cdot Y_n(f_n(\cdot);j\omega)$ for n=1,...,N and the sensitivity Equations (4.6, 4.8) can all be obtained. This provides a numerical method to compute the output spectrum and its each order component which are now determined as analytical polynomial functions of any interested nonlinear parameters. Thus the analysis and design of the output performance of nonlinear systems can now be conducted in terms of these model parameters. Compared with the direct computation by using (3.8 or 3.11, 3.10, 3.2-3.5) and (4.1-4.3), the computational complexity is reduced. And compared with the results in Lang et al 2007, the parametric characteristic analysis of this study provides an explicit analytical expression for the relationship between system output spectrum and model parameters with detailed polynomial structure up to any order and each order output spectrum component can also be determined. Moreover, let

$$G_n(C(M,K,n)_{\omega=\omega_1+\cdots+\omega_n} = \frac{CE(H_n(\cdot))\cdot Y_n(f_n(\cdot);j\omega)}{\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int\limits_{\omega_1+\cdots+\omega_n=\omega}\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega} \tag{4.12}$$

This is the parametric characteristic function of the nth-order nonlinear output frequency response function defined in Lang and Billings (2005), which can be used for the fault diagnosis of engineering systems and structures. □

4.1.1 Parametric characteristics with respect to some specific parameters in $C_{p,q}$

As discussed before, the parametric characteristic vector $CE(H_n(\cdot))$ for all the model nonlinear parameters can be obtained according to Proposition 3.1 or (3.17) in Corollary 3.1, and if there are only some parameters of interest, the computation can be conducted by only replacing other nonzero parameters with 1. In many cases, only several specific model parameters, for example parameters in $C_{p,q}$, are of interest for the analysis of a specific nonlinear system. Thus, the computation of the parametric characteristic vector in (3.17) and (4.8) can be simplified greatly. This section provides some useful results for the computation of parametric characteristics with respect to one or more specific parameters in $C_{p,q}$, which can effectively facilitate the determination of the OFRF and the analysis based on the OFRF that will be discussed later.

Let

$$\delta(p) = \begin{cases} 1 & \text{if } p=0 \\ 0 & \text{else} \end{cases}, \text{ and } pos(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{else} \end{cases} \tag{4.13}$$

Proposition 4.2. Consider only the nonlinear parameter $C_{p,q}=c$. The parametric characteristic vector of the nth-order GFRF with respect to the parameter c is

$$CE(H_n(j\omega_1,\cdots,j\omega_n)) = \begin{bmatrix} 1 & c & c^2 & \cdots & c^{\left\lfloor \frac{n-1}{p+q-1} \right\rfloor - \delta(p)\cdot pos(n-q)} \end{bmatrix} \tag{4.14}$$

where $\lfloor \cdot \rfloor$ is to get the integer part of (.). □

The Proof of Proposition 4.2 is given in Section 4.4. Note that here c may be one parameter or a vector of some parameters of the same nonlinear degree and type in C_{pq}. Also note that $c^n = \underbrace{c \otimes \cdots c \otimes c}_{n}$ and $\otimes$ is the reduced Kronecker product defined in Chapter 2, when c is a vector. Proposition 4.2 establishes a very useful result to study the effects on the output frequency response from a specific nonlinear degree and type of nonlinear parameters. Note also that if some other nonlinear parameters in model (1.5) or (1.6) are zero, only part terms in (4.14) take an effective role. The detailed form of $CE(H_n(j\omega_1,\cdots,j\omega_n))$ can be derived from Proposition 3.1 or (3.17) in Corollary 3.1. However, a direct use of equation (4.14) does not affect the final result.

Corollary 4.2. If all the other nonlinear parameters are zero except $C_{p,q}$=c. Then the parametric characteristic vector of the nth-order GFRF with respect to the parameter c is: if (n>p+q and p>0), or (n=p+q), and if additionally $\frac{n-1}{p+q-1}$ is an integer, then

$$CE(H_n(j\omega_1,\cdots,j\omega_n)) = c^{\frac{n-1}{p+q-1}}$$

else

$$CE(H_n(j\omega_1,\cdots,j\omega_n)) = 0$$

which can be summarized as

$$CE(H_n(j\omega_1,\cdots,j\omega_n)) = c^{\frac{n-1}{p+q-1}} \cdot \delta\left(\frac{n-1}{p+q-1} - \left\lfloor \frac{n-1}{p+q-1} \right\rfloor \right) \cdot (1 - \delta(p)pos(n-q)) \tag{4.15}$$

Proof. The results are directly followed from Propositions 3.1 and 4.2. □

Corollary 4.2 provides a more special case of nonlinear Volterra systems described by (1.5) or (1.6). There are only several nonlinear parameters of the same nonlinear type and degree in the considered system. This result will be demonstrated in the simulation studies in Section 4.3. The following results can be obtained for the output frequency response.

Proposition 4.3. Consider only the nonlinear parameter $C_{p,q}$=c. The parametric characteristic vector of the output spectrum in (4.1) with respect to the parameter c can be written as

$$CE(Y(j\omega)) = \bigoplus_{n=1}^{N} CE(H_n(\cdot)) = \begin{bmatrix} 1 & c & c^2 & \cdots & c^{\left\lfloor \frac{N-1}{p+q-1} \right\rfloor - \delta(p)\cdot pos(N-q)\cdot \delta\left(\frac{N-1}{p+q-1} - \left\lfloor \frac{N-1}{p+q-1} \right\rfloor\right)} \end{bmatrix} \tag{4.16}$$

Then there exists a complex valued function vector $F(j\omega_1,\cdots,j\omega_n;C(M,K)\setminus c)$ with appropriate dimension such that

$$Y(c)_{\omega;C(M,K)\setminus c} = CE(Y(j\omega)) \cdot F(j\omega_1,\cdots,j\omega_n;C(M,K)\setminus c) \tag{4.17}$$

If all the other nonlinear parameters are zero except that $C_{p,q}$=$c \neq 0$ (p+q>1). Then the parametric characteristic vector of the output spectrum in (4.1) with respect to the parameter c is: if p=0

$$CE(Y(j\omega)) = 1 \oplus CE(H_q(\cdot)) \cdot (1 - pos(q-N)) = \begin{bmatrix} 1 & c\cdot(1 - pos(q-N)) \end{bmatrix} \tag{4.18}$$

else

$$CE(Y(j\omega)) = \mathop{\oplus}_{i=0}^{\lfloor N-1/p+q-1 \rfloor} CE(H_{(p+q-1)i+1}(\cdot)) = \begin{bmatrix} 1 & c & c^2 & \cdots & c^{\lfloor N-1/p+q-1 \rfloor} \end{bmatrix} \tag{4.19}$$

□

The proof of Proposition 4.3 is given in Section 4.4. From Corollary 4.2 and Proposition 4.3, it can be seen that different nonlinearities will result in a quite different polynomial structure for the output spectrum, and thus affect the system output frequency response in a different way. By using the results established above, the effect from different nonlinearities on system output frequency characteristics can now be studied. This will be further studied in the following sections.

Moreover, the results above involve the computation of c^n. If c is an I-dimension vector, there will be many repetitive terms involved in c^n. To simplify the computation, the following lemma can be used.

Lemma 4.2. Let be $c=[c_1,c_2,\ldots,c_I]$ which can also be denoted by $c[1:I]$, and $c^n = \underbrace{c \otimes c \cdots \otimes c}_{n}$, "⊗" is the reduced Kronecker product defined in Jing et al (2006), $n \geq 1$ and $I \geq 1$. Then

$$c^n = \left[c^{n-1} \cdot c_1, \cdots, c^{n-1}[s(1)_n - s(i)_n + 1 : s(1)_n] \cdot c_i, \cdots, c^{n-1}[s(1)_n] \cdot c_I\right]$$

where $s(i)_n = \sum_{j=i}^{I} s(j)_{n-1}$, $s(.)_1=1$, and $1 \leq i \leq I$. Moreover, $DIM(c^n) = s(1)_{n+1}$, and the location of c_i^n in c^n is $s(1)_{n+1}-s(i)_{n+1}+1$. □

The Proof of Lemma 4.2 is given in Section 4.4.

4.1.2 An example

To illustrate the results above and introduce the basic idea of the parametric characteristics based output spectrum analysis that will be discussed in the next section, an example study is given in this section. Consider a nonlinear system,

$$a_1\ddot{x} = -a_2 x - a_3\dot{x} - c_1\dot{x}^3 - c_2\dot{x}^2 x - c_3 x^3 + bu(t) \tag{4.20}$$

which is a simple case of model (1.5) with M=3, K=2, $c_{10}(2) = a_1$, $c_{10}(1) = a_3$, $C_{10}(0) = a_2$, $c_{30}(111) = c_1$, $c_{30}(110) = c_2$, $c_{30}(000) = c_3$, $c_{01}(0) = -b$, all other parameters are zero. The GFRFs for system (4.20) can be computed according to Equations (3.8 or 3.11, 3.10, 3.2-3.5). In the following, the parametric characteristics of the GFRFs for system (4.20) are discussed firstly. As will be seen, the parametric characteristics of the GFRFs provide a useful guidance to the analysis and computation of system frequency response functions.

When all the other nonlinear parameters are zero except $C_{p,q}$, it can be obtained from Corollary 4.2 that the parametric characteristic of the nth-order GFRF with respect to $C_{p,q}$ is

$$CE(H_n(j\omega_1,\cdots,j\omega_n)) = C_{p,q}^{\frac{n-1}{p+q-1}} \cdot \delta\left(\frac{n-1}{p+q-1} - \left\lfloor \frac{n-1}{p+q-1} \right\rfloor\right) \cdot (1 - \delta(p)pos(n-q)) \tag{4.21}$$

For system (4.20), note that a_1, a_2, a_3 and b are all linear parameters, and the nonzero nonlinear parameters are $C_{30} = [c_{30}(000) \quad c_{30}(110) \quad c_{30}(111)] = [c_3 \quad c_2 \quad c_1]$. Hence,

$$CE(H_n(j\omega_1,\cdots,j\omega_n)) = C_{3,0}{}' = [c_3 \quad c_2 \quad c_1]^i \text{ for n=2i+1, i=1,2,3,...,}$$
$$\text{else } CE(H_n(j\omega_1,\cdots,j\omega_n)) = 0. \qquad (4.22)$$

It is easy to compute from (4.22) as follows:

For n=3, $CE(H_3(j\omega_1,\cdots,j\omega_3)) = [c_3 \quad c_2 \quad c_1]$;

For n=5, $CE(H_5(j\omega_1,\cdots,j\omega_5)) = [c_3 \quad c_2 \quad c_1]^2 = [c_3 \quad c_2 \quad c_1]\otimes[c_3 \quad c_2 \quad c_1]$
$=[c_3^2, c_3c_2, c_3c_1, c_2^2, c_2c_1, c_1^2]$;

For n=7, $CE(H_7(j\omega_1,\cdots,j\omega_7)) = [c_3 \quad c_2 \quad c_1]^3 = [c_3 \quad c_2 \quad c_1]\otimes[c_3 \quad c_2 \quad c_1]\otimes[c_3 \quad c_2 \quad c_1]$
$=[c_3^3, c_3^2c_2, c_3^2c_1, c_3c_2^2, c_3c_2c_1, c_3c_1^2, c_2^3, c_2^2c_1, c_2c_1^2, c_1^3]$

From Proposition 3.1, there must exist a complex valued function vector $\bar{f}_n(j\omega_1,\cdots,j\omega_n)$ with appropriate dimension, such that for n= 2i+1, i=1,2,3,...,

$$H_n(c_1,c_2,c_3)_{(\omega_1,\cdots,\omega_n)} = [c_3 \quad c_2 \quad c_1]^i \cdot \bar{f}_n(j\omega_1,\cdots,j\omega_n) \qquad (4.23)$$

else
$$H_n(c_1,c_2,c_3)_{(\omega_1,\cdots,\omega_n)} = 0.$$

When there is only one parameter for example c_1 is of interest for analysis, the parametric characteristic can be obtained by simply letting $C_{3,0}= c_1$ in (4.22), i.e., the parametric characteristic vector is: for n=2i+1 and i=1,2,3,...

$$CE(H_n(j\omega_1,\cdots,j\omega_n)) = [1 \quad c_1 \quad c_1^2 \quad \cdots \quad c_1^i] \qquad (4.24)$$

else
$$CE(H_n(j\omega_1,\cdots,j\omega_n)) = 0 \qquad (4.25)$$

Thus the parametric characteristic function with respect to the parameter c_1 is: for n=2i+1 and i=1,2,3,...

$$H_n(c_1)_{(\omega_1,\cdots,\omega_n;c_2,c_3)} = [1 \quad c_1 \quad c_1^2 \quad \cdots \quad c_1^i]\cdot \bar{f}_n(j\omega_1,\cdots,j\omega_n;c_2,c_3) \qquad (4.26)$$

else
$$H_n(c_1)_{(\omega_1,\cdots,\omega_n;c_2,c_3)} = 0 \qquad (4.27)$$

where, $\bar{f}_n(j\omega_1,\cdots,j\omega_n;c_2,c_3)$ is a complex valued function vector with appropriate dimension. The sensitivity of the nth-order GFRFs for n=2i+1 and i=1,2,3,... with respect to the parameter c_1 can also be obtained as

$$\frac{\partial H_n(c_1)_{(\omega_1,\cdots,\omega_n;c_2,c_3)}}{\partial c_1} = [0 \quad 1 \quad 2c_1 \quad \cdots \quad ic^{i-1}]\cdot \bar{f}_n(j\omega_1,\cdots,j\omega_n;c_2,c_3) \qquad (4.28)$$

Consider the output spectrum of system (4.20). From Proposition 4.3,

$$CE(X(j\omega)) = \overset{\lfloor N-1/2 \rfloor}{\underset{i=0}{\oplus}} CE(H_{2i+1}(\cdot)) = \overset{\lfloor N-1/2 \rfloor}{\underset{i=0}{\oplus}} C_{30}{}^i \qquad (4.29)$$

Suppose the output function of interest is

$$y= a_2x + a_3\dot{x} - c_1\dot{x}^3 - c_2\dot{x}^2x - c_3x^3 \qquad (4.30)$$

It will be shown in Chapter 8 that

$$CE(Y(j\omega)) = CE(X(j\omega)) \qquad (4.31)$$

Then from Proposition 4.3, the parametric characteristic function for the output frequency response $Y(j\omega)$ of system (4.20) with respect to nonlinear parameters c_1, c_2 and c_3 is

$$Y(c_1,c_2,c_3)_\omega = \sum_{i=0}^{\lfloor N-1/2 \rfloor} C_{30}{}^i \cdot Y_i(f_i(\cdot);j\omega)$$
$$= \left(\overset{\lfloor N-1/2 \rfloor}{\underset{i=0}{\oplus}} C_{30}{}^i \right) \cdot \left[Y_0(f_0(\cdot);j\omega) \quad Y_1(f_1(\cdot);j\omega)^T \quad \cdots \quad Y_{\lfloor N-1/2 \rfloor}(f_{\lfloor N-1/2 \rfloor}(\cdot);j\omega)^T \right]^T \qquad (4.32)$$

For convenience, consider a much simpler case. Let c_2=c_3=0, then $C_{30}=c_{30}(111)=c_1$. Therefore the parametric characteristic function in this simple case is

$$\begin{aligned} Y(c_1)_\omega &= Y_0(f_0(\cdot);j\omega)+c_1\cdot Y_1(f_1(\cdot);j\omega)+\cdots+c_1^{\lfloor N-1/2\rfloor}\cdot Y_{\lfloor N-1/2\rfloor}(f_{\lfloor N-1/2\rfloor}(\cdot);j\omega) \\ &= [1\ \ c_1\ \cdots\ c_1^{\lfloor N-1/2\rfloor}]\cdot[Y_0(f_0(\cdot);j\omega)\ \ Y_1(f_1(\cdot);j\omega)\ \ \cdots\ \ Y_{\lfloor N-1/2\rfloor}(f_{\lfloor N-1/2\rfloor}(\cdot);j\omega)]^T \end{aligned} \tag{4.33}$$

As mentioned in Remark 2.1 and Remark 4.1, $[Y_0(f_0(\cdot);j\omega)\ \ Y_1(f_1(\cdot);j\omega)\ \ \cdots\ \ Y_{\lfloor N-1/2\rfloor}(f_{\lfloor N-1/2\rfloor}(\cdot);j\omega)]^T$ can be computed by a numerical method for a specific input u(t) and at a specific frequency ω. The idea is to obtain $\lfloor N-1/2\rfloor+1$ system output frequency responses from $\lfloor N-1/2\rfloor+1$ simulations or experimental tests on the system (4.20) under $\lfloor N-1/2\rfloor+1$ different values of the nonlinear parameter c_1 and the same input u(t), then yielding

$$\begin{bmatrix} Y(j\omega)_0 \\ Y(j\omega)_1 \\ \vdots \\ Y(j\omega)_{\lfloor N-1/2\rfloor} \end{bmatrix} = \begin{bmatrix} 1 & c_1(0) & \cdots & c_1(0)^{\lfloor N-1/2\rfloor} \\ 1 & c_1(1) & \cdots & c_1(1) \\ 1 & \vdots & \ddots & \vdots \\ 1 & c_1(\lfloor N-1/2\rfloor) & \cdots & c_1(\lfloor N-1/2\rfloor)^{\lfloor N-1/2\rfloor} \end{bmatrix} \cdot \begin{bmatrix} Y_0(f_0(\cdot);j\omega) \\ Y_1(f_1(\cdot);j\omega) \\ \vdots \\ Y_{\lfloor N-1/2\rfloor}(f_{\lfloor N-1/2\rfloor}(\cdot);j\omega) \end{bmatrix} \tag{4.34}$$

Hence,

$$\begin{bmatrix} Y_0(f_0(\cdot);j\omega) \\ Y_1(f_1(\cdot);j\omega) \\ \vdots \\ Y_{\lfloor N-1/2\rfloor}(f_{\lfloor N-1/2\rfloor}(\cdot);j\omega) \end{bmatrix} = \begin{bmatrix} 1 & c_1(0) & \cdots & c_1(0)^{\lfloor N-1/2\rfloor} \\ 1 & c_1(1) & \cdots & c_1(1) \\ 1 & \vdots & \ddots & \vdots \\ 1 & c_1(\lfloor N-1/2\rfloor) & \cdots & c_1(\lfloor N-1/2\rfloor)^{\lfloor N-1/2\rfloor} \end{bmatrix}^{-1} \cdot \begin{bmatrix} Y(j\omega)_0 \\ Y(j\omega)_1 \\ \vdots \\ Y(j\omega)_{\lfloor N-1/2\rfloor} \end{bmatrix} \tag{4.35}$$

Then equation (4.33) is determined explicitly, which is an analytical function of the nonlinear parameter c_1. The system output frequency response can therefore be analyzed and optimized in terms of the nonlinear parameters. And also from (4.33), the sensitivity of the system output frequency response with respect to the nonlinear parameter, and the nonlinear output frequency response function defined in (4.12) can both be studied. For more complicated cases, a similar process can be followed to conduct a required analysis and design in terms of multiple nonlinear parameters for model (1.5). Compared with the previous results in Lang et al (2007), since the detailed polynomial structure for the output spectrum up to any order can explicitly be determined, this can greatly reduce the simulation amount needed in the numerical method when multiple parameters are considered.

4.2 The parametric characteristics based output spectrum analysis

For more clarity, (4.8a) can be simply rewritten as

$$Y(j\omega)=\psi\cdot\Phi(j\omega)^T \tag{4.36}$$

where $\psi=\bigoplus_{n=1}^{N} CE(H_n(\cdot))$, $\Phi(j\omega)=[\phi_1(j\omega)\ \ \phi_2(j\omega)\ \ \cdots\ \ \phi_N(j\omega)]$. Note that $\phi_1(j\omega)=H_1(j\omega)$ is the first order GFRF, which represents the linear part of model (1.5) or (1.6), and $\phi_n(j\omega)=Y_n(f_n(\cdot);j\omega)$.

As discussed before, Equation (4.8) or (436) provide a more straightforward analytical expression for the relationship between system time-domain model parameters and system output frequency response. By using this explicit relationship, the system output frequency response can therefore be analyzed in terms of any model parameters of interest. Hence, it can considerably facilitate the analysis and design of the output frequency response characteristics of nonlinear Volterra systems in the frequency domain. As demonstrated in Section 4.1.2, the main idea for the parametric characteristics based output spectrum analysis proposed in this Chapter is that, given the model of a nonlinear system in the form of model (1.5) or (1.6), $CE(H_n(\cdot))$ can be computed according to Proposition 3.1 or Corollary 3.1, and $\varphi_n(j\omega)$ can be obtained according to a numerical method which is mentioned before and will be discussed in more detail later, thus the OFRF (Lang et al 2007) of the nonlinear system subject to any specific input function can be obtained, which is an analytical function of nonlinear parameters of system model, and finally frequency domain analysis for the nonlinear system can be conducted in terms of the specific model parameters of interest.

In this section, the parametric characteristics based output spectrum analysis for a Volterra system described by (1.5) or (1.6) is discussed in general in Section 4.2.1. In order to conduct the parametric characteristics based output spectrum analysis, a general procedure is provided in Section 4.2.2, where several basic algorithms and related results are discussed.

4.2.1 A new frequency domain method

The parametric characteristics based output spectrum analysis for Volterra systems described by (1.5) or (1.6) is totally a new frequency domain method for nonlinear analysis. The most noticeable advantage of this method is that any system model parameters of interest can be directly related to the interested engineering analysis and design objective which is dependent on system output frequency response, and thus the system output frequency response can be analysed in terms of some model parameters of interest in an easily-manipulated manner. This method does not restrict to a specific input signal and can be used for a considerable larger class of nonlinear systems. These are the main differences of this method from the other existing methods such as Popov-theory based analysis, describing functions and harmonic balance methods as discussed in Chapter 1.

One important step of this method is to determine the OFRF for the system under study. This will be discussed in more detailed in the following section. Once the system OFRF is obtained, based on the result in Proposition 4.3 and Equation (4.36), the output frequency response function with respect to a specific parameter c can be written as

$$Y(j\omega)=\bar{\varphi}_0(j\omega)+c\bar{\varphi}_1(j\omega)+c^2\bar{\varphi}_2(j\omega)+\cdots+c^{\ell}\bar{\varphi}_{\ell}(j\omega)+\cdots \tag{4.37a}$$

Since $Y(j\omega)$ is also a function of c, therefore, (14a) is rewritten more clearly as

$$Y(j\omega;c)=\bar{\varphi}_0(j\omega)+c\bar{\varphi}_1(j\omega)+c^2\bar{\varphi}_2(j\omega)+\cdots+c^{\ell}\bar{\varphi}_{\ell}(j\omega)+\cdots \tag{4.37b}$$

$Y(j\omega;c)$ is in fact a series of an infinite order, ℓ is a positive integer which can be determined by Proposition 4.3, $\bar{\varphi}_i(j\omega)$ is the complex valued function corresponding to the coefficient c^i in Equation (4.36). If all the other degree and type of nonlinear parameters are zero except that $C_{p,q}=c\neq 0$ (p+q>1), then $\bar{\varphi}_{i+1}(j\omega)=\varphi_i(j\omega)$ ($\varphi_i(j\omega)$ is defined in Equations (4.8), (4.10)-(4.11), and (4.36)). Based on Equations (4.37ab), the following analysis can be conducted.

- **Sensitivity of the output frequency response to nonlinear parameters**
  Based on Equations (4.37ab), this can be obtained easily as

$$\frac{\partial Y(j\omega;c)}{\partial c}=\overline{\varphi}_1(j\omega)+2c\overline{\varphi}_2(j\omega)+\cdots+\ell c^{\ell-1}\overline{\varphi}_\ell(j\omega)+\cdots \tag{4.38}$$

Similarly, the sensitivity of the magnitude of the output frequency response with respect to the nonlinear parameters can also be derived. Note that

$$\begin{aligned}|Y(j\omega;c)|^2 &= Y(j\omega;c)Y(-j\omega;c)\\ &=(\overline{\varphi}_0(j\omega)+c\overline{\varphi}_1(j\omega)+c^2\overline{\varphi}_2(j\omega)+\cdots)(\overline{\varphi}_0(-j\omega)+c\overline{\varphi}_1(-j\omega)+c^2\overline{\varphi}_2(-j\omega)+\cdots)\\ &=\overline{\varphi}_0\cdot\overline{\varphi}_0^*+\sum_{\ell=1}^{\infty}\left(c^\ell\sum_{i=0}^{\ell}\overline{\varphi}_i\cdot\overline{\varphi}_{\ell-i}^*\right):=p_0+cp_1+c^2p_2+\cdots+c^{2\ell}p_{2\ell}+\cdots\end{aligned} \tag{4.39}$$

It is obvious that the spectral density of the output frequency response is still a polynomial function of the parameter c. Equation (4.39) can also be directly derived by following Process C that will be discussed later. Thus, the sensitivity of the magnitude of the output spectrum to the parameter c can be obtained as

$$\frac{\partial|Y(j\omega;c)|}{\partial c}=\frac{1}{2|Y(j\omega;c)|}\frac{\partial|Y(j\omega;c)|^2}{\partial c}=\frac{1}{2|Y(j\omega;c)|}\sum_{\ell=1}^{\infty}\left(\ell c^{\ell-1}\sum_{i=0}^{\ell}\overline{\varphi}_i\cdot\overline{\varphi}_{\ell-i}^*)\right) \tag{4.40a}$$

Given (4.38), (4.40a) can also be computed as

$$\begin{aligned}\frac{\partial|Y(j\omega;c)|}{\partial c}&=\frac{1}{2|Y(j\omega;c)|}\frac{\partial|Y(j\omega;c)|^2}{\partial c}=\frac{1}{2|Y(j\omega;c)|}\left(\frac{\partial Y(j\omega;c)}{\partial c}Y(-j\omega;c)+Y(j\omega;c)\frac{\partial Y(-j\omega;c)}{\partial c}\right)\\ &=\Re(\frac{\partial Y(j\omega;c)}{\partial c}\cdot\frac{Y(-j\omega;c)}{|Y(j\omega;c)|})\end{aligned} \tag{4.40b}$$

The sensitivity function for system output spectrum with respect to a nonlinear parameter provides a useful insight into the effect on system output performance of specific model parameters. This will be illustrated in Section 4.3. Another possible application of the sensitivity function is vibration suppression. In many engineering practice, the effect of vibrations should be considerably suppressed. From equations (4.40ab), it can be seen that if $Y(j\omega,c)$ represents the spectrum of a vibration, in order to suppress the vibration, it should be ensure that $\frac{\partial|Y(j\omega;c)|}{\partial c}<0$ for some c. Consider Equation (4.39), the following conclusion is obvious.

(a) $\frac{\partial|Y(j\omega;c)|}{\partial c}<0$ for some $c\Rightarrow\exists$some $n>0,\sum_{i=0}^{n}sign(c^{n-1})\overline{\varphi}_i\cdot\overline{\varphi}_{n-i}^*<0$

(b) p_1= $\mathrm{Re}(\overline{\varphi}_0(j\omega)\cdot\overline{\varphi}_1(-j\omega))<0\Rightarrow$ there exists ε >0 such that $\frac{\partial|Y(j\omega;c)|}{\partial c}<0$ for $0<c<\varepsilon$ or $-\varepsilon<c<0$. Where $sign(x)=\begin{cases}1 & x\geq 0\\ -1 & \text{else}\end{cases}$, Re(·) is to get the real part of (.). If a nonlinear parameter c satisfies p_1=$\mathrm{Re}(\overline{\varphi}_0(j\omega)\cdot\overline{\varphi}_1(-j\omega))<0$, then it can be utilized for the vibration suppression objective.

- **Evaluation of the radius of convergence for the output frequency response with respect to nonlinear parameters**

It is followed from (4.37ab) that the radius of convergence is given by

$$R=\lim_{\ell\to\infty}\left|\frac{\overline{\varphi}_{\ell-1}(j\omega)}{\overline{\varphi}_\ell(j\omega)}\right| \tag{4.41}$$

Obviously, if $|c|<R$, then the series is convergent. Define a Ratio Function

$$R(\ell;c)=\left|\frac{\overline{\varphi}_{\ell-1}(j\omega)_c}{\overline{\varphi}_{\ell}(j\omega)_c}\right| \tag{4.42}$$

which is a function of ℓ and also varies with different nonlinear parameters. It can be seen that, if

$$\frac{\Delta R(\ell;c_1)}{\Delta\ell}>\frac{\Delta R(\ell;c_2)}{\Delta\ell} \tag{4.43}$$

then the output spectrum has a larger radius of convergence with respect to c_1 than that with respect to c_2. Equation (4.42) and inequality (4.43) can be used as an evaluation of the effect on the convergence of the Volterra series expansion for the nonlinear system under study from a model parameter and the comparative advantage between different parameters. Note that divergence of the Volterra series expansion can sometimes imply the instability of the system under study or the nonexistence of a Volterra series expansion. Thus this analysis can provide some useful information for the design of system output frequency response in terms of different model parameters.

- **Optimization of the output frequency response in terms of nonlinear parameters**

Given a desired magnitude of the output frequency response Y^*, an optimal c^* in ∂S_c can be found such that

$$\min_{c\in\partial S_c}(|Y(j\omega;c)|-Y^*) \tag{4.44}$$

A systematic method for this purpose is yet to be developed, which will be discussed in the future study.

4.2.2 Determination of the OFRF based on its parametric characteristics

As mentioned before, an important step for output spectrum analysis based on the parametric characteristics is to obtain the parametric characteristic function of system output spectrum, which is referred to as the OFRF in Lang et al (2007). In this section, a general procedure for the determination of the OFRF for a given model (1.5) or (1.6) is proposed, and useful algorithms and techniques are provided.

4.2.2.1 Computation of the parametric characteristics of OFRF

This step is to derive $\psi=\bigoplus_{n=1}^{N}CE(H_n(\cdot))$ in (4.36).

- **Determination of the largest order N**

To derive the parametric characteristics of OFRF, the first task is to compute the largest order, *i.e.*, N, of the Volterra series expansion for the nonlinear system, which is basically determined by the significance of the truncation error in the Volterra series expansion of finite order. This can alternatively be done by evaluating the magnitude of the nth-order output frequency response $Y_n(j\omega)$. For example, the magnitude bound of $Y_n(j\omega)$ for the NARX model (1.6) can be evaluated by (Jing et al 2007)

$$|Y_n(j\omega)|\le\alpha_n\cdot b_n\cdot\hbar_n^T \tag{4.45}$$

where $\alpha_n, \hbar_n$ are complex valued functions, and b_n is a function vector of the system model parameters. For the detailed definitions for $\alpha_n, b_n, \hbar_n$ refer to Jing et al (2007). If the magnitude bound of a certain order of $Y_n(j\omega)$ is less than a predefined value (for instance 10^{-8}), then the largest order N is obtained. It should be noted that the magnitude bound is a function of the model nonlinear parameters, therefore, the largest ranges of interest for each nonlinear parameter should be considered in the evaluation of $|Y_n(j\omega)|$.

- **Determination of the parametric characteristics**

Once the largest order N is determined, the next step is to derive the parametric characteristics of GFRFs for the nonlinear system, *i.e.*, $CE(H_n(\cdot))$ from n=2 to N, which will be used in the computation of $\psi = \bigoplus_{n=1}^{N} CE(H_n(\cdot))$. Note that $CE(H_n(\cdot))$ is computed in terms of the parameter vectors $C_{p,q} = [c_{p,q}(0,\cdots,0), c_{p,q}(0,\cdots,1),\cdots, c_{p,q}(\underbrace{K,\cdots,K}_{p+q})]$ for some p,q in (3.17).

Basically, for some specific parameters to be analysed for a system, $CE(H_n(\cdot))$ can be recursively computed by Equation (3.17) with respect to these parameters of interest with other nonzero nonlinear parameters being 1. Alternatively, $CE(H_n(\cdot))$ can also be determined directly without recursive computations by using the results in Proposition 3.1. Based on Proposition 3.1, the parametric characteristic $CE(H_n(\cdot))$ can be obtained as follows, which is referred to as Process A: for $0 \le k \le n-2$,

(A1) Generate all the combinations (r_0, r_1, r_2..., r_k) satisfying $r_0 + \sum_{i=1}^{k} r_i = n+k$ and $2 \le r_i \le n-k$ with respect to a specific value of k;

(A2) Generate all the possible combinations (p_i,q_i) with respect to each r_i satisfying $p_i+q_i = r_i$, and note that when it is for r_0, $1 \le p_0 \le n-k$;

(A3) All the possible combinations can now be generated based on Step (A1) and (A2), then remove all the repetitive terms;

(A4) $CE(H_n(\cdot))$ is obtained in terms of the parameter vectors $C_{p,q}$ for some p,q, which can be stored for any future usage. For a specific nonlinear system, $CE(H_n(\cdot))$ can be obtained only by replacing the corresponding parameter vector $C_{p,q}$ of interest with respect to the specific nonlinear system, and the other parameters in $CE(H_n(\cdot))$ are set to be zero if it is zero or set to be 1 if it is not of interest;

(A5) Achieve the final result by manipulating $CE(H_n(\cdot))$ according to the operation rules of "$\oplus$" and "$\otimes$" (See Chapter 2), and removing the repetitive terms.

By this process, the parametric characteristic $CE(H_n(\cdot))$ can be obtained without recursive computations. For a summary, the parametric characteristic vector $\psi = \bigoplus_{n=1}^{N} CE(H_n(\cdot))$ can be computed by following the process below, which is referred to as Process B:

(B1) Determine the set of the nonlinear parameters of interest, denoted by S_C;

(B2) Determine the largest possible ranges for the nonlinear parameters of interest , denoted by ∂S_C ;

(B3) Determine the largest order N of the Volterra series expansion according to (4.45) and the discussions there.

(B4) Computation of $CE(H_n(\cdot))$ with respect to the parameters S_C of interest following Process A or Equation (3.17) from n=2 to N.

(B5) Combine the final parametric characteristic vector $\psi = \overset{N}{\underset{n=1}{\oplus}} CE(H_n(\cdot))$.

Therefore, based on Process A and Process B, the parametric characteristics of the output frequency response with respect to any specific model parameters of interest , which are the coefficients of the polynomial function (4.36), can be determined. Thus the structure of the polynomial (4.36) is explicitly determined at this stage. Note that, the parametric characteristic vector $CE(H_n(\cdot))$ for all the model nonlinear parameters in (3.13) can be obtained according to (3.17) or Process A, and if there are only some parameters of interest, the computation can be conducted by only replacing other nonzero parameters with 1 as mentioned above.

4.2.2.2 A numerical method

This step is mainly to determine $\Phi(j\omega) = [\phi_1(j\omega) \quad \phi_2(j\omega) \quad \cdots \quad \phi_N(j\omega)]$ in (4.36), then the OFRF in (4.36) is obtained consequently. Since the system model is supposed to be known, the parametric characteristic vector $\psi = \overset{N}{\underset{n=1}{\oplus}} CE(H_n(\cdot))$ is achieved, and note that $\Phi(j\omega)$ is invariant with respect to $\psi = \overset{N}{\underset{n=1}{\oplus}} CE(H_n(\cdot))$, thus $\Phi(j\omega)$ can be derived with respect to any a specific input by following a numerical method as follows, which is referred to as Process C:

(C1) Choose a series of different values of the nonlinear parameters of interest, which are properly distributed in ∂S_C, to form a series of vectors $\psi_1 \cdots \psi_{\rho(N)}$, where $\rho(N) = |\psi|$ denotes the dimension of vector ψ, such that

$$\Psi = [\psi_1{}^T \cdots \psi_{\rho(N)}{}^T]^T \text{ is non-singular} \tag{4.46}$$

(C2) Given a frequency ω where the output frequency response of the nonlinear system is to be analysed or designed. Excite the system using the same input under different values of the nonlinear parameters $\psi_1 \cdots \psi_{\rho(N)}$; collect the time domain output $y(t)$ for each case, and evaluate the output frequency response $Y(j\omega)_1 \cdots Y(j\omega)_{\rho(N)}$ at the frequency ω by FFT technique.

(C3) Step 2 yields

$$\Psi \cdot \Phi(j\omega)^T = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_{\rho(N)} \end{bmatrix} \cdot \begin{bmatrix} \varphi_1(j\omega) \\ \varphi_2(j\omega) \\ \vdots \\ \varphi_{\rho(N)}(j\omega) \end{bmatrix} = \begin{bmatrix} Y(j\omega)_1 \\ Y(j\omega)_2 \\ \vdots \\ Y(j\omega)_{\rho(N)} \end{bmatrix} =: YY(j\omega) \tag{4.47}$$

Hence,

$$\begin{aligned}\Phi(j\omega)^T &= [\phi_1(j\omega) \quad \phi_2(j\omega) \quad \cdots \quad \phi_N(j\omega)]^T \\ &= [\varphi_1(j\omega) \quad \varphi_2(j\omega) \quad \cdots \quad \varphi_{\rho(N)}(j\omega)]^T = \Psi^{-1} \cdot YY(j\omega)\end{aligned} \tag{4.48}$$

In Step C1, $\rho(N)$ different values of the parameter vector ψ in the parameter space ∂S_C, such that $\det(\Psi) \neq 0$ can be obtained by choosing a grid of parameter values of the nonlinear parameters of interest properly spanned in ∂S_C, or using a stochastic-based searching method or other optimization search methods such as GA to generate a non-singular matrix Ψ. In practices, it is not difficult to find such a matrix with a proper inverse, which will be illustrated in Section 4.3. In Step C2, given the largest order N of the system output spectrum, it can be seen that this algorithm needs $\rho(N)$ simulations to obtain $\rho(N)$ output frequency responses under different parameter values. Note from Step C1 that $\rho(N) = |\psi| = \left|\bigoplus_{n=1}^{N} CE(H_n(\cdot))\right|$, which is not only a function of the largest order N but also dependent on the number of parameters of interest . This implies the simulation burden will become heavier if the number of the parameters of interest and the largest order N are becoming larger. In Step C3, since $\det(\Psi) \neq 0$, the complex valued function vector $\Phi(j\omega)$ in (4.48) is unique, which implies the result in (4.48) can sufficiently approximate their real values if the truncation error incurred by the largest order N of the Volterra series is sufficiently small.

Therefore, by following Process C, the complex valued function vector $\Phi(j\omega)$ can accurately be obtained for the specific input function used in Step C2 and at the given frequency ω. Consequently, the OFRF (4.36) subject to the specific input function is now explicitly determined by following the method discussed above for the nonlinear system of interest. Although the function vector $\Phi(j\omega)$ is obtained by using the numerical method above and consequently the obtained OFRF is not an analytical function of the frequencies and the input, the achieved relationship between the output spectrum and model nonlinear parameters is analytical and explicit for the specific input function at the given frequency ω. Moreover, note that since $CE(H_n(\cdot))$ is known, and $\Phi(j\omega) = [\varphi_1(j\omega) \quad \varphi_2(j\omega) \quad \cdots \quad \varphi_N(j\omega)]$ is determined, then $Y_n(j\omega) = CE(H_n(\cdot)) \cdot \varphi_n(j\omega)^T$ is also determined, which represents the analytical function for the nth-order output frequency response of nonlinear systems.

It shall also be noted that, the proposed method above as demonstrated in this section enable the OFRF to be obtained directly in a concise polynomial form as (4.36) without the complex integration in the high-dimensional super-plane $\omega = \omega_1 + \cdots + \omega_n$ especially when the nonlinearity order n is high. By using the proposed method above, the OFRF can be determined up to a very high order with respect to any specific model parameters of interest and any specific input signal at any given frequency. The cost may lie in that the new method needs $\rho(N)$ simulations.

Once the OFRF is obtained, the analysis and design of nonlinear systems described by model (1.5) or (1.6) can be carried out in terms of the model parameters of interest which define system nonlinearities and may represent some structural and controllable factors of a practical engineering system. For example, the sensitivity of system output frequency response with respect to a nonlinear parameter can be studied based on the analytical expression (4.36). By using the link between the nonlinear terms of interest and the components of a practical engineering system and structure, the OFRF may provide a useful insight into the design of nonlinear components in the system to achieve a desired output performance. Therefore, the OFRF based analysis method provides a novel approach to the analysis and synthesis of a considerably wide class of nonlinear systems subject to any specific input signal in the frequency domain.

4.3 Simulations

To demonstrate the application of the new frequency domain analysis method proposed in this Chapter, a nonlinear spring-damping system is studied as shown in Figure 4.1. The system has two nonlinear passive components and one nonlinear active unit. The active unit is described by $F = c_1\dot{x}^2x + c_2x\dot{x}^2$, the output property of the spring satisfies $F = \hat{K}x + c_3x^3$, and the damper $F = B\dot{x} + c_4\dot{x}^3$. $u(t)$ is the external input force. The system dynamics can be described by

$$\hat{M}\ddot{x} = -\hat{K}x - B\dot{x} - c_1\dot{x}^2x - c_2\dot{x}x^2 - c_3x^3 - c_4\dot{x}^3 + u(t) \tag{4.49a}$$

Let the output be

$$y = \hat{K}x \tag{4.49b}$$

This may represent a vibration isolator system with nonlinear spring and damping characteristics. The task for this case study is to investigate how the nonlinear terms included both in passive and active unites affect the output and what the effect might be, and thus to provide a useful insight into the design of corresponding nonlinear parameters to achieve a desired output frequency response.

For clarity in discussion, let $\hat{M} = 240$, $\hat{K}$ =16000, and B=296, then (4.49ab) can be rewritten as

$$240\ddot{x} = -16000x - 296\dot{x} - c_1\dot{x}^2x - c_2\dot{x}x^2 - c_3x^3 - c_4\dot{x}^3 + u(t) \tag{4.50a}$$

$$y=16000x \tag{4.50b}$$

(4.50a) is a simple case of the NDE model (1.5) with $\hat{M}$ =3, $\hat{K}$ =2, $c_{10}(2) = 240$, $c_{10}(1) = 296$, $C_{10}(0) = 16000$, $c_{30}(111) = c_4$, $c_{30}(110) = c_1$, $c_{30}(100) = c_2$, $c_{30}(000) = c_3$, $c_{01}(0) = -1$, and all the other parameters are zero. Therefore, what is of interest for this study is to analyse the effect of the nonlinear terms with coefficients c_1, c_2, c_3 and c_4 on the system output frequency response. To achieve this objective, the procedure proposed in Section 4.2.2 are adopted to derive the OFRF of system (4.50), and the results in Section 4.1 will be used for the computation of the parametric characteristic of the OFRF with respect to the nonlinear parameters c_1, c_2, c_3 and c_4. Moreover, though the method proposed in this paper is suitable for a general input function u(t), for convenience in discussion, the input of system (4.50) is considered to be a sinusoidal function $u(t) = 100\sin(8.1t)$. To illustrate the new results more clearly, first only the effect of parameter c_2 is considered and it is assumed that c_1=c_3=c_4=0. Then complicated cases where the effect of more than one nonlinear parameters is involved will also be investigated.

4.3.1 Determination of the parametric characteristics of OFRF

Note that all the parameters of interest belong to C_{30}, and the other degrees of nonlinear parameters are all zero. Thus Corollary 4.3 and Proposition 4.3 can be utilised directly. Therefore,

$$CE(H_n(j\omega_1,\cdots,j\omega_n)) = c^{\frac{n-1}{2}} \cdot \delta\left(\frac{n-1}{2} - \left\lfloor\frac{n-1}{2}\right\rfloor\right) \cdot (1-\delta(3)pos(n)) = c^{\frac{n-1}{2}} \cdot \delta\left(\frac{n-1}{2} - \left\lfloor\frac{n-1}{2}\right\rfloor\right)$$

$$\psi = CE(Y(j\omega)) = \bigoplus_{i=0}^{\left\lfloor N-1/p+q-1\right\rfloor} CE(H_{(p+q-1)i+1}(\cdot)) = \begin{bmatrix}1 & c & c^2 & \cdots & c^{\left\lfloor N-1/p+q-1\right\rfloor}\end{bmatrix} \tag{4.51}$$
$$= \begin{bmatrix}1 & c & c^2 & \cdots & c^{\left\lfloor N-1/2\right\rfloor}\end{bmatrix}$$

where c=c_2. To derive the detailed form for ψ, the largest order N should be determined first according to Process B in Section 4.2.2. In order to have a larger range in which the parameters can vary, in this case let $c_2 \in (0,10^8)$. The magnitude bound of $Y_n(j\omega)$ can then be evaluated as mentioned in Process B. However, for paper limitation, the detailed computation is omitted in this case. It can be verified that N=23 is enough for use in this case. Therefore,

$$\psi = \begin{bmatrix}1 & c & c^2 & \cdots & c^{\left\lfloor 23-1/2\right\rfloor}\end{bmatrix} = [1, c_2, c_2^2, c_2^3, c_2^4, c_2^5, \ldots, c_2^{11}] \tag{4.52}$$

4.3.2 Determination of $\Phi(j\omega)$ for the OFRF

Following Process C, the matrix $\Psi = [\psi_1^T \cdots \psi_\rho^T]^T$ should be constructed first. In this case, for any 12 different values of c_2, the matrix Ψ is a Vandermonde matrix and thus non-singular. Note that in many cases, the parameters may be set to be some large values and cover a large range. This will make the element values in the matrix Ψ extraordinarily large. Then when the inverse of matrix Ψ is computed, there may be some computation error involved in Matlab. To overcome this problem, ψ can be written as

$$\psi = \bigoplus_{i=0}^{\left\lfloor N-1/p+q-1\right\rfloor} kCE(H_{(p+q-1)i+1}(\cdot))/k = \begin{bmatrix}1 & k(c/k) & k^2(c/k)^2 & \cdots & k^{\left\lfloor N-1/2\right\rfloor}(c/k)^{\left\lfloor N-1/2\right\rfloor}\end{bmatrix} \tag{4.53}$$

Then equation (4.36) can be written as

$$Y(j\omega;c) = \psi \cdot \Phi(j\omega)^T = \begin{bmatrix}1 & (c/k) & (c/k)^2 & \cdots & (c/k)^\ell\end{bmatrix}\begin{bmatrix}\varphi_1(j\omega) & k\varphi_2(j\omega) & \cdots & k^\ell\varphi_\ell(j\omega)\end{bmatrix}^T \tag{4.54}$$

where $\ell = \left\lfloor N - 1/2 \right\rfloor$. Moreover, the range for each parameter can be divided into several sub-range, and the final result is the combination of these results obtained for each sub-range. In this study, let k=10^5, then $\bar{c}_2 = c_2/k \in [0,1000]$. Choose $\bar{c}_2$ to be the following values to construct $\Psi = [\psi_1^T \cdots \psi_\rho^T]^T$, *i.e.*, 0.1,1,50,65,80,100,150,200,250, 300,350,400,450,500,550,600,650,700,750,800,850,900,950,980,1000. The output frequency response

$$YY(j\omega) = \begin{bmatrix}Y(j\omega)_1 & Y(j\omega)_2 & \cdots & Y(j\omega)_\rho\end{bmatrix} \tag{4.55}$$

of system (4.50) at $\omega = 8.1$ rad/s corresponding to different values of c_2 can be obtained through FFT of the time-domain output response. Then using (4.54), it can be derived from (4.48) that

$$\Phi(j\omega)^T = \begin{bmatrix}\varphi_1(j\omega) & k\varphi_2(j\omega) & \cdots & k^\ell\varphi_\ell(j\omega)\end{bmatrix}^T = (\Psi^T\Psi)^{-1}\Psi^T \cdot YY(j\omega) \tag{4.56}$$

Therefore, the output frequency response function of system (4.50) with respect to nonlinear parameter c_2 in the case of c_1=c_3=c_4=0 is obtained as

$$Y(j\omega;c_2) = (2.060893505718041e+002 - 2.402014548824790e+002i)$$
$$+ k^{-1}(-5.14248529981906 + 5.35676372314361i)\, c_2$$

$$
\begin{aligned}
&+ k^{-2} (0.08589533966805 - 0.08827649204263i)\, c_2^{2} \\
&+ k^{-3} (-8.068953639113292e\text{-}004 + 8.248154776018186e\text{-}004i)\, c_2^{3} \\
&+ k^{-4} (4.598423724418538e\text{-}006 - 4.686570228695798e\text{-}006i)\, c_2^{4} \\
&+ k^{-5} (-1.679591261850433e\text{-}008 + 1.708497491564935e\text{-}008i)\, c_2^{5} \\
&+ k^{-6} (4.056287337706451e\text{-}011 - 4.120496550333245e\text{-}011i)\, c_2^{6} \\
&+ k^{-7} (-6.544911009113156e\text{-}014 + 6.641760366680977e\text{-}014i)\, c_2^{7} \\
&+ k^{-8} (6.976300614229155e\text{-}017 - 7.073928662624432e\text{-}017i)\, c_2^{8} \\
&+ k^{-9} (-4.713366512185836e\text{-}020 + 4.776287453573993e\text{-}020i)\, c_2^{9} \\
&+ k^{-10} (1.827866445826756e\text{-}023 - 1.851299290299388e\text{-}023i)\, c_2^{10} \\
&+ k^{-11} (-3.098310700824303e\text{-}027 + 3.136656793561425e\text{-}027i)\, c_2^{11}
\end{aligned} \tag{4.57}
$$

Based on this function, (4.39) can be further computed as

$$
\begin{aligned}
|Y(j\omega;c)|^2 &= p_0 + cp_1 + c^2 p_2 + \cdots + c^{2\ell} p_{2\ell} + \cdots\cdots \\
&= (1.001695593467675e\text{+}005) + k^{-1} (-4.693027791051078e\text{+}003) c_2 \\
&\quad + k^{-2} (1.329525858242289e\text{+}002) c_2^{2} + k^{-3} (-2.55801250200731) c_2^{3} \\
&\quad + k^{-4}\, 0.03645314106899 c_2^{4} + k^{-5} (-3.968756773045435e\text{-}004) c_2^{5} \\
&\quad + k^{-6}\, 0.01517275811829 c_2^{6} + \ldots
\end{aligned} \tag{4.58}
$$

Note that this is an alternating series and it holds that $|p_i| > |p_{i+1}|$ and $|p_i| \to 0$. Hence the series may keep decreasing when c is going larger and within its radius of convergence. By following the similar method demonstrated above, the output frequency response functions of system (4.50) with respect to nonlinear parameters c_1, c_2, c_3 and c_4 of different cases can all be obtained, for instance $Y(j\omega;c_1)$, $Y(j\omega;c_3)$, and $Y(j\omega;c_4)$ (The other nonlinear parameters are zero if not appearing in the function). The results are shown in Figure 4.2-4.4.

Figure 4.2 shows that the variation of the magnitude of the output frequency response functions with respect to each nonlinear parameter. It can be seen that there is a good matching between the theoretical computation results and the simulation results to which they have been fitted, and there is also a good match between the theoretical computation results and the simulation tests (for parameter c_3) which are independent of the fitted simulation results. From both Figure 4.2 and Figure 4.3 it can also be seen that the system output frequency response is much more sensitive to the variation of the nonlinear parameters when they are small. Once the value of a nonlinear parameter is sufficient large, then the sensitivity will tend to be zero. From the comparison of these four nonlinear terms, it can be concluded that the system output frequency response is more sensitive to the variation of the nonlinear parameter c_4 when the values are small; however when the values of each nonlinear parameters are sufficient large, the system output spectrum is more sensitive to the nonlinear parameter c_2. From Figure 4.4 it can be seen that the convergence of the output frequency response functions are all very fast.. It is noted that the ratio functions of c_2 and c_3 go up much faster than that of c_1, especially c_2. This implies that the radius of convergence of the output spectrum corresponding to c_2 should be larger. Simulation tests verify that the system is still stable when $c_2=10^{17}$ where the magnitude of the output spectrum is 0.0216, while the system may tend to be unstable when c_1 tends to be larger than 10^8.

From the analysis above for the four nonlinear parameters of nonlinear degree 3, respectively, it can be seen that

- The computed system output spectrum has a larger radius of convergence with respect to c_2, c_3 and c_4.
- The system output spectrum is more sensitive to c_4 and less sensitive to c_3;
- If the output spectrum with respect to a nonlinear parameter is an alternating series satisfying $|p_i| > |p_{i+1}|$ and $|p_i| \to 0$, then the system output spectrum may be reduced to zero if additionally the radius of convergence for this parameter is sufficiently large.
- The magnitude of output spectrum decreases with the increase of the values of the nonlinear parameters. Thus an introduction of some simple nonlinear terms into a linear system may greatly improve the performance of output frequency response, and the stability of a nonlinear system is not necessarily deteriorated with increasing the values of nonlinear parameters; This also shows that a larger value of the parameter for a nonlinear term may not lead to a bad performance of a system.
- For system (4.50), the nonlinear parameters c_3 and c_4 can be designed to be large enough to achieve a sufficiently small transmitted force since they correspond to passive elements, and several nonlinear terms in the active part can work together to achieve a better performance.

To demonstrate further the advantage of the OFRF based analysis and to show more clearly the effect on the system output spectrum from several nonlinear parameters, the OFRF with respect to c_1, c_2 and c_3, i.e., $Y(j\omega; c_1, c_2, c_3)$ is derived. Let $c_1 \in [0,10^5], c_2 \in [0,6\cdot10^5], c_3 \in [0,5\cdot10^5]$, c_4=-500, and the largest order N of the output spectrum is determined to be 11, then the parametric characteristic can be obtained as ($c = [c_1, c_2, c_3]$)

$$\psi = \begin{bmatrix} 1 & c & c^2 & \cdots & c^{\lfloor 11-1/2 \rfloor} \end{bmatrix} = [1, c_1, c_2, c_3, c_1^2, c_1c_2, c_1c_3, c_2^2, c_2c_3, c_3^2, c_1^3, c_1^2c_2, c_1^2c_3, c_1c_2^2, c_1c_2c_3, c_2^3, c_2^2c_3, c_2c_3^2, c_3^3, c_1^4, c_1^3c_2, c_1^3c_3, c_1^2c_2^2, c_1^2c_2c_3, c_1^2c_3^2, c_1c_2^3, c_1c_2^2c_3, c_1c_2c_3^2, c_1c_3^3, c_2^4, c_2^3c_3, c_2^2c_3^2, c_2c_3^3, c_3^4, c_1^5, c_1^4c_2, c_1^4c_3, c_1^3c_2^2, c_1^3c_2c_3, c_1^3c_3^2, c_1^2c_2^3, c_1^2c_2^2c_3, c_1^2c_2c_3^2, c_1^2c_3^3, c_1c_2^4, c_1c_2^3c_3, c_1c_2^2c_3^2, c_1c_2c_3^3, c_1c_3^4, c_2^5, c_2^4c_3, c_2^3c_3^2, c_2^2c_3^3, c_2c_3^4, c_3^5] \quad (4.59)$$

In order to construct the non-singular matrix Ψ, the series of $\rho(N)$=55 different points $c = [c_1, c_2, c_3]$ in $\partial S_C = \{ c = [c_1, c_2, c_3] \mid c_1 \in [0,1], c_2 \in [0,6], c_3 \in [0,5] \}$ can be obtained by using a simple stochastic-based searching method. In simulations, it is noticed that is easy to find such a series of points that $\det(\Psi) \neq 0$. For example, a series of points $c = [c_1, c_2, c_3]$ are illustrated in Figure 4.5, and it can be obtained in this case that $\det(\Psi)$=0.08608811188201. It can be seen from simulations that it is easy to find a non-singular matrix Ψ with a proper inverse.

Then following the same procedure as demonstrated above, the OFRF $Y(j\omega; c_1, c_2, c_3)$ in this case can be obtained. The results are shown in Figure 4.6-4.7. It can be seen that

- By using the OFRF, the output spectrum can be plotted and analyzed under different combinations of the nonlinear parameters c_1, c_2 and c_3. This provides a straightforward understanding of the relationship between system output spectrum and model parameters which define nonlinearities.
- The OFRF varies with different values of c_1, c_2 and c_3. The effect on the output spectrum from any two nonlinear terms is not necessarily the simple combination

of the contributions from each term respectively. Thus the parameters can be analyzed in order to get the best output frequency response performance. The OFRF provides a useful basis for this kind of analysis and optimization.

From the discussions above, it can be concluded that the OFRF based analysis provides a novel, effective and useful approach to the analysis and design of nonlinear Volterra systems in the frequency domain.

