NONLINEAR ROBUST H_∞ CONTROL

A Doctoral Thesis Presented

by

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To My Dearly Loved Family:

Your Belief and Support of My Dreams has Made Them Possible.

Thank you for your love

ABSTRACT

A new theory is proposed for the full-information finite and infinite horizon-time robust H_∞ control that is equivalently effective for the regulation and/or tracking problems of the general class of time-varying nonlinear systems under the presence of exogenous disturbance inputs. The theory employs the sequence of linear-quadratic and time-varying approximations, that were recently introduced in the optimal control framework, to transform the nonlinear H_∞ control problem into a sequence of linear-quadratic robust H_∞ control problems by using well-known results from the existing Riccati-based theory of the maturing classical linear robust control. The proposed method, as in the optimal control case, requires solving an approximating sequence of Riccati equations (ASRE), to find linear time-varying feedback controllers for such disturbed nonlinear systems while employing classical methods. Under very mild conditions of local Lipschitz continuity, these iterative sequences of solutions are known to converge to the unique viscosity solution of the Hamilton-Jacobi-Bellman partial differential equation of the original nonlinear optimal control problem in the weak form (Çimen, 2003); and should hold for the robust control problems herein. The theory is analytically illustrated by directly applying it to some sophisticated nonlinear dynamical models of practical real-world applications. Under a γ-iteration sense, such a theory gives the control engineer and designer more transparent control requirements to be incorporated *a priori* to fine-tune between robustness and optimality needs. It is believed, however, that the automatic state-regulation robust ASRE feedback control systems and techniques provided in this thesis yield very effective control actions in theory, in view of its computational simplicity and its validation by means of classical numerical techniques, and can straightforwardly be implemented in practice as the feedback controller is constrained to be linear with respect to its inputs.

Research Head: Stephen Paul Banks

Title: Professor of Systems Theory

VITA

Born December 19th, 1978, Eng. Sherif F.F. Fahmy received his Bachelor of Science in Mechanical Engineering, with an overall very good Grade Point Average, with double options in Industrial and Materials & Manufacturing engineering and double minors in Computer Science and Electronics from the American University in Cairo, Egypt, in June 2002. He subsequently pursued a Masters of Science in engineering degree in Control Systems at the University of Sheffield, United Kingdom, and obtained it with distinction in September 2003. He then embarked upon the most challenging and thrilling task of his life-long career to date; that is obtaining his Doctoral of Philosophy[†] degree at the University of Sheffield under the supervision of one of the most highly regarded professors in Nonlinear Control theories, Professor Stephen Paul Banks. By the time of submitting his thesis, he was also on the verge of obtaining his teaching qualification, the Postgraduate Certificate in Higher Education (PCHE), from the Department of Educational Studies & the Learning Development and Media Unit at the University of Sheffield.

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PUBLICATIONS

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- *Some of the research material in this thesis is still to be submitted to various prestigious journals for publication.*

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STATEMENT OF ORIGINALITY

Unless otherwise stated in the text, the work described in this thesis was carried out solely by the candidate. None of this work has already been accepted for any degree, nor is it concurrently submitted in candidature for any degree.

Candidate: ____________________

PhD Student

Supervisor: ____________________

PhD Supervisor

CONTENTS

PART III: DETERMINISTIC H_∞ CONTROL

PART IV: PRACTICAL APPLICATIONS

PART V: CONCLUSIONS & FURTHER RESEARCH

APPENDICES

ACRONYMS, NOTATIONS, AND SYMBOLS

A standard conventional notation will be used throughout this thesis unless otherwise stated. This set of symbols which is very common in control theories' publications is presented hereunder.

t	Time
t_0	Initial (starting) time
t_f	Final time
J	Cost (performance or payoff) functional to be minimized
x	$n-$dimensional state vector of a dynamic system ($x_1,\ldots,x_n$)
u	$m-$dimensional ($m \le n$) system control input vector ($u_1,\ldots,u_m$)
y	$l-$dimensional ($l \le n$) measurement (or output) vector
z	$l-$dimensional desired output vector (or controlled variable)
$\dot{x}(t)$	The derivative of $x(t)$ with respect to time t
w	The exogenous disturbance input
Φ	The state transition matrix of a linear dynamical system
A	$n\times n$ dynamic coefficient matrix of continuous linear differential equations defining a dynamical system
B	$n\times m$ input coupling matrix of continuous linear differential equations defining a dynamical system
C	$l\times n$ measurement sensitivity matrix, defining the linear relationship between the states and the measurements that can be made
D	$l\times m$ input-output coupling matrix (throughput or feedthrough or feedforward matrix)
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	The transfer function realized by $D+C(sI-A)^{-1}B$ (with respect to z)
Q	$l\times l$ state weighting matrix
R	$m\times m$ input weighting matrix
P	$n\times n$ Riccati matrix
$:=$	The left hand side defined by the expression in the right hand side
□ OR (■)	End of the proof

$\Re$ OR $(\mathbb{R})$	Field of real numbers
$\Re^*$	Adjoint of $\Re$
$\Re^{-1}$	Bounded inverse of $\Re$ (for boundedly invertible $\Re$)
$L^{2,m}$	Hilbert space of square norm Lebesgue integrable $\Re^m$-
$L_2(-\infty,\infty)$	Time domain Lebesgue space
$L_\infty(-\infty,\infty)$	Ditto
H_2	Hardy space
H_∞ OR (H^∞)	Ditto
$\langle\cdot,\cdot\rangle$	Inner product (the Hilbert space is implicit)
$\lVert x\rVert$	Euclidean norm of the vector $x \in \Re^n$
$\lVert\bullet\rVert_2$	Euclidean norm on L_2
$\lVert\bullet\rVert_\infty$	Euclidean norm on L_∞
I_n	$n \times n$ identity matrix
A^T	Transpose of matrix A
A^{-1} OR (A')	Inverse of matrix A (for A invertible)
A^*	Complex conjugate transpose of matrix A
$\lambda M - N$	Matrix pencil

Part I

General Introduction

CHAPTER 1

Introduction

1.1. Overview

Many control systems of practical importance are inherently nonlinear and so the need to take into account the nonlinearities of a system has become more and more important as the demands for better performances and more sophisticated requirements increased over the past few years. Additionally, it is without doubt that the last few decades have witnessed tremendous research efforts in analyzing and designing nonlinear control systems in broad areas such as aircraft and spacecraft control, robotics, process control, and biomedical engineering; to name only a few. In particular, many researchers and designers have recently shown an active interest in developing and applying nonlinear control methodologies to such various practical fields with the aid of the differential geometric approaches. Among the various state-of-the-art nonlinear synthesis techniques are the method of feedback linearization and the notion of zero dynamics with their applications to a variety of control problems such as in asymptotic stabilization of minimum phase systems, output regulation and feedback equivalence to a passive system (see [Banks, 1986a; Isidori, 1995; Isidori, 1999; and Marino, 1995]).

Nonetheless, one of the main driving forces behind such a rapid growth, particularly noticeable in modern control theory, was the realization that controllers designed solely from optimization concerns exhibited a lack of robustness with respect to modelling uncertainty, both in theory and in practice. Small mismatches from the model used for design to the actual plant could cause a serious loss of performance or even loss of stability. For linear control theory in the 1980's, a major field of research was H_∞ control (and related topics), which addressed these robustness issues. The control algorithms, that were developed, amounted to sophisticated generalizations of classical design methodologies, and have proven effective in practice, especially for multi-input multi-output systems.

The nonlinear control theory, however, still lacks many of the mathematical tools which are available to the maturing and well-understood classical linear control theory. The robust control theory for linear systems is usually approached using input/output or operator theoretic methods, and so there has been renewed interest in the study of nonlinear input/output systems; and it would be groundbreaking research to make a nonlinear system appear linear. Of course, the success of linear robust control methods has led to an interest in extending such work to nonlinear systems mainly through local linearization techniques about an equilibrium or operating point.

The most common difficulty of analyzing generic nonlinear systems leads to the idea of restricting the class of systems studied due to the key assumption in linearizing a nonlinear system in which the range of operation is assumed to be small for the linearized model to be valid (Slotine & Li, 1991). Unlike linear controllers, nonlinear controllers handle nonlinearities in a much larger operation range while compensates for the parametric model uncertainties that are often neglected in their counterparts. The advantage of nonlinear controllers not only depends on a simple design that is often deeply rooted in the physics of plants but also may permit their implementation with less expensive actuators and sensors that exhibit nonlinear characteristics. Therefore, inherent (natural) and intentional (artificial) nonlinearities, which can be referred to mathematically as continuous or discontinuous nonlinearities, should not be disregarded in the design of control systems, as it is common practice.

As will be seen in later chapters of this dissertation, it appears that the key to extending H_∞ methods to nonlinear systems, while taking full advantage of the classical linear theory, is the ability to solve various sequences of Riccati equations over the state space. Although global linearization techniques can be used (see for instance, [Banks & Yew, 1985; Banks, 1986b; Banks, 1992]) in this context, the problems seem to be generally quite difficult to solve numerically, and this constrains the practical effectiveness of such analysis.

It is conjectured that the recently developed optimal nonlinear control theory (see [Çimen, 2003]) often seems over-idealized when it comes to dealing with stochastic disturbances and noisy measurements and may not always be as efficient in dealing with exogenous uncertainties as its robust control counterpart. So the focus of the research in this thesis represents an elucidation to this problem.

In the following sections a few aims of the present research will be highlighted. Starting with the motivations behind embarking upon a robust control theoretical framework is presented in §1.2. The theoretical idea behind an existing approximation theory is given in §1.3. A brief historical account of the state-space H_∞ control is given in §1.4; while the subsections to follow give both an overview of the current research efforts behind the nonlinear H_∞ theory, and presents the reader with the general H_∞ control problem. Last but not least, §1.5 gives an overview of the necessary prerequisites. Finally, a brief description of this dissertation along with the aims of this research are specified in §1.6.

1.2. Why Robust Control?

Uncertainty and disturbances such as noise are inherent to any real-world practical system. In the deterministic case, the signals and the mathematical model of a system are known without uncertainty and the time-varying behaviour can be reproduced by repeated experimentation. In the stochastic case, this is not possible due to the uncertainty that exists either in its model parameters or in its signals or in both. The values of the signals or the variables occurring in the system can only be estimated with the help of the methods of probability and statistics; and the results are presented as expected values together with the bounds of error.

In reality, despite efforts by identification and parameter estimation, system models are neither precisely known nor are guaranteed to remain the same under the different conditions of operation. While adaptive techniques automatically tune the control action to meet mainly the latter contingency, the issue of uncertainty as well as noise is tackled by robust control techniques. Here, the controller is designed for a nominally specified plant model by taking uncertainties and un-modelled plant dynamics such that the resulting control guarantees satisfactory control under the limitations of knowledge of the plant model. Moreover, the control law is said to be robust if it is valid over the whole range of admissible uncertainty.

From a control point of view, when modelling systems, several sources of uncertainty can be classified as:

A. non-parametric (unstructured) uncertainty

1. un-modelled physical dynamics
2. truncated high frequency modes
3. nonlinearities
4. effects of linearization and time-variation

B. parametric (structured) uncertainty

1. physical parameters vary within given bounds
2. interval uncertainty (L_∞)
3. ellipsoidal uncertainty (L_2)
4. diamond uncertainty (L_1)

The first kind corresponds to inaccuracies or underestimation of the system order; while the second kind corresponds to inaccuracies in the terms actually included in the model. Note that the model imprecision may occur as a result of unknown plant parameters, or from a purposeful choice of a simplified mathematical representation of the system's dynamics, *e.g.* modelling friction as linear or neglecting structural modes in a reasonably rigid mechanical system.

However, the *sine qua non* of robust control techniques is the ability to directly address the inheriting presence of such above-mentioned uncertainties while maintaining the system response and error signals to within prescribed tolerances despite the presence of noise.

1.3. The Approximation Theory

The ultimate objective of feedback control is to use the principle of feedback to cause the output variable of a dynamic process to follow a desired reference variable accurately regardless of the reference variable's path and of any external disturbances or any changes in the dynamics of the process. However, in order to meet this complex goal, a dynamical model of the process to be controlled has first to be physically and mathematically modelled. To a control engineer, a dynamical system, or a dynamic model, is a given process mathematically quantified with state, rate variables and parameters that are functions of time. Because such variables can either be continuous or discrete in time, dynamical systems are mathematically expressed with either differential equations or difference equations. The state-space representation (also known as the "time-domain approach") provides a convenient and compact way to model and analyze systems with multiple inputs and outputs; making it a more appealing method of describing the dynamics of controlled processes over the high-order differential equation representation. Hence, an n^{th} –order differential equation can be conveniently written as a set of n first–order simultaneous differential equations in vector form. To abstract from the number of inputs, outputs and states, the variables are expressed as vectors and the differential and algebraic equations are written in matrix form lending itself to computer analysis. It is worth pointing out that a given dynamical system has a unique dynamical equation model whereas its state-space representation is not unique.

Consider the following nonlinear control systems for continuous-time models which are modelled via finite dimensional deterministic ordinary differential equations of this general time-varying form:

$$\left.\begin{array}{ll}(a)\ \dot{x}(t)=f\big(x(t),w(t),u(t),t\big); & x(0)=x_0 \\ (b)\ y(t)=h\big(x(t),w(t),u(t),t\big) & \end{array}\right\}. \tag{1.1}$$

Alternatively, for discrete-time systems

$$\left.\begin{array}{ll}(a)\ x(k+1)=f\big(x(k),w(k),u(k),k\big); & x(0)=x_0 \\ (b)\ y(k)=h\big(x(k),w(k),u(k),k\big) & \end{array}\right\}. \tag{1.2}$$

Linear models form special cases of the general continuous-time model of (1.1) and discrete-time model of (1.2) which can be given respectively by:

$$\left.\begin{array}{ll}(a)\ \dot{x}(t)=A(t)x(t)+B(t)u(t)+E(t)w(t); & x(0)=x_0 \\ (b)\ y(t)=C(t)x(t)+D(t)u(t)+E_2(t)w(t) & \end{array}\right\}; \tag{1.3}$$

$$\left.\begin{array}{ll}(a)\ x(k+1)=A(k)x(k)+B(k)u(k)+E(k)w(k); & x(0)=x_0 \\ (b)\ y(k)=C(k)x(k)+D(k)u(k)+E_2(k)w(k) & \end{array}\right\}. \tag{1.4}$$

Excluding the exogenous disturbance input, the solution of (1.3)(a) is usually known from the fundamental transition matrix solutions of the inhomogeneous equations (for a definition see §A.2.2). Conversely, the dynamical equations of the discrete-time systems, (1.4)(a), are usually derived from the corresponding synthesized model in the continuous-time domain, as in the general form. A difference equation of the form

(1.2)(a) can not be accurately synthesized in practice unless approximate discrete-time model-based techniques are used, such as the Runga-Kuta algorithm for instance. However, in view of such an approximation, discrete-time systems are not convenient to treat dynamical equations (Borrie, 1992). Note that the difficulty does not arise from the algebraic equation (1.2)(b) since it is plainly a discrete-time version of (1.1)(b). While a large number of systems can be theoretically and practically modelled by means of the set of linear time-varying difference equations (1.4), the nonlinear difference equations, (1.2), form an ideal rather than a practical model. Thus, it is usually impractical to model either a stochastic or a deterministic nonlinear system by difference equations unless, of course, continuous-time models are discretized. It is for this reason that this dissertation will only focus on continuous-time nonlinear systems of the form (1.3).

In order to give the reader an insight about the approximation theory, special attention is drawn around the historical perspective behind the optimal control setting in which this theory first emanated.

Consider a general time-invariant nonlinear system of the form

$$\dot{x} = f(x,u) \tag{1.5}$$

with a linear-quadratic cost function

$$\min J(u) = \frac{1}{2} x^T(t_f)\, F\, x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} \left\{ x^T(t)\, Q\, x(t) + u^T(t)\, R\, u(t) \right\} dt; \tag{1.6}$$

this cost functional can be solved in principle by using the Lie series and infinite-dimensional bilinear systems theory (Banks & Yew, 1985; Banks, 1986b; and Banks, 1992). But due to the complexity in implementing this solution, some fundamental contributions took place in the past few decades to the theory of nonlinear dynamical systems especially in the field of pseudo-linear systems taking the form

$$\dot{x} = A(x)x + B(x)u. \tag{1.7}$$

Since the early 1960's, a number of researchers have proposed nonlinear control algorithms which involve application of linear design methods to linear-like 'factored' representations of a nonlinear system for continuous-time, state feedback, input-affine, autonomous nonlinear dynamic systems in (1.7) (see [Pearson, 1962; Burghart, 1969; Wernli & Cook, 1975; Ehrler & Vadali, 1988; and Hammett, *et al.*, 1998]). However a different approach using the so-called 'freezing' technique that was introduced by Banks & Mahana in 1992 to develop locally optimal and locally asymptotically stabilizing controllers using form (1.7) proved very successful. This technique was further adopted by many authors to approximate nonlinear optimal controllers based on solving the "State-Dependent Riccati Equation" (SDRE) while using it to regulate and control a variety of practical applications (see [Cloutier, *et al.*, 1996; Mracek & Cloutier, 1998; Hammett, *et al.*, 1998; and McCaffrey & Banks, 2001b]). The only limitation to this SDRE feedback is that it can only be applied to finite-time autonomous regulator problems as well as to tracking problems; and not to the more difficult infinite-time problem since this requires solving an infinite-time algebraic Riccati equation for which the theory is not available as yet.

The recursive technique for the nonlinear optimal control problem that was introduced by Banks & McCaffrey (1998) considered systems of the form

$$\dot{x} = A(x)x, \tag{1.8}$$

where the authors presented this system as the limit of linear time-varying (LTV) approximations

$$\dot{x}^{[i]}(t) = A\left(x^{[i-1]}(t)\right)x^{[i]}(t). \tag{1.9}$$

These sequences were shown to converge in the space of continuous functions under very mild conditions. Technically speaking, the convergence holds provided the function $x \to A(x)$ is locally Lipschitz, *i.e.* the minimum condition required for the uniqueness of solutions. The main advantage of this approximation theory which was also employed by Chanane (1998) is that nonlinear systems can be closely approximated by linear ones – a fact which brings all the classical linear tools and machinery to hand. Although LTV systems are much more complex when compared to autonomous systems, recent developments by Banks (2002) led to the representation of the solutions (1.9) by means of the Lie algebra of $A\left(x^{[i-1]}(t)\right)$ (*i.e.* bracketed matrices).

Note that the approximation theory can be readily applied to the dynamical nonlinear system given by the pseudo-linear systems in (1.7)

$$\dot{x}^{[i]}(t) = A\left(x^{[i-1]}(t)\right)x^{[i]}(t) + B\left(x^{[i-1]}(t)\right)u^{[i]}(t), \tag{1.10}$$

to determine control actions by any classical or modern control techniques, such as optimal control (see [Çimen, 2003; Banks & Dinesh, 2000]), robust H_∞ control as considered in this thesis, and many other nonlinear problems.

The approximation theory is not simply a numerical method, instead it was extensively exploited in the study of chaotic motion, Lie algebras and even nonlinear delay systems (see [Banks & McCaffrey, 1998; and Banks, 2002]). With a range of appealing and demanding applications, (1.10) proved very effective in formulating and controlling aircraft systems (Banks, *et al.*, 2000; and Salamci, *et al.*, 2000) including an F8-crusader (Çimen & Banks, 2004); it was also applied to dynamic ship-positioning systems (Çimen, 2003; and Çimen & Banks, 2005); to controlling flexible space structures (Zheng, et al., 2005); and in nonlinear solitary wave motions (Banks, 2001a). Additionally, the approximation technique found its usage in the design of sliding mode controls with optimally selected sliding surfaces for an autopilot design for a missile (Salamci, *et al.*, 2000).

1.4. The H_∞ Control Theory

1.4.1. The State-Space H_∞ Control Theory: A Historical Perspective

Since the central subject of this thesis is the *state-space* H_∞ optimal control, similar, but not quite, to the book by Stoorvogel (1992): *The* H_∞ *Control Problem: a State Space Approach*, in contrast to the approach adopted in the famous book by Francis (1987): *A Course in* H_∞ *Control Theory*; it may be helpful to provide some historical perspective of the state-space H_∞ control theory. This section, however, is not intended as a literature review in H_∞ theory or robust control but rather only an attempt to outline some of the major work in this almost matured field.

The state-space H_∞ techniques deal with providing the multi-input multi-output (MIMO) dynamical system with a feedback control verifying robust stability and robust performance. In the frame of these methods, a description must be done for both the nominal system and the uncertainties associated with the model. Consequently, in this context, ***Robust Stability*** means that whatever the real system inside the boundaries defined by the uncertainties around the nominal plant, the controller is able to stabilize it. Whereas ***Robust Performance*** means that whatever the real system inside the boundaries defined by the uncertainties around the nominal plant, the controller is able to guarantee that the real plant satisfies the required performance.

A fundamental problem in control theory is to design controllers which give satisfactory performance in the presence of uncertainties such as unknown model parameters and disturbances which enter the system dynamics. Consequently, one of the main motivations behind the original introduction of the H_∞ theory in the frequency domain by Zames (1981) was to bring the plant uncertainty back into centre-stage. In other words, the H_∞ control theory originated in an effort to codify classical control methods; where the frequency response functions are shaped to meet certain performance objectives. The linear H_∞ control theory has developed extensively since the early 1980s, and effective numerical methods have been developed for practical implementation in engineering applications (see [Dym, 1994] for some historical remarks).

The linear H_∞ control theory can be considered in either a frequency domain, input-output formulation or a time domain, state-space formulation. Mathematical tools of the linear theory in an input-output setting involve such techniques from operator-theoretic methods (see [Sarason, 1967; Adamjan, et al., 1978; Ball & Helton, 1983]) and complex function theory involving analytic functions such as the Nevanlinna-Pick interpolation and inner-outer factorizations (see [Helton & James, 1999; and Zhou, *et al.*, 1996]). Unfortunately, the standard frequency domain approaches to the H_∞ control problem can neither mathematically nor computationally deal with MIMO systems, much as the Linear Quadratic Gaussian (LQG) theory proved in the early 50s (Zhou, *et al.*, 1996).

Not surprisingly, introduced by Doyle (1984), the first solution to the general MIMO H_∞ control was formulated in the state-space while heavily relying upon inner/outer and coprime factorizations of transfer function matrices that reduced the problem to a Nehari/Hankel norm. This method, although in a mathematical sense "solved" the general rational problem (Francis, 1987; and Francis & Doyle, 1987), it had a main disadvantage in the computational complexity of solving high-order Riccati equations which questioned its realism. As a remedy, model reduction techniques played a centre stage in addressing this problem in particular. The self-contained state-space treatment exploiting the balanced realizations proposed for the model reduction by Moore (1981) can be found in Glover (1984). While the dual realization, where the linear time-varying H_∞ formulation is cast as a linear time-invariant frequency domain counterpart in terms of compensators' existence for example, proved highly successful in addressing a variety of H_∞ control problems: *e.g.* the signature condition based on the Kalman-Popov-Yakubovich approach, and the minimum entropy formulation (see [Stoorvogel, 1992; and Ionescu & Stoica, 1999]).

However, a simpler and more direct state-space H_∞ controller formulae relied on solving an algebraic Riccati equation and completing the square (see [Khargonekar, *et al.*, 1990; and Khargonekar, *et al.*, 1988]). Matrix Riccati equations have also played a key role (see [Doyle, et al., 1989; Barabanov & Ghulchak, 1996; and Ichikawa & Katayama, 1999]). Nonetheless, relations between the H_∞ control and many other topics in control were also exploited: *e.g.* risk sensitivity control (see [Whittle, 1981; and Whittle 1990]); differential games (see [Başar & Bernhard, 1991; Limebeer, *et al.*, 1992; and Green & Limebeer, 1995]); *J*-lossless factorization (see [Green, 1992]); the maximum entropy methods (see [Dym & Gohberg, 1986; and Mustafa & Glover, 1990]); Linear-Matrix-Inequality formulations (see [Chen, *et al.*, 2004]); Hamiltonian-based skew-Toepliz-type solutions to the H_∞ problem (Hirata, *et al.*, 2000); and infinite-dimensional H_∞ Riccati equations (Ichikawa, 2000). In addition, more generalizations were undeniably noticeable in broadly expanding the H_∞ formulation from time-invariant to time-varying, from finite-horizon to infinite-horizon, from finite-dimensional to infinite-dimensional, and even from linear to nonlinear designs.

1.4.2. The Nonlinear H_∞ Control Theory

In contrast to the linear H_∞ control theory, the nonlinear H_∞ control theory is formulated in the time domain much like the dynamical systems' setting. In their book Helton & James (1999) showed that the key to expanding the H_∞ control to nonlinear systems depended on ideas and methods of differential games and nonlinear partial differential equations (or partial differential inequalities). Where for the state-feedback setting, the available storage fitness function satisfies, in the viscosity sense, a first order nonlinear Partial Differential Equation of Hamilton-Jacobi-Bellman (HJB) type;

and where the central controller is obtained by taking argmax over possible controls in the Isaacs equation. While for the output-feedback setting, the problem is reformulated by defining an "information state" that evolves forward in time according to a HJB partial differential equation, also interpreted in the viscosity sense. The same problem, however, has been tackled previously from a different perspective in Isidori & Astolfi, (1992) by non-hyperbolic equilibria that were assumed for the Hamiltonian systems associated with the two Hamilton-Jacobi-Issacs equations; *i.e.* the problem is extended by means of a Hamilton-Jacobi-Inequality.

Another technique is that of the inner-outer factorization, in which the inner factor is dissipative and the outer factor satisfies a weak invertibility condition (Helton & James, 1999). Another way to avoid the infinite dimensional PDE framework is to consider "certainty-equivalent" controllers; which corresponds to considering the solution of the infinite dimensional HJB equation; and under suitable assumptions, including uniqueness of the argmax, the certainty equivalent and central information state controllers agree (Helton & James, 1999).

Nonetheless, the nonlinear extension of the H_∞ optimization problem was further thoroughly investigated in the recent literature in the L_2 framework through polynomial expansions (Foias & Tannenbaum, 1989); dissipative techniques/nonlinear differential game arguments (Başar & Bernhard, 1991); in terms of nonlinear matrix inequalities (Lu & Doyle, 1995); while linear H_∞ methods were also applied to systems perturbed by nonlinear uncertainties (Becker, *et al.*, 1993). Differentiable/incremental and the weighted incremental norms were also used to extend the H_∞ approach to the nonlinear context (Georgiou, 1993; Formion, *et al.*, 1999; and Fromion, *et al.*, 2001); as well as receding-horizon methodology is extended to design H_∞ robust controller in Magni, *et al.*, (2001).

1.4.3. The H_∞ Control Problem

The H_∞ goal is usually as follows: *Given a nominal LTV description of the control plant together with bounds on an appropriate uncertainty model and on the performance objectives, design a LTV controller that meets at least the nominal* *performance requirements and that achieves robust stability.* The fact that an optimization is involved enables the designer a certain amount of scope to investigate the inherent trade-offs between performance and robust stability, and to get some idea of how good a given design is, relative to what is theoretically possible.

The H_∞ optimization can be applied in a variety of different ways, so it is helpful to have a generalized framework in which most controller design problems can be formulated. Such a framework has been developed and widely researched, and is shown in Figure (1.1),

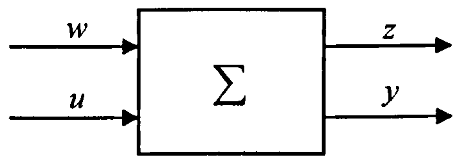


Figure 1.1.: The H_∞ Schematic Representation.

where the continuous-time system or plant, Σ, considered in this thesis is linear time-varying and is studied in the state-space domain. Note that Σ contains the nominal dynamics of the plant, combined within an interconnection which in general will incorporate one or more suitable uncertainty model structures and performance goals; as will become clearer in later chapters.

The two kinds of inputs to the system are:

- u is the control input to the system, containing all the inputs that are generated by the controller,
- w is the exogenous input to the system, containing all the other signals entering the system (in particular the reference and the disturbances inputs acting on the system); and

with the following outputs:

- y contains the measurement outputs (used to choose u, the control input; which in turn is the tool to minimize the effect of w on z),
- z contains all the outputs to the system that must be regulated and whose dependence on the exogenous disturbance input w is to be minimized.

The mapping from y to u is usually constrained such that the closed-loop system is internally stable – a natural requirement that ensures the states do not become too large while regulating performance requirements. Whereas the closed-loop effect of w on z is theoretically measured in terms of the supremum over all disturbances of the quotient of the energy that is flowing out of the system, and the energy flowing into the system; or in other terms by the H_∞ norm.

1.5. Prerequisites

A number of mathematical preliminaries, results and techniques have been compiled in Appendix A for ease of reference. For an understanding of the materials contained in the chapters to follow the reader is required to have such mathematical preliminaries including an understanding of the description and analysis of dynamical systems in particular. Note that the treatment of these prerequisites are briefly compiled in the appendix, and can be considered as incomplete; in the sense that only the materials used in this thesis are covered. It is, however, assumed that the reader also has the necessary background knowledge in relation to the following alphabetically-sorted topics:

- Advanced linear algebra
- Basics of ordinary and partial differential equations and calculus
- Classical control theory
- Functional analysis
- Linear operators norms theory
- Mathematical analysis
- Matrix algebra
- Robust control theory

1.6. Aims and Structure of this Research

The foremost objectives of this research are:

(i) To solve the nonlinear deterministic robust regulator and/or tracking control problem for the general nonlinear form (1.5) under the presence of exogenous disturbance inputs. Where the affine state-space representation is considered in the form: $\dot{x}(t) = A(x,u)x + B(x,u)u + w(x,t)$.

(ii) To extend the nonlinear H_∞ optimal control problem by means of the approximating sequences of linear-time varying systems and solving the corresponding Riccati equations based on the defined H_∞-norm given some robust performance criteria. More specifically, this objective is met by studying the LTV systems of the form:

$$\dot{x}^{[i]}(t) = A\left(x^{[i-1]}(t), u^{[i-1]}(t)\right)x^{[i]}(t) + B\left(x^{[i-1]}(t), u^{[i-1]}(t)\right)u^{[i]}(t);$$

and applying classical deterministic linear H_∞ theories to these approximations.

(iii) To validate the theories by means of numerical simulations of the controlled responses of some practical nonlinear dynamical systems.

This thesis is structurally divided into five main parts as listed below.

PART I: GENERAL INTRODUCTION

This part constitutes the introductory first *Chapter.*

PART II: ROBUST STABILIZATION

Chapter 2 introduces the reader to some of the basic theoretical robust control techniques for continuous-time and finite-dimensional deterministic systems; where particular attention is nominally placed on the general class of control problems that involve linear time-varying plants. It is recommended, however, to skip to the succeeding chapter provided that the reader is already acquainted with such concepts.

In *Chapter 3*, the deterministic robust control problem is studied for generally perturbed linear time-varying systems. By means of a few realistic mathematical constraints, robust stability is proved while guarantying a robust performance of the devised state-feedback controller. The results are then extended to nonlinear systems. For a clearer insight of the proposed hypothetical nonlinear theory, a nonlinear oscillator example is provided to clarify the stated concepts.

PART III: DETERMINISTIC H_∞ CONTROL

Chapter 4 is devoted to give a deeper technical insight behind the H_∞ control problem for linear time-varying systems. It also serves as an introductory chapter to the continuous-time linear time-varying state-regulator optimal control problem while setting up the basic mathematical results that are needed in subsequent chapters.

Chapter 5 extends some already published concepts, which appeared in the literature for the H_∞ control of semi-linear systems in Hilbert spaces, to nonlinear systems. It turns out that only a full-information Riccati operator equation needs to be solved while completing the square to guarantee stability of the given state-affine disturbed nonlinear system. The renowned classical inverted pendulum on a cart control example is considered as a practical realization of the provided theory.

While in *Chapter 6* the approximation theory and the Approximating Sequence of Riccati Equations are directly applied to extend the *Min-Max* finite-horizon linear time-varying H_∞ control problem to its nonlinear complement. This chapter is in fact a straightforward simple extension that yielded very promising practical results to the considered applications herein.

PART IV: PRACTICAL APPLICATIONS

Although the deterministic robust control theory offers a range of wide applicability to problems from diverse areas of engineering, economics and management science only a few practical examples are included in *Chapter 7* making use of the theories provided in Part II and III. The applications under study involve the stabilization of a magnetic levitation steel ball, the control of a highly nonlinear helicopter model at hovering condition, controlling the wing rock phenomenon including yawing motion, and the stabilization of a hypersonic aircraft about the trim condition. The computer simulated results of the developed theories, when applied to both simple as well as highly complex nonlinear systems, were shown to be equally very effective in providing efficient control signals.

PART V: CONCLUSIONS & FURTHER RESEARCH

The final part of this thesis consists of *Chapter 8* where the reader is presented with a discussion about the contributions, results and propositions given in this research. Also some recommendations are specified as a possible extension to this research.

APPENDICES

While *Appendix A* is devoted to providing some essential background material relevant to this thesis' content and for the reader's convenience; *Appendix B*, introduces a theoretical study into representing nonlinear Ordinary Differential Equations by linear Partial Differential Equations, thus giving another possible approach to robust control of nonlinear systems (see for example [Curtain & Zwart, 1995]).

PART II

ROBUST STABILIZATION

CHAPTER 2

Robust Control of Linear Time-Varying Systems

2.1. Introduction

The mathematical techniques of the classical robust control theory have been elaborately discussed by many authors (see, for example, [Marino & Tomei, 1995; and Zhou, *et al.*, 1996]). The approach of this chapter will be to introduce robust control concepts in a general setting and to summarize some of the main results for the continuous-time linear time-varying systems; for completeness. In §2.2 the robustness analysis control scenario is considered. A summary of some of the classical robust control mathematical methodology for linear time-varying systems is presented and the necessary and sufficient conditions for robustness are also given in §2.3. An account of the robustness control methodologies is presented in §2.4.While in §2.5 the mathematical treatment of the state-regulator robust control problem associated with dynamical systems is defined and summarized.

2.2. Robustness Analysis

In order to define the significance of robustness analysis in control theories a scenario is proposed in example (2.1); which can be found in Stoorvogel (2001).

EXAMPLE 2.1. *Assume a paper-making machine having four inputs: water-diluted wood-pulp, water, pressure and steam. The simplified process consists of having the water pressed out of the mixture to allow for the fibres' web formation to dry on steam-heated cylinders; and where the final product is paper. More precisely, there are two outputs going out of the plant: the thickness of the paper and the mass of the fibres per unit area (indicating the desired paper's quality). Thus the control objective is to have both outputs regulated about some desired values while deviations from such values are* *kept as small as possible to ensure a tolerable paper production to meet the ISO specifications.*

□

The first step is to find a mathematical model describing the dynamical behaviour of the paper-making plant. The second step is to use the classical mathematical tools to find suitable inputs to the plant based on a subset or all output measurements. An inexact or rather simple plant description is inferred to allow for the mathematical manipulations of step 2 to be implemented.

Once these inputs are identified they are applied to the plant and not to mathematical model. But because the mathematically predicted plant behaviour might significantly differ from the actual system response; the inputs will in general not be suitable for the plant and the obtained behaviour might be completely surprising. Hence, it is vital to ensure that the control law is robust vis-à-vis the simplicity and inaccuracy within the mathematical model and the real dynamical model.

This realistic constraint leads to the so-called 'robustness analysis' of the plant and the control action. It is, hence, desirable to ensure that the system stability will tolerate against structured and/or unstructured uncertainties; *i.e.* the robustness of the given plant is maintained over its entire operating range regardless of perturbations.

2.3. The Concept of Robust Control

2.3.1. The General Robust Control Problem

In pure model-based robust control (such as, *e.g.*, the sliding control methodology) the robust controller is designed based on the consideration of both the nominal plant and some characterization of the model uncertainties; and the concept of robust control can be stated as follows: Given the time-varying operator $M \in L\left(\ell^2\right)$ and any bounded set of operators $\mathbf{\Delta} \subset L\left(\ell^2\right)$, consider Figure (2.1) for any $\Delta \in \mathbf{\Delta}$.

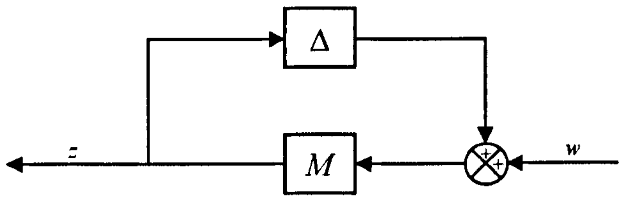


Figure 2.1.: Interconnection of a Nominal Plant, M, with an uncertainty Block, Δ.

The closed-loop map from w to z is given by

$$z = (I - M\Delta)^{-1} Mw. \tag{2.1}$$

Robust stability is related to the existence of the inverse of the $(I - M\Delta)$ term in (2.1) for each $\Delta \in \mathbf{\Delta}$: the nominal requirement is that $(I - M\Delta)$ is invertible. However, if the system is additionally required to be well-defined when any exogenous input enters the uncertainty block then this requirement necessitates the invertibility of $(I - M\Delta)$ on all of ℓ^2; leading to the closely related concepts of uniform stability and robust stability.

DEFINITION 2.1. *An operator M is robustly stable to the bounded set of uncertainty operators, $\mathbf{\Delta}$, if for every $\Delta \in \mathbf{\Delta}$, the inverse, $(I - M\Delta)^{-1}$, exists.*

DEFINITION 2.2. *An operator M is uniformly robustly stable to the bounded set of uncertainty operators, $\mathbf{\Delta}$, if in addition to definition (2.1), (i.e. the operator M is robustly stable), the following holds*

$$\sup_{\Delta \in \mathbf{\Delta}} \left\| (I - M\Delta)^{-1} \right\| < \infty. \tag{2.2}$$

There are numerous robust control definitions and mathematical manipulations to recast the above-mentioned concept. In their paper, Hinrichsen, *et al.*, (1989), for example, the authors introduced the concept of stability radius for time-varying linear systems where invariance properties of the stability radius are analysed for the group of Bohl transformations. The relationship between the stability radius, the norm of a certain perturbation operator, and the solvability of a non-standard differential Riccati equation were also explored in Hinrichsen, *et al.*, (1989).

Another approach to robust control analysis for linear time-varying systems is the operator theoretic line of attack which is self-contained in Feintuch (1998), and where results from the theories of Toeplitz operators and Nest algebras lead to the input-output operators definitions of LTV systems. Robustness in that case is considered from both a fractional representation and a 'time-varying gap' metric viewpoints; but is beyond the scope of this thesis.

2.3.2. The Necessary and Sufficient Conditions for Robustness

The necessary and sufficient conditions for the robustness of linear time-varying systems with structured norm-bounded uncertainty have already been considered in Khammash, (1993); and will be revisited in this section.

The problem of robust stability for this class of systems is studied by considering, M, the part of the system resulting from the interconnection of the nominal plant(s) and controller(s) in Figure (2.2).

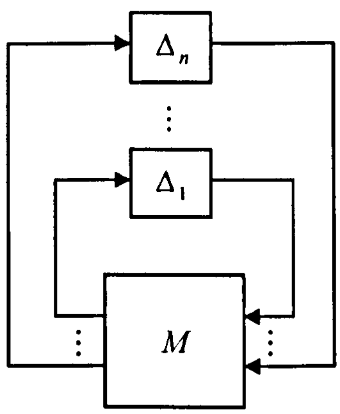


Figure 2.2.: Stability Robustness Problem for Linear Time-Varying Systems.

M is a linear map, but is allowed to be time-varying. It is also assumed to be causal. Connected to M are n perturbation blocks, $\Delta_1,\ldots,\Delta_n$, where each block, Δ_i, is linear and norm bounded (the norm is the induced L_∞ norm). Without loss of generality it is also assumed that $\|\Delta_i\| \leq 1$. Therefore, each Δ_i belongs to the class

$$\Delta := \left\{ \Delta : \Delta \text{ is strictly causal, and } \sup_{u \neq 0} \frac{\|\Delta u\|_\infty}{\|u\|_\infty} \leq 1 \right\}. \tag{2. 3}$$

Following a vector notation, the class of admissible perturbations is

$$\mathscr{D}(n) := \{\Delta = \text{diag}(\Delta_1, \ldots, \Delta_n) : \Delta_i \in \Delta\}. \tag{2. 4}$$

THEOREM 2.1 (Necessary and Sufficient Conditions for Stability Robustness of Time-Varying Systems). *Given an interconnection of a linear time-varying stable system M and n norm-bounded perturbation blocks as in Figure (2.2) over the space of the set of all real valued function on $[0, \infty)$, the following are equivalent:*

1) The system in Figure (2.2) is robustly stable.

2) $(I - M\Delta)^{-1}$ is L_∞-stable for all $\Delta \in \mathscr{D}(n)$.

3) The system of n inequalities: $x_i \leq \left\| M_i^{(T)} X \right\|$, $\quad i = 1, \ldots, n$;

where $X = \text{diag}(x_1, \ldots, x_n)$ has no solution in $(\mathbb{R}^+)^n \backslash \{0\}$ which holds for all $T > 0$.

4) For some $T > 0$, $\sup_t \rho\left(\hat{M}_i^{(T)}\right) < 1$; where $\rho(\cdot)$ denotes the spectral radius and

$$\hat{M}_i^{(T)} := \begin{bmatrix} \left\|M_{11}^{(T)}(t_1)\right\|_{\Im} & \cdots & \left\|M_{1n}^{(T)}(t_1)\right\|_{\Im} \\ \vdots & & \vdots \\ \left\|M_{n1}^{(T)}(t_n)\right\|_{\Im} & \cdots & \left\|M_{nn}^{(T)}(t_n)\right\|_{\Im} \end{bmatrix}$$ *(i.e. belonging to the space $\Im$)*

5) $\inf_{R \in \mathbb{R}} \inf_{T \geq 0} \left\| \rho\left(R^{-1} M^{(T)} R\right) \right\| < 1.$

PROOF: See Khammash (1993).

■

2.4. The Robustness Methodologies

Starting in the early 1960s, the classical approach to the robustness problem, discussed in §2.2 & §2.3, was with the assistance of the Linear Quadratic Gaussian (LQG) theory (see [Stoorvogel, 2001; and Garteur, 1997]). In this approach the uncertainty is added as an extra input to the system and is modelled as a white noise Gaussian process. The major drawback with this approach is that white noise cannot always accurately model this exogenous disturbance.

Although the measurement noise can, in theory, be quite suitably described by a random process, parameter uncertainty can not; in view of the deterministic nature of the error involved. Furthermore, the size of the errors in the parameter uncertainty is relative to the size of the inputs and can only be modelled as an extra input in a non-linear framework_ a fact which adds to the unsuitability of the LQG techniques in dealing with robustness issues.

In the last few years several approaches to robustness have been studied mainly for one goal: to obtain internal stability, where instead of trying to obtain this goal for one system, it is necessary to be fulfilled for a class of systems simultaneously. It is then hoped that a controller which stabilizes all elements of this class of systems also stabilizes the plant itself; leading to the convenient formulation of the post-modern H_∞ robust control theory which resulted in many positive developments in robust control theory.

2.5. Linear Time-Varying Systems with Quadratic Cost

Because an H_∞ optimal control problem can be reduced to the problem of designing a state-regulator for a linear system with quadratic constraints it is essential to cover this regulator problem in this section. The state-regulator problem for a multiple-input multiple-output dynamical system is a special case of the general class of problems where the linear time-varying systems are subject to quadratic costs. In this section, the general problem is stated and the physical motivation behind the choice of the cost function is outlined; then the results for the finite-time linear-quadratic regulator problem are presented. While the tracking problem (servomechanism type of problems) can similarly be considered, it is not covered in this thesis. For the complete proof and a more elaborate discussion on the subject area considered in this section, the reader is referred to any standard textbooks in the field (see *e.g.* [Slotine & Li, 1991]).