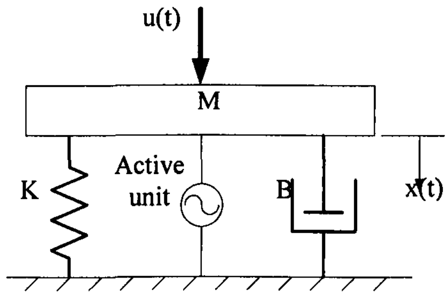

Fig. 4.1 A mechanical system

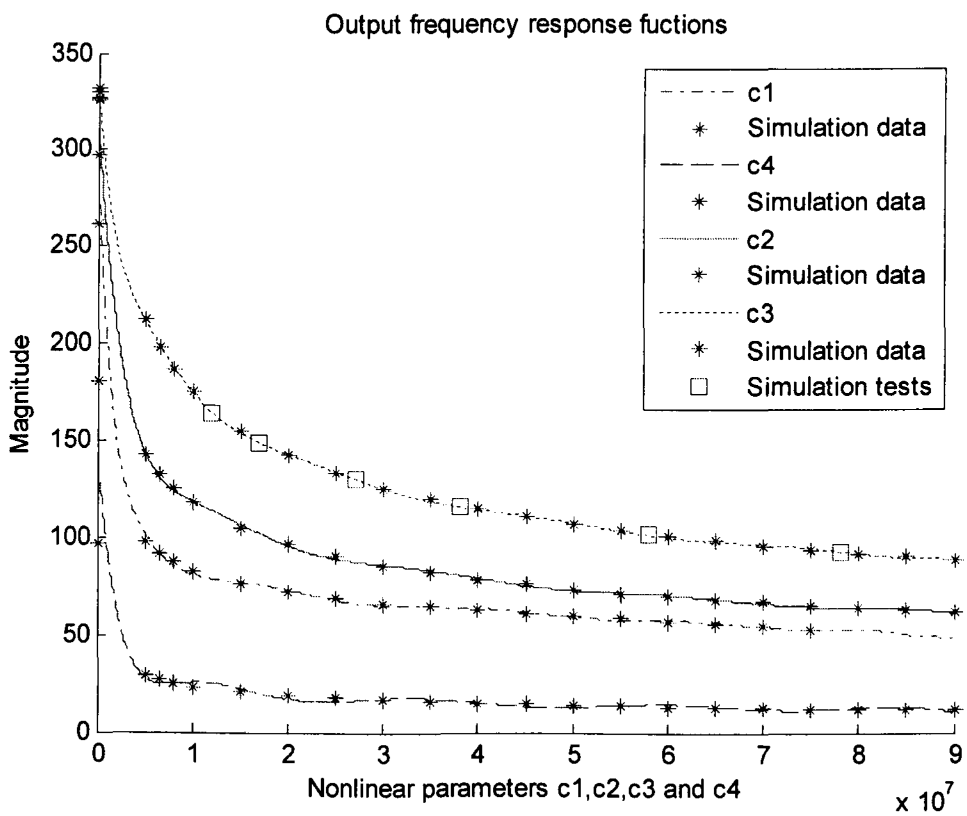

Fig. 4.2 Output frequency response functions with respect to c_1 to c_4 respectively

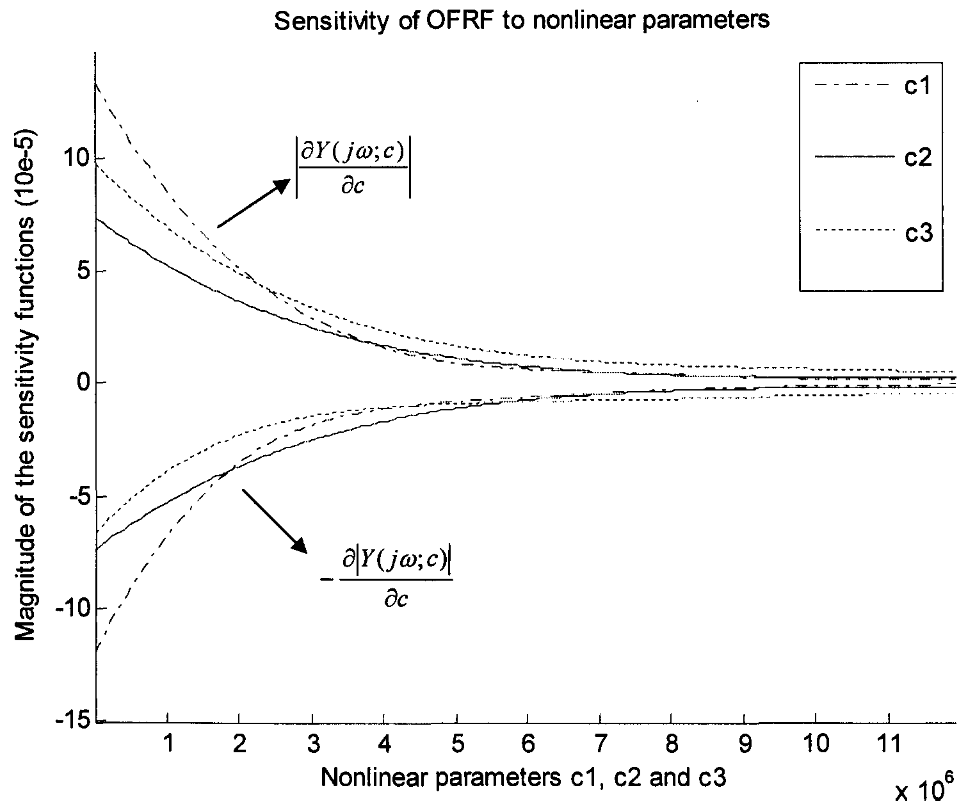

Fig. 4.3 Sensitivity function of the OFRFs with respect to c_1 to c_3 respectively

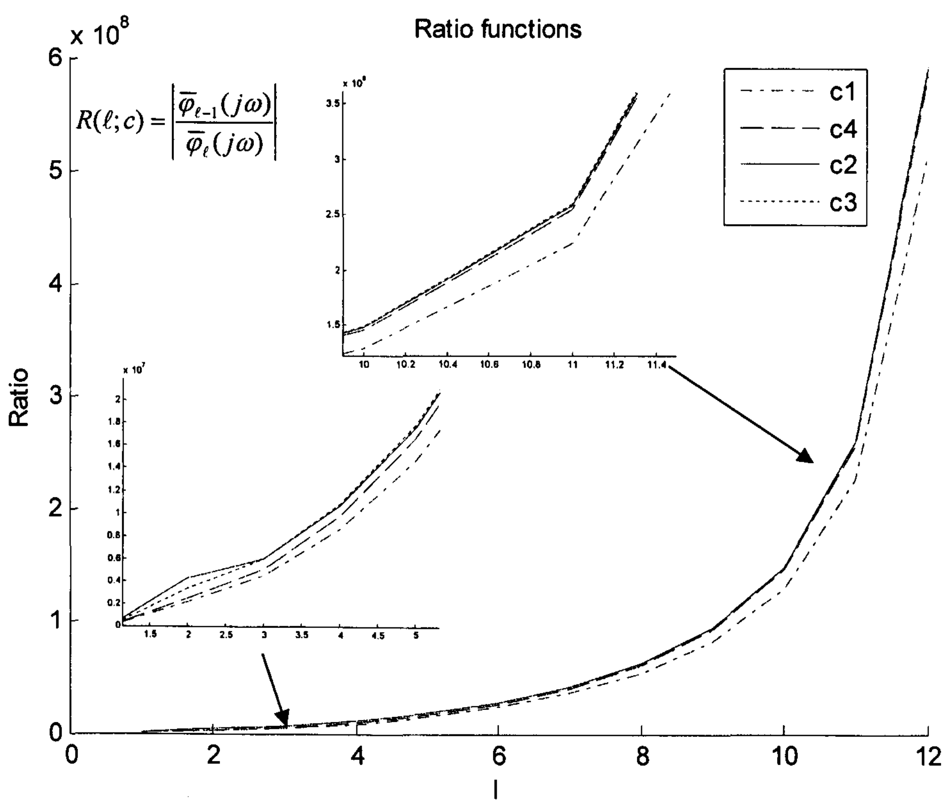

Fig. 4.4 Ratio functions with respect to c_1 to c_4 respectively

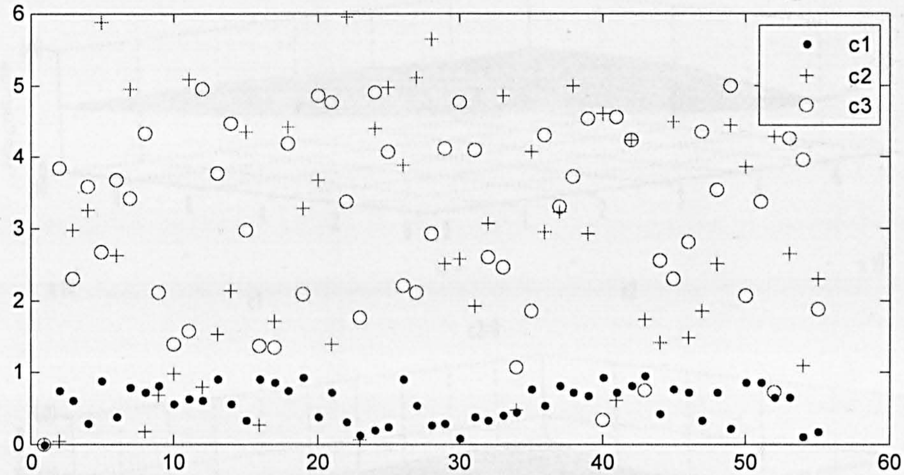

Fig. 4.5 A series of 55 points $c=[c_1,c_2,c_3]$ by random generation in {[0,1], [0,6], [0,5]} where the y-axis is the value of different parameters and the x-axis is the number of different point in the series

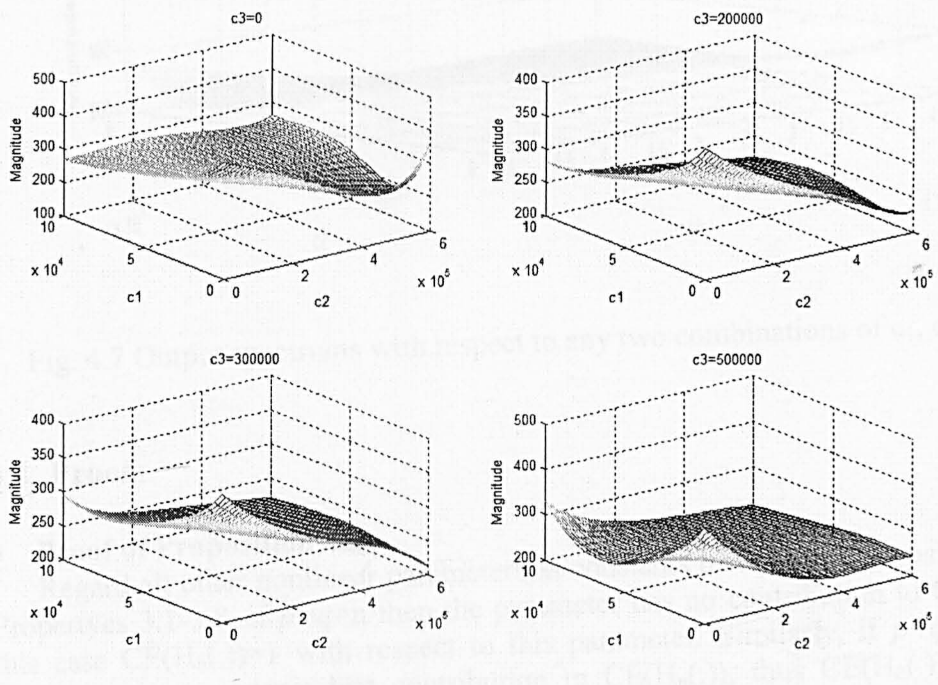

Fig. 4.6 Output spectrums with respect to c_1, c_2 and c_3

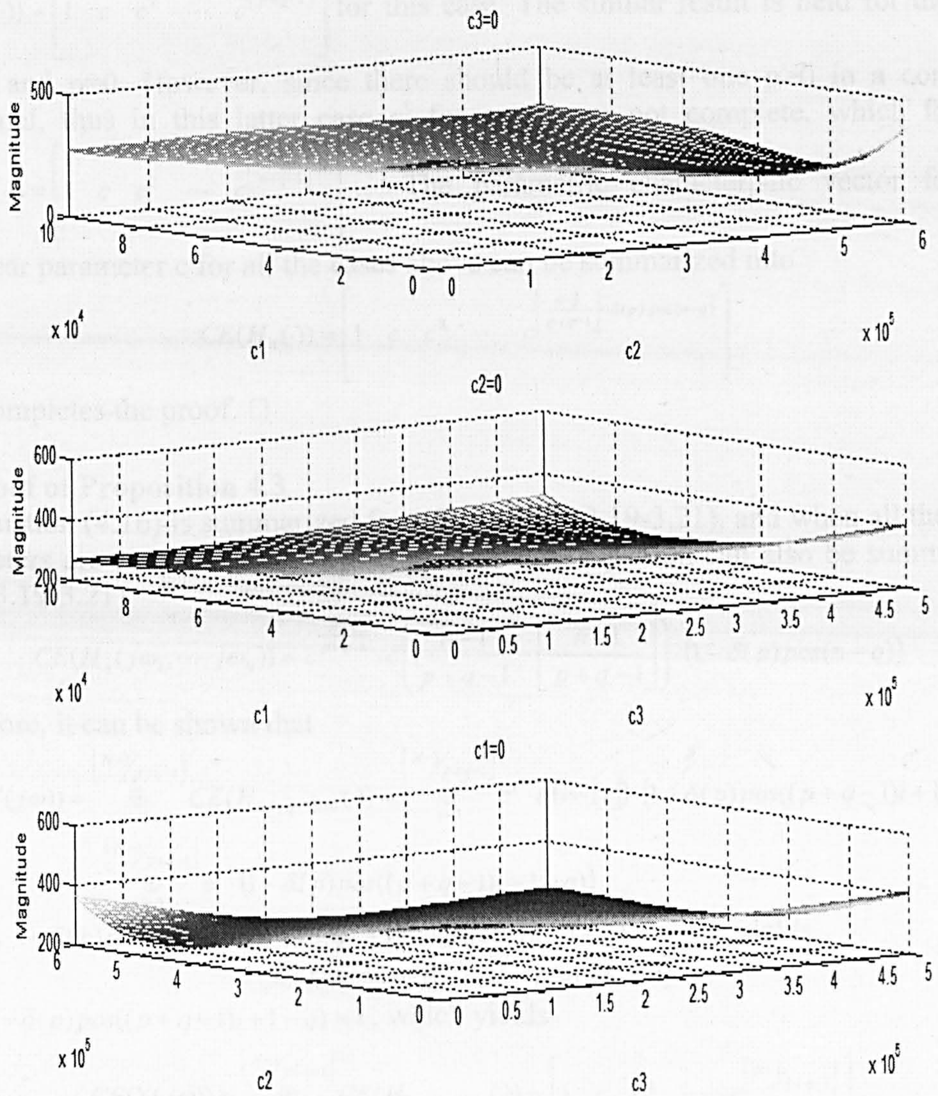

Fig. 4.7 Output spectrums with respect to any two combinations of c_1, c_2 and c_3

4.4 Proofs

- **Proof of Proposition 4.2**

Regard all other nonlinear parameters as constants or 1. From Proposition 3.1 and Propertyes 3.1-3.5, if $p+q$>n then the parameter has no contribution to CE(H_n(.)), in this case CE(H_n(.))=1 with respect to this parameter. Similarly, if $p+q$=n then the parameter is an independent contribution in CE(H_n(.)), thus CE(H_n(.))=[1 c] with respect to this parameter in this case. If p+q<n and p>0, then the independent contribution in CE(H_n(.)) for this parameter should be $c^{\left\lfloor \frac{n-1}{p+q-1} \right\rfloor}$, and the monomials c^x for $0 \le x < \left\lfloor \dfrac{n-1}{p+q-1} \right\rfloor$ are all not independent contributions in CE(H_n(.)). Hence

$CE(H_n(\cdot)) = \begin{bmatrix} 1 & c & c^2 & \cdots & c^{\left\lfloor \frac{n-1}{p+q-1} \right\rfloor} \end{bmatrix}$ for this case. The similar result is held for the case p+q<n and p=0. However, since there should be at least one p>0 in a complete monomial, thus in this latter case c^x for any x are not complete, which follows $CE(H_n(\cdot)) = \begin{bmatrix} 1 & c & c^2 & \cdots & c^{\left\lfloor \frac{n-1}{p+q-1} \right\rfloor -1} \end{bmatrix}$. The parametric characteristic vector for the nonlinear parameter c for all the cases above can be summarized into

$$CE(H_n(\cdot)) = \begin{bmatrix} 1 & c & c^2 & \cdots & c^{\left\lfloor \frac{n-1}{p+q-1} \right\rfloor - \delta(p)\cdot pos(n-q)} \end{bmatrix}$$

This completes the proof. □

- **Proof of Proposition 4.3**

Equation (4.16) is summarized from Equations (3.19-3.21), and when all the other parameters are zero except $c=c_{p,q}(.)$, the following equation can also be summarized from (3.19-3.21)

$$CE(H_n(j\omega_1,\cdots,j\omega_n)) = c^{\frac{n-1}{p+q-1}} \cdot \delta\left(\frac{n-1}{p+q-1} - \left\lfloor \frac{n-1}{p+q-1} \right\rfloor\right) \cdot (1-\delta(p)pos(n-q))$$

Therefore, it can be shown that

$$CE(Y(j\omega)) = \bigoplus_{i=0}^{\left\lfloor N-1/_{p+q-1} \right\rfloor} CE(H_{(p+q-1)i+1}(\cdot)) = \bigoplus_{i=0}^{\left\lfloor N-1/_{p+q-1} \right\rfloor} c^i \cdot \delta(i - \lfloor i \rfloor) \cdot (1-\delta(p)pos((p+q-1)i+1-q))$$

$$= \bigoplus_{i=0}^{\left\lfloor N-1/_{p+q-1} \right\rfloor} c^i \cdot (1-\delta(p)pos((p+q-1)i+1-q))$$

If p=0, $1-\delta(p)pos((p+q-1)i+1-q) = 1-pos((q-1)i+1-q)$, which yields,

$$CE(Y(j\omega)) = [1 \quad c\cdot(1-pos(q-N))]$$

else, $1-\delta(p)pos((p+q-1)i+1-q) = 1$, which yields

$$CE(Y(j\omega)) = \bigoplus_{i=0}^{\left\lfloor N-1/_{p+q-1} \right\rfloor} CE(H_{(p+q-1)i+1}(\cdot)) = \begin{bmatrix} 1 & c & c^2 & \cdots & c^{\left\lfloor N-1/_{p+q-1} \right\rfloor} \end{bmatrix}$$

This completes the proof. □

- **Proof of Lemma 4.2**

The lemma is summarized by the following observation. For clarity, let I=3.

$c^n = \underbrace{c \otimes c \cdots \otimes c}_{n}$ $\qquad$ $s(i)_n = \sum_{j=i}^{I} s(j)_{n-1}$ for i=1,2,3

n=1 $[c_1\ c_2\ c_3]$ $\qquad$ 1 1 1

n=2 $[c_1\ c_2\ c_3] \otimes [c_1\ c_2\ c_3]$ $\qquad$ 3 2 1

$=[c_1^2\ c_1c_2\ c_1c_3\ c_2^2\ c_2c_3\ c_3^2]$

n=3 $[c_1^2\ c_1c_2\ c_1c_3\ c_2^2\ c_2c_3\ c_3^2] \otimes [c_1\ c_2\ c_3]$ $\qquad$ 6 3 1

$=[c_1^3\ c_1^2c_2\ c_1^2c_3\ c_1c_2^2\ c_1c_2c_3\ c_1c_3^2\ c_2^3\ c_2^2c_3\ c_2c_3^2\ c_3^3]$

n=4 $[c_1^3\ c_1^2c_2\ c_1^2c_3\ c_1c_2^2\ c_1c_2c_3\ c_1c_3^2\ c_2^3\ c_2^2c_3\ c_2c_3^2\ c_3^3] \otimes [c_1\ c_2\ c_3]$ $\qquad$ 10 4 1

$=[c_1^4\ c_1^3c_2\ c_1^3c_3\ c_1^2c_2^2\ c_1^2c_2c_3\ c_1^2c_3^2\ c_1c_2^3\ c_1c_2^2c_3\ c_1c_2c_3^2\ c_1c_3^3$
$c_2^4\ c_2^3c_3\ c_2^2c_3^2\ c_2c_3^3\ c_3^4]$

n=5 $[c_1^4\ c_1^3c_2\ c_1^3c_3\ c_1^2c_2^2\ c_1^2c_2c_3\ c_1^2c_3^2\ c_1c_2^3\ c_1c_2^2c_3\ c_1c_2c_3^2\ c_1c_3^3$
$c_2^4\ c_2^3c_3\ c_2^2c_3^2\ c_2c_3^3\ c_3^4] \otimes [c_1\ c_2\ c_3]$ $\qquad$ 15 5 1

$=[c_1^5\ c_1^4c_2\ c_1^4c_3\ c_1^3c_2^2\ c_1^3c_2c_3\ c_1^3c_3^2\ c_1^2c_2^3\ c_1^2c_2^2c_3\ c_1^2c_2c_3^2\ c_1^2c_3^3$

$c_1c_2^4\ c_1c_2^3c_3\ c_1c_2^2c_3^2\ c_1c_2c_3^3\ c_1c_3^4\ c_2^5\ c_2^4c_3\ c_2^3c_3^2\ c_2^2c_3^3\ c_2c_3^4\ c_3^5]$

To complete the proof, the complete mathematical induction can be adopted. An outline for this proof is given here. Note that

$$c^n = \left[c^{n-1}\cdot c_1,\cdots,c^{n-1}[s(1)_n - s(i)_n + 1 : s(1)_n]\cdot c_i,\cdots,c^{n-1}[s(1)_n]\cdot c_I\right]$$

includes all the non-repetitive terms of form $c_1^{k_1}c_2^{k_2}\cdots c_I^{k_I}$ with $k_1+k_2+\cdots+k_I=n$ and $0\le k_1,k_2,\cdots,k_I\le n$. These terms can be separated into *I* parts, the *i*th part of which, i.e., $c^{n-1}[s(1)_n - s(i)_n + 1 : s(1)_n]\cdot c_i$, includes all the non-repetitive terms of degree *n* which are obtained by the parameter c_i timing the components of degree *n*-1 in c^{n-1} from $s(1)_n - s(i)_n + 1$ to $s(1)_n$. Assume that the lemma holds for step *n*. Then for the step *n*+1, the *i*th part of the components in c^{n+1} must be $c^n[s(1)_{n+1} - (s(i)_n + \cdots + s(I)_n) + 1 : s(1)_{n+1}]\cdot c_i$ which is $c^n[s(1)_{n+1} - s(i)_{n+1} + 1 : s(1)_{n+1}]\cdot c_i$. This completes the proof of Lemma 4.2. □

4.5 Summary

The parametric characteristic analysis is performed for the output spectrum of Volterra systems described by NDE models or NARX models in this Chapter and some fundamental results for the parametric characteristics of system output spectrum are established. Based on these results, the parametric characteristic based output spectrum analysis for nonlinear Volterra systems is proposed. This method provides a novel and effective approach to the analysis and design of nonlinear Volterra systems in the frequency domain by using the explicit relationship between the system output frequency response and model parameters. The OFRF is characterized by its parametric characteristic timing a complex valued frequency dependent function vector. Thus in stead of the direct analytical computation of the OFRF, the proposed method simplifies the computation of the OFRF by splitting the computation procedure into two parts ------ one is the computation of the parametric characteristics for the OFRF, which is analytical in the determination of the relationship between the output spectrum and model parameters, and simpler to be implemented, and the other is the determination of the complex valued frequency dependent function vectors, which are obtained by using the Least square method. Some fundamental results, techniques, and a general procedure for the determination of the OFRF for a given NDE or NARX model subject to any specific input signal are provided. Although the proposed method needs $\rho(N)$ simulation data for the numerical method of Process C, and the OFRF obtained by the proposed method is not analytical with respect to the input signal and frequency variants at present, the case study for a simple mechanical system shows that the OFRF analysis based on its parametric characteristic is a useful approach to the analysis and design of nonlinear Volterra systems in the frequency domain.

Chapter 5
MAPPING FROM PARAMETRIC CHARACTERISTICS TO THE GFRFS

Based on the parametric characteristic of the *n*th-order GFRF (Generalised Frequency Response Function) for nonlinear systems described by an NDE (nonlinear differential equation) model, a mapping function from the parametric characteristics to the GFRFs is established, by which the *n*th-order GFRF can directly be written into a more straightforward and meaningful form in terms of the first order GFRF, i.e., an *n*-degree polynomial function of the first order GFRF. The new expression has no recursive relationship between different order GFRFs, and demonstrates some new properties of the GFRFs which can explicitly unveil the linear and nonlinear factors included in the GFRFs, and reveal clearly the relationship between the *n*th-order GFRF and its parametric characteristic, and also the relationship between the *n*th-order GFRF and the first order GFRF. The new results provide a useful insight into the frequency domain analysis and design of nonlinear systems based on the GFRFs.

5.1 Introduction

As discussed in Chapter 1, frequency domain methods for nonlinear systems have been studied for many years (Taylor 1999, Solomou 2002, Pavlov 2007). The frequency domain theory for nonlinear Volterra systems was initiated by the concept of the GFRF (George 1959). Thereafter, many significant results relating to the estimation and computation of the GFRFs and analysis of output frequency response for practical nonlinear systems have been developed (Bendat 1990, Billings and Lang 1996, Chua and Ng 1979, Jing et al 2007).

To compute the GFRFs of nonlinear systems, Bedrosian and Rice (1971) introduced the "harmonic probing" method. By applying the probing method (Rugh 1981), algorithms to compute the GFRFs for nonlinear Volterra systems described by the NDE model and NARX model were derived, which enable the *n*th-order GFRF to be recursively obtained in terms of the coefficients of the governing NARX or NDE model (Peyton-Jones and Billings 1989, Billings and Peyton-Jones 1990, Chen and Billings 1989). Based on the GFRFs, frequency response characteristics of nonlinear systems can then be investigated (Peyton Jones and Billings 1990, Yue et al 2005). These results are important extensions of the well known frequency domain methods for linear systems such as transfer function or Bode diagram, and provide a method to the analysis of nonlinear systems in the frequency domain.

Although these progresses have been made and the GFRFs of nonlinear systems described by NARX models and NDE models can be determined effectively, it can be seen that the existing recursive algorithms for the computations of the GFRFs and system output spectrum can not explicitly and simply reveal the analytical relationship between system time domain model parameters and system frequency response functions in a clear and straightforward manner. Therefore, many problems remain unsolved, such as how the frequency response functions are influenced by the parameters of the underlying system, and the connection to complex non-linear behaviours, etc. In order to solve these problems, the parametric characteristics of the

GFRFs were studied in Chapter 2 and Chapter 3, which effectively build up a mapping from the GFRF to its parametric characteristic and provide an explicit expression for the analytical relationship between the GFRFs and system time-domain model parameters. The significance of the parametric characteristic analysis of the nth-order GFRF is that it can clearly reveal what model parameters contribute to and how these parameters affect system frequency response functions including the GFRFs and output frequency response function. This provides an effective approach to the analysis of the frequency domain characteristics of nonlinear systems in terms of system time domain model parameters.

The study in this chapter is based on the results in Chapter 3. It is shown in Chapter 3 and Chapter 4 that the nth-order GFRF and output spectrum of a nonlinear Volterra system can both be written as an explicit and straightforward polynomial function in terms of nonlinear model parameters, and this polynomial function is characterized by its parametric characteristic with its coefficients being complex valued functions of frequencies and dependent on the system linear characteristics and input (for output spectrum). Note that, the parametric characteristics can be analytically determined by the results in Chapter 3. The focus in this study is to analytically determine the complex valued functions related to the parametric characteristics. An inverse mapping function from the parametric characteristics of the GFRFs to the GFRFs is studied. By using this new mapping function, the nth-order GFRF can directly be recovered from its parametric characteristic as an n-degree polynomial function of the first order GFRF, revealing an explicit analytical relationship between the higher order GFRFs and the system linear frequency response function. Compared with the existing recursive algorithm for the computation of the GFRFs, the new mapping function enables the nth-order GFRF to be explicitly expressed in a more straightforward and meaningful way. Note from previous results that the higher order GFRFs are recursively dependent on the lower order GFRFs. This recursive relationship often complicates the qualitative analysis and understanding of system frequency characteristics. The new results can effectively overcome this problem, and unveil the system's linear and nonlinear factors included in the nth-order GFRF more clearly. This provides a useful insight into the frequency domain analysis and design of nonlinear systems based on the GFRFs, and can be regarded as an important extension of the parametric characteristic theory established in previous chapters. Several examples are given to illustrate these results.

5.1.1 Some notations for this chapter

Some notations are listed here especially for readers' convenience in understanding of the discussions in this Chapter, although some of these notations have already appeared in previous chapters and will also be used in the following chapters.

$c_{p,q}(k_1,\cdots,k_{p+q})$	A model parameter in the NDE model, k_i is the order of the derivative, p represents the order of the involved output nonlinearity, q is the order of the involved input nonlinearity, and $p+q$ is the nonlinear degree of the parameter.
$H_n(j\omega_1,\cdots,j\omega_n)$	The nth-order GFRF

$C_{p,q}=[c_{p,q}(0,\cdots,0),c_{p,q}(0,\cdots,1),\cdots,c_{p,q}(\underbrace{K,\cdots,K}_{p+q=m})]$ A parameter vector consisting of all the nonlinear parameters of the form $c_{p,q}(k_1,\cdots,k_{p+q})$

$CE(.)$ The coefficient extraction operator (Chapter 2)

$CE(H_n(j\omega_1,\cdots,j\omega_n))$ The parametric characteristics of the nth-order GFRF

$f_n(j\omega_1,\cdots,j\omega_n)$ The correlative function of $CE(H_n(j\omega_1,\cdots,j\omega_n))$

$\otimes$ The reduced Kronecker product defined in the CE operator

$\oplus$ The reduced vectorized summation defined in the CE operator

$c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ A monomial consisting of nonlinear parameters

$s_{x_1}s_{x_2}\cdots s_{x_p}$ A p-partition of a monomial $c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$

s_{x_i} A monomial of x_i parameters of $\{c_{p_0,q_0}(\cdot),\cdots,c_{p_k,q_k}(\cdot)\}$ of the involved monomial, $0\le x_i\le k$, and $s_0=1$

$\varphi_n:S_C(n)\to S_f(n)$ A new mapping function from the parametric characteristics to the correlative functions, $S_C(n)$ is the set of all the monomials in the parametric characteristics and $S_f(n)$ is the set of all the involved correlative functions in the nth order GFRF.

$n(s_x(\bar{s}))$ The order of the GFRF where the monomial $s_x(\bar{s})$ is generated

$\bar{\lambda}_n(\omega_1,\cdots,\omega_n)$ The maximum eigenvalue of the frequency characteristic matrix Θ_n of the nth-order GFRF

5.2 The *n*th-order GFRF and its parametric characteristic

In this chapter, consider nonlinear Volterra systems described by the NDE model in (1.5), similar results can be extended to the NARX model (1.6). For convenience, some basic results are restated in this section as follows.

Using the definitions in (3.10), i.e.,

$$H_{0,0}(\cdot)=1,\ H_{n,0}(\cdot)=0 \text{ for } n>0,\ H_{n,p}(\cdot)=0 \text{ for } n<p, \text{ and } \prod_{i=1}^{q}(\cdot)=\begin{cases}1 & q=0,p>1\\ 0 & q=0,p\le 1\end{cases} \tag{5.1}$$

The nth-order GFRF for (1.5) can be written as (3.11), i.e.,

$$H_n(j\omega_1,\cdots,j\omega_n)=\frac{1}{L_n(j\sum_{i=1}^{n}\omega_i)}\sum_{q=0}^{n}\sum_{p=0}^{n-q}\sum_{k_1,k_{p+q}=0}^{K}c_{p,q}(k_1,\cdots,k_{p+q})(\prod_{i=1}^{q}(j\omega_{n-q+i})^{k_{p+i}})H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}) \tag{5.2}$$

The parametric characteristic of the *n*th-order GFRF can be simply computed as (See Corollary 3.1 for details)

$$CE(H_n(j\omega_1,\cdots,j\omega_n))=C_{0,n}\oplus\left(\mathop{\oplus}_{q=1}^{n-1}\mathop{\oplus}_{p=1}^{n-q}C_{p,q}\otimes CE(H_{n-q-p+1}(\cdot))\right)\oplus\left(\mathop{\oplus}_{p=2}^{n}C_{p,0}\otimes CE(H_{n-p+1}(\cdot))\right) \tag{5.3}$$

Moreover, $CE(H_n(j\omega_1,\cdots,j\omega_n))$ can also be determined by following the results in Proposition 3.1, which allows the direct determination of the parameter characteristic vector of the *n*th-order GFRF without recursive computations and provides a sufficient and necessary condition for which nonlinear parameters and how these parameters are included in $CE(H_n(j\omega_1,\cdots,j\omega_n))$.

Based on the parametric characteristic analysis in Chapter 2 and Chapter 3, the *n*th-order GFRF can be expressed as

$$H_n(j\omega_1,\cdots,j\omega_n) = CE(H_n(j\omega_1,\cdots,j\omega_n))\cdot f_n(j\omega_1,\cdots,j\omega_n) \tag{5.4}$$

where $f_n(j\omega_1,\cdots,j\omega_n)$ is a complex valued function vector with an appropriate dimension, which is referred to as the correlative function of the parametric characteristic $CE(H_n(j\omega_1,\cdots,j\omega_n))$ in this study.

Equation (5.4) provides an explicit expression for the analytical relationship between the GFRFs and the system time-domain model parameters. Based on these results, system nonlinear characteristics can be studied in the frequency domain from novel perspectives including frequency characteristics of system output frequency response, nonlinear effect from specific nonlinear parameters, and parametric sensitivity analysis etc as demonstrated in the previous chapters. In this chapter , an algorithm is provided to explicitly determine the correlative function $f_n(j\omega_1,\cdots,j\omega_n)$ in (5.4) directly in terms of the first order GFRF $H_1(j\omega_1)$ based on the parametric characteristic vector $CE(H_n(j\omega_1,\cdots,j\omega_n))$. To achieve this objective, a mapping from $CE(H_n(j\omega_1,\cdots,j\omega_n))$ to $H_n(j\omega_1,\cdots,j\omega_n)$ is established such that the *n*th-order GFRF can directly be written into the parametric characteristic function (5.4) in an analytical form by using this mapping function, and some new properties of the GFRFs are developed. These results are an extension of the previous established parametric characteristic theory and allow higher order GFRFs and, consequently, the OFRF to be analytically expressed as a functional of the system linear FRF (i.e., the first order GFRF). These provide a new advance for the frequency domain analysis of nonlinear Volterra systems.

5.3 Mapping from the parametric characteristic to the *n*th-order GFRF

The parametric characteristic vector $CE(H_n(j\omega_1,\cdots,j\omega_n))$ of the *n*th-order GFRF can be recursively determined by equation (5.3), which has elements of the form $C_{p,q}\otimes C_{p_1,q_1}\otimes C_{p_2,q_2}\otimes\cdots\otimes C_{p_k,q_k}$ (n-2 $\geq k \geq 0$), and each element of which has a corresponding complex valued correlative function in vector $f_n(j\omega_1,\cdots,j\omega_n)$. For example, $c_{0,n}(k_1,\cdots,k_n)$ corresponds to the complex valued function $(j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n}$ in the *n*th-order GFRF.

From Proposition 3.1, an element in $CE(H_n(j\omega_1,\cdots,j\omega_n))$ is either a single parameter coming from pure input nonlinearity such as $c_{0n}(.)$, or a nonlinear parameter monomial function of the form $C_{p,q}\otimes C_{p_1,q_1}\otimes C_{p_2,q_2}\otimes\cdots\otimes C_{p_k,q_k}$ satisfying (3.15), and the first parameter of $C_{p,q}\otimes C_{p_1,q_1}\otimes C_{p_2,q_2}\otimes\cdots\otimes C_{p_k,q_k}$ must come from pure output nonlinearity or input-output cross nonlinearity, *i.e.*, $c_{pq}(.)$ with $p\geq 1$ and $p+q>1$. For this reason, the following definition is given.

Definition 5.1. A parameter monomial of the form $C_{p,q}\otimes C_{p_1,q_1}\otimes C_{p_2,q_2}\otimes\cdots\otimes C_{p_k,q_k}$ with $k\geq 0$ and $p+q>1$ is said to be effective or an effective combination of the involved nonlinear parameters for $CE(H_n(j\omega_1,\cdots,j\omega_n))$ if $p+q=n(>1)$ for $k=0$, or (3.15) is satisfied for $k>0$. □

From Definition 5.1, it is obvious that all the monomials in $CE(H_n(j\omega_1,\cdots,j\omega_n))$ are effective combinations. The following lemma shows further that what an effective monomial should be in certain order GFRF and how it is generated in this order GFRF.

Lemma 5.1. For a monomial $c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ with k>0, the following statements hold:

(1) it is effective for the Z^{th}-order GFRF if and only if there is at least one parameter $c_{p_i,q_i}(.)$ with p_i>0, where $Z=\sum_{i=0}^{k}(p_i+q_i)-k$.

(2) if there are l different parameters with p_i>0, then there are l different cases in which this monomial is produced by the recursive computation of the Z^{th}-order GFRF.

Proof. (1) This is directly from Definition 5.1. Z can be computed according to Lemma 1, *i.e.*, $p_0+q_0+\sum_{i=1}^{k}(p_i+q_i)=Z+k$, which yields $Z=\sum_{i=0}^{k}(p_i+q_i)-k$. (2) From the second and third terms in the recursive algorithm of Equation (3.8), *i.e.*,

$$\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K} c_{p,q}(k_1,\cdots,k_{p+q})(\prod_{i=1}^{q}(j\omega_{n-q+i})^{k_{p+i}})H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})$$
$$+\sum_{p=2}^{n}\sum_{k_1,k_p=0}^{K}c_{p,0}(k_1,\cdots,k_p)H_{n,p}(j\omega_1,\cdots,j\omega_n) \tag{5.5}$$

it can be seen that all the nonlinear parameters with p>0 and $p+q\le n$ are involved in the nth-order GFRF, and each of these parameters must correspond to the initial parameter in an effective monomial of $CE(H_n(j\omega_1,\cdots,j\omega_n))$. Hence, if there are l different parameters with p_i>0 in the monomial $c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$, then there will be l different cases in which this monomial is produced in the Zth order GFRF. This completes the proof. □

Definition 5.2. A (p,q)-partition of $H_n(j\omega_1,\cdots,j\omega_n)$ is a combination $H_{r_1}(w_{r_1})H_{r_2}(w_{r_2})\cdots H_{r_p}(w_{r_p})$ satisfying $\sum_{i=1}^{p}r_i=n-q$, where $1\le r_i\le n-p-q+1$, and w_{r_i} is a set consisting of r_i different frequency variables such that $\bigcup_{i=1}^{p}w_{r_i}=\{\omega_1,\omega_2,\cdots,\omega_n\}$ and $w_{r_i}\cap w_{r_j}=\phi$ for $i\neq j$. □

For example, $H_1(\omega_1)H_1(\omega_2)H_3(\omega_3\cdots\omega_5)$ and $H_1(\omega_1)H_2(\omega_2,\omega_3)H_2(\omega_4,\omega_5)$ are two (3,0)-partitions of $H_5(j\omega_1,\cdots,j\omega_5)$.

Definition 5.3. A p-partition of an effective monomial $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ is a combination $s_{x_1}s_{x_2}\cdots s_{x_p}$, where s_{x_i} is a monomial of x_i parameters in $\{c_{p_1,q_1}(\cdot),\cdots,c_{p_k,q_k}(\cdot)\}$, $0\le x_i\le k$, s_0=1, and each non-unitary s_{x_i} is an effective monomial satisfying $\sum_{i=1}^{p}x_i=k$ and $s_{x_1}s_{x_2}\cdots s_{x_p}=c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$. □

The sub-monomial s_{x_i} in a p-partition of an effective monomial $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ is denoted by $s_{x_i}(c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot))$. Suppose that a p-partition for 1 is still 1, i.e., $\underbrace{1\cdot 1\cdots 1}_{p}=1$.

Obviously $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot) = s_{x_1}s_{x_2}\cdots s_{x_p}(c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)) = s_k(c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot))$. For example, $s_1(c_{1,1}(\cdot))s_2(c_{2,1}(\cdot)c_{3,0}(\cdot))$ and $s_2(c_{1,1}(\cdot)c_{2,1}(\cdot))s_1(c_{3,0}(\cdot))$ are two 2-partitions of $c_{1,1}(\cdot)c_{2,1}(\cdot)c_{3,0}(\cdot)$. Moreover, note that when s_0 appear in a p-partition of a monomial, it means that there is a $H_1(.)$ which appears in the corresponding (p,q)-partition for $H_n(.)$.

For an effective monomial $c_{p,q}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ in $CE(H_n(j\omega_1,\cdots,j\omega_n))$, without speciality, suppose the first parameter $c_{p,q}(\cdot)$ is directly generated in the recursive computation of $H_n(j\omega_1,\cdots,j\omega_n)$, then the other parameters must be generated from the lower order GFRFs that are involved in the recursive computation of $H_n(j\omega_1,\cdots,j\omega_n)$. From Equations (3.1-3.5) it can be seen that each parameter in a monomial corresponds to a certain order GFRF from which it is generated. The following lemma shows how a monomial is generated in $H_n(j\omega_1,\cdots,j\omega_n)$ by using the new concepts defined above. This provides an important insight into the mapping from a monomial to its correlative function.

Lemma 5.2. If a monomial $c_{p,q}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ is effective, and $c_{p,q}(\cdot)$ is the initial parameter directly generated in the xth-order GFRF and p>0, then

(1) $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ comes from (p,q)-partitions of the xth-order GFRF, where $x=p+q+\sum_{i=1}^{k}(p_i+q_i)-k$;

(2) if additionally s_0 is supposed to be generated from $H_1(.)$, then each p-partition of $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ corresponds to a (p,q)-partition of the xth-order GFRF, and each (p,q)-partition of the xth-order GFRF produces at least one p-partition for $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$;

(3) the correlative function of $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ is the summation of the correlative functions from all the (p,q)-partitions of the xth-order GFRF which produces $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$, and therefore is the summation of the correlative functions corresponding to all the p-partition of $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$.

Proof. See Section 5.5 for the proof. □

Remark 5.1. From Lemma 5.2, it can be seen that all the (p,q)-partitions of the xth-order GFRF which produce $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ are all the (p,q)-partitions corresponding to all the p-partitions for $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$. Therefore, to obtain all the (p,q)-partitions of interest, all the p-partitions for $c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ is needed to be determined. □

Based on the results above, in order to determine the mapping between a parameter monomial $c_{p,q}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ and its correlative function in $f_n(j\omega_1,\cdots,j\omega_n)$, the following operator is defined.

Definition 5.4. Let $S_C(n)$ be a set composed of all the elements in $CE(H_n(j\omega_1,\cdots,j\omega_n))$, and let $S_f(n)$ be a set of the complex-valued functions of the frequency variables $j\omega_1,\cdots,j\omega_n$. Then define a mapping

$$\varphi_n : S_C(n) \to S_f(n) \tag{5.6a}$$

such that in $\omega_1,\cdots,\omega_n$

$$H_n^{sym}(j\omega_1,\cdots,j\omega_n) = \frac{1}{n!} \sum_{\substack{\text{all the permutations}\\ \text{of } \{1,2,\ldots,n\}}} CE(H_n(j\omega_1,\cdots,j\omega_n)) \cdot \varphi_n(CE(H_n(j\omega_1,\cdots,j\omega_n))) \tag{5.6b}$$

□

That is, by using the mapping function above, an asymmetric GFRF can be obtained as

$$H_n(j\omega_1,\cdots,j\omega_n) = CE(H_n(j\omega_1,\cdots,j\omega_n)) \cdot \varphi_n(CE(H_n(j\omega_1,\cdots,j\omega_n)))$$

The existence of this mapping function is obvious. For example, $\varphi_n(c_{0,n}(k_1,\cdots,k_n)) = (j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n}$. The task is to determine the complex valued correlative function $\varphi_n(c_{p,q}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot))$ for any nonlinear parameter monomial $c_{p,q}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$ $(0 \le k \le n\text{-}2)$ in $CE(H_n(j\omega_1,\cdots,j\omega_n))$.

Based on Lemma 5.1-5.2, the following result can be obtained.

Proposition 5.1. For an effective nonlinear parameter monomial $c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$, let $\bar{s} = c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$, $n(s_x(\bar{s})) = \sum_{i=1}^{x}(p_i+q_i) - x + 1$, where x is the number of the parameters in s_x, $\sum_{i=1}^{x}(p_i+q_i)$ is the summation of the subscripts of all the parameters in s_x, $\sum_{i=1}^{x}(.) = 0$ if x<1 and $n(1)$=1. Then for $0 \le k \le n(\bar{s})$ -2

$$\begin{aligned}
&\varphi_{n(\bar{s})}(c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}) \\
&= \sum_{\substack{\text{all the 2-partitions}\\ \text{for } \bar{s} \text{ satisfying}\\ s_1(\bar{s})=c_{p,q}(\cdot) \text{ and } p>0}} \Big\{ f_1(c_{p,q}(\cdot),n(\bar{s});\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}) \cdot \sum_{\substack{\text{all the p-partitions}\\ \text{for } \bar{s}/c_{pq}(\cdot)}} \ \sum_{\substack{\text{all the different}\\ \text{permutations}\\ \text{of } \{s_{x_1},\cdots,s_{x_p}\}}} \Big[f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(\bar{s}/c_{p,q}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)}) \\
&\qquad \cdot \prod_{i=1}^{p} \varphi_{n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot)))}(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot));\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot))))}) \Big] \Big\}
\end{aligned} \tag{5.7a}$$

or simplified as

$$\begin{aligned}
&\varphi_{n(\bar{s})}(c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}) \\
&= \sum_{\substack{\text{all the 2-partitions}\\ \text{for } \bar{s} \text{ satisfying}\\ s_1(\bar{s})=c_{p,q}(\cdot) \text{ and } p>0}} \Big\{ f_1(c_{p,q}(\cdot),n(\bar{s});\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}) \cdot \sum_{\substack{\text{all the p-partitions}\\ \text{for } \bar{s}/c_{p,q}(\cdot)}} \Big[f_{2b}(s_{x_1}\cdots s_{x_p}(\bar{s}/c_{p,q}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)}) \\
&\qquad \cdot \prod_{i=1}^{p} \varphi_{n(s_{x_i}(\bar{s}/c_{p,q}(\cdot)))}(s_{x_i}(\bar{s}/c_{p,q}(\cdot));\omega_{l(X(i)+1)}\cdots\omega_{l(X(i)+n(s_{x_i}(\bar{s}/c_{p,q}(\cdot))))}) \Big] \Big\}
\end{aligned} \tag{5.7b}$$

the terminating condition is k=0 and $\varphi_1(1;\omega_i) = H_1(j\omega_i)$, where,

$$\bar{X}(i) = \sum_{j=1}^{i-1} n(s_{\bar{x}_j}(\bar{s}/c_{pq}(\cdot))) \text{ or } X(i) = \sum_{j=1}^{i-1} n(s_{x_j}(\bar{s}/c_{pq}(\cdot))) \tag{5.8a}$$

$$f_1(c_{p,q}(\cdot),n(\bar{s});\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}) = \left(\prod_{i=1}^{q}(j\omega_{l(n(\bar{s})-q+i)})^{k_{p+i}}\right) \Big/ L_{n(\bar{s})}\left(j\sum_{i=1}^{n(\bar{s})}\omega_{l(i)}\right) \tag{5.8b}$$

$$f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(\bar{s}/c_{p,q}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)}) = \prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot))))})^{k_i} \tag{5.8c}$$

$$f_{2b}(s_{x_1}\cdots s_{x_p}(\bar{s}/c_{p,q}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)})=\frac{n_x^*}{n_k^*}\sum_{\substack{\text{all the different}\\ \text{permutations}\\ \text{of } \{k_1,\cdots,k_p\}}}\prod_{i=1}^{p}(j\omega_{l(X(i)+1)}+\cdots+j\omega_{l(X(i)+n(s_{x_i}(\bar{s}/c_{pq}(\cdot))))})^{k_i}$$

(5.8d)

Moreover, $\{s_{\bar{x}_1},\cdots s_{\bar{x}_p}\}$ is a permutation of $\{s_{x_1},\cdots s_{x_p}\}$, $\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}$ represents the frequency variables involved in the corresponding functions, $l(i)$ for i=1…$n(\bar{s})$ is a positive integer representing the index of the frequency variables, $n_k^*=\frac{p!}{n_1!n_2!\cdots n_c!}$, $n_1+\ldots+n_c=p$, c is the number of distinct differentials k_i appearing in the combination, n_i is the number of repetitions of the ith distinct differential k_i, and a similar definition holds for n_x^*.□

Proof. See Section 5.5 for the proof. □

Remark 5.2. Equations (5.7ab) are recursive. The terminating condition is k=0, which is also included in (5.7ab). For k=0, it can be derived from (5.7b) that

$$\varphi_{n(\bar{s})}(c_{p,q}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})=\varphi_{p+q}(c_{p,q}(\cdot);\omega_{l(1)}\cdots\omega_{l(p+q)})$$
$$=f_1(c_{p,q}(\cdot),p+q;\omega_{l(1)}\cdots\omega_{l(p+q)})$$
$$\cdot\sum_{\substack{\text{all the } p\text{-partitions}\\ \text{for } 1}}f_{2b}(s_{x_1}\cdots s_{x_p}(1);\omega_{l(1)}\cdots\omega_{l(p+q-q)})\prod_{i=1}^{p}\varphi_{n(s_{x_i}(1))}(s_{x_i}(1);\omega_{l(X(i)+1)}\cdots\omega_{l(X(i)+n(s_{x_i}(1)))})$$
$$=f_1(c_{p,q}(\cdot),p+q;\omega_{l(1)}\cdots\omega_{l(p+q)})\cdot f_{2b}(\underbrace{11\cdots1}_{p};\omega_{l(1)}\cdots\omega_{l(p)})\cdot\prod_{i=1}^{p}\varphi_1(1;\omega_i)$$
$$=\frac{1}{L_{p+q}(j\sum_{i=1}^{p+q}\omega_{l(i)})}(\prod_{i=1}^{q}(j\omega_{l(p+i)})^{k_{p+i}}\cdot\prod_{i=1}^{p}(j\omega_{l(i)})^{k_i}\cdot\prod_{i=1}^{p}H_1(j\omega_{l(i)})\qquad(5.9)$$

Note that in this case, p+q=$n(\bar{s})$ from (3.15), and $\bar{s}=c_{p,q}(\cdot)$ corresponding to a specific recursive terminal. Hence, (5.9) can be written as

$$\varphi_{n(\bar{s})}(c_{p,q}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})=\frac{1}{L_{n(\bar{s})}(j\sum_{i=1}^{n(\bar{s})}\omega_{l(i)})}(\prod_{i=1}^{q}(j\omega_{l(p+i)})^{k_{p+i}}\cdot\prod_{i=1}^{p}(j\omega_{l(i)})^{k_i}\cdot\prod_{i=1}^{p}H_1(j\omega_{l(i)})\qquad(5.10)$$

In order to verify this result, let n=$n(\bar{s})$= p+q, it can be obtained from (5.2) that for a parameter $c_{p,q}(\cdot)$, its correlative function is

$$\frac{1}{L_{n(\bar{s})}(j\sum_{i=1}^{n(\bar{s})}\omega_i)}\prod_{i=1}^{q}(j\omega_{p+i})^{k_{p+i}})H_{p,p}(j\omega_1,\cdots,j\omega_p)$$

From (3.5), $H_{p,p}(j\omega_1,\cdots,j\omega_p)=\prod_{i=1}^{p}(j\omega_i)^{k_i}\cdot\prod_{i=1}^{p}H_1(j\omega_i)$. This is consistent with (5.10). To further understand the results in Proposition 5.1, the following figure can be referred, which demonstrates the recursive process in the new mapping function and the structure of the theoretical results above (See Figure 5.1). □

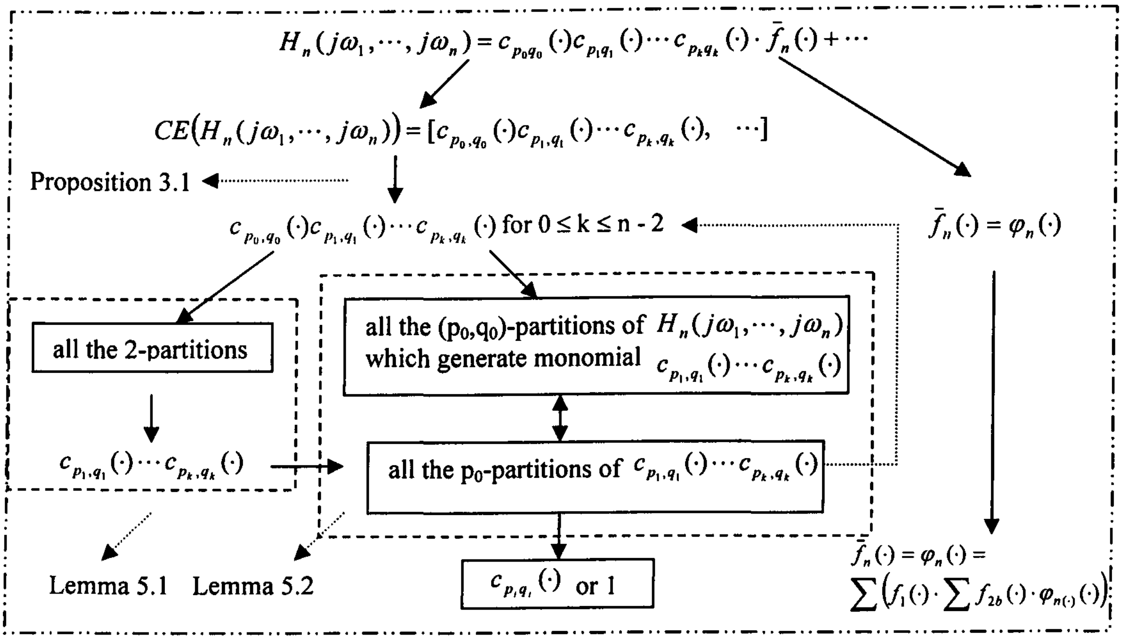

Figure 5.1. An illustration of the relationships in Proposition 5.1

To further demonstrate the results, the following example is given.

Example 5.1. Consider the 4th-order GFRF. The parametric characteristic of the 4th-order GFRF can be obtained from Proposition 3.1 that

$$\begin{aligned}CE(H_4(j\omega_1,\cdots,j\omega_4))= &C_{0,4}\oplus C_{1,3}\oplus C_{3,1}\oplus C_{2,2}\oplus C_{4,0}\oplus C_{1,1}\otimes C_{0,3}\oplus C_{1,1}\otimes C_{1,2}\\ &\oplus C_{1,1}\otimes C_{2,1}\oplus C_{1,1}\otimes C_{3,0}\oplus C_{1,2}\otimes C_{0,2}\oplus C_{1,2}\otimes C_{2,0}\oplus C_{2,0}\otimes C_{0,3}\\ &\oplus C_{2,0}\otimes C_{2,1}\oplus C_{2,0}\otimes C_{3,0}\oplus C_{2,1}\otimes C_{0,2}\oplus C_{3,0}\otimes C_{0,2}\\ &\oplus C_{1,1}\otimes C_{0,2}^2\oplus C_{1,1}^2\otimes C_{0,2}\oplus C_{1,1}\otimes C_{0,2}\otimes C_{2,0}\oplus C_{1,1}^3\oplus C_{1,1}^2\otimes C_{2,0}\\ &\oplus C_{1,1}\otimes C_{2,0}^2\oplus C_{2,0}\otimes C_{0,2}^2\oplus C_{2,0}^2\otimes C_{0,2}\oplus C_{2,0}^3\end{aligned}$$

By using Proposition 5.1, the correlative function of each term in $CE(H_4(j\omega_1,\cdots,j\omega_4))$ can all be obtained. For example, for the term $c_{1,1}(.)c_{0,2}(.)c_{2,0}(.)$, it can be derived that

$$\begin{aligned}&\varphi_{n(\bar{s})}(c_{1,1}(\cdot)c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})=\varphi_4(c_{1,1}(\cdot)c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_4)\\
&=f_1(c_{1,1}(\cdot),4;\omega_1\cdots\omega_4)\\
&\cdot\left[f_{2b}(s_2(c_{0,2}(\cdot)c_{2,0}(\cdot));\omega_1\cdots\omega_3)\cdot\varphi_{n(s_2(c_{0,2}(\cdot)c_{2,0}(\cdot)))}(s_2(c_{0,2}(\cdot)c_{2,0}(\cdot));\omega_{X(1)+1}\cdots\omega_{X(1)+n(s_2(c_{0,2}(\cdot)c_{2,0}(\cdot)))})\right]\\
&+f_1(c_{2,0}(\cdot),4;\omega_1\cdots\omega_4)\\
&\quad\cdot\Big[f_{2b}(s_0s_2(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4)\cdot\varphi_{n(s_0(c_{1,1}(\cdot)c_{0,2}(\cdot)))}(s_0(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_{X(1)+1}\cdots\omega_{X(1)+n(s_0(c_{1,1}(\cdot)c_{0,2}(\cdot)))})\\
&\qquad\cdot\varphi_{n(s_2(c_{1,1}(\cdot)c_{0,2}(\cdot)))}(s_2(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_{X(2)+1}\cdots\omega_{X(2)+n(s_2(c_{1,1}(\cdot)c_{0,2}(\cdot)))})\\
&\quad+f_{2b}(s_1s_1(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4)\cdot\varphi_{n(s_1(c_{1,1}(\cdot)))}(s_1(c_{1,1}(\cdot));\omega_{X(1)+1}\cdots\omega_{X(1)+n(s_1(c_{1,1}(\cdot)))})\\
&\qquad\cdot\varphi_{n(s_1(c_{0,2}(\cdot)))}(s_1(c_{0,2}(\cdot));\omega_{X(2)+1}\cdots\omega_{X(2)+n(s_1(c_{0,2}(\cdot)))})\Big]\\
&=f_1(c_{1,1}(\cdot),4;\omega_1\cdots\omega_4)\\
&\cdot\left[f_{2b}(c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_3)\cdot\varphi_{n(c_{0,2}(\cdot)c_{2,0}(\cdot))}(c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_{0+1}\cdots\omega_{0+n(c_{0,2}(\cdot)c_{2,0}(\cdot))})\right]\\
&+f_1(c_{2,0}(\cdot),4;\omega_1\cdots\omega_4)\\
&\cdot\Big[f_{2b}(s_0s_2(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4)\cdot\varphi_{n(1)}(1;\omega_1\cdots\omega_{n(1)})\varphi_{n(c_{1,1}(\cdot)c_{0,2}(\cdot))}(c_{1,1}(\cdot)c_{0,2}(\cdot);\omega_{n(1)+1}\cdots\omega_{n(1)+n(c_{1,1}(\cdot)c_{0,2}(\cdot))})\\
&+f_{2b}(s_1s_1(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4)\cdot\varphi_2(c_{1,1}(\cdot);\omega_{X(1)+1}\cdots\omega_{X(1)+n(s_1(c_{1,1}(\cdot)))})\\
&\quad\cdot\varphi_2(c_{0,2}(\cdot);\omega_{X(2)+1}\cdots\omega_{X(2)+n(s_1(c_{0,2}(\cdot)))})\Big]\end{aligned}$$

$$= f_1(c_{1,1}(\cdot),4;\omega_1\cdots\omega_4)\cdot\left[f_{2b}(c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_3)\cdot\varphi_3(c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_3)\right]$$
$$+ f_1(c_{2,0}(\cdot),4;\omega_1\cdots\omega_4)\cdot\left[f_{2b}(s_0s_2(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4)\cdot\varphi_1(1;\omega_1)\varphi_3(c_{1,1}(\cdot)c_{0,2}(\cdot);\omega_2\cdots\omega_4)\right.$$
$$\left.+ f_{2b}(s_1s_1(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4)\cdot\varphi_2(c_{1,1}(\cdot);\omega_1,\omega_2)\cdot\varphi_2(c_{0,2}(\cdot);\omega_3,\omega_4)\right] \quad (5.11)$$

To proceed with the recursive computation, it can be derived that

$$f_1(c_{1,1}(\cdot),4;\omega_1\cdots\omega_4) = \prod_{i=1}^{1}(j\omega_{3+i})^{k_{1+i}}\Big/L_4(j\sum_{i=1}^{4}\omega_i) = (j\omega_4)^{k_2}\Big/L_4(j\sum_{i=1}^{4}\omega_i) \quad (5.12a)$$

$$f_1(c_{2,0}(\cdot),4;\omega_1\cdots\omega_4) = 1\Big/L_4(j\sum_{i=1}^{4}\omega_i) \quad (5.12b)$$

$$f_{2b}(s_{x_1}(c_{2,0}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_3) = (j\omega_1+\cdots+j\omega_3)^{k_1} \quad (5.12c)$$

$$f_{2b}(s_0s_2(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4) = \sum_{\substack{\text{all the different}\\ \text{permutations}\\ \text{of } \{k_1,\cdots,k_p\}}}\prod_{i=1}^{2}(j\omega_{X(i)+1}+\cdots+j\omega_{X(i)+n(s_{x_i}(\bar{s}/c_{pq}(\cdot)))})^{k_i}$$
$$= (j\omega_1)^{k_1}(j\omega_2+\cdots+j\omega_4)^{k_2} + (j\omega_2+\cdots+j\omega_4)^{k_1}(j\omega_1)^{k_2} \quad (5.12d)$$

$$\varphi_3(c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_3)$$
$$= f_1(c_{2,0}(\cdot),3;\omega_1\cdots\omega_3)\cdot f_{2b}(s_{x_1}s_{x_2}(c_{0,2}(\cdot));\omega_1\cdots\omega_3)\prod_{i=1}^{2}\varphi_{n(s_{x_i}(\bar{s}/c_{pq}(\cdot)))}(s_{x_i}(c_{0,2}(\cdot));\omega_{X(i)+1}\cdots\omega_{X(i)+n(s_{x_i}(c_{0,2}(\cdot)))})$$
$$= f_1(c_{2,0}(\cdot),3;\omega_1\cdots\omega_3)\cdot f_{2b}(s_{x_1}s_{x_2}(c_{0,2}(\cdot));\omega_1\cdots\omega_3)\varphi_1(1;\omega_1)\varphi_2(c_{0,2}(\cdot);\omega_2,\omega_3)$$
$$= \frac{1}{L_3(j\sum_{i=1}^{3}\omega_i)}\cdot\left((j\omega_1)^{k_1}(j\omega_2+j\omega_3)^{k_2}+(j\omega_3+j\omega_2)^{k_1}(j\omega_1)^{k_2}\right)\cdot H_1(j\omega_1)$$
$$\cdot\frac{1}{L_2(j\omega_2+j\omega_3)}(j\omega_2)^{k_1}(j\omega_3)^{k_2} \quad (5.12e)$$

$$\varphi_3(c_{1,1}(\cdot)c_{0,2}(\cdot);\omega_2\cdots\omega_4)$$
$$= f_1(c_{1,1}(\cdot),3;\omega_2\cdots\omega_4)\cdot f_{2b}(s_{x_1}(c_{0,2}(\cdot));\omega_2,\omega_3)\cdot\varphi_{n(s_{x_1}(c_{0,2}(\cdot)))}(s_{x_1}(c_{0,2}(\cdot));\omega_2,\omega_3)$$
$$= f_1(c_{1,1}(\cdot),3;\omega_2\cdots\omega_4)\cdot f_{2b}(c_{0,2}(\cdot);\omega_2,\omega_3)\cdot\varphi_2(c_{0,2}(\cdot);\omega_2,\omega_3) \quad (5.12f)$$
$$= \frac{(j\omega_4)^{k_2}}{L_3(j\omega_2+\cdots+j\omega_4)}\cdot(j\omega_2+j\omega_3)^{k_1}\cdot\frac{1}{L_2(j\omega_2+j\omega_3)}(j\omega_2)^{k_1}(j\omega_3)^{k_2}$$

Using equations (5.12a-f) in (5.11) yields

$$\varphi_4(c_{1,1}(\cdot)c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_4)$$
$$= f_1(c_{1,1}(\cdot),4;\omega_1\cdots\omega_4)\cdot\left[f_{2b}(c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_3)\cdot\varphi_3(c_{0,2}(\cdot)c_{2,0}(\cdot);\omega_1\cdots\omega_3)\right]$$
$$+ f_1(c_{2,0}(\cdot),4;\omega_1\cdots\omega_4)\cdot\left[f_{2b}(s_0s_2(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4)\cdot\varphi_1(1;\omega_1)\varphi_3(c_{1,1}(\cdot)c_{0,2}(\cdot);\omega_2\cdots\omega_4)\right.$$
$$\left.+ f_{2b}(s_1s_1(c_{1,1}(\cdot)c_{0,2}(\cdot));\omega_1\cdots\omega_4)\cdot\varphi_2(c_{1,1}(\cdot);\omega_1,\omega_2)\cdot\varphi_2(c_{0,2}(\cdot);\omega_3,\omega_4)\right]$$
$$= \frac{(j\omega_4)^{k_2}(j\omega_1+\cdots+j\omega_3)^{k_1}\left((j\omega_1)^{k_1}(j\omega_2+j\omega_3)^{k_2}+(j\omega_3+j\omega_2)^{k_1}(j\omega_1)^{k_2}\right)(j\omega_2)^{k_1}(j\omega_3)^{k_2}}{L_4(j\omega_1+\cdots+j\omega_4)L_3(j\omega_1+j\omega_2+j\omega_3)L_2(j\omega_2+j\omega_3)}\cdot H_1(j\omega_1)$$
$$+\frac{\left((j\omega_1)^{k_1}(j\omega_2+\cdots+j\omega_4)^{k_2}+(j\omega_2+\cdots+j\omega_4)^{k_1}(j\omega_1)^{k_2}\right)(j\omega_4)^{k_2}(j\omega_2+j\omega_3)^{k_1}(j\omega_2)^{k_1}(j\omega_3)^{k_2}}{L_4(j\omega_1+\cdots+j\omega_4)L_3(j\omega_2+\cdots+j\omega_4)L_2(j\omega_2+j\omega_3)}H_1(j\omega_1)$$
$$+\frac{\left((j\omega_1+j\omega_2)^{k_1}(j\omega_3+j\omega_4)^{k_2}+(j\omega_3+j\omega_4)^{k_1}(j\omega_1+j\omega_2)^{k_2}\right)(j\omega_4)^{k_2}(j\omega_3)^{k_1}(j\omega_1)^{k_1}(j\omega_2)^{k_2}}{L_4(j\omega_1+\cdots+j\omega_4)L_2(j\omega_3+j\omega_4)L_2(j\omega_2+j\omega_1)}H_1(j\omega_1)$$
$$(5.13)$$

Therefore, the correlative function of the parameter monomial $c_{1,1}(\cdot)c_{0,2}(\cdot)c_{2,0}(\cdot)$ is obtained. It can be verified that the same result can be obtained by using the recursive algorithm in (5.2, 3.2-3.3, 5.1). For the sake of brevity, this is omitted. By following the same method, the whole correlative function vector $\varphi_4(CE(H_4(j\omega_1,\cdots,j\omega_4)))$ can be determined. Thus the 4^{th}-order GFRF $H_4(j\omega_1,\cdots,j\omega_4)$ can directly be written into a parametric characteristic form which can provide a straightforward and meaningful insight into the relationship between $H_4(j\omega_1,\cdots,j\omega_4)$ and nonlinear parameters, and also between $H_4(j\omega_1,\cdots,j\omega_4)$ and $H_1(j\omega_1)$. □

Remark 5.3. From Example 5.1, it can be seen that Proposition 5.1 provides an effective method to determine the correlative function for an effective monomial $c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$, and the computation process should be able to be carried out automatically without manual intervention. Therefore, Proposition 5.1 provides a simplified method to determine the *n*th-order GFRF directly into a more meaningful form as (5.4) which can demonstrate the parametric characteristic clearly and describe the *n*th-order GFRF in terms of the first order GFRF $H_1(j\omega)$ and nonlinear parameters. This reveals a more straightforward insight into the relationships between $H_n(j\omega_1,\cdots,j\omega_n)$ and nonlinear parameters, and between $H_n(j\omega_1,\cdots,j\omega_n)$ and $H_1(j\omega)$. Note that the high order GFRFs can represent system nonlinear frequency response characteristics (Billings and Peyton Jones 1990, Yue et al 2005) and $H_1(j\omega)$ represents the linear part of the system model. Hence, the results in Proposition 5.1 not only facilitate the analysis of the connection between system frequency response characteristics and model linear and nonlinear parameters, but also provide a new perspective on the understanding of the GFRFs and on the analysis of nonlinear systems based on the GFRFs. □

5.4 Some new properties

Based on the mapping function φ_n established in the last section, some new properties of the *n*th-order GFRF are discussed in this section.

5.4.1 Determination of FRFs based on parametric characteristics

There are several relationships involved in this paper. $H_n(j\omega_1,\cdots,j\omega_n)$ is determined from the NDE model in terms of the model parameters. The CE operator is a mapping from $H_n(j\omega_1,\cdots,j\omega_n)$ to its parametric characteristic, which can also be regarded as a mapping from the nonlinear parameters of the NDE model to the parametric characteristics of $H_n(j\omega_1,\cdots,j\omega_n)$. Thus there is a bijective mapping between $H_n(j\omega_1,\cdots,j\omega_n)$ and the NDE model. The function φ_n can be regarded as an inverse mapping of the CE operator such that the *n*th-order GFRF can be reconstructed from its parametric characteristic, which can also be regarded as a mapping from the nonlinear parameters of the NDE model to $H_n(j\omega_1,\cdots,j\omega_n)$. This can refer to Figure 5.2, where "•" represents the point multiplication between the parametric monomial and its correlative function.

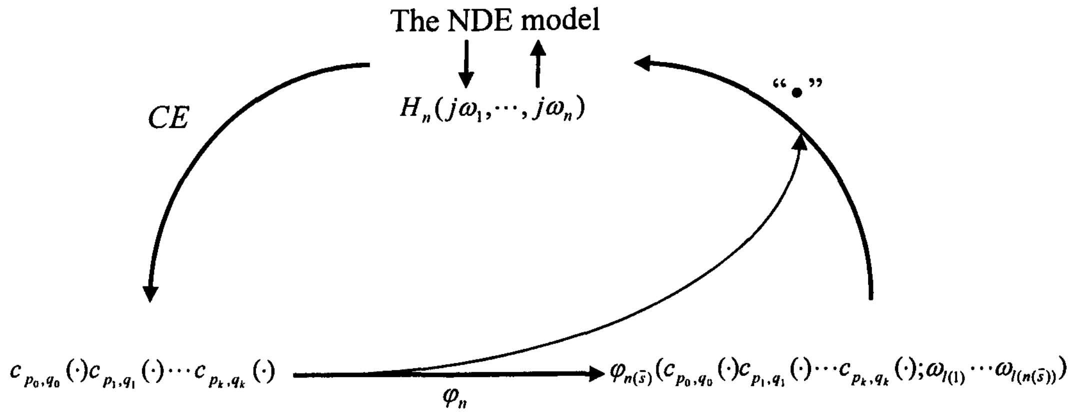

Figure 5.2. Relationship between φ_n and CE

It can be seen from Figure 5.2 that

$$H_n(j\omega_1,\cdots,j\omega_n)=CE(H_n(\cdot))\cdot\varphi_n(CE(H_n(\cdot))) \tag{5.14}$$

From (5.14), the inverse of the operator CE can simply be written as ($x=CE(H_n(\cdot))$)

$$CE^{-1}(x)=x\cdot\varphi_n(x)$$

which constructs a mapping directly from the parametric characteristic of the nth-order GFRF to the nth-order GFRF itself. Note that $CE(H_n(\cdot))$ includes all the nonlinear parameters of degree from 2 to n of the nonlinear system of interest, and $\varphi_n(CE(H_n(\cdot)))$ is a complex valued function vector including the effect of the complicated nonlinear characteristics and also the effect of the linear part of the nonlinear system. Hence, Equation (5.14) reveals a new perspective on the computation and understanding of the GFRFs as discussed in Section 5.3, and also provides a new insight into the frequency domain analysis of nonlinear systems based on the GFRFs.

From the results in Chapters 3 and 4, the output spectrum for model (1.5) can now be determined as

$$Y(j\omega)=\sum_{n=1}^{N}CE(H_n(j\omega_1,\cdots,j\omega_n))\cdot\hat{F}_n(j\omega) \tag{5.15a}$$

when the input is a general input $U(j\omega)$,

$$\hat{F}_n(j\omega)=\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}\varphi_n(CE(H_n(j\omega_1,\cdots,j\omega_n)))\cdot\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega \tag{5.15b}$$

when the input is a multi-tone function $u(t)=\sum_{i=1}^{K}|F_i|\cos(\omega_i t+\angle F_i)$,

$$\hat{F}_n(j\omega)=\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}\varphi_n(CE(H_n(j\omega_{k_1},\cdots,j\omega_{k_n})))\cdot F(\omega_{k_1})\cdots F(\omega_{k_n}) \tag{5.15c}$$

It is obvious that Equation (5.15a) is an explicit analytical polynomial functions with coefficients in $S_C(1)\cup\cdots\cup S_C(N)$ and the corresponding correlative functions in $S_f(1)\cup\cdots\cup S_f(N)$. This demonstrates a direct analytical relationship between system output spectrum and system time-domain model parameters. The effects on system output spectrum from the linear parameters are included in $S_f(1)\cup\cdots\cup S_f(N)$, and the effects from the nonlinear parameters are included in $S_C(1)\cup\cdots\cup S_C(N)$ and also embodied in $S_f(1) \cup \cdots \cup S_f(N)$. This will facilitate the analysis of output frequency response characteristics of nonlinear systems. For example, for any parameters of model (1.5) of interest, which may represent some specific physical characteristics, the output spectrum can therefore directly be written as a polynomial in terms of these parameters. Then how these parameters affect the system output spectrum need only be investigated by studying the frequency characteristics of the new mapping functions involved in the polynomial and simultaneously optimizing the values of these nonlinear parameters. Further study in this topic will be introduced in a later chapter

5.4.2 Magnitude of the *n*th-order GFRF

Based on Equation (5.14), the magnitude of the *n*th-order GFRF can be expressed in terms of its parametric characteristic.

Corollary 5.1. Let $CE_n = CE(H_n(\cdot))$, $\Theta_n = \varphi_n(CE(H_n(\cdot))) \cdot \varphi_n(CE(H_n(\cdot)))^*$, $\varphi_n = \varphi_n(CE(H_n(\cdot)))$, and $\Lambda_n = CE(H_n(\cdot))^T CE(H_n(\cdot))$, then

$$|H_n(j\omega_1,\cdots,j\omega_n)|^2 = CE_n \Theta_n CE_n^T \tag{5.16a}$$

$$|H_n(j\omega_1,\cdots,j\omega_n)|^2 = \varphi_n^* \Lambda_n \varphi_n \tag{5.16b}$$

Proof. It can be derived from (5.14) that

$$\begin{aligned}|H_n(j\omega_1,\cdots,j\omega_n)|^2 &= H_n(j\omega_1,\cdots,j\omega_n) \cdot H_n^*(j\omega_1,\cdots,j\omega_n)\\ &= CE(H_n(\cdot)) \cdot \varphi_n(CE(H_n(\cdot))) \cdot (CE(H_n(\cdot)) \cdot \varphi_n(CE(H_n(\cdot))))^*\\ &= CE(H_n(\cdot)) \cdot \left(\varphi_n(CE(H_n(\cdot))) \cdot \varphi_n(CE(H_n(\cdot)))^*\right) \cdot CE(H_n(\cdot))^T = CE_n \Theta_n CE_n^T\end{aligned}$$

The result in equation (5.16b) can also be achieved by following the same method. This completes the proof. □

From Corollary 5.1, the square of the magnitude of the *n*th-order GFRF is proportional to a quadratic function of the parametric characteristic and also proportional to a quadratic function of the corresponding correlative function. Corollary 5.1 provides a new property of the *n*th-order GFRF, which reveals the relationship between the magnitude of $H_n(j\omega_1,\cdots,j\omega_n)$ and its nonlinear parametric characteristic, and also the relationship between the magnitude of $H_n(j\omega_1,\cdots,j\omega_n)$ and the correlative functions which involve both the system linear and nonlinear characteristics. Given a requirement on $|H_n(j\omega_1,\cdots,j\omega_n)|$, the condition on model parameters can be derived by using equations (5.16ab). This may provide a new technique for the analysis and design of nonlinear systems based on the *n*th- order GFRF in the frequency domain.

Moreover, it can be seen that the frequency characteristic matrix Θ_n is a Hermitian matrix, whose eigenvalues are the positive real valued functions of the system linear parameters but invariant to the values of the system nonlinear parameters in $CE(H_n(\cdot))$. Thus different nonlinearities may result in different frequency characteristic matrix Θ_n, but the same nonlinearities will have an invariant matrix Θ_n. This property of the nth-order GFRF provides a new insight into the nonlinear effect on the high order GFRFs from different nonlinearities. For this purpose, define a new function

$$\bar{\lambda}_n(\omega_1,\cdots,\omega_n) = \lambda_{\max}(\Theta_n) \tag{5.17}$$

which is the maximum eigenvalue of the frequency characteristic matrix Θ_n. As mentioned, the frequency spectrum of this function can act as a novel insight into the nonlinear effect on the GFRFs from different nonlinearities, since this function is only dependent on different nonlinearities but independent of their values. However, the frequency response spectrum of the GFRFs will change greatly with the values of the involved nonlinear parameters, which can not provide a clear insight into the nonlinear effects between different nonlinearities.

Moreover, the following results can be obtained for the bound evaluation for the nth-order GFRF based on the results above.

Proposition 5.2.

$$\sup_{\omega_1,\cdots,\omega_n} |H_n(j\omega_1,\cdots,j\omega_n)| \le \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} |h_n(\tau_1,\cdots,\tau_n)| d\tau_1\cdots d\tau_n \le \sqrt{\sup_{\omega_1,\cdots,\omega_n}(\lambda_{\max}(\Theta_n))}\cdot\|CE_n\| \tag{5.18a}$$

$$\sup_{\omega_1,\cdots,\omega_n} |H_n(j\omega_1,\cdots,j\omega_n)| \le \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} |h_n(\tau_1,\cdots,\tau_n)| d\tau_1\cdots d\tau_n \le \sqrt{\lambda_{\max}(\Lambda_n)}\cdot \sup_{\omega_1,\cdots,\omega_n}(\|\varphi_n\|) \tag{5.18b}$$

Proof. See Section 5.5 for the proof. □

From Equations (5.18ab), it can be seen that the magnitude of the *n*th-order GFRF is proportional to a quadratic function of the parametric characteristic and also proportional to a quadratic function of the corresponding correlative function. These results demonstrate a new property of the *n*th-order GFRF, which reveals the relationship between the magnitude of $H_n(j\omega_1,\cdots,j\omega_n)$ and its nonlinear parametric characteristic, and also the relationship between the magnitude of $H_n(j\omega_1,\cdots,j\omega_n)$ and the correlative functions which include the linear (the first order GFRF) and nonlinear characteristics. Given a requirement on $|H_n(j\omega_1,\cdots,j\omega_n)|$, the condition on model parameters or the first order GFRF can be derived by using these results. Proposition 5.2 also shows that the absolute integration of the *n*th-order Volterra kernel function in the time domain is bounded by a quadratic function of the parameter characteristic. This reveals the relationship between the model parameters and the stability of Volterra series. Obviously, these may provide a new insight into the analysis and design of nonlinear systems based on the *n*th- order GFRF in the frequency domain.

5.4.3 Relationship between $H_n(j\omega_1,\cdots,j\omega_n)$ and $H_1(j\omega_1)$

As illustrated in Example 5.1, $H_n(j\omega_1,\cdots,j\omega_n)$ can directly be determined in terms of the first order GFRF $H_1(j\omega)$ based on the novel mapping function φ_n according to its parametric characteristic. The following results can be concluded.

Corollary 5.2. For an effective parametric monomial $c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot)$, its correlative function is a ρ-degree function of $H_1(j\omega_{l(1)})$ which can be written as a symmetric form

$$\varphi_{n(\bar{s})}(c_{p_0,q_0}(\cdot)c_{p_1,q_1}(\cdot)\cdots c_{p_k,q_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})$$

$$= \frac{(n(\bar{s})-\rho)!\rho!}{n(\bar{s})!} \sum_{\substack{\text{all the combinations of } \rho \text{ integers } \{r_1,r_2,\cdots,r_\rho\} \\ \text{taken from } \{1,2,\cdots,n(\bar{s})\} \text{ without repetition} \\ j \text{ is for different combination}}} \mu_j(\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\prod_{i=1}^{\rho} H_1(j\omega_{\bar{l}(i)})$$

where $\rho = n(\bar{s}) - \sum_{i=0}^{k} q_i = \sum_{i=0}^{k} p_i - k$, $\bar{l} = [r_1, r_2, \cdots, r_\rho]$, and $\mu_j(\omega_{l(1)} \cdots \omega_{l(n(\bar{s}))})$ can be determined by equations (5.7-5.8). Therefore, the *n*th-order GFRF can be regarded as an *n*-degree polynomial function of $H_1(j\omega_{l(1)})$. □

Proof. See Section 5.5 for the proof. □

Corollary 5.2 demonstrates the relationship between $H_n(j\omega_1, \cdots, j\omega_n)$ and $H_1(j\omega)$, and reveals how the first order GFRF, which represents the linear part of system model, affects the higher order GFRFs, together with the nonlinear dynamics. Note that for any specific parameters of interest, the polynomial structure of the FRFs is explicitly determined in terms of these parameters, thus the property of this polynomial function is greatly dependent on the "coefficients" of these parameter monomials in the polynomial, which correspond to the correlative functions of the parametric characteristics of the polynomial and are determined by the new mapping function. Hence, Corollary 5.2 is important for the qualitative analysis of the connection between $H_n(j\omega_1, \cdots, j\omega_n)$ and $H_1(j\omega)$, and also between nonlinear parameters and high order GFRFs .

Example 5.2. To demonstrate the theoretical results above, consider a simple mechanical system shown in Figure 5.2.

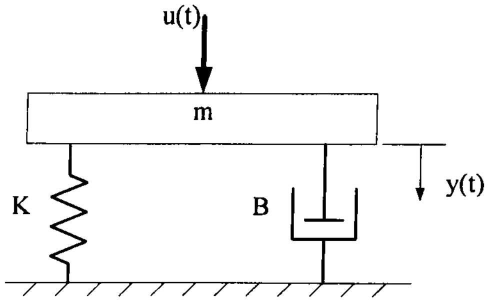

Figure 5.2. A mechanical system

The output property of the spring satisfies $F = Ky + c_1 y^3$, and the damper $F = B\dot{y} + c_2 \dot{y}^3$. *u*(t) is the external input force. The system dynamics can be described by

$$m\ddot{y} = -Ky - B\dot{y} - c_1 y^3 - c_2 \dot{y}^3 + u(t) \tag{5.19}$$

which can be written into the form of NDE model (1.5) with *M*=3, *K*=2, $c_{1,0}(2) = m$, $c_{1,0}(1) = B$, $c_{1,0}(0) = K$, $c_{3,0}(000) = c_1$, $c_{3,0}(111) = c_2$, $c_{0,1}(0) = -1$, and all the other parameters are zero.

There are two nonlinear terms $c_{3,0}(000) = c_1$ and $c_{3,0}(111) = c_2$ in model (5.19), which are all pure output nonlinearity and can be written as $C_{3,0} = [c_1, c_2]$. The parametric characteristics of the GFRFs of model (5.19) with respect to nonlinear parameter $C_{3,0}$ can be obtained according to equation (5.3) or Proposition 3.1 as

$CE(H_{2i+1}(.)) = C_{3,0}^{\ i}$ for *i*=0,1,2,…,, otherwise $CE(H_{2i}(.))=0$ for *i*=1,2,3,…

Therefore,

$CE(H_1(.))=1$;

$CE(H_3(.))=C_{3,0} = [c_1 \ \ c_2]$;

$CE(H_5(.))=C_{3,0}\otimes C_{3,0}=[c_1^2\ \ c_1c_2\ \ c_2^2];$
$CE(H_7(.))=C_{3,0}\otimes C_{3,0}\otimes C_{3,0}=[c_1^3\ \ c_1^2c_2\ \ c_1c_2^2\ \ c_2^3]\ldots\ldots$

By using (5.7-5.10), it can be obtained that

$$\varphi_3(c_{3,0}(000);\omega_1,\omega_2,\omega_3)=\frac{1}{L_3(j\sum_{i=1}^{3}\omega_i)}\cdot\prod_{i=1}^{3}(j\omega_i)^0\cdot\prod_{i=1}^{3}H_1(j\omega_i)=\frac{1}{L_3(j\sum_{i=1}^{3}\omega_i)}\cdot\prod_{i=1}^{3}H_1(j\omega_i)$$

$$\varphi_3(c_{3,0}(111);\omega_1,\omega_2,\omega_3)=\frac{1}{L_3(j\sum_{i=1}^{3}\omega_i)}\cdot\prod_{i=1}^{3}(j\omega_i)\cdot\prod_{i=1}^{3}H_1(j\omega_i)=\frac{\prod_{i=1}^{3}(j\omega_i)}{L_3(j\sum_{i=1}^{3}\omega_i)}\cdot\prod_{i=1}^{3}H_1(j\omega_i)$$

$$\varphi_5(c_{3,0}(000)c_{3,0}(000);\omega_1,\cdots,\omega_5)$$
$$=f_1(c_{3,0}(000),5;\omega_1,\cdots,\omega_5)\cdot\sum_{\substack{\text{all the 3-partitions}\\ \text{for } c_{3,0}(000)}}\ \sum_{\substack{\text{all the different}\\ \text{permutations of }\{0,0,1\}}}\Big[f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(c_{3,0}(000));\omega_1\cdots\omega_5)$$
$$\cdot\prod_{i=1}^{3}\varphi_{n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot)))}(s_{\bar{x}_i}(c_{3,0}(000));\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot))))})\Big]$$
$$=f_1(c_{3,0}(000),5;\omega_1,\cdots,\omega_5)\cdot\begin{pmatrix}f_{2a}(s_0s_0s_1(c_{3,0}(000));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_1(1;\omega_2)\varphi_3(c_{3,0}(000);\omega_3\cdots\omega_5)\\+f_{2a}(s_0s_1s_0(c_{3,0}(000));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_3(c_{3,0}(000);\omega_2\cdots\omega_4)\varphi_1(1;\omega_5)\\+f_{2a}(s_1s_0s_0(c_{3,0}(000));\omega_1\cdots\omega_5)\varphi_3(c_{3,0}(000);\omega_1\cdots\omega_3)\varphi_1(1;\omega_4)\varphi_1(1;\omega_5)\end{pmatrix}$$
$$=\frac{1}{L_5(j\sum_{i=1}^{5}\omega_i)}\cdot\begin{pmatrix}H_1(\omega_1)H_1(\omega_2)\prod_{i=3}^{5}H_1(j\omega_i)\Big/L_3(j\sum_{i=3}^{5}\omega_i)\\+H_1(\omega_1)\prod_{i=2}^{4}H_1(j\omega_i)H_1(\omega_5)\Big/L_3(j\sum_{i=2}^{4}\omega_i)\\+\prod_{i=1}^{3}H_1(j\omega_i)H_1(\omega_4)H_1(\omega_5)\Big/L_3(j\sum_{i=1}^{3}\omega_i)\end{pmatrix}$$
$$=\frac{1}{L_5(j\sum_{i=1}^{5}\omega_i)}\cdot\Big(\frac{1}{L_3(j\sum_{i=3}^{5}\omega_i)}+\frac{1}{L_3(j\sum_{i=2}^{4}\omega_i)}+\frac{1}{L_3(j\sum_{i=1}^{3}\omega_i)}\Big)\cdot\prod_{i=1}^{5}H_1(j\omega_i)$$

$$\varphi_5(c_{3,0}(111)c_{3,0}(111);\omega_1,\cdots,\omega_5)$$
$$=f_1(c_{3,0}(111),5;\omega_1,\cdots,\omega_5)\cdot\sum_{\substack{\text{all the 3-partitions}\\ \text{for } c_{3,0}(111)}}\ \sum_{\substack{\text{all the different}\\ \text{permutations of }\{0,0,1\}}}\Big[f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(c_{3,0}(111));\omega_1\cdots\omega_5)$$
$$\cdot\prod_{i=1}^{3}\varphi_{n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot)))}(s_{\bar{x}_i}(c_{3,0}(111));\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot))))})\Big]$$
$$=f_1(c_{3,0}(111),5;\omega_1,\cdots,\omega_5)\cdot\begin{pmatrix}f_{2a}(s_0s_0s_1(c_{3,0}(111));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_1(1;\omega_2)\varphi_3(c_{3,0}(111);\omega_3\cdots\omega_5)\\+f_{2a}(s_0s_1s_0(c_{3,0}(111));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_3(c_{3,0}(111);\omega_2\cdots\omega_4)\varphi_1(1;\omega_5)\\+f_{2a}(s_1s_0s_0(c_{30}(111));\omega_1\cdots\omega_5)\varphi_3(c_{30}(111);\omega_1\cdots\omega_3)\varphi_1(1;\omega_4)\varphi_1(1;\omega_5)\end{pmatrix}$$
$$=\frac{1}{L_5(j\sum_{i=1}^{5}\omega_i)}\cdot\left(\frac{(j\sum_{i=3}^{5}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=3}^{5}\omega_i)}+\frac{(j\sum_{i=2}^{4}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=2}^{4}\omega_i)}+\frac{(j\sum_{i=1}^{3}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=1}^{3}\omega_i)}\right)\cdot\prod_{i=1}^{5}H_1(j\omega_i)$$

$$
\begin{aligned}
&\varphi_5(c_{3,0}(000)c_{3,0}(111);\omega_1,\cdots,\omega_5)\\
&= f_1(c_{3,0}(000),5;\omega_1,\cdots,\omega_5)\cdot \sum_{\substack{\text{all the 3-partitions}\\ \text{for } c_{3,0}(111)}} \sum_{\substack{\text{all the different}\\ \text{permutations of }\{0,0,1\}}} \Big[f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(c_{3,0}(111));\omega_1\cdots\omega_5)\\
&\qquad\qquad \cdot\prod_{i=1}^{3}\varphi_{n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot)))}(s_{\bar{x}_i}(c_{3,0}(111));\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot))))})\Big]\\
&+ f_1(c_{3,0}(111),5;\omega_1,\cdots,\omega_5)\cdot \sum_{\substack{\text{all the 3-partitions}\\ \text{for } c_{3,0}(000)}} \sum_{\substack{\text{all the different}\\ \text{permutations of }\{0,0,1\}}} \Big[f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(c_{3,0}(000));\omega_1\cdots\omega_5)\\
&\qquad\qquad \cdot\prod_{i=1}^{3}\varphi_{n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot)))}(s_{\bar{x}_i}(c_{3,0}(000));\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot))))})\Big]\\
&= f_1(c_{3,0}(000),5;\omega_1,\cdots,\omega_5)\cdot\begin{pmatrix} f_{2a}(s_0s_0s_1(c_{3,0}(111));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_1(1;\omega_2)\varphi_3(c_{3,0}(111);\omega_3\cdots\omega_5)\\ +f_{2a}(s_0s_1s_0(c_{3,0}(111));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_3(c_{3,0}(111);\omega_2\cdots\omega_4)\varphi_1(1;\omega_5)\\ +f_{2a}(s_1s_0s_0(c_{3,0}(111));\omega_1\cdots\omega_5)\varphi_3(c_{3,0}(111);\omega_1\cdots\omega_3)\varphi_1(1;\omega_4)\varphi_1(1;\omega_5)\end{pmatrix}\\
&+ f_1(c_{3,0}(111),5;\omega_1,\cdots,\omega_5)\cdot\begin{pmatrix} f_{2a}(s_0s_0s_1(c_{3,0}(000));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_1(1;\omega_2)\varphi_3(c_{3,0}(000);\omega_3\cdots\omega_5)\\ +f_{2a}(s_0s_1s_0(c_{3,0}(000));\omega_1\cdots\omega_5)\varphi_1(1;\omega_1)\varphi_3(c_{3,0}(000);\omega_2\cdots\omega_4)\varphi_1(1;\omega_5)\\ +f_{2a}(s_1s_0s_0(c_{3,0}(000));\omega_1\cdots\omega_5)\varphi_3(c_{3,0}(000);\omega_1\cdots\omega_3)\varphi_1(1;\omega_4)\varphi_1(1;\omega_5)\end{pmatrix}\\
&= \frac{1}{L_5(j\sum_{i=1}^{5}\omega_i)}\cdot\left(\frac{1+(j\sum_{i=3}^{5}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=3}^{5}\omega_i)}+\frac{1+(j\sum_{i=2}^{4}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=2}^{4}\omega_i)}+\frac{1+(j\sum_{i=1}^{3}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=1}^{3}\omega_i)}\right)\cdot\prod_{i=1}^{5}H_1(j\omega_i)
\end{aligned}
$$

Hence, it can be obtained that

$$
\varphi_3(CE(H_3(\cdot)))=\frac{1}{L_3(j\sum_{i=1}^{3}\omega_i)}\cdot\left[\frac{1}{\prod_{i=1}^{3}(j\omega_i)}\right]\cdot\prod_{i=1}^{3}H_1(j\omega_i)
$$

$$
\varphi_5(CE(H_5(\omega_1,\cdots,\omega_5)))=
$$

$$
\frac{1}{L_5(j\sum_{i=1}^{5}\omega_i)}\cdot\left[\begin{matrix}\dfrac{1}{L_3(j\sum_{i=3}^{5}\omega_i)}+\dfrac{1}{L_3(j\sum_{i=2}^{4}\omega_i)}+\dfrac{1}{L_3(j\sum_{i=1}^{3}\omega_i)}\\[2ex] \dfrac{1+(j\sum_{i=3}^{5}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=3}^{5}\omega_i)}+\dfrac{1+(j\sum_{i=2}^{4}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=2}^{4}\omega_i)}+\dfrac{1+(j\sum_{i=1}^{3}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=1}^{3}\omega_i)}\\[2ex] \dfrac{(j\sum_{i=3}^{5}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=3}^{5}\omega_i)}+\dfrac{(j\sum_{i=2}^{4}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=2}^{4}\omega_i)}+\dfrac{(j\sum_{i=1}^{3}\omega_i)\prod_{i=1}^{5}(j\omega_i)}{L_3(j\sum_{i=1}^{3}\omega_i)}\end{matrix}\right]\cdot\prod_{i=1}^{5}H_1(j\omega_i)
$$

By using equation (5.14), the GFRFs for n=3 and 5 of system (5.19) can be obtained. Proceeding with the computation process above, any higher order GFRFs of system (5.19) can be derived and written in a much more meaningful form. It can be seen that, the correlative function of a monomial in the parametric characteristic of the nth-order GFRF is an n-degree polynomial of the first order GFRF as stated in Corollary 5.2, and so the nth-order GFRF is. Based on equation (5.14), the first order parametric sensitivity of the GFRFs with respect to any nonlinear parameter can be studied as

$$\frac{\partial H_n(j\omega_1,\cdots,j\omega_n)}{\partial c}=\frac{\partial CE(H_n(\cdot))}{\partial c}\cdot\varphi_n(CE(H_n(\cdot)))$$

For example,

$$\frac{\partial H_3(j\omega_1,\cdots,j\omega_3)}{\partial c_1}=\frac{\partial CE(H_3(\cdot))}{\partial c_1}\cdot\varphi_3(CE(H_3(\cdot)))=[1,0]\cdot\varphi_3(CE(H_3(\cdot)))=\prod_{i=1}^{3}H_1(j\omega_i)\Big/L_3(j\sum_{i=1}^{3}\omega_i)$$

Similarly,

$$\frac{\partial H_5(j\omega_1,\cdots,j\omega_5)}{\partial c_1}=\frac{\partial CE(H_5(\cdot))}{\partial c_1}\cdot\varphi_5(CE(H_5(\cdot)))=[2c_1,c_2,0]\cdot\varphi_5(CE(H_5(\cdot))).$$

Similar results can also be obtained for parameter c_2. It can be seen that the sensitivity of the third order GFRF with respect to the nonlinear spring c_1 and nonlinear damping c_2 is constant which is dependent on linear parameters, but the sensitivity of the higher order GFRFs will be a function of these nonlinearities and the linear parameters. Note that for a Volterra system, the system output is usually dominated by its several low order GFRFs (Boyd and Chua 1985). Hence, in order to make the system less sensitive to these nonlinearities, the linear parameters should properly be designed.

Moreover, the magnitude of $H_n(j\omega_1,\cdots,j\omega_n)$ can also be evaluated readily according to Corollary 5.1. For example, for n= 3

$$|H_3(j\omega_1,\cdots,j\omega_3)|^2=CE_3\Theta_3CE_3^T=\left|\frac{\prod_{i=1}^{3}H_1(j\omega_i)}{L_3(j\sum_{i=1}^{3}\omega_i)}\right|^2\cdot\begin{bmatrix}c_1\\c_2\end{bmatrix}^T\begin{bmatrix}1 & \prod_{i=1}^{3}(j\omega_i)\\ -\prod_{i=1}^{3}(j\omega_i) & \prod_{i=1}^{3}(\omega_i^2)\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

As mentioned above, instead of studying the Bode diagram of $H_3(j\omega_1,\cdots,j\omega_3)$, the frequency response spectrum of the maximum eigenvalue of the third order frequency characteristic matrix defined in Corollary 5.1 can be investigated. See Figures 5.3-5.4. Different values of the linear parameters will result in a different view. An increase of the linear damping enables the magnitude to increase for higher $\omega_1+\omega_2+\omega_3$ along the line $\omega_1+\omega_3=0$. Note that the system output spectrum (5.15a-c) involves the computation of the GFRFs along a super-plane $\omega_1+\cdots+\omega_n=\omega$. The frequency response spectra of the maximum eigenvalue on the plane $\omega_1+\cdots+\omega_3=\omega$ with different output frequency ω are given in Figures 5.5-5.6. The peak and valley in the figures can represent special properties of the system. Understanding of these diagrams can follow the method in Yue et al (2005), and further results are under study.

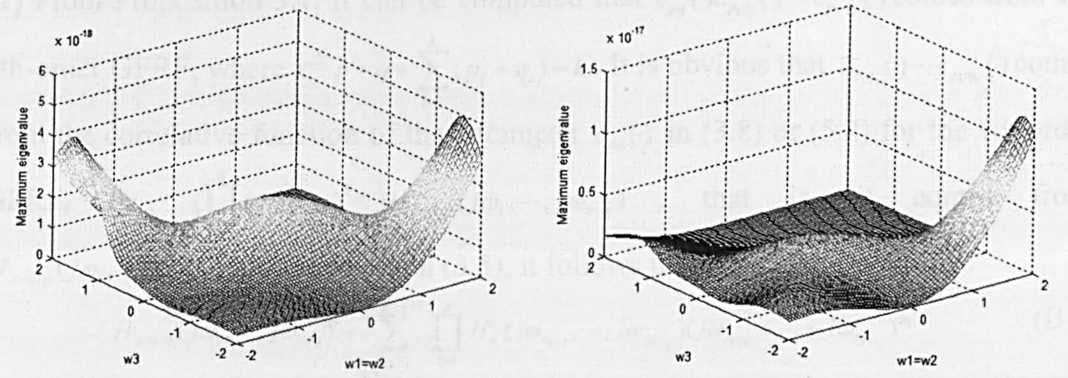

Figure 5.3. Frequency response spectrum of the maximum eigenvalue when m=24, B=2.96(left) or 29.6(right), K=160

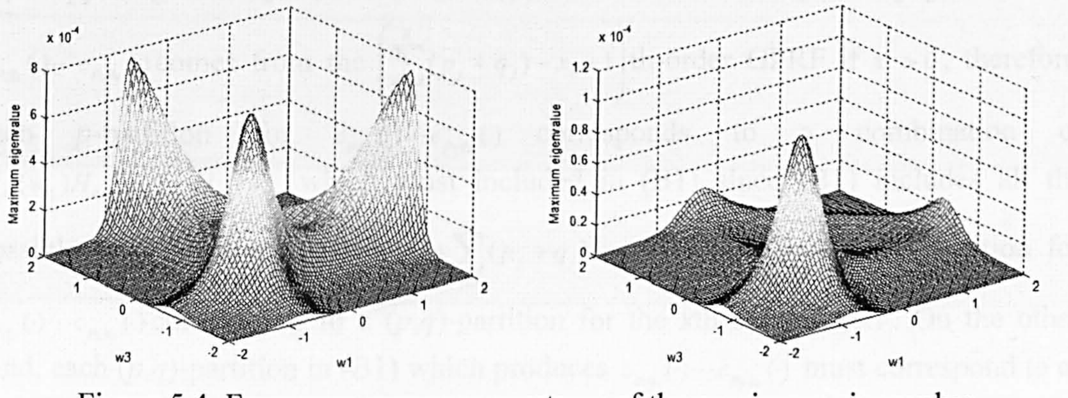

Figure 5.4. Frequency response spectrum of the maximum eigenvalue when m=2.4, B=2.96, K=1.6 and $\omega_1+\omega_2+\omega_3=0.8$ (left) or 1.5(right)

The system output spectrum can also be studied. For example, suppose the system is subject to a harmonic input $u(t)=F_d\sin(\omega_0 t)\quad(F_d>0)$, then the magnitude of the third order output spectrum can be evaluated as (Jing et al 2007a)

$$|Y_3(j\omega)| \le \frac{1}{2^3}\sum_{\omega_{k_1}+\cdots+\omega_{k_3}=\omega}|H_3(j\omega_{k_1},\cdots,j\omega_{k_3})||F(\omega_{k_1})\cdots F(\omega_{k_3})| \le \frac{F_d^3}{2^3}\sum_{\omega_{k_1}+\cdots+\omega_{k_3}=\omega}|H_3(j\omega_{k_1},\cdots,j\omega_{k_3})|$$

From corollary 5.1, $|H_3(j\omega_1,\cdots,j\omega_3)| \le \sqrt{\lambda_3(j\omega_1,\cdots,j\omega_n)}\|CE_3^T\|$. Therefore,

$$|Y_3(j\omega)| \le \frac{F_d^3}{2^3}\sum_{\omega_{k_1}+\cdots+\omega_{k_3}=\omega}\sqrt{\lambda_3(j\omega_1,\cdots,j\omega_n)}\|CE_3^T\| = \frac{F_d^3}{2^3}\sqrt{c_1^2+c_2^2}\sum_{\omega_{k_1}+\cdots+\omega_{k_3}=\omega}\sqrt{\lambda_3(j\omega_1,\cdots,j\omega_n)}$$

For ω =0.8 and m=2.4, B=29.6, K=1.6, it can be obtained that $\sqrt{\lambda_3(j\omega_1,\cdots,j\omega_n)}$ ≤ 0.006055896. Hence, in this case

$$|Y_3(j\omega)| \le 0.00227096F_d^3\sqrt{c_1^2+c_2^2}$$

Obviously, given a requirement on the bound of $|Y_3(j\omega)|$, the design restriction on the nonlinear parameters c_1 and c_2 can further be derived. □

5.5 Proofs

- **Proof of Lemma 5.2**

(1) From Proposition 3.1, it can be computed that $c_{pq}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ comes from the xth-order GFRF, where $x=p+q+\sum_{i=1}^{k}(p_i+q_i)-k$. It is obvious that $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ comes from the correlative function of the parameter $c_{pq}(\cdot)$ in (3.8) or (5.2) for the xth-order GFRF, *i.e.*, $(\prod_{i=1}^{q}(j\omega_{x-q+i})^{k_{p+i}})H_{x-q,p}(j\omega_1,\cdots,j\omega_{x-q})$, that is, it comes from $H_{x-q,p}(j\omega_1,\cdots,j\omega_{n-q})$. From equation (3.5), it follows that

$$H_{x-q,p}(j\omega_1,\cdots,j\omega_{x-q})=\sum_{\substack{r_1\cdots r_p=1\\ \sum r_i=x-q}}^{x-p-q+1}\prod_{i=1}^{p}H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})(j\omega_{r_{X+1}}+\cdots+j\omega_{r_{X+r_i}})^{k_i} \quad \text{(B1)}$$

Obviously, $\prod_{i=1}^{p}H_{r_i}(j\omega_{r_{X+1}},\cdots,j\omega_{r_{X+r_i}})$ is a (p,q)-partition for the xth-order GFRF.

(2) Supposing that s_0 comes from $H_1(.)$, each monomial s_{x_i} in a p-partition for $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ comes from the $\left(\sum_{j=1}^{x_i}(p_j+q_j)-x_i+1\right)$th-order GFRF if $x_i>0$, therefore, each p-partition for $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ corresponds to a combination of $H_{r_1}(w_{r_1})H_{r_2}(w_{r_2})\cdots H_{r_p}(w_{r_p})$ which must included in (B1) since (B1) includes all the possible (p,q)-partitions, where $r_i=\sum_{j=1}^{x_i}(p_j+q_j)-x_i+1$. That is, each p-partition for $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ corresponds to a (p,q)-partition for the xth-order GFRF. On the other hand, each (p,q)-partition in (B1) which produces $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ must correspond to at least one p-partition for $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$.

(3) Equation (B1) includes all the (p,q)-partitions for the xth-order GFRF which produce $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$, thus the correlative function of $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ are the summation of all the correlative functions of each (p,q)-partition. Note that each (p,q)-partition may produce more than one p-partition for $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$. This implies there are more than one cases in the same (p,q)-partition to produce $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$. Therefore, the correlative function of $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ should be the summation of the correlative functions corresponding to all the cases where $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ are produced. This completes the proof. □

- **Proof of Proposition 5.1**

Considering the recursive equation (5.2), the recursive structure in (5.7a) is directly followed from Lemma 5.1 (2) and Lemma 5.2 (3). That is, the correlative function of $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$ are the summation of the correlative functions with respect to all the cases by which this monomial is produced in the same $n(\bar{s})$th-order GFRF, in each case it should include all the correlative functions corresponding to all the p-partition for $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$, and for each p-partition of $c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$, the correlative function should include all the permutations of $x_1x_2\ldots x_p$, since the correlative function $f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(\bar{s}/c_{pq}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)})$ is different with each different permutation which can be seen from (3.5). $f_1(c_{p,q}(\cdot),n(\bar{s});\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})$ is a part of the correlative function for $c_{p,q}(k_1,\cdots,k_{p+q})$ except for $H_{n(\bar{s})-q,p}(j\omega_1,\cdots,j\omega_{n(\bar{s})-q})$, which directly follows from (5.2).

$f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(\bar{s}/c_{pq}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)})$ is a part of the correlative function with respect to a permutation of a p-partition $s_{x_1}\cdots s_{x_p}(\bar{s}/c_{pq}(\cdot))$ of the monomial $\bar{s}/c_{pq}(\cdot)$ which corresponds to a (p,q)-partition for the $n(\bar{s})$th-order GFRF, and it is followed from (3.5). Equation (5.7b) has a similar structure with equation (5.7a), and is an optimised one which simplifies the computation of (5.7a) for the reason that $\prod_{i=1}^{p}\varphi_{n(s_{x_i}(\bar{s}/c_{pq}(\cdot)))}(s_{x_i}(\bar{s}/c_{pq}(\cdot));\omega_{l(X(i)+1)}\cdots\omega_{l(X(i)+n(s_{x_i}(\bar{s}/c_{pq}(\cdot))))})$ is identical to each other under each permutation of a p-partition for the monomial $\bar{s}/c_{pq}(\cdot)$, and therefore the contribution from each permutation is included in $f_{2b}(s_{x_1}\cdots s_{x_p}(\bar{s}/c_{pq}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)})$ which can be obtained from (3.5) and is also given in Peyton-Jones (2007). This completes the proof. □

- **Proof of Proposition 5.2**

From Equation (1.2), it can be obtained that

$$|H_n(j\omega_1,\cdots,j\omega_n)|\le\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|h_n(\tau_1,\cdots,\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))|d\tau_1\cdots d\tau_n$$

which further gives

$$\sup_{\omega_1,\cdots,\omega_n}|H_n(j\omega_1,\cdots,j\omega_n)|\le\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|h_n(\tau_1,\cdots,\tau_n)|d\tau_1\cdots d\tau_n$$

Suppose at point $(\omega_1^*,\cdots,\omega_n^*)$, it holds that

$$\sup_{\omega_1,\cdots,\omega_n}|H_n(j\omega_1,\cdots,j\omega_n)|=|H_n(j\omega_1^*,\cdots,j\omega_n^*)|=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|h_n(\tau_1,\cdots,\tau_n)|d\tau_1\cdots d\tau_n$$

From (5.16a), it can be obtained that

$$|H_n(j\omega_1,\cdots,j\omega_n)|^2\le\lambda_{\max}(\Theta_n)\cdot CE_nCE_n^T$$

Thus it holds that

$$|H_n(j\omega_1^*,\cdots,j\omega_n^*)|^2\le\lambda_{\max}(\Theta_n(\omega_1^*,\cdots,\omega_n^*))\cdot CE_nCE_n^T$$

Hence, $\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|h_n(\tau_1,\cdots,\tau_n)|d\tau_1\cdots d\tau_n\le\sqrt{\sup_{\omega_1,\cdots,\omega_n}(\lambda_{\max}(\Theta_n))}\cdot\|CE_n\|$. Following a similar process, Equation (5.18b) can be obtained. This completes the proof. □

- **Proof of Corollary 5.2**

From (5.10), for a parameter corresponding to a pure input nonlinear term $c_{0,q}(.)$, it can be derived that

$$\varphi_{n(\bar{s})}(c_{0q}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})=\frac{1}{L_{n(\bar{s})}(j\sum_{i=1}^{n(\bar{s})}\omega_{l(i)})}(\prod_{i=1}^{q}(j\omega_{l(i)})^{k_i}$$

There is no $H_1(j\omega_{l(1)})$ appearing in the correlative function. That is, the degree of $H_1(j\omega_{l(1)})$ in the correlative function of this kind of nonlinear parameters is zero. For a parameter corresponding to a pure output nonlinear term $c_{p,0}(.)$, it can be derived that

$$\varphi_{n(\bar{s})}(c_{p0}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})=\varphi_{n(\bar{s})}(c_{n(\bar{s})0}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})=\frac{1}{L_{n(\bar{s})}(j\sum_{i=1}^{n(\bar{s})}\omega_{l(i)})}\prod_{i=1}^{n(\bar{s})}(j\omega_{l(i)})^{k_i}\cdot\prod_{i=1}^{n(\bar{s})}H_1(j\omega_{l(i)})$$

The degree of $H_1(j\omega_{l(1)})$ in the correlative function of this kind of nonlinear parameters is $n(\bar{s})$. For a parameter corresponding to a pure input-output nonlinear term $c_{p,q}(.)$, it can be seen from equation (5.10) that the degree of $H_1(j\omega_{l(1)})$ in the correlative function of this kind of nonlinear parameters is $n(\bar{s})$ -q. Hence, after recursive computation, for a monomial $c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot)$, the degree of $H_1(j\omega_{l(1)})$ in the correlative function is $n(\bar{s})-\sum_{i=0}^{k}q_i=\sum_{i=0}^{k}(p_i+q_i)-k-\sum_{i=0}^{k}q_i=\sum_{i=0}^{k}p_i-k$. It is also noted that the largest order is $n(\bar{s})$ when all q_i=0 corresponding to the parametric monomial whose parameters are all from pure output nonlinearity, and the smallest order is zero when $n(\bar{s})=\sum_{i=0}^{k}q_i$ corresponding to the parametric monomial whose parameters are all from pure input nonlinearity. Therefore, $H_n(j\omega_1,\cdots,j\omega_n)$ can be regarded as an n-degree polynomial function of $H_1(j\omega_{l(1)})$. This completes the proof. □

5.6 Conclusions

A mapping function from the parametric characteristics to the GFRFs is established. The nth-order GFRF can directly be written into a more straightforward and meaningful form in terms of the first order GFRF and model parameters based on the parametric characteristic, which explicitly unveils the linear and nonlinear factors included in the GFRFs and can be regarded as an n-degree polynomial function of the first order GFRF. The new results demonstrate some new properties of the GFRFs, which can reveal clearly the relationship between the nth-order GFRF and its parametric characteristic, and also the relationship between the higher order GFRF and the first order GFRF. These provide a novel and useful insight into the frequency domain analysis and design of nonlinear systems based on the GFRFs. Note that the results of this study are established for nonlinear systems described by the NDE model, similar results can be extended to discrete time nonlinear systems described by NARX model. The frequency characteristics of system output frequency response of nonlinear systems will be studied by using these new results in the next chapter. Moreover, further study will also focus on some detailed issues relating to the application of the theoretical results developed in the present study.

Chapter 6
NONLINEAR EFFECT ON SYSTEM OUTPUT SPECTRUM I ------ ALTERNATING SERIES

The nonlinear effect on system output spectrum is studied for nonlinear Volterra systems. It is shown for the first time that under certain conditions the system output spectrum can be described as an alternating series with respect to some nonlinear parameters. This alternating series has some interesting properties by which system output spectrum can be suppressed easily. The sufficient (and necessary) conditions of this nonlinear effect are studied. These results reveal a novel frequency domain characteristic of the nonlinear effect on a system, and provide a novel insight into the analysis and design of nonlinearities in the frequency domain.

6.1 Introduction

It is known that, the transfer function of a linear system provides a coordinate-free and equivalent description for system characteristics, by which it is convenient to conduct the system analysis and design. Thus frequency domain methods are widely applied in engineering practice. However, as mentioned, although the analysis and design of linear systems in the frequency domain have been well established, the frequency domain analysis for nonlinear systems is not straightforward. Nonlinear systems usually have very complicated output frequency characteristics and dynamic behaviour such as harmonics, inter-modulation, chaos and bifurcation. Investigation and understanding of these nonlinear phenomena in the frequency domain are far from full development.

In this study, understanding of nonlinearity in the frequency domain is investigated from a novel viewpoint for nonlinear Volterra systems. The system output spectrum is shown to be an alternating series with respect to some model nonlinear parameters under certain conditions. This property has great significance in that the system output spectrum can therefore be reduced by a proper design of these model parameters. The sufficient (and necessary) conditions in which the output spectrum can be transformed into an alternating series are studied. These results are illustrated by two examples which involve a spring-damping system with a cubic nonlinear damping. The results established in this study reveal a significant nonlinear effect on the system behaviours in the frequency domain, and provide a novel insight into the analysis and design of nonlinear systems.

The content of this chapter is organised as follows. Section 6.2 provides a simple explanation for the background of this study. The novel nonlinear characteristic and its influence are discussed in Section 6.3. Section 6.4 gives a sufficient and necessary condition under which system output spectrum can be transformed into an alternating series. A conclusion is given in Section 6.5.

6.2 An outline of frequency response functions of nonlinear systems

For convenience, an outline is given in this section for some results discussed in the previous chapters relating to frequency response functions that form the basis of this study. As mentioned, a wide class of nonlinear systems can be approximated by the Volterra series up to a maximum order N around the zero equilibrium (Boyd and Chua 1985) described by (1.1). In this study, consider nonlinear Volterra systems described by the NDE model (1.5). The computation of the nth-order generalized frequency response function (GFRF) for the NDE model (1.5) can be conducted by following Equations (3.8 or 3.11, 3.10, 3.2-3.5). The output spectrum of model (1.6) can be evaluated by (4.1-4.4), i.e.,

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega} H_n(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega \tag{6.1}$$

where,

$$H_n(j\omega_1,\cdots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))d\tau_1\cdots d\tau_n \tag{6.2}$$

is known as the nth-order GFRF defined in George (1959). When the system input is a multi-tone function described by (1.3), the system output frequency response can be described as:

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}H_n(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n}) \tag{6.3}$$

where $F(\omega_{k_i})$ can be explicitly written as

$$F(\omega_{k_i})=\left|F_{|k_i|}\right|e^{j\angle F_{|k_i|}\cdot\mathrm{sig}(k_i)}\ \text{for}\ k_i\in\{\pm1,\cdots,\pm\overline{K}\} \tag{6.4}$$

instead of (4.4), where $\mathrm{sgn}(a)=\begin{cases}1 & a\ge 0\\ -1 & a<0\end{cases}$, and $\omega_{k_i}\in\{\pm\omega_1,\cdots,\pm\omega_{\overline{K}}\}$.

In order to reveal the relationship between the system frequency response functions and the model parameters, the parametric characteristics of the GFRFs and output spectrum are studied in Chapter 3 and Chapter 4. The results show that the nth-order GFRF can be expressed as a more straightforward polynomial function of the system nonlinear parameters, i.e.,

$$H_n(j\omega_1,\cdots,j\omega_n)=CE(H_n(j\omega_1,\cdots,j\omega_n))\cdot f_n(j\omega_1,\cdots,j\omega_n) \tag{6.5}$$

where, $CE(H_n(j\omega_1,\cdots,j\omega_n))$ is referred to as the parametric characteristic of the nth-order GFRF $H_n(j\omega_1,\cdots,j\omega_n)$, which can recursively be determined by (3.17) or (5.3), and $f_n(j\omega_1,\cdots,j\omega_n)$ is a complex valued vector with the same dimension as $CE(H_n(j\omega_1,\cdots,j\omega_n))$. In Chapter 5, a mapping $\varphi_n(CE(H_n(\cdot));\omega_1,\cdots,\omega_n)$ from the parametric characteristic $CE(H_n(j\omega_1,\cdots,j\omega_n))$ to its corresponding correlative function $f_n(j\omega_1,\cdots,j\omega_n)$ is established as

$$\begin{aligned}&\varphi_{n(\bar{s})}(c_{p_0q_0}(\cdot)c_{p_1q_1}(\cdot)\cdots c_{p_kq_k}(\cdot);\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\\ &=\sum_{\substack{\text{all the 2-partitions}\\ \text{for } \bar{s} \text{ satisfying}\\ s_1(\bar{s})=c_{pq}(\cdot)\text{ and } p>0}}\Bigg\{f_1(c_{p,q}(\cdot),n(\bar{s});\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\cdot\sum_{\substack{\text{all the p-partitions}\\ \text{for } \bar{s}/c_{pq}(\cdot)}}\ \sum_{\substack{\text{all the different}\\ \text{permutations}\\ \text{of } \{s_{x_1},\cdots,s_{x_p}\}}}\Bigg[f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(\bar{s}/c_{pq}(\cdot));\omega_{l(1)}\cdots\omega_{l(n(\bar{s})-q)})\\ &\qquad\cdot\prod_{i=1}^{p}\varphi_{n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot)))}(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot));\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot))))})\Bigg]\Bigg\}\end{aligned} \tag{6.6a}$$

where the terminating condition is k=0 and $\varphi_1(1;\omega_i)=H_1(j\omega_i)$ (which is the first order GFRF, i.e., transfer function when all nonlinear parameters are zero), $\{s_{\bar{x}_1},\cdots s_{\bar{x}_p}\}$ is a permutation of $\{s_{x_1},\cdots s_{x_p}\}$, $\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))}$ represents the frequency variables involved in the corresponding functions, $l(i)$ for i=1… $n(\bar{s})$ is a positive integer representing the index of the frequency variables, $\bar{s} = c_{p_0 q_0}(\cdot)c_{p_1 q_1}(\cdot)\cdots c_{p_k q_k}(\cdot)$, $n(s_x(\bar{s})) = \sum_{i=1}^{x}(p_i + q_i) - x + 1$, x is the number of the parameters in s_x, $\sum_{i=1}^{x}(p_i + q_i)$ is the summation of the subscripts of all the parameters in s_x. Moreover, $\bar{X}(i) = \sum_{j=1}^{i-1} n(s_{\bar{x}_j}(\bar{s}/c_{pq}(\cdot)))$, $L_n(j\omega_1 + \cdots + j\omega_n) = -\sum_{r_1=0}^{K} c_{1,0}(r_1)(j\omega_1 + \cdots + j\omega_n)^{r_1}$, and

$$f_1(c_{p,q}(\cdot), n(\bar{s}); \omega_{l(1)} \cdots \omega_{l(n(\bar{s}))}) = (\prod_{i=1}^{q}(j\omega_{l(n(\bar{s})-q+i)})^{r_{p+i}} \Big/ L_{n(\bar{s})}(j\sum_{i=1}^{n(\bar{s})}\omega_{l(i)}) \tag{6.6b}$$

$$f_{2a}(s_{\bar{x}_1} \cdots s_{\bar{x}_p}(\bar{s}/c_{pq}(\cdot)); \omega_{l(1)} \cdots \omega_{l(n(\bar{s})-q)}) = \prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)} + \cdots + j\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{pq}(\cdot))))})^{r_i} \tag{6.6c}$$

The mapping function $\varphi_n(CE(H_n(\cdot)); \omega_1, \cdots, \omega_n)$ enables the complex valued function $f_n(j\omega_1, \cdots, j\omega_n)$ to be analytically and directly determined in terms of the first order GFRF and model nonlinear parameters. Therefore, the nth-order GFRF can directly be written into a more straightforward and meaningful polynomial function in terms of the first order GFRF and model parameters by using the mapping function $\varphi_n(CE(H_n(\cdot)); \omega_1, \cdots, \omega_n)$ as

$$H_n(j\omega_1, \cdots, j\omega_n) = CE(H_n(j\omega_1, \cdots, j\omega_n)) \cdot \varphi_n(CE(H_n(\cdot)); \omega_1, \cdots, \omega_n) \tag{6.7}$$

Using Equation (6.8), Equations (6.1) can be written as

$$Y(j\omega) = \sum_{n=1}^{N} CE(H_n(j\omega_1, \cdots, j\omega_n)) \cdot \bar{F}_n(j\omega) \tag{6.8a}$$

where $\bar{F}_n(j\omega) = \frac{1}{\sqrt{n}(2\pi)^{n-1}} \int_{\omega_1 + \cdots + \omega_n = \omega} \varphi_n(CE(H_n(\cdot); \omega_1, \cdots, \omega_n) \cdot \prod_{i=1}^{n} U(j\omega_i) d\sigma_\omega$. Similarly, Equation (6.3) can be written as

$$Y(j\omega) = \sum_{n=1}^{N} CE(H_n(j\omega_{k_1}, \cdots, j\omega_{k_n})) \cdot \tilde{F}_n(\omega) \tag{6.8b}$$

where $\tilde{F}_n(j\omega) = \frac{1}{2^n} \sum_{\omega_{k_1} + \cdots + \omega_{k_n} = \omega} \varphi_n(CE(H_n(\cdot)); \omega_{k_1}, \cdots, \omega_{k_n}) \cdot F(\omega_{k_1}) \cdots F(\omega_{k_n})$.

As discussed in Chapter 5, it can be seen from Equations (6.7) and (6.8) that the mapping function $\varphi_n(CE(H_n(\cdot)); \omega_1, \cdots, \omega_n)$ can facilitate the frequency domain analysis of nonlinear systems so that the relationship between the frequency response functions and model parameters, and the relationship between the frequency response functions and $H_1(j\omega_{l(1)})$ can be explicitly revealed, and some new properties of the GFRFs and output spectrum can be clearly demonstrated.

In this study, a novel property of the nonlinear effect on system output spectrum is revealed by using the new mapping function $\varphi_n(CE(H_n(\cdot)); \omega_1, \cdots, \omega_n)$ and frequency response functions defined in Equations (6.7-6.8). It is shown that under certain conditions, the nonlinear terms in a system can drive the system output spectrum to be an alternative series of specific model parameters. This reveals a significant nonlinear effect on the system output frequency responses.

6.3 Alternating phenomenon in the output spectrum and its influence

The alternating phenomena and its influence are firstly discussed in this section to point out the significance of this novel property, and then the conditions under which system output spectrum can be expressed into an alternating series are studied in the following section.

For any specific nonlinear parameter c in model (1.5), the output spectrum (6.8a,b) can be expanded with respect to this parameter into a power series as

$$Y(j\omega) = F_0(j\omega) + cF_1(j\omega) + c^2F_2(j\omega) + \cdots + c^{\rho}F_{\rho}(j\omega) + \cdots \tag{6.9}$$

Note that when c represents a pure input nonlinearity, (6.9) may be a finite series; in other cases, it is definitely an infinite series, and if only the first ρ terms in the series (6.9) are considered, there is a truncation error denoted by $o(\rho)$. $F_i(j\omega)$ for i=0,1,2,... can be obtained from $\bar{F}_i(j\omega)$ or $\tilde{F}_i(j\omega)$ in (6.8a,b) by using the mapping $\varphi_n(CE(H_n(\cdot));\omega_1,\cdots,\omega_n)$. Clearly, $F_i(j\omega)$ dominate the property of this power series. Thus the property of this power series can be revealed by studying the property of $\varphi_n(CE(H_n(\cdot));\omega_1,\cdots,\omega_n)$. This will be discussed in detail in the next section. In this section, the alternating phenomenon of this power series and its influence are discussed.