First, consider the following system equations for a continuous-time linear time-varying undisturbed system,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad x(t_0) = x_0; \tag{2. 5}$$

$$y(t) = C(t)x(t), \tag{2. 6}$$

where $x(t)$ is the state, $u(t)$ is the control input, $y(t)$ is the measured output, and the system matrices, $A(t)$, $B(t)$ and $C(t)$, are matrices of appropriate dimensions. Let $z(t)$ denote the desired output, and define $e(t)=z(t)-y(t)$ to denote the error.

Define the quadratic scalar performance index or cost functional by,

$$J(u)=\frac{1}{2}\langle e(t_f),Fe(t_f)\rangle+\frac{1}{2}\int_{t_0}^{t_f}\langle e(t),Q(t)e(t)\rangle+\langle u(t),R(t)u(t)\rangle dt, \quad (2.7)$$

where $\langle\cdot,\cdot\rangle$ indicates the inner-product, and t_0 is the initial time and t_f is the final time.

ASSUMPTION 2.1. The weighting matrices F and $Q(t)$ are positive semi-definite.

ASSUMPTION 2.2. The weighting matrix $R(t)$ is positive-definite.

Each term in the cost functional mathematically captures various physical specifications. The first term $\frac{1}{2}\langle e(t),Q(t)e(t)\rangle$ is nonnegative for all $e(t)$ and is zero for $e(t)=0$; implying that the consequence of the quadratic nature penalizes large errors much more severely than small ones. Similarly, $\frac{1}{2}\langle u(t),R(t)u(t)\rangle$ penalizes the system more severely for large control efforts compared to small controls. It is worthy noticing that the control effort is not constrained point-wise in time. While assumption (2.2) on $R(t)$ ensures the physically realistic constraint that the cost of the control effort is always positive for $u(t)\neq 0$. Finally, the so-called terminal-cost, $\frac{1}{2}\langle e(t_f),Fe(t_f)\rangle$, guarantees that at the terminal time t_f, the error $e(t_f)$ is small.

It is now possible to turn to the state-regulator control problem. The solution of this problem leads to an optimal feedback control system. In different terms, this is the property where the state vector components, $x(t)$, are kept as small as possible near zero without excessive expenditure of control effort or energy. In this particular case $C(t)=I$, and the system Equations in (2.5) & (2.6) reduce to,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad x(t_0) = x_0; \quad (2.8)$$

$$y(t) = x(t). \quad (2.9)$$

Because the desired output, or state, is to be maintained at zero then $z(t) = 0$, and $y(t) = x(t) = -e(t)$. The cost function (2.7) then reduces to,

$$J_1(u) = \frac{1}{2}\langle x(t_f), Fx(t_f)\rangle + \frac{1}{2}\int_{t_0}^{t_f}\langle x(t), Q(t)x(t)\rangle + \langle u(t), R(t)u(t)\rangle dt . \quad (2.10)$$

The solution to this problem has been classically obtained by resorting to the methods from the calculus of variation. Conveniently, a scalar function, H, alternatively known as the Hamiltonian, can be defined as follows,

$$H(x(t), u(t), \lambda(t), t) = L(x(t), u(t), t) + \langle \lambda(t), f(x(t), u(t), t)\rangle, \quad (2.11)$$

where L denotes the integrand in the cost function, $\lambda(t)$ can be thought of as a Lagrange-multiplier imposing that dynamical system (2.8) is satisfied point-wise in time when minimizing the cost function; and the right-hand-side of (2.8) is in fact $f(x(t), u(t), t)$ representing the open-loop dynamics of the system. By excluding the dependence of the various terms on time for brevity, the Hamiltonian in (2.11) therefore reduces to,

$$H(x, u, \lambda) = \frac{1}{2}\left[\langle x, Qx\rangle + \langle u, Ru\rangle\right] + \langle \lambda, Ax + Bu\rangle . \quad (2.12)$$

A set of conditions known as the Euler-Lagrange equations can be obtained by considering the variation in J_1 due to deviations in the control law $u(t)$, for fixed initial and final times, t_0 and t_f, and choosing the multiplier functions $\lambda(t)$ such that the coefficients of the variation in x disappear. So these Euler-Lagrange equations are given by,

$$\dot{x}(t) = \frac{\partial H}{\partial \lambda} = Ax + Bu, \quad (2.13)$$

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial \lambda} = -\left(\frac{\partial f}{\partial x}\right)^T \lambda - \left(\frac{\partial L}{\partial x}\right)^T, \quad (2.14)$$

where the control $u(t)$ is determined by the optimal trajectory,

$$\frac{\partial H}{\partial u} = 0. \quad (2.15)$$

The boundary conditions for the Euler-Lagrange equations are split, *i.e.* some are given for $t = t_0$ and some for $t = t_f$. This problem is known as the Two Point Boundary Value Problem (TPBVP) and is expressed as

$$x(t_0) \quad given; \tag{2. 16}$$

$$x(t_0) = Fx(t_f). \tag{2. 17}$$

Most of the difficulty in solving the TPBVP arises from the boundary conditions; but in the case of the state-regulator the problem can be solved and the optimal control can be obtained. The optimal control is unique and is given by,

$$u(t) = -R^{-1}(t)B^T(t)P(t)x(t), \tag{2. 18}$$

where the symmetric $n \times n$ matrix, $P(t)$, is the unique solution of the Riccati equation (2.19) satisfying the boundary conditions, $P(t_f) = F$.

$$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) + P(t)B(t)R^{-1}(t)B^T(t)P(t) - Q(t). \tag{2. 19}$$

It is obvious that the control is a linear time-varying feedback of the state and that the solution of the Riccati equation is independent of the state. By integrating the Riccati equation, that is subject to the boundary condition $P(t_f) = F$, backwards in time, the solution, $P(t)$, of the Riccati equation is obtained. Figure (2.3) represents the block diagram of the optimal tracking configuration where the state-regulator configuration is contained by the dashed box.

THEOREM 2.2 (LQR Optimal Control). *Given the linear system (2.8) and the quadratic cost functional (2.10) where $u(t)$ is unconstrained, the final time, t_f, is specified, F and $Q(t)$ are positive semi-definite, and $R(t)$ is positive-definite; then a linear time-varying optimal state-feedback exists and is unique. This feedback control is given by (2.18) where the $n \times n$ symmetric matrix $P(t)$ is the unique solution of the Riccati equation (2.19). The state of the optimal system then becomes the solution of the linear differential equation:*

$$\dot{x}(t) = \left[A(t) - B(t)R^{-1}(t)B^T(t)P(t)\right]x(t), \qquad x(t_0) = x_0.$$

REMARK 2.1. The optimal cost is a scalar function given by

$$J(x_0, t_0) = \frac{1}{2} x_0^T P(t_0) x_0,$$

which is the solution of the Hamilton-Jacobi-Bellman equation (Athans & Falb, 1966).

■

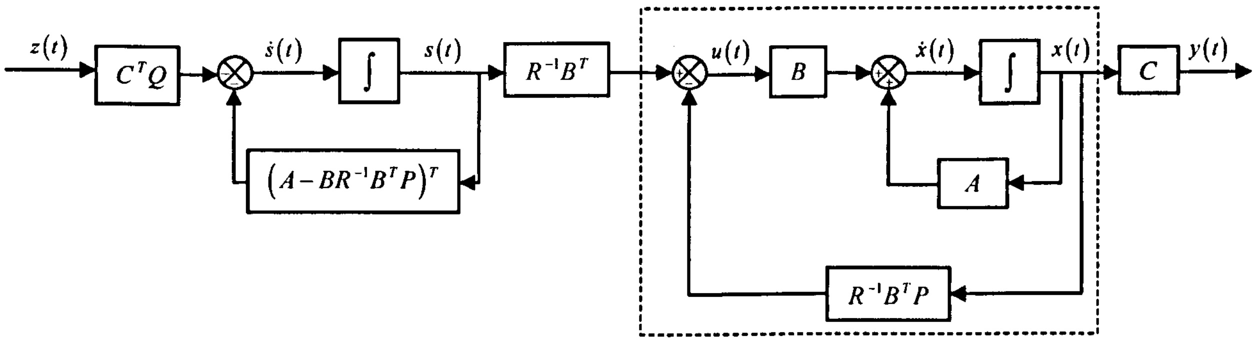


Figure 2.3.: The State-Regulator Control System.

There are, however, certain limitations in the calculus of variations approach discussed above making it unsuitable for application to problems of considerable engineering importance. Such limitations include the differentiability of L that restricts important choices for the cost functional, along with the unsuitability of this approach in handling inputs that are constrained point-wise in time. Accordingly, the Pontryagin's Minimum Principle resolved such limitations (for a mathematical treatment refer to [Athans & Falb, 1966; and Banks, 1986a] and the references therein). It is important to note that the Pontryagin's Minimum Principle, also known as Pontryagin's Maximum Principle, only gives the necessary conditions to be satisfied by the optimal controller; while it does not suggest a method to obtain such a control. In fact, the existence of the optimal controller that would not necessarily be a state feedback is not even guaranteed by the Pontryagin's Minimum Principle.

It is for this reason, along with the difficulty in analytically solving the time-optimal problems with the Pontryagin's Minimum Principle, that it is often preferred to formulate problems for linear time-varying systems with quadratic costs. Not only is this problem mathematically tractable using quadratic costs but it also results in a linear state-regulator optimal feedback controller; and hence very convenient for practical implementations.

CHAPTER 3

Robust Stabilization of Disturbed Nonlinear Plants

3.1. Introduction

The fact that most systems in nature are nonlinear dynamical entities has concerned scientists as well as engineers over the last few decades in an attempt to devise the best control action. Unfortunately, as yet, even though modern control theories, and post-modern control methodologies, have become very sophisticated, there is no one best solution for this problem, and indeed a trade-off will usually be the case.

There has, of course, been a great deal of work on both structured and unstructured uncertainties in the literature (see, for example, [Feintuch, 1998; Stoorvogel, 1992; and Slotine & Li, 1991]); and the stabilization of uncertain dynamical systems has also received lots of attention during the last few decades (see [Francis, 1987; and Petersen & Urgrinovskii, 2000]). However, it is interesting to discern that nonlinearity and time-variations are often ignored in dealing with systems arising in practice. Nonetheless, due to the desire to achieve better quality and accuracy in a wide range of applications, there is an increased interest in including those particular effects when analyzing a system, or when designing controllers and observers. Hence, the main motivation behind this chapter is to consider a different mathematical technique to consider nonlinear time-varying plants in the presence of unstructured disturbances under the roof of the robust regulation concepts that were discussed in the preceding chapter. As it will become clearer, this proposed technique, straightforwardly employs the Approximation theory to guarantee the robust asymptotic stability of the devised feedback control system.

The nonlinear control systems considered in this chapter are modelled via the standard finite-dimensional deterministic ordinary differential equations of this general form:

$$\left.\begin{array}{ll}\dot{x}(t)=f\big(x(t),w(t),u(t)\big); & x(t_0)=x_0,\\ y(t)=h\big(x(t)\big) & \end{array}\right\}, \qquad (3.1)$$

in which $x(t)\in\Re^n$ is the state, x_0 is the initial condition, $u(t)$: $\Re^+\rightarrow\Re^m$ is the control input, $w(t)$: $\Re^+\rightarrow\Re^p$ is the exogenous disturbance input, and $y(t)\in\Re^q$ is the output vector of the controlled variables.

But in order to consider the finite dimensional multivariable systems (3.1) while applying the Approximation theory, the linear time-varying systems will be first considered and stabilized by means of a conventional pole assignment technique. Accordingly, this chapter is organized as follows; in §3.2, the mathematical *modus operandi* is introduced to regulate and control the uncertain linear time-varying plants. While in §3.3, the results are extended to include the more general nonlinear case represented in (3.1). In §3.4, the method is being validated via direct application to a simple example of a nonlinear oscillator; the aim of which is to clarify the previously stated concepts by the help of some simulated results. The chapter then winds up with some conclusions and recommendations in §3.5.

3.2. Control of Linear Time-Varying Uncertain Systems

3.2.1. Eigenstructure Assignment

Since the control methodology that is used in the following sub-section is based on the conventional pole placement by state-feedback techniques; it is essential to give a very brief preface on some of the most important research efforts in this control area. For many years researchers have attempted to generalize the conventional notions of eigenvalues and eigenvectors for linear time-invariant systems to linear time-varying systems. Starting with Wu (1974) who proposed the extended-eigenvalue (X-eigenvalue) and extended-eigenvector (X-eigenvector) notions and where the essence of being 'eigen-' was lost. Richards (1983) gave better understanding of performance and stability of linear periodic time-varying systems; a method that involved Floquet characteristic exponent. Kamen (1988) developed notions on poles and zeros for linear time-varying systems; and Zhu & Morales (1992) introduced a notion of co-eigenvalue.

Tsakalis & Ioannou (1993) extended the pole placement control objective to linear time-varying plants. While an expanded version of the Frobenius form was established for multiple-input multiple-output cases in Valášek & Olgac (1995). Choi, *et al.*, (2001) introduced a novel differential algebraic eigenvalue theory for linear time-varying systems, and proposed an eigenstructure assignment scheme for this class of systems via a differential Sylvester equation. It can be concluded that research efforts in the pole assignment for regulating linear time-varying systems is quite vast and inclusive. However, it turns out that by means of a standard mathematical assumption on the feedback control law, as studied in the succeeding sub-section, a more clear-cut and transparent classical approach to this pole assignment problem of this class of systems is achieved.

3.2.2. Pole Assignment for Uncertain Linear Time-Varying Systems

In the sequel, the general class of continuous-time linear time-varying systems under the presence of disturbances is considered:

$$\dot{x}(t)=A(t)x(t)+B(t)u(t)+w(t), \qquad x(t_0)=x_0; \qquad (3.2)$$

$$y(t)=C(t)x(t).$$

Here $x(t)$ is the state, $u(t)$ is the control input, $y(t)$ is the measured output, and $w(t)$ is the disturbance input. Also the system matrices, $A(t)$, $B(t)$ and $C(t)$, are matrices of appropriate dimensions.

The memory-less linear state-feedback control law is given by,

$$u(t)=-F(t)x(t). \qquad (3.3)$$

By feeding back all the system states to achieve the desired improvement in the system performance, the closed-loop system can be obtained. That is by substituting the feedback law of (3.3) into the general LTV system in (3.2), yielding

$$\dot{x}(t)=\left(A(t)-B(t)F(t)\right)x(t)+w(t). \qquad (3.4)$$

The general state-regulator control system block diagram is given in Figure (3.1).

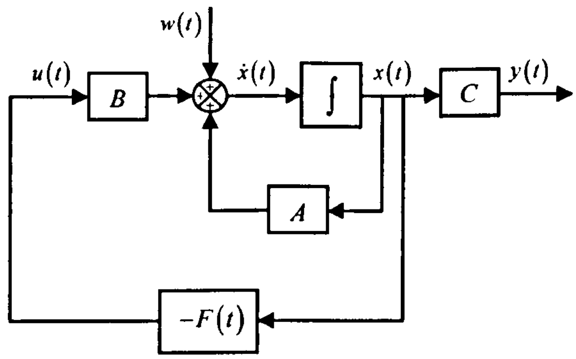


Figure 3.1.: The Robust State-Regulator Configuration.

where the feedback control matrix $F(t)$ can be written as an additive term of both a constant part and a time-varying one:

$$F(t) = F_1 + F_2(t). \qquad (3.5)$$

ASSUMPTIONS: *In order to choose and design the feedback control matrix shown above in (3.5), two assumptions need to be made.*

A.1. The input matrix can be written in this form:

$A(t) = A_1 + A_2(t)$,

A.2. The control matrix can be written in this form:

$B(t) = B_1 + B_2(t)$;

and where (A_1, B_1) is a stabilizable pair.

While assumptions **A.1** & **A.2** hold, the constant part F_1 of the control matrix in (3.5) can be chosen to arbitrarily place the poles of the system following the standard concept of pole assignment methodology.

Following this logical realm of thought, the closed-loop system of (3.4) can be written in this compacted format:

$$\dot{x}(t) = \wp x(t) + \vartheta(t)\, x(t) + w(t), \qquad (3.6)$$

with a linear time-invariant $\wp$,

$$\wp = A_1 - B_1 F_1; \qquad (3.7)$$

and a linear time-varying $\vartheta(t)$,

$$\vartheta(t)=A_2(t)-B_1F_2(t)-B_2(t)F_1-B_2(t)F_2(t). \tag{3. 8}$$

The solution, $x(t)$, of (3.6) is known by means of the variation of constants formula (please refer to *Appendix A*); and is given by,

$$x(t)=e^{\wp t}x_0+\int_0^t e^{\wp(t-\tau)}\left(\vartheta(\tau)x(\tau)+w(\tau)\right)\,d\tau. \tag{3. 9}$$

This solution can be expressed in an alternative normative expression by direct application of the triangle inequality (please refer to *Appendix A*):

$$\|x(t)\|\leq\|e^{\wp t}x_0\|+\int_0^t\|e^{\wp(t-\tau)}\|\left(\|\vartheta(\tau)\|\ \|x(\tau)\|+\|w(\tau)\|\right)d\tau. \tag{3. 10}$$

ASSUMPTION: *The uncertainty* $w(t)$ *satisfies the following assumption*

A.3. given any $\varepsilon>0$, then there exists a $\zeta>0$ such that:

$$\|w(t,x)\|<\varepsilon\|x\|; \tag{3. 11}$$

for all $\|x(t)\|<\zeta$.

REMARK 3.1. Assumption **A.3** is a common and standard assumption in the robust control theory where the disturbance is bounded by the system's states (see *e.g.* [Slotine & Li, 1991]).

Now, suppose that for a given $\mu\geq 1$, the following inequality holds,

$$\|e^{\wp t}\|\leq\mu\, e^{-\omega t}. \tag{3. 12}$$

Then by substituting equations (3.12) & (3.11) in the solution expressed by (3.10), the following norm-bounded inequality is reached,

$$\|x(t)\|\leq\mu\, e^{-\omega t}\ \|x_0\|+\int_0^t\mu\, e^{-\omega(t-\tau)}\left(\|\vartheta(\tau)\|+\ \varepsilon\right)\|x(\tau)\|\ d\tau. \tag{3. 13}$$

However, in order to obtain the best estimate for $\|x(t)\|$, $F_2(t)$ can be chosen to minimize $\|\vartheta(\tau)\|$; *i.e.* by setting:

$$\delta_{F_2}=\min_{F_2(t)}\|\vartheta(t)\|. \tag{3. 14}$$

Recall the measured output equation, $y(t)=C(t)x(t)$, that can alternatively be written as,

$$\|y(t)\| \leq e^{\omega t} \|x(t)\|. \tag{3.15}$$

Fittingly, it can be seen that by making use of (3.13), the output, $y(t)$, in (3.15) is bounded by

$$\|y(t)\| \leq e^{\omega t} \|x(t)\| \leq \mu \|x_0\| + \mu \int_0^t (\|\vartheta(\tau)\| + \varepsilon) e^{\tau} \|x(\tau)\| \, d\tau. \tag{3.16}$$

But $\|\vartheta(\tau)\|$ from (3.14) can be directly substituted in (3.16) while noticing that $y(\tau) \leq e^{\tau} \|x(\tau)\|$,

$$\|y(t)\| \leq \mu \|x_0\| + \mu \int_0^t (\delta_{F_2} + \varepsilon) y(\tau) \, d\tau. \tag{3.17}$$

By applying the Gronwall-Bellman Inequality (please refer to *Appendix A*), (3.17) can be re-written as:

$$\|y(t)\| \leq \mu \|x_0\| \exp\left[\int_0^t \mu (\delta_{F_2} + \varepsilon) \, d\tau\right], \tag{3.18}$$

or the following form is also attained,

$$\|y(t)\| \leq \mu \|x_0\| e^{\mu(\delta_{F_2}+\varepsilon)t}. \tag{3.19}$$

Similarly, the norm of the input signal (3.13) can be manipulated by the Gronwall-Bellman Inequality,

$$\|x(t)\| \leq \mu e^{-\omega t} \|x_0\| + \int_0^t \mu e^{-\omega(t-\tau)} (\delta_{F_2} + \varepsilon) \|x(\tau)\| \, d\tau; \tag{3.20}$$

i.e.

$$\|x(t)\| \leq \mu e^{-\omega t} \|x_0\| \exp\left[\int_0^t \mu e^{-\omega(t-\tau)} (\delta_{F_2} + \varepsilon) \, d\tau\right], \tag{3.21}$$

or

$$\|x(t)\| \leq \mu \|x_0\| e^{(-\omega + \mu(\delta_{F_2}+\varepsilon))t}. \tag{3.22}$$

THEOREM 3.1. *The linear continuous-time dynamical systems in equation (3.2) under the presence of the disturbance* $w(t)$ *in (3.11) and which is subject to the given initial condition* x_0*, is stabilizable provided there exists a constant* F_1 *and a time-varying* $F_2(t)$*; such that* $\omega > \mu(\delta_{F_2} + \varepsilon)$ *if the initial state* x_0 *satisfies* $\mu\|x_0\| < \zeta$.

PROOF: Given that $\mu \geq 1$ (from (3.12)) and $\mu\|x_0\| < \zeta$, it follows that $x_0 \in \beta(0,\zeta)$ in Figure (3.1); in other words, the initial state belongs to the open ball of radius ζ and centre 0.

At this instant, suppose that the solution $x(t)$ does not remain in $\beta(0,\zeta)$. By continuity, there is a first time T when $\|x(T; x_0)\| = \zeta$. By the remarks preceding theorem 3.1, recall that:

$$\|x(t)\| \leq \mu \|x_0\| e^{(-\omega + \mu(\delta_{F_2} + \varepsilon))t},$$

for $t \in [0,T)$. Since $\mu\|x_0\| < \zeta$, this implies that $\|x(t)\| < \zeta$ on $[0,T]$ given that $\omega > \mu(\delta_{F_2} + \varepsilon)$, which is a contradiction. Hence the solution always remains in $\beta(0,\zeta)$, which means that equation (3.22) is true for all t, and asymptotic stability is reached.

■

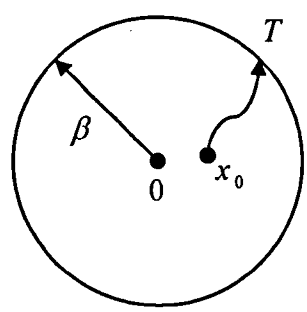


Figure 3.1.: $\beta(0,\zeta)$.

3.3. Control of Uncertain Nonlinear Time-Varying Systems

By making use of the recently introduced Approximation theory that was discussed in the *Chapter 1* and which replaces a nonlinear system by a sequence of linear time-varying approximations, classical linear control techniques can be applied to solve the general nonlinear robust control problem.

In this section, the previous results of §3.2.2 are extended and applied to the general affine nonlinear dynamical systems represented in the state-space factored form:

$$\dot{x}(t) = A(x(t))x(t) + B(x(t))u(t) + w(t), \qquad x(t_0) = x_0. \quad (3.23)$$
$$y(t) = C(t)x(t)$$

The feedback control law is given by,

$$u(t) = -F(x(t))x(t). \quad (3.24)$$

The following sequence of approximations can be introduced (see Banks & McCaffrey, 1998) to the dynamics of generalized plant in (3.23) as follows:

$$\dot{x}^{[i]}(t) = A\left(x^{[i-1]}(t)\right)x^{[i]}(t) + B\left(x^{[i-1]}(t)\right)u^{[i]}(t) + w(t), \text{ with } x^{[i]}(t_0) = x_0. \quad (3.25)$$

Note that,

$$A\left(x^{[i-1]}(t)\right) = A_1 + A_2\left(x^{[i-1]}(t)\right), \quad (3.26)$$

and

$$B\left(x^{[i-1]}(t)\right) = B_1 + B_2\left(x^{[i-1]}(t)\right). \quad (3.27)$$

While the linear control law is given by:

$$u^{[i]}(t) = -F\left(x^{[i-1]}(t)\right)x^{[i]}(t); \quad (3.28)$$

with

$$F\left(x^{[i-1]}(t)\right) = F_1 + F_2\left(x^{[i-1]}(t)\right); \quad (3.29)$$

for $i \geq 0$.

It is essential to note that the factored representation in (3.23) is non-unique and the approximating sequences (3.24) can also accommodate for extra control terms in $A\left(x^{[i-1]}(t), u^{[i-1]}(t)\right)$ and $B\left(x^{[i-1]}(t), u^{[i-1]}(t)\right)$, for instance.

For equations (3.25 – 3.29), the first approximation in these sequences, that is when $i=0$ is given by,

$$\dot{x}^{[0]}(t)=A(x_0)x^{[0]}(t)+B(x_0)u^{[0]}(t)+w(t), \text{ with } x^{[0]}(t_0)=x_0. \tag{3.30}$$

Here, for the first approximation, $x^{[i-1]}(t)$ has been assumed to be x_0 as in Banks & Dinesh (2000). This is indeed the obvious choice given that only the states are available for measurement at the initial time. A second assumption required for the initialization is that $u^{[i-1]}(t)=0$ for $i=0$. However, other values may be chosen depending on available information regarding the initial control.

Now each approximating problem in (3.25) is linear time-varying (with the exception of the first approximation) and quadratic. Hence any classical linear control technique can be used to devise a control law; but for convenience the robust pole assignment discussed in the previous subsection will be followed.

So by redefining,

$$\vartheta(t)=A_2\,(x(t))-B_1F_2\,(x(t))-B_2\,(x(t))F_1-B_2\,(x(t))F_2\,(x(t)); \tag{3.31}$$

and where,

$$\delta_{F_2}=\min_{F_2(x(t))}\|\vartheta(t)\|. \tag{3.32}$$

for each $x(t)\in\beta(0,\zeta)$; then Theorem 3.1 can be directly generalized to prove stability.

THEOREM 3.2. *The nonlinear perturbed dynamical system in equation (3.23) is stabilizable provided there exists a constant F_1 and a time-varying state-dependent $F_2\,(x(t))$ such that $\omega>\mu(\delta_{F_2}+\varepsilon)$ if the initial state x_0 satisfies $\mu\|x_0\|<\zeta$.*

PROOF: This result follows directly by applying Theorem 3.1 to the sequence of linear time-varying systems in equations (3.23 – 3.29).

■

3.4. A Worked Example

3.4.1 Control of a Nonlinear Time-Varying Uncertain Oscillator

In this section a simple example is included to illustrate the effectiveness of this robust control method. However, a different but more practical example of a magnetic levitation ball is considered in chapter 7, §7.1, that makes use of the same proposed methodology.

Suppose that a given nonlinear oscillator under the presence of an exogenous disturbance input $w(t)$ is given by:

$$\ddot{x}(t)+\kappa(x(t))\dot{x}(t)+g(x(t))=u(t)+w(t); \tag{3.33}$$

where

$$\kappa(x(t))=\kappa_1+\kappa_2\ (x(t)). \tag{3.34}$$

To get the phase-plane portrait of the given oscillator in (3.33), the following direct substitutions are used

$$x_1(t)=x(t), \tag{3.35}$$

and

$$x_2(t)=\dot{x}(t); \tag{3.36}$$

alternatively,

$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix}=\begin{pmatrix}0 & 1\\ -3+\alpha(x(t)) & 2+\gamma(x(t))\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}u(t)+\begin{pmatrix}0\\1\end{pmatrix}w(t). \tag{3.37}$$

Where, for simplicity, $B(t)$ is assumed to be constant; this, in fact, occurs in a wide class of real-life systems which only have a nonlinearity in $A(x,u\ ;t)$; *i.e.* $B_2(t)=0$. So $\vartheta(t)$ in (3.31) simplifies to,

$$\vartheta(t)=A_2\ (x(t))-B_1F_2\ (x(t)). \tag{3.38}$$

Recall from (3.26) that

$$A(x(t))=A_1+A_2\ (x(t)). \tag{3.39}$$

Then by comparing equations (3.39) & (3.37), the system matrices can be written as

$$A_1 = \begin{pmatrix} 0 & 1 \\ -3 & 2 \end{pmatrix}, \tag{3.40}$$

and

$$A_2\left(x(t)\right) = \begin{pmatrix} 0 & 0 \\ \alpha\left(x(t)\right) & \gamma\left(x(t)\right) \end{pmatrix}. \tag{3.41}$$

Also recalling from (3.29) that

$$F\left(x(t)\right) = F_1 + F_2\left(x(t)\right); \tag{3.42}$$

then

$$F_1 = \begin{pmatrix} f_1 & f_2 \end{pmatrix}, \tag{3.43}$$

and

$$F_2\left(x(t)\right) = \begin{pmatrix} f_3\left(x(t)\right) & f_4\left(x(t)\right) \end{pmatrix}. \tag{3.44}$$

The state-feedback control law is assumed to be of the form

$$u(t) = -F\left(x(t)\right)x(t); \tag{3.45}$$

or

$$u(t) = -\begin{pmatrix} f_1 + f_3\left(x(t)\right) & f_2 + f_4\left(x(t)\right) \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}. \tag{3.46}$$

Hence from (3.46) & (3.38), $\vartheta(t)$ is,

$$\vartheta(t) = \begin{pmatrix} 0 & 0 \\ \alpha\left(x(t)\right) & \gamma\left(x(t)\right) \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} f_3\left(x(t)\right) & f_4\left(x(t)\right) \end{pmatrix}; \tag{3.47}$$

reducing to

$$\vartheta(t) = \begin{pmatrix} 0 & 0 \\ \alpha\left(x(t)\right) - f_3\left(x(t)\right) & \gamma\left(x(t)\right) - f_4\left(x(t)\right) \end{pmatrix}. \tag{3.48}$$

Intuitively, from (3.48), $\left\|\vartheta(t)\right\| = 0$ by choosing $f_3\left(x(t)\right) = \alpha\left(x(t)\right)$ and $f_4\left(x(t)\right) = \gamma\left(x(t)\right)$.

Consider that the desired poles of the unstable open-loop linear system (A_1, B_1) are λ_1 and λ_2. Then the characteristic equation is expressed by

$$(s - \lambda_1)(s - \lambda_2) = s^2 - (\lambda_1 + \lambda_2)s + \lambda_1\lambda_2. \tag{3.49}$$

The Eigenvalues of the closed-loop state feedback system are roots of

$$\Delta'(\lambda) \triangleq \left| \lambda I-(A_1-B_1F_1)\right| = \left| \lambda I - \begin{pmatrix} 0 & 1 \\ -3 & 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} f_1 & f_2 \end{pmatrix} \right| ; \qquad (3.50)$$

then

$$\Delta'(\lambda) \triangleq s^2-(\lambda_1+\lambda_2)s+\lambda_1\lambda_2 . \qquad (3.51)$$

It follows that

$$f_1 = (\lambda_1\lambda_2)-3 \quad \text{and} \quad f_2 = 2-(\lambda_1+\lambda_2) . \qquad (3.52)$$

Now by setting

$$\tilde{A} = A_1 - B_1 F_1 ; \qquad (3.53)$$

to find a bound on $e^{\tilde{A}t}$, $\tilde{A}$ can be diagonalized,

$$P^{-1}\tilde{A}P = \Lambda . \qquad (3.54)$$

Thus

$$e^{\tilde{A}t} = P e^{\Lambda t} P^{-1} ; \qquad (3.55)$$

i.e.

$$\left\| e^{\tilde{A}t} \right\| \le \|P\| \left\| e^{\Lambda t} \right\| \left\| P^{-1} \right\| , \qquad (3.56)$$

where

$$e^{\Lambda t} = \left\| \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \right\| . \qquad (3.57)$$

So

$$e^{\Lambda t} \le e^{-\min(\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2)t} . \qquad (3.58)$$

Assuming that the uncertainty $w(t,x)$ is bounded by the state, *i.e.* satisfying:

$$\|w(t,x)\| \le \|x(t)\| , \qquad (3.59)$$

for all $\|x\| \le 1$; for example.

Then from the general theory in §3.2.2,

$$\|x(t)\| \le \mu \, \|x_0\| \, e^{(-\omega + \mu(\delta_{F_2} + \varepsilon))t} ; \qquad (3.60)$$

where

$$\mu \|x_0\| < 1 , \qquad (3.61)$$

needs to be satisfied; and where,

$$\mu = \|P\|\|P^{-1}\|. \tag{3.62}$$

REMARK 3.2.

a) If $\min(\mathrm{Re}\lambda_1, \mathrm{Re}\lambda_2) > \|P\|\|P^{-1}\|$, then uniform stability is guaranteed.

b) In the case of a hard constraint on the control, say $|u| < u_{\max}$, then there will be a relationship between the size of the control, the size of the uncertainty and the initial state, since setting $f_3(x(t)) = \alpha(x(t))$ and $f_4(x(t)) = \gamma(x(t))$ might be impossible.

3.4.2. Simulations and Results

Recall the nonlinear oscillator of equation (3.37),

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3+\alpha(x(t)) & 2+\gamma(x(t)) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w(t). \tag{3.63}$$

Then from the discussion in the previous section, §3.4.1, the closed-loop Eigenvalues can be placed at

$$\lambda_1 = -4 \quad \text{and} \quad \lambda_2 = -1. \tag{3.64}$$

And so (3.52) reduces to,

$$f_1 = 1 \quad \text{and} \quad f_2 = 7. \tag{3.65}$$

By choosing:

$$f_3(x(t)) = \alpha(x(t)) = a\cos(x(t)), \tag{3.66}$$

and

$$f_4(x(t)) = \gamma(x(t)) = \frac{b}{1+(x(t))^2}; \tag{3.67}$$

where a and b are constants chosen to be 2 and 5 respectively. Then by substituting for the control law of equations (3.65 – 3.67) in the state-space representation of the nonlinear oscillator (3.63), this alternative form is realized,

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3+\alpha(x(t)) & 2+\gamma(x(t)) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} f_1 & f_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w(t) \tag{3. 68}$$

Consequently, under the presence of white Gaussian noise $w(t)$ and with initial conditions, say, 2 and 2, Figure (3.2) is obtained using MATLAB®. On the shown time interval from 0 to 7 seconds, it is obvious that the system is unstable.

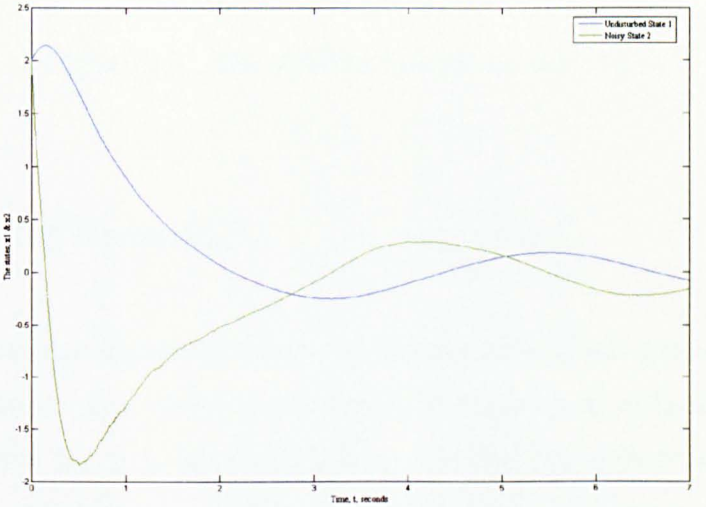


Figure 3.2.: The unstable nonlinear oscillator.

However, the disturbed nonlinear dynamical system can be stabilized by correct choice of the design parameters, f_1 and f_2 which are manipulated by hand. Figure (3.3) shows the controlled nonlinear oscillator for one plausible choice of $f_1 = 5$ and $f_2 = 9$.

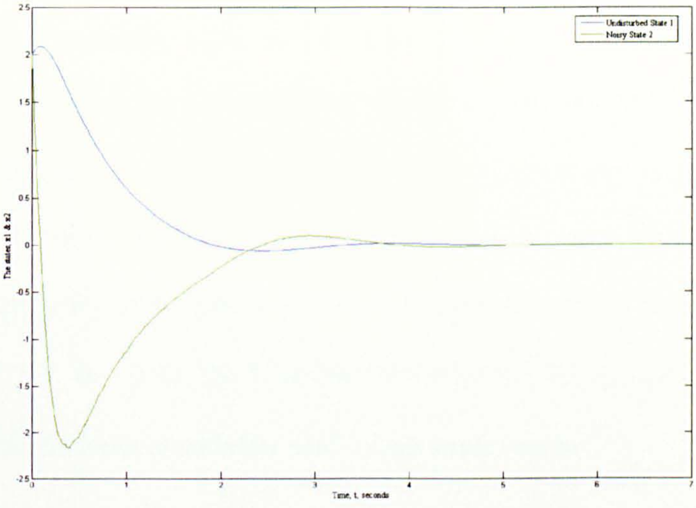


Figure 3.3.: The stabilized nonlinear oscillator.

3.5. Concluding Remarks

It is known that eigenvalues and eigenvectors of a linear transformation (of its matrix representation) play very important roles in the analysis of linear time-invariant dynamical systems; but it is also well known that the eigenvalues of a linear time-varying $A(t)$ do not determine stability of the given dynamical system (Choi, *et al.*, 2001). However, in this chapter, it has been shown that, by means of some standard mathematical inequalities, robust stability can be established for the general class of continuous-time multiple-input multiple-output dynamical systems.

Furthermore, in this chapter the introduced linear time-varying robust stabilization technique was further extended to study nonlinear dynamical systems in the presence of unstructured uncertainty by means of the approximating sequences of linear time-varying problems. The proposed stabilization technique, however, relied heavily on the pole assignment and the eigenvalue notions; and more sophisticated classical control theories (such as the H_∞ control methodology) can relax the proposed assumptions, and yield better performance requirements.

It is to be noted that, as expected, there is a compromise between the control signal, the system's parameters, the size of the disturbance and the poles of the closed-loop system. This technique is simple to implement compared with geometric methods and requires mild conditions such as the local Lipschitz continuity condition. The drawback, however, is that the time-varying nonlinear state-dependent matrix $A(x(t))$ is assumed to be divisible into a constant linear time-invariant part and a nonlinear time-varying one which might also be control dependent (*i.e.* $A(x(t),u(t),t)$); and as a remedy one can apply the approximation theory to include a wider class of systems (*i.e.* $A(x^{[i-1]}(t),u^{[i-1]}(t))$). In short, by bounding the uncertainty, a state-feedback control can be obtained which ensures stability and robust performance.

Part III

Deterministic H_∞ Control

CHAPTER 4

The H_∞ Control Problem: A State-Space Approach

4.1. Technical Introduction

This chapter provides some preliminary but well-established results for the classical linear dynamical plants which follow the conventional modern state-space description,

$$\left.\begin{aligned} \dot{x}(t) &= A(t)x(t)+B(t)u(t)+E(t)w(t), \qquad x(t_0)=x_0, \\ z(t) &= C_1(t)x(t)+D_{12}(t)u(t)+D_{11}(t)w(t), \\ y(t) &= C_2(t)x(t)+D_{22}(t)u(t)+D_{21}(t)w(t); \end{aligned}\right\} \tag{4. 1}$$

in which $x(t)\in\mathbb{R}^n$ is the state, x_0 is the initial condition of the system, $u(t):\mathbb{R}^+\to\mathbb{R}^m$ is the control input, $w(t):\mathbb{R}^+\to\mathbb{R}^p$ is the exogenous disturbance input, $y(t)\in\mathbb{R}^q$ is the measured (or sensor) outputs, and $z(t)\in\mathbb{R}^q$ is the regulated outputs and sometimes called a penalty variable which may include a tracking error; *i.e.* $z(t)$ is the difference between the actual plant output and its desired reference behaviour, expressed as a function of some of the exogenous variables $w(t)$, as well as a cost of the input $u(t)$ needed to achieve the prescribed control goal. The system matrices, in the quadruple representation, $(A,B,C_2 \text{ and } D_{22})$, are assumed to have entries that are continuous functions of time; with a stabilizable (A,B), and detectable (C_2,D_{22}).

A few **remarks** on the feedthrough, or throughput, matrix, $D_{22}(t)$, appearing in the measurement output equation, $y(t)$:

- Often the feedthrough happens to be a zero matrix as it is common in any given physical plant.
- However, adding a feedthrough term to the truncated finite-dimensional mathematical and physical model compensates for the neglected dynamics in this particular model, which is important in ensuring the stability of the closed-loop response of the system.
- In fact, this direct feedthrough term is related to the 'non-dynamic' variables of the system as pointed out in Verghese, *et al.*, (1981); and is usually incorporated for robust H_∞ performance requirements, if desired.

As concisely described in Chapter 1, the H_∞ techniques are devoted to the robust stabilization and control of systems affected by bounded energy inputs; *i.e.* by the disturbances $w(t)$. Realistically, the perturbations affecting any real plants are mathematically modelled as either additive perturbations; or multiplicative perturbation at the input or output and are schematically represented in Figures (4.1) & (4.2) respectively; along with perturbations in the realization of the system. Note that the multiplicative perturbations at the input refer to sensor(s) uncertainty while the multiplicative perturbations at the output suggest actuator(s) uncertainty.

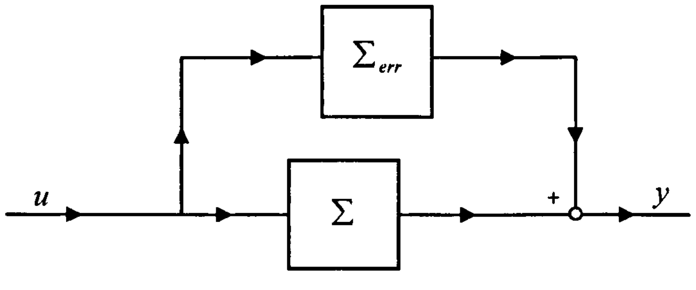


Figure 4.1.: Additive Perturbations.

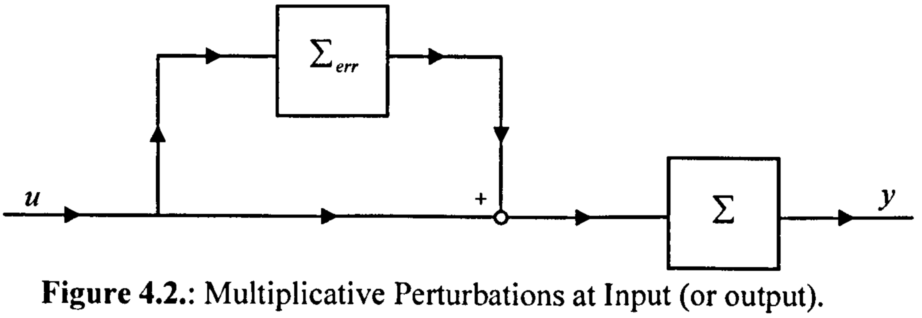


Figure 4.2.: Multiplicative Perturbations at Input (or output).

In the above interconnections in Figure (4.1) & (4.2), Σ_{err} is some arbitrary system representing the uncertainty affecting the original dynamical plant, Σ, and is by definition, unknown. Note that the unstructured uncertainty is only characterized by an upper bound on its magnitude, but no detailed information on its origin from the different plant parameters is available. This implies that no information is available about the form of the perturbation matrix.

Each of these three types of uncertainties can be cast as an H_∞ control problem (see [MacFarlane & Glover, 1990; Vidyasagar, 1985; and Hinrichsen & Pritchard, 1990]). In general multiplicative perturbations as well as perturbations in the realization of a plant often result in the so-called *singular* H_∞ control problems. *Singular* problems simply mean that the feedthrough matrix between the control input and the controlled output, $D_{22}(t)$, in not full column rank; in this case the original finite-horizon state-feedback H_∞ control problem for linear time-varying systems, for instance, is equivalent to another H_∞ problem related to a *reduced order* system. In their paper, Amato, *et al.*, (2000) suggested an iterative reduction procedure to render the matrix full column rank; and where the trivially *reduced order* system is non-*singular*, or regular, and can be solved by standard techniques.