For any $\upsilon \in \mathbb{C}$, define an operator as

$$\mathrm{sgn}_c(\upsilon) = [\mathrm{sgn}_r(\mathrm{Re}(\upsilon)) \quad \mathrm{sgn}_r(\mathrm{Im}(\upsilon))] \tag{6.10}$$

where $\mathrm{sgn}_r(x) = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$ for $x \in \mathbb{R}$.

Definition 6.1 (Alternating series). Consider a power series of form (6.9) with c>0. If $\mathrm{sgn}_c(F_i(j\omega)) = -\mathrm{sgn}_c(F_{i+1}(j\omega))$ for i=0,1,2,3,..., then the series is an alternating series.

The series (6.9) can be written into two series as

$$\begin{aligned} Y(j\omega) &= \mathrm{Re}(Y(j\omega)) + j(\mathrm{Im}(Y(j\omega))) \\ &= \mathrm{Re}(F_0(j\omega)) + c\,\mathrm{Re}(F_1(j\omega)) + c^2\,\mathrm{Re}(F_2(j\omega)) + \cdots + c^{\rho}\,\mathrm{Re}(F_{\rho}(j\omega)) + \cdots \\ &\quad + j(\mathrm{Im}(F_0(j\omega)) + c\,\mathrm{Im}(F_1(j\omega)) + c^2\,\mathrm{Im}(F_2(j\omega)) + \cdots + c^{\rho}\,\mathrm{Im}(F_{\rho}(j\omega)) + \cdots) \end{aligned} \tag{6.11}$$

From Definition 6.1, if $Y(j\omega)$ is an alternating series, then $\mathrm{Re}(Y(j\omega))$ and $\mathrm{Im}(Y(j\omega))$ are both alternating. When (6.9) is an alternating series, there are some interesting properties summarized in Proposition 1. Denote

$$Y(j\omega)_{1\to\rho} = F_0(j\omega) + cF_1(j\omega) + c^2F_2(j\omega) + \cdots + c^{\rho}F_{\rho}(j\omega) \tag{6.12}$$

Proposition 6.1. Suppose (6.9) is an alternating series for c>0, then:

(1) if there exist T>0 and R>0 such that for i>T

$$\min\left\{-\frac{\mathrm{Re}(F_i(j\omega))}{\mathrm{Re}(F_{i+1}(j\omega))}, -\frac{\mathrm{Im}(F_i(j\omega))}{\mathrm{Im}(F_{i+1}(j\omega))}\right\} > R$$

then (6.9) has a radius of convergence R, the truncation error for a finite order $\rho > T$ is $|o(\rho)| \le c^{\rho+1}|F_{\rho+1}(j\omega)|$, and for all $n \ge 0$,

$|Y(j\omega)| \in \Pi_n = [|Y(j\omega)_{1\to T+2n+1}|, |Y(j\omega)_{1\to T+2n}|]$ and $\Pi_{n+1} \subset \Pi_n$;

(2) $|Y(j\omega)|^2 = Y(j\omega)Y(-j\omega)$ is also an alternating series with respect to parameter c; Furthermore, $|Y(j\omega)|^2 = Y(j\omega)Y(-j\omega)$ is alternating only if $\mathrm{Re}(Y(j\omega))$ is alternating;

(3) there exists a $\bar{c} > 0$ such that $\frac{\partial |Y(j\omega)|}{\partial c} < 0$ for $0 < c < \bar{c}$.

Proof.

(1) $Y(j\omega)$ is convergent if and only if $\mathrm{Re}(Y(j\omega))$ and $\mathrm{Im}(Y(j\omega))$ are both convergent. Since $Y(j\omega)$ is an alternating series, $\mathrm{Re}(Y(j\omega))$ and $\mathrm{Im}(Y(j\omega))$ are both alternating from Definition 6.1. Then according to Bromwich (1991), $\mathrm{Re}(Y(j\omega))$ is convergent if $|\mathrm{Re}(c^i F_i(j\omega))| > |\mathrm{Re}(c^{i+1} F_{i+1}(j\omega))|$ and $\lim_{i\to\infty} |\mathrm{Re}(c^i F_i(j\omega))| = 0$. Therefore, if there exists T>0 such that $|\mathrm{Re}(c^i F_i(j\omega))| > |\mathrm{Re}(c^{i+1} F_{i+1}(j\omega))|$ for i>T and $\lim_{i\to\infty} |\mathrm{Re}(c^i F_i(j\omega))| = 0$, the alternating series $\mathrm{Re}(Y(j\omega))$ is also convergent. Now since there exist T>0 and R>0 such that $-\frac{\mathrm{Re}(F_i(j\omega))}{\mathrm{Re}(F_{i+1}(j\omega))} > R$ for i>T and note c<R, it can be obtained that for i>T

$$-\frac{\mathrm{Re}(c^{i+1} F_{i+1}(j\omega))}{\mathrm{Re}(c^i F_i(j\omega))} = -\frac{\mathrm{Re}(c F_{i+1}(j\omega))}{\mathrm{Re}(F_i(j\omega))} = \left|\frac{\mathrm{Re}(c F_{i+1}(j\omega))}{\mathrm{Re}(F_i(j\omega))}\right| < \frac{c}{R} < 1$$

i.e., $|\mathrm{Re}(c^i F_i(j\omega))| > |\mathrm{Re}(c^{i+1} F_{i+1}(j\omega))|$ for i>T and c<R. Moreover, it can also be obtained that for n>0

$$|\mathrm{Re}(F_{T+n}(j\omega))| < \frac{1}{R^n} |\mathrm{Re}(F_T(j\omega))|$$

It further yields that

$$|\mathrm{Re}(c^{T+n} F_{T+n}(j\omega))| < (\frac{c}{R})^n c^T |\mathrm{Re}(F_T(j\omega))|$$

That is, $\lim_{n\to\infty} |\mathrm{Re}(c^{T+n} F_{T+n}(j\omega))| = 0$. Therefore, $\mathrm{Re}(Y(j\omega))$ is convergent. Similarly, it can be proved that $\mathrm{Im}(Y(j\omega))$ is convergent. This proves that $Y(j\omega)$ is convergent. The truncation errors for the real convergent alternating series $\mathrm{Re}(Y(j\omega))$ and $\mathrm{Im}(Y(j\omega))$ are

$$|o_R(\rho)| \le c^{\rho+1} |\mathrm{Re}(F_{\rho+1}(j\omega))| \text{ and } |o_I(\rho)| \le c^{\rho+1} |\mathrm{Im}(F_{\rho+1}(j\omega))|$$

Therefore, the truncation error for the series $Y(j\omega)$ is

$$|o(\rho)| = \sqrt{o_R(\rho)^2 + o_I(\rho)^2} \le c^{\rho+1} |F_{\rho+1}(j\omega)|$$

It can be shown that for $\mathrm{Re}(Y(j\omega))$ and $\mathrm{Im}(Y(j\omega))$, for $n \ge 0$

$$|\mathrm{Re}(Y(j\omega)_{1\to T+1})| < \cdots < |\mathrm{Re}(Y(j\omega)_{1\to T+2n+1})| < |\mathrm{Re}(Y(j\omega))| < |\mathrm{Re}(Y(j\omega)_{1\to T+2n})| < \cdots < |\mathrm{Re}(Y(j\omega)_{1\to T})|$$

$$|\mathrm{Im}(Y(j\omega)_{1\to T+1})| < \cdots < |\mathrm{Im}(Y(j\omega)_{1\to T+2n+1})| < |\mathrm{Im}(Y(j\omega))| < |\mathrm{Im}(Y(j\omega)_{1\to T+2n})| < \cdots < |\mathrm{Im}(Y(j\omega)_{1\to T})|$$

Therefore, $|Y(j\omega)_{1\to T+1}| < \cdots < |Y(j\omega)_{1\to T+2n+1}| < |Y(j\omega)| < |Y(j\omega)_{1\to T+2n}| < \cdots < |Y(j\omega)_{1\to T}|$.

(2)

$$\begin{aligned} |Y(j\omega)|^2 &= Y(j\omega)Y(-j\omega) \\ &= (F_0(j\omega) + cF_1(j\omega) + c^2 F_2(j\omega) + \cdots)(F_0(-j\omega) + cF_1(-j\omega) + c^2 F_2(-j\omega) + \cdots) \\ &= \sum_{n=0,1,2,\ldots} c^n \sum_{i=0}^{n} F_i(j\omega) F_{n-i}(-j\omega) \end{aligned}$$

It can be verified that the (2k)th terms in the series are positive and the (2k+1)th terms are negative. Moreover, it needs only the real parts of the terms in $Y(j\omega)$ to be alternating for $|Y(j\omega)|^2 = Y(j\omega)Y(-j\omega)$ to be alternating.

(3)

$$\frac{\partial|Y(j\omega)|}{\partial c} = \frac{1}{2|Y(j\omega)|}\frac{\partial|Y(j\omega)|^2}{\partial c}$$

$$= \frac{1}{2|Y(j\omega)|}\left\{\mathrm{Re}(F_0(j\omega)F_1(-j\omega)) + c\sum_{n=1,2,\dots} nc^{n-1}\sum_{i=0}^{n} F_i(j\omega)F_{n-i}(-j\omega)\right\}$$

Since $\mathrm{Re}(F_0(j\omega)F_1(-j\omega))$ <0, there must exist $\bar{c} > 0$ such that $\frac{\partial|Y(j\omega)|}{\partial c} < 0$ for $0 < c < \bar{c}$. This completes the proof. □

Proposition 6.1 shows that if the system output spectrum can be expressed as an alternating series with respect to a specific parameter *c*, it is always easier to find a *c* such that the output spectrum is convergent and its magnitude can always be suppressed by a properly designed *c*. Moreover, it is also shown that the low limit of the magnitude of the output spectrum that can be reached is larger than $|Y(j\omega)_{1\to T+2}|$ and the truncation error can also be easily evaluated, if the output spectrum can be expressed into an alternating series.

An example is given to illustrate these results.

Example 6.1. Consider a SDOF spring-damping system with a cubic nonlinear damping which can be described by the following differential equation,

$$m\ddot{y} = -k_0 y - B\dot{y} - c\dot{y}^3 + u(t) \qquad (6.13)$$

Note that k_0 represents the spring characteristic, *B* the damping characteristic and *c* is the cubic nonlinear damping characteristic. This system is a simple case of NDE model (1.5) and can be written into the form of NDE model with *M*=3, *K*=2, $c_{10}(2) = m$, $c_{10}(1) = B$, $c_{10}(0) = k_0$, $c_{30}(111) = c$, $c_{01}(0) = -1$ and all the other parameters are zero.

Note that there is only one output nonlinear term in this case, the *n*th-order GFRF for system (6.13) can be derived according to the algorithm in (3.8 or 3.11, 3.10, 3.2-3.5), which can recursively be given as

$$H_n(j\omega_1,\cdots,j\omega_n) = \frac{c_{3,0}(1,1,1)H_{n,3}(j\omega_1,\cdots,j\omega_n)}{L_n(j\omega_1+\cdots+j\omega_n)}$$

$$H_{n,3}(\cdot) = \sum_{i=1}^{n-2} H_i(j\omega_1,\cdots,j\omega_i)H_{n-i,2}(j\omega_{i+1},\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_i)$$

$$H_{n,1}(j\omega_1,\cdots,j\omega_n) = H_n(j\omega_1,\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_n)$$

Proceeding with the recursive computation above, it can be seen that $H_n(j\omega_1,\cdots,j\omega_n)$ is a polynomial of $c_{30}(111)$, and substituting these equations above into (6.8) gives another polynomial for the output spectrum. By using the relationship (6.7) and the mapping function $\varphi_n(CE(H_n(\cdot));\omega_1,\cdots,\omega_n)$, these results can be obtained directly as follows.

For simplicity, let $u(t) = F_d \sin(\Omega t)$ $(F_d > 0)$. Then $F(\omega_{k_l}) = -jk_l F_d$, for $k_l = \pm 1, \omega_{k_l} = k_l \Omega$, and $l = 1, \cdots, n$ in (6.8b). By using (5.3) or Property 3.3, it can be obtained that for n=0,1,2,3,...

$$CE(H_{2n+1}(j\omega_{k_1}, \cdots, j\omega_{k_{2n+1}})) = (c_{3,0}(1,1,1))^n \text{ and } CE(H_{2n}(j\omega_{k_1}, \cdots, j\omega_{k_{2n}})) = 0 \tag{6.14}$$

Therefore, for n=0,1,2,3,...

$$H_{2n+1}(j\omega_{k_1}, \cdots, j\omega_{k_{2n+1}}) = c^n \cdot \varphi_{2n+1}(CE(H_{2n+1}(\cdot)); \omega_{k_1}, \cdots, \omega_{k_{2n+1}}) \text{ and } H_{2n}(j\omega_{k_1}, \cdots, j\omega_{k_{2n}}) = 0 \tag{6.15}$$

Then the output spectrum at frequency Ω can be computed as

$$Y(j\Omega) = \sum_{n=0}^{\lfloor N-1/2 \rfloor} c^n \cdot \widetilde{F}_{2n+1}(\Omega) \tag{6.16}$$

where $\widetilde{F}_{2n+1}(j\Omega)$ can be computed as

$$\begin{aligned}
\widetilde{F}_{2n+1}(j\Omega) &= \frac{1}{2^{2n+1}} \sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\Omega} \varphi_{2n+1}(CE(H_{2n+1}(\cdot)); \omega_{k_1}, \cdots, \omega_{k_{2n+1}}) \cdot (-jF_d)^{2n+1} \cdot k_1 k_2 \cdots k_{2n+1} \\
&= \frac{1}{2^{2n+1}} \sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\Omega} \varphi_{2n+1}(CE(H_{2n+1}(\cdot)); \omega_{k_1}, \cdots, \omega_{k_{2n+1}}) \cdot (-1)^{n+1} j (F_d)^{2n+1} \cdot (-1)^n \\
&= -j(\frac{F_d}{2})^{2n+1} \sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\Omega} \varphi_{2n+1}(CE(H_{2n+1}(\cdot)); \omega_{k_1}, \cdots, \omega_{k_{2n+1}})
\end{aligned} \tag{6.17}$$

and $\varphi_{2n+1}(CE(H_{2n+1}(\cdot)); \omega_{k_1}, \cdots, \omega_{k_{2n+1}}) = \varphi_{2n+1}(c_{30}(1,1,1)^n; \omega_{k_1}, \cdots, \omega_{k_{2n+1}})$ can be obtained according to Equations (6.6a-c). For example,

$$\varphi_3(c_{30}(1\,1\,1); \omega_{k_1}, \omega_{k_2}, \omega_{k_3}) = \frac{1}{L_3(j\sum_{i=1}^{3}\omega_{k_i})} \cdot \prod_{i=1}^{3}(j\omega_{k_i}) \cdot \prod_{i=1}^{3} H_1(j\omega_{k_i}) = \frac{\prod_{i=1}^{3}(j\omega_{k_i})}{L_3(j\sum_{i=1}^{3}\omega_{k_i})} \cdot \prod_{i=1}^{3} H_1(j\omega_{k_i})$$

$$\begin{aligned}
&\varphi_5(c_{3,0}(1\,1\,1)c_{3,0}(1\,1\,1); \omega_{k_1}, \cdots, \omega_{k_5}) \\
&= f_1(c_{3,0}(1\,1\,1), 5; \omega_{k_1}, \cdots, \omega_{k_5}) \cdot \sum_{\substack{\text{all the 3-partitions} \\ \text{for } c_{3,0}(111)}} \sum_{\substack{\text{all the different} \\ \text{permutations of } \{0,0,1\}}} \Big[f_{2a}(s_{\bar{x}_1} \cdots s_{\bar{x}_p}(c_{3,0}(1\,1\,1)); \omega_{k_1}, \cdots, \omega_{k_5}) \\
&\qquad \cdot \prod_{i=1}^{3} \varphi_{n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot)))}(s_{\bar{x}_i}(c_{3,0}(1\,1\,1)); \omega_{l(\bar{X}(i)+1)} \cdots \omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(\bar{s}/c_{p,q}(\cdot))))}) \Big] \\
&= f_1(c_{3,0}(1\,1\,1), 5; \omega_{k_1}, \cdots, \omega_{k_5}) \cdot \begin{pmatrix} f_{2a}(s_0 s_0 s_1(c_{3,0}(1\,1\,1)); \omega_{k_1}, \cdots, \omega_{k_5}) \varphi_1(1; \omega_{k_1}) \varphi_1(1; \omega_{k_2}) \varphi_3(c_{3,0}(1\,1\,1); \omega_{k_3} \cdots \omega_{k_5}) \\ + f_{2a}(s_0 s_1 s_0(c_{3,0}(1\,1\,1)); \omega_{k_1}, \cdots, \omega_{k_5}) \varphi_1(1; \omega_{k_1}) \varphi_3(c_{3,0}(1\,1\,1); \omega_{k_2} \cdots \omega_{k_4}) \varphi_1(1; \omega_{k_5}) \\ + f_{2a}(s_1 s_0 s_0(c_{30}(1\,1\,1)); \omega_{k_1}, \cdots, \omega_{k_5}) \varphi_3(c_{30}(1\,1\,1); \omega_{k_1} \cdots \omega_{k_3}) \varphi_1(1; \omega_{k_4}) \varphi_1(1; \omega_{k_5}) \end{pmatrix}
\end{aligned}$$

$$= \frac{1}{L_5(j\sum_{i=1}^{5}\omega_{k_i})} \cdot \left(\frac{(j\sum_{i=3}^{5}\omega_{k_i})\prod_{i=1}^{5}(j\omega_{k_i})}{L_3(j\sum_{i=3}^{5}\omega_{k_i})} + \frac{(j\sum_{i=2}^{4}\omega_{k_i})\prod_{i=1}^{5}(j\omega_{k_i})}{L_3(j\sum_{i=2}^{4}\omega_{k_i})} + \frac{(j\sum_{i=1}^{3}\omega_{k_i})\prod_{i=1}^{5}(j\omega_{k_i})}{L_3(j\sum_{i=1}^{3}\omega_{k_i})} \right) \cdot \prod_{i=1}^{5} H_1(j\omega_{k_i})$$

where $\omega_{k_i} \in \{\Omega, -\Omega\}$, and so on. Substituting these results into Equation (6.16), the output spectrum is clearly a power series with respect to the parameter c. When there are more nonlinear terms, it is obvious that the computation process above can directly result in a straightforward multivariate power series with respect to these nonlinear parameters. To check the alternating phenomenon of the output spectrum, consider the following values for each linear parameter: m=240, k_0=16000, B=296, F_d=100, and $\Omega = 8.165$. Then it is obtained that

$$
\begin{aligned}
Y(j\Omega) &= \tilde{F}_1(\Omega) + c\tilde{F}_3(\Omega) + c^2\tilde{F}_5(\Omega) + \cdots \\
&= -j(\frac{F_d}{2})H_1(j\Omega) + 3(\frac{F_d}{2})^3 \frac{\Omega^3 |H_1(j\Omega)|^2 H_1(j\Omega)}{L_1(j\Omega)} \\
&\quad + 3(\frac{F_d}{2})^5 \frac{\Omega^5 |H_1(j\Omega)|^4 H_1(j\Omega)}{L_1(j\Omega)} (\frac{j6\Omega}{L_1(j\Omega)} + \frac{j3\Omega}{L_1(j3\Omega)} + \frac{-j3\Omega}{L_1(-j\Omega)}) + \cdots
\end{aligned}
$$

=(-0.02068817126756 + 0.00000114704116i)
+(5.982851578532449e-006 -6.634300276113922e-010i)c
+(-5.192417616715994e-009 +3.323565122085705e-011i)c^2+... (6.18a)

The series is alternating. In order to check the series further, computation of $\varphi_{2n+1}(c_{3,0}(1,1,1)^n;\omega_{k_1},\cdots,\omega_{k_{2n+1}})$ can be carried out for higher orders. It can also be verified that the magnitude square of the output spectrum (6.18a) is still an alternating series, *i.e.*,

$|Y(j\Omega)|^2$ = (4.280004317115985e-004)-(2.475485177721052e-007)c
+(2.506378395908398e-010)c^2-... (6.18b)

As pointed in Proposition 6.1, it is easy to find a c such that (6.18a-b) are convergent and their limits are decreased. From (6.18b) and according to Proposition 6.1, it can be computed that $0.01671739<|Y(j\Omega)|<0.0192276<0.0206882$ for c=600. This can be verified by Figure 6.1. Figure 6.1 is a result from simulation tests, and shows that the magnitude of the output spectrum decreases when c increases. This property is of great significance in practical engineering systems for output suppression through structural characteristic design or feedback control.

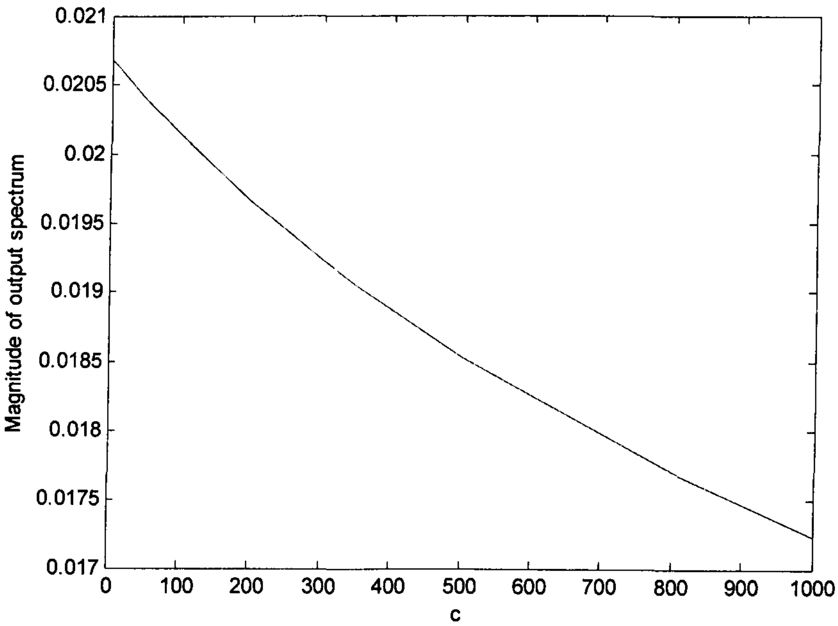

Figure 6.1. Magnitude of output spectrum

6.4 Alternating conditions

In this section, the conditions under which the output spectrum described by Equation (6.9) can be expressed as an alternating series with respect to a specific nonlinear parameter are studied. Suppose the system subjects to a harmonic input $u(t) = F_d \sin(\Omega t)$ $(F_d > 0)$ (The results for this case can be extended to the general input one) and only the output nonlinearities (*i.e.*, $c_{p,0}(.)$ with $p \geq 2$) are considered. For convenience, assume that there is only one nonlinear parameter $c_{p,0}(.)$ in model (1.5) and all the other nonlinear parameters are zero.

Under the assumptions above, it can be obtained from the parametric characteristic analysis in Chapter 3 and Chapter 4 as demonstrated in Example 6.1 and Equation (6.8b) that

$$\begin{aligned} Y(j\Omega) &= Y_1(j\Omega) + Y_p(j\Omega) + \cdots + Y_{(p-1)n+1}(j\Omega) + \cdots \\ &= \tilde{F}_1(\Omega) + c_{p,0}(\cdot)\tilde{F}_p(\Omega) + \cdots + c_{p,0}(\cdot)^n \tilde{F}_{(p-1)n+1}(\Omega) + \cdots \\ &= \tilde{F}_1(\Omega) + c_{p,0}(\cdot)\tilde{F}_p(\Omega) + \cdots + c_{p,0}(\cdot)^n \tilde{F}_{(p-1)n+1}(\Omega) + \cdots \end{aligned} \quad (6.19a)$$

where $\omega_{k_l} \in \{\pm\Omega\}$, $\tilde{F}_{(p-1)n+1}(j\Omega)$ can be computed from (6.8b), and n is a positive integer. Noting that $F(\omega_{k_l}) = -jk_l F_d$, $k_l = \pm 1$, $\omega_{k_l} = k_l\Omega$, and $l = 1, \cdots, n$ in (6.8b),

$$\tilde{F}_{(p-1)n+1}(j\Omega) = \frac{1}{2^{(p-1)n+1}} \sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega} \varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n; \omega_{k_1}, \cdots, \omega_{k_{(p-1)n+1}}) \cdot (-jF_d)^{(p-1)n+1} \cdot k_1 k_2 \cdots k_{(p-1)n+1} \quad (6.19b)$$

If p is an odd integer, then (p-1)n+1 is also an odd integer. Thus there should be (p-1)n/2 frequency variables being $-\Omega$ and (p-1)n/2+1 frequency variables being Ω such that $\omega_{k_1} + \cdots + \omega_{k_{(p-1)n+1}} = \Omega$. In this case,

$$(-jF_d)^{(p-1)n+1} \cdot k_1 k_2 \cdots k_{(p-1)n+1} = (-1) \cdot j \cdot (j^2)^{(p-1)n/2} \cdot (F_d)^{(p-1)n+1} \cdot (-1)^{(p-1)n/2} = -j(F_d)^{(p-1)n+1}$$

If p is an even integer, then (p-1)n+1 is an odd integer for n=2k (k=1,2,3,...) and an even integer for n=2k-1 (k=1,2,3,...). When n is an odd integer, $\omega_{k_1} + \cdots + \omega_{k_{(p-1)n+1}} \neq \Omega$ for $\omega_{k_l} \in \{\pm\Omega\}$. This gives that $\tilde{F}_{(p-1)n+1}(j\Omega)$=0. When n is an even integer, (p-1)n+1 is an odd integer. In this case, it is similar to that p is an odd integer. Therefore, for n>0

$$\tilde{F}_{(p-1)n+1}(j\Omega) = \begin{cases} -j\left(\dfrac{F_d}{2}\right)^{(p-1)n+1} \displaystyle\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega} \varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n; \omega_{k_1}, \cdots, \omega_{k_{(p-1)n+1}}) & \text{if p is odd or n is even} \\ 0 & \text{else} \end{cases} \quad (6.19c)$$

From Equations (6.19a-c) it is obvious that the property of the new mapping $\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n; \omega_{k_1}, \cdots, \omega_{k_{(p-1)n+1}})$ plays a key role in the series. To develop the alternating conditions for series (6.19a), the following results can be obtained.

Lemma 6.1. That $\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n; \omega_{k_1}, \cdots, \omega_{k_{(p-1)n+1}})$ is symmetric or asymmetric has no influence on $\tilde{F}_{(p-1)n+1}(j\Omega)$.

This lemma is obvious since $\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega}(\cdot)$ includes all the possible permutations of $(\omega_{k_1}, \cdots, \omega_{k_{2n+1}})$. Although there are many choices to obtain the asymmetric

$\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{k_1},\cdots,\omega_{k_{(p-1)n+1}})$ which may be different at different permutation $(\omega_{k_1},\cdots,\omega_{k_{(p-1)n+1}})$, they have no different effect on the analysis of $\tilde{F}_{(p-1)n+1}(j\Omega)$.

Lemma 6.2. Consider parameter $c_{p,q}(r_1,r_2,\ldots,r_{p+q})$.

(a1) If $p\ge 2$ and q=0, then

$$\varphi_{n(\bar{x})}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l(n(\bar{x}))})=\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})$$

$$=\frac{(-1)^{n-1}\prod_{i=1}^{(p-1)n+1}H_1(j\omega_{l(i)})}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of } \{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\} \text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\, 0\le\bar{x}_i\le n-1}}\left[\prod_{i=1}^{p}\varphi'_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})\right.$$

$$\left.\cdot\frac{n_x^*(\bar{x}_1,\cdots,\bar{x}_p)}{n_r^*(r_1,\cdots,r_p)}\cdot\sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \{k_1,\cdots,k_p\}}}\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{r_i}\right]$$

where,

$$\varphi'_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})$$

$$=\frac{-1}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of } \{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\} \text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\, 0\le\bar{x}_i\le n-1}}\left[\prod_{i=1}^{p}\varphi'_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})\right.$$

$$\left.\cdot\frac{n_x^*(\bar{x}_1,\cdots,\bar{x}_p)}{n_r^*(r_1,\cdots,r_p)}\cdot\sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \{r_1,\cdots,r_p\}}}\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{r_i}\right]$$

the termination is $\varphi_1'(1;\omega_i)=1$. $n_r^*(r_1,\cdots,r_p)=\frac{p!}{n_1!n_2!\cdots n_e!}$, $n_1+\ldots+n_e=p$, e is the number of distinct differentials r_i appearing in the combination, n_i is the number of repetitions of r_i, and a similar definition holds for $n_x^*(\bar{x}_1,\cdots,\bar{x}_p)$.

(a2) If $p\ge 2$, q=0 and $r_1=r_2=\ldots=r_p=r$, then

$$\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})$$

$$=\frac{(-1)^{n-1}\prod_{i=1}^{(p-1)n+1}[(j\omega_{l(i)})^r H_1(j\omega_{l(i)})]}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}$$

$$\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of } \{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\} \text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\, 0\le\bar{x}_i\le n-1}} n_x^*(\bar{x}_1,\cdots,\bar{x}_p)\cdot\prod_{i=1}^{p}\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$$

where,

if $\bar{x}_i$=0, $\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})=1$, otherwise,

$$\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$$

$$=\frac{(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^r}{-L_{(p-1)\bar{x}_i+1}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})}\cdot$$

$$\sum_{\substack{\text{all the different combinations}\\ \text{of } \{x_1,x_2,\ldots,x_p\} \text{ satisfying}\\ x_1+\cdots+x_p=\bar{x}_i-1,\, 0\le x_j\le\bar{x}_i-1}} n_x^*(x_1,\cdots,x_p)\cdot\prod_{j=1}^{p}\varphi''_{(p-1)x_j+1}(c_{p,0}(\cdot)^{x_j};\omega_{l(\bar{X}'(j)+1)}\cdots\omega_{l(\bar{X}'(j)+(p-1)x_j+1)})$$

The recursive terminal of $\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$ is $\bar{x}_i=1$.

Proof.

$$\varphi_{n(\bar{s})}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})=\varphi_{(p-1)n+1}(c_{p,0}(\cdot)c_{p,0}(\cdot)\cdots c_{p,0}(\cdot);\omega_{l(1)}\cdots\omega_{(l(p-1)n+1)})$$

$$=\sum_{\substack{\text{all the 2-partitions}\\ \text{for } \bar{s} \text{ satisfying}\\ s_1(\bar{s})=c_{p,0}(\cdot)}}\Big\{f_1(c_{p,0}(\cdot),(p-1)n+1;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})\cdot\sum_{\substack{\text{all the p-partitions}\\ \text{for } \bar{s}/c_{3,0}(\cdot)}}\ \sum_{\substack{\text{all the different}\\ \text{permutations}\\ \text{of } \{s_{x_1},\cdots,s_{x_p}\}}}$$

$$\Big[f_{2a}(s_{\bar{x}_1}\cdots s_{\bar{x}_p}(c_{p0}(\cdot)^{n-1});\omega_{l(1)}\cdots\omega_{l(n(\bar{s}))})\cdot\prod_{i=1}^{p}\varphi_{n(s_{\bar{x}_i}(c_{p,0}(\cdot)^{n-1}))}(s_{\bar{x}_i}(c_{p,0}(\cdot)^{n-1});\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(c_{p,0}(\cdot)^{n-1})))})\Big]\Big\}$$

$$=\frac{1}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}\cdot\sum_{\substack{\text{all the p-partitions}\\ \text{for } \bar{s}/c_{p,0}(\cdot)}}\ \sum_{\substack{\text{all the different}\\ \text{permutations}\\ \text{of } \{s_{x_1},\cdots,s_{x_p}\}}}\Big[\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(c_{p,0}(\cdot)^{n-1})))})^{r_i}$$

$$\cdot\prod_{i=1}^{p}\varphi_{n(s_{\bar{x}_i}(c_{p,0}(\cdot)^{n-1}))}(s_{\bar{x}_i}(c_{p,0}(\cdot)^{n-1});\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+n(s_{\bar{x}_i}(c_{p,0}(\cdot)^{n-1})))})\Big]$$

$$=\frac{1}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of } \{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\} \text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\ 0\le\bar{x}_i\le n-1}}\ \sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \text{each combination}}}\Big[\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{r_i}$$

$$\cdot\prod_{i=1}^{p}\varphi_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})\Big]$$

Note that different permutations in each combination have no difference to $\prod_{i=1}^{p}\varphi_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$, thus $\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_1\cdots\omega_{(p-1)n+1})$ can be written as

$$\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_1\cdots\omega_{(p-1)n+1})$$

$$=\frac{1}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of } \{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\} \text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_3=n-1,\ 0\le\bar{x}_i\le n-1}}\ \prod_{i=1}^{p}\varphi_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$$

$$\cdot\sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \text{each combination}}}\ \prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{r_i}$$

$$=\frac{1}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of } \{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\} \text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\ 0\le\bar{x}_i\le n-1}}\ \prod_{i=1}^{p}\varphi_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$$

$$\cdot\frac{n_x^*(\bar{x}_1,\cdots,\bar{x}_p)}{n_r^*(r_1,\cdots,r_p)}\cdot\sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \{r_1,\cdots,r_p\}}}\ \prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{r_i}$$

$n_x^*(\bar{x}_1,\cdots,\bar{x}_p)$ and $n_r^*(r_1,\cdots,r_p)$ are the numbers of the corresponding combinations involved, which can be obtained from the combination theory and can also be referred to Peyton-Jones (2007). From an inspection of the recursive relationship in the equation above, it can be seen that there are (p-1)n +1 $H_1(j\omega_i)$ with different frequency variable at the end of the recursive relationship. Thus they can be brought out as a common factor. This gives

$$\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})=(-1)^n\prod_{i=1}^{(p-1)n+1}H_1(j\omega_{l(i)})\cdot\varphi'_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})\quad(6.20a)$$

where,

$$\varphi'_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})$$
$$=\frac{-1}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of }\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\}\text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\,0\le\bar{x}_i\le n-1}}\prod_{i=1}^{p}\varphi'_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$$
$$\cdot\frac{n_x^*(\bar{x}_1,\cdots,\bar{x}_p)}{n_r^*(r_1,\cdots,r_p)}\cdot\sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \{r_1,\cdots,r_p\}}}\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{r_i} \tag{6.20b}$$

the termination is $\varphi'_1(1;\omega_i)=1$. Note that when $\bar{x}_i=0$, there is a term $(j\omega_{l(\bar{X}(i)+1)})^{r_i}$ appearing from $\frac{n_x^*(\bar{x}_1,\cdots,\bar{x}_p)}{n_r^*(r_1,\cdots,r_p)}\cdot\sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \{r_1,\cdots,r_p\}}}\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{r_i}$. It can be verified that in each recursion of $\varphi'_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})$, there may be some frequency variables appearing individually in the form of $(j\omega_{l(\bar{X}(i)+1)})^{r_i}$, and these variables will not appear individually in the same form in the subsequent recursion. At the end of the recursion, all the frequency variables should have appeared in this form. Thus these terms can also be brought out as common factors if $r_1=r_2=\ldots=r_p$. In the case of $r_1=r_2=\ldots=r_p=r$,

$$\frac{n_x^*(\bar{x}_1,\cdots,\bar{x}_p)}{n_r^*(r_1,\cdots,r_p)}\cdot\sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \{r_1,\cdots,r_p\}}}\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{r_i}$$
$$=n_x^*(\bar{x}_1,\cdots,\bar{x}_p)\cdot\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{r_i}$$

Therefore (6.20ab) can be written, if $r_1=r_2=\ldots=r_p$, as

$$\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})$$
$$=(-1)^n\prod_{i=1}^{(p-1)n+1}[(j\omega_{l(i)})^r H_1(j\omega_{l(i)})]\cdot\varphi'_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)}) \tag{6.21a}$$

$$\varphi'_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})$$
$$=\frac{-1}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of }\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\}\text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\,0\le\bar{x}_i\le n-1}}\prod_{i=1}^{p}\varphi'_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$$
$$\cdot n_x^*(\bar{x}_1,\cdots,\bar{x}_p)\cdot\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{r_i(1-\delta(\bar{x}_i))} \tag{6.21b}$$

(6.21b) can be further written as

$$\varphi'_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})$$
$$=\frac{-1}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}$$
$$\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of }\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\}\text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\,0\le\bar{x}_i\le n-1}} n_x^*(\bar{x}_1,\cdots,\bar{x}_p)\cdot\prod_{i=1}^{p}\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)}) \tag{6.22a}$$

where, if $\bar{x}_i=0$,

$$\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})=1,$$

otherwise,

$$\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$$
$$=(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)x_i+1)})^{r}\varphi'_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})$$
$$=\frac{(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{r}}{-L_{(p-1)\bar{x}_i+1}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})}\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of }\{x_1,x_2,\ldots,x_p\}\text{ satisfying}\\ x_1+\cdots+x_p=\bar{x}_i-1,\,0\le x_i\le\bar{x}_i-1}} n_x^{*}(x_1,\cdots,x_p)$$
$$\cdot\prod_{i=1}^{p}(j\omega_{l(\bar{X}'(i)+1)}+\cdots+j\omega_{l(\bar{X}'(i)+(p-1)x_i+1)})^{r_i(1-\delta(x_i))}\varphi'_{(p-1)x_i+1}(c_{p,0}(\cdot)^{x_i};\omega_{l(\bar{X}'(i)+1)}\cdots\omega_{l(\bar{X}'(i)+(p-1)x_i+1)})$$
$$=\frac{(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{r}}{-L_{(p-1)\bar{x}_i+1}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})}\cdot$$
$$\sum_{\substack{\text{all the different combinations}\\ \text{of }\{x_1,x_2,\ldots,x_p\}\text{ satisfying}\\ x_1+\cdots+x_p=\bar{x}_i-1,\,0\le x_i\le\bar{x}_i-1}} n_x^{*}(x_1,\cdots,x_p)\cdot\prod_{i=1}^{p}\varphi''_{(p-1)x_i+1}(c_{p,0}(\cdot)^{x_i};\omega_{l(\bar{X}'(i)+1)}\cdots\omega_{l(\bar{X}'(i)+(p-1)x_i+1)}) \tag{6.22b}$$

The recursive terminal of (6.22b) is $\bar{x}_i=1$. Replacing (6.20b) into (6.20a) and replacing (6.22ab) into (6.21a), the lemma can be obtained. This completes the proof. □

For convenience, define an operator "*" for sgn$_c$(.) satisfying

$$\mathrm{sgn}_c(\upsilon_1)*\mathrm{sgn}_c(\upsilon_2)=[\mathrm{sgn}_r(\mathrm{Re}(\upsilon_1\upsilon_2))\quad \mathrm{sgn}_r(\mathrm{Im}(\upsilon_1\upsilon_2))]$$

for any $\upsilon_1,\upsilon_2\in\mathbb{C}$. It is obvious $\mathrm{sgn}_c(\upsilon_1)*\mathrm{sgn}_c(\upsilon_2)=\mathrm{sgn}_c(\upsilon_1\upsilon_2)$.

The following lemma is straightforward.

Lemma 6.3. For $\upsilon_1,\upsilon_2,\nu\in\mathbb{C}$, suppose $\mathrm{sgn}_c(\upsilon_1)=-\mathrm{sgn}_c(\upsilon_2)$. If $\mathrm{Re}(\nu)\mathrm{Im}(\nu)=0$, then $\mathrm{sgn}_c(\upsilon_1\nu)=-\mathrm{sgn}_c(\upsilon_2\nu)$. If $\mathrm{Re}(\nu)\mathrm{Im}(\nu)=0$ and $\nu\neq0$, then $\mathrm{sgn}_c(\upsilon_1/\nu)=-\mathrm{sgn}_c(\upsilon_2/\nu)$. □

Proposition 6.2. The output spectrum in (6.19a) is an alternating series with respect to any specific parameter $c_{p,0}(r_1,r_2,\ldots,r_p)$ satisfying $c_{p,0}(.)>0$ and $p=2\bar{r}+1$ for $\bar{r}=1,2,3,\ldots$

(a1) if and only if

$$\mathrm{sgn}_c\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega}(-1)^{n-1}\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^{n};\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})\right)=const\text{, i.e.,}$$

$$\mathrm{sgn}_c\left(\begin{array}{l}\dfrac{H_1(j\Omega)}{L_{(p-1)n+1}(j\Omega)}\displaystyle\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega}\ \sum_{\substack{\text{all the different combinations}\\ \text{of }\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\}\text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\,0\le\bar{x}_i\le n-1}}\left[\prod_{i=1}^{p}\varphi'_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})\right.\\ \left.\cdot\dfrac{n_x^{*}(\bar{x}_1,\cdots,\bar{x}_p)}{n_r^{*}(r_1,\cdots,r_p)}\cdot\displaystyle\sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \{r_1,\cdots,r_p\}}}\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{r_i}\right]\end{array}\right)$$
$$=const \tag{6.23}$$

(a2) if $r_1=r_2=\ldots=r_p=r$ in $c_{p,0}(.)$, $\mathrm{Re}(\frac{H_1(j\Omega)}{L_{(p-1)n+1}(j\Omega)})\mathrm{Im}(\frac{H_1(j\Omega)}{L_{(p-1)n+1}(j\Omega)})=0$, and

$$\mathrm{sgn}_c\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega}\ \sum_{\substack{\text{all the different combinations}\\ \text{of }\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\}\text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\,0\le\bar{x}_i\le n-1}}\left[n_x^*(\bar{x}_1,\cdots,\bar{x}_p)\cdot\prod_{i=1}^{p}\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})\right]\right)=const \quad (6.24)$$

where *const* is a two-dimensional constant vector whose elements are +1, 0 or -1.

Proof. (a1). From Lemma 6.1, any asymmetric $\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{k_1},\cdots,\omega_{k_{(p-1)n+1}})$ is sufficient for the computation of $\tilde{F}_{(p-1)n+1}(j\Omega)$. It can be obtained that

$$\mathrm{sgn}_c(\tilde{F}_{(p-1)n+1}(j\Omega))=\mathrm{sgn}_c(-j(\frac{F_d}{2})^{(p-1)n+1})*\mathrm{sgn}(\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega}\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{k_1},\cdots,\omega_{k_{(p-1)n+1}}))$$

From Lemma 6.3, $\mathrm{sgn}_c(-j(\frac{F_d}{2})^{(p-1)n+1})$ has no effect on the alternating nature of the sequence $\tilde{F}_{(p-1)n+1}(j\Omega)$ for n=1,2,3,.... Hence, (6.19a) is an alternating series with respect to $c_{p,0}(.)$ if and only if the sequence $\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega}\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{k_1},\cdots,\omega_{k_{(p-1)n+1}})$ for n=1,2,3,... is alternating. This is equivalent to

$$\mathrm{sgn}_c\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega}(-1)^{n-1}\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{l(1)}\cdots\omega_{l((p-1)n+1)})\right)=const$$

In the equation above, replacing $\varphi_{(p-1)n+1}(c_{p,0}(\cdot)^n;\omega_{k_1},\cdots,\omega_{k_{(p-1)n+1}})$ by using the result in Lemma 6.2 and noting (p-1)n+1 is an odd integer, it can be obtained that

$$\mathrm{sgn}_c\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega}\frac{\prod_{i=1}^{(p-1)n+1}H_1(j\omega_{l(i)})}{L_{(p-1)n+1}(j\omega_{l(1)}+\cdots+j\omega_{l((p-1)n+1)})}\cdot\sum_{\substack{\text{all the different combinations}\\ \text{of }\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\}\text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\,0\le\bar{x}_i\le n-1}}\left[\prod_{i=1}^{p}\varphi'_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})\cdot\frac{n_x^*(\bar{x}_1,\cdots,\bar{x}_p)}{n_r^*(r_1,\cdots,r_p)}\cdot\sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \{r_1,\cdots,r_p\}}}\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{r_i}\right]\right)$$

$$=\mathrm{sgn}_c\left(\frac{H_1(j\Omega)\prod_{i=1}^{(p-1)n/2}|H_1(j\Omega)|^2}{L_{(p-1)n+1}(j\Omega)}\cdot\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega}\ \sum_{\substack{\text{all the different combinations}\\ \text{of }\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\}\text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\,0\le\bar{x}_i\le n-1}}\left[\prod_{i=1}^{p}\varphi'_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})\cdot\frac{n_x^*(\bar{x}_1,\cdots,\bar{x}_p)}{n_r^*(r_1,\cdots,r_p)}\cdot\sum_{\substack{\text{all the different}\\ \text{permutations of}\\ \{r_1,\cdots,r_p\}}}\prod_{i=1}^{p}(j\omega_{l(\bar{X}(i)+1)}+\cdots+j\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})^{r_i}\right]\right)=const$$

Note that $\prod_{i=1}^{(p-1)n/2}|H_1(j\Omega)|^2$ has no effect on the equality above from using Lemma 6.3, then the equation above is equivalent to (6.23).

(a2). If additionally, $r_1=r_2=\ldots=r_p=r$ in $c_{p,0}(.)$, then using the result in Lemma 6.2, (6.23) can be written as

$$\mathrm{sgn}_c\left(\frac{(j\Omega)^r H_1(j\Omega)}{L_{(p-1)n+1}(j\Omega)}\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega}\ \sum_{\substack{\text{all the different combinations}\\ \text{of }\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\}\text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\,0\le\bar{x}_i\le n-1}}\left[n_x^*(\bar{x}_1,\cdots,\bar{x}_p)\cdot\prod_{i=1}^{p}\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})\right]\right)=const$$

From Lemma 6.3, $(j\Omega)^r$ has no effect on this equation. Then the equation above is equivalent to

$$\mathrm{sgn}_c\left(\frac{H_1(j\Omega)}{L_{(p-1)n+1}(j\Omega)}\sum_{\omega_{k_1}+\cdots+\omega_{k_{(p-1)n+1}}=\Omega}\ \sum_{\substack{\text{all the different combinations}\\ \text{of }\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_p\}\text{ satisfying}\\ \bar{x}_1+\cdots+\bar{x}_p=n-1,\,0\le\bar{x}_i\le n-1}}\left[n_x^*(\bar{x}_1,\cdots,\bar{x}_p)\cdot\prod_{i=1}^{p}\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i};\omega_{l(\bar{X}(i)+1)}\cdots\omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)})\right]\right)=const$$

If $\mathrm{Re}(\frac{H_1(j\Omega)}{L_{(p-1)n+1}(j\Omega)})\mathrm{Im}(\frac{H_1(j\Omega)}{L_{(p-1)n+1}(j\Omega)})=0$, then $\frac{H_1(j\Omega)}{L_{(p-1)n+1}(j\Omega)}$ has no effect, either. This gives Equation (6.24). The proof is completed. □

Proposition 6.2 provides a sufficient and necessary condition for the output spectrum series (6.19a) to be an alternating series with respect to a specific nonlinear parameter $c_{p,0}(r_1,r_2,\ldots,r_p)$ satisfying $c_{p,0}(.)>0$ and $p=2\bar{r}+1$ for $\bar{r}=1,2,3,\ldots$. Similar results can also be established for any other nonlinear parameters. Regarding nonlinear parameter $c_{p,0}(r_1,r_2,\ldots,r_p)$ satisfying $c_{p,0}(.)>0$ and $p=2\bar{r}$ for $\bar{r}=1,2,3,\ldots$, it can be obtained from (6.19a) that

$$Y(j\Omega)=\tilde{F}_1(\Omega)+c_{p,0}(\cdot)^2\tilde{F}_{2(p-1)+1}(\Omega)+\cdots+c_{p,0}(\cdot)^{2n}\tilde{F}_{2(p-1)n+1}(\Omega)+\cdots$$

$\tilde{F}_{2(p-1)n+1}(\Omega)$ for n=1,2,3,... should be alternating so that $Y(j\Omega)$ is alternating. This yields that

$$\mathrm{sgn}_c\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{2(p-1)n+1}}=\Omega}\varphi_{2(p-1)n+1}(c_{p,0}(\cdot)^{2n};\omega_{l(1)}\cdots\omega_{l(2(p-1)n+1)})\right)$$

$$=-\mathrm{sgn}_c\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{2(p-1)(n+1)+1}}=\Omega}\varphi_{2(p-1)(n+1)+1}(c_{p,0}(\cdot)^{2(n+1)};\omega_{l(1)}\cdots\omega_{l(2(p-1)(n+1)+1)})\right)$$

Clearly, this is completely different from the conditions in Proposition 6.2. It may be more difficult for the output spectrum to be alternating with respect to $c_{p,0}(.)>0$ with $p=2\bar{r}$ than $c_{p,0}(.)>0$ with $p=2\bar{r}+1$.

Note that Equation (6.19a) is based on the assumption that there is only nonlinear parameter $c_{p,0}(.)$ and all the other nonlinear parameters are zero. If the effects from the other nonlinear parameters are considered, Equation (6.19a) can be written as

$$Y(j\Omega)=\tilde{F}_1'(\Omega)+c_{p,0}(\cdot)\tilde{F}_p'(\Omega)+\cdots+c_{p,0}(\cdot)^n\tilde{F}'_{(p-1)n+1}(\Omega)+\cdots \quad (6.25a)$$

where

$$\tilde{F}'_{(p-1)n+1}(\Omega) = \tilde{F}_{(p-1)n+1}(\Omega) + \delta_{(p-1)n+1}(\Omega; C_{p',q'} \setminus c_{p,0}(.)) \tag{6.25b}$$

$C_{p',q'}$ includes all the nonlinear parameters in the system. Based on the parametric characteristic analysis in Chapter 3 and the new mapping function $\varphi_n(CE(H_n(\cdot)); \omega_1, \cdots, \omega_n)$ defined in Chapter 5, (6.25b) can be determined consequently. For example, suppose p is an odd integer larger than 1, then $\tilde{F}_{(p-1)n+1}(j\Omega)$ is given in (6.19c), and $\delta_{(p-1)n+1}(\Omega; C_{p',q'} \setminus c_{p,0}(.))$ can be computed as

$$\delta_{(p-1)n+1}(\Omega; C_{p',q'} \setminus c_{p,0}(.)) = \sum_{\substack{\text{all the monomails consisting of the parameters in } C_{p',q'} \setminus c_{p,0}(\cdot) \\ \text{satisfying } np+\sum(p_i'+q_i') \text{ is odd and less than N}}} \left[-j\left(\frac{F_d}{2}\right)^{n(c_{p,0}{}^n s(\cdot))} \cdot \sum_{\omega_{k_1}+\cdots+\omega_{k_{n(c_{p,0}{}^n s(\cdot))}}=\Omega} \varphi_{n(c_{p,0}{}^n s(\cdot))}(c_{p,0}{}^n s(C_{p',q'} \setminus c_{p,0}(\cdot)); \omega_{k_1} \cdots \omega_{k_{n(c_{p,0}{}^n s(\cdot))}}) \right]$$

where $s(C_{p',q'} \setminus c_{p,0}(\cdot))$ denotes a monomial consisting of some parameters in $C_{p',q'} \setminus c_{p,0}(\cdot)$.

It is obvious that if (6.19a) is an alternating series, then (6.25a) can still be alternating under a proper design of the other nonlinear parameters (For example, these parameters are sufficiently small). Moreover, from the discussions above, it can be seen that whether the system output spectrum is an alternating series or not with respect to a specific nonlinear parameter is greatly dependent on the system linear parameters.

Example 6.2. To demonstrate the theoretical results above, consider again the model (6.13) in Example 6.1. Let $u(t) = F_d \sin(\Omega t)$ $(F_d > 0)$. The output spectrum at frequency Ω is given in (6.16-6.17). From Lemma 6.2, it can be derived for this case that

$$\varphi_{2n+1}(c_{3,0}(\cdot)^n; \omega_{l(1)} \cdots \omega_{l(2n+1)}) = \frac{(-1)^{n-1} \prod_{i=1}^{2n+1} [(j\omega_{l(i)})^r H_1(j\omega_{l(i)})]}{L_{2n+1}(j\omega_{l(1)} + \cdots + j\omega_{l(2n+1)})} \cdot \sum_{\substack{\text{all the different combinations} \\ \text{of } \{\bar{x}_1,\bar{x}_2,\bar{x}_3\} \text{ satisfying} \\ \bar{x}_1+\bar{x}_2+\bar{x}_3=n-1,\, 0\le \bar{x}_i \le n-1}} n_x^*(\bar{x}_1,\bar{x}_2,\bar{x}_3) \cdot \prod_{i=1}^{3} \varphi''_{2\bar{x}_i+1}(c_{3,0}(\cdot)^{\bar{x}_i}; \omega_{l(\bar{X}(i)+1)} \cdots \omega_{l(\bar{X}(i)+2\bar{x}_i+1)}) \tag{6.26a}$$

where, if $\bar{x}_i$=0, $\varphi''_{(p-1)\bar{x}_i+1}(c_{p,0}(\cdot)^{\bar{x}_i}; \omega_{l(\bar{X}(i)+1)} \cdots \omega_{l(\bar{X}(i)+(p-1)\bar{x}_i+1)}) = 1$, otherwise,

$$\varphi''_{2\bar{x}_i+1}(c_{3,0}(\cdot)^{\bar{x}_i}; \omega_{l(\bar{X}(i)+1)} \cdots \omega_{l(\bar{X}(i)+2\bar{x}_i+1)}) = \frac{(j\omega_{l(\bar{X}(i)+1)} + \cdots + j\omega_{l(\bar{X}(i)+2\bar{x}_i+1)})^r}{-L_{2\bar{x}_i+1}(j\omega_{l(\bar{X}(i)+1)} + \cdots + j\omega_{l(\bar{X}(i)+2\bar{x}_i+1)})} \cdot \sum_{\substack{\text{all the different combinations} \\ \text{of } \{x_1,x_2,x_3\} \text{ satisfying} \\ x_1+x_2+x_3=\bar{x}_i-1,\, 0\le x_j \le \bar{x}_i-1}} n_x^*(x_1,x_2,x_3) \cdot \prod_{j=1}^{3} \varphi''_{2x_j+1}(c_{3,0}(\cdot)^{x_j}; \omega_{l(\bar{X}'(j)+1)} \cdots \omega_{l(\bar{X}'(j)+2x_j+1)}) \tag{6.26b}$$

Note that the terminal condition for (6.26ab) is

$$\varphi''_{2\bar{x}_i+1}(c_{3,0}(\cdot)^{\bar{x}_i}; \omega_{l(\bar{X}(i)+1)} \cdots \omega_{l(\bar{X}(i)+2\bar{x}_i+1)})\Big|_{\bar{x}_i=1} = \varphi''_3(c_{3,0}(\cdot); \omega_{l(1)} \cdots \omega_{l(3)}) = \frac{(j\omega_{l(1)} + \cdots + j\omega_{l(3)})^r}{-L_3(j\omega_{l(1)} + \cdots + j\omega_{l(3)})} \tag{6.26c}$$

Therefore, from (6.26a-c) it can be shown that $\varphi_{2n+1}(c_{3,0}(\cdot)^n; \omega_1 \cdots \omega_{2n+1})$ can be written as

$$\varphi_{2n+1}(c_{3,0}(\cdot)^n;\omega_1\cdots\omega_{2n+1})$$
$$=\frac{(-1)^{n-1}\prod_{i=1}^{2n+1} j\omega_i H_1(j\omega_i)}{L_{2n+1}(j\omega_1+\cdots+j\omega_{2n+1})}\cdot\sum_{\substack{\text{all the combination }(x_1,x_2,\ldots,x_{n-1})\\ \text{satisfying } x_i\in\{2j+1|1\le j\le n-1\}\\ x_1\ge x_2\ge\ldots\ge x_{n-1},\text{ and}\\ \text{"=" happens only if } x_i+x_{i+1}\le 2n-2}} r_X(x_1,x_2,\ldots,x_{n-1})\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})} \quad (6.27)$$

where $r_X(x_1,x_2,\ldots,x_{n-1})$ is a positive integer which can be explicitly determined by (6.26ab) and represents the number of all the involved combinations which have the same $\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}$. Therefore, according to Proposition 6.2, it can be seen from (6.27) that the output spectrum (6.16) is an alternating series only if the following two conditions hold:

$$\text{(b1)}\ \mathrm{Re}\Big(\frac{H_1(j\Omega)}{L_{2n+1}(j\Omega)}\Big)\mathrm{Im}\Big(\frac{H_1(j\Omega)}{L_{2n+1}(j\Omega)}\Big)=0$$

$$\text{(b2)}\ \mathrm{sgn}_c\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\Omega}\ \sum_{\substack{\text{all the combination }(x_1,x_2,\ldots,x_{n-1})\\ \text{satisfying } x_i\in\{2j+1|1\le j\le n-1\}\\ x_1\ge x_2\ge\ldots\ge x_{n-1},\text{ and}\\ \text{"=" happens only if } x_i+x_{i+1}\le 2n-2}} r_X(x_1,x_2,\ldots,x_{n-1})\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}\right)=const$$

Suppose $\Omega=\sqrt{\frac{k_0}{m}}$ which is a natural resonance frequency of model (6.13). It can be derived that

$$L_{2n+1}(j\Omega)=-\sum_{k_1=0}^{K}c_{1,0}(r_1)(j\Omega)^{r_1}=-(m(j\Omega)^2+B(j\Omega)+k_0)=-jB\Omega$$
$$H_1(j\Omega)=\frac{-1}{L_1(j\Omega)}=\frac{1}{jB\Omega}$$

It is obvious that condition (b1) is satisfied if $\Omega=\sqrt{\frac{k_0}{m}}$. Considering condition (b2), it can be derived that

$$\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}=\frac{j\varepsilon(x_i)\Omega}{-L_{x_i}(j\varepsilon(x_i)\Omega)} \quad (6.28a)$$

where $\varepsilon(x_i)\in\{\pm(2j+1)|0\le j\le\lceil n+1\rceil\}$, and $\lceil n+1\rceil$ denotes the odd integer not larger than n+1. Especially, when $\varepsilon(x_i)=\pm1$, it yields that

$$\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}=\frac{\pm j\Omega}{-L_{x_i}(\pm j\Omega)}=\frac{\pm j\Omega}{\pm jB\Omega}=\frac{1}{B} \quad (6.28b)$$

when $|\varepsilon(x_i)|>1$,

$$\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}=\frac{j\varepsilon(x_i)\Omega}{-L_{x_i}(j\varepsilon(x_i)\Omega)}=\frac{j\varepsilon(x_i)\Omega}{m(j\varepsilon(x_i)\Omega)^2+B(j\varepsilon(x_i)\Omega)+k_0}$$
$$=\frac{j\varepsilon(x_i)\Omega}{(1-\varepsilon(x_i)^2)k_0+j\varepsilon(x_i)\Omega B}=\frac{1}{B+j(\varepsilon(x_i)-\frac{1}{\varepsilon(x_i)})\sqrt{k_0m}} \quad (6.28c)$$

If $B<<\sqrt{k_0m}$, then it gives

$$\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}\approx\frac{1}{j(\varepsilon(x_i)-\frac{1}{\varepsilon(x_i)})\sqrt{k_0m}} \quad (6.28d)$$

Note that in all the combinations involved in the summation operator in (6.27) or condition (b2), *i.e.*,

$$\sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\Omega} \sum_{\substack{\text{all the combination } (x_1,x_2,\ldots,x_{n-1}) \\ \text{satisfying } x_i\in\{2j+1|1\le j\le n-1\} \\ x_1\ge x_2\ge\ldots\ge x_{n-1},\text{ and} \\ \text{"=" happens only if } x_i+x_{i+1}\le 2n-2}} (\cdot)$$

There always exists a combination such that

$$\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}=\frac{1}{B^{n-1}} \tag{6.29}$$

Note that (6.28b) holds both for $\varepsilon(x_i)=\pm 1$, thus there is no combination such that

$$\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}=-\frac{1}{B^{n-1}}$$

Noting that $B<<\sqrt{k_0 m}$, these show that

$$\max_{\substack{\text{all the involved}\\ \text{combinations}}}\left(\left|\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}\right|\right)=\frac{1}{B^{n-1}}$$

which happens in the combination where (6.29) holds.