Once an H_∞- norm bound has been decided, provided a solution exists, the computational burden associated with finding all H_∞ controllers is essentially the same as that required in solving the linear quadratic Gaussian regulator problem discussed in §2.5. As in the deterministic optimal control theory, H_∞ control problems in which perfect information is assumed may be solved using a single Riccati equation with dimension equal to that of the original system; while the output-feedback problem often requires the solution of two Riccati equations; known as the 'two Riccati equation' formula (Limebeer, *et al.*, 1992).

However, in this thesis only regular deterministic state-feedback H_∞ problems are considered since singular problems can in principle be reduced to the former general class of problems. Accordingly, this chapter is organized as follows, in §4.2, the H_∞ finite-horizon case is summarized for the finite-dimensional linear time-varying continuous-time plants. Also a few practical examples from the literature that make use of the established H_∞ methodologies are revealed in §4.3.

4.2. The Finite-Horizon H_∞ Problem

The classical, relatively simple, ideas of the Linear Quadratic optimization were used in a time-domain treatment of the standard H_∞ problem in Tadmor (1993); however, to clarify the standard H_∞ problem, the finite-horizon case under the same LQ concepts, as in Tadmor (1990), is revisited in the discussion to follow. For a more elaborate discussion with further technical details the reader is referred to Tadmor (1990) where the author treats both the infinite-horizon time-invariant case and the finite-horizon time-varying one under the generic Linear Quadratic optimal control approach.

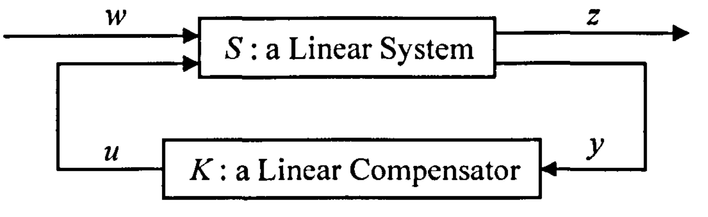


Figure 4.3.: Pictorial Description of the Standard Problem.

Given the pictorial set-up of Figure (4.3), the designer's goal is to minimize the closed-loop effect of the disturbances w on the output z by an appropriately chosen compensator K. This impact is measured in terms of the induced operator norm relative to L_2 signal norms. In other words, the closed-loop stability must be satisfied by means of an optimal γ_0,

$$\gamma_0 = \min_K \max_{w \in L_2} \frac{\|z\|}{\|w\|}. \tag{4.2}$$

Rigorously, (4.2) can be expressed alternatively by the following definition, given $\gamma \in \mathbb{R}$,

$$J_\gamma(x_0, w, u) = \gamma^2 \|w\|^2 - \|z\|^2, \tag{4.3}$$

where x_0 is the initial state in S and z is the output trajectory corresponding to the initial state, w is the disturbance and u is the control input. Note that J_γ in (4.3) is a quadratic form of its three variables.

REMARK 4.1. $\gamma > \gamma_0$ if and only if there exists some internally stabilizing compensator K such that $J_\gamma(x_0 = 0, w, u = Ky)$ becomes a uniformly positive definite form in w, such that,

$$J_\gamma(x_0 = 0, w, u = Ky) \geq \delta^2 \|w\|^2, \tag{4. 4}$$

for some fixed $\delta \neq 0$ and all $w \in L_2$.

This can be written as a MinMax problem,

$$\underset{w \in L_2}{\text{Min}} \, \underset{u \in L_2}{\text{Max}} \, J_\gamma \, (x_0, w, u), \tag{4. 5}$$

where good controls increase J_γ while bad disturbances penalize it.

Let T be a bounded operator on L_2, having a linear time-varying admissible feedback operators that can be realized as input-output mapping of the following linear systems, over $[t_0, t_1]$,:

$$\dot{P} = MP + Nf, \tag{4. 6}$$

$$g = QP + Rf; \tag{4. 7}$$

where $M, N, Q,$ and, R are L_∞ matrices.

ASSUMPTIONS: Recall the dynamical system (4.1) that is restricted to some finite-time interval $[t_0, t_1]$ where without loss of generality the following assumptions hold:

A(i) $D_{22} = 0$,

A(ii) $D_{11} = 0$,

A(iii) $D'_{12} \, [D_{12}, C_1] = [0, I]$,

A(iv) $D_{21}[E', D'_{21}] = [0, I]$,

A(v) $C'_1 C_1 \geq \varepsilon_1 I$ for some $\varepsilon_1 > 0$,

A(vi) $E'E \geq \varepsilon_2 I$ for some $\varepsilon_2 > 0$.

THEOREM 4.1 (Finite-Horizon, Time-Varying Case, Tadmor (1990)).

(i) *The value $\gamma > 0$ is strictly suboptimal in (4.1) over $[t_0, t_1]$ if and only if there exists uniformly bounded, negative definite solutions, P_1 and P_2, to the following two dynamic matrix Riccati equations:*

$$\dot{P}_1 = C'_1 C_1 - P_1 A - A' P_1 - P_1 \left(BB' - \frac{1}{\gamma^2} EE' \right), \qquad P_1(t_1) = 0; \qquad (4.8)$$

$$\dot{P}_1 = -BB' + P_2 \left(-\frac{1}{\gamma^2} EE'P_1 \right)' + \left(A - \frac{1}{\gamma^2} EE'P_1 \right) P_2 + P_2 \left(C'_2 C_2 - \frac{1}{\gamma^2} P_1 BB' P_1 \right) P_2$$

$$P_2(t_0) = 0; \qquad (4.9)$$

(ii) *An admissible compensator assures the closed-loop norm bound $\|T_K\| < \gamma$ if and only if it can be realized in the form:*

$$\dot{P} = (A_1 + P_2 C'_2 C_2) P - \left(I + \frac{1}{\gamma^2} P_2 P_1 \right) Bv + P_2 C'_2 y, \qquad P(t_0) = 0; \quad (4.10)$$

$$A_1 = A + \left(BB' - \frac{1}{\gamma^2} EE' \right) P_1, \qquad (4.11)$$

$$q = C_2 P + y, \qquad (4.12)$$

$$v = K_0 q, \qquad (4.13)$$

$$u = -B' P_1 P + v, \qquad (4.14)$$

where K_0 is an admissible feedback operator with $\|K_0\| < \gamma$.

(iii) *If the system's state is available; i.e., if Assumptions A(iv) and A(vi) are replaced by "$C_2 = I$ and $D_{11} = 0$", then γ is strictly suboptimal if and only if P_1 exists, as claimed, in which case the state feedback:*

$$u = B' P_1 x; \qquad (4.15)$$

assures $\|K_0\| < \gamma$ in a closed loop.

PROOF: See Tadmore (1990).

■

4.3. A Few Examples of Application

This section is intended to simply reveal a few practical applications of the H_∞ control methodologies that the reader might find of significance. Without doubt the pool of applications is quite vast, and this section is by far restricted given that it only specifies a small selection of the many physical examples considered in the literature over the last few decades.

- Balas, *et al.*, (1993) present how to use the H_∞ methodology for the Analysis and Synthesis of a controller for the longitudinal dynamics of an A/C, when using the MIMO H_∞ Direct Problem. Also, they present how to synthesize a lateral-directional controller for a Space Shuttle using the Standard H_∞ Optimization Problem.
- Biss & Woodgate (1990) provide an H_∞ control synthesis for a gas turbine.
- Grimble (2001) considers a variety of practical advanced industrial control systems including power generation and transmission, metal processing, marine control, and aero-engines and flight control designs.
- Jung, *et al.*, (2005) present a detailed investigation on the effect of the uncertainty parameterization type and the performance of H_∞ robust controllers for diesel engine air-path control.
- Li, *et al.*, (1992) present the methodology to work the mixed sensitivity problem as a standard disturbance rejection H_∞ optimization problem.
- Lin (1994) provides useful examples about the application of the H_∞ methodologies for a pitch autopilot design, a roll-yaw autopilot design, a helicopter flight control system, and an integrated flight control system.

- Maciejowski (1989) provides also the synthesis of a flight control system using the H_∞ methodology, and the results are compared with those obtained using the LQG/LTR technique.
- Marcos, *et al.*, (2005) applied an H_∞ Fault detection and isolation to the longitudinal motion of a LTI model of a Boeing 747-100/200 aircraft.
- McFarlane & Glover (1990) consider both a vertical plane dynamics of an aircraft and an attitude control of a flexible space platform using the H_∞ control techniques.
- Postlethwaite & Skogestad (1993) consider a case study of an advanced H_∞ control of high performance helicopters with design objectives and handling quality assessments.
- Ruiz-Velazquez, *et al.*, (2004) provide the robust tracking H_∞ problem for blood glucose control for *type I* diabetes mellitus.
- Safonov, *et al.*, (1988) & (1991) implement an H_∞ control synthesis for a large space structure.
- Safonov & Chiang (1988) and Safonov, *et al.*, (1981) present an aircraft autopilot design.
- Van Crevel (1989) provides a control design for a 90 MW coal fired fluidized bed boiler.

CHAPTER 5

Nonlinear H_∞ Control in Hilbert Spaces

5.1. Introduction

Methodically speaking, the main objective of the H_∞ control problem is to construct a filter that guarantees the optimization of the H_∞-norm from the exogenous uncertainty input to the filtered error output as previously signalized in different terms. Since the early 80's considerable effort took place to extend the H_∞ control concepts and objectives to robustly stabilize the general description of affine nonlinear dynamical uncertain systems. With various rigorous and dogmatic mathematical theories covering finite-dimensional control systems, such as linear matrix inequalities or algebraic/differential Riccati-like equations, it seems that the infinite-dimensional nonlinear H_∞ control problem is still under impelling scrutiny since this problem is, by nature, theoretically complex and requires advanced techniques from the semigroup theory (Phat, 2003).

However, in this chapter, a novel approach is studied for devising control action for infinite-dimensional nonlinear uncertain systems. The work herein rigorously employs the simple and very effective Approximating Sequences of Riccati Equations (ASRE) technique to further extend the standard modern Riccati-based linear time-varying H_∞ control. The introduced sets of decoupled linear time-varying systems and Riccati operator equations enable the usage of standard linear H_∞ control methods to robustly stabilize the general class of disturbed continuous-time state-affine nonlinear dynamical systems, where the choice of γ ensures that the closed-loop H_∞-norm is minimized in the energy sense.

The theoretical framework in this chapter builds on the bilinear setting that was proposed in Phat (2003) to obtain local solutions to the nonlinear H_∞ control problem in infinite-dimensional spaces by means of a linearization argument. That is to say that only the linear time-varying state-space H_∞ controller formulae is expanded while relying on solving a full-information Riccati operator equation and completing the square, *i.e.* a mathematically standard approach as it first appeared in Khargonekar, *et al.*, (1990) and Khargonekar, *et al.*, (1988).

Firstly, a problem statement for the state-affine nonlinear H_∞ methodology that is used in the sequel is presented in §5.2. In §5.3, the linear time-varying H_∞ controller is derived based on the Riccati operator equation in the Hilbert space. A theorem is presented that ensures the robust stabilization of this class of continuous-time linear time-varying systems. In §5.4 the linear theory is extended to include nonlinear disturbed dynamical plants by means of the ASRE technique; it also includes an expanded theorem that ensures the robust stabilization of the given plants. A global convergence Lemma is also considered for the robust nonlinear H_∞ optimal control problems. A practical example of an inverted pendulum on a cart is considered in §5.5.1 followed by some simulated results in §5.5.2. Finally, some concluding remarks are given in §5.6.

5.2. Problem Statement

Consider a continuous-time state-affine nonlinear uncertain dynamical system having the following form:

$$\dot{x}(t)=A(x(t))x(t)+B(x(t))u(t)+E(x(t))w(t); \qquad x(t_0)=x_0; \qquad (5.1)$$

$$z(t)=C(x(t))x(t)+D(x(t))u(t); \qquad (5.2)$$

$\forall t \in \Re^+$.

Here $x(t)\in X$ is the state, $u(t)\in U$ is the control, $w(t)\in W$ is the input disturbance and $z(t)\in Z$ is the observation output; with X, U, Z & W being real Hilbert spaces (observe that although the theory is valid for infinite-dimensions, the operators can also be considered as real Euclidean spaces of appropriate dimensions for finite-dimensional problems).

Then the H_∞ control problem is as follows:

Given a scalar $\gamma > 0$, find a linear feedback control law $u(t) = -B^*(t)P(t)x(t)$ such that:

1. *The given disturbed nonlinear system is robustly stabilizable;*
2. *There exists a scalar* $c_0 > 0$ *such that*

$$\sup \frac{\int_0^\infty \|z(t)\|^2 dt}{c_0 \|x(0)\|^2 + \int_0^\infty \|w(t)\|^2 dt} \leq \gamma; \qquad (5.3)$$

with the supremum taken over all $x_0 \in X$ and all non-zero admissible disturbances $w(t)$.

5.3. Robust H$_\infty$ Control of Linear Time-Varying Uncertain Systems

In this section, an idea introduced by Phat (2003), is extended to design an H_∞ controller in a Hilbert space (see also [Phat, 2004; and Phat, 2001]). Initially, the following general dynamical linear system in the state-space representation is considered:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + E(t)w(t); \qquad (5.4)$$

with the observation output given by:

$$z(t) = C(t)x(t) + D(t)u(t). \qquad (5.5)$$

The unstructured disturbance $w(t)$ is defined by $w \in L_2([0,\infty), W)$.

ASSUMPTION:

A.1. *The functions:* $B(\cdot)u$, $E(\cdot)w$, $C(\cdot)x$ *and* $D(\cdot)u$ *are bounded and defined by* $b = \sup_{t\in R^+}\|B(t)\|$, $e = \sup_{t\in R^+}\|E(t)\|$, $c = \sup_{t\in R^+}\|C(t)\|$ *and* $d = \sup_{t\in R^+}\|D(t)\|$ $\{\exists b,e,c,d > 0\}$.

The linear time-varying state-feedback control law is given by,

$$u(t) = -B^*(t)P(t)x(t); \qquad (5.6)$$

with the operator $(.)^*$ referring to the adjoint.

Now consider the Riccati operator equation (ROE):

$$\dot{P}(t) = -A^*(t)P(t) - P(t)A(t) + P(t)B(t)B^*(t)P(t) - Q(t), \tag{5.7}$$

with $P(t)$ being the solution of the ROE defined by $p = \sup_{t\in R^+} \|P(t)\|$; and where

$$Q(t) = C^*(t)C(t) + I. \tag{5.8}$$

Letting

$$V = \langle P(t)x(t), x(t) \rangle, \tag{5.9}$$

with the inner product $\langle \cdot,\cdot \rangle$ being defined over a complex or real field F as a map $\langle \cdot,\cdot \rangle : X \times X \to \mathrm{F}$.

Differentiating (5.9),

$$\dot{V}(t, x(t)) = \langle \dot{P}(t)x(t), x(t) \rangle + 2\langle P(t)\dot{x}(t), x(t) \rangle; \tag{5.10}$$

Expanding (5.10) by substituting Equations (5.7) & (5.4),

$$\begin{aligned}\dot{V}(t, x(t)) = &\Big\langle \big(-A^*(t)P(t) - P(t)A(t) + P(t)B(t)B^*(t)P(t) - Q(t)\big)x(t), x(t) \Big\rangle \\ &+ 2\big\langle P(t)\big(A(t)x(t) + B(t)u(t) + E(t)w(t)\big), x(t) \big\rangle.\end{aligned} \tag{5.11}$$

Alternatively, (5.11) simplifies to,

$$\begin{aligned}\dot{V}(t, x(t)) = &\big\langle -A^*(t)P(t)x(t) - P(t)A(t)x(t) + P(t)B(t)B^*(t)P(t)x(t) \\ &- Q(t)x(t), x(t) \big\rangle + 2\big\langle P(t)A(t)x(t) + P(t)B(t)u(t) + P(t)E(t)w(t), x(t) \big\rangle.\end{aligned} \tag{5.12}$$

By substituting the control law (5.6) in (5.12), the following equality is obtained,

$$\begin{aligned}\dot{V}(t, x(t)) = &\big\langle -A^*(t)P(t)x(t) - P(t)A(t)x(t) + P(t)B(t)B^*(t)P(t)x(t) \\ &- Q(t)x(t), x(t) \big\rangle + \big\langle 2P(t)A(t)x(t) - 2P(t)B(t)B^*(t)P(t)x(t) \\ &+ 2P(t)E(t)w(t), x(t) \big\rangle.\end{aligned} \tag{5.13}$$

However, (5.13) can be further expanded,

$$\begin{aligned}\dot{V}(t, x(t)) = &\big\langle -\big(A^*(t)P(t) + P(t)A(t)\big)x(t), x(t) \big\rangle \\ &+ \big\langle P(t)B(t)B^*(t)P(t)x(t), x(t) \big\rangle - \big\langle Q(t)x(t), x(t) \big\rangle + \big\langle 2P(t)A(t)x(t), x(t) \big\rangle \\ &- \big\langle 2P(t)B(t)B^*(t)P(t)x(t), x(t) \big\rangle + \big\langle 2P(t)E(t)w(t), x(t) \big\rangle.\end{aligned} \tag{5.14}$$

By making use of $P(t)=P^{*}(t)$ and substituting (5.8), (5.14) reduces to the following equality,

$$\dot{V}(t,x(t))=-\langle(C^{*}(t)C(t)+I)x(t),x(t)\rangle-\langle P(t)B(t)B^{*}(t)P(t)x(t),x(t)\rangle \\ +\langle 2P(t)E(t)w(t),x(t)\rangle. \tag{5.15}$$

By re-arranging (5.15),

$$\dot{V}(t,x(t)) =-\langle Ix(t),x(t)\rangle-\langle P(t)B(t)B^{*}(t)P(t)x(t),x(t)\rangle-\langle C^{*}(t)C(t)x(t),x(t)\rangle \\ +\langle 2P(t)E(t)w(t),x(t)\rangle. \tag{5.16}$$

By a further simplification,

$$\dot{V}(t,x(t)) =-\|x(t)\|^{2}-\langle P(t)B(t)B^{*}(t)P(t)x(t),x(t)\rangle-\langle C^{*}(t)C(t)x(t),x(t)\rangle \\ +\langle 2P(t)E(t)w(t),x(t)\rangle. \tag{5.17}$$

Remark 5.1.

$\langle C^{*}(t)C(t)x(t),x(t)\rangle$ and $\langle P(t)B(t)B^{*}(t)P(t)x(t),x(t)\rangle$ are positive-definite since,

$$\left.\begin{aligned} &\langle C^{*}(t)C(t)x(t),x(t)\rangle=\langle C(t)x(t),C(t)x(t)\rangle=\|C(t)x(t)\|^{2}\geq 0,\\ &\text{and}\\ &\langle P(t)B(t)B^{*}(t)P(t)x(t),x(t)\rangle=\langle B^{*}(t)P(t)x(t),B^{*}(t)P^{*}(t)x(t)\rangle\\ &=\|B^{*}(t)P(t)x(t)\|^{2}\geq 0 \end{aligned}\right\} \tag{5.18}$$

So recalling (5.17) and substituting (5.18),

$$\dot{V}(t,x(t))\leq-\|x(t)\|^{2}+\langle 2P(t)E(t)w(t),x(t)\rangle; \tag{5.19}$$

$$\dot{V}(t,x(t))\leq-\|x(t)\|^{2}+2\|P(t)E(t)w(t)\|\ \|x(t)\|; \tag{5.20}$$

$$\dot{V}(t,x(t))\leq-\|x(t)\|^{2}+2\|P(t)\|\ \|E(t)\|\ \|w(t)\|\ \|x(t)\|, \tag{5.21}$$

conversely (5.21) reduces to,

$$\dot{V}(t,x(t)) \leq -\|x(t)\|^2 + 2pe\|w(t)\| \, \|x(t)\|. \tag{5.22}$$

Integrating both sides of (5.22) from 0 to t, yields,

$$\int_0^t \dot{V}(s,x(s))\, ds \leq \int_0^t -\|x(s)\|^2 ds + \int_0^t 2pe \, \|w(s)\| \, \|x(s)\| \, ds, \tag{5.23}$$

or

$$\langle P(t)x(t),x(t)\rangle - \langle P(0)x(0),x(0)\rangle \leq -\delta_1 \int_0^t \|x(s)\|^2 ds + 2\delta_2 \int_0^t \|w(s)\| \, \|x(s)\| \, ds; \tag{5.24}$$

with $\delta_1 = 1$ and $\delta_2 = pe$.

REMARK 5.2. It follows from the definition of the $\|\cdot\|$ that

$$\int_0^t \|w(s)\| \, \|x(s)\| \, ds \leq \left\{ \int_0^t \|w(s)\|^2 \, ds \right\}^{1/2} \left\{ \int_0^t \|x(s)\|^2 \, ds \right\}^{1/2}. \tag{5.25}$$

Given that $w \in L_2([0,\infty),W)$, then by making use of remark (4.2), Equation (5.24) reduces to the subsequent inequality,

$$\int_0^t \|x(s)\|^2 ds \leq \frac{2\delta_2}{\delta_1} \int_0^t \|w(s)\| \, \|x(s)\| \, ds - \frac{1}{\delta_1}\langle P(t)x(t),x(t)\rangle + \frac{1}{\delta_1}\langle P(0)x(0),x(0)\rangle. \tag{5.26}$$

By making two definitions as follows,

(i) $$\delta_3 = \frac{1}{\delta_1}\langle P(0)x(0),x(0)\rangle, \tag{5.27}$$

and

(ii) $$\delta_4 = \frac{\delta_2}{\delta_1} \left\{ \int_0^\infty \|w(s)\|^2 \, ds \right\}^{1/2}; \tag{5.28}$$

then, (5.26) can be re-written as,

$$\int_0^t \|x(s)\|^2 ds \leq \delta_3 + 2\delta_4 \left\{ \int_0^t \|x(s)\|^2 \, ds \right\}^{1/2} - \frac{1}{\delta_1}\langle P(t)x(t),x(t)\rangle. \tag{5.29}$$

By setting,

$$\alpha = \left\{ \int_0^t \|x(s)\|^2 \, ds \right\}^{1/2}; \tag{5.30}$$

and substituting it back in (5.29),

$$\alpha^2 \le \delta_3 + 2\delta_4 \alpha - \frac{1}{\delta_1} \langle P(t)x(t), x(t) \rangle. \tag{5.31}$$

Remark 5.3. It is clear that

$$\langle P(t)x(t), x(t) \rangle = \left\langle \sqrt{P(t)}x(t), \sqrt{P(t)}x(t) \right\rangle = \left\| \sqrt{P(t)}x(t) \right\|^2 \ge 0. \tag{5.32}$$

By making use of remark (4.3), the inequality in (5.31) reduces to,

$$\alpha^2 - 2\delta_4 \alpha \le \delta_3. \tag{5.33}$$

So by completing the square,

$$(\alpha - \delta_4)^2 - \delta_4{}^2 \le \delta_3, \tag{5.34}$$

or equivalently,

$$\alpha^2 \le \delta_4 + \sqrt{\delta_3 + \delta_4{}^2}. \tag{5.35}$$

Recall the definition of α in (5.30) then (5.35) is in fact,

$$\int_0^t \|x(s)\|^2 \, ds \le \delta_4 + \sqrt{\delta_3 + \delta_4{}^2}, \qquad \forall t \in R^+. \tag{5.36}$$

Now consider the following equality:

$$\int_0^\infty \left[\|z(t)\|^2 - \gamma \|w(t)\|^2 \right] dt = \int_0^\infty \left[\|z(t)\|^2 - \gamma \|w(t)\|^2 + \dot{V}(t, x(t)) \right] dt - \int_0^\infty \dot{V}(t, x(t)) dt. \tag{5.37}$$

Equally,

$$\begin{aligned} \int_0^\infty \left[\|z(t)\|^2 - \gamma \|w(t)\|^2 \right] dt &= \int_0^\infty \left[\|z(t)\|^2 - \gamma \|w(t)\|^2 + \dot{V}(t, x(t)) \right] dt \\ &- \langle P(t)x(t), x(t) \rangle + \langle P(0)x(0), x(0) \rangle. \end{aligned} \tag{5.38}$$

Since the initial condition $P(0)$ is chosen such that $P(0) \ne 0$, Equation (5.38) can be written as,

$$\int_0^\infty \left[\|z(t)\|^2 - \gamma \|w(t)\|^2 \right] dt \le \int_0^\infty \left[\|z(t)\|^2 - \gamma \|w(t)\|^2 + \dot{V}(t, x(t)) \right] dt + \langle P(0)x(0), x(0) \rangle. \tag{5.39}$$

Also recall that the closed-loop state-space representation of the observation output in (5.5) is

$$z(t) = C(t)x(t) - D(t)B^*(t)P(t)x(t). \tag{5.40}$$

Therefore,

$$\|z(t)\|^2 = \langle C(t)x(t) - D(t)B^*(t)P(t)x(t), C(t)x(t) - D(t)B^*(t)P(t)x(t) \rangle. \tag{5.41}$$

By expanding (5.41),

$$\|z(t)\|^2 = \langle C(t)x(t), C(t)x(t) \rangle - \langle C(t)x(t), D(t)B^*(t)P(t)x(t) \rangle - \langle D(t)B^*(t)P(t)x(t), C(t)x(t) \rangle + \langle D(t)B^*(t)P(t)x(t), D(t)B^*(t)P(t)x(t) \rangle, \tag{5.42}$$

or

$$\|z(t)\|^2 = \langle C^*(t)C(t)x(t), x(t) \rangle - \langle x(t), C^*(t)D(t)B^*(t)P(t)x(t) \rangle - \langle C^*(t)D(t)B^*(t)P(t)x(t), x(t) \rangle + \langle P(t)B(t)D^*(t)D(t)B^*(t)P(t)x(t), x(t) \rangle. \tag{5.43}$$

ASSUMPTION: *Two common assumptions are made as it is frequent practice in modern control (see for example [Bittanti, 1991]),*

A.2. $C^*(t)D(t) = 0,$

A.3. $D^*(t)D(t) = I, \quad \forall t \ge 0.$

So under assumptions A.2 and A.3, Equation (5.43) reduces to

$$\|z(t)\|^2 = \langle C^*(t)C(t)x(t), x(t) \rangle + \langle P(t)B(t)B^*(t)P(t)x(t), x(t) \rangle. \tag{5.44}$$

Substituting both (5.44) & (5.16) in the inequality given by (5.39),

$$\int_0^\infty \left[\|z(t)\|^2 - \gamma \|w(t)\|^2 \right] dt \le \int_0^\infty \begin{bmatrix} \langle C^*(t)C(t)x(t), x(t) \rangle \\ + \langle P(t)B(t)B^*(t)P(t)x(t), x(t) \rangle \\ -\gamma \langle w(t), w(t) \rangle - \langle Ix(t), x(t) \rangle \\ - \langle P(t)B(t)B^*(t)P(t)x(t), x(t) \rangle \\ - \langle C^*(t)C(t)x(t), x(t) \rangle + \langle 2P(t)E(t)w(t), x(t) \rangle \end{bmatrix} dt + \langle P(0)x(0), x(0) \rangle. \tag{5.45}$$

This leads to,

$$\int_0^\infty \left[\|z(t)\|^2 - \gamma \|w(t)\|^2 \right] dt \le \int_0^\infty \left[-\gamma \|w(t)\|^2 - \|x(t)\|^2 + 2\|P(t)\| \|E(t)\| \|w(t)\| \|x(t)\| \right] dt$$
$$+ \langle P(0)x(0), x(0) \rangle.$$

(5. 46)

Finally, this inequality directly follows,

$$\int_0^\infty \left[\|z(t)\|^2 - \gamma \|w(t)\|^2 \right] dt \le \int_0^\infty \left[-\gamma \|w(t)\|^2 - \|x(t)\|^2 + 2pe\|w(t)\| \|x(t)\| \right] dt + \|P(0)\| \|x(0)\|^2. \quad (5.\ 47)$$

THEOREM 5.1. *Suppose that assumption A.1 holds then the* H_∞ *optimal control problem has a solution if:*

$$1 - \frac{p^2 e^2}{\gamma} > 0. \quad (5.\ 48)$$

PROOF:

Recall from (5.46) that

$$\int_0^\infty \left[\|z(t)\|^2 - \gamma \|w(t)\|^2 \right] dt \le \int_0^\infty \left[-\gamma \|w(t)\|^2 - \|x(t)\|^2 + 2pe\|w(t)\| \|x(t)\| \right] dt + \|P(0)\| \|x(0)\|^2. \quad (5.\ 49)$$

Then by completion of the square,

$$\int_0^\infty \left[-\gamma \|w(t)\|^2 - \|x(t)\|^2 + 2pe\|w(t)\| \|x(t)\| \right] dt = -\int_0^\infty \left[\left(\sqrt{\gamma}\ \|w(t)\| - \frac{pe}{\sqrt{\gamma}} \|x(t)\| \right)^2 \right] dt$$
$$+ \int_0^\infty \left[\frac{p^2 e^2}{\gamma} \|x(t)\|^2 - \|x(t)\|^2 \right] dt.$$

(5. 50)

It follows that (5.50) reduces to,

$$\int_0^\infty \left[\|z(t)\|^2 - \gamma \|w(t)\|^2 \right] dt \le \int_0^\infty \left[\left[-1 + \frac{p^2 e^2}{\gamma} \right] \|x(t)\|^2 \right] dt + \|P(0)\| \|x(0)\|^2. \quad (5.\ 51)$$

Therefore,

$$\int_0^\infty \left[\|z(t)\|^2 - \gamma \|w(t)\|^2 \right] dt < \|P(0)\| \|x(0)\|^2. \tag{5.52}$$

By dividing both sides of (5.52) by γ and rearranging,

$$\frac{1}{\gamma}\int_0^\infty \|z(t)\|^2 dt < \int_0^\infty \|w(t)\|^2 \, dt + \frac{\|P(0)\|}{\gamma}\|x(0)\|^2, \tag{5.53}$$

and setting $c_0 = \dfrac{\|P(0)\|}{\gamma}$,

then

$$\frac{1}{\gamma}\int_0^\infty \|z(t)\|^2 dt < \int_0^\infty \|w(t)\|^2 \, dt + c_0 \|x(0)\|^2, \tag{5.54}$$

and finally,

$$\frac{\int_0^\infty \|z(t)\|^2 dt}{c_0 \|x(0)\|^2 + \int_0^\infty \|w(t)\|^2 \, dt} < \gamma. \tag{5.55}$$

■

5.4. Nonlinear Optimal H$_\infty$ Control

In this section both the sequence of time-varying linear approximations and the ASRE are applied to extend the previous section. These sequences converge to the solution of the nonlinear H_∞ control problem.

Consider the following nonlinear dynamical system under the presence of disturbance:

$$\left.\begin{aligned} \dot{x}(t) &= A(x(t))x(t) + B(x(t))u(t) + E(x(t))w(t), \qquad x(0) = x_0 \\ z(t) &= C(x(t))x(t) + D(x(t))u(t); \end{aligned}\right\}. \tag{5.56}$$

Then the following sequence of linear time-varying approximations can be introduced,

$$\begin{aligned} \dot{x}^{[0]}(t) &= A(x_0)x^{[0]}(t) + B(x_0)u^{[0]}(t) + E(x_0)w^{[0]}(t), \qquad x^{[0]}(0) = x_0. \\ &\vdots \\ \dot{x}^{[i]}(t) &= A\left(x^{[i-1]}(t)\right)x^{[i]}(t) + B\left(x^{[i-1]}(t)\right)u^{[i]}(t) + E\left(x^{[i-1]}(t)\right)w^{[i]}(t), \qquad x^{[i]}(0) = x_0. \end{aligned} \tag{5.57}$$

with the index "i" referring to the iteration.

Now using the theory of §5.3 for each linear time-varying system in (5.57), the linear feedback control is updated while the ROE is solved backwards in time at each iteration.

These sequences of robust feedback control laws are given by the limit of

$$u^{[i]}(t) = -B^{*}\left(x^{[i-1]}(t)\right)P^{[i]}(t)x^{[i]}(t); \qquad i \geq 0,; \qquad (5.58)$$

where the $n \times n$ symmetric matrix $P(t)$ is the unique solution of the Approximating Sequences of Riccati Equations (ASRE):

$$\begin{aligned}\dot{P}^{[i]}(t) &= -A^{*}\left(x^{[i-1]}(t)\right)P^{[i]}(t) - P^{[i]}(t)A\left(x^{[i-1]}(t)\right) \\ &+ P^{[i]}(t)B\left(x^{[i-1]}(t)\right)B^{*}\left(x^{[i-1]}(t)\right)P^{[i]}(t) - Q(t)\end{aligned} \qquad (5.59)$$

THEOREM 5.2. *Suppose that the conditions:* $b = \sup_{x \in R^{+}} \left\| B(x) \right\|$, $e = \sup_{x \in R^{+}} \left\| E(x) \right\|$, $c = \sup_{x \in R^{+}} \left\| C(x) \right\|$ *and* $d = \sup_{x \in R^{+}} \left\| D(x) \right\|$ $\{\exists b, e, c, d > 0\}$ *hold. Then the* H_{∞} *optimal control problem has a solution if:*

$$1 - \frac{p^{2}e^{2}}{\gamma} > 0. \qquad (5.60)$$

PROOF: This result directly follows by direct application of Theorem 5.1, and the convergence holds.

■

PROPOSITION 5.1. *Given any initial state* $x(t_0) = x_0$, *the sequence of linear-quadratic, time-varying approximations (5.57) obtained by classical linear-quadratic methods for the nonlinear* H_{∞} *optimal control problem converges uniformly on some small time interval* $\left[t_0, t_f\right]$, *where the final time,* t_f, *might depend on* x_0.

This in fact was proved in the space of continuous functions for a small compact time interval, under the local Lipschitz continuity condition of the nonlinear dynamical operators $A(x)$, $B(x)$ and $C(x)$ (see Tomas-Rodriguez & Banks, 2003). But because there were no clear indications on how small this interval should be taken for the method to hold, a global convergence theory was needed to remove this strong restriction; which in essence was obtained for the global nonlinear optimal control problem in Çimen, (2003) by applying the similar principle that appeared in Tomas-Rodriguez & Banks (2003) for general nonlinear homogeneous equations. That is to say, if a solution to the nonlinear optimal control problem exists, for which the cost is finite and the trajectory is bounded on the interval $[t_0,\tau]\subseteq\mathbb{R}$, then the approximating sequences converge on this finite-time interval (see [Çimen & Banks, 2004; and Çimen, 2003]). But for the convenience of the reader, the global convergence Lemma (5.1) for the nonlinear optimal control problem as appeared in Çimen & Banks (2004) is restated below since it should also hold for the nonlinear robust H_∞ optimal control problems in this thesis. However, it should be pointed-out that the Lemma was slightly tailored to fit the context.

Lemma 5.1 (Global Convergence). *Suppose that the robust nonlinear H_∞ optimal control problem has a continuous feedback control on the interval $[t_0,\tau]$. Then the controlled sequence of functions $\{x^{[i]}(t)\},\{y^{[i]}(t)\}$ and feedback controls $\{u^{[i]}(t)\}$ defined by the linear-quadratic, time-varying approximations converge uniformly on $[t_0,\tau]$.*

PROOF: It is shown in Çimen & Banks, (2004), that for the nonlinear optimal control problem, t_f can be chosen to be locally constant, that is, for any $\bar{x}$ there exists a neighbourhood $B_{\bar{x}}$ of $\bar{x}$ such that the sequences of linear-quadratic, time-varying approximations with initial state $x_0 \in B_{\bar{x}}$ converge uniformly on some interval $[t_0,t_{\bar{x}}]$, where $t_{\bar{x}}$ is independent of x_0. Now by contradiction, suppose that the result is false, so that there is a maximal time interval $[t_0,\bar{t})$ such that for any $t_f<\bar{t}$, the quadratic and LTV sequences converge uniformly on $[t_0,t_f]$.

Let us consider the controlled trajectory $x(t;x_0)$ of the original nonlinear H_∞ optimal control problem on the interval $[t_0,\tau]$.

Define the set

$S \triangleq \{x(t;x_0) | t \in [t_0,\tau]\}$.

For each $\bar{x} \in S$, choose a neighbourhood $B_{\bar{x}}$ as above, that is, the sequences of LTV and quadratic approximations converge uniformly on the interval $[t_0,t_{\bar{x}}]$ for any $x_0 \in B_{\bar{x}}$ and for $t_{\bar{x}}$ independent of the initial state x_0. Since S is compact and $\bigcup_{\bar{x}\in S} B_{\bar{x}}$ is an open cover of S, there exists a finite sub-cover $\{B_{\bar{x}_1},\ldots,B_{\bar{x}_p}\}$ with corresponding times $\{t_{\bar{x}_1},\ldots,t_{\bar{x}_p}\}$.

Let

$t_{\min} \triangleq \min\{t_{\bar{x}_1},\ldots,t_{\bar{x}_p}\}$.

Now since, by assumption, the approximating sequence of Riccati operators (5.59) and feedback controls (5.58) converge on $[t_0,t_f]$, the controlled sequence $x^{[i]}(t)$ converges uniformly on $[t_0,\bar{t}-t_{\min}/2]$.

Let

$x_{0,i} \triangleq x^{[i]}(\bar{t}-t_{\min}/2)$.

Since these converge to $x(\bar{t}-t_{\min}/2)$, they can be assumed to belong to $B_{\bar{x}_p}$, so that another sequence of solutions given by the linear-quadratic, time-varying approximations can be obtained from the initial states $x_{0,i}$ and which uniformly converge to the corresponding solutions of the nonlinear H_∞ optimal control problem (with initial state $x_{0,i}$) on the finite-time interval $[\bar{t}-t_{\min}/2,\bar{t}+t_{\min}/2]$. Such solutions can be denoted: $x^{[i,j]}(t)$, which converge to $x^{[i]}(t)$ on the interval $[\bar{t}-t_{\min}/2,\bar{t}+t_{\min}/2]$ as shown in Figure (5.1).

Now let us use a Cantor-like diagonal argument. Consider the functions:

$$\aleph^{[i]}(t) \triangleq \begin{cases} x^{[i]}(t), & t_0 \le t \le \bar{t}-t_{\min}/2, \\ x^{[i,i]}(t), & \bar{t}-t_{\min}/2 \le t \le \bar{t}+t_{\min}/2. \end{cases}$$

Then $\aleph^{[i]}(t)$ converges uniformly to $x(t)$ on $[t_0, \bar{t} + t_{\min}/2]$ and is arbitrarily close to $x^{[i]}(t)$ on $[t_0, \bar{t}]$, which contradicts the assumption that $\{x^{[i]}(t)\}$, and therefore $\{y^{[i]}(t)\}$, is not uniformly convergent on $[t_0, \bar{t}]$. Also since the controls are expressed in a feedback form, it follows that $\{u^{[i]}(t)\}$ also converges on $[t_0, \bar{t}]$.

□

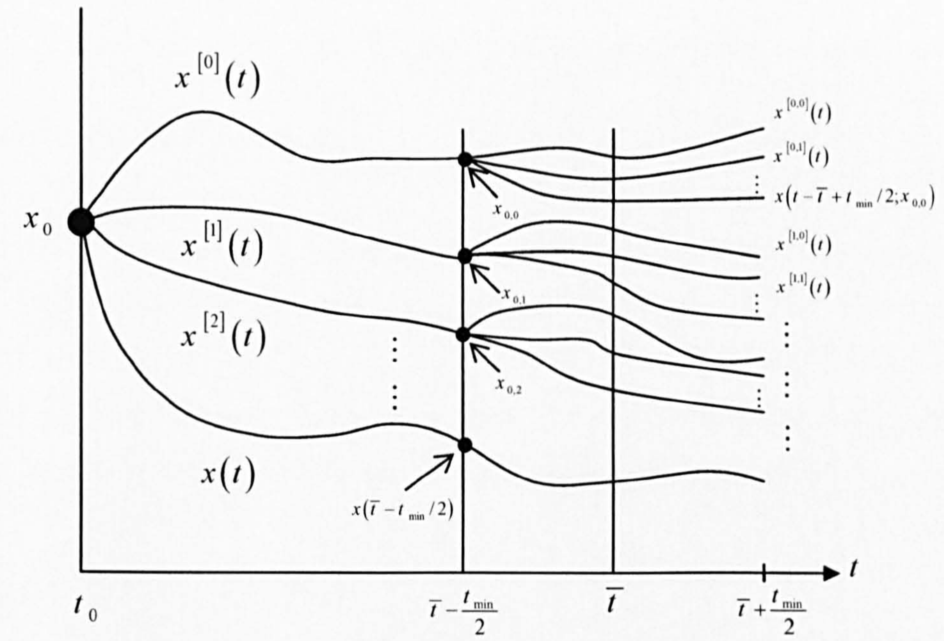


Figure 5.1.: The Approximating Sequence.

5.5. A Design Example: An Inverted Pendulum on a Cart

Nomenclature

M	Mass of the cart in Kg
m	Mass of the pendulum arm in Kg
r	Distance between the centre of the hinge and the pendulum arm's centre of gravity in m.
x	The cart's position in m.
θ	The pendulum arm's deflection from the vertical axis (clockwise direction [positive]) in *rad.*
j	The pendulum arm's moment of inertia in $Kg\ m^2$
f	The cart's coefficient of friction in Kg/s
c	The coefficient of the viscous rotational friction in the hinge supporting the pendulum arm in $Kg\ m^2\ /s$
g	The acceleration due to gravity in m/s^2
F	The input control force applied to the cart in N.

5.5.1. Dynamical Equations

Fittingly, inasmuch as the theoretical applicability of the proposed theory to real-life applications, it is immensely all-encompassing; and the reader is referred to the applications chapter, chapter 7, for more insights. However, for illustrative purposes the following renowned physical model of an inverted pendulum on a cart, shown in Figure (5.2), is considered and where all motions are assumed to be in the plane.

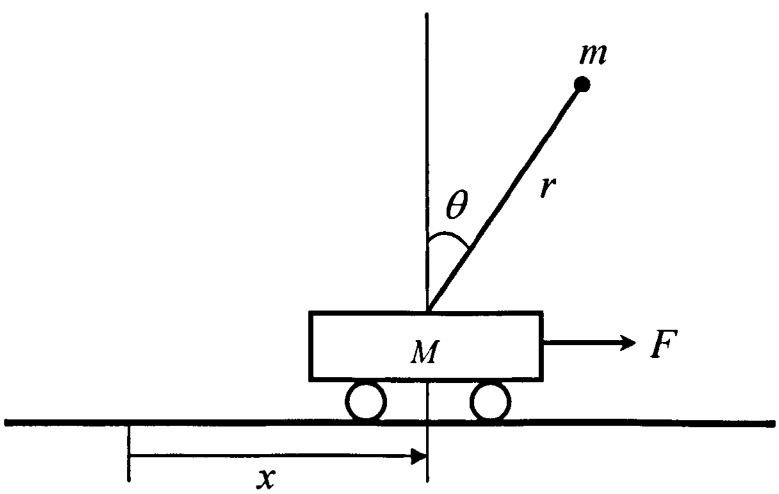


Figure 5.2.: The Inverted Pendulum on a Cart.

This classical control problem which consists of a single pendulum arm attached to a motor-driven cart has indeed found its analogous usage in the 3-dimensional practical real world and is often used to test new controller designs; *i.e.* it is often used as a benchmark test system. With a main control objective to vertically balance and stabilize the pendulum arm while maintaining it in an upright position by moving the cart back and forth over a finite-length track by means of a force, F, the inverted pendulum on a cart system depicts the behaviour of many realistic applications.
Some of these applications include, but are not restricted to, the following:

- Controlling the vertical deviation(s) of space shuttles during take-offs.
- Balancing rockets during launching.
- Maintaining a walking biped robot in an upright position.
- Balancing overhead cranes in an industrial environment.
- Designing a control mechanism for earthquake-resistant buildings.
- Controlling artificial limbs that are usually modelled by means of a double-inverted pendulum model.

The nonlinear mathematical model describing the dynamics of the system is (see for instance [Çimen, 2003]):

$$(M+m)\ddot{x}(t)+mr\ddot{\theta}(t)\cos\theta(t)+f\,\dot{x}(t)-mr\dot{\theta}^2(t)\sin\theta(t)=F. \quad (5.61)$$

$$mr\ddot{x}(t)\cos\theta(t)+\left(j+mr^2\right)\ddot{\theta}(t)+c\dot{\theta}(t)-mgr\sin\theta(t)=0. \quad (5.62)$$

By defining the state vector $x(t)$ of the inverted pendulum as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \triangleq \begin{bmatrix} x(t) \\ \theta(t) \\ \dot{x}(t) \\ \dot{\theta}(t) \end{bmatrix} \quad (5.63)$$

and excluding disturbances for the time-being, the dynamical model {in Equations (5.61) & (5.62)} can be represented in the following factored state-affine form $A(x)x$ as

$$\dot{x}(t)=f(x,u)=A(x)x(t)+B(x)u(t). \quad (5.64)$$

where t is an independent time variable, $u = F$, and the non-unique A and B are nonlinear time-invariant matrices functions in x given by

$$A(x) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_{32}(x) & a_{33}(x) & a_{34}(x) \\ 0 & a_{42}(x) & a_{43}(x) & a_{44}(x) \end{bmatrix} \tag{5. 65}$$

and

$$B(x) = \begin{bmatrix} 0 \\ 0 \\ b_3(x) \\ b_4(x) \end{bmatrix} \tag{5. 66}$$

where the parameters $a_{ij}(x)$ and $b_i(x)$ are

$$a_{32}(x) = -\Gamma m^2 r^2 g \cos x_2 \operatorname{sinc} x_2$$

$$a_{33}(x) = -\Gamma\left(j + mr^2\right) f$$

$$a_{34}(x) = \Gamma\left[c \cos x_2 + \left(j + mr^2\right) x_4 \sin x_2\right] mr$$

$$a_{42}(x) = \Gamma(M + m) mgr \ \operatorname{sinc} x_2$$

$$a_{43}(x) = \Gamma mrf \cos x_2$$

$$a_{44}(x) = -\Gamma\left[(M + m)c + \frac{1}{2} m^2 r^2 x_4 \sin(2x_2)\right]$$

$$b_3(x) = \Gamma\left(j + mr^2\right)$$

$$b_4(x) = -\Gamma mr \cos x_2$$

with

$$\Gamma \triangleq \frac{1}{\left(M + m\sin^2 x_2\right) mr^2 + (M + m) j} \quad \text{and} \quad \operatorname{sinc} x_2 = \begin{cases} 1, & x_2 = 0 \\ \dfrac{\sin x_2}{x_2}, & x_2 \neq 0 \end{cases}$$

The measurement vector equals $y(t) = \left(x(t), \dot{x}(t), \theta(t)\right)$. It is required that the controller yields a robustly stable system with respect to the several uncertainties that affect this given system:

(i) Uncertainties in the parameters $F, m, M \,\&\, r$.

(ii) Flexibility in the pendulum.

(iii) Exogenous additive disturbance inputs.

Also, the limits on the bandwidth and the gains of the controller have to be taken into account; this is essential due to limitations on the sampling rate for the digital implementation as well as the limitation in the speed of the actuators.

The following interconnection can be extracted from Stoorvogel (2000), to clarify the setup and where the weighted integrated tracking error is to be minimized.

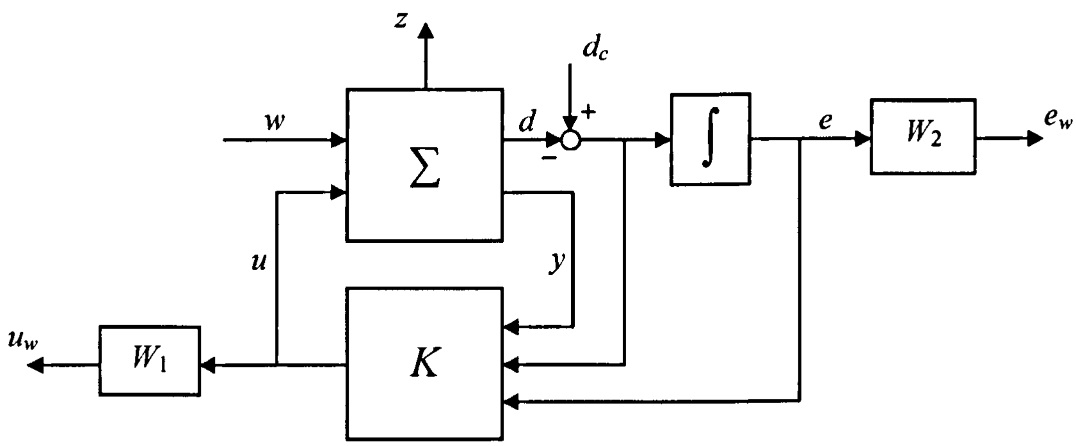


Figure 5.3.: The Interconnection of the Physical Model.

Here d_c is the command signal for the position d and W_1 is a first-order weight of the form

$$W_1(s) := \varepsilon \frac{1+\alpha s}{1+\beta s}. \tag{5. 67}$$

The first-order weight, W_1, expresses the interest in only tracking low-frequency signals and where $\alpha << \beta$ is chosen to obtain a low-pass filter. The incorporation of a scalar ε in the weight is also used to express the relative importance of tracking over other goals that are to be met. Also to guarantee zero steady-state tracking error an integrator is used; particularly important for low frequencies.

Similarly, W_2 has the same structure as W_1 in order to minimize u_w, which is the weighted control input. And conversely by choosing $\alpha >> \beta$ a high-pass filter is obtained and which facilitates the digital implementation of the control law. Indeed this high-pass filter also prevents pushing the actuators beyond their working range and capabilities. Finally, it also prevents the controller from capturing high-frequency uncertainties in the system dynamics, such as bending modes of the pendulum (Stoorvogel, 2000).

Note that z and w are new inputs and outputs to be added to the system Σ to express robustness requirements, *i.e.* the system Σ is of the form:

$$\Sigma: \begin{cases} \dot{x}(t) = & A(x(t))x(t) + B(x(t))u(t) + Ew(t), \\ y(t) = & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x(t), \\ z(t) = & C_2 x(t). \end{cases} \tag{5.68}$$

The matrices A and B are as defined before in Equations (5.65) & (5.66) respectively. On the other hand E and C_2 still have to be chosen. Stoorvogel (2000) used the technique of complex stability radii to design for such matrices. For instance, to guard against fluctuations in the parameters F and m, the friction and the mass of the pendulum, then E and C_2 are chosen as (see Stoorvogel, 2000):

$$E := \begin{pmatrix} 0 \\ -1/M \\ 0 \\ 1/(rM) \end{pmatrix} \qquad C_2 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \end{pmatrix} \tag{5.69}$$

Although a similar definition can be made to guard against fluctuations in all parameters of the two differential equations due to the discarded nonlinearities or the flexibility of the beam, it is not required in this context since all nonlinearities are already incorporated due to the approach of the quadratic linear time-varying sequences that is adopted. In other words, unlike Stoorvogel (2000), there is no need to design E and C_2 to guard against all fluctuations in the two differential equations.

5.5.2. Simulations and Results

On the basis of the above, the controller K is designed, based on the interconnection shown in Figure (3.2) to minimize the H_∞ norm from w to z and not as in Stoorvogel (2000) where the controller K was designed to minimize the H_∞ norm from (w,d_c) to (z,u_w,e_w) and where the parameters of the weights W_1 and W_2 were manipulated, by hand, on the basis of the properties of the designed controllers. That is to say that the filters W_1 and W_2, in Equation (5.67), are not included in the simulation results.

Recalling the inverted pendulum model {in Equations (5.61) & (5.62)} in its factored form (5.64),

$$\dot{x}(t)=A(x)x(t)+B(x)u(t)+E(x)w(t) \tag{5. 70}$$

with the specifications shown in Table (5.1) (in *SI* units); where it should be noted that the pendulum arm's moment of inertia, the cart's coefficient of friction, and the coefficient of the viscous rotational friction in the hinge supporting the pendulum arm, are all incorporated in the simulation results to follow.

M	$1\ Kg$
m	$0.1\ Kg$
r	$0.5\ m$
g	$9.81\ m/s^2$
j	$0.1\ Kg\ m^2$
f	$0.01\ Kg/s$
c	$0.01\ Kg\ m/s^2$

Table 5.1.: Specifications.

The inputs to the system, *i.e.* the initial conditions to the plant, are given in Table (5.2).

$x_1(0)$	Initial value of the cart's position $=0\ m$
$x_2(0)$	Initial pendulum's angle $=\pi/3$
$x_3(0)$	Initial value of the cart's velocity $=0\ m/s$
$x_4(0)$	Initial value of the pendulum's angular velocity $=0\ rad/s$

Table 5.2.: Inputs.

By introducing the sequence of linear time-varying approximations (5.57) and the ASRE (5.59) to the nonlinear dynamical system of the inverted pendulum on a cart (5.70), as discussed in the previous sections, with the parameters of table (5.1) and the inputs of table (5.2), and initially setting the disturbance term to zero and assuming a frictionless setting, then by using MATLAB® Figure (5.4) is obtained. It is worth noticing that the nonlinear solution of the optimal H_∞ controller is successively reached as the iterations proceed (hereunder, the simulated results are shown for $i=1$, $i=2$ and $i=6$).

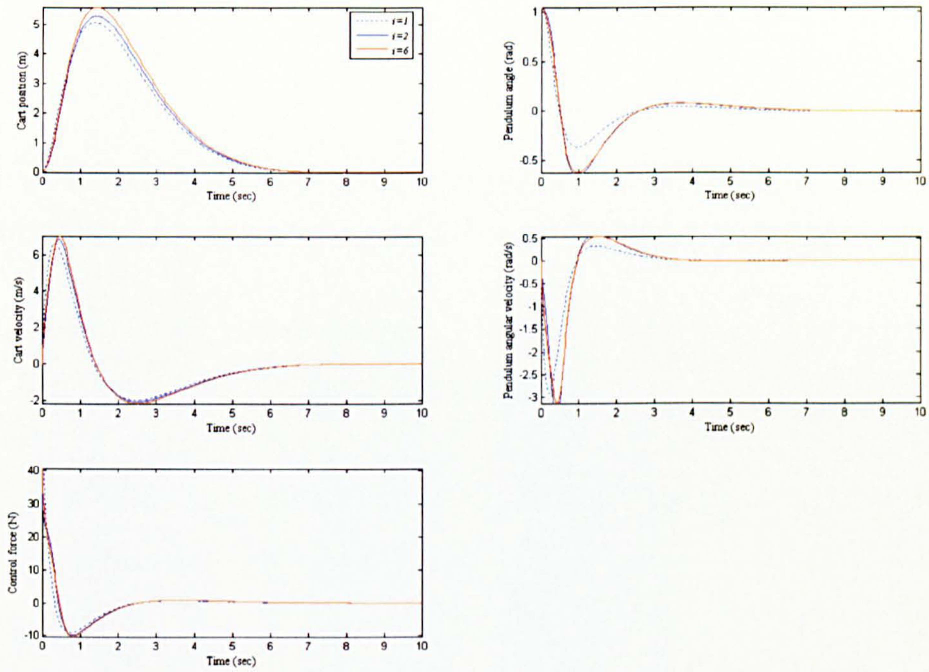


Figure 5.4.: The Optimal H_∞ Controller Responses.

Now under the same inputs and parameters above, let us consider the more general case with the inclusion of the $\alpha \times Ew$ term; with α being some scalar.

Recalling Equation (5. 70), the following case is initially considered:

Case 1:

$$\alpha = 2 \qquad \& \qquad E := \begin{pmatrix} 0 \\ -1/M \\ 0 \\ 1/(rM) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix}. \tag{5. 71}$$

The controlled nonlinear system is shown in Figure (5.5); however due to the large disturbance's magnitude, the pendulum's angle is virtually not controlled even though stability is only reached over an infinitesimal closed-time interval, which realistically speaking, might not be realizable in view of the physical constraints. Note that the disturbance input, w, for the both cases shown below is taken as I_1.

Hence by decreasing the disturbance size to; say:

Case 2:

$$\alpha = 0.1 \qquad \& \qquad E := \begin{pmatrix} 0 \\ -1/M \\ 0 \\ 1/(rM) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 2 \end{pmatrix}; \tag{5. 72}$$

a better response that matches the control objectives is obtained and is shown in Figure (5.5). Note that these two simulated results are obtained after 5 iterations, *i.e.* $i = 5$.

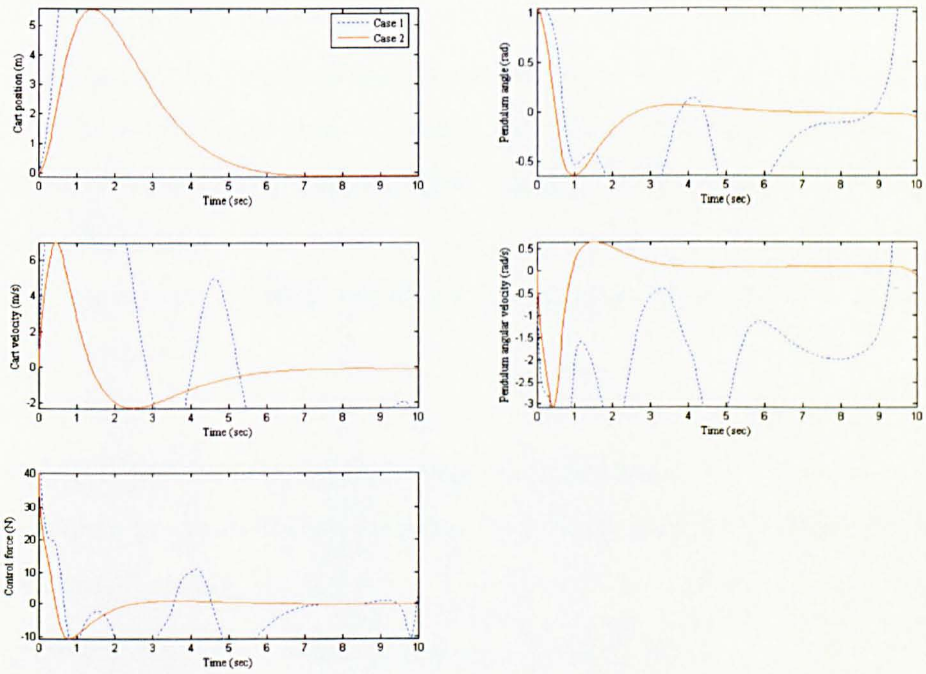


Figure 5.5.: The Disturbed Inverted Pendulum on a Cart System.

It is worth stressing that the above-mentioned designs incorporated a γ-iteration to design for the robust controllers. It turns out that for implementing these controllers γ is chosen to be approximately 10% larger than the infimum over all stabilizing controllers of the closed-loop H_∞ norm.

5.6. Concluding Remarks

In this chapter, an infinite-dimensional nonlinear H_∞ optimal controller was devised using the machinery of the well-established approximation theory in a Hilbert space. The theoretical linear time-varying H_∞ credentials proposed in this chapter build upon the standard Riccati-like equation and completing the square to prove robust stability and devise a linear time-varying state-regulator control law. The closed-loop norm-bounded H_∞ controller then attenuates disturbances under appropriate design parameters. It turns out that, as expected, there is a compromise between robustness and optimality. In other terms, the larger γ the more optimal the controller is and vice versa. The mathematical theory in Euclidean spaces was successfully validated via direct application to the highly nonlinear dynamical model of an inverted single-arm pendulum on a motor-driven cart. Of course, the finite-dimensional dynamical system considered in this chapter involved Euclidean spaces but Hilbert spaces can also be used for PDEs and delay systems.

This chapter is concluded with a conceptual algorithm for a clarification of the proposed H_∞ methodology.

Nonlinear State-Feedback H_∞ optimal control algorithm:

Given a nonlinear mathematical model describing the dynamics of a dynamical system with initial conditions $x(t_0)=x_0$,

1. Express the dynamical system in a state-affine form as:

$$\dot{x}(t)=f(x,u,w)=A(x)x(t)+B(x)u(t)+Ew(t).$$

Note that the general operators $A(x(t),u(t),t)$ and $B(x(t),u(t),t)$ are non-unique.

2. Design the disturbance input matrix, E, in the state-affine form in (1).

3. Design the parameter(s), C and D, in the regulated outputs equation:

$$z(t)=Cx(t)+Du(t).$$

Note that $C^*(t)D(t)=0$ and $D^*(t)D(t)=I$, $\forall t\geq 0$.

4. Introduce the sequence of linear time-varying approximations

a. For $i=0$:

$$\dot{x}^{[0]}(t)=A(x_0)x^{[0]}(t)+B(x_0)u^{[0]}(t)+Ew^{[0]}(t), \qquad x^{[0]}(0)=x_0.$$

b. For $i>0$:

$$\dot{x}^{[i]}(t)=A\left(x^{[i-1]}(t)\right)x^{[i]}(t)+B\left(x^{[i-1]}(t)\right)u^{[i]}(t)+Ew^{[i]}(t).$$

5. Solve the Riccati operator equation backwards in time,

$$\dot{P}^{[i]}(t)=-A^{*}\left(x^{[i-1]}(t)\right)P^{[i]}(t)-P^{[i]}(t)A\left(x^{[i-1]}(t)\right)$$
$$+P^{[i]}(t)B\left(x^{[i-1]}(t)\right)B^{*}\left(x^{[i-1]}(t)\right)P^{[i]}(t)-Q(t)$$

where $Q(t)=C^{*}(t)C(t)+I$.

6. Update the control law,

$$u^{[i]}(t)=-B^{*}\left(x^{[i-1]}(t)\right)P^{[i]}(t)x^{[i]}(t); \qquad \text{for } i\geq 0.$$

7. Test that for a given sufficiently large γ, the following holds:

$$1-\frac{p^2e^2}{\gamma}>0.$$

8. If (7) holds, the family of stabilizing controllers is reached; if not, change γ and go to (5).

CHAPTER 6

Nonlinear H_∞ Control: A Game Theoretic Approach

6.1. Introduction

The study of the methods and means in which *strategic interactions* among *rational players* produce *outcomes* with respect to the *preferences* (or *utilities*) of those players is the essence of the Game theory. Similarly, in a control systems context, Differential Games concern the balance of optimal strategies utilized by two opposing players with conflicting notions of *best performance* of the dynamical system they are trying to control or to relinquish (as seen in chapter 5). Although the game theory found its initial usage in pursuit-evasion games in a military context, it is now playing a more essential role in the design of robust H_∞ controllers. Because disturbances and model uncertainties can be interpreted as *strategies* of an antagonistic player playing against the controller that is trying to take into account the worst possible actions of such a hostile agent (disturbances), that the differential games approach gave the H_∞ formulae a different meaning and a more general appeal.

The first appearance of a connection between Differential Games and H_∞ control is found in Weiland (1989) and Khargonekar, *et al.*, (1990), these papers, de facto, created the impetus for further research efforts in the interconnections between the two fields. The relationship between indefinite factorization, the game theory and the H_∞ control theory was thoroughly exploited (see [Başar & Bernhard, 1995; Glover & Doyle, 1988; Green, *et al.*, 1990; and Pertersen & Clements, 1988]). The connection between risk-sensitivity optimal control and game theory has also received renewed interest in the H_∞ control theory setting (see [Başar & Bernhard, 1995; Glover & Doyle, 1988; Bernstein & Haddad, 1989; and Tadmor, 1990]). While the discrete game theory to the state-feedback H_∞ control problem appeared in Başar (1991), the continuous-time counterpart appeared in Limebeer, *et al.*, (1992).

However, the aim of this chapter is to develop a new and a practical approach to the design of state-feedback H_∞ controllers of nonlinear dynamical systems based on the theory of differential games. In explicit terms, this chapter builds upon the game theoretic approach to H_∞ control problems for time-varying systems which appeared in Limebeer, *et al.*, (1992) as an extension to Tadmor (1990). The main difference, however, with chapter 5, is that a more inclusive theory is presented for both the state-regulator and the output-feedback control problems. Accordingly, in §6.2 the representation formula for all linear time-varying controllers that satisfy an L^∞-type constraint is derived and extracted from Limebeer, *et al.*, (1992) and relying upon the state-feedback concepts. While in §6.3 the output-feedback formulation for the linear time-varying H_∞ control problems is summarized. In §6.4 the linear-quadratic time-varying approximating sequences are employed to extend the linear framework to the more general nonlinear setting. Some computer simulated results of the nonlinear dynamical system of the inverted pendulum on a cart are given in §6.5. Finally, some closing remarks are presented in §6.6.

6.2. A Representation Formula for all H_∞ Solutions

This section summarizes the more general finite-horizon linear time-varying H_∞ control problem discussed in chapter 4. That is, over an optimization horizon-time interval, $[0,T]$, the representation formula for all state-feedback full-information control laws are studied.

The conventional modern state-space realization is considered,

$$\left.\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_1(t)w(t) + B_2(t)u(t), \qquad x(0)=0,\\ z(t) &= C_1(t)x(t) + D_{11}(t)w(t) + D_{12}(t)u(t),\\ y(t) &= C_2(t)x(t) + D_{21}(t)w(t) + D_{22}(t)u(t); \end{aligned}\right\} \tag{6. 1}$$

in which $x(t)\in\mathbb{R}^n$ is the state, x_0 is the initial condition of the system, $u(t):\mathbb{R}^+\to\mathbb{R}^m$ is the control input, $w(t):\mathbb{R}^+\to\mathbb{R}^l$ is the exogenous disturbance input, $y(t)\in\mathbb{R}^q$ is the measured (or sensor) outputs, and $z(t)\in\mathbb{R}^p$ is the regulated outputs and sometimes called a penalty variable which may include a tracking error; *i.e.*

$z(t)$ is the difference between the actual plant output and its desired reference behaviour, expressed as a function of some of the exogenous variables $w(t)$, as well as a cost of the input $u(t)$ needed to achieve the prescribed control goal. The realization in (6.1) can be written as:

$$\begin{array}{c} \\ n \\ p \\ q \end{array}\begin{bmatrix} \\ \dot{x}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} n & l & m \\ A(t) & B_1(t) & B_2(t) \\ C_1(t) & D_{11}(t) & D_{12}(t) \\ C_2(t) & D_{21}(t) & D_{22}(t) \end{bmatrix} \begin{bmatrix} \\ x(t) \\ w(t) \\ u(t) \end{bmatrix} \qquad x(0)=0, \tag{6. 2}$$

The system matrices are assumed to have entries that are continuous functions of time. The goal is to characterize all linear time-varying controllers satisfying $\|z\|_2 < \gamma \|w\|_2$ for all $w \neq 0$.

REMARK 6.1. The antagonistic player, w, tries to maximize the energy in the output z, while the controller, u-player, is trying to minimize this energy simultaneously.

REMARK 6.2. The time response of a linear time-varying system consists of a transient response (*i.e.* trajectory from the initial state to the final state) and a steady-state response (*i.e.* the manner in which the output behaves as $t \to \infty$). Note that since zero initial conditions were assumed by Limebeer, *et al.*, (1992), in (6.1) & (6.2), hence it follows that the transient response is set to zero, as common in classical linear control (please refer to Appendix A).

ASSUMPTION 6.1. It can be assumed without loss of generality that for all $t \in [0,T]$ (see Limebeer, *et al.*, 1992),

A.1. $D'_{12}D_{12} = I_m$ (*i.e.* D_{12} has full column rank m),

A.2. $D'_{21}D_{21} = I_q$ (*i.e.* D_{21} has full column rank q),

A.3. $D_{11} = 0$,

A.4. $D_{22} = 0$.

With the given system, in (6.1) or (6.2), under a leader-follower game sense, the H_∞ control problem is to find a linear admissible and causal state-feedback control

$$u(t)=K(x(s),w(s),t), \qquad 0\le s\le t, \tag{6.3}$$

such that:

$$\|\wp_{zw}\|=\sup_w\left\{\|\wp_{zw}w\|_2 : w\in\ell^2[0,T],\|w\|_2\le 1\right\}<\gamma; \tag{6.4}$$

for some given $\gamma>0$. The operator $\wp_{zw}$ is a mapping between w and z when the control, $u(t)=K(\cdot,\cdot,\cdot)$, is in place. Note that if $z(t)=\wp_{zw}w(t)$, then $\|\wp_{zw}\|<\gamma$, if and only if the finite-time linear-quadratic cost functional is:

$$J(K,w)=\int_0^T\left(z'(t)z(t)-\gamma^2w'(t)w(t)\right)dt\le-\varepsilon\|w(t)\|_2^2, \tag{6.5}$$

for all $w\in\ell^2\ [0,T]$ and some positive ε (see [Limebeer, *et al.*, 1992]).

The H_∞ control problem consequently has a solution if and only if

$$\min_{K(\cdot,\cdot,\cdot)\in\ell}\ \max_{w(t)\in\ell^2[0,T]}\left\{J\left(K(\cdot,\cdot,\cdot),w(t)\right)\right\}\le-\varepsilon\|w(t)\|_2^2; \tag{6.6}$$

is satisfied.

THEOREM 6.1 (A representation formula for all solutions (Limebeer, *et al.*, 1992)). *Suppose that for the given system (6.1) with* $D_{11}=0$, *the Game Riccati Differential Equation:*

$$\begin{aligned}\dot P(t)=&-P(t)\left(A(t)-B_2(t)D'_{12}(t)C_1(t)\right)-\left(A(t)-B_2(t)D'_{12}(t)C_1(t)\right)'P(t)\\&+P(t)\left(B_2(t)B'_2(t)-\gamma^{-2}B_1(t)B'_1(t)\right)P(t)-C'_1(t)\left(I-D_{12}(t)D'_{12}(t)\right)C_1(t),\end{aligned} \tag{6.7}$$

$$P(T=t_f)=0;$$

has a solution on $[0,T]$.

Then

$$u(t)=u^*(t)+\left(U(w-w^*)\right)(t), \tag{6.8}$$

where

$$u^*(t) = -\left(D'(t)C(t) + B'(t)P(t)\right)x(t), \qquad (6.9)$$

$$w^*(t) = \gamma^{-2}B'_1\,(t)P(t)x(t); \qquad (6.10)$$

resulting in

$$\|z(t)\|_2^2 - \gamma^2\,\|w(t)\|_2^2 = \|u(t) - u^*(t)\|_2^2 - \gamma^2\,\|w(t) - w^*(t)\|_2^2 \qquad (6.11)$$

Then

$$\|\wp_{zw}\| < \gamma. \qquad (6.12)$$

PROOF:

From Limebeer, *et al.*, (1992).

■

All solutions to the H_∞ control problem with perfect information and given by (6.1) and (6.8) are shown in Figure (6.1).

Proposition 6.1. As γ tends to infinity, $(\gamma \to \infty)$, the algebraic Riccati equation (6.7) approaches the standard Riccati equation of the linear-quadratic and time-varying optimal control. This in fact is very satisfactory since that as $\gamma \to \infty$ there is no constraint on the closed-loop H_∞-norm.

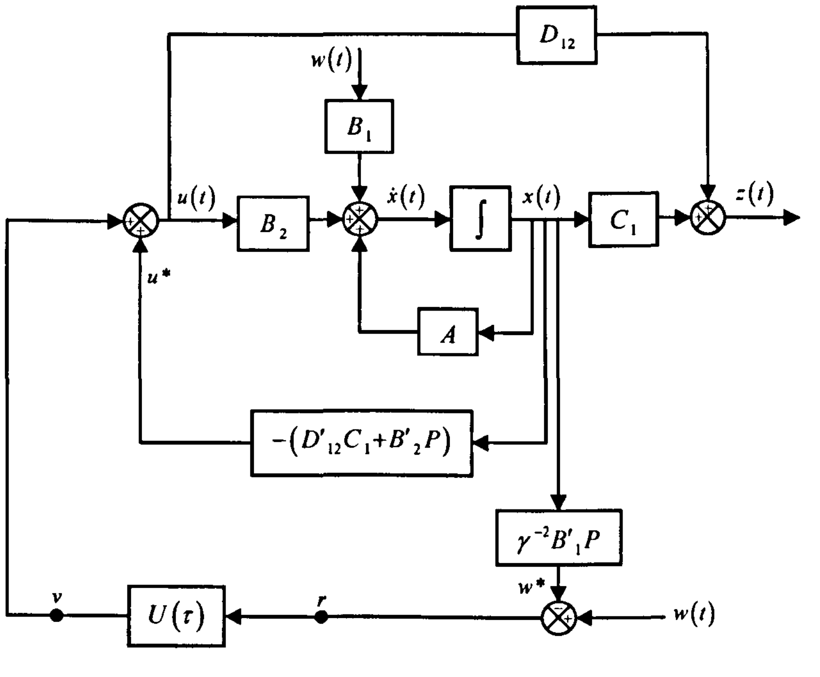


Figure 6.1.: All Solutions with Perfect Information.

REMARK 6.3. By examining Figure (6.1), it can be seen that $r=(w-w^*)$; and that $u=(v+u^*)$ is the input signal to B_2. However, if $w=w^*$, *i.e.* no signal in the $U(\tau)$ channel, then the feedback control law is simply u^*. Whereas if $w \neq w^*$ the u^* control does not have to be used.

In essence the control-player, or the u^*-player, is to ensure that for a given exogenous disturbance input, $w \neq 0 \in \ell^2[0,T]$, the cost functional is $J(K,w)<0$.

6.3. The H_∞ Output-Feedback Control Problem

The output-feedback control problem is the most practical and provides the most valuable solution. The methodology is in fact analogous to those in LQG control and often results in a controller with order equivalent to that of the dynamical plants and disturbance models, plus the order of any dynamic weighting term(s) in the cost functional. The technique is to transform the output feedback problem to a state-estimation one having a remarkably simple observer structure.

In the sequel the following plant is considered

$$\left.\begin{array}{lll} (i) & \dot{x}(t)=A(t)x(t)+B_1(t)w(t)+B_2(t)u(t), & x(0)=0, \\ (ii) & z(t)=C_1(t)x(t)+D_{12}(t)u(t), & \\ (iii) & y(t)=C_2(t)x(t)+D_{21}(t)w(t); & \end{array}\right\} \quad (6.13)$$

where all matrices have entries that are continuous functions of time, and D_{12} & D_{21} are full column rank and full row rank for all $t \in [0,T]$, *i.e.* the regular finite-horizon H_∞ problem is considered.

Limebeer, *et al.*, (1992) solved the problem for three different cases depending on D_{12} & D_{21}; that is if both matrices are square, or either is, or neither is. However, for simplicity, only the second case is considered when either D_{12} or D_{21} is square. In this case the generated feedback controllers are in fact characterized by a single Riccati Differential Equation based on the game theory of §6.2.

REMARK 6.4. If D_{21} is assumed to be square then (6.13) (iii) can be replaced by

$$y(t)=C_2(t)x(t)+w(t). \tag{6.14}$$

THEOREM 6.2 (Problems of the Second Type (Limebeer, *et al.*, 1992)). *Suppose that for the given system (6.13) with (6.13)(iii) replaced by (6.14) then there exists an output-feedback,* $u(t)=\Im y(t)$*, that guarantees* $\|\wp_{zw}\|<\gamma$ *if and only if the Game Riccati Differential Equation:*

$$\begin{aligned}\dot{X}_\infty(t)=&-\left(A(t)-B_2(t)D'_{12}(t)C_1(t)\right)'X_\infty(t)-X_\infty(t)\left(A(t)-B_2(t)D'_{12}(t)C_1(t)\right)\\&+X_\infty(t)\left(B_2(t)B'_2(t)-\gamma^{-2}B_1(t)B'_1(t)\right)X_\infty(t)-C'_1(t)D_\perp(t)D'_\perp(t)C_1(t),\end{aligned} \tag{6.15}$$

has a solution on $[0,T]$ *with terminal condition* $X_\infty(T)=0$ *(where* $D_\perp$ *is a continuous extension to* D_{12}*); and generated by:*

$$\begin{aligned}\dot{\hat{x}}(t)=&\left(A(t)-B_1(t)C_2(t)-B_2(t)\left(D'_{12}(t)C_1(t)+B'_2(t)X_\infty(t)\right)\right)\hat{x}(t)\\&+B_1(t)y(t)+B_2(t)v(t)\end{aligned} \tag{6.16}$$

$$u(t)=v(t)-\left(D'_{12}(t)C_1(t)+B'_2(t)X_\infty(t)\right)\hat{x}(t), \tag{6.17}$$

$$r(t)=y(t)-\left(C_2(t)+\gamma^{-2}B'_1(t)X_\infty(t)\right)\hat{x}(t), \tag{6.18}$$

$$v(t)=U(t)r(t). \tag{6.19}$$

PROOF:

See Limebeer, *et al.*, (1992).

∎

6.4. Nonlinear Extension

In this section both the sequence of time-varying linear approximations and the Approximating Sequences of Riccati Equations are applied to extend the previous sections, §6.2 & §6.3, following the same mathematical theory that was proposed in

§5.4. As previously seen, these sequences converge to the solution of the nonlinear H_∞ control problem.

Consider the following nonlinear dynamical system under the presence of disturbance:

$$\left.\begin{aligned} &\dot{x}(t)=A(x(t))x(t)+B(x(t))u(t)+E(x(t))w(t), && x(0)=x_0 \\ &z(t)=C_1\,(x(t))x(t)+D_{12}\,(x(t))u(t); \\ &y(t)=C_2\,(x(t))x(t)+D_{21}\,(x(t))w(t); \end{aligned}\right\}. \tag{6.20}$$

Remark 6.5. The initial conditions, $x(t_0)$, for the nonlinear dynamical model in (6.20) are taken as x_0 given that the transient response can not be ignored as in remark (6.2).

With the following sequence of linear time-varying approximations,

$$\begin{aligned} &\dot{x}^{[0]}(t)=A(x_0)x^{[0]}(t)+B(x_0)u^{[0]}(t)+E(x_0)w^{[0]}(t), \qquad x^{[0]}(0)=x_0. \\ &\vdots \\ &\dot{x}^{[i]}(t)=A\left(x^{[i-1]}(t)\right)x^{[i]}(t)+B\left(x^{[i-1]}(t)\right)u^{[i]}(t)+E\left(x^{[i-1]}(t)\right)w^{[i]}(t), \qquad x^{[i]}(0)=x_0. \end{aligned} \tag{6.21}$$

where the index "i" refers to the iteration step.

Using the theory of §6.3 for each linear time-varying system in (6.14), it is known that the sequences of linear state-feedback H_∞ control are given by the limit of

$$u^{[i]}(t)=-\left(D_{21}'\left(x^{[i-1]}(t)\right)C_1\left(x^{[i-1]}(t)\right)+B'\left(x^{[i-1]}(t)\right)P^{[i]}(t)\right)x^{[i]}(t); \qquad i\geq 0, \tag{6.22}$$

or a full-information output-feedback controller

$$\begin{aligned} u^{[i]}(t)=U^{[i]}(t)\Big(&y\left(x^{[i-1]}(t)\right)-\left(C_2\left(x^{[i-1]}(t)\right)+\gamma^{-2}B_1'\left(x^{[i-1]}(t)\right)X_\infty^{[i]}(t)\right)\hat{x}^{[i]}(t)\Big) \\ &-\left(D_{12}'\left(x^{[i-1]}(t)\right)C_1\left(x^{[i-1]}(t)\right)+E'\left(x^{[i-1]}(t)\right)X_\infty^{[i]}(t)\right)\hat{x}^{[i]}(t) \end{aligned} \tag{6.23}$$

where the $n\times n$ symmetric matrices $P(t)$ & $X_\infty(t)$ are the unique solutions of the Approximating Sequences of Riccati Equations (ASRE), respectively

$$\dot{P}^{[i]}(t) = -P^{[i]}(t)\left(A\left(x^{[i-1]}(t)\right) - B\left(x^{[i-1]}(t)\right)D'_{21}\left(x^{[i-1]}(t)\right)C_1\left(x^{[i-1]}(t)\right)\right)$$
$$-\begin{pmatrix} A\left(x^{[i-1]}(t)\right) \\ -B\left(x^{[i-1]}(t)\right)D'_{21}\left(x^{[i-1]}(t)\right)C_1\left(x^{[i-1]}(t)\right)\end{pmatrix}' P^{[i]}(t)$$
$$+P^{[i]}(t)\left(B\left(x^{[i-1]}(t)\right)B'\left(x^{[i-1]}(t)\right) - \gamma^{-2}E\left(x^{[i-1]}(t)\right)E'\left(x^{[i-1]}(t)\right)\right)P^{[i]}(t)$$
$$-C_1\left(x^{[i-1]}(t)\right)\left(I - D_{21}\left(x^{[i-1]}(t)\right)D'_{21}\left(x^{[i-1]}(t)\right)\right)C_1\left(x^{[i-1]}(t)\right), \tag{6.24}$$

and

$$\dot{X}^{[i]}_{\infty}(t) = -X^{[i]}_{\infty}(t)\left(A\left(x^{[i-1]}(t)\right) - B\left(x^{[i-1]}(t)\right)D'_{21}\left(x^{[i-1]}(t)\right)C_1\left(x^{[i-1]}(t)\right)\right)$$
$$-\begin{pmatrix} A\left(x^{[i-1]}(t)\right) \\ -B\left(x^{[i-1]}(t)\right)D'_{21}\left(x^{[i-1]}(t)\right)C_1\left(x^{[i-1]}(t)\right)\end{pmatrix}' X^{[i]}_{\infty}(t)$$
$$+X^{[i]}_{\infty}(t)\left(B\left(x^{[i-1]}(t)\right)B'\left(x^{[i-1]}(t)\right) - \gamma^{-2}E\left(x^{[i-1]}(t)\right)E'\left(x^{[i-1]}(t)\right)\right)X^{[i]}_{\infty}(t)$$
$$-C_1\left(x^{[i-1]}(t)\right)\left(I - D_{\perp}\left(x^{[i-1]}(t)\right)D'_{\perp}\left(x^{[i-1]}(t)\right)\right)C_1\left(x^{[i-1]}(t)\right). \tag{6.25}$$

THEOREM 6.2. *Suppose that Assumptions 6.2 hold. Then there exists a family of:*

(i) state-feedback H_{∞} controllers over $t \in [0,T]$, given by:

$$u^{[i]}(t) = -\left(D'^{[i]}_{21}(t)C^{[i]}_1(t) + B'\left(x^{[i-1]}(t)\right)P^{[i]}(t)\right)x^{[i]}(t); \qquad i \geq 0,$$

(ii) output-feedback H_{∞} controllers over $t \in [0,T]$, given by:

$$u^{[i]}(t) = U^{[i]}(t)\left(y\left(x^{[i-1]}(t)\right) - \left(C_2\left(x^{[i-1]}(t)\right) + \gamma^{-2}B'_1\left(x^{[i-1]}(t)\right)X^{[i]}_{\infty}(t)\right)\hat{x}^{[i]}(t)\right)$$
$$-\left(D'_{12}\left(x^{[i-1]}(t)\right)C_1\left(x^{[i-1]}(t)\right) + E'\left(x^{[i-1]}(t)\right)X^{[i]}_{\infty}(t)\right)\hat{x}^{[i]}(t)$$

that robustly stabilize the nonlinear system (6.20); with $P^{[i]}(t)$ & $X^{[i]}_{\infty}(t)$ being the unique solution of the ASREs (6.24) & (6.25) respectively; and where the closed-loop system is given by:

$$\dot{x}^{[i]}(t)=\left(A\left(x^{[i-1]}(t)\right)-B\left(x^{[i-1]}(t)\right)\begin{pmatrix}D'\left(x^{[i-1]}(t)\right)C\left(x^{[i-1]}(t)\right)\\+B'\left(x^{[i-1]}(t)\right)P^{[i]}(t)\end{pmatrix}\right)x^{[i]}(t)$$
$$+E\left(x^{[i-1]}(t)\right)w^{[i]}(t).$$

or

$$\dot{\hat{x}}^{[i]}(t)=\begin{pmatrix}A\left(x^{[i-1]}(t)\right)-E\left(x^{[i-1]}(t)\right)C_2\left(x^{[i-1]}(t)\right)\\-B\left(x^{[i-1]}(t)\right)\begin{pmatrix}D'_{12}\left(x^{[i-1]}(t)\right)C_1\left(x^{[i-1]}(t)\right)\\+B'\left(x^{[i-1]}(t)\right)X^{[i]}_{\infty}(t)\end{pmatrix}\end{pmatrix}\hat{x}^{[i]}(t)$$
$$+E\left(x^{[i-1]}(t)\right)y^{[i]}(t)+B\left(x^{[i-1]}(t)\right)v^{[i]}(t).$$

Then the operator $\wp_{zw}$ is bounded by γ for either case: (i) or (ii), such that:

$$\|\wp_{zw}\|<\gamma.$$

Proof: This result directly follows by direct application of Theorem 6.1 and Theorem 6.2, and the convergence from Lemma 5.1 holds.

■

Remark 6.6. In Theorem 6.2 all operators, $A,B,E,C_1,C_2,D_{12},D_{21}$ are given in a nonlinear state-affine form to make the illustration more inclusive since the linear-quadratic sequences can handle such nonlinearities which can also include a control dependence, *i.e.* $A\left(x(t),u(t),t\right)$ etc. But since the operators: E,C_1 & D_{12} are design parameters, they are most often taken as linear time-invariant.

6.5. A Design Example: An Inverted Pendulum on a Cart

In this section the physical nonlinear model of the inverted pendulum on a cart that was discussed in §5.5 is re-considered in the sequel and is re-shown in Figure (6.2).

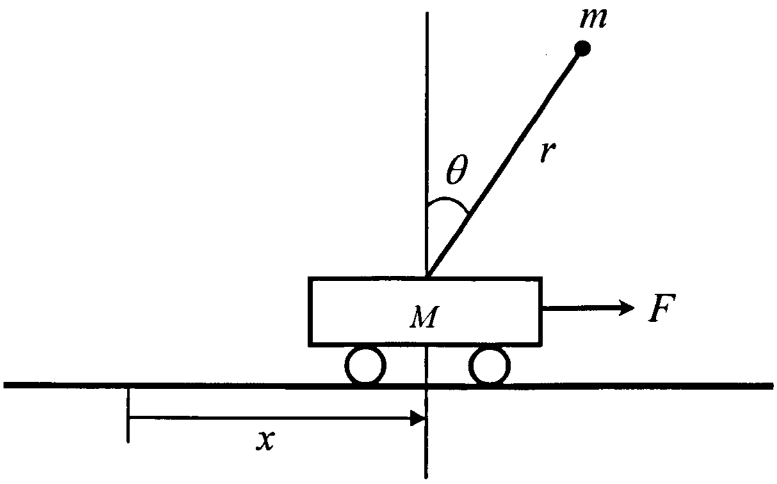


Figure 6.2.: The Inverted Pendulum on a Cart.