Because there are n+1 frequency variables to be +Ω and n frequency variables to be –Ω such that $\omega_1+\cdots+\omega_{2n+1}=\Omega$ in (6.16-17), there are more combinations where $\varepsilon(x_i)>0$ that is $(\varepsilon(x_i)-\frac{1}{\varepsilon(x_i)})\sqrt{k_0 m}>0$ in (6.28c-d). Thus there are more combinations where $\mathrm{Im}(\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})})$ is negative. Using (6.28b) and (6.28d), it can be shown under the condition that $B<<\sqrt{k_0 m}$,

$$\max_{\substack{\text{all the involved}\\ \text{combinations}}}\left(\left|\mathrm{Im}\left(\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}\right)\right|\right)\approx\frac{1}{B^{n-2}(\varepsilon(x_i)-\frac{1}{\varepsilon(x_i)})\sqrt{k_0 m}}\bigg|_{\varepsilon(x_i)=3}=\frac{1}{2.7B^{n-2}\sqrt{k_0 m}}$$

This happens in the combinations where the argument of $\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}$ is either -90^0 or +90^0. Note that there are more cases in which the arguments are -90^0. If the argument is -180^0, the absolute value of the corresponding imaginary part will be not more than

$$\max_{\substack{\text{the combination}\\ \text{whose argument is}\\ -180^0}}\left(\left|\mathrm{Im}\left(\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}\right)\right|\right)\approx\frac{1}{B^{n-4}(\varepsilon(x_i)-\frac{1}{\varepsilon(x_i)})^3\sqrt{k_0 m}^{\,3}}\bigg|_{\varepsilon(x_i)=3}=\frac{1}{2.7^3B^{n-4}\sqrt{k_0 m}^{\,3}}$$

which is much less than $\frac{1}{2.7B^{n-2}\sqrt{k_0 m}}$.

Therefore, if B is sufficiently smaller than $\sqrt{k_0 m}$, the following two inequalities can hold for n>1

$$\mathrm{Re}\Bigg(\sum_{\substack{\text{all the combination } (x_1,x_2,\ldots,x_{n-1}) \\ \text{satisfying } x_i\in\{2j+1|1\le j\le n-1\} \\ x_1\ge x_2\ge\ldots\ge x_{n-1},\text{ and} \\ \text{"=" happens only if } x_i+x_{i+1}\le 2n-2}} r_X(x_1,x_2,\ldots,x_{n-1})\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}\Bigg)>0$$

$$\mathrm{Im}\Bigg(\sum_{\substack{\text{all the combination } (x_1,x_2,\ldots,x_{n-1}) \\ \text{satisfying } x_i\in\{2j+1|1\le j\le n-1\} \\ x_1\ge x_2\ge\ldots\ge x_{n-1},\text{ and} \\ \text{"=" happens only if } x_i+x_{i+1}\le 2n-2}} r_X(x_1,x_2,\ldots,x_{n-1})\prod_{i=1}^{n-1}\frac{j\omega_{l(1)}+\cdots+j\omega_{l(x_i)}}{-L_{x_i}(j\omega_{l(1)}+\cdots+j\omega_{l(x_i)})}\Bigg)<0$$

That is, condition (b2) holds for n>1 under $B << \sqrt{k_0 m}$ and $\Omega = \sqrt{\frac{k_0}{m}}$. Hence, (6.16) is an alternating series if the following two conditions hold:

(c1) B is sufficiently smaller than $\sqrt{k_0 m}$,

(c2) The input frequency is $\Omega = \sqrt{\frac{k_0}{m}}$.

Note that in example 6.1, $\Omega = \sqrt{\frac{k_0}{m}} \approx 8.165$, B=296<< $\sqrt{k_0 m}$ =1959.592. These are consistent with the theoretical results. Therefore the conclusions are verified.

6.5 Conclusions

A novel nonlinear effect on the system output spectrum is revealed in this chapter based on the frequency domain methods established in the previous chapters. It is shown for the first time that under certain conditions the system output spectrum can be described as an alternating series with respect to a specific nonlinear coefficient and this alternating series has some interesting properties which are of significance to engineering practices.

Chapter 7
NONLINEAR EFFECT ON SYSTEM OUTPUT SPECTRUM II ------ OUTPUT FREQUENCIES

For nonlinear Volterra systems, the output frequencies are studied in this chapter. The results show some interesting features of output frequencies of nonlinear systems such as periodicity and opposite properties, and reveal the nonlinear effects on system output spectrum from different nonlinearities. These results have significance in the analysis of nonlinear systems and in the design of nonlinear filters by taking advantage of nonlinearities, and consequently can provide a useful guidance for the practical application of Volterra series theory of nonlinear systems.

7.1 Introduction

As mentioned before, an important phenomenon for nonlinear systems in the frequency domain is that they always have very complicated output frequencies, for example, super-harmonics, sub-harmonics, inter-modulation, and so on. This usually makes it rather difficult to analyze and design the output frequency response behaviour for nonlinear systems. The output frequencies for Volterra systems have been studied by several authors (Raz and Van Veen 1998, Lang and Billings 1997, 2000, Bedrosian and Rice 1971, Wu et al 2007, Wei et al 2007, Bussgang 1974, Frank 1996) by using the frequency domain method based on the Volterra series. These results provide algorithms from different viewpoints for the computation and prediction of the output frequencies for nonlinear systems. It can be seen from the previous results that Volterra systems can effectively be used to account for super-harmonics and inter-modulation in the output spectrum of nonlinear systems.

In this study, some important properties for the output frequencies of the Volterra systems are established. They provide an alternative insight into the super-harmonic and inter-modulation phenomena in the output frequencies of nonlinear systems, especially when the effects from different system nonlinearities are considered. The new properties demonstrate several novel frequency characteristics of the output spectrum for nonlinear systems. They have significance in the analysis and design of nonlinear systems and nonlinear filters in order to achieve a specific output spectrum in a desired frequency band by taking advantage of nonlinearities. These new results can also provide an important guidance to modelling, identification, control and signal processing by using the Volterra series theory in practices. Examples and discussions are provided to illustrate the results.

7.2 Output frequencies for nonlinear Volterra systems

As discussed in Chapter 4, the output spectrum of nonlinear Volterra system (1.1) subject to a general input can be described by (4.1-4.2). For convenience, it is rewritten here as

$$Y(j\omega)=\sum_{n=1}^{N}Y_n(j\omega)$$
$$Y_n(j\omega)=\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}H_n(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega \tag{7.1}$$

where $\int_{\omega_1+\cdots+\omega_n=\omega}(\cdot)d\sigma_\omega$ represents the integration on the super plane $\omega_1+\cdots+\omega_n=\omega$. $Y_n(j\omega)$ is referred to as the nth-order output spectrum. Similarly, when the system is subject to a multi-tone input described by (1.3), the system output spectrum is given in (4.3-4.4), i.e.,

$$Y(j\omega)=\sum_{n=1}^{N}Y_n(j\omega)$$
$$Y_n(j\omega)=\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}H_n(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n}) \tag{7.2}$$

where $F(\omega_{k_i})$ can be written explicitly as $F(\omega_{k_i})=\left|F_{|k_i|}\right|e^{j\angle F_{|k_i|}\cdot \text{sgl}(k_i)}$ for $k_i\in\{\pm1,\cdots,\pm\bar{K}\}$, and $\text{sgn}\,1(a)=\begin{cases}1 & a>0\\ 0 & a=0\\ -1 & a<0\end{cases}$.

From Equations (7.1) and (7.2), it can be seen that the output frequencies corresponding to the nth-order output spectrum, denoted by W_n and simply referred to as the nth-order output frequencies, are completely determined by

$$\omega=\omega_1+\omega_2+\cdots+\omega_n \text{ or } \omega=\omega_{k_1}+\omega_{k_2}+\cdots+\omega_{k_n}$$

which produce super-harmonics and inter-modulation in system output frequencies. Consider any continuous and bounded input function $u(t)$ in $t\geq 0$ with Fourier transform $U(j\omega)$ whose input domain is denoted by V, *i.e.*, $\omega\in V$. Note that V can be any closed set in real. Let $\bar{V}$=-$V\cup V$, whose meaning will be discussed later.

Therefore, for the general input $U(j\omega)$ defined in V, the nth-order output frequencies are

$$W_n=\{\omega=\omega_1+\omega_2+\cdots+\omega_n \mid \omega_i\in\bar{V}, i=1,2,\ldots,n\} \tag{7.3a}$$

or for the multi-tone input (4),

$$W_n=\{\omega=\omega_{k_1}+\omega_{k_2}+\cdots+\omega_{k_n} \mid \omega_{k_i}\in\bar{V}, i=1,2,\ldots,n\} \tag{7.3b}$$

This is an analytical expression for the super-harmonics and inter-modulations in the *n*th-order output frequencies of nonlinear Volterra systems. All the system output frequencies up to order N, denoted by W, can be written as

$$W=W_1\cup W_2\cup\cdots\cup W_N \tag{7.3c}$$

In Equations (7.3abc), $\bar{V}$ represents the input frequency range corresponding to the nth-order output spectrum, V is the original input frequency range corresponding to the first order output spectrum and W_1 represents the output frequencies of linear part in the system. For example, V may be a real set [a,b]$\cup$[c,d], thus $\bar{V}$=[-d,-c]$\cup$[-b,-a]$\cup$[a,b]$\cup$[c,d], where $d\geq c\geq b\geq a$>0. Especially, when the system subjects to the multi-tone input (1.3), then the nth-order input frequency range is $\bar{V}=\{\pm\omega_1,\pm\omega_2,\cdots,\pm\omega_{\bar{K}}\}$, which is obviously a special case of the former one.

7.3 Fundamental properties and the periodicity property

In this section, some fundamental properties of the output frequencies of system (1.1) especially the periodicity of the output frequencies are studied under the assumption that V is a closed set of frequencies in real. Although the computation of the system output frequencies for the case with $V=[a,b]$ and $V=[a_i,b_i]$, i=1,2,...m has been studied in Raz and Van Veen (1998), Lang and Billings (1997), Wu et al (2007) and for the multi-tone case was also studied in Lang and Billings (1997), Wei et al (2007) and Bussgang et al (1974), the properties of the output frequencies of nonlinear systems are established in this study in a uniform manner based on the analytical expressions (7.3abc) for any input domain V. Let max(.) denote the maximum value of the elements in (.), and min(.) the minimum value.

Property 7.1. Consider the nth-order output frequency W_n,
(a) Expansion, i.e., $W_{n-2} \subseteq W_n$;
(b) Symmetry, i.e., $\forall \Omega \in W_n$, then $-\Omega \in W_n$;
(c) n-multiple, i.e., $\max(W_n) = n \cdot \max(V)$ and $\min(W_n) = -n \cdot \max(V)$.
Proof. (a) Consider Equation (7.3a), if let $\omega_{n-1} \equiv -\omega_n$, then

$$\begin{aligned} W_n &= \{\omega = \omega_1 + \omega_2 + \cdots + \omega_n | \omega_i \in \bar{V}, \omega_{n-1} = -\omega_n, i = 1,2,...,n\} \\ &= W_{n-2} = \{\omega = \omega_1 + \omega_2 + \cdots + \omega_{n-2} | \omega_i \in \bar{V}, i = 1,2,...,n-2\} \end{aligned} \quad (7.4)$$

Therefore, $W_{n-2} \subseteq W_n$. The same conclusion also holds for Equation (7.3b).
(b) From the realness of the output spectrum, this property is straightforward. It can also be proved as follows. If $\Omega \in W_n$, then there exists $\omega_i \in \bar{V}$ such that $\Omega = \omega_1 + \omega_2 + \cdots + \omega_n$. Note that $\bar{V}$ is symmetric with respect to 0, thus it must hold that $-\omega_1, -\omega_2, \cdots -\omega_n \in \bar{V}$. Therefore, $-\Omega = -\omega_1 - \omega_2 - \cdots - \omega_n \in W_n$.
(c) This is obvious from $\omega = \omega_1 + \omega_2 + \cdots + \omega_n$ and $\omega = \omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_n}$.
This completes the proof. □

Property 7.1 shows that the output frequency range will expand larger and larger with the increase of the nonlinear order, the expansion is symmetric to zero and its rate is n-multiple of the input frequency range. Property 7.1(a) shows that, the (n-2)th order output frequencies W_{n-2} are completely included in the nth order output frequencies W_n. This property can be used to facilitate the computation of output frequencies for nonlinear systems. That is, only the highest order in odd number and the highest order in even number, to which the corresponding GFRFs are not zero, are needed to be considered in Equation (7.3c). For example, supposing the system maximum order N=10, only W_{10} and W_9 are needed to be computed if $H_{10}(.)$ and $H_9(.)$ are not zero, and the system output frequencies are $W = W_9 \cup W_{10}$ (in case that $H_9(.)$ is zero, W_9 should be replaced by the output frequencies corresponding to the highest odd order of nonzero GFRFs). For the case that $V=[a,b]$, Property 7.1(a) has be shown in Lang and Billings (1997). Here it is shown to hold for any V.

Property 7.1 shows the basic properties of the output frequencies of system (1.1) subject to any input frequencies. The following proposition shows the periodicity of the output frequencies of Volterra systems, providing a new insight into the system output frequency characteristics.

Proposition 7.1 (Periodicity property). The frequencies in W_n can be generated periodically as follows

$$W_n = \bigcup_{i=1}^{\Gamma_n+1} \Pi_i(n) \tag{7.5a}$$

$$\Pi_i(n) = \left\{\omega = \omega_1 + \omega_2 + \cdots + \omega_n \middle| \omega_j \in \bar{V}, \omega_j < 0 \text{ for } 1 \le j \le i-1, \omega_j > 0 \text{ for } j \ge i\right\} \text{ or} \tag{7.5b}$$

$$\Pi_i(n) = \left\{\omega = \omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_n} \middle| \omega_{k_j} \in \bar{V}, \omega_{k_j} < 0 \text{ for } 1 \le j \le i-1, \omega_{k_j} > 0 \text{ for } j \ge i\right\} \tag{7.5c}$$

$$\Gamma_n = n \tag{7.5d}$$

The above process has the following properties

$$\max(\Pi_i(n)) = -(i-1)\min(V) + (n-i+1)\max(V) \text{ and} \tag{7.6a}$$

$$\min(\Pi_i(n)) = -(i-1)\max(V) + (n-i+1)\min(V) \tag{7.6b}$$

$$\max(\Pi_{i-1}(n)) - \max(\Pi_i(n)) = \min(\Pi_{i-1}(n)) - \min(\Pi_i(n)) = \min(V) + \max(V) \tag{7.6c}$$

$$\Delta(n) = \max(\Pi_i(n)) - \min(\Pi_i(n)) = n \cdot (\max(V) - \min(V)) \tag{7.6d}$$

Especially, when the system subjects to a general input $U(j\omega)$ defined in [a,b] or the multi-tone input (1.3) with $\omega_{i+1} - \omega_i = const > 0$ for i=1,..., $\bar{K}$ -1,

$$\Pi_i(n) = \Pi_{i-1}(n) - T \text{ for i=2,..., n+1} \tag{7.6e}$$

where $\Pi_i(n) - T$ is a set whose elements are the elements in $\Pi_i(n)$ minus T, $T = \min(V) + \max(V)$ is referred to as the frequency generation period, and $\Delta(n)$ is referred to as the frequency span in each period.
Proof. See Section 7.6 for the proof. □

For the simple case where V=[a,b], the periodicity above can be easily checked from the result in Lang and Billings (1997).

Property 7.2. Consider the ith frequency generation period $\Pi_i(n)$ in W_n,

(a) If the system input is the multi-tone function (1.3), then for any two frequencies Ω and Ω' in $\Pi_i(n)$ and any two frequencies ω and ω' in V, $\min(\Omega - \Omega') = \min(\omega - \omega')$.

(b) If $\Delta(n)$>T, then $\max(\Pi(n)_{i+1}) > \min(\Pi(n)_i)$ for i=1,..., Γ_n. That is, there is overlap between the successive periods of frequencies in W_n.

Proof. (a) is obvious from the proof for Proposition 7.1. Note that $\max(\Pi(n)_{i+1}) = \max(\Pi(n)_i) - T$, thus it can be derived that $\max(\Pi(n)_{i+1}) - \min(\Pi(n)_i) = \max(\Pi(n)_i) - \min(\Pi(n)_i) - T = \Delta(n) - T > 0$. (b) is proved. □

Proposition 7.1 and Property 7.2 explicitly demonstrate, for the first time, an interesting and useful feature of the output frequencies of nonlinear systems ------ the periodicity. This property can not only be used to simplify the computation of the output frequencies for some special cases as stated in Proposition 7.1 (where only one period of output frequencies need to be computed) but also provide an insight into the computation and understanding of the output frequencies in general case. Some important issues will be discussed further in the following sections. From Proposition 7.1, the following corollary is straightforward.

Corollary 7.1. All the conclusions in Proposition 7.1 and Properties 7.1-7.2 hold for the case that the system subjects to a general input $U(j\omega)$ defined in $\bigcup_{i=1}^{Z}[a+(i-1)\varepsilon, b+(i-1)\varepsilon]$ where $b \ge a, \varepsilon \ge (b-a)$ and Z is a positive integer. □

Note that when V does not satisfy the condition in Corollary 7.1, the property in Equation (7.6e) does not hold. Example 7.1 is given to illustrate the results above.

Example 7.1. Consider a simple nonlinear system as follows

$$y = -0.01\dot{y} + au^2 + bu^3$$

The input is a multi-tone function u(t)=sin(6t)+sin(7t)+sin(8t). The output spectra are given in Figures 7.1-7.2 for different cases. Note that there are only input nonlinearities with order 2 and 3 in the system, thus only the 1st, 2nd and 3rd order GFRFs are not zero and all the other order GFRFs are zero (See Proposition 3.1 and Properties 3.1-3.5 Chapter 3). Hence, the output frequencies of the system are the same as the 2nd and 3rd order output frequencies. That is, when a=1 and b=0, then W=W_2; when a=0 and b=1, then W=W_3; and when a=1 and b=1, then $W = W_2 \cup W_3$. Figures 7.1-7.2 demonstrate clearly the results in Properties 7.1(c)-7.2(a) and Proposition 7.1, and also show that the system output frequencies are simply the accumulation of all the output frequencies corresponding to each order output spectrum when the involved nonlinearities have no crossing effect and no overlap as stated in Property 7.2(b). The overlap of the output frequencies contributed by different orders' system nonlinearity will be discussed in the next section.

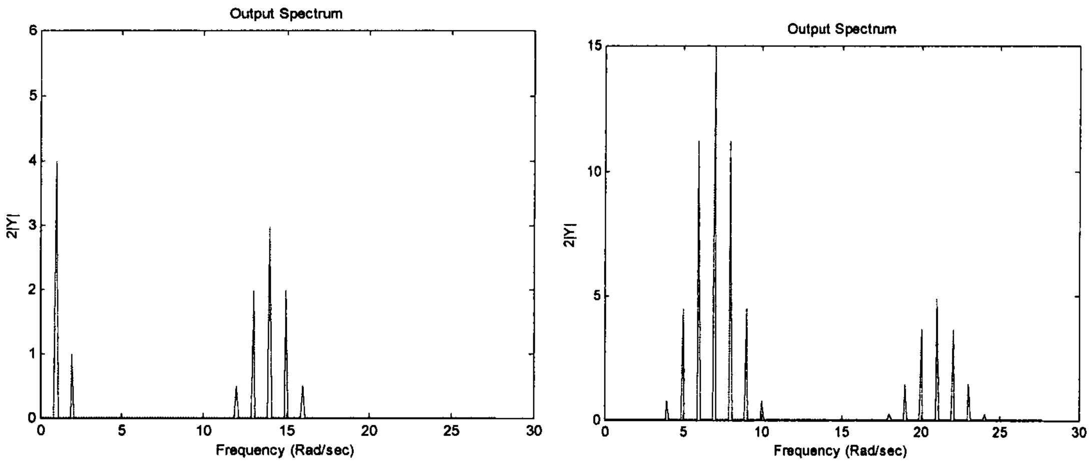

Figure 7.1. Output frequencies when a=1 and b=0 (left) and when a=0 and b=1 (right)

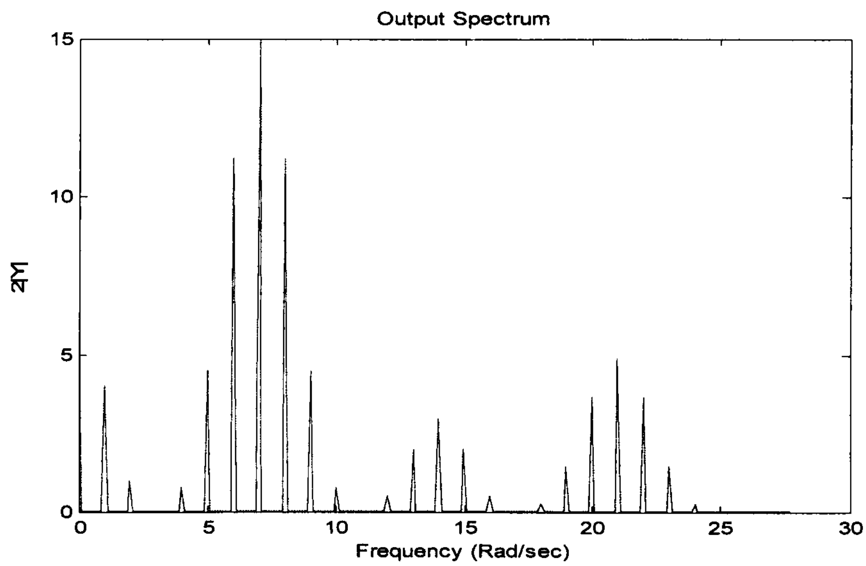

Figure 7.2. Output frequencies when a=1 and b=1

7.4 Nonlinear effect in each frequency generation period

The periodicity of output frequencies is revealed and demonstrated in the previous section. In this section, the nonlinear effect on system output spectrum in each frequency generation period, and especially the nonlinear interaction between different nonlinearities of the same nonlinear degree and nonlinear type are studied.

From (7.1) and (7.2), it can be seen that the operators $\int_{\omega_1+\cdots+\omega_n=\omega}(\cdot)d\sigma_\omega$ and $\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}(\cdot)$ have an important and fundamental role in the frequency characteristics of the nth order output spectrum in each frequency generation period. The following property can be obtained.

Property 7.3. For $\omega\in\Pi_i(n)$ $(1\le i\le\lceil(n+1)/2\rceil)$, $\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}1$ reaches its maximum at the central frequency $(\max(\Pi_i(n))+\min(\Pi_i(n)))/2$ or around the central frequency if the central frequency is not available, and has its minimum value at frequencies $\max(\Pi_i(n))$ and $\min(\Pi_i(n))$, i.e.,

$$\min_{\omega\in\Pi_i(n)}\Big(\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}1\Big)=\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\max(\Pi_i(n))}1=\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\min(\Pi_i(n))}1=C_n^{i-1}$$

Moreover,

$$\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}1>\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=<\omega+T>}1\quad\text{for }\omega\in\Pi_i(n)\ \ (2\le i\le\lceil(n+1)/2\rceil)$$

Especially, for the multi-tone input case with $\omega_{i+1}-\omega_i=const>0$ for i=1,..., $\bar{K}$ -1,

$$\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\max(\Pi_i(n))-k'\cdot const}1=\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\min(\Pi_i(n))+k'\cdot const}1\quad\text{for }0\le k'\le T/const$$

where, $\lceil(n+1)/2\rceil$ is the smallest integer which is not less than $(n+1)/2$, $<\omega+T>$ is the frequency in $\Pi_{i-1}(n)$ which is the most approximate to $\omega+T$. The similar results also hold for the general input case defined in Corollary 7.1 by replacing $\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}1$ as $\int_{\omega_1+\cdots+\omega_n=\omega}1d\sigma_\omega$.

Proof. Note that $\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}1$ is equal to the number of all the combinations satisfying $\omega_{k_1}+\cdots+\omega_{k_n}=\omega$ and with the n frequency variables satisfying the conditions in $\Pi_i(n)$, thus the conclusions in this property can be obtained by using the combination theory, which are straightforward. When the values of ω_1 and $\omega_{\bar{K}}$ are fixed and $\bar{K}$ is approaching infinity such that *const* approaches zero, the multi-tone frequencies will become a continuous closed set $[\omega_1,\omega_{\bar{K}}]$. The input frequencies defined in Corollary 7.1 are further extended from these two cases. Hence, the conclusions holding for the multi-tone case can be easily extended to the input case defined in Corollary 7.1. This completes the proof. □

Property 7.3 shows that in each frequency generation period, the effect of the operator $\int_{\omega_1+\cdots+\omega_n=\omega}(\cdot)d\sigma_\omega$ and $\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}(\cdot)$ on system output spectrum tends naturally to be more complicated at the central frequency. That is, there is only one case for the operator $\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}(\cdot)$ at the two boundary frequency of each period, it reaches the maximum at the central frequency of the same period and tends to decrease in different period with the frequency increasing. These can be regarded as the natural characteristics of the output frequencies that can not be changed (This can be verified by Figure 7.3 in Example 7.2).

Note that different nonlinearities may have quite different effect on system output spectrum. In order to study the nonlinear effect between different nonlinearities of the same nonlinear degree and kind, consider the nonlinear Volterra systems which are described by the NDE model in (1.5), i.e.,

$$\sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{l_1,l_{p+q}=0}^{K} c_{p,q}(l_1,\cdots,l_{p+q})\prod_{i=1}^{p}\frac{d^{l_i}y(t)}{dt^{l_i}}\prod_{i=p+1}^{p+q}\frac{d^{l_i}u(t)}{dt^{l_i}}=0 \tag{7.7}$$

See Chapter 1 after Equation (1.5) for the notations. Similar results discussed in this study can also be easily established for the NARX model in (1.6).

When different categories and degrees of nonlinearities exist in the system, there will be much crossing effects at the same frequency from different nonlinearities. This will make the output spectrum at the frequency of interest to be enhanced or suppressed. For example, different nonlinearities of the same order and the same category can produce the same output frequencies according to Chapter 3. However, the effect from different nonlinearities at the same frequency generation period may counteract with each other such that the output spectrum may be suppressed in some periods and others enhanced. Clearly, this property is of great significance in the design of nonlinear systems for suppressing output vibration (Zhou and Misawa, 2005).

In this study, consider there are only input nonlinearities in the NDE model above with $c_{p,q}(.)=0$ for all $p+q>1$ and $p>0$. In this case, following the results in Chapter 3, the GFRFs can be written as

$$H_n(j\omega_1,\cdots,j\omega_n)=\frac{1}{L_n(j\omega_1+\cdots+j\omega_n)}\sum_{l_1,l_n=1}^{K}c_{0,n}(l_1,\cdots,l_n)(j\omega_1)^{l_1}\cdots(j\omega_n)^{l_n} \tag{7.8}$$

where

$$L_n(j\omega_1+\cdots+j\omega_n)=-\sum_{k_1=0}^{K}c_{1,0}(k_1)(j\omega_1+\cdots+j\omega_n)^{k_1} \tag{7.9}$$

From (7.8-7.9) and (7.2), the nth-order output spectrum under the multi-tone input (1.3) can be obtained

$$\begin{aligned}Y_n(j\omega)&=\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}\left(\frac{F(\omega_{k_1})\cdots F(\omega_{k_n})}{L_n(j\omega_{k_1}+\cdots+j\omega_{k_n})}\sum_{l_1,l_n=1}^{K}c_{0,n}(l_1,\cdots,l_n)(j\omega_{k_1})^{l_1}\cdots(j\omega_{k_n})^{l_n}\right)\\&=\frac{1}{2^nL_n(j\omega)}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}\left(F(\omega_{k_1})\cdots F(\omega_{k_n})\sum_{l_1,l_n=1}^{K}c_{0,n}(l_1,\cdots,l_n)(j\omega_{k_1})^{l_1}\cdots(j\omega_{k_n})^{l_n}\right)\end{aligned} \tag{7.10}$$

To reveal the nonlinear effect from input nonlinearities in each frequency generation period, the following results can be obtained.

Definition 7.1 (Opposite property). Considering two input nonlinear terms of the same degree with coefficients $c_{0,n}(l_1,\cdots,l_n)$ and $c_{0,n}(l_1',\cdots,l_n')$, if there exist two nonzero real number c_1 and c_2 satisfying $c_{0,n}(l_1,\cdots,l_n)=c_1$ and $c_{0,n}(l_1',\cdots,l_n')=c_2$, such that at a given frequency $\Omega>0$,

$$\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\Omega}\left(F(\omega_{k_1})\cdots F(\omega_{k_n})\left(c_1(j\omega_{k_1})^{l_1}\cdots(j\omega_{k_n})^{l_n}+c_2(j\omega_{k_1})^{l'_1}\cdots(j\omega_{k_n})^{l'_n}\right)\right)=0$$

with respect to a multi-tone input (1.3), then the two terms are referred to as opposite at frequency Ω under $c_{0,n}(l_1,\cdots,l_n)$ = c_1 and $c_{0,n}(l'_1,\cdots,l'_n)$ = c_2,, whose effects in the frequency domain counteract each other at Ω.

Note that the concept of the opposite property can be defined similarly for the other categories of nonlinearities. The following result can be concluded for the opposite property of two input nonlinear terms.

Proposition 7.2 (Opposite of input nonlinearity). Consider nonlinear systems with only input nonlinearities subject to multi-tone input, and there are two nonlinear terms with coefficients $c_{0,n}(l_1,\cdots,l_n)$ and $c_{0,n}(l'_1,\cdots,l'_n)$. If there exists a non-negative integer $m\le\lceil(n+1)/2\rceil-1$ such that sgn($F(\omega_{k_1})\cdots F(\omega_{k_n})$) is constant with respect to all the combinations of $\omega_{k_1},\cdots,\omega_{k_n}\in\{\pm\omega_1,\cdots,\pm\omega_{\bar{K}}\}$ satisfying $\omega_{k_1}+\cdots+\omega_{k_n}\in\Pi_{m+1}(n)$, then for the two nonlinear terms,

(1) they can be designed to be opposite at any frequency in the (m+1)th frequency generation period $\Pi_{m+1}(n)$ with proper parametric values of the two coefficients, if and only if $\sum_{i=1}^{n}l_i$ and $\sum_{i=1}^{n}l'_i$ are both odd integers or even integers simultaneously.

(2) for a proper value of $c_{0,n}(l_1,\cdots,l_n)/c_{0,n}(l'_1,\cdots,l'_n)$>0, they are opposite in $\Pi_{m+1}(n)$ if for a real Ω>0,

$$\operatorname{sgn} 1\Big(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=(n-2m)\cdot\Omega\\ \omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}}}(\omega_{k_1})^{l_1}\cdots(\omega_{k_n})^{l_n}\Big)(-1)^{\left\lceil\frac{|l_1-l'_1+\cdots+l_n-l'_n|}{2}\right\rceil}=-\operatorname{sgn} 1\Big(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=(n-2m)\cdot\Omega\\ \omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}}}(\omega_{k_1})^{l'_1}\cdots(\omega_{k_n})^{l'_n}\Big)\quad(7.11)$$

Proof. See Section 7.6 for the proof. □

From Equation (7.10), it can be seen that the magnitude of $Y_n(j\omega)$ depends on $L_n(j\omega)$, $F(\omega_{k_1})\cdots F(\omega_{k_n})\sum_{l_1,l_n=1}^{K}c_{0,n}(l_1,\cdots,l_n)(j\omega_{k_1})^{l_1}\cdots(j\omega_{k_n})^{l_n}$, and $\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}(\cdot)$. $\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}(\cdot)$ represents the system natural effect which can not be changed as mentioned. $L_n(j\omega)$ represents the effect from the linear part of the system and $F(\omega_{k_1})\cdots F(\omega_{k_n})\sum_{l_1,l_n=1}^{K}c_{0,n}(l_1,\cdots,l_n)(j\omega_{k_1})^{l_1}\cdots(j\omega_{k_n})^{l_n}$ represents the nonlinear effect from input nonlinearities. These two later effects can be designed purposely in practice. Therefore, the results in Proposition 7.2 provide guidance to the design of input nonlinearities to achieve a specific output spectrum. Similar results can also be established for the other categories of nonlinearities. The following example illustrates the result in Proposition 7.2.

Example 7.2. Consider a simple nonlinear system as follows

$$y=-0.01\dot{y}+au^5+bu^3\dot{u}^2\quad(7.12)$$

The input is a multi-tone function u(t)=0.8sin(7t)+0.8sin(8t)+sin(9t), which can be written as u(t)=0.8cos(7t-90^0)+0.8cos(8t-90^0)+cos(9t-90^0). Therefore, F($\omega_{\pm1}$)=$\mp$0.8j, F($\omega_{\pm2}$)=$\mp$0.8j and F($\omega_{\pm3}$)=$\mp$j. It can be verified that, sgn($F(\omega_{k_1})\cdots F(\omega_{k_5})$) is constant in each period $\Pi_i(5)$ (i= 1,..,6). This satisfies the condition in Proposition 7.2. The output spectrum under different parameter values are provided in Figures 7.3-7.4. It can be verified that the two nonlinear terms are opposite at the second frequency generation period. For the nonlinear term au^5,

$$\mathrm{sgn}\,1\Big(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=(n-2m)\cdot\Omega\\ \omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}}}(\omega_{k_1})^{l_1}\cdots(\omega_{k_n})^{l_n}\Big)\Big)(-1)^{\left\lceil\frac{|l_1-l_1'+\cdots+l_n-l_n'|}{2}\right\rceil}=\mathrm{sgn}\,1\Big(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_5}=(5-2)\cdot\Omega\\ \omega_{k_1},\cdots,\omega_{k_5}\in\{+\Omega,-\Omega\}}}(\omega_{k_1})^{0}\cdots(\omega_{k_5})^{0}\Big)\Big)(-1)^1$$

$$=\mathrm{sgn}\,1\Big(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_5}=3\cdot\Omega\\ \omega_{k_1},\cdots,\omega_{k_5}\in\{+\Omega,-\Omega\}}}1\Big)\Big)(-1)^1=-1$$

For the nonlinear term $bu^3\dot{u}^2$,

$$-\mathrm{sgn}\,1\Big(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=(n-2m)\cdot\Omega\\ \omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}}}(\omega_{k_1})^{l_1'}\cdots(\omega_{k_n})^{l_n'}\Big)=-\mathrm{sgn}\,1\Big(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_5}=(5-2)\cdot\Omega\\ \omega_{k_1},\cdots,\omega_{k_5}\in\{+\Omega,-\Omega\}}}(\omega_{k_4})^1(\omega_{k_5})^1\Big)$$

$$=-\mathrm{sgn}\,1\Big(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_5}=3\Omega\\ \omega_{k_1},\cdots,\omega_{k_5}\in\{+\Omega,-\Omega\}}}(\omega_{k_1}\omega_{k_2})\Big)$$

Note that there are five combinations for $\omega_{k_1}+\cdots+\omega_{k_n}=3\Omega,\omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}$, i.e.,

-Ω,Ω,Ω,Ω,Ω;Ω,-Ω,Ω,Ω,Ω;Ω,Ω,-Ω,Ω,Ω;Ω,Ω,Ω,-Ω,Ω;Ω,Ω,Ω,Ω,-Ω;

Therefore $-\mathrm{sgn}\,1\Big(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_5}=3\Omega\\ \omega_{k_1},\cdots,\omega_{k_5}\in\{+\Omega,-\Omega\}}}(\omega_{k_1}\omega_{k_2})\Big)=-\mathrm{sgn}\,1(\Omega^2)=-1$. Equation (7.11) is satisfied.

From Figure 7.4 it can be seen that, the counteraction between the effects from the two input nonlinear terms results in suppression of the output spectrum in the second period and enhancement for the first and third periods, compared with the output spectrum under single nonlinear term au^5.

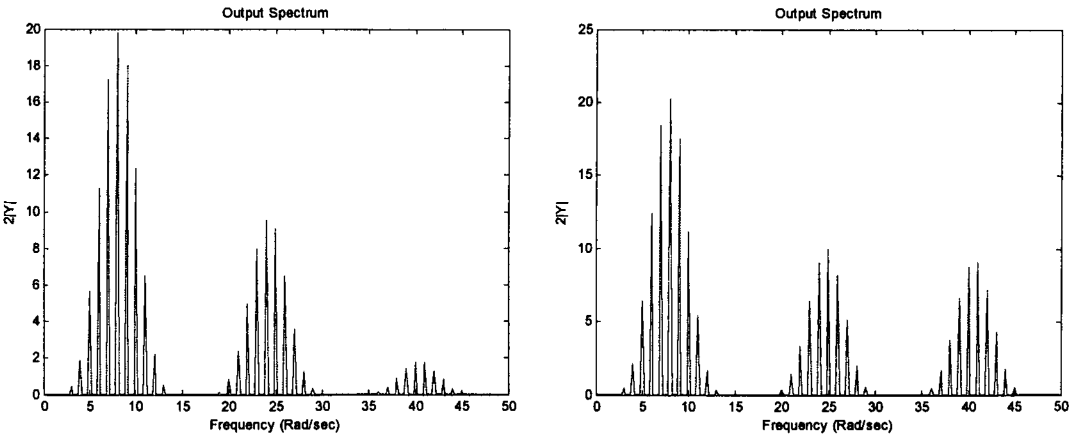

Figure 7.3. Output spectrum when a=1.3 b=0 (left) and a=0,b=0.1(right)

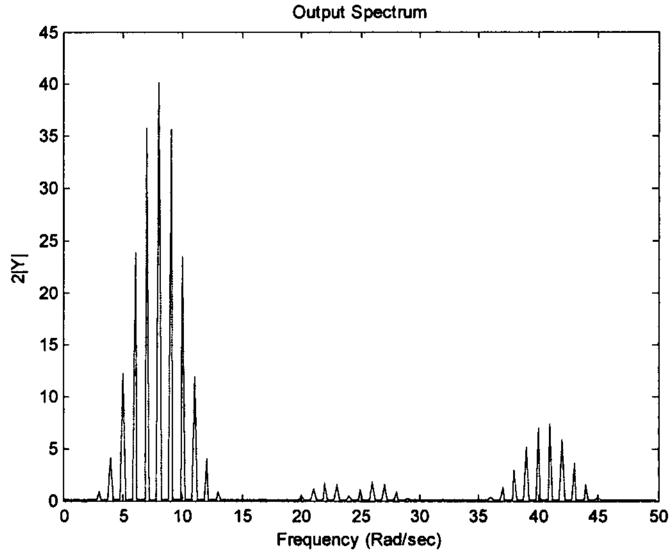

Figure 7.4. Output spectrum when a=1.3 b=0.1

Moreover, it is obvious that given system model and input function, the system output spectrum can be analytically determined from (7.1-7.2). Contrarily, given system model in the multi-tone input case, the input function can be obtained from the output spectrum at a specific frequency generation period for example $\Pi_1(n)$. Because each output frequency in $\Pi_l(n)$ can be explicitly determined, thus a series of equations can be obtained in terms of $F(\omega_{k_1})\cdots F(\omega_{k_n})$, and then $F(\omega_1),\cdots,F(\omega_n)$ can be solved. That is, the original input signal can be recovered from the received signal in a specific frequency generation period. This is another interesting property based on the periodicity and is worth further studying.

7.5 Parametric characteristic of the output frequencies

There are three categories of nonlinearities in model (7.7): input nonlinearity with coefficient $c_{0,q}(.)$ (q>1), output nonlinearity with coefficient $c_{p,0}(.)$ (p>1), and input output cross nonlinearity with coefficient $c_{p,q}(.)$ ($p+q$>1 and p>0) (where p and q are integers). Different category and degree of nonlinearity in a system can bring different output frequencies to the system. How a nonlinear term affects system output frequencies and what the effect is for Volterra systems are a very interesting and important topic. However, few results have been reported for this. This section provides some useful results for this topic based on the properties developed above.

Consider nonlinear Volterra systems described by the NDE model in (7.7). What model parameters contribute to a specific order GFRF and how model parameters affect the GFRFs can be revealed by using the parametric characteristic analysis in Chapter 3. From Equations (7.1, 7.2), it can be seen that the nth-order output frequencies W_n are also determined by the nth order GFRF. If the nth order GFRF is zero, then W_n=[]. It is known from Chapter 3 that the nth order GFRF is dependent on its parametric characteristics, thus the nth-order output frequencies are also determined by the parametric characteristics of the nth-order GFRF. That is, Equations (7.3a-b) can be written as

$$W_n = \{\omega = (\omega_1 + \omega_2 + \cdots + \omega_n)\cdot(1-\bar{\delta}(CE(H_n(\omega_1,\cdots,\omega_n))))|\omega_i \in \bar{V}, i=1,2,\ldots,n\} \quad (7.13a)$$

and

$$W_n = \{\omega = (\omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_n}) \cdot (1 - \bar{\delta}(CE(H_n(\omega_{k_1}, \cdots, \omega_{k_n})))) | \omega_{k_i} \in \bar{V}, i = 1,2,...,n\} \quad (7.13b)$$

where $\bar{\delta}(x) = \begin{cases} 1 & x = 0 \text{ or } 1 \\ 0 & \text{else} \end{cases}$. In Equations (7.13ab), suppose W_n is empty when $\bar{\delta}(CE(H_n(.)))=1$.

Equations (7.13a-b) demonstrate the parametric characteristics of the output frequencies for Volterra systems described by (7.7) and (1.6), by which the effect on the system output frequencies from different nonlinearities can be studied. Since negative output frequencies are symmetrical with positive output frequencies with respect to zero (Property 7.2(b)), thus for convenience only non-negative output frequencies are considered in what follows.

Property 7.4. Regarding nonlinearities of odd and even degrees,

(a) when there are no nonlinearities of even degrees, the output frequencies brought by the nonlinearities with odd degrees happen at central frequencies $(2l+1)T/2$ for l=0,1,2,... with certain frequency span;

(b) when there are only input nonlinearities of even degrees, the output frequencies happen at central frequencies $l \cdot T$ for l=0,1,2,... with certain frequency span;

(c) in other cases, the output frequencies happen at central frequencies $l \cdot T/2$ for l=0,1,2,... with certain frequency span.

The frequency span is $\Delta(n)$ corresponding to the nth order output frequencies if applicable.

Proof. See Section 7.6 for the proof. □

Property 7.4 shows that odd degrees of nonlinearities bring quite different output frequencies to the system from those brought by even degrees of nonlinearities.

Property 7.5. Regarding different categories of nonlinearities,

(a) when there are only input nonlinearities of largest nonlinear degree n, the non-negative output frequencies are in the closed set [0, $n \cdot \max(V)$];

(b) in other cases, the output frequencies span to infinity.

Proof. (a) From Equation (3.17) or Proposition 3.1 in Chapter 3, only the GFRFs of orders equal to the nonlinear degrees of the non-zero input nonlinearities are not zero since there are no other kinds of nonlinearities in the system. Thus the largest order of non-zero GFRFs is n. The conclusion is therefore straightforward from Property 7.1(c). (b) If there are other kinds of nonlinearities, the largest order of nonzero GFRFs will be infinite, because for any parameter $c_{p,q}(.)$ with p>0 and $p+q$>1, it can form a monomial with any high nonlinear degree ($c_{p,q}(.)^n$) and thus contribute to any high order GFRF from Proposition 3.1 in Chapter 3. Thus the output frequencies can span to infinity. This completes the proof. □

The input nonlinearities of a finite nonlinear degree can independently produce output frequencies in a finite frequency band.

Property 7.6. Regarding different categories and degrees of nonlinearities,

(a) when there are only input nonlinearities, a nonlinear term of degree n can only produce output frequencies W_n, and there are no crossing effect on output frequencies between different degrees of input nonlinearities;

(b) in other cases, a nonlinear term of degree n contributes to not only output frequencies W_n but also some high order output frequencies W_m for $m > n$ due to crossing effect with other nonlinearities.

Proof. (a) Considering a nonlinear term $c_{0,n}(.)$, it can be obtained from Equation (3.17) that only $CE(H_n(.))$ is not zero if the other degree and kind of nonlinear parameters are zero. That is, $c_{0,n}(.)$ only contributes to $H_n(.)$ in this case. If there are other input nonlinearities, it can be known from Proposition 3.1 in Chapter 3 that only nonlinear parameters from input nonlinearities can not form an effective monomial which is an element of any order GFRF. That is there are no crossing effects between different degrees of input nonlinearities. (b) When there are output or input-output cross nonlinearities, it can be seen from Proposition 3.1 in Chapter 3 that there are crossing effects between different nonlinearities, and the nonlinear degree of any effective monomial (e.g. $c_{1,q}(.)c_{0,q}(.)^n$ (q>1)) formed by the coefficients from the crossing nonlinearities can be infinity. Thus a nonlinear parameter of degree n, for example $c_{0,n}(.)$, has contribution not only to $H_n(.)$, but also to some higher order GFRFs, for example $c_{1,n}(.)c_{0,n}(.)^z$ is an element of $CE(H_m(.))$ where m=zn+n+1-z. This completes the proof. □

From Property 7.6, the crossing effect usually happens easily between the output nonlinearities and the input-output cross nonlinearities.

Properties 7.4-7.6 provide some novel and interesting results about the output frequencies for nonlinear systems when the effects from different nonlinearities are considered, based on the results from parametric characteristic analysis in Chapter 3. Property 7.4 shows that odd degrees of nonlinearities have quite different effect on system output frequencies from even degrees of nonlinearities. Especially, it is shown from the properties above that input nonlinearities have special effect on system output frequencies compared with the other categories of nonlinearities. That is, input nonlinearities can move the input frequencies to higher frequency bands without interference between different frequency generation periods. These properties may have significance in design of nonlinear systems for some special purposes in practices. For example, some proper input nonlinearities can be used to design a nonlinear filter such that input frequencies are moved to a place of higher frequency or lower frequency as discussed in Billings and Lang (2002). The results in this section have also significance in modelling and identification of nonlinear systems. For example, if a Volterra system has only output frequencies which are odd multiples of the input frequency when subject to a sinusoidal input, the system may have only nonlinearities of odd degree according to Property 7.4. Obviously, the results in this section provide a useful guidance to the structure determination and parameter selection for the design of novel nonlinear filters and also for system modelling or identification.

Example 7.3. Consider a simple nonlinear system as follows

$$y = -0.01\dot{y} + au^5 - by^3 - cy^2$$

The input is a multi-tone function u(t)=sin(6t)+sin(7t)+sin(8t). The output spectra under different parameter values are given in Figures 7.5-7.7, which demonstrate the results in Properties 7.4-7.6. For the input nonlinearity, the readers can also refer to Figures 7.1-7.2.

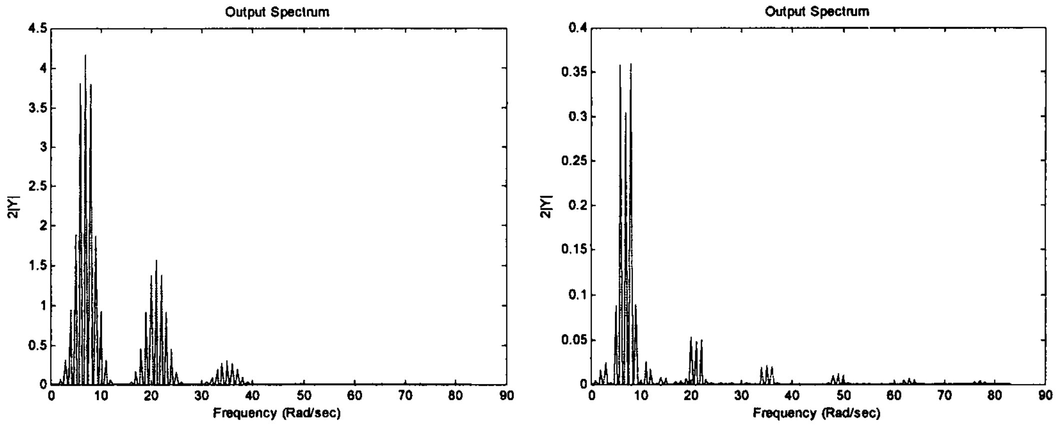

Figure 7.5. Output frequencies when a=0.1, b=0, c=0 (left) and a=0, b=5, c=0 (right)

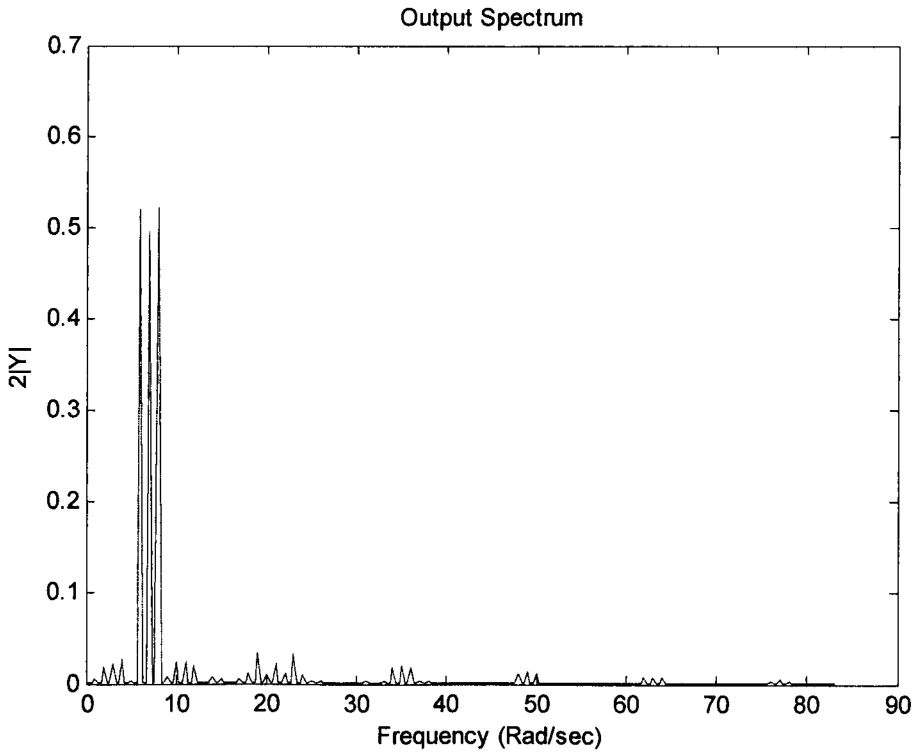

Figure 7.6. Output frequencies when a=0.1, b=5, c=0

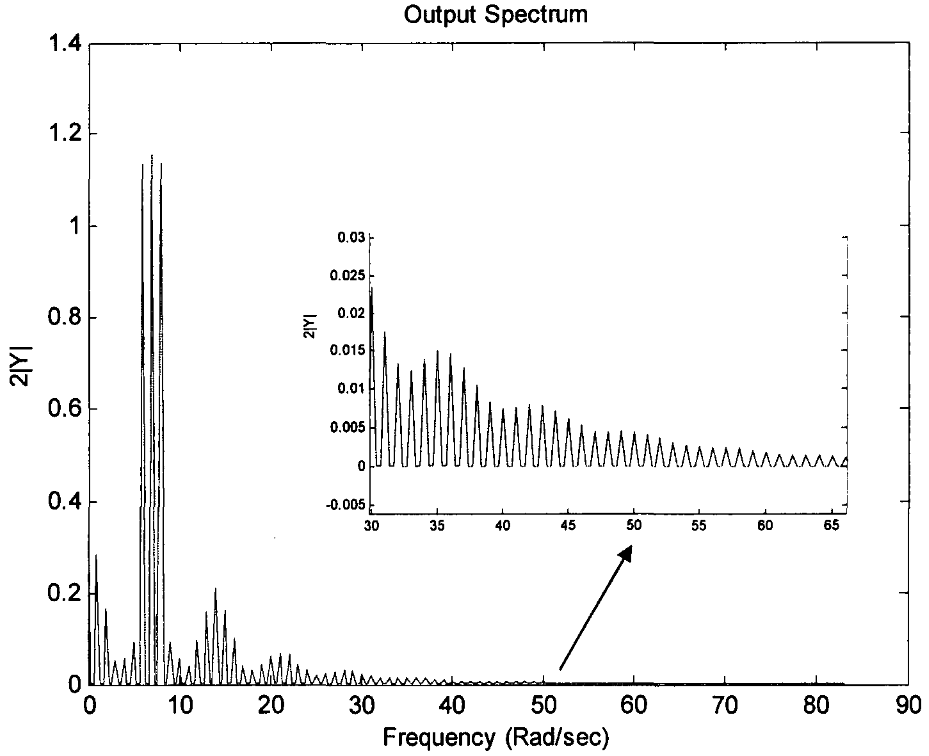

Figure 7.7. Output frequencies when *a*=0, *b*=0, *c*=0.09

When there are only odd nonlinearities, the output frequencies happen at around central frequencies 7*(2k+1). When there are even nonlinearities, the output frequencies appear at around central frequencies 7*k. The input nonlinearities only produce independently the output frequencies within a finite frequency band.. The periodicity of the output frequencies can also be seen clearly from these figures.

Especially, it is worthy pointing out from Figures 7.1, 7.2 and 7.5 that there can be no crossing effects between proper chosen input nonlinearities as mentioned before, which can not be realized by the other categories of nonlinearities. Thus the input frequencies can be moved to higher frequency periodically without interference between different periods and then decoded by using some methods. This property may have significance when a system is designed to achieve a special output spectrum at a desired frequency band in practices by using nonlinearities.

7.6 Proofs

- **Proof of Proposition 7.1**

Consider multi-tone input case only. Then the same results can be extended to the general input case readily. From Equation (7.3b), it can be seen that the frequencies in W_n are determined by $\omega = \omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_n}$. When all the frequency variable $\omega_{k_i} \in \bar{V}$ (for i=1,..., n) are positive, *i.e.*, $\omega_{k_i} > 0$ for i=1,...,n, the computed frequencies are obviously those in $\Pi_1(n)$. Then $\Pi_2(n)$ can be computed by setting that there is only one frequency variable (for example ω_{k_1}) is negative and all the other frequency variables are positive, i.e.,

$$\Pi_2(n) = \left\{\omega = \omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_n} \middle| \omega_{k_i} \in \bar{V}, \omega_{k_1} < 0, \omega_{k_i} > 0, i = 2,3,\ldots,n\right\}$$

Similarly, $\Pi_3(n)$ can be computed by setting that there is only two frequency variables (for example ω_{k_1} and ω_{k_2}) are negative and all the other frequency variables are positive, i.e.,

$$\Pi_3(n) = \left\{ \omega = \omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_n} \middle| \omega_{k_i} \in \overline{V}, \omega_{k_1} < 0, \omega_{k_2} < 0, \omega_{k_i} > 0, i = 2,3,\ldots,n \right\}$$

Proceed with this process until that all the frequency variables are negative. There are totally *n* negative frequencies (or frequency variables) in $\overline{V}$, thus it is obvious that the periodical number of the computation process above is $\Gamma_n = n$.

From Equation (7.5c), it can be obtained that

$$\max(\Pi_i(n)) = -(i-1)\min(V) + (n-i+1)\max(V) \text{ and}$$
$$\min(\Pi_i(n)) = -(i-1)\max(V) + (n-i+1)\min(V)$$

Therefore,

$$\begin{aligned}
&\max(\Pi_{i-1}(n)) - \max(\Pi_i(n)) \\
&= -(i-2)\min(V) + (n-i+2)\max(V) + (i-1)\min(V) - (n-i+1)\max(V) \\
&= \min(V) + \max(V) = T
\end{aligned}$$

and

$$\begin{aligned}
&\min(\Pi_{i-1}(n)) - \min(\Pi_i(n)) \\
&= -(i-2)\max(V) + (n-i+2)\min(V) + (i-1)\max(V) - (n-i+1)\min(V) \\
&= \max(V) + \min(V) = T
\end{aligned}$$

Moreover, the specific width that the frequencies span in $\Pi_i(n)$ is

$$\begin{aligned}
\Delta(n) &= \max(\Pi_i(n)) - \min(\Pi_i(n)) \\
&= -(i-1)\min(V) + (n-i+1)\max(V) + (i-1)\max(V) - (n-i+1)\min(V) \\
&= n \cdot (\max(V) - \min(V))
\end{aligned}$$

which is a constant.

Now consider the case that the input is the multi-tone (1.3) with $\omega_{i+1} - \omega_i = const > 0$ for i=1,..., $\overline{K}$-1. In this case, it can be shown that the difference between any two successive frequencies in $\Pi_i(n)$ is *const*. For example, for any $\Omega \in \Pi_i(n)$, let $\Omega = \omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_n}$. Without speciality, suppose $\min(V) \le \omega_{k_1} < \max(V)$, then the smallest frequency that is larger than Ω must be Ω' which can be computed as $\omega'_{k_1} + \omega_{k_2} + \cdots + \omega_{k_n}$ where $\omega'_{k_1} = \omega_{k_1} + const$. Hence, there exists an integer number $0 \le \alpha \le \Delta(n)/const$ such that $\Omega = \min(\Pi_i(n)) + \alpha \cdot const$ for $\forall\Omega \in \Pi_i(n)$. Considering $\forall\Omega \in \Pi_i(n)$ with $\Omega = \min(\Pi_i(n)) + \alpha\Delta(n)$, it can be obtained that

$$\begin{aligned}
\Omega + T &= \min(\Pi_i(n)) + \alpha\Delta(n) + T \\
&= -(i-1)\max(V) + (n-i+1)\min(V) + \alpha\Delta(n) + \max(V) + \min(V) \\
&= -(i-2)\cdot\max(V) + (n-i+2)\min(V) + \alpha\Delta(n) \\
&= \min(\Pi_{i-1}(n)) + \alpha\Delta(n) \in \Pi_{i-1}(n)
\end{aligned}$$

Therefore, for $\forall\Omega \in \Pi_i(n)$ there exists a frequency $\Omega' \in \Pi_{i-1}(n)$ such that $\Omega' = \Omega + T$ and vice versa. This gives Equation (7.6e). When $\omega_1 = a$, $\omega_{\overline{K}} = b$ and $\overline{K} \to \infty$ such that $\omega_{i+1} - \omega_i = const \to 0$ for i=1,..., $\overline{K}$ -1, it will become the case of a general input $U(j\omega)$ defined in [a,b]. The proposition is proved. □

- **Proof of Proposition 7.2**

(1) When the multi-tone input satisfies that sgn($F(\omega_{k_1})\cdots F(\omega_{k_n})$) is constant with respect to all the combinations of $\omega_{k_1},\cdots,\omega_{k_n} \in \{\pm\omega_1,\cdots,\pm\omega_{\overline{K}}\}$ satisfying

$\omega_{k_1}+\cdots+\omega_{k_n}\in\Pi_{m+1}(n)$ (for example $\bar{K}=1$ or F_i is a real number in (1.3)), then the opposite condition according to Definition 7.1 is that, there exist two nonzero real number c_1 and c_2 such that at a given frequency $\Omega'\in\Pi_{m+1}(n)$,

$$\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\Omega'}\left(c_1(j\omega_{k_1})^{l_1}\cdots(j\omega_{k_n})^{l_n}+c_2(j\omega_{k_1})^{l'_1}\cdots(j\omega_{k_n})^{l'_n}\right)=0 \tag{C0}$$

(C0) can also be written as

$$\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\Omega'}\left(\frac{c_1}{c_2}(j)^{\sum_{i=1}^{n}(l_i-l'_i)}(\omega_{k_1})^{l_1}\cdots(\omega_{k_n})^{l_n}\right)=-\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\Omega'}\left((\omega_{k_1})^{l'_1}\cdots(\omega_{k_n})^{l'_n}\right) \tag{C1}$$

Note that given two specific nonlinear parameters $c_{0,n}(l_1,\cdots,l_n)$ and $c_{0,n}(l'_1,\cdots,l'_n)$, it can be seen that $(\omega_{k_1})^{l_1}\cdots(\omega_{k_n})^{l_n}$ and $(\omega_{k_1})^{l'_1}\cdots(\omega_{k_n})^{l'_n}$ are both nonzero for $\omega_{k_1},\cdots,\omega_{k_n}\in\{\pm\omega_1,\cdots,\pm\omega_{\bar{K}}\}$ satisfying $\omega_{k_1}+\cdots+\omega_{k_n}\in\Pi_{m+1}(n)$, and the right side of (C1) is real, therefore

$$(j)^{\sum_{i=1}^{n}(l_i-l'_i)} \text{ must be nonzero real} \tag{C2}$$

On the other hand, if (C2) holds, whatever the value of $-\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\Omega'}\left((\omega_{k_1})^{l'_1}\cdots(\omega_{k_n})^{l'_n}\right)$ is, there are always exist two real number c_1 and c_2 such that (C1) holds. Hence, the opposite condition above now is equivalent to be that (C2) holds. That (C2) holds is equivalent to be that $\sum_{i=1}^{n}(l_i-l'_i)$ is an even integer. This is further equivalent to be that $\sum_{i=1}^{n}l_i$ and $\sum_{i=1}^{n}l'_i$ are both odd integers or even integers simultaneously.