Recall the nonlinear dynamical model {in Equations (5.61) & (5.62)} which was represented in the following factored state-affine form $A(x)x$ as:

$$\dot{x}(t) = f(x,u) = A(x)x(t) + B(x)u(t). \tag{6.26}$$

where t is an independent time variable, $u = F$, and the non-unique A and B were given by Equations (5.65) & (5.66) respectively. Then it would be vital to investigate the effect of the design parameters on the response of the open-loop unstable system, Σ, and generally represented as:

$$\Sigma : \begin{cases} \dot{x}(t) = & A(x(t))x(t) + B(x(t))u(t) + Ew(t), \\ y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x(t) & + D_{21}w(t), \\ z(t) = & C_1 x(t) + D_{12}u(t). \end{cases} \tag{6.27}$$

As discussed in chapter 5, to guard against fluctuations in the parameters F and m, the friction and the mass of the pendulum, then E and C_1 are chosen as:

$$E := \begin{pmatrix} 0 \\ -1/M \\ 0 \\ 1/(rM) \end{pmatrix} \qquad C_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \end{pmatrix}. \tag{6.28}$$

By applying Theorem 6.2, the family of state-feedback H_∞ controllers over $t \in [0,10]$, can be devised:

$$u^{[i]}(t) = -\left(D_{21}'^{[i]}C_1^{[i]} + B'\left(x^{[i-1]}(t)\right)P^{[i]}(t)\right)x^{[i]}(t); \qquad i \geq 0, \quad (6.29)$$

by solving the Approximating Sequences of Riccati Equations backwards in time and subject to $P(10) = 0$,

$$\dot{P}^{[i]}(t) = -P^{[i]}(t)\left(A\left(x^{[i-1]}(t)\right) - B\left(x^{[i-1]}(t)\right)D_{21}'^{[i]}C_1^{[i]}\right) - \begin{pmatrix} A\left(x^{[i-1]}(t)\right) \\ -B\left(x^{[i-1]}(t)\right)D_{21}'^{[i]}C_1^{[i]} \end{pmatrix}' P^{[i]}(t)$$
$$-C_1'^{[i]}\left(I - D_{21}^{[i]}D_{21}'^{[i]}\right)C_1^{[i]} + P^{[i]}(t)\left(B\left(x^{[i-1]}(t)\right)B'\left(x^{[i-1]}(t)\right) - \gamma^{-2}E^{[i]}E'^{[i]}\right)P^{[i]}(t). \quad (6.30)$$

Using the *SI* specifications in Table (5.1), the introduced sequences of linear time-varying approximations (6.21) and the ASRE (6.30), with the inputs to the system as in Table (5.2), are used to approximate the nonlinear dynamical system of the inverted pendulum on a cart (6.26) and robustly control it for a sufficiently large γ ($\gamma = 8$). Furthermore, the input matrix C_1 in (6.28) was not used since it resulted in poor control actions which are not shown, but instead it was assumed to take the same form as C_2 in (6.27).

Initially, the disturbance input is taken as $w = 0.1$; while the variations in the system's response as a result of changing the disturbance weighting matrix, D_{21}, is studied. Consequently, eight different designs for the linear time-invariant D_{21} were considered as shown in Table (6.1).

Design 1:	$D_{21} := [0 \ \ 0 \ \ 0]^{-1}$	Design 6:	$D_{21} := [1 \ \ 1 \ \ 0]^{-1}$
Design 2:	$D_{21} := [1 \ \ 1 \ \ 1]^{-1}$	Design 7:	$D_{21} := [0 \ \ 1 \ \ 1]^{-1}$
Design 3:	$D_{21} := [1 \ \ 0 \ \ 0]^{-1}$	Design 8:	$D_{21} := [1 \ \ 0 \ \ 1]^{-1}$
Design 4:	$D_{21} := [0 \ \ 1 \ \ 0]^{-1}$		
Design 5:	$D_{21} := [0 \ \ 0 \ \ 1]^{-1}$		

Table 6.1.: The various Designs for the Weighting Matrix D_{21}.

The simulations to follow were carried-out after five iterations, *i.e.* $i = 5$, using MATLAB®; and where designs 1 & 2 are shown in Figure (6.3), designs 1-5 in Figure (6.4), and designs 7 & 8 in Figure (6.5).

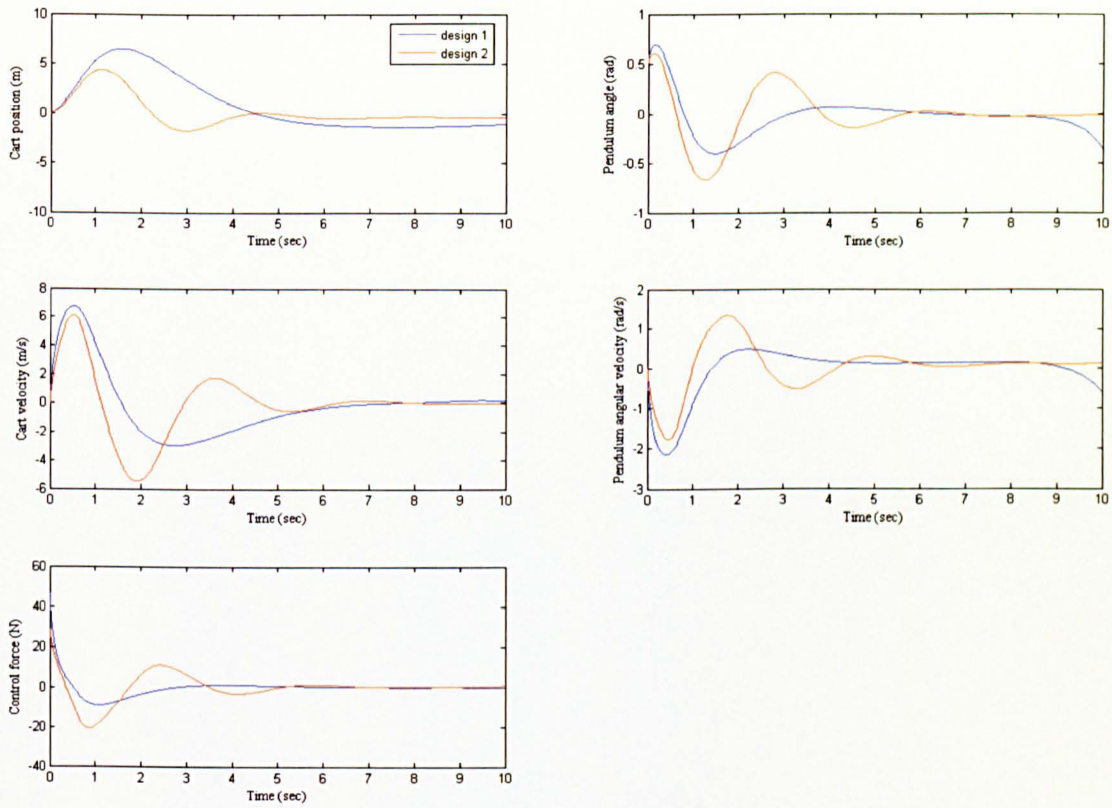
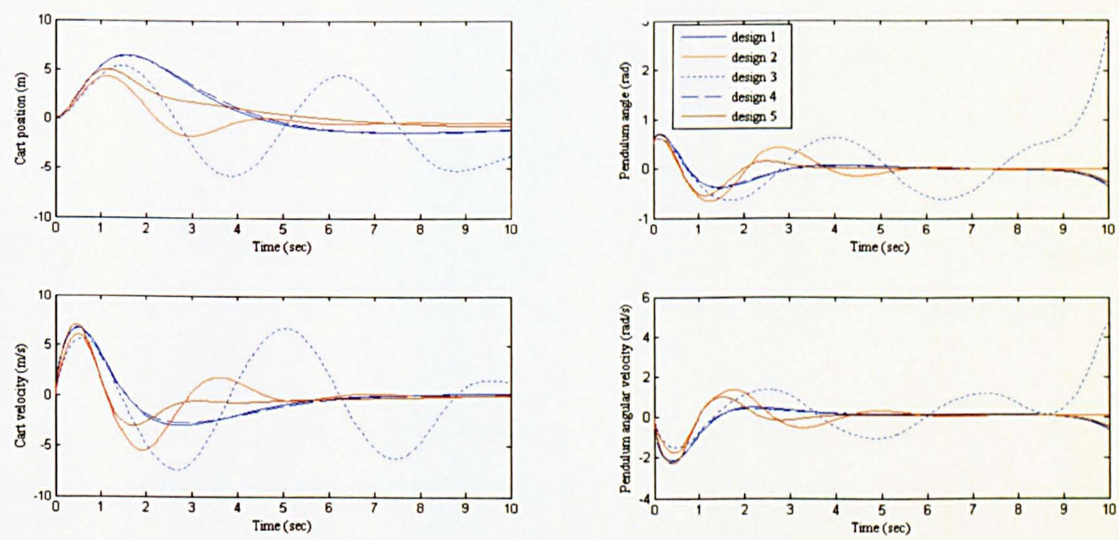


Figure 6.3.: The Inverted Pendulum on a Cart (Designs 1 & 2).



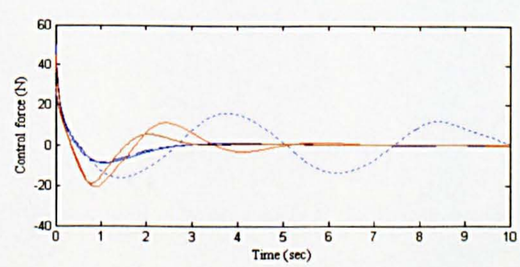


Figure 6.4.: The Inverted Pendulum on a Cart (Designs 1-5).

It appeared that the second design gave the most acceptable responses compared to the others; which logically implied that the larger the weighting on the exogenous disturbance input the better the robustness performance to meet the required specifications.

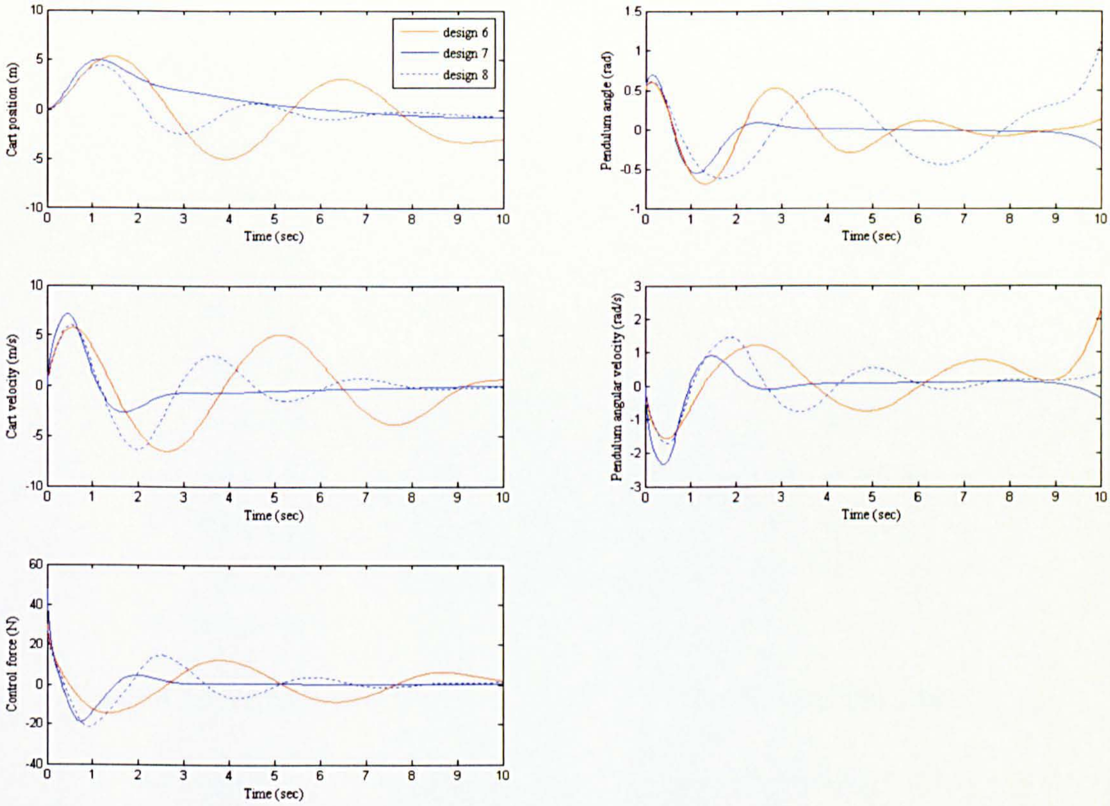


Figure 6.5.: The Inverted Pendulum on a Cart (Designs 6-8).

Having chosen $D_{21} := [1 \quad 1 \quad 1]^{-1}$, the next task would be to deliberately examine the relationship or rather the effect of D_{21} on C_1 and vice-versa. Accordingly, two scalars, δ and ε, are introduced to study their corresponding effect on the measurement output.

$$z(t) = \delta C_1 \, x(t) + \varepsilon D_{11} \, u(t); \tag{6.31}$$

where D_{11} was chosen to be equivalent to D_{21}.

Clearly, the previously shown design scenarios were carried-out with the scalars being set to one with:

$$C_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \& \quad D_{11} := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{6. 32}$$

Allowing δ varying from $0 \rightarrow 2$ and ε from $0 \rightarrow 1.5$, as shown in Tables (6.2) & (6.3) respectively, the different corresponding plots are illustrated in Figures (6.7-6.9).

$\delta = 0$
$\delta = 0.2$
$\delta = 0.5$
$\delta = 0.8$
$\delta = 1$
$\delta = 1.2$
$\delta = 1.3$
$\delta = 1.4$
$\delta = 1.5$
$\delta = 2$

Table 6.2.: Variations in δ.

$\varepsilon = 0$
$\varepsilon = 0.2$
$\varepsilon = 0.5$
$\varepsilon = 0.8$
$\varepsilon = 1$
$\varepsilon = 1.2$
$\varepsilon = 1.3$
$\varepsilon = 1.4$
$\varepsilon = 1.5$

Table 6.3.: Variations in ε.

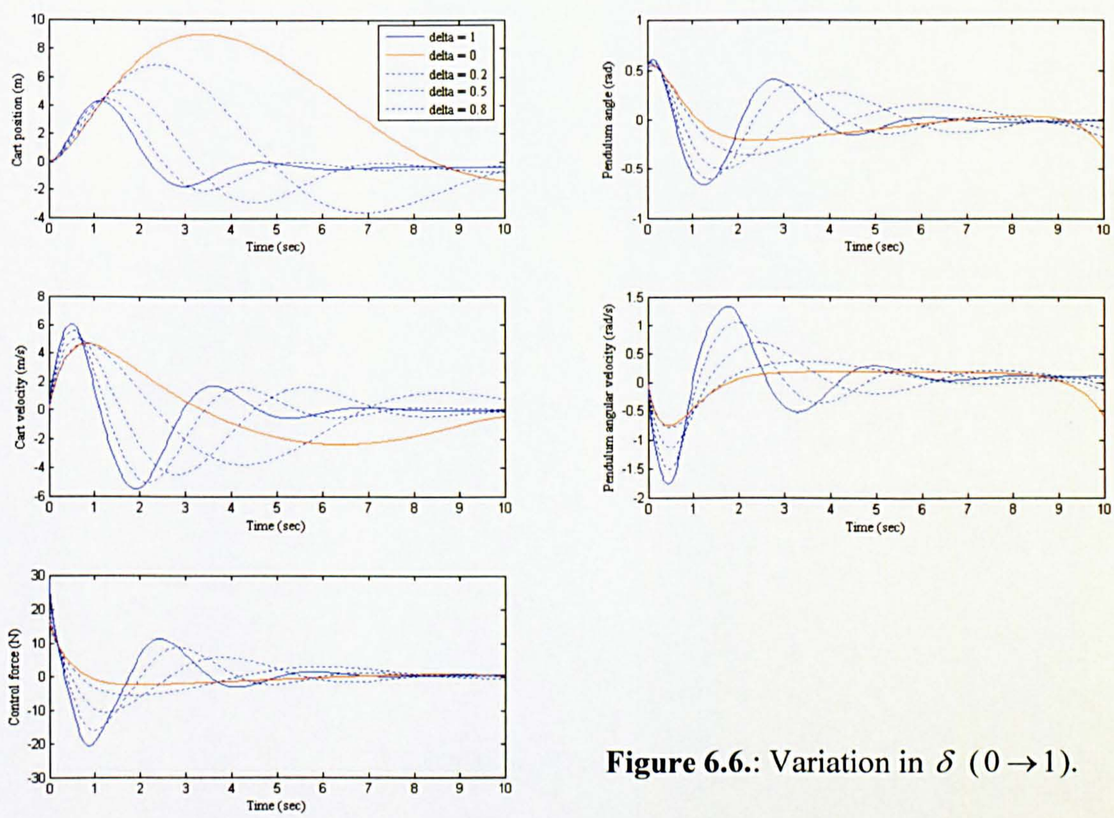


Figure 6.6.: Variation in δ $(0 \rightarrow 1)$.

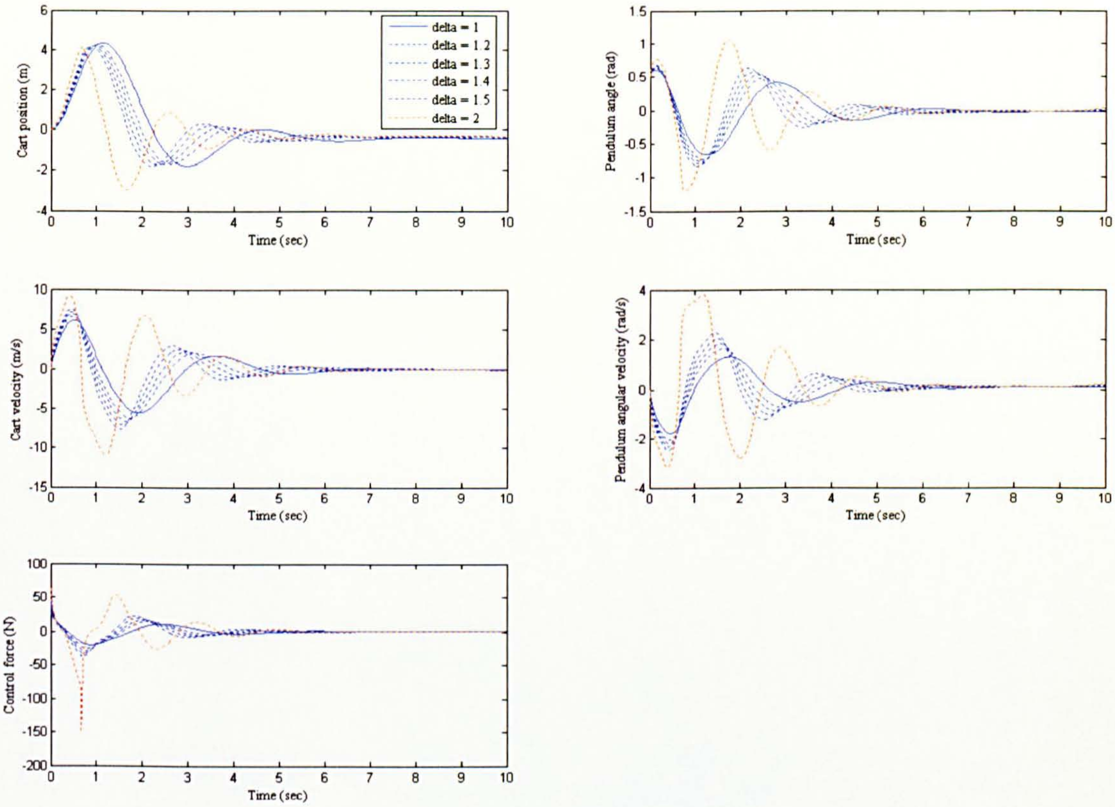


Figure 6.7.: Variation in δ $(1 \rightarrow 2)$.

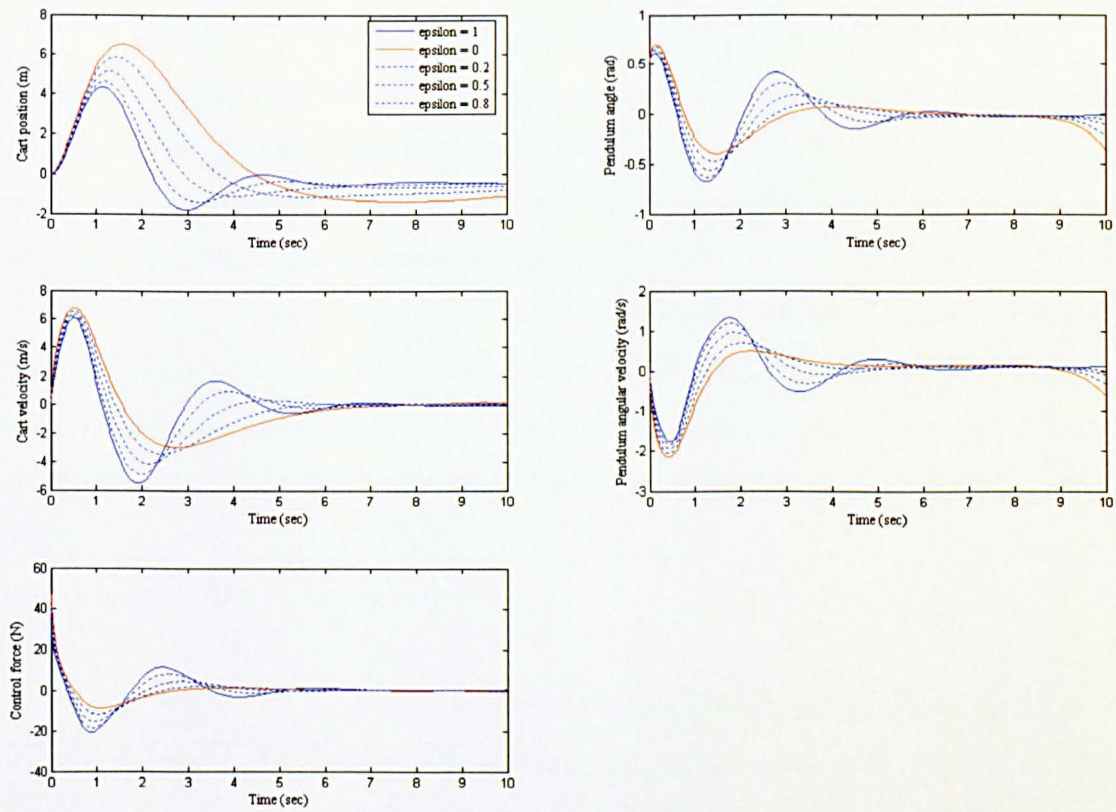


Figure 6.8.: Variation in ε $(0 \rightarrow 1)$.

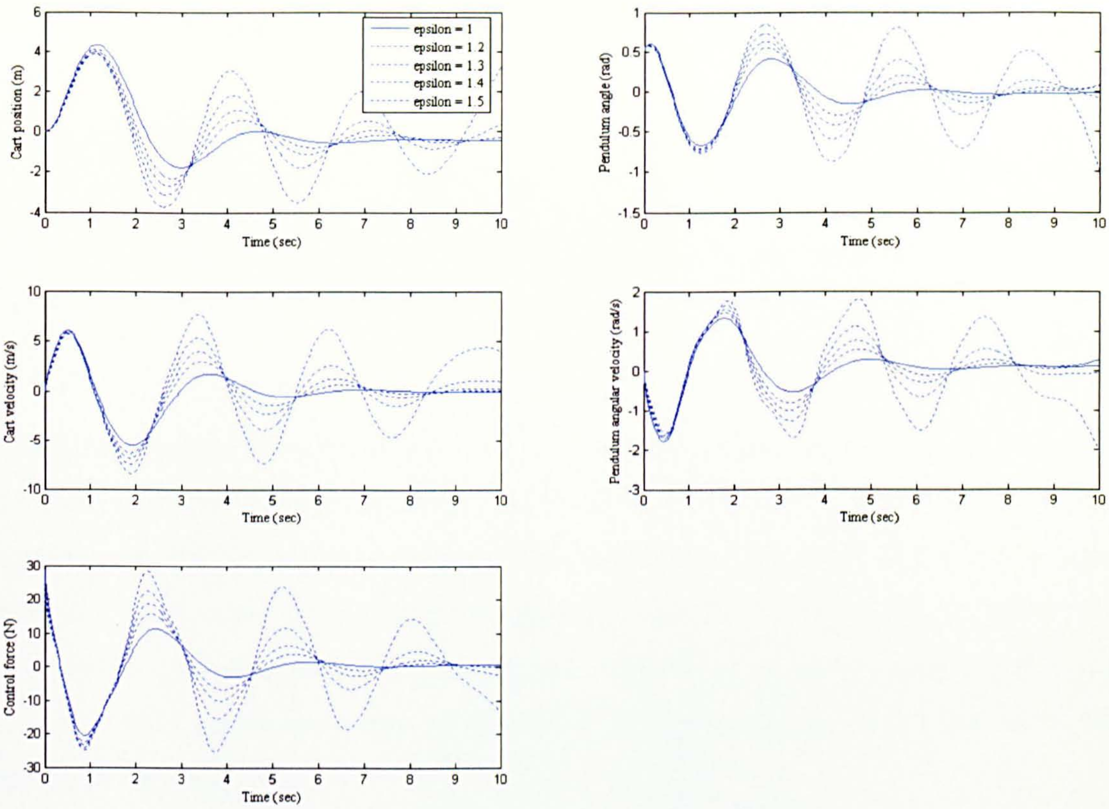


Figure 6.9.: Variation in ε $(1 \rightarrow 1.5)$.

It is clear that the weighting choices resulted in a family of state-regulator H_∞ controllers which had some pros and cons in terms of the robustness and stability requirements; for instance, it can be deduced that:

- as $\delta \rightarrow 0$ longer cart trajectory and time are required to stabilize both the pendulum's arm and cart although the control effort decreases.
- as $\delta \rightarrow 2$ the control effort increases while the resulting responses seem faster even though the same amount of time is required to stabilize the pendulum's arm and cart.
- as $\varepsilon \rightarrow 0$ a longer cart trajectory is needed to stabilize the systems. However, a faster steady-state response for the pendulum's angle is noticeable with a reduced control effort.
- as $\varepsilon \rightarrow 1.5$ the plant reaches instability.

As a final point, it is also imperative to investigate the performance of this designed controller against various realistic disturbances not only relying on the primarily chosen $w = 0.1$, and for this reason, five different values for the disturbance are considered. It is hence desirable to achieve a good level of disturbance attenuation against the following disturbance inputs that were also considered in Pan & Başar (1998):

- *Case 1*: $w = 0$,
- *Case 2*: $w = 0.1$,
- *Case 3*: Band limited white noise signal with power 0.01 and sample rate 5 Hz,
- *Case 4*: $w(t) = 0.1\sin(2\pi t/5)$, and
- *Case 5*: $w(t) = 0.1\cos(\pi t)$.

Figure (6.10) reveals the disturbance attenuation levels achieved by the controller for the various disturbance cases considered above. While the controlled responses for the linear time-varying disturbance in case 5 mimicked the disturbance-free ideal case, case 4 also showed good but oscillatory responses. However, the disturbance attenuations for the band limited white noise, in case 3, were unstable due to the friction coefficients that were already designed for and which added to the system's instability.

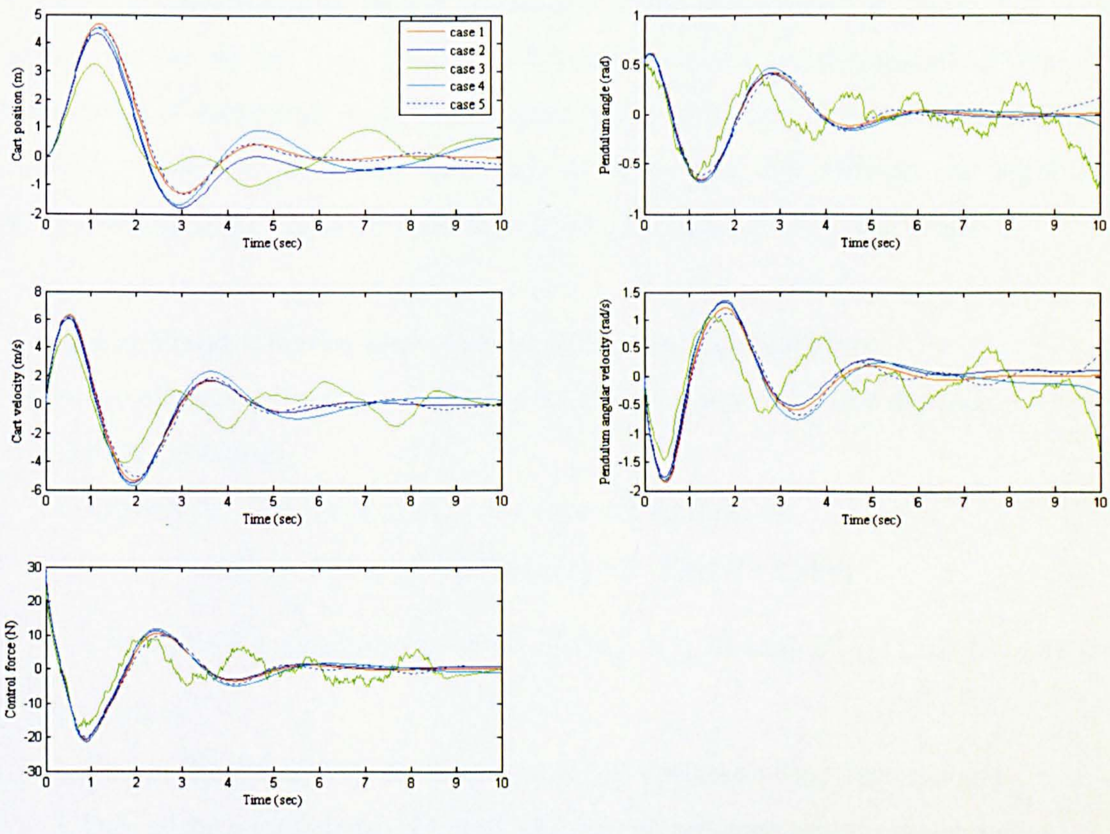


Figure 6.10.: Disturbance Attenuation of Exogenous Inputs.

6.6. Concluding Remarks

In this chapter the continuous-time full-information finite-horizon linear time-varying H_∞ control that appeared in Limebeer, *et al.*, (1992) was extended for nonlinear systems. As in the previous chapter, the approach uses the linear quadratic approximations to devise a state-feedback regulator. The proposed approach under a game theoretic framework takes into account the worst-case disturbances and path-wise constraints, if any. Although not considered in this thesis, constraints representing, for example, actuator saturation or the necessity to avoid dangerous operational regions in a process control or an aeronautics context can also be included in the design requirements if desired.

In the case when the states are not available for feedback, an observer-based controller can always be constructed using similar theoretical arguments, and which will facilitate the implementation of the control algorithms. The explicit construction of the class of robust H_∞ controllers which asymptotically regulate nonlinear systems and achieve pre-specified disturbance attenuation levels with respect to exogenous system inputs proved to be very effective in controlling the pendulum-cart system. The attenuation of exogenous disturbance inputs to the desired performance level(s) over the finite-time interval was also achieved. To conclude this chapter, an algorithmic interpretation of the proposed state-regulation H_∞ technique is given below.

Nonlinear Finite-Horizon State-Feedback H_∞ control algorithm:

Given a nonlinear mathematical model describing the dynamics of a dynamical system with initial conditions,

1. Express the dynamical system in a state-affine form as:

$$\dot{x}(t) = f(x,u,w) = A(x)x(t) + B(x)u(t) + Ew(t).$$

Note that the general operators $A(x(t),u(t),t)$ *and* $B(x(t),u(t),t)$ *are non-unique.*

2. Design the disturbance input matrix, E, in the state-affine form in (1).
3. Design the parameter(s), C_1 and D_{12}, in the regulated outputs equation:

$$z(t) = C_1\left(x(t)\right)x(t) + D_{12}\left(x(t)\right)u(t).$$

4. Choose the output matrix, C_2, and design the disturbance matrix, D_{21}, in the measured output equation:

$$y(t) = C_2\left(x(t)\right)x(t) + D_{21}\left(x(t)\right)w(t).$$

5. Introduce the sequence of linear time-varying approximations

a. For $i = 0$:

$$\dot{x}^{[0]}(t) = A(x_0)x^{[0]}(t) + B(x_0)u^{[0]}(t) + Ew^{[0]}(t), \qquad x^{[0]}(0) = x_0.$$

b. For $i > 0$:

$$\dot{x}^{[i]}(t) = A\left(x^{[i-1]}(t)\right)x^{[i]}(t) + B\left(x^{[i-1]}(t)\right)u^{[i]}(t) + Ew^{[i]}(t).$$

6. Solve the Riccati operator equation backwards in time for a sufficiently large γ,

$$\begin{aligned}
\dot{P}^{[i]}(t) = &-P^{[i]}(t)\Big(A\left(x^{[i-1]}(t)\right) - B\left(x^{[i-1]}(t)\right)D_{21}'\left(x^{[i-1]}(t)\right)C_1\left(x^{[i-1]}(t)\right)\Big) \\
&-\begin{pmatrix} A\left(x^{[i-1]}(t)\right) \\ -B\left(x^{[i-1]}(t)\right)D_{21}'\left(x^{[i-1]}(t)\right)C_1\left(x^{[i-1]}(t)\right)\end{pmatrix}' P^{[i]}(t) \\
&+P^{[i]}(t)\Big(B\left(x^{[i-1]}(t)\right)B'\left(x^{[i-1]}(t)\right) - \gamma^{-2}E\left(x^{[i-1]}(t)\right)E'\left(x^{[i-1]}(t)\right)\Big)P^{[i]}(t) \\
&-C_1\left(x^{[i-1]}(t)\right)\Big(I - D_{21}\left(x^{[i-1]}(t)\right)D_{21}'\left(x^{[i-1]}(t)\right)\Big)C_1\left(x^{[i-1]}(t)\right).
\end{aligned}$$

7. Update the control law over the finite-time interval $t \in \left[t_0, t_f\right]$,

$$u^{[i]}(t) = -\left(D_{21}'^{[i]}C_1^{[i]} + B'\left(x^{[i-1]}(t)\right)P^{[i]}(t)\right)x^{[i]}(t); \qquad \text{for } i \geq 0.$$

8. If (7) results in an acceptable closed-loop response, the family of stabilizing controllers that meet the robustness requirements is reached; if not, change γ and go to (5).

PART IV

PRACTICAL APPLICATIONS

CHAPTER 7

Some Practical Real-World Applications

7.1. Introduction

The aim of this chapter is to tackle a selection of model-based practical applications by means of the extended and/or developed theories that were proposed in previous chapters. It is intended, however, that the selected applications would give the reader a better understanding of the mathematical treatments considered in this thesis.

Accordingly, this chapter is divided into four main sections followed by some concluding remarks. While each main section is divided into three subsections, *i.e.* an introduction about the dynamical system is considered in §7.x.1, followed by the system dynamics' representation and numerical simulations that are given in §7.x.2, and finally, in §7.x.3 some conclusions are discussed.

In more details, the magnetic levitation control problem is considered in §7.2, and where the pole-placement robust stabilization technique of chapter 3 is proposed to stabilize the system. In §7.3 & §7.4 the wing rock lateral-instability model of a simple generic aircraft and a highly nonlinear helicopter model are considered respectively; while the H_∞ control law for both applications was devised by direct application of chapter 5. Last but not least, a hypersonic aircraft model is discussed in §7.5 and controlled with the more inclusive robust H_∞ control method of chapter 6. The chapter is then ended in §7.6 with a brief conclusion.

7.2. The Magnetic Levitation Control Problem

Nomenclature

x	Distance separating the ball from the electromagnet
x_d	Least admissible x (the threshold prior to being attracted by the electromagnet)
u	Input voltage of the amplifier in V
I	Current across the electromagnet in A
e	Voltage of the coil (electromagnet) in V
R	Resistance of the coil in Ω
k	Amplification gain from u to e of the amplifier
g	The acceleration due to gravity ($9.81\ m/s^2$)
m	Mass of the steel ball (0.54 Kg)

7.2.1. Introduction

The problem of robustly controlling (within sensor resolution) the height of a, 25 mm in diameter, steel ball from ground level by levitating it by means of an electromagnet is considered, as schematically shown in Figure (7.2.1), while using the introduced robust stabilization theoretical framework provided in chapter 3.

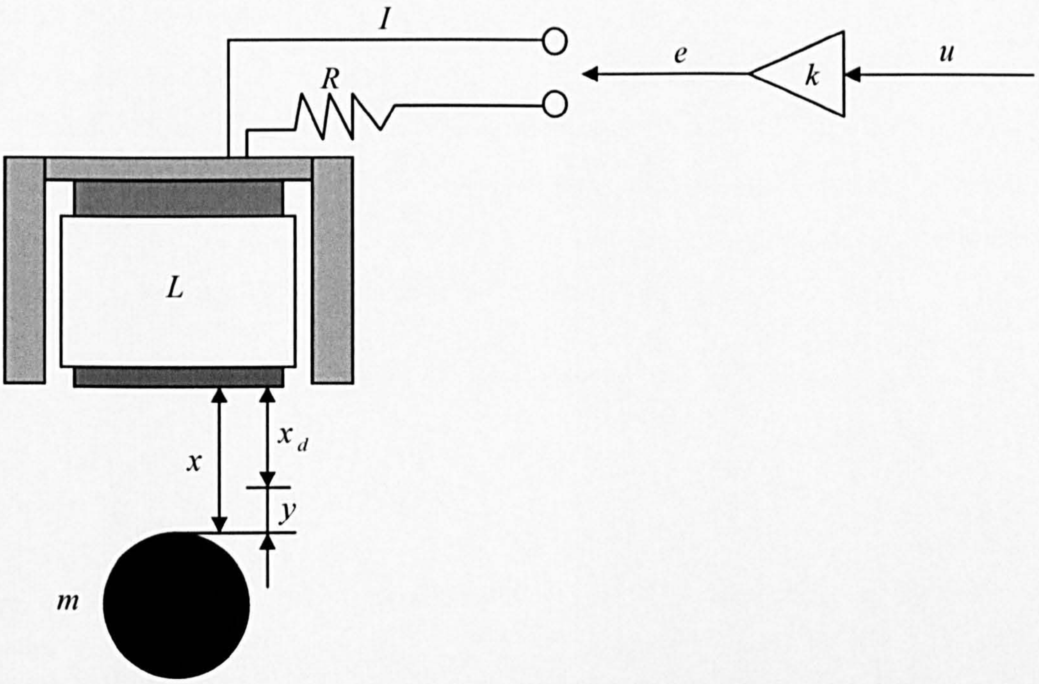


Figure 7.2.1.: The Magnetic Levitation Schematic Representation.

The Electromagnetically levitated and guided systems which follow the same control objective(s) as in the model considered herein are commonly used in the field of high-speed maglev passenger trains, levitation of wind tunnel models, vibration isolations of sensitive machinery, tool machines frictionless bearings and conveyor systems such as in levitating metal slabs during manufacturing for example (see [Barie & Chiasson, 1996] and the references therein). This technology offers many advantages amongst which are a very silent motion in the case of a low and/or high speed people transport vehicles and reduced rail maintenance, for instance.

With its great advantages and benefits, clean or 'green' maglev applications received lots of renewed interests particularly in transport technologies. In the world there are actually two working low speed systems: the Japanese HSST ([Seki, 1995], [Masaaki, 1995]) and the English BAMS (Birmingham Airport MagLev System [Nenadovic & Riches, 1985]). In both these magnetically levitated trains the guidance force needed to keep the vehicles on the track is obtained with the levitation electromagnets thanks to particular shapes of the rails and to a clever placement of the electromagnets with respect to the rails ([Fruechte, *et al.*, 1980]). Nonetheless with both an air drag limitation and an aerodynamic noise generation, modern super-speed maglev trains with their supersonic speeds seemed to compete with airplanes as evident by the pioneered Japanese commercial-type superconductive train.

7.2.2. System Dynamics & Simulations

In fact, the principle behind these above-mentioned technologies, amongst many others, follows the classical open-loop unstable anti-gravity magnetic levitation dynamical nonlinear model that is modelled by the following nonlinear differential equation (see [Zayadine, 1996; and Barie & Chiasson, 1996])

$$m\,\ddot{x}(t) = mg - k\frac{I^2}{x^2(t)}; \tag{7.2. 1}$$

$$e = RI + \frac{d}{dt}(LI), \tag{7.2. 2}$$

where $L(x) = \dfrac{Q}{x_\infty + x} + L_\infty$ with coefficients Q, x_∞, L_∞ that are determined by identification experiments.

It is common practise, to linearize this nonlinear system about its operating point (see [Barie & Chiasson, 1996] and the references therein). However, in the sequel the robust stabilization technique of chapter 3 is used to control the displacement of the steel ball that is governed by the electromechanical equation in (7.2.1), by defining the following relationship,

$$y(t) = x(t) - x_d(t). \qquad (7.2.3)$$

To avoid the problem of phase compensation due to the high inductance of the electromagnet, the active drive to the electromagnet can practically be current driven; that is:

$$u(t) = I^2(t). \qquad (7.2.4)$$

Indeed both position regulation and tracking controllers can be synthesized; however, only the regulation problem about a desired set-point is considered. Then by substituting Equations (7.2.4) & (7.2.3) in (7.2.1),

$$\ddot{y}(t) = g - \left(\frac{k}{m}\right)\left(\frac{u}{(y + x_d)^2}\right). \qquad (7.2.5)$$

Due to the forcing term in (7.2.5), being the gravity term (g), an air damping term $-\kappa(\dot{y})\dot{y}$ is included to render the application more suitable for the theoretical and practical implementation of chapter 3. Accordingly, (7.25) can be written as:

$$\ddot{y}(t) = -\dot{y}\kappa(\dot{y}) - \left(\frac{k}{m}\right)\left(\frac{u}{(y + x_d)^2}\right) + w(t); \qquad (7.2.6)$$

with

$$\kappa(\dot{y}) = (1 - \dot{y}). \qquad (7.2.7)$$

The exogenous disturbance input affecting the dynamical system can in fact be expressed as:

$$w(t) = g(1 - y(t)). \qquad (7.2.8)$$

By substituting Equations (7.2.8) & (7.2.7) in (7.2.6),

$$\ddot{y}(t) = -\dot{y}(t)(1 - \dot{y}(t)) - \left(\frac{k}{m}\right)\left(\frac{u}{(y(t) + x_d)^2}\right) + g(1 - y(t)). \qquad (7.2.9)$$

Remark 7.2.1. Equation (7.2.9) follows a damped harmonic oscillator general form $\ddot{y} + \nu\,\dot{y} = \lambda y$ (see *e.g.* [Smith & Jordan, 1999]).

To get the phase-plane portrait, set

$$y_1(t)=y(t) \qquad \& \qquad y_2(t)=\dot{y}(t); \tag{7.2. 10}$$

and substituting (7.2.10) in (7.2.9), the following state-space representation is reached,

$$\begin{pmatrix}\dot{y}_1(t)\\ \dot{y}_2(t)\end{pmatrix}=\begin{pmatrix}0 & 1\\ g & -\kappa(y_2(t))\end{pmatrix}\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix}+\begin{pmatrix}0\\ -\dfrac{k}{m(y_1+x_d)^2}\end{pmatrix}u(t)+g(1-y_1(t)); \tag{7.2. 11}$$

and the operators, $A(y(t))$ & $B(y(t))$, can be written respectively as,

$$A(y(t))=A_1+A_2(y(t)); \tag{7.2. 12}$$

and

$$B(y(t))=B_1+B_2(y(t)) \;; \tag{7.2. 13}$$

with

$$A_1=\begin{pmatrix}0 & 1\\ g & 0\end{pmatrix} \qquad \& \qquad A_2(y(t))=\begin{pmatrix}0 & 0\\ 0 & -\kappa(y_2(t))\end{pmatrix}; \tag{7.2. 14}$$

$$B_1=\begin{pmatrix}0\\ \dfrac{-k}{m\,x_d^2}\end{pmatrix} \qquad \& \qquad B_2(y(t))=\begin{pmatrix}0\\ -\dfrac{k}{m(y_1+x_d)^2}+\dfrac{k}{m\,x_d^2}\end{pmatrix}. \tag{7.2. 15}$$

The Eigenvalues of the closed-loop state feedback system are roots of

$$|\lambda I-A_1(y)|=\left|\begin{pmatrix}\lambda & -1\\ -g & \lambda\end{pmatrix}\right|=\lambda^2-g=0\Rightarrow\lambda=\pm\sqrt{g}. \tag{7.2. 16}$$

Recall that the control matrix is expressed as,

$$F(y(t))=F_1+F_2\,(y(t));$$

with

$$F_1=(f_1\;\; f_2), \qquad \& \qquad F_2\,(y(t))=(f_3\,(y(t))\;\; f_4\,(y(t))). \tag{7.2. 17}$$

Alternatively, the state-feedback control law takes the form

$$u(t)=-F(y(t))y(t)=-(f_1+f_3\,(y(t))\;\; f_2+f_4\,(y(t)))\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix}. \tag{7.2. 18}$$

Now, from (7.2.18) in (7.2.11),

$$\begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ g & -\kappa(y_2(t)) \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \dfrac{-k\left(f_1 + f_3\,(y(t))\right)}{m(y_1+x_d)^2} & \dfrac{-k\left(f_2 + f_4\,(y(t))\right)}{m(y_1+x_d)^2} \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$
$$+ w(t)\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}. \tag{7.2. 19}$$

But,

$$\vartheta(t) = A_2(y(t)) - B_1F_2(y(t)) - B_2(y(t))F_1 - B_2(y(t))F_2(y(t)). \tag{7.2. 20}$$

So by substituting Equations (7.2.14), (7.2.15) & (7.2.17) in (7.2.20) and re-arranging, the following equality is reached,

$$\vartheta(t) = \begin{pmatrix} \vartheta_{11}(t) & \vartheta_{12}(t) \\ \vartheta_{21}(t) & \vartheta_{22}(t) \end{pmatrix}; \tag{7.2. 21}$$

with

$$\vartheta_{11}(t) = \vartheta_{12}(t) = 0,$$

$$\vartheta_{21}(t) = \frac{k}{m}\left[\left(\frac{1}{(y_1+x_d)^2} - \frac{1}{x_d^2}\right)f_1 + \left(\frac{1}{(y_1+x_d)^2}\right)f_3\,(y(t))\right],$$

and

$$\vartheta_{22}(t) = -\kappa(y_2(t)) + \frac{k}{m}\left[\left(\frac{1}{(y_1+x_d)^2} - \frac{1}{x_d^2}\right)f_2 + \left(\frac{1}{(y_1+x_d)^2}\right)f_4\,(y(t))\right].$$

It follows that to get $\|\vartheta(t)\| = 0$, the controller matrix can be chosen to arbitrarily set $\vartheta_{21}(t) = \vartheta_{22}(t) = 0$ as

$$f_3\,(y(t)) = (y_1+x_d)^2\left(-\frac{1}{(y_1+x_d)^2} + \frac{1}{x_d^2}\right)f_1\ ; \tag{7.2. 22}$$

and

$$f_4\,(y(t)) = (y_1+x_d)^2\left[\frac{m}{k}\kappa(y_2(t)) + \left(-\frac{1}{(y_1+x_d)^2} + \frac{1}{x_d^2}\right)f_2\right]. \tag{7.2. 23}$$

While the linear time-invariant part of the robust controller follows from the closed-loop system $\Delta'(\lambda) \triangleq |\lambda I - (A_1 - B_1F_1)|$, *i.e.*,

$$\left| \begin{pmatrix} \lambda & -1 \\ -g - \dfrac{kf_1}{m\,x_d^2} & \lambda - \dfrac{kf_2}{m\,x_d^2} \end{pmatrix} \right| = \lambda^2 - \frac{kf_2}{m\,x_d^2}\lambda - g - \frac{kf_1}{m\,x_d^2} = s^2 - (\lambda_1 + \lambda_2)s + \lambda_1\lambda_2; \quad (7.2.\ 24)$$

resulting in,

$$f_1 = \frac{m\,x_d^2}{k}(-\lambda_1\lambda_2 - g), \quad (7.2.\ 25)$$

and

$$f_2 = \frac{m\,x_d^2}{k}(\lambda_1 + \lambda_2). \quad (7.2.\ 26)$$

Finally, Equations (7.2.22) & (7.2.23) can be expressed in terms of Equations (7.2.25) & (7.2.26) as follows,

$$f_3\,(y(t)) = \frac{m\,x_d^2}{k}(y_1 + x_d)^2 \left(-\frac{1}{(y_1 + x_d)^2} + \frac{1}{x_d^2} \right)(-\lambda_1\lambda_2 - g); \quad (7.2.\ 27)$$

and

$$f_4\,(y(t)) = \frac{m}{k}(y_1 + x_d)^2 \left[\kappa(y_2(t)) + x_d^2 \left(-\frac{1}{(y_1 + x_d)^2} + \frac{1}{x_d^2} \right)(\lambda_1 + \lambda_2) \right]. \quad (7.2.\ 28)$$

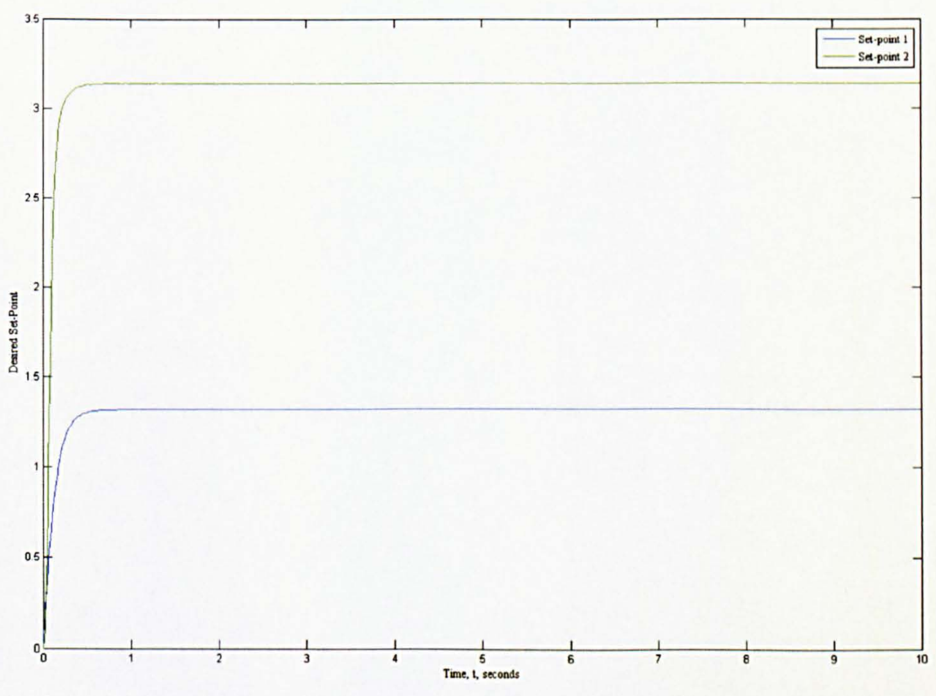


Figure 7.2.2.: Controlled Nonlinear Maglev Problem.

Accordingly, by tuning the amplification gain of the amplifier, the response of the maglev system with zero initial conditions can be simulated using MATLAB® for the following Eigenvalues $\lambda_1=-4$ and $\lambda_2=-7$ with a sufficiently small air damping constant. Figure (7.2.2) shows the controlled response of the system for two different desired set-points, *i.e.* x_d=1.25cm and x_d=3.25cm, over the finite-time interval $[0,10]$.

7.2.3. Conclusion

In this section the maglev nonlinear control problem was robustly controlled using the pole-assignment technique of chapter 3. In view of the simplistic but practical model considered in this section the recursive approximation theory was not employed but instead mathematical manipulations were accomplished by hand. However, in general, it is advisable to adopt the approximating sequences in case higher-dimensions or more sophisticated nonlinear systems are to be considered. As such, the practical implementation in a real-life setting would be rendered much easier. Furthermore, as a future work, servomechanism problems could be considered in case an input signal, *e.g.* a sinusoidal wave, is to robustly be followed.

7.3. The Lynx Helicopter

Nomenclature

u, v, w	Fuselage x, y, z-axis velocity components respectively (m/s)
p, q, r	Fuselage x, y, z-axis angular velocity components respectively (rad/s)
ϕ, θ, ψ	Fuselage attitude, Euler angles (rad)
g	Acceleration due to gravity (m/s^2)
m_s	Mass of helicopter (Kg)
X, Y, Z	External aerodynamic forces acting along the x, y, z-axis (N)
X_R, Y_R, Z_R	Main rotor aerodynamic forces (N)
X_T, Y_T, Z_T	Tail rotor aerodynamic forces (N)
X_F, Y_F, Z_F	Fuselage aerodynamic forces (N)
X_{tp}, Y_{tp}, Z_{tp}	Tail plane aerodynamic forces (N)
X_{fn}, Y_{fn}, Z_{fn}	Fin aerodynamic forces (N)
L, M, N	Aerodynamic moments about the centre of gravity (c.g.)
L_R, M_R, N_R	Main rotor aerodynamic moments about the c.g. (Nm)
L_T, M_T, N_T	Tail rotor aerodynamic moments about the c.g. (Nm)
L_F, M_F, N_F	Fuselage aerodynamic moments about the c.g. (Nm)
L_{tp}, M_{tp}, N_{tp}	Tail plane aerodynamic moments about the c.g. (Nm)
L_{fn}, M_{fn}, N_{fn}	Fin aerodynamic moments about the c.g. (Nm)
θ_0	Main rotor collective pitch (rad)
θ_{0T}	Tail rotor collective pitch (rad)
θ_{1c}	Lateral cyclic pitch (rad)
θ_{1s}	Longitudinal cyclic pitch (rad)

7.3.1. Introduction

As commonly known, helicopters have a dynamical behaviour that is hard to control due to their minimum-phase behaviour. Unlike aircraft mechanisms, helicopters have the ability to hover as well as to move under a fully controlled directional motion. This in fact is due to their propulsive, lift and control forces that can be generated throughout their flights regardless of speed. It follows that helicopters represent a challenging control problem that is highly complex due to their high-dimensional, asymmetric, nonlinear dynamical models.

The complexity of this mechanical system is amplified due to the robustness requirements that are mainly imposed by the ability of those rotorcrafts to withstand the generated vibrations from their rotor assemblies without breakdown (see [Benson & Flowers, 1988]). The controlled physical nonlinear system must behave in a robust manner not only against transient resonances imposed by small perturbations but also against the more important large perturbations that are generated by wind gusts, for example, to avoid limit cycles.

The three different flight modes that range from hovering, vertical flying or forward flying makes designing stable state-feedbacks for such autonomous flying systems, at a theoretical level, a difficult task. Nonetheless, there has been a great deal of publications to stabilize rotorcrafts. For some previous works that were developed for control problems in helicopters the reader is referred to Vilchis, *et al.*, (1997) and the references therein. Additionally, some recent research directions can be found in Shin, *et al.*, (2005) where the authors designed a model-based controller for a fully autonomous small-scale unmanned helicopter system based on the Kalman filter Linear Quadratic Integral (LQI) theory. A nonlinear sliding-mode controller structure for the design of a flight control system for a PUMA helicopter appeared in McGeogh, *et al.*, (2004). Lozano, *et al.*, (2004) presented a discrete-time prediction-based state-feedback controller for the yaw angular displacement of a 4-rotor mini-helicopter. Lee, *et al.*, (2005) designed and evaluated a helicopter trajectory controller using feedback linearization technique relying on the two time-scale separation principle. Moreover, multivariable control of various helicopter motions was considered in Walker (2003); and nonlinear adaptive output-regulations for rotorcrafts appeared in Isidori, *et al.*, (2003). While the H_∞ controllers also played an essential role in stabilizing helicopters (see, for *e.g.*, [Luo, *et al.*, 2003; Postlethwaite, *et al.*, 2005; Postlethwaite, *et al.*, 1998; and Turner, *et al.*, 2001]).

However, in this section, the multi-role Westland Lynx MK7 helicopter which is an under-actuated, highly-agile dynamical system due to its semi-rigid four-bladed main and tail rotor systems, and exhibiting highly nonlinear behaviour with inter-axis coupling (see [Turner, *et al.*, 2001]) is considered using the theoretical approach of chapter 5. The general panorama of some of the aerodynamic forces and torques acting on this helicopter is depicted in Figure (7.3.1).

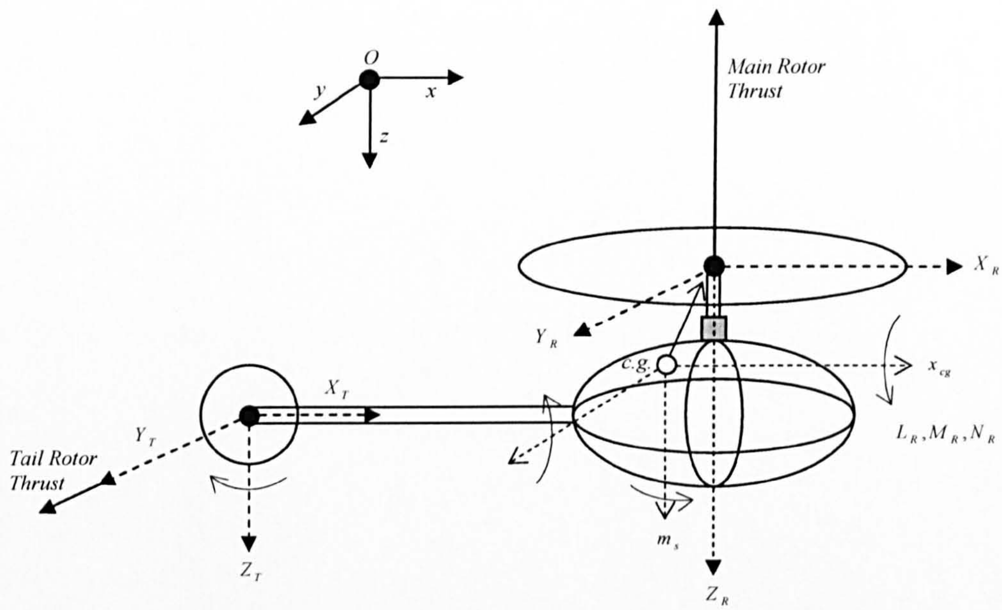


Figure 7.3.1.: The Lynx Helicopter.

7.3.2. System Dynamics & Simulations

The modelled flight mechanics in the sequel assumes rigid rotor blades with sprung hinges at the rotor's centre, as it is common in many other helicopter models (Turner, *et al.*, 2001). The Westland Lynx MK7 helicopter can be modelled by (see [Padfield, 1996; and Luo, *et al.*, 2003):

- **Force Equations**

$$\dot{u} = rv - qw + \frac{X}{m_s} - g\sin\theta , \tag{7.3. 1}$$

$$\dot{v} = pw - ru + \frac{Y}{m_s} + g\sin\phi\cos\theta , \tag{7.3. 2}$$

$$\dot{w} = qu - pv + \frac{Z}{m_s} + g\cos\phi\cos\theta . \tag{7.3. 3}$$

- **Moment Equations**

$$\dot{p} = (c_1 r + c_2 p) + c_3 L + c_4 N , \tag{7.3. 4}$$

$$\dot{q} = c_5 pr - c_6 \left(p^2 - r^2\right) + c_7 M , \tag{7.3. 5}$$

$$\dot{r} = (c_8 p - c_2 r) q + c_4 L + c_9 N . \tag{7.3. 6}$$

- **Attitude Equations**

$$\dot{\phi} = p + (q\sin\phi + r\cos\phi)\tan\theta, \quad (7.3.7)$$

$$\dot{\theta} = q\cos\phi - r\sin\phi, \quad (7.3.8)$$

$$\dot{\psi} = \frac{q\sin\phi + r\cos\phi}{\cos\theta}. \quad (7.3.9)$$

With the coefficients appearing in the moment equations defined by:

$$c_1 = \frac{(I_{yy} - I_{zz})I_{zz} - I_{xz}^2}{\Gamma},$$

$$c_2 = \frac{(I_{xx} - I_{yy} + I_{zz})I_{xz}}{\Gamma},$$

$$c_3 = \frac{I_{zz}}{\Gamma},$$

$$c_4 = \frac{I_{xz}}{\Gamma},$$

$$c_5 = \frac{I_{zz} - I_{xx}}{\Gamma},$$

$$c_6 = \frac{I_{xz}}{I_{yy}},$$

$$c_7 = \frac{1}{I_{yy}},$$

$$c_8 = \frac{I_{xx}(I_{xx} - I_{yy}) + I_{xz}^2}{\Gamma},$$

$$c_9 = \frac{I_{xx}}{\Gamma},$$

$$\Gamma = I_{xx}I_{zz} - I_{xz}^2.$$

The aerodynamic, gravity and propulsion contributions are described by five subsystems:

$$X = X_R + X_T + X_F + X_{tp} + X_{fn}, \quad (7.3.10)$$

$$Y = Y_R + Y_T + Y_F + Y_{tp} + Y_{fn}, \quad (7.3.11)$$

$$Z = Z_R + Z_T + Z_F + Z_{tp} + Z_{fn}, \quad (7.3.12)$$

$$L = L_R + L_T + L_F + L_{tp} + L_{fn}, \quad (7.3.13)$$

$$M = M_R + M_T + M_F + M_{tp} + M_{fn}, \tag{7.3. 14}$$

$$N = N_R + N_T + N_F + N_{tp} + N_{fn}. \tag{7.3. 15}$$

While the pilot's four primary control inceptors are:

- The cyclic stick which is used to control both longitudinal and lateral cyclic, influencing the pitch and roll.
- The collective stick which is used to control the main rotor collective, influencing the vertical flight.
- The pedals which are used to control the tail rotor collective blade angles, influencing the yaw.

The helicopter model can be constructed in the state-space using the configuration data provided in Luo, *et al.*, (2003) and Equations (7.3.1 to 7.3.9) in the affine form:

$$\dot{x}(t) = A\big(x(t)\big)x(t) + B(t)u(t). \tag{7.3. 16}$$

Where

$$x(t) \triangleq \begin{bmatrix} u & v & w & p & q & r & \phi & \theta & \psi \end{bmatrix}^T \tag{7.3. 17}$$

with the non-unique operators $A \in \Re^{9\times 9}$ and $B \in \Re^{9\times 4}$; and plausibly expressed by:

$$A\big(x(t)\big) := \begin{bmatrix}
X'_u & X'_v{+}r & X'_w{-}q & X'_p & X'_q & X'_r & 0 & -g\ \mathrm{sinc}\theta & 0 \\
Y'_u{-}r & Y'_v & Y'_w{+}p & Y'_p & Y'_q & Y'_r & g\ \mathrm{sinc}\phi\cos\theta & 0 & 0 \\
Z'_u{+}q & Z'_v{-}p & Z'_w & Z'_p & Z'_q & Z'_r & -Z'_u{-}Z'_v{-}Z'_w{-}Z'_p{-}Z'_q{-}Z'_r & 0 & 0 \\
L'_u & L'_v & L'_w & L'_p{+}c_2 & L'_q & L'_r{+}c_1 & 0 & 0 & 0 \\
M'_u & M'_v & M'_w & M'_p{-}c_6 p & M'_q & M'_r{+}c_5 p+c_6 r & 0 & 0 & 0 \\
N'_u & N'_v & N'_w & N'_p & N'_q{+}c_8 p-c_2 r & N'_r & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & \cos\phi\tan\theta & q\ \mathrm{sinc}\phi\tan\theta & 0 & 0 \\
0 & 0 & 0 & 0 & \cos\phi & -\sin\phi & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \dfrac{\sin\phi}{\cos\theta} & \dfrac{\cos\phi}{\cos\theta} & 0 & 0 & 0
\end{bmatrix} \tag{7.3. 18}$$

with a linear $B(t)$, for simplicity:

$$B(t)=\begin{bmatrix} X'_{\theta_0} & X'_{\theta_{1s}} & X'_{\theta_{1c}} & X'_{\theta_{0T}} \\ Y'_{\theta_0} & Y'_{\theta_{1s}} & Y'_{\theta_{1c}} & Y'_{\theta_{0T}} \\ Z'_{\theta_0} & Z'_{\theta_{1s}} & Z'_{\theta_{1c}} & Z'_{\theta_{0T}} \\ L'_{\theta_0} & L'_{\theta_{1s}} & L'_{\theta_{1c}} & L'_{\theta_{0T}} \\ M'_{\theta_0} & M'_{\theta_{1s}} & M'_{\theta_{1c}} & M'_{\theta_{0T}} \\ N'_{\theta_0} & N'_{\theta_{1s}} & N'_{\theta_{1c}} & N'_{\theta_{0T}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (7.3.\ 19)$$

The constant subsystems contributions appearing in $A(x(t))$ and $B(t)$ are given in Tables (7.3.2) & (7.3.2) respectively.

$X'_u=-0.0191$	$L'_u=0.0130$
$X'_v=-0.0008$	$L'_v=-0.2290$
$X'_w=0.0170$	$L'_w=0$
$X'_p=-0.3371$	$L'_p=-10.6199$
$X'_q=0.3839$	$L'_q=-3.0470$
$X'_r=0$	$L'_r=-0.0333$
$Y'_u=0.0010$	$M'_u=0.0405$
$Y'_v=-0.0349$	$M'_v=0.0024$
$Y'_w=-0.0017$	$M'_w=-0.0026$
$Y'_p=-0.4032$	$M'_p=0.5281$
$Y'_q=-0.3381$	$M'_q=-1.8394$
$Y'_r=0.1168$	$M'_r=-0.0015$
$Z'_u=0.0136$	$N'_u=0.0020$
$Z'_v=-0.0017$	$N'_v=0.0039$
$Z'_w=-0.2994$	$N'_w=0.0060$
$Z'_p=-0.0257$	$N'_p=-1.8554$
$Z'_q=0.0237$	$N'_q=-0.5412$
$Z'_r=0$	$N'_r=-0.3487$

Table 7.3.1.: Aerodynamic Subsystem Contributions.

$X'_{\theta_0}=5.2424$	$L'_{\theta_0}=7.5007$
$X'_{\theta_{1s}}=-10.3456$	$L'_{\theta_{1s}}=-27.2884$
$X'_{\theta_{1c}}=1.0793$	$L'_{\theta_{1c}}=-156.4425$
$X'_{\theta_{0T}}=0$	$L'_{\theta_{0T}}=-1.069$
$Y'_{\theta_0}=-0.3885$	$M'_{\theta_0}=-1.5019$
$Y'_{\theta_{1s}}=-1.082$	$M'_{\theta_{1s}}=27.09$
$Y'_{\theta_{1c}}=-10.3713$	$M'_{\theta_{1c}}=-4.7239$
$Y'_{\theta_{0T}}=4.7239$	$M'_{\theta_{0T}}=-0.1857$
$Z'_{\theta_0}=-87.010103$	$N'_{\theta_0}=17.7373$
$Z'_{\theta_{1s}}=-0.7293$	$N'_{\theta_{1s}}=-4.8969$
$Z'_{\theta_{1c}}=0.0755$	$N'_{\theta_{1c}}=-27.9728$
$Z'_{\theta_{0T}}=0$	$N'_{\theta_{0T}}=-12.9304$

Table 7.3.2.: Propulsion Subsystem Contributions.

The pitch control is:

$$u(t)=\left[\theta_0 \quad \theta_{1s} \quad \theta_{1c} \quad \theta_{oT}\right]^T \tag{7.3. 20}$$

REMARK 7.3.1. In view of the insufficient exact aerodynamic, gravity and propulsion contributions in the literature as relating to the Lynx helicopter model that is represented by Equations (7.3.1 to 7.3.9), the author made use of the Fourier's linearized system provided in Luo, *et al.*, (2003) based on the configuration data in Table (7.3.3) to calculate the coefficients shown in Tables (7.3.1) & (7.3.2), and include them in the state-affine nonlinear system (7.3.16).

Main Rotor	$a_{0T}=6.0$ $K_\beta=166352$ $C_{D_z}=37.983$	$c=0.391\ m$ $n_b=4$ $\gamma=7.12$	$h_R=1.274\ m$ $R=6.4\ m$ $\gamma_s=0.0698\ rad$	$I_R=2712.56\ Kg\ m^2$ $S=0.0778$ $\lambda_\beta=1.0922$	$I_\beta=678.14\ Kg\ m^2$ $C_{D_x}=0.009$ $\theta_{tw}=-0.14\ rad/m$
Tail Rotor	$a_{0_T}=6.0$ $K_{\beta_T}=16635.2$ $S_T=0.208$ $\lambda_{\beta_T}=1.2236$	$C_T=0.1806\ m$ $l_T=7.66\ m$ $C_{D_{0T}}=0.008$	$g_T=5.8$ $k_{\lambda_T}=0$ $C_{D_{1T}}=5.334$	$h_T=1.146\ m$ $n_{b_T}=4$ $\delta_3=-0.7854\ rad$	$I_{\beta_T}=0.7467\ \mathrm{Kg}\ m^2$ $R_T=1.106\ m$ $\gamma_T=2.66$
Fuselage	$I_{xx}=2767.1\ \mathrm{Kg}\ m^2$ $S_P=19.6047\ m^2$	$I_{xz}=2034.8\ \mathrm{Kg}\ m^2$ $S_s=24.8701\ m^2$	$I_{yy}=13904.5\ \mathrm{Kg}\ m^2$ $l_f=12.06\ m$	$I_{zz}=2034.8\ \mathrm{Kg}\ m^2$ $x_{cg}=-0.0198\ m$	$k_{\lambda_f}=0$ $m_s=4313.7\ Kg$
Tail Plane	$a_{0_{tp}}=3.5$ $\alpha_{tp_0}=-0.0175$	$h_{tp}=0\ m$	$k_{\lambda_{tp}}=0$	$l_{tp}=7.66\ m$	$S_{tp}=1.197\ m^2$
Vertical Fin	$a_{0_{fn}}=3.5$	$h_{fn}=1.274\ m$	$l_{fn}=7.48\ m$	$S_{fn}=1.107\ m^2$	$\beta_{fn_0}=-0.0524$
Engine	$K_3=10000$ $\tau_{e_1}=0.6\ s$	$\Omega_{idle}=35.63\ rad/s$	$Q_{e_{max}}=4591\ Nm$	$\tau_{e_2}=0.025\ s$	$\tau_{e_3}=0.1\ s$

Table 7.3.3.: Configuration Data (*SI* units) for the Westland Lynx Helicopter.

Recalling the theory of Chapter 5, then the nonlinear state-feedback H_∞ algorithm can be applied recursively to update the hovering controller:

$$u^{[i]}(t)=-B^{*}\left(x^{[i-1]}(t)\right)P^{[i]}(t)x^{[i]}(t). \tag{7.3. 21}$$

Using MATLAB®, the simulation in Figure (7.3.2) shows the controlled responses about the hovering trim condition in (7.3.22) after seven iterations for $\gamma=9$ with an exogenous disturbance input matrix, E, that took the designed form of I_9; with a scalar time-invariant disturbance input of $w=0.5$. Whereas, the controlled outputs were θ,ϕ,r; with measured outputs θ,ϕ,r,p,q with corresponding unitary weighting matrices.

$$\begin{bmatrix} u & v & w & p & q & r & \phi & \theta & \Omega & Q_e & \theta_0 & \theta_{1s} & \theta_{1c} & \theta_{0T} \end{bmatrix}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -3.15^\circ \\ 3.43^\circ \\ 34.0261 \\ 20630 \\ 14.73^\circ \\ 0.55^\circ \\ 0.16^\circ \\ 10.26^\circ \end{bmatrix} \quad (7.3.\ 22)$$

where Ω is the main rotor speed in (*rad/sec*) and Q_e is the engine torque in (*Nm*).

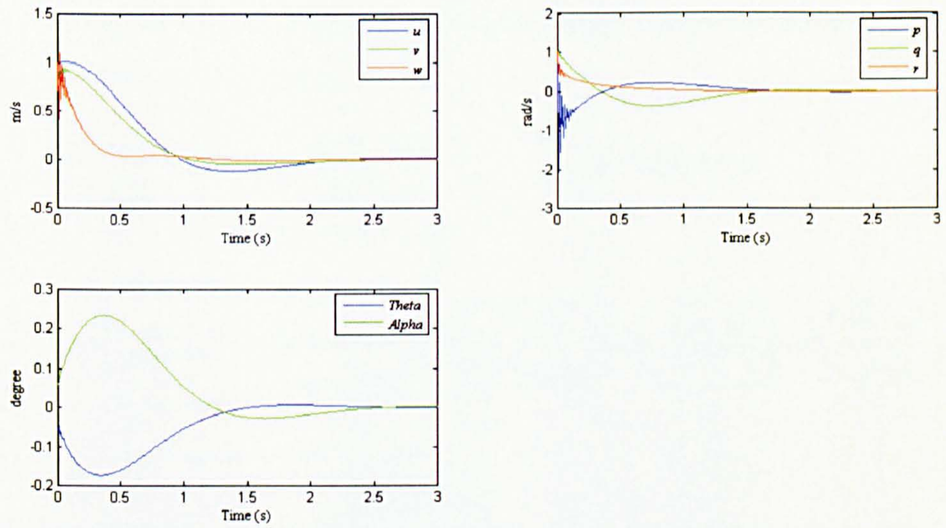


Figure 7.3.2.: Controlled Response at Hovering.

7.3.4. Conclusion

The highly nonlinear six-degrees-of-freedom Lynx helicopter model, with very fast-response dynamics, was controlled using the developed and proposed H_∞ stabilization technique of chapter 5. The response proved highly robust against the disturbance input although some minute oscillations were noticeable at the initiation of the controller. With comparison to Luo, *et al.*, (2003), the simulated steady-state stable responses of the 9-states herein were reached ten-times faster. In different words, by means of the devised state-feedback, the Lynx helicopter reached its stable hovering condition in about two seconds as compared to twenty seconds in Luo, *et al.*, (2003).

Handling qualities' requirements of ADS-33C (AVSCOM, 1989) have provided, over the years, a focus for research efforts with relation to rotorcrafts flight control problems from both industrial and academic perspectives; and it is believed that the proposed flight control architecture in this section would meet such minimum requirements due to the achieved robust stability and robust performance. Furthermore, by adjusting the weighting matrices in the H_∞ formulation extra requirements can be met for an actual implementation and to guard against any unwanted perturbations and/or limit cycles.

The proposed state-feedback controller for the Lynx helicopter can equivalently be applied for an autopilot or a fly-by-wire setting as schematically illustrated in Figure (7.3.3); a setting which was proposed and successfully applied by Postlethwaite, *et al.*, (2005) for the Bell 205 helicopter.

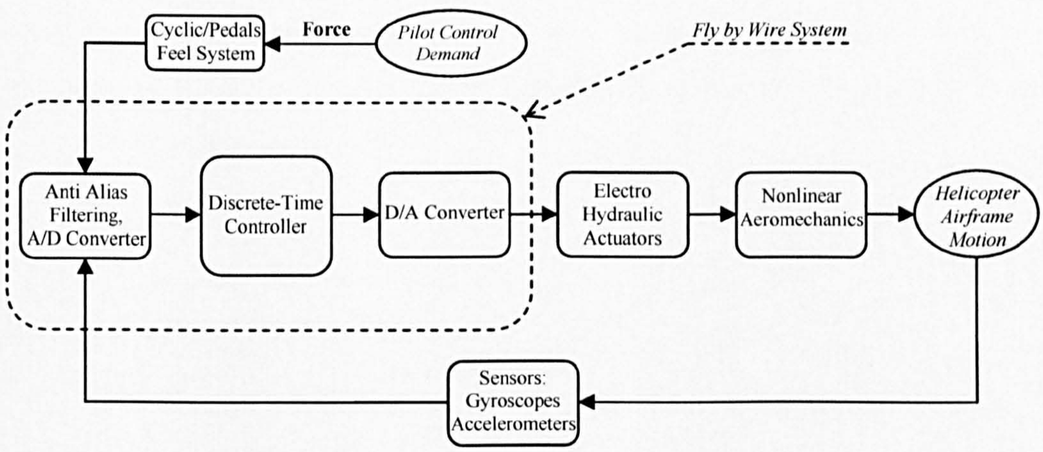


Figure 7.3.3.: Schematic Fly-By-Wire Representation.

To conclude, rotor dynamics in a rotorcraft exhibit highly complex fluid dynamic unsteady behaviours due to vortex flow formations that are described by Navier-Stokes equations (see [Conlisk, 2001; and Le Bouar, *et al.*, 2004]); and as a future work, it would be motivating to include those fundamental aeromechanics in the control architecture for more sophisticated 12 degree-of-freedom models such as the Bell 205 helicopter in Postlethwaite, *et al.*, (2005), for example, while studying different flight modes.

7.4. The Wing Rock Phenomenon Including Yawing Motion

Nomenclature

$x_1(t)$	Bank angle (rad)
$x_2(t)$	Roll-rate (rad/s)
$x_3(t)$	Aileron deflection angle (rad)
$x_4(t)$	Sideslip angle (rad)
$x_5(t)$	Sideslip-rate (rad/s)
k	Aileron-actuator's time-constant

7.4.1. Introduction

Being one type of lateral-directional instability for airplanes flying at subsonic speeds and high angles of attack, wing rock occurs for both low and high-aspect ratio configurations as shown in Figure (7.3.1) (see [Hsu & Lan, 1985]). In fact, the onset of wing rock is often the limiting factor behind the maximum angle of attack an aircraft can exhibit in parts of their flight envelopes, instead of the stall occurrence (see [Konstadinopoulos, *et al.*, 1985]).

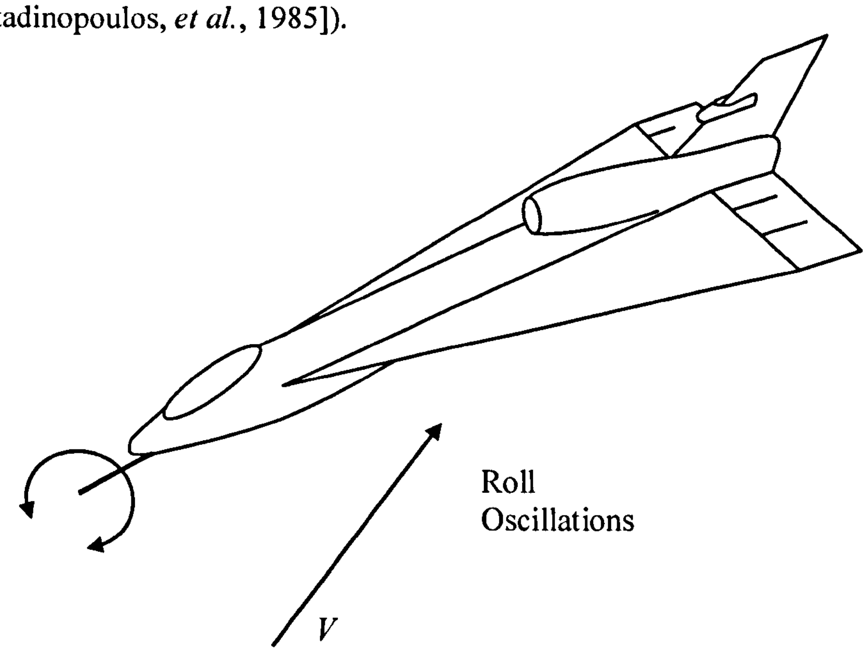


Figure 7.4.1.: The Wing Rock Phenomenon.

It is due to such a hindering aspect as well as the desire for '*super-manoeuvrability*', increased speed and efficiency of military aircrafts, in particular, (see for example [Nelson & Pelletier, 2003]) and/or crew return vehicles (CRVs), that research efforts have been pre-occupied with this phenomenal behaviour for the last few decades.

There has been a constant effort for accurately determining the dynamical and complex model behind the wing rock mainly through flight-tests or wind-tunnel measurements since most available models are mostly constructed based on physical insight (see [Manor & Wentz, 1985; Guglieri & Quagliotti, 1996; Katz, 1999; Saad, *et al.*, 2002; and Tan & Lan, 1996]). Generally speaking, flight dynamic phenomena that limit the aircraft's manoeuvring capability, such as wing rock, wing drop, nose slice and buffet are only discovered during flight testing and resolved using the "quick fix approach" (Nelson & Pelletier, 2003). In that sense, robustness has become essential in order to resolve the model error issue and avoid any plausible degradation of the vehicle's performance.

So far, various control methodologies have been employed in the literature to control the wing rock motion regardless of the type of wings and/or the model's degree-of-freedom. Singh, *et al.*, (1995), used adaptive and neural control techniques for slender delta wings. Sreenatha, *et al.*, (2000), used fuzzy logic for an approximate second order slender delta wing rock. Gain scheduling is used in Ordŏñez and Passino, (2003), to avoid the problem of fixed angle of attack. In Shue, *et al.*, (2000), the robust control problem with state feedback is cast in terms of a Hamilton-Jacobi-Bellman inequality. Shue, *et al.*, (1996), used optimal feedback control. Aruajo, *et al.*, (1998), used variable structure adaptive control. Monahemi & Krstic, (1996), used adaptive feedback linearization. Crassidis, (1999), used model-error control synthesis. Then again, the proposed H_∞ theory of chapter 5 is employed in this framework to control a fighter aircraft's wing rock motion.

Wing rock is an un-commanded roll-yaw oscillation which is initiated either with a sideslip or during a zero-sideslip flight with some asymmetries in the flow over the fighter aircraft (Hsu & Lan, 1985). In other words, the phenomenon is a self-sustaining limit-cycle oscillation with a limited amplitude occurring as a result of the nonlinear coupling between the dynamic response and the unsteady aerodynamic forces as shown in Figures (7.4.2 to 7.4.4). The loss of damping in roll at high angles of attack often characterizes the onset of wing rock. The wing rock motion is usually the result of the coupling of several degrees-of-freedom, adding to the complexity of the motion.

Statistically, the wing rock phenomenon is traced back to some of the early swept-wing fighter airplane and not only limited to a few aircrafts; such as the F-4, F-5, F-14, X-29A, Gnat, Harrier, HP 115, only to mention a few (for a historical account see [Hsu & Lan 1985; and Tan & Lan 1996]). Actually, there has been over thirteen modern aircrafts exhibiting this behaviour (Nelson & Pelletier, 2003).

7.4.2. System Dynamics & Simulations

The five states nonlinear dynamical equations modelling the wing rock dynamics of a fighter aircraft are given by (see [Tewari, 2000]):

- **State Equations**

$$\dot{x}_1(t) = x_2(t), \tag{7.4. 1}$$

$$\begin{aligned} \dot{x}_2(t) = &-w^2 x_1(t) + \mu_1 x_2(t) + \mu_2 x_1^2(t) x_2(t) + b_1 x_2^3(t) + b_2 x_1(t) x_2^2(t) \\ &+ L_\delta x_3(t) + L_\beta x_4(t) - L_r x_5(t) \end{aligned} \tag{7.4. 2}$$

$$\dot{x}_3(t) = -kx_3(t) + ku(t), \tag{7.4. 3}$$

$$\dot{x}_4(t) = x_5(t), \tag{7.4. 4}$$

$$\dot{x}_5(t) = -N_p x_2(t) - N_\beta x_4(t) - N_r x_5(t). \tag{7.4. 5}$$

The wing rock phenomena for this particular fighter aircraft can be constructed in the state-space using the configuration data provided in Tewari (2000) and Equations (5.4.1 to 5.4.5) in the following affine and time-varying form:

$$\dot{x}(t) = A(x(t))x(t) + B(t)u(t), \tag{7.4. 6}$$

where $A \in \Re^{5\times 5}$ and $B \in \Re^{5\times 1}$; and plausibly expressed by (which is obviously non-unique):

$$A(x(t)) = \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{45} \\ 0 & a_{52} & 0 & a_{54} & a_{55} \end{bmatrix} \tag{7.4. 7}$$

where

$a_{12} = 1,$

$a_{21} = -w + \mu_2 x_1(t) x_2(t) + b_2 x_2^2(t),$

$a_{22} = \mu_1 + b_1 x_2^2(t),$

$a_{23} = L_\delta,$

$a_{24} = L_\beta,$

$a_{25} = -L_r,$

$a_{33} = -k,$

$a_{45} = 1,$

$a_{52} = -N_p,$

$a_{54} = -N_\beta,$

$a_{55} = -N_r;$

with a linear $B(t)$, expressed by:

$$B(t) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T \tag{7.4. 8}$$

The constant contributions appearing in the system's dynamics matrix $A(x(t))$ are given in Table (7.4.1) for a particular angle-of-attach (see [Tewari, 2000]).

$w = 0.0201$
$\mu_1 = 0.0105$
$\mu_2 = -0.1273$
$b_1 = 0.0260$
$b_2 = 0.5197$
$L_\delta = 1$
$L_\beta = 0.02822$
$L_r = 0.1517$
$N_p = -0.0629$
$N_\beta = 1.3214$
$N_r = 0.2491$
$k = 20.2020$

Table 7.4.1.: Angle of attack Corresponding Constants.

Recalling the theory of chapter 5, the nonlinear state-feedback H_∞ feedback control law is expressed as an iterative sequence:

$$u^{[i]}(t) = -B^*\left(x^{[i-1]}(t)\right)P^{[i]}(t)x^{[i]}(t), \qquad (7.4.9)$$

while noting that for this particular model the control operator, $B(t)$, is linear as expressed in (7.4.8).

The simulations shown in Figures (7.4.2 to 7.4.4) depict the uncontrolled responses of the bank angle, the phase plane representation of the roll-rate versus the bank angle, and the phase-plane plot of the sideslip-rate versus the bank angle, respectively for this given nonlinear dynamical model.

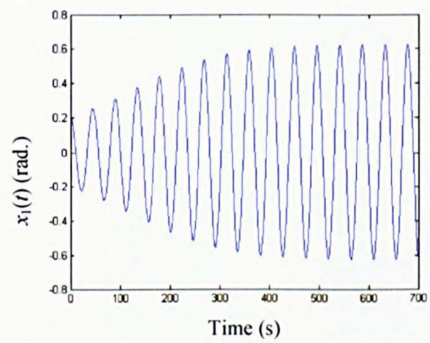


Figure 7.4.2.: Uncontrolled Initial Response of the Bank Angle.

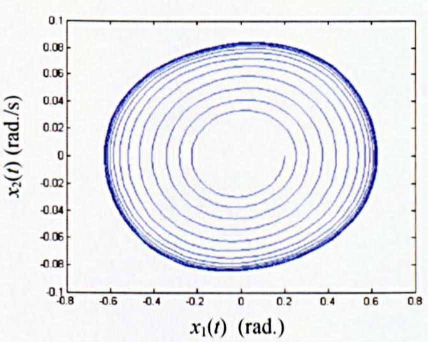


Figure 7.4.3.: Phase-Plane Plot of the Roll-Rate *vs.* the Bank Angle.

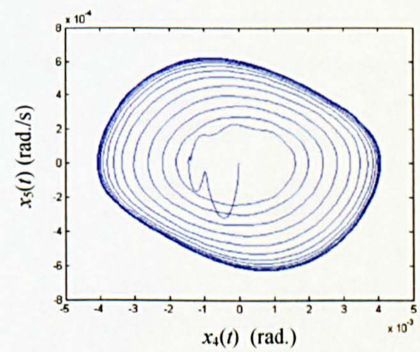


Figure 7.4.4.: Phase-Plane Plot of the Sideslip-Rate *vs.* the Bank Angle.