(2) Let $\mathrm{sgn}(a+bj)=[\mathrm{sgn1}(a),\mathrm{sgn1}(b)]$. Noting that $\sum_{i=1}^{n}(l_i-l'_i)$ is an even integer, then from (7.11), it can be derived that

$$\mathrm{sgn}\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\Omega'}\left(c_1(j\omega_{k_1})^{l_1}\cdots(j\omega_{k_n})^{l_n}\right)\right)=-\mathrm{sgn}\left(\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\Omega'}\left(c_2(j\omega_{k_1})^{l'_1}\cdots(j\omega_{k_n})^{l'_n}\right)\right) \tag{C3}$$

where $\omega_{k_1},\cdots,\omega_{k_n}\in\{+\Omega,-\Omega\}$ and $\Omega'=(n-2m)\Omega$ for any $\Omega>0$. (C3) implies that there exist two nonzero real number c_1 and c_2 satisfying $c_1/c_2>0$ such that at a given frequency $\Omega'\in\Pi_{m+1}(n)=\{(n-2m)\Omega\}$, (C0) holds. Note that $\Pi_{m+1}(n)=\{(n-2m)\Omega\}$ is the case that the input is a single tone function *i.e.*, $\bar{K}=1$. Hence, (7.11) implies that (C0) holds for $\bar{K}=1$. To finish the proof, it needs to prove that, if Equation (7.11) holds, then Equations (C0) holds for all $\Omega'\in\Pi_{m+1}(n)_{\bar{K}>1}$ (note that when $\bar{K}>1$ there are more than one elements in $\Pi_{m+1}(n)_{\bar{K}>1}$). By using the mathematical induction and combination theory, it can be proved that

$$\mathrm{sgn}\left(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=\Omega'\\ \Omega'\in\Pi_{m+1}(n)_{\bar{K}=1}}}\left(c_1(j\omega_{k_1})^{l_1}\cdots(j\omega_{k_n})^{l_n}\right)\right)=\mathrm{sgn}\left(\sum_{\substack{\omega_{k_1}+\cdots+\omega_{k_n}=\Omega'\\ \Omega'\in\Pi_{m+1}(n)_{\bar{K}>1}}}\left(c_2(j\omega_{k_1})^{l_1}\cdots(j\omega_{k_n})^{l_n}\right)\right)$$

For paper limitation, this is omitted. Therefore, if Equation (7.11) holds, Equation (C0) holds for all $\Omega'\in\Pi_{m+1}(n)_{\bar{K}>1}$. □

- **Proof Property 7.4**

(a) According to Proposition 3.1 in Chapter 3, the elements of $CE(H_n(.))$ are monomial functions of the coefficients of the nonlinear terms, i.e., $c_{p_1,q_1}(\cdot)\cdots c_{p_L,q_L}(\cdot)$ for some $L \geq 1$. Note that there are only nonlinearities of odd degrees, i.e., 2k+1 (k=0,1,2,...), thus the nonlinear degree of any monomial in this case is (Proposition 3.1 in Chapter 3) $n = \sum_{i=1}^{L}(p_i + q_i) - L + 1 = \sum_{i=1}^{L}(2k_i + 1) - L + 1 = 2\sum_{i=1}^{L}k_i + 1$. Clearly, n is still an odd number. That is the nonlinearities in the system of this case can only contribute to odd order GFRFs. Thus all the even order GFRFs are zero, i.e., $CE(H_n(.))=0$ for n is even. Therefore, W_n may not be empty only when n is odd, otherwise it is empty.

Suppose n is an odd integer and $CE(H_n(.)) \neq 0$ and 1. That is, there are nonzero elements in $CE(H_n(.))$ and all the elements in $CE(H_n(.))$ consist of the coefficients of some nonlinear terms of the studied case. According to Proposition 7.1, the first period in W_n must be $\Pi_1(n) \subseteq [n \cdot \min(V), n \cdot \max(V)]$, whose central point is obviously $n \cdot T/2$ and of which the frequency span is $\Delta(n)$. Also from Proposition 7.1, the kth period in W_n must be $\Pi_k(n) \subseteq [n \cdot \min(V) - (k-1)T, n \cdot \max(V) - (k-1)T]$, whose central point is obviously $n \cdot T/2$ -(k-1)T=(n-2(k-1))T/2 and of which the frequency span is still $\Delta(n)$. Note that n-2(k-1) is an odd integer for k=1,2,.... The first point of the property is proved.

(b) Consider the case that there are only input nonlinearities of even degrees. In this case, it can be verified from the parametric characteristics in Chapter 3 that only the GFRFs of orders equal to the nonlinear degrees of the non-zero input nonlinearities are not zero. That is, only some GFRFs of even orders are not zero. Suppose n is an even integer and $CE(H_n(.)) \neq 0$ and 1. According to Proposition 7.1, the kth period in W_n must be $\Pi_k(n) \subseteq [n \cdot \min(V) - (k-1)T, n \cdot \max(V) - (k-1)T]$, whose central point is obviously $n \cdot T/2$ -(k-1)T=(n-2(k-1))T/2 and of which the frequency span is $\Delta(n)$. Note that n-2(k-1) is an even integer for k=1,2,.... This second point of the property is proved.

(c) The conclusion is straightforward since there are non-zero GFRFs of even and odd orders. This completes the proof. □

7.7 Conclusions

The super-harmonics and inter-modulations in the output frequencies of Volterra systems, especially of the nonlinear Volterra systems described by the NDE model, are studied, and some interesting properties of the system output frequencies are revealed in a uniform and analytical way. These properties provide several novel insights into the nonlinear behaviour of the Volterra systems such as the periodicity and opposite properties, and reveal the effects of different categories and different degrees of nonlinearities on the system output. These results can be used for the design of nonlinear systems or nonlinear filters to achieve a special output spectrum in a desired frequency band by taking advantage of nonlinearities, and provide an important and significant guidance to the analysis and design of nonlinear systems in the frequency domain by using the Volterra series theories of nonlinear systems.

Chapter 8
AN EXTENSION

For the nonlinear Volterra systems which can be described by a nonlinear state equation with a general nonlinear output function, the system frequency response functions and some related frequency response characteristics are developed and discussed in this Chapter. For this class of nonlinear systems, the new results provide an analytical insight into the relationship between model parameters and the frequency response functions, and the relationship between model parameters and the magnitude bound of frequency response functions, based on the results studied in previous chapters.

8.1 Introduction

As discussed in Chapter 1, great progress has been made in the frequency domain analysis of nonlinear systems based on Volterra series theory (Volterra 1959, Rugh 1981) in the past decades. Based on these results, the parametric characteristic analysis method and its related results are proposed and studied systematically in previous chapters. These new results provide a novel approach to the frequency domain analysis of the nonlinear Volterra systems. It is also noted that most of these results are developed for nonlinear systems which can be described by a simple input output model such as NARX or NDE model as those in (1.5) and (1.6). However, in many cases especially in control literature, the system model is usually described by a state equation with a nonlinear output function of system states. In these cases, many of the frequency domain analysis theory mentioned above can not be directly applied for the analysis. For this reason, some basic results are established for the frequency domain analysis of the nonlinear Volterra systems which can be described by a nonlinear state equation with a nonlinear output function in this chapter. These can be regarded a useful extension of the parametric characteristic theory developed in the previous chapters.

In the following sections, Section 8.2 gives an outline about some related research results that have been studied in the previous sections, and state the problem clearly; Section 8.3 develops the frequency response functions for the general form of nonlinear Volterra systems described by an NARX-type model with a general nonlinear output function; the parametric characteristics and bound characteristics of these frequency response functions are studied in Section 8.4 and Section 8.5; Section 8.6 extends these results for the NARX-type model to an NDE-type model; Some proofs are given in Section 8.8 and a conclusion is provided in Section 8.8.

8.2 Frequency response functions of nonlinear systems described by a simple input-output model

Nonlinear systems can often be modelled as an input-output model referred to as NARX model (Chen 1989), which are given in (1.6). For a wide class of nonlinear systems, this model provides a concise parametric structure and can be identified practically from experimental input-output data by using some well developed methods such as OLS (Billings et al 1988). It is known that, the time-domain input-output relationship of a class of nonlinear systems can be approximated by a Volterra functional series of a finite order in the neighbourhood of the zero equilibrium (Boyd and Chua 1985, and Sandberg 1983), which can be described by (1.1). In this study, consider the class of nonlinear Volterra systems described by the NARX model (1.6), whose GFRFs were given in Peyton-Jones and Billings (1989). Referring to Chapter 3 for the GFRFs given in (3.8 or 3.11, 3.10, 3.2-3.5) for the NDE model (1.5), the GFRFs for NARX model (1.6) can be given as

$$\begin{aligned} & L_n(j(\omega_1+\cdots+\omega_n))\cdot H_n(j\omega_1,\cdots,j\omega_n) \\ & =\sum_{k_1,k_n=1}^{K} c_{0,n}(k_1,\cdots,k_n)\exp(-j(\omega_1 k_1+\cdots+\omega_n k_n)) \\ & +\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_n=1}^{K} c_{p,q}(k_1,\cdots,k_{p+q})\exp(-j\sum_{i=1}^{q}\omega_{n-q+i}k_{p+i})H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}) \\ & +\sum_{p=2}^{n}\sum_{k_1,k_p=1}^{K} c_{p,0}(k_1,\cdots,k_p)H_{n,p}(j\omega_1,\cdots,j\omega_n) \end{aligned} \tag{8.1}$$

$$H_{n,p}(\cdot)=\sum_{i=1}^{n-p+1} H_i(j\omega_1,\cdots,j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)\exp(-j(\omega_1+\cdots+\omega_i)k_p) \tag{8.2}$$

$$H_{n,1}(j\omega_1,\cdots,j\omega_n)=H_n(j\omega_1,\cdots,j\omega_n)\exp(-j(\omega_1+\cdots+\omega_n)k_1) \tag{8.3}$$

where $L_n(j(\omega_1+\cdots+\omega_n))=1-\sum_{k_1=1}^{K} c_{1,0}(k_1)\exp(-j(\omega_1+\cdots+\omega_n)k_1)$. $H_{n,p}(j\omega_1,\cdots,j\omega_n)$ in (8.2) can also be rewritten as

$$H_{n,p}(j\omega_1,\cdots,j\omega_n)=\sum_{\substack{r_1\cdots r_p=1 \\ \sum_{i=1}^{p} r_i=n}}^{n-p+1}\prod_{i=1}^{p} H_{r_i}(j\omega_{X(i)+1},\cdots,j\omega_{X(i)+r_i})\exp(-j(\omega_{X(i)+1}+\cdots+j\omega_{X(i)+r_i})k_i) \tag{8.4}$$

where $X(i)=\sum_{\ell=1}^{i-1} r_\ell$. Moreover, it shall be noted that in Equation (3.8) or (3.11), $c_{p,q}(.)=0$ when $p+q>M$ according to the definition of the NARX model in (1.6).

Note that the expression of nth-order GFRF can be divided into three parts, that is, those arising from pure input nonlinear terms $H_{n_u}(\cdot)$ corresponding to the first part in the right side of equation (8.1), those from cross product nonlinear terms $H_{n_{uy}}(\cdot)$ corresponding to the second part in the right side of equation (8.1), and those from pure output nonlinear terms $H_{n_y}(\cdot)$ corresponding to the last part of equation (8.1). For clarity, (8.1) can also be written as

$$H_n(j\omega_1,\cdots,j\omega_n)=\left(H_{n_u}(\cdot)+H_{n_{uy}}(\cdot)+H_{n_y}(\cdot)\right)/L_n(j(\omega_1+\cdots+\omega_n)) \tag{8.5}$$

Equation (8.5) shows clearly that different categories of nonlinearities produce different contribution to the system GFRFs. Hence, when deriving the GFRFs of a nonlinear system, what is needed is to combine the different contributions from different nonlinearities without directly using the probing method. This property will be used later.

Using the GFRFs above, the system output frequency response can be evaluated as given in (4.1-4.4). It can be seen that these results mentioned provide an important basis for the frequency domain analysis of nonlinear Volterra systems described by model (1.6). However, in many cases especially in the field of control engineering the model of nonlinear systems of interest usually takes a form as

$$\begin{aligned} x(t+1) &= f(x) + g(x,u) \\ y_x &= h(x,u) \end{aligned} \tag{8.6}$$

which is the discrete time nonlinear state space equation, where $x \in \mathbb{R}^n$. It is obvious that frequency domain analysis of this nonlinear system can not be conducted by directly using the results above and some of the results which are developed for NARX model (1.6) in previous chapters. Thus some basic results of the system frequency domain analysis theories for this form of nonlinear Volterra systems are developed in this chapter.

8.3 Frequency response functions for nonlinear Volterra systems with a general nonlinear output function

Consider nonlinear Volterra systems described by the following model in a form similar to model (8.6)

$$x(t) = \sum_{m=1}^{M_1}\sum_{p=0}^{m}\sum_{k_1,k_m=0}^{K} \bar{c}_{p,m-p}(k_1,\cdots,k_m)\prod_{i=1}^{p} x(t-k_i)\prod_{i=p+1}^{m} u(t-k_i) \tag{8.7a}$$

$$y(t) = \sum_{m=1}^{M_2}\sum_{p=0}^{m}\sum_{k_1,k_m=0}^{K} \tilde{c}_{p,m-p}(k_1,\cdots,k_m)\prod_{i=1}^{p} x(t-k_i)\prod_{i=p+1}^{m} u(t-k_i) \tag{8.7b}$$

where M_1, M_2 and K are all positive integers, and $x(t)$, $y(t)$, $u(t) \in \mathbb{R}$. (8.7a) is the system state equation which is still described by a NARX model, and (8.7b) represents the system output which is a nonlinear function of state $x(t)$ and input u(t) in a general polynomial form. This model represents a more useful case than model (1.6), since it is frequently adopted in control literature as mentioned above, although (8.7) can still be written into the form of (1.6). Hence, determination of frequency response functions for model (8.7) is significant. To derive the GFRFs for (8.7), the probing method in Rugh (1981) can be adopted. However, this paper uses an alternative simple method based on the discussions in Section 8.2 for that the structure and nonlinear types of this model are clear.

To derive the GFRFs for model (8.7), system (8.7) can be regarded as a system of one input u(t) and two outputs x(t) and y(t). Therefore, there are two sets of GFRFs for model (8.7) corresponding to the two input-output relationships between input u(t) and two outputs x(t) and y(t) respectively. Considering the GFRFs from input u(t) to output x(t), there are three categories of nonlinearities as mentioned above. Therefore, the nth-order GFRF from input u(t) to output x(t) denoted by $H_n^x(j\omega_1,\cdots,j\omega_n)$ can be directly determined which is the same as (8.1-8.4), i.e.,

$$H_n^x(j\omega_1,\cdots,j\omega_n) = \frac{H_{n_u}^x(j\omega_1,\cdots,j\omega_n) + H_{n_{ux}}^x(j\omega_1,\cdots,j\omega_n) + H_{n_x}^x(j\omega_1,\cdots,j\omega_n)}{L_n(j(\omega_1+\cdots+\omega_n))} \tag{8.8}$$

where, $L_n(j(\omega_1+\cdots+\omega_n)) = 1 - \sum_{k_1=1}^{K}\bar{c}_{1,0}(k_1)\exp(-j(\omega_1+\cdots+\omega_n)k_1)$

$$H_{n_u}^x(j\omega_1,\cdots,j\omega_n) = \sum_{k_1,k_n=0}^{K}\bar{c}_{0,n}(k_1,\cdots,k_n)\exp(-j(\omega_1 k_1+\cdots+\omega_n k_n)) \tag{8.9a}$$

$$\begin{aligned} &H_{n_{ux}}^x(j\omega_1,\cdots,j\omega_n) \\ &= \sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K}\bar{c}_{p,q}(k_1,\cdots,k_{p+q})\exp(-j(\omega_{n-q+1}k_{p+1}+\cdots+\omega_n k_{p+q}))H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}) \end{aligned} \tag{8.9b}$$

$$H_{n_x}^{x}(j\omega_1,\cdots,j\omega_n)=\sum_{p=2}^{n}\sum_{k_1,k_p=0}^{K}\bar{c}_{p,0}(k_1,\cdots,k_p)H_{n,p}(j\omega_1,\cdots,j\omega_n) \tag{8.9c}$$

$$H_{n,p}(j\omega_1,\cdots,j\omega_n)=\sum_{i=1}^{n-p+1}H_i^{x}(j\omega_1,\cdots,j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)\exp(-j(\omega_1+\cdots+\omega_i)k_p) \tag{8.9d}$$

$$H_{n,1}(j\omega_1,\cdots,j\omega_n)=H_n^{x}(j\omega_1,\cdots,j\omega_n)\exp(-j(\omega_1+\cdots+\omega_n)k_1) \tag{8.9e}$$

Similarly, consider the GFRFs from input u(t) to output y(t). There are also three categories of nonlinearities in terms of input u(t) and output x(t) similar to those from input u(t) to output x(t), and there is one linear output y(t). Note that there are no nonlinearities in terms of y(t), and all the nonlinearities come from input u(t) and output x(t). For this reason, the GFRFs from u(t) to y(t) are dependent on the GFRFs from u(t) to x(t). Therefore, in this case the *n*th-order GFRF from input u(t) to output y(t) denoted by $H_n^{y}(j\omega_1,\cdots,j\omega_n)$ is,

$$H_n^{y}(j\omega_1,\cdots,j\omega_n)=H_{n_u}^{y}(j\omega_1,\cdots,j\omega_n)+H_{n_{ux}}^{y}(j\omega_1,\cdots,j\omega_n)+H_{n_x}^{y}(j\omega_1,\cdots,j\omega_n) \tag{8.10}$$

where the corresponding terms in (8.10) are

$$H_{n_u}^{y}(j\omega_1,\cdots,j\omega_n)=\sum_{k_1,k_n=0}^{K}\tilde{c}_{0,n}(k_1,\cdots,k_n)\exp(-j(\omega_1 k_1+\cdots+\omega_n k_n)) \tag{8.11a}$$

$$\begin{aligned}&H_{n_{ux}}^{y}(j\omega_1,\cdots,j\omega_n)\\&=\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K}\tilde{c}_{p,q}(k_1,\cdots,k_{p+q})\exp(-j(\omega_{n-q+1}k_{p+1}+\cdots+\omega_n k_{p+q}))H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})\end{aligned} \tag{8.11b}$$

$$H_{n_x}^{y}(j\omega_1,\cdots,j\omega_n)=\sum_{p=1}^{n}\sum_{k_1,k_p=0}^{K}\tilde{c}_{p,0}(k_1,\cdots,k_p)H_{n,p}(j\omega_1,\cdots,j\omega_n) \tag{8.11c}$$

Note that p is counted from 1 in equation (8.11c), different from equation (8.9c) where p is counted from 2, and $H_{n,p}(j\omega_1,\cdots,j\omega_n)$ in (8.11bc) is the same as that in (8.9b-d) because the nonlinearities in equation (8.7b) have no relationship with y(t) but x(t). Note also that these results can also be derived by following the method in Swain and Billings (2001). However, the results are developed in a more straightforward manner here and provide a concise analytical expression of the GFRFs for model (8.7).

From the GFRFs of model (8.7), the output frequency response of (8.7) can also be derived readily by extending the results in (4.1-4.4). Regard x(t) and y(t) as two outputs actuated by the same input u(t), then

$$X(j\omega)=\sum_{n=1}^{N}\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}H_n^{x}(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega \tag{8.12a}$$

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}H_n^{y}(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega \tag{8.12b}$$

When the system input is a multi-tone signal (1.3), then the system output frequency response can be similarly derived as:

$$X(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}H_n^{x}(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n}) \tag{8.13a}$$

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}H_n^{y}(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n}) \tag{8.13b}$$

where $F(\omega)=\begin{cases}|F_i|e^{j\angle F_i} & \text{if } \omega\in\{\omega_k, k=\pm1,\cdots,\pm K\}\\ 0 & \text{else}\end{cases}$, $\omega_{\pm k}=\pm\omega_k$.

It can be seen from the results above that the frequency response functions for nonlinear systems are quite different from those for linear systems. It is known that in a linear system, frequency response functions of different parts can be combined together by addition or multiplication. This is not the case for nonlinear systems. For instance, if x(t) is only regarded as an input in equation (8.7b) independent of (8.7a), then the GFRFs $H_n^y(j\omega_1,\cdots,j\omega_n)$ and therefore the output spectrum $Y(j\omega)$ will all be changed completely for n>1, since in this case there are only input nonlinearities in (8.7b) and no output nonlinearities. Even so, it can also be seen from (8.12-8.13) that the output frequency ranges for both x(t) and y(t) are the same one, *i.e.*,

$$\bigcup_{n=1}^{N}\{\omega|\omega=\omega_1+\cdots+\omega_n,\omega_i\in R_\omega\} \tag{8.14}$$

where R_ω represents the input frequency range, for example $R_\omega=\{\omega_k,k=\pm1,\cdots,\pm K\}$ for the multi-tone signal (1.3).

These frequency response functions obtained above for model (8.7) provide a useful basis for the frequency domain analysis of nonlinear Volterra systems described by model (8.6). In the following sections, some important frequency response characteristics of these frequency response functions for nonlinear Volterra system (8.7) are further established and discussed.

Example 8.1. Consider the following nonlinear system,

$$\begin{aligned}&mx(t-2)+a_1x(t-1)+a_2x^2(t-1)+a_3x^3(t-1)+kx(t)=u(t)\\&y(t)=a_1x(t-1)+a_2x^2(t-1)+a_3x^3(t-1)+kx(t)\end{aligned} \tag{8.15}$$

which can be written into the form of model (8.7) with parameters K=2, $\bar{c}_{1,0}(2)=-m/k,\bar{c}_{1,0}(1)=-a_1/k,\bar{c}_{2,0}(11)=-a_2/k,\bar{c}_{3,0}(111)=-a_3/k,\bar{c}_{0,1}(0)=1/k$

$\tilde{c}_{1,0}(1)=a_1,\tilde{c}_{2,0}(11)=a_2,\tilde{c}_{3,0}(111)=a_3,\tilde{c}_{1,0}(0)=k$, and all the other parameters are zero.

The GFRFs can be computed according to (8.8-8.11). For example,

$$H_{1_u}^x(j\omega_1)=\sum_{k_1=0}^{2}\bar{c}_{0,1}(k_1)\exp(-j\omega_1k_1)=\bar{c}_{0,1}(0)=1/k,\ H_{1_u}^y(j\omega_1)=0,$$

Because there are no input nonlinearities and cross nonlinearities, thus

$$H_{n_u}^x(j\omega_1,\cdots,j\omega_n)=0 \text{ and } H_{n_u}^y(j\omega_1,\cdots,j\omega_n)=0 \text{ for n>1}$$

$$H_{n_{ux}}^x(j\omega_1,\cdots,j\omega_n)=0 \text{ and } H_{n_{ux}}^y(j\omega_1,\cdots,j\omega_n)=0 \text{ for all } n$$

Regarding the output nonlinear terms,

$H_{1_x}^x(j\omega_1)=0$,

$$H_{2_x}^x(j\omega_1,j\omega_2)=\sum_{p=2}^{2}\sum_{k_1,k_p=1}^{2}\bar{c}_{p,0}(k_1,\cdots,k_p)H_{2,p}(j\omega_1,j\omega_2)$$

$$=\sum_{k_1,k_p=1}^{2}\bar{c}_{2,0}(k_1,k_2)H_{2,2}(j\omega_1,j\omega_2)=\sum_{k_1,k_p=1}^{2}\bar{c}_{2,0}(k_1,k_2)H_1^x(j\omega_1)H_{1,1}(j\omega_2)\exp(-j\omega_1k_2)$$

$$=\sum_{k_1,k_p=1}^{2}\bar{c}_{2,0}(k_1,k_2)H_1^x(j\omega_1)H_1^x(j\omega_2)\exp(-j\omega_2k_1)\exp(-j\omega_1k_2)$$

$$=-\tfrac{a_2}{k}H_1^x(j\omega_1)H_1^x(j\omega_2)\exp(-j\omega_2)\exp(-j\omega_1)$$

$$H_{1_x}^y(j\omega_1)=\sum_{k_1}^{2}\tilde{c}_{1,0}(k_1)H_{1,1}(j\omega_1)=\sum_{k_1}^{2}\tilde{c}_{1,0}(k_1)H_1^x(j\omega_1)\exp(-j\omega_1 k_1)$$
$$=a_1H_1^x(j\omega_1)\exp(-j\omega_1)+kH_1^x(j\omega_1)$$
$$H_{2_x}^y(j\omega_1,j\omega_2)=\sum_{p=1}^{2}\sum_{k_1,k_p=0}^{2}\tilde{c}_{p,0}(k_1,\cdots,k_p)H_{2,p}(j\omega_1,j\omega_2)$$
$$=\sum_{k_1=0}^{2}\tilde{c}_{1,0}(k_1)H_{2,1}(j\omega_1,j\omega_2)+\sum_{k_1,k_2=0}^{2}\tilde{c}_{2,0}(k_1,k_2)H_{2,2}(j\omega_1,j\omega_2)$$
$$=\sum_{k_1=0}^{2}\tilde{c}_{1,0}(k_1)H_2^x(j\omega_1,j\omega_2)\exp(-j(\omega_1+\omega_2)k_1)$$
$$+\sum_{k_1,k_2=0}^{2}\tilde{c}_{2,0}(k_1,k_2)H_1^x(j\omega_1)H_1^x(j\omega_2)\exp(-j\omega_2k_1)\exp(-j\omega_1k_2)$$
$$=kH_2^x(j\omega_1,j\omega_2)+a_1H_2^x(j\omega_1,j\omega_2)\exp(-j(\omega_1+\omega_2)k_1)$$
$$+a_2H_1^x(j\omega_1)H_1^x(j\omega_2)\exp(-j\omega_2)\exp(-j\omega_1)$$

Note that

$$L_n(j(\omega_1+\cdots+\omega_n))=1-\sum_{k_1=1}^{2}\tilde{c}_{1,0}(k_1)\exp(-j(\omega_1+\cdots+\omega_n)k_1)$$
$$=1+\tfrac{a_1}{k}\exp(-j(\omega_1+\cdots+\omega_n))+\tfrac{m}{k}\exp(-j2(\omega_1+\cdots+\omega_n))$$

Hence, by following similar process as above, the GFRFs for x(t) and y(t) can all be computed recursively up to any high orders. For example,

$$H_1^x(j\omega_1)=\frac{H_{1_u}^x(j\omega_1)+H_{1_{ux}}^x(j\omega_1)+H_{1_x}^x(j\omega_1)}{L_1(j\omega_1)}=\frac{1/k}{1+\tfrac{a_1}{k}\exp(-j\omega_1)+\tfrac{m}{k}\exp(-j2\omega_1)}$$
$$H_2^x(j\omega_1,j\omega_2)=\frac{H_{2_u}^x(j\omega_1,j\omega_2)+H_{2_{ux}}^x(j\omega_1,j\omega_2)+H_{2_x}^x(j\omega_1,j\omega_2)}{L_2(j(\omega_1+\omega_2))}$$
$$=\frac{-\tfrac{a_2}{k}H_1^x(j\omega_1)H_1^x(j\omega_2)\exp(-j\omega_2)\exp(-j\omega_1)}{1+\tfrac{a_1}{k}\exp(-j(\omega_1+\omega_2))+\tfrac{m}{k}\exp(-j2(\omega_1+\omega_2))}$$
$$H_1^y(j\omega_1)=H_{1_u}^y(j\omega_1)+H_{1_{ux}}^y(j\omega_1)+H_{1_x}^y(j\omega_1)=k+a_1H_1^x(j\omega_1)\exp(-j\omega_1)$$
$$H_2^y(j\omega_1,j\omega_2)=H_{2_u}^y(j\omega_1,j\omega_2)+H_{2_{ux}}^y(j\omega_1,j\omega_2)+H_{2_x}^y(j\omega_1,j\omega_2)$$
$$=a_1H_2^x(j\omega_1,j\omega_2)\exp(-j(\omega_1+\omega_2))+a_2H_1^x(j\omega_1)H_1^x(j\omega_2)\exp(-j\omega_2)\exp(-j\omega_1)$$

It can be verified that the first order GFRFs are frequency response functions in *z*-space of the linear parts of model (8.15). By using the GFRFs above, the output spectrum can also be computed according to (8.12-8.13).

8.4 Parametric characteristics

The parametric characteristic analysis was proposed and studied in Chapters 2-4. It is used to reveal which model parameters contribute to and how these parameters affect the system frequency response functions. By using the parametric characteristic analysis, some useful characteristics of system frequency response can be obtained, and the explicit relationship between system frequency response and system time domain model parameters can be unveiled. In this section, the parameter characteristics of the output frequency response function relating to the output y(t) of model (8.7) with respect to model nonlinear parameters are studied, and the model nonlinear parameters in equation (8.7a) are focused since nonlinear parameters in equation (8.7b) has no effect on system dynamics. In what follows, let

$\overline{C}(n)=\{\overline{c}_{p,q}(k_1\cdots k_{p+q})|1<p+q\le n, 0\le k_i\le K, 1\le i\le p+q\}$ denotes all the nonlinear parameters in equation (8.7a) with degree from 2 to n, and similarly denote all the parameters in equation (8.7b) with degree from 2 to n as: $\widetilde{C}(n)=\{\widetilde{c}_{p,q}(k_1\cdots k_{p+q})|1<p+q\le n, 0\le k_i\le K, 1\le i\le p+q\}$. All the (*p*+*q*)th degree nonlinear parameters in (8.7) of form $c_{p,q}(.)$ construct a vector denoted by

$$C_{p,q}=[c_{p,q}(0,\cdots,0),c_{p,q}(0,\cdots,1),\cdots,c_{p,q}(\underbrace{K,\cdots,K}_{p+q})]$$

In what follows, $CE(H_{CF})_{\vartheta}$ means to only extract the parameters in the set ϑ from H_{CF}, and without specialty $CE(H_{CF})$ means to extract all the nonlinear parameters (i.e., its nonlinearity degree >1) appearing in H_{CF}.

8.4.1 Parametric characteristic analysis for $H_n^x(j\omega_1,\cdots,j\omega_n)$

Application of the CE operator to a complicated series for its parametric characteristics can be performed by directly replacing the addition and multiplication in the series by "$\oplus$" and "$\otimes$" respectively.

The parametric characteristic of the nth-order GFRF $H_n^x(j\omega_1,\cdots,j\omega_n)$ with respect to model nonlinear parameters $\overline{C}(n)$ is

$$CE(H_n^x(j\omega_1,\cdots,j\omega_n))=CE\left(\frac{H_{n_u}^x(j\omega_1,\cdots,j\omega_n)+H_{n_{ux}}^x(j\omega_1,\cdots,j\omega_n)+H_{n_x}^x(j\omega_1,\cdots,j\omega_n)}{L_n(j(\omega_1+\cdots+\omega_n))}\right)$$

$$=CE(H_{n_u}^x(j\omega_1,\cdots,j\omega_n))\oplus CE(H_{n_{ux}}^x(j\omega_1,\cdots,j\omega_n))\oplus CE(H_{n_x}^x(j\omega_1,\cdots,j\omega_n)) \quad (8.16)$$

$$=\overline{C}_{0,n}\oplus\left(\bigoplus_{q=1}^{n-1}\bigoplus_{p=1}^{n-q}\overline{C}_{p,q}\otimes CE(H_{n-q,p}(\cdot))\right)\oplus\left(\bigoplus_{p=2}^{n}\overline{C}_{p,0}\otimes CE(H_{n,p}(\cdot))\right)$$

where

$$CE(H_{n,p}(\cdot))=\bigoplus_{i=1}^{n-p+1}CE(H_i^x(\cdot))\otimes CE(H_{n-i,p-1}(\cdot)) \text{ or } CE(H_{n,p}(\cdot))=\bigoplus_{\substack{r_1\cdots r_p=1\\ \sum r_i=n}}^{n-p+1}\bigotimes_{i=1}^{p}CE(H_{r_i}^x(\cdot)) \quad (8.17)$$

$$CE(H_{n,1}(\cdot))=CE(H_n^x(\cdot))) \quad (8.18)$$

Note that in (8.16), $E(1/L_n(j(\omega_1+\cdots+\omega_n)))=1$ since there are no nonlinear parameters (in the set $\overline{C}(n)$) in $1/L_n(j(\omega_1+\cdots+\omega_n))$. It is shown in Chapter 3 that

$$CE(H_{n,p}(\cdot))=CE\left(H_{n-p+1}^x(\cdot)\right) \quad (8.19)$$

and thus (8.16) is simplified as

$$CE(H_n^x(j\omega_1,\cdots,j\omega_n))$$
$$=\overline{C}_{0,n}\oplus\left(\bigoplus_{q=1}^{n-1}\bigoplus_{p=1}^{n-q}\overline{C}_{p,q}\otimes CE(H_{n-q-p+1}^x(\cdot))\right)\oplus\left(\overline{C}_{n,0}\oplus\bigoplus_{p=2}^{\lfloor n+1/2\rfloor}\overline{C}_{p,0}\otimes CE(H_{n-p+1}^x(\cdot))\right) \quad (8.20)$$

From (8.20), $CE(H_n^x(j\omega_1,\cdots,j\omega_n))$ has no relationship with $\widetilde{C}(n)$. With the parametric characteristics (8.20), it can be concluded (referring to Chapter 3) that there must exist a complex valued function vector $f_n(j\omega_1,\cdots,j\omega_n)$ with appropriate dimension, such that

$$H_n^x(j\omega_1,\cdots,j\omega_n)=CE\left(H_n^x(j\omega_1,\cdots,j\omega_n)\right)\cdot f_n(j\omega_1,\cdots,j\omega_n) \quad (8.21)$$

Equation (8.21) provides an explicit expression for the relationship between nonlinear parameters $\overline{C}(n)$ and the nth-order GFRF from u(t) to x(t). For any parameter of interest, how its effect is on the GFRFs can be revealed by checking $CE\left(H_n^x(j\omega_1,\cdots,j\omega_n)\right)$. From (8.21), $H_n^x(j\omega_1,\cdots,j\omega_n)$ is in fact a polynomial function of parameters in $\overline{C}(n)$ which define system nonlinearities, thus some qualitative properties of $H_n^x(j\omega_1,\cdots,j\omega_n)$ can also be indicated by $CE(H_n^x(j\omega_1,\cdots,j\omega_n))$. Moreover, using (8.21), (8.12a) can be written as

$$X(j\omega)=\sum_{n=1}^{N}CE(H_n^x(j\omega_1,\cdots,j\omega_n))\cdot\overline{F}_n(j\omega) \tag{8.22}$$

where $\overline{F}_n(j\omega)=\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}f_n(j\omega_1,\cdots,j\omega_n)\cdot\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega$. This is the parametric characteristic function expression for the output $X(j\omega)$. By using this expression, $X(j\omega)$ can be obtained by following a numerical method without complicated computation that involved in (8.8-8.9, 8.12a,8.13a) (for more detailed, refer to Chapter 3 and Chapter 4). More detailed discussion about the potential application of the parametric characteristic analysis can also refer to Chapter 3 and Chapter 4.

8.4.2 Parametric characteristic analysis for $H_n^y(j\omega_1,\cdots,j\omega_n)$

To study the parametric characteristic of the nth-order GFRF $H_n^y(j\omega_1,\cdots,j\omega_n)$ with respect to only model nonlinear parameters in $\overline{C}(n)$, the parametric characteristic with respect to model parameters in $\overline{C}(n)$ and $\widetilde{C}(n)$ are derived first and then the case with respect only to nonlinear parameters in $\overline{C}(n)$ is discussed.

Applying the CE operator to (8.10) yields,

$$CE(H_n^y(j\omega_1,\cdots,j\omega_n))=CE(H_{n_u}^y(j\omega_1,\cdots,j\omega_n))\oplus CE(H_{n_{ux}}^y(j\omega_1,\cdots,j\omega_n))\oplus CE(H_{n_x}^y(j\omega_1,\cdots,j\omega_n))$$

$$=\widetilde{C}_{0,n}\oplus\left(\bigoplus_{q=1}^{n-1}\bigoplus_{p=1}^{n-q}\widetilde{C}_{p,q}\otimes CE(H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}))\right)\oplus\left(\bigoplus_{p=1}^{n}\widetilde{C}_{p,0}\otimes CE(H_{n,p}(j\omega_1,\cdots,j\omega_n))\right)$$

using (8.19), which further gives

$$CE(H_n^y(j\omega_1,\cdots,j\omega_n))$$
$$=\widetilde{C}_{0,n}\oplus\left(\bigoplus_{q=1}^{n-1}\bigoplus_{p=1}^{n-q}\widetilde{C}_{p,q}\otimes CE(H_{n-q-p+1}^x(j\omega_1,\cdots,j\omega_{n-q}))\right)\oplus\left(\bigoplus_{p=1}^{n}\widetilde{C}_{p,0}\otimes CE(H_{n-p+1}^x(j\omega_1,\cdots,j\omega_n))\right) \tag{8.23}$$

Thus the parametric characteristic of $H_n^y(j\omega_1,\cdots,j\omega_n)$ with respect to both nonlinear parameters in $\overline{C}(n)$ and $\widetilde{C}(n)$ is obtained.

Especially, if $\widetilde{C}(n)$ is independent of $\overline{C}(n)$, the parametric characteristic of $H_n^y(j\omega_1,\cdots,j\omega_n)$ with respect to nonlinear parameters in $\widetilde{C}(n)$ can be written as

$$CE(H_n^y(j\omega_1,\cdots,j\omega_n))_{\widetilde{C}(n)}=\widetilde{C}_{0,n}\oplus\left(\bigoplus_{q=1}^{n-1}\bigoplus_{p=1}^{n-q}\widetilde{C}_{p,q}\right)\oplus\left(\bigoplus_{p=2}^{n}\widetilde{C}_{p,0}\right) \tag{8.24}$$

Therefore, in this case $H_n^y(j\omega_1,\cdots,j\omega_n)$ can be expressed as a polynomial function of $\widetilde{C}(n)$ as

$$H_n^y(j\omega_1,\cdots,j\omega_n;\widetilde{C}(n))=CE(H_n^y(j\omega_1,\cdots,j\omega_n))_{\widetilde{C}(n)}\cdot f_n(j\omega_1,\cdots,j\omega_n;\overline{C}(n)) \tag{8.25}$$

where $f_n(j\omega_1,\cdots,j\omega_n;\overline{C}(n))$ is a complex valued function vector with an appropriate dimension, which is also a function of the parameters in $\overline{C}(n)$ in this case. From (8.24), it can be seen that $CE(H_n^y(j\omega_1,\cdots,j\omega_n))_{\widetilde{C}(n)}$ is a vector which is composed of all the elements in $\widetilde{C}(n)$. That is, the nth-order GFRF is a polynomial function of all the parameters in $\tilde{C}(n)$ if $\tilde{C}(n)$ is independent of $\bar{C}(n)$. This conclusion is straightforward. The case where $\tilde{C}(n)$ is dependent on $\bar{C}(n)$ will be discussed in the following section.

8.4.2.1 Parametric characteristics of $H_n^y(j\omega_1,\cdots,j\omega_n)$ with respect to $\bar{C}(n)$

What is of more interest is the parametric characteristic of $H_n^y(j\omega_1,\cdots,j\omega_n)$ with respect to nonlinear parameters in $\bar{C}(n)$ which define system nonlinear dynamics. Consider two cases as follows.

(1) $\tilde{C}(n)$ has no relationship with $\bar{C}(n)$

In this case, it can be derived from (8.23) that

$$CE(H_n^y(j\omega_1,\cdots,j\omega_n))_{\bar{C}(n)} = \left(\mathop{\oplus}_{q=1}^{n-1}\mathop{\oplus}_{p=1}^{n-q}(1-\delta(\tilde{C}_{p,q}))\cdot CE(H_{n-q-p+1}^x(j\omega_1,\cdots,j\omega_{n-q}))\right) \oplus\left(\mathop{\oplus}_{p=1}^{n}(1-\delta(\tilde{C}_{p,0}))\cdot CE(H_{n-p+1}^x(j\omega_1,\cdots,j\omega_n))\right) \tag{8.26}$$

where $\delta(C_{p,q})=\begin{cases}0 & C_{p,q}\neq 0\\ 1 & C_{p,q}=0\end{cases}$. From (8.26) it can be seen that $CE(H_n^y(j\omega_1,\cdots,j\omega_n))_{\bar{C}(n)}$ is the summation by "$\oplus$" of parametric characteristics of some GFRFs for x(t) from the 1st order to the nth order. From the definition of operation "$\oplus$", the repetitive terms should not be counted. Therefore, (8.26) is simplified as

$$CE(H_n^y(j\omega_1,\cdots,j\omega_n))_{\bar{C}(n)} = \mathop{\oplus}_{p=1}^{n}\chi(n,p)\cdot CE(H_{n-p+1}^x(j\omega_1,\cdots,j\omega_n))_{\bar{C}(n-p+1)} \tag{8.27}$$

where

$$\chi(n,p)=1-\delta\left(\sum_{\substack{0\le q\le n-1,1\le p'\le n-q\\ p'+q=p}}(1-\delta(\tilde{C}_{p',q}))\right) \tag{8.28}$$

(8.28) means that if there is at least one nonzero $\tilde{C}_{p',q}$ then the corresponding $CE(H_{n-q-p+1}^x(j\omega_1,\cdots,j\omega_{n-q}))$ will be counted in (8.27). According to Proposition 3.1 in Chapter 3, it follows from (8.27) that the nth-order GFRF for y(t) has relationship with all the nonlinear parameters in $\bar{C}(n)$ of degree from 2 to n' in this case, where $n'\le n$.

(2) $\tilde{C}(n)$ has linear relationship with $\bar{C}(n)$ by $\tilde{c}_{p,q}(\cdot)=\tilde{\alpha}+\tilde{\beta}\bar{c}_{p,q}(\cdot)$ for some real number α and β

Note that applying the CE operator to $\tilde{c}_{p,q}(\cdot)=\tilde{\alpha}+\tilde{\beta}\bar{c}_{p,q}(\cdot)$ for the nonlinear parameter $\bar{c}_{p,q}(\cdot)$ gives $CE(\tilde{c}_{p,q}(\cdot))=CE(\tilde{\alpha}+\tilde{\beta}\bar{c}_{p,q}(\cdot))=\bar{c}_{p,q}(\cdot)$, *i.e.*, $CE(\tilde{C}_{p,q})=\bar{C}_{p,q}$. Hence, in this case (8.23) should be

$$CE(H_n^y(j\omega_1,\cdots,j\omega_n)) = \bar{C}_{0,n}\oplus\left(\mathop{\oplus}_{q=1}^{n-1}\mathop{\oplus}_{p=1}^{n-q}\bar{C}_{p,q}\otimes CE(H_{n-q-p+1}^x(j\omega_1,\cdots,j\omega_{n-q}))\right)\oplus\left(\mathop{\oplus}_{p=1}^{n}\bar{C}_{p,0}\otimes CE(H_{n-p+1}^x(j\omega_1,\cdots,j\omega_n))\right) \tag{8.29}$$

(8.29) can be further written as

$$
\begin{aligned}
&CE(H_n^y(j\omega_1,\cdots,j\omega_n)) \\
&= \overline{C}_{0,n} \oplus \left(\mathop{\oplus}_{q=1}^{n-1} \mathop{\oplus}_{p=1}^{n-q} \overline{C}_{p,q} \otimes CE(H_{n-q-p+1}^x(j\omega_1,\cdots,j\omega_{n-q})) \right) \oplus \left(\mathop{\oplus}_{p=2}^{n} \overline{C}_{p,0} \otimes CE(H_{n-p+1}^x(j\omega_1,\cdots,j\omega_n)) \right) \\
&\quad \oplus \overline{C}_{1,0} \otimes CE(H_n^x(j\omega_1,\cdots,j\omega_n)) \\
&= CE(H_n^x(j\omega_1,\cdots,j\omega_n)) \oplus \overline{C}_{1,0} \otimes CE(H_n^x(j\omega_1,\cdots,j\omega_n))
\end{aligned} \tag{8.30}
$$

In the derivation of (8.30), equations (8.16) and (8.19) are used. (8.30) can reveal that how the model parameters in equation (8.7a) affect system output frequency response. When only nonlinear parameters are considered under the assumption that linear parameters are fixed in the model, then (8.30) is simplified as

$$
CE(H_n^y(j\omega_1,\cdots,j\omega_n))_{\overline{C}(n)} = CE(H_n^x(j\omega_1,\cdots,j\omega_n))_{\overline{C}(n)} \tag{8.31}
$$

(8.31) indicates that the parametric characteristics of the GFRFs for y(t) and x(t) are the same with respect to model nonlinear parameters in $\overline{C}(n)$. Note that equation (8.31) has a relationship with all the parameters in $\overline{C}(n)$ from degree 2 to *n*, which is different from (8.27). In this case both $X(j\omega)$ and $Y(j\omega)$ can be expressed as a polynomial function of model nonlinear parameters in $\overline{C}(n)$ with the same polynomial structure.

8.4.2.2 Some further results and discussions

The following results can be summarized based on Section 8.4.2.1.

Proposition 8.1. Considering system (8.7), there exists a complex valued function vector $\tilde{f}_n(j\omega_1,\cdots,j\omega_n)$ with appropriate dimension which is a function of linear parameters, such that

$$
H_n^y(j\omega_1,\cdots,j\omega_n) = CE\left(H_n^y(j\omega_1,\cdots,j\omega_n)\right)_{\overline{C}(n)} \cdot \tilde{f}_n(j\omega_1,\cdots,j\omega_n) \tag{8.32}
$$

and the output spectrum of system (8.7) can be written as

$$
Y(j\omega;\overline{C}(N)) = \sum_{n=1}^{N} CE\left(H_n^y(j\omega_1,\cdots,j\omega_n)\right)_{\overline{C}(n)} \cdot \tilde{F}_n(j\omega) \tag{8.33}
$$

where $\tilde{F}_n(j\omega) = \frac{1}{\sqrt{n}(2\pi)^{n-1}} \int_{\omega_1+\cdots+\omega_n=\omega} \tilde{f}_n(j\omega_1,\cdots,j\omega_n) \cdot \prod_{i=1}^{n} U(j\omega_i) d\sigma_\omega$. If the input of system (8.7) is the multi-tone signal (1.3), then the OFRF of system (8.7) can be expressed as

$$
Y(j\omega;\overline{C}(N)) = \sum_{n=1}^{N} CE\left(H_n^y(j\omega_{k_1},\cdots,j\omega_{k_n})\right)_{\overline{C}(n)} \cdot \tilde{\tilde{F}}_n(j\omega) \tag{8.34}
$$

where $\tilde{\tilde{F}}_n(j\omega) = \frac{1}{2^n} \sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} \tilde{f}_n(j\omega_{k_1},\cdots,j\omega_{k_n}) \cdot F(\omega_{k_1}) \cdots F(\omega_{k_n})$. $CE\left(H_n^y(j\omega_1,\cdots,j\omega_n)\right)_{\overline{C}(n)}$ is given in (8.27) or (8.31).

Proof. The results are straightforward from the discussions above and the results in Chapter 3 and Chapter 4. □

Proposition 8.2. Under the same assumption as Proposition 8.1 for system (8.7). If $\tilde{C}(n)$ has either no relationship or linear relationship with $\overline{C}(n)$, then $CE\left(H_n^y(j\omega_1,\cdots,j\omega_n)\right)_{\overline{C}(n)}$ is given in (8.27) or (8.31), and the parametric characteristic vector for $Y(j\omega)$ can both be written as

$$
CE(Y(j\omega))_{\overline{C}(N)} = \mathop{\oplus}_{n=1}^{N} CE(H_n^x(j\omega_1,\cdots,j\omega_n))_{\overline{C}(n)} \tag{8.35}
$$

That is, there exists a complex valued function vector $\hat{F}(j\omega_1,\cdots,j\omega_n)$ with appropriate dimension, which is a function of nonlinear parameters in $\tilde{C}(N)$, linear parameters and the input, such that

$$Y(j\omega;\overline{C}(N))=\left(\bigoplus_{n=1}^{N}CE\left(H_n^x(j\omega_1,\cdots,j\omega_n)\right)_{\overline{C}(n)}\right)\cdot\hat{F}(j\omega) \tag{8.36}$$

Proof. See the proof in Section 8.7. □

From Proposition 8.2, both of the two mentioned cases have the same parametric characteristics for the output spectrum $Y(j\omega)$. If $\tilde{C}(n)$ has no relationship with $\overline{C}(n)$, (8.35) may be conservative since some terms in (8.35) have no contribution. However, this does not affect the result of (8.36) because the corresponding terms in the complex valued vector will actually be zero after numerical identification. Once the parametric characteristics $CE\left(H_n^y(j\omega_1,\cdots,j\omega_n)\right)_{\overline{C}(n)}$ are derived, the polynomial structure of the parametric characteristic expression for $Y(j\omega)$ is determined, and then as mentioned above, (8.33) and (8.34) can be determined by using a numerical method. Therefore, analysis, design and optimization of system output frequency response can be conducted based on this explicit polynomial expression in terms of model nonlinear parameters in $\overline{C}(N)$.

Example 8.2. Consider nonlinear system (8.15) again. Note that there are only two nonlinear parameters in $\overline{C}(n)$, *i.e.*, $\overline{c}_{2,0}(11)=-a_2/k,\overline{c}_{3,0}(111)=-a_3/k$, and the nonlinear parameters in $\tilde{C}(n)$ are linear functions of the corresponding parameters in $\overline{C}(n)$. Let $\overline{C}_{2,0}=-a_2/k,\overline{C}_{3,0}=-a_3/k$. The GFRFs up to the 5th orders are computed according to (8.31) as follows,

$$CE(H_1^y(j\omega_1))=1 \tag{8.37}$$

$$\begin{aligned}CE(H_2^y(j\omega_1,j\omega_2))_{\overline{C}(2)}&=CE(H_2^x(j\omega_1,j\omega_2))_{\overline{C}(2)}=\overline{C}_{2,0}\oplus\bigoplus_{p=2}^{\lfloor 2+1/2\rfloor}\overline{C}_{p,0}\otimes CE(H_{2-p+1}^x(\cdot))\\&=\overline{C}_{2,0}\oplus 0=\overline{C}_{2,0}=-a_2/k\end{aligned} \tag{8.38}$$

$$\begin{aligned}CE(H_3^y(j\omega_1,\cdots,j\omega_3))_{\overline{C}(3)}&=CE(H_3^x(j\omega_1,\cdots,j\omega_3))_{\overline{C}(3)}=\overline{C}_{3,0}\oplus\bigoplus_{p=2}^{\lfloor 3+1/2\rfloor}\overline{C}_{p,0}\otimes CE(H_{3-p+1}^x(\cdot))\\&=\overline{C}_{3,0}\oplus\overline{C}_{2,0}\otimes CE(H_2^x(\cdot))=\overline{C}_{3,0}\oplus\overline{C}_{2,0}{}^2=[-\frac{a_3}{k},\frac{a_2^2}{k^2}]\end{aligned} \tag{8.39}$$

$$\begin{aligned}CE(H_4^y(j\omega_1,\cdots,j\omega_4))_{\overline{C}(4)}&=CE(H_4^x(j\omega_1,\cdots,j\omega_4))_{\overline{C}(4)}=\overline{C}_{4,0}\oplus\bigoplus_{p=2}^{\lfloor 4+1/2\rfloor}\overline{C}_{p,0}\otimes CE(H_{4-p+1}^x(\cdot))\\&=0\oplus\overline{C}_{2,0}\otimes CE(H_3^x(\cdot))=\overline{C}_{2,0}\otimes(\overline{C}_{3,0}\oplus\overline{C}_{2,0}{}^2)\\&=\overline{C}_{2,0}\otimes\overline{C}_{3,0}\oplus\overline{C}_{2,0}{}^3=[\frac{a_2a_3}{k^2},-\frac{a_2^3}{k^3}]\end{aligned} \tag{8.40}$$

$$\begin{aligned}CE(H_5^y(j\omega_1,\cdots,j\omega_5))_{\overline{C}(5)}&=CE(H_5^x(j\omega_1,\cdots,j\omega_5))_{\overline{C}(5)}=\overline{C}_{5,0}\oplus\bigoplus_{p=2}^{\lfloor 5+1/2\rfloor}\overline{C}_{p,0}\otimes CE(H_{5-p+1}^x(\cdot))\\&=0\oplus\overline{C}_{2,0}\otimes CE(H_4^x(\cdot))\oplus\overline{C}_{3,0}\otimes CE(H_3^x(\cdot))\\&=\overline{C}_{2,0}{}^2\otimes\overline{C}_{3,0}\oplus\overline{C}_{2,0}{}^4\oplus\overline{C}_{3,0}{}^2=[\frac{a_2^2a_3}{k^3},\frac{a_2^4}{k^4},\frac{a_3^2}{k^2}]\end{aligned} \tag{8.41}$$

The parametric characteristic of the output spectrum up to the 5th order can be obtained as

$$CE(Y(j\omega))_{\overline{C}(5)} = \bigoplus_{n=1}^{5} CE(H_n^y(j\omega_1,\cdots,j\omega_n))_{\overline{C}(n)} = \left[1,-\frac{a_2}{k},-\frac{a_3}{k},\frac{a_2^2}{k^2},\frac{a_2a_3}{k^2},-\frac{a_2^3}{k^3},\frac{a_2^2a_3}{k^3},\frac{a_2^4}{k^4},\frac{a_3^2}{k^2}\right] \quad (8.42)$$

Then according to Proposition 8.2, there exists a complex valued function vector $\hat{F}(j\omega_1,\cdots,j\omega_5)$ such that

$$Y(j\omega;a_2,a_3)) = \left[1,-\frac{a_2}{k},-\frac{a_3}{k},\frac{a_2^2}{k^2},\frac{a_2a_3}{k^2},-\frac{a_2^3}{k^3},\frac{a_2^2a_3}{k^3},\frac{a_2^4}{k^4},\frac{a_3^2}{k^2}\right]\cdot \hat{F}(j\omega_1,\cdots,j\omega_5) \quad (8.43)$$

It should be noted that the system output spectrum in (8.43) is only approximated up to the 5th order. In order to have a higher accuracy, higher order approximation might be needed in practice. To obtain the explicit relationship between system output spectrum and the nonlinear parameters a_2 and a_3 at a specific frequency of interest, $\hat{F}(j\omega_1,\cdots,j\omega_5)$ in (8.43) can be determined by using a numerical method as mentioned before. The idea is to obtain Z system output frequency responses from Z simulations or experimental tests on the system (8.15) under Z different values of the nonlinear parameters (a_2 a_3) and the same input u(t), then yielding

$$Y_Z = [Y(j\omega;a_2,a_3)_1 \quad Y(j\omega;a_2,a_3)_2 \quad \cdots \quad Y(j\omega;a_2,a_3)_Z]^T = \Phi\cdot\hat{F}(j\omega_1,\cdots,j\omega_5) \quad (8.44)$$

where

$$\Phi = \begin{bmatrix} 1,-\frac{a_2(1)}{k},-\frac{a_3(1)}{k},\frac{a_2^2(1)}{k^2},\frac{a_2(1)a_3(1)}{k^2},-\frac{a_2^3(1)}{k^3},\frac{a_2^2(1)a_3(1)}{k^3},\frac{a_2^4(1)}{k^4},\frac{a_3^2(1)}{k^2} \\ 1,-\frac{a_2(2)}{k},-\frac{a_3(2)}{k},\frac{a_2^2(2)}{k^2},\frac{a_2(2)a_3(2)}{k^2},-\frac{a_2^3(2)}{k^3},\frac{a_2^2(2)a_3(2)}{k^3},\frac{a_2^4(2)}{k^4},\frac{a_3^2(2)}{k^2} \\ \vdots \\ 1,-\frac{a_2(Z)}{k},-\frac{a_3(Z)}{k},\frac{a_2^2(Z)}{k^2},\frac{a_2(Z)a_3(Z)}{k^2},-\frac{a_2^3(Z)}{k^3},\frac{a_2^2(Z)a_3(Z)}{k^3},\frac{a_2^4(Z)}{k^4},\frac{a_3^2(Z)}{k^2} \end{bmatrix} \quad (8.45)$$

Then

$$\hat{F}(j\omega_1,\cdots,j\omega_5) = (\Phi^T\Phi)^{-1}\Phi^T Y_Z \quad (8.46)$$

Therefore, equation (8.43) can be determined, which is an explicitly analytical function of the nonlinear parameters a_2 and a_3. By using this method, the system output frequency response can thus be analyzed and designed in terms of model nonlinear parameters of interest. For the detailed discussion of the numerical method can refer to Chapter 4. □

8.5 Magnitude bound characteristics

This section provides an evaluation of the magnitude bound of $Y(j\omega)$, which is significant in many cases where only the magnitude of $Y(j\omega)$ is needed to obtain some information of a system without computing the complicated analytical functions in (8.12-8.13) in multi-dimensional complex space.

It can be derived from (8.12b) that

$$\begin{aligned}|Y(j\omega)| &= \left|\sum_{n=1}^{N}\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega} H_n^y(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega\right| \\ &= \left|\sum_{n=1}^{N}\frac{H_n^y(j\omega_1^*,\cdots,j\omega_n^*)}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega\right| \\ &\le \sum_{n=1}^{N}\frac{|H_n^y(j\omega_1^*,\cdots,j\omega_n^*)|}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}\prod_{i=1}^{n}|U(j\omega_i)|d\sigma_\omega \\ &= \sum_{n=1}^{N}\frac{1}{(2\pi)^{n-1}}|H_n^y(j\omega_1^*,\cdots,j\omega_n^*)|\underbrace{|U|*\cdots*|U(j\omega)|}_{n}\end{aligned} \tag{8.47a}$$

Denote $Y_n(j\omega)=\frac{1}{\sqrt{n}(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega} H_n^y(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_\omega$ representing the nth-order output frequency response. Then

$$|Y_n(j\omega)| \le \frac{1}{(2\pi)^{n-1}}|H_n^y(j\omega_1^*,\cdots,j\omega_n^*)|\underbrace{|U|*\cdots*|U(j\omega)|}_{n} \tag{8.47b}$$

Note that $\underbrace{|U|*\cdots*|U(j\omega)|}_{n}$ can be computed by an algorithm in Billings and Lang (1996).

Thus from (8.47), it can be seen that $|H_n^y(j\omega_1,\cdots,j\omega_n)|$ should be evaluated first in order to obtain the magnitude bound for $Y(j\omega)$. For this purpose, the following notations are introduced.

$$\tilde{C}(p,q)=\begin{cases}\sum_{k_1,k_{p+q}=0}^{K}|\tilde{c}_{p,q}(k_1,\cdots,k_{p+q})|, & 1\le q\le n-1, 1\le p\le n-q\\ \sum_{k_1,k_n=0}^{K}|\tilde{c}_{0,n}(k_1,\cdots,k_n)|, & q=n, p=0\\ \sum_{k_1,k_p=0}^{K}|\tilde{c}_{p,0}(k_1,\cdots,k_p)|, & q=0, 1\le p\le n\\ 0, & else\end{cases} \tag{8.48}$$

$\overline{C}(p,q)$ has the similar definition as (8.48), except $\overline{C}(1,0)=0$. Let

$$\underline{L}=\inf_{\omega=\omega_1+\cdots+\omega_n}\{|L_n(\omega)|\} \tag{8.49}$$

Moreover, let

$$\begin{cases}\overline{H}_{n,p} = \sup_{\omega_1\cdots\omega_n\in R_\omega}\left(|H_{n,p}(\cdot)|\right), H_{0,0}(\cdot)=1\\ H_{n,0}(\cdot)=0 \text{ for } n>0\\ H_{n,p}(\cdot)=0 \text{ for } n<p\\ \overline{H}_n^x = \sup_{\omega_1\cdots\omega_n\in R_\omega}\left(|H_n^x(\cdot)|\right)\end{cases} \tag{8.50}$$

where R_ω is the input frequency range. Furthermore, two operations "$\bullet$" and "$\circ$" are needed in the evaluation of magnitude bound, which was first defined in Jing et al (2007) and is restated in Section 8.7.