Now, for the following given initial conditions:

$$x(0)=[1.0 \text{ rad.} \quad 0.5 \text{ rad./s} \quad 0 \text{ rad.} \quad 0 \text{ rad.} \quad 0 \text{ rad.}/s]^T \qquad (7.4.\ 10)$$

the ASRE can be solved backwards in time, as discussed in chapter 5, following the proposed algorithm, while at each iteration the feedback control law (7.4.9) is updated along with the state-affine nonlinear system in (7.4.6). Figure (7.4.5), depicts the controlled response after six iterations using MATLAB®. Robust performance was achieved for $\gamma = 6$ with an exogenous disturbance input matrix, E, that took the designed form of I_5; with a scalar time-invariant disturbance input of $w = 0.5$. Both the controlled outputs and the measured outputs were the bank angle, the aileron deflection angle and the sideslip angle (*i.e.* $x_1(t)$, $x_3(t)$, and $x_4(t)$) with corresponding unitary weighting matrices. Figure (7.4.5), shows the control input signal needed to suppress this model-based nonlinear aerodynamic phenomena.

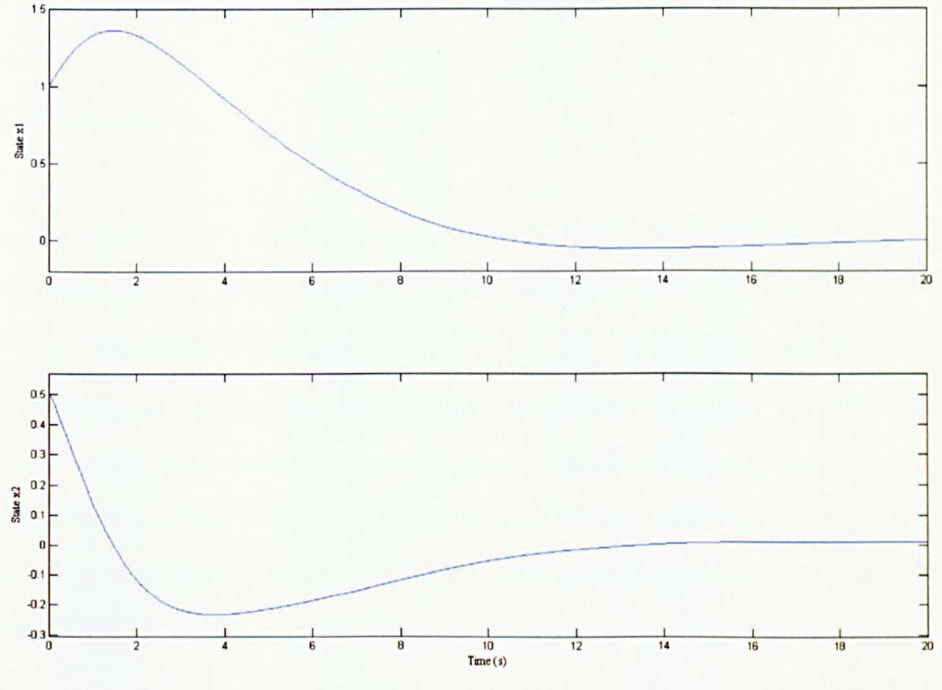


Figure 7.4.5.: Controlled Response of the Bank Angle and the Roll-Rate.

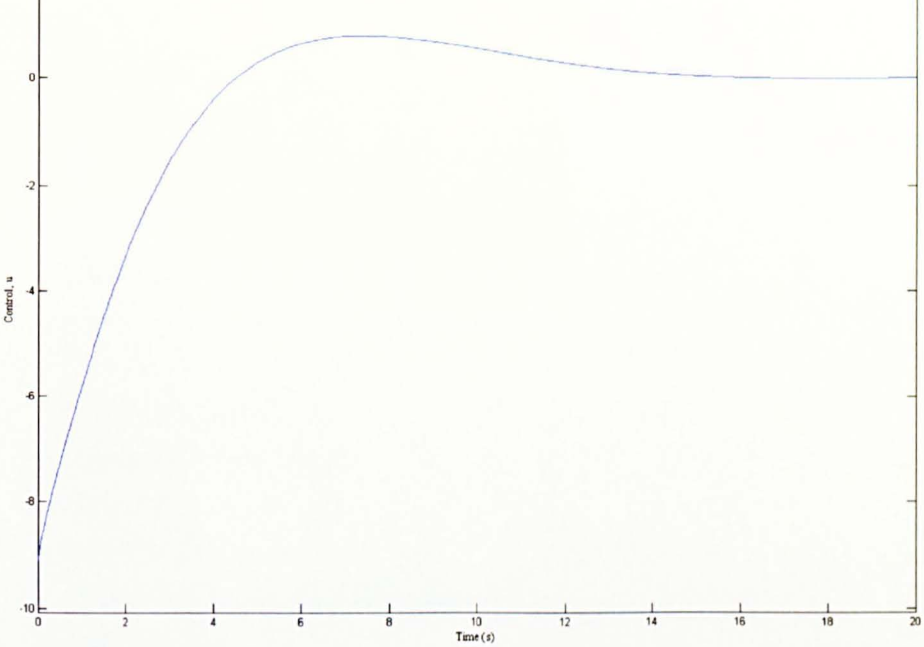


Figure 7.4.6.: Nonlinear Controller Command.

7.4.3. Conclusion

In this section, the nonlinear aerodynamic wing rock behaviour of a fighter aircraft's simple model was controlled and suppressed using the developed and proposed H_∞ regulation technique of chapter 5. The simulated controlled responses seemed promising in view of the realistically small control effort as well as the fast response time to reach the steady-state behaviour. As a future work, more sophisticated analytical models for the wing rock motion can be investigated, such as the three-degree-of-freedom model by Go, *et al.*, (2004) and/or the two-degree-of freedom model by Go, *et al.*, (2002).

7.5. A Hypersonic Aircraft

Nomenclature

a	= speed of sound, ft/s
C_D	= drag coefficient
C_L	= lift coefficient
$C_M(q)$	= pitching moment coefficient due to pitch rate
$C_M(\alpha)$	= pitching moment coefficient due to angle of attack
$C_M(\delta E)$	= pitching moment coefficient due to elevator deflection
C_T	= thrust coefficient
$\bar{c}$	= reference length, $80\,ft$
D	= drag, lbf
h	= altitude, ft
I_{yy}	= moment of inertia, $7\times10^6\ slug\cdot ft^2$
J	= cost function
L	= lift, lbf
M	= Mach number
M_{yy}	= pitching moment, $lbf\cdot ft$
m	= mass, $9375\ slugs$
q	= pitch rate, rad/s
R_E	= radius of the Earth, $20{,}903{,}500\,ft$
r	= radial distance from Earth's centre, ft
S	= reference area, $3603\,ft^2$
T	= thrust, lbf
V	= velocity, ft/s
α	= angle of attack, rad
α_0	= angle of attack at trim condition, rad
γ	= flight-path angle, rad
δE	= elevator deflection, rad
δT	= throttle setting, $\%/100$
μ	= gravitational constant, $1.39\times10^{16}\ ft^3/s^2$
ρ	= density of air, $slugs/ft^3$

7.5.1. Introduction

Humans have long been fascinated with speed _ an allure that led to the development of the supersonic aircraft when the Bell Aircraft Corporation with its rocket propelled XS-1 research aircraft was first to break the mythical sound barrier in

1947 (Williamson, 2005). And ever since, supersonic flights received many technological developments both for civil passenger aircrafts, such as the Concorde, and for military fighters, *e.g.* the Mach-3 Blackbird spy plane. Nonetheless, the first-free flying hypersonic aircraft, NASA's X-43A, based on its scramjet engines recorded Mach 9.8 in 2004 (Williamson, 2005). While, for example, the Japanese Hypersonic Flight Experimental Vehicle (Hyflex) project in 1996 demonstrated great performance and highly successful with their hypersonic lifting conceptual vehicles (Sakurai, *et al.*, 1997)

The future of hypersonic flights lies on having a fully reusable single-stage space-plane that could take off horizontally from an ordinary airport runway, to deliver its payload to orbit and land. While a sibling concept is a hypersonic passenger aircraft that could fly on a sub-orbital trajectory. Indeed the promise of a hypersonic travel will never cease to fuel humans' imagination. In addition, new advances in hypersonic propulsion systems and long-lived structural models are opening the way for the possibilities of developing "a new type of commercial aircraft—the hypersonic transport" (Kirkham & Hunt, 1977). In fact, NASA's Next Generation Launch Technology (NGLT) program with its conceptual flight vehicles is paving the way for more safer and economical launch systems in the not too distant future (Moses, *et al.*, 2004).

Over more than six decades a remarkable achievement was accomplished in hypersonic flights owing to the rigorous research and development by multidisciplinary scientists and engineers. Operating in a 'harsh' and a 'non-forgiving' environment, hypersonic flights often face many unknown problems which designers were unaware of at the first place, *e.g.* the viscous/inviscid interactions and various other problems (Bertin & Cummings, 2003).

Even so, new technological developments are still as promising in designing, building, testing prototypes in air-tunnel (Cox & Crabtree, 1965; Cox, 1964; and Holden, 1993) and flying hypersonic aircrafts albeit the complexity and costly endeavours that are involved with this technology (Bertin & Cummings, 2003). The current technological progress and findings with respect to propulsion systems have constantly improved over the years. Improvement of aerodynamics and jet-energetic parameters of air to-to-space aircrafts and their engines using plasmoid formation was discussed in Durdakov, *et al.*, (1996). While more recent schools of thoughts focused on air-breathing concepts using magneto-hydrodynamic (MHD) energy bypass injector ramjet engines (Lee, *et al.*, 2004).

But from a control standpoint, controlling hypersonic flights presents one of the most challenging and difficult challenges in existence especially for control designs to meet performance and robustness objectives. A hypersonic aircraft drastically differs with aircrafts of conventional subsonic and supersonic speed regimes. With a significant integrated airframe/propulsion system, hypersonic aircrafts, with their high kinetic energy levels at such speeds, feature a high nonlinear coupling between aerodynamics, propulsion and the vehicle dynamics (Sachs, 1998). And it is due to the high velocity flights that hypersonic aircrafts become very sensitive to attitude and velocity changes. For instance, at a speed of 15 Mach and an altitude of $110,000$ ft, a 1-degree increase in the aircraft's angle of attack generates a normal acceleration of $11.5\ \mathrm{ft}/s^2$ which is equivalent to a $1/3\,g$ load factor (Marrison & Stengel, 1998). The difficulties in controlling this highly nonlinear dynamical system are even further magnified due to the inaccuracy in measuring atmospheric properties and aerodynamic characteristics (Wang & Stengel, 2000). Nonetheless, hypersonic aero-elasticity and aero-thermo-elasticity have constantly received considerable attention using piston theory and approximate aerodynamic models (see [Friedmann, *et al.*, 2004; and Weiland, *et al.*, 1993] and the references therein); as well as using computational fluid dynamics (CFD) method (see [Papadopoulos, *et al.*, 1999]). Hypersonic vortex formation and flow computations from a perturbed hypersonic flow also received closer rigorous studies and flow-field simulations (see [Lin & Shen, 1997; and Hemdan, 1990]); as well as interferometric experimental investigations of flow field formation around spheres in free flights (Sedney & Kahl, 1961).

It consequently follows that robustness of the synthesized flight control system is crucial and essential to accommodate for unknown perturbations and uncertainties affecting this particular dynamical system both in theory and in practice. Most control theoretical research communications with relation to hypersonic vehicles revolve around robust methodologies. In their paper, Marrison & Stengel (1998) used the Monte Carlo evaluation (MCE) and Genetic Algorithms (GA) to design a robust controller for a hypersonic aircraft. While, in Wang & Stengel (2000), the authors combined nonlinear dynamic inversion (NDI) with stochastic robustness to produce a control system for a hypersonic aircraft as an extension to the NDI approach in Lane & Stengel (1988). Sliding mode control designs for hypersonic aircrafts were also considered in Xu, *et al.*, (2001a) and Xu, *et al.*, (2001b). A neural adaptive controller was devised for hypersonic aircrafts in Xu, et al., (2003).

In this section, however, the conceptual finite-horizon continuous-time state-feedback nonlinear H_∞ theory of chapter 6 is used to control and robustly stabilize a hypersonic aircraft about its trimmed operating flying condition. Some of the main forces acting on the system are illustrated below in Figure (7.5.1).

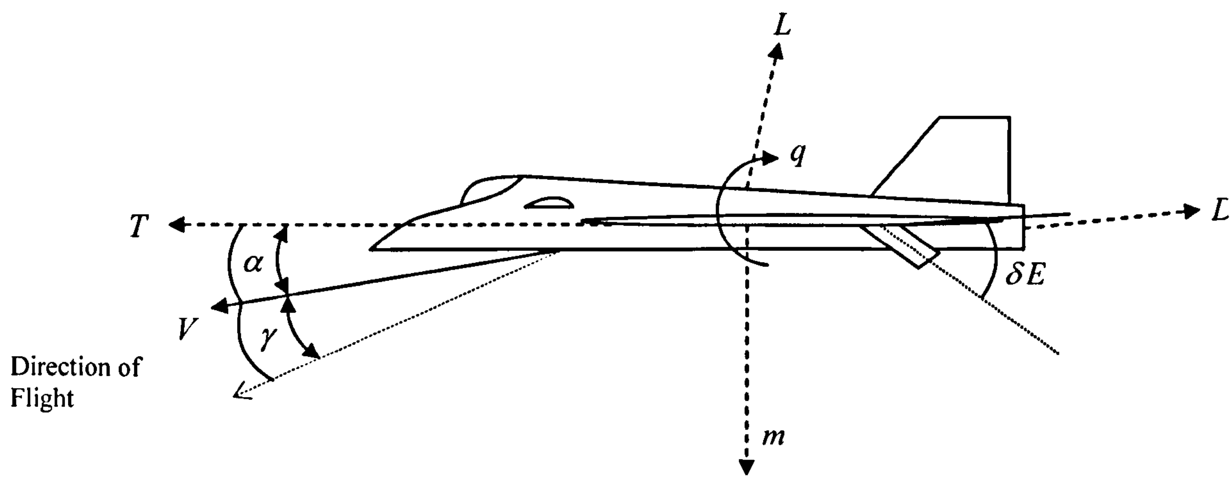


Figure 7.5.1.: Forces Acting on the Hypersonic Aircraft.

7.5.2. System Dynamics & Simulations

The highly nonlinear system dynamics of a hypersonic aircraft can be modelled by fifth-order Ordinary Differential Equations for the velocity, flight-path angle, altitude, angle of attack, and pitch rate, respectively, as follows (see, *e.g.*, [Marrison & Stengel, 1998; and Wang & Stengel, 2000]):

$$\dot{V} = \frac{T\cos\alpha - D}{m} - \frac{\mu \sin\gamma}{r^2}, \qquad (7.5.1)$$

$$\dot{\gamma} = \frac{L + T\sin\alpha}{mV} - \frac{\left(\mu - V^2 r\right)\cos\gamma}{Vr^2}, \qquad (7.5.2)$$

$$\dot{h} = V\sin\gamma, \qquad (7.5.3)$$

$$\dot{\alpha} = q - \dot{\gamma}, \qquad (7.5.4)$$

$$\dot{q} = \frac{M_{yy}}{I_{yy}}; \tag{7.5. 5}$$

where the lift, drag, thrust, pitching moment, and radius from the Earth's centre are modelled, respectively, by

$$L = \frac{1}{2}\rho V^2 SC_L, \tag{7.5. 6}$$

$$D = \frac{1}{2}\rho V^2 SC_D, \tag{7.5. 7}$$

$$T = \frac{1}{2}\rho V^2 SC_T, \tag{7.5. 8}$$

$$M_{yy} = \frac{1}{2}\rho V^2 S\bar{c}\left[C_M(\alpha) + C_M(\delta E) + C_M(q)\right], \tag{7.5. 9}$$

$$r = h + R_E. \tag{7.5. 10}$$

The thrust coefficient C_T is a function of throttle setting δT,

$$C_T = \begin{cases} w_{16}0.0105\left[1 - w_{17}164(\alpha-\alpha_0)^2\right](1+w_{18}17/M)(1+w_{19}0.15)\delta T, \text{ if } \delta T < 1 \\ w_{16}0.0105\left[1 - w_{17}164(\alpha-\alpha_0)^2\right](1+w_{18}17/M)(1+w_{19}0.15\delta T), \text{if } \delta T \geq 1 \end{cases} \tag{7.5. 11}$$

The aerodynamic coefficients and the atmospheric model, which are functions of the states and the control, are assumed uncertain, with w denoting an element of the uncertainty vector. These disturbed parameters are given by:

$$m = w_1 m, \tag{7.5. 12}$$

$$I_{yy} = w_2 I_{yy}, \tag{7.5. 13}$$

$$S = w_3 S, \tag{7.5. 14}$$

$$\bar{c} = w_4 \bar{c}, \tag{7.5. 15}$$

$$a = w_5\left(w_6 8.99\times10^{-9}h^2 - w_7 9.16\times10^{-4}h + 996\right), \tag{7.5. 16}$$

$$M = V/a, \tag{7.5. 17}$$

$$\rho = 0.00238\, e^{-h/w_8 24000}, \tag{7.5. 18}$$

$$C_L = w_9\alpha\left(0.493 + w_{10}1.91/M\right), \tag{7.5. 19}$$

$$C_D = w_{11}0.0082\left(w_{12}171\alpha^2 + w_{13}1.15\alpha + 1\right)\times\left(w_{14}0.0012M^2 - w_{15}0.054M + 1\right), \tag{7.5. 20}$$

$$C_M\ (q)=(\bar{c}/2V)qw_{24}\ (-w_{25}0.025M+1.37)\times(-w_{26}6.83\alpha^2+w_{27}0.303\alpha-0.23), \quad (7.5.21)$$

$$C_M\ (\delta E)=w_{28}0.0292(\delta E-\alpha). \quad (7.5.22)$$

The robust flight controller can be designed for this generic aircraft by first constructing the system dynamics in the state-space form using Equations (7.5.1 to 7.5.5) in the following affine and time-varying form:

$$\dot{x}(t)=A(x(t),u(t))x(t)+B(x(t))u(t)=\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ 0 & a_{32} & 0 & 0 & 0 \\ a_{41} & 0 & 0 & 0 & a_{45} \\ a_{51} & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} V(t) \\ \gamma(t) \\ h(t) \\ \alpha(t) \\ q(t) \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ b_{21} & 0 \\ 0 & 0 \\ b_{41} & 0 \\ 0 & b_{55} \end{bmatrix}\begin{bmatrix} \delta T_{com} \\ \delta E \end{bmatrix} \quad (7.5.23)$$

where the non-unique nonlinear state and control dependent operator $A(x(t),u(t))$, and the state-dependent $B(x(t))$ operator are respectively defined as in (7.5.23) by the following terms:

$$a_{11}=-\frac{1}{2m}\rho\, SC_D\Psi(V)-\frac{1}{2m}\rho\, SC_T\Psi(V)\vartheta(u)\cos(\alpha+\alpha_0),$$

$$a_{12}=-\frac{\mu}{(h+h_0+R_E)^2}\varsigma(\gamma),$$

$$a_{21}=\left(\frac{1}{2m}\rho\, SC_L+\frac{\cos(\gamma+\gamma_0)}{(h+h_0+R_E)}\right)\xi(V),$$

$$a_{22}=-\frac{\mu\psi(\gamma)}{(V+V_0)^2(h+h_0+R_E)^2},$$

$$a_{32}=(V+V_0)\varsigma(\gamma),$$

$$a_{41}=-a_{21},$$

$$a_{42}=-a_{22},$$

$$a_{45}=1,$$

$$a_{51}=\frac{\rho\ (V+V_0)\ S\ \bar{c}\left[C_M(\alpha)+C_M(q)-0.0292w_{28}(\alpha+\alpha_0)\right]}{2I_{yy}};$$

and

$$b_{21}=\frac{1}{2m}\rho\ (V+V_0)\ SC_T\sin(\alpha+\alpha_0),$$

$$b_{41}=-b_{21},$$

$$b_{55}=\frac{\rho\ (V+V_0)^2\ S\ \bar{c}\times 0.0292w_{28}}{2I_{yy}},$$

with the following functions,

$$\psi(\gamma):=\begin{cases} 0 & \text{for } \gamma=0 \\ \dfrac{\cos\gamma-1}{\gamma} & \text{for } \gamma\neq 0 \end{cases};$$

$$\varsigma(\gamma):=\begin{cases} \sin(\gamma_0) & \text{for } \gamma=0 \\ \dfrac{\sin(\gamma+\gamma_0)}{\gamma} & \text{for } \gamma\neq 0 \end{cases},$$

$$\Psi(V):=\begin{cases} V_0 & \text{for } V=0 \\ \dfrac{V+V_0}{V} & \text{for } V\neq 0 \end{cases},$$

$$\xi(V):=\begin{cases} V_0 & \text{for } V=0 \\ \dfrac{V_0}{V} & \text{for } V\neq 0 \end{cases},$$

and

$$\vartheta(u):=\begin{cases} C_T=0.0105w_{16}\left(1-164w_{17}(\alpha-\alpha_0)^2\right)(1+17w_{18}/M)(1+0.15w_{19}) & \text{for } \delta T<1. \\ C_T=0.0105w_{16}\left(1-164w_{17}(\alpha-\alpha_0)^2\right)(1+17w_{18}/M)\left(\dfrac{1}{\delta T}+0.15w_{19}\right) & \text{for } \delta T\geq 1. \end{cases}$$

The measurement output of this generic aircraft is in fact composed of both the altitude and velocity measurements, *i.e.*,

$$y=\begin{bmatrix} V \\ h \end{bmatrix}. \qquad (7.5.24)$$

As with previous sections, it is convenient to consider the exogenous disturbances from a stochastic viewpoint; that is by means of a white noise input to appropriate colouring filters to generate the disturbance spectrum. In the absence of good disturbance model information white noise was scaled by a gain matrix as also considered in Grimble (2001). Accordingly, the disturbance model had the form:

$$w=diag\{0.01,0.01,0.01,0.01,0.01\}/s. \qquad (7.5.25)$$

Now, recalling the nonlinear finite-horizon full-information state-feedback H_∞ control algorithm of chapter 6, the robust controller having the form:

$$u^{[i]}(t) = -\left(D_{21}'^{[i]}C_1^{[i]} + B'\left(x^{[i-1]}(t)\right)P^{[i]}(t)\right)x^{[i]}(t); \qquad \text{for } i \geq 0, \quad (7.5.\,26)$$

can be synthesised; and where the designed disturbance weighting matrix, E, and the regulated output matrices, C_1 and D_{12}, took an identity unitary form of appropriate dimensions. While $P^{[i]}(t)$, the solution of the Riccati equation considered in chapter 6, was solved backwards in time for a sufficiently small Euler step-length of 0.002 increments.

The flight control system considered must provide the control demands for the elevator deflection angle and the forward thrust by means of the throttle setting, to stabilize the aircraft about its trimmed hypersonic cruising flight condition. The engine dynamics of this generic hypersonic aircraft takes a second-order form,

$$\ddot{\delta T} = \mathrm{K}_1\dot{\delta T} + \mathrm{K}_2\delta T + \mathrm{K}_3\delta T_{command}. \qquad (7.5.\,27)$$

where choosing $\mathrm{K}_1 = \mathrm{K}_2 = 0$ and $\mathrm{K}_3 = 1$ provides a suitable model (Wang & Stengel, 2000); these dynamics, however, were not incorporated in the simulations to follow. Nonetheless, two simulated scenarios using MATLAB® are considered in the sequel to test the designed trim controller in (7.5.26) about the trim condition in Table (7.5.1); where the 28 inertial and aerodynamic uncertain parameters (w_i) were assumed to randomly vary from $0.10 \rightarrow 0.0010$.

$M = 15$,
$V = 15{,}060\, ft/s$,
$h = 110{,}000\, ft$,
$\alpha = 0.0315\ rad$,
$\gamma = 0\ rad$,
$q = 0\ rad/s$,
$\delta T = 0.183$,
$\delta E = -0.0066\ rad$,
$T = 4.6853 \times 10^4\ lbf$,

Table 7.5.1.: The Trimmed Cruise Condition.

- **Scenario 1:**

For a sufficiently large γ, that is $\gamma = 10$, the first control objective is to stabilize the hypersonic aircraft about its operational trimmed cruise condition for any change in velocity and altitude. Accordingly, initializing the controller with a $4\,ft/s$ velocity change (*i.e.* $V = 15,056\,ft/s$) and an altitude change of $2000\,ft$ (*i.e.* $h = 112,000\,ft$); it is desirable for the controller to stabilize the system and bringing it back to its stable operational condition, as shown in (7.5.3) for two different iterations ($i = 2$ & $i = 4$). While the thrust input control command is shown in (7.5.4). The responses of the close-loop controlled system seemed realistic while the steady-state was reached within just 5 sec and the angle of attack's variation was also within an acceptable range for a realistic thrust and elevator deflection inputs.

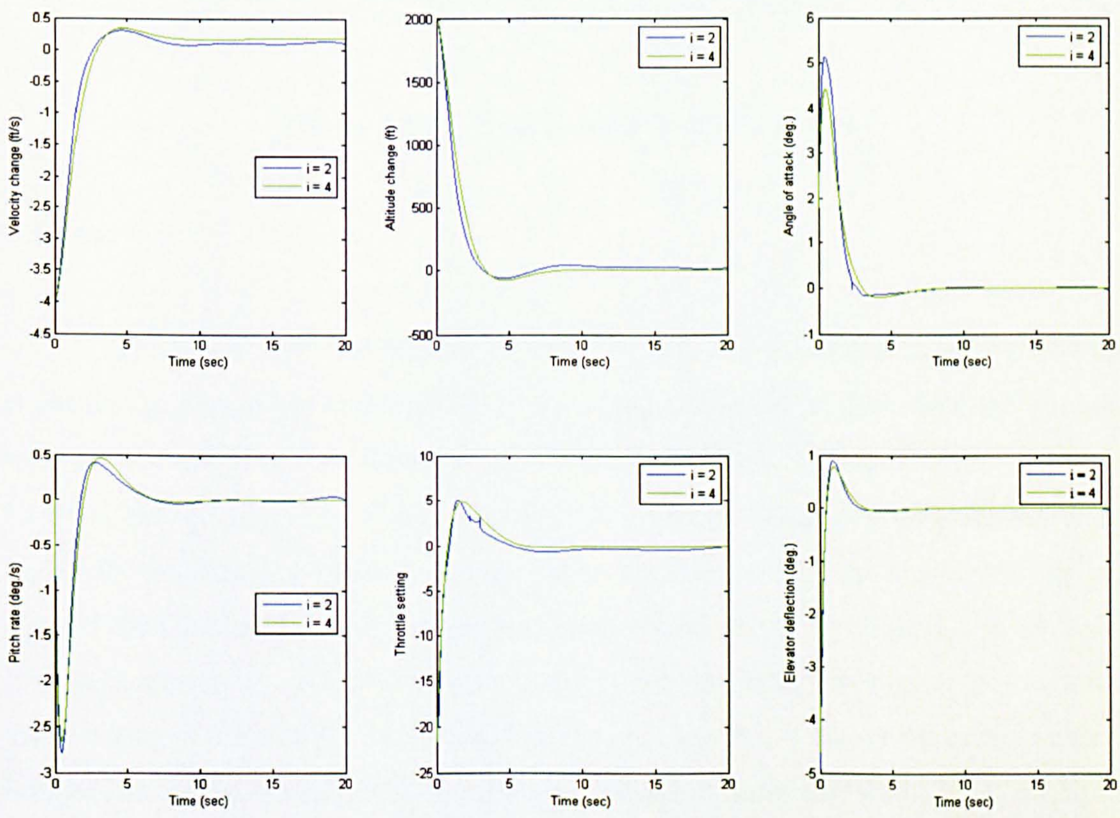


Figure 7.5.3.: Response to an Altitude and a Velocity Change with a Nonlinear H_∞ Controller.

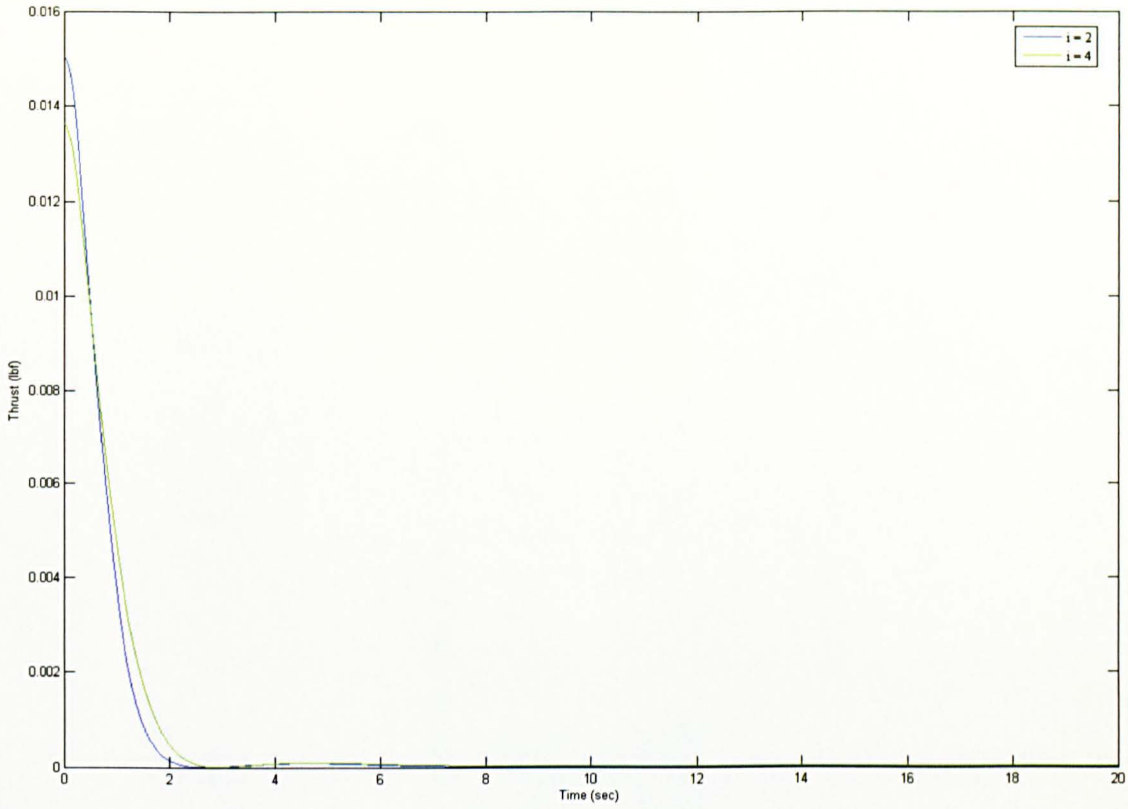


Figure 7.5.4.: Thrust Commanded Response.

- **Scenario 2:**

For only an altitude change of $2000\,ft$ (*i.e.* $h = 112{,}000\,ft$), the same flight controller is engaged to stabilize the system. The responses of this simulated scenario were carried out after four iterations ($i = 4$), and are shown in Figure (7.5.5), with the thrust command shown in Figure (7.5.6). The required thrust and elevator deflection inputs to stabilize the altitude change were sensibly small, and the responses also showed the coupling between the altitude and velocity changes. Overall, the controlled responses appeared to adhere to a good and a highly promising system response amidst the presence of uncertainty. Note that, Figures (7.5.3 to 7.5.6) depict the relative change between the actual system and/or controller response as compared to the trim condition.

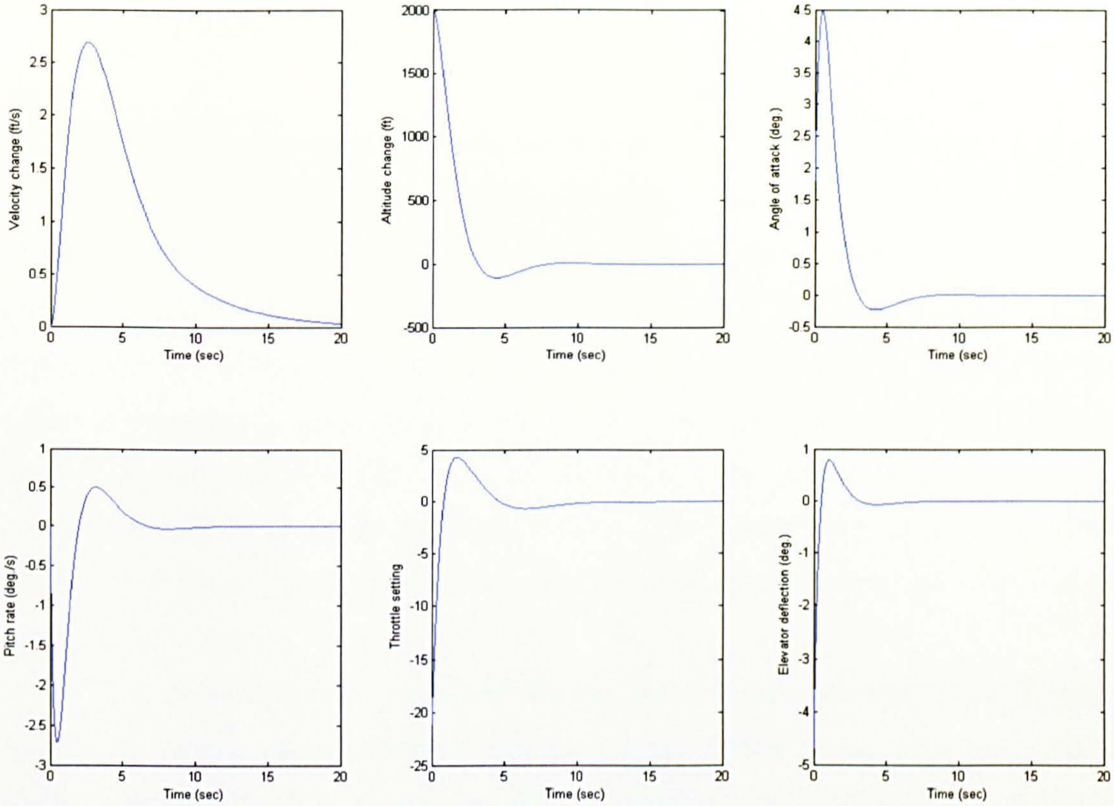


Figure 7.5.5.: Response to a 2000 ft Altitude Change with a Nonlinear H_∞ Controller.

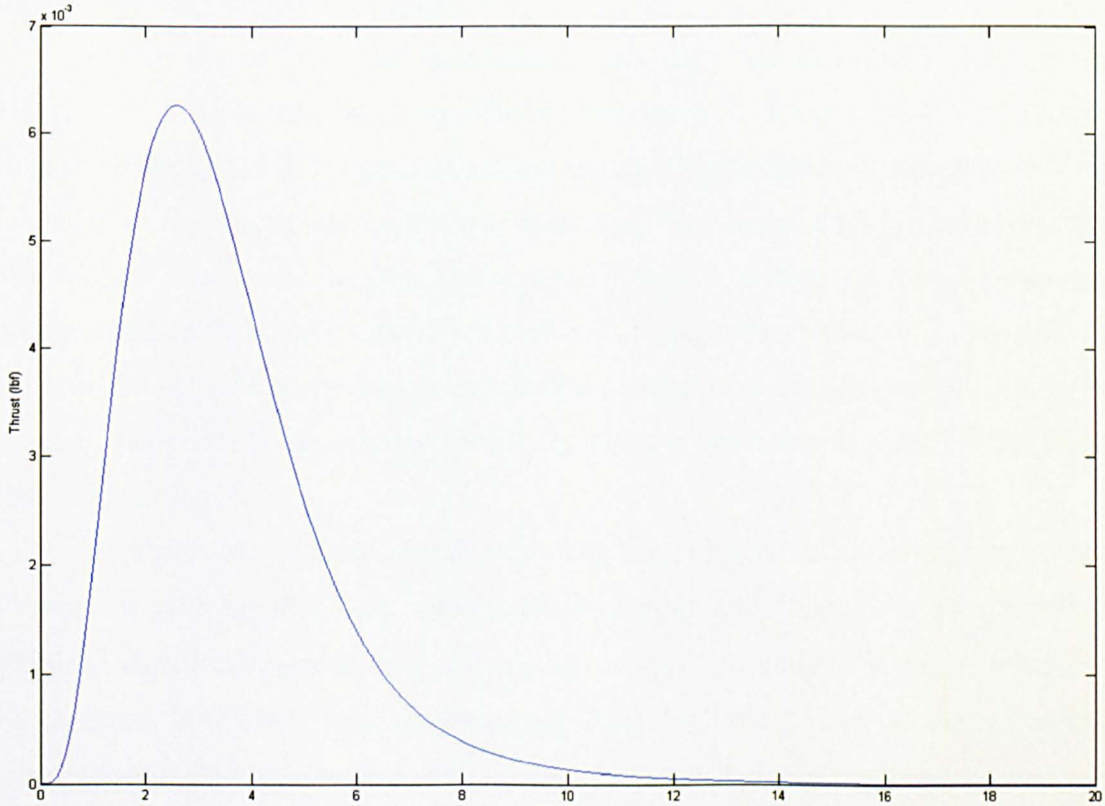


Figure 7.5.6.: Thrust Commanded Response.

7.5.3. Conclusion

Under the regulation problem framework, the extended nonlinear method of chapter 6 was used to control the highly nonlinear longitudinal dynamical equations of a realistic hypersonic aircraft containing 28 uncertain parameters. With stability being of prime importance, the designed flight control system with its simple structure makes it suitable for a commanded input from both a pilot and/or an autopilot. Two simulated different scenarios of the controlled attitude and velocity responses of the hypersonic aircraft around the nominal cruising condition were provided to illustrate the effectiveness of the proposed technique in view of the realistically very fast and stable responses. Tradeoffs between using less thrust and tolerating longer rise times can also easily be examined.

The devised robust control mechanism for the complete unrestrained generic hypersonic vehicle that resembles a reusable launch vehicle makes robustness profiles easily adjusted by tuning the weighting matrices. Accordingly, the theoretical framework of chapter 6 makes it more transparent to include such requirements into the design phase as compared to chapter 5.

While the aerodynamic coefficients were extracted from the NASA Langley Hypersonic Vehicle Simulation Model (see [Marrison & Stengel, 1998; and Wang & Stengel, 2000]); they still represent a main source of parametric uncertainties affecting hypersonic vehicles. In fact, parametric uncertainty that arises in flight control problems also include engine and actuator models (see [Grimble, 2001]). And as a future work such models could be investigated more closely; along with un-modelled dynamics that can also be considered by means of relevant unstructured uncertainty models. In that case, a multiplicative uncertainty model for sensors and actuators may be considered (Grimble, 2001).

Furthermore, without doubt, the ongoing efforts in modelling hypersonic vehicles of arbitrary any shape to directly incorporate within their mathematical and physical dynamical models the various structural, dynamical, aero-dynamical, and coupled aero-elasticity effects are promising. And it is believed that the general solution technique provided in this section makes it suitable to numerically simulate and solve the equations of motion of any other vehicle shape as made available.

7.6. Concluding Remarks

In this chapter the reader was presented with four different practical model-based nonlinear dynamical applications. The utilized control methodologies built upon the proposed theoretical frameworks of chapters 3, 4 & 5, not only for validation purposes but also to provide the reader with the all-encompassing possibilities of applying the proposed robust state-feedback modern theories to the practical real-world.

Although most of the included applications within this chapter were revolving around aeronautical and space technologies, in fields where no man-made errors are tolerated and where robustness compensates for the unknown and where safety is of prime importance, other industrial fields can also be considered as discussed in the concluding chapter.

PART V

CONCLUSIONS & FURTHER RESEARCH

CHAPTER 8

Conclusion of This Thesis

8.1. Contributions of this Dissertation

By its own definition the basic purpose of *control* is to modify an event, a process, or a plant to perform a desired task. In fact, the foundational developments by theoreticians such as Huygens, Maxwell, Routh, Minorsky, Nyquist and Black (to name a few) were motivated by real-world applications. In the 1950s and 1960s, in the hands of renowned mathematicians such as Wiener, Bellman, Lefschetz, Kalman and Pontryagin (again, to name a few) control theory developed as a branch of applied mathematics, *i.e.* independent of its potential application(s) to engineering problems.

Historically speaking, some tenuous arguments were typically invoked to provide some practical motivation or real-life applicability to engineering contexts behind the research on this so-called mathematical control theory. For example, the study of the state-space triple operators, (A,B,C), was rationalized as the study of the linearization of an arbitrary nonlinear system about its local operating equilibrium point – an argument that had some truth, partially due to the inevitable loss of vital global nonlinear dynamical behaviours. But by the end of the 1980s, a fairly complete body of knowledge and theoretical understanding that included powerful techniques of controller synthesis for the general linear systems paradigm was reached; while spectacular applications that fitted practical situations were also reported.

The Classical linear control theory has unquestionably developed extensively over the years. While the linear time-invariant frameworks with their broad concepts that are not only limited to transfer functions and loop-shaping techniques are still applied extensively, the linear time-varying school of thought also proved as effective. Nonetheless, many researchers were enticed to mimic the developments of linear systems theory by extending the basic linear concepts to the general and more comprehensive nonlinear case. Such basic extensions that included controllability, observability, and realizability (to name a few) were, in verity, crowned with great success; but the controller synthesis problem, on the other hand, proved to be much more elusive and difficult. Despite some significant progress, to date, general techniques and methodologies for the stabilization and control of nonlinear systems developed over the years but proved to only be valid for special classes of nonlinear systems while required very restrictive assumptions and conditions to hold. This is, of course, due to the expected daunting complexity of the behaviour of nonlinear dynamical systems.

Conversely, new technological developments had created engineering problems where certain nonlineararities and uncertainties had to be taken into account during the design phase. Namely, the robust control theory developed for general nonlinear systems could not successfully deal with them, basically because of the highly complex controller structures that not only involved a highly complex analytical manipulations and computations; but also the locally admissible control actions were not always guaranteed to work in practise. Accordingly, the material reported in this thesis is an attempt in this direction.

The ultimate goal for a control system designer is to build a controller architecture that will work in a real environment (*i.e.* nonlinear) and in which operating conditions may vary with time. The control system must also be able to withstand other imposed factors such as noise, disturbances and uncertainties. The mathematical representation of dynamical systems, however, often involves simplifying assumptions on the system's nonlinearities and/or high-frequency dynamics, which in principle are either unknown and hence can not be mathematically modelled, or are modelled but ignored during the design stage for simplicity. As a result, in practice, control systems that are designed based on such simplifications may not work in real environments. The *sine qua non* of a control system to properly operate in a realistic setting irrespective of all sorts of exogenous disturbance inputs and modelling assumptions is dependent on the robustness characteristics of the closed-loop system. Mathematically speaking, the controller must perform satisfactory for a family of nonlinear plants and not only the plant under consideration. In that sense, if the designed controller stabilizes the system regardless of parametric changes within prescribed limits then robust stability is reached. Most often some control specifications are also to be satisfied, *i.e.* steady-state tracking, speed of response and disturbance rejection; and if met then the controller is said to have achieved robust performance.

The problem of designing controllers that satisfy both robust stability and robust performance requirements has been addressed by means of one of the cornerstones of the modern control theory, the H_∞ control theory. Over the past decades, since it was introduced by Zames (1981), a proliferation of literature was witnessed on the H_∞ control methodologies as previously discussed.

While for general multi-input multi-output, or multivariable, nonlinear dynamical systems, feedback control and especially robustness issues are still research topics. The urgency of such a drift has been rendered more acute by the recent development of machines with challenging nonlinear dynamics, such as robot manipulators, high-performance aircrafts, industrial processes, advanced underwater and space vehicles, for instance. It became noticeable that to meet the control objectives of these newly emerging and challenging engineering problems, the "*find an application for my theory*" approach had to be discarded due its invalidity; and a new tailor-made general nonlinear theory had to be worked out to globally and robustly stabilize any given technological problem irrespective of the application context.

Accordingly, this Doctoral thesis is an elucidation to the above-mentioned challenge and an attempt to tackle it. More specifically, a new model-based nonlinear control methodology, which can be viewed as an extension to the 'Approximation Theory', was proposed to replace the nonlinear system with a sequence of continuous-time, de-coupled, linear time-varying, non-autonomous, and quadratic ones which converge to the solution of the nonlinear dynamical problem, but are not only locally valid but also globally valid. This means that highly nonlinear problems were solved via a sequence of linear approximations. As such, the classical linear control theory tools were rigorously used throughout to obtain globally robust and optimal nonlinear dynamical controllers; and most often this involved iteratively solving Approximating Sequences of Riccati Equations (ASRE).

Consequently, throughout this thesis particular emphasis was given to the state-feedback controller type where an observer-based design structure was assumed for a practical implementation – this means that state(s) measurement(s) are accessible to achieve the control objectives. Under this architecture, various nonlinear robust control techniques and theories were proposed; ranging from a standard and a simple pole-placement controller technique to the more challenging and appealing H_∞ controller type.

Since the time-varying singular H_∞ control problem, *i.e.* the direct-feedthrough matrix $D(\cdot)$ does not satisfy the full-column rank assumption, was already solved in the literature by means of a recursive procedure (Amato, *et al.*, 2000); only the continuous-time, linear time-varying, full-information H_∞ control theory was extended relying on the Approximation Theory and solving the ASRE and completing the square. The control problem was investigated from both a finite-horizon time and infinite-horizon time H_∞ formulae. It can be concurred, however, that the full-information, finite-horizon H_∞-norm discussed in previous chapters makes it easier and more transparent to incorporate several performance requirements in the cost criterion, especially those performance requirements which are directly related to robustness. That is given a nonlinear finite-dimensional dynamical system on a bounded time-interval $[0,T]$ together with a positive real number γ, the necessary and sufficient conditions for the existence of a dynamic controller were ensured by the theoretical frameworks, *i.e.*, the L_2-induced norm(s) of the resulting closed-loop operator is smaller than γ.

Although most of the papers on the H_∞ control problem mentioned in the previous chapters discussed the "standard" H_∞ problem (that is minimize the $L_2[0,\infty)$-induced operator norm of the closed-loop operator over all internally stabilizing feedback controllers), it is believed that the proposed theoretical framework herein that built upon the standard single full-information Riccati equation framework provided very efficient practical results to nonlinear dynamical systems under the presence of disturbance. The theory, in fact, gives the control engineer and designer more transparent control requirements to be incorporated *a priori* to fine-tune between robustness and optimality needs regardless of technological control problem in-hand.

Furthermore, the proposed nonlinear H_∞ design methods in this thesis marked a significant stage in the development of control systems to practical real-world applications that worked in theory and is expected to comply in practice, since the following was obvious:

- more consistent performance over wider operating conditions,

- very fast responses with minimal control efforts,
- a simple controller structure that is constrained to be linear with respect to its inputs and hence more reliable software implementation and system integrity; although look-up tables are to be calculated off-line,
- less sensitive to faults and sensor or actuator degradation;
- the possibility of using lower specification hardware but meeting the same performance requirements.

There are, however, a few arguments for developing H_∞ control systems (see [Grimble, 2001]). Namely, existing systems are designed assuming adequate models are available – in practice this argument has a grain of truth and is seldom ever true and the outcome is that either poor control or long tuning periods must be accepted. The counter argument is to allow for modelling errors and to then obtain more realistic model-based designs. Undeniably, the nonlinear H_∞ optimal control in this thesis provided a simple method of achieving a robust controller and is expected to guard against such foreseeable parametric uncertainties.

It is worth adding that the numerically simulated control systems in this thesis, (which included the following systems: the inverted pendulum, the magnetic levitation, the wing-rock phenomena, the Lynx helicopter, and a hypersonic aircraft), were conducted using commercially available software that facilitated the visualization of the proposed model-based controlled actions. Indeed, with a model-based controller design, today's engineering teams are building radically more complex systems faster and more reliably than through traditional and more conventional approaches – a fact which enables rigorous testing to the various designs prior to a real-life implementation. A good example is the Mars Rovers that were autonomously successfully landed – two missions that went exactly as simulated under thousands of atmospheric disturbances by means of MATLAB® & SIMULINK® platforms (Petrosky & Flynn, 2004).

It was shown in this thesis that the proposed linear time-varying ASRE robust controller designs are simple and effective. The simplicity is due to the applied well-known classical integration techniques as opposed to the more tedious and laborious algebraic techniques from such controllers obtained by the conventional Hamilton-Jacobi Bellman principle and Taylor series expansion, for instance.

Needless to say, practical implementations of the proposed nonlinear model-based robust stabilization methodologies provided in this thesis are advantageous since the devised linear controllers are simple to put into practice but will involve more components to be incorporated for an actual implementation.

In this thesis, various universal robust controllers were proposed in this thesis which stabilize and regulate problems of the general deterministic autonomous and/or non-autonomous nonlinear systems under the mild Lipschitz continuity condition and provided that the origin of the nonlinear system is an equilibrium point. The monolithic synthesis theory was based on a sequence of the linear time-varying approximation approach and the linear time-varying and quadratic modern control theory.

To summarize, the main contributions of this thesis were to extend the already published 'Approximation Theory' to address and include robustness in general. It can be deduced that the all-inclusive Min-Max H_∞ theory that appeared in chapter 6 proved very efficient when compared with the optimal H_∞ theory of chapter 5 as well as the simple robust methodology of chapter3. More specifically, the Min-Max H_∞ theory not only yielded more robust results when applied to the inverted pendulum on a cart model, for instance, but also enabled the inclusion of more robust performance specifications.

8.2. Recommendations for Future Work

It is believed that the theoretical state-space time-varying H_∞ control problem and its structure are well understood at the moment and even reached a maturity state. However, the practical issue of the design of weights for multi-input, multi-output dynamical systems is still a research subject since it lacks a systematic approach. A lot of work still needs to be done to translate any robustness criteria into a well-formulated and a more systematic and coherent H_∞ problem. Furthermore, another possible theoretical extension to the H_∞ theory given in this thesis would be to incorporate some other more specific performance constraints, such as: steady-state disturbance rejection or fast roll-off (to name a few). These conditions are all based on the requirement that the closed-loop system satisfies certain constraints.

There have been some efforts in addressing the choice of the weights for the H_∞ problem. However, these methods are still relatively *ad-hoc* and need a more thorough foundation before guaranteeing the kind of problems for which these methods can be used. The loop-shaping design method with robustness requirements as well as the small gain theorem both can be classified under such efforts. It is believed, though, that the theoretical framework provided by these methods is only valid for linear time-invariant systems and lots of work still needs to be done to extend the methodology to a linear time-varying context.

The research established in this thesis on nonlinear robust control proved to be theoretically very effective and where particular emphasis was given to the state-regulator problem. However, it is believed that another plausible theoretical extension to this thesis will be to consider the more realistic and more challenging output-feedback strategies which were not addressed herein.

In addition, a constant matrix rank was assumed throughout this thesis mainly because all the considered practical applications conformed to this assumption although the rank variation was not noticed between the ASRE iterations regardless of neither the chosen compact time interval nor the Euler step-length. It is worth pointing out that only the control matrix which could be state- and control-dependent seemed to change ranks with iterations (*i.e.* $i=1 \rightarrow rank\left(B(x,u,t)\right)=2$ & $i=2 \rightarrow rank\left(B(x,u,t)\right)=3$).

However, it can be argued that the constant rank assumption fails in most practical linear time-varying applications (a good example is a C^∞ function). So an extension to both the singular and regular H_∞ control laws; the time-varying rank deficiency assumption can be taken into consideration. That is, the more general case where the ranks of the quadruple matrix representation, A, B, C, D, bounded, piecewise continuously differentiable functions over $t \in \Omega := [t_0, t_f]$ can be assumed to vary in the Ω subspace. Of course there are various ways to resolve this problem with the most obvious choice would be by modifying the controlled variable of the original system by introducing a sufficiently small fictitious parameter ε (at the point in time when the rank drops), so as to render the rank deficient matrix a full-column rank one. However, this approach, as known, leads to a bad conditioned (stiff) differential Riccati equation which renders this kind of approach impracticable (Amato, *et al.*, 2001). And perhaps the techniques provided in the book by Dewilde & der Veen, (1998), could be used instead to avoid this problem. In their book, the authors used the concepts of the linear time-varying system theory to treat and analyse a large class of basic algorithms in a more general setting; and where the time-varying rank deficiency was always avoided or even treated.

With regards to the approximation theory, there are still some natural research directions to be considered. The non-uniqueness of the operators, $A(\cdot)$ & $B(\cdot)$, is still to be addressed. That is, a systematic approach to the design of such operators needs to be researched – the choice of these operators undeniably affects the convergence rate of the quadratic sequences as well as the speed of the controlled response of the given system. Furthermore, the theory does not hold for infinite-dimensional systems, and hence a more comprehensive framework to include this generalization needs to be worked out and which might involve PDEs and HJB equations. Last but not least, the general robust control problems do not usually have continuous solutions and unfortunately, the convergence of the ASRE approach was only proved in the space of continuous functions and so is only valid for a set of control problems (*i.e.* continuous solutions are implicit). Finally, the difficulty arises in implementing the ASRE technique in real-time (*i.e.* on-line) and especially for fast-response dynamical systems; and it naturally follows that further research is essential to develop a more suitable technique for a real-time computer implementation and a real-life operation (in terms of hardware and software).

In terms of practical engineering and technological applications, it is with no doubt that the range of the possible application of the proposed synthesis techniques in this thesis to realistic models is all encompassing. At a glimpse, there is a great deal of interest at the moment in the design of controllers for uncertain time-varying flexible structures, particularly in the area of satellites and where the dynamical equations are highly nonlinear and are usually linearised to apply classical linear design methods (see for example [Ballas, 2002; Bakker & Annaswamy, 1996; Kelkar & Joshi, 1996; and Zheng, *et al.*, 2005]). These methods only apply in a small region around the operating point and it would be enticing to apply the theory in this thesis to such highly nonlinear systems.

During the past few decades, more stringent robust performance requirements were posed by modern systems, such as flight vehicles, large space structures, unmanned air and ground vehicles (UAV/UGV), crew return vehicles (CRV), robots, and chemical processes, for example. But with the availability of low-cost computing power, the successful applications of the proposed modern control theory in this thesis to solving such real-world problems will also be another expected direction with the hope of bridging the gap between academia and industry.

To briefly conclude this thesis, it is worth mentioning a final research direction that will be significant to explore under the field of robust control. Since the early 1990s the linear matrix inequalities (LMIs) have emerged as a functional tool for solving a number of control problems especially with the development of interior-point methods for solving semi-definite programming (SDP) problems. The basic idea of the LMI method in control is to cast the given problem as an optimization one with linear objectives and positive semi-definite constraints that involve symmetric matrices that are affine in the decision variables. In control theory, to borrow words from Doyle, *et al.*, (1991):

"LMIs play the same central role in the postmodern theory as Lyapunov function and Riccati equations played in the modern, and in turn various graphical techniques such as Bode, Nyquist and Nichols plots played in the classical".

Indeed the LMI-based approach constitutes the basis for a post-modern control theory which allows for robust and multi-criteria synthesis (see [Laurent & Niculescu; 2000]). This approach combines both the benefits of classical and modern control methodologies in terms of clear physical interpretation of design parameters and simplicity of numerical solutions with competing specifications.

It is, however, to the knowledge of the author that although LMI-based control methods have reached a certain degree of maturity there are still many control areas that are yet unexplored in this field. Although both theoretical grounds and efficient algorithms exist and leading to more and more industrial applications, it seems that lots of research efforts need to be placed on extending the theoretical framework to deal with time-varying multivariable linear and nonlinear systems which are contemporary starting to emerge.

APPENDICES

APPENDIX A

Some Mathematical Preliminaries

'A system is said to be deterministic when, given certain data e_1, e_2,..., e_n at times t_1, t_2,..., t_n respectively, concerning this system, if E_t is the state of the system at any time t, there is a functional relation of the form $E_t = f(e_1, t_1, e_2, t_2, ..., e_n, t_n, t)$. A system which is part of a deterministic system I shall call determined; one which is not part of any such system I shall call capricious'.

Bertrand Russell 'On the Notion of Cause'

A.0. Abstract

This appendix is a very brief review of some of the fundamental materials that were directly or indirectly used in this thesis. Such concepts are very briefly discussed herein; and for more elaborate proofs and details the reader is referred to standard textbooks (*e.g.* [Desoer & Vidyasagar, 1975; Feintuch, 1998; Francis, 1987; Huston & Pym, 1980; *etc.*]). It is also worth citing Çimen (2003) since some fitting material was directly extracted from his doctoral thesis appendices.

A.1. Linear Functional Analysis

A.1.1. Normed Vector Spaces

DEFINITION **A.1.** *A* ***metric space*** *is a pair* $(\aleph, d)$, *where* $\aleph$ *is a set and* d *is a real-valued metric (or distance) function on* $\aleph$, *that is, a function defined on* $\aleph \times \aleph$ *such that* $\forall x, y, z \in \aleph$:

(i) d *is real-valued, finite and non-negative,*

(ii) $d(x,y) = 0$ *iff* $x = y$,

(iii) $d(x,y) = d(y,x)$ *(symmetry),*

(iv) $d(x,y) \le d(x,z) + d(z,y)$ *(triangle inequality)*

where $d(x,y)$ is often referred to as "the distance between x and y".

Definition A.2. *A* ***linear space*** *(or* ***vector space****) over a field F (F is the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$) is a nonempty set of elements $x, y, \ldots$ (called vectors) together with two algebraic operations. These operations are called vector addition and multiplication of vectors by scalars, that is, by elements of F ($\mathbb{R}$ or $\mathbb{C}$).*

Definition A.3. *A* ***norm*** *over a real vector space $\aleph$ (of finite or infinite dimension) is any non-negative real-valued function $\|\bullet\| : \aleph \to \mathbb{R}_+$, which defines a metric d over $\aleph$, given by*

$$d(x,y) = \|x - y\|,\ x, y \in \aleph.$$

A ***normed space*** *is a vector space with a specified norm that is defined over it, and so all normed vector spaces are metric spaces. The normed space is denoted by $(\aleph, \|\bullet\|)$ or simply by $\aleph$.*

For any scalar $\alpha \in \mathbb{R}$ and elements x and y of the linear space $\aleph$ (vectors),

(i) $\|x\| \geq 0$ *with* $\|x\| = 0$ *iff* $x = 0$,

(ii) $\|\alpha x\| = |\alpha| \|x\|$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$,

where the quantity $\|x\|$ is called the norm of x, which is a measure of the size or length of the vector x, over the normed vector space $\aleph$.

Notice that inequality (iii) above is the **triangle inequality**. Hereafter, $\aleph$ is assumed to be a normed linear space. The norm induces a topology over the linear space, which is used to define continuity and convergence. Norms are also used in numerical analysis for establishing error bounds, and in sensitivity analysis for bounding sensitivities.

The above constraints are rather loose, and many possible norms can be defined for a particular linear space. If $\aleph = \mathbb{R}^n$, the p-norm of a vector $x = (x_1, \ldots, x_n)$ is defined as:

$$\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p \right)^{\frac{1}{p}}, \ x = (x_j) \in \mathbb{R}^n, 1 \le p < \infty.$$

And $\|x\|_\infty = \max_j |x_j|$. In particular when $p = 2$,

$$\|x\|_2 = \left(\sum_{j=1}^{n} x_j^2 \right)^{\frac{1}{2}}, \ x = (x_j) \in \mathbb{R}^n;$$

is called the **Euclidean norm.**

An $m \times n$ matrix A of real elements defines a linear mapping $y = Ax$ from $\mathbb{R}^n$ to $\mathbb{R}^m$. The $(p\text{-})$ norm of A is defined in terms of an associated vector norm by

$$\|A\| = \sup_{x \ne 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|.$$

$\|Ax\|$ is called the **induced norm** of the linear map A, which for $p = 2$ is given by

$$\|A\| = \left[\max \sigma\left(A^T A\right) \right]^{\frac{1}{2}};$$

where $\sigma\left(A^T A\right)$ is the maximum eigenvalue of $A^T A$. The important properties of the induced matrix norms are

(i) $\|Ax\| \le \|A\| \|x\|$,

(ii) $\|\alpha A\| \le |\alpha| \|A\|$,

(iii) $\|A + B\| \le \|A\| + \|B\|$,

(iv) $\|AB\| \le \|A\| \|B\|$.

DEFINITION A.4. *Suppose that the vectors* $x_0, x \in \aleph$ *and* r *is a number such that* $0 < r < \infty$. *Then the set of points*

$$B_r(x_0) = B(x_0, r) = \{ x \in \aleph \mid \|x - x_0\| < r \};$$

and

$$B_r(x_0) = B(x_0, r) = \{ x \in \aleph \mid \|x - x_0\| \le r \},$$

are called ***open*** *and* ***closed balls*** *respectively with centre* x_0 *and radius* r.

DEFINITION A.5. *A subset S of $\aleph$ is said to be* ***bounded*** *iff it is contained in some ball of finite radius. If S is bounded, its diameter is the diameter of the closed ball of the smallest radius containing S. The distance of a point x_0 from S is the number* $dist(x_0,S)=\inf_{x\in S}\|x-x_0\|^2$.

Therefore, for a linear operator A, if $\|A\|<\infty$ then A is **bounded**. On the other hand, if $\|A\|=\max_{\|x\|=1}\|Ax\|=\infty$, then A is said to be **unbounded**.

A.1.2. Banach Spaces

DEFINITION A.6. *A sequence $\{x_k\}_{k=1}^{\infty}\subset\aleph$* ***converges*** *to $x\in\aleph$ if*

$$\lim_{k\to\infty}\|x_k-x\|=0;$$

where x is called the limit of the sequence and may be written as

$$x_k\to x \text{ or } \lim_{k\to\infty}x_k=x.$$

The limit is unique, for if $x_k\to x$ and $x_k\to x'$, then by the triangle inequality

$$\|x-x'\|=\|x-x_k+x_k-x'\|\le\|x-x_k\|+\|x_k-x'\|,$$

and the right-hand side of the equality above tends to zero as $k\to\infty$, hence $x=x'$.

DEFINITION A.7. *A sequence $\{x_k\}_{k=1}^{\infty}\subset\aleph$ is called a* ***Cauchy sequence*** *iff*

$$\lim_{k,l\to\infty}\|x_k-x_l\|=0,$$

that is, provided for each $\varepsilon>0$ there exists $N=N(\varepsilon)>0$ (depending on ε) such that

$$d(x_k,x_l)=\|x_k,x_l\|<\varepsilon,\ \forall k,l\ge N.$$

It is an obvious consequence of the inequality

$$\|x_k-x_l\|=\|x_k-x+x-x_l\|\le\|x_k-x\|+\|x_l-x\|.$$

If $\{x_k\}_{k=1}^{\infty}$ is convergent then it is Cauchy. Every convergent sequence is a Cauchy sequence but not vice-versa.

Definition A.8. *The normed vector space* $\aleph$ *is said to be* ***complete*** *if every Cauchy sequence in* $\aleph$ *converges, i.e. whenever* $\{x_k\}_{k=1}^{\infty}$ *is a Cauchy sequence, there exists* $x \in \aleph$ *such that* $\{x_k\}_{k=1}^{\infty}$ *has limit that converges to* x. *The Euclidean space* $\mathbb{R}^n$, *for example, with the Euclidean distance function is complete.*

Definition A.9. *A Banach space* $\aleph$ *is a complete, normed linear space. Every sequence* $\{x_k\}_{k=1}^{\infty}$ of $\aleph$ converges strongly to a point x *of* $\aleph$*:*

$$\lim_{k\to\infty}\|x_k - x\| = 0,$$

the limit of x *if it exists is uniquely determined following the triangle inequality*

$$\|x - x'\| \le \|x - x_k\| + \|x_k - x'\|.$$

The simplest Banach spaces are $\|x_k - x_l\| = \|x_k - x + x - x_l\| \le \|x_k - x\| + \|x_l - x\|$ *and, with any norm.*

A.1.3. Hilbert Spaces

Definition A.10. *Let* $\aleph$ *be a linear space over the field* $\mathbb{C}$ *of complex numbers. An* ***inner product*** *on* $\aleph$ *is a function* $(x,y) \to \langle x,y\rangle$ *from* $\aleph\times\aleph$ *to* $\mathbb{C}$, *and having the four properties:*

(i) $\langle x,x\rangle$ *is real and* ≥ 0,

(ii) $\langle x,x\rangle = 0$ *iff* $x = 0$,

(iii) *the function* $y \to \langle x,y\rangle$ *from* $\aleph$ *to* $\mathbb{C}$ *is linear,*

(iv) $\overline{\langle x,y\rangle} = \langle x,y\rangle$.

Such an inner product on $\aleph$ *induces a norm, namely,* $\|x\| := \langle x,x\rangle^{1/2}$. *With respect to this norm* $\aleph$ *may or may not be complete. A (complex) Hilbert space is a linear space over* $\mathbb{C}$ *which has an inner product and is complete. A mapping from one Hilbert space to another is called a* ***Hilbert space isomorphism***.

Note that a real symmetric matrix M is **positive definite** if $\langle x, Mx \rangle > 0$ for all $x \neq 0$ and M is **positive semi-definite** if $\langle x, Mx \rangle \geq 0$ for all $x \neq 0$.

DEFINITION A.11. *Let a signal $x(t)$ be defined for all time $-\infty < t < \infty$, and taking values in C^n. Then x is a function $(-\infty, \infty) \to C^n$; and by restricting x to be square **(Lebesgue)** integrable:*

$$\int_{-\infty}^{\infty} \|x(t)\|^2 \, dt < \infty;$$

*the set of all such signals is the **Lebesgue space** $L_2(-\infty, \infty)$ which equals to zero for almost all $t < 0$ is a closed subspace, denoted $L_2[0, \infty)$. Its orthogonal complement (zero for almost all $t > 0$) is denoted $L_2(-\infty, 0]$.*

A.2. Differential Equations

A.2.1. Solution of Differential Equations

DEFINITION A.12. (Vector-Valued Functions). *Let Ω be a subset of $\mathbb{R}^n$, and suppose that f is a complex-valued function defined over Ω. Then f is said to be **continuous** at the point $x_0 \in \Omega$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $x \in \Omega$ and $\|x - x_0\| < \delta$, and for each $\{x_k\}_{k=1}^{\infty}$ in Ω with limit x_0, $\lim_{k \to \infty} f(x_k) = f(x_0)$. f is said to be **absolutely continuous** iff it is continuous at every point in Ω. f is said to be **uniformly continuous** in Ω iff for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $x, x_0 \in \Omega$ and $\|x - x_0\| < \delta$, i.e. ε and δ are independent of x_0. f is said to be **piecewise continuous** in Ω if for every bounded subinterval $\Omega_0 \subset \Omega$, f is continuous for all $x \in \Omega_0$ except, possibly, at a finite number of points where f may have discontinuities.*

Linear differential equations are often written in the form

$$\dot{x} = Ax + Bu\,. \tag{A.2. 1}$$

The solution to the homogeneous portion of (A.2.1), that is Ax, is found by assuming a solution of the form

$$x(t) = e^{At}x(0) \qquad \text{for } t \geq 0;$$

alternatively, the convolution integral $y(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$, can be used to write the total solution as:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}x(t_0)\,Bu(\tau)d\tau\,.$$

Note that the system output equation (with a Laplace transform $Y(s) = G(s)U(s)$) can be written as:

$$y = C^T x = C^T e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} C^T e^{A(t-\tau)}x(t_0)\,Bu(\tau)d\tau\,,$$

this expression in fact represents an addition of the transient response $C^T e^{A(t-t_0)}x(t_0)$, and the steady-state response $\int_{t_0}^{t} C^T e^{A(t-\tau)}x(t_0)\,Bu(\tau)d\tau$, of the given system. Hence for zero initial conditions the first term cancels out.

A.2.2. Lipschitz Condition

The importance of absolutely continuous function (see Definition A.10) lies in the fact that, for example, if $f:[a,b] \to \mathbb{R}$ is absolutely continuous, then its derivative, $\dot{f}(t)$, exists and is finite almost everywhere.

A general set of nonlinear state-equations can be expressed as follows:

$$\dot{x}(t) = f\big(x(t), u(t), t\big); \tag{A.2. 2}$$

where $x(t)$ is the state-vector, $u(t)$ is the input vector, and $f\big(x(t), u(t), t\big)$ denotes a nonlinear function involving the state variables, the inputs, and time, t. The solution, $x(t)$ of (A.2.2) with the initial condition, $x(t_0) = x_0$ may not always exist. The existence of solution of nonlinear differential equations requires that the nonlinear function, $f\left(x(t),u(t),t\right)$, should be defined and *continuous* for all *finite* times, $t \geq t_0$. Also, it is required that $f\left(x(t),u(t),t\right)$ must satisfy the following condition, known as the ***Lipschitz continuity*** or ***condition***.

$$\left|f\left(x(t),u(t),t\right)-f\left(x^*(t),u(t),t\right)\right| \leq \kappa\left|x(t)-x^*(t)\right|, \tag{A.2. 3}$$

where $x^*(t)$ is a vector different from $x(t)$, κ is a constant, and $|\ \Xi\ |$ denotes a vector consisting of the absolute value of each element of the vector Ξ.

In fact, for a mathematical model to predict the future state of a given system from its current state at time t_0, the initial value problem of (A.2.2) must have a unique solution; raising the question of uniqueness of differential equations' solutions and which is resolved through the Lipschitz condition. The function definition in (A.2.3) can be written as

$$\left\|f(x_1,t)-f(x_2,t)\right\| \leq L\left\|x_1-x_2\right\|, \tag{A.2. 4}$$

for all (x_1,t) and (x_2,t) in some neighbourhood of (x_0,t_0) is said to be **Lipschitz** in x, and the positive constant L is called a **Lipschitz constant**. In one dimension, a function which satisfies a Lipschitz condition is absolutely continuous, and hence, differentiable almost everywhere (but not necessarily). The words **locally Lipschitz** and **globally Lipschitz** are also used to indicate the domain over which the Lipschitz condition holds. A function $f(x,t)$ is said to be **locally Lipschitz** in x on a domain (open and connected set) $D\times[a,b]\subset\mathbb{R}^n\times\mathbb{R}$ if each point $x\in D$ has a neighbourhood D_0 such that f satisfies the Lipschitz condition (A.2.4) for all points on $D_0\times[a,b]$ with the same Lipschitz constant L_0. Then $f(x,t)$ is said to be locally Lipschitz in x on $D\times[t_0,\infty)$ if it is locally Lipschitz in x on $D\times[a,b]$ for every compact interval $[a,b]\subset[t_0,\infty)$. A function $f(x,t)$ is Lipschitz in x on $W\times[a,b]$ if it satisfies (A.2.4) $\forall t\in[a,b]$ and all points in the set W, with the same Lipschitz constant L. A function $f(x,t)$ is said to be **globally Lipschitz** if it is Lipschitz on $\mathbb{R}^n$.

A.2.3. The Fundamental (Transition) Matrix

Consider the linear homogeneous differential equation

$$\dot{x}(t) = A(t)x(t), \qquad x(t_0) = x_0; \tag{A.2. 5}$$

where $x(t) \in \mathbb{R}^n$ and $A(t)$ is a continuous matrix-valued function. It is easily shown that (A.2.5) has a unique solution of the form

$$x(t) = \Phi(t,t_0)x_0;$$

where $\Phi(t,t_0)$ is known as the **transition** or **fundamental matrix** of the equation, which has the following properties

(i) $\Phi(t,t_0) = I$,

(ii) $\Phi(t,t_1)\Phi(t_1,t_0) = \Phi(t,t_0), \qquad \forall t,t_1,$

(iii) $\Phi(t,t_0)$ is invertible (i.e. nonsingular) $\forall t,t_1$; or $\Phi^{-1}(t,t_0) = \Phi(t_0,t)$,

(iv) $\frac{d\Phi}{dt}(t,t_0) = \dot{\Phi}(t,t_0) = A(t)\Phi(t,t_0)$.

If A is LTI or constant as in the case of autonomous systems given by $\dot{x}(t) = Ax(t)$, then the transition matrix takes the form

$$\Phi(t,t_0) = \exp\left[A(t-t_0)\right]$$

and the solution becomes

$$x(t) = \exp\left[A(t-t_0)\right]x_0.$$

As for the inhomogeneous equation:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad x(t_0) = x_0,$$

with a forcing term, $u(t)$, it has a solution given by **the variation of constants formula**

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t}\Phi(t,s)B(s)u(s)ds,$$

Hence the solution, $x(t)$, is the summation of the homogenous part of the system $\Phi(t,t_0)x_0$ and the forcing term $\Phi(t,t_0)\int_{t_0}^{t}\Phi^{-1}(s,t_0)B(s)u(s)ds$.

A.3. Inequalities

A.3.1. Gronwall-Bellman Inequality

The following lemma provides the general form of the Gronwall-Bellman inequality, which is extracted from Desoer & Vidyasagar (1975).

LEMMA A.1. *Let* $\lambda:\mathbb{R}_+\to\mathbb{R}$ *be continuous (locally integrable, that is,* λ *is integrable over any bounded interval such as* $[t_0,t]$ *with* $0\le t_0\le t<\infty$, $g,\mu:\mathbb{R}_+\to\mathbb{R}$ *be continuous and non-negative, and* $g\mu$ *be locally integrable over* $\mathbb{R}_+$. *Under these conditions, if a continuous function* $f:\mathbb{R}_+\to\mathbb{R}$ *satisfies*

$$f(t)\le\lambda(t)+g(t)\int_{t_0}^{t}\mu(s)f(s)ds,\ [t_0,t]\in\mathbb{R}_+;$$

then over the same time interval, $[t_0,t]\in\mathbb{R}_+$,

$$f(t)\le\lambda(t)+g(t)\int_{t_0}^{t}\mu(s)\lambda(s)\exp\left[\int_{s}^{t}\mu(\tau)g(\tau)d\tau\right]ds,\ [t_0,t]\in\mathbb{R}_+.$$

A special case of this inequality is reached if $\lambda(t)\equiv\lambda$ *and* $g(t)=0$,

$$f(t)\le\lambda+\int_{t_0}^{t}\mu(s)f(s)ds,\ \forall t\ge t_0.$$

Then

$$f(t)\le\lambda\exp\left[\int_{t_0}^{t}\mu(\tau)d\tau\right],\ \forall t\ge t_0.$$

If in addition, $\mu(t)\equiv\mu\ge 0$ *is a constant, then*

$$f(t)\le\lambda\exp\left[\mu(t-t_0)\right].$$

A.4. Partial Differential Equations

A partial differential equation (PDE) is an equation that involves an unknown function of two or more variables and some of its partial derivatives.

DEFINITION A.13. *If $k \geq 1$ is a fixed integer and Ω denotes an open subset of $\mathbb{R}^n$, then the following expression*

$$F\left(D^k v(x), D^{k-1} v(x), \ldots, Dv(x), v(x), x\right) = 0, \; \left(x \in \Omega\right), \qquad \text{(A.3. 1)}$$

is called a k^{th}-order ***partial differential equation****, where*

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R};$$

with the unknown being

$$v : \Omega \to \mathbb{R}.$$

DEFINITION A.14. *The PDE (A.3.1) is called* ***linear*** *if it has the form*

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha v(x) = g(x);$$

for the given functions a_α $\left(|\alpha| \leq k\right)$, g. The PDE is said to be homogeneous if $g = 0$.

DEFINITION A.15. *The PDE (A.3.1) is called* ***semilinear*** *if it has the form*

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha v(x) + a_0 \left(D^{k-1} v(x), \ldots, Dv(x), v(x), x\right) = 0.$$

DEFINITION A.16. *The PDE (A.3.1) is called* ***quasilinear*** *if it has the form*

$$\sum_{|\alpha| \leq k} a_\alpha \left(D^{k-1} v(x), \ldots, Dv(x), v(x), x\right) D^\alpha v(x) + a_0 \left(D^{k-1} v(x), \ldots, Dv(x), v(x), x\right) = 0$$

DEFINITION A.17. *The PDE (A.3.1) is called fully* ***nonlinear*** *if it depends nonlinearily upon the highest order derivatives.*

APPENDIX B

Modelling Nonlinear Finite-Dimensional Systems by Linear PDEs

B.1. Abstract

In this appendix, a novel concept is introduced for representing nonlinear Ordinary Differential Equations (ODEs) by linear Partial Differential Equations (PDEs). Indeed the literature embraces various techniques for treating nonlinear ODEs (see for example [Banks & Moser, 1993; Grimshaw, 1990; and Smith & Jordan, 1999]); however, this technique is straight-forward and only requires mild conditions such as analyticity.

B.2. Introduction

The solutions of nonlinear ODEs are modelled by 'sections' of the solutions of linear PDEs with the hope of applying the known classical theory of control of linear PDEs to the control of nonlinear finite-dimensional systems. This is done by an appropriate choice of the spectrum of the linear PDE. In §B.3 the basic idea for approximating linear PDEs is given. Proceeding to §B.4, the essential theory behind this appendix is covered where linear PDEs are related to nonlinear ODEs. Finally, in §B.5 a simple example is presented to clarify the stated concepts; and then some closing remarks are given in §B.6.

B.3. Linear PDEs

Consider the linear partial differential equation

$$\frac{\partial \phi(\zeta,t)}{\partial t} = A\phi(\zeta,t), \tag{B. 1}$$

where

$\zeta \in \Omega \subseteq R^n$, and $\phi(\zeta,0)=\phi_0(\zeta)\in L^2$.

With the operator A being a strongly elliptic operator on Ω with some boundary conditions such that A has a spectrum $\{\lambda_n\}$ and a corresponding complete set of Eigen Functions $\{\psi_n(\zeta)\}$, then by theorem (see for instance [Grimshaw, 1990]) the solution is given by

$$\phi(\zeta,t)=\sum_n c_n \psi_n(\xi)\, e^{-\lambda_n t}, \tag{B. 2}$$

with

$$c_n=\langle \phi_0, \psi_n \rangle. \tag{B. 3}$$

More specifically, the well-known dissipative parabolic heat equation can be chosen for illustrative purposes,

$$\frac{\partial \phi(\zeta,t)}{\partial t}=\frac{\partial^2 \phi(\zeta,t)}{\partial \zeta^2}. \tag{B. 4}$$

It is recognized that the solution of (B.4) can be written in this form:

$$\phi(\zeta,t)=\wp(\zeta)\vartheta(t). \tag{B. 5}$$

So by substituting (B.5) into (B.4),

$$\dot{\vartheta}(t)\wp(\zeta)=\vartheta(t)\ddot{\wp}(\zeta). \tag{B. 6}$$

By re-arranging (B.6),

$$\frac{\dot{\vartheta}(t)}{\vartheta(t)}=\frac{\ddot{\wp}(\zeta)}{\wp(\zeta)}=\mu=\text{constant}. \tag{B. 7}$$

It then follows directly that

$$\dot{\vartheta}(t)=-\mu\, \vartheta(t), \tag{B. 8}$$

and

$$\ddot{\wp}(\zeta)=-\mu\, \wp(\zeta). \tag{B. 9}$$

The solutions of (B.8) and (B.9) can respectively be expressed as,

$$\vartheta(t)=-e^{-\mu t}, \tag{B. 10}$$

and

$$\wp(\zeta)=A\cos\zeta\sqrt{\mu}+B\sin\zeta\sqrt{\mu}. \tag{B. 11}$$

It can then be easily shown that for the boundary curve C of rectangular solution domain defined by zero boundary conditions, *i.e.* **Dirichlet boundary conditions** (see, *e.g.*, [Schwartz, 1959]), the spectrum is:

$$\mu = n^2\pi^2; \qquad (n = 1,\ 2,\ldots). \tag{B. 12}$$

And the Eigenvalues become:

$$\lambda = \begin{pmatrix} -\pi^2 & 0 & 0 & . & . & . & . & 0 \\ 0 & -4\pi^2 & & & & & & 0 \\ 0 & & -9\pi^2 & & & & & . \\ . & & & . & & & & . \\ . & & & & . & & & . \\ . & & & & & . & & . \\ . & & & & & & . & . \\ 0 & . & . & . & . & . & . & -n^2\pi^2 \end{pmatrix} \tag{B. 13}$$

In general, it was shown that the solution of any PDE can be approximated by an infinite series (Schwartz, 1959) as

$$F(t) = \sum \alpha\, e^{-\lambda t}, \tag{B. 14}$$

which converges to the PDE's solution provided that

$$\lim_{\kappa\to\infty} \sum_{\lambda=1}^{\kappa} \frac{1}{\lambda} = \infty. \tag{B. 15}$$

The solution of the heat equation, can also be expressed by means of (B.14) as,

$$\phi(\zeta,t) = \sum \left(B_n \sin n\pi\,\zeta\right) e^{-n^2\pi^2 t}, \tag{B. 16}$$

subject to the initial condition

$$\phi(\zeta,0) = \phi_0 = \sum \left(B_n \sin n\pi\,\zeta\right); \tag{B. 17}$$

where B_n has been deliberately written in this form to differentiate it from B.

But condition (B.15) when applied to (B.16), as,

$$\lim_{\kappa\to\infty} \sum_{\mu=1}^{\kappa} \frac{1}{\mu} = \lim_{\kappa\to\infty} \sum_{n=1}^{\kappa} \frac{1}{n^2\pi^2} = 2, \tag{B. 18}$$

converges to 2.

However, by theorem, (B.18) can be made to diverge

IFF: $\sum \frac{1}{n^s}$ for $s < 2$ (see [Friedman, 1962]).

But for simplicity, (B.16) can be tailored to satisfy the above stated condition to approximate the parabolic PDE in (B.4). That is, at a specific point ζ inside the boundary curve C, one set of plausible solutions to (B.16) can be shown to be

$$\phi(\zeta,t)=\sum\left(B_n \sin\sqrt{\lambda_n}\ \zeta\right) e^{-\lambda_n t}. \qquad \text{(B. 19)}$$

Alternatively, (B.19) can expressed as

$$\phi(\zeta,t)=\sum \beta_n e^{-\lambda_n t}, \qquad \text{(B. 20)}$$

with

$$\beta_n = \frac{B_n}{\sin\sqrt{\lambda_n}\ \zeta}. \qquad \text{(B. 21)}$$

And in that case

$$\lim_{\kappa\to\infty} \sum_{\lambda=1}^{\kappa} \frac{1}{\lambda^2} = \lim_{\kappa\to\infty} \sum_{n=1}^{\kappa} \frac{1}{n\pi} = \infty; \qquad \text{(B. 22)}$$

where $\lambda = \sqrt{\lambda_n}$.

Therefore, the equality in (B.20), the initial condition at time $t = 0$ is

$$\phi(\zeta,0)=\sum \beta_n. \qquad \text{(B. 23)}$$

B.4. Nonlinear ODEs

Consider an analytic nonlinear ODE:

$$\dot{x} = f(x), \qquad \text{(B. 24)}$$

subject to an initial condition

$$x(0) = x_0; \qquad \text{(B. 25)}$$

the solution of (B.24) can be written in terms of the Lie series (Banks, 1988) as

$$x(t;x_0) = e^{t f(x)\frac{\partial}{\partial x}} x \Big|_{x=x_0} \qquad \text{(B. 26)}$$

This solution in (B.26) can be expanded by Taylor formula (Friedman, 1962) as

$$x(t;x_0)=\sum_{i=0}^{\infty}\frac{t^i}{i!}\left(f(x)\frac{\partial}{\partial x}\right)^i x\Bigg|_{x=x_0} \tag{B. 27}$$

Subsequently, $x(t)$ can be written in terms of the spectrum $\left(e^{-\lambda_n t}\right)$ for appropriate choice of λ_n; and the formal expansion of (B.26) abridges to

$$x(t;x_0)=\sum \gamma_n\, e^{-\lambda_n t}. \tag{B. 28}$$

B.5. Relating Nonlinear ODEs with Linear PDEs

At this instant, the coefficients of the linear PDE in (B.20) can be solved for by equating them to those of the nonlinear ODE in (B.28), as follows,

$$\beta_n = \gamma_n; \qquad \text{for } (n=1,\ 2,\ldots), \tag{B. 29}$$

where B_n can now be determined from Equations (B.20) & (B.21); which then specifies the initial function $\phi(\zeta,0)$ of the PDE, via the initial condition in (B.17).

B.6. Example

Consider an analytic nonlinear ODE of the general form

$$\dot{x}=f(x), \qquad \text{with } \ x\,(0)=x_0, \tag{B. 30}$$

where, say (see for example [Grimshaw, 1990]),

$$f(x,t)=tx^3, \tag{B. 31}$$

is a continuous function for all x and t, and satisfies a Lipschitz condition in any bounded region.

The solution of (B.31) is known to take the following form (see [Smith and Jordan, 1999]):

$$x=\frac{x_0}{\sqrt{\left(1-x_0^2\, t^2\right)}}; \tag{B. 32}$$

and is only defined for

$$|\, t\,|<x_0^{-1}. \tag{B. 33}$$

But alternatively (refer to Equation B.28), it was shown that all such solutions in (B.32) are periodic and can be expressed by

$$x(t;x_0) = \sum \gamma_n e^{-\lambda_n t}, \tag{B. 34}$$

where $\lambda_n = n\pi$.

Setting

$$s = e^{-\pi t}; \tag{B. 35}$$

or equivalently, $t = -\frac{\ln s}{\pi}$, then (B.28) can be considered as a polynomial expansion of the form

$$x(t;x_0) = \sum \gamma_n s^n; \tag{B. 36}$$

and at a point $x = \zeta$, (B.36) is

$$x\left(-\frac{\ln s}{\pi}, x_0\right) = \sum_{i=1}^{n} \gamma_i s^i \cong \gamma_1 s + \gamma_2 s^2 + \cdots + \gamma_n s^n. \tag{B. 37}$$

As a result, the γ_n coefficients can be obtained from the polynomial expansion in (B.36). Over a specified time interval $t \in \left[0, x_0^{-1}\right)$, the solution of the nonlinear ODE can always be model by the linear PDE. So, recalling that the target is to find the linear PDE's coefficients, then from (B.28), B_n can be solved for,

$$\beta_n = \gamma_n, \qquad \text{for } (n = 1,\ 2, \ldots); \tag{B. 38}$$

where

$$\beta_n = \frac{B_n}{\sin\sqrt{\lambda_n}\ \zeta}. \tag{B. 39}$$

Hence the necessary initial condition for the heat equation to model the system is

$$\phi(\zeta, 0) = \phi_0 = \sum \left(B_n \sin n\pi\ \zeta\right). \tag{B. 40}$$

Note that:

ζ must be chosen so that $\sin\sqrt{\lambda_n}\ \zeta \neq 0$, for all n.

B.7. Concluding Remarks

In this appendix, a simple technique is given to solve nonlinear ODEs by a linear parabolic PDE model. The enclosed example is indeed one-dimensional, but it is easy to generalize the theory to higher-dimensional systems. It would also be of interest to study the regions of validity for the proposed approximations. Future research could be to devise controllers for such a class of systems making use of classical linear control theories for PDEs.

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