Proposition 8.3. Considering system (8.7), for $\omega_1+\cdots+\omega_i\neq 0$ (i =1,2,...,n), the magnitude of $H_n^y(j\omega_1,\cdots,j\omega_n)$ for system (8.7) is bounded by

$$|H_n^y(j\omega_1,\cdots,j\omega_n)| \le \tilde{C}(0,n)+\left(\mathop{\circ}_{q=0}^{n-1}\mathop{\circ}_{p=0}^{n-q}\tilde{C}(p,q)\cdot\mathop{\circ}_{\substack{\sum r_i=n-q\\ 1\le r_1\cdots r_p\le n-p-q+1}}\left(\mathop{\bullet}_{i=1}^{p}b_{r_i}\right)\right)\cdot h_n \tag{8.51}$$

where

$$h_n = [1 \quad (\overline{H}_1^x)^1 \quad \cdots \quad (\overline{H}_1^x)^n] \text{ and } b_{r_i} = [b_{r_i,0} \quad b_{r_i,1} \quad \cdots \quad b_{r_i,r_i}] \tag{8.52}$$

b_{nk} for $0 \le k \le n$ can be recursively computed as follows,

$$b_{nk} = \frac{1}{\underline{L}}\overline{C}(k,n-k) + \frac{1}{\underline{L}}\left(\mathop{\circ}_{m=2}^{n} \mathop{\circ}_{\substack{p+q=m \\ 0\le p,q\le m}} \left(\overline{C}(p,q) \cdot \mathop{\circ}_{\substack{\sum r_i = n-q \\ 1\le r_1 \cdots r_p \le n-m+1}} \left(\mathop{\bullet}_{i=1}^{p} b_{r_i}\right)\right)\right)(k) \tag{8.53}$$

$$b_2 = [b_{20}, b_{21}, b_{22}] = \left[\frac{1}{\underline{L}}\overline{C}(0,2), \frac{1}{\underline{L}}\overline{C}(1,1), \frac{1}{\underline{L}}\overline{C}(2,0)\right] \tag{8.54}$$

$$b_1 = [b_{10}, b_{11}] = [0,1] \tag{8.55}$$

Moreover, $\mathop{\bullet}_{i=1}^{p} b_{r_i} = 0$ if $p<1$, and $\mathop{\circ}_{m=2}^{n}(\cdot) = 0$ if $n<2$.

Proof. See the proof in Section 8.7. □

The bound in (8.51) provides another explicit analytical expression for the relationship between system GFRFs and model parameters as the parametric characteristic function in (8.32). The magnitude bound of the nth-order GFRF can directly be described by an n-degree polynomial function of $\overline{H}_1$. Different order of the GFRFs has a different degree polynomial of $\overline{H}_1$, and has no crossing effect with each other. Using (8.47) and (8.51), it can be derived that

$$\begin{aligned}
|Y(j\omega)| &\le \sum_{n=1}^{N}\left\{\frac{1}{(2\pi)^{n-1}}\underbrace{|U| * \cdots * |U(j\omega)|}_{n} \cdot \left(\widetilde{C}(0,n) \circ \mathop{\circ}_{q=0}^{n-1}\mathop{\circ}_{p=0}^{n-q}\widetilde{C}(p,q) \cdot \mathop{\circ}_{\substack{\sum r_i = n-q \\ 1\le r_1\cdots r_p \le n-p-q+1}} (\mathop{\bullet}_{i=1}^{p} b_{r_i})\right) \cdot h_n\right\} \\
&= \left\{\mathop{\circ}_{n=1}^{N}\frac{1}{(2\pi)^{n-1}}\underbrace{|U| * \cdots * |U(j\omega)|}_{n} \cdot \left(\widetilde{C}(0,n) \circ \mathop{\circ}_{q=0}^{n-1}\mathop{\circ}_{p=0}^{n-q}\widetilde{C}(p,q) \cdot \mathop{\circ}_{\substack{\sum r_i = n-q \\ 1\le r_1\cdots r_p \le n-p-q+1}} (\mathop{\bullet}_{i=1}^{p} b_{r_i})\right)\right\} \cdot h_N \\
&= \left(\mathop{\circ}_{n=1}^{N}(\alpha_n \cdot B_n)\right) \cdot h_N
\end{aligned} \tag{8.56}$$

$$|Y_n(j\omega)| \le \alpha_n \cdot B_n \cdot h_n \tag{8.57}$$

where

$$\alpha_n = \frac{1}{(2\pi)^{n-1}}\underbrace{|U| * \cdots * |U(j\omega)|}_{n} \tag{8.58}$$

$$B_n = \widetilde{C}(0,n) \circ \mathop{\circ}_{q=0}^{n-1}\mathop{\circ}_{p=0}^{n-q}\widetilde{C}(p,q) \cdot \mathop{\circ}_{\substack{\sum r_i = n-q \\ 1\le r_1\cdots r_p \le n-p-q+1}} (\mathop{\bullet}_{i=1}^{p} b_{r_i}) \tag{8.59}$$

Similarly, when the input of (8.7) is a multi-tone signal (1.3), then the output spectrum of system (8.7) is bounded by

$$|Y(j\omega)| \le \left(\mathop{\circ}_{n=1}^{N}(\beta_n \cdot B_n)\right) \cdot h_N \tag{8.60}$$

$$|Y_n(j\omega)| \le \beta_n \cdot B_n \cdot h_n \tag{8.61}$$

$$\beta_n = \frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} F(\omega_{k_1})\cdots F(\omega_{k_n}) \tag{8.62}$$

The magnitude of a frequency response function for a system usually reveals some important information about the system, and consequently takes a great role in the convergence or stability analysis of the system and the truncation error of the corresponding series. Therefore, the magnitude bound results developed in this section can be used to measure the significant orders of nonlinearities or to find the significant nonlinear terms, indicating the stability of a system and providing a basis for the analysis and optimization of system output frequency response.

Example 8.3. Consider system (8.15) with a_2=0, i.e.,

$$\begin{aligned} &mx(t-2)+a_1x(t-1)+a_3x^3(t-1)+kx(t)=u(t) \\ &y(t)=a_1x(t-1)+a_3x^3(t-1)+kx(t) \end{aligned} \tag{8.63}$$

and let $u=A\sin(\Omega t)$. Assume that m, a_1, a_3, and k are all positive. There are only two nonlinear parameters, *i.e.*, $\bar{c}_{3,0}(111)=-a_3/k$ and $\tilde{c}_{3,0}(111)=a_3$. Before the magnitude bound of the output spectrum is evaluated, the parametric characteristics of the GFRFs for $y(t)$ are checked first. In this case, the parametric characteristics for the GFRFs can be computed according to (8.31). It is noted from (8.37-8.41) that

$$CE(H_{2i}^{y}(\cdot))=0 \text{ for } i\geq 1 \tag{8.64}$$

thus

$$H_{2i}^{y}(\cdot)=0 \text{ for } i\geq 1 \tag{8.65}$$

according to Proposition 8.1. Hence, only $\left|H_{2i-1}^{y}(\cdot)\right|$ for $i\geq 1$ are needed to be evaluated for the magnitude of $Y(j\omega)$. Since the input is a sinusoidal signal, the magnitude of $Y(j\omega)$ can be evaluated by (8.60-8.62), which can be written in this case as

$$\left|Y(j\omega)\right|\leq\left(\mathop{\circ}_{i=1}^{\lfloor N+1/2\rfloor}(\beta_{2i-1}\cdot B_{2i-1})\right)\cdot h_{\lfloor N+1/2\rfloor} \tag{8.66}$$

and

$$\left|Y_{2i-1}(j\omega)\right|\leq\beta_{2i-1}\cdot B_{2i-1}\cdot h_{2i-1} \tag{8.67}$$

Note that $u=A\sin(\Omega t)$ is a single tone signal, then

$$\beta_n=2^{-n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}\left|F(\omega_{k_1})\cdots F(\omega_{k_n})\right|=\begin{cases}(\frac{A}{2})^n\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}1\ , & \omega\in\left\{\omega_{k_1}+\cdots+\omega_{k_n}\middle|\begin{matrix}\omega_{k_l}=k_l\Omega,k_l=\pm1,\\ 1\leq l\leq n\end{matrix}\right\}\\ 0 & \text{else}\end{cases} \tag{8.68}$$

From (8.53-8.55) it can be obtained that

$$b_{2i}=0 \text{ for i=1,2,3,...} \tag{8.69}$$

and for n=2i-1, i=1,2,3,...

$$b_{nk}=0 \text{ for } 0\leq k<n \tag{8.70}$$

$$b_{11}=1,\ b_{33}=\frac{1}{\underline{L}}\bar{C}(3,0),\ b_{nn}=\frac{1}{\underline{L}}\bar{C}(3,0)\sum_{\substack{\sum r_i=n\\ 1\leq r_1\cdots r_3\leq n-3+1}}\prod_{i=1}^{3}b_{r_ir_i} \text{ for n>3} \tag{8.71}$$

Therefore,

$$B_1=\mathop{\circ}_{q=0}^{0}\mathop{\circ}_{p=0}^{1-q}\mathop{\circ}_{\substack{\sum r_i=1\\ 1\leq r_1\cdots r_p\leq 2-p-q}}(\tilde{C}(p,q)\cdot(\mathop{\bullet}_{i=1}^{p}b_{r_i}))=\tilde{C}(1,0)\cdot b_1=(|\tilde{c}_{1,0}(1)|+|\tilde{c}_{1,0}(0)|\cdot b_1=[0,a_1+k] \tag{8.72}$$

and for n=2i-1, i=2,3...

$$\begin{aligned} B_n&=\mathop{\circ}_{q=0}^{n-1}\mathop{\circ}_{p=0}^{n-q}\mathop{\circ}_{\substack{\sum r_i=n-q\\ 1\leq r_1\cdots r_p\leq n-p-q+1}}(\tilde{C}(p,q)\cdot(\mathop{\bullet}_{i=1}^{p}b_{r_i}))=(\tilde{C}(1,0)\cdot b_n)\circ(\tilde{C}(3,0)\cdot\mathop{\circ}_{\substack{\sum r_i=n\\ 1\leq r_1\cdots r_3\leq n-2}}(\mathop{\bullet}_{i=1}^{3}b_{r_i})) \\ &=((a_1+k)\cdot b_n)\circ(a_3\cdot\mathop{\circ}_{\substack{\sum r_i=n\\ 1\leq r_1\cdots r_3\leq n-2}}(\mathop{\bullet}_{i=1}^{3}b_{r_i})) \end{aligned} \tag{8.73}$$

According to (8.73) and (8.70-8.71), B_n can be computed up to any high orders. For example,

$$\begin{aligned} B_3&=((a_1+k)\cdot b_3)\circ(a_3\cdot\mathop{\circ}_{\substack{\sum r_i=3\\ 1\leq r_1\cdots r_3\leq 3-2}}(\mathop{\bullet}_{i=1}^{3}b_{r_i}))=((a_1+k)\cdot b_3)\circ(a_3\cdot b_1\bullet b_1\bullet b_1) \\ &=((a_1+k)\cdot[0,0,\frac{a_3}{k\underline{L}}])\circ(a_3\cdot[0,0,1])=[0,0,\frac{a_3(a_1+k)}{k\underline{L}}+a_3] \end{aligned} \tag{8.74}$$

Let $B_n=[B_{n0},B_{n1},\ldots,B_{nn}]$. Hence, using (8.71) and (8.73),

$$B_{nk}=0 \text{ for } 0 \le k < n \tag{8.75}$$

$$B_{nn} = ((a_1+k)\cdot\frac{1}{\underline{L}}\bar{C}(3,0)\sum_{\substack{\sum r_i=n\\ 1\le r_1\cdots r_3\le n-3+1}}\prod_{i=1}^{3}b_{r_ir_i})\circ(a_3\cdot \underset{\substack{\sum r_i=n\\ 1\le r_1\cdots r_3\le n-2}}{\circ}(\overset{3}{\underset{i=1}{\bullet}} b_{r_ir_i})) \text{ for } n=2i-1,\ i=1,2,3\ldots \tag{8.76}$$

Since only the last element in B_n is nonzero, (8.66-8.67) can be rewritten as

$$|Y(j\omega)| \le \underset{i=1}{\overset{\lfloor N+1/2 \rfloor}{\circ}}(\beta_{2i-1}\cdot B_{2i-1,2i-1}\cdot(\bar{H}_1^x)^{2i-1}) \tag{8.77}$$

and

$$|Y_{2i-1}(j\omega)| \le \beta_{2i-1}\cdot B_{2i-1,2i-1}\cdot(\bar{H}_1^x)^{2i-1} \tag{8.78}$$

Note from (8.49-8.50) that

$$\underline{L} = \inf\left|1+\frac{m}{k}\exp(-j2(\omega_1+\cdots+\omega_n))+\frac{a_1}{k}\exp(-j(\omega_1+\cdots+\omega_n))\right| \tag{8.79}$$

$$\bar{H}_1^x = \sup\left|\frac{1}{k+m\exp(-j2\omega_1)+a_1\exp(-j\omega_1)}\right| \tag{8.80}$$

Based on (8.77-8.80), the magnitude bound of the output spectrum of system (8.63) can be evaluated readily. For instance,

$$|Y_1(j\Omega)| \le \beta_1\cdot B_{1,1}\cdot\bar{H}_1^x = \frac{A(a_1+k)}{2}\bar{H}_1^x$$

$$|Y_3(j\Omega)| \le \beta_3\cdot B_{3,3}\cdot(\bar{H}_1^x)^3 = \frac{3A^3a_3(a_1+k+k\underline{L})}{8k\underline{L}}(\bar{H}_1^x)^3$$

This process can be conducted for up to any higher orders, which can be used to evaluate some properties of the nonlinear system, such as the truncation error of Volterra series and system stability etc (Jing et al 2007). □

8.6 Extension to continuous time nonlinear systems

The results above can be extended to continuous time nonlinear Volterra systems in a general form of

$$\begin{aligned}\dot{x} &= f(x)+g(x,u)\\ y_x &= h(x,u)\end{aligned} \tag{8.81}$$

For this purpose, consider the following system described by differential equations

$$\sum_{m=1}^{M_1}\sum_{p=0}^{m}\sum_{k_1,k_m=0}^{K}\bar{c}_{p,m-p}(k_1,\cdots,k_m)\prod_{i=1}^{p}\frac{d^{k_i}x(t)}{dt^{k_i}}\prod_{i=p+1}^{m}\frac{d^{k_i}u(t)}{dt^{k_i}}=0 \tag{8.82}$$

$$\sum_{m=1}^{M_2}\sum_{p=0}^{m}\sum_{k_1,k_m=0}^{K}\tilde{c}_{p,m-p}(k_1,\cdots,k_m)\prod_{i=1}^{p}\frac{d^{k_i}x(t)}{dt^{k_i}}\prod_{i=p+1}^{m}\frac{d^{k_i}u(t)}{dt^{k_i}}=y(t) \tag{8.83}$$

where $x(t)$, $y(t)$, $u(t)\in\mathbb{R}$. System (8.82-8.83) has similar notations and structure as system (8.7). It can be regarded as an NDE model with two outputs x(t) and y(t), and one input u(t). Hence, following the same idea, the GFRFs for the relationship from u(t) to y(t) are given as

$$H_n^y(j\omega_1,\cdots,j\omega_n)=H_{n_u}^y(j\omega_1,\cdots,j\omega_n)+H_{n_{ux}}^y(j\omega_1,\cdots,j\omega_n)+H_{n_x}^y(j\omega_1,\cdots,j\omega_n) \tag{8.84}$$

where

$$H_{n_u}^y(j\omega_1,\cdots,j\omega_n)=\sum_{k_1,k_n=0}^{K}\tilde{c}_{0,n}(k_1,\cdots,k_n)(j\omega_1)^{k_1}\cdots(j\omega_n)^{k_n} \tag{8.85}$$

$$H_{n_{ux}}^{y}(j\omega_1,\cdots,j\omega_n)$$
$$=\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=0}^{K}\tilde{c}_{p,q}(k_1,\cdots,k_{p+q})(j\omega_{n-q+1})^{k_{p+1}}\cdots(j\omega_n)^{k_{p+q}}H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}) \tag{8.86}$$

$$H_{n_x}^{y}(j\omega_1,\cdots,j\omega_n)=\sum_{p=1}^{n}\sum_{k_1,k_p=0}^{K}\tilde{c}_{p,0}(k_1,\cdots,k_p)H_{n,p}(j\omega_1,\cdots,j\omega_n) \tag{8.87}$$

$$H_{n,p}(\cdot)=\sum_{i=1}^{n-p+1}H_i^x(j\omega_1,\cdots,j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_i)^{k_p} \tag{8.88}$$

$$H_{n,1}(j\omega_1,\cdots,j\omega_n)=H_n^x(j\omega_1,\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_n)^{k_1} \tag{8.89}$$

where $H_n^x(j\omega_1,\cdots,j\omega_n)$ is the nth-order GFRF from u(t) to x(t), which is the same as that given in (3.8 or 3.11, 3.10, 3.2-3.5).

Example 8.4. Consider a nonlinear mechanical system shown in Figure 8.1.

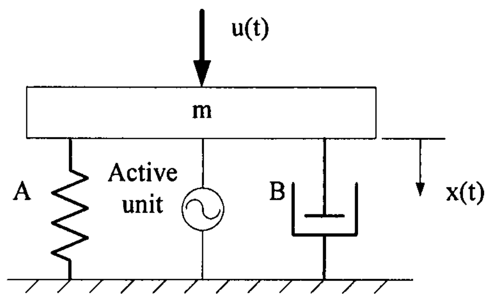

Fig. 8.1 A mechanical system

The output property of the spring satisfies $A=kx$, the damper $F=a_1\dot{x}+a_3\dot{x}^3$, and the active unit is described by $F=a_2\dot{x}^2$. u(t) is the external input force. Therefore, the system dynamics is

$$m\ddot{x}=-kx-a_1\dot{x}-a_2\dot{x}^2-a_3\dot{x}^3+u(t) \tag{8.90}$$

and the output be the transmitted force measured on the base

$$y(t)=a_1\dot{x}+a_2\dot{x}^2+a_3\dot{x}^3+kx(t) \tag{8.91}$$

It can be seen that the continuous time model (8.90-8.91) is similar in structure to the discrete time model (8.15) in Example 8.1. Therefore, similar results regarding the frequency response functions and consequently their related frequency characteristics as demonstrated in Examples 8.1 and 8.2 for the discrete time model (8.15) can be straightforward established for model (8.90-8.91).

Moreover, it can be verified that the results developed by the parametric characteristic analysis above for system (8.7) also hold for system (8.82-8.83).

8.7 Definitions and Proofs

- **Multiplication and addition operators between two vectors of different dimensions**

  Consider two polynomials of degree n and m respectively,

$$f_a=a_0+a_1h+\cdots+a_nh^n=a\cdot\hbar_n^T,\text{ and } f_b=b_0+b_1h+\cdots+b_nh^m=b\cdot\hbar_m^T$$

where the coefficients a_0, a_1, …, a_n; b_0, b_1,…, b_m are all real numbers, h stands for a real or complex valued function, a=[a_0, a_1, …, a_n], b=[b_0, b_1,…, b_m], and $\hbar_i$=[1, h, …, h^i].

Define a multiplication operator "$\bullet$" as $a \bullet b = c$, where c is an $n+m+1$-dimension vector, such that $c(k)= \sum_{\substack{i+j=k \\ 0\le i\le n, 0\le j\le m}} a_i b_j$ for $0 \le k \le m+n$. Denote $(a \bullet b)(k) = \sum_{\substack{i+j=k \\ 0\le i\le n, 0\le j\le m}} a_i b_j$. From this operator it follows that, for example, $f_a \cdot f_b = a \bullet b \cdot \hbar_{n+m}^T$. Similarly, define an addition operator "$\circ$" as $a \circ b = c$, where c is an x-dimension vector, x=max{m,n}, such that c(k)=a(k)+b(k) for $0 \le k \le x$. If k>n or m, then $a(k)$=0 or $b(k)$=0, accordingly. From the operator"$\circ$" it follows that, for example, $f_a + f_b = a \circ b \cdot \hbar_{\max(n,m)}^T$.

These two operators actually define a multiplication operation and an addition operation between two vectors with different dimensions. The operator "$\bullet$" can also be regarded as the Cauchy product between two vectors of different dimensions. A little speciality is that "$\bullet$" produces a new vector from two operated vectors.

- **Proof of Proposition 8.2**

From (8.33) and (8.34), the parametric characteristic vector for $Y(j\omega)$ is

$$CE(Y(j\omega))_{\overline{C}(N)} = \bigoplus_{n=1}^{N} CE(H_n^y(j\omega_1,\cdots,j\omega_n))_{\widetilde{C}(n)} \tag{C1}$$

If $\widetilde{C}(n)$ has a linear relationship with $\overline{C}(n)$, then $CE(H_n^y(j\omega_1,\cdots,j\omega_n))_{\widetilde{C}(n)}$ is given by (8.31). In this case, (8.35) is straightforward by substituting (8.31) into (C1). If $\widetilde{C}(n)$ has no relationship with $\overline{C}(n)$, then substituting (8.27) into (C1) yields

$$CE(Y(j\omega))_{\overline{C}(N)} = \bigoplus_{n=1}^{N}\left(\bigoplus_{p=1}^{n} \chi(n,p)\cdot CE(H_{n-p+1}^x(j\omega_1,\cdots,j\omega_n)_{\overline{C}(n)})\right) \tag{C2}$$

By the definition of operation "$\oplus$", repetitive terms should be removed. Therefore, (C2) further gives

$$CE(Y(j\omega))_{\overline{C}(N)} = \bigoplus_{p=1}^{N} \chi(N,p)\cdot CE(H_{N-p+1}^x(j\omega_1,\cdots,j\omega_N)_{\overline{C}(N)}) \tag{C3}$$

Note that, all the elements in vector $\bigoplus_{p=1}^{N} \chi(N,p)\cdot CE(H_{N-p+1}^x(j\omega_1,\cdots,j\omega_N)_{\overline{C}(N)})$ must be elements in vector $\bigoplus_{n=1}^{N} CE(H_n^x(j\omega_1,\cdots,j\omega_n)_{\overline{C}(n)})$. Hence, the parametric characteristics in (C3) are all included in (8.35). Equation (8.36) is straightforward from Proposition 8.1. □

- **Proof of Proposition 8.3**

It is derived from (8.10) that

$$\begin{aligned}
\left|H_n^y(j\omega_1,\cdots,j\omega_n)\right| &\le \sum_{k_1,k_n=1}^{K}\left|\widetilde{c}_{0,n}(k_1,\cdots,k_n)\right|\left|H_{0,0}(j\omega_1,\cdots,j\omega_n)\right| \\
&+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_n=1}^{K}\left|\widetilde{c}_{p,q}(k_1,\cdots,k_{p+q})\right|\left|H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})\right|+\sum_{p=1}^{n}\sum_{k_1,k_p=1}^{K}\left|\widetilde{c}_{p,0}(k_1,\cdots,k_p)\right|\left|H_{n,p}(j\omega_1,\cdots,j\omega_n)\right| \\
&\le \widetilde{C}(0,n)\overline{H}_{0,0}+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\widetilde{C}(p,q)\overline{H}_{n-q,p}+\sum_{p=1}^{n}\widetilde{C}(p,0)\overline{H}_{n,p}=\sum_{q=0}^{n}\sum_{p=0}^{n-q}\widetilde{C}(p,q)\overline{H}_{n-q,p}
\end{aligned} \tag{D1}$$

From Lemma 1 and Theorem 1 in Jing et al (2007),

$$\overline{H}_{n-q,p} \le \sum_{\substack{r_1\cdots r_p=1 \\ \sum r_i=n-q}}^{n-q-p+1}\prod_{i=1}^{p}\overline{H}_{r_i}^x \quad \text{for } p\ne 0, q\ne n \tag{D2}$$

and

$$\overline{H}_{r_i}^x = b_{r_i,0} + b_{r_i,1}\overline{H}_1^x + \cdots + b_{r_i,r_i}\left(\overline{H}_1^x\right)^{r_i} = b_{r_i} \cdot h_{r_i}^T \tag{D3}$$

where $b_{r_i} = [b_{r_i,0} \quad b_{r_i,1} \quad \cdots \quad b_{r_i,r_i}]$ which can be determined by (8.52-8.55), and $h_{r_i} = [1 \quad \overline{H}_1^x \quad \cdots \quad (\overline{H}_1^x)^{r_i}]$. Then it can be derived from (D2-D3) that

$$\overline{H}_{n-q,p} \le \sum_{\substack{r_1 \cdots r_p = 1 \\ \sum r_i = n-q}}^{n-p-q+1} (\overset{p}{\underset{i=1}{\bullet}} b_{r_i}) \cdot h_{n-q} = \left(\underset{\substack{\sum r_i = n-q \\ 1 \le r_1 \cdots r_p \le n-p-q+1}}{\circ} \left(\overset{p}{\underset{i=1}{\bullet}} b_{r_i} \right) \right) \cdot h_{n-q} \tag{D4}$$

Substituting (D4) into (D1) yields

$$\begin{aligned} \left|H_n^y(j\omega_1,\cdots,j\omega_n)\right| &\le \tilde{C}(0,n) + \sum_{q=0}^{n-1}\sum_{p=0}^{n-q} \tilde{C}(p,q) \left(\underset{\substack{\sum r_i = n-q \\ 1 \le r_1 \cdots r_p \le n-p-q+1}}{\circ} \left(\overset{p}{\underset{i=1}{\bullet}} b_{r_i} \right) \right) \cdot h_{n-q} \\ &= \tilde{C}(0,n) + \left(\underset{q=0}{\overset{n-1}{\circ}} \underset{p=0}{\overset{n-q}{\circ}} \tilde{C}(p,q) \cdot \underset{\substack{\sum r_i = n-q \\ 1 \le r_1 \cdots r_p \le n-p-q+1}}{\circ} (\overset{p}{\underset{i=1}{\bullet}} b_{r_i}) \right) \cdot h_n \end{aligned} \tag{D5}$$

This completes the proof. □

8.8 Conclusions

Some fundamental theoretical results have been established for the frequency domain analysis of nonlinear Volterra systems which can be described by a state space equation with a nonlinear output function. Related frequency characteristics such as the parametric characteristics and bound characteristics for the system frequency response functions are developed and discussed. These results can be regarded as a useful extension of some established results in this topic discussed in previous chapters, and provide an important basis for the frequency domain analysis and design of nonlinear Volterra systems in a more general case. The application of these results to the analysis of practical mechanical systems will be studied in the next chapter

Chapter 9
AN APPLICATION OF THE NEW FREQUENCY DOMAIN METHOD TO OUTPUT VIBRATION SUPPRESSION

Based on the frequency domain theory that is developed in the previous chapters for nonlinear Volterra systems, a frequency domain analysis based nonlinear feedback control approach is proposed. The analytical relationship between system output frequency response and controller parameters is obtained, and a series of associated results and techniques are discussed for the nonlinear feedback controller analysis and design. A general procedure is provided accordingly. The results provide, for the first time, a systematic frequency domain approach to exploiting the potential advantage of nonlinearities to achieve a desired frequency domain performance for active/passive vibration control or energy dissipation systems. The new approach is demonstrated through the design of a nonlinear damping for a vibration suppression problem.

9.1 Introduction

Suppression of periodic disturbances covers a wide range of applications, for example, active control and isolation of vibrations in engineering and vehicle systems. Traditionally, an increase in damping can reduce the response at the resonance. However, this is often at the expense of degradation of isolation at high frequencies (Graham and McRuer 1961). Many methods have been proposed to deal with this problem, such as optimal control, H-infinity control, "skyhook" damper, repetitive learning control, and optimization etc (Graham and McRuer 1961, Housner et al 1997, Karnopp 1995, Lee and Smith 2000). A much more comprehensive and up-to-date survey can refer to (Hrovat 1997). Nonlinear feedback is an approach that has been noted recently by some researchers (Alleyne and Hedrick 1995, Chantranuwathanal and Peng 1999, Zhu et al 2001). It is shown in Lee and Smith (2000) that, although it is not possible to use linear time-invariant controllers to robustly stabilize a controlled plant and to achieve asymptotic rejection of a periodic disturbance, the problem is solvable by using a nonlinear controller for a linear plant subjected to a triangular wave disturbance. Based on the Hamiltonian system theory, an optimal nonlinear feedback control strategy is proposed in Zhu et al (2001) for randomly excited structural systems. It has also been reported many times that existing nonlinearities or deliberately introduced nonlinearities may bring benefits to control system design (Graham and McRuer 1961). Hence, the design of a nonlinear feedback controller to suppress periodic disturbances has great potential to achieve a considerably improved control performance. However, it should be noted that most of these existing methods mentioned above are based on state space and in the time domain, and some of those usually involve a complicated design procedure.

Based on the results discussed in Chapter 3, Chapter 4 and Chapter 8, the OFRF (output frequency response function) for nonlinear Volterra systems can be obtained explicitly, which reveals an analytical relationship between system output spectrum and system model parameters for a wide class of nonlinear systems and provides an important basis for the analysis and design of output behaviour of nonlinear systems in the frequency domain. For a linear controlled plant subject to periodic disturbances, if a nonlinear feedback is introduced to produce a nonlinear closed loop system, the relationship between the disturbance and the system output is nonlinear and can, under certain conditions, be described in the frequency domain by using the OFRF to explicitly relate the controller parameters to the system output frequency response. Therefore, by properly designing the controller parameters based on this explicit relationship, the effect of the periodic disturbance on the system output frequency response could be significantly suppressed. Motivated by this idea, a frequency domain approach to the analysis and design of nonlinear feedback for the exploitation of the potential advantage of nonlinearities is proposed in this study to suppress sinusoidal exogenous disturbances for a linear controlled plant.

This chapter is organized as follows. The problem formulation is given in Section 9.2, which is divided into several basic problems that can be addressed separately. Section 9.3 is concerned with some fundamental issues of the analysis and design of nonlinear feedback corresponding to different basic problems. Some theoretical results and techniques needed to solve these basic problems are established. Section 9.4 illustrates the implementation of the new approach by tackling a simple vibration system. Some proofs for the theoretical results are provided in Section 9.5 and a conclusion is given in Section 9.6.

9.2 Problem Formulation

Consider an SISO linear system described by the following differential equation:

$$\sum_{l=0}^{L} C_x(l) D^l x + b \cdot u + e \cdot \eta = 0 \tag{9.1}$$

$$y = \sum_{l=0}^{L-1} C_y(l) D^l x + d \cdot u \tag{9.2}$$

where, x, y, u, $\eta \in \mathbb{R}^1$ represent the system state, output, control input, and an exogenous disturbance input respectively; η stands for a known, external, bounded and periodical vibration, which can be described by a summation of multiple sinusoidal functions; L is a positive integer; D^l is an operator defined by $D^l x = d^l x / dt^l$. The model of system (9.1-9.2) can also be written in a state-space form:

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{B}u + \mathbf{E}\eta \tag{9.3}$$

$$y = \mathbf{C}\mathbf{X} + du \tag{9.4}$$

where, $\mathbf{X}=[x, D^1x, \ldots, D^{L-1}x]^{\mathrm{T}} \in \Re^L$ is the system state variable, $\mathbf{A}$ and $\mathbf{C}$ are matrixes with appropriate dimensions, $\mathbf{B}=[0_{1\times(L-1)}, \mathrm{b}]^{\mathrm{T}}$, $\mathbf{E}=[0_{1\times(L-1)}, \mathrm{e}]^{\mathrm{T}}$. The problem to be addressed in the present study is:

Given a frequency interval $I(\omega)$ and a desired magnitude level of the output frequency response Y^* over this frequency interval, find a nonlinear feedback control law

$$u = -\varphi(x, D^1x, \ldots, D^{L-1}x) \tag{9.5}$$

such that

$$\max_{\omega \in I(\omega)} (Y(j\omega)Y(-j\omega)) \le Y^* \tag{9.6a}$$

where the feedback control law $-\varphi(x, D^1x, \ldots, D^{L-1}x)$ is generally a nonlinear function of x, D^1x, ..., $D^{L-1}x$, with the linear state/output feedback as a special case; $Y(j\omega)$ is the spectrum of the system output.

For the purpose of implementation, the control objective (9.6a) is transformed to be

$$\max_{\substack{\omega_k \in I(\omega) \\ k=1,2,\ldots,\bar{k}}} (Y(j\omega_k)Y(-j\omega_k)) \le Y^* \tag{9.6b}$$

That is, evaluate the output spectrum at a series frequency point such that the maximum value is suppressed to a desired level. The control law (9.5) should therefore achieve the control objective defined by (9.6b). In the following, assume $I(\omega) = \omega_0$, that is only the output response at a specific frequency is considered. Let $Y = Y(j\omega)Y(-j\omega)|_{(\omega_0,u)}$, then $Y_0 = Y(j\omega)Y(-j\omega)|_{(\omega_0,0)}$ shows the magnitude of the system output frequency response at frequency ω_0 under zero control input. Obviously,

$$Y(j\omega)Y(-j\omega)|_{(\omega_0,u)} \le Y^* < Y_0 = Y(j\omega)Y(-j\omega)|_{(\omega_0,0)} \tag{9.7}$$

To obtain a nonlinear feedback controller, $\varphi(x, D^1x, \ldots, D^{L-1}x)$ is written in a polynomial form in terms of $x, D^1x, \ldots, D^{L-1}x$ as

$$\varphi(x, D^1x, \ldots, D^{L-1}x) = \sum_{p=1}^{M} \sum_{l_1\cdots l_p=0}^{L-1} C_{p0}(l_1,\cdots,l_p)\prod_{i=1}^{p} D^{l_i}x \tag{9.8}$$

where M is a positive integer representing the maximum degree of nonlinearity in terms of $D^ix(t)$ (i=0...L-1); $\sum_{l_1\cdots l_p=0}^{L-1}(\cdot) = \sum_{l_1=0}^{L-1}\cdots\sum_{l_p=0}^{L-1}(\cdot)$. The nonlinear function in (9.8) includes a general class of possible linear and nonlinear functions of D^ix (i=0...L-1). Since $D^ix=e(i+1)^T\mathbf{X}$, where $e(i+1)$ is an L-dimensional column vector whose (i+1)th element is 1 with all other terms zero, $\varphi(x, D^1x, \ldots, D^{L-1}x)$ can also be written as a function of $\mathbf{X}$, *i.e.*, $\varphi(\mathbf{X})$. As mentioned before, for the parameters $C_{p0}(.)$ (p=1,...,M), when $p=1$ the parameters will be referred to as the linear parameters corresponding to the linear terms in (9.8), *e.g.*, $C_{1,0}(2)\frac{d^2x(t)}{dt^2}$. All other parameters in (9.8) will be referred to as nonlinear parameters corresponding to the nonlinear terms $\prod_{i=1}^{p} D^{l_i}x(t)$. p is the nonlinear degree of nonlinear parameter $C_{p0}(\cdot)$. Let

$$C(M,L) = \left(C_{p0}(l_1,\cdots,l_p) \middle| \begin{array}{l} p=1\cdots M \\ l_i = 0\cdots L-1 \\ i=1\cdots p \end{array} \right) \tag{9.9}$$

which includes all the parameters in (9.8). Substituting (9.8) into (9.1) and (9.2) yields the closed loop system as

$$\sum_{p=1}^{M} \sum_{l_1\cdots l_p=0}^{L} \bar{C}_{p0}(l_1,\cdots,l_p)\prod_{i=1}^{p} D^{l_i}x + e\cdot\eta = 0 \tag{9.10a}$$

$$\sum_{p=1}^{M} \sum_{l_1\cdots l_p=0}^{L-1} \tilde{C}_{p0}(l_1,\cdots,l_p)\prod_{i=1}^{p} D^{l_i}x = y \tag{9.10b}$$

where,

$$\bar{C}_{10}(l_1) = C_x(l_1) - bC_{10}(l_1),\ \tilde{C}_{10}(l_1) = C_y(l_1) - dC_{10}(l_1)$$

$$\bar{C}_{p0}(l_1,\cdots,l_p) = -bC_{p0}(l_1,\cdots,l_p),\ \tilde{C}_{p0}(l_1,\cdots,l_p) = -dC_{p0}(l_1,\cdots,l_p),$$

for $p=2\cdots M, l_i = 0\cdots L$, and $i=1\cdots p$. (9.10) is a nonlinear differential equation model, whose generalized frequency response function can be obtained by using the results in Chapter 3. According to the results in Chen and Billings (1989), the model can represent a wide class of nonlinear systems. This implies that the nonlinear control law (9.8) can be used for many control purposes of interests. The task for the nonlinear feedback controller design is to determine M and a range for the controller parameters in (9.9) to make the closed loop system (9.10) bounded stable around its zero equilibrium, and then to determine the specific values for the controller parameters from the OFRF which defines the relationship between the closed loop system output spectrum and controller parameters to achieve the required steady state performance (9.7).

There are generally four fundamental issues to be addressed for the nonlinear feedback design problem as follows:

(a) Determination of the analytical relationship between the system output spectrum and the nonlinear controller parameters.

(b) Determination of an appropriate structure for the nonlinear feedback controller. Only nonlinear terms which are useful for the control purpose are needed in the controller to achieve the design objective..

(c) Derivation of a range for the values of the control parameters over which the stability of the closed loop nonlinear system is guaranteed.

(d) Development of an effective numerical method for the practical implementation of the feedback controller design.

The focus of Section 9.3 is to investigate these fundamental issues. A simulation study will be presented thereafter to illustrate these results.

9.3 Fundamental results for the analysis and design of the nonlinear feedback control

9.3.1 Output frequency response function

In this section, the output frequency response of the closed loop nonlinear system (9.10) is derived. The relationship between the system output spectrum and the controller parameters are investigated to produce some useful results for the nonlinear feedback analysis and design.

9.3.1.1 Output spectrum of the closed loop system

As discussed before, any time invariant, causal, nonlinear system with fading memory can be approximated by a finite Volterra series. With the BIBO stability condition for the controller parameters which will be studied in Section 9.3.3, the relationship between the output y(t) and the input $\eta(t)$ of system (9.10) can be approximated by a Volterra functional series up to a finite order N as described by (1.1), i.e.,

$$y(t)=\sum_{n=1}^{N} y_n(t),\ y_n=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n}\eta(t-\tau_i)d\tau_i \tag{9.11}$$

where $h_n(\tau_1,\cdots,\tau_n)$ is the nth order Volterra kernel of system (9.10) corresponding to the input-output relationship from $\eta(t)$ to y(t). When the input in (9.11) is a multi-tone function in (1.3), i.e.,

$$\eta(t)=\sum_{i=1}^{K}|F_i|\cos(\omega_i t+\angle F_i) \tag{9.12}$$

the system output spectrum can be obtained by extending the result described in (4.3-4.4), as given in (8.12b) and (8.13b), i.e.,

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}H_n(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n}) \tag{9.13}$$

where,
$$F(\omega)=\begin{cases}|F_i|e^{j\angle F_i} & \text{if } \omega\in\{\omega_k, k=\pm1,\cdots,\pm K\}\\ 0 & \text{else}\end{cases} \tag{9.14}$$

$$H_n(j\omega_{k_1},\cdots,j\omega_{k_n})=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)e^{-j(\omega_1\tau_1+\cdots+\omega_n\tau_n)}d\tau_1\cdots d\tau_n \tag{9.15}$$

(9.15) is the *n*th-order generalised frequency response function (GFRF) of system (9.10) for the relationship between $\eta(t)$ and y(t), which can be obtained by directly following the results in Section 8.3 of Chapter 8.

Proposition 9.1. The GFRFs $H_n(j\omega_{k_1},\cdots,j\omega_{k_n})$ from the disturbance $\eta(t)$ to the output y(t) of nonlinear system (9.10) can be determined as

$$H_n(j\omega_1,\cdots,j\omega_n)=\sum_{p=1}^{n}\sum_{l_1\cdots l_p=0}^{L-1}\tilde{C}_{p0}(l_1\cdots l_p)H^1_{np}(j\omega_1,\cdots,j\omega_n) \tag{9.16a}$$

$$H^1_{np}(j\omega_1,\cdots,j\omega_n)=\sum_{i=1}^{n-p+1}H^1_i(j\omega_1,\cdots,j\omega_i)H^1_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_i)^{l_p} \tag{9.16b}$$

$$H^1_{n1}(j\omega_1,\cdots,j\omega_n)=H^1_n(j\omega_1,\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_n)^{l_1},\quad H^1_1(j\omega_1)=e\Big/\sum_{l_1=0}^{L}\overline{C}_{10}(l_1)(j\omega_1)^{l_1} \tag{9.16c}$$

$$H^1_n(j\omega_1,\cdots,j\omega_n)=-\frac{1}{e}H^1_1(j\omega_1+\cdots+j\omega_n)\left(\sum_{p=2}^{n}\sum_{l_1\cdots l_p=0}^{L-1}\overline{C}_{p0}(l_1\cdots l_p)H^1_{np}(j\omega_1,\cdots,j\omega_n)-e\delta(n-1)\right) \tag{9.16d}$$

and $\delta(n)=\begin{cases}1 & n=0\\ 0 & \text{otherwise}\end{cases}$. □

Note that the nth-order GFRF from $\eta(t)$ and x(t) can directly be obtained from (3.8 or 3.11, 3.10, 3.2-3.5) as discussed in Section 8.6, which is denoted by $H^1_n(j\omega_1,\cdots,j\omega_n)$. However, from the study in Chapter 8 it can be seen that, the nth-order GFRF from $\eta(t)$ and y(t) can only be obtained by using the results in Section 8.3 instead of directly applying the results in Billings and Peyton-Jones (1990), because system (9.10) having a nonlinear output is not consistent with the model studied in Billings and Peyton-Jones (1990). From Proposition 9.1, the GFRFs can be computed recursively from the time domain model (9.10), and the output spectrum of system (9.10) can be obtained analytically from (9.13) and (9.16), which are an explicit function of the parameters in the control law (9.8). Therefore, the design of controller (9.8) can be studied in the frequency domain. In order to obtain an analytical relationship between the system output spectrum and model parameters from these recursive computations the OFRF of system (9.10) is expressed as a polynomial function of the nonlinear controller parameters in (9.9) according to Chapter 4, *i.e.*,

$$Y(j\omega)=P_0(j\omega)+a_1P_1(j\omega)+a_2P_2(j\omega)+\cdots \tag{9.17a}$$

where $P_0(j\omega)$ is the linear part of the system output frequency response, $P_i(j\omega)$ ($i\ge1$) represents the effects of higher order nonlinearities, and a_i ($i=1,2,\cdots$) are functions of the nonlinear controller parameters which can be determined by following Chapter 3 and Chapter 4. Moreover, for a nonlinear controller parameter c in (9.9), there exists a series of functions of frequency ω {$\overline{P}_i(j\omega)$, i=0,1,2,3,...} such that

$$Y(j\omega)=\overline{P}_0(j\omega)+c\overline{P}_1(j\omega)+c^2\overline{P}_2(j\omega)+\cdots \tag{9.17b}$$

(9.17b) explicitly shows the relationship between the system output spectrum and the nonlinear controller parameters, and therefore enables the OFRF to be determined by using a simple numerical method which will be discussed in Section 9.3.4. Obviously, this considerably facilitates the analysis and design of the nonlinear feedback controller in the frequency domain. In order to reveal the contribution of the nonlinear controller parameters of different degrees to the output spectrum more clearly and thus shed light on the issue of the structure determination for control law (9.8), some useful results regarding the parametric characteristic of the OFRF are discussed in the following section.

9.3.1.2 Parametric characteristic analysis of the output spectrum

The parametric characteristic analysis of the system output spectrum is to investigate the polynomial structure of OFRF (9.17a), and to reveal how the frequency response functions in (9.13,9.16a-d) depend on the nonlinear controller parameters (*i.e.*, $C_{p0}(.)$ for $p>1$) in (9.9).

Following the results in Section 8.4.2 of Chapter 8, the parametric characteristics of the GFRF $H_n^1(j\omega_1,\cdots,j\omega_n)$ from $u(t)$ to $y(t)$ can be obtained as for n>1

$$\begin{aligned}\mathrm{CE}(H_n^1(j\omega_1,\cdots,j\omega_n)) &= \mathop{\oplus}_{p=2}^{n}\left(C_{p,0}\otimes \mathrm{CE}\left(H_{n,p}^1(j\omega_1,\cdots,j\omega_n)\right)\right) \\ &= \mathop{\oplus}_{p=2}^{n}\left(C_{p,0}\otimes \mathrm{CE}\left(H_{n-p+1}^1(j\omega_1,\cdots,j\omega_n)\right)\right) = C_{n0}\oplus \mathop{\oplus}_{p=2}^{[n+1/2]}\left(C_{p0}\otimes \mathrm{CE}(H_{n-p+1}^1(\cdot))\right)\end{aligned} \tag{9.18}$$

For n=1, $\mathrm{CE}(H_1^1(j\omega_1))=1$. Here, $[n/2]$ means to take the integer part of [.]. From the invariant property of the CE operator, it follows for the nonlinear controller parameters in (9.9) that

$$\mathrm{CE}(\overline{C}_{p0}(l_1,\cdots,l_p)) = C_{p0}(l_1,\cdots,l_{p+q}),\ \mathrm{CE}(\widetilde{C}_{p0}(l_1,\cdots,l_p)) = C_{p0}(l_1,\cdots,l_p)$$

Applying CE operator to Equation (9.16a) for the nonlinear parameters in (9.9),

$$\begin{aligned}\mathrm{CE}(H_n(j\omega_1,\cdots,j\omega_n)) &= \mathrm{CE}\left(\sum_{l_1=0}^{L}\widetilde{C}_{1,0}(l_1)H_{n,1}^1(j\omega_1,\cdots,j\omega_n)+\sum_{p=2}^{n}\sum_{l_1\cdots l_p=0}^{L}\widetilde{C}_{p0}(l_1\cdots l_p)H_{np}^1(j\omega_1,\cdots,j\omega_n)\right) \\ &= \mathrm{CE}\left(\sum_{l_1=0}^{L}(C_y(l_1)-C_{10}(l_1))H_{n1}^1(j\omega_1,\cdots,j\omega_n)+\sum_{p=2}^{n}\sum_{l_1\cdots l_p=0}^{L}(-d)C_{p0}(l_1\cdots l_p)H_{np}^1(j\omega_1,\cdots,j\omega_n)\right) \\ &= \begin{cases}1 & n=1 \\ \mathop{\oplus}_{p=2}^{n}\left(C_{p0}\otimes CE(H_{np}^1(j\omega_1,\cdots,j\omega_n))\right) & n>1\end{cases}\end{aligned} \tag{9.19}$$

Therefore, with respect to the nonlinear parameters in (9.9), the parametric characteristics of the GFRFs $H_n(j\omega_1,\cdots,j\omega_n)$ from $\eta(t)$ to y(t) is the same as those of the GFRFs $H_n^1(j\omega_1,\cdots,j\omega_n)$ from u(t) to y(t), i.e.,

$$\mathrm{CE}(H_n^2(\cdot)) = \mathrm{CE}(H_n^1(\cdot)) \text{ for n>0} \tag{9.20}$$

That is, the effect of the nonlinear parameters in (9.9) on the GFRFs $H_n(j\omega_1,\cdots,j\omega_n)$ is the same as that on the GFRFs $H_n^1(j\omega_1,\cdots,j\omega_n)$. Equations (9.18-9.20) reveal how the GFRFs depend on the nonlinear controller parameters in (9.9). Based on these results, the parametric characteristic of the OFRF can be obtained as

$$\begin{aligned}\mathrm{CE}(Y(j\omega)) &= \mathrm{CE}\left(\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}H_n^2(j\omega_{k_1},\cdots,j\omega_{k_n})F(\omega_{k_1})\cdots F(\omega_{k_n})\right) \\ &= \mathrm{CE}\left(\sum_{n=1}^{N}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}H_n^2(j\omega_{k_1},\cdots,j\omega_{k_n})\right) = \mathrm{CE}\left(\sum_{n=1}^{N}H_n^2(j\omega_{k_1},\cdots,j\omega_{k_n})\right) \\ &= \mathrm{CE}(H_1^2(\cdot))\oplus \mathrm{CE}(H_2^2(\cdot))\oplus\cdots\oplus \mathrm{CE}(H_N^2(\cdot)) = \mathrm{CE}(H_1^1(\cdot))\oplus \mathrm{CE}(H_2^1(\cdot))\oplus\cdots\oplus \mathrm{CE}(H_N^1(\cdot))\end{aligned} \tag{9.21a}$$

Therefore, according to the results in Chapter 8, there exist a complex valued function vector $\widetilde{F}_n(j\omega)$ with appropriate dimension such that

$$Y(j\omega)=(\bigoplus_{n=1}^{N}\mathrm{CE}(H_n^1(j\omega_1,\cdots,j\omega_n)))\cdot\widetilde{F}_n(j\omega) \quad (9.21b)$$

This is the detailed polynomial function of (9.17a). Equation (9.21b) provides an analytical and straightforward expression for the relationship between system output spectrum and the controller parameters. Now the coefficients of the polynomial function (9.17a) can be determined as

$$[a_1 \quad a_2 \quad a_3 \quad \cdots \quad a_K]=\mathrm{CE}(Y(j\omega))=\mathrm{CE}(H_1^1(\cdot))\oplus\mathrm{CE}(H_2^1(\cdot))\oplus\cdots\oplus\mathrm{CE}(H_N^1(\cdot)) \quad (9.21c)$$

where K is the dimension of the vector $\mathrm{CE}(H_1^1(\cdot))\oplus\mathrm{CE}(H_2^1(\cdot))\oplus\cdots\oplus\mathrm{CE}(H_N^1(\cdot))$.

In order to better understand these parametric characteristics, the following results are given, which is a special case of Proposition 3.1.

Proposition 9.2. The elements in CE($H_n^1(j\omega_1,\cdots,j\omega_n)$) include and only include all the parameter monomials (consisting of the nonlinear parameters in (9.9)) in $\mathbf{C}_{p0}\otimes\mathbf{C}_{r_10}\otimes\mathbf{C}_{r_20}\otimes\cdots\otimes\mathbf{C}_{r_k0}$ for $0\le k\le n-2$, satisfying $p+\sum_{i=1}^{k}r_i=n+k$, $2\le r_i\le n-1$, and $2\le p\le n$.□

Proposition 9.2 shows whether and how a nonlinear parameter in (9.9) is included in CE($H_n^1(j\omega_1,\cdots,j\omega_n)$). Different parameters may form one monomials acting as an element in CE($H_n^1(j\omega_1,\cdots,j\omega_n)$), and thus have a coupled effect on $H_n^1(j\omega_1,\cdots,j\omega_n)$. If a nonlinear parameter appears in CE($H_n^1(j\omega_1,\cdots,j\omega_n)$), this implies that it has an effect on $H_n^1(j\omega_1,\cdots,j\omega_n)$ and thus on $Y(j\omega)$. If this nonlinear parameter is an independent element in CE($H_n^1(j\omega_1,\cdots,j\omega_n)$), then it has an independent effect on $Y(j\omega)$. Furthermore, if a parameter frequently appears in CE($H_n^1(j\omega_1,\cdots,j\omega_n)$) with different monomial degrees, this may implies that this parameter has more strong effect on $H_n^1(j\omega_1,\cdots,j\omega_n)$ and thus $Y(j\omega)$. For this reason, the parametric characteristic analysis of $H_n^1(j\omega_1,\cdots,j\omega_n)$ can shed light on the effect of different nonlinear parameters on $H_n^1(j\omega_1,\cdots,j\omega_n)$ and thus $Y(j\omega)$.

From Proposition 9.2 (also referring to Property 3.3 for the general case), the term $(\mathbf{C}_{n0})^i$ should be included in the GFRF $H_m(.)$, where m is computed as $m+k=m+i-1=ni$. Hence, $m= ni - i+1=1+(n-1)i$. It can be seen that, when n is smaller, $\mathbf{C}_{n,0}$ will contribute independently to more GFRFs whose orders are $(n-1)i+1$ for $i=1,2,3,\ldots$; and if n is larger, $\mathbf{C}_{n,0}$ can only affect the GFRFs of orders higher than n. It is known that for a Volterra system, the system nonlinear dynamics could be dominated by low order GFRFs (Boyd and Chua 1985). This implies that the nonlinear terms with coefficient $\mathbf{C}_{n,0}$ of smaller nonlinear degree, e.g., 2 and 3, may play greater roles than other pure output nonlinear terms. This property is significant for the selection of possible nonlinear terms in the feedback design. Moreover, it can be verified from Proposition 9.2 that, If the 2nd and 3rd degree nonlinear control parameters are all zero, *i.e.*, $\mathbf{C}_{20}=0$ and $\mathbf{C}_{30}=0$, then $H_2(.)=0$, and $H_3(.)=0$. However, even if $\mathbf{C}_{n0}=0$ (for $n>3$), the nth order GFRF $H_n(.)$ is not zero, providing there are nonzero terms in $\mathbf{C}_{20}$ or $\mathbf{C}_{30}$. This further demonstrates that the nonlinear controller parameters in $\mathbf{C}_{20}$ and $\mathbf{C}_{30}$ have a more important role in the determination of the GFRFs than other nonlinear parameters, and thus has a more important effect on the output spectrum. These imply that a lower degree nonlinear feedback may be sufficient for some control problems. These provide a guidance for the selection of the candidate terms in (9.9).

9.3.2 The structure of the nonlinear feedback controller

The determination of the structure for the nonlinear feedback controller (9.8) is an important task to be tackled. Firstly, as discussed in Section 9.3.1.2, the structure parameter M in (9.8) should be chosen as small as possible since lower degree of nonlinear terms have greater contributions to the output spectrum. It can be increased gradually until the control objective is achieved. Secondly, after M is determined, whether a term in $\mathbf{C}_{p0}$ is effective or not should be checked. An effective controller must satisfy the inequality (9.7). Thus for the effectiveness of a specific nonlinear controller parameter c, this requirement can be written as

$$\frac{\partial |Y(j\omega_0)|}{\partial c} < 0 \text{ for some c} \tag{9.22}$$

Consider the specific nonlinear controller parameter c in $\mathbf{C}_{p0}$ and let all the other nonlinear controller parameters be zero or assumed to be a constant. Then only the nonlinear coefficient c^i appears in $\mathrm{CE}(H^1_{1+(p-1)i}(.))$ according to Proposition 9.2. Therefore, only the GFRFs for the orders $1+(p-1)i$ (for i=1,2,3,...) need to be computed to obtain the system output spectrum in (9.13). According to (9.21), the output spectrum can be written as

$$Y(j\omega;c) = \overline{P}_0(j\omega) + c\overline{P}_1(j\omega) + c^2\overline{P}_2(j\omega) + \cdots \tag{9.23}$$

It can easily be shown that if $\mathrm{Re}(\overline{P}_0(j\omega)\cdot\overline{P}_1(-j\omega)) < 0$ then there must exist $\varepsilon > 0$ such that $\frac{\partial |Y(j\omega)|}{\partial c} < 0$ for $0 < c < \varepsilon$ or $-\varepsilon < c < 0$, where $\mathrm{Re}(\cdot)$ is to take the real part of (.). This can be used to find the nonlinear terms which are effective. Only the effective nonlinear terms in $C(M)$ is considered. By this way, the structure of the nonlinear function (9.8) can be determined. It shall be noted that, in this process the output spectrum needs to be analytically computed up to at most the third order by using Equations (9.12-9.16). The structure of the control law (9.8) can also be determined by simply including all the possible nonlinear terms of degree up to M. Once the output spectrum is determined by the numerical method in Section 9.3.4, the values of the coefficients of these nonlinear terms can be optimized for the control objective (9.7) in the stability region developed in the following section. If the objective (9.7) can not be achieved after M is enough large, this may implies that the objective (9.7) can not be achieved by the controller (9.8) and a best possible solution can be used for this case.

9.3.3 Stability of the Closed-loop System

As mentioned above, the stability of a nonlinear system should be guaranteed such that the nonlinear system can be approximated by a locally convergent Volterra series. Therefore, a range for the nonlinear controller parameters which can ensure the stability of the closed loop system (9.10) can be determined. For simplicity, (9.10) can also be written in a state space form as

$$\dot{\mathbf{X}} = \mathbf{AX} - \mathbf{B}\varphi(\mathbf{X}) + \mathbf{E}\eta := f(\mathbf{X}) + \mathbf{E}\eta \tag{9.24a}$$

$$y = \mathbf{CX} - \mathbf{D}\varphi(\mathbf{X}) := h(\mathbf{X}) \tag{9.24b}$$

A, **B**, **C**, **D**, **E** are appropriate matrices which are the same as the matrices in (9.3-9.4). Note that the exogenous disturbance in (9.24) is a periodic bounded signal, and the objective in a vibration control is often to suppress the output vibration below a desired level, a concept of asymptotic stability to a ball is adopted in this section. This concept implies that the magnitude of the output for a system is asymptotically controlled to a satisfactory predefined level. Based on this concept, a general result is then derived to ensure the stability of the closed loop nonlinear system (9.24), which can be regarded as an application of some existing theories in Isidori (1999).

A Ball $B_\rho(\mathbf{X})$ is defined as: $B_\rho(\mathbf{X})=\{\mathbf{X}\,\|\mathbf{X}\|\le\rho,\rho>0\}$. A K -function γ (s) is an increasing function of s, and a KL -function β (s,t) is an increasing function of s, but a decreasing function of t. For detailed definitions of K /KL-functions can refer to Isidori (1999).

Asymptotic Stability to a Ball. Given an initial state $\mathbf{X}_0\in\Re^n$ and disturbance input η of a nonlinear system, if there exists a KL -function β such that the solution $\mathbf{X}(t,\mathbf{X}_0,\eta)$ (for $t\ge 0$) of the system satisfies $\|\mathbf{X}(t,\mathbf{X}_0,\eta)\|\le\beta(\|\mathbf{X}_0\|,t)+\rho$, $\forall t>0$, then the system is said to be asymptotically stable to a ball $B_\rho(\mathbf{X})$, where ρ is an upper bound function of η, *i.e.*, there exist a K -function γ such that $\rho=\gamma(\|\eta\|_\infty)$.

Assumption 9.1. There exists a K -function o such that the output function $h(\mathbf{X})$ of the nonlinear system (9.24) satisfies $\|h(\mathbf{X})\|\le o(\|\mathbf{X}\|)$.

Proposition 9.3. If assumption 9.1 holds, then the following statements are equivalent:

(a) There exist a smooth function $V:\Re^L\to\Re_{\ge0}$ and K_∞ -functions β_1,β_2 and K -functions α , γ such that

$$\beta_1(\|\mathbf{X}\|)\le V(\mathbf{X})\le\beta_2(\|\mathbf{X}\|) \text{ and } \frac{\partial V(\mathbf{X})}{\partial\mathbf{X}}\{f(\mathbf{X})+\mathbf{E}\eta\}\le-\alpha(\|\mathbf{X}\|)+\gamma(\|\eta\|_\infty) \quad (9.25)$$

(b) System (9.24) is asymptotically stable to the ball $B_\rho(\mathbf{X})$ with $\rho=\beta_1(2\cdot\beta_2^{-1}\cdot\alpha^{-1}\cdot\gamma(\|\eta\|_\infty))$, and the output of system (9.24) is asymptotically stable to the ball $B_{o(2\rho)}(y)$.□

Proof: See the proof in Section 9.5. □

Note that Proposition 9.3 can guarantee the asymptotical stability to a ball of system (9.24) when subject to bounded disturbance, and asymptotical stability to zero when the disturbance tends to zero. This is just the property of fading memory which is required for the existence of a convergent Volterra series approximation for the system input-output relationship (Boyd and Chua 1985). Although it is not easy to derive a general stability condition for the general controller (9.5), there are always various methods (Ogota 1996) to choose a proper Lyapunov function based on Proposition 9.3 to derive a stability condition for a specific controller.

9.3.4 A numerical method for the nonlinear feedback controller design

The nonlinear controller parameters can be determined by solving equation (9.17) to satisfy the performance (9.6) or (9.7) under the stability condition. However, it can be seen that the analytical derivation of the output spectrum of system (9.10) involves complicated symbolic computation for orders higher than 5. To circumvent this problem, as discussed in Section 9.3.1.1, the numerical method discussed in Section 4.2.2 of Chapter 4 can be used since the detailed polynomial structure of the OFRF can be determined by using the method in Section 9.3.1, which is summarized as follows:

(1) The system output frequency response function can be expressed as $Y(j\omega)Y(-j\omega)=|Y(j\omega)|^2=\mathbf{C}\cdot\tilde{\mathbf{P}}(j\omega)$ according to (9.21) with a finite polynomial degree, where $\tilde{\mathbf{P}}(j\omega)$ is a complex valued function vector,

$$\mathbf{C}=[1\ \ c_1\ \ c_2\ \ c_3\ \ \cdots\ \ c_{K!}]$$
$$=\left(\mathrm{CE}(H_1^1(\cdot))\oplus\mathrm{CE}(H_2^1(\cdot))\oplus\cdots\oplus\mathrm{CE}(H_N^1(\cdot))\right)\otimes\left(\mathrm{CE}(H_1^1(\cdot))\oplus\mathrm{CE}(H_2^1(\cdot))\oplus\cdots\oplus\mathrm{CE}(H_N^1(\cdot))\right)$$

(2) Collect the system time domain steady output $y_i(t)$ under different values of the controller parameters $\mathbf{C}_i=[1\ c_{1i}, c_{2i}, \ldots c_{(K!)i}]$ for $i=1,2,3,\ldots N_i$;

(3) Evaluate the FFT for $y_i(t)$ to obtain $Y_i(j\omega)$, then obtain the magnitude $|Y_i(j\omega_0)|^2$ at frequency ω_0, and finally form a vector $\mathbf{YY}=[|Y_1(j\omega_0)|^2,\cdots,|Y_{N_i}(j\omega_0)|^2]^T$

(4) Obtain the following equation,

$$\begin{bmatrix} 1, & c_{11}, & c_{12}, \cdots, c_{1,K!} \\ 1, & c_{21}, & c_{22}, \cdots, c_{2,K!} \\ \cdots, \cdots, & \cdots, & \cdots, \cdots \\ 1, & c_{N_i1}, & c_{N_i2}, \cdots, c_{N_i,K!} \end{bmatrix} \cdot \begin{bmatrix} \widetilde{P}_0 \\ \widetilde{P}_1 \\ \vdots \\ \widetilde{P}_{K!} \end{bmatrix} = \begin{bmatrix} |Y_1(j\omega_0)|^2 \\ |Y_2(j\omega_0)|^2 \\ \vdots \\ |Y_{N_i}(j\omega_0)|^2 \end{bmatrix} \quad i.e., \ \psi_C \cdot \widetilde{\mathbf{P}}(j\omega_0) = \mathbf{YY}$$

(5) Evaluate the function $\widetilde{P}(j\omega_0)$ by using Least Squares,

$$\widetilde{P}(j\omega_0) = \left(\psi_C^{\ T} \cdot \psi_C\right)^{-1} \cdot \psi_C^{\ T} \cdot YY$$

(6) Finally, the nonlinear controller parameters $\mathbf{C}^*$ for given Y^* at a specific frequency ω_0 can be determined according to

$$Y^* == \mathbf{C}^* \cdot \widetilde{\mathbf{P}}(j\omega_0)$$

The numerical method above is very effective for the implementation of the design of the proposed nonlinear controller parameters, which will be verified by a simulation study in Section 9.5.

Although there are some time domain methods which can address the nonlinear control problems based on Lyapunov stability theory such as the back-stepping technique and feedback linearization (Isidori 1999) *etc*, few results are available for the design and analysis of a nonlinear feedback controller in the frequency domain to achieve a desired frequency domain performance. Based on the analytical relationship between system output spectrum and controller parameters defined by the OFRF, the analysis and design of a nonlinear feedback controller can be conducted in the frequency domain. For a summary, a general procedure for this new method is given as follows.

(A) Derivation of the output spectrum for the closed loop system given M and L.

Given M and L in (9.8), the general output spectrum with respect to the control law (9.8) for the closed loop system (9.10) can be obtained according to Equations (9.13, 9.16a-d). This will be used for the validation of the effectiveness of nonlinear terms in the next step. L is the maximum derivative order which is dependent of the system model, and M is the maximum nonlinearity order which can be given as 2 or 3 at this stage.

(B) Determination of the structure of the nonlinear feedback function in (9.8).

This is to determine the value of M and choose the effective nonlinear controller parameters $C_{p0}(.)$ ($p=2,3,\ldots,M$). Based on the analysis of the parametric characteristics in Section 9.3.1.2, the nonlinear controller parameters included in $\mathbf{C}_{20}$ and $\mathbf{C}_{30}$ take a dominant role in the determination of GFRFs and output spectrum. Hence, M can be chosen as 2 or 3 at the beginning, and increased later if needed. The effectiveness of each nonlinear parameter can be checked by $\Re(\overline{P}_0(j\omega)\cdot\overline{P}_1(-j\omega))<0$, where $\overline{P}_1(-j\omega)$ can be computed from Step(A) by letting the other nonlinear parameters to be zero and $\overline{P}_0(j\omega)$ is the linear part of the output spectrum in this case. If the parameter is not effective, it can be discarded.

(C) Derivation of the region for the nonlinear feedback parameters in $C_{p0}(.)$ for $p=2,3,\ldots,M$.

This is to ensure the stability of the nonlinear closed loop system (9.10), which can be conducted by applying Proposition 9.3 to derive a stability condition for the closed loop system in terms of the nonlinear controller parameters. Although how to develop a systematic method for this purpose for a general nonlinear system is still an open problem, this can be easily done for some special or simple cases.

(D) Determination of the OFRF by using the numerical method and the optimal values for the nonlinear parameters

This is to derive a detailed polynomial expression for the output spectrum according to (9.21) for the maximum nonlinearity order M larger than 3, and use the numerical method provided above to determine the desired value for each nonlinear controller parameter within the stability region to achieve the control objective (9.6) or (9.7).

9.4 Simulation study

Consider a simple case of the model in (9.1) and (9.2), which can be written as

$$\begin{cases} M\ddot{x} = -Kx - a_1\dot{x} + (\eta + u) \\ y = Kx + a_1\dot{x} - u \end{cases}$$

This is the model of a vibration isolation system studied in Daley (2006) (Figure 9.1), where $y(t)$ is the force transmitted from the disturbance $\eta(t)$ to the ground, K and a_1 are the spring and a damping characteristic parameters respectively.

Following the procedure in Section 9.3, a nonlinear feedback active controller $u(t)$ is designed and analysed for the suppression of the force transmitted to the ground. It will be shown that a simple nonlinear feedback can bring much better improvement for the system performance, compared with a linear feedback control. According to the general procedure above, the output spectrum under control law (9.8) for the closed loop system should first symbolically be determined. But for this simple example, it can be left to the next step.

9.4.1 Determination of the structure of the nonlinear feedback controller

Considering the nonlinear feedback in (9.8), for this simple system, M is directly chosen to be 3, and all the other nonlinear controller parameters are chosen to be zero except $C_{30}(111)=a_3$ which represents a nonlinear damping and will be shown to be effective in the later analysis. If $C_{30}(111)=a_3$ is not effective, more other nonlinear terms can be chosen.

The nonlinear feedback control law now is

$$u = -a_3\dot{x}^3$$

and the closed loop system is therefore

$$\begin{cases} M\ddot{x} = -Kx - a_1\dot{x} - a_3\dot{x}^3 + \eta & (9.26a) \\ y = Kx + a_1\dot{x} + a_3\dot{x}^3 & (9.26b) \end{cases}$$

Note that system (9.26) is a very simple case of system (9.10), that is, L=2, $\overline{C}_{10}(2) = M$, $\overline{C}_{10}(1) = a_1$, $\overline{C}_{10}(0) = K$, $\overline{C}_{30}(111) = a_3$, $\overline{C}_{01}(0) = -1$ and $\tilde{C}_{10}(1) = a_1$, $\tilde{C}_{10}(0) = K$, $\tilde{C}_{30}(111) = a_3$; All other parameters in model (9.10) are zero. Moreover, assume the disturbance input is $\eta(t) = F_d \sin(8.1t)$ (8.1 is the interested working frequency of the system), which is a single tone function and a simple case of equation (9.12). Now the task for the nonlinear feedback controller design is to determine a_3 such that system (9.26) satisfies the control objective (9.7).

To verify the effectiveness of this nonlinear control, the output spectrum should be computed up to the 3rd order as discussed in Step(B). Note that only $C_{30}(111)=a_3$ and other nonlinear parameters C_{p0} for $p>2$ are all zero. According to Equations (9.18-9.20), the following parametric characteristics of the GFRFs can be obtained

$$\mathrm{CE}(H_2^1(\cdot)) = \mathbf{C}_{20} \oplus \sum_{p=2}^{[2+1/2]} \mathbf{C}_{p0} \otimes \mathrm{CE}(H_{2-p+1}^1(\cdot)) = \mathbf{C}_{20} = 0,$$

$$\mathrm{CE}(H_3^1(\cdot)) = \mathbf{C}_{30} \oplus \sum_{p=2}^{[3+1/2]} \mathbf{C}_{p0} \otimes \mathrm{CE}(H_{3-p+1}^1(\cdot)) = \mathbf{C}_{30} = a_3$$

$$\mathrm{CE}(H_4^1(\cdot)) = \mathbf{C}_{40} \oplus \sum_{p=2}^{[4+1/2]} \mathbf{C}_{p0} \otimes \mathrm{CE}(H_{4-p+1}^1(\cdot)) = 0,$$

$$\mathrm{CE}(H_5^1(\cdot)) = \mathbf{C}_{50} \oplus \sum_{p=2}^{[5+1/2]} \mathbf{C}_{p0} \otimes \mathrm{CE}(H_{5-p+1}^1(\cdot)) = \mathbf{C}_{30} \otimes \mathrm{CE}(H_3^1(\cdot)) = a_3^2, \;\ldots\ldots$$

It is easy to check from Propositions 9.2 that

$$CE(H_{2n+1}^1(\cdot)) = a_3^n \text{ for } n>0 \text{ and all other } \mathrm{CE}(H_i^1(\cdot)) = 0 \qquad (9.27)$$

This shows that only $H_{2n+1}^1(\cdot)$ for $n>0$ are nonzero and all others are zero. Therefore, the output spectrum can be computed from (9.13, 9.16) with only odd order GFRFs as

$$\begin{aligned}
Y(j\omega) &= \sum_{n=1}^{N} \frac{1}{2^{2n+1}} \sum_{\omega_{k_1}+\cdots+\omega_{k_{2n+1}}=\omega} H_{2n+1}^2(j\omega_{k_1},\cdots,j\omega_{k_{2n+1}}) F(\omega_{k_1})\cdots F(\omega_{k_{2n+1}}) \\
&= \frac{1}{2} H_1^2(j\omega)F(\omega) + \frac{a_3}{8} \sum_{\omega_{k_1}+\cdots+\omega_{k_3}=\omega} G_3^2(j\omega_{k_1}, j\omega_{k_2}, j\omega_{k_3}) F(\omega_{k_1})F(\omega_{k_2})F(\omega_{k_3}) + \frac{a_3^2}{32} \sum_{\omega_{k_1}+\cdots+\omega_{k_5}=\omega} G_5^2(j\omega_{k_1},\cdots,j\omega_{k_5}) F(\omega_{k_1})\cdots F(\omega_{k_5}) + \cdots \\
&:= \overline{P}_0(j\omega) + a_3\overline{P}_1(j\omega) + a_3^2\overline{P}_2(j\omega) + \cdots
\end{aligned}$$

(9.28a)

where

$$\overline{P}_0(j\omega) = \frac{1}{2} H_1^2(j\omega)F(\omega) = \frac{-j(a_1(j\omega)+K)F_d}{2M(j\omega)^2 + 2a_1(j\omega)^1 + 2K}, \quad \overline{P}_1(j\omega) = -\frac{3}{8} MF_d^3\omega^5 \left|H_1^1(j\omega)\right|^2 \left[H_1^1(j\omega)\right]^2$$

$$\overline{P}_2(j\omega) = -\frac{3j}{32} MF_d^5 \left|j\omega H_1^1(j\omega)\right|^4 \left[j\omega H_1^1(j\omega)\right]^2 (j\omega) \cdot \left(j3\omega H_1^1(j3\omega) - j3\omega H_1^1(-j\omega) + j6\omega H_1^1(j\omega)\right)$$

(9.28b)

Note that carrying out the computation above, the analytical relationship between the output spectrum and nonlinear parameter a_3 can be obtained explicitly for up to any high orders. It can be checked that $\mathrm{Re}(\overline{P}_0(j\omega_0)\cdot\overline{P}_1(-j\omega_0))$ = 0.5 ($\overline{P}_0(j\omega_0)\overline{P}_1(-j\omega_0) + \overline{P}_0(-j\omega_0)\overline{P}_1(j\omega_0)$)= -31.132<0 when $a_3>0$, ω_0 =8.1 rad/s and other system parameters as given in the simulation studies. Hence, the nonlinear control parameter a_3 is effective. If there are other nonlinear controller parameters, the same method can be used to check the effectiveness as discussed in Step(B). Only the effective nonlinear terms are used in the controller.

9.4.2 Derivation of the stability region for the parameter a_3

According to Proposition 9.3, the following result can be obtained.

Proposition 9.4. Consider the closed loop system (9.26), and assume the exogenous disturbance input satisfies $\|\eta(t)\| \le F_d$. The system is asymptotically stable to a ball $B_{F_d\sqrt{\lambda_{\min}(Q)^{-1}\varepsilon}}(\mathbf{X})$, if $a_3>0$ and additionally there exist $\mathbf{P}=\mathbf{P}^{\mathrm{T}}>0$, $\beta>0$ and $\varepsilon>0$ such that

$$\mathbf{Q} = \begin{bmatrix} -\mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{A} - \varepsilon^{-1}\mathbf{P}\mathbf{E}\mathbf{E}^T\mathbf{P} & -\beta\mathbf{A}^T\mathbf{C}^T + \mathbf{P}\mathbf{B} - \beta\mathbf{P}\mathbf{E}\mathbf{E}^T\mathbf{C}^T \\ * & +2\beta\mathbf{C}\mathbf{B} - \varepsilon^{-1}\beta^2\mathbf{C}\mathbf{E}\mathbf{E}^T\mathbf{C}^T \end{bmatrix} > 0$$

Moreover, the closed loop system (9.26) without a disturbance input is global asymptotically stable if the above inequality holds with **E**=0. Here, $\mathbf{A}=\begin{bmatrix} 0 & 1 \\ -K/M & -a_1/M \end{bmatrix}$, $\mathbf{B}=[0, 1/M]^T$, $\mathbf{C}=[0,1]$, $\mathbf{E}=[0, 1/M]^T$.

Proof. See the proof in Section 9.5. □

It is noted that the inequality in Proposition 9.4 has no relation with a_3 and is determined by the linear part of system (9.26) which can be checked by using the LMI technique by Boyd et al (1994). This implies that the value of a_3 has no effect on the stability of the system if the inequality is satisfied. Hence, the nonlinear controller parameter a_3 is now only restricted to the region $[0, \infty)$, provided that the linear system satisfies the inequality condition.

9.4.3 Derivation of the OFRF and determination of the desired value of the nonlinear parameter a_3

By using (9.27), the parametric characteristics of the output spectrum of nonlinear system (9.26) can be obtained as

$$\mathrm{CE}(Y(j\omega)) = \mathrm{CE}(H_1^1(\cdot)) \oplus \mathrm{CE}(H_2^1(\cdot)) \oplus \cdots \oplus \mathrm{CE}(H_N^1(\cdot)) = \begin{bmatrix} 1 & a_3 & a_3^2 & \cdots & a_3^Z \end{bmatrix}$$

where $Z = \lfloor N - 1/2 \rfloor$. Therefore, the system output spectrum can be written as a polynomial expression as

$$Y(j\omega) = \overline{P}_0(j\omega) + a_3\overline{P}_1(j\omega) + a_3^2\overline{P}_2(j\omega) + \cdots + a_3^Z\overline{P}_Z(j\omega)$$

Hence,

$$Y(j\omega)Y(-j\omega) = |Y(j\omega)|^2$$

$$= |\overline{P}_0(j\omega)|^2 + a_3\left(\overline{P}_0(j\omega)\overline{P}_1(-j\omega) + \overline{P}_0(-j\omega)\overline{P}_1(j\omega)\right) + a_3^2\left(|\overline{P}_1(j\omega)|^2 + \overline{P}_0(j\omega)\overline{P}_2(-j\omega) + \overline{P}_0(-j\omega)\overline{P}_2(j\omega)\right) + \cdots \tag{9.28c}$$

Clearly, $|Y(j\omega)|^2$ is also a polynomial function of a_3. Given the magnitude of a desired output frequency response Y^* at any frequency ω_0, a_3 can be solved from Equation (9.28c) provided that $|Y(j\omega)|$ can be approximated by a polynomial expression of a finite order. In order to determine a desired value for a_3 to achieve the control objective (9.7), the numerical method proposed in Section 9.3.4 is used. Since Equation (9.28c) is a polynomial function of a_3, $|Y(j\omega)|^2$ can be directly approximated by a polynomial function of a_3 as follows:

$$Y(j\omega)Y(-j\omega) = |Y(j\omega)|^2 \approx a_3^{2Z}\widetilde{P}_{2Z} + \cdots a_3^n\widetilde{P}_n + a_3^{n-1}\widetilde{P}_{n-1} + \cdots + a_3\widetilde{P}_1 + |\overline{P}_0(j\omega)|^2 \tag{9.29a}$$

where $|Y(j\omega)|^2$ can be obtained via evaluating the FFT of the system output response from the system simulations or experimental data. Given $2Z$ different values of a_3, *i.e.*, $a_{31}, a_{32}, \ldots, a_{3,2Z}$, (9,29a) can be further written as (for each values of a_3)

$$|Y(j\omega)_i|^2 \approx a_{3i}^{2Z}\widetilde{P}_{2Z} + \cdots a_{3i}^n\widetilde{P}_n + a_{3i}^{n-1}\widetilde{P}_{n-1} + \cdots + a_{3i}\widetilde{P}_1 + |\overline{P}_0(j\omega)|^2$$

for i=1,2,...,2Z, *i.e.*,

$$\begin{bmatrix} a_{31} & a_{31}^2 & a_{31}^3 & \cdots & a_{31}^{2Z} \\ a_{32} & a_{32}^2 & a_{32}^3 & \cdots & a_{32}^{2Z} \\ & & & \ddots & \vdots \\ a_{3,2Z} & a_{3,2Z}^2 & a_{3,2Z}^3 & \cdots & a_{3,2Z}^{2Z} \end{bmatrix} \cdot \begin{bmatrix} \widetilde{P}_1 \\ \widetilde{P}_2 \\ \vdots \\ \widetilde{P}_{2Z} \end{bmatrix} = \begin{bmatrix} |Y(j\omega)_1|^2 - |\overline{P}_0(j\omega)|^2 \\ |Y(j\omega)_2|^2 - |\overline{P}_0(j\omega)|^2 \\ \vdots \\ |Y(j\omega)_{2Z}|^2 - |\overline{P}_0(j\omega)|^2 \end{bmatrix}$$

Then $\widetilde{P}_1, \widetilde{P}_2, \cdots, \widetilde{P}_{2Z}$ are obtained as

$$\begin{bmatrix} \tilde{P}_1 \\ \tilde{P}_2 \\ \vdots \\ \tilde{P}_{2Z} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{31}^2 & a_{31}^3 & \cdots & a_{31}^{2Z} \\ a_{32} & a_{32}^2 & a_{32}^3 & \cdots & a_{32}^{2Z} \\ & & & \ddots & \vdots \\ a_{3,2Z} & a_{3,2Z}^2 & a_{3,2Z}^3 & \cdots & a_{3,2Z}^{2Z} \end{bmatrix}^{-1} \cdot \begin{bmatrix} |Y(j\omega)_1|^2 - |\overline{P}_0(j\omega)|^2 \\ |Y(j\omega)_2|^2 - |\overline{P}_0(j\omega)|^2 \\ \vdots \\ |Y(j\omega)_{2Z}|^2 - |\overline{P}_0(j\omega)|^2 \end{bmatrix} \tag{9.29b}$$

Consequently, equation (9.29a) is obtained. By using this method, a polynomial expression of $|Y(j\omega)|^2$ in any order can be achieved. Given a desired output frequency response Y^* at a frequency ω_0, a_3 can be solved from (9.29a) to implement the design. Note that roots of equation (9.29a) are multiple. According to Proposition 9.4, the solution a_3 should be a nonnegative real number.

9.4.4 Simulation results

In the simulation study, the parameters of system (9.26) are: K=16000 N/m, a_1=296 N.S/m, M=240 Kg. The resonant frequency of the system is ω_0=8.1 rad/s. In order to show the effectiveness and advantage of the nonlinear feedback controller $u=-a_3\dot{x}^3$, a linear controller $u=-a_2\dot{x}$ will be used for a comparison.

Firstly, let F_d=100 N. We need to obtain the polynomial function (9.29a). In order to have a larger working region of a_3, let Z=6 in (9.29a), and a_3= 500, 1000, 2000, 4000, 6000, 8000, 10000, 12000, 14000, 16000, 18000, 20000. Under these different values of a_3, the output frequency response of the system was obtained and the corresponding output spectrum was determined via FFT operations. Then $\tilde{P}_n(j\omega)$ for n=1…12 were obtained according to (9.29b), which are summarized partly in Table 9.1. For comparisons, the corresponding theoretical results were also computed from equation (9.28abc) and are given partly in Table 9.1. From Table 9.1, it can be seen that there is a good match between the numerical analysis results and the theoretical computations although there are some errors. This result shows that the theoretical computation results are basically consistent with the results from the simulation analyses. It can also be seen from the numerical analysis results in Table 9.1 that equation (9.29a) is in fact an alternative series in this case.

Figure 9.2 shows the results of the system output spectrum under different values of the nonlinear control parameter a_3 and provides a comparison between theoretical computations using polynomial expression (9.28c) up to the 3rd order and the numerical results using the polynomial expression (9.29a) up to the 12th order. This result demonstrates the analytical relationship between the nonlinear control parameter and the system output spectrum, and shows that the theoretical results have a good match with the numerical results when a_3 is small since only up to the 3rd order GFRF are used in the theoretical computations. Hence, with an increase of a_3, the numerical method has to be used in order to give correct results. Moreover, it should be noted that the magnitude of the system output spectrum decreases with the increase of a_3. This verifies that the nonlinear control parameter a_3 is effective for the control problem.

Without a control input, the system output frequency spectrum is as shown in Figure 9.3(b), where $Y(j\omega)|_{\omega_0}=335.71$. Note that the output response spectrum shown in the figures is 2|Y| not |Y|, which is also applied on the plot of the output spectrum using the theoretical computation. This is because 2|Y| represents the physical magnitude of the system output at the frequency ω_0. If the desired output frequency spectrum is set to be Y^*=180, then the calculation according to (9.29ab) and

Proposition 9.4 yields a_3=11869. The output frequency spectrum under the nonlinear feedback control is shown in Figure 9.3 (a), where $Y(j\omega)|_{\omega_0}=180.08$, and hence the result matches the desired result quite well. The system outputs in the time domain without and under the nonlinear feedback control are given in Figure 9.4. It can be seen that the system steady state performance is considerably improved when the nonlinear controller is used.

In order to further demonstrate the advantage of the nonlinear feedback control, consider a linear controller $u=-275\dot{x}$. Under this linear control, the system output frequency response as shown in Figure 9.5 is similar to that achieved under the nonlinear controller. However, when F_d is increased to 200 N, the output frequency response is quite different under the two controllers. The nonlinear feedback controller results in a much smaller magnitude of output frequency response at frequency ω_0, referring to Figure 9.6. Figure 9.7 shows the results of the system outputs in the time domain under the two different control inputs, indicating the nonlinear controller has a much better result than the linear controller. When the input frequency ω_0 is increased to be 15 rad/s, the same conclusions can be reached for the two controllers, referring to Figure 9.8. When the input frequency is decreased to be 5 rad/s, the output spectrums under the two controllers are similar (see Figure 9.9). On the other hand, although increase of the liner damping can also achieve better output performance at the driving frequency, this will degrade the output performance at high frequencies as known in literature (Figure 9.10). However, the nonlinear damping has no obviously such a limitation (Figure 9.11).

TABLE 9.1
COMPARISON BETWEEN SIMULATION AND THEORETICAL RESULTS

Simulation results from (9.29ab)		Theoretical results from (9.28abc)	
$\|\overline{P}_0(j\omega)\|^2$	1.1270e+05	$\|\overline{P}_0(j\omega)\|^2$	1.1257e+05
$\widetilde{P}_1$	-58.9652	$\overline{P}_0(j\omega)\overline{P}_1(-j\omega)$ $+\overline{P}_0(-j\omega)\overline{P}_1(j\omega)$	-62.2641
$\widetilde{P}_2$	0.0423	$\|\overline{P}_1(j\omega)\|^2+\overline{P}_0(j\omega)\overline{P}_2(-j\omega)$ $+\overline{P}_0(-j\omega)\overline{P}_2(j\omega)$	0.0615
$\widetilde{P}_3$	-2.3762e-005	—	—
$\widetilde{P}_4$	9.1382e-009	—	—
$\widetilde{P}_5$	-2.3593e-012	—	—
...	...	...	...

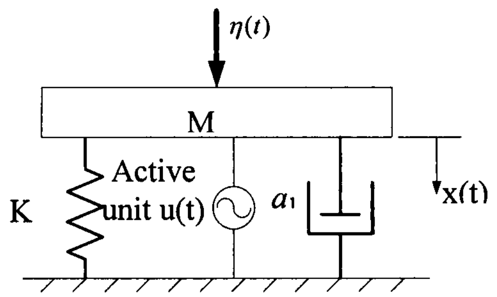

Figure 9.1. A vibration isolation system

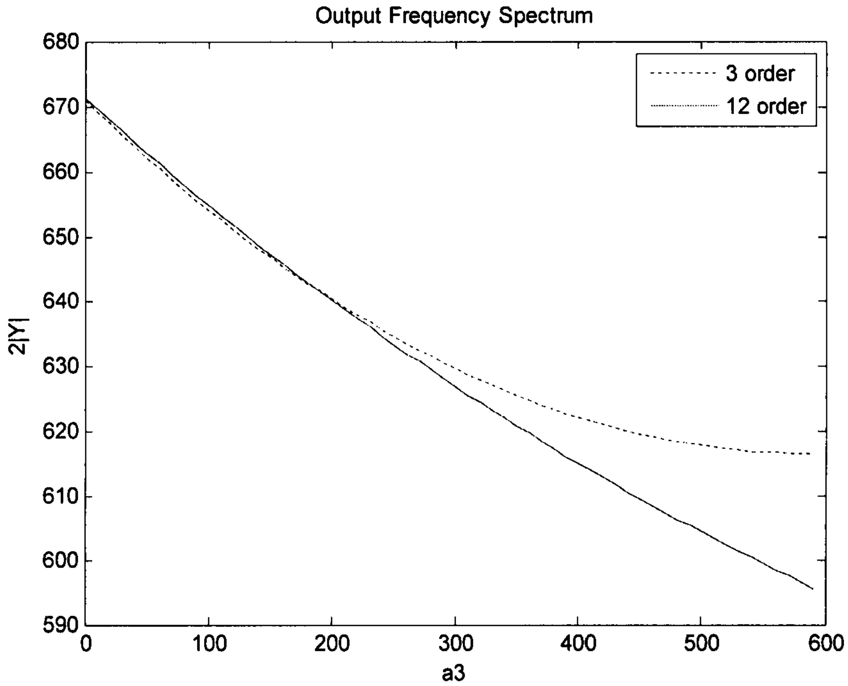

Figure 9.2 Analytical relationship between the system output spectrum and the control parameter a_3

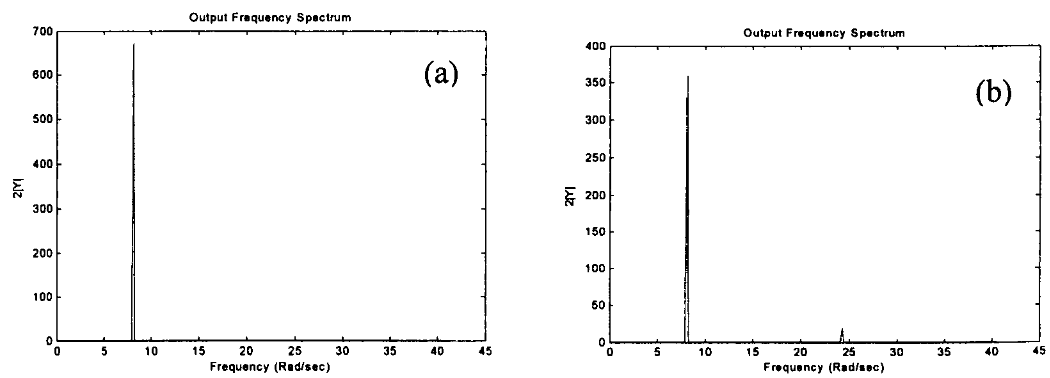

Figure 9.3 Output spectrum (a) without a feedback control, (b) with the designed nonlinear feedback

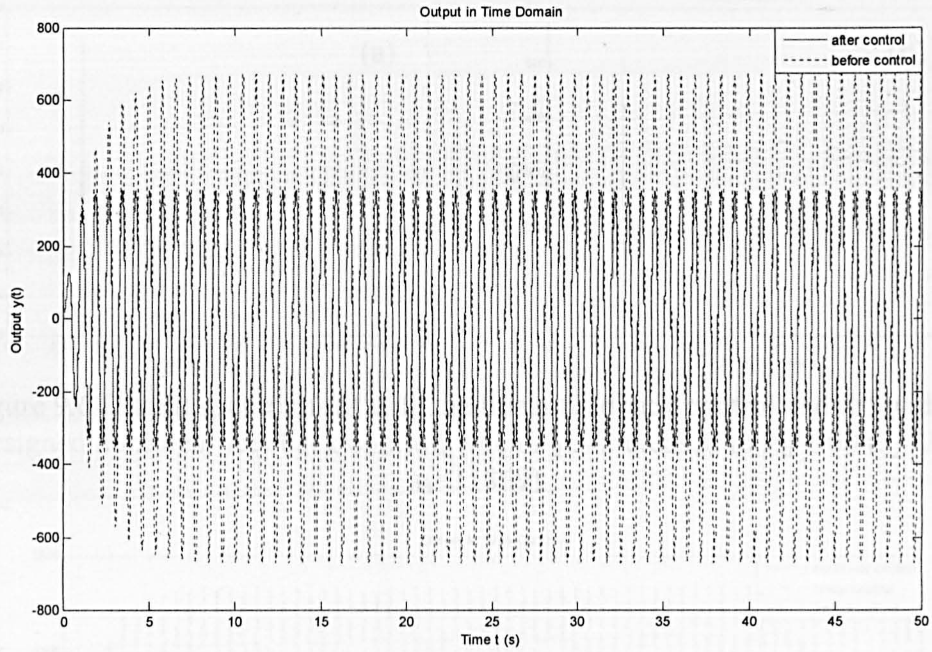

Figure 9.4. System output in time domain: before and after control

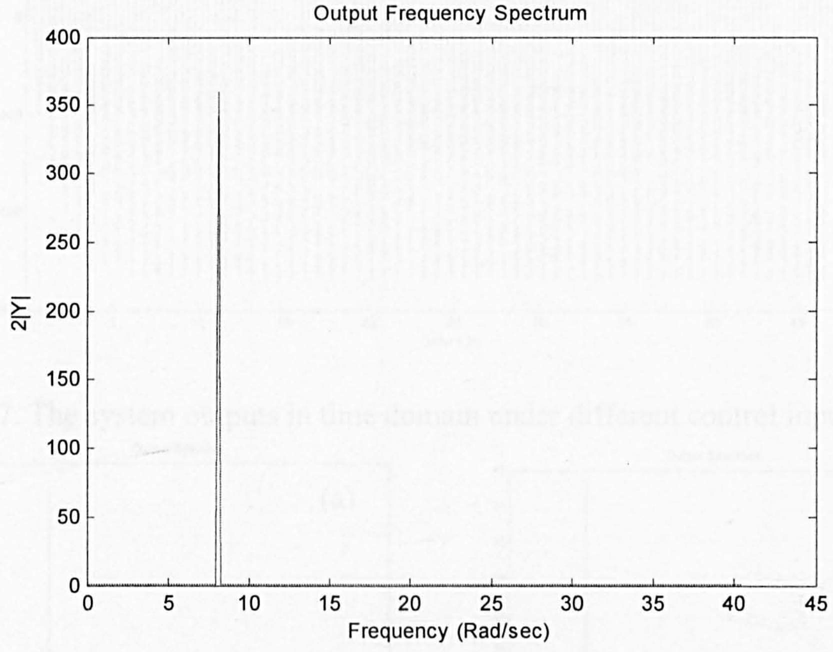

Figure 9.5 Output spectrum with the linear feedback control

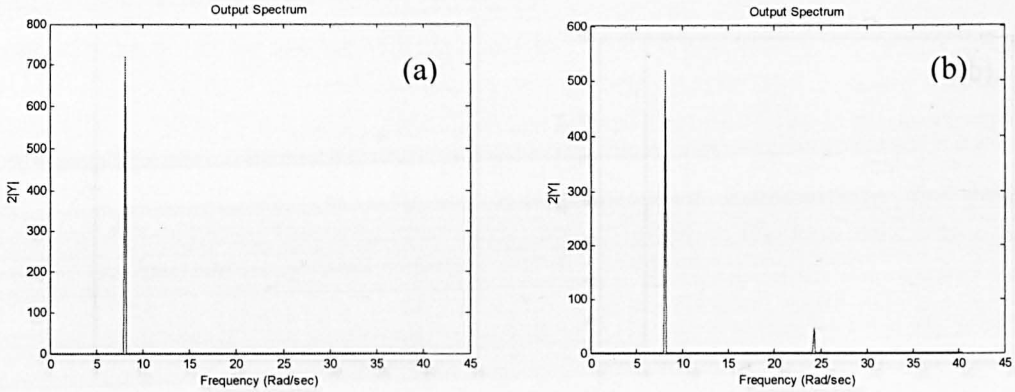

Figure 9.6 Output spectrum (a) with the linear feedback control and (b) with the designed nonlinear feedback control, when F_d is increased to F_d=200 (a_2=275, a_3=11869)

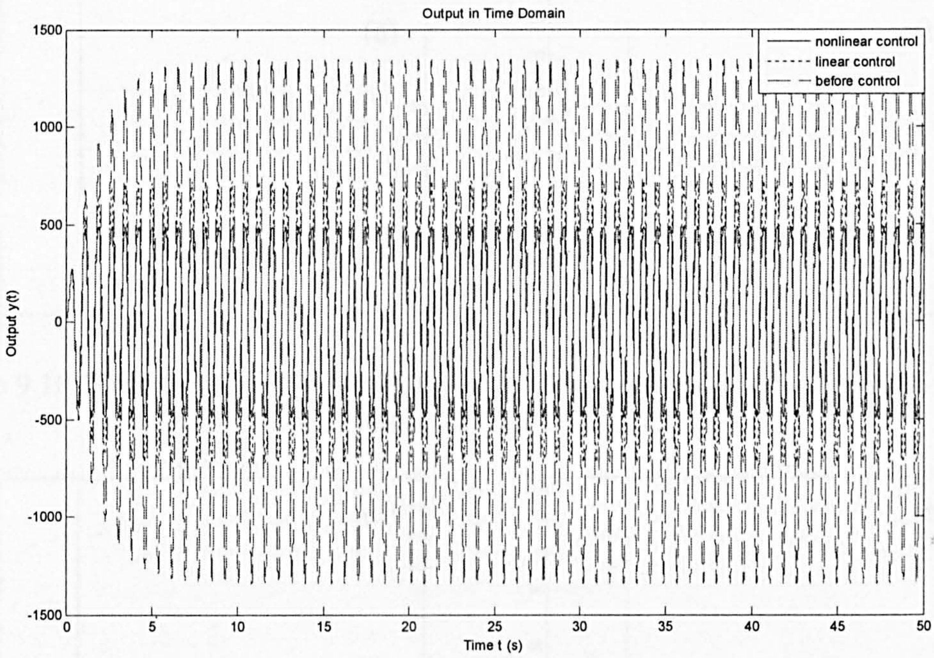

Figure 9.7. The system outputs in time domain under different control inputs (F_d=200)

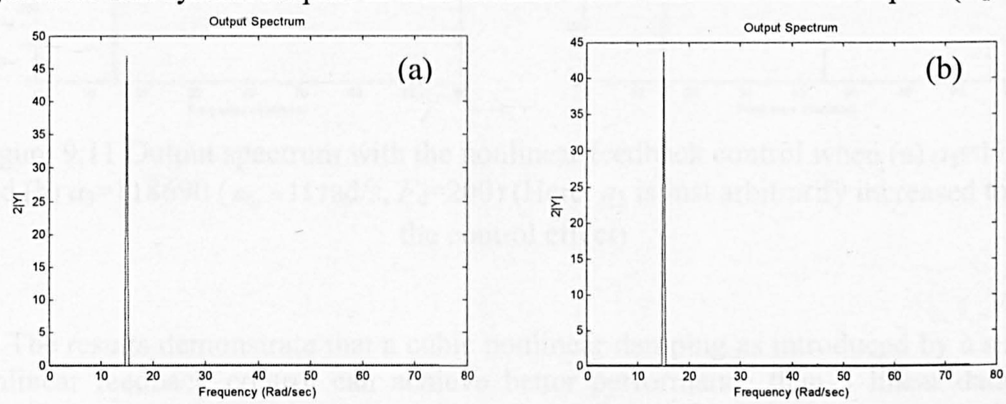

Figure 9.8 Output spectrum (a) with the linear feedback control and (b) with the designed nonlinear feedback control, when $\omega_0 = 15$ rad/s, F_d=100, a_2=275, a_3=11869

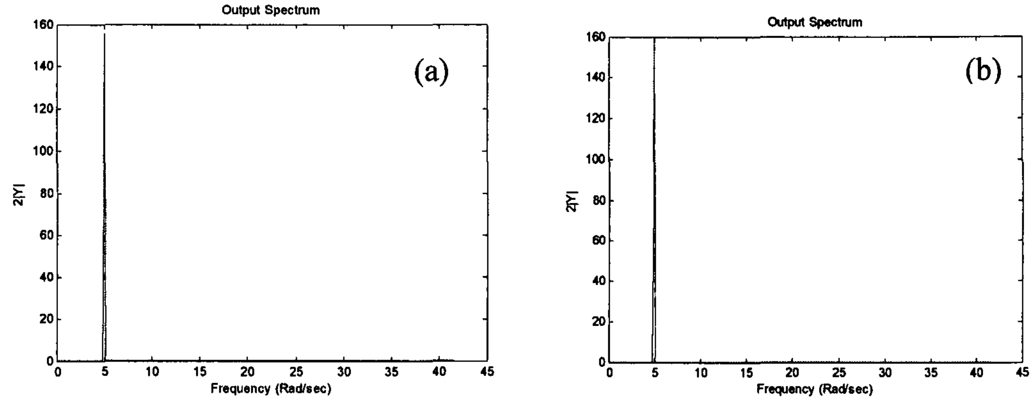

Figure 9.9 Output spectrum (a) with the linear feedback control and (b) with the designed nonlinear feedback control when $\omega_0 = 5$ rad/s, F_d=100, a_2=275, a_3=11869

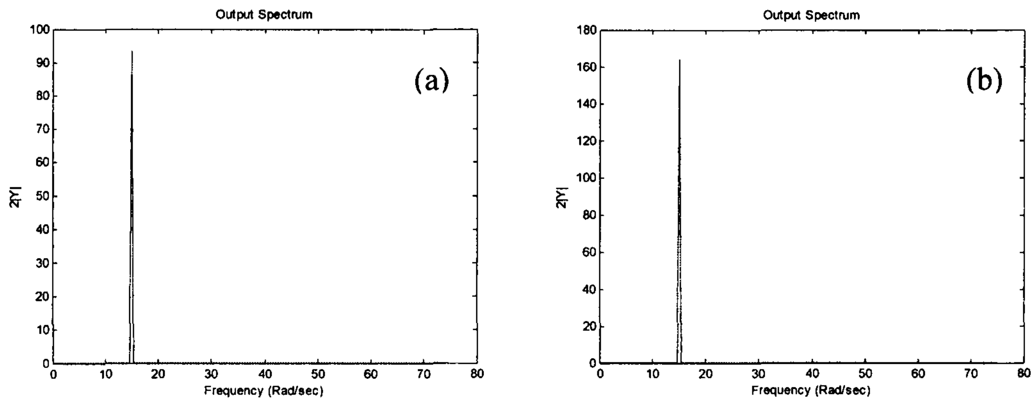

Figure 9.10 Output spectrum with the linear feedback control when (a) a_2=275 and (b) a_2=2750 ($\omega_0 = 15$ rad/s, F_d=200)

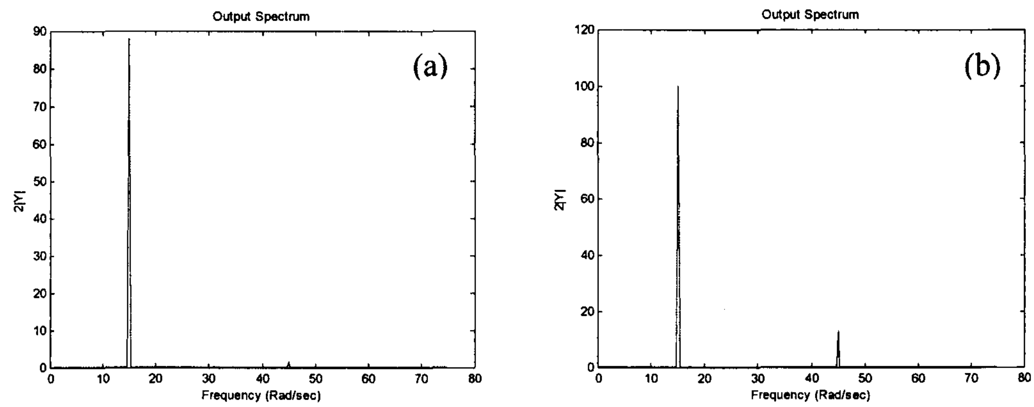

Figure 9.11 Output spectrum with the nonlinear feedback control when (a) a_3=11869 and (b) a_3=118690 ($\omega_0 = 15$ rad/s, F_d=200) (Here, a_3 is just arbitrarily increased to see the control effect)

The results demonstrate that a cubic nonlinear damping as introduced by a simple nonlinear feedback control can achieve better performance than a linear damping control for vibration suppression both in low and high frequencies. The frequency domain method proposed in this study provides an effective approach to the analysis and design of the nonlinear feedback control. Although only a simple case with only one nonlinear term is studied in this simulation, much more complicated cases with multiple nonlinear parameters can also be analysed and designed by following a similar method. It should be noted that there may be some other methods in the literature which can be used to realize the same control purpose of this study, however, the advantage of this method is that it can directly relate the nonlinear controller parameters to system output frequency response and therefore the nonlinear controller or structural parameters can be analysed and designed in the frequency domain, which is a more understandable way in engineering practice. Furthermore, the designed controller, for instance the nonlinear damping designed in the example study above, may also be realized by a passive unite, and the analysis by using this method can be performed directly for a physical characteristics of a structural unite in a system. This will have great significance in practical applications.

9.5 Proofs

- **Proof of Proposition 9.3:**

To prove Proposition 9.3, the following Lemmas are needed.

Lemma 9.3. Consider two positive, scalar and continuous process in time t, $x(t)$ and $y(t)$ satisfying $y(t) \le \alpha(x(t))$ (for $t \ge 0$), where α is a K-function. If $x(t)$ is asymptotically stable to a ball $B_\rho(x)$, then $y(t)$ is asymptotically stable to a ball $B_{\alpha(2\rho)}(y)$.

Proof. There exists a KL-function β, such that function x(t) (for $t \ge 0$) satisfies $x(t) \le \beta(x(0),t)+\rho$, $\forall t > 0$. Therefore,

$y(t)$

$\le \alpha(x(t)) = \alpha(\beta(x(0),t)+\rho) \le \alpha(\max(2\beta(x(0),t),2\rho)) = \max(\alpha(2\beta(x(0),t)),\alpha(2\rho)) \le \alpha(2\beta(x(0),t))+\alpha(2\rho)$

Note that $\alpha(2\beta(x(0),t))$ is still a KL-function of $x(0)$ and t, thus the lemma is concluded. □

From Lemma 9.3, if there exists a K-function o such that the output function $h(\mathbf{X})$ of a nonlinear system satisfies $\|h(\mathbf{X})\| \le o(\|\mathbf{X}\|)$, then the system output is asymptotically stable to a ball if the system is asymptotically stable to a ball.

Lemma 9.4. Consider a scalar differential inequality $\dot{y}(t) \le -\alpha(y(t))+\gamma$, where α is a K-function and γ is a constant and $y(t)$ satisfies Lipschitz condition. Then there exists KL-function β such that

$$y(t) \le \beta(|y(t_0)-\alpha^{-1}(\gamma)|,t)+\alpha^{-1}(\gamma).$$

Proof. Consider the differential equation $\dot{y}(t) = -\alpha(y(t))$. From Lemma 10.1.2 in Isidori (1999) it is known that, there is a KL-function β such that $y(t) = \beta(y(t_0),t)$. Similarly, considering the differential equation $\dot{y}(t) = -\alpha(y(t))+\gamma$, then $y(t) = sign(y(t_0)-\alpha^{-1}(\gamma))\cdot\beta(|y(t_0)-\alpha^{-1}(\gamma)|,t)+\alpha^{-1}(\gamma)$. Thus from the comparison principle and the differential inequality $\dot{y}(t) \le -\alpha(y(t))+\gamma$, the lemma follows. □

Then to prove Proposition 9.3, it follows from (9.25) that

$$\dot{V}(\mathbf{X}(t)) \le -\alpha(\|\mathbf{X}\|)+\gamma(\|\eta\|_\infty) \quad \text{(A1)}$$

Noting $V(\mathbf{X}) \le \beta_2(\|\mathbf{X}\|)$, we have $\|\mathbf{X}\| \ge \beta_2^{-1}(V(\mathbf{X}))$. Substituting this inequality into (A1), we have

$$\dot{V}(\mathbf{X}(t)) \le -\alpha(\beta_2^{-1}(V(\mathbf{X}))) + \gamma(\|\eta\|_\infty)$$

From lemma 9.4, it follows that, there exist a KL-function β, such that

$$V(\mathbf{X}(t)) \le \beta(V_0, t) + \beta_2^{-1} \cdot \alpha^{-1} \cdot \gamma(\|\eta\|_\infty) \tag{A2}$$

where, $V_0 = \left|V(\mathbf{X}(t_0)) - \beta_2^{-1} \cdot \alpha^{-1} \cdot \gamma(\|\eta\|_\infty)\right|$. From (A2), $V(\mathbf{X}(t))$ is asymptotically stable to the ball $B_{\beta_2^{-1}\cdot\alpha^{-1}\cdot\gamma(\|\eta\|_\infty)}(V)$. Noting $\beta_1(\|\mathbf{X}\|) \le V(\mathbf{X})$, we have $\|\mathbf{X}\| \le \beta_1(V(\mathbf{X}))$. From lemma 9.3, $\mathbf{X}(t)$ is asymptotically stable to the ball $B_\rho(\mathbf{X})$. Furthermore, since assumption 9.1 holds, from lemma 9.3, $y(t)$ is asymptotically stable to the ball $B_{o(2\rho)}(y)$. This completes the proof of sufficiency. The proof of the necessity of the proposition can follow a similar method as demonstrated in the appendix of Hu et al (2005). The proof completes. □

- **Proof of Proposition 9.4:**

The state-space equation of system (9.26a) can be written as $\dot{\mathbf{X}} = \mathbf{AX} - \mathbf{B}\phi + \mathbf{E}\eta$, where, $\mathbf{X} = [x, \dot{x}]^T$, $\phi = a_3\sigma^3$, $\sigma = \mathbf{CX}$. Choose a Lyapunov candidate as:

$$V = \mathbf{X}^T\mathbf{PX} + \tfrac{\alpha}{2}\sigma^4 \tag{A3}$$

where, $\alpha > 0$. Equation (A3) further follows

$$\begin{aligned}\dot{V} &= \mathbf{X}^T\mathbf{P}\dot{\mathbf{X}} + \dot{\mathbf{X}}^T\mathbf{PX} + 2\alpha\sigma^3\mathbf{C}\dot{\mathbf{X}} = \mathbf{X}^T(\mathbf{A}^T\mathbf{P} + \mathbf{PA})\mathbf{X} - 2\mathbf{X}^T\mathbf{PB}\phi + 2\mathbf{X}^T\mathbf{PE}\eta + \tfrac{2\alpha}{a_3}\phi\mathbf{C}(\mathbf{AX} - \mathbf{B}\phi + \mathbf{E}\eta) \\ &= \mathbf{X}^T(\mathbf{A}^T\mathbf{P} + \mathbf{PA})\mathbf{X} - 2\mathbf{X}^T\mathbf{PB}\phi + \tfrac{2\alpha}{a_3}\phi\mathbf{CAX} - \tfrac{2\alpha}{a_3}\phi\mathbf{CB}\phi + 2\mathbf{X}^T\mathbf{PE}\eta + \tfrac{2\alpha}{a_3}\phi\mathbf{CE}\eta\end{aligned} \tag{A4}$$

Let $\mathbf{Z} = \begin{bmatrix}\mathbf{X}\\ \phi\end{bmatrix}$, $\mathbf{T} = \begin{bmatrix}\mathbf{PE}\\ \frac{\alpha}{a_3}\mathbf{CE}\end{bmatrix}$, and $\beta = \alpha/a_3$ then equation (A4) follows

$$\dot{V} = \mathbf{Z}^T\begin{bmatrix}\mathbf{A}^T\mathbf{P} + \mathbf{PA} & \beta\mathbf{A}^T\mathbf{C}^T - \mathbf{PB}\\ * & -2\beta\mathbf{CB}\end{bmatrix}\mathbf{Z} + \mathbf{Z}^T\mathbf{T}\eta \le \mathbf{Z}^T\begin{bmatrix}\mathbf{A}^T\mathbf{P} + \mathbf{PA} & \beta\mathbf{A}^T\mathbf{C}^T - \mathbf{PB}\\ * & -2\beta\mathbf{CB}\end{bmatrix}\mathbf{Z} + \varepsilon^{-1}\mathbf{Z}^T\mathbf{TT}^T\mathbf{Z} + \varepsilon\eta^T\eta$$

$$= \mathbf{Z}^T\left(\begin{bmatrix}\mathbf{A}^T\mathbf{P} + \mathbf{PA} & \beta\mathbf{A}^T\mathbf{C}^T - \mathbf{PB}\\ * & -2\beta\mathbf{CB}\end{bmatrix} + \varepsilon^{-1}\mathbf{TT}^T\right)\mathbf{Z} + \varepsilon\eta^2 = -\mathbf{Z}^T\mathbf{QZ} + \varepsilon\eta^2$$

Note that, in the inequality above, the following inequality is used

$$2\mathbf{Z}^T\mathbf{T}\eta \le \varepsilon^{-1}\mathbf{Z}^T\mathbf{TT}^T\mathbf{Z} + \varepsilon\eta^T\eta, \text{ for any } \varepsilon > 0.$$

If $\mathbf{Q}=\mathbf{Q}^T>0$, then $\mathbf{Z}^T\mathbf{QZ} \ge \lambda_{\min}(\mathbf{Q})\|\mathbf{X}\|^2$ is a K-function of $\|\mathbf{X}\|$. Hence, according to Proposition 9.3, the system is asymptotically stable to a ball $B_\rho(\mathbf{X})$ with $\rho = \sqrt{\lambda_{\min}(\mathbf{Q})^{-1}\varepsilon\sup(\|\eta\|^2)} = F_d\sqrt{\lambda_{\min}(\mathbf{Q})^{-1}\varepsilon}$. Additionally, when there is no exogenous disturbance input, and if $\mathbf{Q}=\mathbf{Q}^T>0$ holds with $\mathbf{E}=0$, then it is obvious that the system without a disturbance input is globally asymptotically stable. This completes the proof. □

9.6 Conclusions

A frequency domain approach to the analysis and design of nonlinear feedback controller for suppressing periodic disturbances is studied and some preliminary results in this subject are provided. Although there already are some time domain methods, which can address the nonlinear control problems based on Lyapunov stability theory, few results are available for the design and analysis of a nonlinear feedback controller in the frequency domain to achieve a desired frequency domain performance. Based on the analytical relationship between system output spectrum and controller parameters defined by the OFRF, this study provides a systematic frequency domain approach to exploiting the potential advantage of nonlinearities to achieve a desired output frequency domain performance for the analysis and design of vibration systems. Compared with other existing methods for the same purposes, the method in this chapter can directly relate the nonlinear parameters of interest to the system output frequency response and the designed controller may also be realized by a passive unite in practice. Although the results in this paper are developed for the problem of periodic disturbance suppression for SISO linear plants, the idea can be extended to a more general case (*i.e.*, nonlinear controlled plants) and to address more complicated control problems.

Chapter 10
SUMMARY AND OVERVIEWS

Frequency domain methods can usually provides very intuitive insights into the underlying mechanism of a studied system in a coordinate-free and equivalent manner, compared with the corresponding time domain methods. Thus they are widely applied in engineering practice and extensively studied in literature. Due to the complicated output frequency characteristics and dynamic behaviour of nonlinear systems, a systematic frequency domain theory for the analysis and design of nonlinear systems has been a focused topic in the past several decades. As discussed in Chapter 1, different subjects have been studied in this field and many remarkable results have been achieved both in theory and practice.

In this study, new advances in the characterization and understanding of nonlinear systems in the frequency domain have been achieved based on the Volterra series theories of nonlinear systems. A systematic frequency domain approach for the analysis and design of nonlinear Volterra systems is developed via a novel technique known as parametric characteristic analysis, which is developed for the extraction of parametric characteristics of any parameterized polynomial systems satisfying separable property.

The contributions of this study are:

(a) A parametric characteristic analysis method is proposed for parameterized polynomial systems with separable property, which is to reveal what model parameters affect system frequency response functions and how they do. Based on this technique, it is shown for the first time that, the analytical relationship between high order frequency response functions of Volterra systems and system time-domain model parameters, and also provides a novel method for the understanding of the higher order GFRFs of Volterra systems. Refer to Chapters 2-3 and Chapter 8.

(b) By using the parametric characteristic analysis, the system output spectrum up to any orders can be explicitly expressed as a polynomial function of model parameters of interest which relates the system output frequency response to any model nonlinear parameters such that system output frequency response can be analyzed via these model parameters. This provides a significant basis for the analysis and design of nonlinear Volterra systems in the frequency domain. Refer to Chapter 4 and Chapters 8-9.

(c) A novel mapping function from the parametric characteristics of the nth-order GFRF to itself is established. This result enables the nth-order GFRF and output spectrum to be directly written as a polynomial forms in terms of the first order GFRF and model nonlinear parameters, which is shown to be a new approach to the understanding of higher order GFRFs. Refer to Chapter 5.

(d) It is theoretically shown for the first time that system output spectrum can be expressed as an alternating series with respect to some model nonlinear parameters under certain conditions. The result reveals a significant nonlinear effect on the system behaviours. Refer to Chapter 6.

(e) The nonlinear effects on system output spectrum from different nonlinearities are also studied. This provides some novel insights into the nonlinear effect on system output spectrum in the frequency domain, such as the counteraction between different nonlinearities at some specific frequencies, periodicity property of output frequencies and so on. These results can facilitate the structure selection and parameter determination for system modelling, identification, filtering and controller design. Refer to Chapter 7.

(f) A new method for the vibration control problem is proposed. It is a systematic frequency domain approach to exploiting the potential advantage of nonlinearities to achieve a desired output frequency domain performance for the analysis and design of vibration systems. Refer to Chapter 9.

The significance of these results is that a systematic frequency domain theory for the analysis and design of a class of nonlinear systems is established. In this novel method, (1) it can directly relate the nonlinear model parameters of interest to system frequency response functions, and therefore the nonlinear controller parameters or structural parameters can be analysed and designed in the frequency domain, which is a more understandable way in engineering practice; (2) it can be used not only to design a nonlinear feedback controller for a system by exploiting the potential advantages of nonlinearities for a practical system, but also to analyse and design structural nonlinear characteristics which can be realized in a passive/active manner to achieve a desired passive structural physical characteristics; (3) it provides a novel approach to understanding the nature of a considerably large class of nonlinearities in the frequency domain.

Although interesting and significant results have been achieved, there are still many tasks yet to be done for the full development of a systematic frequency domain method. For example, understanding and characterization of nonlinearities in the frequency domain based on the results developed in this dissertation, optimization and design of nonlinear systems based on the system OFRF, automatic and systematic controller designs for a wider class of nonlinear systems by exploiting nonlinearities, extensions of the results for SISO systems to MIMO systems, development of practical techniques for the applications of these theoretical results, and so on. All these issues are left to the future studies in the direction that is established by the results in this dissertation.

Appendix
Publication List during Studying for PhD Degree

- **Refereed Journal Articles**

[1] Jing X.J., Lang Z.Q. and Billings S.A., Frequency Domain Analysis for Suppression of Output Vibration from Periodic Disturbance using Nonlinearities. Journal of Sound and Vibration, 314, 536 - 557, 2008

[2] Jing X. J., Lang Z. Q., Billings S. A. and Tomlinson G. R., The parametric characteristic of frequency response functions for nonlinear systems. International Journal of Control, Vol. 79, No. 12, 1552–1564, December 2006

[3] Jing X. J., Lang Z.Q., and Billings S.A., New Bound Characteristics of NARX Model in the Frequency Domain. International Journal of Control, Vol 80, No1, 140-149, 2007

[4] Jing X. J., Lang Z.Q., and Billings S.A., Correction on some typos in 'New Bound Characteristics of NARX Model in the Frequency Domain'. International Journal of Control, Vol 80, No3, pp. 492-494, 2007

[5] Jing X. J., Lang Z.Q. and Billings S.A., "Magnitude Bound Characteristics of the GFRFs for NARX Model". Automatica, 44, 838-845, 2008

[6] Jing X. J., Lang Z.Q. and Billings S.A., Frequency domain analysis for nonlinear Volterra systems with a general nonlinear output function. International Journal of Control, 81:2, 235 – 251, 2008

[7] Xing Jian Jing, Zi Qiang Lang, Stephen A. Billings, Mapping from parametric characteristics to generalized frequency response functions of nonlinear systems. International Journal of Control, Vol. 81, No. 7, 1071 - 1088, July 2008

[8] Xing-Jian Jing, Zi-Qiang Lang, Stephen A. Billings. Output Frequency Response Function based Analysis for Nonlinear Volterra Systems. Mechanical Systems and Signal Processing, 22, 102–120, 2008

- **Refereed Conference Proceedings**

[9] Xing Jian Jing and Zi Qiang Lang. Properties of output frequencies of Volterra systems. to appear in International Conference on Control (UKACC), Manchester, U.K., Sep 2-4, 2008

[10]Xing Jian Jing, Zi Qiang Lang, Stephen A. Billings. New Results on the Generalized Frequency Response functions of Nonlinear Volterra Systems Described by NARX model, to appear in IFAC World Congress, Seoul, Korea, July 6-11, 2008

[11]Xing Jian Jing and Zi Qiang Lang. Magnitude Bounds of Generalized Frequency Response Functions of Nonlinear Volterra Systems. Proceedings of the European Control Conference, Kos, Greece, 3068-3073, July 2-5, 2007

[12]Jing X.J., Lang Z.Q. and Billings S.A., Frequency domain analysis based nonlinear feedback control for suppressing periodic disturbance, The 6th World Congress on Intelligent Control and Automation, June 21–23, Dalian, China, 2006

[13]Xing-Jian Jing, Zi-Qiang Lang, Stephen A. Billings. Output Frequency Response Function for NARX model of Nonlinear Volterra Systems. Proceedings of the 12th Chinese Automation & Computing Society Conference in the UK, Loughborough, England, 16 September 2006

- **Research Reports in University**

[1] Jing X. J., Lang Z. Q., Billings S. A., and Tomlinson G. R., The Parametric Characteristics of Frequency Response Functions for Nonlinear Systems. Department of Automatic Control and Systems Engineering, University of Sheffield, Research Report 932, Aug 2006

[2] Jing X. J., Lang Z. Q., Billings S. A., and Tomlinson G. R., A New Approach to Nonlinear Feedback Control for Suppressing Periodic Disturbances, Part 1. Fundamental Theory. Department of Automatic Control and Systems Engineering, University of Sheffield, Research Report 933, Aug 2006

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